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AN ECONOMIC MODEL OF OIL EXTRACTION:
THEORY AND ESTIMATION

by

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ABSTRACT

Although the field of Natural Resource Economics is relatively young, its growth has been rapid and it is now of substantial size. In that part of the field devoted to exhaustible resources, however, the literature is primarily qualitative with relatively little attention having been directed towards empirical testing of the qualitative predictions. The purpose of this dissertation is to construct a dynamic model of extraction for a specific exhaustible resource, to obtain empirical estimates of the extraction technology and to perform hypothesis tests of the model's predictions. The specific resource chosen is oil and the empirical work is based on oil reservoir data from the Province of Alberta.

The building of the oil-reservoir extraction model draws on the principles of oil-reservoir engineering. Under the assumption that a rational agent manages the reservoir, the empirical implications for the number of wells to be used and the strategy for pressure maintenance activities are derived. A dual, restricted cost function forms the basis of the empirical work. Estimation of the parameters of this cost function through its implied factor demand equations permits information about the extraction cost characteristics of individual reservoirs to be obtained and hypothesis tests on the structure and characteristics of the cost function to be performed. It is found that oil pools producing in the sample year (1973) in Alberta are not homogeneous with respect to cost. Rather, the pools in the sample show a high degree of variation in geological factors that significantly affect extraction costs. The evidence strongly suggests that marginal extraction costs are a non-

increasing function of extraction rates in the range of observations. In addition, marginal extraction costs vary systematically across pools with variation in key geological factors. Since the current system of prorationing in Alberta allocates monthly demand among the producing pools in the province, the above results imply that a marginal reallocation which increases the share of demand produced by the relatively low-cost pools will lead to an efficiency gain.

The empirical results are found to support the model's predictions regarding the behaviour of the shadow price (or costate variable) for pool pressure. Finally, the results are used to test and conditionally confirm the hypothesis that oil reservoirs in Alberta have been exploited in order of declining quality over time.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

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CHAPTER 1

Introduction

Although the field of Natural Resource Economics is relatively young, its growth has been rapid and it is now of substantial size. In that part of the field devoted to exhaustible resources, however, the literature is primarily qualitative with relatively little attention having been directed towards empirical testing of the qualitative predictions.[1] This is surprising since most of the predictions are conditional upon the quantitative nature of the extraction technology as will be explained in this chapter. The purpose of this dissertation is to construct a dynamic model of extraction for a specific exhaustible resource, to obtain empirical estimates of the extraction technology and to perform hypothesis tests of the model's predictions. The specific resource chosen in this application is oil and the empirical work is based on oil reservoir data from the Province of Alberta, the major oil-producing region in Canada.

In positive or normative models of resource extraction, it is commonly assumed that the objective is to choose the depletion path which maximizes the present-value of net returns from extracting the resource. The cost of extracting the resource at any point in time is often assumed to be a non-decreasing function of the rate of extraction and possibly a non-increasing function of the stock of remaining reserves. The first assumption allows for the possibility that marginal extraction costs may

1. Some notable exceptions, Lasserre (1982), Slade (1982), Uhler (1979a) and Cairns (1981), will be discussed in this chapter. In addition there are empirical models which are not designed to perform hypothesis tests such as Bradley (1980) and Campbell and Scott (1980).

rise with faster extraction rates, holding the stock level constant. The second assumption allows for the possibility that for a given extraction rate, the cost function may shift upwards as the stock of reserves is depleted. The qualitative characteristics of the optimal path of extraction rates can then be derived. What is interesting is that many of these qualitative results depend upon the quantitative nature of the cost function.

To see this and to review some of the basic predictions of exhaustible resource economics, consider the following representation of the cost function which is discussed above:

$$C(w(t),Q(t),S(t)) \tag{1.1}$$

where w(t) is a vector of factor prices in period t, Q(t) is the extraction rate in period t, and S(t) is the remaining stock of reserves at time t. If the cost function is meant to be an aggregate representation of individual cost functions, then the decreasing-in-S assumption embodies the additional assumption that it is optimal to utilize individual deposits in increasing cost order.[2] If one assumes that (1.1) is the cost function for an individual deposit, then the decreasing-in-S assumption is meant to capture the increasing difficulty with which additional units are extracted as the deposit is depleted.

The importance of the quantitative nature of the extraction cost function becomes apparent by examining the characteristics and predictions of the solution to the typical optimal depletion problem.[3] Suppose for

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2. This is a basic result in Resource Economics. See, for example, Hartwick (1978), Solow and Wan (1976), Solow (1974), Ulph (1978) and if Kemp and Long (1980) then also Lewis (1982). However, the conditions under which an aggregate representation exists are very restrictive: Blackorby and Schworm (1982).

3. These characteristics and predictions are well known. There are many references but three that are particularly useful are Dasqupta and Heal (1979), Levhari and Liviatin (1977) and Peterson and Fisher (1977).

example there is perfect foresight, complete information, constant prices, an initial stock of reserves of size S_0 and a discount rate, $\delta > 0$. Under these conditions, the output supply function or extraction rate may be increasing or decreasing over time, depending on the quantitative nature of the cost function:

$$dQ/dt = [C_{QS} - \delta(P - C_Q) - C_s]/C_{QQ} \quad (1.2)$$

where P is the output price, time arguments have been suppressed and a subscripted C denotes the partial derivative of the cost function with respect to the variable in the subscript. If depletion does not affect extraction costs then $C_S = C_{QS} = 0$. Consequently, $dQ/dt < 0$ since $(P - C_Q) > 0$. This is the case of the "tilting" of the extraction profile towards the present first made explicit in the theory of the extractive firm by Scott (1967). If, however, $C_S < 0$ and, as is reasonable to assume, $C_{QS} < 0$ (so that depletion causes total and marginal costs to rise), then the sign of dQ/dt is ambiguous. If the absolute value of C_S is small (relative to $\delta(P - C_Q)$) then $dQ/dt < 0$. Conversely, if the absolute value of C_S is large, then $dQ/dt > 0$. The reasoning for this result is this: The preference for present over future profits, as reflected in the positive discount rate, tends to tilt the extraction profile toward the present. The depletion effect, however, tends to tilt the extraction profile toward the future so that the higher extraction costs can be postponed, thereby reducing the present value of costs. The net effect of the two opposing forces depends upon the size of

C_S relative to the size of $\delta(P - C_Q)$.[4]

The total volume of reserves that is economically recoverable also depends critically upon the quantitative nature of the extraction cost function. If C_S is equal to (close to) zero, then all of the reserves will be (may be) extracted. If $C_S < 0$, it will normally be optimal to leave some of the resource in the ground, the amount depending on the size of C_S. Moreover, the extent to which price and tax changes affect the total volume of recoverable reserves depends upon the quantitative nature of the cost function.[5]

The above results hold whether (1.1) is viewed as an aggregate cost function or as the cost function for an individual deposit. There has been a limited amount of empirical estimation of extraction technologies which are conditional upon the state of depletion as in (1.1). At an aggregate industry level, Uhler (1979a) found the state of depletion of explorable land to have a significant impact on the outcome of current exploration activity. Zimmerman (1977) assumed the state of depletion of coal could be represented by declining quality of characteristics such as seam thickness. At the level of an individual deposit, Slade (1982) and Lasserre (1982) have provided evidence which contributes substantially to the understanding of the supply behaviour of extractive firms. In Slade (1982), estimation of a cost function for an individual copper deposit yielded the result that $C_S < 0$ and $C_{QS} < 0$. The estimated cost function was then used to simulate optimal supply paths under a variety of price and tax regimes. In Lasserre (1982), factor demand and output supply functions were estimated using data on a number of individual hard

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4. See Levhari and Liviatin (1977, 187) for an example in which $dQ/dt > 0$ over the entire life of the deposit.
5. See Conrad and Hool (1981), Heaps (1982), Levhari and Liviatin (1977), and Slade (1982).

rock mineral deposits. Again, the stock of reserves of the deposits were found to have a significant influence on the behaviour of the extractive firm.

Understanding the supply behaviour of the extractive firm is important because it is predicted to be different from the supply behaviour of the standard static firm. Testing whether a set of data supports this prediction, however, is more difficult and has not previously been done. The extractive firm is predicted to make current supply decisions by weighing the current costs and benefits of extraction against the opportunity cost of reducing the finite stock size. Stated in terms of a first-order condition, the competitive extractive firm equates marginal cost plus the shadow price of the stock to market price in each period. Thus, the shadow price is the only variable that would make the behaviour of the extractive firm appear to differ from the behaviour of the standard static firm. If the predictions are correct, the shadow price is positive, but its size depends inversely on the size of the stock of reserves, and its time path depends on the discount rate and the quantitative nature of the cost function. In an interesting paper, Cairns (1981) calculated a shadow price for Canadian nickel reserves and found it to be very small relative to market price. The implication of this is that firms would not make serious optimization errors if they ignored the shadow price and just behaved as static profit maximizers. While an ingenious approach, it does not permit testing of the theory of the extractive firm.

In the extraction of oil, a key stock variable is reservoir pressure. The associated shadow price of pressure behaves like the reserve shadow price, is an inverse function of the stock of pressure, and is equal to zero if the firm behaves as a static profit maximizer. In this dissertation, a variety of hypothesis tests are performed on the shadow price of pressure, thereby permitting one to test the predictions of the theory of the extractive firm as applied to the case of reservoir oil.

The predicted characteristics of a competitive industry equilibrium are conditional upon the nature of the extraction cost functions of the individual deposits that make up the industry and in particular on whether and in what way they are heterogeneous. While few would argue that the assumption of cost homogeneity is realistic, it is another matter to determine what factors contribute to and the extent to which they contribute to, cost heterogeneity in a cross-section of deposits. For example, a mineral deposit may be of very low grade and yet be a low-cost deposit owing to its size and proximity to the surface.[6] In this dissertation, an extraction cost function is estimated which can be used to calculate the extent of cost heterogeneity in a cross-section of oil pools and the extent to which variance in key geological factors contribute to this heterogeneity.

The nature of inter-deposit cost heterogeneity influences the character of the competitive equilibrium over time. Suppose, for example, there are R deposits, each with a constant but different unit extraction cost and there is complete information by all firms. Under these conditions, the deposits will be sequentially exploited in increasing cost order. The testable predictions are that one should observe a rise in the unit costs of deposits brought into production over time and one should not observe any variation in unit costs in a cross-section of deposits at any point in time.

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6. See Bradley (1980) for example.

If, on the other hand, individual deposits have rising marginal cost functions then industry equilibrium at any point in time is characterized by the following condition:

$$C_Q^i + \lambda^i = C_Q^j + \lambda^j \quad ; \; i,j = 1,2,\ldots,R \tag{1.3}$$

where λ^i is the shadow price of the i^{th} deposit. Thus, (1.3) implies there may be variation in marginal extraction costs in a cross-section of deposits at any point in time. Over time, the predictions are less precise than when unit costs are constant, but one would still expect to observe a trend towards the use of higher cost deposits. These hypotheses of equilibrium behaviour are tested in this dissertation for the Alberta oil industry.

The dissertation is organized as follows. In Chapter 2, a necessary overview of the principles of oil reservoir engineering is provided and the extraction function that forms the basis of the oil extraction model is developed.

In Chapter 3, the dynamic model of extraction of oil from an underground reservoir is formulated and analyzed. A one-period, restricted or variable extraction cost function, dual to the one-period technology set, is derived. This function forms the basis for the empirical work.

The empirical model is specified in Chapter 4. This involves the specifying of a functional form for the extraction cost function and the derivation of the estimation equations. The econometric problems associated with these equations are discussed and solved and the data are discussed.

The empirical results are presented and analyzed and hypothesis tests are performed in Chapter 5.

The dissertation is concluded in Chapter 6 by drawing on the results of Chapter 5 to test the hypothesis that deposits of higher cost have been brought into production over time in the Alberta oil industry. In addition, the chapter contains concluding comments.

Appendix A contains the technical derivations to the econometric problems in Chapter 4. Appendix B contains a listing of data sources and a listing of the names of the oil pools used in the sample.

CHAPTER 2

Principles of Oil Production[1]

2.0 Introduction

The purpose of this chapter is to provide an elementary overview of oil reservoir engineering principles and to derive the implicit representation of an oil reservoir production function that will form the basis of the oil extraction model.

The flow of fluids and gases through a porous medium such as an oil reservoir is a highly complex process dependent on many physical properties of the rock such as its porosity (the ratio of the volume of pore space to the total volume of the oil bearing rock), pore geometry, the distribution and size of the channels connecting the pores, wettability (the degree to which water or oil adheres to the surface of the rock across which it flows), viscosity, and temperature. The combined effect of all of these factors is captured in a single characteristic called permeability. Measured in a unit called the darcy, permeability is a measure of the ease with which a fluid or gas can flow through some medium.

Reservoir pressure is the driving force of oil production. When a pressure differential is created in the reservoir by the sinking of a well, hydrocarbons are forced towards the point of relatively low pressure until the pressure differential is eliminated. The velocity of the flow in response to a given pressure differential is determined by the

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1. The primary sources for this chapter are Dake(1978) and Skinner(1981).

permeability of the reservoir. In most cases, the pressure differential at the well bore is maintained by causing the reservoir's hydrocarbons to flow up the well to the surface either by gas lift or by a pumping system. (Reservoir pressure is normally sufficient to cause fluids to flow into the well but only occasionally all the way to the surface). In almost all cases, the natural reservoir pressure is depleted as the contents of the reservoir are extracted, eventually to a point where it is insufficient to overcome the natural resistance to flow provided by the host rock. The exception to this rule is the pure water-drive mechanism in which there is a virtually endless supply of water from a connected aquifer to fill the spaces vacated by the hydrocarbons, thereby preventing pressure decline. In practice, however, most reservoirs have a combination of drive mechanisms: solution gas expansion, gas cap expansion, and a water drive mechanism. In these cases, reservoir pressure declines with extraction. It is possible, however, to inhibit pressure decline with artificial pressure maintenance schemes, sometimes referred to as secondary recovery techniques.

There are three basic techniques for maintaining reservoir pressure. The first is simply to reduce the rate of fluid extraction from the reservoir. The second is to inject fluids (normally water) into the reservoir rock some distance from a producing well. The injected fluids replace the extracted fluids thereby inhibiting pressure decline. An interface between the injected fluids and the reservoir hydrocarbons is formed. The success of this interface, called a floodfront, in displacing or pushing the hydrocarbons towards the producing wells depends largely on the relative permeability of the fluids. The relative permeability of water to oil, for example, is the ratio of the absolute water permeability to the absolute oil permeability. If the reservoir is highly permeable to water relative to oil, water flows more easily than oil through the reservoir. In this case, the floodfront would have little success in displacing oil since the injected water would simply bypass the majority of the oil. Conversely, if the reservoir has a low water-oil relative permeability, the floodfront would displace a much larger fraction of the reservoir oil.

A third technique involves injecting gas or natural gas liquids (captured in production) back into the reservoir to create or enhance a gas cap. Sufficient gas injection raises the reservoir pressure and forces the fluids towards the producing wells.

Many of the hydrocarbons can appear in gaseous or liquid form in the reservoir depending on the temperature and pressure. Consider a cylinder containing 100% ethane (C_2H_6). For a given temperature, as pressure falls from high to low, the ethane changes from a liquid to a gas. The point at which this transition occurs is called the vapour pressure point. This is shown as a function of pressure and temperature in Figure 1a.

The same relationship is shown in Figure 1b for a cylinder containing 100% heptane (C_7H_{16}). The diagram indicates that under the same temperature and pressure conditions where ethane is a gas, heptane is a liquid. Thus, when both are present in the cylinder, there are regions in which both are gases, both are liquids, and one is a gas and the other is a liquid. The latter is the two-phase region. As shown in Figure 1c, the bubble-point line separates the liquid phase region from the two-phase region. It is so named because as pressure falls from high to low, the first bubbles of gas appear at this line. The dew line separates the gas from the two-phase region. This is where the first drops of liquid or dew

FIGURE 1a

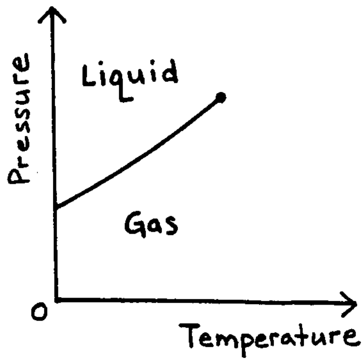


FIGURE 1b

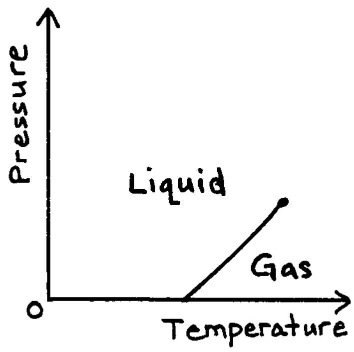


FIGURE 1c

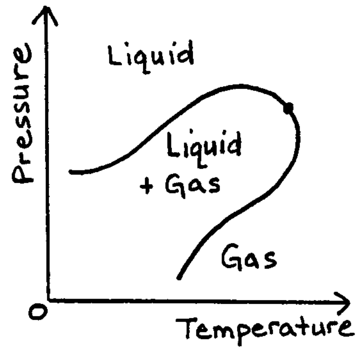
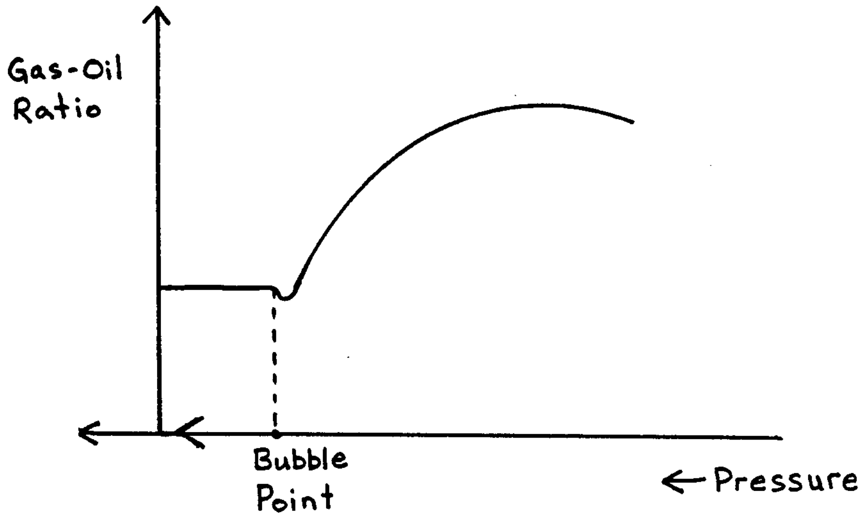


FIGURE 2



appear as pressure rises from low to high.

The bubble point has significance because of the dramatically different characteristics of reservoir production above and below it. Above the bubble point, each reservoir barrel of oil contains a certain volume of dissolved gas, R_s. As this barrel is produced and brought to the surface, the lower atmospheric pressure will cause the gas to separate yielding a gas-oil ratio equal to R_s. However, if reservoir pressure falls below the bubble-point, some of the dissolved gas is released and becomes free gas in the reservoir. Gas has a far lower viscosity than oil and therefore a far higher velocity. Thus, when gas is freed, it travels to the producing wells at a far higher speed than oil. As a result, the producing gas-oil ratio, GOR, will rise dramatically as shown in Figure 2.[2]

The higher producing GOR implies that reservoir pressure declines far more rapidly per barrel of oil produced than it does if the reservoir is kept at or above the bubble-point.

The bubble-point has important implications for pressure maintenance techniques such as water injection. As pressure falls, the amount of water that must be injected to replace produced oil so as to maintain constant pressure, say, takes a sudden jump upwards at the bubble-point since the volume that was occupied by the large amount of produced gas must also be replaced.

2. The temporary dip in GOR at the bubble-point is due to a lag caused by the gas having to overcome friction before flowing freely.

2.1 The Reservoir Production Function

The above discussion makes it clear that it is pressure differentials that cause oil to be produced, production that promotes pressure decline and injection that inhibits pressure decline. A natural way to model production then is in terms of the pressure differential at the well bore, a variable controlled by the rate of pumping at the surface. However, the data required to make this approach operational are not available. An alternative approach must therefore be adopted. In this section, such an alternative to modelling production, based on the material balance equation of reservoir engineering, is derived.

The material balance equation is based on the law that fluids and gases occupy less space when under greater pressure. Removal of fluids from a reservoir causes a finite drop in reservoir pressure. If one were able to take the produced fluids back down to the reservoir at the lower pressure, they would occupy a larger volume of space. The difference in the space occupied by the fluids at the lower pressure is identically equal to the production, expressed in terms of reservoir pressure. This identity will be used to form the basis of the implicit functional relationship determining the production of oil.

In order to understand the material balance equation, some definitions are required:

R = initial oil in place in stock tank barrels(stb) ,ie evaluated at standard surface conditions

v = initial hydrocarbon volume of the gas cap divided by the initial hydrocarbon volume of the oil

q = oil production in stb (cumulative over a finite period of time)

R_p = cumulative gas-oil ratio in standard cubic feet (scf) per stb

R_s = the gas content of a barrel of oil in scf per stb

B_g = the volume in reservoir barrels that one scf of gas will occupy as free gas in the reservoir.

B_0 = the number of barrels occupied by 1 stb of oil at reservoir pressure in reservoir barrels (rb) per stb. Note: $B_0 \geq 1$ due to the fact that at higher pressure, more gas dissolves in the oil thereby increasing its volume.

Assume that reservoir pressure falls by an amount $dP=P_0 - P_1 > 0$. The resulting change in the reservoir volume occupied by the reservoir fluids is equal to the sum of the following three sources of expansion:

First, the oil plus the originally dissolved gas expands when pressure drops. Call this amount of expansion A. A is a function of the initial oil reserves, R, the size of the pressure drop, dP, and a number of reservoir specific parameters which are not available in the data set but which are all functions of reservoir pressure. Thus A is implicitly written as a function of R, P_0, and dP.

Second, the gas cap, if there is one, expands by an amount B, which is a function of v,R,P_0, and dP.

Third, the hydrocarbon pore space shrinks because the reservoir water expands by an amount C, which is also a function of v,R,P_0, and dP.

Surface production is observed to be q stb of oil plus qR_p scf of gas. If these two volumes are taken down to the lower reservoir pressure, P_1, some of the gas will dissolve in the q stb of oil yielding a volume occupied of qB_0. All that is known about the total gas that was produced is that qR_s will dissolve in the oil and the remainder, $q(R_p - R_s)$

will be free gas, occupying a volume of $q(R_p - R_s)B_g$. Thus, the total oil and gas production, evaluated at reservoir pressure, P_1, is

$$q[B_0+(R_p - R_s)B_g]$$

and this must be equal to the sum of the three sources of expansion caused by the pressure drop. This equality is the material balance equation:

$$q[B_o+(R_p-R_s)B_g] = A(R,dP,P_o) + B(v,R,dP,P_o) + C(v,R,dP,P_o)$$

Therefore,

$$q = D(v,R,dP,P_o)/\{B_o+(R_p-R_s)B_g\} \qquad (2.1)$$

If a gas cap is not present, v=0 and $R_p = R_s$, so that the above expression simplifies. These parameters as well as B_o and B_g are not available in the data set but are also functions of reservoir pressure. Thus it must be assumed that the following implicit relationship is representative of the material balance equation:

$$q = G(R,P_o,dP) \qquad (2.2)$$

It is assumed that (2.2) represents the relationship for an individual well since it is only through a well that production can be realized. Thus, the arguments of (2.2) are well-specific. Although the oil reserves per well that are capable of influencing a well's production are not observable, reserves-per-well is a function of variables which are observable,namely the pay thickness of the reservoir into which the well is drilled and the water saturation level of the reservoir. Pay thickness is simply the thickness of the oil-bearing portion of the porous rock. It varies from as little as 1 metre to as much as 100 metres in the sample of oil pools to be used in the empirical work. It is without doubt the most important determinant of reserves-per-well. Water saturation is also an important variable - a reservoir that is highly saturated with water therefore has a low oil saturation.

Finally, it is assumed that reserves-per-well is a function of the number of wells in the reservoir that are competing for the migratory reservoir fluids. Thus, the production relation for an individual well is given by:

$$q = h(N, W, P_0, dP, Z)$$

where W = water saturation level

Z = pay thickness

N = number of wells in the reservoir

Assuming these arguments are constant throughout the reservoir,[3] the total output of oil from the reservoir as a whole is simply the output per well multiplied by the number of wells in the reservoir:

$$Q = q \cdot N = f(N, W, P_0, dP, Z) \qquad (2.3)$$

This function implicitly represents the production relationship for the reservoir oil. The controllable variable factors of productions are N and

3. Data availability make this assumption necessary.

dP. The production function is assumed to be increasing at a non-increasing rate in these two variables. Because q must be increasing in R, then Q must be decreasing in W and increasing in Z. Finally, the production relation may have the property of decreasingness in P ,due to the fact that under greater pressure, a cubic metre of reservoir oil will contain less surface oil (evaluated at surface conditions) because of its higher dissolved gas content.

The production relation for a hypothetical oil reservoir in (2.3) forms the basis for the dynamic model of oil extraction that is developed and analyzed in the next chapter.

CHAPTER 3

An Oil-Reservoir Extraction Model

3.0 Introduction

Building upon the aspects of basic oil reservoir engineering outlined in Chapter 2, the purpose of this chapter is to construct an empirically testable model of oil extraction. In the first part of the chapter, Section 3.1, the literature which deals specifically with the problem of oil-reservoir extraction modelling is reviewed. In Section 3.2, an alternative model is constructed and its behavioural implications are examined. In Section 3.3, a restricted or variable one-period cost function is constructed which is consistent with the dynamic and technological structure of the model of Section 3.2. The term 'restricted' implies that the cost function embodies cost-minimizing factor use subject to some restrictions on the choice set. The form such restrictions commonly take is that at least one factor of production remain fixed. In this chapter, there are two types of restrictions placed on the choice set. The first is of the fixed-factor type although a somewhat broader view of what constitutes a factor of production is taken. The second restriction is of a different type, involving a restriction on the path of a state variable. The reason for imposing this restriction will be made clear. It is through this restricted or variable one-period cost function that the oil-extraction model will be empirically tested. In order to examine the dependence of the implied optimal depletion behaviour on the parameters of the restricted cost function, the dynamic optimization problem is reformulated in terms of the cost function and analyzed in Section 3.4.

3.1 Review of Oil Extraction Models

There are two characteristics which distinguish the problem of modelling oil extraction from that of other exhaustible resources: the common property nature of oil and the pressure dynamics of oil reservoirs. The common property problem is similar to the problem first analyzed by Gordon (1954) in the context of the fishery. It has been analyzed recently in the context of oil industry behaviour by Eswaran and Lewis (1982). The problem of pressure dynamics becomes important when one is modelling the extraction behaviour from individual reservoirs as is done in this thesis. In only two studies, Uhler (1979) and Kuller and Cummings (1974), has this important characteristic been incorporated into oil extraction models. In these cases, the common property problem is avoided either by making the model normative so as to derive the conditions for optimal extraction as in Kuller and Cummings (1974) or if the model is positive, as in Uhler (1979) and this thesis, by assuming the reservoir is under unitized management.[1] Before proceeding with the development of the extraction model based on the results of Chapter 2, a brief review of these two papers is presented.

In Kuller and Cummings (1974), care is taken to incorporate technological aspects of oil reservoir engineering into the model. They consider an individual reservoir from which n firms are extracting oil

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1. This is not an unrealistic representation of modern oil reservoir management practices.

and their problem is to determine the optimal management policy of the reservoir as a whole; that is, a policy which maximizes the present value of joint profits. Their primary concern is to incorporate the following three features into the model: (i) the extraction rate from a well depends on the number of wells and the extraction rates of other wells, (ii) investment in pressure maintenance, by augmenting pressure, can increase extraction rates, and (iii) total recoverable reserves depend on the time path of extraction. The contribution of the paper is the finding that in determining optimal extraction rates, more than the traditional user cost [Scott (1967)] should be taken into account. Rather, there is a user cost associated with each of the above external effects of a firm's extraction rate. In addition, the optimal investment stategy for firms should also take account of its external effects on reservoir pressure and production capabilities of other firms. The model is too general and too arbitrary in the sense that all functional relationships are implicit to permit the derivation of any behavioural predictions or to permit empirical application. The main value of the paper is in drawing attention to the many positive and negative externalities that exist in oil extraction.

Uhler (1979), on the other hand, formulates a model which makes explicit use of pressure dynamics equations. While the stock of recoverable reserves is not written as an arbitrary function of the path of extraction rates, as in Kuller and Cummings (1974), it is shown that the extraction path, through its effect on pressure decline, can affect the stock of reserves that are ultimately recovered. It is also shown that this undesirable effect can be reduced if not totally eliminated when pressure maintenance through water injection is a technical possibility.

While the model to be developed in the next section is more in the spirit of the Uhler model than the Kuller and Cummings model, it differs from both on a fundamental level: the specification of the production relation. In both of the models under review, it is assumed that there exists a physical upper limit on the extraction rate of the following type:

$$Q \leqq h(P,X)$$

where P is the level of reservoir pressure and X is a vector of capital inputs such as the number of wells. Ignoring the question of whether or not such a limit exists, an empirical problem arises in trying to estimate the parameters of extraction technology implied by this specification. As Uhler shows, if marginal extraction costs are rising or if extraction rates affect ultimate reserves directly as for Kuller and Cummings, Q will seldom be allowed, if ever, to reach this upper limit. It is therefore impossible to estimate the extraction technology implied by this specification. This is not the case for the production relation derived in Chapter 2. Rather, it is amenable to empirical estimation and is firmly based on the principles of reservoir engineering.

The assumption of an upper extraction limit which is a function of reservoir pressure, however, is a clever analytic device which captures the crucial effects of presure dynamics. The implication of the assumption is that water injection will never occur unless the constraint is to become binding at some point in time. Even then, as Uhler shows, water injection will not necessarily occur. For example, in the special case of linear extraction costs, Uhler finds that the constraint will become immediately binding which has the effect of causing the shadow price of pressure to rise continually as the oil is extracted at the maximum, but declining, rate possible. Only if the shadow price becomes sufficiently high at some date will water injection commence so as to inhibit pressure and, hence, production decline. As will be shown, similar results are obtained in the model to be presented. One difference is that the shadow price of pressure is predicted to rise during an initial phase of extraction and to fall thereafter until the terminal date. The implied path of reservoir pressure is first rising or falling, depending on the initial stock of pressure and is ultimately falling. These two phases may be separated by a third phase in which water injection is used to maintain a constant stock of pressure.

3.2 The Extraction Model

Two approaches to modelling oil extraction were surveyed in the previous section. In this section, the reservoir-specific production relation developed in Chapter 2 is used to form the basis of an alternative extraction model.

In Chapter 2, it was argued that the production relation for the r^{th} reservoir could be represented in the following way.

$$Q_r(t) = f[-\dot{P}_r(t), N_r(t), P_r(t), Z_r] \tag{3.1}$$

where $Q_r(t)$ = reservoir output in period t

$-\dot{P}_r(t)$ = pressure change in period t

$P_r(t)$ = pressure level in period t

$N_r(t)$ = number of oil wells in the reservoir in period t

Z_r = vector of natural factors of production specific to reservoir r (water saturation and pay thickness)

Letting f_i be the partial derivative with respect to the i^{th} argument, the production relation has the following properties:

$f_1 > 0 \quad f_{11} \lesseqqgtr 0$

$f_2 > 0 \quad f_{22} \lesseqqgtr 0$

$f_3 < 0$

$f_4 > 0 \quad f_{44} \lesseqqgtr 0$

The partial derivatives with respect to the 4^{th} argument hold only if water saturation is indexed negatively and pay thickness is indexed positively.

Reservoir pressure is viewed as a stock of capital which can be maintained through water injection or utilized through production. If one permits water injection to augment pressure, it is not the net pressure change during t, $-\dot{P}(t)$, which enters the production relation but the gross pressure change before injection that must enter. Define the gross pressure change as

$$u_r(t) = g[m_r(t), P_r(t), Z_r] - \dot{P}_r(t) \qquad (3.2)$$

where the g function measures the extent to which water injection in the r^{th} reservoir in year t, $m_r(t)$, augments the r^{th} stock of pressure given $P_r(t)$ and Z_r.

Thus, the production relation for the r^{th} reservoir becomes

$$Q_r(t) = f[u_r(t), N_r(t), P_r(t), Z_r] \tag{3.3}$$

The convention of referring to $u_r(t)$ as the utilization rate of the stock of pressure in the r^{th} reservoir is adopted. In addition the r subscripts and time arguments will be suppressed except where it creates ambiguity. To derive behavioural implications, the model is given more structure with the following simplifying assumptions. An individual reservoir of known depth and physical characteristics is assumed to contain a known quantity of recoverable oil. Common property problems are eliminated by assuming that the reservoir is under unitized management.[2] It is assumed that the objective of the reservoir manager is to maximize the present-value of profits, taking the well-head price of oil as constant. Finally, it is initially assumed that the two man-made factors of production, N and m, are completely variable and their constant prices are w_1 and w_2, respectively.

Thus, the optimization problem facing the manager of the r^{th} reservoir is the following:

$$\underset{\langle u,N,m,T\rangle}{\text{Maximize}} \int_0^T e^{-\delta t}\{w_0 \cdot f(u,N,P,z) - w_1 N - w_2 m\}dt$$

$$\begin{aligned} \text{subject to} \quad & \dot{P} = g(m,P,z) - u \\ & \dot{S} = -f(u,N,P,z) \\ & P(0) = P_0 > 0 \\ & S(0) = S_0 > 0 \\ & P,S,u,N,m \geq 0 \end{aligned}$$

---

2. Given that quite recent Alberta data is to be used to test the model, this is not an unreasonable assumption.

where P_0 and S_0 are the initial stocks of pressure and oil reserves, respectively.

It is instructive to analyze, to the extent possible, the solution to the optimal depletion problem. While the general model and not even special cases of it can be explicitly solved, a good deal can be learned about the nature of the solutions by making use of optimal control theory to generate the equations of motion of the system and phase diagrams to analyze this motion.

Letting λ_1 and λ_2 be the costate variables associated with the state variables, P and S respectively, the present-valued Hamiltonian function for this problem is given by:

$$H = e^{-\delta t}\{w_0 \cdot f(u,N,P,z) - w_1 N - w_2 m + \lambda_1[g(m,P,z) - u] - \lambda_2 \cdot f(u,N,P,z)\}$$

Assuming the existence of an interior solution, the following conditions must hold at every point in time in order to maximize the H function at every point in time:

$$\partial H/\partial u = 0 \rightarrow (w_0 - \lambda_2)f_u - \lambda_1 = 0 \tag{3.4}$$

$$\partial H/\partial N = 0 \rightarrow (w_0 - \lambda_2)f_N - w_1 = 0 \tag{3.5}$$

$$\partial H/\partial m = 0 \rightarrow -w_2 + \lambda_1 g_m = 0 \tag{3.6}$$

$$\frac{d}{dt}\lambda_1 e^{-\delta t} = -[(w_0 - \lambda_2)f_p + \lambda_1 g_p]e^{-\delta t} \tag{3.7}$$

$$\frac{d}{dt}\lambda_2 e^{-\delta t} = 0 \tag{3.8}$$

For given terminal conditions (to be determined using transversality conditions) these five equations determine the optimal time paths of all variables. Interpretation of these first-order conditions is facilitated by obtaining an expression for λ_1. Using (3.5) and (3.6), (3.7) becomes:

$$d\lambda_1 e^{-\delta t}/dt = - [(f_p w_1/f_N) + (w_2/g_m)g_p)]e^{-\delta t} \quad (3.7')$$

Integrating both sides of (3.7') over time from t to T, an approach developed by Levhari and Liviatin (1977), yields,

$$\int_t^T \lambda_1 e^{-\delta\tau} d\tau = - \int_t^T e^{-\delta\tau}[(f_p w_1/f_N) + (w_2/g_m)g_p]d\tau$$

so that

$$\lambda_1(T)e^{-\delta T} - \lambda_1(t)e^{-\delta t} = - \int_t^T e^{-\delta\tau}[(f_p w_1/f_N) + (w_2/g_m)g_p]d\tau$$

At this point it is useful to impose a subset of the transversality conditions, namely:

$$\lambda_1(T)P(T) = 0$$

These conditions imply that if it is optimal to leave a positive stock of the pressure, then it must yield zero value to the optimal program at the terminal time. Assuming that a positive stock is left unexploited, then,

$$\lambda_1(t) = \int_t^T e^{-\delta(\tau-t)}[(f_p w_1/f_N) + (w_2/g_m)g_p]d\tau \qquad (3.9)$$

In equation (3.9), the first term in brackets can be interpreted as the change in oil production due to a marginal change in the stock of pressure (f_P) multiplied by the opportunity cost of producing one more unit of oil (w_1/f_N) and hence the shadow value of a unit of oil. Thus, the first term in brackets is the marginal value product of pressure.

The change in pressure may also affect the efficiency of injection. The second term in brackets in (3.9) expresses the marginal valuation of this effect. More precisely, it is the marginal pressure product (g_P) multiplied by the opportunity cost of altering the stock of pressure (w_2/g_m) and hence the shadow valuation of that change.

The sum of the two terms in brackets in (3.9) then is the instantaneous total value of a marginal change in the stock of pressure. If the change occurs at time t, its effects are felt in all time periods following t until the termination date, T. Expression (3.9) shows that $\lambda_1(t)$ is the present-valued sum of the instantaneous values of all these future effects caused by a change in the stock of pressure at time t. Thus, $\lambda_1(t)$ can be interpreted as the marginal value or the shadow price of reservoir pressure at time t. Note from (3.7) that this shadow price does not follow any simple time path. While it may be increasing or decreasing at any particular point in time, it must terminate at zero. Its time path will be examined in more detail for some special cases of the general model below.

The other costate variable, λ_2, is the shadow price of oil reserves or marginal user cost. In this model, it follows a very simple time path: it must grow exponentially at the rate of discount. Its positive terminal value is determined by the condition that $S(T)=0$.

The first-order conditions can be understood more fully by taking the ratio of (3.4) and (3.5) to obtain:

$$f_u/f_N = \lambda_1/w_1 \qquad (3.10)$$

Since λ_1 is the shadow price of pressure, it is the endogenous factor price for the factor of production, u. Thus (3.10) merely states that there must be equality between the marginal rate of substitution (between u and N) and relative factor prices. In most models of optimal depletion, factor price ratios are constant over time unless one allows for exogenous time trends. However, in this model, there is an endogenous time trend in the factor price ratio. While λ_1 is falling, the utilization of pressure relative to the number of wells in the reservoir must rise.

The condition determining the optimal level of pressure maintenance, (3.6), states that the marginal cost of augmenting pressure, w_2/g_m, must equal the marginal value of pressure, λ_1. Thus, the shadow price of pressure is instrumental in determining the optimal level of water injection into the reservoir. This point is emphasized here because it will be repeatedly referred to in the next section and in subsequent chapters.

The remaining transversality condition, to determine the optimal length of the depletion program, is that depletion should continue as long as $H(t) > 0$,but should stop when $H(t) = 0$. Denoting this time by T then gives:

$$H(T) = 0 \rightarrow (w_0-\lambda_2)f(u,N,P,z) - w_1N - w_2m = 0 \qquad (3.11)$$

where all variables are evaluated at $t=T$.

There is reason to believe that $m(T)=0$. Indeed, one would expect that it is usually the case that pressure maintenance is terminated sometime before the end of the depletion program, after which pressure is merely depleted. This hypothesis is substantiated later in a special case of this general model. It is possible to show now that $m(T)>0$ is inconsistent with the terminal condition $\lambda_1(T)=0$ but that $m(t)>0$ for $t<T$ is possible. At time $t=T$, it is true that,

$$-w_2/g_m \leq \lambda_1 \quad ; \quad m \geq 0$$

where the inequalities hold with complementary slackness. When $\lambda_1=0$, it must be true that $-w_2/g_m<0$ as long as $g_m>0$. Thus, $m=0$. Since $\lambda_1(T)=0$, then $m(T)=0$. If $\lambda_1(t)>0$ for $t<T$, $m(t)>0$ for $t<T$ if g_m approaches infinity as m approaches zero. Otherwise, it is possible for $m(t)=0$ for some finite period of time at the end of the depletion program as $\lambda_1(t)$ approaches zero.

Substituting $m(T)=0$ into (3.11) and combining it with (3.5) evaluated at $t=T$ yields the following condition:

$$f_N = f/N \tag{3.12}$$

This condition states that the marginal contribution of a well to reservoir output must equal average reservoir output or output per well at the terminal moment. This condition can be thought of as determining the optimal number of wells with which to extract the final barrel of oil.

Because the shadow price of pressure is zero at T, there is no incentive to conserve on its use. Hence, it will be used up to the point of zero marginal returns, $f_u = 0$, at the terminal moment. This does not imply, however, that the stock of pressure is exhausted since as P becomes small, so does f(u,N,P,z) thereby making H(t) approach zero. Thus, it will seldom pay to utilize all remaining pressure.

A common problem with non-linear optimal control models is that it is impossible to characterize their solutions with much precision. The problem presented here is no exception. Linear models, however, are amenable to more precise characterizations and often provide insight into the solution of the general, non-linear model. For this reason, two special (linear) cases of the above general model are presented below. In both cases, it is assumed that the models are linear in the control variables. Specifically, it is assumed that

$$g(m,P,z) = \xi m \quad , 0 \leq m \leq \overline{m} \qquad , \xi > 0$$

$$f(u,N,P,z) = uQ(N,P) \quad , 0 \leq u \leq \overline{u}$$

In the first case examined, it is additionally assumed that $Q_P=0$ while in the second case $Q_P > 0$. Finally, it is assumed that N is fixed and exogenous to the problem at hand. One can think of N as being fixed capital which is optimally chosen at t=0. All subsequent decisions are then conditional on the fixed value of N.

Case 1

These modifications make the Hamiltonian, written below, linear in the control variables and independent of the state of the system.

$$H = \{w_0 uQ(N) - w_1 N - w_2 m + \lambda_1(\xi m - u) - \lambda_2 uQ(N)\}e^{-\delta t}$$

The Hamiltonian is maximized with respect to u amd m at every point in time by adhering to the following 'bang-bang' rules:

$$u^* = \begin{cases} \bar{u} \\ u \varepsilon [o, \bar{u}] \\ o \end{cases} \quad \text{as} \quad \lambda_1 \begin{cases} < (w_0 - \lambda_2)Q(N) \\ = (w_0 - \lambda_2)Q(N) \\ > (w_0 - \lambda_2)Q(N) \end{cases}$$

$$m^* = \begin{cases} \bar{m} \\ m \varepsilon [o, \bar{m}] \\ o \end{cases} \quad \text{as} \quad \lambda_1 \begin{cases} > w_2/\xi \\ = w_2/\xi \\ < w_2/\xi \end{cases}$$

The costate variables must follow the paths given by:

$$\dot{\lambda}_1 - \delta\lambda_1 \leq 0 \quad ; \quad P \geq 0$$

$$\dot{\lambda}_2 - \delta\lambda_2 \leq 0 \quad ; \quad S \geq 0$$

which must hold with complementary slackness.

As is usual, for given terminal conditions, these rules determine the optimal paths of all variables. There are nine sub-regions through which these paths may travel. Three regions are defined by the function,

$$A(t) = (w_0 - \lambda_2(t))Q(N)$$

and three regions are defined by the constant w_2/ξ, the marginal cost of pressure maintenance. As long as $S(t) > 0$, the A function is falling over time as follows:

$$\dot{A}(t) = -\delta\lambda_2(t)Q(N) < 0$$

Region A1 is defined by $\lambda_1(t) < A(t)$. Since $\dot{\lambda}_1(t) = \delta\lambda_1(t) > 0$ as long as $P(t) > 0$, then $\lambda_1(t)$ is rising and $A(t)$ is falling so that $A(t)-\lambda_1(t)$ is positive but diminishing in Region A1.

Region A2 is defined by $\lambda_1(t) = A(t)$ and can hold only for an instant while $P(t) > 0$.

Region A3 is defined by $\lambda_1(t) > A(t)$. Since $\lambda_1(t)$ rises while $P(t) > 0$ and $A(t)$ falls, this region, once entered, will never be left. In region A3, $u^* = 0$ so it is the region in which it is never profitable to exploit the reservoir. This region does not require examination and therefore, three sub-regions can be eliminated from the analysis.

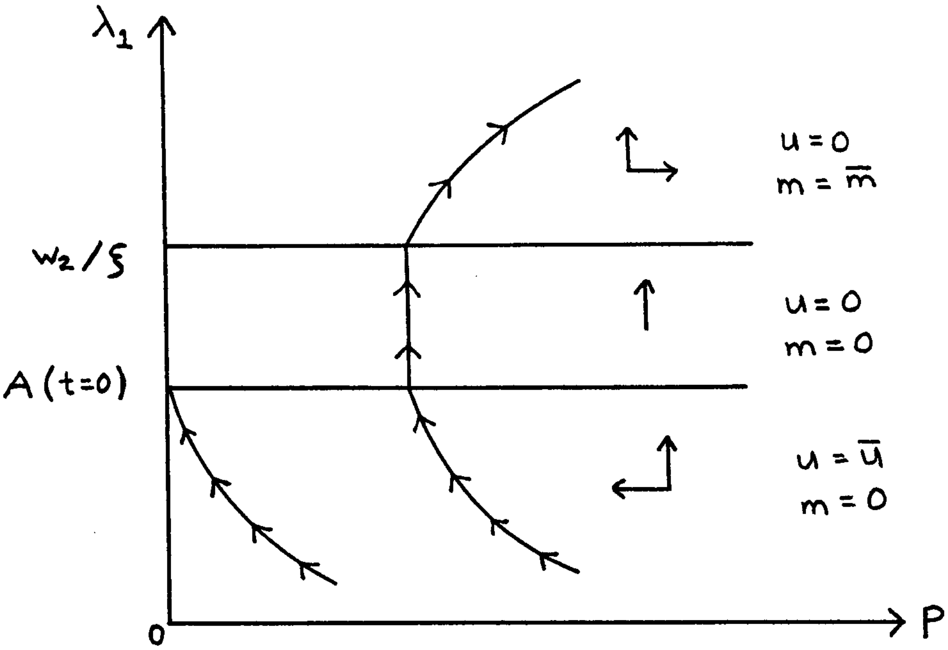
The conditions determining the 'bang-bang' solution values for the control variables are fairly easily interpreted. If the marginal shadow value of pressure, λ_1, exceeds (is exceeded by) the marginal value product of pressure, $(w_0 - \lambda_2)Q(N)$, then set u equal to its minimum (maximum) value. If there is equality between marginal shadow value and marginal value product, then the value of the Hamiltonian is independent of the value of u.

If the shadow value of pressure also exceeds (is exceeded by) the marginal cost of augmenting pressure, w_2/ξ, set injection, m, equal to its maximum (minimum) value. Equality between λ_1 and w_2/ξ implies the value of the Hamiltonian is independent of the value of m.

Region A3 has been ruled out as uninteresting leaving six sub-regions to analyze. A phase diagram will be used to analyze the motion of the system through these regions. In order to construct the phase diagram, one must know the relative size of A(t) and w_2/ξ. One possibility is that $A(t=0) < w_2/\xi$ in which case $A(t) < w_2/\xi$ for all $t \varepsilon (0,T)$. The phase diagram, Figure 3, then applies.

The region above A(t=0) is A3 where $u^* = 0$. Thus, interest focuses only on optimal programs that begin with $\lambda_1 < A(t=0)$. In this sub-region, $u^* = \overline{u}$ and $m^* = 0$ so that $\dot{P} < 0$ and $\dot{\lambda}_1 = \delta\lambda_1$ as long as $P > 0$. The motion of the system through all regions is depicted in the phase diagram. As is apparent, given any $P_0 > 0$, the optimal program must start and finish with $\lambda_1(t) < A(t)$. Moreover, it must finish with $P(T) = 0$ since it would be suboptimal to leave some pressure unexploited as long as there are oil reserves left to be extracted. However, if this is true, then pressure is depleted before reserves are depleted which means this cannot be an optimal solution. Thus, if one assumes that the initial stock of natural reservoir pressure is insufficient to deplete oil reserves, then $A(t) < w_2/\xi$ cannot be the case. More precisely, the time required to deplete pressure is $\tilde{T} = P_0/\bar{u}$, which, if less than the time required to deplete reserves, $\hat{T} = R_0/\bar{u}Q(N)$, implies that $A(t=0) > w_2/\xi$. This is the more interesting of the two possibilities since it will involve pressure maintenance in the optimal solution. However, the case depicted in Figure 3 is a real possibility if the size of initial pressure relative to reserves is large and indeed one does observe oil reservoirs being depleted without ever having undergone pressure maintenance.

FIGURE 3

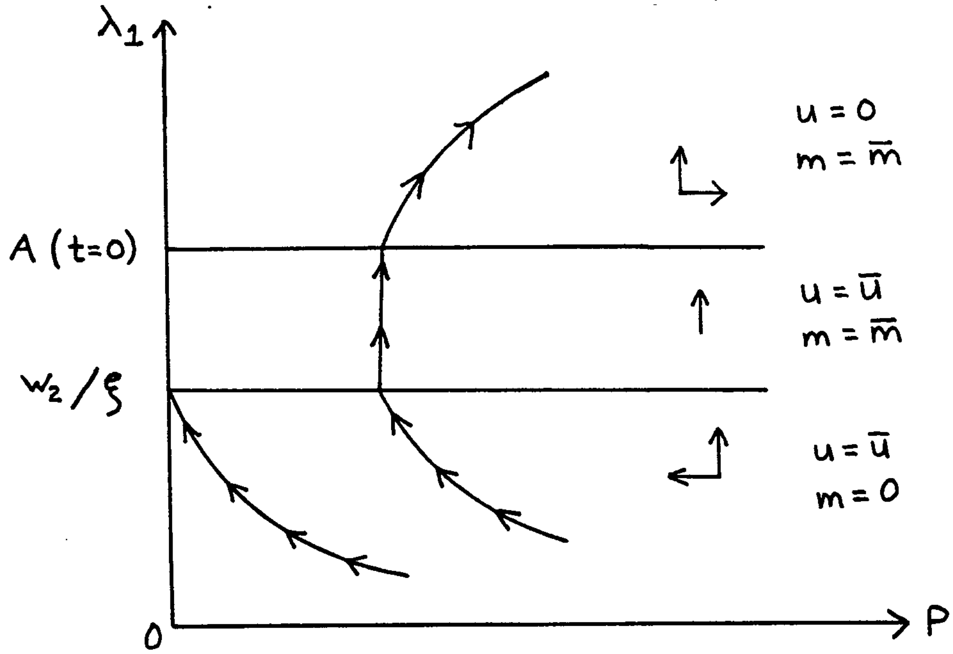


The more interesting case for analysis, where $A(t=0) > w_2/\xi$, is depicted in the phase diagram in Figure 4. This diagram is drawn under the assumption that $\bar{u} = \xi\bar{m}$ so that $\dot{P} = 0$ for $w_2/\xi < \lambda_1(t) < A(t)$. Moreover, it is drawn holding $A(t)$ constant over time while it is known that $\dot{A}(t) < 0$, so that the upper boundary tends to fall over time. It may be deduced, however, that $A(t) > w_2/\xi$ for all $t\varepsilon(0,T)$. To see this, suppose that $A(t) < w_2/\xi$. This implies $m(t) = 0$ since $\lambda_1(t) < A(t)$ for $u^* > 0$. By assumption, the region above w_2/ξ must be entered since injection is required to deplete reserves. Thus, if $A(t)$ were to fall below w_2/ξ at any point in time, the system would be in the region above both $A(t)$ and w_2/ξ which is where $u^* = 0$ and $m^* = \bar{m}$. Since this is the region that is never left once entered, it is suboptimal to enter it and therefore it is suboptimal to have $A(t) < w_2/\xi$ at any time if $A(t=0) > w_2/\xi$.

Since $P(T) = 0$ is a terminal condition, a path must be chosen which reaches the vertical axis in Figure 4. The only optimal path that satisfies this condition is the one where λ_1 reaches w_2/ξ just as P is exhausted. At this moment it becomes optimal to begin injecting water at the maximum rate. Call this point in time T'. Then for $t < T'$, $\lambda_1(t) = (w_2/\xi)e^{-\delta(T_1-t)}$, where $T' = P_0/\bar{u}$.

At time T', $P(T') = 0$ and $\lambda_1(T') = w_2/\xi$. It must also be the case that $A(T') > w_2/\xi$. If $A(T') = w_2/\xi = \lambda_1(T')$, then it would make no difference to profits whether the remaining oil were extracted or not which cannot be the case since the Hamiltonian is independent of the state of depletion. Thus, it must be the case that $u = \bar{u}$ and $m = \bar{m}$ at $t = T'$. Since $\dot{P} = 0$, then $P(t) = 0$ and $\dot{\lambda}_1(t) < \delta\lambda_1(t)$ for $t \geq T'$. It is possible that the system remains at the point where $\lambda_1 = w_2/\xi$ and $P = 0$ until time T when $S(T) = 0$.

FIGURE 4



Since $Q = \bar{u}Q(N)$, the remaining reserves at T' are

$$S(T') = S_0 - P_0 \cdot Q(N)$$

Finally, it can easily be shown that the final depletion date, T, is equal to $S_0/(\overline{u}Q(N))$.

As one would expect, the total depletion time is independent of the the initial stock of pressure and the marginal cost of pressure maintenance in this simple model. The first stage of the optimal depletion program during which pressure is depleted and oil is produced at the maximum rate with no pressure maintenance, is of length T' which does depend positively on the initial stock of pressure and inversely on the maximum utilization rate of pressure.

The fact that it is optimal to deplete pressure to its minimum level before beginning pressure maintenance is not suprising. It is, in fact, similar to a result in the Resource Economics literature regarding the optimal timing of exploration for reserves from which to extract. The analogous result is that it never pays to discover reserves prematurely, before existing reserves are depleted as long as exploration costs are linear and extraction costs are independent of the stock of known reserves. (See Pindyck (1978))

Thus, if one were to make pressure maintenance a non-linear function of the rate of water injection as in the general model, one would expect it to be optimal to begin pressure maintenance before pressure reaches its minimum level in order to reduce the present value of injection costs.

A more interesting reason for there to be pressure maintenance when the stock of pressure is positive, however, is the existence of stock effects in production. These are said to exist whenever the production possibilities are influenced by the stock of pressure.

Case 2

The production function is now permitted to depend on the stock of pressure so that $Q = uQ(N,P)$ with $Q_P > 0$ and $Q_{PP} < 0$. How will this modification alter the optimal depletion program?

The switching functions determining the optimal controls do not change but A(t) is now dependent upon P(t) as well as $\lambda_2(t)$ so that

$$\dot{A}(t) = -\dot{\lambda}_2(t)Q(N,P) + (w_0 - \lambda_2(t))Q(N,P)\dot{P}$$

Moreover, $\lambda_1(t)$ now follows the time path given by

$$\dot{\lambda}_1(t) = \delta\lambda_1(t) - Q_P(N,P)$$

Thus, $\lambda_1(t)$ may be rising or falling, but assuming $P(T) > 0$ then it must eventually fall to zero at time T. The following results suggest that it may be optimal for $\lambda_1(t)$ to change direction during the program.

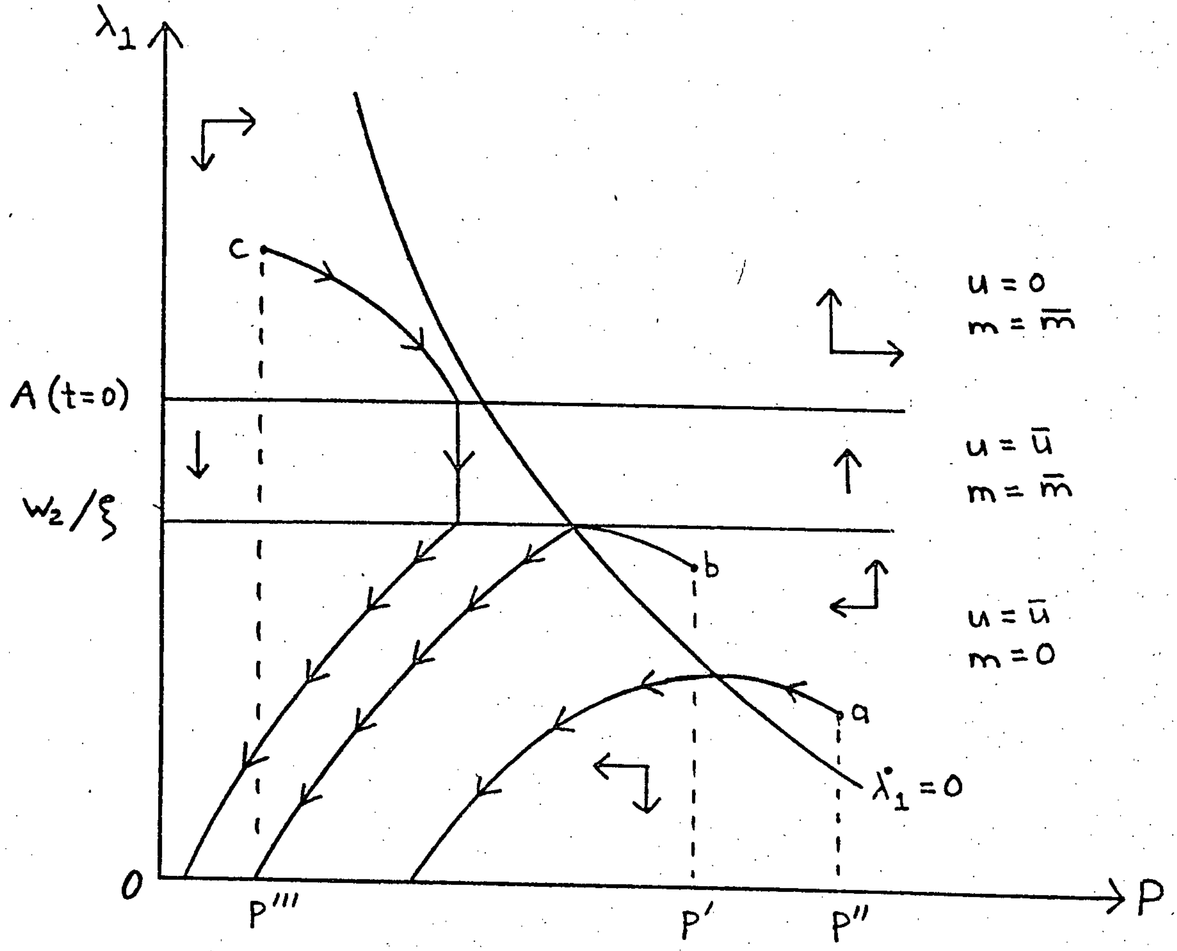
Drawing a phase diagram for this case requires finding the locus of points in λ_1,P - space at which $\dot{\lambda}_1 = 0$. This occurs wherever

$$\delta\lambda_1 = Q_P(N,P)$$

which implies $\left.\delta\frac{d\lambda_1}{dP}\right|_{\dot{\lambda}_1=0} = Q_{PP}(N,P) < 0$

Thus, the isocline is negatively sloped and does not touch either axis if we assume that Q_P tends to infinity as P approaches zero and to zero only as P approaches infinity. This isocline is shown in Figure 5.

FIGURE 5


λ_1
c
A (t=0)
w_2/ξ
b
a
$\dot{\lambda}_1 = 0$
$u = 0$
$m = \bar{m}$
$u = \bar{u}$
$m = \bar{m}$
$u = \bar{u}$
$m = 0$
0
P'''
P'
P''
P

Because $\partial\dot{\lambda}_1/\partial\lambda_1 = r > 0$, then λ_1 is increasing above and decreasing below the $\dot{\lambda}_1 = 0$ locus. Combining this information with the information about the optimal controls in the various regions produces the motion that is depicted in Figure 5. Only the case having $A(t=0) > w_2/\xi$ is examined and the diagram is drawn holding A(t) constant even though it is known to move over time. A(t) will fall whenever $\lambda_1(t) < A(t)$ but may rise or fall when $\lambda_1(t) > A(t)$.

The optimal depletion program now depends critically on the initial stock of pressure. Suppose it is large such as at P''. The optimal trajectory starting at point a requires setting $u^* = \overline{u}$ and $m^* = 0$ for the entire program. The shadow price first rises and then falls, eventually to zero. Alternatively, had the initial stock of pressure been lower, at P' say, the optimal trajectory starting at point b approaches the intersection of the $\dot{\lambda}_1 = 0$ locus and the w_2/ξ line where the value of m is switched from zero to $\bar{m}$ so that $\dot{P} = 0$ and $\dot{\lambda}_1 = 0$. The system may remain in this position for some finite period of time until remaining reserves decline sufficiently to make it optimal to leave this stationary point by setting $m^* = 0$. The trajectory then descends into the lower region finishing finally where $\lambda_1(T) = 0$. This implies an interesting optimal time path for reservoir pressure as shown in Figure 6.

During the initial phase of depletion, pressure declines at the maximum possible rate until it reaches some critical level, $\hat{P}$. This critical pressure level may be maintained over much of the life of the reservoir provided economic conditions do not change. As exhaustion of the oil reserves approaches, pressure maintenance is terminated and pressure decline at the maximum rate is once again observed and continues until the terminal date.

FIGURE 6

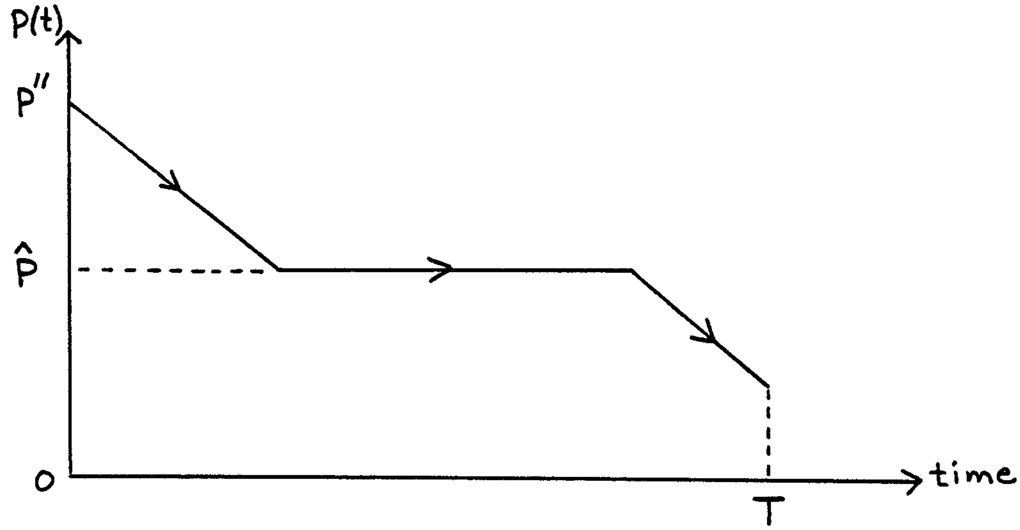
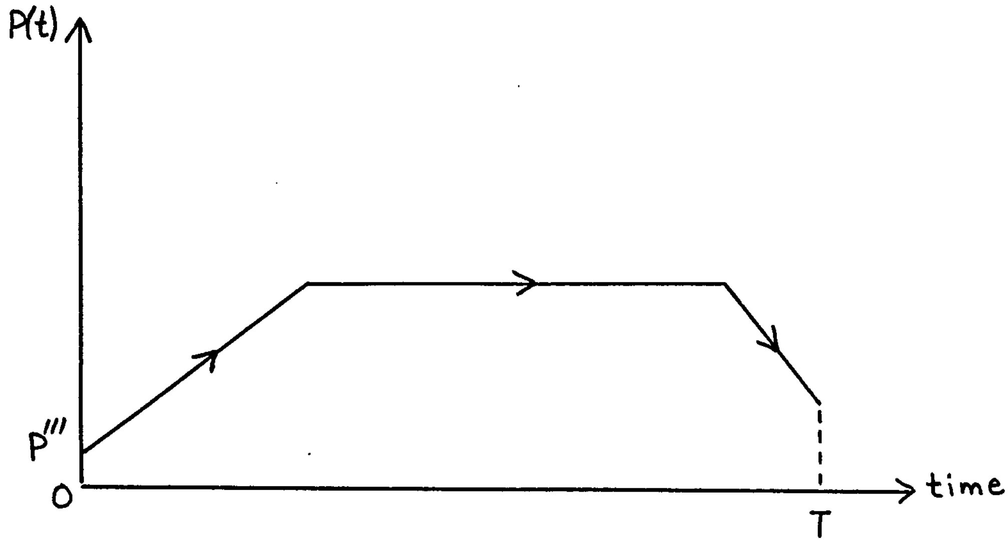


FIGURE 7



A third possibility occurs if the initial stock of pressure is very low, such as at P'''. If initial oil reserves are large enough to warrant the investment, the optimal trajectory starts at point c where $u^* = 0$ and $m^* = \overline{m}$. Thus, in this case, there is an initial period of pressure buildup with no extraction of oil from the reservoir. This continues until pressure reaches another critical level. At that point, extraction of oil commences and pressure maintenance continues so that $u^* = \overline{u}$ and $m^* = \overline{m}$ and there follows a period of time in which the pressure level of the reservoir is kept constant. As before, this situation ends when oil reserves have become so low that continued pressure maintenance is not optimal. At this point, the trajectory enters the lower region where $u^* = \overline{u}$ and $m^* = 0$ so that $\dot{P} = -\overline{u}$ until $S(T)=0$. This will normally occur with $P(T) > 0$. The implied path of reservoir pressure is shown in Figure 7.

The final point to be made is that in this case the total length of the depletion program does depend upon the initial stock of pressure and the marginal cost of pressure maintenance.

To summarize, this special case of the general model implies that the optimal depletion of the reservoir may or may not involve pressure maintenance. It was found that this will depend largely on the stock of initial pressure relative to the stock of oil reserves. If pressure maintenance is warranted, one would expect to observe a phase in the depletion program during which pressure is maintained at a constant level. This is to be followed by a phase of pressure decline as the oil reserves near exhaustion. The initial phase may involve pressure decline or pressure buildup, depending upon the initial stocks of pressure and oil reserves. Of additional interest is the fact that the lower the cost of pressure maintenance, the more likely is the optimal trajectory to include a phase of pressure maintenance. This result is due to the dependence of the production function on the level of reservoir pressure.

3.3 The Variable Cost Function

The objective of this section is to generate a one-period cost function which embodies the dynamic nature and the technology inherent in the model of oil extraction developed and analyzed above. The purpose of compacting this information into a static cost function is to facilitate the testing of the model and the obtaining of information about the determinants of the optimal extraction policies of oil reservoirs. Estimates of the parameters of the static cost function can be obtained using available data on individual oil reservoirs in the Province of Alberta. An additional important feature of this procedure is that it permits one to estimate the state-dependent cost of oil extraction as a function of the exogenous 'natural factors of production' that differentiate reservoirs.

A cost function embodies the cost-minimizing choice of factors of production at a point in time that produce a given level of output at that point in time. The complication at hand is that factor use at time t affects not only costs at time t but also in all time periods thereafter through their effect on the state variable, P, of the system. In order to eliminate this complication, it is necessary to restrict the choice set of the factors of production in the cost-minimization exercise in such a way that this state variable follows some exogenously determined path during the period. In this way, the dynamic optimization problem is made a two-stage optimization problem. The first stage is to find the minimum cost at any point in time as a function of the exogenous change in the state variable. The second stage is to find the maximum present value of the program by determining the optimal changes in the state variables at each point in time.

Another way of stressing the need for the restrictions on the technology set in order to generate the cost function is as follows. To obtain a cost function, one chooses the cost-minimizing input bundle that produces some output level, given factor prices. Because the factors affect the state of pressure, however, their true factor prices consist of the market purchase price plus the value of their effect on the optimal program by affecting the state of the system. That is, one needs to know the endogenously determined shadow price of pressure in order to choose the cost-minimizing input bundle. This is unobservable, however, so that it is impossible to generate the cost function in the standard way. By restricting the choice set to conform to an exogenous change in the state of pressure, one prevents the choice of an input bundle from affecting the state of the system since it is exogenously given both at the beginning and at the end of the period. Thus one eliminates the need for shadow prices in generating the cost function. It is, however, a restricted cost function.[3]

The instantaneous cost-minimization problem is written formally below showing the restrictions on the technology set.

3. In different contexts, this technique has been utilized by Berndt, Fuss and Waverman (1977), and Diewert and Lewis (1981) and has been reviewed by Berndt, Morrison and Watkins (1981).

$$\underset{\langle u,N,m\rangle}{\text{Minimize}} \quad (w_1N + w_2m) \tag{3.13}$$

$$\text{Such that} \quad Q = f(u,N,P,Z) \tag{3.14}$$

$$\dot{P} = g(m,P,z) - u = -\theta \tag{3.15}$$

where Q,θ are given constants.

The two constraints implicitly define the instantaneous technology set (Diewert and Lewis (1981)) in that the factors of production must be chosen so as to satisfy the constraints. By substituting (3.15) into (3.14), the unobservable factor of production, u, can be eliminated from the minimization problem. The restriction on the input bundle now reduces to

$$f[g(m,P,z) + \theta,N,P,z] = Q \tag{3.16}$$

which has the properties of a standard production function. Thus, the problem is reduced to a standard cost-minimization exercise which is known to yield a dual cost function with known properties.[4] Thus, the following is the one-period variable cost function for oil extraction:

$$C(w_1,w_2;P,\theta,Q,z) \tag{3.17}$$

All non-price arguments of the restricted cost function are treated as 'netputs' following Diewert(1974) and McFadden(1978). Netputs may be inputs or outputs and are treated symmetrically. Here, the sign

4. Diewert (1973,1974,1978), Diewert and Lewis(1981), and McFadden(1978).

convention adopted is that inputs are indexed with positive numbers and outputs are indexed with negative numbers. Thus, the cost function is positive and is non-increasing in the netput vector. For example, output is indexed negatively. A larger output means a smaller Q (now a negative number) and hence higher costs if cost is decreasing in netputs. Diewert (1973) shows that if the production technology exhibits constant returns to scale, the (negative of the) restricted cost function will be homogeneous of degree one in the netput vector. The restricted cost function is also non-decreasing, quasi-concave and homogeneous of degree one in factor prices.

It is well known that a cost function satisfying the proper regularity conditions embodies all of the technological parameters of its dual production function. It is also the case that a similar dual relationship exists between a restricted cost function and its underlying production function. Indeed, as will be shown, all of the technological parameters needed to find the solution to the dynamic optimization problem are embodied in the static, variable cost function. Moreover, these parameters can be estimated through the restricted factor demand equations which are easily obtained using Shephard's Lemma:

$$\begin{aligned} N^*(w_1,w_2;P,\theta,Q,z) &= \partial C(w_1,w_2;P,\theta,Q,z)/\partial w_1 \\ m^*(w_1,w_2;P,\theta,Q,z) &= \partial C(w_1,w_2;P,\theta,Q,z)/\partial w_2 \end{aligned} \qquad (3.18)$$

One need only specify a functional form for the restricted cost function which has the properties described above, apply Shephard's Lemma to obtain the factor demand equations and estimate the parameters of the cost function through the two factor demand equations using data on the variables listed in (3.18) for individual oil reservoirs.

Before specifying a functional form for the cost function and proceeding with its estimation, it is useful to undertake the second stage of the two-stage optimization problem to see how the static, variable cost function fits into the dynamic optimization problem and through a brief analysis of the solution, to see that it contains all of the information needed to solve the overall maximization problem.

The second stage of the optimization problem is written as follows:

$$\underset{\langle\theta,Q,T\rangle}{\text{Maximize}} \int_{o}^{T} e^{-rt}\{w_0Q(t) - C[w_1,w_2;P(t),\theta(t),Q(t),Z]\}dt$$

$$\begin{aligned} \text{subject to} \quad \dot{P}(t) &= -\theta(t) \\ \dot{S}(t) &= -Q(t) \\ P(0) &= P_0 > 0 \\ S(0) &= S_0 > 0 \end{aligned}$$

The present-valued Hamiltonian for this problem is

$$H = e^{-\delta t}\{w_0Q - C[w_1,w_2;P,\theta,Q,z] - \lambda_1\theta - \lambda_2Q\} \qquad (3.19)$$

where time arguments have been suppressed and λ_1 and λ_2 are the costate variables associated with P and S, respectively. If an interior solution exists, then maximization of H at each point in time implies that the following conditions hold at each point in time:

$$\partial H/\partial Q = 0 \rightarrow w_0 - C_Q - \lambda_2 = 0 \qquad (3.20a)$$

$$\partial H/\partial\theta = 0 \rightarrow -C_\theta - \lambda_1 = 0 \tag{3.20b}$$

$$d\lambda_1 e^{-\delta t}/dt = C_p e^{-\delta t} \tag{3.20c}$$

$$d\lambda_2 e^{-\delta t}/dt = 0 \tag{3.20d}$$

Given T and P(T) the above four conditions determine the time paths of Q,θ,P, and the costate variables. Interpretation of the first-order conditions proceeds as in the previous section by imposing a subset of the transversality conditions ($\lambda_1(T)=0$) and manipulating (3.20c) to obtain the following expression for λ_1.

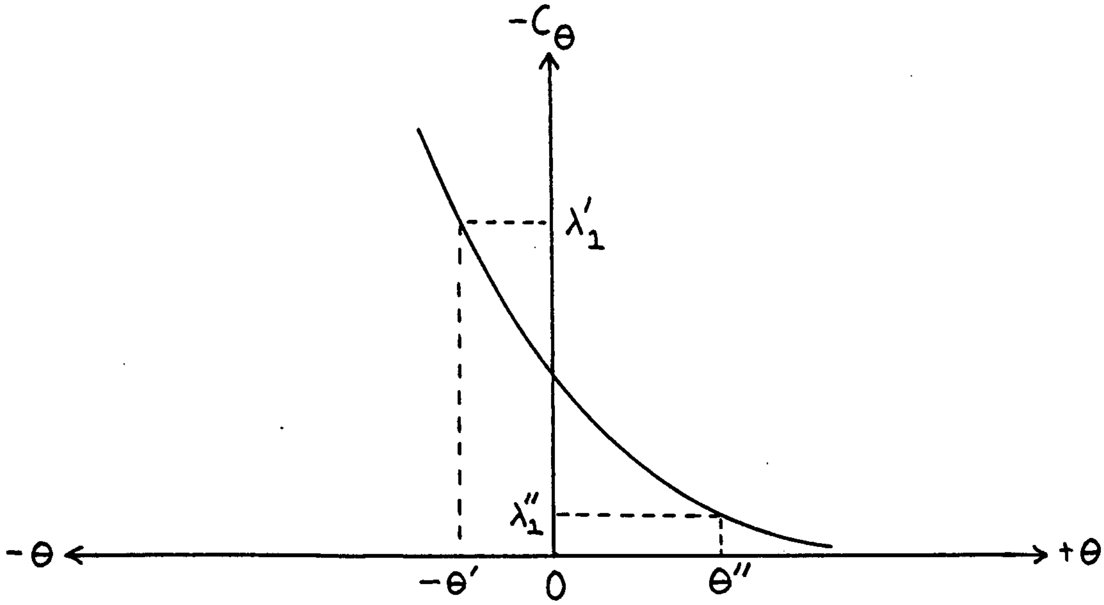
$$\lambda_1(t) = -\int_t^T e^{-\delta(\tau - t)} C_p d\tau \tag{3.20c'}$$

Since $C_p \leq 0$, then $\lambda_1(t) \geq 0$. As is apparent from (3.20c') this costate variable is the present value of the change in all future costs that result from a marginal change in the current stock of reservoir pressure. It can therefore be interpreted as the shadow price of pressure.

From (3.20b), the optimal solution, if it is interior, requires equating $-C_\theta$ (the negative of the current marginal cost of a change in the stock of pressure) with the shadow price of pressure. (Recall that $C_\theta < 0$). Figure 8 depicts this relationship.

Recall that when $\theta > 0$, pressure is falling (an input) and when $\theta < 0$ pressure is rising (an output). If pressure has a very high shadow price such as λ_1', it is optimal to inject so much that reservoir pressure actually rises by the absolute value of $-\theta'$. If the shadow price is low

FIGURE 8



such as λ_1'', it is optimal to reduce pressure by θ''. This may or may not require positive injection of fluid into the reservoir. For extremely small values of λ_1, pressure decline will be high and there will almost certainly be zero injection.

The position of the curve in Figure 8 depends upon the values of all other arguments in the cost function. In particular, it depends upon P, the stock of reservoir pressure. Since λ_1 is the costate variable for P, it must be a decreasing function of P. Thus, if two reservoirs are identical in every respect except for the levels of pressure, λ_1 would be higher in the reservoir with the lower pressure making it optimal to have a slower rate of pressure decline in that reservoir which may imply a higher level of water injection.

Because $-\lambda_1 = C_\theta$ at an interior solution, one can test the above hypothesis empirically by estimating the cross-partial derivative, $C_{\theta P}$ since

$$-\partial\lambda_1/\partial P = C_{\theta P}$$

The partial derivatives of the restricted cost function contain sufficient information to permit one to empirically estimate the shadow price of pressure and reserves and to test hypotheses about their signs. These partial derivatives can be estimated through the factor demand equations.

The dependence of the solution to the optimal depletion problem on the partial derivatives of the restricted cost function can be demonstrated by taking total time derivatives of (3.20a) and (3.20b). Solving these for $\dot{Q}$ and $\dot{\theta}$ yields the following:

$$\dot{Q} = \{C_{\theta\theta}[C_{QP}\cdot\theta + \delta(C_Q - w_0)] - C_{Q\theta}[C_{\theta P}\cdot\theta - C_P + \delta C_\theta\}/(C_{QQ}C_{\theta\theta} - C_{\theta Q}^2) \quad (3.21)$$

$$\dot{\theta} = \{C_{QQ}[C_{\theta P}\cdot\theta - C_P + \delta C_\theta] - C_{\theta Q}[C_{QP}\cdot\theta - \delta(w_0 - C_Q)]\}/(C_{QQ}C_{\theta\theta} - C_{\theta Q}^2) \quad (3.22)$$

As is apparent, the direction of change of both control variables is dependent upon the parameters of the cost function. In (3.21), if one assumes that the state of pressure has no influence on cost and sets all partial derivatives involving P equal to zero, one obtains the following simpler expression:

$$\dot{Q} = \frac{-\delta C_{\theta\theta}(w_0 - C_Q) - \delta C_\theta \cdot C_{Q\theta}}{C_{QQ}C_{\theta\theta} - C_{\theta Q}^2} \quad (3.23)$$

which is negative if $C_{Q\theta} < 0$ since $C_\theta < 0$ and $C_{\theta\theta} > 0$. This corresponds to the standard result of the simple Hotelling-type model of optimal depletion but requires a stronger assumption ($C_{Q\theta} < 0$) to obtain the negatively-sloped production profile. Corresponding to this result is the condition that the shadow prices of the two state variables must rise (in absolute value) at the rate of discount over time.

Under the same assumptions, one obtains a simpler expression for the change over time in θ:

$$\dot{\theta} = \frac{\delta C_\theta C_{QQ} + \delta(w_0 - C_Q)C_{\theta Q}}{C_{QQ}C_{\theta\theta} - C_{\theta Q}^2} \quad (3.24)$$

which is negative indicating that it is optimal for the rate of pressure decline to diminish over time.

Thus, if the level of reservoir pressure does not affect the cost of extraction, one expects to observe pressure decline at a declining rate and a declining extraction rate throughout the life of the reservoir. The cost function provides a method of empirically testing for these conditions.

The optimal time paths of Q and θ are more complicated when the cost of extraction is dependent on the state of pressure. Associated with this case is the result that the time rate of change of λ_1 may be positive, negative, or zero.

To obtain some insight into this problem, it is necessary to once again simplify and resort to the use of phase-diagram analysis. Assume that extraction costs are linear in the rate of extraction of oil. Then the first-order conditions for a maximum become

$$-C_\theta - \lambda_1 = 0$$

$$Q^* = \begin{cases} \bar{Q} \\ Q\varepsilon[o,\bar{Q}] \\ o \end{cases} \text{as} \quad \lambda_2 \begin{cases} < W_O - C_Q \\ = W_O - C_Q \\ > W_O - C_Q \end{cases}$$

where C_Q is the (assumed) constant marginal cost of extraction.

$$\dot{\lambda}_1 = \delta\lambda_1 + C_P$$

$$\dot{\lambda}_2 = \delta\lambda_2$$

$$\dot{P} = -\theta$$

$$\dot{S} = -Q$$

To draw a phase-diagram in θ,P - space, use the following two equations which describe their motion:

$$\dot{P} = -\theta \tag{3.25}$$

$$C_{\theta\theta}\dot{\theta} = \delta C_{\theta} - C_{P} - C_{\theta P}\dot{P}$$

where it is assumed that $\dot{Q} = 0$. This will be true except when the system switches from one region where $\lambda_2 < W_O - C_Q$ to where $\lambda_2 > W_O - C_Q$. At this point, Q will make a discrete change of magnitude $\overline{Q}$ and then return to $\dot{Q} = 0$. Setting $\dot{\theta} = 0$ to find the locus of points where pressure change is zero yields:

$$\delta C_{\theta} - C_{P} + C_{\theta P} \cdot \theta = 0$$

In order to determine the slope of this isocline one requires information about cross-partial and third derivatives. In the absence of this information, assume that third derivatives are zero. It is known that $C_{\theta P} > 0$ since $\partial\lambda_1/\partial P = -C_{\theta P} < 0$. It is reasonable to assume that the numerator of the expression for the slope of the isocline, given below, is positive for small δ since $C_{PP} > 0$. Formally, it is assumed that:

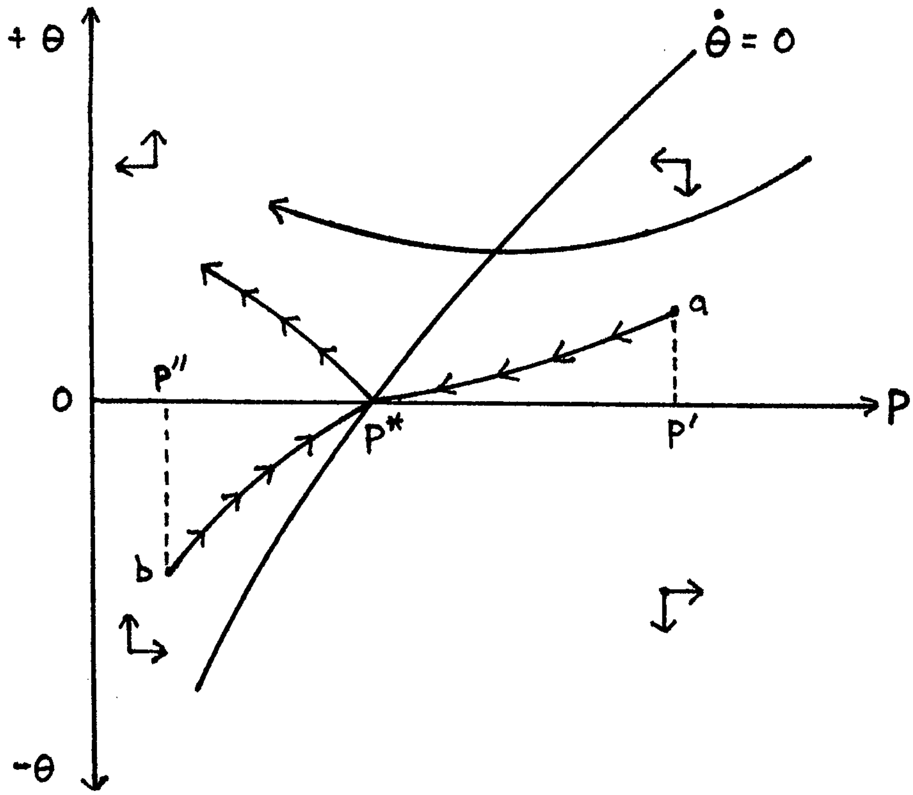
$$\left.\frac{\partial\theta}{\partial P}\right|_{\dot{\theta}=0} = \frac{C_{PP} - \delta C_{\theta P}}{\delta C_{\theta\theta}} > 0$$

This assumption then generates the isocline and motion shown in Figure 9.

A saddlepoint equilibrium occurs at the pair $(P^*,0)$ where $P^* > 0$. Depending on the time available, T, which depends on the stock of initial reserves, and depending upon the initial stock of pressure, the system may reach this stationary point and remain there for some finite amount of time. For example, suppose the system starts at the point a with an initial stock of pressure equal to P'. The trajectory follows $\dot{\theta} < 0$ and $\dot{P} < 0$ until $(P^*,0)$ is reached. It remains there until reserves are sufficiently depleted that continued pressure maintenance is not warranted. It then leaves in a northwesterly direction as shown with $\dot{\theta} > 0$ and $\dot{P} < 0$ until exhaustion of the oil reserves.

Alternatively, had the system started with an initial stock of pressure of P'', the optimal trajectory starts at point b with $\theta < 0$, $\dot{\theta} > 0$ and $\dot{P} > 0$ until $(P^*,0)$ is reached. During the initial phase of pressure buildup, it may be optimal for simultaneous extraction of oil reserves to occur provided $\lambda_2 < W_0 - C_Q$. This possibility was ruled out in the case analyzed in Section 3.2 by the assumption that the production function was linear in u and pressure maintenance was linear in m. The corresponding assumption here would be that cost is linear in θ. By not making this assumption, one allows for the possibility of simultaneous extraction and injection at variable rates so that pressure may rise, fall or remain constant. As before, after a finite period of time, the system must leave the stationary point in a northwesterly direction with $\dot{\theta} > 0$ and $\dot{P} < 0$. Note that while $\dot{P} > 0$ implies $m > 0$, $\dot{P} < 0$ does not imply $m = 0$. Thus, the final phase of the depletion program may involve simultaneous extraction and injection but in such a way that there is a

FIGURE 9



steady decline in reservoir pressure until oil reserves are exhausted.

The stationary point occurs at $P^* > 0$ due to the assumption that $C_P \neq 0$. If, on the other hand, it were assumed that extraction costs were independent of the state of pressure so that $C_P = 0$ then the $\dot{\theta} = 0$ isocline would intersect the $\dot{P} = 0$ isocline at the origin. This implies that the system cannot come to rest until pressure is completely depleted and is analagous to the first case of the linear model examined in Section 3.2 where injection does not occur before pressure is depleted to its minimum level. This result follows from (3.25) which now reduces to:

$$C_{\theta\theta}\dot{\theta} = \delta C_{\theta} < 0 \quad \text{for} \quad P > 0.$$

Thus, $\dot{\theta} < 0$ as long as $P > 0$ so that the system cannot come to rest at a positive level of pressure. On the other hand, when $P = 0$, this condition becomes $C_{\theta\theta}\dot{\theta} \geq \delta C_{\theta}$ which is consistent with $\dot{\theta} = 0$ so that the system, if it does come to rest for some period of time, must do so at $P = 0$. The slope of the $\dot{\theta}$ isocline is undefined in θ, P - space but one can think of it as being vertical and coincident with the vertical axis.

It has been demonstrated that the extraction cost function contains the information needed to solve the dynamic optimization problem. In general, one does not expect all reservoirs to be identical and therefore not to have identical optimal extraction policies. As discussed earlier, inter-reservoir quality differences are captured in the cost function by differences in the vector of natural factors of production. A cross-sectional view of reservoirs at a point in time will reveal differences in the components of the z vector and in the values of the state variable P(t). The latter can be expected to give rise to cost differences at a point in time but is controllable over time. Of greater interest are uncontrollable or exogenous sources of cost variations across reservoirs - the natural factors of production or the components of the z vector.

In terms of the phase diagrams, inter-reservoir differences caused by z differences will lead to shifts in isoclines and trajectories. In terms of first-order conditions for a maximum, these differences will lead to differences in marginal extraction costs, C_Q, the marginal cost of changing the stock of pressure, C_θ, stock effects, C_P, shadow prices λ_1 and λ_2 and, of course, the level of total costs. All of this information is embodied in the variable cost function and can be estimated empirically.

One of the more interesting effects of z-differences is on the shadow prices. Because one observes a large variation in pressure maintenance practices in a cross-section of oil reservoirs, one naturally wonders whether this is due to the fact that the pools are at different states of depletion or if there are fundamental quality differences in the reservoirs caused by differences in the components of the z vector that explain this fact. One can attempt to answer this question using the information obtained through estimating the parameters of the cost function. As discussed in an earlier section, the primary determinant of water injection is the shadow price of reservoir pressure. In the next chapter, the hypothesis that the shadow price of pressure varies in a systematic way with variation in natural factors of production across reservoirs thereby explaining the observed wide variation in pressure maintenance practices across oil reservoirs is tested.

The z vector can also be a source of rent differentials across oil reservoirs through its effect on the cost of extraction. This information is also contained in the restricted cost function.

In the actual circumstances of the Province of Alberta, the oil extraction industry is subject to strict government regulations on the extraction rates of individual reservoirs. The dynamic optimization problem of the reservoir manager can be modified to accomodate these additional restrictions in the following way. The objective is now to minimize the cost of producing an exogenously given stream of extraction rates given by the vector $\tilde{Q}$.

$$\underset{\{\theta\}}{\text{Minimize}} \int_{o}^{T} e^{-\delta t} C(w_1 w_2; P, Q, \theta, Z) dt$$

$$\text{s.t.} \quad \dot{P} = -\theta$$

$$P(0) = P_0$$

$$S(0) = S_0$$

Given S_0, T is determined by $\tilde{Q}$. The only problem therefore, is to choose the time path of θ. The present-valued Hamiltonian is given by

$$H = e^{-\delta t} \{C(w_1, w_2; P, \tilde{Q}, \theta, Z) - \lambda\theta\}$$

One wishes to minimize H at each point in time which implies that the following conditions must hold:

$$C_\theta - \lambda = 0$$

$$\dot{\lambda} - \delta\lambda = -C_P$$

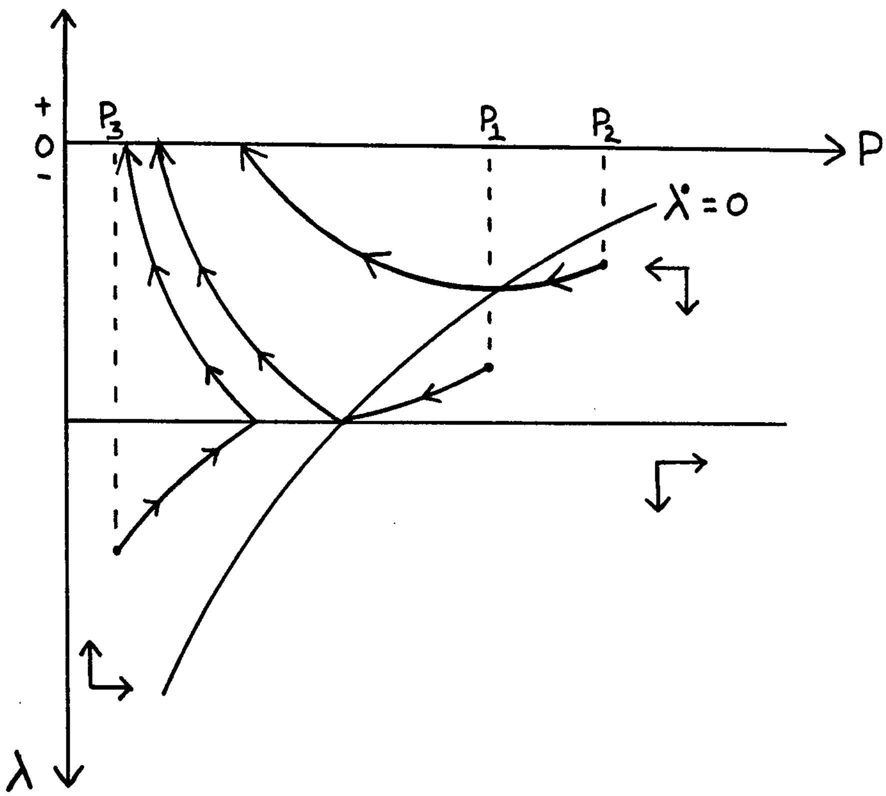
where λ is the shadow price of reservoir pressure. Thus, this modified problem is similar to that already analyzed but simpler because there is only one state variable. A special case of the problem not yet analyzed occurs when the variable cost function is linear in θ and Q is constant. The optimal controls for θ are then given by the following:

$$\text{if } \begin{cases} C_\theta < \lambda \\ C_\theta = \lambda \\ C_\theta > \lambda \end{cases} \quad \text{set} \quad \begin{array}{l} \theta = \theta_{max} \\ \theta = \theta \\ \theta = -\theta_{max} \end{array} \tag{3.26}$$

where it is assumed that a physical limit of θ_{max} exists on the rate at which pressure can be altered. Recalling that C_θ is negative, then λ is also negative in this minimization problem. Thus (3.26) says that if the absolute value of C_θ, the marginal cost of augmenting pressure, exceeds the absolute value of λ, the marginal benefit of augmenting presure, set $\theta = \theta_{max}$. This implies rapid depletion of reservoir pressure and no water injection. Alternatively, if the absolute value of C_θ is less than the absolute value of λ, set $\theta = -\theta_{max}$. This implies a rapid buildup of reservoir pressure and, hence, a positive rate of water injection. The equations of motion, combined with (3.26) can be used to derive the following phase diagram in Figure 10.

If the initial stock of pressure is particularly large such as P_2, the optimal trajectory remains in the region of maximum pressure decline which implies that pressure maintenace is never undertaken. With a lower initial level of pressure, P_1, the optimal trajectory first involves a period of rapid pressure decline and zero water injection, followed by a

FIGURE 10



period of constant pressure at the quasi-stationary point, followed by a final period of rapid pressure decline. As before, the system must terminate with $\lambda(T)=0$.

In this chapter a model of oil extraction has been developed and its implications for depletion behaviour have been analyzed. A one-period variable cost function has been constructed which is dual to the one-period technology set of the model. The cost function can be used to empirically test the model and to obtain extraction cost information. In the empirical application, two approaches to modelling the man-made factors of production are taken. The first approach is to assume that both N and m are variable factors and that each is optimally chosen at every point in time. In this case, the variable cost function contains both factor prices as arguments. In the second approach, it is assumed that N is a fixed factor of production that is optimally chosen at t=0. All subsequent decisions are then conditional on the fixed stock of oil wells in place. In this case, the variable cost function is a factor-requirements function for m, and does not contain factor prices but does contain the stock of wells, N, as an argument.

CHAPTER 4

The Empirical Specification and Estimation Procedures

4.0. Introduction

In the previous chapter, a model of oil reservoir depletion was developed and the arguments of the variable extraction cost model were specified. It was argued that the parameters of this function which, given appropriate data, could be empirically estimated could convey information not only about the optimal depletion strategy of oil reservoirs but also about inter-reservoir extraction cost heterogeneity.

Two approaches to modelling oil well capital have been adopted. The first is the "putty-putty" approach in which an oil well is a completely variable factor of production. In this case, Model I, the two variable factors of production (the number of oil wells and the rate of water injection) are chosen in any period so as to minimize the one-period variable extraction costs. The second is the "putty-clay" approach. In this case, Model II, the number of oil wells is chosen only in the initial period so as to minimize the present-value of the cost of producing an exogenous output stream over an endogenous period of time. Thereafter, all variable input decisions are made subject to the existence of a fixed stock of oil wells. The only remaining variable input in the restricted technology set is the rate of water injection. Thus, the variable cost function of Model II is equivalent to a factor-requirements function.

In this chapter, the data and econometric procedures employed to obtain estimates of the variable cost function parameters for both models are presented and discussed. This includes the specification of functional forms for the variable cost functions, derivation of the estimation equations and discussion of and solutions to the econometric problems associated with these equations. The chapter is organized into two sections. In Section 1, Model I is completely specified and analyzed. A discussion of the data, most of which is the same as that used for Model II, is included in this section. Model II is completely specified and analyzed in Section 2. Appendix A contains the technical derivations used to obtain some of the results in the chapter and Appendix B documents the data sources.

4.1. Model I

The variable extraction cost function of Model I is implemented by assuming that the units of time are discrete one-year periods and that the non-price netput vector includes the following observable variables.

W_r = the water saturation level in the r^{th} reservoir

P_{rt} = the pressure level in the r^{th} reservoir at the beginning of year t

θ_{rt} = the observed change in the pressure level in the r^{th} reservoir during year t

Q_{rt} = the observed production of oil from the r^{th} reservoir during year t

Z_r = the pay thickness of the r^{th} reservoir.

Let the vector $X = (x_1, x_2, \ldots, x_5)$ represent the above list of netputs and let w_1 be the input price per oil well and w_2 be the input price per unit of water injection. The functional form for the variable extraction cost function is specified as the following quadratic:

$$C(w_1, w_2; X) = \sum_{i=1}^{2} \sum_{j=1}^{5} \alpha_{ij} w_i X_j + \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{h=1}^{5} \beta_{ij} \sqrt{w_i} \sqrt{w_j} X_h \qquad (4.1)$$

$$+ 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{h=1}^{2} \gamma_{ij} X_i X_j w_h$$

The suppression of time and reservoir subscripts, which will be continued hereafter, should not cause any confusion: unless otherwise stated, all observations are made in the same year and in all cases, the reservoir subscript r is attached to each component of the X vector and the two dimensional vector of variable factors of production.

The specification in (4.1) guarantees that the variable cost function is homogeneous of degree one in factor prices but does not automatically satisfy the properties of concavity or non-decreasingness in factor prices. Similarly, the regularity conditions with respect to the netput vector are not automatically satisfied but must be numerically checked: variable costs must be non-decreasing in X_i if X_i is an output and non-increasing in X_i if X_i is an input. The variable cost function need not satisfy the conditon of convexity in the netput vector since the possibility of increasing returns to scale is not ruled out by the specification in (4.1).

The parameters in equation (4.1) cannot be estimated directly because cost data are not available on a reservoir-by-reservoir basis.

However, the parameters can be estimated by using Shephard's Lemma on (4.1) to obtain the variable-cost-minimizing factor demand equations, data for which are available. These are given by

$$N = \sum_{j=1}^{5} \alpha_{1j}X_j + [\beta_{11} + \beta_{12}(w_2/w_1)^{1/2}] \sum_{j=1}^{5} X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \gamma_{ij}X_iX_j$$

$$m = \sum_{j=1}^{5} \alpha_{2j}X_j + [\beta_{22} + \beta_{12}(w_1/w_2)^{1/2}] \sum_{j=1}^{5} X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \gamma_{ij}X_iX_j \tag{4.2}$$

where N and m are the variable-cost-minimizing demands for the number of oil wells and the rate of water injection, respectively.

Because the sample is cross-sectional and for the reasons given in the data section below, the relative factor price w_1/w_2 is a constant across all observations in the sample. Thus, the β_{ij} terms combine with the α_{ij} terms to yield estimates of parameters which are composites of these underlying parameters. The resulting estimation equations are written as the system in (4.3) where it is assumed that any errors, reflected in the e and u terms, respectively, are completely random.

$$N = \sum_{j=1}^{5} a_{1j}X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \gamma_{ij}X_iX_j + e$$

$$m = \sum_{j=1}^{5} a_{2j}X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \gamma_{ij}X_iX_j + u \tag{4.3}$$

In addition it is assumed that the error terms e and u are jointly distributed normal random variables with zero mean and covariance matrix Σ where

$$\Sigma = \begin{bmatrix} \sigma_e I & \rho\sigma_u\sigma_e I \\ \rho\sigma_u\sigma_e I & \sigma_u I \end{bmatrix}$$

where I is the identity matrix. Thus, each equation is assumed to have a constant variance.

The system of equations in (4.3) forms the basis of the econometric model used to estimate the parameters of the variable cost function.[1] There are, however, three features of the system in (4.3) that require special attention in the estimation of its parameters. First, there are across-equation restrictions on the γ_{ij} parameters that must be satisfied. Second, a significant percentage of the observations on the dependent variable, m, occur at its lower limiting value of zero thereby creating the potential for limited dependent variable bias. Third, the "exogenous" variable, $X_3 = \theta$, is known to be an endogenous variable in the dynamic stage of the cost minimization problem thereby creating the potential for simultaneity bias. These three econometric issues will be dealt with thoroughly after a discussion of the data set that will be used to generate the parameter estimates.

4.1.1 Data

All observations were made for the year 1973 on 80 oil pools situated throughout the Province of Alberta. The size of the sample and the pools included in the sample were determined by the following factors. Of all the oil pools in the Province that were in operation in

1. Although the structural parameters, α_{ij} and β_{ij}, cannot be estimated, this is of little consequence as long as the cost function is not applied to data in a year other than that used to generate the parameter estimates.

1973, all those that began operation before 1962 were excluded. This practice, which eliminates the majority of potential observations, was adopted due to the perverse regulatory framework that existed in Alberta before 1962: the rate of extraction that any pool was allowed was closely linked to the number of oil wells drilled into the pool.[2] Hence, the extraction rate, Q, cannot be treated as an exogenous variable in the variable cost function for pools developed before 1962.

Because 1973 is the observation year, pools developed in 1971 or later were also excluded to ensure that all of the pools in the sample were fully operational. Of the remaining pools (approximately 300), a complete set of observations could be obtained for only 80. A detailed discussion of the data sources is provided in Appendix B.

N: The number of oil wells

This dependent variable is the total (or cumulative) number of oil wells observed to be in place by the year 1973. Across the 80 pools in the sample, N ranges from a minimum of 1 well to a maximum of 256 wells with the average number of wells per pool being 15.74 and the standard deviation being 35.77.

Casual observation of time profiles of N on a pool-by-pool basis cannot produce firm conclusions about whether oil wells are variable or fixed factors, but can offer some evidence in support of one or the other views. If one looks at these time profiles, it appears that in the majority of cases, the build-up of oil wells occurs fairly rapidly (over a one or two year period) and is then followed by a relatively long period in which the number of wells is constant. There are deviations from this

2. Watkins (1977)

phenomenon, however, in which there is a gradual build-up and sometimes a decline in the number of wells. An explanation of the deviations which is consistent with the view that an oil well is a fixed factor is that at development time the true size of the oil pool is unknown. The drilling of some wells can sometimes lead to the knowledge that the pool is larger than believed and, hence, to the drilling of additional (step out) wells. Thus, one may observe a gradual build-up of wells in a pool because of incomplete information about the reservoir and not because oil wells are variable factors.

On the other hand, there are arguments in support of the view that an oil well is a variable factor. First, given that the decision to sink a well is irreversible, it is still a simple matter to postpone the investment until any time desired thereby creating some degree of variability in the decision regarding the optimal number of wells to hold at any time. Second, it is possible, and is frequently practised, to convert an oil well to a water injection well at any time, again adding some variability to the decision regarding the optimal number of wells to hold at any particular time.

It is clear that to keep the model tractable, the assumption that an oil well is either "putty-putty" or "putty-clay" has to be maintained. Casual observation, however, suggests that an oil well is more correctly viewed as a quasi-fixed factor of production in which elements of both "putty" and "clay" are present and that either view will, at best, act as an approximation. While it is not possible to test the hypothesis that an oil well is variable against the alternative that it is fixed, the reasonableness of the empirical results for Model I versus those for Model II will be helpful in choosing one over the other.

m: The rate of water injection

This dependent variable is the quantity (m^3) of water observed to have been injected into a pool during 1973. Over the 80 pools in the sample, m ranges from its limit value of zero up to a maximum of 3.937 million m^3 of water. Its mean value of 214.0 thousand m^3 and standard deviation of 638.6 m^3 are deceiving measures because 61.2% of the observations on m occur at the limiting value of zero. The fact that only 38.8% of the pools in the sample were under water injection in 1973 is consistent with the results of theoretical analysis of optimal reservoir depletion in Chapter 3 where it was argued that an initial period of zero water injection is optimal under certain conditions on the production technology.

W: Water saturation

This exogenous variable is the percentage of the liquid volume of the reservoir that is water. It ranges in value from 5% to 50% with a mean and standard deviation of 21.35% and 11.18, respectively.

P: Pressure

This variable is measured in pounds per square inch absolute at the beginning of 1973. It ranges from 160 to 4600 PSIA with a mean value of 1841.4 and a standard deviation of 739.1.

θ: Change in pressure

The observed change in reservoir pressure during 1973, measured in PSIA, has a mean value of -6.0, a standard deviation of 176.91 and a range from -500 to +1000 PSIA. Note that, unlike the analytical model, θ here is not measured as the negative of pressure change. The control variable, θ, was indexed negatively in the analytical model to facilitate interpretation of the shadow price of pressure.

Q: Extraction rate of oil

This exogenous variable, measured in m^3, is the observed production of crude oil over the entire year of 1973. It ranges from 1922 to 2,926,100, has a mean of 182,150 and standard deviation of 490,960.

Z: Pay thickness

This is the average thickness of the oil bearing rock in the reservoir and is measured in metres. Its mean value across the reservoirs in the sample is 22.3 metres, it has a standard deviation of 28.57 and ranges in value from 1 to 100 metres.

w_1 and w_2: Factor prices

The rental price of an oil well is clearly a function of the depth to which it must be drilled to reach the reservoir and perhaps other location-specific factors such as rock hardness. These factors can therefore lead to a significant level of variation in the price of oil wells across pools.

The cost of drilling injection wells is equivalent to that of drilling production wells and is affected in the same way by depth and rock hardness as is the cost of production wells. Therefore, it will be argued that although there may be variation in the levels of factor prices across pools at a point in time, there is not variation in the ratio of factor prices across pools at a point in time. The assumptions required to draw this conclusion are made explicit in the following argument.

While it is possible to obtain data for the price of oil production wells, no such data exist for the price of water injection. One reason for this is that the true price of water injection is an unobservable shadow price as will be explained below. It is possible, however, to infer the value of this shadow price from available data.

It is common practice for water injection to occur through wells that have been converted from production to injection. Thus, the appropriate factor price for water injection in any time period includes a shadow price which depends upon the optimal number of conversions in the period. This is given by a simple first-order condition: production wells will be converted to injection wells up to the point that the marginal value product of production wells is just equal to the marginal value product of injection wells (net of conversion costs). Since the optimal number of production wells to hold at any point in time is determined by equating the marginal value product of the wells with the (market-determined) rental price, one can deduce that the appropriate factor price for injection wells is the rental price of oil production wells plus conversion costs. If one then assumes that the level of water injection per injection well is a constant so that $m = \alpha \cdot N_I$ where m = level of water injection, N_I = number of injection wells and α = constant, then the price of water injection relative to the price of oil wells is equal to $1/\alpha$, a constant across pools, assuming zero conversion costs.

Following these assumptions, the average value of α over the sample is $1.7 \times 10^8 m^3$. Because m has been scaled to have units of $10^8 m^3$ in the estimation, the price of water injection relative to the price of oil wells is therefore, 0.59. This information is not required to obtain the parameter estimates but it is needed, as will be seen, to compute the predicted values of the partial derivatives of the variable cost function and the predicted variable costs.

In order to compute the predicted variation in variable cost over reservoirs, additional assumptions must be made about the price-depth relationship. The assumption adopted here is the simplest one possible: the capital price per well is a linear function of depth as shown below.

$$w_1 = \overline{w}_1 \cdot \text{DEPTH}$$

where $\overline{w}_1$ is the observed capital price per metre of oil wells. This unit price is calculated by dividing total industry expenditures in Alberta in 1973 on development well drilling and related surface equipment by the total development metres drilled by the industry in 1973. The capital price can then be converted to a rental price by making assumptions about the average life of oil wells or one can simply normalize by setting the rental price per metre equal to unity. The data sources are described in Appendix B.

4.1.2. Econometric Issues

As was suggested above there are three econometric features of the system of equations in (4.3) that require special attention. These are discussed in increasing order of complexity in this section but the technical details are relegated to Appendix A.

Parameter Restrictions

It is a common characteristic of variable cost or variable profit functions which contain several fixed netputs that the implied system of factor demands satisfy several equality restrictions on the parameters in different equations.[3] In (4.3) the equality restrictions are clearly indicated: the γ_{ij} i=1,2,...,5; j=1,2,...,5 parameters are the same in the two equations. Thus , ordinary least-squares cannot be used to estimate the parameters and satisfy the restrictions. An iterative approach, combined with OLS would work but the method adopted here is maximum likelihood estimation. The SHAZAM[4] econometrics package easily accommodates this problem. The equality restrictions are imposed at each iteration.

Limited Dependent Variable Bias

The second econometric problem is more difficult than the first but is not new. Tobin (1958) first dealt with the statistical problems caused by a clustering of dependent variable observations at a limiting value in the context of a single-equation model. Wales and Woodland (1980) extended this to a more general case of a two-equation model. In either case, ordinary least squares will yield biased estimates because the assumptions of the linear regression model are not satisfied. In particular, a necessary assumption is that the expected value of each error term is equal to zero. However, this is not possible when the dependent variable assumes its limit value of zero, say, since only positive (and never negative) errors are possible due to the truncation from below. When the expected value of the errors is not zero, OLS is

3. See Diewert (1973) and (1974) for example.
4. White (1978)

known to yield biased estimates.

The solution to this problem proposed by Tobin was to obtain maximum likelihood estimates of the parameters, taking care to correctly specify the likelihood of a set of observations by taking account of the truncation from below. This will yield estimates which are consistent and efficient. The following is the modification of the system of equations in (4.3) used to eliminate the limited dependent variable bias.

Using the notation of Wales and Woodland (1980), rewrite the right-hand sides of the equations in (4.3) as $g(x;\ a_1,\gamma) + e$ and $h(x;\ a_2,\gamma) + u$, respectively. The estimation model is:

$$N = g(x;\ a_1,\gamma) + e$$

$$m = \begin{cases} 0 & \text{if } h(x;\ a_2,\gamma) + u < 0 \\ h(x;\ a_2,\gamma) + u & \text{if } h(x;\ a_2,\gamma) + u \geq 0 \end{cases} \tag{4.4}$$

This model requires that if the desired water injection rate is negative, it must be set at its minimum permissable value, zero. Let $F(v)$ be the cumulative unit normal distribution function and let $f(v)$ be the unit normal density function for a random variable, v. Let $\eta(u,e)$ be the joint normal density function for the error terms with covariance matrix Σ, as given above. Without loss of generality, order the data so that the first q observations are those for which m assumes the value zero and the remaining R-q observations are those for which m exceeds zero.

As is shown in Appendix A, the likelihood function for a sample of R observations is given by

$$L = \prod_{i=1}^{q} a_e f(a_e e_i)\cdot F(y_i)\cdot \prod_{j=q+1}^{R} \eta(u_j,e_j) \tag{4.5}$$

where $a_e = 1/\sigma_e$

$a_u = 1/\sigma_u$

$$y_i = -[h_i(x;\ a_2,\gamma)\cdot a_u + e_i\rho a_e]/(1-\rho^2)^{1/2}$$

The objective is to maximize (the logarithm) of (4.5) with respect to a_e, a_u, ρ, and the parameters of the variable cost function. Even though the estimation equations in (4.4) are linear in the parameters, the first derivatives of the logarithm of the likelihood function are non-linear making an analytical solution impossible. Hence, a numerical, non-linear optimization routine must be used.[5] The Hessian matrix of the likelihood function, evaluated at its maximum, yields standard errors and t-statistics, for the parameters, which are asymptotically valid for hypothesis testing.

Simultaneity Bias

The third econometric issue that must be addressed is that of the potential for simultaneity bias in the estimation of the variable cost function parameters. The source of this potential bias is the endogeneity of θ, the change-in-pressure variable, in the dynamic cost minimization problem and its treatment as an exogenous variable in the variable extraction cost function. As will be shown, there is good reason to suspect that θ is correlated with exogenous variables not present in the

5. University of British Columbia's "Monitor for Nonlinear Optimization," a routine which permits the user to call any one of a variety of optimization routines and to monitor its performance interactively, was used to maximize (4.5). The optimization routines that were used are FLETCH and FNMIN, both of which are quasi-Newton methods. Analytical derivatives were not supplied, but were numerically computed by the Monitor.

estimation equations. The effects of these excluded variables could be entering through the error terms thereby leading to a correlation between θ and the errors and this, in turn leads to estimation bias.

The variable extraction cost function includes θ as an argument by design. While explicit solutions for dynamic factor demands are obtainable in the context of infinite-time horizon models of the firm, this is not the case in the context of a finite-time horizon model of the extractive firm. Instead, it is impossible to obtain an explicit solution for θ in an endogenous and finite time horizon model.[6] For this reason, θ is held constant so that the variable cost function can be explicitly defined.

It is possible, however, to obtain an explicit solution for θ under certain conditions: first, the objective function must be quadratic or linear and second, the time horizon must be assumed to be exogenous.[7]

If a reduced form equation for θ can be obtained, it can be used to eliminate the source of potential simultaneity bias in the regresion model in (4.4). The remainder of this section, then, is devoted to the derivation of a reduced form equation for θ as a function of an assumed, exogenous time horizon.

It is assumed that the objective of the reservoir manager is to minimize the present value of the cost of producing an exogenously determined, constant output rate, Q, over a finite and exogenous time horizon, T and that all prices are constant. Formally, the problem is to

6. The assumption of a quadratic objective function implies exhaustion must occur in finite time.
7. In essence, this makes the problem of solving for θ similar to the problem of solving for dynamic factor demands in the infinite time horizon models of the firm where quadratic objective functions are also employed. The practical difference created by the finiteness of the time horizon, however, is substantial as can be verified by looking at Appendix A.

$$\underset{\{\theta\}}{\text{Minimize}} \int_{o}^{T} e^{-\delta t} c(w_1, w_2; W, P, \theta, Q, Z)\,dt \tag{4.6}$$

$$\text{subject to } \dot{P} = -\theta$$
$$P(0) = P_0$$
$$P(t) \geq 0$$

Time subscripts have been and will continue to be suppressed where it does not cause ambiguity. In (4.6), only θ and P are functions of time. Factor prices, which are constant, are suppressed in the following. The Hamiltonian function is

$$H = e^{-\delta t}\{c(W,P,\theta,Q,Z) - \lambda\theta\}$$

Assuming an interior solution exists, the following conditions must hold at every point in time to minimize H:

$$C_\theta - \lambda = o \tag{4.7}$$
$$\dot{\lambda} - \delta\lambda = -C_p$$

where the arguments of the cost function have been suppressed. Equations (4.7) determine the optimal time paths of P and θ, given any starting or finishing points for λ. An optimal finishing point is to have the level of pressure such that its shadow price is zero at time T. This requirement is written as condition (4.8)

$$\lambda(T) = 0 \tag{4.8}$$

Together, these three conditions completely solve the problem given $P(0) = P_0$ and $\dot{P} = -\theta$. However, to obtain an explicit solution is difficult even when the objective function is quadratic.

The procedure for obtaining the explicit solution is based on solving a system of simultaneous linear differential equations. This system is obtained in the following way. Since $C(\cdot)$ is quadratic, the first equation of (4.7) can be used to explictly solve for θ as a function of λ and P, say $\theta(\lambda,P)$. Using $\dot{P} = -\theta$ and the second equation in (4.7) the system of linear differential equations becomes:

$$\begin{aligned} \dot{\lambda} &= \delta\lambda - C_p[W,P,\theta(\lambda,P),Q,Z] \\ \dot{P} &= -\theta(\lambda,P) \end{aligned} \qquad (4.9)$$

The endogenous variables are λ and P which, because the equations in (4.9) are linear in λ and P, can be explicitly solved as functions of time, P_0, and λ_0, the initial shadow price. This puts the solution for θ in terms of an unobservable variable, λ_0. This variable can be eliminated, however, by making use of the transversality condition. Since (4.9) can be used to solve for $\lambda(t)$ as a function of λ_0, it also yields $\lambda(T)$ as a function of λ_0. Since $\lambda(T)$ must equal zero, λ_0 can be solved for as an explicit function of T. Substitution of this result then yields $\lambda(t)$ as an explicit function of T and therefore provides the explicit solution for θ as a function of T, P_0, all of the constant variables and all of the parameters of the quadratic objective function.

This exercise, which is carried out in Appendix A, yields the following reduced form equation for $\theta(t)$:

$$-\theta(t) = A_1\lambda(t) + B_1P(t) + C_{11}W + C_{12}Q + C_{13}Z + C_{14} \qquad (4.10)$$

where,

$$\lambda(t) = h_o(P_o - \Omega_1)\{e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}\} +$$
$$\Omega_2\{B_1[(r_1-A_1)e^{r_1t}-(r_2-A_1)e^{r_2t}]-B_2[(r_1-A_1)e^{r_1t}-(r_2-A_1)e^{r_2t}]\}/ \qquad (4.11)$$
$$\{B_1[(r_1-A_1)e^{r_1T}-(r_2-A_1)e^{r_2T}]\}$$

$$\text{and } \Omega_1 = h_{11}W + h_{12}Q + h_{13}Z + h_{14}$$

$$\Omega_2 = h_{21}W + h_{22}Q + h_{23}Z + h_{24}$$

All of the parameters that appear in (4.10) and (4.11) are functions of the structural parameters of the variable cost function and the exact relationships are written out in Appendix A.

The reduced form equation for $\theta(t)$ in (4.10) is a non-linear function of the parameters of the variable cost function, the initial level of reservoir pressure, P_o, the exogenous time horizon T, the current "age" of the reservoir, t, and the other three exogenous variables of the model: W, Q, and Z.

It was stated earlier that there is reason to suspect that θ is correlated with exogenous variables that do not appear in the regression model in which θ is taken to be exogenous, thereby creating the potential for simultaneity bias. The source of this suspicion is made clear in (4.10) and (4.11) where it is apparent that the optimal choice for θ at time t depends, in a systematic way, on three variables, t, T and P_o, that do not appear in the regression model in which θ is taken as exogenous. Thus, the potential does exist for θ to be correlated with the residuals in that regression model.

At the conceptual level, the reduced form equation for θ in (4.10) can be estimated simultaneously with the equations for the original two dependent variables, N and m, in (4.3) or (4.4), by imposing the complicated parameter restrictions across equations that are written in Appendix A. This procedure will yield consistent estimates of the structural parameters. At a practical level, however, this procedure is unlikely to be successful given the large number of parameters and the non-linearity of the estimation equations combined with the additional non-linearity of the likelihood function created by the limited dependent variable problem. Moreover, the benefit probably does not justify the cost of carrying out this procedure when one considers the following alternative.

At a cost of not being able to obtain estimates of the structural parameters but only reduced form parameters, the non-linear parameter restrictions need not be imposed. The simultaneity bias problem is still eliminated and the reduced estimation cost is probably well worth the sacrifice of information implied by this procedure.

Without imposing parameter restrictions across equations, only the distinct parameters of the reduced form equation for $\theta(t)$ can be estimated. As is shown in Appendix A this yields the following highly simplified version of the reduced form equation:

$$-\theta(t) = \alpha_0[e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}](P_0-W-Q-Z-1)$$
$$+ \alpha_1 W + \alpha_2 Q + \alpha_3 Z + \alpha_4 + B_1 P \quad (4.12)$$

The next step that might be considered is to estimate the resulting unrestricted, three-equation system. This is unlikely to be successful however, because of the very large number of parameters. An alternative is to make the substitution of (4.12) into (4.3) or (4.4) to eliminate $\theta(t)$ and then to estimate the resulting reduced form parameters. The resulting reduced form equations are quadratic as before, but no longer contain θ and cannot be used to identify the structural parameters.

Redefine X_3 in the following way:

$$X_3 = \{e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}\}(P_0-W-Q-Z-1)$$

Then substitution of (4.12) into (4.3) yields the following estimation equations:

$$\begin{aligned} N &= \xi_{01} + \sum_{j=1}^{5} \xi_{1j}X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \psi_{ij}X_iX_j + e_1 \\ m &= \xi_{02} + \sum_{j=1}^{5} \xi_{2j}X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \psi_{ij}X_iX_j + u_1 \end{aligned} \tag{4.13}$$

The practical difference between the system in (4.13) and that in (4.3) is that the former is non-linear in the parameters and contains 4 additional parameters, ξ_{01}, ξ_{02}, β_1, and β_2 and 4 additional variables, a constant, P_0, t, and T.

Although the structural parameters cannot be identified this is not a serious concern: the parameters in (4.13) will still yield estimates of the effects of the exogenous variables on extraction costs and the estimates are free of simultaneity bias. The drawback of the system in (4.13) is the requirement of observations on T which has been treated as an exogenous variable. Since T really is an endogenous variable one might reasonably argue that while one source of simultaneity bias has been eliminated by substituting θ out of the model, another has been introduced by including T in the model. While this is true, if one were able to solve for T, it would be a function only of the exogenous variables already included in the regression model. Thus, there is less reason to suspect T to be systematically correlated with the residuals.

Direct observations on T are not available. This information is inferred from other data. In particular, the variable T is approximated by dividing observations on reported recoverable oil reserves by Q, the observed 1973 production of oil. This procedure undoubtedly leads to measurement errors as is obvious from the fact that the maximum value of T calculated in this way is 712 years. However, the mean value of 34.05 years is very reasonable as is the minimum value of 6 years and standard deviation of 78.5 years. One is forced to assume the measurement errors are random and indeed, there is no reason to suspect otherwise.

The simplest way of dealing with the simultaneity bias problem is a "two-stage" approach in which the ordinary least-squares predicted values of θ obtained from a regression of θ on all of the exogenous variables in the system are used in place of the observed values of θ in the two-equation system for N and m. The predicted values of θ are uncorrelated with the error terms in the equations for N and m. Hence, the maximum-likelihood estimates of the parameters in these two equations using the predicted values of θ are consistent.

Finally, it should be noted that the full system of three equations for θ, N and m is block recursive in that the first block consisting of one equation for θ depends only on exogenous variables and the second block consisting of the two equations for N and m depends only on exogenous variables plus an endogenous variable, θ, determined in the first block. Thus, if the error term in the first block is uncorrelated with the error terms in the second block, the simultaneity problem vanishes. Thus, maximum likelihood estimates of the parameters in the two-equation system for N and m, using observed values for θ, would be consistent.

4.2. Model II

As an alternative to treating an oil production well as a variable factor of production, the model is reformulated here under the assumption that an oil well is a fixed factor, the number of which is optimally chosen only during the initial development phase of the oil pool. For analytical purposes, this phase is assumed to occur instantaneously at t=0. For practical purposes in data collection, this development phase is permitted to take up to 5 years.

As was stated in the introduction to this Chapter, after setting the factor price for water injection to one, the variable cost function of Model II is equivalent to a factor requirements function which shows the quantity of water injection required to produce some output rate Q, given θ, N and the other fixed factors. It is non-increasing in arguments which are inputs and non-decreasing in arguments which are outputs. The following quadratic function is specified for the factor requirements function.

$$m(x) = b_o + \sum_{i=1}^{6} b_i X_i + 1/2 \sum_{i=1}^{6} \sum_{j=1}^{6} g_{ij} X_i X_j \qquad (4.14)$$

where, X_1 = water saturation level, W

X_2 = pressure, P

X_3 = pressure change, θ

X_4 = extraction rate, Q

X_5 = pay thickness, Z

X_6 = number of oil wells, N.

The parameters of (4.14) can be estimated directly using the one-equation version of the limited dependent variable model already described. This eliminates the limited dependent variable bias but not the source of potential simultaneity bias. The latter can be handled in exactly the same way as it was for Model I. The objective of the reservoir manager is assumed to be to minimize the present-value of the cost of producing a constant, exogenous output stream over an exogenous time period:

$$\underset{\{\theta,N\}}{\text{Minimize}} \int_0^T e^{-\delta t} m(W,P,\theta,Q,Z,N)dt + \phi \cdot N \tag{4.15}$$

$$\text{subject to} \quad \begin{aligned} \dot{P} &= -\theta \\ P(0) &= P_0 \\ P(t) &\geq 0 \\ N &\geq 0 \end{aligned}$$

where ϕ is the market price of an oil well. The Hamiltonian for this problem is:

$$H = m(W,P,\theta,Q,Z,N) - \lambda\theta \tag{4.16}$$

Except for the addition of N in the Hamiltonian, (4.16) is identical to the Hamiltonian for Model I associated with the minimization problem (4.6). Since N is exogenous in (4.16), the solving of the reduced form for $\theta(t)$ is carried out in precisely the same way and the solution appears exactly the same except for the addition of one exogenous, constant. Thus the reduced form is

$$-\theta(t) = \alpha_0[e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}](P_0-W-Q-Z-N-1) + \alpha_1 W + \alpha_2 Q + \alpha_3 Z + \alpha_4 N + \alpha_5 + B_1 P \qquad (4.17)$$

where, although the notation is the same, the parameters in (4.17) are not the same as those in (4.12).

To find the reduced form equation that is to be estimated, redefine X_3 as

$$X_3 = [e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}](P_0-W-Q-Z-N-1)$$

Substitution of $\theta(t)$ into (4.14) then yields:

$$m(x) = \Gamma_0 + \sum_{i=1}^{6} \Gamma_i X_i + 1/2 \sum_{i=1}^{6} \sum_{j=1}^{6} \Gamma_{ij} X_i X_j + e_2 \qquad (4.18)$$

which is the reduced form estimation equation after the addition of error term e_2, a random normal variable with zero mean and constant variance.

The Optimal Number of Wells

Just as it was possible to find an explicit solution to the dynamic optimization problem, it is possible, although more difficult, to solve (4.15) for the optimal level of initial investment in N, the number of oil wells. The resulting reduced form parameters can be estimated with available data.

Assuming an interior solution to the minimization of (4.15) with respect to N, the following first-order condition must hold at an optimum:

$$\partial/\partial N \int_0^T e^{-\delta t} m(W,P,\theta,Q,Z,N)dt + \phi = 0 \tag{4.19}$$

If the first stage of the minimization problem has already been carried out so that $\theta(t)$ and $P(t)$ are known, then (4.19) can be explicitly solved for N as long as the objective function is quadratic.

The solution of (4.19) is carried out in Appendix A. However, because it is known that the structural parameters of (4.19) will not be identified in the reduced form equation for N, no attempt has been made in the derivation to preserve them.

The resulting reduced form equation determining the optimal number of wells is given below.

$$\begin{aligned} N = \phi/D + \; & W(f_{10}q_0 + f_{11}q_1 + f_{12}q_2 + f_{13}q_3 + f_{14}q_4 + f_{15})/D \; + \\ & Q(f_{20}q_0 + f_{21}q_1 + f_{22}q_2 + f_{23}q_3 + f_{24}q_4 + f_{25})/D \; + \\ & Z(f_{30}q_0 + f_{31}q_1 + f_{32}q_2 + f_{33}q_3 + f_{34}q_4 + f_{35})/D \; + \\ & P_0(f_{51}q_1 + f_{52}q_2 + f_{53}q_3 + f_{54}q_4 + f_{55})/D \; + \\ & (f_{60}q_0 + f_{61}q_1 + f_{62}q_2)/D \end{aligned} \tag{4.20}$$

where,

$$D = -\{f_{40}q_0 + f_{41}q_1 + f_{42}q_2 + f_{43}q_3 + f_{44}q_4 + f_{45}q_5\}$$

and,

$$q_0 = (1 - e^{-\delta T})/\delta$$

$$q_1 = [e^{(r_1-\delta)T}-1]/(r_1-\delta)$$

$$q_2 = [e^{(r_2-\delta)T}-1]/(r_2-\delta)$$

$$q_3 = [e^{(2r_1-\delta)T}-1]/(2r_1-\delta)$$

$$q_4 = [e^{(r_1+r_2-\delta)T}-1]/(r_1+r_2-\delta)$$

$$q_5 = [e^{(2r_2-\delta)T}-1]/(2r_2-\delta)$$

The parameters to be estimated are r_1, r_2 and the f_{ij}. The fact that the relative price of oil wells enters the regression equation correctly suggests that a different data set is used to estimate the parameters of (4.20). The data set to be used here is somewhat larger than that used in previous regressions but the main difference is that the observations on the dependent variable N are now dated. An observation on the variable N now includes the number of wells drilled in the initial development of the pool[8] and the date at which the development began. The sample includes most of the 80 observations used in the previous

8. Some pools in the sample had two or more distinct development phases separated by several years of a constant N. In these cases, only the initial development period was used as the observation.

regressions[9] plus additional observations on pools developed up to and including 1975 for a total of 110 observations. Table I shows the total number and the percentage of observations occuring at each date, the average number of wells per pool, N, and the average well depth per pool in metres $\overline{DEP}$.

To calculate ϕ_0, the relative price of oil wells at the time N is chosen, it is assumed that the price of water injection in year 0, is expected to remain at that level for all subsequent periods. The assumptions required to calculate a price series for water injection and the method for actually calculating ϕ are made clear in the following chapter. The data sources are documented in Appendix B.

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9. Excluded were pools which had no distinct development phase but displayed a steady and gradual rise in N over time and pools which had no development phase but produced with only the original discovery well. Approximately 20% of the pools fell into these categories.

TABLE I

DATE	NO.	%	NTOT	$\overline{N}$	$\overline{DEP}$
1963	6	5.4	149	24.8	4611.8
1964	9	8.2	343	38.1	1883.2
1965	11	10.0	276	25.1	1449.5
1966	19	17.3	127	6.7	2591.9
1967	10	9.1	56	5.6	1631.2
1968	10	9.1	89	8.9	1309.9
1969	8	7.3	54	6.8	1644.5
1970	4	3.6	65	16.3	1709.3
1971	6	5.4	44	7.3	1241.4
1972	6	5.4	28	4.7	3060.0
1973	3	2.7	76	25.0	1397.1
1974	5	4.5	157	31.4	1393.5
1975	1	0.9	3	3.0	1095.5
1976	7	6.4	281	40.1	1278.3
1977	5	4.5	96	19.2	846.1

NO. = NUMBER OF OBSERVATIONS

% = PERCENTAGE OF TOTAL NUMBER OF OBSERVATIONS

NTOT = TOTAL NUMBER OF WELLS DRILLED

$\overline{N}$ = AVERAGE NUMBER OF WELLS DRILLED PER POOL

$\overline{DEP}$ = AVERAGE WELL DEPTH PER POOL (METRES)

CHAPTER 5

Empirical Results

5.0 Introduction

In the previous chapter, two empirical models of variable extraction costs were formulated, the estimation equations associated with each model were derived and the estimation problems and procedures were described. Recall that in Model I, the estimation equations consist of a system of two equations determining the two variable factors of production at time t (the number of oil wells and the rate of water injection). Model II, which differs in that oil wells are assumed to be fixed rather than variable factors of production, consists of one equation determining the one variable factor of production at time t (the rate of water injection). An additional but independently estimated reduced form equation in Model II is that determining the optimal number of oil wells to be drilled in the initial time period, t=0.

In this chapter, the empirical results obtained from estimating the parameters of these two models are presented and analyzed. The remaining part of the chapter is organized into 4 sections. In Section 5.1 the results obtained for Models I and II when θ is treated as an exogenous variable are presented and analyzed. The effects of limited dependent variable bias are examined in this section in that two sets of results are presented for Model I: the first set is obtained by ignoring the limited dependent variable problem and the second set is obtained from consistent, maximum likelihood estimation. In Section 5.2, the results obtained for

Models I and II when θ is treated as an endogenous variable are presented and analyzed. Section 5.3 contains the results of estimating the reduced form equation from Model II determining the optimal number of oil wells to be drilled at t=0. The chapter is concluded in Section 5.4 with a summary of the empirical findings and the implications for (i) the optimal depletion strategy of oil reservoirs and (ii) the extent and determinants of extraction cost heterogeneity among the oil reservoirs in the sample.

5.1 The Variable Extraction Cost Function: Exogenous θ

The complete set of parameter estimates for Models I and II are presented in Table III. The first column contains the estimates of Model I parameters obtained when the limited dependent variable problem is ignored. The second column contains estimates of the same parameters obtained by maximum likelihood estimation which takes account of the limited dependent variable problem. The third column of Table III shows the parameter estimates for Model II obtained by maximum likelihood estimation which takes account of the limited dependent variable problem. In each column, the asymptotic t-statistics are shown in parentheses beside the associated parameter estimate and the logarithm of the value of the maximized likelihood function, L, is printed at the bottom of the column.[1]

1. Because Model II was estimated using the SHAZAM econometrics package, the reported t-statistics are actually the t-statistics for the "normalized" regression coefficients of Tobin's (1958) original article. The normalization is obtained by dividing all coefficients by the standard deviation of the residuals. These t-statistics can, nevertheless, be used to perform the standard (asymptotically valid) hypotheses tests.

To facilitate the interpretation of the results in Table III, the estimation models associated with each of the columns in Table III are summarized in Table II. Note that the subscripts denoting the observation index have been suppressed but that each oil pool provides one observation and that all 80 observations occur in the year 1973. Also not shown in Table II is the fact that for estimation purposes, the data terms in the equations were scaled to make the parameter estimates of similar orders of magnitude.

Limited Dependent Variable Bias

Comparison of the estimates in Columns 1 and 2 of Table III demonstrates the effect of the limited dependent variable bias which is inherent in the estimates of Column 1. One would expect the maximum likelihood estimates in Column 2 to be considerably different from those in Column 1 due to the large number of observations which occur at the limit value. Visual inspection of the first two columns of Table III substantiates this expectation.

In Column 2, 12 parameter estimates are lower and 13 are higher in value than those in Column 1. The average percentage change in values is 1088.3%. The largest decrease is -3811.4% (a_{25}), the largest increase is 27,559.3% (γ_{15}) and the smallest change is -3.4% (a_{14}).

One can use the asymptotically valid likelihood ratio (LR) test to perform a test of the hypothesis that the parameters in the two columns are equal. This test was performed by computing the log of the likelihood function (used in estimating the parameters in Column 2) when all of the parameters (except those in the covariance matrix) are restricted to equal the values contained in Column 1. This produced a function value of

Table II

Summary of Estimation Models

1. MODEL I (column 1 of Table 5.1)

$$N = g(x,a_1,\gamma) + e = a_{11}W + a_{12}P + a_{13}\theta + a_{14}Q + a_{15}Z + 1/2\gamma_{11}W^2$$
$$+ \gamma_{12}WP + \gamma_{13}W\theta + \gamma_{14}WQ + \gamma_{15}WZ + 1/2\gamma_{22}P^2 + \gamma_{23}P\theta + \gamma_{24}PQ + \gamma_{25}PZ$$
$$+ 1/2\gamma_{33}\theta^2 + \gamma_{34}\theta Q + \gamma_{35}\theta Z + 1/2\gamma_{44}Q^2 + \gamma_{45}QZ + 1/2\gamma_{55}Z^2 + e$$

$$m = h(x,a_2,\gamma) + u = a_{21}W + a_{22}P + a_{23}\theta + a_{24}Q + a_{25}Z + 1/2\gamma_{11}W^2$$
$$+ \gamma_{12}WP + \gamma_{13}W\theta + \gamma_{14}WQ + \gamma_{15}WZ + 1/2\gamma_{22}P^2 + \gamma_{23}P\theta + \gamma_{24}PQ + \gamma_{25}PZ$$
$$+ 1/2\gamma_{33}\theta^2 + \gamma_{34}\theta Q + \gamma_{35}\theta Z + 1/2\gamma_{44}Q^2 + \gamma_{45}QZ + 1/2\gamma_{55}Z^2 + u$$

2. MODEL II (column 2 of Table 5.1)

$$N = g(x,a_1,\gamma) + e$$

$$m = \begin{cases} h(x,a_2,\gamma) + u & \text{if} \quad h(x,a_2,\gamma) + u \geqq 0 \\ 0 & \text{if} \quad h(x,a_2,\gamma) + u < 0 \end{cases}$$

3. MODEL II (column 3 of Table 5.1

$$m = b_0 + b_1W + b_2P + b_3\theta + b_4Q + b_5Z + b_6N + 1/2g_{11}W^2 + g_{12}WP$$
$$+ g_{13}W\theta + g_{14}WQ + g_{15}WZ + g_{16}WN + 1/2g_{22}P^2 + g_{23}P\theta + g_{24}PQ + g_{25}PZ$$
$$+ g_{26}PN + 1/2g_{33}\theta^2 + g_{34}\theta Q + g_{35}\theta Z + g_{36}\theta N + 1/2g_{44}Q^2 + g_{45}QZ$$
$$+ g_{46}QN + 1/2g_{55}Z^2 + g_{56}ZN + 1/2g_{66}N^2$$

Table III

Model I and Model II Parameter Estimates: Exogenous θ

	1. Model I		2. Model I			3. Model II	
a_{11}	0.2395	(1.75)	2.5448E-2	(0.13)	b_1	-0.1560	(-0.97)
a_{12}	0.3159	(1.39)	0.7177	(1.69)	b_2	0.5574	(2.35)
a_{13}	0.2264E-1	(0.16)	6.1258E-2	(0.31)	b_3	0.1073	(0.87)
a_{14}	1.1781	(10.03)	1.1381	(4.40)	b_4	-0.1820	(-0.65)
a_{15}	-0.3745	(-3.67)	-0.5795	(-2.49)	b_5	-0.3301	(-2.20)
a_{21}	-0.2749E-1	(-0.51)	-0.2696	(-1.83)	b_6	-0.1055	(-0.68)
a_{22}	-0.6743E-1	(-0.59)	0.2754	(0.74)	g_{11}	0.6013E-1	(1.45)
a_{23}	-0.9154E-1	(-1.38)	-5.3030E-2	(-0.33)	g_{12}	-0.8929	(-2.15)
a_{24}	0.6851	(6.16)	0.6553	(2.57)	g_{13}	0.6274	(2.09)
a_{25}	-0.5461E-2	(-0.08)	-0.2136	(-0.96)	g_{14}	1.6322	(2.15)
γ_{11}	0.2309E-2	(0.11)	7.8638E-2	(1.52)	g_{15}	0.1259	(3.03)
γ_{12}	0.2006	(0.95)	-0.1344	(-0.23)	g_{16}	-0.8616	(-0.32)
γ_{13}	0.3475	(1.97)	0.6638	(1.62)	g_{22}	-3.8696	(-2.58)
γ_{14}	-0.9640	(-2.75)	-0.6190	(-0.85)	g_{23}	-1.5633	(-2.69)
γ_{15}	0.3484E-1	(2.08)	9.6365	(1.72)	g_{24}	0.5839	(0.74)
γ_{22}	0.4518	(0.81)	-1.3592	(-0.64)	g_{25}	0.2997	(0.86)
γ_{23}	0.3367E-1	(0.16)	-0.4461	(-0.55)	g_{26}	10.884	(1.65)
γ_{24}	-1.6335	(-5.97)	-1.3336	(-1.87)	g_{33}	-4.5317	(-3.17)
γ_{25}	-0.4035	(-2.05)	-0.4409	(-0.80)	g_{34}	0.1332	(1.01)
γ_{33}	-1.1502	(-1.33)	-1.6662	(-0.58)	g_{35}	3.8046	(3.56)
γ_{34}	-0.2583E-1	(-0.25)	8.0319E-2	(0.37)	g_{36}	0.2353	(1.31)
γ_{35}	1.7009	(2.40)	2.1293	(0.97)	g_{44}	0.5793E-1	(0.79)
γ_{44}	0.8800E-1	(1.79)	-8.0732E-3	(-0.08)	g_{45}	0.2477E-2	(0.03)
γ_{45}	-0.2209	(-4.06)	-0.2834	(-2.64)	g_{46}	-3.1498	(-2.40)
γ_{55}	1.5961	(2.82)	4.7496	(2.77)	g_{55}	4.2358	(3.51)
					g_{56}	-0.2958	(-4.59)
L	-459.82		-419.047		g_{66}	2.8005	(1.36)
					b_0	0.9464	(0.36)
					L	-59.57	

Column 1: Model I: Limited dependent variable problem (LDV) ignored.
Column 2: Model I: Maximum likelihood taking account of LDV problem.
Column 3: Model II: Maximum likelihood taking account of LDV problem.
Asymptotic t-statistics in parentheses.

-440.880 which is smaller than the maximum value of -419.047. The LR test statistic is λ' where

$$\lambda' = 2[\ln(L_0) - \ln(L_1)]$$

where $\ln(L_0)$ and $\ln(L_1)$ are the logarithms of the likelihood functions under the null (restricted) hypothesis and alternative (unrestricted) hypothesis, respectively. The test statistic, λ', is asymptotically distributed as a $\chi^2(K)$ where K = the number of restrictions. The LR test statistic for the current hypothesis test is 43.66 which exceeds the critical value for $\chi^2(25)$ of approximately 37.6 at the 5% level of significance as well as the critical value of 40.6 at the 1% level of significance. Thus, the null hypothesis that the parameters in Columns 1 and 2 are equal can be rejected.

It is clear that the statistical effect of the limited dependent variable bias is to cause large changes in the individual parameter estimates and a significant reduction in the likelihood of the sample. Moreover, the estimates in Column 1 are inferior on economic grounds as will be shown in the discussion of regularity conditions.

Hypothesis Tests on the Exogenous Variables

The statistical significance of the individual exogenous variables is investigated with the LR test. The test of the hypothesis that an individual exogenous variable, X_i, has no influence on variable extraction costs is a test of the joint null hypotheses that each of the parameters in the following partial derivative are equal to zero:

Model I: $\partial c(w_1, w_2; X)/\partial X_i = \sum_{j=1}^{2} a_{ji} w_j X_i + \sum_{h=1}^{2} \sum_{j=1}^{5} \gamma_{ij} X_j w_h$

Model II: $\partial m(X)/\partial X_i = b_i + \sum_{j=1}^{6} g_{ij} X_j$

The test on an individual X_i is carried out by maximizing the likelihood function subject to the restrictions that the above parameters are equal to zero. The LR test statistics obtained from doing this for both models (including both sets of Model I estimates) and for each of the X_i are summarized in Table IV.

The critical $\chi^2(7)$ at a 5% significance level is 14.07. Thus, all null hypotheses can be rejected in Model II and all but one can be rejected in Model I. There is, therefore, agreement among all estimates that the controllable netputs-extraction rate, pressure, and number of oil wells in Model II - have significant effects on variable extraction costs, and the natural netputs - pay thickness and the water saturation level of reservoirs - also have significant effects on variable extraction costs. There is conflict between the results of Models I and II, however, over the significance of the effect of the controllable netput, θ (change-in-pressure).

Acceptance of the null hypothesis in Model I that θ does not affect variable extraction costs is a surprising and undesireable result. It suggests that while the level of reservoir pressure affects variable costs, the rate at which pressure is diminished does not. It implies that there is no unique solution to the dynamic cost minimization problem. Moreover, it implies that in the production relation which is dual to the variable cost function, θ has no impact on the rate of oil production.

TABLE IV

Likelihood Ratio Test Statistics

	Column of Table III	Water Saturation	Pressure	Change In Pressure	Extraction Rate	Pay Thickness	No. of Wells
MODEL I	1	20.64	44.18	8.92	278.18	59.08	---
MODEL I	2	19.42	18.22	7.76	180.29	44.47	---
MODEL II	3	25.44	14.96	20.00	101.52	53.04	47.84

The principles of oil reservoir engineering, outlined in Chapter 2, suggest that this result is incorrect.

Two arguments can be put forward to explain the conflicting results concerning the effect of θ on variable extraction costs. The first is that multicollinearity between θ and the other exogenous variables may exist thereby causing the effect of θ to appear insignificant. From Chapter 4 it is known that the reduced form equation for θ (obtained from solving the dynamic minimization problem) includes some terms which are linear in the other exogenous variables of the model. There is, therefore, justification for suspecting the existence of multicollinearity. However, the strength of this argument is diminished by the fact that θ is statistically significant in Model II. If there did exist a serious multicollinearity problem, its effects ought to be observed in Model II as well as Model I. The fact that they are not suggests a second argument.

If the true model of variable extraction costs is Model II where an oil well is a fixed factor of production, then the dependent variable N in Model I will not be influenced by θ, the observed pressure change in one year of a reservoir's life. Because of the across-equation restrictions in the estimation of the two factor demands in Model I, the two equations are not independent. It is therefore possible that the known dependence of water injection on θ was not sufficient to overcome the possible independence of the number of wells on θ.

It was stated earlier that it is not within the scope of the thesis to perform formal testing of hypotheses regarding the nature of oil wells as factors of production, fixed or variable. However, the above analysis suggests that, on the grounds of the reasonableness of the results analyzed so far, Model II is preferable to Model I.

The likelihood ratio tests have indicated that all of the arguments of the variable cost function (with the exception of θ in Model I) have a significant effect on the variable cost of extracting oil. The qualitative nature of each effect is determined by computing the predicted values for each of the partial derivatives of the variable cost function with respect to its arguments. This exercise must also be carried out to determine whether the function that has been estimated is indeed a variable cost function.

Regularity Conditions

The necessary conditions for the estimated functions in Models I and II to be a variable cost function and a factor requirements function, respectively are that they be increasing or decreasing in their arguments depending on whether they are outputs or inputs, respectively. Each partial derivative of a quadratic function depends upon the values of various parameters and the values of all of the exogenous variables. Thus, the regularity conditions cannot be globally satisfied but should be satisfied at least in the neighbourhood of the sample means of the exogenous variables or preferably at all data points.

The sufficient conditions for the estimated function in Model I to be a variable cost function are automatically satisfied. A common sufficient condition is that a cost function be concave in input prices. Because the effects of input prices are not discernable in the cross-section sample used to generate the parameter estimates in Table III, this condition cannot be violated. An additional sufficiency condition is that a variable cost function be convex in the vector of fixed netputs if the underlying technology exhibits constant returns to scale. However, since the possibility of increasing returns to scale is not ruled out in the empirical models, this is not a condition that must be satisfied. Thus, only the first-order regularity conditions, explained below, must be satisfied.

Inspection of the variable cost function for Model I in Chapter 4 or the factor demands for Model I in Table II indicates that each partial derivative with respect to one of the X_i exogenous variables has the following form:

$$C_i = a_{1i}w_1 + a_{2i}w_2 + (w_1+w_2)\sum_{j=1}^{5} \gamma_{ij}X_j \tag{5.1}$$

where C_i denotes the partial derivative of the cost function with respect to X_i.

To satisfy the first-order regularity conditions, the estimated function must be non-increasing in inputs and non-decreasing in outputs. Since it is only the sign of the partial derivative that need be known, the dependence of equation (5.1) on the level of factor prices can be eliminated by dividing through by (w_1+w_2). This yields normalized partial derivatives, the signs of which depend only upon relative factor prices in the observation year. Following the arguments made in Chapter 4, $w_1/w_2 = 1.69$ so that $w_1/(w_1+w_2) = 0.63$ and $w_2/(w_1+w_2) = 0.37$. It is these normalized partial derivatives that are calculated to determine the qualitative effects of the exogenous variables.

For Model II, the partial derivatives depend only upon the price of water injection. Setting this price equal to unity then yields the normalized partial derivatives for Model II.

These partial derivatives were calculated at all data points and at sample means for the functions associated with Table III. Table V below contains the predicted values of the normalized partial derivatives at sample means for each of the three columns in Table III.

Table V

Normalized Partial Derivatives at Sample Means

Model	Column of Table III	C_W	C_P	C_θ	C_Q	C_Z	C_N
Model I	1	-1.51	0.32	-0.17	0.61	-2.66	--
Model I	2	0.16	0.10	0.14	0.52	-0.27	--
Model II	3	0.35	0.01	0.10	0.23	-0.36	-0.57

All partial derivatives should be positive with the exception of C_Z, C_N and possibly C_P which can be positive or negative for the reasons given below. Thus, the function estimated in column 1 of Model I does not satisfy the regularity conditions for a variable cost function. Conversely, the function estimated in Column 2 of Model I (elimination of limited dependent variable bias) does satisfy the regularity conditions. Similarly, the variable cost or factor requirements function of Model II satisfies all regularity conditions at sample means.

That the parameter estimates of Model I in column 1 (limited dependent variable problem ignored) are inferior on economic grounds to those in column 2 of Table III (limited dependent variable bias eliminated) is apparent from the above discussion of the results of the regularity conditions checks. In the following discussions of each of the partial derivatives, therefore, column 1 estimates are ignored. Thus, from this point, all references to the estimates of the parameters of

Model I mean the consistent, maximum likelihood estimates contained in column 2 of Table III.

Water Saturation: W

The estimated functions in Models I and II are increasing in the water saturation level of the reservoir in the neighbourhood of sample means but not at all data points. This relationship is intuitively explained as follows. Because higher water saturation means lower oil reserves per well and because oil reserves per well are a natural factor of production, more inputs must be used and hence a higher cost incurred to obtain a given level of extraction when water saturation is higher.

It is interesting to note that the water saturation variable is probably capturing two effects: the reserves effect discussed above and a relative permeability effect. The latter is explained as follows. The permeability of water relative to oil is an increasing function of the water saturation level of the reservoir. The larger is this relative permeability, the less resistance is there to the flow of water through the reservoir relative to oil and hence the less successful is the displacement of oil by water injection. Thus, the greater the water saturation level, the larger the volume of water that must be injected to displace a given volume of oil production.

Pressure: P

The estimated functions are increasing in the current level of reservoir pressure. One might reasonably believe this to be an incorrect result. However, recollection of the production relationship developed in Chapter 2 will explain the apparent error. There it was shown that for a given change in reservoir pressure, θ, the surface production of oil may be a decreasing function of reservoir pressure: under higher pressure, more gas and less oil is contained in a unit of reservoir oil. When that unit is brought to the surface, the oil content is lower, the greater the reservoir pressure from which it originated. This effect causes variable costs to increase with pressure.

Conversely, the amount of water injection required to maintain or augment pressure is a decreasing function of pressure at or below the "bubble point" of the reservoir as discussed in Chapter 2. This effect causes cost to decrease with pressure.

Thus, the net effect of pressure on cost could be positive or negative. However, since it is generally regarded as bad reservoir management to operate a reservoir below the "bubble point" most of the observations in the sample are probably at or above bubble point. This hypothesis is substantiated by the positive predicted values for the normalized partial derivatives at sample means and at 63 out of 80 and 41 out of 80 observations in Models I and II, respectively.

One must be cautious not to infer from the fact that the predicted variable cost is increasing in pressure that the lower the pressure, the better off is the reservoir owner. The fallacy of this inference is due to two facts. The first is the bubble-point-induced discontinuity described above. The second is that the lower the stock of pressure, the fewer are the units available for depletion in the dynamic optimization model.

Change-in Pressure: θ

The estimated functions are increasing in θ, the change-in-pressure variable.[2] When θ is positive it is an output (pressure rises) and when θ is negative it is an input (pressure falls). An increase in pressure decline, for example, leads to a decrease in the variable costs required to achieve a given extraction rate, other things equal. The estimated functions satisfy this regularity condition at sample means and at all observations but one in Model I and all but 19 in Model II.

The Extraction Rate: Q

As required, variable costs are increasing in the rate of extraction of oil for both of Models I and II. For Model I, this condition was also satisfied at all observations and at all but 5 observations in Model II.

Pay Thickness: Z

An increase in a reservoir's pay thickness leads to an increase in oil reserves per well and should therefore lead to a reduction in variable extraction costs. This condition is satisfied by Models I and II at sample means, for all observations for Model I, and all but 25 observations in Model II.

Stock of Oil Wells: N

In Model II, an increase in this fixed factor of production should lead to a decrease in the requirement of the variable factor, water injection and hence a decrease in variable costs. This condition is

2. For the empirical model, this variable is definded as $\theta = P_t - P_{t+1}$ and is therefore equal to the observed pressure change during 1973. For the analytical model, θ was defined as the negative of pressure change.

satisfied at sample means and at all but 11 observations. Thus, water injection and oil production wells are substitutable in the restricted technology set.

Predicted Value of Variable Extraction Costs

The final regularity condition that must be satisfied is that the predicted value of the estimated function be non-negative. This condition is satisfied at sample means and the majority of data points for Model I and for all data points in Model II.

Models I and II satisfy all regularity conditions, at least in the neighbourhood of sample means. In the following sections, the estimated parameters of these functions are exploited by way of hypothesis testing to extract information about (i) the characteristics of the optimal solution to the dynamic cost minimization problem or the optimal depletion policy of an oil reservoir and (ii) the factors giving rise to inter-pool cost heterogeneity.

Constant Returns to Scale

Constant returns to scale in the restricted production technology is equivalent to linear homogeneity of the variable cost function and the factor requirements function in the vector of fixed netputs. Thus, the null hypothesis of constant returns to scale is directly testable through these functions. A likelihood ratio test is used to perform this test. The estimated function is linear homogeneous in Model I iff $\gamma_{ij} = 0$ for all i and j and in Model II iff $g_{ij} = 0$ and $b_0 = 0$ for all i and j. Imposing these restrictions and maximizing the likelihood functions over the remaining parameters produces LR test statistics of 45.27 and 92.42

for Models I and II, respectively, both of which exceed their respective critical values of 25 and approximately 36 for $\chi^2(15)$ and $\chi^2(22)$. Thus, the null hypothesis of constant returns to scale is soundly rejected. This result is interesting not only in itself but also because it implies that one cannot rule out the possibility of non-increasing marginal extraction costs.

The Marginal (Variable) Cost of Extraction

Non-increasing marginal variable extraction costs imply that an increase in extraction rates will lead to an increase in the rents earned by a reservoir, assuming exogenous prices. With respect to the policy practice of the Alberta government of determining the allocation of aggregate provincial production among the oil pools operating in the province, non-increasing marginal variable extraction costs imply that a more efficient allocation could be achieved by increasing the allowable extraction rates to each of a reduced total number of oil pools. It is current practice in the Province of Alberta to pro-rate provincial crude oil supply to market demand. Each month, aggregate demand for crude oil is determined by asking refineries to reveal their desired purchases for the month. This aggregate is then allocated among the operating pools in the province in a manner that is considered equitable. In particular, each pool's share of aggregate production is set equal to its share of aggregate crude oil reserves, provided the implied extraction rate does not exceed a maximum permissable rate determined by government engineers. This allocation rule is unlikely to maximize aggregate rents. A rent-maximizing rule would allocate production so as to equate marginal extraction cost plus marginal user cost across pools. That is, assuming an interior solution, the rent-maximizing rule requires that

$$C_Q^i + uc^i = C_Q^j + uc^j \quad ; \quad i,j = 1,2,\ldots,R$$

This allocation rule must hold at every point in time. Marginal cost in the i^{th} pool is a function of the optimally chosen θ^i and uc^i is the marginal user cost of the reserves in the i^{th} pool. This allocation rule implicitly determines the optimal extraction rate for each pool such that the sum of allocations equals aggregate demand. It is apparent that the optimal extraction rate for a pool cannot occur on a flat or decreasing portion of the marginal cost function (assuming it eventually turns upward at $Q <$ aggregate demand). Thus, if the estimated functions imply flat or decreasing marginal costs in the range observed, one may conclude that the existing allocation in Alberta is not maximizing aggregate rents (unless marginal cost is constant and equal across pools for all $Q \leq$ aggregate demand). Information about the marginal cost functions within and across pools is obtained with the following hypothesis tests.

The null hypothesis that marginal variable extraction costs are a constant function of the extraction rate is tested with an asymptotically valid t-test of the null hypothesis that $\gamma_{44} = 0$ for Model I and $g_{44} = 0$ for Model II, in Table III. The null hypothesis cannot be rejected in either case.

The policy implications of this result are reinforced by the fact that θ, a factor of production, has been held fixed. If marginal costs are non-increasing when a factor of production is held fixed they are _a fortiori_ non-increasing when that factor is allowed to vary. However, the policy implications are moderated by the fact that the result only applies to marginal changes in extraction rates in the range observed in the sample.

To achieve an efficient allocation of an aggregate level of production among pools requires the knowledge of whether marginal extraction costs, which may be non-increasing in the extraction rates of pools, differ for different pools. The results in Table III indicate that marginal variable extraction costs vary in a systematic way with variation in the natural factors of production. In Model I, marginal costs are not significantly affected by differences in water saturation levels (γ_{14}) but are significantly lower in pools that have a larger pay thickness (γ_{45}). The opposite is true for Model II: marginal costs are significantly higher in pools that have a higher level of water saturation (g_{14}) but are not significantly affected by differences in pay thickness (g_{45}).

While these results are consistent with the hypothesis that marginal costs are significantly affected by the level of oil reserves per well, the minor conflict in the results again suggest that one of the models is a better description of the process by which the data were generated than the other but is not helpful in determining which model is better. However, an explanation of the conflict is available. If the water saturation level is acting primarily as a proxy for the relative permeability of water to oil, the conflict in results is understandable. In Model II where the number of wells is an explanatory variable, the reserves-per-well effect is probably captured by the number of wells in place so that the effect of pay thickness is insignificant and the effect of water saturation is as a proxy for relative permeability. In Model I, the reserves-per-well effect, via pay thickness, is an important determinant of the number of wells but relative permeability, via water saturation, does not influence the choice of the number of wells. The latter effect must dominate the estimation of γ_{14} which, recall, is restricted to be equal in the two factor demand equations.

The null hypothesis that marginal (variable) extraction costs are independent of θ, the change-in-pressure variable cannot be rejected for either Model I (γ_{34}) or Model II (g_{34}). The effect of the level of reservoir presure on marginal (variable) extraction costs, a priori, is ambiguous depending on whether the reservoir is above or below the "bubble point". A two-tailed t-test cannot reject the null hypothesis that $\gamma_{24}=0$ in Model I and $g_{24}=0$ in Model II at the 5% level of significance. Thus, the level of reservoir pressure does not appear to have a significant impact on marginal (variable) extraction costs.

In Model II, the effect of the stock of oil wells on marginal (variable) extraction costs is given by g_{46}. The null hypothesis that $g_{46}=0$ can be rejected at the 5% significance level. Thus, marginal (variable) extraction costs are significantly lower if the stock of oil wells in place is higher.

The Shadow Price of Pressure

Recall that in the dynamic cost minimization model, θ (change in pressure) is the control variable. In Chapter 3 it was shown that, if an interior solution exists, the Hamiltonian function is maximized by equating (the absolute value of) $\partial C/\partial\theta$ with the (absolute value of) the shadow price of pressure λ, at each point in time. Thus, one ought to be able to infer from the predicted values of $\partial C/\partial\theta$ and $\partial m/\partial\theta$ interesting information about the shadow prices of pressure for the various oil pools in the sample. To do this, however, requires the assumption of an interior solution. A solution can be interior only if the Hamiltonian function, and therefore the variable cost or factor requirements function, is non-linear in the control variable θ. This is a testable hypothesis: the results in Table III show that the null hypothesis that the variable cost function is linear in θ cannot be rejected in Model I (γ_{33}) but can be rejected for the factor requirements function in Model II (g_{33}). Thus, the assumption of an interior solution is reasonable in Model II but not in Model I.

Linearity of the Hamiltonian function in θ implies a "bang-bang" control or corner solution in which there is non-equality between $\partial C/\partial\theta$ and the shadow price, λ. This makes it difficult to make inferences about λ from the estimated characteristics of $\partial C/\partial\theta$ and $\partial m/\partial\theta$. In particular, a prediction of the reservoir depletion model developed in Chapter 3 is that the higher is an oil pool's shadow price for pressure, the more likely is the pool to be under water injection. While it would be interesting to test this prediction by testing if, indeed, the calculated shadow prices for pools under water injection are higher than those not under water injection, it does not seem possible without assuming an interior solution, an assumption which cannot reasonably be made for Model I. The following argument, however, explains how the results can still be used to test the hypothesis even for Model I.

Suppose the cost function is linear in θ, as in Model I, so that the optimal control for θ is of the "bang-bang" type. From the analysis of the solution to the dynamic problem in Chapter 3, it is known that if a pool is not under water injection, then $|\partial C/\partial\theta| > |\lambda|$: the (absolute value of the) marginal cost of augmenting pressure is greater than (the absolute value of) the marginal benefit. Conversely, a pool which is under water injection necessarily has $|\partial C/\partial\theta| < |\lambda|$. The observations have been ordered so that the first 49 observations are all of the pools which are not under water injection and the remaining 31 are all of the pools which are under water injection. Define μ_0 as the average predicted value of $\partial C/\partial\theta$ over the first 49 observations and μ_1 as the average predicted value of $\partial C/\partial\theta$ over the remaining 31 observations. If $\mu_0 < \mu_1$, then it is necessarily true that, on average, the shadow prices for pressure are higher for the pools which are under water injection than for the pools which are not under water injection.

Thus, the null hypothesis to be tested is that $\mu_0 = \mu_1$ against the alternative that $\mu_0 < \mu_1$. If the null can be rejected for both of Models I and II, then one has found that λ is, on average, significantly higher for pools that are under water injection. If the null hypothesis cannot be rejected, no conclusions can be drawn about λ in Model I, but one can conclude there is no significant difference in the shadow price between the two groups of pools (recalling that it can be assumed that $\lambda = \partial C/\partial\theta$ in Model II).

The hypothesis test is performed by considering the first 49 and the remaining 31 predicted values of $\partial C/\partial\theta$ as two independent samples. The t-statistic is computed as

$$\hat{t} = (\mu_1 - \mu_0)/(\frac{s_1^2}{31} + \frac{s_0^2}{49})^{1/2}$$

where s_i^2 is the sample variance of the i^{th} sample, $i = 1,2$. The relevant statistics are given below in Table VI.

TABLE VI

Testing for Equality of Shadow Prices

	Model I	Model II
μ_0	0.128	0.059
μ_1	0.163	0.160
$\hat{t}$	1.60	2.22

At approximately a 7% level of significance, the null hypothesis can be rejected in Model I but not at a 5% level of significance. In Model II, the null hypothesis is decisively rejected. These results, therefore, imply that the shadow price of pressure is significantly higher for oil pools which are under water injection than for pools which are not under water injection.

The results in Table III also indicate that for Model II at least, where the assumption of an interior solution is reasonable, the shadow price of pressure varies systematically with observed differences in pressure, water saturation level, and pay thickness over the oil reservoirs in the sample. The shadow price is significantly lower for pools with higher pressure (non-increasingness is a sufficient conditon for a minimum in the dynamic cost minimization problem). The shadow price is significantly higher when the water saturation level is higher, a result consistent with the hypothesis that water saturation is acting primarily as a proxy for relative permeability and not for reserves per well in Model II. A higher level of water saturation reduces the effectiveness of water injection thereby requiring a higher value to be placed on pressure in order to make it optimal to inject a greater volume of water to displace a given volume of oil. The shadow price of pressure is significantly higher in pools which have a greater pay thickness. This result is consistent with the phase-diagram analysis of the dynamic model in Chapter 3 where it was argued that a shadow price would be set high only if oil reserves were sufficiently large to warrant water injection. However, it is not consistent with the earlier hypothesis that the effect of reserves-per-well is captured by the number of wells and not pay thickness in Model II. In addition, the results in Table III show that the shadow price is not significantly affected by the number of wells in place. It is possible, therefore, that the effect of reserves on shadow price is being captured by pay thickness and not the number of wells in place.

The results in Table III are consistent with the dynamic model of reservoir depletion in Chapter 3 and suggest that differences in the key physical characteristics of reservoirs lead to a significant level of heterogeneity in extraction costs of reservoirs in operation at the same point in time. This latter issue will be returned to after presenting and analyzing the results obtained when the reduced-form equation for θ is used to substitute θ out of the estimation models.

5.2 The Variable Extraction Cost Function: Endogenous θ

It was argued in Chapter 4 that the inclusion of θ as an exogenous variable in the estimation models is a source of potential sumultaneity bias. While it is impossible to test the hypothesis of no correlation between error terms since it is not possible (in practice) to estimate the three-equation model, the purpose of this section is to attempt two ways of eliminating the simultaneity problem if it exists. The first is to use the solution for the optimal θ to eliminate θ from the estimation equations and then to obtain parameter estimates of the resulting reduced-form equations, (4.13) and (4.18), which, for convenience, are reproduced in Table VII. The second is to use OLS predicted values for θ instead of observed values for θ in the two-equation estimation model.

Certain empirical restrictions had to be imposed in order to obtain results for the estimation models in Table VII. These models have two complicating features not present in the estimation models which have already been presented. The first is the larger number of parameters to be estimated (4 more in Model I and 3 more in Model II) bringing the total number of parameters to 33 and 31 for Models I and II, respectively. The second is the non-linearity of the equations in Table VII - all estimated equations have been linear in parameters to this point. These additional complications reduce the chance of achieving a numerical solution to the maximum likelihood problem, a problem which is already inherently non-linear due to the limited dependent variable problem.

In an attempt to estimate Models I and II, satisfactory convergence was not achieved. A satisfactory convergence is one where the function value cannot be improved by more than a specified tolerance level and both

Table VII

Estimation Equations: θ Endogenous

Model I

$$N = \xi_{01} + \sum_{j=1}^{5} \xi_{1j}X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \psi_{ij}X_iX_j + e_1$$

$$m = \xi_{02} + \sum_{j=1}^{5} \xi_{2j}X_j + 1/2 \sum_{i=1}^{5} \sum_{j=1}^{5} \psi_{ij}X_iX_j + u_1$$

where $$X_3 = \{e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}\}(P_o-W-Q-Z-1)$$

Model II

$$m(x) = \Gamma_o + \sum_{i=1}^{6} \Gamma_iX_i + 1/2 \sum_{i=1}^{6} \sum_{j=1}^{6} \Gamma_{ij}X_iX_j + e_2$$

where $$X_3 = [e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}](P_o-W-Q-Z-N-1)$$

the first and second order conditions for a maximum are satisfied. This problem was dealt with by imposing the restrictions that certain parameters equal zero. The following paragraph explains how this was done.

In the attempt to maximize the likelihood function over all 33 parameters for Model I, the function value moved rapidly initially to a value of approximately -419. From this point, a very large number of iterations produced very small increases in the function value. When the convergence criterion was reduced from 10^{-8} to 10^{-6}, an apparent convergence was achieved and the first-order partial derivatives were all extremely small. However, the second-order partial derivatives did not satisfy the conditions for a maximum. The step-size parameter used to numerically calculate these derivatives was increased in size but the problem remained. Consequently, the following procedure was adopted for determining which parameters could be set to zero. The set of 33 parameters was partitioned into 3 overlapping subsets. The likelihood function was maximized with respect to one subset at a time, holding constant all parameters in the complement set, thereby reducing the effective number of parameters over which the function was being maximized. In practical terms, this is quite important since the numerical optimization routines are known to not work well with more than 20 parameters. It was then possible to achieve successful convergence with respect to a subset of parameters, conditional on the values of the remaining parameters. The computed asymptotic (but conditional) t-statistics could be compared over subsets. The t-statistics which were stable across subsets were used to provide information about which parameters could be set to zero without significantly affecting the value of the likelihood function. A group of parameters (2 or 3) would then be set to zero and the whole process restarted in an attempt to achieve a successful convergence over the full set of non-zero parameters. In Model I, a total of 10 parameters had to be set to zero before a satisfactory convergence could be achieved with respect to the remaining 23 parameters.[3] In Model II, 10 parameters were set to zero leaving 21 parameters. In both cases, the restrictions do not appear to have significantly affected the value of the likelihood function. The final results are presented in Table VIII. The analysis of these results, while less exhaustive, follows the pattern set in Section 5.1.

Regularity Conditions

The first-order partial derivatives of the estimated functions are somewhat more complicated to calculate than previously since the X_3 term involves each of the exogenous variables except pressure at time t. The predicted values at sample means of the normalized partial derivatives are presented in Table IX.

The most striking result in Table IX is that the estimated function in Model I fails to satisfy the cost-function regularity conditions in 2 out of 4 cases. Variable costs are predicted to be decreasing functions of the extraction rate and the water saturation level. These are unacceptable results.

A second notable result is that the estimated function in Model II still satisfies all regularity conditions. This provides further evidence in support of the preference for Model II over Model I as a description of the process by which the data were generated.

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3. Nine zeroes appear in Table VIII - the tenth is ρ.

Table VIII

Maximum Likelihood Estimates: θ Endogenous

Model I			Model II		
ξ_{01}	0.0		Γ_0	0.0	
ξ_{11}	0.0		Γ_1	0.0	
ξ_{12}	0.4749	(2.79)	Γ_2	0.1127	(1.19)
ξ_{13}	0.0		Γ_3	0.0	
ξ_{14}	1.2554	(7.12)	Γ_4	0.0	
ξ_{15}	-0.3674	(-3.60)	Γ_5	-0.1918	(-2.23)
ξ_{02}	0.0		Γ_6	0.0	
ξ_{21}	-6.2441E-2	(-0.96)	Γ_{11}	1.9501E-3	(0.11)
ξ_{22}	-0.1177	(-0.72)	Γ_{12}	-0.3397	(-0.93)
ξ_{23}	-99.336	(-1.15)	Γ_{13}	-11.793	(-2.35)
ξ_{24}	0.9330	(5.01)	Γ_{14}	1.2901	(5.62)
ξ_{25}	0.2354	(2.23)	Γ_{15}	9.400E-2	(3.14)
ψ_{11}	0.0		Γ_{16}	0.0	
ψ_{12}	0.0		Γ_{22}	-1.3921	(-1.90)
ψ_{13}	2.1033	(1.17)	Γ_{23}	7.6933	(1.25)
ψ_{14}	-1.3048	(-3.75)	Γ_{24}	0.0	
ψ_{15}	0.0		Γ_{25}	0.0	
ψ_{22}	0.7999	(0.84)	Γ_{26}	5.0135	(2.00)
ψ_{23}	2.5304	(1.05)	Γ_{33}	19.576	(1.89)
ψ_{24}	-1.9324	(-2.88)	Γ_{34}	5.9167	(3.84)
ψ_{25}	-1.4415	(-3.14)	Γ_{35}	-1.4085	(-3.89)
ψ_{33}	-209.91	(-0.67)	Γ_{36}	-4.8828	(-0.93)
ψ_{34}	25.340	(1.06)	Γ_{44}	0.0	
ψ_{35}	1.0012	(1.31)	Γ_{45}	0.0	
ψ_{44}	0.0		Γ_{46}	-4.1806	(-4.68)
ψ_{45}	-0.4756	(-4.37)	Γ_{55}	6.3320	(4.03)
ψ_{55}	0.0		Γ_{56}	-0.2292	(-4.91)
β_1	-3.8975	(-7.36)	Γ_{66}	5.2227	(5.81)
β_2	-0.6906	(-1.21)	β_1	-9.0769E-2	(-5.60)
			β_2	-0.1296	(-5.65)
L	-419.519				
			L	-71.6673	

Asymptotic t-statistics in parentheses

Table IX

Normalized Partial Derivatives

	C_W	C_P	C_Q	C_Z	C_N
Model I	-1.342	-0.196	-0.825	-1.565	--
Model II	0.316	-0.60	0.274	-0.298	-0.430

A third result in Table IX is the negative sign on the predicted partial derivatives with respect to pressure. This is a perfectly acceptable result and is consistent with the findings in Section 5.1: because θ is no longer held constant in the cost function, there is no reason for variable costs to be an increasing function of reservoir pressure as is the case when θ is held constant.

The predicted negative signs on C_W and C_Q in Model I are undoubtedly related to the parameter restrictions that had to be imposed to obtain maximum likelihood results. Of the 10 parameters set equal to zero, 5 of them involve terms in W or Q. All of the individual parameters that make up these partial derivatives except ψ_{14} have the correct sign. The negative sign on ψ_{14} implies that marginal costs are lower when water saturation is higher. This result is not consistent with the theory, nor is it consistent with the finding in Model II, where Γ_{14} is significantly greater than zero. It is, however, consistent with the negative (but insignificant) estimate of γ_{14} in Section 5.1. While this probably <u>explains</u> the failure of Model I to satisfy the regularity conditions, it does not <u>justify</u> it and one is still inclined to favour Model II over Model I.

The Marginal Cost of Extraction

Are marginal extraction costs still a non-increasing function of the extraction rate as was found in Section 5.1? There was strong evidence in support of the hypothesis that ψ_{44} and Γ_{44} are not significantly different from zero which is why these were among the parameters that were set to zero to obtain the results in Table VIII. However, the derivative of the marginal cost function involves more terms than just these single parameters since X_3 involves all exogenous variables except P(t). Thus the slope of the marginal extraction cost curve is given by

$$\text{Model I:} \quad \partial^2 C/\partial Q^2 = \psi_{44} + \psi_{33}(e^{x_1}-e^{y_1})^2$$

$$\text{Model II:} \quad \partial^2 C/\partial Q^2 = \Gamma_{44} + \Gamma_{33}(e^{x_2}-e^{y_2})^2$$

where $(e^{x_1}-e^{y_1})^2$ and $(e^{x_2}-e^{y_2})^2$ are the (squares) of the terms involving T and t and have the values $1.2\text{x}10^{-5}$ and $1.421\text{x}10^{-2}$ at sample means for Models I and II, respectively. Thus, the slope of the marginal cost curve depends on the signs of ψ_{33} and Γ_{33}. Referring to Table VIII, the estimates of these parameters are not significantly different form zero at the 5% significance level, but Γ_{33} is significantly different form zero at the 10% level. Moreover, it has a positive sign which implies an upward-sloping marginal variable extraction cost curve. This result contradicts the finding in Section 5.1 when θ is held constant. Simultaneity bias is a possible explanation of the conflict. However, another explanation which is perhaps more realistic is the fact that the X_3 term involves a conglomeration of all exogenous variables, thereby restricting the second-order marginal effects of Q (since $\Gamma_{44} = 0$) to be felt through Γ_{33} which also reflects part of the second-order effects of all other exogenous variables except initial pressure. This result is not considered good evidence in support of the hypothesis of an upward-sloping marginal cost curve.

In Model I marginal extraction costs are still predicted to be significantly lower for pools with higher pressure and a larger pay thickness. In Model II, marginal extraction costs are significantly higher in pools with a higher water saturation level and are significantly lower in pools with a larger stock of oil wells [4] reinforcing the earlier evidence of substitutability between water injection and production wells.

Pay thickness appears to have a significant positive effect on marginal extraction costs, at the 10% level. However, this result again is felt only through the Γ_{33} term since Γ_{45} is one of the coefficients set to zero. Thus, as for the effect of Q on marginal costs, not much confidence is placed on the estimated effect of Z on marginal costs through the Γ_{33} term alone.

The exponent coefficients are significantly different from zero in Model II but only one is in Model I. It is interesting to note that the roots of the characteristic equation of the differential equation system can be simply calculated since $r_1 = \beta_1 + \beta_2$ and $r_2 = \beta_1 - \beta_2$ where r_1 and r_2 are the roots. Thus, both roots are negative in Model I but are of opposite sign in Model II.

The second method of dealing with the simultaneity problem is a two-stage procedure. In the first stage, an ordinary least squares regression of θ on all of the exogenous variables in the system including P_0, T-t, and in Model II, N, was performed. The resulting predicted values for θ were then used in the second stage in which the parameters of

4. The latter result is given by $\Gamma_{36} + \Gamma_{33}(e^{x_2}-e^{y_2})^2$ which is negative.

Models I and II were estimated using the maximum likelihood techniques already discussed. The consistent parameter estimates obtained from applying this procedure are reported in Table X.

The estimates in Table X are directly comparable to those in columns 2 and 3 of Table III. The estimates for Model I satisfy all regularity conditions at sample means. Moreover, there are no notable changes in the results in Table X as compared to those in column 2 of Table III. The same cannot be said for Model II.

There are some significant changes in the results obtained from Model II when the two-stage approach is used. The most significant of these relates to the marginal variable cost of extraction. The results in Table X show that g_{45} and g_{46} are both significantly greater than zero indicating that marginal costs or the marginal factor requirement is increasing in pay thickness and the number of wells. These results differ from the results of Table III and are contrary to expectations. An additional difference in results, but one that would be considered acceptable, is that g_{44} is significantly greater than zero indicating that marginal costs are increasing in the extraction rate. All of these results are unacceptable, however, due to the fact that this estimate of Model II fails to satisfy regularity conditions. In particular, it is estimated that the factor requirement is decreasing in the rate of extraction and increasing in the level of pay thickness at sample means. For this reason, the model was re-estimated, excluding the pay thickness variable from the regression. The reasons for this are twofold: first, as has been suggested, it is believed that the effect of reserves-per-well is being captured, to some extent, by the number of wells so that the number of wells is interfering with the intended effect of pay thickness.

Table X

Maximum Likelihood Parameter Estimates: Using Predicted Values for θ

	1. Model I			2. Model II		3. Model II	
a_{11}	0.1236E-1	(0.01)	b_1	-0.2681E-1	(-0.20)	0.0780	(0.39)
a_{12}	0.6711	(1.53)	b_2	0.2239	(0.57)	0.1860	(0.53)
a_{13}	0.3447	(0.55)	b_3	-0.1262	(-0.27)	0.3234	(0.92)
a_{14}	1.3798	(6.53)	b_4	0.3150	(1.06)	0.6628	(1.73)
a_{15}	-0.5027	(-2.16)	b_5	0.6160E-2	(0.04)	0.0	
a_{21}	-0.2740	(-1.98)	b_6	0.9118E-1	(0.60)	-0.2457	(-1.20)
a_{22}	0.1924	(0.50)	g_{11}	-0.1323E-1	(-0.32)	-0.0177	(-0.35)
a_{23}	0.1765	(0.29)	g_{12}	0.5236	(0.44)	-0.1996	(-0.25)
a_{24}	0.8974	(4.31)	g_{13}	0.7770	(0.41)	-0.0939	(-0.10)
a_{25}	-0.1177	(-0.55)	g_{14}	1.2396	(1.85)	0.3763	(0.40)
γ_{11}	0.1093	(1.61)	g_{15}	0.2808E-1	(0.50)	0.0	
γ_{12}	-0.8683	(-0.55)	g_{16}	-5.0970	(-1.57)	1.1184	(0.26)
γ_{13}	-1.5800	(-0.59)	g_{22}	-3.8511	(-1.64)	-1.1854	(-0.58)
γ_{14}	-1.2032	(-2.75)	g_{23}	-2.2481	(-1.19)	-0.3041	(-0.23)
γ_{15}	0.1076	(1.26)	g_{24}	-10.952	(-2.46)	-2.6903	(-2.21)
γ_{22}	-0.5896	(-0.29)	g_{25}	0.5099	(0.61)	0.0	
γ_{23}	-0.3734	(-0.19)	g_{26}	41.212	(2.94)	3.5968	(0.46)
γ_{24}	-0.6935	(-0.44)	g_{33}	-11.002	(-1.11)	-4.0539	(-1.58)
γ_{25}	-0.7936	(-0.92)	g_{34}	-15.949	(-2.57)	-0.6620	(-2.13)
γ_{33}	2.7480	(0.24)	g_{35}	12.375	(1.25)	0.0	
γ_{34}	2.2069	(0.72)	g_{36}	6.3087	(2.93)	-1.7244	(-1.61)
γ_{35}	-1.4281	(-0.13)	g_{44}	0.9374E-1	(2.12)	0.0945	(1.66)
γ_{44}	-0.1215	(-0.55)	g_{45}	4.4959	(2.50)	0.0	
γ_{45}	-0.9088	(-1.09)	g_{46}	7.4310	(1.59)	-2.8766	(-1.56)
γ_{55}	5.0355	(1.04)	g_{55}	-1.8114	(-0.39)	0.0	
			g_{56}	-0.4295	(-3.73)	0.0	
L	-421.93		g_{66}	-5.2932	(-1.22)	6.2435	(2.03)
			b_0	-2.4763	(-0.93)	-2.9758	(-0.80)
			L	-62.74		-82.44	

Asymptotic t-statistics in parentheses

Second, it is known from the results to be presented in the next section that the number of wells is strongly correlated with the extraction rate (positively) and pay thickness (inversely). If multicollinearity is affecting the results adversely, eliminating one of the variables may improve the results. Pay thickness is chosen as the variable to be eliminated because on a priori grounds, it is the least likely variable to have an important influence on water injection requirements. The results are reported in the third column of Table X.

With pay thickness eliminated, the estimate of Model II satisfies all regularity conditions at sample means. An interesting finding is that water injection is an increasing function of the number of wells, at sample means. This result, which is contrary to that in Table III, indicates that water injection is a complement of and not a substitute for production wells. This finding can be explained by a feature of oil reservoir production practices that is not captured in Model I or Model II: production wells can be and often are converted to injection wells. Even though production wells and water injection may be substitutes in the production of oil flow from the reservoir, this feature may tend to create a positive relationship between the number of conversions and the number of production wells at any point in time. Thus, when the econometric model does not standardize for this conversion feature, it is possible to observe complementarity rather than substitutability between the rate of fluid injection and the number of production wells.

The most interesting of the results for Model II concerns the hypothesis of a flat marginal extraction cost curve. The g_{44} parameter is significantly positive in the second column of Table X but is not meaningful since marginal cost is negative in this case, as was discussed above. In column 3, when pay thickness is excluded, the g_{44} parameter scarcely changes in value, but the asymptotic t-statistic falls sufficiently that the null hypothesis can be rejected only at about the 10% level of significance. The size of the g_{44} coefficient, however, indicates that the marginal cost curve is very nearly flat: at sample means, the implied elasticity of the marginal cost curve is .08. Thus, although there is some evidence to reject the hypothesis of non-increasing marginal costs, the implied slope of the marginal cost curve is, for practical purposes, flat.

5.3 The Optimal Number of Oil Wells: Model II

Model II is incomplete until the problem of the optimal number of oil wells to drill at time zero is solved and the parameters of the resulting reduced-form equation are estimated. The purpose of this section is to present and analyze the parameter estimates. For convenience, the reduced-form equation determining the optimal number of oil wells is reproduced in Table XI.

Referring to Table XI, one sees that the parameters to be estimated are r_1, r_2 and the f_{ij}. It turns out, however, that there is an extremely high degree of multi-collinearity among the $q_0, q_1, \ldots q_5$ data terms. A variety of starting values for r_1 and r_2 were tried, including the values estimated in the previous section, but there remained nearly perfect collinearity among the $q_0, q_1, \ldots q_5$ terms. It was therefore impossible to estimate any of the parameters in Table XI. Instead, assume that the $q_0, q_1 \ldots q_5$ terms are perfectly correlated with the term $e^{\beta_2 T}$, where β_2 is an unknown parameter. Using this in Table XI yields the following

Table XI

Estimation Equation for Number of Wells: Model II

$$N^* = -\phi/D - W(f_{10}q_0 + f_{11}q_1 + f_{12}q_2 + f_{13}q_3 + f_{14}q_4 + f_{15}q_5)/D$$
$$- Q(f_{20}q_0 + f_{21}q_1 + \ldots + f_{25}q_5)/D$$
$$- Z(f_{30}q_0 + f_{31}q_1 + \ldots + f_{35}q_5)/D$$
$$- P_0(f_{51}q_1 + \ldots + f_{55}q_5)/D$$
$$- (f_{60}q_0 + f_{61}q_1 + f_{62}q_2)/D$$

where $D = f_{40}q_0 + f_{41}q_1 + \ldots + f_{45}q_5$

$$q_0 = (1 - e^{-\delta T})/\delta$$

$$q_1 = [e^{(r_1-\delta)T}-1]/(r_1-\delta)$$

$$q_2 = [e^{(r_2-\delta)T}-1]/(r_2-\delta)$$

$$q_3 = [e^{(2r_1-\delta)T}-1]/(2r_1-\delta)$$

$$q_4 = [e^{(r_1+r_2-\delta)T}-1]/(r_1+r_2-\delta)$$

$$q_5 = [e^{(2r_2-\delta)T}-1]/(2r_2-\delta)$$

reduced-form equation for N, the parameters of which can be estimated:

$$N = \beta_0 + \beta_1(\phi e^{-\beta_2 T}) + \beta_3 W + \beta_4 Q + \beta_5 Z + \beta_5 P_0 + e_3 \qquad (5.2)$$

where e_3 is a normally distributed random error term. Before analyzing the maximum likelihood estimates of the parameters in equation (5.2), an explanation of ϕ, the price term is necessary.

In Chapter 4, it was argued that the relative price of oil wells and water injection is constant across reservoirs at any point in time. It was also argued that the price of water injection was at least partially a shadow price determined by the optimal conversion of production wells to injection wells: wells will be converted up to the point that the marginal value product of injection wells equals the marginal cost of injection wells (which is equal to the marginal value product of production wells plus the marginal conversion cost). It is possible for there to be variation in the relative price over time periods. However, data limitations require that it be assumed that any variation in the relative price over time is caused by variation in either the price of oil wells or the market price of conversion but not by the shadow price component of the full factor price for injection. This is equivalent to assuming that, if an oil well is converted to injection, it is converted when its marginal value product in oil production reaches a certain level, a level which is constant over the years in the sample.

Having made this assumption, it is possible to use data showing total industry capital expenditures on pressure maintenance and secondary recovery by year plus data showing the net addition to the stock of pressure maintenance wells in the province (this includes new wells plus conversions) to obtain an average market price per injection well.[5] This capital price should be converted to a flow price per unit of water injected. However, because it is assumed that injection per well is constant and because the flow-price conversion factor is assumed to be constant across pools for empirical purposes, variation in the relative capital prices described above will be equivalent to variation in the relative factor prices. Thus, ϕ, for estimation purposes is the average price per oil well divided by the average price per injection well; it is constant across pools in any year, but varies over the years in the sample. It is assumed that ϕ is expected to remain constant over the life of the pool by the decision-maker at t=0.

Parameter estimates are presented in Table XII. Asymptotic t-statistics appear in parentheses beside the associated estimate.

Table XII

Maximum Likelihood Parameter Estimates

β_0	23.557	(2.53)
β_1	-3.3090	(-1.42)
β_2	-0.41693	(1.23)
β_3	0.072387	(0.33)
β_4	0.33011	(12.20)
β_5	-0.36419	(-3.28)
β_6	-0.36094	(-0.96)
L	-498.9081	

These estimates were obtained using the SHAZAM econometrics package.

5. The sources for these data are discussed in Appendix B.

One expects the relative price of oil wells to have a negative impact on the number of wells drilled. If it is expected that the relative price is to remain high, it would be desirable <u>ex ante</u> to substitute water injection at some future date for wells drilled now. The estimate of the relative price effect is given by β_1 in Table XII. The parameter estimate is significantly less than zero at the 10% level of significance but not at the 5% level. The implied price elasticity of demand at sample means is -0.39. Hence a 10% rise in the relative price of oil wells leads to approximately a 4% fall in the number of wells drilled to produce a given rate of output, indicating an inelastic demand for oil wells. This is not a surprising result in view of the fact that oil wells are absolutely essential to extract oil from a reservoir. These results indicate that it would take a doubling of the relative price of oil wells to lead to a reduction of 6 wells on the average pool of 17 wells, in order to produce a given output rate.

The number of wells drilled to produce a given output rate is responsive to the pay thickness of the reservoir. The estimate of this effect, β_5, is significantly less than zero at even the 1% level of significance. At sample means, the estimate of β_5 implies an elasticity with respect to pay thickness of -0.33. It is not uncommon for one reservoir to have a pay thickness double that of another. The results here indicate that, all other things equal, 33% fewer wells would be drilled in the average reservoir with twice the pay thickness of another.

The hypothesis that the number of wells is independent of the planned average extraction rate is decisively rejected. It is apparent from Table XII that the estimate of β_4 is significantly greater than zero.

The water saturation level of reservoirs does not appear to have a significant impact on the number of wells drilled. This finding substantiates the hypothesis that water saturation is not acting as a proxy for reserves per well as was intended in the model - this effect appears to be captured in pay thickness. Rather, water saturation, in view of its significant impact on water injection, appears to be acting as a proxy for the relative permeability of water to oil.

The greater is the initial pressure in a reservoir, the fewer are the wells required to produce a given rate of output since the rate of pressure decline can be increased to take advantage of the high natural pressure. While the estimate of β_6 has a negative sign in Table XII, it is significantly less than zero only at about the 17% level of significance.

The coefficient on the constant term is significantly greater than zero and the coefficient on the exponent, β_2, is not significantly different from zero except at the 20% level of significance, using a two-tailed test. This does not imply that T does not have a significant impact on N, however, since the constant term is known to be a function of T.

This concludes the presentation of parameter estimates. The implications of the findings in this section and the previous sections for oil reservoir cost heterogeneity and for optimal depletion policies are discussed in the next section.

5.4 Implications and Summary of Results

The empirical results presented in the two previous sections are reasonably consistent with both theoretical extraction models developed in Chapter 3. Model II, however, performed better under empirical testing than Model I.

Of the two major econometric issues, the limited dependent variable problem appears to be the more important and the more easily handled. Two approaches for dealing with the simultaneity problem were adopted. In the first approach, the attempt to simultaneously estimate the reduced-form equation for θ (change-in-pressure) was not completely successful, primarily due to the highly non-linear nature of the reduced form equation. In the second, two-stage approach, greater success was achieved. In both cases, the majority of results were qualitatively similar to those obtained when θ was treated as an exogenous variable. The notable exception was the issue of whether or not marginal extraction costs are a non-increasing function of the extraction rate. It was consistently found that this hypothesis could not be rejected when θ was treated as an exogenous variable but some evidence to the contrary was found when θ was treated as an endogenous variable. The latter is not considered to be strong evidence, however, for the reasons explained in Section 5.3.

The policy implications of evidence in support of the hypothesis that marginal extraction costs are non-increasing in the extraction rate are important given the fact that allowable extraction rates for individual pools are determined by regulation. The results here suggest that a reallocation of aggregate output among pools will lead to efficiency gains. This finding could be given more substance if detailed cost data were available on a pool-by-pool basis.

The traditional argument in favour of slower extraction rates is that overly rapid extraction leads to a rapid pressure loss and hence a reduction in the volume of ultimately recoverable oil reserves. The modern technology of pressure maintenance through water injection, however, makes this traditional argument far less appropriate now than in the early days of the oil industry. The results of this thesis are consistent with this view. Pressure decline manifests itself through an increase in the shadow price of pressure in the theoretical model. Thus, if pressure decline becomes a problem, the resulting higher shadow price should lead to a greater desire to artificially maintain pressure by injecting water into the reservoir. The empirical findings support this hypothesis in that it was found that the implied shadow price of pressure was significantly higher for pools under water injection than for pools not under water injection. Thus water injection was found to respond positively to higher shadow prices which in turn respond inversely to the levels of pressure. Rapid extraction rates, therefore, do not necessarily lead to a loss of recoverable reserves.

The results suggest there is a great deal of variation in extraction costs and marginal extraction costs across pools. In a competitive equilibrium, one would observe equality among the sum of marginal extraction and marginal user costs across pools.[6] One could invoke the assumption of competitive equilibrium and then infer the user cost distribution from the predicted distribution of marginal extraction costs. However, the Alberta market is clearly not in competitive

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6. This is true if one assumes upward-sloping marginal extraction cost curves.

equilibrium since the output rate choice for each producer is not determined by the producer. While one could infer the rent (as a residual earning) distribution from the predicted distribution of marginal costs, one could not make any inferences about the efficiency of the allocation of production across pools. However, the latter is not the case if marginal extraction costs are non-increasing in the extraction rate. In this case, differences in marginal extraction costs are due to differences in the quality of reservoir-specific characteristics (not marginal user costs) and an efficient allocation would involve favouring of the low-cost pools. For example, if marginal extraction cost functions for reservoirs are non-increasing in extraction rates but are at different levels for different reservoirs, then an efficiency gain could be made from a marginal reallocation of the fixed aggregate output level away from the high-cost towards the low-cost reservoirs. A closer examination of the predicted variation in unit extraction costs is made below.

Reservoir pay thickness has consistently been found to have a significant negative impact on variable extraction costs. In view of the extreme variation in this natural factor of production among the oil pools in the sample, one wonders how the pools with a low pay thickness compete with the pools endowed with a pay thickness perhaps 50 times as large. There are two explanations. The first is the fact, once again, that allocation of production among pools is not necessarily efficient. The second is the possibility of compensating variation in pool depth, a natural factor of production not yet analyzed. The trade-off between depth and pay thickness which maintains constant unit extraction costs will also be examined below.

In order to compute the predicted variation in unit extraction costs and to calculate the depth-pay thickness trade-off, a number of simplifying assumptions will be made. First, all calculations will be made using the results for Model II when θ is treated exogenously. Second, the capital cost of oil wells will be converted to a fixed cost per year, a task that requires other assumptions that will be made clear. Third, it will be assumed that unit operating costs in any year (fuel, labour, maintenance and miscellaneous but not injection) are constant across pools.[7] This is a restrictive assumption but is the best available. Its effect will be to reduce the predicted variation in unit costs across pools. However, since the primary objective is to demonstrate that there is a great deal of variation in unit extraction costs across pools, this assumption will only strengthen the argument by forcing more smoothness on the predicted variations than really exists.

Let ϕ_t be the (real) capital price per metre of wells drilled in year t. Let D_{rt} be the average depth of the rth reservoir which was developed in year t. Then the capital cost of developing the r^{th} reservoir is

$$K_{rt} = \phi_t \cdot D^e_{rt} \cdot N_{rt}$$

where e is a coefficient which allows for the possibility that the cost of wells is not linear in depth, N_{rt} is the number of wells drilled into the r^{th} reservoir in the development year and K_{rt} is the capital cost.[8] Assuming a T-year life and a discount rate of δ, this is

7. The data used for this calculation are documented in Appendix B.
8. The parameter e is not estimated but the calculations that follow are done for the reasonable range of values of from 0.9 to 1.3.

converted to a fixed cost per year of k_r in the following way.

$$K_r = \int_0^T e^{-\delta t} k_r dt$$

where t has been set to zero. Solving for k_r yields

$$k_r = \delta \cdot K_r / (1 - e^{-\delta T})$$

In the computations, δ is set equal to 0.15 which means that the term on the denominator rapidly approaches the value of one as T gets large. (For T = 20, it equals 0.95 and for T = 30 it equals 0.99).

A flow price per unit of water injection is calculated in roughly the same manner. This will result in a somewhat crude approximation of the unit water injection costs since the only data available are development expenditure data on pressure maintenance. However, as will be seen, water injection costs are small relative to development costs so calculation errors will not drastically affect the total cost per unit estimates. Moreover, the estimates of injection costs here are consistent with the range of values estimated in Watkins (1977) using industry - supplied cost data for a small number of specific oil pools.

If ϕ_I is the capital cost per metre of injection wells and α is injection per well, then the flow cost per unit of injection is

$$w_{2r} = \phi_I \cdot D_r^e \cdot \delta / [\alpha(1 - e^{-\delta T})]$$

If the output rate of the r^{th} reservoir in 1973 is Q_r, then the calculated extraction cost per unit of oil produced is

$$C_r = k_r/Q_r + w_{2r} \cdot m_r/Q_r + oc \qquad (5.3)$$

where m_r is the water injection rate of the r^{th} reservoir and oc is the (assumed constant) operating cost per unit of output. The extraction cost is converted to the units of 1973 dollars per barrel. The results are presented in Figure 11.

Figure 11 shows the unit extraction costs across pools using the predicted values for N and m. The average well-head cost per barrel is \$2.33. The average water injection cost for pools actually under water injection is \$0.22 per barrel of oil produced. It is interesting to note that the average unit extraction cost for pools under injection is \$1.53 per barrel, lower than the overall average. The operating cost is estimated to be \$0.412 per barrel for each reservoir.

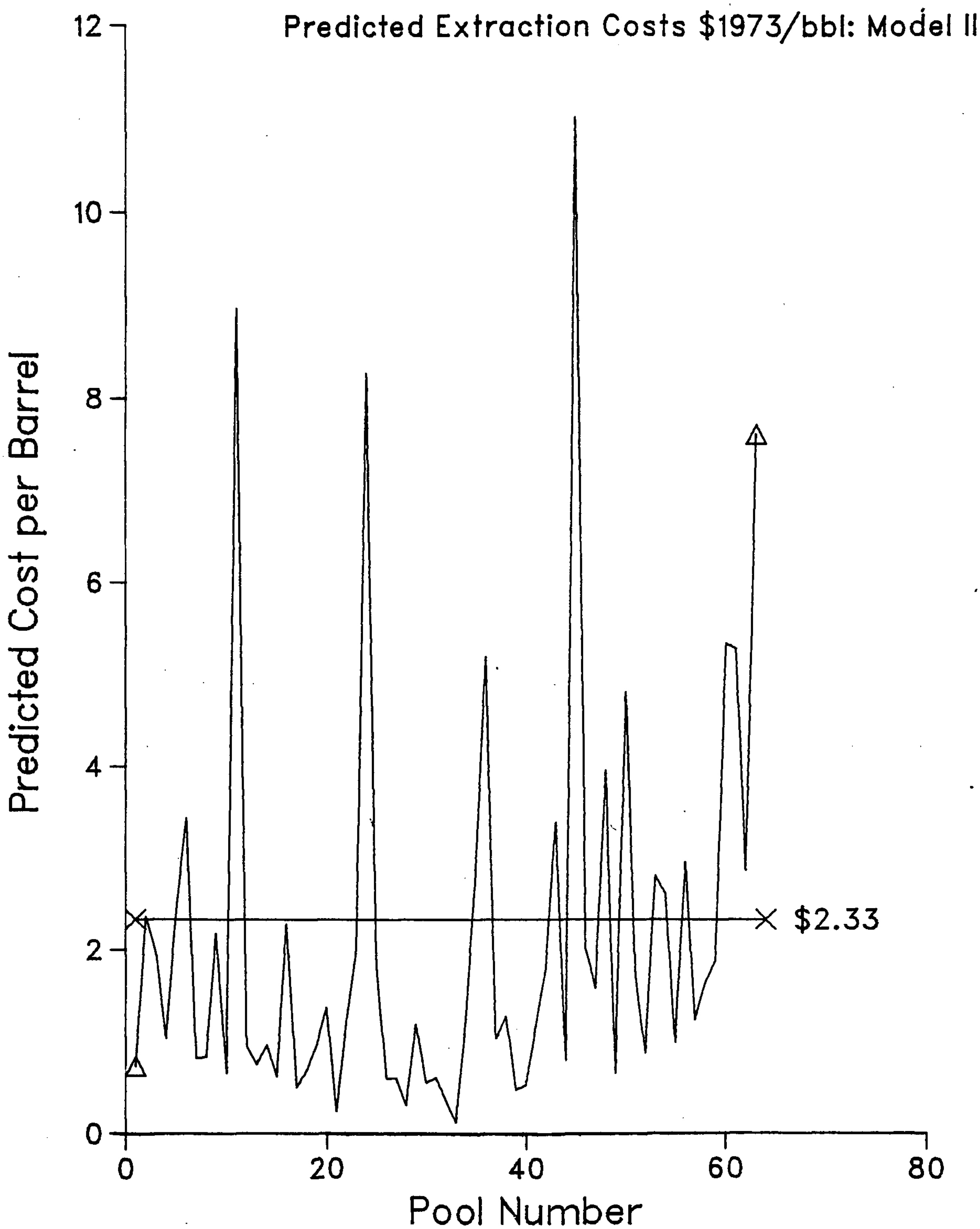
In generating this plot, the following 2 observations were removed to improve the scaling of the Y-axis:

1. Predicted unit cost = -19.43
2. Predicted unit cost = 15.91

However, these observations were included in computing the averages that appear above. The number of pools used to generate this plot, 64, is the number in common between the 80 used to estimate the injection equation and the 110 used to estimate the development well equation.

As can be seen in Figure 11, there is a very large degree of variation of unit costs across pools implying a large degree of variation in the rents accruing to various pools. If the hypothesis of

FIGURE 11


Predicted Extraction Costs $1973/bbl: Model II
12
10
8
6
4
2
0
Predicted Cost per Barrel
$2.33
0
20
40
60
80
Pool Number

non-increasing marginal extraction costs is correct, then the results shown in Figure 11 imply that a more efficient inter-pool allocation of provincial output could be achieved. They also imply that rents were distributed widely among pools in 1973 that would undoubtedly have been excluded in an efficient allocation.

The unit cost calculations can be used in demonstrating the depth-pay thickness trade-off. At the average value of $2.33, the distribution of unit cost among its 3 components is the following:

Oil well fixed cost: 74.2%

Water injection cost: 9.0%

Operating cost: 16.8%

Holding oc constant, a 10% increase in the average depth of a reservoir leads to a 7.5% increase in the calculated extraction cost per barrel if e = 0.9 and a 10.8% increase if e = 1.3.

It was found that the elasticity of the number of oil wells drilled with respect to pay thickness is -0.33 at sample means. Thus, a 10% increase in the pay thickness of a reservoir leads to a 3.3% decrease in the fixed oil well costs per barrel on average. It can also be calculated that a 10% increase in pay thickness leads to a 25.6% reduction in unit injection costs on average. Combining the two effects, a 10% increase in pay thickness reduces extraction costs per barrel by 4.76% on average.

Thus, a 15.8% increase in pay thickness if e = 0.9 or a 22.7% increase if e = 1.3 will reduce costs by the same amount that a 10% increase in depth will increase costs, all other things held constant. At sample averages, this means that a 2.36 metre increase in pay thickness for e = 0.9 or a 3.40 metre increase for e = 1.3 is required for a 124.8 metre increase in pool depth to keep extraction costs from rising. If the behavioural hypothesis of the competitive model that the pools brought into production in the sixties were in approximately the same cost or quality category, then one would expect the pools in the sample to reflect, on average, this approximate trade-off of between 52.9 and 36.7 metres of depth for each 1 metre of pay thickness. An ordinary least squares regression of depth on pay thickness, forced through the origin, produces a slope coefficient of approximately 37 metres. With a t-statistic of 7.95, it is not significantly different from 52.9. There is, of course a great deal of variation about this regression line probably due to the regulatory influence and this is what causes a good deal of the variation in unit extraction costs across pools.

It is now quite clear that reservoir depth and pay thickness are the two major determinants of extraction cost heterogeneity. It is satisfying that the data observed at a point in time are consistent with there being a positive trade-off on average of the two factors. In the next chapter, the relationship between these two factors over time is examined.

CHAPTER 6

Oil Extraction Costs: Concluding Comments

6.0 Introduction

A basic prediction of the Economic Theory of Natural Resources is that, given positive discount rates, firms will tend to exploit exhaustible resources in order of increasing cost. In the Province of Alberta then, one would expect to observe the depletion of oil resources being manifested in a trend toward the use of lower quality oil pools over time. Drawing on the implications of some of the empirical results of this thesis, this depletion hypothesis is put to the test in Section 6.1. In Section 6.2, the results of the research undertaken in this thesis are summarized, conclusions are drawn and directions for future research are suggested.

6.1 The Depletion Hypothesis

In the previous chapter, a significant inverse relationship was found to exist between extraction costs and pay thickness. It was hypothesized that extraction costs are an increasing function of pool depth. Thus, if oil pools are developed in order of increasing cost, there should exist, in any period of time, a positive trade-off between the depths and pay thicknesses of pools brought into production. A significant relationship of this type was found to exist, on average, among the pools studies in Chapter 5. Moreover, if pools are developed in order of increasing cost, one would observe a worsening of this trade-off over time as lower quality pools are brought into production.

The data set used to test the depletion hypothesis consists of observations on all oil pools[1] discovered in the Province of Alberta between the years 1910 and 1978. Each observation consists of the average pool depth, the average pay thickness and the discovery date of the pool. Ideally one would like to date each observation by development date rather than discovery date. However, the former is not as readily available as the latter. Moreover, discovery date is an excellent measure of development date in most cases since a pool is commonly recorded as a discovery only when development is a profitable undertaking.

The simplest way in which the hypothesis of declining quality can be supported is for the data to show a trend of increasing depth and decreasing pay thickness over time for oil pools brought into production. To test for this possibility the following calculations were made. For each discovery year from 1910 to 1978, the average depth and pay thickness over all pools reported discovered in that year were calculated and plotted over time in Figures 12 and 13.

Only 7 pools were reported discovered before 1940 and for some of these pools pay thickness was not recorded. Thus, in Figure 12, the first positive observation for pay thickness occurs in 1931. From the early 1940's until the late 1960's, pay thickness displays an increasing, not a decreasing, trend followed by a decreasing trend until 1978. An ordinary least squares line, when fitted to this data, has a significantly positive slope.

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1. There were a total of 1567 oil pools and the data were taken taken from a magnetic tape described in Appendix A.

FIGURE 12

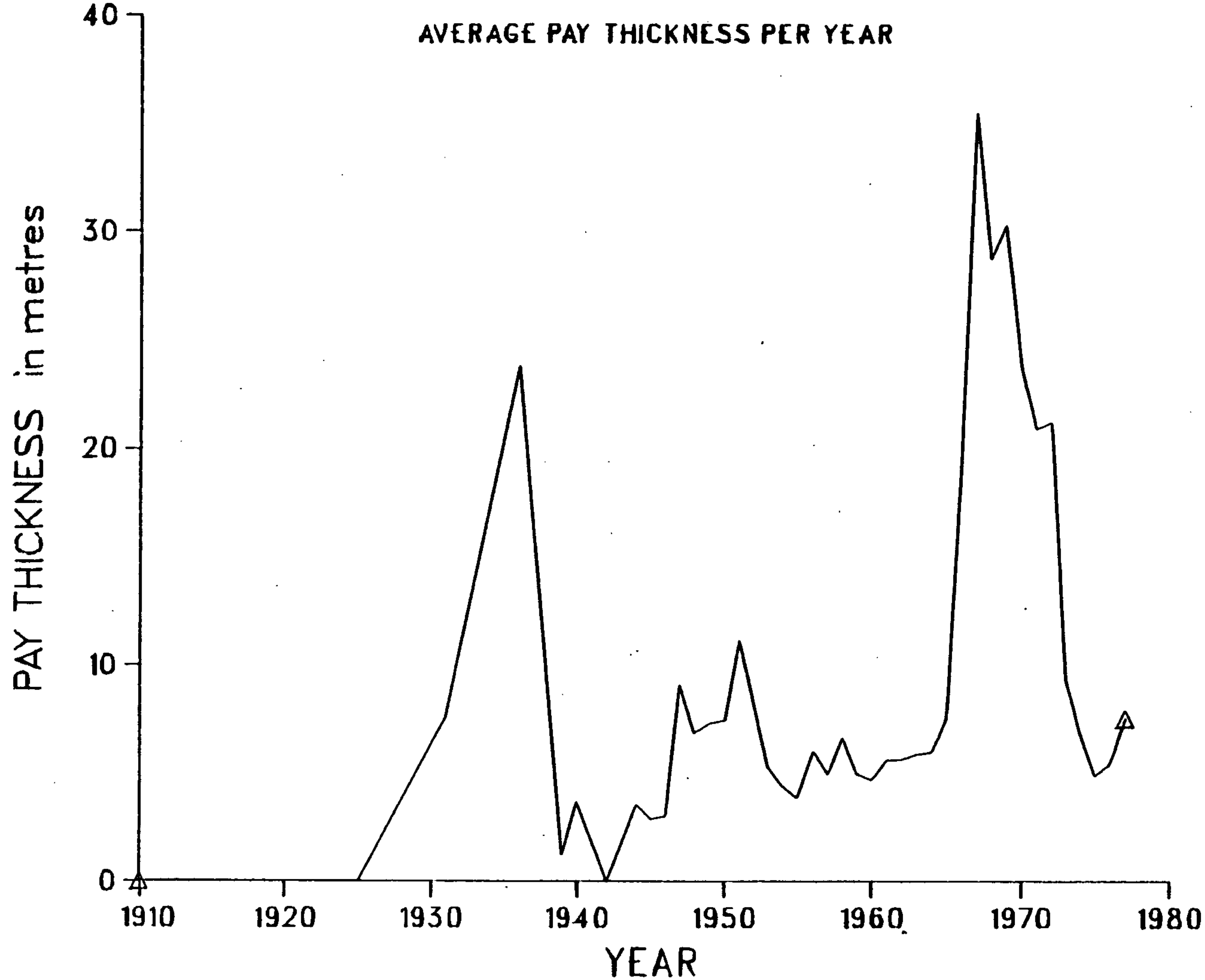
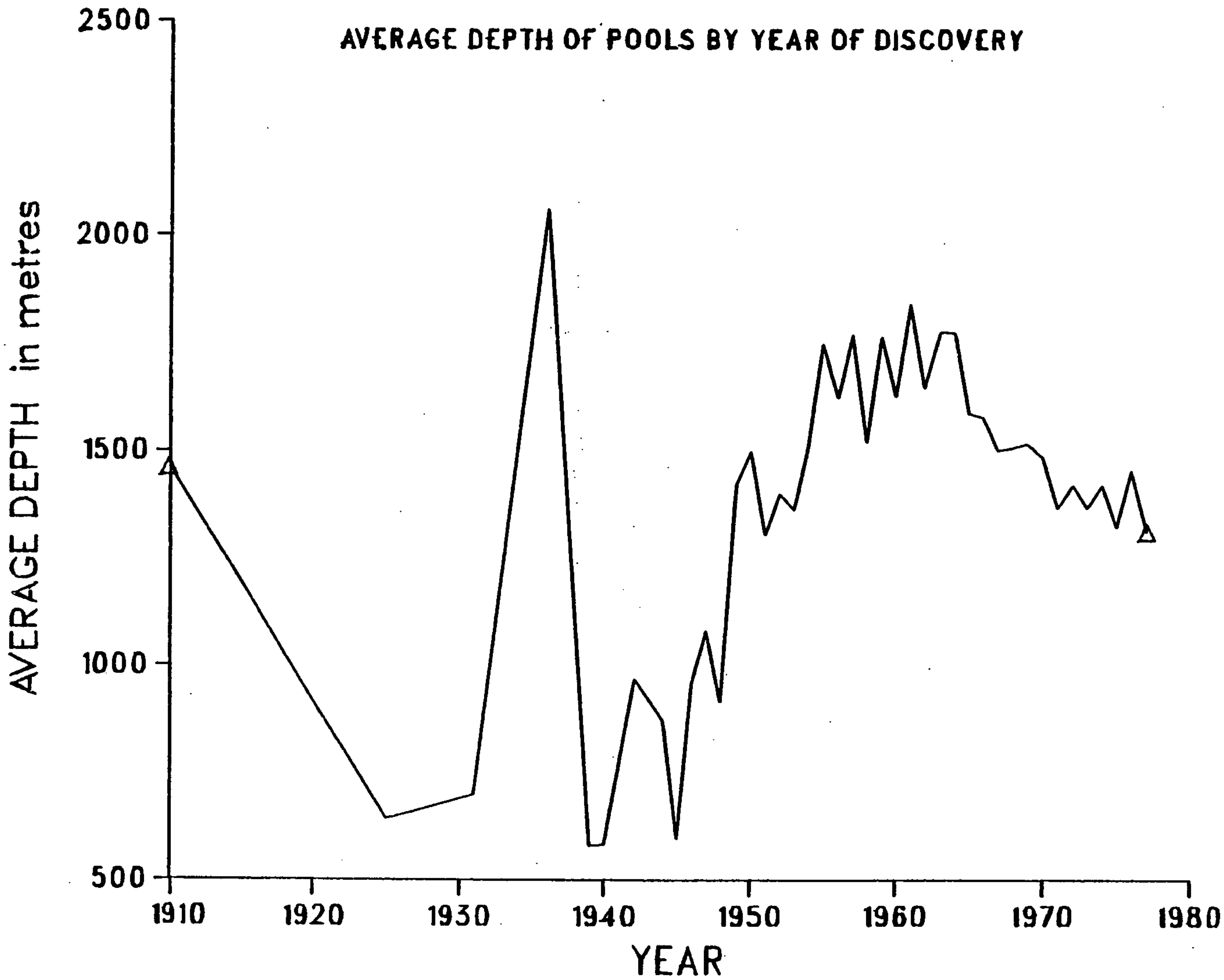

AVERAGE PAY THICKNESS PER YEAR
PAY THICKNESS in metres
40
30
20
10
0
1910
1920
1930
1940
1950
1960
1970
1980
YEAR

FIGURE 13



AVERAGE DEPTH OF POOLS BY YEAR OF DISCOVERY
2500
2000
1500
1000
500
AVERAGE DEPTH in metres
1910
1920
1930
1940
1950
1960
1970
1980
YEAR

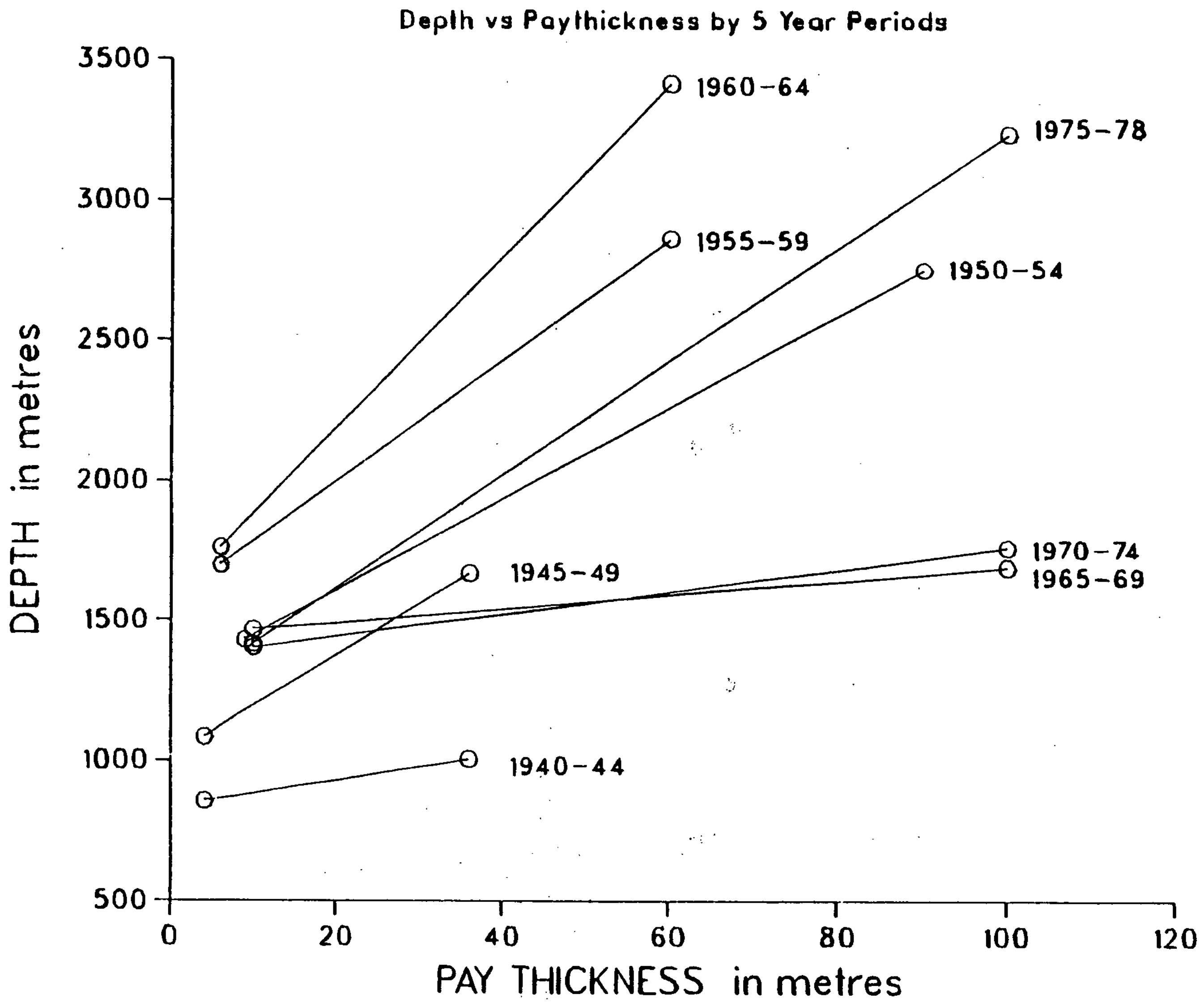
Figure 13 shows an upward trend in the average depths of pools discovered between 1940 and the mid 1960's, and a gradually declining trend thereafter.

Two observations can be made at this point. The evidence in Figures 12 and 13 does not appear to support the hypothesis of declining quality. Moreover, it is apparent that the mid to late 1960's is a key period in which something happened to reverse the upward trends of depth and pay thickness.

A second way in which the data can support the hypothesis of declining quality is to show a worsening over time of the positive trade-off between depth and pay thickness among pools being brought into production. If the hypothesis is correct, plots of depth on the y-axis against pay thickness on the x-axis, which have a positive slope, should shift upwards over time. This behaviour was looked for in the data by fitting least-squares regression lines between the depth and pay thickness of pools discovered in each time period. Time periods were taken to be five-year intervals starting in 1940. The final period was a four-year interval from 1975-1978, producing a total of 8 time periods. For each time period, the predicted values of depth were calculated using a range of pay thickness which did not exceed the observed range for that period. The resulting regression lines are plotted in Figure 14.

It is apparent in Figure 14 that, for the first 5 time periods, the data do display the upward-shifting trade-off behaviour expected under the hypothesis of declining quality. However, the trade-off lines take a dramatic downward shift for the 1965-69 and 1970-74 periods, after which the upward-shifting behaviour is renewed with the 1975-78 period.

FIGURE 14



Depth vs Paythickness by 5 Year Periods
DEPTH in metres
PAY THICKNESS in metres
3500
3000
2500
2000
1500
1000
500
0
20
40
60
80
100
120
1960-64
1975-78
1955-59
1950-54
1945-49
1970-74
1965-69
1940-44

While the sudden reversal in the late 1960's of the 25 year trend towards the use of lower quality oil pools may appear perplexing, the explanation is quite simple. The exploration-discovery relationship is subject to randomness. At any point in time, there is a positive probability that a relatively high-quality, low cost pool or group of pools will be discovered. This is precisely what happened in Alberta in 1966 and continuing into the early 1970's with the discovery of the Zama, Rainbow and Virgo oil fields. The discoveries in this period were dominated by the Zama field in which 100 pools were discovered in 1967 alone. The majority of these pools were relatively shallow with a large pay thickness.

Thus, the data are consistent with the hypothesis of declining quality. However, one must recognize that there is a certain degree of randomness in discovery patterns and hence in the position of a depth-pay thickness trade-off curve at any point in time. Moreover, one must recognize the possiblity that the probability of discovery is positively correlated with the quality of the discovery. One can hypothesize that had the existence of the Zama, Rainbow and Virgo fields been known, their pools would have been exploited long before many of the pools that actually were exploited between 1950 and 1966.

While the data are consistent with the depletion hypothesis of declining quality, this finding does not imply that Alberta oil has become increasingly scarce. If scarcity is defined in economic terms so that extraction cost is a relevant measure, the cost-increasing effect of declining quality over time may or may not have been offset by the cost-reducing effect of technological progress. Moreover, Figure 14 suggests that even in the absence of technological change, average real extraction costs of new pools in the early 1970's were probably less than those of the early 1950's.[2] Moreover, by 1975-78, average real extraction costs of new pools were probably scarcely any higher than in the early 1950's, even in the absence of technological improvements.

These findings suggest that a direction future research might take is to construct indices of real extraction costs between time periods. This could be done by estimating the parameters of the variable extraction cost function for a base period and for some later period and then constructing an aggregate average cost index for each time period using weighted averages of the arguments of the cost function. The ratio of the two indices is a measure of relative scarcity between the two periods. If its value is greater than (less than) (equal to) one, the resource is more (less)(equally) scarce in the later time period. A technical difficulty with this project applied to Alberta oil is the pre-1962 distortions induced by the regulatory framework that probably caused extraction costs to be higher than otherwise.

6.2 Summary and Conclusions

The general concern in writing this dissertation has been to provide some empirical content to Natural Resource Economics and to test some basic predictions of the theory. The specific resource chosen in this application is oil in the Province of Alberta. A dynamic model of oil extraction was constructed, drawing heavily on the principles of oil reservoir engineering and taking care to make it empirically operational. The empirical implications of the model under the assumption of rational

2. Assuming the real prices of factors of production did not display substantial increases.

or optimizing behaviour of a reservoir manager were analyzed. A variable cost function, dual to a restricted one-period technology set was defined and formed the basis of the empirical work. Two versions of the extraction model were estimated. The second model, in which an oil well is viewed as a fixed factor of production chosen at the initial development date and in which water injection is a variable factor chosen in each period proved superior to the first model in which both are viewed as variable factors of production chosen optimally in each period. In both cases, a limited dependent variable problem was successfully overcome in the estimation. The potential simultaneity bias problem created by the restriction that had to be imposed on the one-period technology set was solved analytically, but due to the complexity of the resulting estimation problem, less practical success was achieved.

Estimation of the parameters of the variable cost function permitted information about the extraction technology of individual reservoirs to be obtained and hypothesis tests about that technology and about some theoretical predictions to be performed. It was found that oil pools are not homogeneous with respect to cost and hence, technology. Rather, the pools in the sample showed a high degree of variation in geological factors that significantly affect extraction costs.

The evidence strongly suggests that marginal extraction costs are a non-increasing function of extraction rates in the range of observations. Moreover, marginal extraction costs showed systematic variations across pools in the sample. These results imply that a more efficient allocation of provincial production among pools could be achieved by increasing the shares of the relatively low cost pools.

The empirical results were found to support the testable predictions of the dynamic extraction model. It was possible to infer information about the shadow price of reservoir pressure from the estimates of the variable cost function. The results supported the hypothesis that the shadow price is inversely related to the level of reservoir pressure. Moreover, the results supported the prediction that pressure maintenance activities are more likely to be undertaken in pools with higher shadow prices. Interestingly, one would not expect to obtain these results if the assumption of unitized reservoir management was incorrect. In the absence of cooperative behaviour, the common property problem would make most of the benefits of pressure maintenance external to the individual firm, thereby making the perceived shadow price of pressure very small or equal to zero.

Finally, the evidence of a statistically significant positive trade-off between the depth and pay thickness of pools brought into production at a point in time suggests that the hypothesis that deposits will be exploited in roughly sequential order is supportable. The high degree of variation of predicted unit extraction costs observed at a point in time is consistent with this finding because of the presence of regulatory controls under which relatively inefficient pools are permitted a share of provincial production. The data support the hypothesis of a worsening over time of the positive trade-off between depth and pay thickness with the condition that the trade-off is subject to random improving shifts due to the random nature of the exploration-discovery relationship.

BIBLIOGRAPHY

Berndt, E.R., M.A. Fuss and L. Waverman (1979), "A Dynamic Model of Costs of Adjustment and Interrelated Factor Demands, with an Empirical Application to Energy Demand in U.S. Manufacturing," U.B.C. Discussion Paper No. 79-30.

------, C.J. Morrison and G.C. Watkins (1981), "Dynamic Models of Energy Demand: An Assessment and Comparison" in Berndt, E.R. and B.C. Field (eds) Modeling and Measuring Natural Resource Substitution, The MIT Press: Cambridge, Massachusetts.

Blackorby, C. and Schworm, W. (1982), "Rationalizing the Use of Aggregates in Natural Resource Economics," in W. Eichorn, R. Henn, K. Neumann, and R.W. Shephard (eds), Economic Theory of Natural Resources, Physica Verlag, Wurzburg-Wien.

Bradley, Paul G., (1980) "Modelling Mining: Open Pit Copper Production in British Columbia," Resources Policy, 6, 44-59.

Cairns, Robert D., (1981) "An Application of Depletion Theory to a Base Metal: Canadian Nickel," Canadian Journal of Economics, XIV, 635-648.

Campbell, H. and Scott, A.D., (1980) "Postponement vs Attenuation: An Analysis of Two Strategies for Predicting and Mitigating the Environmental Damage of Large-Scale Uranium Mining Projects," Economic Record, March.

Conrad, R. and B. Hool (1981), "Resource Taxation with Heterogeneous Quality and Endogenous Reserves," Journal of Public Economics 16, 17-33.

Craig, Forrest F. Jr. (1971), The Reservoir Engineering Aspects of Waterflooding, Society of Petroleum Engineers of AIME, New York.

Dake, L.P. (1978), "Fundamentals of Reservoir Engineering, Elsevier Scientific Publishing Company, Amsterdam-Oxford-New York.

Dasgupta, P. and G. Heal (1979), Economic Theory and Exhaustible Resources, Cambridge Economic Handbooks.

Diewert, W.E. (1973), "Functional Forms for Profit and Transformation Functions," Journal of Economic Theory 6, 284-316.

-------, (1974), "Applications of Duality Theory," in M.D. Intriligator and D.A. Kendrick (eds), Frontiers in Quantitative Economics, Volume II, North Holland, Amsterdam.

-------, (1978), "Duality Approaches to Microeconomic Theory," in K.J. Arrow and M.D. Intriligator (eds), Handbook of Mathematical Economics, North Holland, Amsterdam.

-------, and T.R. Lewis (1981), "The Comparative Dynamics of Efficient Programs of Capital Accumulation and Resource Depletion," UBC, PNRE Paper No. 59.

Eswaran, M. and T.R. Lewis (1982), "Exhaustible Resources, Property Rights and Alternative Equilibrium Concepts," U.B.C. Discussion Paper No. 82-34.

Gordon, H.S., (1954) "The Economic Theory of a Common Property Resource: The Fishery," Journal of Political Economy, 124-142.

Hartwick, J.M. (1978), "Exploitation of Many Deposits of An Exhaustible Resource," Econometrica 46, 201-217.

Heaps, T. (1982), "The Taxation of Nonreplenishable Natural Resources Revisited," Paper presented at the Canadian Economic Theory Conference, Edmonton, 1982.

Jones, Park J. (1946), Petroleum Production, Volume I: The Optimum Rate of Production. Reinhold Publishing Corp., New York.

Kemp, M.C. and Long, N.V., (1980), "On Two Folk Theorems Concerning the Extraction of Exhaustible Resources," Econometrica, 48, 663-674.

Kuller, R.G. and R.G. Cummings (1974), "An Economic Model of Production and Investment for Petroleum Reservoirs," American Economic Review LXIV, 66-79.

Lasserre, Pierre, (1982), "Factor Demands and Output Supply Decisions of the Extractive Firm," Unpublished Ph.D. Dissertation, Dept. of Economics, University of British Columbia.

Levhari, D. and N. Liviatan (1977), "Notes on Hotelling's Economics of Exhaustible Resources," Canadian Journal of Economics X, 177-192.

Lewis, Tracy R. (1982), "Sufficient Conditions for Extracting Least Cost Resource First," Econometrica, 50, 1081-1083.

McFadden, D. (1978), "Cost, Revenue, and Profit Functions," in M. Fuss and D. McFadden, Production Economics: A Dual Approach to Theory and Applications, Volume 1. North Holland, Amsterdam.

Peterson, F.M. and A.C. Fisher (1977), "The Exploitation of Extractive Resources: A Survey," Economic Journal, 681-721.

Pindyck, Robert S. (1978), "Optimal Exploration and Production of Nonrenewable Resources," Journal of Political Economy.

Scott, Anthony D. (1967) "The Theory of the Mine under Conditions of Certainty," in Mason Gaffney, (ed) Extractive Resources and Taxation, University of Wisconsin Press.

Skinner, D.R. (1981), Introduction to Petroleum Production, Volume 1. Gulf Publishing Co., Texas.

Slade, Margaret E., (1982), "Tax Policy and the Supply of Exhaustible Resources: Theory and Practice," Resources Paper No. 83, Dept. of Economics, University of British Columbia.

Smith, Charles Robert (1966), Mechanics of Secondary Oil Recovery, Reinhold Publishing Corporation, New York.

Solow, Robert M., (1974), "The Economics of Resources or the Resources of Economics," American Economic Review, 64, 1-14.

Solow, R. and F.Y. Wan (1976), "Extraction Costs in the Theory of Exhaustible Resources," Bell Journal of Economics, 359-378.

Tobin, J. (1958), "Estimation of Relationships for Limited Dependent Variables," Econometrica 26, 24-36.

Uhler, R.S. (1979), "The Rate of Petroleum Exploration and Extraction" in R.S. Pindyck (ed), Advances in the Economics of Energy and Resources, Volume 2, J.A.I. Press, Greenwich, Ct.

Uhler, R.S. (1979a), Oil and Gas Finding Costs, Canadian Energy Research Institute, No. 7, Calgary.

Ulph, A.M. (1978), "A Model of Resource Depletion with Multiple Grades," Economic Record, 54 (147) 334-345.

Wales, T.J. and A.D. Woodland (1980), "Sample Selectivity and the Estimation of Labour Supply Functions," International Economic Review, 21, 437-468.

Watkins, G.C. (1975), "A Note on the Economics of Oil-Enhanced Recovery Projects," in A.D. Scott (ed), Natural Resource Revenues: A Test of Federalism, U.B.C. Press, Vancouver.

-------, (1977), "Conservation and Economic Efficiency: Alberta Oil Proration," Journal of Environmental Economics and Management 4, 40-56.

Watkins, G.C., (1977), "Costs of Supply: Oil Enhanced-Recovery Schemes in Alberta," Bulletin of Canadian Petroleum Geology, 25, 433-440.

White, K.J. (1978), "A General Computer Program for Econometric Methods - SHAZAM", Econometrica 46, 239-240.

Zimmerman, M.B. (1977), "Modeling Depletion in a Mineral Industry: The Case of Coal," Bell Journal of Economics and Management Science 8, 41-65.

APPENDIX A

Econometric Problems

A.1. Limited Dependent Variable Bias

The likelihood function (4.5) is derived as follows. The probability of observing m=o in the r^{th} oil pool, given e_r, is

$$\text{Prob}(m_r = o:e_r) = \text{Prob}(u_r < -h_r(x;a_2,\gamma):e_r)$$

which, using the joint distribution for u and e becomes

$$= \int_{-\infty}^{-h_r(x;a_2,\gamma)} n(u_r,e_r)du_r$$

$$= \int_{-\infty}^{-h_r(x;a_2,\gamma)} n(u_r:e_r)\cdot n(e_r)du_r$$

where n(v) is a normal distribution (the marginal) and $n(u_r:e_r)$ is the conditional normal. This becomes

$$= 1/\sigma_e f(e_r/\sigma_e)\cdot \int_{-\infty}^{-h_r(x;a_2,\gamma)} n(u_r:e_r)du_r$$

$$= a_e f(a_e e_r)\cdot F(y_r)$$

where $y_r = -[h_r(x;a_2,\gamma) + e_r\rho\sigma_u/\sigma_e]/\sigma_u(1-\rho^2)^{1/2}$

The probability of observing $m_r > 0$ is simple given by

$$\text{Prob}(m_r > 0 : e_r) = n(u_r, e_r)$$

Thus, the likelihood of q + R-q observations is the product of the probabilities of observing each observation as given above, and this yields (4.5).

A.2. Simultaneity Bias

The reduced form solution for $\theta(t)$ which appears in (4.10) is derived below. The first order conditions (4.7) are

$$w_1 a_{13} + w_2 a_{23} + [\gamma_{13}W + \gamma_{23}P + \gamma_{33}\theta + \gamma_{34}Q + \gamma_{35}Z](w_1+w_2) - \lambda = 0$$

$$\dot{\lambda} - \delta\lambda = -\{w_1 a_{12} + w_2 a_{22} + [\gamma_{12}W + \gamma_{22}P + \gamma_{23}\theta + \gamma_{24}Q + \gamma_{25}Z](w_1+w_2)\}$$

Thus, the first equation can be explicitly solved for θ and substituted into the $\dot{\lambda}$ equation to yield

$$\begin{aligned}\dot{\lambda} = {} & (\delta - \gamma_{23}/\gamma_{33})\lambda + 1.6(\gamma_{23}^2/\gamma_{33} - \gamma_{22})P + 1.6(\gamma_{23}\gamma_{13}/\gamma_{33} - \gamma_{12})W \\ & + 1.6(\gamma_{23}\gamma_{34}/\gamma_{33} - \gamma_{24})Q + 1.6(\gamma_{23}\gamma_{35}/\gamma_{33} - \gamma_{25})Z \\ & + \gamma_{23}/\gamma_{33}(a_{13} + 0.6a_{23}) - (a_{12} + 0.6a_{22}) \qquad \text{(A4.1)}\end{aligned}$$

where, w_1/w_2 have been set equal to 1.6. Using the solution for θ in $\dot{P} = -\theta$ yields

$$\begin{aligned}\dot{P} = {} & -\lambda/1.6\gamma_{33} + \gamma_{23}/\gamma_{33}P + 1/\gamma_{33}(0.6a_{13} + 0.4a_{23} \\ & + \gamma_{13}W + \gamma_{34}Q + \gamma_{35}Z) \qquad \text{(A4.2)}\end{aligned}$$

Thus (A4.1) and (A4.2) form the explicit analogue of the system in (4.9). This system can be simplified as follows:

$$\dot{P} = A_1\lambda + B_1P + G_1$$
$$\dot{\lambda} = A_2\lambda + B_2P + G_2 \qquad (A4.3)$$

where,

$$A_1 = -0.6\gamma_{33}$$
$$B_1 = \gamma_{23}/\gamma_{33}$$
$$G_1 = C_{11}W + C_{12}Q + C_{13}Z + C_{14}$$
$$C_{11} = \gamma_{13}/\gamma_{33}$$
$$C_{12} = \gamma_{34}/\gamma_{33}$$
$$C_{13} = \gamma_{35}/\gamma_{33}$$
$$C_{14} = (0.6a_{13} + 0.4a_{23})/\gamma_{23} \qquad (A4.4)$$
$$A_2 = \delta - \gamma_{23}/\gamma_{33}$$
$$B_2 = 1.6(\gamma_{23}^2/\gamma_{33} - \gamma_{22})$$
$$G_2 = C_{21}W + C_{22}Q + C_{23}Z + C_{24}$$
$$C_{21} = 1.6(\gamma_{23}\gamma_{13}/\gamma_{33} - \gamma_{12})$$
$$C_{22} = 1.6(\gamma_{23}\gamma_{34}/\gamma_{33} - \gamma_{24})$$
$$C_{24} = \gamma_{23}(a_{13} + 0.6a_{23})/\gamma_{33} - (a_{12} + 0.6a_{22})$$

(A4.3) is a non-homogeneous system of linear differential equations. The solution is tedious but straightforward:

$$P(t) = c_1e^{r_1t} + c_2e^{r_2t} + \Omega_1 \qquad (A4.5)$$

$$\lambda(t) = 1/B_1\{(r_1-A_1)c_1e^{r_1t} + (r_2-A_1)c_2e^{r_2t}\} + \Omega_2 \qquad (A4.6)$$

where,

$$\Omega_1 = (B_2G_1 - A_1G_2)/(A_1A_2 - B_1B_2)$$
$$\Omega_2 = (B_1G_2 - A_2G_1)/(A_1A_2 - B_1B_2) \qquad (A4.7)$$
$$r_1,r_2 = 1/2(A_1+B_1) \pm [(A_1+B_1)^2 - 4(A_1B_2-B_1A_2)]^{1/2}$$
$$c_1 = \{(r_2-A_1)(P_0-\Omega_1) - B_1(\lambda_0-\Omega_2)\}/(r_2-r_1)$$
$$c_2 = \{B_1(\lambda_0-\Omega_2) - (r_1-A_1)(P_0-\Omega_1)\}/(r_2-r_1)$$

Notice that c_1 and c_2 were determined from the initial conditions for the two endogenous variables; ie. $P(0)=P_0$ and $\lambda(0)=\lambda_0$. Also note that r_1 and r_2 are the roots of the characteristic equation.

While P_0 is observable, λ_0 is not. It can be eliminated however by evaluating (A4.6) at t=T and setting $\lambda(T)=0$ and solving the resulting equation for λ_0. Substituting this expression back into (A4.6) yields:

$$\lambda(t) = (A_1B_2-A_1B_1-A_2B_1)(P_0-\Omega_1)\{e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}\}/B_1 + \Omega_2/B_1\{B_1[(r_1-A_1)e^{r_1T} - (r_2-A_1)e^{r_2T}] - B_2[(r_1-A_1)e^{r_1T} - (r_2-A_1)e^{r_2T}]\}/\{(r_1-A_1)e^{r_1T} - (r_2-A_1)e^{r_2T}\} \qquad (A4.8)$$

where,

$$\beta_1 = (A_1+B_1)/2$$
$$\beta_2 = 1/2[(A_1+B_1)^2 - 4(A_1B_2-A_2B_1]^{1/2}$$

Substitution of (A4.7) into

$$\dot{P}(t) = -\theta(t) = A_1\lambda(t) + B_1P(t) + G_1$$

yields the reduced form solution for θ as a function of exogenous variables and the structural parameters of the variable cost function. This appears as (4.10) and (4.11) in Chapter 4 where the following parameter simplifications were made:

$$
\begin{aligned}
h_0 &= (A_1B_2-A_1B_1-A_2B_1)/B_1 \\
h_{11} &= (B_2C_{11}-A_1C_{21})/(A_1A_2-B_1B_2) \\
h_{12} &= (B_2C_{12}-A_1C_{22})/(A_1A_2-B_1B_2) \\
h_{13} &= (B_2C_{13}-A_1C_{23})/(A_1A_2-B_1B_2) \\
h_{14} &= (B_2C_{14}-A_1C_{24})/(A_1A_2-B_1B_2) \\
h_{21} &= (B_1C_{21}-A_2C_{11})/(A_1A_2-B_1B_2) \\
h_{22} &= (B_1C_{22}-A_2C_{12})/(A_1A_2-B_1B_2) \\
h_{23} &= (B_1C_{23}-A_2C_{13})/(A_1A_2-B_1B_2) \\
h_{24} &= (B_1C_{24}-A_2C_{14})/(A_1A_2-B_1B_2)
\end{aligned}
\tag{A4.9}
$$

If the system of equations is to be estimated without imposing the parameter restrictions in (A4.4), (A4.7) and (A4.9) then only the distinct paramters of (A4.8) can be estimated. To determine what these are, carry out the substitution of $\lambda(t)$ in the $-\theta(t)$ equation to obtain:

$$
\begin{aligned}
-\theta(t) = {} & A_1h_0P_0Y_1 + [C_{11} - A_1h_0h_{11}Y_1 + A_1h_0h_{21}Y_2]W + \\
& [C_{12} - A_1h_0h_{12}Y_1 + A_1h_0h_{22}Y_2]Q + \\
& [C_{13} - A_1h_0h_{13}Y_1 + A_1h_0h_{23}Y_2]Z + \\
& [C_{14} - A_1h_0h_{14}Y_1 + A_1h_0h_{24}Y_2] + B_1P
\end{aligned}
\tag{A4.10}
$$

where $Y_1 = [e^{\beta_1(T+t)+\beta_2(T-t)} - e^{\beta_1(T+t)-\beta_2(T-t)}]$

$$Y_2 = \frac{(r_1-A_1)e^{r_1t}-(r_2-A_1)e^{r_2t}-B_2/B_1[(r_1-A_1)e^{r_1t}-(r_2-A_1)e^{r_2t}]}{(r_1-A_1)e^{r_1t} - (r_2-A_1)e^{r_2t}}$$

There are clearly too many paramters in (A4.10) to obtain unique estimates of each. Some will have to be almalgamated with others. There is more than one way to do this but the following is the one chosen:

$$-\theta(t) = \alpha_0 P_0 Y_1 + [\alpha_1-\alpha_0 h_{11}Y_1]W + [\alpha_2-\alpha_0 h_{12}Y_1]Q + [\alpha_3-\alpha_0 h_{13}Y_1]Z + [\alpha_4-\alpha_0 h_{14}Y_1] + B_1P \qquad (A4.11)$$

where
$$\alpha_0 = A_1h_0$$
$$\alpha_1 = C_{11} + A_1h_0h_{21}Y_2$$
$$\alpha_2 = C_{12} + A_1h_0h_{22}Y_2 \qquad (A4.12)$$
$$\alpha_3 = C_{13} + A_1h_0h_{23}Y_2$$
$$\alpha_4 = C_{14} + A_1h_0h_{24}Y_2$$

Because there still remain non-unique parameters in (A4.11) the following restrictions must be imposed:

$$h_{11} = h_{12} = h_{13} = h_{14} = 1$$

The resulting reduced form equation below, which also appears as (4.12), has 8 unique parameters:

$$-\theta(t) = \alpha_0 P_0 Y_1 + [\alpha_1-\alpha_0 Y_1]W + [\alpha_2-\alpha_0 Y_1]Q + [\alpha_3-\alpha_0 Y_1]Z + [\alpha_4-\alpha_0 Y_1] + B_1P \qquad (A4.13)$$

A.3. The Optimal Number of Wells

In the following, (4.19) is solved for N. No attempt is made to preserve the relationships between structural and reduced form parameters.

Using (4.14)

$$
\begin{aligned}
\partial m/\partial N(\cdot) &= b_6 + g_{16}W + g_{26}P + g_{36}\theta + g_{46}Q + g_{56}Z + g_{66}N \\
&+ [b_2 + g_{12}W + g_{22}P + g_{23}\theta + g_{24}Q + g_{25}Z + g_{26}N]\partial P/\partial N \\
&+ [b_3 + g_{13}W + g_{23}P + g_{33}\theta + g_{34}Q + g_{35}Z + g_{36}N]\partial \theta/\partial N \qquad \text{(A4.14)}
\end{aligned}
$$

where,

$$
\begin{aligned}
P(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t} + \Omega_1 \\
\dot{P}(t) &= -\theta(t) = r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t} \\
c_1 &= a_{10}P_0 + a_{11}W + a_{12}Q + a_{13}Z + a_{14}N + a_{15} + a_{16}\lambda_0 \\
c_2 &= a_{20}P_0 + a_{21}W + a_{22}Q + a_{23}Z + a_{24}N + a_{25} + a_{26}\lambda_0 \\
\lambda_0 &= a_0(P_0 - \Omega_1)[e^{r_1 T} - e^{r_2 T}] - a_1\Omega_2 \\
\Omega_1 &= b_{11}W + b_{12}Q + b_{13}Z + b_{14}N + b_{15} \\
\Omega_2 &= b_{21}W + b_{22}Q + b_{23}Z + b_{24}N + b_{25}
\end{aligned}
$$

Let $y_1 = e^{r_1 T} - e^{r_2 T}$ and notice that it is constant over time. Thus,

$$
\begin{aligned}
c_1 &= [a_{10}+a_0 y_1]P_0 + [a_{11}-b_{11}y_1]W + [a_{12}-b_{12}y_1]Q \\
&\quad + [a_{13}-b_{13}y_1]Z + [a_{14}-b_{14}y_1]N + [a_{15}-b_{15}y_1] - a_1\Omega_2
\end{aligned}
$$

A similar expression exists for c_2. The parameters in these expressions will not be identifiable so are amalgamated at this stage. This yields:

$$c_1 = d_{10}P_0 + d_{11}W + d_{12}Q + d_{13}Z + d_{14}N + d_{15}$$

$$c_2 = d_{20}P_0 + d_{21}W + d_{22}Q + d_{23}Z + d_{24}N + d_{25}$$

Therefore,

$$\partial P(t)/\partial N = d_{14}e^{r_1 t} + d_{24}e^{r_2 t} + b_{14}$$

$$\partial\theta/\partial N = -\{r_1 d_{14}e^{r_1 t} + r_2 d_{24}e^{r_2 t}\}$$

Now, rewrite P(t) and θ(t) using the definitions of c_1,c_2 and Ω_1

$$\begin{aligned} P(t) = {} & [d_{10}e^{r_1 t}+d_{20}e^{r_2 t}]P_0 + [b_{11}+d_{11}e^{r_1 t}+d_{21}e^{r_2 t}]W + \\ & [b_{12}+d_{12}e^{r_1 t}+d_{22}e^{r_2 t}]Q + [b_{13}+d_{13}e^{r_1 t}+d_{23}e^{r_2 t}]Z + \\ & [b_{14}+d_{14}e^{r_1 t}+d_{24}e^{r_2 t}]N + [b_{15}+d_{15}e^{r_1 t}+d_{25}e^{r_2 t}] \end{aligned}$$

$$\begin{aligned} \theta(t) = {} & -[r_1 d_{10}e^{r_1 t}+r_2 d_{20}e^{r_2 t}]P_0 - [r_1 d_{11}e^{r_1 t}+r_2 d_{21}e^{r_2 t}]W - \\ & [r_1 d_{12}e^{r_1 t}+r_2 d_{22}e^{r_2 t}]Q - [r_1 d_{13}e^{r_1 t}+r_2 d_{23}e^{r_2 t}]Z - \\ & [r_1 d_{14}e^{r_1 t}+r_2 d_{24}e^{r_2 t}]N - [r_1 d_{15}e^{r_1 t}+r_2 d_{25}e^{r_2 t}] \end{aligned}$$

The next step is to substitute these expressions into the expression for $\partial m(\cdot)/\partial N$ and then combine like terms. One need only do this for one of the terms because of the similarity of the coefficients on terms. This is done below for the W term:

$$\partial m(\cdot)/\partial N = \ldots + W\{g_{16}+(d_{14}e^{r_1t}+d_{24}e^{r_2t}+b_{14})$$

$$[g_{12}+g_{22}(b_{11}+d_{11}e^{r_1t}+d_{21}e^{r_2t}) - g_{23}(r_1d_{11}e^{r_1t}+r_2d_{21}e^{r_2t})] -$$

$$(r_1d_{14}e^{r_1t}+r_2d_{24}e^{r_2t})[g_{13}+g_{23}(b_{11}+d_{11}e^{r_1t}+d_{21}e^{r_2t}) -$$

$$g_{33}(r_1d_{11}e^{r_1t}+r_2d_{21}e^{r_2t})]\} + \ldots$$

Collecting like exponent terms in this coefficient and almalgamating parameters yields:

$$\partial m(\cdot)/\partial N = \ldots + W\{f_{10}+f_{11}e^{r_1t}+f_{12}e^{r_2t}+f_{13}e^{2r_1t}+f_{14}e^{(r_1+r_2)t}+f_{15}e^{2r_2t}\}+\ldots$$

The partial derivative $\partial m/\partial N$ can now be integrated and then solved for N. Define the following variables obtained upon integration:

$$q_0 = (1 - e^{-\delta T})/\delta$$

$$q_1 = [e^{(r_1-\delta)T}-1]/(r_1-\delta)$$

$$q_2 = [e^{(r_2-\delta)T}-1]/(r_2-\delta)$$

$$q_3 = [e^{(2r_1-\delta)T}-1]/(2r_1-\delta)$$

$$q_4 = [e^{(r_1+r_2-\delta)T}-1]/(r_1+r_2-\delta)$$

$$q_5 = [e^{(2r_2-\delta)T}-1]/(2r_2-\delta)$$

Using these definitions, and carrying out the integration, the first order condition (4.19), solved for N yields:

$$
\begin{aligned}
N = & -\phi/D - W(f_{10}q_0+f_{11}q_1+f_{12}q_2+f_{13}q_3+f_{14}q_4+f_{15}q_5)/D \\
& - Q(f_{20}q_0+f_{21}q_1+f_{22}q_2+f_{23}q_3+f_{24}q_4+f_{25}q_5)/D \\
& - Z(f_{30}q_0+f_{31}q_1+f_{32}q_2+f_{33}q_3+f_{34}q_4+f_{35}q_5)/D \qquad \text{(A4.16)} \\
& - P_0(f_{51}q_1+f_{52}q_2+f_{53}q_3+f_{54}q_4+f_{55}q_5)/D \\
& - \{f_{60}q_0+f_{61}q_1+f_{62}q_2\}/D
\end{aligned}
$$

where, $D = f_{40}q_0+f_{41}q_1+f_{42}q_2+f_{43}q_3+f_{44}q_4+f_{45}q_5$ (A4.17)

and the f_{ij} are the reduced form parameters to be estimated.

APPENDIX B

B.1. Data Sources

The raw reservoir data were obtained form publications and computer tapes prepared by the Energy Resources Conservation Board of Alberta (ERCB). A modified version[1] of the General Well Data File provides detailed information about each oil well drilled in the province. Each well is classified by type according to the Lahee system (development well, service well, wildcat etc.) and by status according to a 99-digit ERCB code. The date at which the status is reported, the date of drilling completion, the name, depth and the codes defining the field and reservoir into which the well was completed are also reported. Also available on this tape is detailed information about the physical properties (such as pay thickness and porosity) of the reservoir into which the well was completed. An additional tape provides annual production and injection data for all oil pools in the province.[2] From the data on this tape, observations on annual oil production and annual water, gas and natural gas liquids injection were drawn.

The data for the pressure variable and the annual change in pressure variable were obtained from ERCB publication 76-10: Reservoir Performance Charts: Oil Pools. This publication was also used to supplement the data for pay thickness and porosity as some observations were missing from the computer tape.

The size of the sample was determined by the number of observations that were common to all three data sources. For most variables the data

1. The extensive modification, made by Russell Uhler, linked each observation in the General Well Data File with detailed physical reservoir data for the reservoir to which the observed well is attached.
2. This tape was also kindly made available to me by Russell Uhler.

could be used in their raw form as drawn from the tapes or publication. However, to generate the data for the number of wells by type (according to the Lahee classification), status and date for each pool required extensive programming.

Additional data used to estimate the reduced form equation in Model II for the number of wells were obtained from ERCB publication 78-10: Reservoir Performance Charts: Oil Pools and ERCB publication 78-18: Reserves of Crude Oil, Gas, Natural Cas Liquids and Sulfur, Province of Alberta.

To calculate the relative price of oil wells series for Model II, the capital price per oil well was calculated by dividing total industry expenditures on development well drilling and related surface equipment by the total number of completions for each year. These data were obtained from Canadian Petroleum Association Statistical Handbook, 1981 and ERCB 80-17, Alberta Oil and Gas Industry Annual Statistics. The capital price per injection well was calculated in the same way. Total development expenditures by the Alberta industry by year is taken from the Canadian Petroleum Association Statistical Handbook. The net change in the stock of water and gas injection wells by year was calculated from the "Statement of Service and Capped Gas Wells as at Dec. 31" in ECRB 80-17. Whenever required, prices were converted to real terms using the price index for machinery and equipment from: "Price indices for capital expenditure on plant and equipment by industry (1974): Mines, Quarries and Oil Wells" which is contained in Statistics Canada 13-568 Occasional, Fixed Capital Flows and Stocks 1926-1978, pp. 279-280.

In the estimation of the reduced form equation for N in Model II, the oil production variable is not the same as the variable used in the

1973 cross-section estimation equations. The values for oil production in the former, in the units of m^3 per day, are average daily extraction rates observed in the years following development and are meant to reflect the expected rates at the time of development. The source of these data was ERCB 78-10: Reservoir Performance Charts.

The calculation of 1973 average operating costs per barrel in Chapter 5, was made by dividing total industry operating expenditures in 1973, available in the CPA Statistical Handbook by total Alberta oil production, available in ERCB, Alberta Oil and Gas Industry Annual Statistics.

B.2. List of Pool Names

The following is a list which shows the Field Code, Pool Code, Field Name and Pool Name of each pool used in the 1973 sample.

047	644001	Alexis - Banff A
092	248004	Bantry - Mannville D
144	788001	Black - Keg River A
157	500003	Boundary Lake South - Triassic C
157	500005	" " " - Triassic E
185	244006	Campbell Namao - Namao Blairmore F
212	294004	Chauvin South - Lloydminster D
213	644001	Cherhill - Banff A
259	250008	Countess - Upper Mannville H
269	638004	Crossfield East - Elkton D
273	294001	David - Lloydminster A

320	176002	Edson - Cardium B
377	176004	Ferrier - Cardium D
377	176005	" - Cardium E
377	176007	" - Cardium G
405	248002	Garrington - Mannville B
423	758001	Golden - Slave Point A
425	744001	Goose River - Beaverhill Lake A
430	250002	Grand Forks - Upper Mannville B
456	310001	Hays - Lower Mannville A
457	322001	Hayter - Dina A
457	322002	" - Dina B
486	300002	Hussar - Glauconitic B
486	320015	" - Basal Mannville O
500	250005	Jenner - Upper Mannville E
500	250006	" - Upper Mannville F
500	250015	" - Upper Mannville O
547	250001	Lathom - Upper Mannville A
560	250004	Little Bow - Upper Mannville D
560	250007	" " - Upper Mannville G
571	276007	Lloydminster - Sparky G
571	276163	" - Sparky D and Gen Pete B
604	300001	Medicine River - Glauconitic A
604	642005	" " - Pekisko E
605	696001	Meekwap - D-2A
615	778001	Mitsue - Gilwood A
620	720002	Morinville - D-3B
644	778001	Nipisi - Gilwood A

644	778003	Nipisi - Gilwood C
644	789001	" - Keg River SS A
650	334002	Niton - Basal Quartz B
668	658001	Olds - Wabamun A
682	126001	Peco - Belly River A
685	126021	Pembina - Key Belly River U
685	126024	" - Key Belly River X
685	126027	" - Belly River AA
753	782006	Rainbow - Muskeg F
753	788001	" - Keg River A
753	788002	" - Keg River B
753	788005	" - Keg River E
753	788006	" - Keg River F
753	788007	" - Keg River G
753	788008	" - Keg River H
753	788009	" - Keg River I
753	788011	" - Keg River K
753	788015	" - Keg River O
753	788019	" - Keg River S
753	788020	" - Keg River T
753	788021	" - Keg River U
753	788027	" - Keg River AA
753	788031	" - Keg River EE
753	788035	" - Keg River II
753	788038	" - Keg River LL
753	788105	" - Keg River EEE
753	788117	" - Keg River QQQ

754	788001	Rainbow South - Keg River A
754	788002	" " - Keg River B
754	788005	" " - Keg River E
754	788007	" " - Keg River G
764	758001	Red Earth - Slave Point A
764	758005	" " - Slave Point E
764	976005	" " - Granite Wash E
785	176001	Ricinus - Cardium A
785	176004	" - Cardium D
785	176012	" - Cardium L
886	696001	Swalwell - D-2 A
891	638003	Sylvan Lake - Elkton C
891	642003	" " - Pekisko C
894	346001	Taber North - Taber A
895	248001	Taber South - Mannville A