CHARACTERISTICS OF UNSKILLED, SKILLED AND HIGHLY SKILLED MENTAL CALCULATORS

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ABSTRACT

This study was concerned with the identification of the processes and procedures which characterized unskilled, skilled, and highly skilled mental calculation performance of high-school students. Fifteen skilled and fifteen unskilled mental calculators were selected from 280 senior secondary mathematics students as a result of their performances on a mental multiplication test. One highly skilled 13-year-old was later added to the sample. These three skill groups were compared on a number of dimensions including the strategies used to determine the mental products of multi-digit factors, ability to recall numerical equivalents, and short-term memory capacity.

The study identified twelve mental calculation strategies used by these students. The majority of unskilled subjects made extensive use of strategies more suited to written than mental calculation tasks. The more proficient subjects tended to employ strategies based upon properties suggested by the factors.

There were statistically significant differences between the mean performance of each group on a multiplication basic fact recall test. A statistically significant but weak linear relationship between recall and mental multiplication performance existed. The proficient subjects retrieved significantly more large numerical equivalents during the solution of a mental calculation than did the unskilled subjects. The highly-skilled subject could recall quickly the majority of 2-digit squares and many 3-digit squares.

Statistically significant differences between the performances of the skilled and unskilled groups on four measures of short-term
memory capacity were found. Further analysis indicated that only a weak linear relationship between capacity and mental calculation performance existed. Certain aspects of the initially stated calculative task and the interim calculations were found to be particularly susceptible to forgetting by mental calculators.

These suggestions for further research were made: studying the characteristics of subjects who differ in the ability to determine mental sums, differences, and quotients; evaluating the short-term memory demands imposed by different mental calculation strategies; determining which strategies can be used to improve the mental calculation abilities of lesser skilled subjects.
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CHAPTER I

THE PROBLEM

Introduction to the Problem

In the everyday world of the consumer and worker, the need for a mental calculation will often arise. A series of recent British studies reviewed by the Committee of Inquiry into the Teaching of Mathematics in Schools (Cockcroft, 1982) concluded that many adults cannot meet this need because they are unable to perform even the most elementary of mental calculations. The reviewers claimed that:

There are indeed many adults in Britain who have the greatest difficulty with even such apparently simple matters as adding up money, checking their change in shops or working out the cost of five gallons of petrol. Yet these adults are not just the unintelligent or the uneducated. They come from many walks of life and some are very highly educated indeed, but they are hopeless at arithmetic...(Cockcroft, p. 5)

The difficulties of adults in performing mental calculations have been reported by the National Assessment of Educational Progress (NAEP, 1977, 1983a, 1983b). Asked to multiply 90 and 70 "in the head," 45% of 17-year-olds were unable to do so (NAEP, 1983b, p. 2). As well, 55% of the 17-year-olds were unable to calculate mentally 4 x 625 (NAEP, 1983a, p. 32). On orally presented mental division exercises, almost 40% of 17-year-olds were unable to determine the solution to either of the items, 480/16 or 3500/35, within a 10 second time period (NAEP, 1983a, p. 11). Over 60% of the 17-year-olds could not select the correct estimate to 3.04 x 5.3: the choices were 1.6, 16, 160, and 1600 (NAEP, 1983b, p. 3).

As in other forms of human reasoning, the ability to determine
a mental calculation varies greatly among individuals. Some people are so adept at calculating mentally that they have been described as "lightning calculators." A.C. Aitken (Hunter, 1962), Bidder (1856) and Dase (Scripture, 1891) were a few such individuals. The following account of Aitken's mental decimalization of 1/851 illustrates the magnitude of some expert's powers:

The instant observation was that 851 is 23 times 37. I use this fact as follows. 1/37 is 0.027027027027... and so on repeated. This I divide mentally by 23. (23 into 0.027 is 0.001 with remainder 4). In a flash I can see that 23 into 4027 is 175 with remainder 2, and into 2027 is 88 with remainder 3, and into 3027 is 131 with remainder 14, and even into 14,027 is 609 with remainder 20. And so on like that. Also before I ever start this, I know how far it is necessary to go in this manner before reaching the end of the recurring period: for 1/37 recurs at three places, 1/23 recurs at twenty-two places, the lowest common multiple of 3 and 22 is 66, whence I know that there is a recurring period of 66 places. (Hunter, 1962, p. 245)

Such a demonstration of recondite knowledge and reasoning must appear particularly freakish in this age of dime-store calculators.

Although certainly not approaching the performances of the expert mental calculators, some individuals do seem to be quite skilled. In an unpublished study conducted by the writer (Hope, Note 1), several young adults could determine the mental products of such calculative tasks as 9 x 742, 25 x 25, and 25 x 48.

Why are there such apparent individual differences in mental calculation performance? Researchers such as Hunter (1962, 1978, 1979), Hitch (1977, 1978), and Howe and Ceci (1979) have argued that how individuals make use of their memory during a mental calculation will be an important factor in determining their level of proficiency. According to Hunter, a mental calculation makes demands on memory that are of these three distinguishable kinds: memory for calculative
method, memory for numerical equivalents, memory for interrupted working.

The first two demands are met mainly by calling on one's long-term memory to provide both calculative strategies and a set of useful numerical equivalents. Calculative method or strategy is a "schematic procedure which decomposes your working into a pre-arranged sequence of steps" (Hunter, 1978, p. 340). Numerical equivalents are the basic constituent parts of the calculation. The numerical equivalents commonly retrieved by the average person to solve a calculation will be the basic facts of addition, subtraction, multiplication and division.

The third demand is met by some form of temporary storage or "working memory" (Hitch, 1977, 1978; Howe & Ceci, 1979). In performing a calculation, a subject must find a way to record each step of the calculation as well as to keep track of the calculative route. This demand on working memory is virtually eliminated for written calculations because the subject can employ ciphers to keep track of the current and past states of the calculation. However, for mental calculations, the demand must be met by the calculator's own internal working memory resources.

Thus, individual differences in one or more aspects of these memory demands could likely be the reason for the great variation in mental calculation performance. What remains to be done by researchers is to particularize these individual differences in memory demands. Do individuals who differ in mental calculation performance use the same or different calculative strategies? Do they retrieve the same or different numerical equivalents during the solution of a mental
calculation task? Do they possess the same or different capacities to store and process temporary calculative information? The purpose of the present study was to study individuals who differed in mental calculation performance in order to provide some answers to these questions.

Statement of the Problem

The study attempted to identify the processes and procedures which characterized unskilled, skilled, and highly skilled mental calculation performance during the solution of calculation tasks involving multi-digit factors. Specifically, the following research questions were used to guide the investigation:

Questions About Calculative Strategies

1. Can individuals who differ in mental calculation performance be characterized by the types of calculative strategies used to solve a task? In particular,

   1-1. Which strategies are most frequently applied by each of the skill groups?

   1-2. Which skill group most frequently changes from one strategy to another in response to a change in the calculative task?

   1-3. Is there a relationship between the type of calculation task and the type of strategy selected to solve the task?

   1-4. Can any general characteristics be identified that appear to distinguish efficient from inefficient strategies?
Questions About Retrieval of Numerical Equivalents

2. Can individuals who differ in mental calculation performance be characterized by the types of numerical equivalents retrieved to solve a mental calculation task? In particular,

2-1. Do the skill groups differ in their ability to quickly and accurately recall the basic facts of multiplication?

2-2. Do skilled mental calculators retrieve a greater number of large numerical equivalents to solve mental multiplication tasks than do unskilled mental calculators?

Questions About Short-Term Memory Processes

3. Can individuals who differ in mental calculation performance be characterized by the efficiency of their short-term memory systems? In particular,

3-1. Are there differences among the skill groups on measures of short-term memory capacity?

3-2. Do skilled mental calculators employ different memory devices than do unskilled mental calculators to minimize the forgetting of the initial or interim calculations?

3-3. What is the effect on performance when mental rather than written methods must be used to solve computational tasks?

3-4. Which stages of a mental calculation are particularly susceptible to forgetting.
Definition of Terms

Mental calculation refers to the cognitive processes required to solve numerical calculation tasks without benefit of any external memory-aids, including pencil and paper. For the purposes of the study, mental multiplication was the process that was investigated.

Skilled, unskilled, and highly skilled mental calculators refer to those individuals who exhibited extremes of performance in a test of mental multiplication. The tests and procedures used to select these subjects are described in Chapter III.

Calculative strategy refers to the type of procedure used to decompose a task into a series of more tractable calculations. Several strategies that may have been used by expert mental calculators have been identified through a thorough review of the literature. These are discussed in Chapter II. The strategies used by the subjects who participated in the study are presented in Chapters III and IV.

Retrieval of a numerical equivalent describes the numerical information that a subject retrieves, but does not calculate, during the process of solving a calculation task.

Short-term memory (STM) refers to a hypothetical system whose function is to process information held for temporary periods. This memory system has been referred to as working memory, primary memory, immediate memory, and memory for interrupted working.

Long-term memory refers to a hypothetical system whose function is to store information in a more permanent form than short-term memory. This system has been referred to as secondary memory.
Processing capacity refers to the supposedly limited capability of the short-term memory system to process and retain incoming information. Researchers have used the terms structural resources and short-term memory capacity to describe processing capacity.

Information-processing refers to the models of memory which attempt to analyze the flow of information within the human organism. The digital computer has been used as an analogy to describe this approach and descriptions of the processing of information through registers, loops, memory stores, and retrieval routines often accompany such models. The limited capacity of short-term memory to process information has been an important feature of most information-processing theories.

Discussion of the Problem

Hunter's (1962, 1978, 1979) analyses of expert mental calculators has led him to the conclusion that the type of calculative strategy a subject applies to a mental calculation will be a major determiner of performance. He has argued that the more proficient mental calculators are able to select and apply an "efficient" strategy which minimizes their reliance on temporary memory. Thus, individual differences in mental calculation performance should reflect, according to Hunter's argument, differences in the choice of strategy: the skilled mental calculator selecting more efficient strategies than the unskilled mental calculator.

Evidence in support of Hunter's argument has been sparse. His conclusions have been based almost entirely upon his investigations of A.C. Aitken's extraordinary calculative performances. The subject was
so atypical that any generalizations about lesser skilled calculative performance will be, at best, speculative. Moreover, Hunter made no attempt to specify these strategies. Aitken's methods were described in general terms such as "unconventional" (Hunter, 1978) or "novel" (Hunter, 1962).

To help guide the present investigation, a tentative list of strategies was developed by the researcher through examining the accounts of expert mental calculation provided by Hunter (1962, 1977, 1978), Scripture (1891), Mitchell (1907), Ball (1956), Jakobsson (1944), Bidder (1856), Gardner (1977), and Smith (1983). Since this list of strategies was based upon a highly unrepresentative sample of mental calculators, it was not clear which strategies, if any, could be used to discriminate between the unskilled, skilled, and highly skilled mental calculators selected to participate in the study.

Other researchers such as Levine (1982) and Maier (1977) have examined the calculative strategies of less atypical subjects. They have reported that unskilled mental calculators could be characterized by their use of a mental analogue of the conventional pencil-and-paper algorithm. Levine's evidence should be considered as only suggestive because estimation and not mental multiplication was the focus of her study. Maier, on the other hand, has provided no empirical evidence to support his claims. Thus, no study has identified the methods of solution used by subjects who have exhibited differing levels of mental calculation performance.

Hunter has presented convincing evidence that expert calculators can retrieve large numerical equivalents to aid in a mental calculation: for example, A. C. Aitken, the expert calculator
studied by Hunter, could retrieve instantly such equivalents as $123 \times 45 = 5535$ without appearing to perform any intermediate calculations (1962, p. 249). Retrieval of large numerical equivalents by expert calculators has been reported by Jakobsson (1944), Scripture (1891), Gardner (1977), and Ball (1956). Such evidence can be considered only suggestive because what is true for experts might not necessarily be true for skilled but non-expert subjects.

The great contrast in performance between calculations done by pencil-and-paper and mental methods has led some researchers to conclude that proficient mental calculation makes great demands on the limited resources of short-term memory. Evidence provided by Howe and Ceci (1979), Hitch (1977, 1978, 1980), and Merkel and Hall (1982) has demonstrated that mental calculation does make great demands on short-term memory.

Assuming that STM processes are a factor in mental calculation performance, it follows that individual differences in mental calculation could reflect the differing processing capacities that each individual brings to bear upon the calculative task. Thus, those subjects who are skilled should possess greater STM capacities than those subjects who are unskilled mental calculators. No researchers have investigated the relationship between mental multiplication performance and processing capacity.
Significance of the Study

In his 1958 short story *The Feeling of Power*, Isaac Asimov described a future society in which even the most elementary calculations were solved by a computer. People had become so dependent upon machines that calculation was a virtually forgotten science. A low-grade Technician, Aub, startled this futuristic scientific world by managing to reconstruct the ancient methods of calculation. Such a discovery did not go unnoticed by a select group of politicians and scientists who quickly grasped the significance of Aub's discoveries: the potential liberation of human thought from the intellectual servility imposed by the machine.

Shuman, the chief Programmer of this advanced technological society, surmised that the possessor of such calculative knowledge would be provided with great intellectual and political power:

Nine times seven, thought Shuman with deep satisfaction, is sixty-three, and I don't need a computer to tell me so. The computer is in my head.
And it was amazing the feeling of power that gave him.

Unlike Asimov's future society, our contemporary society has not yet progressed to the point where traditional methods of calculation are threatened with extinction. Many studies including NACOME (NCTM, 1975) and the British Columbia Mathematics Assessments (Robitaille & Sherrill, 1977; Robitaille, 1981) have reported that schools still spend a considerable amount of time teaching children how to compute. However, the nearly universal availability of hand-held calculators seems destined to challenge the favoured position enjoyed by the traditional methods of pencil-and-paper computation currently taught in schools.
The calculator far surpasses any other previous invention in reducing the memory requirements needed by the user. The calculation of a product such as $123 \times 456$, for example, requires the recall of neither numerical equivalents nor calculative method. Other than knowing which keys to push, the order in which to push the keys, and recalling which entries have been made, the user has been freed from the more taxing memory demands normally associated with calculation.

Since the calculator holds an obvious advantage over other more conventional techniques of calculation, these questions become important: has mental calculation as an educational goal become obsolete or can a study of a possible anachronism be justified on either practical or theoretical grounds? The remainder of this chapter presents several arguments to answer these important questions.

Mental Calculation: A Practical Life-Skill

A curriculum can be evaluated in any number of ways. One popular method of assessment has been to examine the social utility of a curriculum's content. According to this viewpoint, a good curriculum should be based upon those skills needed to solve problems encountered in everyday activity. Commonly such skills have been identified through the study of people as they proceed through their regular activities. Several studies (Brown, 1957; Flournoy, 1957, 1959; Sauble, 1955; Wandt & Brown, 1957) have attempted to identify the routine uses of calculation, including those requiring a mental calculation.

In one study (Wandt & Brown, 1957), individuals were asked to note their calculative activities over a 24 hour period. The
researchers' intention was to determine those activities not directly connected with on-the-job performance. Each reported use of a calculation was placed in one of four categories: mental-exact, mental-approximate, pencil-and-paper-exact, and pencil-and-paper approximate. The study reported that approximately 75 percent of the calculations were done using mental procedures while the remaining 25 percent required pencil-and-paper methods. Even in an era where calculators were virtually non-existent, most people had more need of mental than written methods to solve tasks requiring a calculation. The study also found that mental-exact calculations were much more numerous than mental-approximate methods. Wandt and Brown concluded that "considerable emphasis should be placed on mental-exact and mental-approximate mathematics at both elementary and secondary levels" (p. 153).

Certainly curriculum developers must exercise caution when studies of routine behaviour are used to judge the merit of a curriculum. Many writers (Niss, 1981; Tyler, 1970) have argued that deriving content from studies of contemporary life can lead to a rapidly outmoded curriculum. The identification of a set of mathematical skills whose utility will remain timeless is a particularly difficult, if not impossible, task.

There is no doubt that the ubiquitous hand-held calculator would affect the findings of a replication of Wandt and Browns' early study. Today's citizens would likely not use mental calculation methods as frequently as Wandt and Browns' subjects did 25 years ago. Nevertheless, mental calculation can still have great practical utility despite the continuous proliferation of calculators. It will
remain a convenient tool used both to determine an exact solution for a narrow range of numerical tasks and to estimate the results of calculations produced by non-mental methods.

Mental calculations and exact solutions. Teachers spend a great deal of time in teaching children how to calculate. That teachers consider student competence in computation to be the most important goal of arithmetic has been well-documented. Major surveys including Priorities in School Mathematics (NCIM, 1981), the National Advisory Committee on Mathematical Education (NCIM, 1975), and the British Columbia Mathematics Assessments (Robitaille & Sherrill, 1977; Robitaille, 1981) have concluded that most teachers have spent the majority of instructional time in arithmetic in the pursuit of this goal.

To teachers, computational facility has meant usually proficiency with the conventional pencil-and-paper algorithms. These methods of "ciphering" have been taught with the belief that they are life-skills essential for functioning in the world of the adult. Yet the skills taught in school are not necessarily the skills that people use outside of school.

Wandt and Brown (1957) found that the great majority of everyday calculation tasks were solved using mental rather than the pencil-and-paper techniques so emphasised in school arithmetic programs. Maier (1977) claimed that adults apply methods quite different from those taught in school mathematics classes to solve most everyday calculative tasks. Referring to these unconventional and often untaught procedures as "folk math," he wrote:
Some of the general differences between school math and folk math are clear. One is that school math is largely paper-and-pencil mathematics. Folk mathematicians rely more on mental computations and estimations and on algorithms that lend themselves to mental use. When computations become too difficult or complicated to perform mentally, more and more folk mathematicians are turning to calculators and computers. In folk math, paper and pencil are a last resort. Yet they are the mainstay of school math. (p. 86)

Are Maier's observations correct? If they are, the modern calculator and the expected advances in calculative technology will pose more of a threat to pencil-and-paper than to mental methods of calculation. If a calculation task seems complex, the user is more likely to reach for a calculator than a pencil and paper. If the calculation task is reasonable, what can be more convenient than a mental calculation. As Maier has suggested, "Other computation tools may not always be available, but folk mathematicians always carry their brains with them" (p. 89).

Some recent evidence indicates a renewed interest in elevating the role of mental arithmetic in mathematics programs. The British report Mathematics Counts (Cockcroft, 1982) outlined the mathematics required in higher education, employment, and adult life, generally. The report suggested that mental calculation should be given a far more prominent position in mathematics programs than has been accorded in the recent past. The report concluded that the "decline of mental and oral work within mathematics classrooms represents a failure to recognize the central place which working 'done in the head' occupies throughout mathematics" (p. 75).

Admittedly, mental calculation is a limited tool for solving most types of calculative tasks, especially when compared to the capabilities of the calculator. Few individuals will use mental
methods to solve computations such as 123 x 456, for example, when a calculator is available. On the other hand, it does not seem to be an unreasonable goal for mathematics educators to expect that most people will be able to use some form of mental calculation to solve tasks such as 9 x 300, $1.99 + $5.99, 480/16, 1000 - 501, 2 x 555, and even 25 x 48. Exact mental calculation will remain as a practical method of calculation: the use of this type of calculation will be determined both by the nature of the calculative task and by the capabilities of the user. The challenge for teachers will be to help children develop the necessary capabilities.

**Estimation as a form of mental calculation.** Estimation as a method of determining the reasonableness of a proposed solution to a computation has become increasingly important, especially in the age of calculators. The NCTM's *An Agenda for Action* (1980) recommended that:

> Teachers should incorporate estimation activities into all areas of the program on a regular and sustaining basis, in particular encouraging the use of estimating skills to pose and select alternatives and to assess what a reasonable answer may be. (p. 7)

Trafton (1978), Denmark and Kepner (1980), and Levin (1981) have made similar recommendations.

Despite the claimed importance of estimation, many studies (NAEP, 1977, 1979, 1983a; Carpenter, Corbitt, Kepner, Lindquist & Reys, 1981, Robitaille & Sherrill, 1977; Robitaille, 1981; Keeves & Bourke, 1976) have demonstrated that both children and adults are not very good at estimating solutions to calculation tasks. For example, only 54% of 17-year-olds and 64% of young adults could estimate a sum to the nearest million dollars given four hypothetical government
expenditures (Carpenter, Coburn, Reys & Wilson, 1978, p. 30). Even more surprising was the finding that only 37% of the 17-year-olds could provide a reasonable estimate to the item $12/13 + 7/8$ (NAEP, 1979, p. 39). This latter result seems particularly disheartening since the choices were 1, 2, 19, and 21!

Why do children and adults have such apparent difficulty determining quantitative estimates? Some authors (Skvarcius, 1973) have concluded that estimation simply has not been taught in schools. If this conclusion is correct, the disappointing findings reported by many studies merely reflect a lack of instructional opportunity.

Other studies have attributed the difficulties in estimating to the failure of the subject to employ useful estimation strategies. One study (Reys, Rybolt, Bestgen & Wyatt, 1982), which attempted to identify and describe the processes used by good estimators, reported that good estimators could be characterized by: (Note: underline emphasis is mine)

...the quick and efficient use of mental computation to produce accurate numerical information with which to formulate estimates. All estimators exhibited well-developed skill with multiples of 10 or a limited number of digits, and many others were fluent in mentally computing with larger numbers, more digits, and even different types of numbers (e.g., fractions). On some problems, subjects resorted to mental computation rather than using an estimation technique. For these problems and these students, it was more efficient for the person to compute mentally rather than estimate. (p. 197-198)

The close relationship between estimation and mental calculation established by Reys et al. implies that no instructional program purporting to teach estimation skills can afford to ignore the teaching of mental calculation. Estimation, after all, can be thought of as no more than a form of "less precise" mental calculation.
Potential Contributions to Theory and Practice

It has been argued that despite the advances in electronic calculative aids, mental calculation will continue to be a worthwhile goal in mathematics education. If, as Begle and Gibb (1980) have suggested, the purposes of research in mathematics education are to "find out how and why something works and then to see what works in practice" (p. 8), the present study could be considered a necessary first step in improving the proficiency of unskilled mental calculators.

The study may demonstrate that the skilled and unskilled mental calculators differ in their choices of strategies. Such a finding would suggest that the performance of the lesser skilled mental calculators could be improved if teachers provide them with instruction on how and when to use the strategies of the more skilled mental calculators. Before such instruction is planned, however, it would be wise to compare the skilled and unskilled subjects' methods for clues about the efficiency of particular techniques. For example, some methods of the skilled subjects might be efficient because the amount of information that must be dealt with at one time can be reduced. An examination of the unskilled subjects' apparently less efficient techniques could reveal mental operations which are difficult to carry out, especially during particularly taxing mental calculations. Knowing what the learner should do or should not do during a mental calculation has obvious implications for designing effective instruction.

The possible finding that the skill groups differ in some fundamental arithmetic processes such as the ability to recall quickly
and accurately basic multiplication facts would be additional useful information for practitioners. Several researchers including Resnick and Ford (1981), Case (1978), and Gagne (1983a, 1983b) are convinced that the information-processing demands of complex tasks can be reduced if the component behaviours have been developed to the point of automaticity. As Gagne has argued, automatization of basic skills ensures that the scarce cognitive resources of attention can be devoted to the more intricate and complex parts of the calculative task (1983a, p. 15). Thus, many skilled subjects' mental calculation performances might be improved by ensuring that their recall of basic facts becomes highly automatized.

On a less optimistic note, skilled mental calculation could be found to be associated with short-term memory resources far beyond those possessed by the average young adult. Such a finding would suggest that proficient mental calculation performance requires some basic and essentially unalterable mechanistic processes which some individuals can never possess. If that is so, the study could help educators identify individuals who would not likely benefit from instruction in mental calculation. External memory-aids might always be a necessity for these subjects.

A direct practical consequence is not the only measure of the significance of a study. A study can make a significant contribution to mathematics education if, through the attempt to explain how people learn and use a specific skill such as mental calculation, insight is gained into how people use memory and cognition, in general. This type of information could be important because many researchers including Allport (1980a, 1980b), Baddeley (1981), Claxton (1980), Hitch (1980),
Neisser (1978), and Nilsson (1979) have argued that little progress has been made towards understanding how memory contributes to cognition in everyday settings.

As an important step towards reform, researchers have been urged to move from the study of memory as it relates to conventional "laboratory tasks" to the study of memory as it relates to "real-world" cognition. The view that memory research should, so to speak, "come out of the lab and see the real world," has been expressed by Neisser (Claxton, 1980):

A psychology that cannot interpret ordinary experience is ignoring almost the whole range of its natural subject matter. It may hope to emerge from the lab some day with a new array of important ideas, but that outcome is unlikely unless it is already working with principles whose applicability to natural situations can be foreseen. (p. 20)

Other researchers have expressed similar sentiments. Hitch (1980) argued that there is a "need to move beyond the bounds of typical laboratory experimentation by considering the issue of ecological validity" (p. 157). To Hitch, an ecological valid study means that the focus of the study should be upon tasks that are more representative of normal everyday cognition.

Neisser's (1978) proposed reforms for memory research require a major shift of thinking for many researchers. He suggested, "We should be careful in what we say about memory in general until we learn more about these many memories in particular" (p. 19). The present study, which investigated how different individuals used memory to cope with the demands of mental multiplication, appears to be both timely and worthwhile.

A study of how individuals manage to cope with everyday
situations which require some use of memory can help researchers identify those concepts developed in traditional experimental studies which might be fundamentally inadequate to explain the more prosaic uses of memory. This knowledge can be used to refine particular theories about memory. For example, Cole, Hood and McDermott (1982) have argued that because "everyday life contexts for thinking differ in important ways from the contexts assumed to have been obtained in laboratory tasks" (p. 373), these differences could undermine any attempts to generalize from one research setting to another.

Cole et al. used an interesting analogy to demonstrate why conclusions reached in an experimental setting do not necessarily apply to everyday instances of memory use. They stated:

In brief, one is advised to think of our examples of everyday cognitive tasks as bearing a relation to closed experimental tasks that is analogous to the the relation of a sieve to a bowl. If the bowl is an environment which completely constrains its contents, a sieve is more open space than netting; there is enough metal netting to provide the sieve with a recognizable shape of a bowl. But, like a sieve, and unlike a bowl, our specification of task and behaviour in everyday life cannot hold water. (p. 372)

Empirical investigations of mental calculation processes have been based on the performance of subjects on "closed" experimental tasks. In such a setting little opportunity exists for the resourceful person to lessen the memory load by selecting efficient calculative strategies and effective memory techniques. The generalizations about individual differences in calculative performance based on these limited task environments might not apply to the more tractable mental multiplication tasks used in the present study.

Consequently, this study could identify some aspects about memory as it is used in solving mental multiplication tasks that have
not been apparent in experimental studies of mental calculation. At the very least, this knowledge could:

...prevent us making hasty statements about what people can't do that they do in fact do in their normal (extra-experimental) lives. And very likely it would stimulate new and significant areas of research as we watch and puzzle over cognition in its natural habitat. (Claxton, 1980, p. 19)
CHAPTER II

REVIEW OF RELATED LITERATURE

Introduction to the Review

Since one of the objectives of the present study was to determine if skilled and unskilled mental calculators could be characterized by their differential uses of memory, a thorough review of memory research was considered to be necessary. Because the sheer magnitude and diversity of memory research is beyond the scope of this study, this review of research has been restricted to those conceptual approaches which appeared to hold the greatest promise for explaining individual differences in mental calculation. Information-processing theories seemed to hold that promise.

These theories place great emphasis on sequentially ordered events, actions, and manipulation of information: the very processes necessary for proficient mental calculation. Resnick and Ford (1981) have stated the focus of information-processing theories is on:

...the structure of knowledge within the mind and on the mechanisms by which knowledge is manipulated, transformed, and generated in the process of solving the myriad problems humans face both in adapting to practical demands of their environments and in following their more intellectual pursuits. (p. 197)

The approach has been used recently by researchers to study a variety of the more complex forms of human cognition. For example, Pascual-Leone (1970), Case (1972, 1974a, 1974b, 1975, 1978) and, Romberg and Collis (1981) have provided a different perspective from which to view Piagetian research. The theory has been used to analyze these other areas of human functioning: problem solving (Newell &
Simon, 1972), intelligence (Hunt, 1980), arithmetic problem solving (Hiebert, Carpenter, & Moser, 1982), individual differences in place-value comprehension (Brockmann, 1978), learning disabilities in mathematics (Nason & Redden, 1983). Only a few researchers such as Hitch (1977, 1978), Merkel and Hall (1982), Whimbey, Fischhof and Silikowitz (1969) and, Dansereau and Gregg (1966) have attempted to examine mental calculation from an information-processing perspective.

This chapter will identify and explain the prominent features of the human information-processing system. Since most empirical studies of mental calculation performance have used this theoretical framework, a review of the major concepts and issues should provide the reader with the necessary background material.

This review proved difficult because there seemed to be as many theories as there were researchers. Often the important concepts of one researcher seemed to have been easily dismissed or ignored by another researcher. There are so many variations that Tulving and Madigan (Shallice, 1979, p. 258) suggested, somewhat facetiously, that these numerous theories could be discriminated only by factor analysis! In spite of this difficulty, information processing does provide a useful framework within which to examine the topic of the study: the characteristics of subjects differing in mental calculation performance.
The Human Information-Processing System

Although there are many different theories of information processing, they are all based on an analogy with a digital computer. Just as a computer needs several types of resources to perform tasks, the human performer needs them as well. Researchers such as Hunt (1980) and Newell and Simon (1972) have argued that structural and strategy resources are of particular importance to any problem solving machine, be it computer or human. Since these two resources have figured prominently in many explanations of proficient cognitive performance, including mental calculation, each will be discussed in detail under separate headings.

Structural Resources and Mental Functioning

Structural resources refer to those "mechanical capacities for storing, retrieving and transforming information" (Hunt, 1980). Just as the hardware characteristics of a digital computer set limits on the machine's ability to reason, structural resources "set limits on the effectiveness of specific information-processing functions" (Hunt, p. 471).

But what are these human structural resources? Essentially, they involve the processes postulated to be endemic to the short-term or working-memory system. This memory system, or memory structure as it has sometimes been called, was developed as a construct to explain why human beings have such apparent difficulty in retaining information over short periods of time.

William James (1956) was one of the first theorists to distinguish between consciousness and the more permanent long-term
memories that all human beings possess. He used the term primary memory to describe the "just past" (p. 646). The term secondary memory was reserved to describe memory proper: the "knowledge of a former state of mind after it has already dropped from consciousness" (p. 648).

Modern theorists postulate that there is something different about the way humans remember information over shorter periods of time as opposed to the way they remember information over longer periods of time. Such differences are believed to reflect different memory systems with differing processing mechanisms. The terms short-term memory, short-term store, primary memory, immediate memory, and working memory have been used by different theorists to describe the hypothetical system that processes ephemeral information. Long-term memory and secondary memory have been used to refer to the more indelible forms of memory.

Broadbent (1958) was one of the first researchers to use the computer metaphor to explain human memory processes. He proposed that humans, like any other physical system, are limited in the rate at which they can process information. His central thesis was that this, quite theoretical, upper limit on information-processing capacity, in bits-per-second, was the direct physical basis for the selective nature of conscious attention and acted as a "bottleneck" to transmission flow (Allport, 1980b, p. 114). This idea of a highly specialized memory system with a limited capacity to store information has become a distinguishing characteristic of a "structural model of memory" (Zechmeister & Nyberg, 1982, p. 49).

Researchers such as Waugh and Norman (1965), and Atkinson and
Shiffrin (1968, 1971) soon extended and elaborated upon Broadbent's earlier model. In Atkinson and Shiffrins' three-store model, short-term memory was given, "a position of pivotal importance" (1971, p. 82). The authors explained that "the short-term store is considered a working memory: a system, in which decisions are made, problems are solved and information flow is directed" (p. 83).

The working memory system in Atkinson and Shiffrins' early model was considered to be more than a simple store for temporarily held information. The authors argued that control processes, or cognitive abilities that directed the flow of information, were also required for short-term memory retention. The authors believed that control processes such as rehearsal and coding had "enormous consequences for performance" (p. 82).

Undoubtedly, these models have had great influence on information-processing research, but most theorists have abandoned these early multi-store models. The experimental failures cited by researchers such as Allport (1980a, 1980b), Claxton (1980), Crowder (1982), and Craik and Lockhart (1972) and the apparent inability to explain complex but everyday human cognition (Neisser, 1978; Nilsson, 1979; Hitch, 1980) have considerably weakened support for these models.

Despite the shortcomings of the multi-store models of memory, the concept of a limited-capacity working memory is still regarded as sound by many memory researchers. After all, "any credible theory of memory must account for the fact that people can consciously process only a limited amount of material at once" (Lachman, Lachman & Butterfield, 1979, p. 272). For this reason, the various successors
developed by Anderson (1980), Newell and Simon (1972), Baddeley (1981), Nason and Redden (1983), and Baddeley and Hitch (1974) have retained the notion that every human has a limited capacity to retain and process temporary information.

What is the capacity of the short-term memory system? Experimental estimates of short-term memory capacity have proved elusive because, for one reason, researchers are not always clear what they mean by capacity limitations. What is clear is that how researchers picture short-term memory will influence their attempts to ascertain its capacity.

**Estimating storage capacity.** Short-term memory capacity has been referred to as the quantity of information a memory system can hold. Researchers who have held this view of capacity picture working memory as a type of container similar to the storage buffer of a digital computer, although a more folksy image of a "bucket" would also be appropriate. Several writers have appeared to hold this view. Wortman and Loftus (1981) asked this question: "How much information does short-term memory hold?" (p. 182). Resnick and Ford (1981), in discussing the limited processing capabilities of working memory, stated, "no one knows exactly how many 'pieces' of information it takes to 'fill up' working memory..." (p. 31). Similarly, Atkinson, Atkinson, and Hilgard (1983) suggested:

Given this fixed capacity, it is tempting to think of short-term memory as a sort of mental box with roughly seven slots. Each item entering short-term memory goes into its own slot. So long as the number of items does not exceed the number of slots, we can recall items perfectly. (p. 225)

For those researchers who have likened short-term memory to a container, it naturally follows that capacity would be measured by
somehow "filling the memory container with information" until some degradation in recall occurs. The most popular method of estimating storage capacity has been digit span. It has been used to assess "higher mental processes" since about 1885 when the first experiments on memory span were recorded by Ebbinghaus (Dempster, 1981). Memory span is commonly measured by requiring a subject to recall series of digits that have been presented orally, usually at a rate of about one digit-per-second. The maximum number of digits a subject can recall in perfect order is taken as the subject's "immediate memory span." Many standardized tests including the various editions of the Wechsler Adult Intelligence Scale have incorporated digit span as a subtest.

What is the span of the average person? Unfortunately, estimates vary. Because there has been no consensus regarding the type of the information used to fill short-term memory to its capacity, different techniques using different tasks yield different estimates for different ages of subjects.

An estimate of capacity is influenced by a subject's store of long-term knowledge. One subject could view the digits 1, 4, 2, 8, 5, 7 as a series of six random digits while another subject could recognize the series as the recurring period for the decimal fraction 1/7. For the former subject, there are six units of information; for the latter subject, there is but one unit.

Miller (1956) would say that each subject employed different "chunking" procedures. Miller's estimate of the average person's storage capacity has been cited in the famous form: "Seven, plus or minus two." This has meant that the average person can hold seven "chunks" or units of information for temporary periods of time. But,
to use Miller's words, "we are not very definite about what constitutes a chunk of information" (p. 93). This ambiguity about the size of a chunk has caused great difficulties in determining estimates of the capacity of working memory.

Zechmeister and Nyberg (1982), in an attempt to provide a clearer definition, have defined a chunk as a "unit of information organized according to a rule or corresponding to some familiar pattern" (p. 43). This definition still suffers from some ambiguity since the familiar pattern will vary from subject to subject depending upon the type of information that each subject can retrieve from long-term memory.

The influence of long-term memory explains some of the variable estimates obtained for short-term memory capacity. Depending on the measure used, the short-term store seems capable of maintaining from 2 to 20 words (Lachman, Lachman & Butterfield, 1979, p. 268). This fact that long-term memory processes can affect the estimate of the capacity of the short-term store has not been very satisfying for those theorists who have postulated the relative independence of each memory system.

That chunking can greatly influence estimates of memory span was well demonstrated by a subject participating in a recent study. Ericsson, Chase, and Faloon (1980) reported the performance of one subject who used chunking to enlarge his memory span. His knowledge and interest in the sport of long-distance running enabled him to group digits according to a discernible "running time." Consequently his memory span increased from a average of seven digits to a phenomenal 79 digits! This subject was atypical in other ways as well:
he was willing to spend a total of some hundreds of hours practising. Not many subjects participating in studies of memory have demonstrated such a willingness to attain proficiency in what most people would consider to be a questionable skill. Interestingly his span for letters remained at seven.

Other factors besides a subject's background knowledge influence estimates of working-memory capacity. Case (1972, 1974a, 1974b, 1978) and Pascual-Leone (1970) have argued that capacity can change as children grow older. Obviously it is difficult to determine a numerical value for a limit when the limit seems to be constantly changing.

Dempster (1978) has another point of view. He has argued that age differences do not affect span but method of presentation and stage of practice do. Thus, the age of the subject, the nature of the tasks, and the subject's familiarity with the material can be confounding factors that have to be taken into account when estimating the capacity of working memory.

What can be concluded about estimating short-term memory storage capacity? Presently, no more than a fluid estimate of a subject's capacity can be obtained. Perhaps Anderson's comment says it best:

Thus, while it is clear that the capacity of short-term memory is limited, how to measure that capacity is not at all clear. Cognitive psychology is still working at developing adequate measures of the capacity of short-term memory. (1980, p. 167)

Estimating attentional capacity. The container analogy has been a useful metaphor for viewing short-term memory capacity. However, other views lead to different methods for establishing the capacity of
short-term memory. One view sees capacity as the ability to perform certain processing operations while attending to new information (Wingfield, 1979, p. 43). According to researchers who hold this view, there are physical limits to attending to or processing information. The natural analogy that presents itself for this notion of capacity is not a passive container, but rather a "limited power supply." Cognitive performance is limited because, as Allport (1980b) explained, "Once the supply is fully loaded, any more watts consumed by one part of the system means less for all the rest, regardless of what they want it for" (p. 116).

The analogy at first glance seems reasonable enough. Most people have experienced situations where performance on some task has suffered because of some concurrently competing task. As Norman and Bobrow (1975) have explained, "The processing resources for any system are limited, and when several processes compete for the same resources, eventually there will be a deterioration of performance" (p. 44).

As there is a lack of agreement in measuring storage capacity, there is a similar lack of agreement about estimating attentional capacity. Allport (1980b) has been especially critical of the notion of attentional capacity and its role in structural models of memory. Commenting on the work of Kahneman, who has devoted much attention and effort on the study of "attention and effort," Allport quipped:

...how can we measure the non-specific demand on general capacity, made by a particular task, and from which predictions of residual capacity, or performance on some other simultaneous task or performance could be derived? Kahneman calls this a "basic problem for experimental psychology." Well, it is certainly a basic problem for his theory. (p. 117)
Despite the difficulties in measuring and even defining the working-memory processes required for higher mental functioning, most researchers still agree that working memory has a limited capacity to hold and process information. However, the older notion of equating digit span with short-term memory capacity seems to have been discarded. Researchers who study short-term memory and its contribution to higher mental functioning now incorporate several methods of estimating capacity (see, for example, Martin, 1978 and Merkel & Hall, 1982). This eclectic approach to estimating capacity was used in the present study.

**Strategies and Mental Functioning**

Strategy is another important resource needed by both human and electronic problem solvers. Put simply, the computer uses programs to solve problems; humans use strategies. Accordingly, individual differences in reasoning could reflect differences either in structural resources, or in the choice of strategy, or in both. It is no exaggeration to say that many information-processing studies have attempted to explain differences in functioning by studying differences in structural resources in isolation from a subject's choice of strategy. As some researchers (Hunt, 1980; Newell & Simon, 1972) have pointed out, this paradigm has not been particularly fruitful for explaining differences in complex human reasoning.

Newell and Simon (1972) have argued that strategies must be considered in evaluating individual differences in cognitive performance. They warned, "A few, and only a few, gross characteristics of the human IPS (information processing system) are invariant over task and problem solver" (p. 788). To argue that
differences in structural resources rather than choice of strategy is the most important factor in explaining differences in human reasoning is to argue that the hardware characteristics of a computer are more important to its functioning than its software. To drive this point home, consider the analogy to basketball playing offered by Hunt (1980):

If you tried to predict a basketball player's scoring potential from isolated physical characteristics, you would have only limited success. Extreme weakness or lack of stature would be associated with very poor performance, but once the person moved into the "above normal" field, correlations with physiological measures break down. The reason is that there are two quite different ways of scoring points in basketball. Some players score by muscling their way underneath the basket, then jumping up and slamming the ball down into the goal. For players who use this strategy, height and weight are good predictors of success, while hand-eye coordination and depth perception are not. The other strategy for scoring is to step backwards, away from your opponent, and toss a high, arcing shot up into the goal, over the heads and hands of the opposition. Players who use this strategy need not be particularly large or strong, but must be quick and have excellent depth perception. (p. 456)

The point Hunt makes is this: the relationship between task performance and information-processing capabilities depends upon the individual's choice of how the task is to be done. Reasoning is seen as an orchestration of structural capacities and strategies. The use of a strategy will be determined both by the nature of the task and by the structural resources possessed by a subject.

But what is meant by a "strategy"? The answer depends upon both the situation and the investigator: different situations and different investigators call for different meanings. Some discussion of these meanings seems necessary since, as Allport (1980b) has remarked, "so much of the difficulty of psychology has to do with getting clear what it is we are talking about" (p. 147).
Gagne (1977, 1984) has been one of many information-processing psychologists who has used the term "cognitive strategy" to describe the very global ways used by individuals to focus their knowledge and skills on problem situations that have not been previously encountered. Strategies are "ways of 'using one's head'" (Gagne, 1977, p. 167).

What are some specific ways of "using one's head"? No clearly defined and agreed upon list of strategies has been developed because, according to Gagne (1977), "the strategies themselves have not been identified and described" (p. 37). Despite this obvious shortcoming, a partial list of important cognitive strategies has been identified.

Gagne (1977), himself, has identified attending and selective perceiving, coding for long-term storage, retrieval, and problem solving as important internal processes associated with learning and remembering (p. 167). Dempster (1981) identified rehearsal, grouping, chunking, and retrieval as important strategic variables (p. 64). Howe and Ceci (1979, p. 71) suggested several cognitive strategies a person can acquire including labelling, elaborating, various kinds of coding processes, using mediators, organizing items, rehearsal and various other planned activities. To these theorists, cognitive strategies describe types of general thinking skills required by individuals whenever a need to process information arises.

Because cognitive strategies are supposedly content-free, a mental calculation strategy would not be regarded by many memory researchers as an instance of a cognitive strategy. Instead, such a mental process would be considered knowledge about a learned rule or procedure: in other words, what Gagne (1977, 1984) would call an
"intellectual skill" and what Anderson (1980) would term "procedural knowledge." Thus generalizations about the information-processing characteristics of content-free cognitive strategies may not necessarily apply to content-specific mental calculation strategies.

Not all researchers seem to make a rigid distinction between strategies as general thinking processes and strategies as specific learned rules or algorithms. Hitch, for example, has used the terms "cognitive strategy" (1977, p. 337) and "information-processing strategy" (1978, p. 319) to refer to the variety of mental addition methods he has attempted to investigate.

Hitch seems to have taken the position that cognitive strategies cannot be divorced from the specific rules used to attack a task because different rules may foster or inhibit the use of different cognitive strategies. In the case of mental calculation, some calculative strategies (learned rules) may be more efficient than others because they provide the user with more opportunity to apply cognitive strategies such as chunking, rehearsal, and the like. Thus, to take Hitch's viewpoint, "good" calculative strategies like "good" cognitive strategies minimize the deleterious effects of short-term forgetting.

Whether or not generalizations about the information processing characteristics of cognitive strategies can be applied to mental calculation strategies is beyond the scope of the present study. However, it does seem possible to wed the two notions of strategy so that the risks of invalid generalization can be minimized. Consequently, for the purposes of the present study, a mental calculation strategy is assumed to involve the recall of a learned
procedure as well as the attendant cognitive strategies used to minimize forgetting during the application of the procedure.

In summary, it has been argued that performance in mental calculation, as in other forms of human information processing, depends upon both the choice of an efficient strategy and adequate structural resources. A number of questions regarding individual differences in mental calculation suggest themselves. Can different mental calculation strategies be identified? Which of these strategies, if any, seem to be associated with proficient mental calculation performance? Do proficient mental calculators possess structural resources that are substantially different from the norm?

The remainder of the chapter will be used to explore these questions. Two sets of studies which have investigated some aspect of mental calculation will be reviewed: (1) studies on expert mental calculators; (2) experimental studies on the role of STM in mental calculation.

The Characteristics of Expert Mental Calculators

One fruitful method used to study any form of human reasoning is to examine the highly proficient behaviours demonstrated by experts. Often an analysis of expert behaviour has provided valuable insight into how a trait such as memory contributes to reasoning. Although they have had little practical significance, studies of expert behaviour in chess (Chase & Simon, 1973), Go (Reitman, 1976), and the abacus (Hatano, Miyake, & Binks, 1977; Hatano & Osawa, 1983), have increased the knowledge about information processing during complex reasoning.
A valuable adjunct to the analysis of highly skilled behaviour is to study unskilled or novice behaviour as well. Often the strategies and processes needed for proficient behaviour in the specified area of cognition under study can be identified by comparing the behaviour of both the expert and non-expert. Researchers may be able to use this knowledge about experts' strategies and processes to develop instruction designed to help less skilled individuals become at least more proficient, if not expert, in the skill under study. Case (1975, 1978) and Shulman (1976) are two researchers who have been strong advocates of the instructional implications of novice-expert comparisons.

The analysis of expert behaviour can reveal some underlying processes essential to highly efficient cognitive behaviour, in general. Theoretical extensions to other similar forms of human reasoning are then possible. Scripture realized this potential for theory generalization as early as 1891. Referring to the study of expert calculative behaviour, he optimistically thought "we can perhaps gain light on the normal processes of the human mind by a consideration of such exceptional cases" (p. 1).

Very few researchers have examined expert mental calculation; even fewer researchers have analyzed this form of proficient thinking from an information-processing viewpoint. Most studies of mental calculation have been historical accounts based not on direct observation but on secondary sources whose reliability, at times, appears to be questionable. Few researchers have examined the nature of mental calculation; most researchers have been content to report accomplishments rather than identifying and describing techniques of
Fortunately, there are studies of expert mental calculators which have been incisive and revealing. Bidder's (1856) analysis of his own expert behaviour proved to be an extremely insightful and useful source. Perhaps the most comprehensive early paper was Scripture's (1891). Later accounts of expert calculative behaviour including Mitchell (1907), Gardner (1977), and Ball (1956) have been based upon this earlier work. Although the studies of Hunter (1962, 1977, 1978, 1979) were restricted to the analysis of the introspective reports provided by A.C. Aitken, a very proficient mental calculator, Hunter's insights about the role of memory in mental calculation have been invaluable. A recent book by Smith (1983), which provides a valuable overview of expert mental calculation, seems to be the most comprehensive work on the subject.

The Calculative Accomplishments of Expert Mental Calculators

The calculative performances of expert mental calculators were often, to use a now hackneyed but, nevertheless, accurate term, awesome. A few examples should suffice to illustrate the extent of these experts' powers.

Dase, a 19th century German prodigy, possessed incredible calculating powers. According to Ball (1956), Dase once calculated the correct product of 79 532 853 and 93 758 479 in 54 seconds. In answer to a request to find the product of two numbers, each of twenty digits, he took 6 minutes; to find the product of two numbers each of forty digits, he took 40 minutes (p. 476). There were reports that Dase once extracted the square root of a number of a hundred digits in 52 minutes (p. 477)! Ball believed that Dase's feats:
far surpass all other records of the kind, the only calculations comparable to them being Buxton's squaring of a number of thirty-nine digits, and Wallis' extraction of the square root of a number of fifty-three digits. (p. 477)

Zerah Colburn was regarded as a backward child until he demonstrated some facility with multiplication. When Colburn was 8 years old, he impressed his father's friends with the following exhibition:

... he undertook and succeeded in raising the number 8 to the sixteenth power, 281,474,976,710,656. He was then tried as to other numbers, consisting of one figure, all of which he raised as high as the tenth power, with so much facility that the person appointed to take down the results was obliged to enjoin him not to be too rapid. (Scripture, 1891, p. 14)

A. C. Aitken was both an expert mental calculator and a talented mathematician. An example of his calculative abilities was provided by a mathematical colleague who related a story about their joint venture to an exhibition of desk calculators (Gardner, 1977):

The salesman-type demonstrator said something like "We'll now multiply 23,586 by 71,283." Aitken said right off "And get..." (whatever it was). The salesman was too intent on selling even to notice, but his manager, who was watching, did. When he saw Aitken was right, he nearly threw a fit (and so did I). (p. 75)

The most recent account of extraordinary calculative power has been attributed to Shankuntal Devi. According to the 1982 Guiness Book of Records, she demonstrated the multiplication of two 13-digit numbers: namely, 7 686 369 774 870 and 2 465 099 745 779. Each number was constructed at random by a computer. Her correct answer of 18 947 668 177 995 426 462 773 730 took only 28 seconds (Smith, 1983, p. 97)!

These talents could be viewed disdainfully as merely "mechanical" or "unthinking" exercises. That calculators such as Aitken were not unthinking automatons and were very flexible in their
methods is highlighted in the following partial account of his attempt to determine the square root of 851 (Hunter, 1962). The reader is reminded that the following reasoning was entirely "mental."

I at once perceive that 841 is 29 squared. So 29 is a good first approximation. At once I have noted the remainder 10 and halved it (by my rule) and noted mentally that 5/29 is 0.172.... At once, I risk 29.172 as an answer which is almost certainly correct to five significant digits. But already I am off on another track, because 29.172 is nearly 29 1/6, that is, 700/24. And almost before having formulated the procedure in a rational manner, I have divided 851 by 700/24, that is, multiplied it by 24/700. So I feel (rather than see or hear) 20,424/700. But then some experience tells me that 700 times 29 1/6 is 20,416.66666.... Averaging at speed 20,424 and 20,416.66666... getting 20,420.33333, dividing by 700, and placing the decimal point in the proper place - and all of this in one continuous follow-through like a good golf stroke - I have 29.17190476 (p. 251)

He continued on in this manner until after less than 15 seconds, he had carried the calculation to 8 decimal places where he was "satisfied to go no further" (p. 251). This solution was done by no robot: this was ingenuity, skill, and intelligence in action.

Some calculative prodigies were delightfully eccentric. Buxton, who Mitchell (1907) reported "was very stupid even from boyhood" (p. 63), must have been an interesting character. He could give from memory an itemized account of all the free beer he had been given from the age of 12 on. How such a claim can be validated is not clear.

Using apparently clumsy and laborious calculative methods, he was reported to have mentally squared a number of 39 digits over a period of 2 1/2 months (Mitchell, p. 63). At least his perseverance and indefatigability, if not his good judgement, have to be admired.

Scripture (1891) related an incident which suggests that Buxton's calculative talents bordered on the obsessive and
monomaniacal. His visit to London, where he was taken to see a performance of King Richard III, illustrates his exaggerated zeal for numbers and calculations:

During the dance he fixed his attention upon the number of steps; he attended to Mr. Garrick only to count the words that he uttered. At the conclusion of the play they asked him how he liked it. He replied "such an actor went in and out so many times and spoke so many words; another so many, etc...." (p. 5)

Reverend H. W. Adams provided another illustration of curious behaviour demonstrated by an expert mental calculator. Safford's antics as he calculated the product of $365 \times 365 \times 365 \times 365 \times 365 \times 365$ were described in the following manner:

He flew around the room like a top, pulled his pantaloons over the top of his boots, bit his hand, rolled his eyes in their sockets, sometimes smiling sometimes talking, and then seeming to be in agony, until, in not more than one minute, said he, 133,491,850,208,566,925,016,658,299,941,583,225!

(Scripture, 1891, p. 30)

J.R. Newmann wryly commented: "An electronic computer might do the job a little faster but it wouldn't be as much fun to watch."

It would be a mistake to surmise that eccentric behaviour was the norm. Some mental calculators such as Buxton might be properly labelled as "retarded savants": brilliant in handling immense numbers; dull in everything else, including mathematics. Generally speaking, most expert calculators were very talented and literate individuals. Ampere, Gauss, and Wallis were highly proficient calculators who were more well known for their mathematical and scientific accomplishments than for their calculative prowess. Bidder was a professional engineer. A.C. Aitken was a professional mathematician. As is often the case with expertise in most fields of human endeavour, calculative prodigies exhibited wide differences as regards heredity, education,
general intelligence, retention of their powers over many years, success in other fields, longevity, and the like.

Despite these differences in background, there were some characteristics that appeared to be common to all of these expert calculators. Of particular significance to this study was evidence which suggested that expert calculators could be differentiated from the non-expert by their memory for calculative strategies, memory for numerical equivalents, and enormous structural resources.

The Calculative Strategies of Expert Mental Calculators

The strategies used by expert mental calculators cannot be easily determined simply by examining the reports of their calculative performances. Some reports were vague and rarely did the writer identify the particular method of solution. As well, since many early calculators earned a living by staging exhibitions of their powers, they were very reluctant to divulge their methods of reasoning (Gardner, 1977).

A few proficient calculators seemed unable to explain their reasoning. Ball (1956) wrote of Buxton: "It was only in rare cases that he was able to explain his methods of work, but enough is known of them to enable us to say that they were clumsy" (p. 469).

Despite these limitations, there seems sufficient evidence to identify a tentative list of some experts' methods of solution. While no claim is made that the list of strategies is complete, the list does seem extensive.

Likely the reader will be intrigued by the unconventionality of the methods used by these expert mental calculators. Certainly, they
contrast sharply with the still common algorithms used in school to solve written computations. Most calculators claimed that these mental methods were largely self-taught.

An attempt has been made to label each strategy with a conventional mathematical term that appears to denote the method's more salient features. A brief explanation with an accompanying sample calculation will be given for each listed strategy. Some documented use of the strategy by a mental calculator will be provided wherever possible. No claim is made that the categories are mutually exclusive since many strategies can be considered as special cases of other strategies. Finally, it should be mentioned that often a calculative task would be solved through an application of several calculative strategies used serially rather than through the exclusive application of one strategy.

The list has been ordered to reflect what the writer believes is each strategy's generalizability. That is, the strategies which can be applied to a wider range of calculative tasks have been placed earlier in the list than the more specialized strategies found later in the list.

The list contains only those strategies which were used to calculate products. There were several reasons why the analysis was restricted to mental multiplication strategies. In the first place, most of the introspective analyses reported by observers have been restricted to multiplication. Expert calculators seemed to find, as Ball (1956) reported, "Of the four fundamental processes, addition and subtraction present no difficulty and are of little interest" (p. 481). Mitchell (1907) held a similar view: "... multiplication and
not addition seems to be the fundamental and favorite operation in mental calculation" (p. 103). As well, the evidence suggests that division tasks seemed to be solved through the intelligent use of multiplication rather than the application of some conventional division algorithm.

Finally multiplication is a process whereby properties and attributes abound. After all, the mathematical field of number theory is essentially no more than the study of the properties of multiplication. Given that most calculators were intrigued by number properties, perhaps it should be no surprise to find that expert calculators thrived on challenging multiplication tasks.

**Mental analogue of the pencil-and-paper algorithm.** Expert mental calculation could involve no more than the application of a mental analogue of the conventional paper-and-pencil algorithm taught in schools. What could be more natural than an apparent simple extension of a learned procedure to a different setting? Scripture (1891) believed that there was evidence which suggested that Buxton and Fuller used such techniques. He reported that:

> It is said of Buxton that he preserved the several processes of multiplying the multiplicand by each figure of the lower line in their relative order, and place as on paper until the final product was found. (p. 58)

In theory, at least, the procedure could be employed to solve complex calculations if the calculator possessed a prodigious short-term memory resource. Such a calculator would be, a "little calculator with a big memory" (Mitchell, 1907, p. 117). Any person capable of mentally squaring a 39-digit number over a period of 2 1/2 months without losing track of the calculation, as Buxton apparently
did, could have the necessary resources to use the seemingly awkward mental analogue of the conventional algorithm for written computation. However, the literature has been ambiguous on this point and one is left to speculate.

**Distribution.** Perhaps the most well documented method of solution used by expert calculators has been distribution. Often used when easier methods were not apparent, the calculator typically proceeded by partitioning one or both factors into either a series of sums or differences, carrying out a set of multiplications, and adding or subtracting each of these partial products to obtain a final solution.

Bidder's (1856) solution of the task 89 x 73 typifies the process. Although he solved the task "instantly," the speed belied a lengthy mental process. He explained that the calculation was accomplished by using this series: 80 x 70, 80 x 3, 9 x 70, 9 by 3 (p. 256).

He claimed he always started with the left-hand or most significant digit (p. 260). This left-to-right procedure seemed to be a method common to most expert calculators (Smith, 1983, p. 109). Bidder reported that each of the partial products in a calculation was successively added to produce a running sum rather than postponing the addition of all partial products to the end of the calculation.

During a public exhibition of his powers, Bidder used the instance of 373 by 279 to illustrate his method. After quickly announcing the product of 104,067, he explained:

...now the way I arrive at the result is this - I multiply 200 into (sic) 300 = 60,000, then multiplying 200 into 70, gives 14,000, I then add them together, and obliterating the previous figures from my mind, carry forward 74,000;
multiply 3 by 200 = 600, and I add that on and carry forward
74,600. I then multiply 300 by 70 = 21,000, which added to
74,600, the previous result, gives 95,600, and I obliterate
the first. Then multiplying 70 by 70 = 4,900 and adding that
amount, gives 100,500. Then multiplying 70 by 3 = 210, and
adding as before, gives 100,710. I then have to multiply 9
into 300 = 2,700, and pursuing the same process brings the
result to 103,410; then multiplying 9 into 70 = 630, and
adding again = 104,040; then multiplying 9 into 3 = 27, and
adding as before, gives the product, 104,067. This is the
process I go through in my mind. (p. 260)

Bidder could use distribution to multiply a 12-digit number by
another 12-digit number! Not surprisingly Bidder explained that the
task "required much time, and was a great strain upon the mind" (p.
256). For practical purposes, he felt that the use of the
distribution procedure was restricted to calculating the products of,
at most, 3 digit factors. Apparently he did not fix this limit
without reason. He explained:

Each set, or series of 3 figures, constitutes a step in
numbers, 787 is one series, - the second series is 787
thousand, the next series 787 millions, the next 787 thousand
millions, and the next 787 billions. Therefore, at the
change beyond each third figure, another idea (emphasis is
mine) must be seized by the mind; and though it is but one
idea, yet with all the training I have had, when I pass three
figures, and jump from 787 to 1,787, I cannot realize to
myself that it is but one idea; - in fact there are two, and
this increases the strain on the registering powers of the
mind. (p. 263)

To put Bidder's comments in a more contemporary light, he
seemed to be able to extend the limits of his short-term memory
through "chunking." An "idea" or chunk was related to a scale whose
radix was 1000: ones, thousands, millions and, so on. For example, to
calculate 1787 by 787, the task was apprehended by Bidder as 787 by
787 and 787 by 1000. This tendency to calculate with large "number
chunks" rather than individual digits can be said to characterize
proficient calculators (Hunter, 1978; Smith, 1983).
Colburn (Scripture, 1891) and the two subjects studied by Jakobsson (1944) were reported to have used a method similar to that of Bidder's. Mitchell (1907) reported that Diamandi and Bidder's son, who like his father was an expert calculator, used a complex variation which involved cross multiplication. Their calculations progressed from right to left instead of left to right.

Some proficient calculators used the distribution strategy by chunking in a very flexible manner. For example, Aitken's solution of 123 by 456 involved the partial products of 45 by 123 and 123 by 6 (Hunter, 1962, p. 249).

Distribution can be used in a subtractive as well as in an additive sense. Subtractive distribution was used by one of Jakobsson's (1944) subjects to solve these calculative tasks:

If the multiplier is a number in the vicinity of some round number, one proceeds in the following manner: \( 127 \times 359 = 127 \times (360 - 1) = 45720 - 127 = 45593; 546 \times 784 = 784 \times (550 - 4) = 431200 - 3136 = 428064. \) (p. 187)

As Jakobsson suggested, the subtractive form of distribution was frequently applied when one or more of the factors were perceived by the subject as being close to a "round number": in other words, some multiple of a power of 10. Ball (1956) stated that Inaudi made use of the strategy to solve \( 27 \times 729 \) by thinking \( 27 \times 730 - 27 \) (p. 482).

The ability to use the annexation algorithm (determining the product when one factor is a multiple of a power of 10 by "annexing zeroes") seems to be essential for using distribution to solve mental calculation tasks. Bidder's solution of 373 \( \times 279 \) described earlier involved several such applications of the algorithm. Often the expert
A calculator used an abridgement to annex zeroes. For example, during Aitken's solution of 123 x 456, the partial product 123 x 450 was expressed as "123 x 45 = 5535." Hunter (1962) commented: "This thinker does not 'burden the mind with zeros'" (p. 249).

Another distribution strategy used by some expert calculators involved an application of the well-known algebraic equivalence for the difference of squares: \(a^2 - b^2 = (a + b) \times (a - b)\). To calculate the product of 51 and 49, one merely has to think 2500 - 1: that is, 51 x 49 = (50 + 1) (50 - 1) = 50^2 - 1 = 2500 - 1. Ingemar, Jakobsson's subject (1944), solved many tasks in this manner including 78 x 84 = 81^2 - 3^2 = 6561 - 9 = 6552 (p. 186).

Notice that this variation of distribution is useful in those instances where the two factors are of a similar magnitude and the mean value of the two factors is a convenient reference number such as a quickly recalled square. Since most expert calculators could readily recall or calculate squares, this strategy was particularly useful. Likely this ability to recall or calculate squares quickly was a natural outcome of the expert's interest in calculating square roots (Smith, 1983, p. 123).

**Factoring.** Mental multiplying by factoring one or both factors and then applying the associative law is a useful and ingenious mental calculation strategy. Evidence provided by Ball (1956), Gardner (1977), Mitchell (1907), Scripture (1891), and Smith (1983) suggests that many expert calculators favoured factoring over most other strategies. Because of the calculator's almost instant apprehension of the factors needed for the calculation, many calculation tasks could be solved quickly through factoring. Even a person possessing a
hand-held calculator could not hope to approach an expert's rate of working.

The readiness with which a presented number can lead to factor apprehension has been well documented. When Colburn (Ball, 1956) was "asked for the factors of 247,483 he replied 941 and 263; asked for the factors of 171,395 he gave 5, 7, 59 and 83; asked for the factors of 36,083 he said there were none" (p. 471). Possibly with tongue-in-cheek, Ball wrote that Colburn, "however, found it difficult to answer questions about the factors of numbers higher than 1,000,000" (p. 471). On another occasion when asked for the product of 21 734 by 543, Colburn immediately replied: "11 801 562." He explained that he had arrived at this solution by multiplying 65 202 by 181 (Ball, 1956, p. 472).

Aitken possessed an aptitude for factoring which must have equalled if not excelled Colburn's. That he could use factoring to aid in very complex calculations was evidenced by the incredible response he gave when he was asked to multiply 987 654 321 and 123 456 789:

I saw in a flash that 987,654,321 by 81 is 80,000,000,001; and so I multiplied 123,456,789 by this, a simple matter, and divided the answer by 81. Answer: 121,932,631,112,635,269. The whole thing could hardly have taken more than half a minute. (p. 251)

Hunter believed that Aitken's apprehension of numerical attributes was "immediate, simultaneous and often autonomous" (p. 246). Aitken seemed not to have to set himself to apprehend numbers; "rather he must set himself to prevent such apprehending" (p. 246). That this apprehension was often autonomous is illustrated in the following anecdote offered by Aitken (Hunter, 1962):

If I go for a walk and if a motor car passes and it has the registration number 731, I cannot but observe that it is 17
times 43. But as far as possible, I shut that off because it interferes with thought about other matters. And after one or two numbers like that have been factorized, I am conditioned against it for the rest of my walk. (p. 247)

There is evidence that Bidder (1856), the two subjects studied by Jakobsson (1944), Gauss, Safford (Mitchell, 1907), and many other calculators had a similar aptitude for factoring and its use in calculation.

Aliquot parts and iteration seemed to be two popular factoring strategies known by expert mental calculators. The aliquot parts factoring strategy involves a series of steps. First, one factor is multiplied by a multiple, call it "a," of the remaining factor. The resulting product is divided by "a" to obtain the product of the original factors. As so often happens in explaining mathematical procedures, an illustration can be more illuminating than this admittedly awkward verbal description. The calculation could proceed in the following manner to multiply 25 by 48, for example: $48 \times 25 = 48 \times (100/4) = 4800/4 = 1200$. This technique is most useful when calculating products with factors such as 25, 50 and 125 because the reference numbers are 100 (25 x4), 100 (50 x 2), and 1000 (125 x 8), respectively.

Some calculators used other reference numbers to aid in calculation. For example, Gullan (Jakobsson, 1944) often used 1001 as a reference point. She determined the product of 143 and 674 in the following manner: $143 \times 674 = (674 \times 1001) / 7 = 674 674 / 7 = 96 382$ (p. 188).

She extended the technique to include products such as 125 by 575 in the manner described by Jakobsson: $125 \times 575 = (575 / 8) \times$
1000. Realizing that 575 divided by 8 is 71 with a remainder of 7, she added 875 (7 x 125) to 71 000 (71 x 1000) and obtained the correct product 71 875 (p. 187).

The use of iterative techniques was used frequently by some experts to calculate with powers of two or three. The solution to 27 x 32, for example, can be found by doubling the factor 27, 5 times: 54, 108, 216, 432, 864. For those calculators who can determine "successive triples" quickly, the technique can be extended to calculating products where the factors are powers of three: To calculate the product of 27 and 32, for example, the factor 32 can be tripled successively as 96, 288 and 864.

The number of iterations needed to determine the product of larger factors makes this technique cumbersome to use. However, this fact did not seem to deter some calculators. Mitchell (1907), who fancied himself to be a capable calculator, offered the following reasoning for squaring 162: "...to square 162, again, the stages would be 486, 1458, 2916, 8748, 26244, i.e. multiplying successively by 3, 3, 2, 3, 3" (p. 92).

Buxton's attempt to multiply by 378 seems particularly awkward (Mitchell, p. 63). He multiplied the other factor a by 5, 20, and 3 to obtain 300a, multiplied a by 5 and 15 to obtain 75a, and, finally, multiplied a by 3 to obtained the required product of 300a + 75a + 3a. Despite Buxton's enormous calculative powers, the obvious use of the annexation algorithm in this calculation escaped him.

Mitchell surmised that Buxton often used iterative techniques which involved counting multiples of the multiplicand. The illiterate slave Tom Fuller was believed to have used counting techniques which
were similar to Buxton's (Mitchell, p. 63). Such evidence suggests the possibility that, in order to remember the numerous calculative details, these calculators may have possessed enormous structural resources and, thus, were able to get by with these seemingly inefficient techniques. This possibility will be discussed later in the chapter.

Expert Mental Calculators' Memory for Numerical Equivalents

Hunter (1978) concluded that expert calculators acquired a store of numerical equivalents in order to save a lot of calculative effort. These experts, unlike the average adult, have a "vastly large fund of numerical equivalents upon which they can draw with speed and accuracy" (p. 340). If his conclusion is valid, what could be a more efficient strategy than the simple retrieval of a "fact" from long-term memory.

Jakobsson (1944) concluded that experts acquired and used what he called an "extended multiplication table." His studies of two precocious calculators led him to surmise that:

...they make use of 'extended multiplication Tables' (sic), that is to say, multiplication Tables up to 30 by 30. These products as well as several products by higher numbers which had become fixed in their memory, they knew by heart. (p. 186)

Scripture (1891) believed that Mondeux knew "almost by heart the squares of all the entire numbers less than 100..." (p. 48). Scripture suspected that Colburn, Buxton, and Dase also possessed tables which reached to products as high as 100 x 100. He asked rhetorically: "Did any of the prodigies possess such a table?" He answered: "Considering their enormous powers of memory it would be
almost unexplainable if they did not" (p. 46).

Gardner (1977) concluded that proficient calculators possessed a library of numerical data which could be accessed at will. He said:

Aitken's skull housed an enormous memory bank of data. This is typical of the lightning calculators; I doubt that there has ever been one who did not know the multiplication table through 100, and some authorities have suspected that Bidder and others knew it to 1,000 but would not admit it. (p. 73)

However, other authorities such as Mitchell have taken an opposing viewpoint. He argued:

There is no warrant, then, for supposing that any of the prodigies except Gauss and Dase used a multiplication table larger than that of ordinary mortals; and even in these two cases there is no direct evidence, only a bare possibility;... (p. 110)

He thus concluded:

We may therefore dismiss the theory of enlarged multiplication tables, at least until its advocates have brought forward further and more definite evidence than any that has yet been produced. (p. 111)

One of his arguments to support his skepticism seemed rather weak. For some reason, he believed that the only way information such as multiplication products can be committed to memory is through deliberate study. As he explained:

...but in dealing with the enlarged multiplication table theory, we must insist that the only legitimate interpretation of the theory is that such a table is deliberately committed to memory (emphasis is Mitchell's) by the calculator.... (p. 106)

He seemed unwilling to accept the fact that people who are highly interested in a subject, as calculators obviously were, can memorize a lot of information without a conscious and deliberate attempt to commit anything to memory.

Undoubtedly, Hunter (1978) would have disagreed with Mitchell's
viewpoint. In attempting to explain the acquisition of Aitken's formidable store of knowledge, he stated: "We might suppose that the acquisition of this store involves a great deal of deliberate memorization" (p. 343). But he argued that the supposition would be wrong as "there is relatively little deliberate memorization bleakly undertaken for its own sake: if there were, the likely outcome would be boredom and discouragement" (p. 343).

Bidder believed that he did not possess an enlarged multiplication table. Rather, he thought that his ability for extremely rapid calculation gave observers the mistaken impression that he possessed such a library of numerical facts (p. 256). He stated, "I acquired the whole multiplication table up to 10 times 10; beyond which I never went; it was all that I required" (p. 258). As stated earlier, Gardner (1977) did not believe Bidder's claim. Ball (1956) has expressed a similar skepticism of Bidder's self denial.

Smith (1983) argued that since most educated people possess part of an extended multiplication table, the important question to determine is the extent of the expert calculator's table. He believed that there was a dearth of evidence to support what he thought were the extravagant claims about the size of the expert's memory store. According to Smith, Wim Klein is the only calculator known to have used a multiplication table of 100 by 100 (p. 60).

Expert Mental Calculators' Short-Term Memory Capacity

Information-processing theorists such as Hunt (1980), Newell and Simon (1972), Resnick and Ford (1981), Romberg and Collis (1981), and Anderson (1980) have all expressed views about the role played by
working memory in limiting the performance on any series of mental operations. Normally the limited capacity of working memory does not inhibit performance in tasks where the problem information is continuously available for visual inspection. For example, during written calculation there is little demand on short-term memory processes. All stages of the calculation are recorded and, thus, are available for continual examination.

Mental calculation can be another matter: the luxury of an external memory-aid is not available. Consequently a calculation can involve a great deal of processing by short-term memory. Therefore, it could be argued that expert mental calculation would require processing capabilities far exceeding the limited resources possessed by the average person. Is there any evidence which indicates that expert calculators possessed such memory resources?

Alfred Binet studied many of the great 19th century calculators and attempted to estimate their memory spans. Although he used slightly different techniques for measuring span than are used today, the conclusion that these calculators had large spans seems inescapable. Binet set the limit of Inaudi's memory span at 42 (Mitchell, 1907, p. 71), for example. This figure stands in stark contrast to the average adult digit span of about 7.

Mitchell reported that one unidentified calculator was said to be able to repeat 150 figures in order after a single hearing! He was also able to repeat the series backwards and name the 30th or 50th figure from either end (p. 86). One could imagine that this subject would have surpassed with ease the ceiling of instruments designed to measure such notions of capacity as backward digit span and probed
Perhaps the most reliable assessment of a great calculator's capacity was conducted by Hunter (1977). The subject, A.C. Aitken, was reported to have had a span of 13 with auditory presentation of random digits and a span of 15 with visual presentation of random digits (p. 156).

Of course, it would be tempting to attribute a casual relationship between span and proficient calculative thinking. But, which would be the cause and which the effect? There could be a reciprocal relationship in which increasingly proficient mental calculation results in increasingly higher estimates of span which, in turn, begets increasingly proficient mental calculation.

Suggestive evidence about the existence of such a reciprocal relationship has been provided by studying grand experts in "abacus-derived mental calculation." Hatano and Osawa (1983) found that these abacus experts had digit spans which far exceeded their memory for letters and common fruit names. They concluded that:

We reasonably assume that it was through extensive practice in abacus and mental computation that their digit span was extended to double the "norm," because of the domain-specificity of the extension. (p. 102)

**Implications for Studies of Non-Expert Mental Calculation**

An attempt has been made to review those studies which have investigated some aspect of expert mental calculation. In particular, it was hoped to identify those factors which could be used to characterize skilled and unskilled performance. There is at least suggestive evidence that individual differences in mental calculation performance could reflect several differences in these uses of memory:
the choices of calculative strategies, the recall of useful numerical equivalents, the ability to process information in the short-term memory system.

However, since expert mental calculators are a very select and unrepresentative group of people, any conclusions about individual differences in mental calculation performance must remain tentative. These experts could be as different from skilled mental calculators as skilled calculators are from unskilled subjects. Skilled mental calculators could gain their advantage over the unskilled through some unidentified process.

Little is known about unskilled mental multiplication. Few researchers have attempted to identify the strategies used by unskilled mental calculators. In particular, no researchers have examined the difficulties that some subjects have with mental multiplication tasks. Maier (1977) has claimed that unskilled mental calculators rely on procedures similar to pencil-and-paper mental reproductions. A procedure which, he has argued, does not lend itself to mental use. However, no evidence was provided to support his claim.

Levine's study (1982) has provided some indirect evidence that unskilled mental calculators may favour the pencil-and-paper mental analogue to solve mental arithmetic computations. She found that by far the most popular strategy used by the lowest-scoring subjects on a test measuring the ability to estimate products and quotients was the pencil-and-paper mental analogue. In fact, she reported that almost 10% of the college student sample participating in her study made exclusive use of this estimation strategy.

The literature was also inconclusive on whether or not skilled
and unskilled mental calculators could be characterized by their ability to recall useful numerical equivalents during the solution of a mental multiplication task. In fact, there was some disagreement among researchers about whether experts possessed a larger than normal storehouse of useful "number facts." Whether non-expert but skilled mental calculators do possess an extensive library of numerical equivalents remains to be demonstrated.

Despite the obvious fact that some mental calculations require a great deal of short-term memory processing, it is not clear if skilled and unskilled mental calculators can be characterized by the efficiency of their short-term memory stores. Although expert calculators often possessed extraordinary structural resources, one can only speculate that skilled mental calculators have greater STM capacities than unskilled subjects.

Experimental Studies of STM and Mental Calculation

Many researchers (Hunt 1980; Lindsay & Norman, 1977; Resnick & Ford, 1981) have argued that any task requiring a great deal of information processing will suffer degraded performance because of the limited processing capabilities available to a subject. Hitch (1977, 1978), Merkel and Hall (1982), Whimbey, Fischhof, and Silikowitz (1969), and Dansereau and Gregg (1966) have all demonstrated that mental calculation requires a great deal of short-term memory processing.

Hitch's studies (1977, 1978) have been among the best attempts to examine mental calculation under experimental conditions. He studied how short-term forgetting contributed to errors of mental
calculation. The tasks he used were of low to medium complexity. The range of difficulty is best illustrated by the examples, \(325 + 431\) and \(6345 \times 5\).

He postulated that mental calculation proceeds serially and that the following series of mental operations are involved: retrieval of some information, usually in the form of digits; an arithmetic combination retrieved from long-term memory; storage of the "carried" digit, if applicable; and the storage of the sum or, in the case of a sum greater than 10, the separate storage of both the decade digit - the "carry" - and the concomitant unit digit. He evaluated a number of hypotheses about the role of short-term forgetting in mental calculation performance using this elementary information-processing analysis.

First, he hypothesised that the later steps in a calculation would show more errors since information stored in working memory is lost rapidly during any activity interpolated between presentation and recall (1977, p. 333). The data suggested that "the forgetting of individual items is increased by interpolated activity during the interval prior to retrieval and combination" (p. 334).

He also attempted to determine if the difficulty of recalling an item might be increased by the presence of other stored items. He hypothesised that the more information being processed by working memory, the greater the amount of forgetting. Hitch found that this "storage load," as he described it, did contribute to loss of recall.

Dansereau and Gregg (1966) attempted to analyze the underlying processes involved in mental multiplication. Because their findings were based on the study of a single subject, generalizations about the
process of mental calculation are limited. They found that the time taken in the mental solution of various multidigit products correlated highly with the total number of processing stages, corresponding to "add," "multiply," "carry," and "hold." No attempt was made to identify the types of calculative strategies used by the participating subject.

These few studies have been important in that they have provided evidence suggesting that mental calculation involves the same underlying processes as more conventional short-term memory tasks. However, they leave a number of interesting questions unanswered about the contribution of short-term memory to individual differences in mental calculation performance. If, as Hitch (1978) has implied, there was some variance in performance even on these relatively simple tasks, how can such differences be explained?

It seems unlikely the differences in performance reflect choice of calculative strategy: the subjects were asked to use the same strategy and all subjects reported that they implemented the requested strategy. It seems very unlikely that erroneous addition combinations retrieved from long-term memory contributed to the variation in performance: all subjects were adults and could be expected to have good recall of "basic addition facts."

A plausible explanation is that the differences in performance reflected differences in STM capacity. Hitch did not report any measures of his subjects' capacities so there is no way of testing the hypothesis. Nevertheless, Hitch (1978) noted and suggested that subsequent studies of mental calculation should incorporate other assumed characteristics of short-term memory such as its limited
capacity to hold items (p. 322).

Only a few researchers have attempted to determine if there is a relationship between short-term memory capacity and mental calculation performance. One difficulty in investigating this relationship has been the selection of the instrument used to measure STM capacity. As has been discussed, there has been little agreement among researchers as to the most valid method of capacity assessment.

Perhaps the most reasonable approach for a researcher to take in the absence of a consensus would be to identify first the key processes required in a mental calculation and then select those measures of capacity which seem most likely to measure such processes. This was the approach taken to select the STM measures that were used in the present study. A review of the research literature seemed to suggest that ordering and transforming were two STM processes that could be necessary for proficient mental calculation.

Ordering and Mental Calculation

Ordering is one process that seems to be required for success in both mental calculation and many short-term memory tasks. Almost all the initial and interim information in a mental calculation must be stored in an ordered manner. Haphazard storage of information is to be avoided at all costs. When multidigit factors are presented to a subject, they must be stored in a way that preserves the place value of each digit. In a similar manner, the order of each stage of the calculative strategy must be remembered. The subject must constantly consider these questions: what have I done and what have I left to do? Thus, a basic process underlying all mental calculations could be
ordering or what some researchers (Merkel & Hall, 1982; Martin, 1978) have preferred to call memory for order.

Perhaps the most common measure of this process has been forward digit span. The technique has been included as a subtest in many psychometric measures of intellectual function including the Weschler Intelligence Scale for Children - Revised (Weschler, 1974) and the Weschler Adult Intelligence Scale (Weschler, 1955).

Dempster (1981) has cited studies which found that forward digit span correlated highly with various mathematics achievement measures, including an r = 0.77 on SAT - Math (p. 65). He claimed, "there is ample evidence that the factors underlying span also underlie performance in more complex domains and that memory span taps some basic aspect of human information processing" (p. 65).

Memory for order has been used in a few studies to study tasks which involve some type of mental calculation. Merkel and Hall (1982) attempted to determine if memory for order was related "to the amount of mental manipulation involved in tasks" (p. 428). They chose a very simple form of mental calculation as a measure of "mental manipulation." Each subject was required to solve 12 mental addition tasks involving 2 or 3 digit addends. Each task was classified as either involving a "carry" operation (MMC) or "not involving a carry" operation (MMNC). The MMC tasks were assumed to require more processing and thus were expected to induce a higher memory load than were the MMNC tasks.

A basic assumption not made explicit by the researchers was that each subject used the strategy suggested by their analysis. Hitch (1977) has suggested that this is not always a reasonable
assumption because some subjects can reduce the processing demand by choosing and using an efficient strategy. If a subject can reduce the processing demand of a calculative task, STM capacity likely will not be a significant factor in determining performance in that task.

Merkel and Hall (1982) reported correlations between forward digit span on MMC and on MMNC of 0.34 and 0.03, respectively (p. 434). These correlations are not particularly high and neither correlation was statistically significant. Not surprisingly, these mental addition tasks were particularly easy for the college-level subjects and, consequently, little variance in performance resulted. Because of this ceiling effect, the reliabilities for MMC and MMNC were quite low: MMNC was 0.36 and MMC was 0.52 (p. 432). A stronger relationship may have been detected if more reliable measures were used.

Merkel and Hall replicated the study using fifth graders instead of adults. The correlations of forward digit span on MMNC and on MMC were 0.42 and 0.28, respectively. Only the correlation between MMNC and forward digit span was statistically significant.

Merkel and Hall attempted to explain the different results for the younger and older subjects by arguing that "short-term or working memory increases or changes in nature with age" (p. 440). Another plausible explanation not identified by the researchers is that the college-level subjects could have used more efficient mental calculation strategies than the younger subjects. Since the study did not report the types of strategies used by the subjects in solving the mental addition tasks, this explanation cannot be validated.

Another measure that has been used to estimate the ability of a subject to process temporary information is delayed digit span. This
measure is similar to forward digit span in that a series of digits is presented to a subject. Instead of immediately recalling the digit series as is required in estimating forward digit span, the subjects must first attend to a short intervening activity before recalling the series. This task would seem to require the use of similar memory processes as mental calculation: both tasks require a subject to store some information for a short period of time while other information is being received or processed.

Whimbey, Fischhof and Silikowitz (1969) tried to determine if a correlational relationship existed between delayed digit span and a special form of mental addition. Each adult subject was asked to solve these types of tasks: "You have 8A, 3B, 2C and 5D and you add this to 2B and 5D. How many of each category do you now have?" (p. 57). Whimbey et al. obtained a correlation of 0.77 and, in a later replication, they reported a correlation of 0.67. Correcting the correlation of 0.77 for the unreliability of each measure produced a correlation of 0.95. Like Merkel and Hall (1982), they also concluded that some basic process measured by digit span and not general intelligence was responsible for these high correlations.

Whether or not a similarly high relationship between delayed digit span and mental multiplication performance would exist is not clear. In everyday instances of mental multiplication, there is at least an opportunity to apply a wide variety of strategies. In contrast, the mental calculation task incorporated by Whimbey et al. provides a subject with little opportunity to apply a diversity of calculative strategies. And, as Hunt (1980) has argued:

Performance in such a situation will be more determined by mechanistic information-processing functions than by choice
of a problem-solving strategy simply because of the limited range of strategies possible. (p. 458)

Transforming and Mental Calculation

Transforming is another short-term memory process often identified as basic to tasks requiring information processing. In addition to an initial storage of information, many tasks require that the original store be transformed to a form more in keeping with the mental operations required to solve the task. Backward digit span has been a sub-test in many psychometric measures including the WAIS (Wechsler, 1955) and the WISC-R (Weschler, 1974). These sub-tests have been used to estimate a subject's capacity to transform information. An estimate of this short-term memory capacity is obtained by requiring a subject to recall a serial list of digits in the reverse order of presentation.

Researchers such as Hiebert, Carpenter and Moser (1982), Romberg and Collis (1981), Scardamalia (1977), and Case and Globerson (1974) have included backward digit span as part of a battery of memory measures used to estimate "M-capacity": a construct developed by Pascual-Leone (1970) and which has been postulated to be a measure of the basic intellectual limitation of children.

Romberg and Collis (1981) have argued that estimates of backward digit span measure a basic process underlying many more complex information-processing tasks since the task requires a form of "short-term memory and an information operation or transformation. The numbers in the task not only have to be held in mind, they have to be held in mind while operating on them in some way" (p. 6).

Similar processing may take place during a mental calculation.
Those subjects who tend to use a series of "right-to-left" procedures to solve a calculative task have to transform the originally stored digits into a reversed listing, solve the various sub-tasks and, finally, output the product again by reversing a serially stored set of digits. Thus, backward digit span could be a good predictor for those types of mental calculation tasks where a subject must transform a series into a reversed arrangement.

Although no studies of mental calculation have incorporated backward digit span as an experimental variable, a few studies have examined the relationship between the basic processes underlying backward digit span and some form of mathematical reasoning. Hiebert et al. investigated, among other things, the relationships between the processes that first-grade children used to solve verbal addition and subtraction tasks and an information-processing task: in this case, the information-processing task was backward digit span. They found that of the four cognitive variables included in the study, backward digit span was the variable most consistently related to arithmetic performance. They concluded that:

 Viewing children's mathematical behaviour from an information processing perspective is a relatively new approach, and the search continues for appropriate measures of processing capacity as well as analysis procedures that specify the processing demands of individual tasks. (p. 97)

**Implications for Studies of Mental Calculation Tasks**

These few experimental studies have demonstrated that mental calculation requires a great deal of STM processing. Whether or not STM capacity is related to performance on all types of mental calculation tasks is not clear. The studies that have established a
relationship between capacity and mental calculation performance have
used tasks for which a variety of approaches was not possible.

Whether a relationship exists between capacity and performance
on mental calculation tasks such as multiplication where more
strategies could be available to deal with the demands of the
calculative tasks remains to be seen. One of the purposes of the
present study was to examine the relationship between several measures
of STM capacity and mental multiplication performance.
CHAPTER III

METHODS AND PROCEDURES

Outline of the Study

The three phases of the study were the pilot testing phase, the screening phase, and the interview and assessment phase. Each phase can be characterized by differing purposes and procedures.

The purpose of the pilot testing phase was to evaluate the feasibility of some testing procedures and, in particular, to help develop several instruments designed to assess proficiency in mental multiplication. The information gained from the pilot testing was used to refine many aspects of the subsequent phases of the study.

One of the major goals of the study was to acquire a detailed analysis of the processes involved in mental calculation. The methods of solution used by subjects to solve a task requiring a mental calculation were of particular interest. However, rarely can methods of solution be identified through an analysis of either a written or stated solution alone. A far more useful technique of analysis is to gather introspective reports either as a subject is attempting to solve a calculation task or immediately after a solution has been completed.

The collection of introspective reports requires that the researcher meet with individual subjects in a private setting. As can be appreciated, such a procedure is very time-consuming. To make the data collection and analysis more manageable, a relatively small sample by psychometric standards was selected: 30 high school students
and one elementary-school student. Thus, the purpose of the screening phase was to select the interview and assessment phase sample from a larger sample of 280 senior high school mathematics students.

If the study was to be able to identify the characteristics of subjects who exhibited differing levels of proficiency in mental calculation, a means of ensuring that the participating subjects did differ in calculative performance had to be found. Consequently the results of a screening test CAL1 were used to categorize subjects as either skilled or unskilled mental calculators. Fifteen skilled and fifteen unskilled subjects were selected by this method and each subject agreed to participate in the interview and assessment phase. One highly skilled elementary-school student was selected to participate in this phase of the study as well.

The purpose of the interview and assessment phase was to identify characteristics that appeared to distinguish skilled from unskilled mental calculators. To meet this purpose, several instruments were administered to all participating subjects. A thirty-item mental multiplication test CAL2 was administered individually to each subject. After stating a solution to a mental calculation task, the subject was asked to explain the method of solution in as much detail as possible. The researcher asked probing questions if the report seemed to lack the detail needed to classify the method of solution. These explanations were recorded on audio tape for later transcription and analysis. The time needed by a subject to complete each task was also measured.

Two tests of arithmetic fundamentals were also administered to individual subjects. One test of written paper-and-pencil
computational performance WPP required a subject to solve ten multi-digit multiplication tasks using any written algorithm. A test of basic fact recall BFR was used to assess each subject's ability to recall 100 basic multiplication facts. The times taken to complete each of these tests were also measured and recorded for all subjects.

Four measures designed to estimate a subject's short-term memory processing capacity were administered. Two measures incorporated in the Weschler Adult Intelligence Scale (Weschler, 1955), forward digit span FDS and backward digit span BDS, were administered individually. Two other measures, letter span LS and delayed digit span DDS, were administered to all the subjects in small groups of 4 to 5 subjects. DDS and LS were not administered to the one highly skilled subject.

Eleven subjects who demonstrated a high level of proficiency during the interviews were given the more challenging mental multiplication test CAL3. This test contained 15 items that had been identified through the pilot testing as being very difficult. The test procedures were similar to those used during the administration of CAL2. Each subject was asked to explain the method of solution and these explanations were recorded, reviewed, and later transcribed. Each subject's solution time was also measured.

A great deal of qualitative data was gathered during this phase of the study. The physical gestures of some subjects as they performed a calculation and their accompanying comments provided additional insights into the process of mental calculation.

In order to help the reader keep track of the various phases, samples, instruments, and procedures, an overview of the study has
been included in Table I.

Selection of the Subjects

With the exception of one subject, all subjects who participated in the screening as well as the interview and assessment phases of the study were senior high-school students currently enrolled in a grade 11 or 12 mathematics course. A high-school sample seemed preferable to an elementary-school sample because it was expected that older students would demonstrate a greater variation in mental multiplication performance than would younger students. This expected greater variation in performance seemed attractive for a study investigating individual differences.

In the screening phase, a test of mental multiplication CALL was administered to 286 students enrolled in ten different classes of senior-level mathematics. This sample included five grade 11 mathematics classes containing 139 students and five grade 12 mathematics classes containing 147 students. All subjects were studying mathematics courses that were part of the academic university entrance program. No subject was enrolled in either the less rigorous Alternative Mathematics program or the more practically-oriented General Mathematics program. All grade 11 students had completed Algebra 9 and 10, and all grade 12 students had completed Algebra 9, 10, and 20. The majority of students had also completed Geometry 10. Thus, all the high-school subjects who participated in the study had reasonably comparable mathematical opportunities.

Two secondary schools, or collegiates as they are called in Saskatchewan, agreed to participate in the study. Four classes
<table>
<thead>
<tr>
<th>Sample</th>
<th>Instruments and Procedures</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The Pilot Testing Phase</strong></td>
<td></td>
</tr>
<tr>
<td>180 university students</td>
<td>Five pilot tests; 80 mental multiplication items; 30-second item-presentation rate.</td>
</tr>
<tr>
<td>61 university students</td>
<td>Two trial-forms of a 20-item mental multiplication screening test; 20-second item-presentation rate.</td>
</tr>
<tr>
<td><strong>The Screening Phase</strong></td>
<td></td>
</tr>
<tr>
<td>280 senior high school mathematics students</td>
<td>CAL1: 20-item mental multiplication test; 20-second item-presentation rate. Used to select the skilled and unskilled sample.</td>
</tr>
<tr>
<td><strong>The Interview and Assessment Phase</strong></td>
<td></td>
</tr>
<tr>
<td>11 most skilled students</td>
<td>CAL3: 15-item difficult mental multiplication test. Used as a supplement to CAL2.</td>
</tr>
<tr>
<td>1 highly skilled 13-year-old girl</td>
<td>All tests and procedures administered except for DDS and LS. Knowledge of primes and squares also was ascertained.</td>
</tr>
</tbody>
</table>
containing 121 students were enrolled in school A which was under the jurisdiction of the Saskatoon Separate (Catholic) School Board. Six classes containing 165 students were enrolled in school B which was under the jurisdiction of the Saskatoon Public School Board.

Each school used a form of ability streaming for the mathematics classes. School A used the nomenclature A and B to designate the top two streams while school B used the nomenclature AA, A, and B to represent the top three streams. In school A, the A and B streams constituted approximately 30% and 50%, respectively, of all the grade 11 and 12 students. The remainder of subjects were either enrolled in a non-academic mathematics program or they were not taking any mathematics courses.

According to the mathematics department head of school B, the enrollment in the three top streams varies greatly from year to year. She reported that the present AA, A, and B streams constituted about 10%, 25%, and 50%, respectively, of the grade 11 and 12 student enrollment. As was the case for school A, the remainder of students were either taking a non-academic mathematics class or were not enrolled in any mathematics courses. The sampling distribution that was used during the screening phase is presented in Table II.

The subjects who participated in the screening phase were eligible to participate in the subsequent interview and assessment phase only if their performance on CALL met the standards established for this selection purpose. These standards will be discussed later in this chapter.

The original sample of 286 subjects was reduced to thirty subjects by applying these selection standards to the results of the
### TABLE II

**SAMPLING DISTRIBUTION OF SUBJECTS PARTICIPATING IN THE SCREENING PHASE**

<table>
<thead>
<tr>
<th>School</th>
<th>Ability Stream</th>
<th>Grade 11</th>
<th>Grade 12</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>27</td>
<td>34</td>
<td>61</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>32</td>
<td>28</td>
<td>60</td>
</tr>
<tr>
<td>B</td>
<td>AA</td>
<td>24</td>
<td>27</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>28</td>
<td>24</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>28</td>
<td>34</td>
<td>62</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>139</td>
<td>147</td>
<td>286</td>
</tr>
</tbody>
</table>
screening test call. Fifteen subjects were designated as skilled mental calculators and fifteen as unskilled mental calculators. Nineteen subjects were chosen from school A and eleven from school B. Originally twenty subjects attending school A were selected to participate in the final phase of the study but one subject found it difficult to make arrangements to meet with the researcher. By mutual agreement, he withdrew from the study.

All skilled subjects were enrolled in the top stream in each school. Likewise, all unskilled subjects were from the lower mathematics streams. Seven grade 11 and eight grade 12 students formed the skilled group and seven grade 11 and eight grade 12 students formed the unskilled group. Thirteen male and two female subjects formed the skilled group and five male and ten female formed the unskilled group.

These thirty subjects met individually with the researcher where the purpose of the study was explained and all procedures were described. At this time, an eligible subject was invited to participate in the interview and assessment phase of the study. All thirty subjects volunteered to participate in the study and each signed the subject consent form included in Appendix A.

One additional subject was later invited to participate in the study as a result of a fortuitous event. A grade 12 student approached the researcher after a screening test administration and explained that she had a young friend who she believed was an exceptionally skilled mental calculator. Fortunately, her offer to make the necessary arrangements was accepted because this 13-year-old subject proved to have calculative powers far beyond any of the
skilled subjects who participated in the study. The analysis of her calculative techniques led to a better understanding of the process of mental calculation. In this study, the term highly skilled will be used to describe the performance of this one subject.

To summarize, a group of 15 unskilled subjects, a group of 15 skilled subjects, and one highly skilled subject participated in the interview and assessment phase of the study.

Test Instruments and Procedures

The Screening Instrument: CAL1

Item selection. The process of item selection for all three mental calculation tests deviated somewhat from conventional practice. To develop a test for a study of written computational skills, the researcher typically selects items by first restricting the item domain to a series of 1-digit by 2-digit products, 1-digit by 3-digit products and, so forth. Often the items are categorized further by considering which place values would involve a "carried digit": typically, 1-digit by 2-digit tasks, no carries; 1-digit by 2-digit tasks, carry a ten; and, so forth. Once such a scheme has been developed, the items for the test can be created by selecting place value digits at random. In the case of written calculation, this procedure works well because the researcher makes the reasonable assumption that almost all subjects will apply the same strategy to solve each computation.

Such a scheme could not be used to select items for a test of mental calculation performance. The major purpose of the study was to determine if skilled and unskilled mental calculators differed in
their choices of calculative strategies. Thus, items had to be selected to ensure that at least an opportunity existed for the subjects to apply a diversity of strategies. Since the research literature on expert mental calculation suggested that apprehended number properties were a useful calculative aid, the majority of items were selected according to the number properties suggested by the factors.

The following scheme was created by the researcher to ensure that a large proportion of items would cue the hypothesized strategies if some subjects did in fact possess a knowledge of such strategies. All items selected for the study met one of the following conditions (A few examples are given to clarify the item selection procedures):

1. The prime factorization of both factors included only 2, 3, and 5 as prime factors. (Examples: 25 x 32, 12 x 15, 64 x 250, 125 x 125).

2. The item did not meet condition 1 and at least one factor was either a multiple of 10 or a multiple of 25. (Examples: 50 x 39, 8 x 70, 25 x 65, 75 x 201).

3. The item did not meet either condition 1 or 2 and either factor was one greater or one less than a multiple of a power of 10. (Examples: 8 x 99, 11 x 79, 49 x 51).

4. The item did not meet either condition 1, 2, or 3. (Examples: 87 x 23, 8 x 4211, 73 x 83, 13 x 13).

Condition 1 items were thought to have the best chance of eliciting a variety of calculative strategies and, therefore, two-thirds of the items were of this type. The condition 4 items, on the other hand, seemed least likely to elicit a variety of strategies
and, consequently, only 9 items were chosen to meet this condition. The numbers of items chosen to meet all four conditions are included in Table III.

In order to reduce the size of the item pool, an item was considered for selection only if the factors were of the following forms: 1-digit by 2-digit, 1-digit by 3-digit, 1-digit by 4-digit, 2-digit by 2-digit, 2-digit by 3-digit, 3-digit by 3-digit. The number of item forms included in each category is summarized in Table III. The majority of the items were of the 2-digit by 2-digit form while the other forms were used sparingly.

There were a number of reasons why the great majority of items selected for testing contained 2-digit by 2-digit factors. First, 2-digit by 2-digit factors were expected to provide a subject with more opportunities to use a variety of methods for determining a mental product than 1-digit by 2-digit factors. Second, finding the mental products of either 2-digit by 3-digit or 3-digit by 3-digit factors was expected to be a much more difficult task than finding the mental product of 2-digit by 2-digit factors. In order to ensure that less skilled subjects did not become overly discouraged, these more difficult items were used sparingly.

Five pilot tests were constructed from the 80 item pool. Four tests contained 15 items and one test contained 20 items.

Some special precautions had to be taken to ensure that all subjects used only mental calculation procedures. The group administration of any instrument designed to evaluate proficiency in mental calculation poses special problems: each subject needs a pencil to record a solution but a pencil cannot be used as a calculative aid.
TABLE III

DISTRIBUTION OF ITEMS SELECTED FOR THE PILOT TESTING PHASE

<table>
<thead>
<tr>
<th>Conditions</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-digit by 2-digit factors</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>1-digit by 3-digit factors</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>1-digit by 4-digit factors</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2-digit by 2-digit factors</td>
<td>39</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>52</td>
</tr>
<tr>
<td>2-digit by 3-digit factors</td>
<td>9</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>3-digit by 3-digit factors</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Total                53  6  12  9  80
Persuasion was the method used to prevent a subject from using a pencil to calculate. All subjects were informed that there was little to be gained from using a pencil in their calculations since the results of the test would have no effect on their coursework grades. The researcher emphasized that a pencil was to be used only to record the solution. They were instructed to record neither the factors nor any intermediate calculations.

As a check to determine if the test procedures had been followed, the answer sheets were carefully examined for any evidence indicating that some stage of the calculation had been recorded. As a result of this scrutiny, the results of ten subjects were removed before any analysis of the pilot testing data was conducted. Of course, there was no way to determine with certainty that a subject did not make some surreptitious record of a calculation.

There was an additional unconventional testing requirement: the time given to complete each item had to be controlled to some degree. For the purposes of this study, skilled mental calculators were defined as being more capable and faster than less skilled subjects in completing mental calculations. The decision to use speed of calculation as an additional criterion of mental calculation performance was based upon both experimental precedent and practical considerations. Reys, Rybolt, Bestgen, and Wyatt (1982) used response time to select good computational estimators. Levine (1982) recommended that the effect of time constraints on estimation ability be examined in future studies. Furthermore, since a "quick" mental calculation is often a practical necessity, the incorporation of a solution time constraint seemed both appropriate and desirable.
In a group situation, however, it is very difficult to determine the solution times of a large number of subjects who would complete each task likely at differing times. A practical method to estimate solution times in a large group setting is to control the item-presentation rate so that the time for each item is the same for all subjects. If the item-presentation rate is sufficiently rapid, only the most skilled subjects should be able to complete each calculation.

These methods of presenting the items and controlling the item-presentation rate were considered: audiotapes, videotapes, 35 mm slides (see Reys, Rybolt, Bestgen & Wyatt, 1982, p. 185), a microcomputer controlled monitor. After careful consideration of these possible modes of item presentation, a recorded audiotape presentation was used. The convenience and simplicity of the medium are obvious. As well, the researcher had used this presentation medium before (Hope, Note 1) and the participating subjects appeared to have little difficulty in remembering the factors despite the very short duration of the spoken word.

The five pilot tests were recorded on audio tape using an initial item-presentation rate of 30 seconds. Each item was expressed using the term "times": for example, "Item 1. Nine times fifty-one." The item order for each test was scrambled so that the beginning and the end of the test contained equally difficult items. This item arrangement was expected to encourage the less skilled subjects to attempt the later presented items.

The five pilot tapes were administered to 180 undergraduate university students who were currently enrolled in mathematics
education classes. There were several reasons why university rather than high school-students were used to pilot the test items. An important consideration was subject accessibility. University students were far easier for the researcher to access than were high-school students. A further and more important consideration was that the item results for each sample were expected to be very similar since the university and high-school students did have comparable mathematical backgrounds. When each sample's results on a number of common items were compared, the performance patterns of each sample were found to be very similar. Thus, the university subject pool seemed to be an adequate sample for pilot-testing purposes.

Each test administration was supervised by the researcher. Prior to each administration, the general nature and purpose of the study were explained. All students were offered the opportunity to withdraw from the testing. They were also given the choice of identifying their answer sheet or not. The total time needed to explain the testing procedures and administer the test to each class was about 15 minutes. No more than one pilot test was administered to a class but one test was administered to two classes.

Each item solution was scored as either correct or incorrect. A difficulty index (p-value) and a discrimination index (point-biserial r) were calculated for every item used in the pilot study. The conventional practice of selecting items with difficulties between 0.3 and 0.7 (Allen & Yen, 1979, p. 121) seemed inappropriate for a test designed to distinguish between two extremes of performance. Thus, it was decided to select items that were either very difficult or very easy. Specifically, an item was considered for
inclusion if it was either difficult \((0.03 < p < 0.30)\) or easy \((p > 0.85)\). The inclusion of very easy and very difficult items was expected to provide good discrimination between extreme and non-extreme performances (Allen & Yen, 1979, p. 121). The inclusion of a relatively large number of very easy items \((p > 0.85)\) was to ensure that only the most unskilled mental calculators would be unable to answer at least one or two items correctly.

Two potential trial-forms of CALI were constructed from the pool of 80 items that had been administered in the preliminary stages of the pilot study. Each trial-form contained 10 difficult items designated as Part A and 10 easy items designated as Part B. Part A had different items but the items of Part B were identical for each trial-form.

The items 25 x 25, 25 x 48, and 16 x 16 had difficulty indices of 0.42, 0.41, and 0.34, respectively. Although these difficulty indices were not within the range stipulated for inclusion, the items were included in a trial-form because each item contained many attributes that were expected to aid in a mental calculation.

The final procedure in the development of CALI was to administer each of these trial-forms to another pilot sample. Sixty-one university students agreed to complete both trial-forms. The test procedures were similar to those used in the preliminary stages of the pilot testing. However, an item-presentation rate of 20 seconds was used instead of 30 seconds. This change in item-presentation rate was expected to further increase the difficulty of each trial-form.

The analysis of the test results indicated that each trial-form
was equally difficult (10.95 and 10.56) and the variance was about the same (3.96 and 5.06) for each test. The most reliable trial-form was retained as CALL. The final trial-form of CALL selected to be used in the screening phase is included in Appendix B.

The results of these two trial-forms were also used to evaluate the selection criteria that had been established for subject participation in the interview and assessment phase. Essentially, an attempt was made to set a basal level for skilled performance and to set both a basal and a ceiling level for unskilled performance. Specifically, the screening criteria were:

1. Any subject who answered 7 or more of the 10 difficult CALL items correctly and who also answered at least 9 of the 10 easy items correctly was defined as a skilled mental calculator.

2. Any subject who answered at least 2 but less than 10 of the easy CALL items correctly and who also answered no more than one of the difficult items correctly was defined as an unskilled mental calculator.

These external standards will provide any researcher planning a replication with more suitable information than provided by relative standards such as "the skilled are those subjects whose scores were in the top 10%." The basal level established for the skilled subjects seemed reasonable for such an exploratory study. The basal level set for unskilled performance was chosen to ensure that all subjects participating in the later interview and assessment phase would have at least some minimal proficiency in mental calculation.

Fewer than two percent of the university subjects sampled in the pilot study phase met the proposed standards established for
skilled mental calculation performance. When the screening standards were applied to the results of one trial-form of CAL1, only 1 of the 61 university subjects demonstrated skilled performance while the performances of 6 university subjects were designated as unskilled. After the standards were applied to the results of the second trial-form of CAL1, no subject was found who met the skilled performance standards while 17 subjects were designated as unskilled mental calculators.

Test administration. The screening phase, which involved the administration of CAL1 to high-school students, began in school A during mid-October and was completed in school B by late November. All testing took place during a regularly scheduled mathematics class. The researcher supervised every test administration.

All subjects and teachers were most cooperative and no major problems were encountered. There were the expected minor difficulties such as an unscheduled fire drill in the middle of one test administration.

Despite the researcher's emphasis, a few individuals either misunderstood or chose to ignore the instructions not to use a pencil as a calculative aid. This deviation from the test instructions was apparent both from observing a subject's behaviour during the test administration and from the evident pencil marks on a subject's answer sheet. Six subjects ignored this procedural requirement and their results were removed from any analyses.

Scoring. All answers were scored either as correct or incorrect. For each subject, a total score, a score on Part A, and a score for Part B were recorded.
When the selection standards were applied to these results, 16 of the 280 subjects (approximately 6%) met the standards set for skilled performance. Ninety-four subjects (approximately 34%) were classified as unskilled mental calculators. Because the statistical techniques used in the study assume large differences in performance between the two extreme groups, the subjects demonstrating the most extreme performances were selected. Each extreme group contained 15 subjects.

CAL1 was also administered to the highly skilled subject. She easily met the criteria established for skilled performance.

On the advice of a teacher, some unskilled subjects were not asked to participate in the interview and assessment phase of the study. Subjects who demonstrated comparable performance were chosen as replacements.

The Probing Instrument: CAL2

Item selection. CAL2 was the instrument used in the interview and assessment phase to determine each participating subject's method of solution. The 30 multiplication tasks were all selected from the 80 item pool used in the pilot testing phase.

The item selection process for CAL2 as in the construction of CAL1 was guided more by the expected strategies that an item might elicit than by its psychometric properties. However, an attempt was made to select a cross-section of items that did vary in difficulty. Some easy items were included so that the unskilled mental calculators would be encouraged to attempt the more difficult items. Difficult items were needed to challenge and maintain the interest of the
skilled mental calculators. About 50% of the items selected had difficulty indices between 0.3 and 0.7, 20% had indices greater than 0.7, and the remaining 30% had indices less than 0.3. Five items selected for CAL2 had also been included in CAL1. Appendix B contains a listing of all 30 items included in CAL2.

Test administration. Unlike CAL1 which was administered to large groups of subjects, CAL2 was administered to individuals. Each subject met with the researcher in a private setting. After explaining the purpose of the test, the researcher told the subject that a calculation task requiring mental multiplication would be stated orally. As soon as both factors were heard, the subject would be expected to make an attempt to determine the solution by using any mental method that seemed "natural." If it was natural to talk out loud or to use some aid such as fingers, the subject was instructed to do so. Each subject was encouraged to ignore the presence of the researcher as a calculation attempt was made.

All subjects were told that accuracy was more important than speed, but they were instructed to complete the calculation as quickly as possible. For the first two test administrations, a stop watch in full view of the subject was used to assess a subject's solution time for each item. However, this method of timing was abandoned because it proved to be very distracting for subjects. Instead, the solution times were calculated from the audio record made of each interview.

Each subject was told that the researcher would not indicate whether any answer was correct or not. Such information was withheld to prevent the less skilled mental calculators from becoming too discouraged. However, some exceptions to this procedural rule had to
be made: a few unskilled subjects were encouraged by the knowledge that a solution was correct and appeared more willing to attempt subsequent tasks.

Subjects were often asked to make more than one attempt at a solution. In fact, some subjects made up to four attempts. Any subject who requested another attempt was always allowed to do so but only the first attempt was used to determine the accuracy and speed of a solution. A subject's subsequent attempts were used by the researcher to help identify the method of solution used in the initial attempt and to provide additional information about the difficulties a subject was experiencing during a mental calculation.

Whether or not a correct solution was stated, each subject was asked to explain the calculative method that was applied. Usually a subject's method of solution was evident from the introspective report that was provided. If the method could not be clearly identified, the researcher would ask the subject to provide more detail. For example, the question, "Did you multiply the 25 and 8 by doing 8 x 5, carry the 4, and so on or did you just think 200?," was posed to some subjects after an attempt to solve the task 25 x 48.

At times, some students needed a pencil and paper to describe a method of solution but such external aids were NOT available during the attempt to solve a given task. When the researcher was convinced that either the method of solution had been clearly identified or further probing would be futile, he progressed to the next item.

The order of item presentation during the interview corresponds to the item order listed in Appendix B. This order was used with all but one subject. This unskilled subject was unwilling to attempt any
item that seemed difficult, especially in the early stages of the interview. In order to offer encouragement, the easiest items were administered first. As a result of this deviation from the usual testing procedures, she appeared to relax and made an attempt to solve all items with the exception of 25 x 480.

The amount of time needed to complete the administration of CAL2 varied greatly from subject to subject. Typically, with skilled subjects, about one hour was needed to administer CAL2 and discuss each subject's method of solution. However, the slow responses and numerous attempts at re-calculation greatly increased the time needed to complete the interviews with the unskilled subjects. The interview of one unskilled subject, for example, took about three one-hour sessions to complete.

An attempt was made to ensure that consistent standards of procedure were maintained throughout the interview and assessment phase: specific tasks, task sequences, instructions, and questions were common to all interviews. Nevertheless, the researcher did not hesitate to deviate from a planned interview if an opportunity to gain additional insight into the process of mental calculation arose. Some subjects were presented with unique tasks and questions: only 6 subjects were given the mental calculation 123 x 456; only one subject was asked to multiply mentally 8 x 25. Often the information gained from such spontaneous decisions helped improve the quality of the subsequent interviews.

Appendix C contains a partial CAL2 interview with one skilled subject. This interview provides further clarification of the procedures used in this study to determine a subject's methods of
solution.

Scoring. Each item of CAL2 was scored as either correct or incorrect. If a subject had requested that the factors be repeated after the calculation had begun, the item was scored as incorrect. But if the subject had requested confirmation of the task by responding with a question such as, "Was it 12 x 15?", the item was scored on the basis of the subject's completed solution.

If a subject immediately corrected an erroneous response without any prompting from the interviewer, the item was scored as correct. One subject's response to 8 x 4211 should clarify this scoring procedure. He said, "32 611... No, wait... It's 33 688 rather." A maximum CAL2 proficiency score of 30 was possible.

The time taken by a subject to complete a solution was measured by using a stopwatch. Only the first attempt at a task was timed. This timing information was gathered during the analysis of the audio records made for each subject. Timing was initiated as soon as the researcher had stated the question and terminated as soon as the subject had stated a solution. Each solution time was estimated to the nearest second.

Often a subject's solution time was increased because of a protracted statement of the solution. For example, one subject responded, "6000, 600, and 78." This statement took 4 seconds. The solution time was increased for those subjects who immediately corrected a stated solution. This increase in solution time is well illustrated by the following subject's response to the task 32 x 500. He said, "1600, No, wait, 2600, No, wait. 32 x 500 is 1500, 15 000, 16 000!"
The Challenge Test: CAL3

Item selection. CAL3 was a mental multiplication test which was designed to challenge the most skilled mental calculators participating in the study. The purpose of the test was to determine if these subjects would use other strategies not identified during the administration of CAL2. The challenge test included 15 items which had been identified during the pilot testing phase as being very difficult (p < 0.10). A list of the items included in CAL3 is presented in Appendix B.

Test administration. The procedures used to administer CAL3 were identical to those used for the administration of CAL2. The test was given to only one subject at a time. Each calculative task was read to a subject and immediately after a solution was stated, the subject was asked to provide an explanation of the method of solution. All responses were recorded on audio tape for later analysis.

The test was administered to 11 skilled subjects and the one highly skilled subject. CAL3 was not administered to four skilled subjects because they were not enthusiastic about further participation. Since the only purpose of CAL3 was to provide supplementary information about the types of mental calculation strategies known by skilled calculators and no comparisons between skilled and unskilled performers were to be made, 11 subjects seemed an adequate sample size. CAL3 was not administered to any unskilled subject.

Scoring. Each item was scored and recorded as either correct or incorrect. The solution time for each correct response was also estimated from the audio records.
The Test of Written Paper-and-Pencil Computational Skills: WPP

Item selection. The purpose of WPP was two-fold. First, the results of WPP were to be used to eliminate any subjects who could not correctly apply a written algorithm. For the purposes of the present study, there seemed to be little point in studying such dysfunctional subjects. No subjects were eliminated from the study because of their poor performance on WPP.

A more important purpose of WPP was to compare mental and written calculation performances. Any differences in performance resulting from the absence of the permanent memory-store served by the written page were assumed to reflect the additional burden on the limited resources of the short-term memory system.

To ensure that mental and written calculation performance could be compared, the same difficult items included in CAL1 (the A-items) were included in WPP. These 10 items are listed in Appendix B.

Test administration. WPP was administered individually to each subject. WPP could have been administered as a group test. Since each subject took no more than 2-3 minutes to complete the 10 written items, WPP was administered individually at the mid-point of each CAL2 interview.

All 10 items were listed on a sheet in a scrambled presentation order. Each subject was instructed to solve these multiplication questions by applying any written method of calculation. They were asked to record all steps in the calculation.

Instructions were given to complete the calculation as quickly as possible as the time taken to complete the test would be recorded. However, the researcher also emphasized that accuracy should not be
sacrificed for an increase in speed.

A minor problem arose when a few subjects misunderstood the directions and tried to determine a mental rather than a written solution. In these cases, the subject was stopped and the procedures were clarified. Despite this additional instruction in procedures, a few skilled subjects continued to use mental calculation to solve tasks such as $16 \times 16$. No comment was made about this minor deviation from procedure and the subjects were allowed to complete the test without any further interruptions.

Scoring. Each item in WPP was scored and recorded as either correct or incorrect. The maximum possible score was 10. The time needed to complete the test was recorded for each subject.

The Test of Recall of Basic Facts: BFR

Item selection. This instrument was designed to determine how quickly and accurately a subject could recall the basic facts of multiplication. Since these numerical equivalents are the basic "building blocks" of most calculations, an assessment of recall seemed vital to any analysis of individual differences in mental calculation performance.

BFR included all 100 multiplication items formed from single digit factors. The order of these items was scrambled to ensure that the solution of one item could not be used to aid in the recall of a successive item.

Test administration. BFR was administered immediately after the administration of WPP was completed. A subject was given a sheet listing the 100 items and instructed to state the solution to each
item as quickly as possible. The subject was reminded that, although
the time taken to complete the test would be measured, accuracy was
considered to be more important than speed. The interviewer recorded a
subject's incorrect responses as the testing proceeded.

Scoring. Each item was scored as either correct or incorrect. The maximum possible score on BFR was 100. The total access time,
defined as the time needed to complete BFR, was also estimated to the
nearest second and recorded for each subject.

Forward Digit Span: FDS

Item selection. A subtest of the Wechsler Adult Intelligence
Scale (Wechsler, 1955) was used to estimate forward digit span FDS.
This subtest contains two lists (Trial I and II) of seven series of
digits. Each series of digits varies in length from 3 to 9 digits.

Test administration. FDS was administered individually to all
subjects. The subjects were instructed that a series of digits would
be read to them. As soon as a series was read and the researcher
paused, a subject was expected to recall the digits in the same order
as the presentation. Each series was read at a rate of one digit-per-
second.

Beginning with the shortest series listed in Trial I, the
researcher continued to present successively longer series until an
error was made by a subject. Thereafter, the researcher presented a
series of equivalent length from Trial II. This process continued
until either a list was completed or a subject failed on both trials
of a given series.

Scoring. A subject's score on FDS was the number of digits in
the largest series repeated without error in Trial I or II. The maximum possible score for FDS is 9. But in the event a subject "reached the ceiling" of either FDS or BDS, longer series were to be administered. These longer series were needed for only one subject's re-test of FDS.

**Backward Digit Span: BDS**

**Item selection.** The subtest of the WAIS (Wechsler, 1955) was used to assess backward digit span BDS. This subtest contains two trials of seven series of digits whose length ranges from 2 to 8 digits.

**Test administration.** The test procedures were similar to those used to administer FDS with the exception that each subject was instructed to "say the series backwards." To ensure that all subjects understood this change in procedure, a practice series was given.

**Scoring.** A score for each subject was calculated by determining the number of digits in the longest series repeated backwards without error in Trial I or II. The maximum possible score for BDS was 8.

**Delayed Digit Span: DDS**

**Item selection.** The items for DDS used in the present study were constructed using the procedures outlined in a study conducted by Whimbey and Lieblum (1967). Six trials of 5 series of digits were constructed. The digits in each series were selected at random. The length of each series varied from 4 to 8 digits. The order of presentation of each series was scrambled for each trial. All series included in DDS are presented in Appendix B.
Test administration. DDS was administered to groups ranging in size from 3 to 6 students. Small group sizes were necessary to ensure that all subjects followed the somewhat "unconventional" testing procedures.

Each subject was given a sheet to record the attempts at recall. On each sheet was a section which listed the following word-letter pairs: bear-Q, bird-L, crab-V, fish-H, frog-R, flea-J. The presentation of each series of digits was followed immediately by a statement of one of these six words. All subjects were instructed to locate the word-letter section after hearing any one of these words, find the word, and write the corresponding letter in the designated place on the record sheet before attempting to write the digit series. A practice trial was given to ensure that all subjects understood the directions.

Each series of digits was read at a rate of 2 digits-per-second. The researcher listened through an earphone to an electronic metronome to help pace the reading of each series.

Scoring. In scoring DDS, a series was considered correct if it was reproduced in the order presented. A subject's score was determined by alloting 4 points for each 4-digit series correct, 5 points for each 5-digit series correct, and so forth. This scoring procedure was identical to that used by Whimbey and Lieblum (1967). The maximum possible score for DDS was 180.

Letter Span: LS

Item selection. Another measure commonly used to assess STM capacity is letter span LS. The items for LS were constructed using
procedures similar to those used by Brown and Kirsner (1980). Four trials containing 9 series of letters were developed by randomly selecting consonants from B to M. The absence of vowels ensured that a series contained no recognizable English words which could be used by a subject to foster retention. Each series length varied from 3 to 11 letters. The order of each series in a trial was scrambled.

Test administration. LS was administered to small groups of subjects in the same testing session as DDS. Each subject was given a sheet designed to record a given series.

The subjects were instructed to attempt to recall each series in the same order as the presentation. Each letter in a series was read at a rate of one letter-per-second. One trial list was completed to ensure that all subjects understood the instructions.

Scoring. Each series was considered correct if it was recalled in the order presented. A score was determined by assigning 3 points for a 3-letter series, 4 points for a 4-letter series, and so forth. The maximum possible score for LS was 252.

Classification of Methods of Solution

Since one of the major purposes of the study was to identify the calculative strategies used by high-school students to solve mental multiplication tasks, a strategy classification scheme was needed. It became evident after a somewhat fruitless search of the literature that the researcher would have to develop a scheme.

The analyses of experts' calculative methods outlined in Chapter II did guide the research in the preliminary stages of the study. Nevertheless, the continual intertwining of data collection and
analysis helped the researcher develop and refine the classification scheme used to answer the research questions.

An example will illustrate how analysis of the accumulated data led to a refinement of the strategy classification scheme. In the early stages of the interview and assessment phase the unskilled subjects almost always used the pencil-and-paper analogue while the skilled subjects rarely used this method of solution. These patterns of approach to the task continued until the last interview with a skilled subject. Unlike the other skilled subjects, she made almost exclusive use of this method for even particularly difficult tasks. This "exception to the rule" forced the researcher to elaborate on the pencil-and-paper mental analogue. As a result, four specific paper-and-pencil mental strategies were identified and included in the classification scheme.

To facilitate the classification of the calculative procedures used by a subject to solve the CAL2 and CAL3 mental calculation tasks, all audio records of the interviews were transcribed. Approximately 300 typed pages resulted from this transcription process. A few recorded solutions for two subjects were destroyed by the tape-recorder: specifically, an unskilled subject's reported solutions for 5 items and a skilled subject's reported solutions for 3 items. In these cases, the written notes made during the CAL2 and CAL3 interviews were substituted for the missing audio records.

If the interviewer thought that the subject had not provided sufficient details for classifying the applied strategy, the subject was questioned for more details. Although the subjects might not remember the specific numbers that were involved in a computation,
they usually had no problem remembering and describing the method of solution.

As is often the case with any classification scheme, a certain amount of ambiguity had to be tolerated. In particular, some strategies were difficult to classify under a single heading because the subject seemed to combine several strategies. Completing a calculative task by combining strategies was especially true of the skilled subjects during the CAL3 interviews. For those situations where a combination of strategies was used, the strategy was classified according to what the interviewer believed was the primary or dominant strategy. If a subject appeared to have arranged the task so that a factoring strategy could be applied, for example, but completed a stage of the calculation using another strategy such as additive distribution, the solution attempt was classified as factoring.

The annexation algorithm —calculating with factors which are multiples of a power of 10 by annexing zeroes— is a form of factoring. But annexation played a secondary rather than a primary role in solving the types of calculative tasks included in this study. For example, even though many subjects began to calculate 25 x 480 by reasoning, "take off the zero and add it later," another strategy was needed to solve the difficult portion of the computational task, in this case, 25 x 48. There were no items included in CAL2 and CAL3 which could be solved through the exclusive application of the annexation algorithm.

The analysis of the introspective reports revealed that four general methods of solution were used by the subjects to solve the
CAL2 and CAL3 mental multiplication tasks. Eleven specific calculative strategies were also identified. The relationship between the general methods and the specific strategies reported by some or all subjects is summarized in Table IV.

**Pencil-and-paper mental analogue (P&P).** This general method of mental calculation involves the application of the conventional pencil-and-paper algorithm to solve mental calculation tasks. These four main variations were identified: (1) no partial product retrieved (P&PO); (2) one partial product retrieved (P&P1); (3) two partial products retrieved (P&P2); (4) stacking. The digits 0, 1, and 2 in P&PO, P&P1, and P&P2, respectively, represent the number of times during a calculation a subject who used the pencil-and-paper mental analogue retrieved a numerical equivalent larger than a basic fact to determine a partial product.

1. **No partial product retrieved (P&PO).** A subject using this strategy will make no attempt to adapt pencil-and-paper methods to a mental medium. Regardless of the type of task presented, the calculation will proceed in a digit-by-digit, right-to-left manner. Each partial product will be calculated and no numerical equivalents larger than basic facts will be retrieved during the calculation.

One unskilled subject's unsuccessful attempt to solve $25 \times 480$ provides a good illustration of an application of this strategy:

Let's see. 480 on the top and 25 on the bottom. $5 \times 0$, $5 \times 8$ is 40, carry 4, and 4 is 24. I have to realize that the second number is one over. $2 \times 0$, $2 \times 8$ is 16, carry 1; $2 \times 4$ is 8 and 1 is 9 so 960, 9600. So 9600 and 2400 is 0, 0,.....19 thousand and....860.

2. **One partial retrieved (P&P1).** Rather than completing the calculation of each partial product by proceeding digit by digit, a
TABLE IV

GENERAL METHODS AND SPECIFIC STRATEGIES USED TO SOLVE MENTAL MULTIPLICATION TASKS

<table>
<thead>
<tr>
<th>General Method</th>
<th>Specific Strategy</th>
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<tr>
<td>Pencil-and-Paper</td>
<td>P&amp;PO: No partial product retrieved</td>
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<tr>
<td>Mental Analogue</td>
<td>P&amp;P1: One partial product retrieved</td>
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<tr>
<td></td>
<td>P&amp;P2: Two partial products retrieved</td>
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<tr>
<td></td>
<td>Stacking</td>
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<td>Distribution</td>
<td>Additive</td>
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<td>Fractional</td>
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<td>Half-and-double</td>
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<td>Aliquot parts</td>
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<td>Exponential</td>
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<tr>
<td>Retrieval of a</td>
<td>Numerical Equivalent</td>
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<td>Numerical Equivalent</td>
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</table>
subject who uses P&P1 will retrieve a numerical equivalent for one partial product. The following report involved an application of P&P1 to calculate 25 x 48: "5 x 48 is... 5 x 8 = 40, carry 4, 24, 240. And 2 x 48 is 96. I know that. I think of 96 brought over one. So it's 1200." Notice that the partial product 240 was calculated digit by digit but the second partial product 96 was retrieved as a numerical equivalent.

3. Two partials retrieved (P&P2). In this variation of the pencil-and-paper mental analogue, the calculation will be arranged so that two partial products can be retrieved as numerical equivalents rather than calculated digit by digit. One subject's explanation of 12 x 250 illustrates an application of this strategy. She explained, "2 x 250 is 500 and 1 x 250 is 250. Move over one, 3000."

4. Stacking. This strategy can be applied to questions where the factors are of a 1-digit by x-digit form. Essentially, each partial product will be completed digit-by-digit but, instead of carrying, each product will be visualized in a stacked arrangement. One reported solution of 8 x 999 is a good example: "I thought 8 x 9 is 72, 72, and 72, right across." The subject who provided this report said that she never "carried" during the calculation. She determined the total sum by adding from left to right. To describe her thought processes to the researcher, she drew the diagram below:

```
  72
  72
  72
  7992
```
Distribution. This method of calculation is initiated by transforming one or more factors into a series of either sums or differences. The calculation proceeds by applying one of these following four strategies:

1. **Additive.** This strategy is based on the principle of the distributive law of multiplication over addition. As the calculation progresses, each partial product will be added successively to produce a running sum. The following solution of $8 \times 4211$ involved an application of additive distribution: "$8 \times 4000$ is 32 000, $8 \times 200$ is 1600, so it's 33 600. $8 \times 11$ is 88, so the answer is 33 688."

Unlike the use of P&PO where the calculation proceeds digit by digit, additive distribution involves calculating frequently with multiples of powers of ten. Thus, knowledge of the annexation algorithm is essential to the successful application of this strategy.

For some calculations a factor will not be transformed into a series of addends whose terms involve multiples of a power of ten. Instead, the computation will be arranged so that known numerical equivalents can be retrieved. The skilled subject who solved $15 \times 16$ by thinking, "15 squared is 225 which is a fact I know and another 15 is 240," illustrates such an application: i.e., $15 \times 16 = 15 \times (15 + 1)$.

There were some reported solutions where P&P2 and P&P3 were difficult to distinguish from additive distribution. In these few cases, the researcher had to rely on additional information to guide the classification decision. The language used by a subject to describe a calculation procedure was often helpful in distinguishing between these two strategies. Terms characteristic of written
multiplication procedures such as "12 on top, and 15 on the bottom," "carry the 1," and "move over one" were often used by subjects to describe the pencil-and-paper strategies. Such terms were usually absent in reported uses of distribution.

Generally speaking, the calculations for pencil-and-paper and distribution strategies proceeded in opposite "directions": pencil-and-paper calculations proceeded from the least to the most significant digits (right to left) and distribution proceeded from the most to the least significant digits (left to right). This difference in direction often was used to distinguish P&P2 and P&P3 from additive distribution.

Another distinguishing characteristic of additive distribution was that most calculations usually involved an application of the annexation algorithm which was expressed by the user in terms such as "9 x 7 is 63 and add 3 zeroes." An application of pencil-and-paper strategies usually incorporated some abridgement such as using 12 to represent 120.

2. **Fractional.** For those factors which have a unit digit equal to 5, fractional distribution can be applied. A task such as 15 x 48 can be calculated as, "10 x 48, 480 and 1/2 of 480 is 240; so it's 720." Larger factors such as 125 x 125 can be reasoned as, "100 x 125, 12 500 and 1/4 of 12 500 is 3125. So it's 15 625."

Although factoring is needed to complete a portion of the calculation, this strategy was classified as a type of distribution rather than as a type of factoring because the calculation proceeds by partitioning each factor initially into a sum rather than a product.

3. **Subtractive.** This strategy is based upon the distributive
principle of multiplication over subtraction. Before the calculation can proceed, the subject must be able to express a factor as a difference between two numbers which the subject thinks will make the calculation more tractable. For example, $8 \times 999$ was solved by the majority of skilled subjects who reasoned $8000 - 8 = 7992$: i.e.,

$$8 \times 999 = 8 \times (1000 - 1).$$

A few subjects can apply subtractive distribution by incorporating a retrieval of a large numerical equivalent. For example, one skilled subject solved $15 \times 16$ by thinking, "16 squared is 256 and then minus 16, so 240": i.e.,

$$15 \times 16 = (16 - 1) \times 16.$$  

4. **Quadratic.** Certain properties of quadratics can be used by some subjects as a calculative aid. There were three forms of quadratic distribution identified in the study.

If the square of the mean value of the two factors in a multiplication task can be retrieved by a subject, the calculation can be completed by using the algebraic identity for the difference of squares: i.e.,

$$(x - y)(x + y) = x^2 - y^2.$$  

For $49 \times 51$, one subject reasoned, "50 x 50 minus 1" and, for $23 \times 27$, he thought, "25 squared minus 2 squared, so it's 625 - 4 = 621."

The remaining two variations of quadratic distribution were not used by any subject in the CAL2 interviews. However, they were used to solve some CAL3 items. One strategy involved binomial expansion: i.e.,

$$(x + y)^2 = x^2 + 2xy + y^2.$$  

To determine a square, one subject reasoned, 

"$48 \times 48 = (50 - 2)^2 = 2500 - 200 + 4."$$  

This same subject used quadratic distribution to calculate $125 \times 125$ by reasoning, $$(100 + 25)^2 = 10000 + 5000 + 625.$$  

The application of another variation of quadratic distribution
is limited to those squares whose unit digit is 5. For example, \(75^2\) can be calculated by reasoning, \(7 \times 8 = 56\) and add (annex) 25, so it's 5625. Such a strategy is based on the knowledge that \((10x + 5)^2 = 100x(x + 1) + 25\).

Factoring. This method of solution differs from distribution in that one or more factors in the presented calculative task can be transformed into a series of products or quotients rather than into a series of sums or differences. Several types of factoring strategies were identified.

1. General. To apply this strategy, the user must be able to factor one or more factors before applying the associative law for multiplication. To solve \(25 \times 48\), one subject factored 25 into \(5 \times 5\) and reasoned, "\(5 \times 48\) is 240 and \(5 \times 240\) is 1200." This subject also applied additive distribution to determine the intermediate calculations \(5 \times 48\) and \(5 \times 240\): i.e., \(5 \times 48 = 5 \times 40 + 5 \times 8\) and \(5 \times 240 = 5 \times 200 + 5 \times 40\), respectively. Despite the subject's additional use of additive distribution, the reasoning was classified as general factoring rather than distribution because the subject transformed the computation initially into a series of products rather than sums. Additive distribution can be thought of as playing an ancillary rather than primary calculative role during an application of factoring.

The remaining variations of factoring could all be considered, in a mathematical sense, as special cases of general factoring. However, since these variations did seem to differ with regard to their range of applications, any use of these variations were reported as separate strategies.
2. **Half-and-double.** If one factor in a multiplication task is a multiple of 2, a special form of factoring can be employed. The calculation proceeds usually by "halving" the multiple of 2 and "doubling" the remaining factor. This process of halving and doubling continues to a point where the subject can complete the calculation by applying another strategy.

The following examples illustrate how two subjects who both used the half-and-double factoring strategy differed in their method of terminating this process. One subject reported that to calculate 12 x 16, "I did doubling and halving again. I did 1/2 of 12 is 6. Doubling 16 gives 32. And 6 x 30 is 180, so 192." Another subject explained, "I just decided to multiply by 2, or 12 x 2, four times."

3. **Aliquot parts.** Instead of transforming a factor into two or more factors as is the case with other factoring strategies, a subject using aliquot parts transforms one factor into a quotient. For example, although she erred in calculating 25 x 480, one subject's explanation illustrates how aliquot parts was applied. She explained, "When I multiply by 25 I think of money, like quarters. So I divided 480 by 4, 120 and then changed it to 1200."

This strategy is applied frequently to those computations where one factor "f" is a factor of a power of 10 "p" and the remaining factor is a multiple of the quotient "p/f". Items such as 25 x 48 and 12 x 250, for example, can be solved through an application of aliquot parts: i.e., 25 x 48 = 100/4 x 48 = 100 x 48/4 = 100 x 12 = 1200 and 12 x 250 = 12 x 1000/4 = 12/4 x 1000 = 3 x 1000 = 3000.

4. **Exponential.** This form of factoring can be used to calculate the products of powers through the application of an
exponential rule. To solve $32 \times 32$, one subject reasoned: "I solved it by thinking powers of 2. 32 is 2 to the fifth, so squared is 2 to the tenth which I just know is 1024."

There were some reports that could have been classified as applications of either half-and-double or exponential factoring. For example, $32 \times 64$ was reasoned by one subject as "32 x 32 is 1024, which is a power of 2, so double it, 2048." Further questioning revealed that the subject knew 1024 was a power of 2 but he did not know that 1024 was equal to $2^{10}$. If the subject seemed to incorporate some knowledge of exponential arithmetic into the calculation, the strategy was classified as exponential factoring.

Retrieval of a numerical equivalent. To determine some products, a subject appears to do no calculation and solves the task by retrieving the product from memory. For example, many skilled subjects stated immediately "625" when presented with the task $25 \times 25$.

A subject's method of solution was classified as retrieval only if both the subject and researcher were convinced that no calculation had taken place. For example, the response, "I did $5 \times 25$, then I kind of remembered that $25 \times 25$ is 625," was classified as general factoring rather than retrieval because the subject admitted later that some calculation had been necessary. A quick response (less than 3 seconds) was considered as additional evidence that retrieval rather than calculation had taken place.
Reliability of the Instruments and the Procedures

All instruments and procedures used in the various phases of the study were designed for differing purposes and functions. Consequently, a variety of methods was needed to assess the "reliability" of these various measures.

One problem inherent in extreme group designs is calculating reliability coefficients that can be interpreted with some meaning. All subjects who participated in the interview and assessment phase were not selected at random from the population. Rather the selection of the sample was based upon the performance standards established for this purpose. Any reliability coefficient based on the combined scores of the skilled and unskilled subjects would apply only to this type of extreme sample. An attempt was made, however, to incorporate some method of assessing the reliability of each measure.

Reliability of CAL1

CAL1 served two major functions in the study. First, this test was used to select subjects who differed in mental calculation performance. Second, CAL1 was the instrument used to determine the strength of relationship between mental multiplication performance and various other measures used in the study. Therefore, it was important to calculate some index of the instrument's reliability.

Two estimates of the reliability of CAL1 were calculated. To estimate the internal consistency, the Kuder-Richardson 20 formula was applied and a coefficient of 0.80 was obtained. This suggested that CAL1 was a relatively homogenous test.

Furthermore, as CAL1 was used to select students who differed
in mental calculation performance, it seemed important to determine if an individual's score remained uniform over a period of time and to ensure, at the very least, that the relative standings of the subjects were maintained. CAL1 would not be a particularly reliable screening instrument if a subsequent re-administration of the test indicated that some unskilled subjects should be re-classified as skilled and, conversely, some skilled subjects were identified as unskilled.

To determine the stability of a subject's score on CAL1 over a period of time, a large sample of the subjects who participated in the initial screening phase was re-tested. Specifically, CAL1 was re-administered to four classes totalling 105 students about nine weeks after the first test administration. Such a period of time seemed sufficiently long to minimize the potential problem of a "carry-over effect" (Allen & Yen, 1979, p. 77) which could reduce an estimate of the reliability. An index of reliability was determined by calculating the correlation between the 105 pairs of test and re-test scores. The test-retest reliability was calculated to be 0.79 which seemed to indicate that a subject's relative standing remained reasonably stable over a period of time.

Reliability of CAL2 and CAL3

The function of both CAL2 and CAL3 was to identify the procedures that a subject used to solve a variety of mental multiplication tasks. A subject's scores on these two tests, though determined and reported in Chapter IV, were not used in any statistical analyses. Thus, reliability coefficients for the CAL2 and CAL3 performance scores were not necessary. The reliability of the
strategy classification scheme which was based on the tasks used in CAL2 and CAL3 was of particular interest. The procedures used to estimate this form of reliability will be discussed later in the chapter.

Reliability of WPP and BFR

Both the 10-item test of written multiplication skills WPP and the 100-item test of basic fact multiplication recall BFR contained tasks that were expected to be easy particularly for senior-high school students. This assumption proved to be correct: the mean performances on WPP and BFR for the entire 30-subject sample were 8.7 (s_x = 1.28) and 98.3 (s_x = 2.21), respectively. Expressed another way, the average p-values for the items in WPP and BFR were 0.87 and 0.98, respectively. Any estimate of reliability based on item variance such as KR20 would have yielded an extremely low coefficient since the item variance was minimal. An accumulation of scores at the floor or ceiling of a test will always lower the reliability of a test (Allen & Yen, 1979, p. 214).

Each subject's performances on basic fact recall and written computation skills were later assessed by re-administering WPP and BFR several weeks after the initial testing had taken place. As might be expected with the highly overlearned skills assessed by these tests, little change in performances was observed. The mean score on the re-test of WPP changed by 0.30 (3%) and the mean score on the re-test of BFR changed by only 0.37 (0.37%). Furthermore, over the two testing sessions, 80% of the subjects' WPP scores did not vary by more than a score of 1. Likewise, 70% of the subjects' BFR scores did not change
by more than 1. These results seemed to indicate that all subjects' performances on these selected arithmetic fundamentals were quite consistent. In this sense, WPP and BFR could be considered reliable.

Time needed to complete the basic fact recall test BFR was found to remain relatively constant over separate testing sessions. The mean change in solution time was about 8 seconds: this represented an average change of about 8%. Eighty percent of the subjects' solution times never changed by more than 10 seconds over the two trials. Thus, access time or the time required to complete the 100 items of BFR seemed to remain relatively stable over time.

Reliability of FDS, BDS, DDS, and LS

Each subject's forward digit span and backward digit span were assessed by using the appropriate Weschler Adult Intelligence Scale (Weschler, 1955) subtests. The test manual reported a digit span reliability coefficient of \( r = 0.71 \) for the standardized sample of young adults. Unfortunately, this reported reliability was based on the combined rather than the separate scores of FDS and BDS. However, Dempster (1981, p. 65) suggested that most psychometric tests of digit span are moderately reliable (about \( r = 0.66 \)).

Each subject was retested on FDS and BDS in order to determine the stability of a subject's score over a period of several weeks. The mean changes over the two assessments of FDS and BDS were calculated to be 0.6 and 0.8, respectively: representing average changes of about 9% and 15% for FDS and BDS scores, respectively. Furthermore, 90% of the subjects' FDS scores and 80% of their BDS scores did not deviate by more than 1 from one testing session to the other.
Some skilled subjects increased their scores on these two memory measures by changing strategies over the two testing sessions. One subject commented, "Last time (the first testing session) I tried to remember 2-digit series and this time (the second testing session) I tried to remember groups of three. It seemed to work better." This apparent change in "chunking" strategy by some subjects and not others will change some subjects' scores in an unpredictable manner. This type of measurement error could reduce the reliability of these measures.

The other measures used to provide an estimate of STM capacity were delayed digit span DDS and letter span LS. The construction of DDS was based on Whimbey and Leiblum's (1967) recommendations. DDS is usually more reliable than either FDS or BDS because more trials are used to estimate capacity. The researchers reported a split-half reliability, corrected for the full length of the test, of $r = 0.88$. The measure of letter span LS was developed from the suggestions made by Brown and Kirsner (1980) who reported a split-half reliability of 0.92 based upon a sample of college-level students.

The Reliability of the Introspective Reports

The accuracy of a subject's introspective report is not easily verified. The likelihood exists that the verbal reports bear no close relationship to the actual cognitive processes used by a subject to complete a task. There are precautions that can be undertaken, however, to ensure that introspective reports are a credible source of data.

Careful consideration must be given to the procedures used to
gather the verbal data. How the introspective reports are generated will determine to a large degree the types of information that can be reported reliably. Two data collection procedures considered for the present study were concurrent and retrospective verbalization (Ericsson & Simon, 1980).

Concurrent verbalization or "thinking-aloud" refers to information verbalized at the time a subject is attending to a task. An advantage of this procedure over other forms of introspection is that the information under the conscious attention of the subject can be traced directly and, hence, the researcher is provided with an indirect assessment of the internal stages of the cognitive process under study. There were several reasons why this type of introspective reporting was not used in the present study to determine a subject's method of solution.

First, as Ericsson and Simon (1980) have pointed out, one usual consequence of thinking-aloud is a slight decrease in the speed of task performance (p. 226). Although for the purposes of the present study, a reduction in calculation speed would not have been an intolerable outcome, a comparison of each skill group's ability to calculate quickly seemed desirable.

There was a more important reason that concurrent introspection was not used in the present study. The intent of the study was to identify the processes and procedures that people normally used to cope with the demands of a mental calculation. To have required subjects to engage in overt verbalization during a calculative task when such an activity was not part of their normal calculative routine seemed an unnecessary and undesirable obtrusion that could have
affected the course of a mental calculation.

Retrospective rather than concurrent verbalization seemed to be the preferred method of gathering the required information about each subject's calculative methods. Obviously this form of reporting cannot protract the time needed for an already completed calculation. Whether post hoc explanations of one calculation can affect the course of a subsequent calculation is subject to debate. Nevertheless, retrospective verbalization seemed less likely than concurrent verbalization to interfere with a subject's normal approach to a mental calculation.

An obvious weakness of retrospective reporting is the potential loss of calculative detail associated with the forgetting of temporarily-held information. Since unattended information stored in STM is rapidly forgotten, each subject in the study was asked to report on the method of solution immediately after completing the calculation task. A short delay between the completion of the calculation and the start of the verbal report ensured that some previously heeded information would still reside in STM (Ericsson & Simon, p. 226).

Some loss of information was perfectly acceptable because an abundance of calculative detail often was not necessary to classify a subject's strategy. For example, the fact that a subject could recall only that "some number was carried" but could not remember the value of the carry was sufficiently detailed for the purpose of strategy classification.

Attempts were made to corroborate each subject's report: by ensuring that both the researcher and subject understood the meanings
of the various terms involved in the discussion and explanations; by using probing questions to elicit additional information not reported initially by the subject. Although the accuracy of a subject's report can never be fully demonstrated, the approaches used in the study seemed to ensure that all subjects gave as complete and as accurate accounts of their thought processes as possible.

Reliability of the Classification Scheme.

Once all introspective reports had been compiled and classified, the reliability with which the reports had been classified needed to be estimated. Since classification schemes can be ambiguous, an attempt was made to ensure that the classification of the introspective reports would not vary markedly from researcher to researcher.

In order to ensure that a subject's reported method of solution was accurately classified, another mathematics educator was provided with a written description of the classification scheme and asked to classify a 10 percent sample of the 900 introspective reports. There was mutual agreement on 95 percent of these reports.

The Design of the Study and Methods of Analysis

The Logic of an Extreme Groups Design

Comparing the performances of extreme groups on a variety of measures is a research methodology which has been used often in the exploratory phase in the investigation of a psychological construct. Such investigations are initiated by evaluating a relatively large random sample of subjects via some measure of the construct under
study. High and low performing subgroups are selected by using the distribution of scores which results. These two extreme subgroups are either assigned to a treatment condition or evaluated on a variety of dimensions that the researcher suspects contribute to the observed differences in performance (Feldt, 1970, p. 133).

By comparing each group's performance on these other dimensions, the researcher hopes to identify some variables that can be incorporated into subsequent experimental studies. If no differences exist between the extreme groups on these secondary measures, the existence of a monotonic relationship between the criterion and secondary performance measures seems unlikely. Through this comparison process, possible sources of variation can be eliminated from further theoretical consideration.

Differences between the extreme groups must be interpreted with more caution. Although the extreme groups were selected to differ on one dimension, a researcher has no way of knowing whether or not the groups differ on a multitude of other dimensions, as well. Hence, there is difficulty in proving that any one particular difference between the groups on a secondary measure of performance is the source of the observed difference in the performances of each group on the criterion measure.

The need for prudent analysis and interpretation in comparative group research has been expressed by Cole and Means (1981):

...we do want to underscore the need for extreme caution. The researcher risks error when: (1) he goes beyond the behaviour observed under a particular set of circumstances to hypothesize about general cognitive processes; (2) he tries to interpret differences in performance manifested by groups that vary in unspecified but certainly numerous ways... (p. 11).
Since this study employed a comparative or extreme groups design, some ways of minimizing spurious inferences about group differences in mental calculation performance had to be found.

One way of attempting to eliminate alternative explanations of group differences is to select the comparison groups so that they are matched in terms of characteristics that are not under study but whose presence could also produce the same behavioural outcomes (Cole and Means, p. 39). In this study, both the skilled and unskilled groups were reasonably matched for age and level of mathematics studied. Admittedly, this matching does not necessarily solve the problem of guarding against other relevant and unaccounted for variables such as intelligence, quality of mathematics instruction, or sex. However, this matching process was considered adequate for an initial inquiry into the nature of individual differences in mental calculation performance.

Another approach used to improve the interpretability of the findings of the present study was to consider the patterns of performance within the selected groups. In particular, an attempt was made to determine the distinguishing characteristics of high and low scoring members of the same skill group. For example, the types of strategies used by low and high scoring unskilled subjects were identified and compared.

The identification and explanation of deviant behaviour was also another method of analysis employed in this study. The researcher attempted to examine the behaviour of those subjects who were more similar on one dimension to the members of the other skill group than they were to the members of their own skill group. For example, it
seemed important to understand why some subjects who used the methods of solution characteristic of the unskilled calculators were able to attain a level of skilled performance.

Examining performance patterns of a group over a number of different calculation tasks was also another method of inquiry employed in the study. One purpose of the difficult tasks included in the challenge test CAL3 was to make the skilled subjects perform like the unskilled subjects, the logic being that the connection between such processes as methods of solution and calculation performance could become more apparent.

The Statistical Analysis of Extreme Group Data

Estimating the magnitude of a hypothesised linear relationship among the variables selected for study poses special problems for extreme group researchers. The nature of the design precludes the direct application of the analytic techniques so common in experimental correlational research.

All subjects in the present study were selected because they met some standard of mental calculation performance. This selection procedure in effect removed the "middle part" of the population distribution from any subsequent statistical analyses. The removal of this particular range of scores invalidates any conclusions based upon conventional correlational or regression analyses. If the results of the skilled and unskilled subjects on some variable Y were simply combined, any calculated correlation coefficient would, typically, exaggerate the size of the relationship between variable Y and the selection variable X (see, for example, Allen & Yen (1979, p. 36) for
Thus, linear relationships detected by the indiscriminate use of correlational and regression analysis will be more likely artifactual than real.

A number of researchers (Pearson, 1903; Peters & Van Voorhis, 1940; Feldt, 1970; Alf & Abrahams, 1975; Abrahams & Alf, 1978) have presented a variety of statistical techniques to analyze information gathered from "widespread classes." The valid use of each technique rests on several assumptions.

First, it must be assumed that both high and low scoring groups have been selected from a relatively large sample (preferably N > 100) based upon the performance scores on a variable X. In the case of the present study, X represents a subject's performance on the screening test CALl.

Second, estimates of performance on a variable Y are assumed to have been obtained for only those subjects with high and low scores on X. In the present study, the various instruments used in the interview and assessment phase and described in this chapter were used to obtain scores on the variable Y.

An important further assumption that underlies the statistical techniques for analysing extreme group data is that the variables X and Y are normally distributed. If all of these assumptions have been met or at least reasonably approximated, there are several methods that can be used to estimate the relationship between X and Y.

Perhaps the simplest statistical method is to compare the Y score means of the two extreme groups using a t test for the significance of the difference between the means. The reasoning is this: any relationship in the population between X and Y will be
reflected in the significant difference between the Y distribution means for the "high" and "low" subgroups (Feldt, 1970, p. 133). In the present study, t tests were used to compare the means of the skilled and unskilled groups in an attempt to determine the presence of a relationship between CALL mental multiplication performance and the various secondary measures of performance used in the interview and assessment phase.

Using a t test to compare the extreme groups' performances can provide useful but somewhat incomplete information about the relationships under study. McNemar (1960) has argued that a comparison of the means of two extreme subgroups can provide information about the presence only and not the strength of a relationship. Without some estimate of the degree of association between two variables, the importance of trivial relationships can easily be exaggerated. Furthermore, a t test, as McNemar pointed out, is "particularly fallacious in case the underlying relationship happens, unbeknownst, to be nonlinear" (p. 298).

Fortunately, there are statistical methods available to estimate the strength of a linear relationship existing in the "intact" population when statistics about only the extreme subgroups' performances are available. To estimate the full-range correlation between CALL mental multiplication performance and performance on the other cognitive tasks used in the study, the "covariance information statistic" was used (Alf & Abrahams, 1975; Abrahams & Alf, 1978; Garg, 1983). The calculations needed to estimate R, the correlation in the intact population, are:
\[ R = r' \frac{S_x/s_x}{\sqrt{1 - (r')^2 + (r')^2 \left\{ \frac{S_x/s_x}{s_x} \right\}^2}} \]

where:

- \( r' \) is the correlation between \( X \) and \( Y \) in the sample consisting of the combined high and low groups.
- \( S_x \) is the standard deviation of \( X \) in the population or random sample.
- \( s_x \) is the unbiased standard deviation of \( X \) in the sample consisting of combined high and low groups.

In the present study, \( S_x \) refers to the standard deviation of the scores of the 280 subjects who completed the CALL screening test and \( s_x \) refers to the standard deviation of the combined CALL scores of the 15 skilled and 15 unskilled mental calculators. The value of \( S_x/s_x \) was \( 3.25/6.72 = 0.484 \).

Garg (1983) designed a Monte Carlo study to investigate empirically the efficacy of several strategies used to estimate the degree of relationship in an extreme group setting. He concluded that the covariance information strategy was superior to the other proposed strategies in terms of estimation of correlation, power, and mean square error for all values of the population coefficient and the proportion of subjects in each "tail" of the distribution (p. 370). As well, the covariance information strategy is less cumbersome to calculate than the strategies developed by Peters and Van Voorhis (1940), and Feldt (1970).

To test the significance of the correlation \( r' \) (and, thus, the significance of \( R \)), the following \( t \) test can be used (Alf & Abrahams, 1975, p. 565):
\[ t = r' (N_o - 2)^{1/2} (1 - (r')^2)^{-1/2} \]

where \( N_o \) is the number of pairs of observations in the combined upper and lower groups. The significance of the \( t \) test is determined using \( N_o - 2 \) degrees of freedom.

The Adequacy of the Sample Size

The collection and analysis of individual introspective reports formed an integral part of the present study. This procedural requirement had to be kept in mind when the selection of a sample was being considered. The size of the sample had to be large enough to ensure that the proposed quantitative analyses would be meaningful, yet small enough to ensure that each interview would yield adequately detailed qualitative information. The researcher is placed on the horns of a dilemma: was the chosen sample size of 30 subjects adequate to ensure the validity of both the proposed quantitative and qualitative analyses?

Viewed from the perspective of the standard tenets of statistical analyses, the sample size can be considered small. If sample statistics such as means, standard deviations, and correlations are to generalize to a population, an adequately large sample size must be chosen. Generally speaking, the larger the sample size, the more confidence one has that the sample statistics are reliable estimates of the population parameters.

Compared to the very large sample sizes used by many survey studies such as the National Assessment of Educational Progress, the selection of 30 subjects seems trifling. On the other hand, if the experimental studies of information-processing reviewed in Chapter II
and which formed the theoretical foundations of the present study are examined, a sample size of 30 subjects appears to be about the norm (for example, see Hitch, 1978, N = 30; Whimbey, Fischhoff & Silikowitz, 1969, N = 24; Merkel & Hall, 1982, N = 30; Dansereau & Gregg, 1966, N = 1).

White (1980) has suggested that for inferences involving measures of association, the optimum sample size can be answered only when the researcher comes to grip with the question: "What degree of association between the two variables will have implications that are educationally interesting or significant?" (p. 49). The exploratory nature of the study means that almost any statistically significant relationship between mental calculation performance and each of the variables incorporated in the study could help identify potential sources of individual difference in mental calculation performance. What would have to be the degree of association in the parent population in order for a particular sample correlation to reach statistical significance?

Using the correlational formulas described in this chapter, the correlation between a variable Y and CAL1 mental calculation performance would have to be at least 0.19 for a sample size of 30 (df = 28) and 0.12 for an sample size of 62 (df = 60) to reach statistical significance at the p = 0.05 level. Thus, a relatively weak linear relationship could be detected using a sample size of 30. Considering the fact that some data collection techniques and analyses would have had to be eliminated if the sample size had been doubled, the apparent gain in statistical power resulting from such an increased sample size seems poor compensation.
Viewed from a case study perspective, the sample size chosen for the present study would be seen as excessively large. The detailed analyses often associated with typical "N = 1" case studies cannot be attained when the sample size becomes excessively large: the more subjects, the more diffuse the qualitative data.

But if the researcher had chosen to focus the study on the behaviour of a very small number of students, the generality of the quantitative data would have been, at best, greatly reduced or, at worst, completely illusory. In a study proposing to investigate individual differences in mental calculation, information about inter-subject variability is an essential requirement. To have focussed on the behaviour of a single case would have been, to say the least, a reductio ad absurdum.

Considering the purposes of the present study, a sample size of 30 subjects seemed to be an adequate compromise: large enough to place some confidence in the statistical analyses; small enough to ensure that an adequate amount of qualitative data could be amassed. The choice of the particular sample size provided the researcher with an opportunity to ally the analytic techniques of case study and statistical research. This alliance seemed to exploit the strengths of each rather than compound their weaknesses.
CHAPTER IV

PRESENTATION OF THE FINDINGS

Proficiency in Mental Calculation

Performance on CALL, the Screening Test

The mean performance of the 280 grade 11 and 12 students for the 20-item CALL screening test was 11.1 (S_x = 3.25). Scores ranged from a low of 2 to a high of 20. The mean performance of the skilled group on CALL was 18.6 (s_x = 1.24) while the mean performance of the unskilled group was 5.7 (s_x = 1.54). Compared with the usually high scores obtained in written computation (see for example: NAEP, 1977, 1983a; Robitaille and Sherrill, 1977) by this age group, mental calculation is a significantly more difficult task.

The distribution of scores on CALL for the 280 subjects is illustrated in Figure 1. The formula used to calculate the correlation from widespread classes assumes normality in the total distribution of the population trait. An examination of the distribution reveals the assumption of normality seemed tenable.

The difficulty indices for each item used in CALL are listed in Table V. Comparisons are difficult to make because few studies have reported any findings regarding mental multiplication performance. The most recent National Assessment of Educational Progress did report the results of one released item, 90 x 70, but the test conditions were not given. Fifty-five percent of the 17-year-old NAEP sample solved this mental calculation (1983a, p. 32) and 92 percent of the sample participating in the screening phase of this study solved the
Figure 1. Frequency distribution of CAL1 mental multiplication scores for 280 grade 11 and 12 mathematics students.

$\bar{X} = 11.11 \quad S_x = 3.25$
<table>
<thead>
<tr>
<th>Easy Items</th>
<th>p-value</th>
<th>Difficult Items</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 x 51</td>
<td>0.95</td>
<td>32 x 64</td>
<td>0.08</td>
</tr>
<tr>
<td>30 x 200</td>
<td>0.95</td>
<td>27 x 32</td>
<td>0.09</td>
</tr>
<tr>
<td>8 x 70</td>
<td>0.94</td>
<td>16 x 72</td>
<td>0.12</td>
</tr>
<tr>
<td>12 x 12</td>
<td>0.94</td>
<td>15 x 48</td>
<td>0.20</td>
</tr>
<tr>
<td>60 x 40</td>
<td>0.93</td>
<td>25 x 65</td>
<td>0.22</td>
</tr>
<tr>
<td>70 x 90</td>
<td>0.92</td>
<td>16 x 45</td>
<td>0.23</td>
</tr>
<tr>
<td>7 x 511</td>
<td>0.86</td>
<td>15 x 64</td>
<td>0.26</td>
</tr>
<tr>
<td>12 x 500</td>
<td>0.81</td>
<td>24 x 24</td>
<td>0.27</td>
</tr>
<tr>
<td>8 x 99</td>
<td>0.79</td>
<td>16 x 16</td>
<td>0.36</td>
</tr>
<tr>
<td>2 x 592</td>
<td>0.79</td>
<td>25 x 48</td>
<td>0.40</td>
</tr>
</tbody>
</table>
same item.

Performance on CAL2, the Probing Test

The CAL2 performance levels of the skilled and unskilled subjects differed substantially. The mean performances for the skilled and unskilled groups were 24.3 ($s_x = 3.92$) and 9.00 ($s_x = 5.08$), respectively. On this 30-item test, the skilled subjects' scores ranged from a high of 29 to a low of 17. The unskilled subjects' scores ranged from a high of 19 to a low of 1. The highly skilled subject obtained a perfect score of 30 on CAL2.

A slight overlap in the distribution of each group's scores existed. The most proficient subject in the unskilled group obtained a score that was equivalent to two skilled subjects' scores and greater than one skilled subject's score. This overlap in the distribution of scores did not exist for CAL1, the screening instrument used to select the two groups. Such a discrepant finding could be used to question the validity of CAL1 as a screening instrument. Besides normality in the total distribution of the population in both variables, the statistical technique used to estimate a correlation in the intact population from the statistics of "widespread classes" assumes sharp truncation of the tails of the distribution (Peters & Van Voorhis, 1940, p. 386).

This discrepancy can be accounted for by the differing procedures used to administer each instrument. Each item in CAL1 was presented so that a subject had no more than 20 seconds to complete and record a solution. No such time limit was imposed during the administration of CAL2. Thus, the apparent improvement in performance
by this one unskilled subject could be due to the increased time available for a solution. This seems to be a reasonable explanation since the average solution time for this subject was 34 seconds. Thus, if time to calculate a solution is considered in the determination of mental calculation proficiency, no overlap in the distribution of CAL2 scores for each group existed.

As a general rule, the unskilled group could be characterized as being slow to calculate a solution. This finding is not surprising since time to calculate a solution formed part of the criteria for selection of subjects. Nevertheless, the contrast between the solution times of the two groups was considerable. Over 30 percent of the correct responses of the unskilled group were completed in 30 seconds or longer. For some items, a great deal of time was needed to complete a calculation. For example, one unskilled subject solved 49 x 51 in 93 seconds while another solved 17 x 99 in 124 seconds. In contrast, two skilled subjects solved 49 x 51 and 17 x 99 in 5 and 6 seconds, respectively.

Figure 2 demonstrates graphically the vast differences in calculative performance and solution times between the various skill groups. The data points for the graph were obtained by calculating each subject's score on CAL2 for a number of different time limits. For example, if the imposed time limit was 30 seconds, a subject's previously correct response was regarded as correct only if the solution time for that item was less than or equal to 30 seconds. After each item was scored according to this timing criterion, the mean performance for each group was calculated. Thus, the vertical axis represents the mean score of each group and the horizontal axis
Figure 2. Comparative performance of each skill group on CAL2 scored under differing time limits.

○ Unskilled  △ Skilled  ◊ Highly skilled
represents the imposed time limit.

An examination of the response curve for the unskilled group reveals that performance dropped gradually as more demanding time limits were imposed. Greatly degraded performance resulted when the time limits were less than 30 seconds. At a 20 second time limit the mean performance had dropped to only 4 (13%). This low performance level compares favourably with the results of CAL1 where the mean performance for this group was 5.67 (28%).

As can be seen by examining the graph in Figure 2, the performance level of the skilled group did not seem to suffer appreciably until a limit of less than 20 seconds was reached. The performance level of the skilled subjects with a 6 second limit still exceeded that of the unskilled group's regardless of the imposed time limit.

The response curve of the highly skilled subject portrays dramatically her ability to respond quickly and accurately to a mental calculation task. An examination of the curve in Figure 2 reveals that her perfect score on CAL2 remained intact until a 6 second limit was imposed. Even with a limit of 4 seconds, her score of 26 was greater than the mean score of the skilled group for every value of the imposed time limit.

Performance on CAL3, the Challenge Test

As was expected, the 15-item challenge test CAL3 proved to be more difficult than CAL2. The mean score for those 11 skilled subjects selected to take the test was 9.27 (62%). The CAL2 mean score for these same subjects was 25.91 (86%). The standard deviation
calculated for this sample was 3.07 for CAL3 and 3.05 for CAL2.

The calculation items chosen for CAL3 had an appreciable effect on this sample's solutions times. The median solution time for correctly answered items was 26.9 seconds. An examination of Figure 3 demonstrates how the mean score on CAL3 was affected by reducing the time limit. A visual comparison of the response curves in Figures 2 and 3 demonstrates that performance on CAL3 diminished much more rapidly with a reduced time limit than performance on CAL2.

The response curve of the highly skilled subject indicates how rapid her solution times were relative to the other skilled subjects. Seven items including 75 x 75, 32 x 64, 18 x 72, 48 x 48, 125 x 125, 64 x 250, and 64 x 64 were all answered in 2 seconds or less. For comparison purposes, the response curve for the most skilled subject has been included in Figure 3. It is probably no exaggeration to state that the highly skilled subject's ability to calculate mentally was as superior to those subjects who were identified as skilled as their ability was superior to those subjects who were identified as unskilled.

Evidence has been provided to demonstrate that the mental calculation performance levels of each of the groups who participated in the interview and assessment phase differed substantially. The remainder of this chapter will be used to present evidence that the groups differed on a number of other dimensions as well. Each skill group could be characterized by differences in the following: (1) choice of methods of solution used to solve a mental calculation; (2) retrieval of numerical equivalents useful for a mental calculation; (3) short-term memory capacity. Though the logic of a comparative
Figure 3. Comparative performance of 11 skilled subjects, the most skilled subject, and 1 highly skilled subject on CAL3 scored under differing time limits.

△ Skilled □ Most skilled ◇ Highly skilled subject
groups design precludes any attempt to attribute the differences in mental calculation performance to any variable investigated in the study, any proposed explanation of mental calculation performance will have to take some of these variables into account.

Choice of Method of Solution and Calculative Strategy

It became evident after a number of interviews had been completed that skilled and unskilled mental calculators could be characterized by the methods chosen to solve calculation tasks. The frequency and proportion of general methods and specific strategies reported by the unskilled and skilled groups during the administration of CAL2 are summarized in Table VI. The analysis of the strategies and methods used by the 11 skilled subjects and the one highly skilled subject during the administration of the challenge test CAL3 is contained in Table VII.

Invalid approaches to a calculative task added to the difficulty of the classification of the methods of solution and the calculative strategies. For instance, to solve 15 x 15, several subjects reasoned, "10 x 10 = 100 and 5 x 5 = 25, so 125." Though this calculative approach will yield incorrect solutions, a partial understanding of additive distribution seemed evident. In order to ensure that all subjects' calculative approaches could be tabulated, this faulty "strategy" was incorporated into the classification scheme. These partial forms of distribution have been designated in Tables VI and VII as incomplete distribution.

In response to the computation 16 x 16, one unskilled subject said, "244. It just came into my head." Her approach was not
### TABLE VI

FREQUENCY AND PERCENTAGE OF GENERAL METHODS AND SPECIFIC STRATEGIES USED BY SKILLED AND UNSKILLED SUBJECTS TO SOLVE CAL2 MENTAL MULTIPLICATION TASKS

<table>
<thead>
<tr>
<th>General methods and specific strategies</th>
<th>Unskilled</th>
<th>Skilled</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frequency</td>
<td>Percentage</td>
</tr>
<tr>
<td>Pencil-and-paper mental analogue</td>
<td>387</td>
<td>86</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No partial retrieved</td>
<td>320</td>
<td>71</td>
</tr>
<tr>
<td>One partial retrieved</td>
<td>52</td>
<td>12</td>
</tr>
<tr>
<td>Two partials retrieved</td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td>Stacking</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Distribution</td>
<td>53</td>
<td>12</td>
</tr>
<tr>
<td>Additive</td>
<td>36</td>
<td>8</td>
</tr>
<tr>
<td>Fractional</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Subtractive</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Quadratic</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Incomplete</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>Factoring</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>General</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Half-and-double</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Aliquot parts</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Exponential</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Retrieval of a numerical equivalent</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Guess</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Totals</td>
<td>450</td>
<td>100</td>
</tr>
<tr>
<td>General methods and specific strategies</td>
<td>Skilled</td>
<td></td>
</tr>
<tr>
<td>----------------------------------------</td>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td></td>
<td>Frequency</td>
<td>Percentage</td>
</tr>
<tr>
<td>Pencil-and-paper mental analogue</td>
<td>34</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No partial retrieved</td>
<td>16</td>
<td>10</td>
</tr>
<tr>
<td>One partial retrieved</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>Two partials retrieved</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Stacking</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Distribution</td>
<td>102</td>
<td>62</td>
</tr>
<tr>
<td>Additive</td>
<td>68</td>
<td>41</td>
</tr>
<tr>
<td>Fractional</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Subtractive</td>
<td>24</td>
<td>15</td>
</tr>
<tr>
<td>Quadratic</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>Incomplete</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Factoring</td>
<td>29</td>
<td>18</td>
</tr>
<tr>
<td>General</td>
<td>13</td>
<td>8</td>
</tr>
<tr>
<td>Half-and-double</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Aliquot Parts</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Exponential</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Retrieval of a numerical equivalent</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Totalsa</td>
<td>165</td>
<td>101</td>
</tr>
</tbody>
</table>

*Due to rounding, some totals do not equal 100%.*
classified as retrieval but instead was tabulated as a *guess*. For an explanation of the classification scheme used in this study, the reader is referred to Chapter III.

To determine if the choices of general methods of solution were different for skilled and unskilled groups, a chi-square test for independent samples was used. Specifically, this method of analysis was used to test the null hypothesis that the proportion of general methods of solution chosen to solve mental multiplication tasks did not differ between skilled and unskilled groups. For the purposes of this analysis, the number of invalid responses, "incomplete distribution" and "guess," were not included. Since these responses accounted for only 2% of the total responses, this loss of information seemed insignificant.

Since the magnitude of the calculated chi-square was 393 (df = 3, p < 0.001), the null hypothesis that the two groups did not differ in their choices of methods was rejected. Thus, it was concluded that skilled and unskilled mental calculators differed significantly in the methods of solution they chose and used to solve mental multiplication tasks.

One consequence of temporary storage during a mental calculation is that interim information produced in the course of computation will undergo rapid forgetting if not utilized immediately. This suggests that those mental calculation strategies which require a large number of calculation stages should be susceptible to a great deal of forgetting and, consequently, relatively poor performance would be expected. To carry the argument further, it can be hypothesized that the digit-by-digit, pencil-and-paper strategy P&P0
with its greater number of steps compared to other strategies should be more closely associated with poor performance.

An analysis of the data seems to support this hypothesis: the unskilled subjects made much more frequent use of P&PO than the skilled subjects. The mean number of uses of P&PO by the skilled and unskilled groups was 1.9 ($s_x = 2.61$) and 21.3 ($s_x = 8.46$), respectively. The difference between the means was highly significant ($t_{28} = -8.49; p < 0.001$).

The $t$ test for the difference between means assumes that the two underlying populations have equal variances. The large difference between the variances of the scores of the skilled and unskilled groups suggested that this assumption may not have been reasonable. However, if unequal population variances are assumed and the significance of the difference between the means is determined using the method outlined by Snedecor and Cochran (1967, p. 115), the difference between the means remains significant at the $p < 0.001$ level.

The possible existence of a negative correlation between mental multiplication performance and the frequency of the selection of P&PO was determined in a series of steps. As a first step in determining the strength of this hypothesized linear relationship, a scatter plot was constructed. Figure 4 is a scatter plot where the x-axis represents the number of times a subject selected P&PO to solve the CAL2 items and the y-axis represents that subject's score on CAL1, the screening test. An examination of the scatter plot in Figure 4 indicates that a moderately strong linear relationship existed between the application of P&PO and performance on CAL1.
Figure 4. Scatter plot of frequency of selection of the digit-by-digit, right-to-left calculative strategy (P&PO) to solve CAL2 tasks and score on CAL1.

- Unskilled
- Skilled
The two groups were selected on the basis of their performance on CAL1 while the methods of solution were determined as each subject solved an item on CAL2. Thus, to measure the association between mental calculation performance and choice of method of solution, the proportion of times a subject chose P&PO to solve the CAL2 items must be assumed to be an accurate estimate of the proportion of times the strategy would be used to solve the CAL1 items. This assumption did not seem to be unreasonable, especially in the case of the unskilled subjects who rarely deviated from the use of P&PO.

Alf and Abrahams' covariance information statistic (see Chapter III for discussion) was used to determine the magnitude of the apparent linear relationship. The following information was used to calculate $R$: $r' = -0.86$, $S_x/S_x = 3.25/6.72 = 0.484$. The estimated $R$ for the entire intact population was $-0.63$ which was highly significant ($t_{28} = -8.92; p < 0.001$). Thus, a moderately strong linear relationship existed since approximately 40% ($R^2 = 0.40$) of the variance in mental multiplication performance can be explained by knowing the degree to which a subject seems to depend upon the digit-by-digit, right-to-left form of the pencil-and-paper mental analogue to solve mental calculation tasks.

Useful information about how frequently each skill group used particular mental calculation strategies to solve the CAL2 mental multiplication tasks is provided by Table VI. What cannot be determined from this tabulation is how frequently individual subjects within each skill group changed strategies in response to a change in the calculative task. Other analyses were required to make this determination.
As a first step in the analysis, the number of strategies used to solve the 30 CAL2 items was calculated for each subject. The mean number of strategies used by the skilled and unskilled groups was 6.4 ($s_x = 1.80$) and 2.8 ($s_x = 1.32$), respectively. The difference between the means was highly significant ($t_{28} = 6.25; p < 0.001$).

Figure 5 represents how the number of strategies chosen by each subject was distributed over the two skill groups. An examination of the distribution demonstrates how the skilled subjects were much more likely to change strategies to correspond to a change in the nature of a calculative task. This finding suggested that a possible linear relationship between the number of strategies a subject used to solve a series of calculative tasks and mental calculation performance existed.

Further evidence of a linear relationship can be found by examining the scatter plot included in Figure 6. The x-axis represents the number of strategies selected by a subject to solve the 30 CAL2 items and the y-axis represents that subject's score on CAL1. A visual inspection of the scatter plot indicates a linear trend. The strength of the apparent linear relationship was determined by using Alf and Abrahams' covariance information statistic. The correlation $r'$ was 0.76 and the estimated $R$ for the intact population was estimated to be 0.49 ($t_{28} = 6.19; p < 0.001$). Thus, a statistically significant and moderate linear relationship existed.

An analysis of the introspective reports provided by the subjects suggested that there were also important qualitative differences between the two groups that could not be revealed through a quantitative analysis alone. A detailed discussion of each group's
Figure 5. Frequency distribution of number of strategies used by skilled and unskilled subjects to solve 30 CAL2 mental multiplication tasks.

- Unskilled
- Skilled
Figure 6. Scatter plot of number of strategies used by subjects to solve 30 CAL2 mental multiplication tasks and scores on CAL1.

- Unskilled
- Skilled
methods of solution, accompanied by many illustrative examples, will be presented.

The following nomenclature will be used in this study to refer to subjects participating in the interview and assessment phase. The letters U and S will refer to an unskilled and skilled subject, respectively. The number refers to the rank of that subject's score on CAL2 without regard for the tied scores that existed: 1 is the highest rank and 15 is the lowest rank. The single highly skilled subject will be referred to as HS.

The Unskilled Subjects' Choices of Calculative Methods

Use of pencil-and-paper mental analogue. The unskilled subjects strongly favoured the use of the pencil-and-paper mental analogue to solve a calculative task. This method was applied to 86% of the attempted tasks. Dividing the unskilled group into high and low scoring subjects based on CAL2 performance revealed that the low scoring subjects relied almost entirely on P&PO and P&P1 to solve mental multiplication tasks. These two strategies were applied by these least skilled subjects to 94% of the calculation tasks used in the interview and assessment phase.

The subjects who relied on P&PO and P&P1 referred to each calculative stage often in spatial terms more suited to a written than a mental medium. Subject U13's attempted solution of 4 x 625 contains numerous such references. She reasoned, "625 is on the top, 4 on the bottom, 4 x 5 is 20, put down the 0, carry the 2. 4 x 2, 8, and 2 is 10. Carry 1 and then...5 x 6 is 30...30 and 1 is... Put down the 0... Can I start again?"
Little attempt was made by the unskilled subjects who favoured this method to examine the calculative task for any number properties or relationships which could be used as an aid in calculation. Even the most transparent properties were not heeded. For example, 11 of the 30 items in CAL2 included factors which contained 1 as a digit in the units or tens place value position: 12 x 81, 17 x 99, 12 x 15, and so forth. Thus, the subjects who applied the pencil-and-paper method were not expected to calculate each partial product digit by digit to solve these types of items. Rather, it was thought that the partials would be determined by either reasoning, "1 x a = a" or by using an equivalent process such as thinking, "bring down the a." Surprisingly, 61% of the unskilled group's attempts to solve these 11 items did not incorporate this elementary principle to expedite the calculation. It was found that 32% and 85% of the high and low scoring unskilled subjects' attempts, respectively, did not use the identity property of multiplication as an calculative aid for these 11 items.

Some subjects who employed P&PO did arrange the factors to seemingly take advantage of this obvious property. However, instead of retrieving the partial product, they went through a digit-by-digit calculative process. Subject U13's attempted solution of 49 x 51 is an example of such calculative behaviour. She thought, "51 on the bottom, 49 on the top. It seems easier that way. 1 x 9, 1 x 4, 49. Put down the 0..."

Subject U2 was asked why she went through a digit-by-digit process to calculate 13 x 13. She responded, "Why would I multiply through? To be sure. Because the last time I was doing a problem like this I just brought the number down." Her reference to "the last time
I brought a number down" was to the item 16 x 16 where she calculated
the first partial as 96 and used this value instead of 16 (160) as the
second partial product. Using the first partial as a substitute value
for the second partial product was a relatively common error made by
unskilled subjects. The various types of mental calculation errors
will be discussed later in this chapter.

Those subjects who depended upon P&PO to solve most calculation
tasks sometimes introduced calculations which, though correct, can be
considered redundant and unnecessary. For example, subject U10
explained that her solution of 50 x 64 included a partial product
"with all 0's on the top." Further probing revealed that she had
visualized the calculation arranged as illustrated below:

\[
\begin{array}{c}
64 \\
\times 50 \\
\hline
00 \\
3200 \\
\end{array}
\]

Perhaps the most graphic example of over-dependence on the use
of P&PO was subject U13's attempted solution of 20 x 30. She
explained, "30 is on the top, 20 is on the bottom. 0 x 0 is 0, 0 x 3
is 0. Put down the 0. And 2 x 0 is 0 and 2 x 3 is 6. And then you
add them together and you'd get ....6....600?" Her calculation took
34 seconds to complete.

Most unskilled subjects used a digit-by-digit, right-to-left
calculative process during the additive phase of a mental calculation.
As U1 explained, "I do it (addition) from right to left and then read
the answer out backwards."

This tendency of unskilled subjects to fragment a calculation
could have been one of the reasons that even the most obvious sums
were calculated rather than retrieved. Subject Ull, in calculating 12 x 250, was heard to say, "So five hundred and two thousand five hundred" before beginning the process of addition. But instead of retrieving a sum of 3000, she immediately began to say, "OK. 5, 0, 0, and 2, 5, 0, 0,... would be ....0, 5, 7, 2....2, 7, 5, 0?". She was asked later in the interview to add 2500 and 500 and again used a digit-by-digit, right-to-left additive process.

The tendency of unskilled subjects to view calculation in terms of an imaginary spatial arrangement similar to the format used during written computation became very evident during the additive phase. U9 would use expressions such as, "Move over one, 4, that's under the 6, and 6, that's under the 1..." Subject U3 commented that to remember the position of each digit he had to repeat constantly, "something like 4 over the 6, 2 over the 9, etc."

Some unskilled subjects appeared to become disenchanted with this type of protracted additive process and began, instead, to use more flexible methods which were based on some recognized number property of the addends. Subject U2, at the midpoint of the CAL2 interview, attempted to solve 17 x 99. She said:

I got 990 and 693 but I had problems adding. I'd do 9 and 9 is 18, carry the 1,... but I'd forget what I'd finished. Finally for 990 and 693, I took 10 off the 693, and got 1000 and then 1683. At first I thought this addition method might be harder but it was easier. First time I've done this method before.

The use of P&PO and P&P1 was accompanied often by motions which indicated that the subject was attempting to "write" each stage of the calculation. Fingers were used to form calculations either in the air or on a desk. One subject used a pencil poised just above the table to
record her calculative stages. Eleven of the fifteen unskilled subjects were observed using an imaginary "writing instrument" to solve the majority of mental calculation tasks. Similar physical gestures were observed during the screening phase of the study. When CAL1 was administered to one class of 34 students, 11 students were observed using fingers as an apparent calculative aid.

All subjects who used "writing" believed the practice helped them cope with the demands of the calculation. Subject U2 explained her fingers helped her "remember" a completed partial product. For example, to solve 12 x 15, she "did 120 and then as soon as I did 60 I forgot 120. The second time I put an emphasis on the 60 by pressing my finger down harder." Subject U9 explained that fingers helped him "see the problem" and improved his concentration.

The forgetting of the position of each digit of a partial product during the additive phase of a mental calculation was a great source of error for unskilled subjects. Subject U6 believed the use of fingers helped her remember these positions. She explained she aligned the fingers of each hand to represent the place-value position of each digit in the two partials and then tried to visualize a digit on each finger.

The use of distribution. The unskilled group made little use of this strategy: 12% of this group's attempted solutions involved an application of this method. Two subjects, U4 and U7, accounted for almost 80% of the number of reported applications of distribution. Nine subjects never used the method once.

The one unskilled subject who did favour additive distribution over all other methods was subject U7. He was an unusual subject.
Despite his sound reasoning, he often erred in the fundamentals. For example, to solve $9 \times 742$, he attempted, "$9 \times 742, 9 \times 7 = 72$, so 7200 and $9 \times 4$ is 28. So 280 and 7200 is 7480. $9 \times 2$ is 18, so 7498." In the latter stages of the interview, he was asked to recall $9 \times 7$. He replied, "Let's see. $9 \times 9$ is 81, 72, 63. Well, there go the last two questions." Frequently, he had to calculate rather than recall the larger basic facts. He commented he "didn't bother to remember larger numbers" because he relied on his calculator instead.

There were some subjects who seemed to be attempting additive distribution but who failed to complete the necessary calculations. This error frequently occurred when a subject applied additive distribution to only the decades and units of each factor. For example, subject U15 calculated $25 \times 25$ as $20 \times 20$ and $5 \times 5$. He and subject U7 accounted for over 70% of the unskilled subjects' attempts to apply this form of "incomplete distribution."

A comment made by U15 suggests that this invalid procedure was used in an apparent attempt to break away from the unsuccessful attempts to apply pencil-and-paper methods to this mental medium. After applying P&PO to the first 11 items in CAL2, he tried to solve $25 \times 25$ by reasoning, "400 and 25." He commented, "I don't feel too confident because I don't know if my method works or not. I haven't tried it before." Apparently, he never did determine that the method was invalid since he later applied the method to six other items.

Despite their considerable mathematical experience, two unskilled subjects could not correctly apply annexation as a calculative aid. The researcher noticed that neither subject used annexation to help determine the solution to the CAL2 item, $32 \times 500$. 
Ull was given the task 50 x 700 to test her understanding. She explained, "you multiply the 5 and the 7 and add 5 zeroes." Apparently she thought that the number of annexed zeroes was determined by counting the number of digits rather than the "zeroes" in each factor. As a further test of her understanding, she was given 300 x 800. She responded, "24 and six zeroes."

Subject U15 could annex the correct number of zeroes but used the following faulty logic to complete the calculation of 30 x 40. He explained, "That's easy, 120. I just timesed the two numbers (3 and 4) and there's two zeroes there. So it's (the product) in the 100's." Obviously these two subjects would have difficulty using additive distribution since the correct application of the annexation algorithm is a necessary requirement.

Subtractive distribution was the only other type of distribution applied by the unskilled subjects and its use was restricted to only one subject. She used the strategy to calculate 8 x 99 and 17 x 99 successfully but her attempts to solve 8 x 999 and 15 x 48 were unsuccessful.

The use of factoring. The use of the factoring method by the unskilled group was very infrequent. Only two percent of their attempted solutions involved some factoring strategy. Eleven of the 15 unskilled subjects made no attempts at factoring.

Although their first attempts to factor were unsuccessful, the use of the strategy by two unskilled subjects suggested that the method was a recent discovery inspired by an apparent dissatisfaction with pencil-and-paper methods. Subject U9 reported that he obtained 4000 as the solution to 12 x 250 by trying a "different" method. His
introspective report indicated that this method was aliquot parts.

He explained, "I pulled a fast one here! I said 250, how many times does it take to get to 1000? It's 4 and 3 sets of 4 in 12. So 3000. Did I say 4 (4000) the first time?" Subject U12's first answer in calculating 25 x 120 was 1500 because she "divided 8 into 120 and added two zeroes." Asked why she chose to divide 120 by 8, she said "because...oh, I should have divided by 4. Four 25's are 100. So 4 into 120 is 4..., 30. So 3000." When asked if she had ever used this method before, she replied, "No, I just discovered it today because of all these questions."

The Skilled Subjects' Choices of Calculative Methods

The use of the pencil-and-paper mental analogue. The pencil-and-paper mental analogue was used infrequently by the skilled group. Approximately 22% of the skilled group's introspective reports were classified under this method. Five skilled subjects used this method far more frequently than any other subjects. These five subjects accounted for about 85% of the method's use by the skilled group. At the other extreme were 6 skilled subjects who never used the method to solve any items in CAL2.

Some skilled subjects did rely on this method to solve mental multiplication questions but, unlike the unskilled subjects, they tended to avoid the use of the digit-by-digit P&PO strategy. While 71% of the unskilled group's responses were classified as P&PO, only 6% of the skilled group's responses were so classified.

Subject S3 was the most proficient subject to make extensive use of the pencil-and-paper method. However, she used P&P2 and P&P3
rather than P&P0. She relied on her ability to retrieve numerical equivalents which seemed to minimize the number of calculative steps needed to solve a computation. For example, instead of calculating each partial product for 15 x 16 digit by digit, she thought, "15 x 16. 80 and 16. Move one over, 160. So 240."

These retrievals of numerical equivalents had the effect of decreasing the solution time for many calculation tasks. The solution to 15 x 16 took S3 less than 4 seconds. In contrast, the successful solutions of subjects U3 and U4 who used P&P0 took 45 and 52 seconds, respectively. S3's left-to-right method of adding the partial products also deviated from the right-to-left procedures commonly used by unskilled subjects.

S3 used a pencil-and-paper strategy which was termed stacking (see Chapter III for example and discussion). She used this strategy to solve calculation tasks of the form 1-digit by x-digit: specifically, 8 x 99, 9 x 74, 8 x 625 and 8 x 999. She commented that she liked the strategy because "I don't have to carry with this method."

An interesting pattern emerged when each CAL2 item was ranked according to the number of times a pencil-and-paper strategy was applied by the skilled group. Pencil-and-paper strategies were used most frequently to solve these items: 8 x 4211, 9 x 742, 8 x 612, 32 x 500, 9 x 74, 50 x 64, 32 x 500, 25 x 65. The format of four of these items is 1-digit by x-digit and, with the exception of 25 x 65, the factors of the remaining items, 50 x 64 and 32 x 500, can be easily transformed to this format by an application of the annexation algorithm.
Mental products involving a 1-digit factor did seem to elicit the most frequent applications of the pencil-and-paper method but there were a few important exceptions. Only one skilled subject applied the method to solve 4 x 625 and 8 x 625. Subject S3 who applied this method far more frequently than the other skilled subjects was asked why she did not use the pencil-and-paper strategy to solve 4 x 625. She replied, "usually something tells me whether it's easier to go right to left (apply pencil-and-paper) or go left to right (apply distribution). My knowledge of 4 x 25 = 100 helped me make the decision." Subject S13 made a similar comment: "I started to do it with the 625 on top of the 8 and then I lost my numbers. So I thought 8 x 25 is 200 and 8 x 600 is 4800. Then I added these two together. I saw it was a 25 type of problem."

Generally speaking, skilled subjects seemed to use the pencil-and-paper method either for very "easy" calculative tasks or for tasks which have no apparent properties. The relationship between the absence of discernible number properties and the application of the pencil-and-paper method became much more evident during the CAL3 interviews. Two of the most difficult items (p = 0.33) for the sample of skilled subjects were 87 x 23 and 73 x 83. For these two items, the number of subjects who applied the pencil-and-paper method increased to 5 from a median of 2.

The researcher asked those subjects who normally avoided the use of the pencil-and-paper method why the method had been applied to these particular items and not to other items. Subject S1 explained that he had more difficulty with 87 x 23 than with the other items "because I had kind of a gimmick to solve the other questions." He
believed that for such items, "one method was as good as another." Subject S9 said he couldn't solve 87 x 23 "because he had to multiply it out just like on paper." He chose this method because the "problem doesn't have any numbers that you can work with." After his unsuccessful attempt to solve 87 x 23, subject S11 was asked if he normally avoided the use of the pencil-and-paper method. He commented, "Yes, I only use it with really weird numbers like that." He also said that he was forced to use the method to solve 73 x 83 because he "couldn't think of any way of doing it such as double-and half. The numbers (factors) weren't close to some convenient numbers."

The unskilled subjects applied the pencil-and-paper method almost habitually but the skilled subjects' use was much more discriminating. Generally speaking, the pencil-and-paper method was applied to a calculative task by the skilled subjects only if no discernible properties were apparent. In those cases where this method was applied, the skilled subjects made attempts to retrieve rather than calculate the partial products.

The use of distribution. Distribution was the method used most frequently by the skilled group. Over one-half of the skilled subjects' attempted solutions involved a distribution strategy. The data in Table VII indicate that distribution was the most frequently reported method in the CAL3 interviews, as well.

The number of uses of additive distribution to solve the CAL2 calculative tasks ranged from a high of 23 to a low of 1. The median number of uses was 11. The additive distribution strategy was always applied in a series of stages. The calculation was usually initiated
by determining the product of the most significant digits of each factor and annexing the appropriate number of zeroes. The direction of the calculation progressed from the more significant to the less significant digits in each factor.

In a long series of such calculations, the skilled subjects would determine a running sum of the partial products rather than delay the addition until all partials had been calculated. Subject S8's solution of $8 \times 4211$ illustrates both this left-to-right sequence of calculations and progressive addition. He thought, "$8 \times 4211, 8 \times 4000$ is $32,000$, $8 \times 200, 1600$, so $33,600$. And $8 \times 11$ is 88 so 33 688."

Many skilled subjects believed that progressive addition helped minimize forgetting. Subject S5 explained, "I add as I go along. It works better that way. Otherwise you get too many numbers and get confused." Subject S8 said, "I try to add the numbers as I go along and forget the others. I usually start at the left, add, and drop those numbers." When asked to explain why he added in this manner, he said, "there was too much to remember otherwise."

One subject explained that consistently "working from left to right" helped him to "keep track" of the completed and yet to be completed intermediate calculations. He solved the CAL3 item $24 \times 625$ by reasoning, "$20 \times 600 = 12,000$, $20 \times 25 = 500$, 12 500 and filed that away; $4 \times 600 = 2400$ and $4 \times 25$ is 100, so 2500. Add to 12 500, so 15 000." To explain his method of track-keeping, he took a piece of paper and diagrammed each stage in the manner indicated below:

**Visualized**

\[ \begin{array}{cccc}
625 \\
times 24 \\
\end{array} \]

**Calculated**

\[ \begin{array}{cccc}
12,000 \\
500 \\
2400 \\
100 \\
\end{array} \]
He also explained that he "tries to break the problem into 'two simple numbers' to work with." He drew the following diagram to illustrate his method:

```
  6 25
  2 4
```

Although additive distribution requires the continual application of some addition method, the conventional right-to-left, digit-by-digit procedure, so common in written computation, was generally ignored by most skilled subjects. Frequently, the addition was completed by using some form of a left-to-right process. S9's solution of 480 + 96 illustrates one such process. He explained, "Bump up the 4 to 5 and 7, 6."

Other sums were determined by arranging one addend so that a multiple of a power of 10 was obtained as a partial sum. Subject S6 added 960 and 64 by thinking, "960 and 40 is 1000, add the 24 which was left over." For those sums where an addend was close to a multiple of a power of 10, some subjects would apply a compensation strategy. S6 reasoned that 480 and 96 equalled 576 because "480 and 100 is 580 then minus 4 is 576."

**Fractional distribution** was a form of additive distribution which incorporated factoring to complete part of the calculation. This strategy was frequently applied to those calculative tasks where one factor had 5 as a unit digit. For example, the item with the largest number of reported applications of this strategy was 15 x 48. All four subjects who used this strategy simply reasoned "480, 1/2 of 480, 240, so 720."
Subject S14 accounted for 12 of the 15 instances of fractional distribution reported during the CAL2 interviews. He obviously favoured this technique because, whenever possible, he tried to change the task into a form that made a subsequent application of fractional distribution more feasible. To solve 24 x 24, he first thought, "25 x 24 - 24," and then he said, "I went 24 x 10, 240. Doubled that, 480. I took 1/2 of 240, 120. Added 120 and 480, so 600. Then I took off a 24, so 576."

S14 was the only subject who applied this strategy to solve a CAL3 item. Although his attempt to calculate 125 x 125 was not successful, his explanation illustrates how he tried to adapt the strategy to solve more complex items. He reasoned, "125 x 10, 1250, double that, 2500. And then I took 1/4 of that and added that onto 125 x 10." To have applied the strategy correctly, he should have used a reasoning process similar to the following: 125 x 125 = (100 + 25) x 125 = (100 x 125) + 1/4 x (100 x 125) = 12500 + (1/4 x 12500) = 12500 + 3125 = 15625.

Subtractive distribution was used by the majority of skilled subjects to solve items such as 8 x 99, 17 x 99, and 8 x 999 where each item included a factor whose magnitude was close to a multiple of a power of 10. For the three items, 8 x 99, 17 x 99, and 8 x 99, 28 of the skilled subjects' solutions incorporated subtractive distribution. In comparison, only 3 of the unskilled subjects' solutions involved this strategy. The CAL3 items which prompted the most attempts of this strategy were 18 x 72 (1440 - 144), 48 x 64 (3200 - 128) and 48 x 48 (2400 - 96).
Subtraction strategies which differed from pencil-and-paper methods always accompanied an application of subtractive distribution. The subtraction was usually accomplished in a series of stages by renaming the subtrahend as either a sum or difference. To solve the computation 8190 - 91, subject S14 thought, "8190 - 90 = 8100, so 1 more off is 8099." He also subtracted 96 from 2400 by subtracting 100 and then compensated by adding 4.

Sixteen instances of quadratic distribution were reported during the CAL2 and CAL3 interviews. Over 60% of the attempts were applied to solve 49 x 51 and 89 x 91. Subject S1 used difference of squares while other subjects used a mathematically equivalent but likely more inefficient procedure. After he solved 49 x 51, S5 explained, "50 x 50 is 2500, and minused 50 because of the 49 and then added 49": i.e., 49 x 51 = (50 - 1)(50 + 1) = 2500 - 50 + 50 - 1 = 2500 - 50 + 49.

Other variations of quadratic distribution were applied to calculating squares. Subject S1 used binomial expansion to solve 48 x 48 by thinking, "50 squared minus 200 and add 4": i.e., 48 x 48 = (50 - 2)^2 = 50^2 - 2 x (2 x 50) + 2^2 = 2500 - 200 + 4. The task 125 x 125 was solved by thinking, "100 squared plus 5000 and add 625": i.e., 125 x 125 = 100^2 + 2 x (25 x 100) + 25^2 = 10 000 + 5000 + 625.

Subject S11 was aware of the rule for squaring factors ending in 5 but he was unable to determine a correct solution in his initial attempts. In calculating 75 x 75, he annexed a 5 instead of a 25: i.e., he calculated 75 x 75 = 7 x 8 x 100 + 5 = 5605 instead of calculating 7 x 8 x 100 + 25. For 125 x 125, he calculated 13 x 13 x 100 + 25 instead of 12 x 13 x 100 + 25. As he explained his
reasoning, he realized and corrected these errors.

The use of factoring. In contrast to the infrequent use of factoring by the unskilled subjects, the majority of skilled subjects used factoring to solve some calculative tasks. Factoring was involved in 14% of the skilled group's attempted solutions. The number of applications reported for each subject varied from a high of 10 to a low of 0. However, only three skilled subjects did not report any attempts at factoring.

General factoring accounted for about 3% of the skilled group's reported strategies. Generally speaking, its use was restricted to those items which contained factors that were multiples of either 2 or 5. At least one attempt at general factoring was reported for these items: 8 x 625, 25 x 48, 25 x 480, 25 x 120, 12 x 250, 25 x 32, 15 x 48.

The higher incidence of general factoring reported for these items as compared to other items such as 87 x 23 was due to the fact that each calculative task can be reformulated to involve multiples of powers of 10. For example, the four subjects who solved 12 x 250 by using general factoring reasoned "since 12 = 3 x 4 then 4 x 250 is 1000, so 3 x 1000 or 3000." Similarly, subject S4 applied the strategy to solve 25 x 120 by thinking "5 x 120 is 600 and 5 x 600 is 3000." Subject S8 used general factoring for those products that could be decomposed into what he described as "easy combinations." To solve 75 x 240, a CAL3 item, he said, "I broke up the 75 into groups of 25. And I broke up the 240 into groups of 4. When I worked out the combinations I got the number of 100's."

General factoring was also applied to two items, 25 x 25 and 25
x 65, where each factor is a multiple of 5 but neither factor is a multiple of 2. The final stage of the calculation, in these cases, was completed by applying some other strategy such as additive distribution. The computation 25 x 25 was reasoned as "5 x 25, 125; 5 x 120 is 600 and 25, 625." And 25 x 65 was reasoned as "25 x 65 = 25 x (64 + 1), so 25 x 64 is 4 x 25, 100, 'times' 16, so 1600, and add 25, 1625."

Half-and-double is a factoring strategy which was frequently used to solve calculative tasks where at least one factor was an even number. If the second factor is a multiple of 5, multiple applications of half-and-double will eventually produce a factor which is a multiple of a power of 10. Subject S11, who accounted for most of the reported applications of this strategy during the CAL2 interviews and for all the reported instances during the CAL3 interviews, used two consecutive applications of half-and-doubling to solve 24 x 625. He explained, "I doubled and halfed until 1250 x 12. Which I thought would be 15 000. However, I checked by doubling and halving one more time. So 2500 x 6 which is 15 000."

The most common factoring strategy applied by the skilled group was aliquot parts. One-half of the number of attempted applications of this strategy involved three items: namely, 25 x 48, 25 x 120, and 25 x 32. Subject S5 explained that if a task had a factor of 25 or 50, he would apply the strategy by dividing the remaining factor by either 4 or 2, respectively, and complete the calculation by annexing the appropriate number of zeroes. When asked to elaborate on this strategy's range of usefulness, he replied, "If it's a large number (the second factor in the given product) then I would try to use it
but not with items such as 50 x 20. It's especially useful if one factor is an even number but odd factors can be confusing."

However, some subjects did apply aliquot parts to items with odd factors. Subject S1 solved the calculative task 25 x 65 in a novel manner. He incorporated decimal arithmetic into his calculation by reasoning "65 divided by 4 is 16.25 Move the decimal two places over. So 1625." Subject S5 solved the same item by applying aliquot parts to calculate 64 x 25 and adding 25 to this result.

In the CAL3 interviews, there were 7 instances of aliquot parts reported and every attempt was directed at solving 64 x 250. Interestingly, subject S14 solved this item by applying aliquot parts to only part of the calculation. He explained, "I took 64 x 10, 640. Multiplied by 10, so 6400; tripled that, 19 200; and then subtracted 1/2 of 6400." The following analysis demonstrates the logic behind his method: 250 x 64 = 25 x 10 x 64 = 25 x 640 = (30 - 5) x 640 = 3 x 10 x 640 - 5 x 640 = 3 x 6400 - 6400/2 = 19 200 - 3200 = 16 000. The tendency of skilled subjects to incorporate several calculative strategies to solve the difficult CAL3 items added to the problems of strategy classification.

Several subjects commented that they applied aliquot parts only after reformulating the task into more concrete terms. Money analogies were mentioned by several subjects. Subject S13 always described her applications of this strategy by making such monetary references. She commented, "when I multiply by 25, I think of money, like quarters. So I divided by 4 to calculate dollars." She thought that her knowledge of this strategy was acquired through a practical life-experience. She explained, "When I was in Brownies, we always
used to sell cookies and they were $1.25. So I got used to adding, subtracting, and multiplying by quarters." The comments of some unskilled subjects indicated that they were aware that such analogies could possibly have aided their calculative attempts. After applying the pencil-and-paper method to solve 25 x 25, subject U1 said, "Maybe I should have counted quarters."

There was only one instance of the application of exponential factoring recorded during the CAL2 interview. Subject S1 solved 32 x 32 by reasoning that the product equalled 2 to the tenth which he knew was equivalent to 1024. Although subject S5 realized that the product would be a power of 2, he could not retrieve the numerical equivalent. When first presented with this task, he said, "I suppose 2 to the 10th is not good enough." When the researcher said no, he correctly calculated the product by applying additive distribution: i.e., 32 x 30 + 32 x 2.

There were six instances of exponential factoring reported during the CAL3 interviews. Three subjects used the strategy to solve the items 32 x 64 and 64 x 64. In each case, 1024 was used as a reference point. Subject S9, who did not know 1024 was 2 to the 10th, applied an interesting tactic to complete his calculation of 32 x 64. He surmised that 64 x 64 would be "a power of 2 ending in 6." Since he knew that 1024 was large power of 2 close to but less than the square of 64, he "doubled 1024 twice and got 4096."
The Highly Skilled Subject's Choices of Calculative Methods

HS relied heavily on an apparent ability to quickly discern number properties useful for calculation. Factoring and retrieval were the most frequently applied methods of solution. When the reports obtained during the administration of CAL1, CAL2, and CAL3 were combined, she solved over one-half of the items by using either factoring or retrieval. For the remaining items, she applied some form of distribution. She never once applied a pencil-and-paper strategy to solve any mental products given in the study.

She seemed capable of applying sophisticated strategies without any apparent mental effort: for example, she solved 87 x 23 by instantly reasoning "69 x 30 - 69." The complexity of this solution is illustrated by the following analysis: 87 x 23 = (29 x 3) x 23 = 29 x (3 x 23) = 29 x 69 = 69 x (30 - 1) = 69 x 30 - 69 = 2070 - 69 = 2001.

The greatest challenge to her calculative abilities was the item 123 x 456 which was also administered to 5 skilled subjects. All subjects who attempted this item commented that the calculative details were difficult to remember and they had to re-calculate several times. This difficulty in retaining the numbers in this calculation greatly protracted the solution times. The solution times for the four skilled subjects who correctly answered this item were 50, 115, 228, and 350 seconds.

Each skilled subject reported using either distribution or a pencil-and-paper method. Subject S3, who favoured the pencil-and-paper method, used progressive addition as each partial product was calculated. All skilled subjects reported that there were
no recognizable number properties other than the obvious identity property ("1" x 456) that could be used to expedite the calculation.

HS completed this task in 50 seconds and offered the following explanation:

I first thought that 456 x 123 is 152 x 41 x 9. Then I thought 19 x 41 x 8 x 9. Did 779 x 8 by thinking 5600, 560, and so 6160. 6160 and 72 is 6232. Finally I multiplied 6232 by 9 by thinking 9 x 62 is 558, which I know. So 55800 and 9 x 32. 55800 and 288 is 56088.

The reader is reminded that all this reasoning was done "in the head" without any opportunity to refresh the factors by a visual inspection.

HS differed from the majority of skilled subjects because she seemed able to sense immediately whether the numbers in a calculative product could be readily factored. Her acute sense of "number factorability" became quite evident during an interview designed to assess her knowledge of prime numbers.

Hunter (1962) assessed Aitken's ability to determine whether a number was prime or composite as a demonstration of the "readiness with which a presented number leads on to numerical properties" (p. 247). This task of factoring is, at least, more time-consuming if not more difficult than mental multiplication because of the greater number of calculations that must be done to ascertain the factorability of a number.

A list of 27 numbers containing both primes and composites was constructed. The primes were chosen at random from a table of prime numbers. Six primes were selected because they were used by Hunter. The composite numbers were selected to ensure that no even composites were included. Many composites with relatively large prime factors such as 899 and 667 were selected to increase the difficulty of
factorization.

HS was instructed to state immediately if she thought a given number was either prime or composite. She was to determine at least one prime factor if the number was composite. She correctly identified 20 of 27 numbers as being either prime or composite. Her greatest error was stating a composite number as prime.

The following is a partial account of the number factorability assessment. The reader should note how she was able to use various number theory concepts such as divisibility checks to determine if the given number was composite. The numbers in the parentheses refer to her solution times; R refers to the researcher.

R: Try 507.
HS: That's not prime (1 s).
R: What are the factors?
HS: Let me see...Oh, 169 and 13 x 13.
R: How did you know it wasn't prime? Did you try different factors?
HS: Well, I knew immediately it was not prime because it's a multiple of 3. The sum of the digits are divisible by 3.

R: Try 599.
HS: That's prime (1 s).

R: How about 187?
HS: That's not prime.
R: How did you know?
HS: Well, like 1 and 7 are 8, so I knew it was a multiple of 11.
R: Were you taught this divisibility rule?
HS: Well, I knew them before I was taught them.
R: How did you learn them?
HS: Just playing around with numbers.

R: Try 833.
HS: That's not prime (2 s).
R: What are the factors?
HS: 7, 219,...no, 119.
R: So 7 and 119?
HS: Yes, and 49 and 17.

R: Try 529.
HS: That's the square of 23 (1 s).

R: Try 667.
HS: That's not prime (2 s). Do you want the factors?
R: Yes, if possible.
HS: Let's see..it's 23 x 29 (4 s).

R: Try 1063.
HS: That's not prime (4 s).
R: What are the factors?
HS: Uhm... (10 s). I don't know but it's not prime.
R: It is prime.
HS: Oh, I didn't think it would be.

R: Try 301.
HS: That's prime (2 s).
R: No, it's...
HS: Oh, yes. It's 43 x 7.

R: Try 179.
HS: It's prime (2 s).

R: Try 509.
HS: That's prime (1 s).

R: Try 533.
HS: That's prime (2 s).
R: No.
HS: OK. This will be a bad one. Could I have a pen, please?...Wait, did you say 553?
R: No, 533.
HS: Oh, well then, it's 13 and 41.

Retrieval of Numerical Equivalents

Two processes involved in mental calculation are the initial selection of a calculative strategy and the subsequent retrieval of a series of numerical equivalents. The purpose of a strategy is to decompose and organize a calculation into a series of more tractable sub-calculations. This process of decomposition and organization continues until the subject is able to retrieve a needed numerical equivalent from memory. Evidence has been provided which indicated that the skill groups differed in their choices of calculative strategies to solve a mental multiplication task: there were also a number of ways that the skill groups differed in the ability to
retrieve numerical equivalents useful for mental multiplication.

Retrieval of the Basic Multiplication Facts

The most commonly accessed numerical equivalents were the basic facts of multiplication. The ability to solve a mental calculation task when the most popular strategies such as P&PO and additive distribution were applied depended upon a ready access to these numerical equivalents. To determine if there were any differences between skilled and unskilled subjects in retrieving this type of calculative information, a test of the 100 basic multiplication facts BFR was administered to all subjects.

Accuracy of basic fact recall. Each group exhibited close to perfect basic fact recall. More than 70% of the all the participating subjects made less than three errors on BFR. The median score for the combined groups was 99.06. The mean scores on BFR for the skilled and unskilled groups were 99.9 (s_x = 0.35) and 96.7 (s_x = 3.10), respectively. Despite the small difference between the group means, this difference was statistically significant (t_{28} = 3.97; p < 0.001). If the variances of the two populations are assumed to be unequal, the difference between the means remained significant at the p < 0.01 level.

The covariance information strategy was used to determine the strength of the relationship existing between basic fact multiplication recall and mental multiplication performance. The correlation r' for the combined groups was 0.58; the correlation for the intact group R was estimated to equal 0.33 (t_{28} = 3.77; p < 0.001). Thus, a statistically significant but weak relationship (R^2 <
0.11) existed between basic fact recall and mental multiplication performance.

However, basic fact recall may be a more difficult task during a mental calculation. In the test of recall BFR, the subject had to attend to only one task: namely, the task of recalling (or possibly calculating) a basic fact of multiplication. This is not the case during mental calculation where a subject must attend to a variety of tasks as a retrieval process is under way. The given factors of the calculative task, the calculated partial products, and the current stage of the strategy must all be remembered while the subject attempts to recall some fact. Because the ability to recall numerical equivalents may be affected by the extra requirements of mental calculation, an attempt was made to assess each subject's ability to recall basic facts under the conditions of calculation by examining the introspective reports.

The number of basic fact errors made during a mental calculation was estimated for each CAL2 item. The total number of basic fact retrieval errors assessed in this manner was estimated at 1 for the skilled group and 18 for the unskilled group.

These seven items accounted for all the basic fact recall errors: 9 x 74, 15 x 16, 8 x 625, 8 x 612, 16 x 16, 9 x 742, 8 x 4211. The three facts 8 x 6, 9 x 7, and 9 x 4 accounted for over 70 percent of these basic fact errors. As might be anticipated, those subjects who made the greatest number of errors on BFR also made the most retrieval errors during a mental calculation.

These findings suggest that the inability to retrieve a basic
fact during a mental calculation contributed minimally to differences in mental calculation performance. After all, the combined number of errors in mental calculation made by the skilled and unskilled groups was 400. By far the majority of errors (95%) made during mental calculation must be have been due to some processes other than the failure to recall a basic fact. Thus, differences in basic fact recall performance can be discounted as a major source of variation in the mental calculation performance of young adults.

**Time to access a basic fact.** The unskilled and skilled subjects appeared to differ more greatly on the time needed to access a basic fact. Total access time, defined as the time needed to complete the 100 items on BFR, was determined for each subject. The mean total access time for the skilled and unskilled groups was 79.8 seconds ($s_x = 14.74$) and 118.9 seconds ($s_x = 32.44$), respectively. The difference between these mean total access times was highly significant ($t_{28} = -4.25; p < 0.001$).

Figure 7 contains a scatter plot in which total access time has been plotted against performance on CAL1. Examination of the scatter plot reveals that total basic fact access time seems to be a weak predictor of mental calculation performance. Those subjects with the "faster" total access times tended to have higher scores on CAL1; those with the "slower" times tended to have lower CAL1 scores.

The degree of association between total access time and performance on CAL1 was estimated. The correlation for the combined groups $r'$ was calculated to be $-0.61$. The correlation $R$ for the intact sample was $-0.35$ which was statistically significant ($t_{28} = -4.07, p < 0.001$). Thus, about 12% of the variance in mental calculation
Figure 7. Scatter plot of total time needed to recall 100 basic multiplication facts (access time) and score on CALL.

○ Unskilled  ▲ Skilled
performance can be explained by the total time needed to retrieve 100 basic multiplication facts.

**Reconstruction of basic facts.** Many unskilled subjects paused noticeably for certain items such as $8 \times 6$ during the administration of BFR. The researcher suspected that these pauses indicated that a subject's answers were reconstructed rather than retrieved. Some subjects admitted they had to calculate rather than recall some basic facts. For example, subject Ull made no errors on BFR but she frequently paused for items which included 8 as a factor. These pauses greatly inflated the estimate of her total access time. When questioned after the BFR administration, she admitted that her "8's tables were weak." Rather than retrieving 48 for $8 \times 6$, she had reasoned, "6 x 8 is 48 because 7 x 8 is 56."

There was some direct observational evidence collected during the administration of CAL2 which suggested that reconstruction during a mental calculation contributed to the forgetting of numerical information. Those subjects who had the greatest access times often lost track of the calculation as they attempted to reconstruct rather than retrieve a basic fact. If these subjects were able to complete the calculation, the additional reconstructive steps increased their solution times.

The relationship between total access time, reconstruction, and errors in mental calculation becomes more evident if a few introspective reports are analysed. The reports of the subjects U5, U7, and Ull are particularly illuminating.

The slowest subject U5, after attempting to solve $8 \times 625$ for 67 seconds, reported that he had lost the numbers. He gave the
following report as he calculated (the series of dots is used to represent a relative measure of time): "8 x 25, 40, ... 8, 16 and 2 is 18. And 8 x 6 is ... uhm... 48. I have to go back. 8 x 625 is... 40, so...4000... I forgot the numbers." The lengthy pause after 8 x 6 suggests that a reconstructive process had taken place and this process may have contributed to the forgetting of previously stored calculations.

U7, who made the greatest number of errors on BFR (10 errors), admitted that for the "longer basic facts I have to reason them out." His calculation of 8 x 625 illustrates how calculative details can be forgotten during the reconstruction of a basic fact. He tried to count out multiples of 600 to determine the partial product of 8 x 600. After 28 seconds into the calculation, he asked the researcher to repeat the question. This was the full report he gave: "Well. 600, ... 4 times is 1200. No, it would be 6, 12, 18, ... No, it would be 4800. So the 8 times 600 is 4800 and the last two numbers were... 44?"

The reconstructive process needed to calculate 8 x 6 also hindered his calculation of 8 x 612. He thought, "8 x 5 is ... 40 and 6 is... 46. So 460 and 8 times 10 is 80... Can't remember the next digit... So 4600, ..., 4680, and 16 is 4696." He completed this reasoning in 60 seconds. In contrast, the typical skilled subject correctly solved this calculative task in about 5 seconds.

Even a fact such as 8 x 4 was reconstructed rather than retrieved by U7. To calculate 8 x 4211, he reported, "8 x 4000 is 24000, ... No, 8 x 3 is 24... and 8 must be 32. It's 32000... Can I have the next digit please?" Thus, after 31 seconds, some main
feature of the task, in this case the digits of a factor, had been forgotten.

Although subject Ull had perfect recall on BFR, she reconstructed some facts to solve some mental calculation tasks. For those calculative tasks where there was evidence of reconstruction, her response times were slow and she was unable to retain some features of the calculation. Because of the time she needed to reconstruct 8 x 6, her incorrect solution of 8 x 625 took a lengthy 56 seconds to complete. She reported, "8 x 5 is 40, and....8 x 6 is....can't remember my 8's table. 8 x 6 is....48 and 8 x 2 is 16, carry 1, 28, 29,..so,...2, 9, 4, 0." In this task, the calculated value of 48 was replaced by 28: this substitution of incorrect numbers for a previously correct calculation was an error made by many unskilled subjects.

Ull gave an answer of 5792 after attempting to calculate 8 x 612 for 74 seconds! Again, she had to reconstruct 8 x 6: "2 x 8, 16, carry 1, 8, 9, and 8 x 6 is....My times tables....36?....(very long pause)....So it would be 6 x 7 = 42,...56. No, 57....92." She commented, "I had to reason out 8 x 6. It's 56 because 6 x 7 = 42 and add 8,...no, 6. So, it's 48. I must have been thinking of 8 x 7?"

The Retrieval of Large Numerical Equivalents

Blocking. Many skilled calculative thinkers used what they described as "blocking" to reduce the number of calculations needed to solve a calculative task. For example, subject S7 explained that before calculating 125 x 125, he organized the calculative task into these three easily determined blocks: 100 x 125, 25 x 100, 25 x 25.
He completed the calculation by annexing zeroes to obtain the products 12,500 and 2,500 and by retrieving the large numerical equivalent 625 from memory.

The ability to organize a computation into a smaller number of blocks seemed to be associated with a subject's ability to access large numerical equivalents. If a subject could access only basic facts, the computation had to be partitioned into a larger number of blocks reflecting this limited storage of information. Thus, the popularity of P&PO amongst the unskilled subjects could have been due partially to the fact that each block necessitated the retrieval of only basic facts.

On the other hand, if a subject had access to a store of large numerical equivalents, there was an increased opportunity to organize a computation into a smaller number of easily determined blocks. In fact, some subjects used unit blocking (organizing a computation into one block) when the computation was recognized as a numerical equivalent that could be retrieved from memory. Forty-six responses were classified as unit blocking: that is, the calculative task had been solved entirely by retrieving one large numerical equivalent.

To ensure that the solution was retrieved and not calculated, each subject was always asked if any intermediate calculations had been attempted. Typically, a subject would respond to this query with phrases such as, "I just know it," "I memorized it," "It's a fact," or as one subject said, "It's common knowledge; everyone knows that." The very short solution times (1 to 4 seconds) which were associated with these responses also suggested a retrieval rather than a calculative process.
The skilled subjects accounted for 44 instances of unit blocking while the unskilled accounted for two instances. The differences were so large that a test for statistical significance seemed unnecessary. Thus, skilled subjects solve more calculative tasks by retrieving a large numerical equivalent than unskilled subjects.

The calculative tasks solved by retrieving rather than calculating are displayed in Table VIII. Examination of Table VIII indicates that 44 of the 46 tasks solved by retrieval were squares. The only square in CAL2 not solved by retrieving a numerical equivalent was 32 x 32, the largest square included in CAL2.

Although not solved by a single retrieval of a numerical equivalent, there were many calculative tasks which were solved by organizing the computation into several blocks and by retrieving a large numerical equivalent associated with at least one of the blocks. Several solutions to the item 15 x 16 illustrate this blocking process: retrieving $15^2$ and calculating "225 + 15"; retrieving $16^2$ and calculating "256 - 16"; retrieving 5 x 16 and calculating "160 + 80." If the numerical equivalents of 1 x 15 and 1 x 16 are excluded, no unskilled subject retrieved a numerical equivalent larger than a basic fact for this same task.

No skilled subject used unit blocking to solve a CAL3 item but several computations were blocked so that a large numerical equivalent could be retrieved. The most frequently accessed large numerical equivalent was "$25^2 = 625" which was retrieved 8 times. This numerical equivalent was recalled during these computations: 75 x 75 = $25^2$ x 3 x 3, 24 x 625 = 24 x $25^2$, 125 x 125 = $(100 + 25)^2$. Several subjects knew
**TABLE VIII**

PRODUCTS DETERMINED BY RETRIEVING RATHER THAN CALCULATING

<table>
<thead>
<tr>
<th>Task</th>
<th>Skilled Number of Subjects</th>
<th>Unskilled Number of Subjects</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 x 48</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>12 x 15</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>16 x 16</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>25 x 25</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>13 x 13</td>
<td>13</td>
<td>0</td>
</tr>
<tr>
<td>24 x 24</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>15 x 15</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>44</strong></td>
<td><strong>2</strong></td>
</tr>
</tbody>
</table>
"2^{10} = 1024" and applied this knowledge to solve the CAL3 items 32 x 64 and 64 x 64. Subject S1 applied his knowledge that "36^2 = 1296" to calculate the solution to the CAL3 item 36 x 72: he said, "I doubled 1296."

The large numerical equivalents that were retrieved and applied to a calculation frequently involved squares and other powers. Those subjects who could recall these large equivalents were often asked why they could remember such apparent esoteric information. Several subjects commented that powers of 2 such as 16^2 and 32^2 were memorized because of "working with computers and the binary system." Subject S9, who was asked why he knew immediately that 1024 was a large power of 2, replied:

> When I was a little kid we'd go on these long trips and I used to sit in the back of the car and think 1 + 1 is 2, 2 + 2 is 4, and go all the way....Just for something to occupy my mind. I would recognize that 1024 is a power of 2 but I wouldn't know that it was 2 to the 10th.

Subject S1 was asked why he could remember that 36^2 equalled 1296. He explained, "Probably working with probability. In rolling dice you have a chance of 1/6 of rolling a 1 so if you do that 4 times in a row it would be 1/6 to the 4th or 1 in 1296." Their answers suggest that this numerical information was memorized through the pursuit of some activity which interested them rather than through a deliberate intention to commit these facts to memory.

There was some evidence, however, which suggested that some skilled subjects take pleasure in "memorizing, merely for the sake of memorization." Subject S1 was asked if he had ever taken a memory test. He replied "I'm not sure but once a friend and I had a contest to see how far we could each memorize pi. I got as far as 250 places
and then I lost interest in the problem." At a subsequent interview, he was asked to recite this sequence. He recalled 46 digits in the series correctly without any preparation for this task. No other subject in the study reported a similar interest in number recitation.

Several subjects, including some unskilled subjects, commented that when a calculative task required the determination of a square, they often thought that they "should know that." This sense of "knowing" often had the effect of protracting the solution times because the subject seemed to go through a quick and unsuccessful retrieval process before selecting some other method of solution. There were many instances of this "calculative deja vu."

When SI was given the item 24 x 24, there was a noticeable pause punctuated by the phrase, "Hmmm." When questioned about the pause, he replied, "I had to multiply it out (using distribution) but I felt I should know that one." Subject S11 claimed that an unsuccessful search for 16 x 16 was made before applying distribution. After the calculation was completed, he said, "256 rang a bell. I should have known that 16² is a fact." Even 7 unskilled subjects believed that the squares of 13, 15, 16, and 25 should have been retrieved rather than calculated. One unskilled subject explained, "It would be handy to know these for calculating square roots in algebra."

The 12's Facts. The unskilled subjects would rarely retrieve a "12's fact," the product of 12 and another factor, to aid a mental calculation. The skilled subjects, on the other hand, often solved calculative tasks through these retrievals. For example, 12 x 81 was solved by several skilled subjects by thinking, "8 x 12 is 96; so 960
and 12 is 972." To determine if the infrequent retrieval of these specific multiplication facts by the unskilled subjects was either a matter of choice or a matter of memory, each subject's ability to recall products from the "12's multiplication table" was assessed.

Each subject was asked to recall quickly the solution to $5 \times 12$, $6 \times 12$, $7 \times 12$, $8 \times 12$, $9 \times 12$, $11 \times 12$, and $12 \times 12$. All these items were presented randomly. The easy items $1 \times 12$, $2 \times 12$, $3 \times 12$, $4 \times 12$, and $10 \times 12$ were not presented. If the subject responded quickly and claimed that no calculation was needed, the solution was classified as retrieval of a large numerical equivalent. The results are summarized in Table IX.

Examination of the table indicates that large differences existed between the two groups in the ability to retrieve these selected numerical equivalents. These results provide further evidence that the ability to retrieve numerical equivalents beyond the basic facts of multiplication seems to be more characteristic of skilled than unskilled mental calculators.

The Highly Skilled Subject's Memory for Numerical Equivalents

The highly skilled subject HS had a remarkable ability to retrieve large numerical equivalents. She solved 16 of the 45 CAL2 and CAL3 items by an immediate retrieval of the solution. For those items where she did not recall the product immediately, she frequently retrieved large numerical equivalents to solve some part of the calculation. Nine items were solved in this manner. Her calculations involved numerical equivalents that no other subject in the study recalled. Several examples can be used to illustrate her retrieval and calculation processes: $23 \times 27$ was solved through retrieving $23^2$ and
<table>
<thead>
<tr>
<th>Groups</th>
<th>Unskilled</th>
<th>Skilled</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tasks</td>
<td>Number of subjects correctly recalling</td>
<td>Number of Subjects correctly recalling</td>
</tr>
<tr>
<td>5 x 12</td>
<td>1</td>
<td>14</td>
</tr>
<tr>
<td>6 x 12</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>7 x 12</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>8 x 12</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>9 x 12</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>11 x 12</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>12 x 12</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>Total</td>
<td>6</td>
<td>99</td>
</tr>
</tbody>
</table>
182

calculating $23^2 + 92$; $18 \times 72$ was solved through retrieving $18^2$ and
calculating $18^2 \times 4$; $27 \times 81$ was solved through retrieving $27^2$ and
calculating $27^2 \times 3$.

This researcher suspected that the items in CAL2 and CAL3 did
not provide an adequate estimate of HS's memory for numerical
equivalents. The following procedures were used to estimate her memory
for squares: all 2-digit squares from $11^2$ to $99^2$, excluding multiples
of 10, were used to test her recall; each item was presented orally;
the presentation order was scrambled to ensure that no item could be
determined in advance.

She correctly recalled a total of 69 (85%) of the 81 2-digit
squares. All her solutions were stated within 1 or 2 seconds after the
square had been presented orally. The reasoning such as "I just know
it" or "It just popped into my head" was considered additional
evidence that a retrieval rather than a calculative process had taken
place.

For the 12 squares that HS claimed were "guesses," her answers
were in error by only a few percent. Frequently, the answer was
correct except for the digit order: for example, she stated 7936
instead of 7396. If given a second attempt, she either calculated or
retrieved the correct solution in a few seconds.

HS stated all 4-digit products as a pair of 2-digit numbers:
for example, 7056 was stated as "Seventy, fifty-six." Speculating
that some subtle memory process could underlie this method of stating
the answer, the researcher asked her to explain why the products were
expressed as pairs. She said, "It's easier to say 'seventy,
fifty-six' than 'seven thousand fifty-six'." Another potentially
prolific research area nipped in the bud!

To expedite the process of trying to estimate how many more squares HS could retrieve, the following procedures were used in a second interview. Starting at the number 101, she was asked to state instantly that number's square. If she couldn't recall this square, she was instructed not to bother with any calculation and, instead, she was to proceed to the next square in the progression. This procedure was terminated at 349² because she began to tire of the activity. Table X lists the squares that she could recall instantly.

As can be seen by examining this table, her knowledge of squares was exceptional. She correctly recalled 46 3-digit squares and the one 4-digit square she recalled was the square of 1024 or 1,048,576! Often she would respond with a statement such as "I'm supposed to know that one," "I used to know that one," or "I knew it but I can't remember it now" when she could not recall a square. These expressions of recognition suggest that she may have been able to recall even more squares in the past. Certainly her parents thought that she was not as knowledgeable now as she had been in the past.

For several squares, she would surmise immediately that her stated answer was incorrect and state an alternative. To check the feasibility of an answer, she often applied reasoning which was sophisticated and surprisingly rapid. Her response to 315² illustrates her rapid appraisal of a potential answer. She said, "90 125, no that's wrong it's 99 225." Asked why she thought that 90 125 was incorrect, she explained, "Because it's not a multiple of 9." The reader is reminded that both the retrieval of two reasonable responses and the reason for rejecting one of these choices took place within 5
<table>
<thead>
<tr>
<th>Interval</th>
<th>Recalled squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>101-109</td>
<td>All</td>
</tr>
<tr>
<td>111-119</td>
<td>All except $115^2$ and $118^2$</td>
</tr>
<tr>
<td>121-129</td>
<td>$121^2$, $123^2$, $124^2$, $125^2$, $126^2$, $128^2$</td>
</tr>
<tr>
<td>131-139</td>
<td>$135^2$</td>
</tr>
<tr>
<td>141-149</td>
<td>$144^2$</td>
</tr>
<tr>
<td>151-159</td>
<td>$152^2$</td>
</tr>
<tr>
<td>161-169</td>
<td>$162^2$</td>
</tr>
<tr>
<td>171-179</td>
<td>$171^2$</td>
</tr>
<tr>
<td>181-189</td>
<td>None recalled</td>
</tr>
<tr>
<td>191-199</td>
<td>None recalled</td>
</tr>
<tr>
<td>201-209</td>
<td>$201^2$, $202^2$, $204^2$</td>
</tr>
<tr>
<td>211-219</td>
<td>None recalled</td>
</tr>
<tr>
<td>221-229</td>
<td>$225^2$, $228^2$</td>
</tr>
<tr>
<td>231-239</td>
<td>None recalled</td>
</tr>
<tr>
<td>241-249</td>
<td>$242^2$</td>
</tr>
<tr>
<td>251-259</td>
<td>$252^2$, $256^2$</td>
</tr>
<tr>
<td>261-269</td>
<td>$261^2$, $264^2$</td>
</tr>
<tr>
<td>271-279</td>
<td>$275^2$</td>
</tr>
<tr>
<td>281-289</td>
<td>$288^2$</td>
</tr>
<tr>
<td>291-299</td>
<td>$294^2$</td>
</tr>
<tr>
<td>301$^+$</td>
<td>$301^2$, $304^2$, $312^2$, $315^2$, $326^2$, $343^2$, $1024^2$</td>
</tr>
</tbody>
</table>
Further questioning revealed that she was able to apply such divisibility rules as "casting out nines" to check the reasonableness of her calculations. She used reasoning similar to the following to determine the reasonableness of the answers 90 125 and 99 225:

The sum of the digits in 315 is divisible by 9 so that the square must be a multiple of 9. But the sum of the digits in 90 125 is not divisible by 9 and, therefore, it cannot be the square of 315. Consequently, 99 225 seemed the most plausible solution.

The following anecdote is another illustration of her remarkably quick and acute power of reasoning. I remarked quite incorrectly that her initial response of 4356 to 66 was incorrect. Her response to the re-administration of this item was revealing.

R: Try 66 squared again.
HS: 4356 (instantly).
R: OK. That one's wrong, but you're close.
HS: 4536 (instantly).
R: Yes. Does that happen sometimes, you get the right digits but in the wrong ..... 
HS: It's 4356.
R: No, it's 4536
HS: No, because 4536 is 504 x 9! I know it's 4356.
R: (checks calculation) My apologies, you're correct.

At a subsequent interview, her memory of this event was refreshed and she was asked to explain her reasoning. She said, "4536 couldn't be the square of 66 because 9 is not a factor of 66 but it is a factor of both 504 and 9." The apparent logic of her reasoning follows:

4536 can be factored into 504 and 9. 504 has a factor of 9, therefore, 9 x 9 is a factor of 4536. On the other hand, 9 is not a factor of 66 so 9 x 9 cannot be a factor of 66 squared. Therefore, 4536 is not the square of 66.

Both these calculation and verification activities took place during a time interval of a few seconds.

During one of the interviews, HS commented that she could
recall most 2-digit squares "but some of them I get caught on, especially prime numbers like 59." When asked what she would do if she forgot the square of 59, she said, "It's 59 x 60 - 59." Thus, calculative strategies were also used to check the reasonableness of "retrieved" squares. The researcher was able to identify some other ways that she used to determine the reasonableness of her answers. These methods are best illustrated through citing a few examples.

The following discussion relates to the task $62^2$:

HS: 37...let me see, 3844 (5 seconds).
R: I noticed a pause there, what were you thinking.
HS: I was thinking 3724 but I knew that was wrong so I figured it out again.
R: So you thought 3724 but felt it was wrong. Why would you think it's wrong?
HS: It's too close to the square of 61.

Subsequent discussion revealed this thinking process: "It might be 3724 but $61^2$ is 3721 which is too close to 3724 so it can't be 3724."

To complete the process, she thought $62 \times 60$ and $62 \times 2$. Both the estimation and calculation activities were completed in less than 5 seconds.

Often she had to decide between two likely answers. When asked to recall the square of 97, she answered 9049 instantly. As soon as she was told that the answer 9049 was incorrect, she immediately responded, "Then it's 9409." She said, "I guessed" to explain this second answer. Apparently, she suspected that either 9049 or 9409 was correct. Therefore, once she obtained the information that one answer was incorrect, she knew that the other alternative was correct.

There were times when she recalled a number that seemed unrelated to the square. This process can be illustrated by this example:
R: Try 88 squared.
HS: 7744 (5 seconds).
R: How did you determine that?
HS: I just thought for a moment.
R: Do you mean you just concentrated until the answer appeared?
HS: Yes.
R: Did you do any calculations at all?
HS: No.
R: Were any other numbers thought of before you said 7744?
HS: Yes, 7004.
R: Why would you think of this number?
HS: I don't know...Well 88 x 8 with a zero in the middle of it.

In other words, she thought 88 x 8 which equals 704. And 7004 can be thought of as "704 with a '0' in the middle of it." Perhaps such an involuntary calculative response was cued by the digits contained in $88^2$. She commented that this response had "hurt" the speed of the calculation.

**Short-Term Memory Processes in Mental Calculation**

**Types of Forgetting During Mental Calculation**

One of the major types of errors made by all subjects as they performed a mental calculation was failing to remember some aspect of either the initial calculative task or the interim calculations. The researcher had hoped to be able to categorize the different types of forgetting and assess the relative frequency of each error type. However, a reliable assessment of each type of error and its frequency was not possible for several reasons.

Although all subjects could remember and describe the method of solution, they often had difficulty recalling the numerical details of a calculation after a solution had been stated. When probed for these details, some subjects could provide only ambiguous responses such as "I think I got lost in the adding," "I may have forgotten to carry,"
or "It was memory again." Even less detail was provided by those students who became frustrated, confused, and simply stated, "I don't know," "I lost track," or "I'm lost."

A second difficulty in classification was the tendency for some subjects to make several types of errors during a calculation. Even if all the errors could be identified, the present study's research design could not be used to determine which error type singularly affected the calculation performance.

However, there were some types of forgetting which subjects could recall with sufficient detail for reasonably unambiguous classification. Many errors were also identified as a subject calculated "out loud." These common errors of forgetting were identified: (1) forgetting the factors of the initial calculative task; (2) forgetting a completed calculation; (3) losing track of the direction of the calculation; (4) forgetting the order of a series; (5) forgetting to "carry" or carrying the wrong number; (6) misalignment during addition.

The data provided on error classification are "soft" and any conclusions should be treated with caution. Nevertheless, the qualitative data do provide simple confirmation that forgetting some aspect of temporarily-held information is a much more important source of error in mental than in written calculation tasks.

Forgetting the initial factors. The unskilled subjects reported that the retention of the initial factors was a difficult task. The researcher could identify this type of error when a subject asked either a question of verification such as "Is it 8 x 612?" or a question about the task such as "What was the question again?" The
skilled and unskilled subjects differed in the number of times they asked the researcher to either verify or repeat the question: 9 times for the skilled and 64 times for the unskilled.

Subject U6 had particular difficulties in remembering the factors in a calculative task. Her response to 8 x 612 illustrates this difficulty: "...Is it 612? (5 seconds into the calculation) ...612 x 8? (20 seconds into the calculation)...1296?" As can be seen from this example, this difficulty in completing the calculation was not due to misunderstanding the task. This difficulty occurred after several seconds had passed and substantial calculative progress had been made.

Two calculation tasks, in particular, had the highest incidence of this type of forgetting. Five unskilled and 4 skilled subjects, and 7 unskilled and 1 skilled subjects, asked the researcher to either repeat or verify the factors of 9 x 742 and 8 x 4211, respectively. Sl's comment, after he solved 8 x 4211, seemed instructive. He commented, "I have tricks for some problems but I find it hard to remember these long ones (1-digit by x-digit factors). I calculate the numbers and then I have to remember a string of them as I keep calculating."

Forgetting a completed calculation. Retaining the numbers of a calculation was another difficult operation for the unskilled subjects. This type of forgetting occurred frequently as the subject attempted to calculate another partial product. After calculating for 47 seconds, U4 gave up in her attempt to solve 23 x 27. She explained:

I got both numbers (the partials) but couldn't seem to remember them. I had 161 and, I think, 460 was the other one. I never did add. I would calculate the second number and lose the first and then go back to calculate the first
and lose the second.

This alternation between calculation and recalculation greatly protracted the solution times of some subjects. U2 correctly calculated 17 \times 99 only after several repeated calculations of the partial products. However, the solution took over 2 minutes to complete!

In order to terminate a lengthy calculation, some subjects used only a partially recalled calculation to determine a solution. After he said 1525 was the solution to 25 \times 65, subject U2 explained, "when I went to add I had forgotten the first one (partial) so I just sort of made up what I thought it could be... I took something out of my memory and it wasn't right."

Often a subject would complete a calculation but inadvertently either reduce or increase this calculation by a multiple of a power of 10. For example, subject S12 applied distribution to solve 23 \times 27 and obtained as a partial product 4600 instead of 460. He said, "I had 460 but it must have changed to 4600." Many subjects complained they frequently forgot to annex zeroes. This difficulty in retaining the number of "zeroes" became apparent during the calculation of 25 \times 480 where seven skilled subjects gave an initial response of 1200 instead of 12000.

Rehearsal was a common technique used by all subjects to help retain a completed calculation. When subject U2 was asked why she repeated a partial product so many times before continuing to calculate the second partial, she replied, "It helps make the numbers firm." S1 made a similar comment after he successfully solved the very difficult calculation 123 \times 456. He commented that he had
obtained a partial product of 10,448 (23 x 456) and "went over that number a couple of times." When asked to give reasons for this repetition, he responded, "So I wouldn't forget it. It reinforces it."

There was some evidence that a few skilled subjects used other memory techniques besides rehearsal to help remember a calculation. Such memory devices became evident only during the administration of CAL3 where the skilled subjects commented that "the numbers seemed more difficult to remember" than the easier items included in CAL2.

S7 employed a number of different memory tactics to help retain his interim calculations. To solve 32 x 64, he applied additive distribution and calculated a partial of 1920. He explained, "Now instead of memorizing (rehearsing) it I just tried to remember it as a date. I thought that it would stick that way better. First time I came across a number that I could remember as a date."

A more common memory tactic for this subject was to examine a number for a recognizable pattern or property: to recall the partial product 2880, he said, "I tried to remember that pattern of numbers 2, 8, 8"; for the calculated partial of 16,800, he said, "I needed a way to recall that so I thought 8 is 1/2 of 16 and stored that as 1000's"; for the partial product of 7280, he remembered, "72 is a multiple of 8"; for the partial 3840, he said, "I tried to remember by saying 4 is 1/2 of 8 and a 3 at the beginning."

His successful solution of the item 123 x 456 is a fascinating illustration of an ability to coordinate a number of different thought processes during a particularly difficult mental calculation. He explained:
100 x 456 is 45 600. I set that to the left of my "mental vision." 20 x 456 is 2 x 450, 900 and 2 x 6, 12 so 912. Somehow it changed to 906 instead of 912. I did it again to get 9120. Added 9000 and 45 000 so 54 000. And 600 + 120 is 720, so 54 720. I then remembered this by thinking they're (54 and 72) quite frequent in the times tables so I tried to remember "54 and 72 in the thousands." I remembered I had blocked 100, 20, and 3 so I had to multiply the remaining 3. 3 x 400, 1200. 3 x 50, 150 so 1350 and 3 x 6 is 18, so 1368. I went over it a couple of times so I wouldn't forget it. Added left to right, so 54 and 1, 5500; 300 and 700, that scrolls up to another 1000. So 55 600 and, finally, 56 088. It's easy to get lost in this calculation.

**Losing track of the direction of the calculation.** Another problem inherent in mental calculation appeared to be keeping track of what calculations had been performed and remained to be performed. There were many instances where subjects lost track of the calculation and proceeded to either miss a step, duplicate a calculative step, or introduce some unnecessary numbers into the computation. Subject U3 took 66 seconds to complete a calculation of 12 x 16. After several recalculation of the partials, she said:

16 on the top and 12 on the bottom. 6 x 2 = 12, carry 1, 2 x 1 is 2, add equals 3.... So 32. Put down the 0. 6 x 1 = 6. Add 1 is 7...so 72. So you have 32 and 72... (begins to add) ...723?

What appeared to happen was that she somehow interchanged the "spatial positions" of the factors 12 and 16. The diagrams below illustrate how she likely viewed the calculation.

\[
\begin{array}{c}
16 \\
\times 12 \\
\hline 32 \\
\end{array}
\quad
\begin{array}{c}
12 \\
\times 16 \\
\hline 72 \\
\end{array}
\]

Several subjects commented that they had "a problem remembering which number was on the top and which was on the bottom." This problem was particularly evident where, as one subject explained, "the numbers
(factors) were close to being equal."

One common track-keeping difficulty was associated with the calculation of squares. This type of forgetting was easy to identify because, instead of calculating the second partial product, the subject repeated the first partial product and incorporated this value into the addition stage of the computation. Subject Ul3's attempt at 16 x 16 illustrates this type of error. She explained: "6 x 6 = 36, carry 3. 6 x 1 = 6, add 3 is 9. So 96. Put down the 0. 960 and 96 is 6 + 0 = 6, 9 + 6 = 15, carry 1,...so 1, 5, 0, 6....1506." There were 26 instances of this type of forgetting reported for unskilled subjects. No skilled subject made this type of error. Almost one-third of the unskilled subjects' responses to the six squares included in CAL2 involved this error. Therefore, repeating rather than calculating a second partial in the determination of a square appeared to be a significant source of error for the unskilled subjects.

There were comments made by some subjects which suggested that the decision to repeat rather than calculate was influenced by the repetition of the factors in a square. Subject U2 who used the partial product of 96 and 960 to calculate 16 x 16 said, "I thought the bottom number (960) should be the same but now I know that it should be 1 x 6 and 1 x 1. I thought that with the repetition of the 16, the top and bottom numbers should be the same." Subject U9 made a similar comment after calculating 25 x 25. He explained, "this time I thought that since 5 x 25 is 125 and the numbers (factors) are the same, it's going to be another 125."

Subject U3 applied a variation of this invalid strategy to solve 32 x 32 and offered the following justification: "32 and 32, 2 x
3 and 3 x 2 then you know it's going to be the opposite because you're doing the same thing in a different order. So instead of 64 it would be 46 and then you kind of shift it over." In other words, he added the partial products of 64 and 460 to determine the incorrect solution of 504.

There was only one instance of this type of error in the pencil-and-paper computation test WPP. Subject Ul1 tried to solve 24 x 24 by substituting 960 for 480 as the second partial product. No other subject made this type of error during written calculation. The fact that this type of forgetting existed in mental but not in written calculation demonstrates how the absence of an external memory-store can affect performance and introduce different types of errors.

The tendency to lose track of a calculation was likely one reason why some attempted solutions seemed very unreasonable. Subject Ul3's solution of 9206 for the item 17 x 99 was such an example. Subject Ul4's solutions were often unreasonable: she made three attempts to solve 24 x 24 and her answers for these attempts were 1056, 4160 and 3460. For the last attempt, she had originally stated "30 thousand and 460" before deciding on 3460.

Since the skilled subjects tended to use different strategies than the unskilled subjects, they committed different types of track-keeping errors. For example, a common error made by subjects who used subtractive distribution was losing track and using the wrong subtrahend. Usually, one of the original factors of the calculative task was used as the subtrahend: for example, S4 used 99 as the subtrahend instead of 17 to calculate 17 x 99.

Some skilled subjects who attempted to use factoring made an
error by incorporating a wrong factor in the calculation. Frequently, this type of failure occurred when aliquot parts was applied. For example, subject Sll calculated 4000 instead of 3000 as the product of 25 x 120. During his introspection, he said, "12 'twenty-fives' are 400....No, 300. So 3000. I think I know what happened I thought 4 into 12, 3 times but then went 4000. It just happened."

Forgetting the order of a series. Losing the order of a series of digits was another type of forgetting in mental calculation. This type of error usually occurred either during the final stages of calculating the product of a 1-digit and a x-digit factor or after the sum of the partials had been determined by using a digit-by-digit, right-to-left addition procedure.

Subject Ull's solution of 15 x 15 illustrates how this problem can arise during a mental calculation. She thought, "5 x 5 is 25, carry 2. And 5 x 1 is 5, and 2 is 7...so 75. And 15,...so 1, 5, 0 and 7, 5 is....5, 6, 7....7, 5, 6...756?" She also encountered this difficulty during a calculation of a partial product. Her initial calculative steps in solving 23 x 27 were "3 x 7 is 21, carry the 2, 3 x 2 is 6,...7, 8, and..86." This difficulty in retaining the order may be related to her low backward digit span of 3.

Subject Ull seemed to forget even the order of the digits in the initially presented factors after she began to calculate. After attempting to solve 25 x 48 for 20 seconds, she forgot her completed calculations, began again, and said, "OK. 54 x 28 is..." The most graphic demonstration of changing the order and values in a series was her calculation of 25 x 32 where she began to calculate 321 x 12!
Carrying errors. There were at least 51 instances where a subject either forgot to "carry" or added a wrong value during a calculation. This error was observed 37 and 14 times for the unskilled and skilled subjects, respectively. For some subjects such as U8, mental carrying was a major source of difficulty. She made at least 7 carry errors during the CAL2 interviews.

Some mental calculation errors occurred because a subject had interchanged the carry and hold digits. For example, as subject U10 attempted to solve 9 x 74, he reasoned, "9 x 4 is 36, carry 3, put down the 6, and 9 x 7 is 63. Add 6 so 69. So 696." Subject U11 frequently added only a carried 1. During the attempt of 25 x 48, she said, "8 x 5 is 40, carry the 1....8 x 2 is 16, 17...170..." Again, these types of errors were unique to mental calculation and were not evident in the pencil-and-paper test.

Those subjects who used their fingers to accompany a calculation would frequently complete the carry operation by skip counting instead of retrieving a basic addition fact. Subjects U11 and U12 often used this process to complete the carry calculation. When asked to calculate 8 x 25, U11 said, "8 x 5 is 40, carry 4, 8 x 2 is 16, 17, 18, 19, 20, 21, 210." She tapped her fingers in unison to this skip counting process and, as can be seen from her proposed solution of 210, she lost count of this additive process.

U12's second attempt at solving 8 x 999 involved this same counting process. She said, "... Should be 72, 72, and 72. So (begins to tap fingers) 73, 74, 75, 76, 77, 78, 79, 80,... 80 002, no 8002." When she was asked later in the interview to calculate the sum of 62 and 7, she instantly responded with the correct solution. Asked
why she used her fingers to count units, she said, "I just get confused. I have to work it out sometimes."

Further questioning revealed that she used a reconstructive rather than retrieval process to solve some basic facts of addition. When asked to solve $9 + 7$, she said, "Oh, I didn't need to calculate because you just go one step lower." She meant that $9 + 7$ could be reconstructed by thinking "$17 - 1 = 16$.

**Misalignment during addition.**

During the right-to-left, digit-by-digit addition process that was typically used by the unskilled subjects, "remembering where the numbers went" was a common complaint. This inability to "align the columns" had many variations.

The most common alignment difficulty seemed to be not remembering "to move one place over." For example, subject U2 correctly calculated each partial product for the item $12 \times 250$ but gave 750 as a solution. In her second calculation attempt, she realized that she had forgotten to "move over one" before adding: in other words, she added 500 and 250, instead of 500 and 2500.

Other subjects had great difficulties in remembering the positions of the digits in each addend. These difficulties resulted in solutions that seemed to reflect a process which could be described as "sliding alignments." For example, subject U9 correctly calculated the partials of 325 and 130 for the item $25 \times 65$. During the addition stage, he reasoned, "$5, 2 + 3$ is 5, $3 + 3$ is 6, 1, so 1655." The diagram below illustrates how the alignment of each partial had the appearance of "sliding" as the calculation proceeded:
This same subject appeared to calculate one sum by annexing rather than adding digits. He correctly calculated the partials of 240 and 96 (960) for the item 25 x 48. In his third attempt at adding 96 and 240, he determined a solution of 9260. The researcher drew a diagram and asked the subject if the calculation had been completed in the manner illustrated below:

He exclaimed, "You know, I think you're right!"

This inability to align the positions of the partials became an insurmountable barrier for some subjects. Ul commented, after spending 78 seconds in calculating 17 x 99, "I can remember 99 and 693 but I can't get them back together. I keep putting the 99 in the wrong place."

Subject Ul2's attempted solution of 15 x 48 is perhaps the best illustration of this alignment difficulty. Her solution included instances of skip counting, finger "writing," rehearsal, and recalculation. She spent 104 seconds in attempting to complete the calculation. She said:

48 x 15 is .... 48 x 15 is....40...I think my brain's done in.... (recalculates). 48 x 15...40...20, 21, 22, 23, 24, ...240....0....48 (begins adding) and 24,...16,...240, ..240,...240,...12, 8,...48, 40,...80 and 240 is (adds again) 12,...24,...24 and 48, 24,...2, 6, 6. Uhh...24 and 48. I can't do this. I know what it is but I can't do it....24 and 48...Oh, it's 700.
A Comparison of Mental and Written Computation

The analysis of the types of forgetting during mental calculation suggested that many important features of a calculation can be lost when a subject has no opportunity either to refresh the factors or to record each completed calculation. By comparing the performance of written and mental computation, an estimate of the degree of forgetting due to the change in presentation and response mode was determined.

To ensure that a valid comparison could be made, 10 items from CAL1 were also used to create WPP, a test of written paper-and-pencil computational performance. The timing requirements used for each test administration, though not identical, did seem comparable. The mental calculation test had a 20-second item-presentation rate. This method of timing was not used in WPP. Instead each subject was instructed to complete the written test quickly and accurately. The longest time taken by a subject to complete WPP was 183 seconds and the total time for presenting the 10-item portion of the mental calculation test was 200 seconds. Thus, all subjects were given at least as much or more time to complete the mental calculation tasks as they had been given to complete the written calculation tasks. Any difference in proficiency should not be due to dissimilar testing conditions.

Table XI includes a summary of the results of each testing medium. An examination of the table reveals that on the written test WPP there was little difference in performances between skilled and unskilled subjects. The difference between the group means was not statistically significant ($t_{28} = 0.48; p > 0.20$). Thus, the superior mental calculation performance of the skilled subjects certainly
<table>
<thead>
<tr>
<th>Testing medium</th>
<th>Skilled</th>
<th>Unskilled</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{x}$</td>
<td>$s_x$</td>
</tr>
<tr>
<td>Written (10 WPP items)</td>
<td>8.9</td>
<td>1.10</td>
</tr>
<tr>
<td>Mental (10 CALL items)</td>
<td>8.5</td>
<td>1.24</td>
</tr>
</tbody>
</table>
cannot be due to any superior knowledge of conventional written computational techniques.

Without the benefit of the external memory store served by the written page, however, the unskilled group's performance dropped dramatically: 14 unskilled subjects did not solve any items mentally; one subject solved only one item mentally. On the other hand, the skilled groups' performance was not significantly reduced by an apparent change in the testing medium.

Alf and Abraham's covariance information statistic was calculated to determine the magnitude of the linear relationship that existed between written and mental multiplication computation. The combined groups' correlation $r'$ was determined to be 0.15 and $R$, the correlation for the intact population, was estimated to be 0.07. Since the resulting $t_{28}$ equalled 0.80 ($p > 0.20$), a linear relationship between the written and mental calculation performance of young adults does not exist.

**Memory Capacity and Mental Multiplication Performance**

Several measures of STM capacity were used to determine if the two groups could be distinguished by differing STM capacities. In particular, forward digit span FDS, backward digit span BDS, delayed digit span DDS, and letter span LS were used to estimate each subject's memory capacity. The results of each group's performance on each of these four tests are presented in Table XII.

An examination of Table XII shows that the skilled group had greater mean scores than the unskilled group on every STM measure used in the study. The difference between the two groups' mean scores was
statistically significant for each capacity measure. Thus, the existence of a linear relationship between STM capacity and mental calculation performance seemed plausible. Scatterplots of CALL mental multiplication performance and each measure of STM capacity were used as an initial test of linearity. Figures 8 through 11 contain these plots.

An examination of each graph indicates that a weak linear relationship existed for each capacity measure used in the study. As further suggestive evidence that STM capacity was not a major source of individual differences in mental calculation performance was the finding that HS had a forward and backward digit span of 8 and 6, respectively. The span estimates of most of the skilled subjects and a few of the unskilled subjects were greater than hers.

The correlation between CALL performance and each measure of STM capacity was determined using the Alf and Abrahams' covariance information statistic. The values and statistical significance of each correlation $R$ are presented in Table XII. Although each correlation was statistically significant, the relationship between mental calculation performance and STM capacity seems to be weak: no STM measure accounted for more than about 11% of the variance in mental calculation performance.


TABLE XII

MEASURES OF STM CAPACITY:

GROUP STATISTICS, SIGNIFICANCE TESTS FOR THE DIFFERENCES
BETWEEN THE MEANS, THE CORRELATIONS AND THEIR SIGNIFICANCE
BETWEEN CALL MENTAL MULTIPLICATION PERFORMANCE AND CAPACITY

<table>
<thead>
<tr>
<th>STM Capacity Measures</th>
<th>Skilled</th>
<th>Unskilled</th>
<th>t&lt;sub&gt;28&lt;/sub&gt;</th>
<th>R</th>
<th>t&lt;sub&gt;28&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>S&lt;sub&gt;X&lt;/sub&gt;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FDS</td>
<td>7.8</td>
<td>1.15</td>
<td>6.3</td>
<td>1.22</td>
<td>3.47**</td>
</tr>
<tr>
<td>BDS</td>
<td>6.2</td>
<td>1.32</td>
<td>4.8</td>
<td>1.26</td>
<td>2.97**</td>
</tr>
<tr>
<td>DDS</td>
<td>65.0</td>
<td>33.55</td>
<td>38.0</td>
<td>30.32</td>
<td>2.31*</td>
</tr>
<tr>
<td>LS</td>
<td>83.1</td>
<td>33.54</td>
<td>52.5</td>
<td>25.67</td>
<td>2.81**</td>
</tr>
</tbody>
</table>

*  p < 0.05

** p < 0.01

*** p < 0.001
Figure 8. Scatter plot of forward digit span and score on CALL.

- ○ Unskilled
- △ Skilled
Figure 9. Scatter plot of backward digit span and score on CALl.

- Unskilled
- Skilled
Figure 10. Scatter plot of delayed digit span and score on CAL1.

- ● Unskilled
- ▲ Skilled
Figure 11. Scatter plot of letter span and score on CAL1.

- • Unskilled
- △ Skilled
CHAPTER V

SUMMARY AND CONCLUSIONS

Introduction

Effect: The second of two phenomena which always occurs together in the same order. The first, called a Cause, is said to generate the other— which is no more sensible than it would be for one who has never seen a dog except in the pursuit of a rabbit to declare the rabbit the cause of the dog.
Ambrose Bierce, The Devil's Dictionary, 1911.

Comparative studies such as this present one can suggest but not conclude. Since the purpose of the study was exploratory, no attempt was made to isolate and control variables which could be a factor in mental calculation performance. Clearly, in a comparative study almost any outstanding feature of a subject's character, background, or knowledge can be interpreted as a possible source of influence: all that is needed is a little imagination. For example, the fact that Scripture (1891) reported that Colburn, an exceptionally proficient mental calculator, had supernumerary digits on each hand and foot suggests that these bodily characteristics were thought to be of some importance. A similar extrapolation was made by the phrenologist Gall who examined the young Colburn and:

...without any previous intimation of his character, readily discovered on the sides of the eyebrows certain protuberances and peculiarities which indicated the presence of a faculty for computation (Scripture, p. 17).

Consequently, attributing group differences to any variable investigated in the study is fraught with difficulties and entails great risk. However, the results of the study can be used to
articulate and clarify questions about individual differences in mental calculation for which answers can be sought by future empirically based studies. Hopefully, researchers who plan to conduct investigations of mental calculation and its close relative, estimation, can benefit from such articulation and clarification.

Summary and Discussion of the Findings

The intent of the study was to identify the processes and procedures which characterized unskilled, skilled, and highly skilled mental calculation performance during the solution of calculation tasks involving multi-digit factors. The data collection and analyses of the study were guided by the following major research questions:

1. Can individuals who differ in mental calculation performance be characterized by the types of calculative strategies used to solve a task?

2. Can individuals who differ in mental calculation performance be characterized by the types of numerical equivalents retrieved to solve a mental calculation task?

3. Can individuals who differ in mental calculation performance be characterized by the efficiency of their short-term memory systems?

Questions About Mental Calculation Strategies

Can skilled mental calculators be distinguished from their unskilled counterparts by their choices of calculation methods or strategies used in mental computations? The evidence suggests an affirmative answer to this question. The study demonstrated that the skilled subjects possessed a large repertoire of calculative plans but
the unskilled subjects possessed few plans.

The strategies. Which strategies were most frequently applied by each of the skill groups? Although additive distribution was the favoured calculative tool of the skilled subjects, the skilled subjects knew and applied many of the non-routine strategies known to have been used by expert calculators. These included the following: fractional distribution, subtractive distribution, quadratic distribution, general factoring, factoring by halving-and-doubling, factoring by aliquot parts, exponential factoring.

Some calculations were determined by applying the pencil-and-paper mental analogue but only rarely would the skilled subjects attempt to apply the right-to-left, digit-by-digit procedure commonly associated with written computational methods. When the pencil-and-paper mental analogue was used, the skilled subjects would attempt to apply an abbreviated form of this method of solution. Such an abridgement took the form of retrieving rather than calculating each partial product.

As was the case for expert calculators, the skilled subjects tended to use a left-to-right sequence of calculations rather than the right-to-left sequence so commonly associated with written computational methods. Progressive addition always accompanied this left-to-right calculation of the partial products. The skilled subjects used such methods in the belief that their memory load would be reduced.

The skilled subjects tended to complete the calculation by using an addition process unlike the conventional right-to-left, digit-by-digit written addition algorithm. Rather than partitioning
each addend into a series of individual digits, the skilled subjects would arrange the addends so that a convenient partial sum could be determined. The calculation would be completed by either adding or subtracting a suitably chosen number to the partial sum. For those strategies such as subtractive distribution which required a subtractive method to complete the calculation, a similar departure from the conventional written methods would be undertaken by the skilled subjects.

HS, the youngest and most highly skilled mental calculator who participated in the study, possessed an acute sense of number factorability. Her quick apprehension of the factors of a presented number combined with an ability to retrieve large numerical equivalents useful for multiplication contributed greatly to her calculative powers. Unlike the majority of the other subjects in the study, she never once used any form of the pencil-and-paper mental analogue to complete a calculation.

HS and the other more skilled subjects worked with numbers rather than individual digits to aid a mental calculation. Numbers are, for these proficient subjects, rich in association and meaning. Hunter remarked that proficient calculators such as A. C. Aitken apprehend a number "as a multiplicity of numerical properties and, so to speak, as bristling with signalling properties" (1962, p. 246). A similar remark could be made of the most proficient subjects in this study and, in particular, the highly skilled subject HS. "Signalling properties" appear to act as retrieval cues: the choice of a calculative strategy being influenced by the properties apprehended by the subject. In a sense, the proficient subjects seemed to view
calculation in much the same manner as an artist might view a painting. Both individuals see relationships unnoticed by the "unskilled eye." Colour, form, and space cue the artist; number properties cue the proficient mental calculator.

The most skilled calculators were able to orchestrate a number of strategies to solve a complex mental calculations. HS's solution of 123 x 456 described in Chapter IV, for example, indicated that she was able to carry through a multiplicity of complex and inter-related activities in a relatively short space of time. Classifying this solution as an application of general factoring seems a gross oversimplication of the complete mental calculation process and, in this sense, the strategy classification scheme used in the study provides an inadequate description of these more complex calculative schemes.

Maier's contention that many young adults "are enslaved to the slow and awkward procedures learned in school" (1977, p. 92) seems to be an accurate description of those unskilled mental calculators who participated in the study. In contrast to the diversity of methods possessed by the skilled subjects, unskilled mental calculators seemed tethered to the right-to-left, digit-by-digit, pencil-and-paper mental analogue.

Regardless of the factors included in a task, an unskilled subject's approach was typically the same, the fragmententation of the calculation into a series of individual digits as a preparatory step for the subsequent application of the right-to-left, digit-by-digit strategy P&P0. Their reluctance to discard even the physical actions and spatial terminology associated with written methods reflects how
the thinking of the unskilled subjects was dominated by this particular calculative strategy.

Generally speaking, the unskilled mental calculators worked not with numbers but with individual digits. Thus, the unskilled subjects were content to attend to only the surface details of the task. Even seemingly obvious properties that might aid the calculation such as the identity principle of multiplication were ignored usually by the majority of the unskilled subjects.

During the additive phase of a calculation, the right-to-left, digit-by-digit pattern continued as the unskilled calculators applied the algorithm used for written addition tasks. This addition process proved to be particularly difficult since the subjects had to remember not only the two series of digits representing the value of the calculated partial products but the "column positions" as well.

The task and choice of strategy. Is there a relationship between the types of calculative tasks and the type of strategy selected to solve those tasks? If the question concerns unskilled calculators, the answer must be stated in the negative since, as has been discussed, different items seemed to elicit the same type of calculative response.

The skilled subjects were much more discriminating in their choice of strategy. Certain general patterns of responses were noted. If the numbers in the calculative task were replete with relatively small prime factors, general factoring would be the most likely approach. The uses of subtractive distribution, factoring by aliquot parts, factoring by halving-and-doubling, fractional distribution, exponential factoring, and the various types of quadratic distribution
were most evident for those tasks where particular "signalling" numbers were present. Factors close to a multiple of a power of 10 such as 999, for example, often acted as a "signal" for subtractive distribution; factors such as 25 often signalled factoring by aliquot parts.

Because additive distribution can be used to solve a wide range of calculative tasks, this strategy was much in evidence. However, there were some subjects such as the highly skilled subject who used additive distribution for only the more prosaic 1-digit factor tasks such as 9 x 742 or those tasks such as 87 x 23 where the presented numbers had few factors. The skilled subjects would use the pencil-and-paper mental analogue to solve a particularly "easy" task such as 9 x 74 or a task such as 73 x 83 that was described by one skilled subject as having "weird numbers."

The ease with which some skilled subjects were able to orchestrate a composition of several calculative plans to complete some difficult calculative tasks tended to obscure the relationship between the selection of a calculative strategy and the type of calculative task. Consequently, the relationships outlined by the researcher should be considered only a rule-of-thumb.

Adapting to the task. Which skill group most frequently changed from one strategy to another in response to a change in the calculative task? Although all subjects were instructed to use whatever methods they thought were naturally suited to the task, large differences existed between the abilities of the skilled and unskilled subjects to change the approach to correspond with a change in the task. A moderately strong linear relationship between the number of
calculation strategies a subject applied to solve a series of
calculative tasks and mental calculation performance was found to
exist.

The proficient mental calculators demonstrated that they not
only possessed a variety of calculative strategies but like the good
estimators studied by Reys, Rybolt, Bestgen, and Wyatt (1982) and the
capable problem solvers studied by Krutetski (1969) they could readily
change from one strategy to another if the task was thought to warrant
the change. The highly skilled subject's almost habitualized
verification activities she demonstrated during the "factorability and
square recalling" assessments are further testimony to the variety of
approaches which some subjects can use to calculate mentally. She
often applied an initial strategy, checked by applying an alternative
strategy, and in some cases used some number theoretic principle such
as a divisibility rule to provide a further check of the calculation
before attempting to state the solution. The selection of a checking
procedure often varied from task to task. Hunter (1962) has made
similar observations about the verification activities of expert
mental calculators.

Do these findings about the flexible approaches towards mental
multiplication tasks demonstrated by skilled subjects indicate that
this behaviour is characteristic of proficiency on all types of mental
calculation tasks? For instance, do highly skilled mental calculators
change strategies more frequently than less skilled subjects when
presented with a series of mental sums? Until further evidence is
provided, any conclusions about the adaptive behaviour of skilled
calculators should be restricted to the types of mental multiplication
tasks used in the present study.

Not all skilled subjects readily switched strategies. A few subjects strongly favoured additive distribution; the very skilled subject S3 favoured the use of the pencil-and-paper mental analogue. S3's mastery of this method was such that she was able to solve even the difficult task 123 x 456. These exceptions to the rule suggest that there is likely no simple relationship between adaptive behaviour and proficient mental calculation. Perhaps, there are many types of calculative expertise.

Hatano and Osawa (1983), for example, have argued that there are experts who are proficient because of the speed, accuracy, and automaticity of their procedural skill. Using Hatano and Osawa's terminology, S3 might be described as a "routine expert." Those calculators who demonstrate expertise through flexible planning and adaptation to new calculative tasks could be described as "adaptive experts." The most proficient skilled subject S1 and the highly skilled subject HS would seem to be representative of this latter form of expertise.

How to distinguish between these types of experts is not clear. Despite the ambiguity in classifying proficient subjects as routine or adaptive, researchers interested in investigating expert cognitive behaviour would be well-advised to consider that different tasks may require differing forms of expertise.

One cannot complete the discussion of adaptation to a calculative task without making some reference to the behaviour of unskilled mental calculators. Unlike the adaptive stance taken by many skilled subjects, the unskilled subjects exhibited behaviour which
could be described as "calculative monomania." Regardless of the change in the nature of the calculative task, most of the unskilled subjects would attempt to take the same and usually unsuccessful digit-by-digit, right-to-left approach.

Researchers who have studied various forms of learning disabilities would likely be tempted to describe the unskilled subjects' calculative behaviour as "perseverative." Woodward (1981) defines perseveration as the "tendency to continue an activity once it has been started and to be unable to modify or stop the activity even though it is acknowledged to have become inappropriate" (p. 189). Glennon (1981) describes the process as "the inability of the individual to switch with ease from one stimulus situation to another" (p. 63). Given the evidence provided by the present study, the term does seem to apply to the behaviour of the unskilled mental calculators.

The inflexible behaviour of the unskilled subjects could be argued to be not a perseverative tendency at all, but simply a Hobson's choice: the type of behaviour one would expect from subjects who have a limited knowledge of a variety of calculative techniques. Regardless of how unsuccessful a subject's approach has proved to be, if the subject knows only this approach, a singularity of approach is not surprising.

Nevertheless, there were some aspects of the unskilled subjects' calculative thinking reported in Chapter IV that seemed to perseverate. U10's revelation that she calculated a partial product "with all 0's on the top" to solve 50 x 64 and U13's persistent use of digit-by-digit calculations to solve even easy items such as 20 x 30
are two instances of a possible perseverative tendency.

The strongest evidence of "rule bound" behaviour was the finding that some subjects always made the same type of error when particular tasks were presented. For example, a few unskilled subjects would substitute the value of the calculated first partial product for the value of the second partial product whenever a mental product of a square was required. Why this type of error would perseverate in mental but not in written calculation is a question that seems worthy of further analysis and investigation.

Efficiency of a calculative strategy. Can any general characteristics be identified that appear to distinguish efficient from inefficient strategies? This study provided suggestive evidence that some mental calculation strategies are more efficient than others. Since a moderately strong relationship between mental calculation performance and a subject's infrequent use of the digit-by-digit, pencil-and-paper mental analogue existed, this strategy could be inefficient. The fact that HS, the highly skilled mental calculator, had an aversion to using this strategy seems to further implicate the strategy as being inefficient.

Why would the digit-by-digit, pencil-and-paper mental analogue be an inefficient mental calculation strategy? This strategy has proven to be particularly useful for written purposes. Because of the design of the study, efficient subjects and efficient techniques cannot be disassociated from one another. Thus, one cannot properly conclude that a strategy used by a skilled subject is necessarily more efficient than a strategy used by an unskilled subject. Only future studies which ensure that all subjects use the same strategy can
answer this question about efficiency. However, the findings of the study seem to suggest that efficient mental calculation strategies have several of these distinguishing characteristics: (1) they eliminate the need for the carry operation; (2) they proceed in a left-to-right manner; (3) they progressively incorporate each interim calculation into a single result.

The reader should be reminded at this juncture that the following discussion is directed towards describing the characteristics of efficient mental and not written computational techniques.

1. Eliminating the carry operation. Research has demonstrated the difficulty of the "carry operation" (Hitch, 1977, 1978; Dansereau & Gregg, 1966; Merkel & Hall, 1982). For example, Hitch found that many errors in mental addition were due to this particular mental operation. Moreover, he found that "subjects are just as prone to forget the absence of carrying as they are to forget its presence" (p. 321). Hitch's findings suggest that a carry operation requires these two working memory operations: storing and retrieving the carry digit; remembering whether carrying was necessary or not. If the calculative task involves only a few carries, the additional burden imposed by a carry operation likely can be handled by working memory. But with some particularly complex calculations, the burden of a carry can become excessive and thus calculation performance can suffer.

An analysis of the strategies used by proficient mental calculators indicates that carrying was conspicuous by its absence. The elimination of the carry operation was accomplished in several ways. First, the skilled subjects rarely used the digit-by-digit,
paper-and-pencil mental analogue, a strategy which can involve many carries both in the multiplicative and additive stages of the calculation. The carry requirement of P&PO seems to be one reason why the skilled subjects avoided the use of this strategy. One skilled subject who never used this strategy made this point: he said, "I don't use the pencil-and-paper method because there's too many carries to remember."

Additive distribution, the "calculative drafthorse" of the skilled calculators, may be more efficient than P&PO because there is usually no need to remember a mental carry. Carrying can be eliminated by arranging the calculation so that each factor has only one significant digit. To calculate the product of 9 and 742, for example, the factors become 9, 700, 40, and 2. No carries are required to calculate the partial products 9 x 700, 9 x 40, and 9 x 2. Those subjects who choose to use the pencil-and-paper mental method, on the other hand, need to use the carry operation twice.

Additive distribution does not eliminate carrying completely in the additive phase but the need for this operation will be infrequent. Each partial product, with the exception of the product of the unit digits, will be a multiple of 10. Since each of these partials include "zeroes" in the right-most place value positions, most carries can be eliminated during the calculation of the sum of two partials. Using 9 x 742 as an example, the calculations will become 6300 + 360 and 6660 + 18. Neither of these sums require the carry operation. An analysis of the other strategies used by proficient subjects reveals a similar absence of the carry operation.

Another way that "carries" during a mental calculation can be
eliminated is by retrieving rather than calculating a partial product. For example, no carry operations were used to complete the calculation of 15 x 16 when one skilled subject reasoned, "80 and 16, move one over, 160. And, 160 and 80 is 240."

2. **Left-to-right calculation.** Another common characteristic of the proficient subjects in the study was their tendency to complete the calculation in a left-to-right fashion. Most researchers including Ball (1956), Bidder (1856), Gardner (1977), Hunter (1962, 1979), Mitchell (1907), Scripture (1891), and Smith (1983) have reported similar findings. The reasons for this behaviour is not clear.

Mitchell (1907) believed that custom rather than efficiency, or what he called "convenience," sanctioned the use of left-to-right procedures. He explained:

> ... if, however, the calculator should accidentally form the habit of beginning with the last figure, it is hard to see where any real inconvenience would result. In mental as in written arithmetic, much depends on custom and habit; it is hard to see any great difference in convenience beginning at the right and beginning at the left, either in mental or in written multiplication. (p. 104)

Thus, Mitchell would accord no particular significance to this feature of expert mental calculation.

One interesting explanation proposed by Gardner (1977) explains why some expert calculators who earned a living by performing mental calculations often preferred a left-to-right method. The advantage for a stage performer, according to Gardner, is that:

> ...they can start calling out a product while still calculating it. This is usually combined with other dodges to give the impression that computing time is much less than it really is. (p. 70)

This clever deception may be true of some professional performers but
this seems an unlikely explanation for the left-to-right calculative
behaviour of the subjects in this study.

A careful analysis of left-to-right methods combined with the
findings of several studies suggests that these methods are possibly
less demanding than right-to-left methods on short-term memory. There
are several features about left-to-right calculation that may minimize
the burden on short-term memory.

First, the likelihood of forgetting the initial calculations
can be greatly lessened using left-to-right methods since portions of
the answer can be spoken before the entire calculation is completed.
The skilled subjects who used additive distribution would state the
answer to 8 x 4211, for example, in a series of stages such as
"Thirty-two..., thirty-three thousand, and... six hundred and...
eighty-eight... It's thirty-three thousand six hundred eighty-eight."
This manner of stating the answer may be efficient since Hitch (1978)
demonstrated that "interim information produced in the course of
computation will undergo rapid forgetting if it is not immediately
utilized" (p. 306). On the other hand, if right-to-left methods are
used, the answer cannot be stated until the entire calculation is
completed.

Furthermore, Hitch's finding that in mental addition fewer
errors were made in the most significant digits of the sum when
subjects proceeded in a left-to-right manner has important
implications for studies of estimation procedures. In the pursuit of
an estimate, errors in calculating the value of the most significant
digits are obviously most undesirable. Thus, it can be hypothesised
that left-to-right methods will improve the accuracy of an estimate.
Further research is needed to evaluate this hypothesis.

A second reason that left-to-right methods seem to be efficient is that a transformation operation can be eliminated. Those subjects who favour right-to-left procedures have an onerous task. They are required to retain not only a series of discrete calculations but, as an additional requirement, the order of the recalled series has to be transformed into a left-to-right sequence more commensurate with the Hindu-Arabic numeration system. According to many unskilled subjects, transforming the order is a somewhat difficult operation. One unskilled subject expressed the difficulty in this manner: "You're going this way and then you have to say it that way."

Since left-to-right calculation seems to correspond more closely to place-value order, there is no need for this "reversal" transformation. The elimination of any unnecessary mental operation becomes especially important during particularly lengthy calculations. Further research is needed to determine if left-to-right and right-to-left methods of mental calculation do impose different memory loads.

3. Retaining a single result. Another feature of proficient mental calculation was the tendency for subjects to incorporate progressively the interim calculations into a single result. In the case of distribution, for example, the retention of a single result was accomplished by continually retrieving a sum, updating by adding a newly calculated partial, and storing this new sum. In the case of factoring, a running product rather than a sum was modified continually.

What seems to be the purpose behind this technique? The
proficient subjects in the study believed that retaining a running total was less demanding on memory than the mathematically equivalent procedure of computing a total in the last stages of the calculation. "There is too much to remember otherwise" said one skilled subject as he explained why he preferred to keep a running sum.

Those who have studied expert mental calculation have made similar comments about the memory advantages of this technique. Bidder (1856) explained that the object of progressive addition in distribution was to "have one fact and one fact only, stored away at any one time" (p. 260). He believed his memory demand could be significantly reduced by focussing on only one result.

Mitchell (1907) arguments were similar Bidder's: "...it is much easier to combine at each separate stage, and relieve the memory of the strain of remembering the partial results throughout the process" (p. 104). Since the unneeded calculations of a proficient calculator are jettisoned as so much excess memory baggage, the load on short-term memory is supposedly lightened. Whether or not the technique does reduce the memory load as its proponents have claimed will have to be left to the judgement of an experimental study.

Questions About Retrieval of Numerical Equivalents

Can individuals who differ in mental calculation performance be characterized by the types of numerical equivalents they retrieve to solve a mental multiplication task? The evidence suggests that the skilled mental calculators had a more extensive library of useful numerical equivalents than the unskilled subjects.
Basic fact recall. How did the two groups compare on the ability to retrieve quickly and accurately the basic facts of multiplication? The basic facts of multiplication were retrieved more quickly and mastered to a higher degree by the skilled than by the unskilled subjects. The differences between the accuracy and speed of recall of each group were statistically significant. However, neither difference was great. Both skill groups had attained a considerably high level of mastery.

A weak linear relationship between mental multiplication performance and recall accuracy was found to exist. A similarly weak relationship between mental multiplication performance and time to retrieve the basic facts of multiplication also existed. These findings indicate that the basic fact mastery of young adults is not an important factor contributing to individual differences in mental calculation performance. Whether variation in basic fact recall contributes to individual differences in mental calculation performance in other samples such as younger children could be the subject of a future study.

There were a few unskilled subjects who were unable to retrieve a selected set of the basic facts of multiplication. Instead, these particular facts had to be reconstructed either by a process of skip counting or by applying distribution. There was some weak evidence which suggested that reconstruction of a basic fact during a mental calculation can contribute to forgetting some important feature of the calculation. What level of automaticity of basic fact recall is necessary for completing mental calculation tasks is suggested as another topic worthy of future investigations.
Recall of large numerical equivalents. How did the skill groups compare with respect to retrieving large numerical equivalents to aid in a mental calculation? Was there any evidence to support the proposition that proficient calculators possess an "extended mental multiplication table?" The skilled calculators were found to have a more extensive "mental multiplication table" than the 10 x 10 basic fact table possessed by the unskilled calculators.

There were differences between the two groups in the ability to recall the so-called "12's table." Almost all the skilled subjects could recall these specific types of numerical information. On the other hand, almost none of these numerical facts could be recalled by the unskilled subjects.

The extended mental multiplication tables of the skilled calculators went only slightly beyond the "12's tables." The majority of the skilled subjects could retrieve rather than calculate a number of squares including $13^2$, $15^2$, $16^2$ and $25^2$. This numerical information was often incorporated in the calculation of more difficult calculation tasks such as $125 \times 125$. In contrast, the unskilled subjects were able to solve a calculative task rarely by "blocking" and accessing these large numerical equivalents.

HS, the highly skilled subject, had by far the most extensive mental multiplication table of those subjects who participated in the study. She could retrieve rather than calculate the great majority of 2-digit squares and a number of 3-digit and 4-digit squares as well. Many of her calculations were solved either through a single retrieval process or by arranging the calculation so that a retrieval of a large numerical equivalent would expedite the calculation.
Questions About Short-Term Memory Processes

Can individuals who differ in mental calculation performance be characterized by the efficiency of their short-term memory systems? The evidence suggests that the skilled subjects had greater STM capacities, employed different memory devices, and were less affected by a change from written to mental methods of calculation than the unskilled subjects. The forgetting of temporarily held information was found to be a major source of error in mental calculation.

STM capacity. Are there differences among the skill groups on measures of STM capacity? On each of the four measures used to assess short-term memory processes, forward digit span, backward digit span, delayed digit span, and letter span, there was a statistically significant difference between the mean spans of the two skill groups. The STM capacity of the highly skilled mental calculator HS was found to be well within the normal range of memory span.

A statistically significant but weak linear relationship between mental multiplication performance and each of these four measures of STM capacity was found to exist. No one measure of capacity predominated as a predictor of mental multiplication performance.

Written versus mental calculation performance. What is the effect on performance when mental rather than written methods must be used to solve computational tasks? There was a negligible and statistically non-significant difference between the performances of the skilled and unskilled subjects on a written multiplication test. Thus, there seems to be no relationship between the written and mental multiplication performance of young adults.
A differential effect between mental and written calculation performance was noted. The unskilled group's performance dropped dramatically while the performance of the skilled group remained relatively undiminished by a change from a written to mental testing medium.

**Forgetting of information.** What aspects of a calculation seem likely to be forgotten during a mental calculation? An analysis of the errors made by the subjects in the study indicated that certain aspects of the initially stated task or the interim calculations seemed to be particularly susceptible to forgetting during a mental calculation. The following types of forgetting were identified: (1) forgetting the factors of the initial task; (2) forgetting the numbers in a partially completed calculation; (3) losing track of the direction of the calculation; (4) forgetting the order of the series; (5) forgetting to carry or adding the wrong carry; (6) mental "misalignment" during the additive phase of the calculation. Except for carry errors, these types of forgetting were not evident during written computation.

**Memory devices.** Do skilled mental calculators employ different memory devices than unskilled mental calculators to minimize the forgetting of the initial or interim calculations? There was some evidence to suggest that the skill groups employed some common, as well as different, memory devices to minimize the forgetting of some feature of the calculation. A tactic used extensively by both the skilled and unskilled subjects was rehearsal, both covert and overt. This memory device was applied in an attempt to remember the initially stated factors and the numbers computed at various stages of the
A visuo-spatial representation of pencil-and-paper computation was another memory device used by most of the unskilled calculators and a few of the skilled calculators. Deciding to put the "larger number on top and the smaller on the bottom" can be thought of as a method of helping a subject keep track of at least the initial stages of a calculation. Another tactic used by these subjects to help retain the order of a series of digits was imagining the series of digits in each partial product as aligned in "columns."

The use of fingers during a mental calculation was used by the majority of unskilled mental calculators in an apparent attempt to enhance retention. This tactile aid was used to write each factor and to record all stages of the calculation. One subject used her fingers to position the digits in each partial product in preparation for the final additive phase. Whether or not this tactile behaviour aids retention can be determined only through future empirical studies.

Because the skilled subjects tended not to use the digit-by-digit, right-to-left strategy so favoured by the unskilled subjects, they employed a somewhat different set of memory devices. Chunking, or what one skilled subject described as "blocking," was a particularly important device used by the skilled subjects. Certainly, arranging a factor into a series of "blocks" as a preparatory step for the subsequent retrieval of a large numerical equivalent seemed to be an effective way to reduce the amount of numerical detail.

The introspective report provided by one skilled subject indicated that associations could be used to foster retention of the partial products during a difficult calculation. Usually such
associations were based upon some number property apprehended by the subject. The data are too weak to conclude that the use of mnemonics during calculation is a prevalent practice among proficient mental calculators. However, Aitken claimed that, "Mnemonics I have never used, and deeply distrust. They merely perturb with alien and irrelevant associations a faculty that should be pure and limpid" (Smith, 1983, p. 61). Further investigation of the memory devices of mental calculators seems to be necessary before any firm conclusions about either their popularity or effectiveness can be reached.

Conclusions about STM processes. What can be concluded about the role of the supposedly limited capacity of the short-term memory store in mental calculation performance? The types of forgetting documented in Chapter IV did seem to be consistent with the notion of a short-term memory system with a limited capacity to store and process temporary information. Many subjects forgot interim calculations, lost track of the calculative direction, and could not remember the order of a series of stored digits. The finding that the unskilled subjects' computational performance dropped dramatically when the external memory-store served by the written page was removed seems to be additional support for the notion of a limited-capacity memory store.

On the other hand, the finding that only a weak relationship existed between mental calculation performance and STM capacity seems to weaken the argument for the important contribution that the short-term memory system supposedly makes to this type of reasoning process. Several explanations for this apparent discrepant finding can be presented.
On the technical side, the low correlations reported in this study could have been attenuated by the somewhat restricted range of the STM capacity scores determined for the subjects of the study. For example, in the case of forward digit span, the span estimate for all subjects including the highly skilled subject only varied from 5 to 9: well within the expected range for normal subjects (Miller, 1956; Dempster, 1981). Perhaps, the relationship between capacity and mental calculation performance becomes evident only when a greater range of span estimates are considered: those subjects with exceptionally low span estimates being more likely to be poor mental calculators than those with particularly high estimates.

However, this lack of a relationship could be more than a statistical one produced solely by studying subjects who differ very little in STM efficiency. Possibly the notion of a limited-capacity memory store is conceptually inadequate to explain individual differences in mental calculation performance.

In the present study, the subjects' choices of calculative strategy were left unconstrained. All subjects were left free to select and use whatever mental methods that seemed suited to the task at hand. Unconstrained choice of strategy has not been a hallmark of most experimental studies of mental calculation. In some studies the control over a subject's use of strategies has been accomplished by presenting tasks which can be attacked in only very few ways. Control over use of strategies has also been exercised by instructing the subjects to solve a task in a particular manner.

Jarman (1978, 1980) has designated those tasks where a variety of solution methods are possible as heterogenous. Those tasks which
are structured so that very little, if any, variation in strategy selection is possible, he has designated as homogeneous. Furthermore, Hunt (1980) has argued that strong relationships between STM capacity and complex cognitive processes are evident only when homogenous tasks are used in a research study. Thus, the strong relationship between capacity and the more homogeneous tasks used by most experimental studies to measure mental calculation performance likely exists when there is only a limited opportunity for a subject to employ a variety of strategies.

The tasks used in this study could be considered heterogeneous because they seemed to foster rather than inhibit strategic variation. Since, as Hitch (1977, 1978) and Hunter (1978) have argued, calculative strategies can vary in terms of information-processing requirements, the resourceful person with an average span can always find a way to ease the burden of memory when presented with a series of heterogeneous tasks. Bidder (1856) made similar arguments about mental calculation. He argued that only through the clever use of a mental calculation strategy can the memory demands of difficult calculations be overcome. He explained, "the process might appear prolix, complicated, and inexpedientious, although it is actually arranged with a view of affording relief to the memory" (p. 254).

Subjects such as the highly skilled subject HS simply do not require large structural resources to solve most types of everyday calculative tasks. How much short-term memory processing is involved in recalling a large numerical equivalent to solve a mental calculation? The memory demand must be surely minimal. Through the judicious selection of a calculative strategy, HS can get by with less
structural resources than the selection of more inefficient strategies would necessitate. Like frugal people who learn not to strain their limited financial resources, proficient calculators learn to rely on a clever choice of strategy designed to place no great strain on their temporary memory resources.

The simultaneous-successive information processing model of cognition described by Das, Kirby, and Jarman (1979) provides another perspective from which to view the cognitive processes which may underlie mental calculation. In brief, the model proposes that information is likely to be processed in one of two ways. Successive synthesis refers to processing in sequential temporal-based forms (Jarman, 1980, p. 76). Any task where information must be arranged and retained in a serial order requires successive processing. Simultaneous integration refers to the synthesis of separate elements into groups; these groups often taking on spatial overtones (Das, Kirby & Jarman, p. 49). Simultaneous processes are involved in tasks where the discernment of relationships and patterns are required.

This study demonstrated that the skilled subjects employed calculative strategies which required them to apprehend numerical relationships and to incorporate this information into the calculation. But the instruments used in this study to estimate STM capacity involve mainly successive information processing. Thus, the low correlations between mental calculation proficiency and memory capacity reported in this study may apply only to measures of successive processing. Perhaps measures of simultaneous processing should be included in any future investigations of individual differences in mental calculation performance.
Furthermore, it can be hypothesised that a relationship between simultaneous-successive processing and mental calculation performance may vary across calculative strategies. Performance on the pencil-and-paper mental analogue and additive distribution would likely require more successive than simultaneous processing since the user must be more concerned with remembering the order and location of calculative information than with determining useful number properties. Those individuals demonstrating high successive processing skills as estimated by measures such as forward digit span and visual short-term memory would be expected to perform better with these types of calculative strategies than those individuals with low successive processing skills.

Conversely, those individuals demonstrating high simultaneous processing skills as estimated by measures such as Raven's Coloured Progressive Matrices (Das, Kirby & Jarman, p. 52) would be expected to perform better with factoring strategies than those individuals with low simultaneous processing skills. Weak relationships between simultaneous processing and performance with the pencil-and-paper mental analogue and between successive processing and performance with factoring would provide further evidence that simultaneous-successive processing requirements vary systematically across calculative strategies.

Further research into the relationship between cognitive processes and performance on other types of mental calculation tasks is also needed. An analysis of mental addition and subtraction operations suggests that neither operation lends itself particularly well to the "shortcuts" so evident in this study of mental
multiplication. Consequently, these addition and subtraction tasks could be less heterogenous than mental multiplication tasks. Attaining proficiency in mental addition, it can be hypothesised, would require different processing skills than a similar level of attainment in mental multiplication performance.

The recent studies of expert abacus operators (Hatano & Osawa, 1983; Hatano, Miyake & Binks, 1977) provide some indirect support for this hypothesis. These experts can calculate a mental sum of fifteen 5-digit to 9-digit addends presented orally. However, unlike expert mental calculators, they did not appear to use any novel strategies or "short-cuts" to accomplish their calculative feats of addition. Instead, according to the experts' verbal reports, sums were determined by calculating on a form of a visualized "mental abacus."

Not surprisingly, the forward and backward digit spans of these subjects were found to average about 15 and 14, respectively. In contrast to the abnormally large digit spans, the letter spans of these individuals were about average. Hatano and Osawa concluded that the enlarged digit spans were developed in response to the specific information-processing requirements of the mental calculation skills that these experts aspired to master. A similar reciprocal relationship between other non abacus-derived mental addition tasks and digit span could be speculated to exist.

The fact that a subject had access to innumerable memory aids not available during the assessments of STM processing efficiency could be another reason why STM capacity was found to be a poor predictor of mental multiplication performance. There is no possible way that a subject can re-create the entire series from a few recalled
digits in a conventional measure of digit span because the series have been designed to discourage these reconstructive process. However, in the case of mental calculation, if a subject forgets a series of stored calculations, the luxury of re-calculation exists for as long as the subject retains the initial factors. Thus, digit span is likely not an accurate parallel of the fundamental memory processes needed during a mental calculation.

Mental calculation can also provide the subject with more opportunities for chunking than provided by conventional memory tests. The skilled subject S7, whose use of associative techniques was reported in Chapter IV, could have greatly increased the retentiveness of interim calculations by relying on his long-term memory to make these associations. The use of long-term memory to increase the information-processing efficiency of a subject's calculations cannot be predicted by the crude STM measures used in this study.

In conclusion, mental calculation may involve processes similar to those which underlie conventional memory tests. But there is a real possibility that subtle, and likely important, differences do exist. Unfortunately, in understanding the everyday uses of mental calculation, Cole, Hood and McDermott's conclusion that "the analytic apparatus we bring to these (everyday) environments from experimental psychology does not apply" (1982, p. 373), could be correct.
Researchers such as Case (1975) and Shulman (1976) have argued that novice-expert comparisons can often lead to new insights regarding mathematics teaching. This study which compared unskilled and skilled mental calculation performance has implications for developing instructional programs designed to improve the performance of lesser skilled individuals.

Likely the simplest route to improved mental calculation performance would be for teachers to provide opportunities for meaningful practice. All subjects who participated in the study were convinced that the topic of mental calculation was never taught in school. Even the most proficient subjects in the study believed that, for all practical purposes, they were self-taught and their rules of working were of their own invention.

The results of the screening phase of the study have demonstrated that very few individuals reach even a moderate level of proficiency if mental calculation is left to develop on its own. This seemingly obvious fact appears to have been overlooked by teachers in their zealous pursuit of student competence in written computation. The fact that so few teachers expect students to calculate mentally seems to explain why so many unskilled subjects could see no purpose to mental calculation. "Why bother," said one subject, "when I can always use my calculator." If any mathematics educators believe that mental calculation should form an integral part of the mathematics program, mental calculation activities should be introduced well before the development of such negative attitudes.
Since the skilled and unskilled mental calculators could be characterized by their different choices of calculative strategies, a clear implication for improving the performance of young adults would be to teach the strategies of the skilled to the unskilled. Further research is needed to determine which calculative strategies are "teachable." However, some educated guesses can be made.

As has been discussed in the study, left-to-right additive distribution appears to be a more efficient strategy than the digit-by-digit, pencil-and-paper mental analogue. Even the most unskilled subjects should be able to master this strategy. There are several reasons to warrant this optimistic conclusion.

First, the number of "mental steps" needed to complete a calculation using distribution does seem to be less than the number needed to complete the digit-by-digit, pencil-and-paper mental analogue. Consequently, there seems to be no reason to suspect that additive distribution involves more processing than the method so favoured by the unskilled subjects.

The observation that some unskilled subjects abandoned the pencil-and-paper mental analogue and, in an apparent burst of inspiration, used an incomplete form of distribution is another reason for optimism. Perhaps with proper instruction and sufficient practice even the most unskilled mental calculators should be able to master the complete form of distribution to solve some types of mental multiplications.

How well the unskilled subjects can learn the other seemingly more complex strategies used by the skilled subjects remains to be seen. The fact that some unskilled subjects began to apply factoring
by aliquot parts during the latter stages of these relatively brief interviews provides suggestive evidence that a reasonable level of proficiency is possible.

The finding that an exceptionally large STM capacity seemed not to be necessary for proficient mental multiplication performance seems to be more good news for those mathematics educators attempting to improve the mental calculation skills of young adults. The selection of an efficient strategy seems to be a much more critical factor than memory span in determining the level of mental multiplication performance.

The finding that forgetting seemed to occur when basic facts were reconstructed rather than instantly recalled has implications for instructing those subjects such as younger children who do not have complete mastery of the basic facts. A great deal of reconstruction is often necessary in the early stages of basic fact mastery (Thornton, 1978). However, this reconstruction could seriously limit a child's ability to calculate mentally. The relationship between mental calculation performance and basic fact recall speed and accuracy amongst younger children should be the subject of an empirical investigation.

Another important factor in developing mental calculation skills could be providing young children with ample opportunities to explore number patterns. Bidder, for example, believed that his incredible powers had very humble origins. As a young child, he would arrange concrete materials such as peas and marbles into rectangular arrays. Through the continued exploration of the number patterns suggested by these arrangements, he gradually acquired a repertoire of
strategies. Moreover, he reported he could recall many basic facts of multiplication well before he knew the word "multiply" (Bidder, 1856, p. 258).

He noticed that larger "sums" could be calculated by a clever re-arrangement of a larger array into smaller size arrays. Hence, the discovery of additive distribution followed from these intuitive explorations provided by structured materials. His proposal that such intuitive notions be well developed before "ciphering" is taught seems very modern.

The highly skilled subject's skills had developed largely from an early fascination with numbers manifested by the continual "practising and playing with numbers." The subject's calculative talents were discovered quite by accident by the parents when the child was about 10 years old. The mother's recollection of the discovery was as follows:

We were in the car one night asking ordinary times tables to some of the smaller children. And she said she wanted to take part in the game but wanted something harder. Only we kept making them harder and harder and she was coming back quicker and quicker than you can do on a calculator. Although this study demonstrated that the young child was still a highly skilled mental calculator, the mother commented that her daughter was now "much slower and less accurate too."

A keen interest in the properties of numbers seemed to be a common experience of many skilled subjects as well. Those subjects who could recall their early days in elementary school commented that they liked to "explore the multiplication tables for interesting patterns." These intuitive explorations occasionally would reveal some relationship that the child would later develop into a mental
calculation technique.

Thus, teachers would do well to use activities which encourage children to explore number patterns and relationships. Developing an understanding of number relationships and properties could help establish a stable network of associated and related concepts which support each other. These inter-connected ideas and rich associations could form a durable base to which new but related numerical information can be fitted.

**Concluding Remarks**

Mathematics has been said to "contain much that will neither hurt one if one does not know it nor help one if one does not know it." Perhaps, this epigram is an apt description of mental calculation. The trend to create devices designed to replace most of one's calculative efforts will no doubt continue and likely there will be little incentive to acquire considerable expertise in mental calculation. Why should one bother learning calculative strategies and memorizing numerical equivalents when this information can be purchased for a few dollars?

This study has demonstrated that proficient mental calculation is much more than a computational tool which can be replaced easily by modern technology. Proficiency demands that the user search for meaning and understanding and any process which depends upon the integration of seemingly disconnected mathematical concepts and rules is well worth retaining as a goal of mathematics education. The benefits of mental calculation, therefore, go beyond the ability to make quick calculations. Max Beberman made this point over 25 years
Mental arithmetic... is one of the best ways of helping children become independent of techniques which are usually learned by strict memorization....Moreover, mental arithmetic encourages children to discover computational short cuts and thus to gain deeper insight into the number system. (Sr. Josephina, 1960, p. 199)

The unskilled subjects' difficulties were not due to a lack of a computational tool: rather, a lack of awareness that different tasks called for different tools appeared to be at the root of their problems. They attempted to solve calculative tasks much in the same manner as an inexpert handyman might use a wrench instead of a hammer to drive a finishing nail. Though both tools can do the job, one seems greatly unsuited to the task at hand.

HS's thoughts on the purposes of mental calculation seem particularly insightful and well worth reporting. When asked if she had ever been given a memory test, she remarked, "I'm not very good at memorizing." After the testing had been completed, this researcher commented that some people might regard her ability to recall squares and to calculate mentally as an exercise which required a great deal of memory. She exclaimed, "But that's not memorizing. That's knowing and thinking."

To close on a somewhat museful note, this study has demonstrated that proficient mental calculation involves a sophisticated form of thinking that hopefully will not be supplanted by advances in calculator technology. The destination of a mental calculation is a correct answer but the path is understanding. The good mental calculator is free to choose many paths; the poor mental calculator can choose few paths. How to open more paths to more people
is a question whose answer is left to future studies.
REFERENCES


Flournoy, M.F. Developing ability in mental arithmetic. Arithmetic Teacher, 1957, 4, 147-150.

Flournoy, M.F. Providing mental arithmetic experiences. Arithmetic...
Teacher, 1959, 6, 133-139.


Hunt, E. Intelligence as an information-processing concept.


Josephina, Sr. Mental arithmetic in today's classroom. Arithmetic Teacher, 1960, April, 199-207.


Maier, E. Folk math. Instructor, Feb., 1977, 84-89, and 92.


Merkel, S.P. and Hall, V.C. The relationship between memory for order and other cognitive tasks. Intelligence, 1982, 6, 427-441.

Miller, G.A. The magical number seven, plus or minus two: Some limits on our capacity for processing information. The Psychological Review, 1956, 63(2), 81-97.


National Assessment of Educational Progress. NAEP Newsletter, 16(2), Spring, 1983b.


Neisser, U. Memory: What are the important questions?. In M.M.


REFERENCE NOTES

Hope, J.A. A study of the mental calculation abilities of education students. Unpublished study, University of Saskatchewan, 1983.
APPENDIX A

SUBJECT CONSENT FORM
APPENDIX B

THE INSTRUMENTS
### The Items of the Screening Test Call

<table>
<thead>
<tr>
<th>Item</th>
<th>Dimensions</th>
<th>Difficulty</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>7 x 51</td>
<td>(B)*</td>
</tr>
<tr>
<td>2.</td>
<td>25 x 48</td>
<td>(A)</td>
</tr>
<tr>
<td>3.</td>
<td>16 x 72</td>
<td>(A)</td>
</tr>
<tr>
<td>4.</td>
<td>8 x 70</td>
<td>(B)</td>
</tr>
<tr>
<td>5.</td>
<td>16 x 16</td>
<td>(A)</td>
</tr>
<tr>
<td>6.</td>
<td>8 x 99</td>
<td>(B)</td>
</tr>
<tr>
<td>7.</td>
<td>32 x 64</td>
<td>(A)</td>
</tr>
<tr>
<td>8.</td>
<td>60 x 40</td>
<td>(B)</td>
</tr>
<tr>
<td>9.</td>
<td>27 x 32</td>
<td>(A)</td>
</tr>
<tr>
<td>10.</td>
<td>24 x 24</td>
<td>(A)</td>
</tr>
<tr>
<td>11.</td>
<td>12 x 500</td>
<td>(B)</td>
</tr>
<tr>
<td>12.</td>
<td>16 x 45</td>
<td>(A)</td>
</tr>
<tr>
<td>13.</td>
<td>30 x 200</td>
<td>(B)</td>
</tr>
<tr>
<td>14.</td>
<td>15 x 48</td>
<td>(A)</td>
</tr>
<tr>
<td>15.</td>
<td>7 x 511</td>
<td>(B)</td>
</tr>
<tr>
<td>16.</td>
<td>12 x 12</td>
<td>(B)</td>
</tr>
<tr>
<td>17.</td>
<td>25 x 65</td>
<td>(A)</td>
</tr>
<tr>
<td>18.</td>
<td>70 x 90</td>
<td>(B)</td>
</tr>
<tr>
<td>19.</td>
<td>15 x 64</td>
<td>(A)</td>
</tr>
<tr>
<td>20.</td>
<td>2 x 592</td>
<td>(B)</td>
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</table>

* A and B refer to the difficult and easy items, respectively.
### THE ITEMS OF THE PROBING TEST CAL2

<table>
<thead>
<tr>
<th>Item</th>
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<tbody>
<tr>
<td>1.</td>
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<td>2.</td>
<td>12 x 15</td>
<td>180</td>
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<tr>
<td>3.</td>
<td>8 x 99</td>
<td>792</td>
</tr>
<tr>
<td>4.</td>
<td>25 x 480</td>
<td>12000</td>
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<tr>
<td>5.</td>
<td>9 x 74</td>
<td>666</td>
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<tr>
<td>6.</td>
<td>4 x 625</td>
<td>2500</td>
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<tr>
<td>7.</td>
<td>12 x 81</td>
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<td>8.</td>
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<td>9.</td>
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<td>10.</td>
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<td>17.</td>
<td>23 x 27</td>
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<td>18.</td>
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<td>20.</td>
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<td>21.</td>
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<tr>
<td>22.</td>
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<td>23.</td>
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<td>29.</td>
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<td>30.</td>
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THE ITEMS OF THE CHALLENGE TEST CAL3

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<tbody>
<tr>
<td>1.</td>
<td>75 x 75</td>
<td>(5625)</td>
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<td>32 x 64</td>
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<td>18 x 72</td>
<td>(1296)</td>
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<tr>
<td>4.</td>
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<td>(15 000)</td>
<td>12.</td>
<td>87 x 23</td>
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<tr>
<td>5.</td>
<td>48 x 64</td>
<td>(3072)</td>
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<td>89 x 91</td>
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<td>6.</td>
<td>27 x 81</td>
<td>(2187)</td>
<td>14.</td>
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<td>7.</td>
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<td>(2304)</td>
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<td>8.</td>
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<td>[15 625]</td>
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THE ITEMS OF THE WRITTEN MULTIPLICATION TEST WPP

1. 64  
   x 15

2. 48  
   x 15

3. 24  
   x 24

4. 32  
   x 64

5. 16  
   x 16

6. 25  
   x 65

7. 45  
   x 16

8. 27  
   x 32

9. 48  
   x 25

10. 72 
    x 16
### The Items of the Basic Multiplication Fact Test BFR

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<td>0 × 9</td>
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<tr>
<td>7 × 8</td>
<td>8 × 7</td>
<td>5 × 3</td>
<td>5 × 0</td>
</tr>
<tr>
<td>6 × 0</td>
<td>3 × 4</td>
<td>5 × 5</td>
<td>1 × 5</td>
</tr>
<tr>
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<td>7 × 3</td>
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<tr>
<td>4 × 3</td>
<td>1 × 1</td>
<td>4 × 2</td>
<td>2 × 5</td>
</tr>
<tr>
<td>6 × 3</td>
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<td>6 × 2</td>
<td>1 × 6</td>
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<tr>
<td>0 × 8</td>
<td>5 × 4</td>
<td>7 × 9</td>
<td>6 × 5</td>
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<tr>
<td>4 × 7</td>
<td>0 × 4</td>
<td>6 × 4</td>
<td>1 × 7</td>
</tr>
<tr>
<td>2 × 8</td>
<td>1 × 3</td>
<td>0 × 3</td>
<td>0 × 1</td>
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<td>4 × 1</td>
<td>3 × 8</td>
<td>5 × 7</td>
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<td>7 × 7</td>
<td>2 × 4</td>
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<td>6 × 8</td>
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<td>7 × 4</td>
</tr>
<tr>
<td>5 × 8</td>
<td>9 × 9</td>
<td>0 × 6</td>
<td>0 × 7</td>
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<tr>
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<td>9 × 4</td>
<td>8 × 2</td>
<td>6 × 6</td>
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<td>7 × 1</td>
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<td>1 × 8</td>
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<td>3 × 0</td>
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<td>3 × 6</td>
<td>3 × 1</td>
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<td>3 × 3</td>
<td>8 × 5</td>
<td>2 × 9</td>
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<td>0 × 0</td>
<td>3 × 2</td>
<td>1 × 0</td>
<td>8 × 9</td>
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### THE ITEMS OF THE DELAYED DIGIT SPAN TEST DDS

#### LIST 1

<table>
<thead>
<tr>
<th>Letter</th>
<th>Digit Series</th>
<th>Letter</th>
<th>Digit Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. J</td>
<td>5,0,1,2,1, flea</td>
<td>1. H</td>
<td>3,0,4,7, fish</td>
</tr>
<tr>
<td>2. V</td>
<td>1,4,7,2,9,7,2,6, crab</td>
<td>2. L</td>
<td>2,9,8,2,4,0, bird</td>
</tr>
<tr>
<td>3. R</td>
<td>2,6,1,1,3,2, frog</td>
<td>3. V</td>
<td>5,8,3,9,2, crab</td>
</tr>
<tr>
<td>4. H</td>
<td>1,8,6,9,8,5,9, fish</td>
<td>4. Q</td>
<td>3,6,9,4,1,3,5, bear</td>
</tr>
<tr>
<td>5. Q</td>
<td>2,7,7,1, bear</td>
<td>5. J</td>
<td>7,5,8,7,9,2,4,1, flea</td>
</tr>
</tbody>
</table>

#### LIST 2

<table>
<thead>
<tr>
<th>Letter</th>
<th>Digit Series</th>
<th>Letter</th>
<th>Digit Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. J</td>
<td>4,8,7,3,6,7, flea</td>
<td>1. L</td>
<td>6,8,2,3,0,4,7, bird</td>
</tr>
<tr>
<td>2. H</td>
<td>3,8,4,6,5,0,4,3, fish</td>
<td>2. V</td>
<td>9,5,8,6,0,9, crab</td>
</tr>
<tr>
<td>3. R</td>
<td>4,1,7,9,3,0,8, frog</td>
<td>3. R</td>
<td>2,9,3,6, frog</td>
</tr>
<tr>
<td>4. Q</td>
<td>6,1,5,7,2, bear</td>
<td>4. Q</td>
<td>6,8,7,5,3,1,8,4, bear</td>
</tr>
<tr>
<td>5. L</td>
<td>4,5,3,4, bird</td>
<td>5. H</td>
<td>8,9,8,1,2, fish</td>
</tr>
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</table>

#### LIST 3

<table>
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<th>Digit Series</th>
<th>Letter</th>
<th>Digit Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. V</td>
<td>4,6,1,9, crab</td>
<td>1. L</td>
<td>3,7,0,8,5,3,4,2, bird</td>
</tr>
<tr>
<td>2. J</td>
<td>2,5,6,3,6,7,3, flea</td>
<td>2. H</td>
<td>9,4,3,4,2,0, fish</td>
</tr>
<tr>
<td>3. R</td>
<td>1,9,2,5,6,3,2,7, frog</td>
<td>3. R</td>
<td>5,1,9,2,7, frog</td>
</tr>
<tr>
<td>4. Q</td>
<td>3,7,9,2,5,2, bear</td>
<td>4. V</td>
<td>8,4,3,9,3,0,5, crab</td>
</tr>
<tr>
<td>5. L</td>
<td>8,5,3,5,1, bird</td>
<td>5. J</td>
<td>8,9,1,3, flea</td>
</tr>
</tbody>
</table>
### The Items of the Letter Span Test LS

#### List 1
1. C,G,C  
3. J,H,G,F  
5. L,J,D,K,J  
6. C,D,M,J,M,L  

#### List 2
2. F,C,C,J  
3. J,D,C,H,K,C,D,L,K  
5. B,F,B,L,C,D,F  
8. H,K,H  
9. L,J,M,L,D

#### List 3
4. K,L,B,B,F  
5. K,D,L,C  
6. F,D,H,H,M,L,J  
7. C,B,L  

#### List 4
1. M,L,F  
2. C,J,D,F,H,J  
3. C,L,J,C,J,C,D  
4. K,J,B,L,B  
7. D,J,H,L,K,C,D,G  
9. K,G,D,G
APPENDIX C

PORTION OF A CAL2 INTERVIEW WITH A SKILLED SUBJECT
PORTION OF A CAL2 INTERVIEW WITH A SKILLED SUBJECT

The following portion of a CAL2 interview with a skilled subject has been included to provide further clarification of the procedures used to gather information about the methods used by subjects to calculate mental products. This particular interview was selected because the subject employed a wide variety of calculative strategies. For the sake of brevity, some items have been excluded from this discussion.

R and S refer to statements made by the researcher and subject, respectively. Comments about these statements follow each discussed item. The solution time for the subject's first attempt at a calculation is presented in the parentheses.

R: Try 9 times 742.
S: 9 times 742 (repeats question to himself).....6 thousand..six hundred and....78 (34 seconds).
R: Good. How was that problem done?
S: I just multiplied it out like I would on pencil and paper.
R: 9 times 2 is 18, and then carrying a 1?
S: Yes.
R: Why do you think it took you so long to do that problem?
S: Well, I find it much easier to do these things (mental calculation) if I can see them on a piece of paper. It's always there and I can't forget it while I'm concentrating on multiplying a different pair of numbers.
R: Did you forget a calculation and have to do it over again?
S: I did it several times to recheck.
R: Any other better ways of doing this problem.
S: No.

COMMENTS: This strategy was classified as the digit-by-digit, right-to-left, pencil-and-paper mental analogue P&P0.

R: Try 12 times 15.
S: 180 (2 seconds).
R: How was that done?
S: I just happen to know that one.
R: Did you do any calculations?
S: No.
R: Why would you know this fact?
S: Well, basically I know my times tables up to 15 times 15 at least.

COMMENTS: This strategy was classified as a retrieval of a numerical equivalent. He was the only skilled subject to claim recall for this item. His claim that he "knew his times table to 15 x 15" was not completely accurate. He needed to calculate rather than recall the products 13 x 14, 15 x 13, and 15 x 14. In these cases, the calculation was very rapid.

R: How about 8 times 99?
S: 792 (4 seconds).
R: How was that one reasoned?
S: I did 8 times 100 and then subtracted 8.
R: When you subtracted 8 from 100, did you visualize the problem as you would in calculating with a pencil and paper?
S: Well. I thought 792. It was obvious.

COMMENTS: This strategy was classified as subtractive distribution.

R: 25 times 480.
S: 1200?.....No, 12 000. (18 seconds)
R: What was your method?
S: Well, I knew that 100 is 25 times 4. So I divided the 480 by 4 to get 120 and multiplied by 100.
R: Why did you say 1200 at first?
S: I lost track of the places..the number of zeroes when I was multiplying.
R: Did you divide 48 by 4 instead of 480 by 4?
S: Yes, I think so.

COMMENTS: This strategy was classified as factoring by aliquot parts. This apparent difficulty in retaining zeroes was a problem common to all subjects.

The items 9 x 74, 4 x 625, 12 x 81, 50 x 64 were presented. They were all answered correctly and rapidly.

R: How about 8 times 625?
S: 5000 (4 seconds)
R: What was your method?
S: I remembered the 4 times 625 I calculated earlier (referring to item 4 x 625 given previously) which is 2500 so I doubled 2500 because 8 is twice as much as 4.
R: You just doubled 2500?
S: Yes.

COMMENTS: This strategy was classified as factoring by halfing-and-doubling. The skilled subjects often were able to recall some previous calculation and to incorporate this value in a later solution without having to resort to a re-calculation.
The item 25 x 48 was presented and answered correctly and rapidly by using aliquot parts.

R: Try 16 times 16.
S: 256 (1 second).
R: How did you know that?
S: I know powers of 2 pretty well.
R: So, it's just a fact and you don't have to multiply?
S: Yes.
R: Why would you remember powers of 2?
S: Possibly with working with computers a lot and the binary system.

COMMENTS: This strategy was classified as a recall of a numerical equivalent.

R: Try 25 times 25.
S: 625 (1 second).
R: How did you determine that? Was it just a fact?
S: Yes.
R: Why would you remember that one?
S: I'm really not sure. Maybe because it seems to come into common usage more than a lot of other multiplications, especially squares like that.
R: Are you familiar with many squares?
S: Some squares. The obvious ones up to about 20.
R: I'll likely test you on that later.

COMMENTS: This strategy was classified as recall of a numerical equivalent. The subject demonstrated that he could recall these squares: 13², 14², 15², 16², 17², 21², 25² and 36².

The items 32 x 500 and 25 x 120 were presented and solved.

R: Try 17 times 99.
S: 1683 (9 seconds).
R: How was that done?
S: By taking 17 times 100 and then subtracting 17 from that.
R: When you were subtracting 17 from 1700, what strategy did you use?
S: I took 1700 and put it on the top and then the 17 below. And then I subtracted it out and then I made a quick check to make sure. 1683 and 17 added to make 1700.
R: Now let me make sure I understand what you did to subtract. When you visualized the 17 under the 1700, did you calculate the difference in the same manner as in pencil-and-paper calculation?
S: No. Well 100 minus 17 I know is 83. So 1700 minus 17 would be 1683.
R: And then you checked by adding a 17.
S: Yes.
R: How did you add to check?
S: Well, I can just see numbers combining like that when they
add up to 100. 83 and 17, it's quite obvious to me that the answer is 100.

R: Do you often check your answers in mental arithmetic?
S: Yes. I use another method if it's convenient to do it that way.

COMMENTS: This strategy was classified as subtractive distribution. Interestingly, he completed the subtraction by aligning the "17 under the 1700" but did not use a mental equivalent of the written algorithm. Perhaps this visual process was used to help retain the minuend and subtrahend.

The following discussion followed the presentation of 17 x 99.

R: When you were young do you remember looking at number patterns or doing mental calculation?
S: When I was quite young I used to practise writing out my own charts. I used to really like numbers. I'd write out my own multiplication tables and charts.
R: Did you examine these charts for patterns and relationships?
S: Yes.
R: Thinking back..I know it's a long time but can you remember any patterns you discovered?
S: Well, one thing I remember discovering is with squares. You can figure out the next square, if you know another square. For example, if you know 20 squared is 400, you can figure out the next square 21 squared by adding the 20 + 21 which is the next number to the square of 20.
R: I think I know what you are trying to describe but, to make sure, write out the example for me.

(the subject took a pencil and paper and wrote $21^2 = 20^2 + 20 + 21 = 400 + 20 + 21$.)

R: Did you ever try to figure out why this pattern works?
S: No, I haven't really.
R: Can you remember when you discovered this calculating pattern?
S: I think it was around grade 5 or so.
R: And you discovered this pattern simply from examining and writing out number charts?
S: Yes.

COMMENTS: His belief that an interest in number patterns was the driving force behind some of his mental calculation methods was common to the most proficient calculators. His rule for calculating squares can be explained as:

$$(x + 1)^2 = x^2 + x + (x + 1)$$

R: Did you ever try to adapt your rule to other squares?
S: Not really.
R: Can you think of a way of adapting the rule to solve 29 squared?
S: Let's see 29 squared,..you could probably work that backwards. I
guess I'd start with 900 and subtract the next number 30 and subtract 29.

R: But you've never used this rule in the past?
S: No, I just thought of it.

COMMENTS: The subject generalized this rule:
\[(x - 1)^2 = x^2 - x - (x - 1)\]

R: Any other patterns you discovered?
S: Well, let's see. I remember one where if you want to multiply, say 14 x 16, you'd think 15 squared is 225... I noticed if one number is greater than the square and the other is one less than a square then the answer is 1 less than the square. So 14 x 16 is 225 - 1 = 224.

R: Do you ever remember getting any instruction in mental arithmetic?
S: No. I just seemed to learn it on my own.

COMMENTS: The rule he explained was the difference of squares:
\[(x + 1)(x - 1) = x^2 - 1\]

As was the case with most skilled calculators, he could not recall any teacher instruction in the topic of mental calculation. They were convinced that, for all practical purposes, they were self-taught.

R: Try 12 times 16.
S: Hmm. I'll have to calculate....192 (7 seconds).
R: How was that done?
S: Well, that one I just decided to go ...well multiply by 2, four times.
R: Just as a check, can you explain your reasoning again?
S: Well 16 is 2 to the power of 4, so I did 12 times 2 is 24, times 2 again is 48, times 2 again is 96, and times 2 again is 192.
R: Why did you choose this method instead of the pencil-and-paper method you used on some of the earlier items?
S: It just seemed easier than multiplying it out the long way.
R: Do you change strategies depending on the problem?
S: Yes.
R: Do you ever find that you try one strategy, go so far into the calculation, and then abandon the strategy in favour of another?
S: Yes...if I start getting lost or the numbers get confusing and I forget them. I just kind of start another one.

COMMENTS: This strategy was classified as factoring by doubling-and-halfing. Certain properties of the factors, in this case powers of 2, seemed to cue particular calculative strategies.

R: Try 23 times 27.
S: 621 (7 seconds).
R: How was that done?
S: Well, using that trick I had here (referring to discussion of
earlier "differences of squares" solution of 14 x 16).
R: Do you mean 25 squared minus 4?
S: Yes, but I'm not sure if this works or not?
R: OK. Try another way to confirm the answer.
S: 23 x 27?
R: Yes.
S: Yes, it's 621 (3 seconds).
R: What method did you use?
S: I did 20 x 27 and 27 x 3.
R: How did you calculate 27 x 3? Did you think 7 times 3 is 21, carry 2.
S: No for that one I thought immediately of 81 because it's powers of 3.
R: How about the 2 x 27?
S: Thought immediately of 54. Then 540.
R: How did you add 540 and 81?
S: Well, I thought of dropping the 1 and then thought 62, so 621.

COMMENTS: This strategy was classified as quadratic distribution (difference of squares). His check involved additive distribution and included the retrieval of the large numerical equivalents 3 x 27 and 2 x 27. This ability to organize a calculation into a series of easily determined "blocks" was a characteristic of the skilled subjects.

He was presented the item 13 x 13 which he retrieved as 169.
R: Hmm... 1024 (4 seconds).
S: Powers of 2.
R: Would you elaborate? What do you mean by powers of 2?
S: Well, 32 is 2 to the 5th, times 2 to the 5th again, so 2 to the 10th, which I just know as a fact. It's 1024.
R: Any reason why you knew that 2 to the 10th is 1024.
S: I just know some powers of 2.
R: Is this probably related to your work with the computer?
S: Possibly.

COMMENTS: This strategy was classified as exponential factoring. He was the only skilled subject to use this strategy to solve a CAL2 item.

He was presented with the items 8 x 612 and 15 x 16 which were solved correctly and rapidly.
R: Try 12 x 250.
S: 750 (13 seconds). Whoops, that's not it. I got lost.
R: Try again.
S: O.K. 3000 (3 seconds).
R: In your first attempt, what did you mean by "getting lost?"
S: Well, I thought 250 is 1000 divided by 4. I'm not sure how I got lost but 12 divided by 4 is 3. So probably I did 3 times 250 instead of 3 times 1000.
R: What did you do in your second attempt?
S: I did 10 times 250 is 2500 and 2 times 250 is 500, so 3000.

COMMENT: His initial strategy was classified as factoring by aliquot parts. This error of using the incorrect factor 250 instead of 1000 was made by several subjects who used aliquot parts.

He was given the item 25 x 32 which he solved correctly and rapidly.

R: How about 15 times 48?
S: ..720 (7 seconds).
R: What was your method?
S: OK. I took the tens part first and thought 480. And since it was 5 times 48 it's just half of 480 which is 240. And then I added 480 and 240.

COMMENTS: This strategy was classified as fractional distribution.

He was given the item 8 x 999 which he solved correctly and rapidly.

R: Try 25 times 65.
S: 1625 (5 seconds).
R: How was that done?
S: Divided the 65 by 4, and got 16.25 and then moved the decimal place over two places.

COMMENTS: This strategy was classified as aliquot parts. He was the only subject who incorporated decimal arithmetic into a calculation.

R: Try 49 times 51.
S: 2499 (4 seconds).
R: How?
S: 50 squared minus 1.

COMMENTS: This strategy was classified as quadratic distribution (difference of squares).

He was presented with 24 x 24 which he solved correctly in 14 seconds using additive distribution. He said he slowed down in this calculation because "I thought I knew that one but I had to calculate anyway."

R: How about 8 times 4211.
S: 33 thousand, 6 hundred, 88
R: Straight multiplication. 8 times 11 is 88 and 8 times 2 is 16 carry the 1.
S: Which type of problem do you prefer: doing a problem with a 1-digit factor like this or a problem like 49 times 51?
R: 49 times 51. I have a trick for those but these other long ones like 8 times 4211 I have to remember a string of calculations as I go along.
COMMENTS: This strategy was classified as pencil-and-paper mental analogue with one non basic fact retrieval (8 x 11). His comment that he preferred problems with factors that have some discernable properties rather than those with "property-less" 1-digit by x-digit factors was echoed by several skilled subjects.

The last presented item was 15 x 15 which he recalled immediately as 225.