TUNNELING RESISTANCE OF A ONE DIMENSIONAL RANDOM LATTICE

By

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE in THE FACULTY OF GRADUATE STUDIES (Department of Physics)

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

September, 1984

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The resistivity of a one-dimensional lattice consisting of randomly distributed conductivity and insulating sites is considered. Tunneling resistance of the form $R_i e^{i e}$ is assumed for a cluster of $i$ adjacent insulating sites. Three different ensembles are considered and compared. In the first ensemble the number of insulating "atoms" is fixed and distributed in a linear chain; in the second one there exists a fixed probability $p$ of having an insulator "atom" occupying a site in a linear chain, and finally the third one consists of a line bent into a circle and the probability $p$ is considered. It is observed that in the thermodynamic limit, the average ensemble resistance per site diverges at the critical filling fraction $p_c = e^{-b}$, while the variance of the resistance diverges at the lower filling fraction $p_{c1} = p_c^2$. Computer simulations of large but finite systems, however, exhibit a much weaker divergence of the resistance per site at $p_c$ and no divergence of the variance at $p_{c1}$. 
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DEDICATION

To My Son
ACKNOWLEDGEMENTS

I would like to thank my supervisor, Professor R. Barrie, for his support and guidance throughout this work.

I would also like to thank Professor B. Bergersen and Dr. P. Palffy-Muhoray for their help with the computer.

Finally, I wish to thank the funding through CAPES, Brasil, and NSERC.
1.1 Motivation for the Work

The properties of composite materials of metals and insulators have been the subject of many studies (ABE75). These materials have three different electronic conduction regimes: metallic regime, dielectric regime and transition regime.

The original motivation for this work was a study of the mechanism of conduction in these granular metals in the dielectric regime. In this regime metal particles are dispersed in a dielectric continuum.

Abeles (ABE75), in his work, calculates the low-field conductivity, \( \sigma_L \), in the dielectric regime of granular metals taking into consideration the tunneling through an insulator, from one isolated metallic grain to the next one. He takes into account the charging energy, \( E_C \), that is necessary to remove an electron from one neutral metal grain and place it on another neutral metal grain. Therefore, the expression for the total low-field conductivity, \( \sigma_L \), is the sum of products of mobility, charge and number density of charge carriers over all possible percolation paths:

\[
\sigma_L \propto \int_0^\infty \beta(s) \exp[-2x_s - (E_C^0/2KT)] \, ds;
\]

where \( \beta(s) \) is the density of percolation paths associated with the value \( s \) and \( s \) is the grain separation. The exponent \( e^{-2x_s} \) is the tunneling probability with \( x = (2m\phi/h^2)^{1/2} \), \( m \) the electron mass, \( \phi \) the barrier
height and $h$ Planck's constant. The exponent $e^{-E^0/2KT}$ is the Boltzmann factor, which is proportional to the number density of all charge carriers with charging energy $E^0_c$.

After making some approximations, Abeles evaluates the expression and obtains:

$$\sigma_L = \sigma_0 e^{-2(C/KT)^{1/2}};$$

where $\sigma_0$ is a constant independent of temperature and $C = \chi s e^0_c$.

In order to isolate the role of tunneling in this work, one can look at zero temperature and low-field conductivity, removing the idea of a charging energy factor. Thus one can consider the conduction to be through a one dimensional series of metal-insulator-metal junctions.

1.2 Description of the Model

The tunneling effect in junctions of metal-insulator-metal has been widely discussed (DUK69). In the present work, the regime of low voltage and the tunneling between two metals of the same kind at zero temperature will be considered.

Since our interest is in the low voltage ohmic regime, the potential barrier will be considered as rectangular.

The expression for the tunneling current, for the above case, was obtained by Sommerfeld and Bethe (SOM33) and is reproduced in the paper by Holm (HOL51). Figure (1) shows the diagram of a trapezoidal barrier in an insulating film between two similar metal electrodes. In the ohmic regime the barrier can be considered as a rectangular one.
**Figure 1**: Diagram of a trapezoidal barrier in an insulating film between two similar electrodes.

The equation for the tunneling resistance per unit cross section area is (HOL51):

\[
R = \left( \frac{h^2}{q^2} \frac{\Delta s}{(2m\phi)^{1/2}} \right) \exp\left( \frac{4\pi \Delta s}{h} \left( 2m\phi \right)^{1/2} \right); \tag{1}
\]

where \( h \) = Planck's constant

\( q \) = electron charge

\( \Delta s \) = thickness of the junction

\( m \) = electron mass

\( \phi \) = work function

For our purposes we will write the barrier thickness as \( \Delta s = i\ell_0 \), where \( \ell_0 \) is the unit of "atomic" spacing and \( i \) corresponds to the number of sites with unit length \( \ell_0 \).

The resistance \( R \) of a single insulating cluster of \( i \) sites due to tunneling will then be of the form:
where \( R_o = h^2 \frac{\epsilon_0^2}{q^2} (2m\phi)^{1/2} \) and \( b = (4 \pi \frac{\epsilon_0}{\hbar}) (2m\phi)^{1/2} \).

To obtain equation (1), the tunneling probability for a rectangular barrier was calculated and this was done by the evaluation of the transmission coefficient \( D(E_x) \) using the WKB approximation.

As a model of the one dimensional granular metal, consider a linear chain of length \( N\ell_o \) of metal and insulator "atoms" distributed randomly. Each site of length \( \ell_o \) can be occupied by a metal or an insulator "atom". A cluster of \( i \) consecutive insulating units, each of unit length \( \ell_o \), will form an island of size \( i \), Figure (2).

\[ R = R_0 i e^{bi} \]  \hspace{1cm} (2)

---

Figure 2: Diagram of a linear chain of metal-insulator-metal junctions.

In the future we will talk about an array of \( N \) sites. The resistances of the metal islands are going to be considered equal to zero and the resistance of an insulating island will be given by equation (2). In other words, the linear array will be composed of a series of metal-insulator-metal junctions. A typical junction of metal-insulator-metal, with unit length \( \ell_o \) equal to 5\( \text{Å} \), would have \( b = 5.125 \phi^{1/2} \). Considering,
for instance, the case of SiO$_2$(ABE75), $b$ would be of the order of 10. In order to illustrate the study of the statistics of the problem, the value of $b$ will be taken to be $\ln 2$. This choice will become clear later (See p. 9).
CHAPTER TWO

STATISTICS OF THE MODEL

The present model will be concerned with the ohmic regime and zero temperature.

Consider a linear chain of sites that is randomly occupied, either by an insulator "atom" or a metal "atom". The end two sites of the chain are filled with metal. Each arrangement of metal-insulator-metal will form a junction. We consider the linear chain to be formed of a set of independent junctions.

Take $n_1$ as the number of dielectric atoms, $(n_2 - 2)$ the number of metal atoms such that $N = n_1 + n_2$ is the number of sites in the linear chain.

The calculation of the resistance and variance for different ensembles, in this model, is described below.

2.1 Fixed Number of Insulator "Atoms" Distributed in the Chain

In the case of a fixed number of insulator "atoms" distributed in a chain with metal ends, the total ensemble average of the resistance will be given by:

$$< R > = R_0 \sum_{i=1}^{n_1} i e^{b_i} < K_i >,$$

where $< K_i >$ is the average number of dielectric clusters of size $i$ ($i = 1, 2, \ldots, n_1$).
In analogy with the distribution of runs (WIL62), the probability of having \( K_i \) dielectric clusters of size \( i \) (\( i = 1, 2, \ldots, n_1 \)) is:

\[
P(K_i, [K_i]) = \frac{n_1!}{K_1! K_2! \cdots K_{n_1}!} \binom{n_2 - 1}{K} \binom{N - 2}{n_1};
\]

where \( [K_i] = \{K_1, K_2, \ldots, K_{n_1}\} \), \( \sum_{i=1}^{n_1} K_i = K \), \( \sum_{i=1}^{n_1} i K_i = n_1 \) and \( K \) is the total number of metal-insulator-metal junctions. The value of \( K \) ranges from 0 to \( n_1 \) and the value of \( n_1 \) from 0 to \( N-2 \).

The expression for \( \langle K_i \rangle \) can then be evaluated

\[
\langle K_i \rangle = \sum_{K=0}^{n_1} \sum_{[K]} \frac{n_1!}{K_1! K_2! \cdots K_{n_1}!} \binom{n_2 - 1}{K} \binom{N - 2}{n_1} K_i.
\]

It is shown in Appendix A that:

\[
\langle K_i \rangle = \binom{n_2 - 1}{N - 2} \binom{N - i - 3}{n_1 - 1} \tag{4}
\]

The total resistance will then be given by:

\[
\langle R \rangle = R_0 \sum_{i=1}^{n_1} i e^{bi} \binom{n_2 - 1}{N - 2} \binom{N - i - 3}{n_1 - i} \tag{5}
\]

We have not been able to express this sum in any significantly simpler fashion. Figures (3a) and (3b) show the graph of the resistance per site versus the fraction \( n_1/N \) for values of \( N = 100 \) and \( N = 200 \).
Figure 3: Dependence of the ensemble average resistance per site on the fraction $n_1/N$. Ensembles with fixed $n_1$ and fixed probability $p$ in a line. (a) $N = 100$ (b) $N = 200$
In the limit of $N$, $n_1$ and $n_2$ tending to infinity, such that
\[ \frac{n_1}{N} = p \text{ and } \frac{n_2}{N} = (1 - p), \]
the expression for the resistance per site is:
\[
\frac{\langle R \rangle}{NR_0} \rightarrow N(1 - p)^2 \frac{pe^b}{(1 - pe^b)^2}.
\]

Therefore, the average resistance per site diverges for an infinite chain, when $p = p_c = e^{-b}$. The value of $b$ is then chosen to be equal to $\ln 2$ such that the filling factor $p_c = 1/2$ and is at the centre of the interval from $p = 0$ to $p = 1$.

In order to determine $\langle R^2 \rangle$, the moment $\langle K_i K_j \rangle$ has to be calculated by using the following expression:
\[
\langle K_i K_j \rangle = \sum_K \sum_{K'} \frac{K!}{K_1! K_2! \ldots K_{n_1}!} \langle K \rangle^i \langle K \rangle^j,
\]
with $\sum_i K_i = K$ and $\sum_i K_i = n_1$.

From Appendix B, the expression for $\langle K_i K_j \rangle$ is:
\[
\langle K_i K_j \rangle = \frac{(n_2 - 1)(n_2 - 2)}{(N - 2)^2} \frac{(N - i - j - 4)}{n_2 - 4} + \delta_{ij} \langle K_i \rangle;
\]
it is to be noted that the first term is zero unless $i + j < n_1$.

This leads to:
\[
\langle R^2 \rangle = R_0^2 \sum_{i,j} i j e^{b(i+j)} \langle K_i K_j \rangle.
\]

The average of the square of the resistance is calculated in
Appendix C to be:

\[
\langle R^2 \rangle = R_0^2 \left\{ \frac{(n_2 - 1)(n_3 - 2)}{(N - 2)} \sum_{m=0}^{n_1} \frac{(4m^3 - m)}{12} e^{bm} \left( \frac{N - m - 4}{n_2 - 4} \right) + \right. \\
+ \frac{(n_2 - 1)}{(n_1)} \sum_{i=1}^{n_1} i^2 e^{2bi} \left( \frac{N - i - 3}{n_2 - 3} \right) \}.
\]

Again, we have not been able to express these sums in any simpler form.

In the limit of \( N, n_1, \) and \( n_2 \) tending to infinity, such that \( n_1/N = p \) and \( n_2/N = (1 - p) \), the expression for the variance is

\[
\frac{\langle R^2 \rangle - \langle R \rangle^2}{R_0^2} = N(1 - p)^2 \frac{pe^{2b}(1 + pe^{2b})}{(1 - pe^{2b})^3}
\]

Therefore, the variance diverges for an infinite chain when

\[
p = p_{c_1} = e^{-2b}.
\]

In order to evaluate the variance, the \( \langle R \rangle^2 \) and \( \langle R^2 \rangle \) are calculated numerically. The behavior of the variance with the ratio \( n_1/N \) is shown in Figure (4).

2.2 Probability \( p \) of Having an Insulator Occupying a Site

In this ensemble, there is a fixed probability \( p \) of having an interior site occupied by an insulator, and probability \( (1-p) \) for such a site to be occupied by a metal. The two end exterior sites are occupied by metal. The probability \( (M0041) \) of having \( n_1 \) dielectric "atoms" distributed in \( K_i \) clusters of size \( i(i = 1, 2, \ldots, n_1) \), is:
Figure 4: Dependence of the variance \((<R^2>-<R>^2)/N^2R_0^2\) on the fraction \(n_1/N\). Ensemble with fixed \(n_1 (N = 100)\).
\[ P(n_1; K; \{ K_i \}) = p^{n_1} (1 - p)^{n_2 - 2} \left( \frac{K!}{K_1! \cdots K_{n_1}!} \right) \binom{n_2 - 1}{K} \] (11)

with \( \sum_{i=1}^{n_1} K_i = K \) and \( \sum_{i=1}^{n_1} i K_i = n_1 \) and \( n_1 \) taking values from zero to \( (N - 2) \).

The evaluation of \( \langle K_i \rangle \) is performed on Appendix D, and it gives:

\[ \langle K_i \rangle = \sum_{n_1=0}^{N-2} \sum_{K=1}^{n_1} \sum_{\{ K_i \}} P(n_1; K; \{ K_i \}) K_i p^{n_1} (1 - p)^{n_2 - 2} \] (12)

with \( \sum_{i=1}^{n_1} K_i = K \) and \( \sum_{i=1}^{n_1} i = n_1 \).

Therefore,

\[ \langle K_i \rangle = \begin{cases} p^i (1 - p) \{(1 - p) (N - i - 3) + 2\}, & i < N - 2 \\ p^{N-2} & i = N - 2 \end{cases} \] (13)

The total resistance is then evaluated

\[ \langle R \rangle = R_o \left\{ \sum_{i=1}^{N-3} i p^i (1 - p) \left[ 2 + (1 - p)(N - 3 - i) \right] e^{bi} \right\} \\
+ (N - 2) (pe^b)^{N-2} \] (14)

After performing the sum over the island sizes:

\[ \langle R \rangle / R_o =\]

\[ (1 - p) \left[ 2 + (1 - p)(N - 3) \right] \left[ 1 - (N - 2)(pe^b)^{N-3} + (N - 3) (pe^b)^{N-2} \right] \]

\[ \times \frac{pe^b}{(1 - pe^b)^2} - (1 - p)^2 \left[ 1 + pe^b - (N - 2)^2 (pe^b)^{N-3} + (2N^2 - 10N + 11) \right] \]

\[ \times (pe^b)^{N-2} - (N - 3)^2 (pe^b)^{N-1} \frac{pe^b}{(1 - pe^b)^3} + (N - 2) (pe^b)^{N-2} \] (15)
The graph of the ensemble average resistance versus the filling factor $p$ is plotted in Figures (3a) and (3b).

In the limit of $N \to \infty$, the geometric series converges only for $p < p_c = e^{-b}$.

At the probability $p_c = e^{-b}$, the expression for the average resistance is:

$$< R > = R_0 \left\{ \frac{N-3}{6} \right\}$$

Therefore,

$$< R_c > = R_0 \left\{ \frac{N-3}{6} \right\} \frac{(N-3)(N-2)}{2}$$

$$- \left(1 - p\right)^2 \frac{(N-3)(N-2)(2N-5) + (N-2)}{6}$$

At the thermodynamic limit, of $N \to \infty$, and at the critical probability $p_c$, the average resistance per site is:

$$\frac{< R_c >}{N} + R_0 \frac{1}{6} (1 - p_c)^2 N^2$$

The calculation of the moment $< K_1 K_j >$, is performed in Appendix E, by performing the sums:

$$< K_1 K_j > = \sum_{n_1=0}^{N-2} \sum_{K=1}^{n_1} \left\{ \begin{array}{c} K! \\ K_1! K_2! \ldots K_{n_1}! \end{array} \right\} (n_2 - 1) \binom{n_2 - 1}{K} K_1 K_j \ p^{n_1}(1 - p)^{n_2 - 2}$$

with $\sum_i K_i = K$ and $\sum_i K_i i = n_1 *$.
Therefore,\(^1\)

\[
\langle K_i K_j \rangle = \{ (N - i - j - 4) (N - i - j - 5) (1 - p)^2 \\
+ 6(N - i - j - 4) (1 - p) + 6 \} p^{i+j} (1 - p)^2 \theta (i + j, N - 3) \\
+ 2(1 - p) p^{N-3} \delta (i + j, N - 3) \\
+ \delta_{ij} [(N - i - 3) (1 - p) + 2] (1 - p) p^i \theta(i, N - 2) \\
+ p^{N-2} \delta_{ij} \delta_{i,N-2}
\]

(20)

where \( \theta(m, n) = \begin{cases} 1 & \text{if } m < n \\ 0 & \text{otherwise} \end{cases} \)

Using the above expression, the average of the square of the resistance can then be evaluated. This is done in Appendix F, and gives:

\[
\langle R^2 \rangle = R_o^2 \left\{ \sum_{i,j} e^{b(i+j)} \langle K_i K_j \rangle \right\}
\]

Therefore,

\[
\frac{\langle R^2 \rangle}{R_o^2} = \frac{(1 - p)^2}{12} [(4A - C) S_3 + BS_2 - 4BS_4 + 4CS_5 - AS_1] \\
+ \frac{1}{3} (pe^{b})^{N-3} (1 - p) (N - 2) (N - 3) (N - 4) \\
+ (1- p) \{[2 + (N - 3) (1 - p)] T_2 - (1 - p) T_3 \} \\
+ (N - 2)^2 (pe^{2b})^{N-2}
\]

(21)

where \( B = 6(1 - p) + (2N - 9) (1 - p)^2 \)

\(^1\)Mood appears to have an error in \( \langle K_i K_j \rangle \).
A = 6 + 6(N - 4) (1 - p) + (N - 4) (N - 5) (1 - p)^2
C = (1 - p)^2

\[ S_k = \sum_{m=0}^{N-4} m^k (pe^b)^m ; \]
\[ T_k = \sum_{i=0}^{N-3} i^k (pe^{2b}) 1 \]

\[ S_1 = \frac{pe^b}{(1 - pe^b)^2} \{ 1 - (N - 3) (pe^b)^{N-4} + (N - 4) (pe^b)^{N-3} \} \] (22)

\[ S_2 = \frac{1}{(1 - pe^b)^3} \{ pe^b - (pe^b)^{N-3} (N - 3)^2 + (pe^b)^{N-2} [2(N - 4)^2 + 2(N - 4) - 1] - (pe^b)^{N-1} (N - 4)^2 + (pe^b)^2 \} \]

\[ S_3 = \frac{pe^b}{(1 - pe^b)^4} \{ 1 - (pe^b)^{N-4} (N - 3)^3 + (pe^b)^{N-3} [3(N - 4)^3 + 6(N - 4)^2 - 4] + (pe^b)^{N-2}[-3(N - 4)^3 - 3(N - 4)^2 + 3(N - 4) - 1] + (pe^b)^{N-1} (N - 4)^3 + 4(pe^b) + (pe^b)^2 \} \]

\[ S_4 = \frac{pe^b}{(1 - pe^b)^5} \{ 1 - (N - 3)^4 (pe^b)^{N-4} + (pe^b)^{N-3} [4(N - 4)^4 + 12(N - 4)^3 + 6(N - 4)^2 - 12(N - 4) - 11] - (pe^b)^{N-2} [6(N - 4)^4 + 12(N - 4)^3 - 6(N - 4)^2 - 12(N - 4) + 11] + (pe^b)^{N-1} [4(N - 4)^4 + 4(N - 4)^3 - 6(N - 4)^2 + 4(N - 4) - 1] - (pe^b)^N (N - 4)^4 + 11(pe^b) + 11(pe^b)^2 + (pe^b)^3 \} \] (25)
\[ S_5 = \frac{pe^b}{(1 - pe^b)^6} \{1 - (N - 3)^5 \} (pe^b)^{N-4} \]

\[ + (pe^b)^{N-3} [5(N - 4)^5 + 20(N - 4)^4 + 20(N - 4)^3 - 50(N - 4)^2 - 26 - 20(N - 4)^2] \]

\[ + (pe^b)^{N-2} [10(N - 4)^5 + 30(N - 4)^4 - 60(N - 4)^2 + 66] \]

\[ + (pe^b)^{N-1} [10(N - 4)^5 - 20(N - 4)^3 + 20(N - 4)^2 - 20(N - 4)^2 + 50(N - 4) - 26] \]

\[ + (pe^b)^{N} [5(N - 4)^5 + 5(N - 4)^4 - 10(N - 4)^3 + 10(N - 4)^2 - 5(N - 4) + 1] \]

\[ + (pe^b)^{N+1} (N - 4)^5 + 26(pe^b) + 66(pe^b)^2 + 26(pe^b)^3 + (pe^b)^4 \]  

(26)

\[ T_2 = \frac{1}{(1 - pe^b)^3} \{pe^2b - (pe^2b)^{N-2} (N - 2)^2 \}

\[ + (pe^2b)^{N-1} [2(N - 3)^2 + 2(N - 3) - 1] - (pe^2b)^{N} (N - 3)^2 + (pe^2b)^2 \]  

(27)

\[ T_3 = \frac{pe^2b}{(1 - pe^2b)^4} \{1 - (pe^2b)(N-3) (N - 2)^3 \}

\[ + (pe^2b)^{N-2} [3(N - 3)^3 + 6(N - 3)^2 - 4] \]

\[ + (pe^2b)^{N-1} [-3(N - 3)^3 - 3(N - 3)^2 + 3(N - 3) - 1] \]

\[ + (pe^2b)^{N} (N - 3)^3 + 4 (pe^2b) + (pe^2b)^2 \]  

(28)

In the limit of \( N \to \infty \), the geometric series all converge only for \( p < e^{-2b} \). Consequently, the variance will have a divergence at the probability \( p_{c_1} \) equal to \( e^{-2b} \).

Considering the leading terms in the variance, at the thermodynamic limit and at \( p_{c_1} = p_c^2 = e^{-2b} \), the variance diverges as:

\[ < R^2 > - < R >^2 + \frac{1}{12} N^4 (1 - p_{c_1})^2 \]  

(29)
The behavior of the variance with the probability \( p \) is shown in Figure (5) and the plot of the variance versus the array size is in Figure (6).

2.3 **Ensemble with Fixed Probability \( p \) and Line Bent into a Circle**

This ensemble consists in taking the linear array, with fixed probability \( p \), and bending it into a circle. The case of \( n_1 = 0 \) (\( n_2 = N \)) and \( n_1 = N \) (\( n_2 = 0 \)) will be excluded. A similar ensemble is described by David and Barton (DAV62).

Therefore, the probability of having \( n_1 \) dielectrics distributed in \( \{ K_1 \} \) clusters of size \( i \) is given by:

\[
P(n_1; K, \{ K_1 \}) = p^{n_1} (1 - p)^{n_2} \frac{N}{K} \left( \frac{n_2 - 1}{K - 1} \right) \frac{K!}{K_1! \cdot K_2! \ldots \cdot K_{n_1}!}
\]

With the above expression, the moment \( \langle K_i \rangle \) can be calculated. This is done in Appendix G.

The final expression for \( \langle K_i \rangle \) is:

\[
\langle K_i \rangle = \begin{cases} 
N p^i (1 - p)^2 & \text{for } i \leq N - 2 \\
N p^{N-1} (1 - p) & \text{for } i = N - 1.
\end{cases}
\]

The total resistance is then evaluated

\[
\frac{\langle R \rangle}{R_0} = N(1 - p)^2 \sum_{i=1}^{N-2} i(\text{pe}^b)^i + N(N - 1)(1 - p)(\text{pe}^b)^{N-1}
\]

Therefore:
Figure 5: Dependence of the variance \((< R^2 > - < R >^2)/N^2 R_0^2\) on the probability \(p\). Ensemble with fixed probability \(p\) in a line \((N = 100)\).
Figure 6: Dependence of the variance \((\langle R^2 \rangle - \langle R \rangle^2) / N^2 \sigma_0^2\) on the array size for three different values of \(p\). Ensemble with fixed probability \(p\) in a line.
\[ \frac{\langle R \rangle}{R_o} = N(1 - p)^2 \frac{pe^b}{(1 - pe^b)^2} \left\{ 1 - (N - 1) (pe^b)^{N-2} + (N - 2) (pe^b)^{N-1} \right\} \]

\[ + (1 - p) N(N - 1) (pe^b)^{N-1}. \]  \hfill (32)

Again, as \( N \to \infty \) the average resistance diverges at a critical probability \( p_c = e^{-b} \).

Analysing the average resistance at \( p_c \) one obtains:

\[ \frac{\langle R \rangle}{R_o} = N(1 - p)^2 \sum_{i=1}^{N-2} i + N(N - 1) (1 - p) \]

Therefore,

\[ \frac{\langle R \rangle}{R_o} = N(1 - p)^2 \frac{(N - 2)(N - 1)}{2} + N(N - 1) (1 - p) \]  \hfill (33)

Figure (7) shows the graph of the ensemble average resistance versus the probability \( p \).

At the thermodynamic limit of large \( N \), the average resistance per site goes as:

\[ \frac{\langle R \rangle}{NR_o} \to \frac{1}{2} N^2 (1 - p)^2. \]  \hfill (34)

The calculation of the moment \( \langle K_i K_j \rangle \) is done in Appendix H.

The expression for \( \langle K_i K_j \rangle \) is then given by:

\[ \langle K_i K_j \rangle = N p^{i+j}(1 - p)^3 \{ (1 - p)(N - i - j - 3) + 2 \} \theta(i + j, N - 2) \]

\[ + N p^{i+j}(1 - p)^2 \delta(i + j, N - 2) + N p^i(1 - p)^2 \theta(i, N-1) \delta_{ij} \]

\[ + N (1 - p) p^i \delta(i, N - 1) \delta_{ij}. \]  \hfill (35)
Figure 7: Dependence of the ensemble average resistance per site on the probability $p$. Ensemble with fixed probability $p$ and line bent into a circle ($N = 100$).
The average of the square of the resistance can then be evaluated. And the final expression which is in Appendix I is:

\[
\frac{\langle R^2 \rangle}{R_0^2} = \frac{N}{12} (1 - p)^3 \left[ 4[2 + (N - 3)(1 - p)] S_3' - [2 + (N - 3)(1 - p)] S_1' \right] + (1 - p)^3 \frac{N}{12} \left[ S_2', (1 - p) - 4(1 - p) S_4' \right] + N(p_e b)^{N-2} (1 - p)^2 \frac{(N - 1)(N - 2)(N - 3)}{6} + N(1 - p)^2 T_2' + (N - 1)^2 (p_e^2 b)^{N-1} N(1 - p) \quad (36)
\]

where

\[
S_\alpha' = \sum_{m=1}^{N-3} m^\alpha (p_e b)^m;
\]

\[
T_2' = \sum_{i=1}^{N-2} i^2 (p_e^2 b)^i
\]

The value of each sum being equal to:

\[
S_1' = \frac{p_e b}{(1 - p_e^2)^2} \left[ 1 - (N - 2)(p_e b)^{N-3} + (N - 3)(p_e b)^{N-2} \right]
\]

\[
S_3' = \frac{p_e b}{(1 - p_e^2)^4} \left[ 1 - (p_e b)^{N-3} (N - 2)^3 + (p_e b)^{N-2} [3(N - 3)^3 + 6(N - 3)^2 - 4] + (p_e b)^{N-1} [-3(N - 3)^3 - 3(N - 3)^2 + 3(N - 3) - 1] + (p_e b)^N (N - 3)^3 + 4p_e b + (p_e^2 b)^2 \right]
\]
\[ S'_4 = \frac{p e^b}{(1 - p e^b)^5} \left[ 1 - (N - 2)^4 (p e^b)^{N-3} \right. \\
\left. + (p e^b)^{-2} [4(N - 3)^4 + 12(N - 3)^3 + 6(N - 3)^2 - 12(N - 3) - 11] \right. \\
\left. - (p e^b)^{-1} [6(N - 3)^4 + 12(N - 3)^3 - 6(N - 3)^2 - 12(N - 3) + 11] \right. \\
\left. + (p e^b)^N [4(N - 3)^4 + 4(N - 3)^3 - 6(N - 3)^2 + 4(N - 3) - 1] \right. \\
\left. - (p e^b)^{N+1} (N - 3)^4 + 11(p e^b)^2 + 11(p e^b) + (p e^b)^3 \right] \]

\[ T'_2 = \frac{1}{(1 - p e^b)^3} \left[ p e^{2b} - (p e^{2b})^{N-1} (N - 1)^2 + \right. \\
\left. + (p e^{2b})^N [2(N - 2)^2 + 2(N - 2) - 1] - (p e^{2b})^{N+1} (N - 2)^2 + (p e^{2b})^2 \right] \]

Once more, as \( N \to \infty \) the variance will diverge at the probability \( p_{c_1} = e^{-2b} \). The variance at \( p_{c_1} = e^{-2b} \) and at the limit of large \( N \), will tend to the limiting value:

\[
\frac{\langle R^2 \rangle - \langle R \rangle^2}{N^2 \langle R^2 \rangle_o} \to \frac{1}{3} (1 - p_{c_1})^2 N^2.
\]

Using the expressions for \( \langle R^2 \rangle \) and \( \langle R \rangle \), the variance was calculated numerically. The result of this calculation is shown in Figure (8).

### 2.4 Comparison Between Ensembles

Figures (3a) and (3b) shows the total resistance per site plotted against the probability \( p \) for the approaches in sections 2.1 and 2.2. The two ensembles are related by the condition that \( \left( n_1/N \right) = p \). For small probability \( p \), the two ensembles agree.
Figure 8: Dependence of the variance \( \left( \langle R^2 \rangle - \langle R \rangle^2 \right)/N^2 R_0^2 \), on the probability \( p \). Ensemble with fixed probability \( p \) and line bent into a circle \((N = 100)\).
The numerical value of the contribution to the resistance per site due to each island size was calculated for the two approaches. The result of it is shown in Figures (9a), (9b) (9c), (9d), for an array of size N = 100. Different values of p were considered. For finite N and large p the two approaches disagree. This discrepancy arises because the ensemble with fixed probability p allows dielectric clusters larger than the corresponding \( n_1 \), of the ensemble with fixed \( n_1 \). The contribution of these large islands is dominant in the total resistance.

Comparing the variances of approaches in Sections 2.1 and 2.2, it was observed that the variance for the ensemble with fixed probability p has a higher value than the one with fixed \( n_1 \). Following we quote the numerical values of the variance for the two ensembles for \( N = 100 \).

<table>
<thead>
<tr>
<th>( n_1/N )</th>
<th>Ensemble fix p</th>
<th>Ensemble fix ( n_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.27</td>
<td>( 1.1 \times 10^5 )</td>
<td>4.5</td>
</tr>
<tr>
<td>0.25</td>
<td>( 4.6 \times 10^2 )</td>
<td>2.3</td>
</tr>
<tr>
<td>0.24</td>
<td>( 5.9 \times 10 )</td>
<td>1.7</td>
</tr>
</tbody>
</table>

This behaviour is again a result of the contribution of the large islands.

The exact way that the resistance diverges, for the ensemble with fixed \( n_1 \), was not obtained. The reason is that it was not possible to perform the sum over the island sizes in the expression for the resistance. Therefore, the thermodynamic limit was taken before the sum was done.

The ensembles of approaches 2.2 and 2.3 allow the comparison of two ensembles with fixed p but different boundary conditions. Comparing equations (13) and (31), it can be seen that the way of putting an island
Figure 9: Dependence of the value of \( \langle RI \rangle / NR_0 \) on the island size.
\( \langle RI \rangle \) is the contribution to \( \langle R \rangle \) from all the islands of size 1. Ensemble with fixed \( n_1 \) (dashed line) and fixed probability \( p \) (solid line) in a line (\( N = 100 \)).

(a) \( p = 0.45 \), \( n_1 = 45 \) \( N = 100 \)
(b) \( p = 0.48 \), \( n_1 = 48 \) \( N = 100 \)
(c) \( p = 0.50 \), \( n_1 = 50 \) \( N = 100 \)
(d) \( p = 0.52 \), \( n_1 = 52 \) \( N = 100 \)
of size $i$ in a line is different from the one in a circle. Equations (13) and (31) differ because the embedding of an island of size $i$ in a line of length $N$ is $(N - i + 1)/N$ whereas in a circle it is 1 (embedding of an island of size $i$ is the number of ways it can be put in divided by the total number of sites). In the limit of $N \to \infty$ these two embeddings are both unity. However, in order to obtain the average value of the resistance, a summation over $i$ has still to be performed. If this summation is done before the limit $N \to \infty$ is taken, then these two ensembles produce different results. In particular, we have noted that for large $N$ the line problem gave the result (18) \( \langle R \rangle / NR_o \approx (1/6)(1 - p_c)^2 N^2 \) whereas the circle problem gave the result (34) \( \langle R \rangle / NR_o \approx (1/2)(1 - p_c)^2 N^2 \). A similar difference occurs for the variance. It is to be remembered that equation (13) is for finite $i$.

Thus by considering all three ensembles we have demonstrated, for this particular problem, the differences that occur for fixed $n_1$ and fixed $p$ and for different boundary conditions.
CHAPTER THREE

COMPUTER SIMULATIONS

It transpires that for the computer simulations it makes little
difference which boundary conditions, metal ends or cyclic boundary
conditions, are used nor whether the randomness is generated according to
the ensemble of paragraph 2.1 or 2.2 of the previous chapter. We quote
here the results for metal ends and filling according to a fixed $p$.
Figure (10) shows $< R >_M^{1/N}$, the sample mean resistance per site for $M$
computer simulations.

Computer simulations for different $N$ were done; they herald a $N^2$ divergen
of $< R >_M$ at $p = p_c$ as $N$ becomes larger (PAL84). The value
of $< R >_M$ at $p_c$ is orders of magnitude smaller than the corresponding
ensemble value obtained by approaches 2.1 or 2.2 of the previous sections.

The question then arises as to the nature of the relation between the
computer simulation of this chapter and the ensemble theory of the previous
one. A previous work (PAL84) describes the problem and what should be the
representative ensemble for the computer simulations.

A summary of the relation between the computer simulation and the
ensemble theory is described in the following paragraph.

What the computer program does is generate repeatedly a set ${K_i}$
and then calculates $R = \sum_{i=1}^{N-2} i \cdot e^{bi} K_i$. It then averages this $R$ over $M$
computer runs. It is found that for a given $M$ the computer each time
produces zero for $K_i$ for $i$ larger than some $i_{\text{max}}$. In the computer
Figure 10: Sample mean resistance per site $\langle R \rangle_{M/NR_0}$ for $M$ runs versus the probability $p$ in a line ($M = 1000, N = 1000$). For each $p$ the 1000 runs were done 3 times.
averaging over $M$ runs, islands of size greater than $i_{\text{max}}$ will not provide their very large contribution to the average resistance.

An acceptable analytic description of the computer simulations is to cut off the sum over $i$ in the ensemble approach, at some $i_{\text{max}}$. In this way, the effect of rare events, islands bigger than $i_{\text{max}}$, are not taken into account. The analytic description approaches the ensemble one when the number of runs is approximately equal to $M = e^{N-L\ln N}$ (PAL84).

In systems like this, where rare events dominate the ensemble arithmetic mean, we do not expect the sample mean to agree with the ensemble arithmetic mean; since as we found the sample mean does not pick up the rare events. The question then arises as to how in the ensemble approach of Chapter 2 we are to predict theoretically what experiment (either computer simulation or laboratory measurements) should expect. It has been suggested that the ensemble geometric mean, since it is not so dominated by rare events might be more comparable to the experiment geometric sample mean. Unfortunately, by approach of Chapter 2 we were not able to calculate the ensemble geometric mean. However, as a check on the related suggestion that the computer simulation geometric means will be less divergent than the arithmetic means, we in fact obtained also the computer simulation geometric mean. Figure (11) shows the geometric mean over $M$ runs, of the resistance per site against the logarithm of the array size. It appears that the geometric mean grows at $p = p_c$ as $\ln(N)$, which is a slower growth than the one of the arithmetic mean at $p = p_c$ which grows as $\ln^2(NM)$ (PAL84).

By comparing three groups of 1000 runs the geometric mean shows smaller fluctuations in comparison with the arithmetic mean.

The plot of the cumulative average resistance per site against the
Figure 11: Computer simulation geometric mean over M runs versus \( \ln(N) \). Ensemble with fixed probability \( p \) in a line.

\( p = 0.5 \) and \( M = 1000 \), for each point such that the fluctuation can be observed.
number of runs for a set of 1000 runs is shown in Figure (12).

From the plot of the cumulative average resistance per site against the number of runs, the behavior of the average value of the resistance per site can be analysed. It can be seen that the mean of the resistance per site exhibits sudden jumps followed by decreases. These jumps are caused by the appearance of large islands in a certain computer simulation. Then as the number of runs increases the importance in the average of those large islands decreases. These decreases will continue until another jump occurs. A similar behavior has already been observed by Montroll and Shlesinger (MON83).

In discussing income distributions, Montroll and Shlesinger (MON83) introduce the process of amplification, amplification of amplification, etc. Starting with a basic distribution, they can then construct a new distribution which allows for the possibility of continuing levels of amplification. Each amplifier class has the same basic distribution function but the mean value of the quantity measured is amplified N times with a small probability \( \lambda \).

Analogously, in the cumulative average resistance per site the mean of the resistance per site suffers amplifications or jumps equal to \( e^b \) and probability \( \lambda = p \).

This behavior is also reflected in the histogram of \( \ln(R/R_0) \), Figure (13), where for large values of the resistance per site the histogram has spikes at constantly spaced intervals. A histogram was done with a different value of \( b \) and it was checked that the distances between the spikes were equal to the new value of \( b \). In other words, each considerable increase in the resistance is mainly \( me^b \), where \( m \) is an integer.
Figure 12: Computer simulation cumulative average resistance per site for $M$ runs. Ensemble with fixed probability $p$ in a line.
Figure 13: Histogram of the logarithm of the sample mean resistance per site for M runs. Ensemble with fixed probability p in a line (M = 1000, N = 1000, p = 0.5). In order to be able to exhibit the whole histogram on one diagram, the horizontal scale is in fact χ with χ = [\ln (\langle R \rangle_{M/NR_0} - (-0.5))] / 0.1 + 1.
CHAPTER FOUR

CONCLUSION

The statistics of the problem of a random series of metal-insulator-metal junctions was studied considering three different ensembles. It was found that the effect of large insulator clusters was dominant in the behavior of the ensemble average resistance and variance. Therefore, the main difference between the ensembles with fixed probability $p$ and fixed $n_1$, distributed in a linear arrangement, was due to the limitation on the second ensemble to pick up larger island sizes.

When comparing two ensembles with fixed probability $p$, of having an insulator occupying a site, but one in a linear arrangement and the other one in an arrangement of a line bent into a circle, a discrepancy is observed in the way the ensemble average resistance diverges for $N \to \infty$ at $p = p_c$. This fact is due to the different embeddings in the two ensembles. If the limit $N \to \infty$ is taken before the summation of island contributions for different size islands, then the two approaches agree.

The divergence of the ensemble average resistance and variance at the limit $N \to \infty$ was observed for the three ensembles at $p = p_c = e^{-b}$ and $p = p_{c1} = e^{-2b}$, respectively. This behavior is also typical of the so-called Griffiths singularities (GRI69). In these, as in our case, in the limit $N \to \infty$ the coherence length (in our case, the average dielectric island size) does not go to infinity until $p = 1$, in one-dimensional systems. Before that, there occurs a divergence in the resistance at $p_c$. 
As a suggestion for future work, the temperature dependence in the tunneling resistivity in a randomly-filled chain problem, should be considered. It may be that at non-zero temperatures the behavior of the divergence is significantly modified.

Another interesting problem is the extension of the present problem to a two-dimensional one, which would be a better approximation to the simulation of granular metals.

In conclusion we would like to point out that this project has made a significant contribution to the study of a one-dimensional random system which displays a sensitivity to statistically rare events.
REFERENCES


GRA Gradshteyn, S./I.M. Ryzhik. Table of Integrals Series and Products, p. 4, equation KR64 (71.1).


APPENDIX A

CALCULATION OF THE MOMENT \( \langle K_1 \rangle \)

FOR FIXED NUMBER OF INSULATOR "ATOMS" DISTRIBUTED IN THE CHAIN

\[
\langle K_1 \rangle = \sum_{K} \sum_{\{K_1\}} \frac{K!}{K_1! \cdot K_2! \cdots K_n!} \left( \frac{(n_2 - 1)}{K} \right) \binom{n}{n_1},
\]

with \( 0 \leq n_1 \leq N - 2 \), \( 0 \leq K \leq n_1 \), \( 0 \leq i \leq n_1 \), \( \{K_i\} = \{K_1, K_2, \ldots, K_n\} \)

\( \sum_i K_i = n_1 \) and \( \sum_i K_i = K \).

But:

\[
\sum_{\{K_1\}} \frac{K!}{K_1! \cdot K_2! \cdots K_n!} \left( \frac{(n_2 - 1)}{K} \right) K_1 = K \sum_{\{K_1\}} \frac{(K - 1)!}{(K_1 - 1)! \cdots (K_i - 1)! \cdots K_n!} \left( \frac{(n_2 - 1)}{K} \right);
\]

with \( \sum_i K_i = n_1 \) and \( \sum_i K_i = K \).

Making the change of variables: \( K' = K - 1 \), \( K'_j = \begin{cases} K_i - 1 & \text{if } j = i \\ K_j & \text{otherwise} \end{cases} \)

\[
\sum_{\{K_1\}} \frac{K!}{K_1'! \cdot K_2'! \cdots K_n'!} \left( \frac{(n_2 - 1)}{K} \right) K_1' = K \sum_{\{K_1'\}} \frac{K_1'!}{K_1' \cdot K_2'! \cdots K_n'!} \left( \frac{(n_2 - 1)}{K} \right);
\]

with \( \{K_1'\} = \{K_1', K_2', \ldots, K_n'\} \), \( \sum_i K_i' = K - 1 \) and \( \sum_i K_i' = n_1 - i \).
Making use of the procedure given in Wilks (Wil62), the above sum can be written as:

\[ \sum_{\frac{K'}{K_1}} \frac{K'!}{K_1! K_1'! \ldots K_1^n!} = \frac{(N_1 - i - 1)!}{(K - 2)! (n_1 - i - K + 1)!} \]

with \( \sum_{i} K_1' = K - 1 \) and \( \sum_{i} i K_1' = n_1 - i \).

Therefore

\[ \langle k_1 \rangle = \sum_{K} K \left( \begin{array}{c} n_2 - 1 \\ K \end{array} \right) \left( \begin{array}{c} n_1 - i - 1 \\ K - 2 \end{array} \right) / \left( \begin{array}{c} N - n_2 \\ n_1 \end{array} \right) \]

It is worth noticing that:

- The range of \( K \) is from \( K = 1 \) to the smaller \( n_2 - 1 \) and \( n_1 - i + 1 \);

- If \( n_1 = 0 \), then \( K = 0 \) which corresponds to \( \langle k_1 \rangle = 0 \);

- For a given \( n_1 > 1 \), the term \( K = 1 \) is zero unless \( i = n_1 \).

Consequently, for a given \( n_1 \) and \( i < n_1 \), the first moment is given by:

\[ \langle k_1 \rangle = \sum_{K=2}^{n_2-1} \left( \begin{array}{c} n_2 - 2 \\ n_2 - 1 - K \end{array} \right) \left( \begin{array}{c} n_1 - i - 1 \\ K - 2 \end{array} \right) \left( \begin{array}{c} n_2 - 1 \\ n_1 \end{array} \right) \]

Setting \( K'' = K - 2 \) and noticing that \( K'' \leq n_2 - 3 \); the sum over \( K \) then reduces to:
\[
\langle K_1 \rangle = \frac{(n_2 - 1)}{\binom{N - n_2}{n_1}} \sum_{k=0}^{n_2 - 3} \binom{n_2 - 2}{n_2 - 3 - k} \binom{n_1 - i - 1}{k}
\]

Since \( \sum_{q=0}^{p} \binom{n}{q} \binom{m}{p-q} = \binom{n+m}{p} \), (GRA65); (A3)

where \( m \) is a natural number. It follows that

\[
\langle K_1 \rangle = \frac{(n_2 - 1)}{\binom{N - n_2}{n_1}} \binom{N - i - 3}{n_1 - i} \text{ with } i \leq N - 3
\]

In the case of \( i = n_1 \), the smaller of \((n_2 - 1)\) and \((n_1 - i + 1)\) is \( (n_1 - i + 1) \), unless \( n_2 = 2 \) when they are equal.

Therefore,

\[
\langle K_{n_1} \rangle = \frac{1}{\binom{N - 2}{n_1}} \binom{n_2 - 1}{1} \binom{-1}{-1} = \frac{(n_2 - 1)}{\binom{N - 2}{n_1}}
\]

The two cases, \( i < n_1 \) and \( i = n_1 \) can be combined as:

\[
\langle K_1 \rangle = \frac{(n_2 - 1)}{\binom{N - 2}{n_1}} \binom{N - i - 3}{n_1 - i} \text{ with } i \leq N - 3 \tag{A4}
\]

Notice that if \( n_1 = N - 2 \), \( \langle K_1 \rangle = 0 \) unless \( i = N - 2 \) when \( \langle K_1 \rangle = 1 \).
APPENDIX B

CALCULATION OF THE MOMENT < \( K_1 K_j \)>

FOR FIXED NUMBER OF INSULATOR "ATOMS" DISTRIBUTED IN THE CHAIN

\[
< K_1 K_j > = \sum_{K} \sum_{\{K_1\}} \frac{K!}{K_1! K_2! \cdots K_n!} \frac{1}{n_1!} \binom{n_2 - 1}{K} \binom{N - 2}{n_1} K_1 K_j, \quad (B1)
\]

with \( \sum_i K_i = K \), \( \sum_i K_i = n_1 \), \( 0 < K < n_1 \) and \( 0 < i < n_1 \).

Separating out the diagonal and off-diagonal terms in (B1), the sum over \( \{K_1\} \) can then be written as:

\[
\begin{align*}
\sum_{\{K_1\}} & \frac{K!}{K_1! K_2! \cdots K_n!} K_1 K_j = \delta_{ij} \sum_{\{K_1\}} \frac{K!}{K_1! K_2! \cdots K_n!} K_1^2 \\
& + (1 - \delta_{ij}) \sum_{\{K_1\}} \frac{K!}{K_1! K_2! \cdots K_n!} K_1 K_j, \quad (B2)
\end{align*}
\]

with \( \sum_i K_i = K \) and \( \sum_i K_i = n_1 \).

Consider the diagonal term in equation (B2)

\[
\begin{align*}
\sum_{\{K_1\}} \frac{K!}{K_1! K_2! \cdots K_n!} K_1^2 = \sum_{\{K_1\}} \frac{K!}{K_1! K_2! \cdots (K_1-1)! \cdots K_n!} ; \quad (B3)
\end{align*}
\]

with \( \sum_i K_i = K \) and \( \sum_i K_i = n_1 \).
Making the change of variables:

\[ K' = K - 1 \]

\[ K'_j = \begin{cases} K_j - 1 & \text{if } j = i \\ K_j & \text{otherwise} \end{cases} \]

Equation (B3) is equal to:

\[
K \sum_{\{K'_1\}} \frac{K'! (K'_1 + 1)}{K'_1! \cdots K'_i! \cdots K'_n!} = K \sum_{\{K'_1\}} \frac{K'! K'_1!}{K'_1! \cdots K'_i! \cdots K'_n!} 
\]

\[
+ K \sum_{\{K'_1\}} \frac{K'!}{K'_1! \cdots K'_i! \cdots K'_n!} \tag{B4}
\]

with \( \sum_i K'_i = K - 1 \) and \( \sum_i K'_i = n_1 - 1 \).

Perform once more, the change of variables in the first equation of

the right-hand side (B4) such as: \( K'' = K' - 1 \), \( K''_j = \begin{cases} K'_j - 1 & \text{if } j = i \\ K'_j & \text{otherwise} \end{cases} \)

Equation (B3) is then written as:

\[
K(K - 1) \sum_{\{K''_1\}} \frac{K''!}{K''_1! \cdots K''_i! \cdots K''_n!} + K \sum_{\{K''_1\}} \frac{K'!}{K'! \cdots K'_i! \cdots K'_n!} ,
\]

with \( \sum_i K''_i = K - 1; \sum_i K''_i = n_1 - 1; \)

\( \sum K'' = K - 2 \) and \( \sum i K''_i = n_1 - 2i. \)

From Appendix A, the value of the sum over \( \{K'_1\} \) can then be obtained. And equation (B3) reduces to:
\[
\sum_{\{k_i\}} \frac{K^2}{K_1! K_2! \cdots K_{n_1}!} = \{K(K - 1)\left(\begin{array}{c} n_1 - 2i - 1 \\ K - 3 \end{array}\right) + K\left(\begin{array}{c} n_1 - i - 1 \\ K - 2 \end{array}\right)\} (B5)
\]

Analogously, the off-diagonal term in (B2) is:

\[
(1 - \delta_{ij}) \sum_{\{k_i\}} \frac{K!}{K_1! K_2! \cdots K_{n_1}!} K_i K_j
\]

\[
= \frac{K(K - 1)}{n} \left(\begin{array}{c} n_1 - i - j - 1 \\ K - 3 \end{array}\right) (1 - \delta_{ij}) \quad (B6)
\]

With (B5) and (B6), the expression for (B1) is then:

\[
< K_i K_j > = \{\sum_{K} K(K - 1) \left(\begin{array}{c} n_1 - 2i - 1 \\ K - 3 \end{array}\right) \left(\begin{array}{c} n_2 - 1 \\ K \end{array}\right) \\
+ \sum_{K} K \left(\begin{array}{c} n_1 - i - 1 \\ K - 2 \end{array}\right) \left(\begin{array}{c} n_2 - 1 \\ K \end{array}\right) \} \frac{\delta_{ij}}{n} \left(\begin{array}{c} n_1 \\ K \end{array}\right) + \{\sum_{K} K(K - 1) \left(\begin{array}{c} n_1 - i - j - 1 \\ K - 3 \end{array}\right) \left(\begin{array}{c} n_2 - 1 \\ K \end{array}\right) \} \frac{(1 - \delta_{ij})}{n} \left(\begin{array}{c} n_1 \\ K \end{array}\right)
\]

(B7)

with \(0 \leq K \leq n_1\).

Consider, first, the sum over \(K\) for the off-diagonal term in (B7):

\[
\sum_{K=0}^{n_1} K(K - 1) \left(\begin{array}{c} n_1 - i - j - 1 \\ K - 3 \end{array}\right) \left(\begin{array}{c} n_2 - 1 \\ K \end{array}\right) =
\]
Setting \( K'' = K - 2 \) and rearranging terms, then equation (B8) is given by:

\[
\sum_{K=0}^{n_1} (K-1) \left( \begin{array}{c} n_1 - 2 \cr K - 3 \end{array} \right) \left( \begin{array}{c} n_2 - 2 \cr K - 1 \end{array} \right) (n_2 - 1) \]

(B8)

with the condition \( 0 \leq K'' \leq n_1 - i - j \).

By relation (A3), the final expression for (B8) is

\[
(n_2 - 1)(n_2 - 2) \sum_{K''=0}^{n_1-i-j} \left( \begin{array}{c} n_1 - i - j - 1 \cr K'' \end{array} \right) \left( \begin{array}{c} n_2 - 3 \cr K'' \end{array} \right)
\]

(B9)

Consider now, the sums over \( K \) for the diagonal terms in (B7). In analogy with equation (B9), the first term is given by:

\[
\sum_{K=0}^{n_1} K(K-1) \left( \begin{array}{c} n_1 - i - j - 1 \cr K - 3 \end{array} \right) \left( \begin{array}{c} n_2 - 1 \cr K \end{array} \right)
\]

\[
= (n_2 - 1)(n_2 - 2) \left( \begin{array}{c} n_1 - i - j - 4 \cr n_2 - 4 \end{array} \right)
\]

(B10)
The second sum over \( K \), in equation (B7), can be simplified as:

\[
\sum_{K=0}^{n_1} \binom{n_1 - i - 1}{K-2} \binom{n_2 - 1}{K} = \sum_{K=0}^{n_1} (n_2 - 1) \binom{n_1 - i - 1}{K-2} \binom{n_2 - 2}{K-1}
\]

Setting \( K' = K - 1 \) in (B11) and using relation (A3), equation (B11) is given by:

\[
\sum_{K'=0}^{n_1-1} \binom{n_1 - i - 1}{K'-1} \binom{n_2 - 2}{K'} (n_2 - 1) = (n_2 - 1) \binom{N - i - 3}{n_1 - i}
\]

Therefore,

\[
\sum_{K=0}^{n_1} \binom{n_1 - i - 1}{K-2} \binom{n_2 - 1}{K} = (n_2 - 1) \binom{N - i - 3}{n_1 - i}
\]

With the relations (B9), (B10) and (B12), the expression for \( \langle K_1 K_j \rangle \) is:

\[
\langle K_1 K_j \rangle = \frac{(n_2 - 1)}{\binom{N - 2}{n_1}} \{ (n_2 - 2) \binom{N - i - j - 4}{n_2 - 4} + (N - i - 3) \delta_{ij} \}
\]
APPENDIX C

CALCULATION OF $\langle R^2 \rangle$

FOR FIXED NUMBER OF INSULATOR "ATOMS" DISTRIBUTED IN THE CHAIN

The average of the square of the resistance is given by:

$$\langle R^2 \rangle = \frac{R_0^2}{\sum \sum_{i,j}^n i j \ e^{b(i+j)} \ K_i K_j},$$

with $0 \leq i + j \leq n_l$.

From equation (B13), the $\langle R^2 \rangle$ is given by:

$$\frac{\langle R^2 \rangle}{R_0^2} = \frac{(n_2 - 1)(n_2 - 2)}{n_1} \sum \sum_{i,j}^n i j \ e^{b(i+j)} \ \frac{(N - i - j - 4)}{n_2 - 4}$$

$$+ \frac{(n_2 - 1)}{n_1} \sum_i^N 2 \ e^{2bi} \ \frac{(N - i - 3)}{n_2 - 3} \quad (C2)$$

Rearranging terms and making a change of variables in (C2) such that:

$i + j = m$ and $i - j = n$, the expression for $\langle R^2 \rangle$ can be written as:

$$\frac{\langle R^2 \rangle}{R_0^2} = \frac{(n_2 - 1)(n_2 - 2)}{n_1} \sum \sum_{m,n}^N \ \frac{(m^2 - n^2)}{4} \ e^{bm} \ \frac{(N - m - 4)}{n_2 - 4}$$

$$+ \frac{(n_2 - 1)}{n_1} \sum_i^N 2 \ e^{2bi} \ \frac{(N - i - 3)}{n_2 - 3},$$

with $-m \leq n \leq m$ and $0 \leq m \leq n_l$. 
Performing the summation over $n$, one obtains:

$$\frac{\langle R^2 \rangle}{R^2_0} = \frac{(n_2 - 1)(n_2 - 2)}{\binom{N - 2}{n_1}} \sum_m \frac{(4m^3 - m)}{12} e^{bm} \left( \binom{N - m - 4}{n_1 - m} \right)$$

$$+ \frac{(n_2 - 1)}{\binom{N - 2}{n_1}} \sum_i \frac{2^2}{2^{2i}} e^{2bi} \left( \binom{N - i - 3}{n_1 - i} \right)$$

with $m = i + j$, $0 \leq m \leq n_1$ and $0 \leq i \leq n_1$. 

(C3)
APPENDIX D

CALCULATION OF $\langle K_1 \rangle$

FOR FIX PROBABILITY $p$ OF HAVING AN INSULATOR OCCUPYING A SITE IN A LINE

The moment $\langle K_1 \rangle$ is given by:

$$\langle K_1 \rangle = \sum_{n_1} \sum_{K} \sum_{\{K_i\}} K! K_1! K_2! \ldots K_n! \binom{n_2 - 1}{K} p^{n_1} (1 - p)^{n_2 - 2},$$

with

$$\sum_i K_i = K, \quad \sum_i i K_i = n_1, \quad 0 < K < n_1,$$

$$0 < i < n_1 \text{ and } 0 < n_1 < N - 2 \quad (D1)$$

The sum over $\{K_i\}$ and the subsequent sum over $K$ in equation (D1) has already been performed in Appendix A, equations (A2) and (A4).

Therefore:

$$\langle K_1 \rangle = \sum_{n_1} (n_2 - 1) \binom{N - i - 3}{n_1 - i} p^{n_1} (1 - p)^{n_2 - 2}, \quad (D2)$$

with $0 < n_1 < N - 2$ and $1 < N - 3$.

Equation (D2) can be rewritten as:

$$\langle K_1 \rangle = \sum_{n_1} p^{n_1} (1 - p)^{N - n_1 - 2} (N - i - 3) \binom{N - i - 4}{n_1 - i}$$

$$+ 2 \sum_{n_1} p^{n_1} (1 - p)^{N - n_1 - 2} \binom{N - i - 3}{n_1 - i}; \quad (D3)$$
with \( i \leq N - 3 \) and \( 0 \leq n_1 \leq N - 2 \).

Making the change of variables in (D3), \( n'_1 = n_1 - i \) and rearranging the limits in the sum over \( n'_1 \) such that \( \binom{N - i - 4}{n'_1} \neq 0 \) and

\[
\binom{N - 1 - 3}{n'_1} \neq 0,
\]

one obtains

\[
\langle K_i \rangle = (1 - p)^2 \ p^i (1 - p)^{N - i - 4} \sum_{n'_1 = 0}^{N - i - 4} \left( \frac{p}{1 - p} \right)^{n'_1} \binom{N - i - 4}{n'_1} (N - i - 3)
\]

\[
+ 2(1 - p) \ p^i (1 - p)^{N - i - 3} \sum_{n'_1 = 0}^{N - i - 3} \left( \frac{p}{1 - p} \right)^{n'_1} \binom{N - i - 3}{n'_1},
\]

with \( i \leq N - 3 \).  

By the Binomial series definition the sums in (D4) can be calculated. Then:

\[
\langle K_i \rangle = p^i (1 - p) \left\{ (1 - p) (N - i - 3) + 2 \right\}, \text{ for } i < N - 2.
\]

(D5)

For the case of \( i = N - 2 \), \( \langle K_i \rangle \) is equal to \( p^{N - 2} \).
APPENDIX E

CALCULATION OF $\langle K_i K_j \rangle$

FOR FIX PROBABILITY $p$ OF HAVING AN INSULATOR OCCUPYING A SITE IN A LINE

$$\langle K_i K_j \rangle = \sum_{n_1} \sum_{K_1} \cdots \sum_{K_n} \frac{K_1! \cdots K_n!}{K_1 \cdots K_n} \left( \begin{array}{c} n_2 - 1 \\ K_i K_j \end{array} \right) p^{n_1} (1 - p)^{n_2 - 2},$$

with $0 \leq n_1 \leq N - 2$, $0 \leq K \leq n_1$, $\sum K_i = K$,

$$\sum_i K_i = n_1, \quad 0 \leq i \leq n_1 \text{ and } \{K_1\} = \{K_1, K_2, \ldots, K_{n_1}\}. \quad (E1)$$

The expression for the sums over $\{K_1\}$ and $K$ can be obtained from Appendix B. Equation (E1) can then be written as:

$$\langle K_i K_j \rangle = \sum_{n_1} \left\{ (n_2 - 1) (n_2 - 2) \left( \frac{N - i - j - 4}{n_2 - 4} \right) + \right.$$  

$$+ \delta_{ij} (n_2 - 1) \left( \frac{N - i - 3}{n_2 - 3} \right) \right\} p^{n_1} (1 - p)^{n_2 - 2},$$

with $0 \leq n_1 \leq N - 2.$ \quad (E2)

Taking into consideration that:

$$\begin{pmatrix} n \end{pmatrix} = 0 \text{ if either } r > n \text{ or } r < 0;$$

equation (E2) is rewritten as:
\[
<K_i K_j> = \left\{ \sum_{n_1} (n_2 - 1) (n_2 - 2) \binom{N - i - j - 4}{n_2 - 4} p_1 (1 - p) \right\}
\]
\[
\times \delta(i + j, N - 3) + 2 (1 - p) p^{N - 3} \delta(i + j, N - 3) +
\]
\[
+ \sum_{n_1} (n_2 - 1) \binom{N - i - 3}{n_2 - 3} p_1 (1 - p)^{n_2 - 2} \delta_{ij} \theta(i, N - 2)
\]
\[
+ \delta_{ij} \delta_{i,N-2} p^{N-2},
\]

with \(0 \leq n_1 \leq N - 2\) and \(\theta(m, n) = \begin{cases} 1 \text{ if } m < n \\ 0 \text{ otherwise} \end{cases}\).

Each sum over \(n_1\), in equation (E3), will be evaluated separately in the next paragraphs.

The first sum over \(n_1\), in equation (E3) can be rewritten as:
\[
\sum_{n_1} p_1 (1 - p)^{N - n_1 - 2} (n_2 - 1) (n_2 - 2) \binom{N - i - j - 4}{n_2 - 4} =
\]
\[
\sum_{n_1} p_1 (1 - p)^{N - n_1 - 2} \left\{ (N - i - j - 4) [(N - i - j - 5) \binom{N - i - j - 6}{n_2 - 6}
\]
\[
+ 6 \binom{N - i - j - 5}{n_2 - 5} + 6 \binom{N - i - j - 4}{n_2 - 4} \right\},
\]

with \(0 \leq n_1 \leq N - 2\).

Consider the first term in the right-hand side of equation (E4).

After making the change of variables \(n'_1 = n_1 - i - j\) and rearranging the limits of the sum, such that the binomial coefficient is not equal to zero, one obtains:
Using the same procedure used to evaluate expression (E5), the others sums in the right-hand side of equation (E4) are given by:

\[
6(N - i - j - 4) (1 - p)^{N-2} \sum_{n_1=0}^{N-2} \left( \frac{p}{1 - p} \right)^{n_1} \left( \frac{N - i - j - 5}{n_2 - 5} \right) = \]

\[
6(N - i - j - 4) (1 - p)^{3} p^{i+j}; \quad (E6)
\]

and

\[
6(1 - p)^{N-2} \sum_{n_1=0}^{N-2} \left( \frac{p}{1 - p} \right)^{n_1} \left( \frac{N - i - j - 4}{n_2 - 4} \right) = 6(1 - p)^{2} p^{i+j}. \quad (E7)
\]

Considering equations (E5), (E6) and (E7), equation (E4) can then be evaluated, being equal to:

\[
\{(N - i - j - 4) ((N - i - j - 5) (1 - p)^{2} + 6(1 - p)) + 6 \} (1 - p)^{2} p^{i+j} \quad (E8)
\]
Now, the second sum in the expression for \( \langle K_i K_j \rangle \), equation (E3), will be calculated.

Therefore

\[
\sum_{n_1} (n_2 - 1) \binom{N - i - 3}{n_2 - 3} p^{n_1} (1 - p)^{n_2 - 2} =
\]

\[
(1 - p)^{N-2} \left\{ \sum_{n_1} (N - i - 3) \left( \frac{p}{1 - p} \right)^{n_1} \binom{N - i - 4}{n_1 - 1} \right\}
\]

\[
+ \sum_{n_1} 2 \left( \frac{p}{1 - p} \right)^{n_1} \binom{N - i - 3}{n_1 - 1} \right\} =
\]

\[
(1 - p)^{N-2} \left\{ (N - i - 3) \sum_{n_1'}=0 \left( \frac{p}{1 - p} \right)^{n_1'+i} \binom{N - i - 4}{n_1'} \right\}
\]

\[
+ 2 \frac{p^i}{(1 - p)^{N-3}} \right\}, \quad \text{with } n_1' = n_1 - i.
\]

Therefore

\[
\sum_{n_1} (n_2 - 1) \binom{N - i - 3}{n_2 - 3} p^{n_1} (1 - p)^{n_2 - 2} =
\]

\[
[N - i - 3 (1 - p) + 2] (1 - p) p^i, \quad \text{(E9)}
\]

\[
\text{with } 0 \leq n_1 \leq N - 2.
\]

With equations (E8) and (E9), equation (E3) can then be evaluated.

It is given by:
\[< K_i K_j > = \{(N - i - j - 4) (N - i - j - 5) (1 - p)^2 + 6(1 - p)\} \frac{1}{2} \frac{p^i}{p^j} \theta(i + j, N - 3) + 2(1 - p) p^{N-3} \delta(i + j, N - 3) + [(N - i - 3) (1 - p) + 2] (1 - p) p^i \delta_{ij} \theta(i, N - 2) + \delta_{ij} \delta_{1, N - 2} p^{N-2}\] (E10)
APPENDIX F

CALCULATION OF $< R^2 >$

FOR FIX PROBABILITY $p$ OF HAVING AN INSULATOR OCCUPYING A SITE IN A LINE

$$< R^2 > = R_o^2 \{ \sum_{i \leq N-2} \sum_{j \leq N-2} i^2 e^{b(i+j)} (1-p)^2 \{ 6 + 6(N - i - j - 4) (1-p) +$$

$$+ (N - i - j - 4) (N - j - 5) (1-p)^2 \} \delta(i+j, N-3) +$$

$$2 p^{N-3} (1-p) \delta(i+j, N-3) + p^i (1-p) [2 + (N - i - 3) (1-p)] x \delta_{ij} \theta(i, N-2) + \delta_{ij} p^{N-2} \delta_{i,N-2} \} \}$$

(F1)

In the next paragraphs each term in equation (F1) will be considered separately.

a) \[ \sum_{i \leq N-2} \sum_{j \leq N-2} i^2 e^{b(i+j)} (1-p)^2 \{ 6 + 6(N - i - j - 4) (1-p) +\]

$$+ (N - i - j - 4) (N - j - 5) (1-p)^2 \} ,$$

with $i + j < N - 3$. \hspace{1cm} (F2)

Performing a change of variables such that $m = i + j; n = i - j$ and rearranging the terms and the limits of the sums; equation (F2) is then written as:
\[ \sum_{m} (pe_b)^m (1 - p)^2 \{ A - 6m (1 - p) - (2N - 9) mC + m^2 C \} \sum_{n} \frac{(m^2 - n^2)}{4} = \]

\[ \frac{(1 - p)^2}{12} \sum_{m} (pe_b)^m \{ (4A - C) m^3 - 4Bm^4 + Bm^2 + 4Cm^5 - Am \} = \]

\[ \frac{(1 - p)^2}{12} \{ (4A - C) S_3 + BS_2 - 4BS_4 + 4CS_5 - AS_1 \}, \quad (F3) \]

where: \(-m \leq n \leq m, \quad 0 \leq m \leq N - 4;\]

\[ A = 6 + 6(N - 4)(1 - p) + (N - 4)(N - 5)(1 - p)^2; \]

\[ B = 6(1 - p) + (2N - 9)(1 - p)^2; \]

\[ C = (1 - p)^2 \]

\[ S_k = \sum_{m=0}^{N-4} (pe_b)^m m^K. \]

b) \[ \sum_{i} \sum_{j} (pe_b)^{N-3} 2 (1 - p) \delta_{i+j, N-3} = \]

\[ 2(pe_b)^{N-3} (1 - p) \frac{1}{4} \{ \sum_{j=0}^{N-3} (N - 3)^2 - \sum_{j=0}^{N-3} [(N - 3) - 2j]^2 \} = \]

\[ \frac{1}{3} (pe_b)^{N-3} (1 - p) (N - 2) (N - 3) (N - 4) \quad (F4) \]

c) \[ \sum_{i} \sum_{j} i^2 (pe_b)^{i+j} (1 - p) \{ 2 + (N - i - 3)(1 - p) \} \delta_{i+j} = \]

\[ \sum_{i=0}^{N-3} i^2 (pe_b)^{i} (1 - p) \{ 2 + (N - i - 3)(1 - p) \} = \]

\[ (1 - p) \{ 2 + (N - 3)(1 - p) \} T_2 - (1 - p)^2 T_3, \quad (F5) \]
where $T_k = \sum_{i=0}^{N-3} i^k (pe^2b)^i$.

\[ \sum_{i} \sum_{j} i^j e^{b(i+j)} p^{(N-2)} \delta_{ij} \delta_{i,N-2} = (N - 2)^2 (pe^2b)^{N-2} \]  \hspace{1cm} (F6)

With equations (F3), (F4), (F5) and (F6) the final expression for $< R^2 >$ is

\[ \frac{< R^2 >}{R_0^2} = \frac{(1 - p)^2}{12} \left\{ (4A - C) S_3 + BS_2 - 4BS_4 + 4CS_5 - AS_1 \right\} \]

\[ + \frac{1}{3} (pe^b)^{N-3} (1 - p) (N - 2) (N - 3) (N - 4) \]

\[ + (1 - p) [2 + (N - 3) (1 - p)] T_2 - (1 - p)^2 T_3 \]

\[ + (N - 2)^2 (pe^2b)^{N-2}. \]  \hspace{1cm} (F7)

with $S_k = \sum_{m=0}^{N-4} (pe^b)^m k^m$

and

\[ T_k = \sum_{i=0}^{N-3} i^k (pe^2b)^i. \]
APPENDIX G

CALCULATION OF $< K_1 >$

FOR FIXED PROBABILITY $p$ OF HAVING AN INSULATOR OCCUPYING A SITE IN A LINE BENT INTO A CIRCLE

$$< K_1 > = \sum_{n_1} \sum_{K} \sum_{\{K_i\}} \frac{K!}{K_1! K_2! \ldots K_n!} K_1 \left( \frac{n_2 - 1}{K - 1} \right) \left( \frac{N}{K} \right)^n p_1^{n_1} (1 - p)^{n_2},$$

with $1 \leq n_1 \leq N - 1$, $1 \leq K \leq n_1$, $\{K_i\} = \{K_1, K_2, \ldots, K_n\}$, $\sum_i K_i = K$,

$$\sum_{i} K_i = n_1 \text{ and } 1 \leq i \leq n_1.$$

(G1)

The sum over $\{K_i\}$ can be performed in the same way as in Appendix A leading to:

$$< K_1 > = \sum_{n_1} \sum_{K} \frac{N}{K} p_1^n (1 - p)^{n_2} \left( \frac{n_1 - i - 1}{K - 2} \right) \left( \frac{n_2 - 1}{K - 1} \right),$$

with $1 \leq n_1 \leq N - 2$ and $1 \leq K \leq n_1$.

(G2)

Performing the change of variables in equation (G2) such that $K' = K - 1$ and rearranging the limits of the sums in order that the binomial coefficient is not zero; equation (G2) then reduces to:

$$< K_1 > = \sum_{n_1} \sum_{K'} N \left( \frac{n_2 - 1}{K'} \right) \left( \frac{n_1 - i - 1}{n_1 - i - K'} \right) p_1^n (1 - p)^{n_2},$$

With $1 \leq n_1 \leq N - 1$ and $1 \leq K' \leq n_1 - i$.

(G3)

From equation (A3), expression (G3) is given by:

$$< K_1 > = \sum_{n_1} N \left( \frac{N - i - 2}{n_1 - i} \right) p_1^n (1 - p)^{n_2},$$

(G4)
with \( 1 \leq n_1 \leq N - 1 \) and \( i \leq N - 2 \).

Making the change of variables such that \( n'_1 = n_1 - i \) and rearranging the limits of the sum, equation (G4) is then equal to:

\[
<K_i> = (1 - p)^N \left( \frac{p}{1 - p} \right)^i N \sum_{n'_1=0}^{N-i-2} \binom{N - 2 - i}{n'_1} \left( \frac{p}{1 - p} \right)^{n'_1} \]

By the multinomial formula it is straightforward that:

\[
<K_i> = N p^i (1 - p)^2 \quad \text{for} \quad i \leq N - 2.
\]

A more general way of writing \( <K_i> \) is:

\[
<K_i> = \begin{cases} 
N p^i (1 - p)^2 & \text{if } i \leq N - 2 \\
N p^{(N-1)} (1 - p) & \text{if } i = N - 1 \text{ and } n_2 = 1.
\end{cases}
\]

(G6)
APPENDIX H

CALCULATION OF \( \langle K_i K_j \rangle \)

FOR FIXED PROBABILITY \( p \) OF HAVING AN INSULATOR OCCUPYING A SITE IN A LINE
BENT INTO CIRCLE

\[
\langle K_i K_j \rangle = \sum_{n_1} \sum_{K} \sum_{\{K_i\}} \frac{K_1! \ldots K_{n_1}!}{K_1! \ldots K_{n_1}!} K_1 K_j \left\{ p_1^{n_1} (1 - p)^{n_2} \frac{N}{K} \binom{n_2}{K-1} \right\},
\]

with \( 1 \leq n_1 \leq N - 1 \), \( 1 \leq K \leq n_1 \), \( \sum \limits_i K_i = K \),
\[
\sum \limits_i K_i = n_1 \text{ and } 1 \leq i \leq n_1
\]

Equation (H1) can be written as:

\[
\langle K_i K_j \rangle = \sum_{n_1} \sum_{K} \sum_{\{K_i\}} \frac{K_1! \ldots K_{n_1}!}{K_1! \ldots K_{n_1}!} K_1 K_j \left\{ p_1^{n_1} (1 - p)^{n_2} \frac{N}{n_2} \binom{n_2}{K} \right\}, \tag{H2}
\]

with the same restrictions as in equation (H1).

Referring to Appendix B, the result for the sum over \( \{K_i\} \) can be obtained leading to:

\[
\langle K_i K_j \rangle = \sum_{n_1} \sum_{K} p_1^{n_1} (1 - p)^{n_2} \frac{N}{n_2} \binom{n_2}{K} [K(K - 1) \binom{n_1 - i - j - 1}{K - 3} + K \binom{n_1 - i - 1}{K - 2} \delta_{ij}], \tag{H3}
\]

with \( 1 \leq n_1 \leq N - 1 \) and \( 1 \leq K \leq n_1 \).

In order to perform first the sum over \( K \), consider now the first term at the right-hand side of equation (H3) and rearrange the terms.
Therefore:

\[
\sum_{K} p_{1} (1 - p)_{2} \frac{N_{2}}{N_{1}} \binom{n_{2}}{K} K(K - 1) \binom{n_{1} - i - j - 1}{K - 3} =
\]

\[
N \sum_{K} (n_{2} - 1) \binom{n_{2} - 2}{K - 2} \binom{n_{1} - i - j - 1}{K - 3}, \text{ with } 1 \leq K \leq n_{1}. \quad (H4)
\]

Making the change of variables in equation (H4), \( K'' = K - 2 \) and using relation (A3); the final expression for equation (H4) is then given by:

\[
\sum_{K} p_{1} (1 - p)_{2} \frac{N_{2}}{N_{1}} \binom{n_{2}}{K} K(K - 1) \binom{n_{1} - i - j - 1}{K - 3} =
\]

\[
\binom{n_{1} - i - j}{N_{1} - i - j - 3} \binom{n_{2} - 2}{K''} \binom{n_{1} - i - j}{K'' - 1} \binom{n_{1}}{p_{1}} (1 - p)_{2}^{n_{2}}
\]

\[
N(n_{2} - 1) \binom{N - i - j - 3}{n_{1} - i - j} \binom{n_{1}}{p_{1}} (1 - p)_{2}^{n_{2}} \quad (H5)
\]

Now take into consideration the second term in the right-hand side in equation (H3) and rearrange the terms.

\[
\sum_{K} p_{1} (1 - p)_{2} \frac{N_{2}}{N_{1}} \binom{n_{2}}{K} \binom{n_{1} - i - 1}{K - 2} =
\]

\[
\sum_{K} p_{1} (1 - p)_{2} \frac{n_{2} - 1}{N} \binom{n_{2} - 1}{K - 1} \binom{n_{1} - i - 1}{K - 2} =
\]

\[
\sum_{K'} \binom{n_{2} - 1}{N} p_{1} (1 - p)_{2} \binom{n_{2} - 1}{K'} \binom{n_{1} - i - 1}{K' - 1},
\]

with 1 \( \leq K \leq n_{1} \) and \( K' = K - 1 \).

Again using relation (A3), one gets:
\[
\sum_{K} p^{n_1} (1 - p)^{n_2} \frac{N}{n_2} K \binom{n_2}{K} \binom{n_1 - i - 1}{K - 2} =
\]

\[
N p^{n_1} (1 - p)^{n_2} \binom{N - i - 2}{n_1 - 1}, \text{ restricted to } 1 \leq K \leq n_1.
\]

Considering equations (H5), (H6) and the limits at which the binomial coefficients are not equal to zero; the expression for \( \langle K_i K_j \rangle \), equation (H3), reduces to:

\[
\langle K_i K_j \rangle = \sum_{n_1=1}^{N-3} p^{n_1} (1 - p)^{n_2} N(N_2 - 1) \binom{N - i - j - 3}{n_1 - i - j} \delta(i + j, N - 2)
\]

\[+ N p^i j (1 - p)^2 \delta(i + j, N - 2)
\]

\[+ \sum_{n_1=1}^{N-2} p^{n_1} (1 - p)^{n_2} N \binom{N - i - 2}{n_1 - i} \delta_{ij} \delta(i, N - 1)
\]

\[+ N(1 - p) p^i \delta_{ij} \delta_{i, N-1} \quad (H7)
\]

In order to evaluate the sum over \( n_1 \) at the first term in equation (H7), perform the change of variables, \( n_1' = n_1 - i - j \). Therefore:

\[
\sum_{n_1=1}^{N-3} p^{n_1} (1 - p)^{N-n_1} N(N - n_1 - 1) \binom{N - i - j - 3}{n_1 - i - j} =
\]

\[
N(1 - p)^N \left( \frac{p}{1 - p} \right)^{i+j} (N - i - j - 1) \sum_{n_1' = 0}^{N-3} \left( \frac{p}{1 - p} \right)^{n_1'} \binom{N - i - j - 3}{n_1'}
\]

\[
+ N - i - j - 3 \sum_{n_1' = 0}^{n_1} \left( \frac{p}{1 - p} \right)^{n_1'} \binom{N - i - j - 3}{n_1'} \quad (H8)
\]
With the relation \( n_1^{'}\binom{N - i - j - 3}{n_1^{''}} = (N - i - j - 3) \binom{N - i - j - 4}{n_1^{''} - 1} \)

and using the definition of the binomial coefficients, equation (H8) reduces to:

\[
N(1 - p)^N \left( \frac{p}{1 - p} \right)^{i+j} \left\{ \frac{(N - i - j - 1)}{(1 - p)^{N-1-j-3}} - \right.
\]

\[
(N - i - j - 3) \sum_{n_1^{''}=0}^{N-1-j-4} \left( \frac{p}{1 - p} \right)^{n_1^{''}+1} \binom{N - i - j - 4}{n_1^{''}} \right\},
\]

where \( n_1^{''} = n_1^{'} - 1 \).

Again using the definition of the binomial coefficients, one gets:

\[
\sum_{n_1^{'}=1}^{N-3} \binom{n_1}{N-n_1} p^{1-j} (1 - p)^{N-(N-n_1)} \binom{N-i-j-3}{n_1} =
\]

\[
N p^{i+j} (1 - p)^3 \left[ (1 - p) (N - i - j - 3) + 2 \right]. \quad (H9)
\]

Consider now the second sum in the right-hand side of equation (H7), and perform a change of variables: \( n_1^{'} = n_1 - i \).

Therefore:

\[
\sum_{n_1^{'}=1}^{N-2} \binom{n_1}{N-n_1} p^{1-j} (1 - p)^{N-(N-n_1)} \binom{N-i-2}{n_1} \delta_{ij} = \theta(i, N - 1) =
\]

\[
N(1 - p)^N \sum_{n_1^{'}=0}^{N-i-2} \left( \frac{p}{1 - p} \right)^{n_1^{'+1}} \binom{N-i-2}{n_1^{'}} = N \ p^i(1 - p)^2 \quad \text{(H10)}
\]

Therefore, with (H9) and (H10), the expression for the moment \( <K_i K_j> \), equation (H7), is given by:
\[ <K_i K_j> = N p^{i+j} (1 - p)^3 \{ (1 - p) (N - i - j - 3) + 2 \} \theta(i + j, N - 2) + \]
\[ + N p^{N-2} (1 - p)^2 \delta(i + j, N - 2) + \]
\[ + N p^i (1 - p)^2 \delta_{ij} \theta(i, N - 1) + N(1 - p) p^{N-1} \delta_{ij} \delta_{i,N-1} \]

\text{(H11)}
APPENDIX I

CALCULATION OF $< R^2 >$

FOR FIX PROBABILITY $p$ OF HAVING AN INSULATOR OCCUPYING A SITE IN A LINE BENT INTO A CIRCLE

$$< R^2 > = R_o \sum \sum_{i,j} i j e^{b(i+j)} < K_i K_j >$$

Substitute equation (H11) into equation (II).

$$\frac{< R^2 >}{R^2_o} = \sum \sum_{i,j} i j (p e^b)^{i+j} N (1 - p)^3 [2 + (N - 3)(1 - p)] \theta(i + j, N - 2) -$$

$$\sum \sum_{i,j} i j (p e^b)^{i+j} N (1 - p)^4 (1 + j) \theta(i + j, N - 2) +$$

$$\sum \sum_{i,j} i j (p e^b)^{i+j} N (1 - p)^2 \delta(i + j, N - 2) +$$

$$\sum \sum_{i,j} i j N p^i e^{b(i+j)} (1 - p)^2 \theta(i, N - 1) \delta_{ij} +$$

$$\sum \sum_{i,j} i j N(1 - p) e^{b(i+j)} p^i \delta(i, N - 1) \delta_{ij}$$

The sums in equation (II) are going to be evaluated separately in the subsequent paragraphs.
a) \[ \sum \sum \ ij \ (p e_b)^{i+j} \ \delta(i + j, N - 2) = \]
\[ \sum_{m=0}^{N-3} \sum_{n=-m}^{m} \frac{m^2 - n^2}{4} (p e_b)^m = \sum_{m=0}^{N-3} \frac{4m^3 - m}{12} (p e_b)^m = \]
\[ \frac{1}{12} \{4S'_3 - S'_1\}; \text{ where } i + j = m, \ i - j = n \]

and \[ S'_k = \sum_{m=0}^{N-3} (p e_b)^m m^k. \]

b) Analogously to the deduction of equation (13), it is obtained that:

\[ \sum \sum \ ij \ (p e_b)^{i+j} (i + j) \ \delta(i + j, N - 2) = \frac{1}{12} [4 S'_4 - S'_2]. \]

(14)

c) \[ \sum \sum \ (p e_b)^{i+j} ij \ \delta(i + j, N - 2) = \]
\[ (p e_b)^{N-2} \sum_{i=0}^{N-2} i(N - 2 - i) = \frac{1}{6} (N - 1) (N - 2) (N - 3) (p e_b)^{N-2}. \]

(15)

d) \[ \sum \sum \ ij \ p^i e_b(i+j) \ \delta_{ij} \ \delta(i, N - 1) = \sum_{i=0}^{N-2} i^2 (p e^{2b})^i = T'_2, \]

(16)

where \[ T'_k = \sum_{i=0}^{N-2} (p e^{2b})^i i^k. \]
Consequently, with (I3), (I4), (I5) and (I6), equation (I2) is given by:

\[
\frac{\langle R^2 \rangle}{R_0^2} = \frac{1}{12} (4S'_3 - S'_1) N(1 - p)^3 [2 + (N - 3)(1 - p)] - \frac{1}{12} (4S'_4 - S'_2) N(1 - p)^4 + N(1 - p)^2 \frac{1}{6} (N - 1)(N - 2)(N - 3)(p e^b)^{N-2} + N(1 - p)^2 T'_2 + N(N - 1)^2 (p e^b)^{N-1} (1 - p)
\]  

(I7)

where \( S'_K = \sum_{m=0}^{N-3} (p e^b)^m \frac{m^K}{m!} \) and \( T'_K = \sum_{i=1}^{N-2} (p e^b)^i \frac{i^K}{i!} \).