THE COMPUTATION OF THE BLOCK-ERROR RATE ON A RAYLEIGH-FADING CHANNEL IN THE PRESENCE OF ADDITIVE WHITE GAUSSIAN NOISE

by

BRIAN HOWARD MARANDA

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Department of Electrical Engineering

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

Date December 2, 1982
Abstract

The problem of computing the probability $P_f(M,N)$ of more than $M$ bit errors in a block of $N$ bits for a Rayleigh-fading channel in the presence of additive white Gaussian noise is considered. In the case of very slow Rayleigh fading, analytical formulas for $P_f(M,N)$ have been derived in the literature, but these formulas are not well suited for numerical computation. Several simple approximations for the special case $P_f(0,N)$ have also appeared in the literature, but apparently no work has been done for the case $M > 0$. In this thesis an accurate approximation for $P_f(M,N)$ is derived, and a bound on the error in this approximation is given. Also included are approximations for $P_f(M,N)$ when selection diversity is employed.

If very slow fading is not assumed, there exist no known analytical methods for the computation of the block-error probability. Simulations are performed on a digital computer for this case. Furthermore, an empirical formula is derived that can be used to estimate easily and accurately the output of the simulator. The consequence is a great saving of time and effort in the computation of a value of $P_f(M,N)$ that is more realistic than that provided under the assumption of very slow fading.
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Note: See page 28 for the significance of a caret $^\wedge$. 
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Chapter 1
Introduction

In a binary digital communication system, information is usually transmitted in blocks, or packets, of \( N \) bits. There are two main techniques for combatting the effects of channel errors: forward error correction (FEC) and automatic repeat-request (ARQ). To evaluate the performance of an FEC scheme that employs an error-correction code capable of correcting \( M \) or fewer bit errors per block, one wishes to know the probability \( P_b(M,N) \) of more than \( M \) bit errors in a block, where \( 0 \leq M < N \). In practice, we are concerned with the case \( M \ll N \).

The overall performance of the ARQ systems, which are based upon the detection of errors at the receiver and a subsequent request for retransmission, depends primarily on the probability of block error, \( P_b(0,N) \). (For most codes the probability of an undetected error is very small; therefore, the probability of a detected error is very close to the block-error probability).

For most binary signalling schemes on an additive white Gaussian noise (AWGN) channel, the probability \( P_b(M,N) \) is easily calculable from the bit-error probability \( p \):

\[
P_b(M,N) = \sum_{m=M+1}^{N} \binom{N}{m} p^m (1-p)^{N-m}
\]

(1.1.1)

\[
= 1 - \sum_{m=0}^{M} \binom{N}{m} p^m (1-p)^{N-m},
\]

with the special case \( P_b(0,N) = 1 - (1-p)^N \). Thus for the AGWN channel under steady-signal conditions, the probability of bit
error is a useful figure of merit.

However, radio transmission in an environment in which there are multiple changing propagation paths is subject to random time variation, or fading, of the signal strength at the receiver. Such multipath interference occurs in many situations. Examples are:

(i) long-distance communication based on ionospheric reflection (used mainly in the HF band).

(ii) trans-horizon communication via tropospheric scatter (used at UHF).

(iii) mobile-radio transmission in urban areas, where often there is no line-of-sight path between the receiver and transmitter — the received signal is due totally to the scattering of the transmitted signal from buildings (used at UHF).

The effect of fading on digital transmission is to cause a time variation of the error probability at the receiver, which in turn results in a "clustering" of errors. There are several techniques that are used, usually in conjunction with coding, to counteract the adverse effects of fading. One method is to interleave the codewords in an attempt to break up the clustering of errors [6]. We shall not consider interleaving here. Another method to help overcome the effects of fading is
to use diversity transmission. The idea behind diversity transmission is to send the same message on a number of independent paths, or branches. The hope is that not all signals will be subject to deep fades, and hence that by suitably combining the received signals the probability of error will be reduced.

Denote the average bit-error probability in the fading case by $p_f$, and similarly denote by $P_f(M,N)$ the fading counterpart of $P_b(M,N)$. The average bit-error probability is no longer a good figure of merit, because the probability $P_f(M,N)$ is not related to $p_f$ through the simple equation (1.1.1). The question of how to compute $P_f(M,N)$ thus arises. In this thesis, this problem is considered for the specific case of Rayleigh fading in the presence of additive white Gaussian noise. The theoretical basis for the use of the Rayleigh model under a variety of fading conditions can be found in [7], [23], and [29]. We subdivide the problem into the following two cases:

(1) In the case of nonselective\(^1\) very slow Rayleigh fading, an expression for $P_f(M,N)$ can be written in terms of integrals, which in the special case of non-coherent frequency-shift keying (NCFSK) can in turn be expanded into finite series [9]. Unfortunately, these series are not convenient for numerical computation, and in [9] the calculation of $P_f(0,N)$ is done by numerical integration. Several approximations for $P_f(0,N)$ have

\(^1\)That is, flat across the spectrum bandwidth.
been made in the literature [21,26], but these approximations are derived in an ad hoc manner, i.e. the expressions are manipulated and terms are neglected until results culled from tables can be easily applied. Also, the case for $M > 0$ is largely ignored in the literature. In this thesis a method of approximation that allows a simple treatment of the case $M > 0$ is presented; this new method yields a much better approximation to $P_f(0,N)$ as well. Furthermore, an approximation is given for $P_f(M,N)$ in the more difficult case of selection diversity [25].

(2) If the requirement of very slow fading is relaxed the analytical methods of (1) are no longer applicable, and $P_f(M,N)$ must be computed by performing simulations. The simulations done for this thesis were performed on a digital computer, and were tailored to the mobile-radio environment. The emphasis here is on mobile radio for the simple reasons that there is a large amount of literature on the statistical properties of mobile radio and that some software has already been published in this area [24]. The main drawback of this approach is the large amount of computer time required and the consequent high cost. To obviate the need for such simulations, an empirical formula is derived that allows an accurate prediction of the simulation results when $M = 0$; this empirical formula permits the easy computation of $P_f(0,N)$ when very slow fading is not assumed.
The thesis has two main chapters, Chapters 2 and 3, which deal with cases (1) and (2) above. Chapter 4 presents the conclusions and a short discussion of possible topics for future investigation.
Chapter 2

Very Slow Rayleigh Fading

In this chapter we examine methods for the computation of $P_f(M,N)$ for a channel subject to very slow Rayleigh fading in the presence of additive white Gaussian noise. Section 2.1 introduces the basic concepts and notation, and is largely a review of what has been done in the literature. As we shall see, almost all the results published to date deal with the special case of $P_f(0,N)$. In Section 2.2 we derive a new approximation to $P_f(0,N)$, and show how the results can be extended to $P_f(M,N)$ for $M > 0$. Moreover, an upper bound on the error in the approximation is provided. In Section 2.3, the formulas of Section 2.2 are expanded into infinite series (the approximations in the literature are essentially first-order expansions). Finally, in Section 2.4, approximations are derived for the computation of $P_f(M,N)$ when selection diversity is employed.
2.1 **Background**

2.1.1 **The AWGN channel**

We begin by reviewing several useful results for the transmission of one of two equally likely signals on the AWGN channel with steady-signal reception. The AWGN channel can be represented by

\[ n(t) \]

\[ s_i(t) \]

\[ r_i(t) \]

**Figure 2.1: The AWGN channel.**

where \( n(t) \) is white Gaussian noise with two-sided power spectral density \( N_0/2 \). Consider two signals \( s_1(t) \) and \( s_2(t) \) which are zero outside the interval \( 0 \leq t \leq T \) and which satisfy

\[
E_s = \int_0^T s_i(t)^2 \, dt. \quad (i = 1, 2)
\]

(For the modulation schemes mentioned below, we may assume that \( s_i(t) \) is of this form.) Assume that with equal a priori probability we send either \( s_1(t) \) or \( s_2(t) \), and that the receiver is optimum. Then the probabilities of error for some important binary signalling schemes are given by the following well-known expressions [23,29]

\[
p(\gamma) = (1/2) \exp(-\alpha \gamma)
\]

\[
\begin{align*}
\alpha &= 1/2, \text{ noncoherent FSK (NCFSK)} \\
\alpha &= 1, \text{ differential PSK (DPSK)}
\end{align*}
\]
and

\[ p(\gamma) = \frac{1}{2} \text{erfc}(\sqrt{a\gamma}) \]

where \( \gamma \triangleq E_s/N_0 \) is the signal-to-noise power ratio (SNR).

Suppose now that we transmit a block of \( N \) bits. For NCFSK, CFSK, and PSK modulations the errors for different signalling elements are independent,\(^2\) and hence the probability of \( m \) errors in a block of \( N \) bits is

\[ \binom{N}{m} p^m (1 - p)^{N-m}, \]

where \( p = p(\gamma) \) for a specific SNR \( \gamma \). The result (2.1.3) does not apply to DPSK modulation, since the differential encoding causes dependence between errors at the receiver. However, for those modulation techniques for which (2.1.3) does apply, we can immediately write down the probability that more than \( M \) bits in a block are in error:

\[ P_b(M,N) = \sum_{m=M+1}^{N} \binom{N}{m} p^m (1 - p)^{N-m} \]

\[ = 1 - \sum_{m=0}^{M} \binom{N}{m} p^m (1 - p)^{N-m}. \]

\(^2\)If \( s_i(t) \) is zero outside the range \( 0 \leq t \leq T \) it must have infinite bandwidth — a signal cannot be both time-limited and band-limited (see Appendix 5B of [29]). Intersymbol interference, which causes some dependence between successive bits, may in practice be ignored on channels with sufficiently large bandwidth.
The dependence of $P_b(M,N)$ on the SNR $\gamma$ is implicit in equation (2.1.4); in those instances where we wish to make this dependence clear we shall write $P_b(M,N;\gamma)$. From equation (2.1.4) we see that even if the probabilities $P_b(M,N)$ are the quantities of most interest, the probability of bit error retains its status as a fundamental unit for the measure of system performance.

2.1.2 The fading channel

Let us now consider the case of nonselective, very slow Rayleigh fading in the presence of additive white Gaussian noise. As discussed in Section 9.2 of [23], nonselective fading can be represented as a purely multiplicative process. The specific model we use is shown in Figure 2.2.

![Figure 2.2: The fading channel.](image)

Here $a(t)$ is a stationary Rayleigh process with first-order density function

$$f_{a(t)}(a) = \frac{2a}{A} \exp(-a^2/A) \quad (a \geq 0) \quad (2.1.5)$$

and $s_i(t)$ and $n(t)$ are as before. The instantaneous SNR is a random process which we define by
Then from equations (2.1.5) and (2.1.6) it follows that

\[ f_{\gamma(t)}(\gamma) = \frac{N_0}{AE_S} \exp\left(-N_0\frac{\gamma}{AE_S}\right) \quad (\gamma \geq 0) \]

\[ = \gamma_0^{-1} \exp\left(-\gamma/\gamma_0\right), \quad (\gamma \geq 0) \]

where \( \gamma_0 \triangleq AE_S/N_0 \) is the average SNR. From now on we shall write simply \( f(\gamma) \) for \( f_{\gamma(t)}(\gamma) \).

Let us now assume that the fading process \( a(t) \) varies so slowly with respect to the signalling rate that it may be considered constant over the duration of each bit; this is what we mean by "slow fading". Under this assumption, the probability of error for a particular bit is just the probability \( p(\gamma) \) of a bit error on the AWGN channel. Thus for the slow-fading case we consider \( p(\gamma) \) to be the probability of a bit error conditioned on the SNR \( \gamma \) that results from a particular value of the fading multiplier. To compute \( p_\gamma \), the average probability of bit error under fading conditions, we then average \( p(\gamma) \) over all fading signal strengths. For the particular model considered here, the average is taken over the distribution \( f(\gamma) \) given in (2.1.7). Then

\[ p_\gamma = \int_0^\infty p(\gamma) f(\gamma) \, d\gamma \]

\[ = \gamma_0^{-1} \int_0^\infty p(\gamma) \exp\left(-\gamma/\gamma_0\right) \, d\gamma. \]

Note that equation (2.1.8) resembles a Laplace transform. To be
precise, \( p_f = \gamma_0^{-1} \tilde{P}(\gamma_0^{-1}) \), where \( \tilde{P} \) is the Laplace transform of \( p(\gamma) \). For \( p(\gamma) \) as in (2.1.1), the integral (2.1.8) is easy to evaluate analytically; for \( p(\gamma) \) as in (2.1.2) the integral is a bit more difficult, and it is easiest just to consult a table of Laplace transforms. The results are [23]

\[
\begin{align*}
\frac{1}{2} \left[ \frac{1}{1 + a\gamma_0} \right] & \quad \text{for } a = 1/2, \text{ noncoherent FSK (NCFSK)} \\
\frac{1}{2} \left[ 1 - \frac{1}{(1 + 1/a\gamma_0)^{1/2}} \right] & \quad \text{for } a = 1, \text{ differential PSK (DPSK)}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} \left[ 1 - \frac{1}{(1 + 1/a\gamma_0)^{1/2}} \right] & \quad \text{for } a = 1/2, \text{ coherent FSK (CFSK)} \\
\frac{1}{2} \left[ \frac{1}{1 + a\gamma_0} \right] & \quad \text{for } a = 1, \text{ PSK}
\end{align*}
\]

2.1.3 The computation of \( P_f(M,N) \): Series expressions

We come now to our main topic: the computation of the probability \( P_f(M,N) \) of more than \( M \) errors in a block of \( N \) bits under fading conditions. Since bit errors are no longer binomially distributed as in (2.1.3), \( P_f(M,N) \) is not connected to \( p_f \), the probability of bit error under fading conditions, through (2.1.4). One approach to the computation of \( P_f(M,N) \) is to extend the above analysis by assuming that the fading process is so slow that the SNR \( \gamma \) remains constant not only over a single bit, but also over an entire block of \( N \) bits, and that the signal-to-noise ratio varies from block to block according to the distribution \( f(\gamma) \) of equation (2.1.7). In this thesis we shall term such fading "very slow". Although the assumption of very slow fading becomes less and less realistic
as the block length becomes larger, we shall see that based on this assumption we can derive results that are quite useful, particularly in Chapter 3, where the assumption of very slow fading is relaxed. An expression for $P_f(M,N)$ that is analogous to the expression (2.1.8) for the average bit-error probability $p_f$ can be written down:

$$P_f(M,N) = \int_0^\infty P_b(M,N;\gamma) f(\gamma) \, d\gamma$$

$$= \gamma_0^{-1} \int_0^\infty P_b(M,N;\gamma) \exp(-\gamma/\gamma_0) \, d\gamma,$$

where $P_b(M,N;\gamma)$ is the probability of error under steady-signal conditions given in equation (2.1.4). Equation (2.1.11) displays the dependence of $P_f(M,N)$ on the average signal-to-noise ratio $\gamma_0$. If we were to follow the notational convention established for $P_b(M,N) = P_b(M,N;\gamma)$, we should write $P_f(M,N) = P_f(M,N;\gamma_0)$ in those cases where we desire to make the dependence on $\gamma_0$ clear; however, we shall soon see that it is more convenient to indicate the dependence on $\gamma_0$ through the quantity $r \triangleq 2/\gamma_0$, and to write $P_f(M,N) = P_f(M,N;r)$. For $\gamma_0$ greater than 2 (i.e. about 3 dB), $r$ is in the range $0 < r < 1$, and for $\gamma_0 >> 1$, $r$ satisfies $0 < r << 1$. Thus we are primarily concerned with small positive values of $r$.

Since (2.1.4) does not apply to DPSK modulation (except in the trivial case $N = 1$), from now on we restrict our attention to NCFSK, CFSK, and PSK. Substitution of (2.1.4) into (2.1.11) gives
\[ P_f(M, N) = \sum_{m=M+1}^{N} \binom{N}{m} \int_0^\infty p(\gamma)^m [1 - p(\gamma)]^{N-m} f(\gamma) \, d\gamma \]

\[ = 1 - \sum_{m=0}^{M} \binom{N}{m} \int_0^\infty p(\gamma)^m [1 - p(\gamma)]^{N-m} f(\gamma) \, d\gamma \quad (2.1.12) \]

\[ = 1 - \gamma_0^{-1} \sum_{m=0}^{M} \binom{N}{m} \int_0^\infty p(\gamma)^m \exp(-\gamma/\gamma_0) \, d\gamma. \]

Using the binomial expansion for \([1 - p(\gamma)]^{N-m}\) we obtain

\[ P_f(M, N) = 1 - \gamma_0^{-1} \sum_{m=0}^{M} \binom{N}{m} \sum_{k=0}^{N-m} (-1)^k \binom{N}{m} \binom{N-m}{k} \int_0^\infty p(\gamma)^{m+k} \exp(-\gamma/\gamma_0) \, d\gamma, \quad (2.1.13) \]

Let us first look at the simplest case, when \(M = 0\). We have

\[ P_f(0, N) = 1 - \gamma_0^{-1} \sum_{k=0}^{N} (-1)^k \binom{N}{k} \int_0^\infty p(\gamma)^k \exp(-\gamma/\gamma_0) \, d\gamma \quad (2.1.14) \]

\[ = \gamma_0^{-1} \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \int_0^\infty p(\gamma)^k \exp(-\gamma/\gamma_0) \, d\gamma, \]

where in the last step the \(k = 0\) term (which is equal to 1) has been cancelled out. For \(N = 1\), we obtain the results for \(P_f = P_f(0, 1)\) given in equations (2.1.9) and (2.1.10). For \(N > 1\), we must evaluate the integral

\[ \gamma_0^{-1} \int_0^\infty p(\gamma)^k \exp(-\gamma/\gamma_0) \, d\gamma \quad (2.1.15) \]

for \(k > 1\). Unfortunately, because of the complex form of \(p(\gamma)\) for CFSK and PSK modulations, there appears to be no analytical result for the integral when \(k > 1\). However, in the case of NCFSK, \(p(\gamma)\) has the simple form \((1/2)\exp(-\gamma/2)\) and (2.1.15) can easily be evaluated:
\[ \gamma_0^{-1} \int_0^{\infty} p(\gamma)^k \exp(-\gamma/\gamma_0) \, d\gamma \]
\[ = \gamma_0^{-1} (1/2)^k \int_0^{\infty} \exp[-\gamma(k/2 + 1/\gamma_0)] \, d\gamma \]
\[ = \left( \frac{1/2}{\gamma_0(k/2 + 1/\gamma_0)} \right)^k = r \left( \frac{1/2}{k + r} \right)^k \]

where \( r = 2/\gamma_0 \). Substitution into (2.1.14) of the last expression appearing in (2.1.16) yields

\[ P_f(0,N;r) = -r \sum_{k=1}^{N} \frac{(-1/2)^k}{k + r}. \]

This equation appears in both [21] and [26] (with obvious differences in notation). Equation (2.1.17) can be used to compute \( P_f(0,N;r) \) for small block lengths, but numerical difficulties begin to arise as \( N \) is increased, and for very large block lengths these difficulties are substantial. The main problem with using the alternating series (2.1.17) to compute \( P_f(0,N;r) \) for large \( N \) is the severe loss of significant figures caused by cancellation between terms. Note also that the terms may become very large in magnitude. For example, take \( N = 100 \) and \( \gamma_0 = 100 = 20 \) dB (then \( r = 2/\gamma_0 = 0.02 \)). Using formulas which are derived in Section 2.2, we find \( P_f(0,100;0.02) \approx 8.57 \times 10^{-2} \). However, the magnitude of the \( k = 50 \) term in (2.1.17) is about \( 3.58 \times 10^{10} \). Then for the sum to yield \( 8.57 \times 10^{-2} \), it is obvious that an enormous amount of cancellation between terms must take place, and hence to evaluate (2.1.17) on a computer to even one or two digits of accuracy requires multiple-precision arithmetic. Tests were run on the Amdahl 470 computer here at UBC to find the values of \( N \)
for which the direct evaluation of the series (2.1.17) is practical. The calculations were performed using double-precision arithmetic, and $\gamma_0$ was taken to be 20 dB. The value of $N$ was steadily increased and the computed values of $P_f(0,N;0.02)$ were compared with the correct values, which were computed using the methods of Section 2.2. For small values of $N$, (2.1.17) gave excellent results, but the accuracy steadily deteriorated as $N$ increased, and for $N \approx 80$ only two significant figures were obtained. For $N > 90$, the effect of cancellation was so dominant that the value of $P_f$ computed via (2.1.17) was useless (for example, the computed value was sometimes negative). If we now consider $0 < M < N$, then, for NCFSK, equation (2.1.13) leads to

$$P_f(M,N) = 1 - r \sum_{m=0}^{M} \left( \frac{1}{2} \right)^m \left( \begin{array}{c} N-m \\ m \end{array} \right) \frac{(-1/2)^k}{k + m + r}$$

(2.1.18)

$$= P_f(0,N) - r \sum_{m=1}^{M} \left( \frac{1}{2} \right)^m \left( \begin{array}{c} N-m \\ m \end{array} \right) \frac{(-1/2)^k}{k + m + r} .$$

The numerical evaluation of (2.1.18) on a computer clearly runs into the same difficulties as before.

2.1.4 The computation of $P_f(M,N)$: Numerical integration

One way to avoid the difficulties in evaluating the summations above is to compute $P_f(M,N)$ by numerically integrating (2.1.12). (In the case of CFSK or PSK modulation, for which the integrals cannot be evaluated analytically, this
approach must be taken.) This procedure is carried out in [9] for \( N = 1, 10, 10^2, 10^3, 10^4, 10^5 \), although only the special case \( M = 0 \) is considered. The numerical integration of (2.1.12) is also done in [21] as a check on approximations to \( P_f(0,N) \) that are derived there [see equations (2.1.24) and (2.1.26)]; the values of \( N \) are 1, 10, 10^2, 10^3, 10^4, and the case \( M = 0 \) is still the only one considered. In later sections of this thesis, we shall be deriving approximations to \( P_f(M,N) \), and naturally we shall want to compare the results of these approximations with the exact results given by numerical integration.\(^3\) However, there are several reasons why the numerical results given in [9] and [21] are insufficient for our purposes. The main reason is that numerical integration is performed in [9] and [21] only for the simplest case \( M = 0 \), whereas we are interested in the case \( M > 0 \) as well. Also, the results in [9] are presented in the form of graphs that are not accurate enough for adequate comparison with the approximations we derive. The results in [21] are given to 4 significant figures, but only for \( \gamma_0 \) in steps of 10 dB. For these reasons, equation (2.1.12) was numerically integrated on the Amdahl 470 for a more suitable range of parameters: \( N = 10, 10^2, 10^3, 10^4 \) [recall that for \( N = 1 \) there are the exact results (2.1.9)]; \( M = 0, 1, 2, 3 \); and \( \gamma_0 \) ranging from 0 to 50 dB in steps of 1 dB. The results for \( M = 0 \) and \( \gamma_0 = 0, 10, 20, 30, 40, 50 \) dB

\(^3\)The results of the numerical integration are exact in the sense that a specified number of significant figures can be guaranteed in the computation of \( P_f(M,N) \). See Appendix A.
were compared with those given in [21], and were found to agree to 4 significant figures in all but one case (in [21] they give $P_f = 5.814 \times 10^{-1}$ at $N = 100$ and $\gamma_0 = 10$ dB, whereas we calculated $P_f = 5.806 \times 10^{-1}$). Some of the numerical results are graphed in Figures 2.3 and 2.4. The regularity of the curves, particularly for high SNR's, suggests that the dependence of $P_f(M,N;r)$ on $M$, $N$, and $r$ can be characterized in a fairly simple manner. The details of the numerical integration are described in Appendix A.

2.1.5 Approximate methods for the computation of $P_f(0,N)$

Another approach to the calculation of $P_f(M,N)$ is to approximate the sum (2.1.18) in such a way that numerical problems are avoided. It appears that approximations have been made in the literature only for the case $M = 0$, i.e. for (2.1.17) instead of (2.1.18). A simple approximation to (2.1.17) is derived in [26]; this approximation is re-derived in [21] in almost precisely the same way. The basis for the approximation is to note that for $\gamma_0 >> 1$ we have $r << 1$, and so

$$P_f(0,N;r) = -r \sum_{k=0}^{N} \binom{N}{k} \frac{(-1/2)^k}{k + r}$$

We now use the identity

$$\sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k} = -\frac{1}{2} \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^{k+1}}{k}.$$
Figure 2.3: $P_f(0,N)$ for different block sizes.
Figure 2.4: $P_f(M,10)$ for $M = 0, 1, 2, 3$. 

Average signal-to-noise ratio $\gamma_0$ in dB.
where \( H_N \triangleq \sum_{k=1}^{N} k^{-1} \) (see exercise 13 of [13], Section 1.2.7).

Substitution of (2.1.20) into (2.1.19) with \( y = 1/2 \) gives

\[
P_f(0,N;r) \approx r \left[ H_N - \sum_{k=1}^{N} (1/k^2) \right] = \frac{2}{\gamma_0} \left[ H_N - \sum_{k=1}^{N} (1/k^2) \right].
\]

The quantity in brackets in (2.1.21) is well suited for numerical computation, even when \( N \) is large. However, for large \( N \) the amount of computation required to evaluate (2.1.21) is sufficiently large that a computer is still required. Thus it would be convenient if we could find some method to avoid the summations in (2.1.21); fortunately it is not hard to do so. From [13] we have the asymptotic expansion

\[
H_N \sim \log N + C + 1/2N - 1/12N^2 + 1/120N^4
\]

with an error less than \( 1/252N^6 \). The logarithm in (2.1.22) is the natural logarithm (logarithm to the base \( e \)). Since we shall have no need to use common logarithms, \( \log(\cdot) \) will always denote the natural logarithm. The letter \( C \) denotes Euler's constant, which is defined by

\[
C = \lim_{N \to \infty} \{ H_N - \log N \}
\]

and which has a numerical value \( C \approx 0.57721 \). The summation \( \sum_{k=1}^{N} (1/k^2) \) in (2.1.21) converges extremely rapidly as \( N \) increases, and for \( N \geq 10 \) is well approximated by its limit
\[ \sum_{k=1}^{\infty} \frac{1}{k2^k} = \log 2 \] (for \( N = 10 \) the relative error in this approximation is about 0.012\%, and for \( N = 15 \) the relative error is about 0.00026\%). We then have

\[ P_f(0, N; r) \approx r \left[ \log(N/2) + C + 1/2N - 1/12N^2 + 1/120N^4 \right]. \tag{2.1.24} \]

Approximations similar to (2.1.24) are given in [21] and [26]. This approximation is good for a fairly wide range of block sizes \( N \) and signal-to-noise ratios \( \gamma_0 \). For a fixed \( N \), the approximation grows arbitrarily good as \( \gamma_0 \to \infty \). However, the approximation is poor for small signal-to-noise ratios, as is shown by Figure 2.5, where the approximation (2.1.24) is graphed for \( N = 100 \), along with the results obtained by numerical integration. If we now hold \( \gamma_0 \) fixed and consider the behavior of (2.1.24) as \( N \to \infty \), we see that the approximation breaks down: the right-hand side of (2.1.24) goes to infinity, although \( P_f(0, N; r) \) is of course less than unity. A slightly different approximation that behaves better for large \( N \) is also derived in [21]. Although we do not go through the details here, the authors use what they call a step function approximation to write

\[ P_f(0, N; r) \approx 1 - \exp \left[ -r \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k} \right]. \tag{2.1.25} \]

Note that the argument of the exponential function in (2.1.25) is just the last expression in (2.1.19). In fact, if the argument of the exponential in (2.1.25) is small, taking the first-order term in the series expansion of the exponential yields (2.1.19). Using the asymptotic approximations just
Figure 2.5: $P_f(0,100)$ computed using (2.1.24) and numerical integration.
developed, we may write

\[ P_f(0,N;r) \approx 1 - \exp\{-r [\log(N/2) + C + 1/2N - 1/12N^2 + \ldots]\} \]  
(2.1.26)

As shown by numerical examples in [21], approximation (2.1.26) is better overall than (2.1.24), particularly when \( N \) is large and (2.1.24) fails.

Let us now consider the more difficult case \( M > 0 \), a case which apparently has not yet been tackled in the literature. From (2.1.18) we see that the problem can be broken up in such a way that we can employ the results already derived for \( P_f(0,N) \). Thus the problem for \( M > 0 \) reduces to approximating the summation in the second equation of (2.1.18). It is difficult, however, to work directly with the summation, and it turns out that results can more easily be derived using a much different method, which we consider in the next section. As a by-product of our investigation of the case \( M > 0 \) we shall find a better approximation for the case \( M = 0 \) as well. Furthermore, whereas there is no analytical estimate of the error in the above approximations, we shall be able to obtain explicit bounds on the error in the approximations derived in the next section.
2.2 A new approach to approximating $P_f(M,N;r)$

2.2.1 Introduction

We shall find it most convenient to work with $P(M,N;r) = 1 - P_f(M,N;r)$, the probability that there are $M$ or fewer errors in a block of $N$ bits. From equation (2.1.12) it follows that

$$P(M,N;r) = \gamma_0^{-1} \sum_{m=0}^{M} \left( \begin{array}{c} N \\ m \end{array} \right) \int_{0}^{\infty} p(\gamma)^m [1 - p(\gamma)]^{N-m} e^{-\gamma/\gamma_0} d\gamma.$$  \hspace{1cm} (2.2.1)

This equation holds for NCFSK, CFSK, and PSK modulations. For reasons stated in the previous section we consider only NCFSK, for which $p(\gamma) = (1/2) \exp(-\gamma/2)$. Furthermore, we consider a generalization of (2.2.1), in which the block length $N$ may be any real number $x \geq 1$:

$$P(M,x;r) = \gamma_0^{-1} \sum_{m=0}^{M} \left( \begin{array}{c} x \\ m \end{array} \right) \int_{0}^{\infty} p(\gamma)^m [1 - p(\gamma)]^{x-m} e^{-\gamma/\gamma_0} d\gamma.$$  \hspace{1cm} (2.2.2)

It may not be clear why we consider non-integer block sizes $x$, but in Chapter 3, where we divide the original block into sub-blocks, it will be a natural extension to think of the block size as an arbitrary real number. Also, it turns out that the analysis for the extended case is only slightly more involved than it would be if we restricted ourselves to integer block sizes $N$.

The binomial coefficient $\left( \begin{array}{c} x \\ m \end{array} \right)$ in equation (2.2.2) is defined for real $x$ by
\begin{equation}
\binom{x}{m} = \frac{x(x-1)(x-2) \cdots (x-m+1)}{m!} \tag{2.2.3}
\end{equation}

\begin{equation}
\frac{\Gamma(1+x)}{\Gamma(1+x-m) \Gamma(1+m)},
\end{equation}

where \( \Gamma(\cdot) \) is the well-known Gamma function. The Gamma function occurs so frequently in what follows that it is worthwhile summarizing some of its properties here [28]. The Gamma function is often defined for complex \( z \) in the right half plane by the integral

\begin{equation}
\Gamma(z) = \int_{0}^{\infty} t^{z-1} \exp(-t) \, dt, \quad \text{Re}(z) > 0 \tag{2.2.4}
\end{equation}

where \( t^{z-1} = \exp((z-1)\log(t)) \) and \( \log(t) \) is real. It can be shown that \( \Gamma(z) \) as defined by the integral (2.2.4) is analytic for \( \text{Re}(z) > 0 \), and that it can be analytically continued to the whole complex plane \( \mathbb{C} \), except for the non-positive integers \( 0, -1, -2, \ldots \), where it has simple poles. Another representation for the Gamma function is

\begin{equation}
1/\Gamma(z) = z \exp(Cz) \prod_{n=1}^{\infty} (1 + z/n) \exp(-z/n), \tag{2.2.5}
\end{equation}

where \( C \) is Euler's constant. The infinite product above converges for all complex \( z \) and defines an entire function (whence it follows that \( \Gamma(z) \) has no zeros). The recurrence relation

\begin{equation}
z \cdot \Gamma(z) = \Gamma(1+z) \tag{2.2.6}
\end{equation}

holds for \( z \neq 0, -1, -2, \ldots \), and using the fact that \( \Gamma(1) = 1 \), we find that \( \Gamma(1+n) = n! \) for non-negative integers \( n \).
Making the substitution \( t = p(\gamma) = (1/2) \exp(-\gamma/2) \) in the integral (2.2.2), we get

\[
P(M, x; r) = r \sum_{m=0}^{M} \binom{x}{m} \int_{0}^{1/2} t^{m+r-1} (1 - t)^{x-m} \, dt, \tag{2.2.7}
\]

where \( r = 2/\gamma_0 \) as before. Equation (2.2.7) can be written in the form

\[
P(M, x; r) = r \sum_{m=0}^{M} \binom{x}{m} B_{1/2}(m+r, 1+x-m), \tag{2.2.8}
\]

where \( B_y(a, b) = \int_{0}^{y} t^{a-1} (1 - t)^{b-1} \, dt \) is the incomplete Beta function. If we take \( y = 1 \) in the definition of the incomplete Beta function, the result is the (complete) Beta function \( B(a, b) = B_1(a, b) \). From equation (2.2.8) we see that the computation of \( P(M, x; r) \) can be boiled down to the evaluation of the incomplete Beta function. This function has been tabulated in [19], but the parameter ranges presented in these tables are insufficient for our purposes. For example, consider

\[
P(0, x; r) = r \, 2^r \, B_{1/2}(r, 1+x). \tag{2.2.9}
\]

In the tables [19], the smallest value listed for the arguments \( a, b \) of \( B_y(a, b) \) is 0.5. But for \( \gamma_0 > 4 \approx 6 \text{ dB} \) we have \( r < 0.5 \), and so these tables cannot be used in the range of most interest. We might try substituting the binomial expansion of the factor \((1 - t)^{x-m}\) into (2.2.7), but this just leads us back to (2.1.18), and it was the difficulties in numerically evaluating (2.1.18) that led us to seek an alternative expression in the first place.

There are some other problems with the expressions above.
Since $P(0,x;r) \to 1$ as $x_0 \to \infty$ [this is intuitively obvious from the physical meaning of $P(0,x;r)$], it follows from (2.2.9) that $r 2^r B_{1/2}(r,1+x) \to 1$ as $r \to 0^+$. This means that $B_{1/2}(r,1+x) \to \infty$ as $r \to 0^+$, and so there could be numerical problems for large signal-to-noise ratios. Consider also the block-error probability

$$P_f(0,x;r) = 1 - r 2^r B_{1/2}(r,1+x). \quad (2.2.10)$$

When $x_0 >> 1$ and $r 2^r B_{1/2}(r,1+x)$ is very close to unity, the right-hand side of (2.2.10) is a subtraction of two nearly equal quantities. The result is a loss of significant figures.

From the foregoing discussion it may appear that (2.1.18) is a better starting point for approximating $P_f(M,x;r)$ than the alternative expression (2.2.8). For example, an equation similar to (2.2.10) is given in [21] (the authors consider only the case $M = 0$), but then (2.1.17) is used as the starting point for deriving an approximation similar to (2.1.24). But we shall see that (2.2.8) can serve as the foundation for excellent approximations to $P_f(M,x;r)$. The most important observation to make, and, in fact, the observation that forms the basis for the approximations to follow, is that the incomplete Beta function in (2.2.8) can be replaced by the Beta function with negligible error over a wide range of parameter values. We first introduce some new notation, letting $Q(m,x;r)$ denote the probability of exactly $m$ bit errors in a block of $x$ bits; it follows that

$$Q(m,x;r) = \binom{x}{m} r 2^r B_{1/2}(m+r,1+x-m). \quad (2.2.11)$$
We also introduce the quantity

\[ \hat{Q}(m,x;r) = \binom{x}{m} r^{2^r} B(m+r,1+x-m). \quad (2.2.12) \]

The definition in (2.2.12) introduces a notational convention to which we will adhere for the rest of this thesis: a quantity with a caret \(^\wedge\) indicates an approximation to the original quantity that has been obtained by replacing the incomplete Beta function by the Beta function. Then (2.2.8) becomes

\[ P(M,x;r) = \sum_{m=0}^{M} Q(m,x;r), \quad (2.2.13) \]

and, using the notational convention described above,

\[ \hat{P}(M,x;r) = \sum_{m=0}^{M} \hat{Q}(m,x;r). \quad (2.2.14) \]

We shall use \( \hat{P} \) to approximate \( P \). Before we consider the advantages (and disadvantages) of doing so, we first examine the error incurred in making this approximation.

### 2.2.2 Error analysis

If we write

\[ Q(m,x;r) = \hat{Q}(m,x;r) - \epsilon(m,x;r) \quad (2.2.15) \]

then it follows from equation (B.9) of Appendix B that \( \epsilon(m,x;r) \) satisfies

\[ 0 < \epsilon(m,x;r) \leq \binom{x}{m} \frac{r}{2^{x-m}(1+x-m)}. \quad (2.2.16) \]

The upper bound in (2.2.16) holds only when \( 0 < r < 1 \) (see Appendix B for all the restrictions under which the above
result holds). We think of $\epsilon(m,x;r)$ as the error incurred by approximating $Q$ with $\hat{Q}$. Substituting (2.2.15) into (2.2.13), we find

$$P(M,x;r) = \sum_{m=0}^{M} \hat{Q}(m,x;r) - \sum_{m=0}^{M} \epsilon(m,x;r)$$

(2.2.17)

$$= \hat{P}(M,x;r) - E(M,x;r),$$

where from (2.2.16) it follows that

$$0 < E(M,x;r) \leq \sum_{m=0}^{M} \epsilon(m,x;r)$$

(2.2.18)

$$\leq \sum_{m=0}^{M} \binom{x}{m} \frac{r}{2^{x-m} (1+x-m)}.$$ 

$E(M,x;r)$ represents the error that results when we replace the incomplete Beta function in (2.2.8) by the Beta function, i.e. when we approximate $P$ by $\hat{P}$.

We now examine how the upper bound in (2.2.18) depends upon its parameters, and then look at several numerical examples in order to get an idea of the magnitude of the numbers involved. As usual, we first consider the simplest case, $M = 0$. Then

$$P(0,x;r) = \hat{P}(0,x;r) - E(0,x;r),$$

where $0 < E(0,x;r) \leq r/2^x(1+x)$. The error bound

$$\frac{r}{2^x(1+x)} = \frac{1}{2^{x-1}(1+x)\gamma_0}$$

(2.2.19)

goest to zero exponentially with increasing block size $x$, and so the error in the approximation rapidly becomes negligible as $x$
increases. Also, the error bound varies inversely with the average signal-to-noise ratio \( \gamma_0 \), and so the error in approximation decreases as \( \gamma_0 \) increases. Note, however, that we refer here to the absolute error, whereas a more meaningful indicator of the error is the relative error — in this case, the error relative to \( P_f \), the quantity we wish to compute. From equation (2.1.23) we see that to a first approximation \( P_f \) also varies inversely with \( \gamma_0 \). Since both \( P_f \) and the error bound depend inversely on \( \gamma_0 \), changing \( \gamma_0 \) while leaving \( x \) fixed has little effect on the relative error, even though \( P_f \) and the error bound may change by several orders of magnitude. The reader may question the validity of this conclusion, because the analysis is based not on the actual error \( E(0,x;r) \), but on the upper bound given in (2.2.19), and a bounding technique different from that of Appendix B might yield a tighter bound on \( E(0,x;r) \) with a different functional dependence on \( \gamma_0 \). However, tests have shown that the error bound in (2.2.19) is quite tight, and the conclusion drawn above does hold. An example is given in Figure 2.6, in which both the relative error \( E(0,x;r)/P_f(0,x;r) \) and the error bound (2.2.19), also divided by \( P_f(0,x;r) \), have been graphed for \( x = 10 \). There are several points to be made.

(i) The upper bound on the error is quite close to the actual error, although it does not hold for \( \gamma_0 \) less than about 3 dB [see the remark following (2.2.16)].

(ii) As \( \gamma_0 \) increases the relative error rapidly approaches a
Figure 2.6: The relative error in the approximation.
constant, and for \( \gamma_0 > 10 \) dB may be considered almost constant. Reference to Figure 2.3 shows that in contrast \( P_f(0,10;r) \) changes by several orders of magnitude over the same range of SNR values.

(iii) The relative error is greatest for small values of \( \gamma_0 \), say \( \gamma_0 < 10 \) dB. From the example in Figure 2.5 we see that (2.1.21) holds only for the straight-line portions of the graphs in Figure 2.3, i.e. for \( \gamma_0 \) greater than about 10 or 15 dB (depending on the block size). Thus the inverse dependence of \( P_f \) with \( \gamma_0 \), a dependence that was used to predict a relative error that is almost constant with respect to the SNR \( \gamma_0 \), does not hold for small values of the SNR.

The reader should bear in mind that although the error, viewed as a function of \( \gamma_0 \), will have the general shape of Figure 2.6 for all block sizes, the exponential dependence of (2.2.19) on \( x \) so totally dominates the behavior of the error term that for block sizes greater than 10 or so the error is negligible regardless of the value of \( \gamma_0 \), i.e. for 0 dB and above. It is only for small block sizes (\( x \) less than about 10) that the increased error for \( \gamma_0 \) less than 10 or 15 dB becomes important, and if we restrict the average SNR to be greater than about 10 dB, the error in the approximation is still quite small (less than 1\% relative error) for \( x \) as small as 4 or 5. The only parameter range for which we expect the error to be appreciable is when both the block size and SNR are small.
From (2.2.18) we see that when \( M > 0 \) the bound on \( E(M,x;r) \) is a bit more tedious to compute than it is for \( M = 0 \). However, the error term \( E \) is usually so small that we may replace the upper bound in (2.2.18) with a much simpler bound that, although weaker than (2.2.18), is more than sufficient for our purposes. In addition to the restrictions on \( r, x, \) and \( m \) given in Appendix B, we assume that \( M \) is an integer satisfying \( 0 \leq m \leq M < x \). It can then be shown using definition (2.2.3) that

\[
(x \choose m) \leq \left( \frac{x}{M} \right)^{M-m}.
\]

We also have that \( \frac{1}{1+x-m} \) < \( \frac{1}{1+x-M} \), and so

\[
\sum_{m=0}^{M} \frac{x}{M} \frac{1}{2^{x-m} (1+x-m)} \leq \left( \frac{x}{M} \right) \frac{1}{2^{x-M} (1+x-M)} \sum_{m=0}^{M} \xi^{M-m},
\]

where we have put \( \xi = M/2(x-M) \). Using the well-known result for the sum of a geometric series, we conclude, on the basis of (2.2.18) and the above inequality, that

\[
0 < E(M,x;r) \leq \left( \frac{x}{M} \right) \frac{r}{2^{x-M} (1+x-M)} \left[ \frac{1-\xi^{M+1}}{1-\xi} \right]. \tag{2.2.20}
\]

If \( \xi = 1 \) the quantity in brackets should be replaced by \( M+1 \). Note that in the important case \( M = 0 \) the upper bounds in (2.2.18) and (2.2.20) are identical.

We now look at some numerical examples. In the tables below, the values of the upper bounds in (2.2.18) and (2.2.20) are given for two different block sizes and average signal-to-
noise ratios, and for $M = 0, 1, 2, 3$. The parameters chosen for Table I, $x = 100$ and $\gamma_0 = 20$ dB, represent typical values of the block size and average SNR. The parameters chosen for Table II, $x = 10$ and $\gamma_0 = 5$ dB, fall into the region where we expect the error in the approximation to start to become large. Also provided are the values of $P_f(M, x; r)$, which were computed using numerical integration as described in Appendix A. The numbers in the tables are rounded off to the number of decimal places presented.

<table>
<thead>
<tr>
<th>$M$</th>
<th>Bound in (2.2.18)</th>
<th>Bound in (2.2.20)</th>
<th>$P_f(M, x; r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1.562 \times 10^{-34}$</td>
<td>$1.562 \times 10^{-34}$</td>
<td>8.567 \times 10^{-2}</td>
</tr>
<tr>
<td>1</td>
<td>$3.171 \times 10^{-32}$</td>
<td>$3.171 \times 10^{-32}$</td>
<td>6.738 \times 10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>$3.187 \times 10^{-30}$</td>
<td>$3.188 \times 10^{-30}$</td>
<td>5.805 \times 10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>$2.114 \times 10^{-28}$</td>
<td>$2.115 \times 10^{-28}$</td>
<td>5.177 \times 10^{-2}</td>
</tr>
</tbody>
</table>

Table I: Upper bounds for parameter values $x = 100, \gamma_0 = 20$ dB ($r = 0.02$).

<table>
<thead>
<tr>
<th>$M$</th>
<th>Bound in (2.2.18)</th>
<th>Bound in (2.2.20)</th>
<th>$P_f(M, x; r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$5.615 \times 10^{-5}$</td>
<td>$5.615 \times 10^{-5}$</td>
<td>6.915 \times 10^{-1}</td>
</tr>
<tr>
<td>1</td>
<td>$1.291 \times 10^{-3}$</td>
<td>$1.304 \times 10^{-3}$</td>
<td>4.969 \times 10^{-1}</td>
</tr>
<tr>
<td>2</td>
<td>$1.364 \times 10^{-2}$</td>
<td>$1.409 \times 10^{-2}$</td>
<td>3.412 \times 10^{-1}</td>
</tr>
<tr>
<td>3</td>
<td>$8.776 \times 10^{-2}$</td>
<td>$9.413 \times 10^{-2}$</td>
<td>2.139 \times 10^{-1}</td>
</tr>
</tbody>
</table>

Table II: Upper bounds for parameter values $x = 10, \gamma_0 = 5$ dB ($r = 0.6325$).

Comparing the values of the two upper bounds given in the preceding tables, we see that the upper bound in (2.2.20) is very tight. It is obvious, though, that for the choice of
parameters in Table I above the bound does not have to be very tight, since the error incurred by replacing the incomplete Beta function in (2.2.8) by the Beta function is smaller than $P_f(M,100;0.02)$ by at least a factor of $10^{25}$. The small magnitude of the error term demonstrates the effect of the exponential dependence on block size. It is clear that the error term will be negligible for larger block sizes, since the effect of increasing $x$ will be to decrease the error estimate while increasing the error probability $P_f(M,x;r)$. The parameters chosen for Table II are in that region where we expect the worst results: both the block size and the average signal-to-noise ratio are small. From the table we see that even in this case the error is small relative to $P_f(M,10;r)$, particularly for small $M$. Using the bound (2.2.18), we can conclude that the relative error in the approximation is less than 0.0082% for $M = 0$, and less than 4.0% for $M = 2$. However, if $x$ is very small we are able to compute $P_f(M,N;r)$ directly from (2.1.17) (assuming an integer block size, $x = N$). Of course, one usually desires to find a single formula that will work over the entire parameter range in which one is interested: it is always a bother to have to consider separate cases. It is evident from the examples provided above that the approximation (2.2.17) is quite robust, in the sense that it works well over a wide range of parameter values. So far we have only shown that we can approximate $P$ by $\hat{P}$ with negligible error over the parameter range of interest — we have yet to demonstrate that $\hat{P}$ is easier to compute than $P$. 
2.2.3 The approximation to $P_r(0,x;r)$

The advantage of working with $\hat{P}$ instead of $P$ is that there is an equation relating the Beta function to the Gamma function [28]:

$$B(a,b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} . \quad \text{Re}(a) > 0 \quad \text{Re}(b) > 0$$ (2.2.21)

Putting this result into (2.2.12) we find

$$\hat{Q}(m,x;r) = \binom{x}{m} r^{2r} \frac{\Gamma(1+x-m) \Gamma(m+r)}{\Gamma(1+x+r)} ,$$ (2.2.22)

and, substituting (2.2.22) into (2.2.14), we obtain

$$\hat{P}(M,x;r) = r 2^r \sum_{m=0}^{M} \binom{x}{m} \frac{\Gamma(1+x-m) \Gamma(m+r)}{\Gamma(1+x+r)} .$$ (2.2.23)

We begin our analysis with the special case $M = 0$, which we consider in detail because, as will be shown later, the general case $M > 0$ can be reduced to this special case. With $M = 0$, equation (2.2.23) becomes

$$\hat{P}(0,x;r) = r 2^r \frac{\Gamma(1+x) \Gamma(r)}{\Gamma(1+x+r)}$$ (2.2.24)

with an error in approximation less than $r/2x(1+x)$. Using the recurrence relation (2.2.6) we can write

$$\hat{P}(0,x;r) = 2^r \frac{\Gamma(1+x) \Gamma(1+r)}{\Gamma(1+x+r)} .$$ (2.2.25)

Moving from (2.2.24) to (2.2.25) is numerically beneficial, because a singularity at $r = 0$ is removed. This singularity of (2.2.24) at $r = 0$ occurs because $\Gamma(r)$ has a pole there (and at $r = -1, -2, -3, \ldots$), and hence $\Gamma(r)$ goes to infinity as $r \to 0^+$ (i.e. as $\gamma_0 \to \infty$). However, the poles of the Gamma
function are simple, and so \( r \cdot \Gamma(r) \) has a removable singularity at \( r = 0 \); in fact
\[
\lim_{r \to 0^+} r \cdot \Gamma(r) = \lim_{r \to 0^+} \Gamma(1+r) = \Gamma(1) = 1.
\]
(In other words, the residue of the \( \Gamma \) function at zero is 1.)

Of course, we should have anticipated such behavior when \( r \) is near zero: while examining the behavior of equation (2.2.9) it was noted that when \( r \to 0^+ \) the incomplete Beta function \( B_{1/2}(r,1+x) \) goes to infinity in such a way that the product \( r \cdot B_{1/2}(r,1+x) \) approaches unity. Since the bound \( r/2^x(1+x) \) on the difference between (2.2.9) and (2.2.24) goes to zero as \( r \to 0^+ \) for a fixed \( x \), the behavior of the two expressions must be the same near \( r = 0 \).

Let us now consider some of the details of numerically evaluating (2.2.25). The most obvious feature of (2.2.25) is that the Gamma function must be evaluated. This makes (2.2.25) inconvenient for use on a hand calculator, but is no real drawback if a computer is available — the Gamma function is a standard function of analysis and so there is usually a packaged program available at any well-equipped computer installation (at UBC for example). However, even if a Gamma function program is available we cannot just blindly evaluate (2.2.25) as it stands, because the terms \( \Gamma(1+x) \) and \( \Gamma(1+x+r) \) may become very large. For example, \( \Gamma(101) = 100! \approx 9.33 \times 10^{157} \) and \( \Gamma(1001) = 1000! \approx 4.02 \times 10^{2567} \), and both these numbers exceed the largest number that is easily representable on most computers. Thus to avoid overflow it is best to work with the logarithm of the Gamma function, and to write
\[ \frac{\Gamma(1+x)}{\Gamma(1+x+r)} = \exp[\log \Gamma(1+x) - \log \Gamma(1+x+r)]. \quad (2.2.26) \]

Fortunately \( \log \Gamma(\cdot) \) has been analyzed almost as extensively as the Gamma function itself (see, for example, [28]), and there are programs available for evaluating this function. There are several reasons why double-precision arithmetic should be used in the evaluation of (2.2.25). First, the subtraction in (2.2.26) will result in a loss of significant figures, particularly for large block sizes. Also, our final goal is the approximation of \( P_f(0,x;r) \) by

\[ \hat{P}_f(0,x;r) = 1 - \hat{P}(0,x;r), \quad (2.2.27) \]

and the subtraction in this equation will also result in a loss of significant figures.

If a pre-written program for \( \Gamma(\cdot) \) or \( \log \Gamma(\cdot) \) is not available, there are several programs listed in the Collected Algorithms from the Association for Computing Machinery that may be used to evaluate these functions. These programs are based on either Chebyshev approximations or asymptotic expansions, and are surprisingly simple and short (20 lines or so). Thus, even if a packaged program is not available, the evaluation of \( \Gamma(\cdot) \) or \( \log \Gamma(\cdot) \) poses no real problem, and can easily be done on a programmable calculator, or can even be done manually on a hand calculator in a matter of minutes.

Equation (2.2.25) was programmed for the Amdahl 470 using DLGAMA, which is a UBC library routine for evaluating \( \log \Gamma(\cdot) \) to double-precision accuracy. Values of \( \hat{P}_f \) were computed for
$N = 10, 10^2, 10^3, 10^4$ and for $\gamma_0$ from 0 to 50 dB using equation (2.2.27), and were then compared with the corresponding values of $P_f$ obtained by numerical integration. There are at least three sources of error that could cause discrepancies between the two sets of results:

(i) the error that results from approximating $P_f$ by $\hat{P}_f$. This source of error is only important for $N = 10$, since for $N = 10^2, 10^3, 10^4$ the error in approximation is so small that, barring all other sources of error, $\hat{P}_f$ would approximate $P_f$ to at least double-precision accuracy (16 significant figures on the Amdahl). For $N = 10$ we see from Figure 2.6 that the relative error in approximation approaches a constant value of about $4 \times 10^{-5}$, and so, ignoring other sources of error, we expect about four significant figures from (2.2.27) when $N = 10$.

(ii) numerical error in the evaluation of (2.2.27), caused by the effects of performing arithmetic using a finite word-length. An example of this type of error is the loss of significant figures discussed at the top of the previous page.

(iii) inaccuracy in the numerical integration. As described in Appendix A, the values of $P_f$ computed via numerical integration are accurate to at least five significant figures.

According to (i) and (iii), for $N = 10$ we can expect the
results to agree to about four significant figures if the error described in (ii) can be ignored. Comparison showed this to be the case. For \( N = 10^2, 10^3, 10^4 \), the only relevant sources of error are those in (ii) and (iii). Comparison showed that the values of \( P_f \) computed using (2.2.27) agreed to five significant figures for almost all values of \( \gamma_0 \) with those calculated using numerical integration. Thus double-precision arithmetic was entirely adequate, and over the parameter range examined the source of error described in (ii) above was never very severe. In fact, in most cases the values of \( P_f \) computed using (2.2.27) were probably much more accurate than those obtained by numerical integration (and thus, in the end, were a better check on the numerical integration than vice versa). Generally, when the block size is greater than about 10 bits the error in the approximation is so small that if we graph both the results obtained by numerical integration and the approximate results on the same graph, the scale of which is as in Figure 2.3, the lines lie on top of one another for the entire range 0 to 50 dB.

2.2.4 The approximation to \( P_f(M,x;r) \)

We start by approximating \( Q(m,x;r) \), the probability of \( m \) bit errors in a block of \( x \) bits, for \( m \geq 0 \). If we express the binomial coefficient in equation (2.2.22) in terms of Gamma functions according to equation (2.2.3), we get
\[ \hat{Q}(m,x;r) = r 2^r \frac{\Gamma(1+x) \Gamma(m+r)}{\Gamma(1+x+r) \Gamma(1+m)}. \quad (2.2.28) \]

For \( m = 1, 2, \ldots \) we can use (2.2.6) to write

\[ \Gamma(m+r) = r(r+1) \cdots (r+m-1) \Gamma(r). \quad (2.2.29) \]

Using Pochhammer's symbol, which is defined by

\[
\begin{align*}
(r)_0 &= 1 \\
(r)_m &= r(r+1) \cdots (r+m-1) \quad \text{for } m = 1, 2, \ldots
\end{align*}
\quad (2.2.30)
\]

(see entry 6.1.22 of [1]), we can extend (2.2.29) to the case \( m = 0 \) and write \( \Gamma(m+r) = (r)_m \Gamma(r) \) for \( m = 0, 1, 2, \ldots \) Then (2.2.28) becomes

\[ \hat{Q}(m,x;r) = r 2^r \frac{\Gamma(1+x) \Gamma(r)}{\Gamma(1+x+r)} \frac{(r)_m}{m!} \quad (2.2.31) \]

\[ = 2^r \frac{\Gamma(1+x) \Gamma(1+r)}{\Gamma(1+x+r)} \frac{(r)_m}{m!}. \]

Note that the expression in front of the term \( (r)_m/m! \) is just the approximation to \( P(0,x;r) \) given in equation (2.2.25). Thus

\[ \hat{Q}(m,x;r) = \hat{P}(0,x;r) \frac{(r)_m}{m!}, \quad (2.2.32) \]

and hence

\[ Q(m,x;r) \approx \hat{P}(0,x;r) \frac{(r)_m}{m!}, \quad (2.2.33) \]

where a bound on the error in the approximation is given in (2.2.16). If \( r \ll 1 \) (i.e. if the average SNR \( \gamma_0 \) is large), it can be shown from (2.2.30) that \( (r)_m/m! \approx r/m \) for \( m \geq 1 \). Then if \( r \ll 1 \) and \( m \geq 1 \) we have that
\[ Q(m, x; r) \approx \hat{P}(0, x; r) \frac{r}{m}. \]  \hspace{1cm} (2.2.34)

Equation (2.2.34) shows that when the SNR is large, the probability that there are \( m \) bit errors in a block varies approximately inversely with \( m \).

It is now easy to write down an approximation for \( P(M, x; r) \) when \( M > 0 \): putting (2.2.32) into (2.2.14) yields

\[ \hat{P}(M, x; r) = \hat{P}(0, x; r) \sum_{m=0}^{M} \frac{Z(m)}{m!}. \]  \hspace{1cm} (2.2.35)

From (2.2.17) we may write

\[ P(M, x; r) = \hat{P}(0, x; r) \sum_{m=0}^{M} \frac{Z(m)}{m!} + E(M, x; r), \]  \hspace{1cm} (2.2.36)

where \( E(M, x; r) \) satisfies (2.2.20). From (2.2.17) we also have that \( \hat{P}(0, x; r) = P(0, x; r) + E(0, x; r) \), and substitution of this equation into (2.2.36) yields

\[ P(M, x; r) = P(0, x; r) \sum_{m=0}^{M} \frac{Z(m)}{m!} + \]

\[ + E(0, x; r) \sum_{m=0}^{M} \frac{Z(m)}{m!} - E(M, x; r). \]  \hspace{1cm} (2.2.37)

Let us get a bound on the absolute value of the term

\[ E(0, x; r) \sum_{m=0}^{M} \frac{Z(m)}{m!} - E(M, x; r). \]  \hspace{1cm} (2.2.38)

We use the following fact: if \( a \) and \( b \) are positive real numbers, then \( |a - b| \leq \max\{a, b\} \). Since both terms in (2.2.38) are positive,
\[ \left| E(0, x; r) \sum_{m=0}^{M} \frac{(r)_m}{m!} - E(M, x; r) \right| \leq \]
\[ \leq \max_{m=0}^{M} \left\{ E(0, x; r) \sum_{m=0}^{M} \frac{(r)_m}{m!}, E(M, x; r) \right\} \]

It is easy to see that for \(0 < r < 1\) the inequality \(0 < \frac{(r)_m}{m!} \leq 1\) holds, and hence, using the bound in (2.2.20), we have

\[ \max_{m=0}^{M} \left\{ E(0, x; r) \sum_{m=0}^{M} \frac{(r)_m}{m!}, E(M, x; r) \right\} \]
\[ \leq \max \left\{ (M+1)E(0, x; r), E(M, x; r) \right\} \]
\[ \leq \max \left\{ \frac{(M+1)r}{2^x(1+x)}, \frac{x}{2^x(M)} \frac{r}{1-x-M} \left[ \frac{1-\xi^{M+1}}{1-\xi} \right] \right\} \]
\[ = \frac{x}{2^x(M)} \frac{r}{1-x-M} \left[ \frac{1-\xi^{M+1}}{1-\xi} \right], \]

where \(\xi = M/2(x-M)\). The last step follows upon noting that \(2^M \geq M+1\) for \(M \geq 0\), that \(1/(1+x-M) \geq 1/(1+x)\), and that both \(\binom{x}{M}\) and the quantity in square brackets are greater than unity. Summarizing, we have

\[ P(M, x; r) \approx P(0, x; r) \sum_{m=0}^{M} \frac{(r)_m}{m!} \]  \hspace{1cm} (2.2.39)

with an error in approximation that is less than the upper bound in (2.2.20) when \(0 < r < 1\) and \(0 \leq M < x\). Thus from our previous investigation we know that the error in approximation in (2.2.39) is negligible for almost all cases of interest.
Equation (2.2.39) is one of the major results of this thesis, because it demonstrates that the computation of $P(M,x;r)$ can be reduced (approximately, at least) to the computation of $P(0,x;r)$: given the value of $P(0,x;r)$, computed by any method at all, we need only compute the "correction factor" $\sum_{m=0}^{M} \frac{(r)_m}{m!}$ in order to obtain $P(M,x;r)$. The factor $\sum_{m=0}^{M} \frac{(r)_m}{m!}$ poses no numerical problems, and for small $M$ can easily be handled on a pocket calculator. Equation (2.2.39) is interesting from a theoretical point of view; in practice, having no method to compute $P(0,x;r)$ exactly, we use (2.2.35). This latter equation was programmed for the Amdahl 470, and the results were compared with the results obtained by numerical integration. The agreement between the results was very good, as our previous error analysis would lead us to expect.
2.3 Infinite series expansions

2.3.1 Introduction and notation

In this section we derive infinite series expansions for $P_f$, again considering only non-coherent FSK. With the approximations of Section 2.2 in mind, one might reasonably wonder about the need to develop infinite series expansions. These expansions are useful in that they show the connection between the approximations developed in the literature (i.e. the approximations discussed in Section 2.1) and those developed in the previous section. In particular, we shall see how approximations (2.1.24) and (2.1.26) follow from the formulas of Section 2.2, and how these approximations can be made more accurate with little added computational effort. Thus the formulas we derive in this section provide a simple computational alternative to the formulas of Section 2.2, although we shall no longer have rigorous bounds on the error in the approximation. We shall also see how to modify approximations (2.1.24) and (2.1.26) for the case $M > 0$. Finally, the formulas we derive here will be used in our discussion of diversity (Section 2.4). Since the mathematical details are somewhat tedious, many of them have been placed in Appendix C: our primary objective is to give those results that the reader may find useful or interesting.

Note that we may think of equation (2.1.21) as a first-order expansion of $P_f$ in terms of the quantity $r = 2/\gamma_0$. To show in what sense this is true, and how we can derive the higher-order terms of the expansion, we first go back to
equation (2.1.17):

$$P_f(0,N;r) = -r \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k + r}.$$  \hfill (2.3.1)

For $|r| < k$, we may write

$$\frac{1}{k + r} = \frac{1}{k(1 + r/k)} = k^{-1} \sum_{j=0}^{\infty} (-r/k)^j.$$

Since the smallest value assumed by the index of summation $k$ in equation (2.3.1) is $k = 1$, the following equations are valid for $|r| < 1$:

$$P_f(0,N;r) = -r \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k} \sum_{j=0}^{\infty} (-r/k)^j \hfill (2.3.2)$$

$$= \sum_{j=0}^{\infty} \left[ (-1)^{j+1} \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k^{j+1}} \right] r^{j+1}$$

$$= \sum_{i=1}^{\infty} \left[ (-1)^{i} \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k^{i}} \right] r^{i}. $$

The interchange in the order of summations in (2.3.2) is clearly valid because one of the sums is finite. Comparison of (2.1.19) and (2.3.2) shows that making the approximation of (2.1.19) is equivalent to taking the first term of the series (2.3.2). Although the first coefficient

$$- \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k} \hfill (2.3.3)$$

is not easy to compute directly for large $N$, we saw in Section 2.1 that the use of identity (2.1.20) puts it into a form that is easily approximated or computed exactly if desired. We
should distinguish here between two different approximations: one approximation consists of ignoring the higher-order terms of the infinite series (2.3.2) [this is the approximation made in (2.1.19)], whereas the other approximation is the one made to the terms themselves [this is the approximation made in going from (2.1.21) to (2.1.24)]. The error in approximating the first term (2.1.21) by (2.1.24) is so small for $N$ greater than about 10 bits that almost all the error may be attributed to the omission of the higher-order terms. If we now desire a better approximation to $P_f$, then the obvious way to proceed is to compute the higher-order terms in (2.3.2). However, the higher-order coefficients

$$(-1)^i \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k^i} \quad (i \geq 2) \quad (2.3.4)$$

are just as difficult to evaluate directly as (2.3.3). One might hope for a general identity analogous to (2.1.20) which will transform the coefficients (2.3.4) into expressions that are more easily computed or approximated. Unfortunately, there appears to be no such identity, and each coefficient must be dealt with in an ad hoc manner. It is possible to get somewhere by attacking the $i = 2$ term directly, but the computations get more and more difficult as $i$ increases.

Using the formulas of Section 2.2 we can largely sidestep the difficulties mentioned above. The key here is that the coefficients in the expansion of $\hat{P}$ are much easier to work with than those in the expansion of $P$. Let us begin our
analysis with some new notation: define \( q_i(m,x) \) and \( \hat{q}_i(m,x) \) by

\[
Q(m,x;r) = \left(\begin{array}{c} x \\ m \end{array}\right) r 2^r B(m+r,1+x-m) = \sum_{i=0}^{\infty} q_i(m,x) r^i
\]

(2.3.5)

\[
\hat{Q}(m,x;r) = \left(\begin{array}{c} x \\ m \end{array}\right) r 2^r B(m+r,1+x-m) = \sum_{i=0}^{\infty} \hat{q}_i(m,x) r^i.
\]

The infinite series in (2.3.5) converge for at least \(|r| < 1\). It is not hard to show that convergence actually holds for \(|r| < m\) when \( m \geq 1 \), but the situation is more difficult for \( m = 0 \) (this is one of the few instances where the case \( m = 0 \) is more difficult than \( m > 0 \)), and the sense in which (2.3.5) holds for \( m = 0 \) is described in Appendix C. We also define

\[
p_i(M,x) = \sum_{m=0}^{M} q_i(m,x),
\]

(2.3.6)

and similarly define \( \hat{p}_i(M,x) \). Then

\[
P(M,x;r) = \sum_{i=0}^{\infty} p_i(M,x) r^i
\]

and

\[
\hat{P}(M,x;r) = \sum_{i=0}^{\infty} \hat{p}_i(M,x) r^i
\]

for \(|r| < 1\). It is not hard to derive exact formulas for the coefficients \( q_i \) and \( p_i \) when \( x = N = \) an integer — for example, using (2.3.2) and the relation \( P = 1 - P_f \), we have in the special case \( M = 0 \) that \( q_0(0,N) = 1 \) and

\[
q_i(0,N) = (-1)^{i+1} \sum_{k=1}^{N} \binom{N}{k} \frac{(-1/2)^k}{k^i}. \quad (i \geq 1)
\]

(2.3.8)

We have already seen that these coefficients are hard to work with, and we expect the situation to be even worse when \( M > 0 \). But we know that \( \hat{P} \) is an excellent approximation to \( P \), and
so it is natural to see what we can learn from the series expansion of \( \hat{P} \). Thus we focus on the coefficients \( \hat{q}_i \) and \( \hat{p}_i \).

Before we proceed, we recall some previous notation, and introduce some more; much of the notation used here is the same as in [13]. In Section 2.1 we defined the quantity \( H_n \) for \( n = 1, 2, 3, \ldots \) by

\[
H_n = \sum_{k=1}^{n} k^{-1}
\]

(the \( H \) stands for "harmonic"). We extend this definition by writing

\[
H_n^{(i)} = \sum_{k=1}^{n} k^{-i}
\]

for \( i = 1, 2, 3, \ldots \). Thus \( H_n = H_n^{(1)} \). We shall also use the convention that \( H_0^{(i)} = 0 \); this convention will allow us to write down several of the expressions that follow without the need to consider special cases. It is an elementary result of analysis that the harmonic series diverges, i.e. \( H_n \to \infty \) as \( n \to \infty \). However, for \( i \geq 2 \) the limit of \( H_n^{(i)} \) exists as \( n \to \infty \), and we put

\[
\zeta(i) = \lim_{n \to \infty} H_n^{(i)} = \sum_{k=1}^{\infty} k^{-i}
\]

for \( i \geq 2 \). The function \( \zeta(i) \) is called the Riemann Zeta function [28].

We also introduce the Bernoulli numbers \( B_n \), which are defined by the infinite series

\[
\frac{z}{\exp(z) - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} , \quad |z| < 2\pi
\]
(The Bernoulli numbers are sometimes defined slightly differently, and so one must be careful when consulting various references.) The first few coefficients in (2.3.12) are

\[ B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6, \quad B_3 = 0, \quad B_4 = -1/30 \]

and it can be shown that all coefficients with odd indices except \( B_1 \) are zero. A table of Bernoulli numbers can be found in [1]. Another fact which we shall use is that

\[ \zeta(2n) = |B_{2n}| \frac{(2\pi)^{2n}}{2(2n)!} \]

for \( n = 1, 2, 3, \ldots \); in particular

\[ \zeta(2) = \sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6} \]

and

\[ \zeta(4) = \sum_{k=1}^{\infty} k^{-4} = \frac{\pi^4}{90}. \]

2.3.2 Extension of formula (2.1.26)

We do not start by immediately deriving the expansion for \( \hat{Q} \) given in (2.3.5); instead we first represent \( \hat{Q} \) in the form

\[ \hat{Q}(m,x;r) = \left( \frac{r}{m} \right) \exp \left[ \sum_{i=0}^{\infty} b_i(m,x) r^i \right] \]

for \( 1 \leq m < x \). (According to our notational convention we should write \( \hat{b}_i(m,x) \) instead of \( b_i(m,x) \), but since we shall not be examining the corresponding expansion for \( Q(m,x;r) \) it is convenient to omit the caret). The reason why we derive an expansion of the form (2.3.15) is because the coefficients \( b_i(m,x) \) can be easily expressed in terms of known functions,
and the $\hat{q}_1$ can in turn be easily represented in terms of the $b_i$. Note that (2.3.15) holds only for $m \geq 1$. It is shown in Appendix C that

\[
\hat{q}(0,x;r) = \exp\left[ \sum_{i=0}^{\infty} b_i(1,x) r^i \right], \quad (2.3.16)
\]

and so finding the coefficients $b_i(m,x)$ for $m \geq 1$ will suffice for the case $m = 0$ as well. In Appendix C exact formulas for these coefficients are obtained; the results are

\[
\begin{align*}
 b_0(m,x) &= 0 \\
 b_1(m,x) &= \log 2 + \psi(m) - \psi(1+x) \\
 b_i(m,x) &= \frac{\psi^{(i-1)}(m) - \psi^{(i-1)}(1+x)}{i!}. \quad (i \geq 2)
\end{align*}
\]  

(2.3.17)

The function $\psi$ in (2.3.17) is the the Psi function (also called the Digamma function), which is defined by [1,28]

\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (2.3.18)
\]

and which is tabulated in [1]. The Psi function is analytic in the same region of the complex plane as the Gamma function, i.e. in $\mathbb{C}\setminus\{0, -1, -2, \ldots\}$. We use $\psi^{(j)}$ to denote the $j$-th derivative of the Psi function. The expansion (2.3.15) can now be used for computational purposes, since we have expressed the coefficients in equation (2.3.15) in terms of known functions. These coefficients can be accurately computed for $x > 10$ or so by means of asymptotic expansions; the details of deriving these expressions are given in Appendix C. There it is shown that for $i = 1$ we have
\( b_1(m,x) \sim H_{m-1} - [\log(x/2) + C + 1/2x - 1/12x^2 + 1/120x^4 - \ldots], \)  
(2.3.19)

and for \( i \geq 2 \) we have

\[
 b_i(m,x) \sim (-1)^i i^{-1}\left\{ \zeta(i) - H^{(i)}_{m-1} - \\
- \left[ \frac{1}{(i-1)x^{i-1}} - \frac{1}{2x^i} + \frac{1}{(i-1)!} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \left( \frac{2n+i-2}{x^{2n+1}} \right) \right] \right\}.
\]
(2.3.20)

Although the preceding formulas may look complicated, they are quite simple to use. For the reader's convenience the asymptotic forms of the first few coefficients are written out explicitly in Table III below.

| \( b_1(m,x) \) | \( H_{m-1} - [\log(x/2) + C + 1/2x - 1/12x^2 + \]
| | \( + 1/120x^4 - 1/252x^6 + \ldots] \) |
| \( b_2(m,x) \) | \( (1/2) \{ \zeta(2) - H^{(2)}_{m-1} - [1/x - 1/2x^2 + 1/6x^3 - \]
| | \( - 1/30x^5 + 1/42x^7 - \ldots] \}) |
| \( b_3(m,x) \) | \( (-1/3)\{ \zeta(3) - H^{(3)}_{m-1} - [1/2x^2 - 1/2x^3 + 1/4x^4 - \]
| | \( - 1/12x^6 + 1/12x^8 - \ldots] \}) |
| \( b_4(m,x) \) | \( (1/4) \{ \zeta(4) - H^{(4)}_{m-1} - [1/3x^3 - 1/2x^4 + 1/3x^5 - \]
| | \( - 1/6x^7 + \ldots] \}) |

Table III: Asymptotic forms of the coefficients.

From (2.3.14) we have the exact values for \( \zeta(2) \) and \( \zeta(4) \); there is no such closed form for the Zeta function evaluated at the odd positive integers, and \( \zeta(3) \) must be computed numerically. From [1] we find that \( \zeta(3) \cong 1.20205. \) For block sizes greater than about 10 bits, the above asymptotic formulas allow easy
computation of the coefficients $b_i$ to considerable accuracy.

Consider now the computation of $\hat{P}(0,x;r) = \hat{Q}(0,x;r)$ using (2.3.16). Fortunately, the convergence of the infinite series in (2.3.16) is sufficiently rapid that we may truncate it after only several terms with little error; reference to approximation (2.1.26), developed in [21], shows that it corresponds to retaining only the first term $b_1(1,x)$ in (2.3.16). At high average SNR's the single term is sufficient, but at lower SNR's it is advisable to take several more. The methods here demonstrate how we can extend the approximation (2.1.26) to higher-order terms and thus increase the accuracy. Retention of only the first four terms, corresponding to the four coefficients written out in detail above, gives good accuracy for a wide range of average SNR's $\gamma_0$.

The computation of $\hat{P}(M,x;r)$ for $M > 0$ by first computing $\hat{Q}(m,x;r)$ for $m = 0, 1, 2, \ldots, M$ using (2.3.15) and (2.3.16), and then substituting these intermediate results into (2.2.14), is not very efficient. A much better approach is to use (2.3.16) to compute $\hat{P}(0,x;r)$ as described above, and then to use equation (2.2.35) to compute $\hat{P}(M,x;r)$.

2.3.3 Extension of formula (2.1.24)

Based on what we have derived so far, it is easy to determine the coefficients $\hat{q}_i(m,x)$ and $\hat{p}_i(M,x)$ of equations (2.3.5) and (2.3.7). The result will be an extension of approximation (2.1.24) to higher-order terms, in the same manner as we extended approximation (2.1.26). The method of
doing this is to express the coefficients $\hat{q}_i$ and $\hat{p}_i$ in terms of the known coefficients $b_i$. We begin by writing

$$
\sum_{i=0}^{\infty} a_i(m,x) r^i = \exp\left[ \sum_{i=1}^{\infty} b_i(m,x) r^i \right] \quad (2.3.21)
$$

where the $b_i$ are known and the $a_i$ are to be determined. We have used the fact that $b_0(m,x) = 0$ in writing $(2.3.21)$. From now on we shall usually suppress the parameters $m$ and $x$ in our notation, and simply write

$$
\sum_{i=0}^{\infty} a_i r^i = \exp\left[ \sum_{i=1}^{\infty} b_i r^i \right]. \quad (2.3.22)
$$

In Appendix C it is shown that there is a recurrence formula which holds between the coefficients $a_i$ and $b_i$. This formula is

$$
\begin{cases} 
  a_0 = 1 \\
  a_i = i^{-1} \sum_{k=1}^{i} k b_k a_{i-k}. \quad (i \geq 1)
\end{cases} \quad (2.3.23)
$$

We easily find that

$$
\begin{align*}
  a_0 &= 1 \\
  a_1 &= b_1 \\
  a_2 &= b_1^2/2 + b_2 \\
  a_3 &= b_1^3/6 + b_1 b_2 + b_3 \\
  a_4 &= b_1^4/24 + b_2 b_1^2/2 + b_2^2/2 + b_1 b_3 + b_4 \\
\end{align*} \quad (2.3.24)
$$

and so on. Now that we have expressed the coefficients $a_i$ in terms of $b_i$, it is a simple matter to express $\hat{p}_i$ in terms of $b_i$. We first consider the case $M = 0$. We have
\[ \hat{P}(0,x;r) = \hat{Q}(0,x;r) = \sum_{i=0}^{\infty} a_i(1,x) r^i \]  

(2.3.25)

from equations (2.3.16) and (2.3.21), and it immediately follows that \( \hat{p}_i(0,x) = \hat{q}_i(0,x) = a_i(1,x) \). Since \( a_0(1,x) = 1 \),

\[ \hat{P}_f(0,x;r) = \sum_{i=1}^{\infty} a_i(1,x) r^i = \sum_{i=1}^{\infty} \hat{p}_i(0,x) r^i. \]  

(2.3.26)

But

\[ a_i(1,x) = b_i(1,x) \sim -[\log(x/2) + C + 1/2x - 1/12x^2 + ...] \]  

(2.3.27)

and so taking the first term in (2.3.26) recovers the approximation (2.1.24) [see also the note after (2.1.26)]. Note that the derivation leading to (2.1.24) depended on the fact that the block size was an integer \( N \); equation (2.3.27) shows that (2.1.24) may be used for non-integral block sizes simply by replacing the integer \( N \) by the real number \( x \) (although the intermediate steps leading to (2.1.24) no longer make sense). The importance of (2.3.26) is that we now have formulas that enable us to compute as many of the coefficients \( \hat{p}_i(0,x) \) as we desire, i.e. we have extended the approximation (2.1.24) to arbitrary order. The computational scheme is slightly more complex than it is for equation (2.3.16), since we must not only compute the coefficients \( b_i(1,x) \), but also must then find the coefficients \( \hat{p}_i(0,x) = a_i(1,x) \) using the recurrence relation (2.3.23).

Lastly, we demonstrate how a slight modification of the first-order approximation (2.1.24) allows it to be used for \( M > 0 \) as well. Comparison of equations (2.3.15) and (2.3.21) yields, for \( m \geq 1 \),
\[ \hat{Q}(m,x;r) = \left( \frac{r}{m} \right) \sum_{i=0}^{\infty} a_i(m,x) r^i = \sum_{i=1}^{\infty} \left[ a_{i-1}(m,x)/m \right] r^i. \]

Hence, when \( m \geq 1 \), we have
\[
\begin{align*}
\hat{q}_0(m,x) &= 0 \\
\hat{q}_i(m,x) &= a_{i-1}(m,x)/m.
\end{align*}
\]

Since \( \hat{q}_i(0,x) = a_i(1,x) \) [see the top of the previous page], we may write
\[
\hat{p}_i(M,x) = \sum_{m=0}^{M} \hat{q}_i(m,x)
\]
\[
= a_i(1,x) + \left( 1 - \delta_{i0} \right) \sum_{m=1}^{M} a_{i-1}(m,x)/m
\]

where \( \delta_{ij} \) is the Kronecker delta. In general we cannot analytically simplify (2.3.29) very much, although we can make some useful simplifications when \( i \) is small. We shall examine only the cases \( i = 0 \) and \( i = 1 \), i.e. we shall consider only a first-order approximation. When \( i = 0 \), \( \hat{p}_0(M,x) = a_0(1,x) = 1 \).

For \( i = 1 \),
\[
\hat{p}_1(M,x) = a_1(1,x) + \sum_{m=1}^{M} a_0(m,x)/m.
\]

From (2.3.24) we find that \( a_1(1,x) = b_1(1,x) \), and that \( a_0(m,x) = 1 \). Then
\[
\hat{p}_1(M,x) = b_1(1,x) + H_M = b_1(M+1,x),
\]

where the last equality follows immediately on reference to Table III. Hence when \( r \) is small,
\[ \hat{P}(M,x;r) = \sum_{i=0}^{\infty} \hat{p}_i(M,x) r^i \]
\[ \approx \hat{p}_0(M,x) + \hat{p}_1(M,x) r \]  
(2.3.32)

\[ = 1 + b_1(M+1,x) r \]

and so

\[ \hat{P}_f(M,x;r) = 1 - \hat{P}(M,x;r) \approx - b_1(M+1,x) r \]
\[ = r \left[ -\log 2 - \psi(M+1) + \psi(1+x) \right] \]  
(2.3.33)

\[ = r \left[ -\chi_M + \log(x/2) + C + 1/2x - 1/12x^2 + ... \right]. \]

The last equation shows the simple modification that must be made to the first-order approximation (2.1.24) to extend it to the case \( M > 0 \).
2.4 Diversity

2.4.1 Introduction

We come now to the last topic in our analysis of very slow Rayleigh fading: the computation of the block-error rate when ideal selection diversity is employed. We assume that there are \( L \) branches, where \( L \) is an integer greater than or equal to 1, and letting \( \gamma_i \) denote the instantaneous signal-to-noise ratio on branch \( i \), we further assume that \( \{ \gamma_i \} \) is a set of independent random variables, each having distribution (2.1.7). In a receiver using selection diversity, the branch with the highest SNR is chosen. In [23] it is shown that the probability distribution function of the largest SNR \( \gamma^* \) is given by

\[
f_{\gamma^*}(\gamma) = L \left[ 1 - \exp(-\gamma/\gamma_0) \right]^{L-1} \gamma_0^{-1} \exp(-\gamma/\gamma_0) \quad (\gamma \geq 0)
\]

(2.4.1)

Under the condition of very slow fading, the SNR's \( \gamma_i \) are assumed to remain constant over an entire block, and hence the same branch will be chosen for an entire block. In other words, \( \gamma^* \) remains constant over an entire block, but varies from block to block according to (2.4.1), and so in analogy with equation (2.1.11) we may immediately write down

\[
P_d(L,M,x) = \int_0^\infty P_b(M,x;\gamma) f_{\gamma^*}(\gamma) \, d\gamma
\]

(2.4.2)

\[
= 1 - \sum_{m=0}^M \binom{x}{m} \int_0^\infty p(\gamma)^m [1 - p(\gamma)]^{x-m} f_{\gamma^*}(\gamma) \, d\gamma,
\]

where \( P_d(L,M,x) \) is the probability that there is more than \( M \) errors in a block of \( x \) bits when \( L \)-branch selection diversity
is used. We shall usually write simply $P_d$ for $P_d(L,M,x)$, and shall use the notation $P_{d,c} = 1 - P_d$ (the subscript $c$ stands for "complement"). Putting (2.4.1) into (2.4.2) gives

$$P_{d,c} = (L/\gamma_0) \sum_{m=0}^{\infty} \binom{x}{m} \int_0^\infty p(\gamma)^m [1 - p(\gamma)]^{x-m}$$

$$\cdot [1 - \exp(-\gamma/\gamma_0)]^{L-1} \exp(-\gamma/\gamma_0) \, d\gamma.$$  

We consider only NCFSK, for which $p(\gamma) = (1/2) \exp(-\gamma/2)$. For the special case $M = 0$ and $x = 1$ (i.e. when we are computing the probability of bit error), see [14] or [20]; we are interested in the more general case. Making the substitution $t = p(\gamma)$, as we did in equation (2.2.2), we find

$$P_{d,c} = L \gamma_0 2^r \sum_{m=0}^{\infty} \binom{x}{m} \int_0^{1/2} t^{m+r-1} (1 - t)^{x-m} [1 - (2t)^r]^{L-1} \, dt.$$  

(2.4.4)

Using the binomial expansion of the term $[1 - (2t)^r]^{L-1}$, we can write the previous equation in the form

$$P_{d,c} = L \gamma_0 2^r \sum_{m=0}^{\infty} \binom{x}{m} \sum_{j=0}^{L-1} \binom{L-1}{j} (-1)^j 2^{jr}$$

$$\cdot \int_0^{1/2} t^{m+(j+1)r-1} (1 - t)^{x-m} \, dt.$$  

(2.4.5)

If we now take $x = N = \text{an integer}$ in (2.4.5) above, and use the binomial expansion of the term $(1 - t)^{N-m}$, we find

$$P_{d,c} = L \gamma_0 2^r \sum_{m=0}^{\infty} \binom{N}{m} \sum_{j=0}^{L-1} \binom{L-1}{j} (-1)^j L^{-m}$$

$$\cdot \sum_{k=0}^{N-m} \binom{N-m}{k} (-1)^k \frac{1}{2^{m+k} [m+(j+1)r+k]}.$$  

(2.4.6)
Equation (2.4.6) is the same as equation (15) of [25] (with a slight rearrangement and obvious changes in notation). As noted in [25], equation (2.4.6) is an alternating series that may be difficult to evaluate numerically when N or M is large. In fact, the largest value of N used in [25] for numerical results is N = 100.

2.4.2 Approximations

We now develop some approximations for \( P_{d,c} \) (and hence for \( P_d \)); these approximations work well for block sizes greater than 10 or so. We begin by writing equation (2.4.5) in the form

\[
P_{d,c} = L r 2^r \sum_{m=0}^{M} \binom{x}{m} \sum_{j=0}^{L-1} \binom{L-1}{j} (-1)^j 2^jr B_{2}(m+(j+1)r,1+x-m).
\]

We now approximate \( P_{d,c} \) in the manner of Section 2.2, i.e. we replace the incomplete Beta function by the Beta function:

\[
\hat{P}_{d,c} = L r 2^r \sum_{m=0}^{M} \binom{x}{m} \sum_{j=0}^{L-1} \binom{L-1}{j} (-1)^j 2^jr B_{2}(m+(j+1)r,1+x-m).
\]

In Appendix B we show that

\[
|P_{d,c} - \hat{P}_{d,c}| = |P_d - \hat{P}_d| \leq \left( \sum_{m=0}^{M} \binom{x}{m} \frac{L r^L}{2^{x-m} (1+x-m)} \right).
\]

This error bound is very similar to those examined in Section 2.2, and it follows that the error in the approximation is
negligible for the parameter range in which we are interested. Equation (2.4.8) may be rearranged to give

\[ \hat{P}_{d,c} = L \sum_{j=0}^{L-1} \frac{(-1)^j}{j+1} \left[ (j+1)r \sum_{m=0}^{M} \binom{x}{m} B(m+(j+1)r,1+x-m) \right] \]

\[ = L \sum_{j=0}^{L-1} \frac{(-1)^j}{j+1} \hat{P}(M,x;(j+1)r) \quad (2.4.10) \]

We now put \( k = j+1 \) to get

\[ \hat{P}_{d,c} = \sum_{k=1}^{L} \frac{(L/k)}{(L-1/k-1)} (-1)^{k+1} \hat{P}(M,x;kr) \quad (2.4.11) \]

Since we already have an efficient way to compute \( \hat{P}(M,x;r) \), equation (2.4.11) can be used to approximate \( P_{d,c} \), and hence \( P_d \), when \( L \) is small. If \( L \) is large, the alternating series (2.4.11) may prove unsuitable for numerical computation. In Figure 2.7, \( \hat{P}_d(L,0,100) \) has been plotted as a function of the average SNR \( \gamma_0 \) for \( L = 1, 2, 3, 4 \). Clearly the effect of diversity is to substantially improve the probability of block error for a given SNR, although the improvement obtained by adding one more branch decreases as the number of branches increases.

We can get more insight into the behavior of \( \hat{P}_d \) by deriving its infinite series expansion. We use (2.3.7) to write
Figure 2.7: $\hat{P}_d(L,0,100)$ for $L = 1, 2, 3, 4$. 

Average signal-to-noise ratio $\tau_0$ in dB.
\[ \hat{P}(M,x;kr) = \sum_{i=0}^{\infty} \hat{p}_i(M,x) r^{i} k^{i} \]

when \(|r| < 1/k\) (recall that the series expansion for \(\hat{P}(M,x;r)\) converges for at least \(|r| < 1\)). Since the largest value assumed by the index of summation \(k\) in (2.4.11) is \(k = L\), we may write

\[ \hat{P}_{d,c} = \sum_{k=1}^{L} \binom{L}{k} (-1)^{k+1} \sum_{i=0}^{\infty} \hat{p}_i(M,x) r^{i} k^{i} \]

\[ = \sum_{i=0}^{\infty} \hat{p}_i(M,x) \left[ \sum_{k=0}^{L} \binom{L}{k} (-1)^{k+1} k^{i} \right] r^{i} \]

(2.4.12)

for \(|r| < 1/L\). Denote the quantity in square brackets by \(S(L,i)\), i.e.

\[ S(L,i) = \sum_{k=0}^{L} \binom{L}{k} (-1)^{k+1} k^{i} = \sum_{k=0}^{L} \binom{L}{k} (-1)^{k} k^{i}. \quad (2.4.13) \]

Let \(\{ \binom{m}{n} \}\) denote Stirling numbers of the second kind \([1,13]\). On page 67 of \([13]\) we find the relation

\[ (-1)^n n! \binom{m}{n} = \sum_{k=0}^{n} \binom{n}{k} k^m (-1)^k \]

(2.4.14)

with the convention that \(0^0 = 1\). Using this same convention, we can combine equations (2.4.13) and (2.4.14) to give

\[ S(L,i) = - \sum_{k=1}^{L} \binom{L}{k} (-1)^{k} k^{i} = \delta_{i0} - \sum_{k=0}^{L} \binom{L}{k} (-1)^{k} k^{i} \]

\[ = \delta_{i0} - (-1)^L L! \binom{i}{L}, \]

where \(\delta_{ij}\) is the Kronecker delta. Thus
\[ \hat{P}_d(L,M,x) = 1 - \hat{P}_{d,c}(L,M,x) \]
\[ = 1 - \sum_{i=0}^{\infty} \hat{P}_i(M,x) S(L,i) r^i \]
\[ = (-1)^L L! \sum_{i=1}^{\infty} \hat{P}_i(M,x) \binom{L}{i} r^i. \]

In the preceding calculations we used the result \( \hat{P}_0(M,x) = a_0(1,x) = 1 \) (see page 56) and the fact that \( S(L,0) = 1 \) to cancel the first term of the series. It can be shown that [13]

\[ \binom{i}{i} = 0 \text{ for } 1 \leq i < L, \]
\[ \binom{L}{L} = 1, \text{ and } \binom{L+1}{L} = \binom{L+1}{L+1}/2. \]

Thus

\[ \hat{P}_d = (-1)^L L! \{ \hat{P}_L(M,x) r^L + [L(L+1)/2] \hat{P}_{L+1}(M,x) r^{L+1} + \ldots \} \]
or, retaining only the first term,

\[ \hat{P}_d \approx (-1)^L L! \hat{P}_L(M,x) r^L \]
\[ = (-2)^L L! \hat{P}_L(M,x)/\gamma_0^L. \]

Thus the effect of diversity is to make the probability of error proportional to \( \gamma_0^{-L} \) for large SNR's, instead of proportional to \( \gamma_0^{-1} \) as in equation (2.1.24). The probability of error therefore drops off much more rapidly as the signal-to-noise ratio increases than it does when diversity is not employed, as shown in Figure 2.7.
Chapter 3
A model for calculating block-error rates on slow Rayleigh fading channels

In the previous chapter we examined several methods for the computation of $P_f(M,N)$ on a channel subject to very slow Rayleigh fading. The purpose of this chapter is to consider the computation of $P_f(M,N)$ when the assumption of very slow fading is relaxed, i.e. when the signal-to-noise ratio does not necessarily remain constant over an entire block. The emphasis will be on the mobile-radio environment.

Without the assumption of very slow fading, the analytical methods of Chapter 2 are no longer applicable, and at present there appears to be no analytical method for computing the probability of block error. Thus we must resort to simulations. In general, there are two types of simulators: hardware and software. Descriptions of hardware simulators for Rayleigh fading can be found, for example, in [2] and [4]. It was decided for the purposes of this thesis to use a software simulator based on a computer program given in [24]. The software simulator was chosen over a hardware version because of the ease of its implementation. A disadvantage of using a simulator on a digital computer is the long running time (and hence high cost) required to guarantee statistically valid results.

To overcome the expense of computer simulations, an attempt was made to find a simple empirical formula, based on the formulas of Chapter 2, that would allow us to predict the
results of the simulations. In Section 3.2.2 we describe a model from which we derive such an empirical formula. This formula turns out to be quite successful in that it allows easy calculation of the probability of block error when very slow fading is not assumed.
3.1 The Simulator

3.1.1 Implementation

The simulator was implemented both on the university's Amdahl 470 and on a Nova 840 that is resident in the department of electrical engineering. Although it is much slower than the Amdahl, the Nova could be run for much longer periods of time because the computer time was free. The simulation programs for both computers were written in FORTRAN, but they differed in many respects due to differences in the machines and their operating systems. For example, the Amdahl has virtual-memory capability, i.e. the system automatically performs transfers between primary and secondary storage when necessary, thereby enabling the programmer to manipulate large data structures easily and reliably. On the Nova, however, storage allocation and transfers between core and disk are largely the responsibility of the programmer. Furthermore, the Amdahl has a large amount of support software, such as random-number generators and fast Fourier transform (FFT) routines. All such programs had to be coded especially for the Nova. Thus, although the simulation programs implemented the same algorithm, they differed in many respects. The fact that the simulators gave identical results in several tests increased the confidence in their correctness.

3.1.2 Description of the algorithm

The simulation program uses a Monte Carlo method, i.e. a
method based on random-number generators (the numbers should more accurately be called pseudo-random, but the term random number is standard). An outline of the algorithm follows; the details will be described in subsequent sections.

The simulation program first generates and stores a fading sequence with the exponential distribution given in equation (2.1.7), and then normalizes it to the desired average SNR \( \gamma_0 \). The sequence is given an exponential distribution instead of a Rayleigh distribution because it is more convenient to work directly with signal-to-noise ratios.\(^4\) Samples are taken from the exponentially distributed sequence one at a time; each sample represents the SNR that is associated with each bit in a simulated stream of transmitted bits. As a first step in deciding whether or not a given bit is in error, the probability of error for that bit is calculated. This calculation is done by substituting the associated SNR from the fading sequence into the appropriate formula for the probability of bit error. That is, if \( \gamma_k \) is the \( k \)-th SNR value in the fading sequence, then the \( k \)-th bit is in error with probability \( p(\gamma_k) \), where \( p(\gamma) \) is chosen from (2.1.1) or (2.1.2) in accordance with the modulation scheme being examined. Next, a random number \( u_k \) is chosen from the uniform distribution on

\(^4\)Of course, we can derive a Rayleigh-distributed sequence from the exponentially distributed one by taking the square root of the sample values. A Rayleigh-distributed sequence is desired for certain tests that are used in Section 3.1.3 to verify the statistical properties of the sequence generator, and for these tests the square root is taken.
the interval \([0,1]\), and is compared with \(p(\gamma_k)\). If \(u_k < p(\gamma_k)\)
we say that the bit is in error; otherwise we say that the bit
is correct. Note that the signal-to-noise ratio is still
assumed to be constant over the duration of a bit, although it
may change from bit to bit. Thus the simulator should give a
bit-error rate equal to the theoretical results for slow fading
given in equations (2.1.9) and (2.1.10).

By segmenting the fading sequence into blocks of \(N\)
samples, we can easily keep track of the number of errors per
block. If a sufficiently large number of blocks are observed,
the block-error rate may then be stated with a high degree of
certainty. To gauge the statistical accuracy of the
simulations, the following procedure was adopted. A fading
sequence is first generated, and the block-error rate for this
sequence is computed as a relative frequency. The results for
this sequence are said to constitute a single trial. Thus if \(n\)
trials are conducted, the result is \(n\) relative frequencies
\(P_1(M,N), P_2(M,N), \ldots, P_n(M,N)\) that approximate the true
probability of block error. The sample mean \(P_s(M,N)\) and the
sample variance are then computed. Assuming that the
distribution of the sample mean is approximately Gaussian, and
replacing the unknown true variance by the sample variance, we
construct a 95% confidence interval about the sample mean using
the \(t\)-distribution. Justification for using the \(t\)-distribution
can be found, for example, in [27]. A 95% confidence interval
means that we are 95% confident that the stated interval
contains the true mean. Finally, we note that the algorithm
above implicitly assumes a lack of intersymbol interference.

3.1.3 Generation of the fading sequence

Before looking at the details of generating the fading sequence, let us consider the statistical properties that we desire the sequence to have. As stated in the introduction, the emphasis here is on mobile-radio communication. The following discussion is based on continuous-time statistics; the simulator of course employs their discrete-time counterparts.

The statistical model that we use for the mobile-radio environment is the one examined by R.H. Clarke in [7]. It is assumed that the received signal contains no direct component, but instead consists entirely of indirect components, each of which has an angle of arrival that is uniformly distributed from 0 to $2\pi$. The phases of the arriving waves are assumed to be uniformly distributed from 0 to $2\pi$ and statistically independent. Clarke showed that the amplitude of the received signal envelope is then Rayleigh distributed, and that the phase is uniformly distributed. Furthermore, when a vertical monopole antenna is used the power spectral density of the Rayleigh-fading envelope is

$$S(f) = \begin{cases} S_0 \left[1 - \left(\frac{f}{f_D}\right)^2\right]^{-1/2}, & |f| < f_D \\ 0, & |f| \geq f_D \end{cases}$$ (3.1.1)

where $S_0$ is a constant and $f_D$ is the Doppler frequency, given by
f_D = v/\lambda. \quad (3.1.2)

In equation (3.1.2) \( v \) is the vehicle speed and \( \lambda \) is the carrier wavelength. The use of a different antenna leads only to a different power spectral density; see [10] for the power spectral densities that result from a variety of antenna configurations.

Several other statistical properties of the Rayleigh-fading envelope have been derived in the literature. In [12] it is shown that the average duration of the fades below the level \( \rho \) is

\[
T(\rho) = \frac{\exp(\rho^2) - 1}{\sqrt{2\pi} f_D \rho}, \quad (3.1.3)
\]

and that the average rate at which the envelope crosses the level \( \rho \) with a positive slope is

\[
N(\rho) = \sqrt{2\pi} f_D \rho \exp(-\rho^2). \quad (3.1.4)
\]

The quantity \( \rho \) in equations (3.1.3) and (3.1.4) is the amplitude of the fading envelope measured relative to the RMS value of the envelope, i.e.

\[
\rho \triangleq \frac{\text{envelope amplitude}}{\text{RMS value of the envelope}}. \quad (3.1.5)
\]

Field tests described in the literature have verified experimentally that the above results hold quite well. It has been verified that the short-term statistics of the fading envelope are well described by the Rayleigh distribution [7,22], and experimental results in [11] and [22] show that the
theoretical formulas (3.1.3) and (3.1.4) adequately predict the average duration of fades and the rate of level crossings.

Let us now return to the generation of the fading sequence. The section of the simulator that generates this sequence was taken almost verbatim from [24]. The computer program in [24] is based on the fact that a Rayleigh-distributed sequence can be generated by adding two independent Gaussian-distributed sequences in quadrature (see Chapter 7 of [18]). A shaping filter is used to give the sequence the desired spectrum (3.1.1). The shaping is done by multiplication in the frequency domain followed by a translation into the time domain using an FFT. The number of points in the FFT was set to 4096, resulting in a simulated bit rate of $R = 4096$ bits/sec. Note that the normalization of the fading sequence makes the actual value of $S_0$ in equation (3.1.1) immaterial; it was set to unity for convenience.

To check the operation of the fading-sequence generator, a sequence was generated that was 500 seconds long, and several statistical quantities based on this sequence were computed. First, the cumulative distribution function (cdf) of the fading sequence was computed and compared with the desired Rayleigh cdf. In Figure 3.1 the solid line is the theoretical Rayleigh cdf, and the plotted points are the values computed from the fading sequence. Clearly the distribution of the output of the fading-sequence generator is very close to the desired Rayleigh distribution. Also computed from the 500-second-long sequence were the average fade duration and the level-crossing rate. The
Level $p$ in dB relative to the RMS value.

Figure 3.1: Rayleigh cdf.
theoretical results given by equations (3.1.3) and (3.1.4) are graphed as solid lines in Figures 3.2 and 3.3 respectively; the curves in these two figures have been normalized to unity Doppler frequency. The plotted points are the values computed from the fading sequence, also normalized to unity Doppler frequency. As one can see, the agreement between the theoretical and the measured results is excellent.

3.1.4 Simulation results

The simulation program was used to compute the probability of block error, $P_s(M,N)$, for NCFSK modulation. The program was run for block sizes $N = 63, 127, 255, 511, 1023, 2047$; for $M = 0, 1, 2, \ldots, 9$; for Doppler frequencies $f_D = 10.5, 15.5, 20.5, 25.5$ Hz; and for average SNR's $\gamma_0 = 5, 10, 15, \ldots, 35$ dB. In Section 3.1.2 the use of a 95% confidence interval to check the statistical accuracy of the simulations was described. Tests were run to determine the number of trials, $n$, required to make this interval sufficiently small that meaningful conclusions could be drawn. It was felt that a reasonable criterion was to make the 95% confidence interval less than about 10% of the computed sample mean $P_s(M,N)$. This criterion could not be met uniformly with respect to the average SNR $\gamma_0$, because for a fixed number of trials the confidence intervals become larger with increasing SNR. This pattern holds because at low SNR's the probabilities are close to unity, and hence many errors will occur during the simulation. The large number of errors ensures good statistical
Level $\rho$ in dB relative to the RMS value.

Figure 3.2: Average fade time $T(\rho)$.

Level $\rho$ in dB relative to the RMS value.

Figure 3.3: Level-crossing rate $N(\rho)$. 
results. At high SNR's the probabilities are small, i.e. we are examining rare events, and so not enough errors occur during the simulation to obtain statistics that are as good as those at low SNR's. When sequences 60 to 90 seconds long per trial were used, the tests showed that taking \( n \) in the range \( 10 \leq n \leq 15 \) was sufficient to meet the criterion given above at \( \gamma_0 = 25 \) dB. The bit-error probability was also computed, and as anticipated it agreed well with the theoretical result \( p_f = 1/(2 + \gamma_0) \) given for NCFSK in equation (2.1.9). Some of the simulation results are presented here in graphical form; the idea is not to catalog all the results produced, but to give representative examples.

Figure 3.4 gives \( P_s(0,127) \) as a function of the average SNR \( \gamma_0 \) for all four Doppler frequencies, and with all other parameters held fixed. The plotted points are those produced by the simulation, and the curves connecting them are plotted using quadratic interpolation. In addition to these four curves, two other curves are given for reference. The lower curve is the value of \( P_f(0,127) \) for very slow fading, computed using the approximate methods of Chapter 2. The upper curve shows the probability of block error \( 1 - (1 - p_f)^{127} \) that would result on an independent-error channel that has a bit-error probability \( p_f = 1/(2 + \gamma_0) \).

From Figure 3.4 we see that the probability of block error computed by the simulation program is larger than the value computed under the assumption of very slow fading. Intuitively one anticipates that allowing faster fading should have an
Figure 3.4: $P_s(0,127)$ at different Doppler frequencies.
adverse effect on the performance, and this conjecture is confirmed. It is further supported by the fact that $P_s(M,N)$ increases with the Doppler frequency $f_D$, i.e. the faster the fading the worse the performance. However, the performance appears to be less sensitive to changes in $f_D$ for larger values of $f_D$, e.g. the curves for 20.5 and 25.5 Hz are closer together than are the curves for 10.5 and 15.5 Hz. Also, the probability of block error is smaller than that on a random-error channel with the same bit-error rate.

Figure 3.5 gives the error probability $P_s(0,N)$ for different block lengths $N$. Comparison with Figure 2.3 shows that the deterioration of performance with increasing block length is worse for the simulation results than it is for the case of very slow fading.

Finally, Figure 3.6 shows $P_s(M,63)$ for several values of $M$ at Doppler frequency 15.5 Hz. For $M \geq 1$, $P_s$ does not appear to have a simple dependence on $\gamma_0$ when $\gamma_0$ is large, i.e. the curves for different values of $M$ are not straight for large SNR's, as they are in the case of very slow fading (see Figure 2.4). This behavior suggests that it might be more difficult to describe analytically the curves for $M \geq 1$ than it was when the fading was very slow.
Figure 3.5: $P_s(0,N)$ for different block sizes at 15.5 Hz.
Figure 3.6: $P_s(M,63)$ for $M = 0, 1, 2, 3$ at 15.5 Hz.
3.2 Derivation of the empirical formula

3.2.1 Introduction

The purpose of this section is to develop an empirical formula relating the results of the computer simulations to the equations of Chapter 2. If such a formula could be found, its usefulness would be obvious: we could easily predict the results that would be obtained by the simulation program without actually running the program. The upshot would be an enormous saving in computer time, while allowing us to compute a more realistic value of the probability of block error than is provided under the assumption of very slow fading. For simplicity we shall restrict ourselves to the case $M = 0$, i.e. to the approximation of $P_s(0,N)$. As we shall see, a simple model allows the derivation of an empirical formula that works quite well. We start by describing this model.

3.2.2 The model

Given a block of $N$ bits, we divide this block into a number of sub-blocks, each of length $x$ bits. The hypothesis is that by choosing the size of the sub-blocks correctly, we may treat the sub-blocks as though they were independent. Furthermore, if the sub-blocks are small enough, it seems reasonable to assume that the fading process is so slow relative to the sub-block duration that the SNR is approximately constant over the sub-block. Thus we may approximate the probability that each sub-block is correct by
using the results for very slow fading. Then, in the notation of Chapter 2, the probability that each sub-block is correct is \( P(0, x; r) \), where \( r = 2/\gamma_0 \). Because the sub-blocks are assumed to be independent, the probability \( P(0, x; r) \) is multiplied together to obtain the probability that the original block of \( N \) bits is correct. Since there are \( N/x \) sub-blocks we use the formula

\[
1 - P(0, x; r)^{N/x}
\]

(3.2.1)

to approximate the probability of block error, \( P_s(0, N) \). We shall now investigate formula (3.2.1) in detail. If it turns out that formula (3.2.1) works well, this is not necessarily due to the reasons given in our description of the model; however, if the formula works well, the actual reasoning that led to it can be ignored.

The problem now is to discover how the sub-block size \( x \) must be chosen in order to make (3.2.1) yield the probability \( P_s(0, N) \) computed by the simulation program. We do not restrict \( x \) to integer values, i.e. the size of the sub-blocks may be non-integral. The derivations of Chapter 2 allowed for this possibility. Recall that, besides the block size \( N \), \( P_s(0, N) \) will depend on the average signal-to-noise ratio \( \gamma_0 \), on the Doppler frequency \( f_D \), and on the bit rate \( R \). To reduce the number of variables we must consider, we define the ratio \( L = R/f_D \). Note that the units of \( L \) are bits when \( R \) is in bits/sec and \( f_D \) is in Hertz. The dependence of \( P_s(0, N) \) on \( R \) and \( f_D \) can be specified entirely by the single quantity \( L \) because it is only the fading rate relative to the bit rate that
determines the error probability \( P_s(0,N) \) when the other parameters are held fixed. Since the fading rate is proportional to the Doppler frequency \( f_D \) [see equation (3.1.4)], the error probability \( P_s(0,N) \) remains constant when \( R \) and \( f_D \) change by the same factor. We think of \( L \) as a fundamental block length that helps to characterize the fading channel for a given \( R \) and \( f_D \). We also introduce a quantity \( d \), which is defined by the requirement that

\[
1 - P(0,x;r)^{N/x} = P_s(0,N) \tag{3.2.2}
\]

where \( x = L/d \). We shall call \( d \) the divisor: it is the number by which we must divide \( L \) to obtain the size of the sub-blocks.

Suppose we are given the following three parameters: the block size \( N \), the quantity \( L = R/f_D \), and the average SNR \( \gamma_0 \), i.e. we are given enough information to specify a value of \( P_s \). The only quantity we must now know in order to apply formula (3.2.1) is the corresponding value of the divisor \( d \), as previously determined by equation (3.2.2), for then we can compute the sub-block size \( x = L/d \) and substitute this directly into (3.2.1). The divisor could theoretically change with all three parameters \( N, L, \) and \( \gamma_0 \). (Since \( R \) was held fixed at 4096 bits/sec during the computer simulations performed for this thesis, any change in \( L \) is due solely to a change in the Doppler frequency \( f_D \). Hence we shall refer to the change in \( d \) with \( f_D \), although it should be remembered that the results we shall derive here do not depend on the particular bit rate used in the simulations.) The usefulness of the formula (3.2.1)
depends largely on how \(d\) fluctuates with \(N\), \(f_D\), and \(\gamma_0\): the simpler the relationship, the more useful the formula. As we shall see, the divisor \(d\) is almost independent of \(N\) and \(f_D\), and has a simple dependence on \(\gamma_0\). In fact, \(d\) changes so little with \(N\) and \(f_D\) that we shall assume it is constant with respect to those two parameters, and focus on characterizing its dependence on \(\gamma_0\). We shall use curve fitting to find an empirical relationship between \(d\) and \(\gamma_0\).

3.2.3 Numerical results

The first step in making our empirical formula quantitative was to compute, for each of the values of \(P_s\) produced by the simulation program, the divisor \(d\) that makes equation (3.2.2) hold. Since simulations were run for 6 block sizes, 4 Doppler frequencies, and 7 average SNR's, \(P_s\) was computed for \(6 \times 4 \times 7 = 168\) different parameter combinations. The procedure for computing the 168 corresponding divisors was to solve equation (3.2.2) for the value of \(x\) that makes it hold for a specific value of \(P_s\), and then to set \(d = L/x\). Since equation (3.2.2) is a complicated nonlinear equation it was not possible to solve it analytically, and the solution had to be found numerically. As a check on the solutions, two different numerical methods were used: Newton's method and a simple bisection method. During these numerical computations, the left-hand side of (3.2.2) was evaluated using the approximations of Section 2.2. To further clarify what is being done here, the expression in (3.2.1) has been graphed in Figure
3.7 as a function of \( x \) for \( N = 255 \) and for several values of the average signal-to-noise ratio. One can find the solution of (3.2.2) for a given \( P_g \) by drawing a horizontal line across Figure 3.7 at the appropriate level until it intersects the correct curve, and by then extending a line downward to find the corresponding value of \( x \). This procedure has been illustrated in Figure 3.7 for the value \( P_g(0,255) \approx 0.246 \), obtained at \( f_D = 15.5 \) Hz and \( \gamma_0 = 25 \) dB using the simulation program. From the figure we find that the sub-block size causing equation (3.2.2) to be satisfied is \( x \approx 15 \) bits (the numerical methods, which are of course much more accurate than the graphical method shown here, yield \( x \approx 14.96 \) bits). The divisor in this case is

\[
\frac{d}{x} = \frac{L}{x} = \frac{R}{(x \cdot f_D)}
\]

\[
\approx \frac{4096.0}{(15.0 \times 15.5)}
\]

\[
\approx 17.6.
\]

There were several difficulties in the numerical computations just described, difficulties which will now be discussed.

One difficulty occurs when the SNR is small and the block size large. In this parameter range the probability of block error is close to unity, and in many cases the simulation program yielded a value \( P_g = 1.0 \) to single-precision accuracy, i.e. the simulation program did not record a single correct block for the duration of the simulation. A problem arises because for small SNR's the formula of (3.2.1) yields a value that is very close to unity for a wide range of \( x \) values, i.e. is relatively insensitive to changes in \( x \) (see Figure 3.7).
Figure 3.7: Formula (3.2.1) as a function of $x$ for different average SNR's $\gamma_0$. 
Hence the value of $x$ computed as a solution of equation (3.2.2) is unreliable when $P_s = 1.0$. On the one hand, this problem is annoying when trying, as we are here, to derive an empirical formula based on (3.2.1), because the values of $x$ for small SNR's will have no consistent trend. On the other hand, this behavior works in our favour when the purpose is to approximate $P_s$ using formula (3.2.1), since even an inaccurate value of $x$ allows a good approximation to the probability of block error when it is near unity. Also, we are not generally interested in the case when the block-error rate is near unity.

Another difficulty stems from the behavior of expression (3.2.1) when $x$ is small. As can be seen from Figure 3.7, the expression in (3.2.1) has a maximum near $x = 2$ bits, and decreases for $x$ below this value (see Figure 3.7). The problem arises because as the average SNR $\gamma_0$ increases the sub-block size $x$ decreases, and for $\gamma_0$ near 35 dB the value of $x$ is about 5 bits or so, i.e. near the peaks of the curves in Figure 3.7. If now $\gamma_0$ is increased further, then the horizontal line corresponding to the probability $P_s$ lies above the peak in the curve, and equation (3.2.2) no longer has a solution. Consequently, when the SNR becomes large the numerical methods tend to "lock in" at $x \approx 2$ bits. This behavior suggests that formula (3.2.1) might not work well for large SNR's (or, equivalently, for small sub-block sizes). Since we shall not be working with SNR's greater than 35 dB we do not need to worry very much about the possible difficulties above 35 dB.

The method of dealing with the above-mentioned problems
was simply to reject those values of \( x \) (and hence \( d \)) that were obviously wrong. This procedure may seem arbitrary to the reader, but it was employed in the absence of any other suitable criterion. (Also, one should not lose sight of the fact that our sole purpose here is to derive an empirical formula that allows us to predict accurately the outcome of our simulations — if the formula we end up with does the job well, then the means by which we obtained it are justified a posteriori.) Of the 168 divisors computed, corresponding to the 168 data points \( P_s \), it was decided to reject 30, leaving 138 divisors for subsequent analysis. In Table IV the divisors are listed; the values that were rejected have been replaced by dashes. The table has been organized to demonstrate clearly the behavior of \( d \) with the changes in the three parameters \((N, f_D, \gamma_0)\) on which it depends.

It is immediately evident from the table that the divisor changes little with either the block size \( N \) or the Doppler frequency \( f_D \), but does change substantially with the average signal-to-noise ratio \( \gamma_0 \). For a fixed \( \gamma_0 \) the divisor seems to decrease slightly with increasing \( N \) and \( f_D \), although this trend is often interrupted by what appear to be statistical fluctuations. There is a much stronger trend in the change of \( d \) with \( \gamma_0 \). Since we desire an empirical formula that is as simple as possible, and since the statistical fluctuations just mentioned could make it hard to characterize correctly the dependence of \( d \) on \( N \) and \( f_D \), it was decided to regard \( d \) as independent of \( N \) and \( f_D \), and to attempt to find an approximate
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Table IV: Divisors
relationship between $d$ and $\gamma_0$ alone:

$$d = F(A_1, A_2, ..., A_k; \gamma_0). \quad (3.2.3)$$

In equation (3.2.3), the coefficients $A_i$ are fixed parameters in the functional relationship $F$ that holds between $d$ and $\gamma_0$. The procedure carried out to determine a suitable $F$ and the numerical values of the parameters was as follows. First, a specific functional form for $F$ was posited on the basis of graphs. Then, the parameters $A_i$ were chosen to obtain a least squares fit to the 138 data points. To make the formulation more precise, let $(\gamma_{0i}, d_i)$ denote the data points for $i = 1, 2, ..., 138$. We then seek the parameters $A_i$ that minimize the expression

$$\sum_{i=1}^{138} r_i^2 = \sum_{i=1}^{138} [F(A_1, ..., A_k; \gamma_{0i}) - d_i]^2 \quad (3.2.4)$$

where $r_i = F(A_1, ..., A_k; \gamma_{0i}) - d_i$ is called the residual. Two different computer programs, both of which were made available by the UBC Computing Centre, were used to carry out the least squares fit. The first program, NL2SOL, was best suited for the least squares minimization, as it required only the coding of $F$ and its first partial derivatives with respect to the parameters $A_i$. The second program, DGRADX, was a program for maximizing general nonlinear functions, and was not tailored specifically to the least squares problem. Thus to minimize the summation appearing in (3.2.4), it was necessary to code the negative of the summation, along with its first and second partial derivatives with respect to the parameters. As a check
on the numerical results, both programs were run for each functional form $F$ that was tried. The results were almost identical in all cases (the differences were so small as to be negligible to the accuracy that we were working).

To show what the function $F$ must look like, the divisor has been graphed in Figure 3.8 as a function of $\gamma_0$ for the specific values $N = 255$ and $f_D = 10.5$ Hz. The shape of the curve, which is much the same for the other block sizes and Doppler frequencies considered here, suggests a parabola of some sort. Two different functional forms were tried:

(a) \[ F(A_1, A_2, A_3; \gamma_0) = A_1 + A_2 \gamma_0^3 \]

The units of $\gamma_0$ were taken to be decibels. The computed values of the parameters were:

\begin{align*}
A_1 &= 5.148 \\
A_2 &= 2.237 \times 10^{-5} \\
A_3 &= 4.102.
\end{align*}

With this choice of parameters, the minimized value of (3.2.4) was 578.0.

(b) \[ F(A_1, A_2, A_3; \gamma_0) = A_1 + A_2 \exp(A_3 \gamma_0) \]

The computed values of the parameters were:

\begin{align*}
A_1 &= 2.850 \\
A_2 &= 6.435 \times 10^{-1} \\
A_3 &= 1.248 \times 10^{-1}.
\end{align*}

The minimized value of (3.2.4) was 557.9.

The exponential curve in (b) provides a slightly better fit than does the power law in (a). Both curves are plotted in Figure 3.9, along with the data shown in Figure 3.8. Clearly
Figure 3.8: Plot of the divisor as a function of $\gamma_0$ at $N = 255$ and $f_D = 10.5$ Hz.
Figure 3.9: Plots of the curves that have been fitted to the data in table IV.
the fit is good no matter which one of the two functional forms we use.

At this point we have completed the derivation of our empirical formula. A computer program was written to evaluate (3.2.1), using the exponential fit in (b) above to find the divisor for a specified average SNR $\gamma_0$. The results produced by the program were compared with the simulation results for all 168 combinations of the parameters $N$, $f_D$, and $\gamma_0$; the comparison showed that formula (3.2.1) accurately predicts the simulator output $P_s(0,N)$. To measure quantitatively the accuracy of the empirical formula, the average of the absolute value of the percentage error was taken over the 168 values of $P_s(0,N)$ computed by the simulation program. The calculated average was only about 1.7%, indicating the high accuracy of the predicted values. Note that the average was taken over all 168 probabilities even though the empirical formula was based on a subset of 138 probabilities. It is, of course, the purpose of the empirical formula to allow us to predict what the output of the simulation program would be for parameter values different from those we have already used. If we make the reasonable assumption that $P_s$ depends smoothly on the parameters $N$, $f_D$, and $\gamma_0$, we can conclude on the basis of the above results that the empirical formula derived here will accurately predict the output of the simulator over a wide range of parameter values.

Presented now are some graphical examples. Figures 3.10 and 3.11 are similar to Figures 3.4 and 3.5 respectively,
except here the solid lines were plotted using formula (3.2.1). The agreement between the simulator results and the values predicted by formula (3.2.1) is obviously good.
Figure 3.10: $P_o(0.127)$ at different Doppler frequencies.

Average signal-to-noise ratio $\gamma$ in dB.

Probability of block error $P_b(0.127)$. Results simulator.

- $\circ = 25.5$
- $\square = 20.5$
- $\times = 15.5$
- $\diamond = 10.5$ Hz
Figure 3.11: $P_s(0,N)$ for different block sizes at 15.5 Hz using the empirical formula.
4.1 Conclusions

In this thesis, several methods for computing the probability of block error on a Rayleigh fading channel in the presence of additive white Gaussian noise were examined. Chapter 2 focussed on the special case of very slow fading; Chapter 3 considered the effects of relaxing the assumption of very slow fading.

The results of Chapter 2 show that when the fading is very slow and the modulation technique is non-coherent FSK, numerical methods exist that allow the accurate approximation of the block-error probability $P_f(M,N)$. An important conclusion is that the computation of $P_f(M,N)$ for $M > 0$ can be reduced approximately to the computation of $P_f(0,N)$ and that of a multiplicative correction factor. Furthermore, rigorous error bounds show that the error incurred by making these approximations is negligible over a wide range of parameter values. Also considered in Chapter 2 is the computation of the block-error rate when L-branch selection diversity is employed; the formulas derived are useful for small values of $L$. It is shown that the probability of block error, considered as a function of the average signal-to-noise ratio $\gamma_0$, falls off as $\frac{1}{\gamma_0^L}$.

In Chapter 3 a more realistic channel is assumed, one in which the SNR is not necessarily constant over an entire block. The block-error rate must now be estimated by simulation; the simulations carried out for this thesis were performed by
digital computer. The statistical properties of the simulator were chosen to be those of the mobile-radio environment. The results produced by the simulator had largely the qualitative properties that one intuitively expects, e.g. the faster the fading, the worse the system performance. However, the simulation results were not considered to be an end in themselves; the primary function of the simulator was to produce a sufficient amount of data to test a specific model of the fading channel when very slow fading is not assumed. A formula based on this model was shown to be successful in predicting the simulator output. Thus the formula of Chapter 3 provides a simple method of computing a more realistic value of the probability of block error than is available under the assumption of very slow fading.
4.2 Future work

There are several areas in which the work of this thesis could be extended. Almost all of the formulas derived here are for the case of non-coherent FSK modulation; this modulation technique was considered because it yields the most tractable equations and because it is the modulation scheme analysed most often in the relevant literature (of course, the second reason mentioned is probably a direct consequence of the first). Attempts should be made to find useful approximations when other modulation techniques are used, although this may be difficult.

Furthermore, the results of Chapter 3 should be extended to cover the case $M > 0$ as well. It is not clear whether this would entail significantly more complex equations, or whether this extension could be accomplished in a straightforward manner.
References


Appendix A

This appendix details the computation of $P_f(M,N)$ using numerical integration. First consider the calculation of

$$P_f(0,N) = 1 - \gamma_0^{-1} \int_0^\infty [1 - p(\gamma)]^N e^{-\gamma/\gamma_0} \, d\gamma \quad (A.1)$$

$$= \gamma_0^{-1} \int_0^\infty [1 - [1 - p(\gamma)]^N] e^{-\gamma/\gamma_0} \, d\gamma,$$

where $p(\gamma) = (1/2) \exp(-\gamma/2)$ for non-coherent FSK. Making the substitution $y = \gamma/\gamma_0$ in (A.1) gives

$$P_f(0,N) = \int_0^\infty [1 - [1 - p(\gamma_0 y)]^N] e^{-y} \, dy. \quad (A.2)$$

Equation (A.2) is the form of the integral that was used for numerical integration. Note that the range of integration in (A.2) is infinite, making the application of the usual quadrature rules difficult. However, the rapidly decreasing exponential in the integrand suggests that there is little contribution from the "tail" of the integral, and this is indeed the case. Thus, given a maximum allowable error $\Delta$, the strategy was to find a number $b$ (depending on $\Delta$, $\gamma_0$, and $N$) such that

$$0 < \int_b^\infty [1 - [1 - p(\gamma_0 y)]^N] e^{-y} \, dy < \Delta, \quad (A.3)$$

and then to compute

$$P_f(0,N) \approx \int_0^b [1 - [1 - p(\gamma_0 y)]^N] e^{-y} \, dy. \quad (A.4)$$

(We ignore for the moment the error involved in the numerical integration of the finite integral in (A.4).) The above
strategy may appear to be trivial, because the convergence of the integral (A.2) guarantees that

$$\lim_{b \to \infty} \int_b^\infty \{1 - [1 - p(\gamma_0 y)]^N\} e^{-y} \, dy = 0,$$

and so given any $\Delta > 0$ there always exists a number $b$ such that

$$\int_b^\infty \{1 - [1 - p(\gamma_0 y)]^N\} e^{-y} \, dy < \Delta.$$  

The point here is that the rapid decrease of the integrand ensures that $b$ can be chosen sufficiently small to make the numerical integration over the finite interval in (A.4) feasible. Just as important is the fact that there is a simple procedure for finding $b$, as we now show. Observing that for fixed $N$ and $\gamma_0$ the function $1 - [1 - p(\gamma_0 y)]^N$ is a monotonically decreasing function of $y$, we have

$$0 < \int_b^\infty \{1 - [1 - p(\gamma_0 y)]^N\} e^{-y} \, dy$$

$$\leq \{1 - [1 - p(\gamma_0 b)]^N\} \int_b^\infty e^{-y} \, dy \quad (A.5)$$

$$= \{1 - [1 - p(\gamma_0 b)]^N\} e^{-b}.$$ 

Now, given any $\Delta > 0$, if we choose $b$ such that

$$\{1 - [1 - p(\gamma_0 b)]^N\} e^{-b} < \Delta, \quad (A.6)$$

then we know that (A.3) holds, and that the error in (A.4) is less than $\Delta$. Note that the last line of (A.5) is just the integrand of (A.4) evaluated at $y = b$. Thus the formulation (A.4) is convenient for use on a computer, because once we have coded the integrand for the numerical integration, we can use
this same code as part of a simple search technique to determine a value of b that makes (A.6) true.

The actual program used for the numerical integration was a double-precision version of SQUANK, which is an adaptive quadrature routine based on Simpson's rule. This program was made available by the UBC Computing Centre. SQUANK is easy to use, because it adaptively tries to meet an error tolerance supplied by the user. Thus if the user specifies a maximum error $\Delta$, SQUANK will come back with an estimate of the integral within $\Delta$ of the true value (the program indicates if it cannot meet the error tolerance supplied by the user).

Combining all the above gives us the following procedure: First a $\Delta > 0$ is chosen, and then a value of b is found such that (A.6) is satisfied. This same $\Delta$ is then supplied to SQUANK as the maximum error tolerance for the integration of (A.4). If SQUANK is able to meet the specified error tolerance, we are then assured that $P_f(0,N)$ has been computed with a total error less than $2\Delta$. In the calculations performed for this thesis, the value of $\Delta$ was 1.0x10^{-9}. Reference to Figure 2.3 shows that the smallest values of $P_f(0,N)$ that we have to deal with are on the order of 10^{-8}. Therefore, in those cases where SQUANK is able to meet the specified error tolerance, we are guaranteed that $P_f(0,N)$ has been computed correctly within at least two digits in the fifth decimal place.

The details of computing $P_f(M,N)$ using numerical integration when $M > 0$ are much the same as when $M = 0$, and are not described here.
Appendix B

B.1 No diversity

In Section 2.2 we defined the quantities

\[ Q(m,x;r) = \binom{x}{m} r^{m} B_{\frac{1}{2}}(m+r,1+x-m) \]

and

\[ \hat{Q}(m,x;r) = \binom{x}{m} r^{m} B(m+r,1+x-m), \]

where the incomplete Beta function \( B_y(a,b) \) is given by

\[ B_y(a,b) = \int_{0}^{y} t^{a-1} (1 - t)^{b-1} \, dt, \]

and \( B(a,b) = B_1(a,b) \). The purpose of this appendix is to derive a bound on the difference between \( Q \) and \( \hat{Q} \). For the rest of this appendix we assume that the parameters \( r, x, \) and \( m \) satisfy the following conditions:

(a) \( r \) is real, and \( 0 < r < 1 \). Since \( r = 2/\gamma_0 \), the restriction \( 0 < r < 1 \) implies that \( 2 < \gamma_0 < \infty \). Thus this restriction sacrifices very little in a practical sense, but it makes the analysis simpler because there are fewer special cases to consider.

(b) \( x \) is real, and \( x > 1 \).

(c) \( m \) is an integer, and \( 0 \leq m < x \).

Let \( \epsilon(m,x;r) = \hat{Q}(m,x;r) - Q(m,x;r) \). Then using the definitions (B.1) and (B.2), we have
\[ \epsilon(m,x;r) = \binom{x}{m} r 2^r \int_{\frac{1}{2}}^{1} t^{m+r-1} (1-t)^{x-m} \, dt > 0. \] (B.3)

We consider the two cases \( m = 0 \) and \( m \geq 1 \) separately. The reason for breaking the problem into these two cases is because the exponent of \( t^{m+r-1} \) is negative for \( m = 0 \) and is positive for \( m \geq 1 \).

**CASE 1:** \( m = 0 \).

For \( m = 0 \) equation (B.3) reduces to

\[ \epsilon(0,x;r) = r 2^r \int_{\frac{1}{2}}^{1} t^{r-1} (1-t)^x \, dt. \]

Since \( r - 1 < 0 \), it follows that \( t^{r-1} \leq (1/2)^{r-1} = 2^{1-r} \) for \( t \in [1/2,1] \), i.e. for \( t \) in the range of integration. Then

\[ 0 < \epsilon(0,x;r) \leq r 2^r 2^{1-r} \int_{\frac{1}{2}}^{1} (1-t)^x \, dt \]

\[ = 2r \int_{\frac{1}{2}}^{1} (1-t)^x \, dt. \] (B.4)

Making the substitution \( u = 1-t \) in the last integral, we get

\[ \int_{\frac{1}{2}}^{1} (1-t)^x \, dt = \int_{0}^{\frac{1}{2}} u^x \, du = \frac{1}{2^{1+x}(1+x)}. \] (B.5)

Substitution of (B.5) into (B.4) yields

\[ 0 < \epsilon(0,x;r) \leq \frac{r}{2^x (1+x)}. \] (B.6)
CASE 2: \( m \geq 1 \).

In this case \( m + r - 1 > 0 \), and so for \( t \in [1/2,1] \) the inequality \( t^{m+r-1} \leq 1 \) holds. Then (B.3) becomes

\[
0 < \epsilon(m,x;r) = \binom{x}{m} r^r \int_{1/2}^{1} t^{m+r-1} (1 - t)^{x-m} \, dt \\
\leq \binom{x}{m} r^r \int_{1/2}^{1} (1 - t)^{x-m} \, dt \\
= \binom{x}{m} r^r \int_{0}^{1/2} u^{x-m} \, du \\
= \binom{x}{m} \frac{r^r}{2^{1+x-m}(1+x-m)}. \tag{B.7}
\]

Summarizing inequalities (B.6) and (B.7),

\[
0 < \epsilon(m,x;r) \leq \begin{cases} 
\frac{r}{2^x(1+x)}, & m = 0 \tag{B.8a} \\
\binom{x}{m} \frac{r^r}{2^{1+x-m}(1+x-m)}, & m \geq 1. \tag{B.8b}
\end{cases}
\]

Of course, it would be nice to have a single bound for the two cases \( m = 0 \) and \( m \geq 1 \). We can obtain such a bound by making the simple observation that the inequality \( 2^r < 2 \) holds for \( 0 < r < 1 \), and so it follows from (B.8b) that

\[
0 < \epsilon(m,x;r) \leq \binom{x}{m} \frac{r}{2^{x-m}(1+x-m)} \tag{B.9}
\]

for \( m \geq 1 \). Note that (B.9) reduces to (B.8a) in the case \( m = 0 \). Thus inequality (B.9) holds for \( m \geq 0 \). The price we pay for the convenience of a single formula that holds for \( m \geq 0 \) is that the bound for \( m \geq 1 \) is worse. However, the bound in (B.9) is larger by at most a factor of 2, and as the examples in Section
2.2 demonstrate, the bound in (B.9) is usually so small that a factor of 2 will not influence our opinion of whether or not the difference between \( Q \) and \( \hat{Q} \) is negligible.

B.2 Diversity

We want to obtain a bound on the difference between \( P_{d,c} \) and \( \hat{P}_{d,c} \), which are defined by equations (2.4.7) and (2.4.8) respectively. It is easy to see that replacing the incomplete Beta function in (2.4.7) by the Beta function is equivalent to extending the upper limit of integration in (2.4.4) from 1/2 to 1. Thus we may write

\[
\hat{P}_{d,c} - P_{d,c} = \]

\[
= L r \sum_{m=0}^{\infty} \binom{M}{m} \int_{\frac{1}{2}}^{1} (1 - t)^{x-m} [1 - (2t)^r]^{L-1} dt,
\]

where \( r \) and \( x \) are as in Section B.1 above, and \( L \geq 1 \) is an integer. For \( L = 1 \) the integral reduces to that of the non-diversity case, and we can apply the results of Section B.1.

For the case \( L > 1 \), we can bound the \([1 - (2t)^r]^{L-1}\) factor of the integrand in such a way that the results of Section B.1 are still applicable. We start by noting that the inequality \((1 + y)^r < 1 + ry\) holds for \(-1 < y \neq 0 \) and \(0 < r < 1\) (see, for example, [15]). Taking \( y = 1 \), we have \(2^r < 1 + r\). Then if \(0 < r < 1\) and \(1/2 \leq t \leq 1\), we have \((2t)^r \leq 2^r < 1 + r\), and hence \(|1 - (2t)^r| < r\). It now follows that
\[ \left| L r^{2r} \int_{ \frac{1}{2} }^{1} t^{m+r-1} (1 - t)^{x-m} \left[ 1 - (2t)^{r} \right]^{L-1} dt \right| \]

\[ \leq L r^{L} 2^{r} \int_{ \frac{1}{2} }^{1} t^{m+r-1} (1 - t)^{x-m} dt \]

\[ \leq \frac{L r^{L}}{2^{x-m} (1+x-m)}. \]

The last inequality above follows from (B.9). Using the preceding result with (B.10) yields

\[ |P_{d,c} - \hat{P}_{d,c}| \leq \sum_{m=0}^{M} \binom{x}{m} \frac{L r^{L}}{2^{x-m} (1+x-m)}. \]  

(B.11)
Appendix C

The purpose of this appendix is to present the mathematical derivations of the results given in Section 2.3. We shall without mention use the notation of the thesis proper, with the exception that we shall replace the quantity \( r = 2/\gamma_0 \) by the complex variable \( z \) throughout this appendix.

C.1 Existence of the expansions

We begin by showing that \( \tilde{Q}(m,x;z) \) and \( \hat{Q}(m,x;z) \), which are defined by

\[
\tilde{Q}(m,x;z) = \binom{x}{m} z 2^z B_{\frac{1}{2}}(m+z,1+x-m)
\]

and

\[
\hat{Q}(m,x;z) = \binom{x}{m} z 2^z B(m+z,1+x-m),
\]

have series expansions

\[
\tilde{Q}(m,x;z) = \sum_{i=0}^{\infty} q_i(m,x) z^i
\]

and

\[
\hat{Q}(m,x;z) = \sum_{i=0}^{\infty} \hat{q}_i(m,x) z^i
\]

that hold for \(|z| < 1\) (the region of convergence may in actuality be larger, but this does not concern us). In equations (C.1) we define \( 2^z \) by \( 2^z = \exp\{z \log(2)\} \), where \( \log(2) \) is real; this same definition will be used if \( 2 \) is replaced by any real number. By the definition of the incomplete Beta function, we have
\[ B_y(m+z,1+x-m) = \int_0^y t^{m+z-1} (1 - t)^{x-m} \, dt , \quad (C.3) \]

where it is assumed that \( m \) is an integer satisfying \( 0 \leq m < x \), and that \( y \) is a real number in the range \( 0 < y \leq 1 \). The integral (C.3) converges for \( \text{Re}(z) > -m \), and defines an analytic function of \( z \) there. Thus, since \( z^2 \) is entire, it follows immediately that \( Q(m,x;z) \) and \( \hat{Q}(m,x;z) \) are analytic functions of \( z \) for \( \text{Re}(z) > -m \), and so if \( m \geq 1 \) these functions are analytic in the unit disk \( D = \{ z : |z| < 1 \} \). Since the analytic functions are precisely those functions that have power series expansions, and vice versa, it follows that (C.2) holds when \( m \geq 1 \). When \( m = 0 \) the issue is not so simple, because then \( Q(m,x;z) \) and \( \hat{Q}(m,x;z) \) as given by (C.1) are defined and analytic only for \( \text{Re}(z) > 0 \), a region which obviously does not contain the unit disk. Thus we must look at the cases

\[ Q(0,x;z) = z^2 B_y(z,1+x) \]
\[ \hat{Q}(0,x;z) = z^2 B(z,1+x) \]

in more detail. Although we do not prove it here, it can be shown that the following expression holds for \( \text{Re}(z) > 0 \):

\[ B_y(z,1+x) = y^k \sum_{k=0}^{\infty} \binom{x}{k} (-1)^k \frac{y^k}{z + k} . \quad (C.4) \]

It can further be shown that the right-hand side of (C.4) defines a function that is analytic for the entire complex plane, except for the points \( z = 0, -1, -2, \ldots \) Thus the right-
hand side of (C.4) is the unique analytic extension of the left-hand side. Using (C.4) it is easy to see that $z 2^z B_y(z,1+x)$ has a removable singularity at $z = 0$; to be precise, we may write

$$z 2^z B_y(z,1+x) = (2y)^z \left[ 1 + \sum_{k=1}^{\infty} \binom{x}{k} (-1)^k \frac{y^k}{z+k} \right], \quad (C.5)$$

where the right-hand side is analytic for the entire complex plane, except for the points $z = -1, -2, \ldots$. Taking $y = 1/2$ and $y = 1$, we can use the right-hand side of (C.5) to analytically extend $Q(0,x;z)$ and $Q(0,x;z)$ to this same region, i.e. to a region containing D. Thus the expansions in (C.2) exist even for $m = 0$.

C.2 Expressions for the coefficients in (2.3.15)

We now derive expressions for the coefficients $b_i(m,x)$ in equation (2.3.15) [the derivation also proves the validity of this expansion]. For $0 \leq m < x$, we have from equation (2.2.28) that

$$Q(m,x;z) = z 2^z \frac{\Gamma(1+x) \Gamma(m+z)}{\Gamma(1+x+z) \Gamma(1+m)}. \quad (C.6)$$

If we omit $m = 0$ from the range considered, we can write

$$Q(m,x;z) = \frac{z 2^z \Gamma(1+x) \Gamma(m+z)}{m \Gamma(1+x+z) \Gamma(m)}. \quad (C.7)$$

for $1 \leq m < x$. For convenience, we let

$$f(m,x;z) = 2^z \frac{\Gamma(1+x) \Gamma(m+z)}{\Gamma(1+x+z) \Gamma(m)}. \quad (C.8)$$

for $1 \leq m < x$. In (C.8) the complex variable $z$ is viewed as the
main independent variable, whereas \( x \) and \( m \) are considered to be parameters that are held fixed during the discussion. Thus we often write simply \( f(z) \) for \( f(m,x;z) \), and any differentiation that occurs is understood to take place with respect to \( z \). Comparison of (C.7) and (C.8) shows that

\[
\hat{Q}(m,x;z) = \left( \frac{z}{m} \right) f(m,x;z) \quad (C.9)
\]

for \( m \geq 1 \); hence in order to find a representation of the form (2.3.15) we want to write \( f \) in the form

\[
f(m,x;z) = \exp \left[ \sum_{i=0}^{\infty} b_i(m,x) z^i \right]. \quad (C.10)
\]

If we put \( m = 0 \) into (C.6), we find

\[
\hat{Q}(0,x;z) = z 2^z \frac{\Gamma(1+x) \Gamma(z)}{\Gamma(1+x+z) \Gamma(1)} = 2^z \frac{\Gamma(1+x) \Gamma(1+z)}{\Gamma(1+x+z)}
\]

\[
= f(1,x;z). \quad (C.11)
\]

The last equation shows that (2.3.16) holds.

We now derive formulas for the coefficients \( b_i(m,x) \) in (C.10). It is understood that for the duration of the discussion \( x \) represents a real number satisfying \( x > 1 \) and \( m \) represents an integer satisfying \( 1 \leq m < x \). Since \( \Gamma(m+z) \) is analytic in the complex \( z \)-plane except at \( z = -m, -m-1, -m-2, \ldots \), and both \( 2^z \) and \( 1/\Gamma(1+x+z) \) are entire, \( f(z) \) is analytic except at \( z = -m, -m-1, -m-2, \ldots \). In particular, \( f(z) \) is analytic in the unit disk \( D = \{ z : |z| < 1 \} \), and it is this region to which we shall restrict our attention. Also, \( f(z) \) has no zeros in \( D \) (its zeros are located at \( z = -x-1, -x-2, -x-3, \ldots \)). We apply the following result (see Corollary 4.16, Ch. IV
of [8]): Let \( G \) be a simply connected region of the complex plane \( \mathbb{C} \), and let \( f: G \to \mathbb{C} \) be an analytic function such that \( f(z) \neq 0 \) for any \( z \) in \( G \). Then there is an analytic function \( g: G \to \mathbb{C} \) such that \( f(z) = \exp(g(z)) \). If \( z_0 \in G \) and \( \exp(w_0) = f(z_0) \), we may choose \( g \) such that \( g(z_0) = w_0 \).

Since \( D \) is simply connected, and \( f \) satisfies the hypotheses of the above theorem, there exists a function \( g(z) = g(m,x;z) \), analytic in the unit disk, such that

\[
f(z) = \exp(g(z)).
\]  

(C.12)

Let \( z_0 = w_0 = 0 \). Then \( \exp(w_0) = 1 = f(z_0) \), and we may choose \( g(z) \) such that \( g(0) = 0 \). Applying the chain rule to (C.12), we find

\[
f'(z) = \exp(g(z)) g'(z) = f(z) g'(z)
\]

where, of course, the prime indicates differentiation. Since \( f(z) \) has no zeros in \( D \), we may write

\[
g'(z) = \frac{f'(z)}{f(z)}.
\]  

(C.13)

Since \( g \) is analytic in \( D \), it has a Taylor series

\[
g(z) = \sum_{i=0}^{\infty} b_i(m,x) z^i
\]  

(C.14)

valid for \( |z| < 1 \). The coefficients \( b_i(m,x) \) are given by

\[
b_i(m,x) = \frac{g^{(i)}(0)}{i!} = \frac{g^{(i)}(m,x;0)}{i!}
\]  

(C.15)

where \( g^{(i)}(z) \) is the \( i \)-th derivative of \( g \) with respect to \( z \), with the special cases \( g^{(0)}(z) = g(z) \) and \( g^{(1)}(z) = g'(z) \). Since \( g(0) = 0 \) we immediately have that \( b_0(m,x) = 0 \). To find
the higher-order coefficients we substitute (C.8) into (C.13) to get

\[ g'(z) = \log 2 + \frac{\Gamma'(m+z)}{\Gamma(m+z)} - \frac{\Gamma'(1+x+z)}{\Gamma(1+x+z)}. \]  

(C.16)

Using the Psi function, equation (C.16) may be written

\[ g'(z) = \log 2 + \Psi(m+z) - \Psi(1+x+z). \]  

(C.17)

By induction, we have, for \( i = 2, 3, 4, \ldots \)

\[ g^{(i)}(z) = \Psi^{(i-1)}(m+z) - \Psi^{(i-1)}(1+x+z), \]  

(C.18)

and so from (C.15) we find

\[ b_{i}(m,x) = \log 2 + \Psi(m) - \Psi(1+x) \]  

(C.19)

\[ b_{i}(m,x) = [\Psi^{(i-1)}(m) - \Psi^{(i-1)}(1+x)]/i! \quad (i \geq 2). \]

These are the results stated in equation (2.3.17).

C.3 The asymptotic expansion of the coefficients

We show how the asymptotic expansions for the coefficients \( b_{i}(m,x) \) are derived. We look at two cases: \( i = 1 \) and \( i \geq 2 \).

CASE 1: \( i = 1 \).

When \( i = 1 \), we have from (C.19) that

\[ b_{1}(m,x) = \log 2 + \Psi(m) - \Psi(1+x). \]

From entry 6.3.2 of [1] we find that
\[ \psi(m) = -C + H_{m-1} \quad (C.20) \]

for \( m = 1, 2, 3, \ldots \). We could use this same formula to write
\[ \psi(1+N) = -C + H_N \]
when \( x = N \) is an integer, but in keeping with our intention of developing formulas that are valid for arbitrary real block sizes, we shall take a different approach and shall instead use an asymptotic expansion for \( \psi(1+x) \). (Our general approach is to view the block size \( x \) as being so large that asymptotic expansions are called for, whereas we view \( m \) as being sufficiently small that it is feasible to compute directly summations involving \( m \) terms.) We first write
\[ \psi(1+x) = \psi(x) + 1/x \quad (C.21) \]
(this immediately follows from the recurrence formula \( \Gamma(z) = \Gamma(1+z) \) and the definition of \( \psi \)). In [1] we find the following asymptotic expansion for \( \psi(z) \):
\[ \psi(z) \sim \log(z) - 1/2z - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n z^{2n}} \quad (C.22) \]
\[ = \log(z) - 1/2z - 1/12z^2 + 1/120z^4 - 1/252z^6 + \ldots \]
as \( z \to \infty \). Using the result [1]
\[ |B_{2n}| \sim 2(2n)!/(2\pi)^{2n}, \quad (C.23) \]
which holds for large \( n \), it is easy to see that the infinite series in (C.22) does not converge; nevertheless, the notation is the standard notation for an asymptotic series. Combining (C.21) and (C.22) yields
\[
\psi(1+x) \sim \log(x) + 1/2x - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n \ x^{2n}}
\]

\[
= \log(x) + 1/2x - 1/12x^2 + 1/120x^4 - 1/252x^6 + \ldots
\]

and finally

\[
b_i(m,x) = \log 2 - C + H_{m-1} - \psi(1+x)
\]

\[
\sim H_{m-1} - [\log(x/2) + C + 1/2x - 1/12x^2 + 1/120x^4 - \ldots].
\]

This is the result given in (2.3.19).

**CASE 2:** \( i \geq 2. \)

We consider

\[
b_i(m,x) = \left[ \psi^{(i-1)}(m) - \psi^{(i-1)}(1+x) \right]/i!
\]

From entry 6.4.3 of [1] we find

\[
\psi^{(i-1)}(m) = (-1)^i(i-1)! [ \xi(i) - H_{m-1}^{(i)} ]
\]

for \( m = 1, 2, 3, \ldots \) and \( i \geq 2. \) Note that the previous equation does not make any sense for \( i = 1, \) since \( \xi(1) \) does not exist.

We again find an asymptotic expansion for the term in \( x, \) starting by repeatedly differentiating (C.21) to get

\[
\psi^{(i-1)}(1+x) = \psi^{(i-1)}(x) + (-1)^{i+1} (i-1)!/x^i.
\]

Entry 6.4.11 of [1] gives the asymptotic expansion
\[ \psi(i-1)(z) \sim (-1)^i \left[ \frac{(i-2)!}{z^{i-1}} + \frac{(i-1)!}{2z^i} + \sum_{n=1}^{\infty} \frac{B_{2n}(2n+i-2)!}{(2n)! z^{2n+i-1}} \right] \]

as \( z \to \infty \). Combining the last three equations gives

\[ b_i(m,x) \sim (-1)^{i-1} \left\{ \delta(i) - H_{m-1}^i \right\} - \frac{1}{(i-1)x^{i-1}} \frac{1}{2x^i} + \frac{1}{(i-1)!} \sum_{n=1}^{\infty} \frac{B_{2n}(2n+i-2)!}{(2n)! x^{2n+i-1}} \} \]

This is the formula given in (2.3.20).

C.4 Derivation of the recurrence formula

We now prove that the recurrence formula (2.3.23) holds. Based on equations (2.3.22) and (C.10) we may write

\[ f(z) = \sum_{i=0}^{\infty} a_i z^i = \exp\left( \sum_{i=1}^{\infty} b_i z^i \right), \quad (C.27) \]

where the parameters \( m \) and \( x \) have not been indicated [see the note after (2.3.21)]. Since \( f(z) \) in analytic at least in the unit disk \( D \), the leftmost power series in (C.27) converges for at least \( |z| < 1 \) (we have already seen that the other power series in (C.27) converges for at least this same region). There are several ways to find formulas for the \( a_i \) in terms of the known coefficients \( b_i \), the most obvious of which is to use the series expansion of \( \exp(\cdot) \) to rearrange the right-hand side of (C.27) into ascending powers of \( z \) and then to equate coefficients. The disadvantage of this method is that it rapidly becomes tedious. A better method begins by recalling that
\[ f'(z) = f(z) g'(z), \quad (C.28) \]

where \( g(z) \) has the series expansion given in equation (C.14).

Since a power series may be differentiated term-by-term in its disk of convergence, the substitution of the series expansions for \( f(z) \) and \( g(z) \) into (C.28) yields

\[
\sum_{i=1}^{\infty} a_i z^{i-1} = \left[ \sum_{i=0}^{\infty} a_i z^i \right] \left[ \sum_{i=1}^{\infty} b_i z^{i-1} \right]. \quad (C.29)
\]

Multiplying both sides of (C.29) by \( z \), we may write

\[
\sum_{i=1}^{\infty} a_i z^i = \sum_{i=1}^{\infty} c_i z^i \quad (C.30)
\]

where \( c_i \) is given by the Cauchy product formula

\[
c_i = \sum_{k=1}^{i} k b_k a_{i-k} = \sum_{k=0}^{i-1} (i-k) a_k b_{i-k}. \quad (C.31)
\]

Taking the Cauchy product is allowed because a power series converges absolutely in its region of convergence. Equating coefficients in (C.30) and using (C.31), we obtain the following recurrence formula:

\[
a_i = c_i / i = i^{-1} \sum_{k=1}^{i} k b_k a_{i-k}. \quad (i \geq 1) \quad (C.32)
\]

To "start" the recurrence formula we need the value of \( a_0 \), a value which can easily be obtained by putting \( z = 0 \) in (C.27) to find \( a_0 = 1 \).