PREFERENCES, ENDOWMENTS AND BELIEFS
AS REVEALED IN MARKET PRICES

by

GORDON ARTHUR SICK

B.Sc., The University of Calgary, 1971
M.Sc., The University of Toronto, 1972
M.Sc., The University of British Columbia
(Business Administration), 1977

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Faculty of Commerce and Business Administration

Department of Division of Finance

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date November 26/80
Abstract

This thesis examines conditions under which prices signal information about agents' preferences, endowments and/or probability information. In a multi-period economy, this information is important because it helps agents to make inferences about future prices. In a single period economy, this is important because, even if agents are only interested in other agents' probability information, it is important for them to be able to distinguish its effect on prices from the effects of preferences and endowments on prices.

Several exchange economy models are constructed to see under what conditions a fully informing rational expectations equilibrium (FRE) exists in which the relevant information is revealed by prices. One class of models is in a two period state preference setting in which preferences exhibit linear risk tolerance (so that aggregate preferences exist). It is shown that a FRE exists that reveals aggregate preference parameters. In another two period state preference model with power utility (in which aggregate preferences do not exist), it is shown that prices generically can reveal local information about the distribution of agents' endowments. In another class of models, in both one period and two period settings with specific distributional assumptions (normal and non-central gamma returns), conditions are found under which prices reveal information about probabilities, aggregate risk preferences and aggregate impatience.
The thesis discusses the notion of a rational expectations equilibrium as a solution of a fixed point problem. It also discusses information in terms of $\sigma$-algebras and partitions of state spaces.
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Chapter 1 Overview

Information and prices

In a dynamic economy, agents will face a sequence of markets for financial assets and consumption goods, and, at any date, will be uncertain about what prices will prevail in those markets at future dates. The uncertainty about future prices stems from uncertainty about exogenous parameters, such as productivity (or the weather), war, arrival of new technology, population shifts, etc. Various agents often have more information about these parameters than others, or agents may simply have different information because, for example, two electronics experts may observe technology breakthroughs in different product markets. At any point in time these types of uncertainty may be classified into two categories. One category relates to what states of the world will occur in the future (e.g., "severe drought in Texas in 1981"). The other category relates to the parameters of the probability distribution of what states of the world will occur (e.g., "expected rainfall in Texas in 1981"). For simplicity, suppose that all agents have the same information and beliefs about what states of the world could occur and that differential information pertains only to differential information about the parameters of the probability distribution ("probability parameters").

Uncertainty about future prices may also stem from uncertainty about agents' preference parameters, such as risk aversion, impatience
and the functional class of agents' utility functions (e.g., exponential, logarithmic). Also, if preferences are not such that an aggregate investor exists, the joint distribution of wealth endowments and preferences also affects (future) prices. Certainly agents have better information about their own preferences than they have about other agents' preferences.

Agents' demands in current markets generally depend upon the information that they have about probability parameters, preferences and endowments. Thus current prices are a function of all of this information and hence reflect, at least to some extent, their information. Agents may then use the endogenous information impounded in prices to refine their own exogenous information. The improved information alters agents' demands and thus alters the way prices depend upon, and hence reflect, all of the agents' information in the first place. Information, demands, and prices are thus involved in a feedback loop, as depicted in Figure 1.1.

\[
\text{information} \rightarrow \text{demands} \rightarrow \text{prices}
\]

Figure 1.1 Information feedback

Because of the feedback, the market may fail to equilibrate. Problems of this sort are discussed in Chapter 2.

The assumption that agents use all available information, including the endogenous information impounded in prices is an
assumption about the rationality of agents. Because of this, the branch of economics dealing with these issues is called the "theory of rational expectations." As discussed in Chapter 2, this involves a confusion between expectations (or beliefs) and information, so that the title is really a misnomer. However, we shall refer to these issues as being about rational expectations, in accordance with the usage in the literature.

The rational expectations literature, which is reviewed briefly in Chapter 2, has generally only dealt with cases where prices convey information about probability parameters, overlooking preference and endowment parameters. This thesis examines models where prices convey information about preferences, endowments, and jointly about preferences and probabilities.

Overview of the following chapters

Chapter 2 provides a general discussion of information and distinguishes information from beliefs. It also defines a rational expectations equilibrium as an equilibrium in an economy where agents use their own exogenous information, as well as the endogenous information impounded in prices. Existence of a rational expectations equilibrium is characterized as a fixed point problem in a function space. Two types of existence problems are discussed, which motivate interest in studying the existence of fully informing
rational expectations equilibria (FRE's). The chapter concludes with a brief survey of some of the micro- and macro-economic literature on rational expectations.

Chapter 3 models several two period state preference economies in which aggregate preferences exist (agents have extended power or log utility, or exponential utility). For such economies, market prices at current and future dates depend on an aggregate risk aversion parameter, so that current prices can reflect the value of the preference parameter, and hence help to resolve some of the uncertainty about future prices. In such markets, FRE's are shown to exist. The strong assumptions made about preferences and market structure enable investors to simplify their initial portfolio choice, however, in such a way that they can form optimal demands in the current market, without having to assess the market's aggregate risk aversion parameter. A model is presented in the Appendix to Chapter 3 in which aggregate preferences exist, but no such simplification is possible for agents, forcing them to infer aggregate risk aversion from current prices.

Chapter 4 studies the question of whether prices can reveal information about agents' endowments in a two period world. When there are I agents, in general there must be at least I prices to convey the information about endowments. The problem of establishing the existence of FRE's that convey multivariate information is more difficult than that of analysing the univariate FRE's of Chapter 3. In this case the results are not constructive and establish
generic existence only. That is, if a FRE does not exist for an economy, a "small perturbation" of the economy yields a FRE, and, if an economy has a FRE, all "sufficiently close" economies have FRE's. The generic existence of FRE's in one-period economies, where probability information is unknown, has been established under certain strong dimensionality assumptions by Allen [1978, 1979]. She uses differential topology to get these results. The results in Chapter 4 study the existence of locally fully informing rational expectations equilibria (LFRE's) only: if the vector of endowments is confined to a suitably small set, prices reveal the endowment vector in equilibrium. By only searching for LFRE's, Chapter 4 establishes generic existence under slightly weaker dimensionality assumptions than Allen, by using only well-known analytic results about implicit functions.

In Chapter 5, one and two period economies with one risky asset are studied to see when prices can reveal information about probability parameters and preference parameters (aggregate risk aversion, and, in the two period economy, impatience). In contrast to the results of Chapter 4, these multivariate information results are constructive. The results are stronger than generic results: they give sufficient, and in some cases, necessary conditions for a FRE to exist. The models are based upon intertemporally additive exponential utility functions and involve two specific probability families for the increments to the social wealth process: normal
and non-central gamma. The conditions for the existence of a FRE typically involve the relationship between agents' rate of impatience and discount bond prices (in the one period models) and the slope of the term structure of interest rates (in the two period models).

Chapter 6 provides some concluding remarks.
Chapter 2  The Rational Expectations Concept

Introduction

This chapter provides a general discussion of rational expectations models, in terms of their definition, existence and general properties. It starts with a general discussion of information, in terms of \( \sigma \)-algebras and partitions in probability spaces, and proceeds to a characterization of rational expectations equilibria as fixed points (in a Banach space). Continuity and measurability problems that create problems for the existence of rational expectations equilibria are discussed. This leads to a study of the existence of fully informing rational expectations equilibria (FRE's), the existence of which is more readily verifiable than the existence of general rational expectations equilibria.

The chapter concludes with a survey of some of the rational expectations models in the literature.

Information, beliefs and rational expectations

Consider a market system for \( S \) goods involving uncertainty about, say, future endogenous or exogenous production levels. Suppose agent \( i \) \((i=1, \ldots, I)\) receives information \( A_i \) about the true value of the uncertain variables. One may think of this in terms of a probability space \( (\Omega, \mathcal{B}, P_i) \), with state space \( \Omega \), \( \sigma \)-algebra \( \mathcal{B} \) and probability measure \( P_i \). Agent i's beliefs are represented by \( P_i \) and his information \( A_i \) is a sub-\( \sigma \)-algebra of \( \mathcal{B} \) upon which he may take conditional expectations.
If the state space $\Omega$ has finitely many elements (states), $B$ is the collection of all subsets of $\Omega$ ($B=2^\Omega$), and $P_i$ is generated by the non-negative probability of each state. Agent $i$'s information $A_i$ may be regarded as a partition of $\Omega$ (actually the partition generates $A_i$) such that agent $i$ can distinguish which member of the partition has occurred, but cannot distinguish which state in the partition member has occurred. For example, if $\Omega = \{1, \ldots, n\}$, suppose $A_i$ is generated by the partition $\{\{1,2\}, \{3\}, \{4,\ldots,n\}\}$. Then, if state 1 occurs, agent $i$ only knows that either state 1 or state 2 occurred. Conditioning on a partition member $B$ involves forming the usual conditional probabilities.

$$P(j|B) = \frac{P(j)}{\Sigma_{k\in B} P(k)} \quad (j \in B).$$

The most important conditional expectations are those used to generate demand functions. Let $\Delta \subseteq \mathbb{R}_+^n$ be a feasible set of prices, where $\mathbb{R}_+^n$ is the non-negative orthant in Euclidean $n$-space. Let $\bar{x}_i(\omega) \in \mathbb{R}_+^S$ be agent $i$'s endowment at $\omega \in \Omega$. The excess demand function:

$$\xi_i: \Delta \times \mathbb{R}_+^S \times \{A_i \subseteq 2^\Omega | A_i \text{ is a sub-}\sigma\text{-algebra of } B\} \to \mathbb{R}_+^S$$

solves, for each $\omega \in \Omega$ and information $A_i$, the expected utility maximization problem. That is,

$$\xi_i(p, \omega, A_i) \equiv \arg \max_{x \in \mathbb{R}_+^S} E_i(U_i(x + \bar{x}_i(\omega)) | A_i).$$

$${\{x \in \mathbb{R}_+^S | p'x = 0\}}$$
Here, "arg max" refers to the maximizing argument of a maximization problem and \( U_i \) is von Neumann-Morgenstern utility. For simplicity, we consider only single-valued demands, with non-satiation. If \( \tilde{x}_i(\omega) \) is a non-constant function of \( \omega \), we require it to be an \( A_i \)-measurable random variable, so that agent \( i \) knows his budget constraint. If utility is sufficiently smooth, etc. then \( \xi_i(p,\omega,A_i) : \Omega \to \mathbb{R}^S \) will be a proper (\( A_i \)-measurable) random variable for all \( p, A_i \). If \( \Omega \) is a finite state space, this means that \( \xi_i(p,\omega,A_i) \) will be constant over members of the partition generating \( A_i \), since agent \( i \) cannot distinguish between states in the same member of the partition generating \( A_i \).

Loosely speaking, for each realization \( \omega \) of the probability state space \( \Omega \), agent \( i \) receives some (but perhaps not all) information about the value of \( \omega \) (he can tell whether or not \( \omega \) is in any given event in \( A_i \), but not whether \( \omega \) is or is not in any smaller events), as well as the value of his endowment. He then solves (2.1) for his (random) excess demand function \( \xi_i(p,\omega,A_i) \).

Fix \( \omega \) and impose the market clearing condition
\[
(2.2) \quad \sum_i \xi_i(p,\omega,A_i) = 0.
\]

This defines the price \( p \), assuming, for simplicity, that equilibrium prices are unique. Set \( p(\omega) = p \). Repeat this for all \( \omega \) to get the market clearing price function \( p \). Under appropriate regularity conditions, \( p \) will be a \( \mathcal{B} \)-measurable function of \( \omega \) and hence a true random variable. Indeed, since \( p \) can only vary according to the information
\{A_i\} communicated to the agents, \(p\) should be \(A\)-measurable where 
\[ A = A_1 \vee A_2 \vee \ldots \vee A_I \] is the smallest \(\sigma\)-algebra containing all the \(A_i\)'s. 
If \(\Omega\) is finite, \(A\) corresponds to the common refinement of all the 
partitions generating the \(A_i\)'s. In some sense, \(A\) is the "social 
endowment" of information.

The important point is that \(p = \rho(\omega)\) varies with the information that (other) individuals receive. A rational individual 
will include the information conveyed by \(\rho\) in selecting his demand 
function. Let \(\rho^{-1}(A)\) denote the \(\sigma\)-algebra generated by \(\rho\) (as defined 
in footnote 2., i.e., the information conveyed by the price function \(\rho\). 
Let \(A_i \vee \rho^{-1}(A)\) be the smallest \(\sigma\)-algebra containing both \(A_i\) and 
\(\rho^{-1}(A)\), i.e., all the information contained in \(A_i\) and \(\rho\). This is 
all the information available to agent \(i\).

This yields the following analogue of the tâtonnement process. 
A rational individual will now have for each \(\omega\), excess demand at 
price \(p\) of \(\xi_i(p, \omega, A_i \vee \rho^{-1}(A))\). Note that now, the random excess de-
mand \(\xi_i\) depends on the whole price function \(\rho : \Omega \to \mathbb{R}^S\), since it 
communicates information, as well as on the price \(p\), since it deter-
mines the budget constraint. Also, \(\xi_i\) depends on \(\omega\) only through the 
information conveyed by \(A_i \vee \rho^{-1}(A)\), so that under appropriate regu-
larity conditions \(\xi_i(p, \omega, A_i \vee \rho^{-1}(A))\) is \(A_i \vee \rho^{-1}(A)\)-measurable.

Now, fix \(\omega\) and impose the market clearing condition

\begin{equation}
\sum_i \xi_i(p, \omega, A_i \vee \rho^{-1}(A)) = 0.
\end{equation}
Set $p'(\omega) = p$. Repeating this for all $\omega \in \Omega$ defines a new price function $p' : \Omega \to \Delta$ which, under appropriate regularity conditions will be $A$-measurable (i.e., random). Of course, since agents have formulated their demands with respect to a new information set $(A_i\nu_\rho^{-1}(A))$, rather than $A_i$, the new price function $p'$ will, in general, be different from $p$. Individuals must now re-revise their expectations using $A_i\nu_\rho'^{-1}(A_i)$ rather than $A_i$ in (2.1). With the market clearing condition, this gives a new price function $p''$, and so on ad infinitum. The question of whether such an analogue of the tâtonnement process will actually converge is, of course, quite difficult and will not be analysed here. The important point is that a fixed point (if it exits) of such a sequence of re-adjustments is called a rational expectations equilibrium.

A rational expectations equilibrium occurs when there exists an $A$-measurable price function (random variable) $\rho : \Omega \to \mathcal{S}_+$ such that if excess demands are defined by (2.1) then

\[(2.4) \quad \Sigma_i \varepsilon_i(\rho(\omega), \omega, A_i\nu_\rho^{-1}(A)) = 0 .\]

(For mathematical rigour, add an appropriate sprinkling of "almost surely.") Thus, agent $i$ uses his own exogenous information $A_i$ and the endogenous information $\rho^{-1}(A)$ conveyed by prices to form random excess demand $\varepsilon_i(\rho(\cdot), \cdot, A_i\nu_\rho^{-1}(A))$, which is $A_i\nu_\rho^{-1}(A)$-measurable.

Note that the price function $\rho$ which agents assume holds in formulating their expectations and demands is actually the one that clears the market. This is the "self-fulfilling" nature of a rational expectations equilibrium. Thus, the consistency that
is implicit in self-fulfilled "expectations" is not really one of beliefs or expectations, for there could be differential beliefs. (\(P_i\) different from \(P_j\), or, even incorrect beliefs \((P_i\) different from Nature's "true" probability law \(P\)). Rational expectations really only imposes a consistency on agents' information \(A_i \mathcal{V} P^{-1}(A)\).

Rational expectations require that agents only forecast the possibility of feasible events— that is, they use the state space \(\Omega\), and have a \(\sigma\)-algebra \(A_i \mathcal{V} P^{-1}(A) \subseteq B\). It also requires that agents correctly use all available information (as in \(A_i\) and \(P^{-1}(A)\)). But, it does not require any homogeneity or correctness of probability beliefs \(P_i\), which are exogenous to the model. In a repeated economy, agents may update \(P_i\) in a Bayesian manner, and over time, \(P_i\) may converge to nature's "true" \(P\). To make rational expectations models yield empirically testable implications, it often seems necessary to add the hypothesis that beliefs are homogeneous and "correct" (coincident with \(P\)). Moreover, if agents have differential beliefs, they must adjust for this in order to infer information from other agents' actions or from prices. For example, suppose a superior stock analyst (characterized by a large \(\sigma\)-algebra \(A_i\), or a very refined partition) is also an incurable optimist and can be only observed to be "buying stock" or "buying lots of stock." A less well-informed agent, but with somewhat more "correct" and pessimistic beliefs, will find it optimal to sell stocks when the analyst merely buys, and to buy stocks when the analyst buys lots of stock.
Clearly, agents may have either homogeneous or differential beliefs while having either homogeneous or differential information. Rational expectations is a misnomer for this theory in that it is really only a theory of the rational use of information, not of beliefs or expectations.

Having opened up the possibility of rational expectations equilibria with heterogeneous beliefs, in what follows, we will assume, for simplicity, that agents have homogeneous beliefs, since assuming heterogeneous beliefs adds many parameters to models and often makes closed form solutions impossible to attain. Heterogeneous beliefs merely cloud the problem of studying differential information.

**Existence of rational expectations equilibria**

With this machinery in place, one can appreciate the difficulties associated with establishing the existence of a rational expectations equilibrium, much less of analyzing problems of stability or dynamics. In standard models of uncertainty without differential information, we have $A_i = A(\pi_i)$ (and $A = B$ in the certainty case), so that the price function conveys no information (i.e., $A_i \varphi^{-1}(A) = A_i = A(\pi_i)$). Existence of equilibrium is established by standard methods, such as in Debreu (1959) or Arrow and Hahn (1971). In this case, price $p$ is a fixed point of a continuous mapping of $\Delta$ into itself, with the property that a fixed point corresponds to a point of zero aggregate excess demand.
For the general rational expectations problem with differential information, we need a whole price random variable $\rho : \Omega \rightarrow \Delta$. That is, we desire a fixed point of some mapping in a function space of random variables, rather than just a fixed point in Euclidean $S$-space. This function space, endowed with an appropriate topology, will be a Banach space. This, in itself, is no problem, for there are fixed point theorems for Banach spaces. The problem is that it may not be possible to postulate enough continuity, compactness and other topological properties on the exogenous parts of the problem to ensure continuity of any useful mapping in the Banach space of price random variables. This occurs because the excess demand random variables $\xi_i(p, \cdot, \Lambda_j, \nu^p, \cdot, \cdot)$ are not, in general, continuous functions, under any appropriate topology, of the price random variable $\rho$. That is, in general, the information communicated by prices, $\rho^{-1}(A)$ is not, in some sense, a continuous function of $\rho$.

To illustrate this, consider a family of functions on $\{\omega : \omega > 0\}$, parameterized by $\theta \in [0, 1]$, and defined by

$$\rho_\theta(\omega) = 1 + \frac{\theta}{\omega + 1} - \frac{1 - \theta}{\omega + 2}.$$

Suppose that agents are attempting to learn something about $\omega$ by observing $\rho_\theta(\omega)$. The invertibility of $\rho_\theta$ depends on its monotonicity and hence on the zeros of its derivatives. There are four cases:

1. $\theta \in [0, 1/5]$. There are no stationary points for $\rho_\theta(\omega)$ for $\omega > 0$. 

Figure 2.1 A family of functions.
2. $\theta = 1/5$. $\rho^i_\theta (\omega) = 0$ only when $\omega = 0$.

3. $\theta \in (1/5, 1/2)$. $\rho^i_\theta (\omega) = 0$ for exactly one $\omega^* > 0$, and $\omega^* \to \infty$ as $\theta \to 1/2^-$.

4. $\theta \in [1/2, 1]$. $\rho^i_\theta (\omega) \neq 0$ for all $\omega > 0$.

This is graphed in Figure 2.1.

For $\theta > 1/2$, $\rho^i_\theta (\omega)$ is an invertible function of $\omega$, but for $\theta < 1/2$ it is not invertible. Call $\omega_1$ and $\omega_2$ confounding if

$$\rho^i_\theta (\omega_1) = \rho^i_\theta (\omega_2) = \rho,$$

say. Since the stationary point $\omega$ of $\rho^i_\theta (\cdot)$ goes to $+\infty$ as $\theta \to 1/2^-$, there are arbitrarily large confounding pairs $\omega_1, \omega_2$ with $|\omega_1 - \omega_2|$ arbitrarily large and with $\rho^i_\theta (\omega_i)$ arbitrarily close to 1. If an agent's preferences are markedly different in states $\omega_1$ and $\omega_2$, the numerical value of the excess demand will be very volatile and discontinuous at $\theta = 1/2$ as $\rho^i_\theta$ suddenly becomes invertible. This information discontinuity at $\theta = 1/2$ is not removable and generally leads to a jump discontinuity of demand at $\theta = 1/2$. These discontinuities tend to foil the application of fixed point theorems to establish the existence of equilibrium.

In addition to these continuity problems, there are other problems (which may be termed "measurability problems") with establishing the existence of a rational expectations equilibrium. We have already argued that agent $i$'s excess demands will be measurable with respect to his information $A_i \text{wp}^{-1}(A)$; that is, his demands will vary only according to the information he actually receives. On the other hand, different pieces of information
will generally lead to different demands, so that his excess demand function will often generate the whole $\sigma$-algebra $A_i \cap \mathcal{V}_p^{-1}(A)$.

For a more concrete interpretation, suppose the state space $\Omega$ is finite, so that we need only consider the partitions that generate the information $\sigma$-algebras. (A finer partition corresponds to superior information, etc.) With an obvious change of notation, let $A_i$ be agent i's exogenous information partition, so that he can distinguish states $\omega \in \Omega$ only if they are in different members of his partition. Let $\rho^{-1}(A)$ be the partition generated by $\rho$—i.e., the partition consisting of equivalence sets of states $\omega$ which map to the same price under $\rho$. Then $A_i \cap \mathcal{V}_p^{-1}(A)$ is the common refinement of $A_i$ and $\rho^{-1}(A)$. To say that agent i's excess demand function is $A_i \cap \mathcal{V}_p^{-1}(A)$-measurable means that it is constant on members of the partition. In general, excess demands will be different on different members of the partition (i.e., will generate the whole partition).

The equilibrium condition is that excess demands sum to zero, for all $\omega \in \Omega$. But excess demands that vary on different partitions (or $\sigma$-algebras) will not, in general, sum to a constant function. For example, in a two-agent economy, we cannot have an equilibrium where one agent varies his excess demands on a finer (or simply, different) partition than the other, because such excess demand cannot sum to zero for all $\omega \in \Omega$. If we can rule out differential information (after taking into account endogenous price information) then we may get an equilibrium. That is, we require of an equilibrium
\( \rho \) that \( A_i \rho^{-1}(A) = A_j \rho^{-1}(A) \) for all \( i, j \). Hence, we have
\[
A_i \rho^{-1}(A) \subseteq A \subseteq (A_i \rho^{-1}(A))v...v(A_i \rho^{-1}(A)) = A_i \rho^{-1}(A),
\]
so that \( A = A_i \rho^{-1}(A) \) for all \( i \). That is in equilibrium, all agents are fully informed. If the \( A_i \) differ sufficiently
this also tends to imply that \( \rho^{-1}(A) = A \), that is, all information is revealed by prices. If a rational expectations equilibrium exists with \( \rho^{-1}(A) = A \), we shall say that a fully informing rational expectations equilibrium (FRE) exits.

This measurability argument suggests that, if agents "like to use a lot of information" in formulating their demands (i.e., \( \xi(\rho(\cdot),\cdot, A_i \rho^{-1}(A)) \) generates the whole \( \sigma \)-algebra \( A_i \rho^{-1}(A) \)), agents must be fully informed for equilibrium to exist, and if the \( A_i \)'s vary significantly across agents, the only way to provide all information is to have \( \rho^{-1}(A) = A \), a FRE. 3)

If under perturbations of \( \omega \), a FRE continues to exist, there are no information discontinuities so that the continuity and measurability problems are solved simultaneously with FRE's. In the following chapters we study the existence of FRE's only.

There is a straightforward procedure for verifying the existence of a FRE. One simply solves for the fully-informed demand functions \( \xi_i(p,\omega,A) \). The market clearing condition then defines \( \rho \), from which it is then, in principle, possible to check whether \( A = \rho^{-1}(A) \). In practice, there may be simple parameters which are sufficient for defining conditional probabilities or summarizing the source of uncertainty, such as sufficient statistics.
drawn from independent, identically distributed random variables. A sufficient condition for the existence of a FRE is that $p$ be an invertible function of these parameters. This is the technique used in the following chapters.

**Other papers concerning rational expectations**

The term "rational expectations" was coined in a paper by Muth [1961]. His model was the first of a series of macroeconomic models developed by various authors. These models generally have homogeneous information ($A_i = A, \forall i$) and homogeneous beliefs ($P_i = P, \forall i$). To enhance the empirical testability of these models, it is usually assumed that $P$ also represents Nature's "true" probability law, so that *ex post* probability laws are the same as *ex ante* probability laws. In these models, the assumption of rational expectations is basically a strong consistency criterion that is used to obtain stronger conclusions in macro models. For a good review of this literature, see Shiller [1978]. In finance, the homogeneous beliefs and information assumption has been used by Cox, Ingersoll and Ross [1978] in models of the term structure of interest rates. In a general equilibrium model of the stochastic process of interest rates, they used the assumption of homogeneous and correct beliefs to relate the interest rate process that agents assume hold when formulating demands to the process that actually results, given agents' demands.
The micro-economic and general equilibrium applications of rational expectations assumptions generally center around differential information and homogeneous (and correct) beliefs. Here the central question is often the existence of equilibrium.

An example of such a differential information rational expectations model is the "lemons" model of used car markets by Akerlof [1970]. In this market, the seller has better information about the quality of his car than has the buyer. In the absence of some method of signalling quality, the owner of a good car will only receive a price corresponding to an average (i.e., inferior) car. He withdraws his car from the market, leaving a lower average quality, and, in a like manner, the owners of the next grade of cars leave the market, finally leaving only the lowest quality "lemons" on the market. In this case, information is conveyed by the fact that a car is offered on the market (for a given price). At any given price, there is positive supply and positive demand, but the two are never equal, so the market fails.

In finance, there are models of capital structure by Leland-Pyle [1977] and Ross [1977, 1978a] and of credit rationing by Jaffee-Russell [1976] and Heinkel [1978]. These models use differential information to explain various empirically observed phenomena that hitherto could only be explained by institutional rigidities like transactions costs. The capital structure models show how it is possible for a firm to have an optimal capital structure in a tax-free world, in
contradiction to the Modigliani-Miller irrelevance theorem. In the Leland-Pyle model, the proportion of equity financed by insiders signals information about the random returns on capital investment. This results in a fully informing rational expectations equilibrium in which entrepreneurs over-invest in their firm relative to what their optimal investment would be if all agents exogeneously have the same information. This occurs because, with differential information the benefits of inside investment in the firm are not only the future returns, but also the current returns of selling part of the firm at the higher price that results when it is classified as a high-return firm. This distortion is due to the differential information, not to a lack of price-taking behaviour.

In the Ross models, managers have superior information about firm type and have compensation based incentive schemes that allow the market to perceive firm type by observing the manager's compensation formula. In several of the Ross models, the compensation scheme is based on the current value of the firm, as well as the future value of the equity. By varying the firm's debt level, managers tailor their compensation scheme to signal firm type, yielding an optimal debt level.

The Heinkel model uses a borrower's choice of the amount of debt to signal his previously private information about how risky his project is. Risky borrowers would like to borrow a large amount, since, with limited liability they can benefit significantly by high project returns, but are not penalized significantly for low returns.
By rationing borrowers (i.e., keeping the face value of debt low), the limited liability feature is of less value of risky borrowers, and lenders will only lend to low risk borrowers. Thus, the borrowers reveal their true type ("self-select") by choosing distinct interest/loan size pairs. However, low-risk borrowers are rationed to a smaller loan size than they would choose if risk classes were exogenously revealed. The Jaffee-Russell paper has some similar notions, but emphasizes "honest" versus "dishonest" borrowers.

These models of debt markets and of capital structure, as well as related models of insurance markets by Rothschild-Stiglitz [1976] have the feature that, although agents are price-takers, informed agents' actions affect the "market's" perceptions of the goods they are offering (e.g., "firm type"), and hence the literature centers on incentive schemes, moral hazard, agency and information signalling behaviour.

There are other models, however, in which agents do not act in ways to explicitly encourage or discourage the dissemination of their private information to the market. For example, Grossman [1976] models a situation in which agents have different information about the next-period value of a risky security. The current price is determined by the aggregate content of that information (which is superior to each agent's private information). Moreover, the relevant aggregate information can be inferred by observing price alone. This leads to one disturbing feature of Grossman's model,
as pointed out by Grossman [1976, p.582], Kraus [1976] and Grossman-Stiglitz [1976]. If the private collection of information is costly, but the observation of prices is costless, agents will not collect private information, expecting that it will be revealed in prices, which, of course, it will not, if nobody collects information.

At least Grossman's model yields an equilibrium. There are models by Green [1977], Rothschild [1976], and Jordan-Radner [1977] in which equilibrium fails to obtain in rational expectations models. For simplicity, these models deal with two (classes of) agents: informed and uninformed. In a finite state-space model, the informed agents can distinguish states in a more refined partition than the uninformed agents. However, probability beliefs are homogeneous in the sense that both agents agree on the probabilities of events in the coarser, uninformed partition.

In these models, the uninformed agents either become informed or not when they observe equilibrium prices. If they are informed, their demands are such that prices are not fully informing, but if they are not informed, only fully informing prices can equilibrate the economy, since informed agents' excess demands are different in each member in the refined partition. Thus, measurability problems force the non-existence of a rational expectations equilibrium. In the Jordan-Radner model, non-existence occurs even under perturbations of the underlying parameters of the economy — that is, non-existence is, in a sense, generic.
On the other hand, there are generic existence results for FRE's by Radner [1978] and Allen [1978, 1979], which are based on theorems in differential topology. The general result is that the dimension of the space of prices must (substantially) exceed the dimension of the information space. These dimensionality restrictions are violated in the Jordan-Radner counterexample.

These results have all addressed the communication of probability information only. Information about preferences and endowments is also important and the question of whether or not this can be conveyed by prices is discussed in Kraus-Sick [1979a, 1979b, 1980]. The following chapters discuss those results as well as extensions of those results.
Footnotes to Chapter 2

1. A rigorous understanding of measure-theoretic probability theory is not required to understand the thrust of the argument presented here. The theory is presented in, for example, Feller [1966, Vol. II, Chapters IV and V] and references therein. Roughly speaking, selection of a state \( \omega \in \Omega \) corresponds to a random realization of the system. The \( \sigma \)-algebra \( \mathcal{B} \) is a family of subsets (called events) of \( \Omega \), which satisfies special conditions. The probability measure \( P_i : \mathcal{B} \to [0,1] \) assigns to each event \( B \in \mathcal{B} \) a non-negative probability of its occurrence, with \( P_i(\Omega) = 1 \). The probability of any subset of \( \Omega \) that is not an event is undefined. A random variable is a function \( \eta : \Omega \to \mathbb{R}^n \) (for some \( n \)) which is \( \mathcal{B} \)-measurable. That is, for any \( (a_1, \ldots, a_n) \in \mathbb{R}^n \),

\[
\{ \omega \in \Omega | \eta_j(\omega) \leq a_j, j=1,\ldots,n \} \in \mathcal{B},
\]

where \( \eta_j(\omega) \) is the \( j \)-th component of \( \eta(\omega) \). Roughly speaking, this means that \( \eta \) does not vary too much -- any change in the value of \( \eta \) can be captured by an event in \( \mathcal{B} \).

2. For example, let \( \eta : \Omega \to \mathbb{R}^n \) be a random variable. Then (see, for example, Feller [1966, Vol. II, pp. 160-162]), the conditional expectation \( E_i(\eta | A_i) \) is the \( A_i \)-measurable random variable \( \xi : \Omega \to \mathbb{R}^n \) such that \( E_i (1_A \xi) \equiv \int_A \xi(\omega) dP_i(\omega) = \int_A \eta(\omega) dP_i(\omega) = E_i (1_A \eta) \) for all \( A \in A_i \), where \( 1_A : \Omega \to \mathbb{R} \) is the set indicator function defined by

\[
1_A(\omega) = \begin{cases} 
1 & \text{if } \omega \in A \\
0 & \text{if } \omega \notin A
\end{cases}
\]
Note that, since $E_i(\eta|A_i)$ is a random variable it depends on the realization $\omega \in \Omega$ and should really be written as, say $E_i(\eta|A_i,\omega)$. Following conventional usage, we suppress the explicit references to $\omega$. If probability densities exist, or probability is discrete, this is in accordance with the usual notion of conditional expectation or probability. Also, if $A \in \mathcal{B}$ is an event, define $A_i \equiv \{\phi, A, \Omega \setminus A, \omega\}$. Since $E_i(\eta|A_i)$ is $A_i$-measurable, it is constant on the event $A$ and the constant is unique if $A$ has positive probability. Let $E_i(\eta|A)$ be that constant value on $A$. This conforms with our conventional notion of the non-random conditional expectation given an event of positive probability.

To condition on a random variable, $\rho$ say, is to condition on the $\sigma$-algebra generated by $\rho$, namely that $\sigma$-algebra generated by the events
\[
\{\omega \in \Omega | \rho(j)(\omega) \leq a_j, j = 1, \ldots, n\}
\]
for arbitrary $(a_1, \ldots, a_n) \in \mathbb{R}^n$. If $\Omega$ is finite, the $\sigma$-algebra generated by $\eta$ is that which is generated by the partition of $\Omega$ generated by inverse images of points in $\mathbb{R}^n$ under $\rho$. The partition is
\[
\{\{\omega : \rho(\omega) = p\} : p \in \mathbb{R}^n\}
\]
The $\sigma$-algebra generated by $\rho$ distinguishes only those points which are mapped to different points by $\rho$, and hence is the information conveyed by observing $\rho$. 
3. This is not to suggest that non-fully informing rational expectations equilibria do not exist. If there are two dimensions of uncertainty ($n = 2^2$) but an informed agent's demand and hence price conveys only a univariate function of the uncertainty, it is possible for equilibrium to exist. This occurs, for example, in Grossman [1977], in which random supply is added to uncertainty about the value of a risky asset. The informed agents' demands are univariate in nature and generate a "one-dimensional $\sigma$-algebra" (i.e., a $\sigma$-algebra generated by one-dimensional translations of a manifold, so that there is only one remaining degree of freedom on which variation can occur in $2^2$). In his model, optimal demands for the uninformed agent are merely conditional expectations of optimal informal demands, and hence can generate the same one dimensional $\sigma$-algebra, allowing excess demands to sum to zero. If uninformed agents have different preferences than informed agents, or have some information that "informed" agents do not have, then uninformed demands may not generate the same $\sigma$-algebra as informed agents, or have some information that "informed" agents do not have, then uninformed demands may not generate the same $\sigma$-algebra as informed demands and the measurability problem prevents the existence of equilibrium. It would be interesting to study the robustness of this non-fully informing rational expectations equilibrium.
Chapter 3  Revelation of Aggregate Preference Parameters

Introduction

In this chapter, we study the question of whether, in a multi-period economy, current prices reveal enough information about the aggregate preference parameters of the economy to resolve some of the uncertainty about future prices. This is done in a rational expectations context: the demands leading to the formation of prices must be consistent with the information conveyed by prices. Motivation for this work comes from three directions.

One line of motivation comes from the Arrow [1953] paper on the optimal allocation of risk-bearing. Arrow first showed that in an economy with C commodities and S states of the world, SC state-contingent ("Arrow-Debreu") securities corresponding to each state-commodity pair will achieve a Pareto efficient equilibrium. He then established that the same allocation can be obtained by S+C markets: S markets for state-contingent wealth and, once a particular state is revealed, C markets for the commodities. An objection, voiced, for example, by Drèze [1970, p. 144] and Nagatani [1975], is that this reduction in the number of markets will result in the same allocation only if agents know what commodity prices will prevail in each state of nature, before it is revealed and hence before the market opens to generate these prices. If current prices can reveal preference parameters, they may reveal these state-conditional future prices, allowing an optimal allocation with a reduced number of markets in the manner envisioned by Arrow.
Another line of motivation is provided by Grossman [1976], in which agents come to a market with diverse information about a risky asset and the market clearing process aggregates that information and conveys the aggregate parameter value, via price signals. In this chapter, agents come to market with diverse preferences and the market price conveys an aggregate preference value rather than aggregate probability information, which is important in predicting next period prices.

A third line of motivation comes from the recent literature on the recoverability of utility functions -- see, e.g., Dybvig [1979], Dybvig and Polemarchakis [1979] and Green, Lau and Polemarchakis [1979]. The general question addressed in this literature is: "Given certain restrictions on the admissible class of utility functions, and an opportunity to observe an agent's demand function on a certain set of prices and incomes, when can the utility function be determined ('recovered')?" For example, Dybvig studies the class of additive utility functions for two goods (e.g., complete markets and von Neumann-Morgenstern utility) and finds conditions under which the utility function is recoverable given two Engel curves (impossible) or given three Engel curves (generically possible) or given demands along two rays through the origin (always possible, providing the slope of the utility function is bounded as consumption goes to zero). Dybvig has analogous results for utility functions with more than two goods (or states). Dybvig also studies the recoverability of von Neumann-Morgenstern utility from demands for lotteries over a single risky asset and finds that the main condition for recoverability is that marginal utility be bounded as consumption goes to zero.
There are two important characteristics of these recoverability results, in contrast to the present research:

1. Utility functions are recovered from demand functions, which are not observable from price data in a general equilibrium setting (unless multiple observations are available for econometric estimation).

2. They require an infinite number of observations to specify whole Engel curves, for example, unless the Engel curves are previously known to have a specific functional form, in which case only enough observations to evaluate all parameters are required. In contrast, we use a general equilibrium setting and recover information by observing one realization of the economy. This is obtained at the expense, of course, of having to previously specify the functional form of the utility function.

The first models in this chapter are in a two period state preference setting arising from complete markets. The last model of the chapter has two periods and incomplete markets with normally distributed security returns.

Notation

We deal with a pure exchange economy and a single good ("money"). There are I agents \(i = 1, \ldots, I\), and three dates \((0,1,2)\). Securities, which are the only traded objects, pay off in money in \(C\) final (date 2) states \((c = 1, \ldots, C)\) or in \(S\) intermediate (date 1) states \((s = 1, \ldots, S)\).

There are homogeneous probability beliefs at date 0 about the joint distribution of intermediate and final states. These beliefs are:

- \(\pi_s = \Pr(\text{intermediate state } s)\)
- \(\pi_{sc} = \Pr(\text{final state } c|\text{intermediate state } s)\)
- \(\pi_c = \Pr(\text{final state } c) = \sum_s \pi_s \pi_{sc}\)
One may think of the intermediate states $s$ as presenting new probability information causing each agent to update his probability estimates about final state $c$ from $\pi_{c}$ to $\pi_{sc}$.

An agent is endowed with money at date 0 and makes sequential portfolio decisions at date 0 and date 1 for, respectively, an initial portfolio of claims on money in intermediate states and a final portfolio of claims on money in final states. These are denoted by:

\begin{align*}
y_{i} & \equiv \text{endowed wealth of agent } i \\
y_{is} & \equiv \text{payoff to agent } i\text{'s initial portfolio if intermediate state } s \text{ prevails at date 1} \\
y_{isc} & \equiv \text{payoff to agent } i\text{'s final portfolio if intermediate state } s \text{ prevails at date 1 (when final portfolio is purchased) and final state } c \text{ prevails at date 2.}
\end{align*}

The aggregate supply of these claims is $X_{c} = \sum_{i}X_{isc}$. Without loss of generality, we assume $X_{c}$ is independent of $s$.

Markets consist of two types of securities:

Type "f" is a riskless bond maturing at the next date with a yield of zero (i.e., cash), with zero net supply. Type "m" is a proportional share of the social endowment at the next date (i.e., the market portfolio), with unit net supply.

Define prices of the market portfolio by:

\begin{align*}
\rho_{0} & \equiv \text{date 0 price of market portfolio in terms of cash numeraire.} \\
\rho_{s} & \equiv \text{date 1 price of market portfolio in state } s \text{, in terms of a state } s \text{ cash numeraire.}
\end{align*}

Let $m_{i}$ = individual $i$'s endowed fractional holding of the market portfolio at date 0 ($\sum_{i}m_{i} = 1$),
\( f_{i0} \) = individual i's dollar holding of cash in the portfolio selected at date 0,
\( m_{i0} \) = individual i's fractional holding of next date social endowment in the portfolio selected at date 0,
\( f_{is}, m_{is} \) = similarly for the final portfolio selected at date 1 in intermediate state s.

Then intermediate and final payoffs are, respectively,
\[
\begin{align*}
(3.1) \quad y_{is} &= f_{i0} + m_{i0}^{ps} \\
(3.2) \quad x_{isc} &= f_{is} + m_{is}^{x_{c}}.
\end{align*}
\]

Budget constraints for dates 0 and 1 are, respectively,
\[
\begin{align*}
(3.3) \quad f_{i0} + \rho^{m_{i0}} &= \rho^{m_{i}} \\
(3.4) \quad f_{is} + \rho^{m_{is}} &= y_{is}.
\end{align*}
\]

Market clearing relations are:
\[
\begin{align*}
(3.5) \quad \Sigma_{i} f_{i0} &= \Sigma_{i} f_{is} = 0 \\
(3.6) \quad \Sigma_{i} m_{i0} &= \Sigma_{i} m_{is} = 1.
\end{align*}
\]

There is no consumption before the final period (and hence no natural inter-temporal discount rate), so that, in market M, there is a need for \( S \) numeraires for date 1 and one numeraire for date 0. In (3.3) and (3.4), these numeraires have been chosen to be the "cash" portfolio relevant for that date and state. For simplicity we may define \( y_{i} = \rho^{m_{i}} \), so that (3.3) becomes:
\[
(3.7) \quad f_{i0} + \rho^{m_{i0}} = y_{i}.
\]

We may think of this market setting as arising at each date from complete markets of securities paying off at the next date.
The complete market structure degenerates to a market for two securities because all agents (will) have linear risk tolerance (LRT) with the same slope. That is, if an agent has von Neumann-Morgenstern utility $U(x)$ for wealth $x$, then his risk tolerance (at $x$) is the inverse of his absolute risk aversion:

$$\frac{1}{R_A(x)} = \frac{-U'(x)}{U''(x)}$$

We shall assume that agents have linear risk tolerance with the same slope coefficient ("cautiousness"), \textit{i.e.}:

$$\frac{U_i'(x)}{U_i''(x)} = \alpha_i + \lambda x$$

This is necessary and sufficient for linear sharing rules and surrogate functions (cf. Wilson [1968]) and hence results in aggregation (cf. Rubinstein [1974] and Brennan-Kraus [1978]) as well as two-fund monetary separation (cf. Cass-Stiglitz [1970]). With two-fund monetary separation, trading in a complete market will degenerate into trading in two portfolios: cash ("money") and the market portfolio of all risky assets. We may assume, without loss of generality, that cash has a net zero supply, because the net supply of cash can be absorbed into the market portfolio, merely shifting its return distribution by a constant.

\textbf{A power utility economy}

Suppose all agents have extended power utility (with the same exponent) for consumption at date 2. Specifically, the utility for agent $i$ for consuming $x$ at date 2 is:

$$U_i(x) \equiv \gamma^{-1} (\theta_i + x)^\gamma$$

$i = 1, \ldots, I$
where $0 \leq \gamma < 1$. This is a concave, increasing utility function displaying decreasing absolute risk aversion. At date 1 in state $s$ agent $i$ chooses a portfolio $(f_{is}, m_{is})$ to solve:

$$\max \, \gamma^{-1} \sum_{c} \pi_{sc} (\theta_{i} + f_{is} + m_{is} X_{c})^{\gamma} \quad \{f_{is}, m_{is}\}$$

subject to:

$$f_{is} + \rho_{s} m_{is} = y_{is}. \quad (3.4)$$

Note that we have used (3.2) to substitute $x_{isc} = f_{is} + m_{is} X_{c}$ into the objective function. The first order conditions yield

$$\rho_{s} = \frac{\sum_{c} \pi_{sc} X_{c} (\theta_{i} + y_{is} + m_{is} (X_{c} - \rho_{s}))^{\gamma-1}}{\sum_{c} \pi_{sc} (\theta_{i} + y_{is} + m_{is} (X_{c} - \rho_{s}))^{\gamma-1}} \quad (3.10)$$

This is a nonlinear equation implicitly giving no closed form solution for the demand $m_{is}$ as a function of $\rho_{s}$ and $y_{is}$. However, it is easy to verify that (3.10) holds for all $i$ if and only if $2$:

$$m_{is} = \frac{\theta_{i} + y_{is}}{\theta_{A} + \rho_{s}} \quad i = 1, \ldots, I \quad (3.11)$$

$$\rho_{s} = \frac{\sum_{c} \pi_{sc} X_{c} (\theta_{A} + X_{c})^{\gamma-1}}{\sum_{c} \pi_{sc} (\theta_{A} + X_{c})^{\gamma-1}} \quad (3.12)$$

where

$$\theta_{A} = \sum_{i} \theta_{i}. \quad (3.13)$$

Summing (3.1) over $i$ and noting (3.5) and (3.6) yields $\sum_{i} y_{is} = \rho_{s}$. Hence, summing (3.11) over $i$ yields $\sum_{i} m_{is} = 1$, so that the market for the risky asset clears at date 1. By Walras' law, the market for the riskless asset also clears, so (3.11) and (3.12) represents an equilibrium, where $f_{is}$ is computed from (3.4). Note that (3.11)
does not give a demand function for the risky asset, since it solves (3.10) only when (3.12) holds. However, in equilibrium, (3.12) does hold and (3.11) gives the correct sharing rule or numerical value of demand.

Another way of obtaining (3.12) is to compute it as the marginal rate of substitution between the risky and the riskless asset for a representative or aggregate investor A with all market wealth \( m_A \) so that \( m_A = 1 \) and \( f_A = 0 \) by (3.5) and (3.6)) and with utility

\[
U_A(x) = y^{-1}(\theta_A + X)^{Y}
\]

where \( \theta_A \) is given by (3.13). Hence, it is appropriate to refer to \( \theta_A \) as a market risk tolerance parameter.

At date 0, all uncertainty about relative prices \( p_s \) in date 1 arises from either uncertainty about the state \( s \) or uncertainty about \( \theta_A \), which, for computing prices, is a "sufficient statistic" for \( \theta_1, \ldots, \theta_I \).

We can now compute the date 1 derived utility for wealth \( y_{is} \), fold back to date 0 and solve for date 0 demands (and hence prices) in the manner of dynamic programming. Substituting (3.4) and (3.11) into the date 1 utility and simplifying yields the derived utility.

\[
\gamma^{-1} \sum_{c} \Pi_{s} \left( \frac{\theta_A + X_c}{\theta_A + p_s} \right)^{Y} (\theta_i + y_{is})^{Y}
\]

As in the discussion of Chapter 2, suppose that \( \theta_A \) and hence \( p_s \) are known to all agents. We will check to see that its revelation can be sustained in equilibrium. Then the date 0 portfolio problem
becomes:
\[
\max_{\{f_{10}, m_{10}\}} \gamma^{-1} \sum_i \pi_i \sum_c \pi_{Sc} \left( \frac{\Theta_A + X_c}{\Theta_A + \rho_s} \right)^\gamma (\Theta_i + \gamma y_{1s}) \gamma
\]
subject to:

(3.1) \[ y_{1s} = f_{10} + m_{10} \rho_s \]
(3.7) \[ f_{10} + \rho_0 m_{10} = y_i \]

This is an extended power utility problem that is exactly analogous to the date 1 problem when the probability weights \( \pi_{Sc} \) are replaced by \( \pi_s \) times the factor in large square brackets above.

As in the date 1 analysis, equilibrium is characterized by:

\[
m_{10} = \frac{\Theta_i + y_i}{\Theta_A + \rho_0}
\]

\[
\rho_0 = \frac{\sum_i \pi_i \sum_c \pi_{Sc} \left( \frac{\Theta_A + X_c}{\Theta_A + \rho_s} \right)^\gamma \rho_s (\Theta_A + \rho_s)^{-1}}{\sum_i \pi_i \sum_c \pi_{Sc} \left( \frac{\Theta_A + X_c}{\Theta_A + \rho_s} \right)^\gamma \rho_s (\Theta_A + \rho_s)^{-1}}
\]

Cancelling factors in \((\Theta_A + \rho_s)\), substituting for \(\rho_s\) from (3.12) and recalling that \(\sum_i \pi_i \sum_c \pi_{Sc} = \pi_c\) yields:

\[
\rho_0 = \frac{\sum_c \pi_c X_c (\Theta_A + X_c)^{-1}}{\sum_c \pi_c (\Theta_A + X_c)^{-1}}
\]

This yields \(\rho_0\) as a function of \(\Theta_A\). If the function is invertible then a fully informing rational expectations equilibrium (FRE) exists, as discussed in Chapter 2. We have:
Thus, \( p_Q \) is an invertible function of \( \theta_A \) and we have 3):

**Theorem 3.1** If all agents have extended power utility (3.9) with decreasing absolute risk aversion and the market is characterized by equations (3.1) to (3.7), then there exists a fully informing rational expectations equilibrium (FRE) in which agents can infer aggregate risk preference \( \theta_A \) from the date 0 price \( p_Q \) and hence correctly infer the prices \( p_s \) that would occur if state \( s \) occurred \((s = 1, ..., S)\).

This model is somewhat akin to the Grossman [1976] model in which individuals are endowed with exponential utility and have independent identically distributed observations about the next period mean value of a risky asset's normally distributed return. Grossman showed that the market price is an invertible function of the sample mean observation and, hence, that a FRE exits.

One feature of his model that was pointed out by Grossman-Stiglitz [1976] and Kraus [1976] was that, since prices convey all relevant information, individuals will not collect private information if it is costly to obtain. But if they do not collect private information, it will never be impounded in prices, so that prices will not convey all available information after all. The model presented here does not have this difficulty, since agents must
assess their own utility functions before coming to market, in order to formulate their demands. This effectively costless, heterogeneous information will come to market. 4)

Corollary 3.2: Under the same hypotheses as for Theorem 3.1, but assuming all agents have extended power utility with increasing absolute risk aversion, viz.

\[ u(x) = - (\theta_i x)^\gamma \]

where \( \gamma > 1 \), there exists a fully informing rational expectations equilibrium where agents can infer \( \theta_A \) from the date 0 price \( p_0 \).

Proof: By reasoning analogous to that of Theorem 3.1, we have, for example,

\[ m_{i0} = \frac{\theta_i - y_i}{\theta_A p_0} \]

\[ \hat{p}_0 = \frac{\sum_{c} \pi_c X_c (\theta_A - x_c)^{\gamma - 1}}{\sum_{c} \pi_c (\theta_A - x_c)^{\gamma - 1}} \]

and \( \frac{d\hat{p}_0}{d\theta_A} > 0 \) \( 5) \). Q.E.D.

A natural question to ask is how the results are affected by consumption at dates 0 and 1, as well as date 2. Suppose all agents have intertemporally additive power utility functions, so that von Neumann-Morgenstern utility becomes:

\[ \gamma^{-1} (\theta_i + x_{i0})^\gamma + \gamma^{-1} \eta_1 \sum_s \pi_s (\theta_i + x_{is})^\gamma + \gamma^{-1} \eta_2 \sum_{sc} \pi_s \pi_{sc} (\theta_i + x_{isc})^\gamma \]

where agent \( i \) consumes \( x_{i0}, x_{is} \) and \( x_{isc} \) respectively at date 0, date 1 (state \( s \)), and date 2 (state \( (s,c) \)). This is analogous to the previous problem with no intermediate consumption, where \( S+1 \)
states, corresponding to date 0 and date 1 consumption have been added. This has similar aggregation and separation properties to the market already studied, and in appropriate settings, yields a FRE. At date 0 and date 1, markets separate into three assets: date 0 or date 1 consumption, a riskless asset and a risky asset. (In fact, two assets will do at each date, since the vector of agents' consumption good holdings in $A^I$ will be spanned by the vectors of riskless asset holdings and risky asset holdings in $A^I$.) Date 0 relative prices will reveal $\theta_A$. However, the date 0 riskless asset will have a special form: it must provide one unit of riskless consumption at date 1. Since the relative prices of these two types of consumption (i.e., the rate of interest) will, in general, be different in different states $s$ at date 1, the date 0 riskless asset cannot simply pay $1 at all date 2 states $s$: it will have a variable payoff in dollar terms. If markets are complete, then clearly such an asset will be provided (i.e., spanned), but its composition depends on the relative date 1 prices, which depend on $\theta_A$. Thus, not knowing $\theta_A$, in complete markets an agent cannot compute which weights to use on prices, in order to compute the relative price of the risky and riskless assets, and hence invert to get $\theta_A$. At date 0, there would be $S$ prices of the financial securities relative to the date 0 consumption good. This is enough to yield the single parameter $\theta_A$, under many conditions, but a study of these does not appear to be very instructive.

**Intermediate labor income**

Another question is how the results are affected if date 1
(labor) income is introduced for agent $i$. Specifically, suppose that, at date 1, agent $i$ receives income of $L_i$, independent of state $s$, the value of which is revealed at date 0 to agent $i$, but to nobody else. Thus, date 1 aggregate wealth is uncertain, at date 0, creating more uncertainty about date 1 prices. We shall use the same notation as before with the following changes:

The date 1 budget constraints become:

$$f_{is} + \rho_s m_{is} = y_{is} + L_i$$  \hspace{1cm} (3.17)

The date 1 market for the riskless asset clears when

$$\sum_i f_{is} = \sum_i L_i = L$$ \hspace{1cm} (3.18)

The reason for having a nonzero aggregate supply of the riskless asset is to ensure that, by purchasing at date 0 the fraction $m_{i0}$ of the risky asset, paying off $m_{i0}\rho_s$ at date 1 in state $s$, agent $i$ is only making a claim to the risky market asset, rather than other agents' labor income. That is, summing (3.17) over $i$ and using (3.18) yields $\rho_s = \sum_i y_{is}$ rather than $\rho_s = \sum_i y_{is} + L$, which would be the case if $\sum_i f_{is} = 0$, which would be aesthetically unappealing. In this setting, the following holds:

**Theorem 3.3:** Under the same assumptions as for Theorem 3.1 (extended power utility, etc.), but substituting (3.17) and (3.18) for (3.4) and (3.5), respectively, where $L_i$ is agent $i$'s date 1 labor income, known to only him at date 0, there exists a fully informing rational expectations equilibrium in which the date 0 price $\rho_0$ reveals $\theta_A + L$, which allows computation of date 1 prices $\rho_s$.

**Proof:** Using the techniques of the proof of Theorem (3.1), one can show that:
\[ \rho_s = \frac{\sum_{c} \Pi_{sc} X_c \left( (\Theta_A + L + X_c)Y^{-1} \right)}{\sum_{c} \Pi_{sc} (\Theta_A + L + X_c) Y^{-1}} \]

and
\[ m_{is} = \frac{\Theta_i + L_i + y_{is}}{\Theta_A + L + \rho_s} \]

At date 1 in state \( s \), the utility for financial wealth \( y_{is} \) is, for agent \( i \),
\[ (3.19) \quad \sum_{c} \Pi_{sc} \left( \frac{\Theta_A + L + X_c}{\Theta_A + L + \rho_s} \right)^Y (\Theta_i + L_i + y_{is})^Y \]

If agents know \( \Theta_A + L \), the date 0 price is
\[ \rho_o = \frac{\sum_{c} \Pi_{sc} X_c \left( (\Theta_A + L + X_c)Y^{-1} \right)}{\sum_{c} \Pi_{sc} (\Theta_A + L + X_c) Y^{-1}} \]

and
\[ m_{io} = \frac{\Theta_i + L_i + y_i}{\Theta_A + L + \rho_o} \]

Thus, all prices are a function of \( \Theta_A + L \) in the same way that prices in Theorem 3.1 were a function of \( \Theta_A \). Just as \( \rho_o \) reveals \( \Theta_A \) in Theorem 3.1, \( \rho_o \) reveals \( \Theta_A + L \) here, and a FRE exists. Q.E.D

An interesting variation on this market structure is to suppose that \( L_i \) is revealed to agent \( i \) at date 1 only, and is random at date 0, denoted by \( \tilde{L}_i \). Suppose, for simplicity, that \( L = \sum_i \tilde{L}_i \) is not random, but known to all agents at date 0. Such a situation could arise, for example, if agents are stevedores who report to a hiring hall every period. Workers are assigned to jobs randomly, since there is not enough work for all. However, all workers know beforehand the total amount of labor to be supplied contractually by the union.
This minor change prevents the existence of a FRE, although the economy may admit a rational expectations equilibrium that is not fully revealing. A FRE would have agents holding at date 0 the fraction:

\[ \bar{m}_{i0} = \frac{\theta_i + L_i + y_i}{\theta_A + L + \rho_0} \]

of the risky asset (from the proof of Theorem 3.3). It would also allow them to compute \( \theta_A + L \) from \( \rho_0 \), so it would require the random demand \( \bar{m}_{i0} \) to generate the whole information \( \sigma \)-algebra \( \bar{L}_i \), although this information is not available to any agent at date 0. Hence a FRE cannot exist. In this market structure, (3.19) still represents agent i's date 1, state s derived utility for wealth \( \bar{y}_{is} \), since \( L_i \) is known then. At date 0, agent i must take expectations over state s and \( \bar{L}_i \), as well as over \( \bar{\theta}_A \), \( \bar{\rho}_s \), and \( \bar{y}_{is} = \bar{m}_{is} + f_{i0} \bar{\rho}_s \), since a FRE doesn't exist. The expected utility becomes:

\[
\sum_s \sum_c \sum_{\bar{L}_i} \sum_{\bar{y}_{is}} \left( \frac{\bar{\theta}_A + L + X_c}{\bar{\theta}_A + L + \bar{\rho}_s} \right)^Y (\bar{\theta}_i + \bar{L}_i + \bar{y}_{is})^Y \left( \bar{\theta}_i + \bar{\rho}_s \right)
\]

The expectation operator \( E \) is the expectation conditional on \( \bar{\rho}_s \) and \( \bar{\theta}_i \), with respect to the random variables \( \bar{\theta}_A \), \( \bar{L}_i \), \( \bar{\rho}_s \), \( \bar{y}_{is} \), and \( \bar{\theta}_i \).

Note that \( \bar{\rho}_s \) provides no information about \( \bar{L}_i \), since no agents have information about \( \bar{L}_i \) at date 0. If a rational expectations equilibrium exists, it may be assumed that the \( \bar{L}_i \) are independent of \( \bar{\theta}_A \), \( \bar{y}_{is} \) and \( \bar{\rho}_s \), unconditionally and conditional on \( \bar{\rho}_0 \). The randomness of \( \bar{L}_i \) serves to make markets incomplete, thereby destroying the separation and
aggregation properties. Another simplifying assumption may be called an "information-taking" assumption about agents, analogous to the price-taking assumption used in the theory of competitive markets. That is, one can assume, as an approximation from some law of large numbers, that the distribution of \( \frac{\bar{\theta} + L + x}{\bar{\theta} + L + \bar{p}} \) is independent of that of \( \bar{\theta} \), even though \( \bar{\theta} = \sum_i \bar{\theta}_i \). That is, if there are enough agents, agent \( i \), will not make any inferences about \( \bar{p} \) from his own \( \bar{\theta}_i \) so that \( \bar{p} \) will depend on \( \bar{\theta}_i \) only through \( \bar{\theta} \).

Computing the first order conditions and substituting for \( f_{i0} \) from the budget constraint yields (with endowment \( y_i = m_i \bar{p} \)), in the general case,

\[
\rho_o = \frac{\sum \sum C \sum \sum \sum \sum \sum E \left( \frac{\bar{\theta} + L + X_c}{\bar{\theta} + L + \bar{p}} \right) \rho_o \left( \bar{\theta} + L + m_i \bar{p} + m_i \left( \rho - \bar{p} \right) \right)^{1/2} \left( \bar{\theta}_i, \bar{m}_i \right)}{\sum \sum C \sum \sum \sum \sum \sum E \left( \frac{\bar{\theta} + L + X_c}{\bar{\theta} + L + \bar{p}} \right) \rho_o \left( \bar{\theta} + L + m_i \bar{p} + m_i \left( \rho - \bar{p} \right) \right)^{1/2} \left( \bar{\theta}_i, \bar{m}_i \right)}
\]

\( i = 1, \ldots, I \).

We also have \( \bar{\theta} = \sum_i \bar{\theta}_i \) and \( \sum m_i = 1 \). Thus, there are \( I+1 \) conditions on the endogenous variables \( \bar{m}_i \) and \( \bar{p} \) that define them, subject to a constraint on the exogenous variables, \( \bar{\theta} \) and \( \bar{\theta}_i \). By substituting for \( f_{i0} \) from the budget constraint, we deal with one market only so Walras' Law does not reduce the information carrying capacity of prices. The existence of a rational expectations equilibrium in this setting has not been established, but assigning probability distributions to \( \bar{\theta} \) and \( \bar{L} \) would result in a well-defined Banach space fixed point problem, which may be studied numerically by computer.
FRE's with other linear risk tolerance utility classes

A natural question is whether a FRE exists when agents have utility functions in the other classes exhibiting linear risk tolerance, namely the extended log and exponential classes. The answer is affirmative, as in the next two theorems:

**Theorem 3.4** Suppose the market structure of Theorem 3.3 holds, except that agents all have extended log utility for date 2 wealth of the form

\[ U_n(x) = \ln (\theta_1 + x) \]

Then a FRE exists in which agents can infer \( \theta_1 + L \) from \( p_0 \), and hence can also infer the prices that will obtain at date 1.

**Proof:** The first order conditions are the same as for the power utility class of Theorem 3.1, where \( \gamma = 0 \), an inadmissible parameter value for power utility (giving constant utility). The proof of Theorem 3.1 applies with \( \gamma = 0 \), needing only elementary modifications. Q.E.D

**Theorem 3.5** Suppose the market structure of Theorems 3.3 and 3.4 holds, except that all agents have exponential utility of the form

\[ U_n(x) = - \exp(-\theta_1 x) \]

Then a FRE exists in which agents can infer \( \theta = (\sum \theta_1)^{-1} \) from \( p_0 \) and hence can also infer the prices that will obtain at date 1.

**Remark:** At date 0, agents cannot and need not make any inferences about aggregate date 1 labor income \( L \). Essentially, this arises because the aggregate investor has exponential utility and hence
constant absolute risk aversion. Wealth does not affect choices among gambles, and hence does not affect the marginal rate of substitution between the risky and safe assets.

Proof: At date 1 in state \( s \), the first order conditions for agent \( i \) yield:

\[
\rho_s = \frac{\sum_c \pi_s \cdot X \cdot \exp \left( -\theta_i \left( f_{is} + m_{is} X_c \right) \right)}{\sum_c \pi_s \cdot \exp \left( -\theta_i \left( f_{is} + m_{is} X_c \right) \right)}
\]

\[
= \frac{\sum_c \pi_s \cdot X \cdot \exp \left( -\theta_i m_{is} X_c \right)}{\sum_c \pi_s \cdot \exp \left( -\theta_i m_{is} X_c \right)}
\]

Note that the factor in \( f_{is} \) is just a wealth effect and drops out.

The first order conditions are the same for all agents when \( \theta_i m_{is} \) is constant \( (i = 1, \ldots, I) \). Since \( \sum_i m_{is} = 1 \), we have the sharing rule:

\[
m_{is} = \frac{\theta}{\theta_i}
\]

and the price becomes

\[
\rho_s = \frac{\sum_c \pi_s \cdot X \cdot \exp \left( -\theta X_c \right)}{\sum_c \pi_s \cdot \exp \left( -\theta X_c \right)}
\]

After computing the derived utility for date 1 wealth \( L_{i1} + y_{is} \), the date 0 first order condition becomes, for agent \( i \),

\[
\rho_0 = \frac{\sum_s \sum_c \pi_s \cdot \pi_c \cdot \exp \left( \theta (\rho_s - X_c) \right) \rho_s \cdot \exp \left( -\theta_i \left( f_{i0} + L_{i1} + m_{i0} \rho_s \right) \right)}{\sum_s \sum_c \pi_s \cdot \pi_c \cdot \exp \left( \theta (\rho_s - X_c) \right) \exp \left( -\theta_i \left( f_{i0} + L_{i1} + m_{i0} \rho_s \right) \right)}
\]

Once again, the factors in \( f_{i0} \) and \( L_i \) cancel and the market clears with the sharing rule:

\[
m_{i0} = \frac{\theta}{\theta_i}
\]
and price:

\[
\rho_o = \frac{\sum_s \varphi(s)^{\gamma} \pi_c \psi(s) \rho_s \exp(-\theta X_c)}{\sum_s \varphi(s)^{\gamma} \pi_c \psi(s) \exp(-\theta X_c)}
\]

\[
= \frac{\sum_c \varphi(c) \psi(c) \exp(-\theta X_c)}{\sum_c \varphi(c) \psi(c)}
\]

Differentiating \( \rho_o \) w.r.t \( \theta \) and re-arranging terms into a sum of squares as in Theorem 3.1 yields:

\[
\frac{d\rho_o}{d\theta} < 0
\]

Hence a FRE can be sustained, since only \( \theta \) and not \( L \) is needed to compute \( \rho_s \). Q.E.D

Rollover of portfolios at date 1

A re-examination of the proof of Theorem 3.5 yields a disturbing observation, namely, that:

\[ m_{i0} = \theta_i / \theta_i = m_{iS} \quad (i = 1, \ldots, I; s = 1, \ldots, S). \]

Thus agent \( i \) will trade at date 0 to a portfolio consisting of the proportion \( \theta_i / \theta_i \) of the risky asset and the remainder of his wealth in the safe asset. At date 1 in state \( s \), this will be worth \( \rho_s (\theta_i / \theta_i) \) which is just enough to purchase the same fraction of the aggregate date 1 risky asset, which also happens to be his optimal holding. The date 0 riskless asset holdings \( f_{i0} \) are added to labor income \( L_i \) to give the optimal riskless asset holdings \( f_i \) for date 1. Thus, at date 0, agent \( i \) may view the risky asset not as an asset paying \( \rho_s \) at date 1 in state \( s \), but as an asset paying \( X_c \) at date 2 in state \((s,c)\). This will allow him to trade to an optimal demand holding \( m_{i0} \), which he merely needs to re-invest
mechanically ("rollover") at date 1 into the same amount of the risky asset. If all agents do this, date 0 prices will be the same as if the date 0 problem is viewed as a conventional two stage problem. Using this portfolio rollover technique, agents need not predict date 1 prices $p_s$, and hence have no need to know $\Theta$. The machinery used to establish the existence of a FRE is not really needed because agents can achieve optimal holdings using the rollover technique. Interpreted broadly, this rollover economy offers a FRE in the sense that agents can behave as if they were fully informed. Alternatively, the market may be dominated by investors with exponential utility, while an infinitesimal investor with a different utility function will not choose to rollover holdings, but will desire to trade at date 1. For such an investor, it is important to infer the aggregate risk aversion that sets prices.

The question arises as to whether or not the same rollover technique works for extended power and log utility economies. For these economies, the optimal risky asset holdings at date 1 (state $s$) and date 0 are, respectively:

$$m_{is} = \frac{\Theta_{i}^+L_{i}^+y_{is}}{\Theta_{A}^+L + \rho_{s}}$$

and,

$$m_{io} = \frac{\Theta_{i}^+L_{i}^+y_{i}}{\Theta_{A}^+L + \rho_{0}}$$

Expanding $m_{is}$, we find that:

$$m_{is} = \frac{\Theta_{i}^+L_{i}^+f_{io}+m_{io}\rho_{s}}{\Theta_{A}^+L + \rho_{s}} \quad \text{by (3.1)}$$
Thus, the rollover simplification is present in the power and log utility economies, as well. The rollover algorithm arises because holdings in the market portfolio at date 0 can be rolled over into identical holdings in any state at date 1. The same holds for the riskless asset: $1 at date 0 brings $1 in whatever state s occurs at date 1, which brings $1 in whatever state c occurs at date 2. (With labor income, the agent merely counts the date 1 L as part of "human wealth" at date 0.) Hence, at date 0, agents can effectively buy claims to the date 2 riskless asset and market portfolio. Since these are the only securities they would purchase if presented with a complete set of date 2 contingent claims, they effectively face a complete market in date 2 contingent claims, although at date 1 they revise their estimates of the probabilities of the occurrence of the final states. Hirshleifer [1971] and Marshall [1974] have shown that the contract curve of an exchange economy with complete markets does not depend on the state probabilities, so that at date 0, agents trade to the contract curve, and do not re-trade at date 1, even though they revise their probability information at date 1.
Note that this rollover feature is not related to the myopia of Mossin [1968] for linear risk tolerance utility functions. The Hirshleifer-Marshall result is in a general equilibrium setting, while myopia is related to a single individual's portfolio demand. By staying on the contract curve and merely rolling over their portfolios of risky and riskless assets, agents are following a stationary investment policy even though yield distributions may not be stationary (e.g., if the probabilities of some states become zero, the number of effective states changes and yield distributions must change, no matter how prices move). This provides a counter-example to Mossin's [1968, p.122] contention that agents will have stationary investment policies only if yield distributions are stationary.

This rollover feature cannot be avoided in a state preference setting without losing the closed form solutions for prices. The closed form prices arise from the aggregation and separation in the linear risk tolerance utility class and the effectively complete markets at dates 0 and 1. Without complete markets, aggregation fails, but with complete markets, the Hirshleifer-Marshall result obtains. This motivates the model of the next section in which date 0 markets are not complete, but separation obtains because returns are normally distributed and prices are readily computed by using constant absolute risk aversion (exponential) utility.
A model in which date 0 prices reveal aggregate preferences and agents re-balance portfolios in the intermediate period.

As in the previous sections, consider a pure exchange economy with I agents and three dates (0, 1, 2). At date 0 there are two securities which have jointly normally distributed date 1 payoffs 

\[ \begin{pmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{pmatrix} \sim N \left( \begin{pmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \right) \]

At date 0 agent i selects a portfolio vector \( (\alpha_{i1}, \alpha_{i2})' \) and realizes returns at date 1 of \( y_{i1} = \alpha_{i1}\tilde{W}_1 + \alpha_{i2}\tilde{W}_2 \). He faces the budget constraint

(3.20) \[ y_{i0} = \alpha_{i1} + p_0\alpha_{i2} \]

where \( y_{i0} \) is his initial endowment and \( p_0 \) is the price of the second asset (with the first as numeraire). Markets at date 0 clear when

(3.21) \[ \sum_i \begin{pmatrix} \alpha_{i1} \\ \alpha_{i2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

At date 1, there is a riskless asset with return \( R \) and a risky asset with a date 2 payoff of

\[ \tilde{V} \sim N(\bar{V}, \sigma_V^2) \]

Agent i realizes date 2 wealth of

\[ y_{i2} = \beta_{iR} R + \beta_{iV} \tilde{V} \]

where \( \beta_{iR} \) is his lending/borrowing and \( \beta_{iV} \) is his holding of the risky asset. He faces the budget constraint

(3.22) \[ y_{i1} = \beta_{iR} + p_1\beta_{iV} \]
where $p_1$ is the price of the (second period) risky asset (with the riskless asset as numeraire). The market clears when

\[
\sum_i \begin{bmatrix} \beta_{iR} \\ \beta_{iV} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]  

(3.23)

Individual $i$ has utility for final wealth of $-\exp(-\theta_i y_{i2})$. At date 1 he maximizes

\[
-E \exp(-\theta_i (\beta_{iR} + \beta_{iV})) = -\exp(-\theta_i (\beta_{iR} + \beta_{iV}) + \frac{\sigma_i^2}{2} \beta_{iV}^2)
\]

subject to (3.22). Solving (3.22) for $\beta_{iR}$ and simplifying the objective yields the first order condition

\[
\theta_i (\bar{V} - p_{1R} - \theta_i \beta_{iV}^2) \exp(-\theta_i (\beta_{iR} + \beta_{iV}) + \frac{\sigma_i^2}{2} \beta_{iV}^2) = 0.
\]

Hence

\[
\beta_{iV} = \theta_i^{-1} \left( \frac{\bar{V} - p_{1R}}{\sigma_i^2} \right).
\]

(3.24)

Summing over $i$ and noting (3.23) yields

\[
p_1 = \frac{1}{R} (\bar{V} - \theta \sigma_i^2)
\]

(3.25)

where $\theta = (\sum_i \theta_i^{-1})^{-1}$. This gives the capital asset pricing model (CAPM) price of the risky asset.

Using (3.24) and (3.25), the derived utility for date 1 wealth $y_{i1}$ becomes

\[
\exp(-\theta_i R y_{i1}) \exp(-\frac{1}{2} \theta_i^2 \sigma_i^2).
\]
Assume that $R$ is known and fixed beforehand, and that $\sigma_Y$ is nonstochastic so that the second factor in the above utility function is a scaling factor that may be ignored. Thus, at date 0, individual $i$ maximizes

$$-E \exp(-\theta_i R \tilde{Y}_{i1}) = -\exp(-\theta_i R (\tilde{w}_1 + \alpha_{i2} \tilde{w}_2) + \theta_i R^2 \sigma_Y^2 (\alpha_{i1} \sigma_Y^2 + \alpha_{i2}^2 \sigma_X^2))$$

subject to the budget (3.20). Substituting for $\alpha_{i1}$ from the budget, the first order conditions become, after simplification:

$$\tilde{w}_1 p_0 - \tilde{w}_2 + \theta_i R (-p_0 (y_{i0} - p_0 \alpha_{i2}) \sigma_1^2 + \alpha_{i2} \sigma_2^2) = 0$$

Divide by $\theta_i$ and sum over $i$, noting from (3.20) and (3.21) that

$$\sum_i y_{i0} = 1 + p_0$$

to get

$$\tilde{w}_1 p_0 - \tilde{w}_2 + \theta R (-p_0 (1 + p_0) \sigma_1^2 + p_0 \sigma_2^2 + \sigma_2^2) = 0$$

so that $\theta = \frac{\tilde{w}_1 p_0 - \tilde{w}_2}{R (\sigma_2^2 - p_0 \sigma_1^2)}$

Since $\tilde{w}_1$, $\tilde{w}_2$, $\sigma_1^2$, $\sigma_2^2$, and $R$ are known, one can compute $\theta$ knowing $p_0$, so that a FRED exists and since date 0 markets are incomplete, agents must re-balance their portfolios at date 1. This yields the following:

**Theorem 3.6** In a two period (three date) economy, where the second period market consists of a normally distributed risky asset and a riskless asset (with known interest rate), the first period market consists of two normally distributed risky assets, and all agents have exponential utility, as described in this section, market prices
depend on the aggregate risk aversion parameter $\theta$ which is revealed by date 0 prices. Hence a fully informing rational expectations (FRE) exists, even though date 0 markets are incomplete and agents must re-balance their portfolios at date 1.

The tractable computations in Theorem 3.6 resulted partly from the portfolio separation that was induced by the normally distributed portfolio returns. Date 0 markets do not have a riskless asset and hence are incomplete so that the exponential utility function itself is not enough to ensure portfolio separation. This suggests that the theorem may also extend to other utility functions for which the utility of negative wealth is well-defined. (The normal distribution has support on the whole real line.) This eliminates utility functions such as extended log and power with irrational- or the inverse of even exponents. For power utility of the form $U_i(x) = \gamma^{-1}(\theta_i + x)^\gamma$ with $\gamma = m/(2n + 1)$; $m,n$ integers, the expected utility is well-defined and can be computed as a sum of two gamma functions. This separates but does not aggregate in the date 0 market, and does aggregate in the date 1 market. Hence, date 1 prices depend on $\Theta_A = \sum_1^T \Theta_i$, but the date 0 price depends on a different function of $(\Theta_1, \ldots, \Theta_T)$. In this case, it is doubtful whether a FRE exists, so that Theorem 3.6 may not be robust with respect to relaxation of the exponential utility class assumption.

This line of reasoning also suggests that it is not adequate to make strong distributional assumptions that ensure portfolio separation, as in Ross [1978b], to ensure tractability, since the aggregation results must also obtain to achieve the parsimony of parameters.
The motivation for the assumptions of Theorem 3.6 came from a desire to use capital asset pricing model (CAPM) results with and without a riskless asset (and hence, with and without complete markets) to get two fund separation. The two fund separation results without a riskless asset follow from Black [1972]. For the CAPM, the market price of risk can be expressed in terms of each agent's "global risk aversion"

$$\frac{E U_i''(\bar{x})}{E U_i'(\bar{x})}$$

(cf. Rubinstein [1973]). Since this is constant only for exponential utility, exponential utility was chosen for the model.

It would appear, then, that the only real generalization that this model admits (that retains the parsimonious aggregation and separation properties) is the generalization to several multivariate normal assets. The results are straightforward, since separation obtains even without the riskless asset (as in the Black [1972] CAPM). It is even hard to incorporate incomplete markets in the second period as well as the first, since then the derived utility for date 1 wealth has a quadratic function in the exponent, yielding a bliss point of maximum global utility. Such an economy would not be meaningful without free disposal beyond the bliss point, and the resulting derived utility becomes intractable.
Conclusion

The models of this chapter were formulated to study whether current prices can reveal enough information about preference parameters to resolve some of the uncertainty about prices in the next period. For simplicity, all the models were constructed so that only an aggregate preference parameter had to be conveyed. Chapter 4 deals with the other major exogenous factor (besides preferences) that affects demands and prices: the allocation of endowments. Since, by definition, aggregation obtains when the allocation of endowments does not affect prices, this factor must be studied in a non-aggregation setting.
Footnotes to Chapter 3.

1. One may regard $\theta_i$ as a risk tolerance parameter, since (3.9) has decreasing absolute risk aversion, so that increasing $\theta_i$ is equivalent to increasing wealth and hence decreasing absolute risk aversion, or increasing risk tolerance.

2. It is valuable to establish that there is a unique price system that equilibrates the economy, so that (3.12) does represent the equilibrium price system. If prices are not unique, agents must have some way of knowing that equilibrium prices are represented by (3.12) rather than some other price system. In economies where all agents have utilities from one class that aggregates (that is, for which aggregate excess demand functions are unaffected by redistributions of endowments amongst individuals), aggregate excess demands and prices are formed as though all wealth were endowed upon a single agent (with suitable aggregate preferences), and, in effect become "one household economies." If in addition, demands are single-valued, suitably continuous and bounded from below, a one-household economy equilibrates with a unique price vector (cf. Arrow-Hahn [1971, pp. 217-220]). In finance, preferences aggregate iff utilities are all exponential, or all extended power with the same exponent or all extended log, as used in Chapters 3 and 5 (cf. Rubinstein [1974] and Brennan-Kraus [1978]). These utilities yield sufficiently well-behaved excess demands so that prices are unique in complete markets or markets where there is
portfolio separation yielding effectively complete markets (and hence admitting aggregation).

Another way to consider this, suggested by Stephen Ross, is to split all wealth between two identical aggregate investors and note that, in the Edgeworth box for a complete securities market, the contract curve is, by symmetry, a line. This line is also an Engel curve, and since the preferences that lead to aggregation in finance are all homothetic (through some point, perhaps \(-\infty\)), the indifference curves all cut the Engel line at the same angle so that only one price hyperplane can equilibrate the economy for any one set of endowments and preferences.

![Edgeworth box with linear contract curve, homothetic indifference curves and endowment e.](image)

Figure 3.1. Edgeworth box with linear contract curve, homothetic indifference curves and endowment e.
3. Checking to see whether the bases of the exponents used throughout this analysis are positive, note that we must assume \( \theta_A + X_c > 0 \), \( c = 1, \ldots, C \). By (3.12) and (3.15), \( \rho_s > 0 \) (\( s = 1, \ldots, S \)) and \( \rho_0 > 0 \). Also, from (3.12) and (3.15),

\[
\frac{\sum_c \pi_{sc} (\theta_A + X_c)^Y}{\sum_c \pi_{sc} (\theta_A + X_c)^{Y-1}} > 0
\]

and

\[
\frac{\sum_c \pi_{sc} (\theta_A + X_c)^Y}{\sum_c \pi_{sc} (\theta_A + X_c)^{Y-1}} > 0
\]

Also, in equilibrium,

\[
\theta_i + x_{isc} = \theta_i + f_{is} + m_{is} X_c
\]

\[=(\theta_i + y_{is}) \left(\frac{(\theta_A + X_c)}{\theta_A + \rho_s}\right).
\]

This is positive iff \( \theta_i + y_{is} > 0 \). By similar reasoning,

\[
\theta_i + y_{is} = (\theta_i + y_{i}) \left(\frac{(\theta_A + \rho_s)}{\theta_A + \rho_0}\right)
\]

which is positive iff \( \theta_i + y_{i} = \theta_i + m_i \rho_0 > 0 \). Since \( \rho_0 > 0 \), this is positive under a wide range of circumstances -- e.g., \( \theta_i > 0, m_i > 0 \). Note that one rationale for extended power and log utility is that \(-\theta_i > 0\) is a minimum subsistence level of consumption, so that \( x_{isc} \) is consumption beyond the subsistence level. This rationale fails if \( \theta_i > 0 \).
Finally, we note that for certain parameter values, some agents may go bankrupt in some states -- i.e., have $x_{isc} < 0$ or $y_{is} < 0$ for some $i$, $s$, $c$. As long as agents are not allowed to default at date 2 if $x_{isc} < 0$ (e.g., negative consumption of financial wealth is feasible if agents have human wealth that they can use to pay off debts at date 2), then, even if $y_{is} < 0$ at date 1, they will choose to hold a portfolio as indicated by the sharing rules (and an "interior" optimum for their first order conditions) rather than default at date 1, so long as $\theta_{i} + y_{is} > 0$, as indicated in Figure 3.2 below.

![Figure 3.2 Interior optimum](image)

The indifference curves for $f_{is}$ and $m_{is}$ induced by the strictly concave von Neumann-Morgenstern utility cross the $m_{is}$ axis, but do yield interior optima satisfying the standard Lagrange equations. To see this, imagine that the $\theta_{i}$ is human wealth to be received at date 2 in the proof of Theorem (3.1), and utility is $-\gamma^{-1}(x)\gamma$.
where \( x \) is total final wealth. See also Theorem 3.3.

4. I am indebted to S. Grossman for reminding me of this nice feature of the model.

5. Now, we require \( \theta_A > X_c \), \( c = 1, \ldots, C \), for \( \rho_0 \) and \( \rho_s \) to be real. As in footnote 3, this ensures that the bases of exponents will be positive, as long as \( \theta_i > y_i, i = 1, \ldots, I \).

6. For example, if asset markets are complete at date 1 in state \( s \), agent \( i \) can be viewed as facing risky returns with \( S+1 \) states (consumption and the \( s \) previous risky states). There is no problem resulting from the failure of the probabilities to sum to 1. By the previous separation results, then,

\[
\begin{align*}
X_{is} &= f_{is} + m_{is} X_s \\
X_{isc} &= f_{isc} + m_{isc} X_{sc}
\end{align*}
\]

where \( X_s \) is the aggregate supply of the consumption good in state \( s \). Hence, only two vectors, \((f_{is}, \ldots, f_{Is}) \in I^I\) and \((m_{is}, \ldots, m_{Is}) \in I^I\), are needed to span the space of all agents optimal holdings of risky and consumption assets.

7. That is, the date 1 market is complete and yields aggregation and separation because of the extended power utility. The date 0 market is incomplete and will not aggregate, even with the extended power derived utility. However, the normally distributed random variables at date 0 induce a mean variance model which, of course separates, even without a riskless asset.
Chapter 4  Revelation of Individual Endowments

Introduction

In the last chapter we examined the possibility of prices revealing preference data (actually an aggregate preference datum), which could be used by agents to make inferences about future prices. Here, we study whether prices can signal information about the distribution of endowments of wealth. Since this information is necessarily multivariate, we study conditions under which a vector of endowments can be signalled by prices. In order to infer the value of a vector in Euclidean n-space ($\mathbb{R}^n$), one must, in general, be able to observe a related vector in $\mathbb{R}^m$ where $m \geq n$. For example, if $(x,y,z)$ denotes a point in $\mathbb{R}^3$, one can infer the position of the point using three spherical coordinates (two angles and a radius) or three cylindrical coordinates (an angle, a height and a radius), but never with two coordinates only. In the absence of degeneracies, it is necessary to have $m \geq n$ in order to convey n-dimensional information with an m-vector. The condition $m \geq n$ is not sufficient and the problem must be studied more carefully in a specific setting. In the last chapter, we had $m = n = 1$, since $p_0 \in \mathbb{R}^1$ and $\theta_A \in \mathbb{R}^1$. In that case, a FRE existed. Most of the rational expectations literature to this date only discusses the existence of a FRE when $n \leq 1$. For example, Radner [1977] studies the revelation of discrete information (n=0) by prices, and Grossman [1976] has n=1 (an aggregate information parameter).
To establish the existence of a FRE, one must show that a map from the parameter space (a subset of $\mathbb{R}^n$) to the price space (a subset of $\mathbb{R}^m$) is invertible. (If the social endowment is nonrandom, we drop one dimension from the price space, by Walras' law.) It is generally much harder to show constructively that a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is one-to-one when $m - n > 1$ than it is when $m = n = 1$. The analyses of multivariate FRE's are by Allen [1978, 1979]. She establishes the genericity of a FRE. A property (of an economy) is generic if arbitrarily small adjustments to the parameters (tastes, endowments, probabilities, etc.) do not destroy the property, and, given an economy without the property, there are arbitrarily close economies that exhibit that property. Thus, a generic property is open and dense in the family of economies under some appropriate topology. The Allen results rely on the Whitney embedding theorem of differential topology (see, e.g., Hirsh [1976]) to get the genericity of the existence of a FRE when $m \geq 2n + 1$. Using a diffuseness assumption on preferences and endowments, Allen [1979] weakens the dimensionality requirement to $m \geq n + 1$, if agents are not concerned that the price map is many to one on a set of probability zero. The results are not robust with respect to relaxation of the requirement that $m \geq n + 1$, for Jordan and Radner [1977] provide a generic nonexistence result when $m = n = 1$. The use of differential topology is somewhat nonconstructive and results using it are hard to grasp intuitively.

In order to be more constructive, and have $m=n$, we shall largely rely on the better known implicit function results which are based on the rank of the Jacobian matrix of a transformation. These results are only of a local nature, so we shall study the existence of
locally fully informing rational expectations equilibria (LFRE) for which there exists some open set of information parameters such that economies restricted to this open set are FRE. It is interesting to note that the Jordan-Radner counterexample is locally fully informing (LFRE) at all but one value of the exogenous parameters of the economy.

The analysis is based on a power utility economy (where agents have different powers, so aggregation fails and distribution of endowments matters). The market structure is merely a complete market version of the three-date market used in Chapter 3, although it is valuable to consider some related parallel economies to get the generic results.

Market structure, notation and preferences

As in Chapter 3, there are three dates, with S states at date 1 and C states at date 2. Transition probabilities are, at date 0, $\pi_s$ ($s = 1, \ldots, S$), and at date 1, state $s$, $\pi_{sc}$ ($c = 1, \ldots, C$). At date 0, agent $i$ has an endowment of $y_i$ and trades to date 1 state $s$ contingent wealth $y_{is}$. At date 1, agent $i$ trades to date 2 state $c$ contingent wealth $x_{isc}$, which he consumes, achieving utility

$U_i(x_{isc}) = y_i^{-1} x_{isc} y_i$ (0 if $y_i<1$) or $U_i(x_{isc}) = \log x_{isc} (y_i=0)$

Exogeneous social endowments of wealth at dates 0, 1, and 2 are, respectively

$Y = \sum_i y_i$

$Y_s = \sum_i y_{is}$

$X_c = \sum_i x_{isc}$
Note that, here, \( Y \) and \( Y_g \) are both given exogenously, in contrast to the models of Chapter 3, where \( Y = \rho_0 \) and \( Y_g = \rho_g \) were endogenous prices. The values of the \( Y_g \) are exogenously revealed to all agents at date 0, although agents do not know the value of \( Y \). Markets provide a complete set of financial claims to wealth contingent on the state of nature at the next date. Prices are:

\[
q_s = \text{date 0 price of a claim on $1 at date 1, contingent on intermediate state } s.
\]

\[
p_{sc} = \text{date 1 price when intermediate state } s \text{ occurs of a claim on $1 at date 2, contingent on final state } c.
\]

If agent \( i \) knows the prices \((p_{11}, ..., p_{sc}, q_1, ..., q_S)\) at date 0, he can solve the usual dynamic programming problem for portfolio demands. Thus at date 1, in state \( s \) he performs

\[
\max_{\{x_{isc}\}} \sum_c P_{sc} U_i(x_{isc})
\]

subject to

\[
\sum_c p_{sc} x_{isc} = y_{is}
\]

At date 0, he performs

\[
\max_{\{y_{is}\}} \sum_s \sum_c \Pi_{sc} U_i(x_{isc})
\]

subject to

\[
x_{isc} \text{ optimal in date 1, state } s \text{ and }
\]

\[
\sum_s q_s y_{is} = y_i
\]

Denote this market regime by \( M \). It is represented by the tree in Figure 4.1.
Figure 4.1 Market Regime M

\[ \sum_i y_i = Y \]

\[ q_s \]

\[ \pi_s \]

\[ \Sigma y_{is} = Y_s \]

\[ \pi_{sc} \]

\[ \Sigma x_{isc} = X_c \]

(date 0 market)

(date 1 market)

Figure 4.2 Market Regime M'

\[ \Sigma_i y_i = Y \]

\[ r_c \]

\[ \bar{\pi}_c = \Sigma_s \pi_s \pi_{sc} \]

\[ \Sigma_i x_{ic} = X_c \]

(date 0 market)

Figure 4.3 Market Regime M''

\[ \Sigma_i y_i = Y \]

\[ \pi_s \]

\[ \Sigma y_{is} = Y_s \]

\[ \pi_{sc} \]

\[ \Sigma_i x_{isc} = X_c \]

(date 0 market)
It is also useful to consider two related market regimes.

First consider the complete market \( M' \) for date 0 claims to money in date 2 state \( c \), and for which there are no intermediate states.

Let \( r_c = \) date 0 price of claim to $1 at date 2, contingent on final state \( c \).

\[ x_{ic} = \text{payoff to individual}\ i's\ portfolio\ if\ final\ state\ c\ prevails\ at\ date\ 2.\]

\[ \bar{\pi}_c = \sum_s \pi_s \cdot \Pi_{sc}, \text{the unconditional probability of state } c \text{ occurring.} \]

The budget constraint of agent \( i \) is

\[ (4.6) \quad \sum_c r_c x_{ic} = y_i \]

Demands by individual \( i \) are determined by (4.6) and the first-order conditions

\[ (4.7) \quad \frac{U'_ic(x_{ic})}{U'_id(x_{id})} = \frac{r_c/\bar{\pi}_c}{r_d/\bar{\pi}_d}. \]

Equilibrium requires that demands satisfy the market clearing condition

\[ (4.8) \quad \sum_i x_{ic} = X_c. \]

Now consider a more refined complete market \( M'' \) for claims to money in final states in which intermediate states exist and the final state claims are also contingent on the identity of the intermediate state as well as the final state.

Let \( \bar{p}_{sc} = \) date 0 price of claim to $1 at date 2, contingent on intermediate state \( s \) at date 1 and final state \( c \) at date 2.

Agent \( i \)'s budget constraint is

\[ \sum_s \sum_c \bar{p}_{sc} x_{isc} = y_i. \]
The first order conditions for agent $i$ are

$$\frac{U'_{ic}(x_{isc})}{U'_{id}(x_{itd})} = \frac{\bar{p}_{sc}/\pi_{sc}}{\bar{p}_{td}/\pi_{td}} (s,t = 1, \ldots, S; c,d = 1, \ldots, C).$$

The market clearing condition is (4.3).

Observe that the equilibrium that results in markets $M'$ also satisfies the equilibrium conditions for market $M''$, where

$$x_{isc} = x_{ic}$$

(4.9) $$\bar{p}_{sc} = (\pi_{sc}/\pi_{c})r_{c}.$$ Markets $M'$ and $M''$ present the same opportunity set to all individuals and so market values must be the same in both. Assume $M''$ has a unique equilibrium. 2)

The reason for introducing markets $M'$ and $M''$ as a step in the analysis of the market $M$ is that Arrow [1953] has shown that, if all individuals are informed of ex post equilibrium prices in market $M$, individuals will face the same opportunity sets and achieve the same final state payoffs in $M$ and in $M''$. Since $M'$ and $M''$ are equivalent, we can analyze $M$ through consideration of the simpler market $M'$.

Arrow has also shown that equilibrium prices in $M$ and $M''$ are related by

$$q_{s}p_{sc} = \bar{p}_{sc}.$$ Summing individual final period budget constraints in $M$ and applying the market clearing relations (4.2) and (4.3) yields

$$\sum_{c}p_{sc}x_{c} = Y_{s}.$$
Therefore, equilibrium prices in M and M' are related by

\begin{align}
q_s &= \left(\pi_s/Y_s\right)\pi_c \left(\bar{\pi}_c/\pi_c\right)r_cX_c \\
p_{sc} &= \left(\pi_s\pi_{sc}/\bar{\pi}_c\right)\left(r_c/q_s\right).
\end{align}

The revelation of information

Assume that all agents know that everyone has power or log utility of the form (4.1), and they all know that all agents have utility exponents from the set \( \{\gamma_1, \ldots, \gamma_I\} \). However, agent i does not know how much wealth \( y_j \) the other agents \( (j=1, \ldots, i; j \neq i) \) have. There may be a class of several agents with the same \( \gamma_i \), in which case the aggregate demand function for the class will be the same as if all wealth of the class were bestowed on a single agent with the same utility function as all members of the class. Thus, although an agent knows his personal wealth, in general, he will not know the aggregate wealth \( y_i \) of his class, so that all agents desire to infer the whole vector of wealths \( (y_1, \ldots, y_I) \in \mathbb{R}^I \).

At date 0, agents do not know the \( y_i \) or the aggregate \( Y \), but do know the aggregate social endowments of date 1 wealth \( \{Y_S\} \) and date 2 wealth \( \{X_C\} \). As in Chapter 3, they have homogeneous probability beliefs \( \{\pi_S, \pi_{SC}\} \). They desire to infer the value of \( (y_i) \) and hence the state conditional date 1 price vector \( (p_{sc}) \).

It is convenient to consider first the market \( M' \) in this setting and then get \( q_s \) and \( p_{sc} \) in M by using (4.10) and (4.11). Agent i's demand function at date 0 in \( M' \) can be derived as

\[ x_{ic} = \left(y_i(r_c/\pi_c)^{\delta_i}\right) \left(\sum_d (r_d^{\delta_i+1} / \pi_d^{\delta_i})\right)^{-1} \]

where \( \delta_i \equiv (\gamma_i - 1)^{-1} \).
Differentiating totally with respect to \( x_{ic}, y_i \) and \( r_c \), summing over individuals and noting that \( X_c \) is fixed, so that \( \sum_i dx_{ic} = dX_c = 0 \), yields the following relation expressed in matrix form:

\[
(A - X'GBX) \, dr + X'G \, dy = 0
\]

where

\[
A = \text{diag} \left[ \sum_i s_i x_{ic} / r_c \right]
\]

\[
X = [x_{ic}]
\]

\[
B = \text{diag} \left[ s_i + 1 \right]
\]

\[
G = \text{diag} \left[ y_i^{-1} \right]
\]

\[
dr = (dr_1, \ldots, dr_C)'
\]

\[
dy = (dy_1, \ldots, dy_I)'
\]

and "diag" denotes a diagonal matrix.

Assuming prices are differentiable functions of wealth endowments, (4.12)

\[
dr = - (A - X'GBX)^{-1} X'G \, dy
\]

Also, (4.10) can be expressed as (4.13)

\[
q = E\Pi F r
\]

where

\[
q = (q_1, \ldots, q_S)'
\]

\[
E = \text{diag} \left[ \pi_s / Y_s \right]
\]

\[
\Pi = [\pi_{sc}]
\]

\[
F = \text{diag} \left[ X_c / \bar{\pi} \right]
\]

\[
r = (r_1, \ldots, r_C)'
\]

Hence, changes in observed prices are related to changes in wealth endowments by (4.14)

\[
dq = Hdy
\]
where
\[ dq \equiv (dq_1, \ldots, dq_S)' \]
\[ H \equiv -E\Pi F(A-X'GBX)^{-1}X'G \]

Applying the implicit function theorem, \((y_i)\) is locally a function of \((q_1, \ldots, q_S)\) if the coefficient matrix \(H\) has rank \(I\). If this is the case, then \(p_{SC}\) is locally a function of \((q_1, \ldots, q_S)\) because \(p_{SC}\) is a function of \(r_c\) and \(q_s\) by (4.11) and \(r_c\) is a function of \((y_1, \ldots, y_1)\) by the uniqueness of prices in economy \(M'\). Hence, a LFRE would exist.

Observe that \(E\Pi F\) is \(SxS\), \((A-X'GBX)^{-1}\) is \(CxS\), and \(X'G\) is \(CxI\). Clearly, rank \((H) < I\) if \(C < I\) or \(S < I\), so that prices \(q\) cannot communicate all of the \(I\)-dimensional \((y_i)\) information. However, this need not rule out the existence of a FRE. Moreover, even if \(C \geq I\) and \(S \geq I\), it is surprisingly hard to establish conditions for rank \((H) = I\). The problem requires an analysis by cases.

**Case 1:** \(C \leq S\).

**Theorem 4.1:** A sufficient condition for the existence of a FRE (global, not just local) is that rank \((\Pi) = C\).

**Proof:** Since \(E\) and \(F\) are nonsingular diagonal matrices, rank \((E\Pi F) = \text{rank } (\Pi) = C\). Thus the mapping (4.13) of \(r\) to \(q\) is invertible. By observing \(q\), an agent may infer \(r\) and hence \(\{p_{SC}\}\) by (4.11). This allows him to select his date 0 portfolio in \(M\) as though he were fully informed.

Q.E.D.

Note that in Theorem 4.1 agents do not learn all information about \(y \equiv (y_1, \ldots, y_1)\), but just enough about it (i.e., about which \(r\) and \(\{p_{SC}\}\) obtains) to behave optimally.
Case 2: $S < \min (I, C)$. With $I > S$ and $C > S$, there are not enough dimensions of variation in $(q_1, \ldots, q_S)$ to inform market participants of the value of either $(r_1, \ldots, r_C)$ or $(y_1, \ldots, y_S)$. Thus, we have:

Theorem 4.2: If $S < \min (I, C)$ neither a LFRE nor FRE exist, unless the market is degenerate to allow aggregation of individuals (thus effectively reducing $I$) or of final states (thus reducing $C$).

Case 3: $I < S < C$. This case is probably the most interesting and certainly the most intractable. The interest stems from the fact that this case places no upper bound on the number of possible states of the world. There are $SC$ prices $p_{sc}$ which an agent would like to know in order to solve his dynamic programming problem in market $M$ at date 0. By knowing the probabilities $\{\pi_{sc}\}$ and $\{\pi_s\}$, an individual can reduce this information collection problem, by using (4.10) and (4.11), to one of finding out $C$ dimensions of information for the $r_c$'s to supplement the information $(q_1, \ldots, q_S)$ which he can observe in the date 0 market. However, if $C > S$, he still cannot learn all of the information he needs unless it actually has a lower dimension than $C$. Since the $r_c$'s are determined by only wealth levels, $y_i$, there are FRE's for arbitrarily large numbers, $C$, of possible final states as long as the number of categories of agents, $I$, does not become larger than $S$, the number of observable signals (prices).

Unfortunately it is very hard to get positive or negative results in this case. First, here is a somewhat negative result.
Counterexample: If $1 \leq S < C$, there are economies for which $H = - E F(A-X'GBX)^{-1}X'G$ has rank less than $I$. For example, let $S = I = 2$, $C = 3$, and let $\eta_1 \neq 0$. and let $\eta_1 + \eta_2 = 0$. One may think of $(\eta_1, \eta_2)'$ as a vector of proportions of wealth reallocations $dy = (dy_1, dy_2)'$ such that total wealth $Y = y_1 + y_2$ is constant ($dY = 0$). Let $\xi = (\xi_1, \xi_2, \xi_3)' = - F(A-X'GBX)^{-1}X'G(\eta_1, \eta_2)'$. It is shown in the Appendix to this chapter that we can select $\Pi$ of full rank such that it is a proper conditional probability matrix and $\Pi \xi = 0$ (and hence rank $H \leq 1 < I$).

The counterexample shows that it is not always the case that the standard hypotheses of the implicit function theorem hold and hence guarantee the existence of a LFRE. It is not necessarily a counterexample to the existence of a LFRE. There are many 1-1 mappings which have a singular Jacobian at some point in their domain. For example, let $f(x) = x^3$. Since $f'(0) = 0$, the Jacobian is singular at 0, even though the function is strictly monotone increasing and invertible. What is needed (and has not been found) is a differentiable path $P$ given by $y(t) = (y_1(t), ..., y_I(t))$ for $t \in [0, 1]$ such that $q(t) - q(0) \equiv - \int_{t=0}^{t} E F(A-X'GBX)^{-1}X'Gdy(\tau) = 0$ ($t \in [0, 1]$) but $r(t_2) - r(t_1) \equiv \int_{t=t_1}^{t_2} (A-X'GBX)^{-1}X'G dy(\tau) \neq 0$ for almost all $t_1 < t_2$.

There would be no LFRE's anywhere along this path since $q$ would be constant along the path, even though $r$ continuously varies. An individual cannot tell where the economy is on the path, but needs the information for his dynamic program.
There are two positive results on the existence of LFRE's, however. For example, they do exist.

**Theorem 4.3**: If, in economy $M'$, the matrix of optimal holdings $X$ has rank $I$ and $I \leq S \leq C$, and agent 1 has $-1 < \gamma_1 \leq 0$, there exist arbitrarily small changes in $\{\pi_s\}$, $\{\pi_{sc}\}$, $\{X_c\}$ and $y_1$ such that the resulting economy has a LFRE (i.e., where $q$ is locally an invertible function of $y$). That is, the set of LFRE economies is dense in the set of economies of full rank $X$ matrices.

Moreover, if a power utility economy $M'$ has rank $(H) = I$, (and hence is a LFRE), it remains a LFRE under arbitrarily small changes in the exogenous parameters of the economy. That is, the implicit function sufficient condition for existence of a LFRE is an open property.

**Comments**: Coupled with the counterexample, this theorem suggests that the matrix $[\pi_s \pi_{sc}]$ of joint intermediate and final state probabilities is crucial in determining whether enough information about market $M'$ prices $r$ can be communicated by date 0 market $M$ prices $q$. This matrix of probabilities constitute a veil that agents may or may not be able to see through at date 0.

The assumption about the rank of $X$ will be weakened in Theorem 4.4.
Proof of Theorem 4.3

First, we shall show that we can perturb the \( \pi 's \) to ensure rank 
(\( H \)) = I, so that a LFRE exists by the implicit function theorem (see, e.g., Dieudonné [1960], Rudin [1964]), provided that \( A-X'GBX \) is 
invertible. Recall that

\[
H \equiv - E \Pi F(A-X'GBX)^{-1}X'G
\]

\[
= - \text{diag}[Y^{-1}][\pi_s|c]D
\]

where

\[
\pi_s|c = \frac{\pi_s\pi_{sc}}{\pi_c} = \text{Pr} \text{ (intermediate state } s|\text{final state } c)\]

and

\[
D \equiv \text{diag}[X_c](A-X'GBX)^{-1}X'G
\]

Clearly, we can vary the \( \pi_s \) and \( \pi_{sc} \) so that the \( \pi_s|c \) change but the

\( \pi_c \) and hence \( X, A, G, B \) and \( D \) are constant, since they are formed
in economy \( M' \), which is unchanged. The rank of \( D \) is I since the
rank of \( X \) is I, and \( D \) is just \( X \) pre- and post-multiplied by invertible
matrices. Let \( \tilde{D} \) be the square submatrix formed by taking the first
I rows of \( D \) and suppose w.l.o.g. that rank \( (\tilde{D}) = I \). Also, we may
delete rows of \( [\pi_s|c] \) so that we can assume w.l.o.g. that \( S = I \).

Then, the determinant of \( [\pi_s|c]D \) is a multinomial in \( \{\pi_s|c \ s = 1, \ldots, \ I - 1; \ c = 1, \ldots, \ C\} \), say

\[
\det([\pi_s|c]D) = m(\pi_s|c)
\]

If we cannot slightly perturb the \( \{\pi_s|c\} \) so that rank \( (H) = I \),
then \( m(\pi_s|c) \equiv 0 \) on some open set (in the product of projections of
\( S - \text{simplices into } \omega^{S-1} \)). As in the discussion of the counterexample
in the Appendix, this can only occur if all the coefficients of \( m \)
are identically 0. But, this cannot be so, for if we define

\[
\pi_s|c = \begin{cases} 
1 & \text{if } s = c, \\
0 & \text{otherwise}
\end{cases}
\]
then $m(\pi_s|_c) = \det(\tilde{D}) \neq 0$. Hence, for some arbitrarily small perturbation of the $\pi_s|_c$'s, $H$ has rank $I$ and a LFRE exists.

To see that the property "rank $(H) = I$" occurs on an open set, note that prices, demands, etc. are all continuous in the exogenous parameters of the economy (if prices are differentiable) and that the set of SXI matrices with full rank (i.e., having submatrices with nonzero determinant) is open in $\mathbb{R}^{SI}$.

Now it only remains to be shown that the invertibility of $(A-X'GBX)$ is an open and dense property itself. Clearly, if $A-X'GBX$ is nonsingular, the behavior of the economy is sufficiently continuous for the nonsingularity to hold under small perturbations of the parameters of the economy.

If $-1 < \gamma_1 \leq 0$ ($i = 1, \ldots, I$), the matrix $-A + X'GBX$ has all positive elements and a dominant diagonal (cf. Gale-Nikaido [1965]) and hence is positive definite and invertible. A $CxC$ matrix $M$ has a dominant diagonal if there exist $\alpha_c > 0$ ($c = 1, \ldots, C$) such that

$$|m_{cc}| \alpha_c > \sum_{d=1}^{C} |m_{cd}| \alpha_d$$

(c = 1, \ldots, C).

In this case, take $\alpha_c = r_c^*$.

Suppose that $-1 < \gamma_1 \leq 0$, even though other agents' exponents may not satisfy this inequality. By increasing the wealth $y_j$ and the social endowments $\{X_c\}$ it is possible to move agent 1 along his Engel curve (a ray through the origin) without varying prices $\{r_c\}$ and other agents' allocations. This occurs if $dX_c = \frac{aX_c}{\partial y_1} dy_1 = \frac{x_1c}{y_1} dy_1$. 


This has the effect of adding more and more of a dominant diagonal matrix to $-A + X'GBX$. At some point in the process, the matrix $-A + X'GBX$ itself must assume a dominant diagonal and become invertible. Moreover, the matrix $-A + X'GBX$ is linear in $y_1$, provided the $X_c$'s vary as above so that $x^c_1$ is linear in $y_1$. Hence $\det (-A + X'GBX)$ is a $C^{th}$ degree polynomial in $y_1$ which does not vanish everywhere. Since its zeroes are isolated, there must be arbitrarily small perturbations of $y_1$ and $(X_c)$ that make $A-X'GBX$ invertible. Q.E.D.

The next theorem weakens the assumption about the rank of $X$.

**Theorem 4.4:** With the same hypotheses as in Theorem 4.3, except that the rank of $X$ may be less than 1, there exist arbitrarily small changes in probability beliefs $\{\pi_s\}$, $\{\pi_{sc}\}$ and endowments $\{y_1\}$, $\{X_c\}$ such that a LFRE exists. Moreover, the economy will be perturbed to a neighbourhood of its exogenous parameters on which rank $(X) = J$ for some constant $J \leq I$, and on which LFRE's exist.

**Comment:** The proof of this result uses the rank theorem (Dieudonné [1960, p. 273], Rudin [1964, p. 198]), which is a generalization of the implicit function theorem to mappings which are of locally constant (but not necessarily full) rank. The rank theorem establishes the existence of a mapping $y \to z \in \mathbb{R}^J$ such that $r$ is an invertible function of $z$. The techniques of Theorem 4.3 can then be used to adjust the joint probabilities $\{\pi_s, \pi_{sc}\}$ so that prices $q$ communicate the value of $z$ and hence $r$. The important point here is that it may not be necessary to signal all the endowment information, since $z$ is an adequate summary statistic.
Rank Theorem: Let \( y^\circ \in N \subseteq \mathbb{R}^I \), where \( N \) is an open set, and \( f : N \to \mathbb{R}^C \) be a continuously differentiable mapping such that, in \( N \), the rank of the Jacobian matrix is a constant \( J \). Then there exist:

1. an open neighbourhood \( U \subseteq N \) of \( y^\circ \) and a function
   \[
g : U \overset{1-1}{\longrightarrow} (-1,1)^I
   \]
such that \( g \) and \( g^{-1} \) are continuously differentiable (here, \((-1,1)^I \equiv \{y \in \mathbb{R}^I \mid |y_i| < 1, i = 1, \ldots, I\} \) is the open unit ball in \( \mathbb{R}^I \)), and

2. an open neighbourhood \( V \supseteq f(U) \) of \( f(y^\circ) \) and a function
   \[
h : (-1,1)^C \overset{1-1}{\longrightarrow} V \text{ with } h \text{ and } h^{-1} \text{ continuously differentiable,}
   \]
such that \( f = h \circ f^\circ \circ g \) where

   \[
f^\circ : (-1,1)^I \to (-1,1)^C \text{ by } f^\circ(z_1, \ldots, z_I) = (z_1, \ldots, z_J, 0, \ldots, 0).
   \]

This theorem says that, if the mapping \( f \) has constant rank \( J \) on some open set, then the action of \( f \) can be summarized by \( J \) variables \( (z_1, \ldots, z_J) \). If \( J = I \), it is the usual implicit function theorem.

Proof of Theorem 4.4

Theorem 4.4 is proved by first perturbing \( y_1 \) and \( (X_C) \) to a point where \( A-X'GBX \) has full rank, and then perturbing \( y \) to some point about
which there is a neighbourhood on which the price map \( f : y \rightarrow r \in \mathbb{C} \) has a Jacobian of constant rank \( J \) while \( A-X'GBX \) stays of full rank as in the proof of Theorem 4.3. The desired result follows by varying \( \pi_s|_C \) as in the proof of Theorem 4.3 to ensure that the prices \( q \) are a 1-1 function of the variables \( z_1, \ldots, z_J \) whose existence is established by the Rank Theorem.

First, perturb \( y_j \) and \( (X_C) \) to a neighbourhood on which \( A-X'GBX \) is invertible. Then note that \( y \) can be perturbed to a point \( y^0 \) about which there is an open neighbourhood \( N \) on which rank \( (X) \) is locally a constant, say \( J \). This occurs because \( \{ y : \text{rank}(X) \geq J' \} \) is open for any \( J' \), due to the continuity of the mapping \( y \rightarrow \det(\tilde{X}) \) in market \( M' \) where \( \tilde{X} \) is any square submatrix of \( X \). Since rank \( (X) \) is bounded above by \( I \), we can choose any \( y^0 \) for which the corresponding matrix \( X^0 \) has rank \( J \) where \( J \) is the lim sup of the ranks of the \( X \) matrices as the endowment vector approaches \( y \).

Let \( f : y \rightarrow r \) be the mapping sending endowments to prices \( r \) in market \( M' \). The Jacobian matrix

\[
\frac{\partial f}{\partial y} = - (A-X'GBX)^{-1}X'G
\]

has constant rank \( J \) in the neighbourhood \( N \), since it is merely the matrix \( X' \) pre- and post-multiplied by invertible matrices.

Now, apply the Rank Theorem. Using the notation \((z_1, \ldots, z_J, 0, \ldots, 0) = f \circ g(y)\), one can see that prices \( r = h(z_1, \ldots, z_J, 0, \ldots, 0) = h(z) \), say, where \( z = (z_1, \ldots, z_J) \).
By (4.13)

\[ q = E F r \]

\[ = \text{diag}[Y_s^{-1}] [\pi_s|_c] \text{diag}[X_c] \circ \tilde{h}(z) \]

where the conditional probabilities \( \pi_s|_c \) are defined in the proof of Theorem 4.3.

\[ \therefore \frac{\partial q}{\partial z} = \text{diag}[Y_s^{-1}] [\pi_s|_c] \text{diag}[X_c] \frac{\partial \tilde{h}}{\partial z} \]

By construction, \( \partial \tilde{h}/\partial z \) has full rank \( J \), so it is possible to use the same type of argument as in the proof of Theorem 4.3 to find an arbitrarily small perturbation of \( \{\pi_s|_c\} \) so that \( \partial q/\partial z \) also has full rank \( J \). That is, \( q \) is locally an invertible function of \( z \).

Thus, knowing the \( q \)'s, agents can infer the \( z \)'s, and from the \( z \)'s they can infer prices \( r = \tilde{h}(z) \), which allows them to select their optimal demand functions for \( \{y_s\} \) at date 0.

Note that agents cannot learn all of the \( y \)'s. They need only know the relevant information impounded in \( z \). Unfortunately, \( z \) may not have any natural interpretation as an observable economic variable. 4)

Q.E.D.

Conclusion

In this chapter, we have studied whether or not there exist locally fully informing rational expectations equilibria (LFRE) in a power utility economy where agents are uncertain about future prices because they are uncertain about the current distribution of wealth. It was shown that LFRE's "usually occur" in the generic sense that they occur on an open and dense set of parameterizations of power utility economies in which at least one agent has a mild
restriction on his utility exponent, provided that there are more states of the world (and hence markets) at date 0 than there are individuals. Prices may not completely reveal endowments, but they tend to reveal enough information to forecast future prices. If there are not more markets than people, then a FRE exists when there are more date 1 states than date 2 states of the world.

For all of these results, a key issue is whether or not the structure of transition probabilities from date 1 states to date 2 states is rich enough to signal information about date 1 prices with date 0 prices.

Note that although the analysis of this model used the complete market structures M' and M", so that the S markets at date 0 in M will, by Arrow's theorem, allow ultimate allocations as if the markets were complete in date 2 goods, the Hirshleifer-Marshall result does not yield the rollover strategy that flawed the last chapter. This happens because agents must know date 1 relative prices to optimally convert the S securities of the first period to the C securities of the second period. The next chapter deals with constructive models that are not flawed by the rollover strategy.
Footnotes to Chapter 4.

1. Openness alone is not a strong condition, since the unit ball is open in &n, but in many senses is an insignificant part of &n. Density alone is not strong since the property may fail under small perturbations. Alternatively, some writers connote genericity with full Lebesgue measure. Even if the underlying topology is the usual metric topology on &n, neither notion of genericity implies the other. Although sets of full Lebesgue measure are dense, they are not necessarily open. On the other hand, there are open and dense sets of less than full Lebesgue measure. As an example, consider a modification of the construction of the Cantor set obtained by deleting shorter and shorter closed intervals from the unit interval in &1 so that in the limit the remainder (a modified Cantor set) has measure ½ (rather than zero, as for the Cantor set). This set is closed and nowhere dense, so that its complement is open and dense, but of less than full measure, cf., Taylor [1973, p. 94].

2. If γ1 = γ2 = ... = γI, preferences aggregate and prices are unique by footnote 2, Chapter 3. Hence, prices should also be unique if the γI's do not vary "too much" cross-sectionally. If prices are not unique, we must assume all agents know which rule the Walrasian auctioneer follows in selecting prices.

3. Alternatively, one may think of one agent per class, so that agent i desires to know only I-1 other endowments, given that social endowments are true social endowments less his demands. Essentially, the same results of this chapter hold under these circumstances with obvious modifications, such as replacing I with I-1.
4. A reasonable conjecture is that, if all agents or classes of agents have different power utility exponents, then \( J = I \) holds generically. That is, the set of endowments and probabilities for which \( \text{rank } (X') = I \) is dense and open in the set of all endowments and probabilities. The conjecture is true for \( I = 2 \), for, suppose agent \( i \) trades to \( x_i \in \&^C \). Then \( \text{rank } (X') < I \) iff \( x_i = \alpha x_2 \) for some \( \alpha \in \& \). That is, \( x_1 \) and \( x_2 \) lie on the same ray through the origin. Since the Engel curves are rays through the origin with power utility that differ only when the powers differ, it must be that \( y_1 = y_2 \) and \( y_1 = \alpha y_2 \).
Appendix to Chapter 4

Details of the counterexample providing a Jacobian of less than full rank

We desire \( \pi_{sc} > 0 \) such that

\[
\begin{align*}
(4A.1) \quad & \pi_1 \pi_{11} + \pi_2 \pi_{21} = \pi_1 \\
(4A.2) \quad & \pi_1 \pi_{12} + \pi_2 \pi_{22} = \pi_2 \\
(4A.3) \quad & \pi_1 \pi_{13} + \pi_2 \pi_{23} = \pi_3 \\
(4A.4) \quad & \xi_1 \pi_{11} + \xi_2 \pi_{12} + \xi_3 \pi_{13} = 0 \\
(4A.5) \quad & \xi_1 \pi_{21} + \xi_2 \pi_{22} + \xi_3 \pi_{23} = 0,
\end{align*}
\]

where we are given \( \{\pi_c\}, \{\pi_s\} \) and \( \{\xi_c\} \). Recall the interpretation that \( \eta_i = dy_i \), so that

\[
\begin{align*}
\eta &= -(A'G B X)^{-1} X'Gdy \\
\xi &= F \eta
\end{align*}
\]

Equations (4A.1) - (4A.3) must be satisfied for the \( \pi_{sc} \) to be conditional probabilities and equations (4A.4) and (4A.5) say that \( \pi F \eta = 0 \) and hence \( \eta \equiv E \pi F \eta = 0 \). Now,

\[
\begin{align*}
\Sigma_{s,c} \pi_{s} \pi_{sc} c = \Sigma_{s,c} \pi_{s} \pi_{sc} X_c / \pi_c dr_c \\
&= \Sigma_c X_c dr_c \quad \text{(by definition of} \ \pi_c) \\
&= \Sigma_i dy_i \quad \text{(by Walras' law or (4.6) and (4.8))} \\
&= 0
\end{align*}
\]

Hence equations (4A.1) - (4A.5) are redundant, so we may drop (4A.5), leaving 4 equations in 6 unknowns. Solving for the other variables in terms of \( \pi_{11} \) and \( \pi_{12} \) one can see that the feasible region is
represented by

(4A.6) \[ 0 \leq \pi_1 \pi_{11} \leq \pi_1 \]

(4A.7) \[ 0 \leq \pi_1 \pi_{12} \leq \pi_2 \]

(4A.8) \[ 0 \leq \pi_1 \xi_1 \pi_{11} - \pi_1 \xi_2 \pi_{12} \leq \xi_3 \pi_3, \]

assuming w.l.o.g. that \( \xi_3 > 0 \). Since \( \xi_s, c \pi_s \pi_{sc} \xi_c = 0 \), the \( \xi_c \)'s are not all of the same sign, so assume w.l.o.g. that \( \xi_1 < 0 \).

Since \( \pi_1, \pi_2 \) and \( \xi_3 \pi_3 \) are all positive, there is a nonempty open subset \( N \) of the positive quadrant of \&^2, having the origin as a limit point, which is contained in the feasible set for \((\pi_{11}, \pi_{12})\) given by (4A.6) and (4A.8). Now, the determinant of the first two columns of \( \Pi \) is a multinominal in \( \pi_{11} \) and \( \pi_{12} \) (after solving for the other \( \pi_{sc} \)'s in terms of these), and hence can vanish everywhere in \( N \) only if it vanishes everywhere in \&^2. (If it vanishes in \( N \), all its derivatives must also vanish, and hence so must the coefficients.) But setting \( \pi_1 \pi_{11} = \pi_1 \) and \( \pi_{12} = 0 \) (so that \( \pi_2 \pi_{22} = \pi_2 \)) gives a nonzero determinant of the first two columns of \( \Pi \).

Hence one can choose \( \Pi \) of full rank to satisfy (4A.1) to (4A.5).
Chapter 5  Prices Revealing Aggregate Risk Aversion, Impatience and Probability Beliefs

Introduction

The two previous chapters showed how prices can reveal aggregate preference and endowment information in an exchange economy. The other major factor influencing prices is probability beliefs (when assets are risky). This chapter studies the question of when prices reveal preferences and probability beliefs. Two types of models are discussed. One involves a one period world with consumption at two dates and three securities: current consumption, a riskless bond and a claim on risky future consumption. The resulting two relative prices may reveal aggregate risk aversion and probability belief parameters. The other model involves two periods, so that the introduction of a term structure in the bond market creates three relative prices that may reveal an impatience parameter, in addition to risk aversion and probability parameters.

Typical rational expectations models in the literature involve the revelation of probability beliefs only, and the two previous chapters involve the revelation of preferences, so it is reasonable to ask why it should be important to analyse the joint revelation of these parameters. Part of the rationale is related to the motivation for interest in revelation of preferences alone. Agents' information about aggregate preferences may be based on econometric observation of prior markets (as in, for example, Friend and Blume [1977]), casual inferences drawn from observation of prior markets, and/or inferences drawn from observation of current market prices.
Only the last method requires rational expectations machinery like that developed in Chapter 3, and it is necessary only if preferences are unstable and cannot be accurately forecast from prior data. The finance literature has never seriously studied the possibility of unstable preferences, presumably because researchers have tended to ascribe variations in prices of risky assets to changes in beliefs or information, but not to changes in preferences. Without specifying functional forms for utilities and security payoff distributions, it is generally hard to distinguish between changes in information and changes in preferences as superior explanations for variations in security prices. As a result, the history of asset prices is usually explained by a sequence of information arrivals, because this is more amenable to empirical analysis than a sequence of changes in preferences.

However, there is some anomalous evidence about security prices that is hard to totally ascribe to a sequence of information arrivals. Shiller [1979] and LeRoy and Porter [1979] have analyzed the question of whether the variance rates in time series of stock prices are too high to be explained solely by information arrivals. For example, Shiller assumes that current stock prices should be the present value of optimal forecasts of future dividend payments. Since the error of an optimal forecast is orthogonal to the forecast, the variance of an actual dividend series should be greater than the variance of the stock prices. He finds this bound to be too seriously violated to be ascribed, for example, to variation in interest rates.
This seems to contradict the hypotheses of market efficiency and rational expectations, unless one also ascribes some of the variation in stock prices to changes in preferences.

Another, somewhat more casual, observation suggesting that preferences may vary over time is provided by van Horne [1978, pp. 155-161] who notes that bond risk premia (interest differentials between high and low grade bonds) vary over the business cycle more than is justifiable solely by changes in relative default risk. He suggests that investor preferences vary over the business cycle (people become more risk averse in recessions).

Related to van Horne's contention is the question of whether the suddenness and severity of the Great Depression are really adequately explained by the traditional explanations related to bank failures, money supply contractions or liquidity traps. A sudden increase in risk aversion could have precipitated the stock market collapse.

Thus, there is merit in studying the question of when variations in preferences can be distinguished from variations in information, given specific functional forms for preferences and beliefs.

Another feature of this chapter is that it provides a constructive analysis of fully revealing rational expectations equilibria (FRE's) in which multivariate information is revealed. This is in contrast to the local implicit function results of the previous chapters and the purely topological results of Allen [1978,1979], that establish generic existence only.
The chapter starts with a general analysis for exponential utility functions and arbitrary probability distributions and progresses to more specific results for two probability families: the normal and gamma families. It is a generalization of the analysis in Kraus-Sick [1980].

The central question of this chapter is whether prices can reveal probability information, risk aversion and impatience parameters. The aggregate risk aversion parameter studied here is derived explicitly from individual risk aversion parameters. However, only a single probability parameter for all agents is considered, although under some circumstances, one may prefer to think of it as an aggregate sufficient statistic of diverse information, as in Grossman [1976]. Alternatively, one may think of the value of the parameter being exogenously revealed to some informed agents, and communicated to other agents by market prices (in a FRE).

Similarly market prices depend on an aggregate impatience parameter \( n \), which with exponential utility may be thought of as a geometric mean of individual impatience parameters, as in Rubinstein [1974], or as a commonly held impatience parameter. In the latter case, one must justify why this parameter must be revealed to agents if they already know their own personal preferences. The inference of \( n \) may be of interest to some infinitesimal agent with an entirely unrelated preference structure, who does not affect market prices but would like to know more about next period
prices. This is the same rationale used in Chapter 3, when dis-
cussing the need for inference of future prices, in the presence
of the rollover algorithm.

Notation and setting

The following notation is similar to that used in the two
previous chapters:

There are I agents \((i = 1 \ldots, I)\)

\(y_{it} = \text{date } t \text{ wealth of agent } i \ (t = 0,1,2)\)

\(c_{it} = \text{date } t \text{ consumption of agent } i \ (t = 0,1,2)\)

\(f_{it\tau} = \text{agent } i\text{'s investment in the date } t \text{ riskless asset}
\text{ that matures at date } \tau \ (t, \tau = 0,1,2; t < \tau)\)

\(m_{it} = \text{agent } i\text{'s investment at date } t \text{ in the risky asset}
\text{ that pays off at date } t + 1 \ (t = 0,1)\)

\(D_{t\tau} = \text{date } t \text{ price of the riskless bond, maturing at}
\text{ date } \tau \ (t, \tau = 0,1,2; t < \tau)\).

\(P_t = \text{date } t \text{ price of the risky asset } (t = 0,1)\)

\(C_t = \text{aggregate date } t \text{ dividend yielded by risky asset}
\text{ } (t = 0,1,2)\).

Conditional on information available at any date prior to \(t\),
any variable with subscript \(t\) is, in general, random, and this
randomness will be denoted by a tilde (\(~\)).

The one period models involve dates 1 and 2 only and the two
period models involve all three dates.

Budget constraints are for dates 0 and 1, respectively:

\[(5.1) \quad y_{i0} = c_{i0} + f_{i01}D_{01} + f_{i02}D_{02} + m_{i0}P_0\]
\[ y_{i1} = c_{i1} + f_{102}D_{12} + m_{i0}P_1 \]  

Date 1 and date 2 wealth are realized as

\[ y_{i1} = f_{101} + f_{102}D_{12} + m_{i0}(P_1+C_1) \]  

\[ y_{i2} = f_{i12} + m_{i1}C_2. \]  

Note that (5.3) reflects the fact that the date 0 purchase of a two period bond \( f_{i02} \) yields \( f_{i02}D_{12} \) at date 1 and that the date 0 purchase of the risky asset \( m_{i0} \) yields \( m_{i0}C_1 \) in consumption dividend at date 1 with \( m_{i0}^P_1 \) remaining in capital value.

All wealth is consumed at date 2, so

\[ c_{i2} = y_{i2}. \]  

The date 0 market clears when

\[ \sum_{i=1}^{I} \begin{pmatrix} c_{i0} \\ f_{101} \\ f_{102} \\ m_{i0} \end{pmatrix} = \begin{pmatrix} C_0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]  

The date 1 market clears when

\[ \sum_{i=1}^{I} \begin{pmatrix} c_{i1} \\ f_{101} \\ m_{i1} \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \\ 1 \end{pmatrix}. \]  

The following assumption is maintained throughout:

(A1) The stochastic process generating \( (C_0, C_1, C_2) \) is a Markov process with independent increments:

\[ \tilde{\Delta}_t = \tilde{C}_t - \tilde{C}_{t-1}, \quad (t = 1, 2). \]  

The distribution of \( \tilde{\Delta}_t \) is assumed to depend on an arbitrary parameter \( \zeta_t \).
One-period models: Exponential utility

This section considers markets that are open at date 1 only.

Assume that agent i has intertemporally additive von Neumann-Morgenstern utility with constant absolute risk aversion for consumption at dates 1 and 2. That is

\( U_{i1}(c_{i1}, f_{i12}, m_{i1}) = -\exp(-\theta_i c_{i1}) - \eta E_{\xi_2} \exp(-\theta_i y_{i12}) \)

where \( y_{i12} \) is a function of \( f_{i12} \) and \( m_{i1} \) by (5.4) and the expectation operator \( E_{\xi_2} \) is for the distribution of \( \tilde{C}_2 \) conditional on date 1 information which includes \( P_1, D_{12} \) and \( C_1 \). The conditional distribution of \( \tilde{C}_2 \) is a function of the parameter \( \xi_2 \). Two distributions are considered here: normal and gamma. The coefficient of absolute risk aversion is \( \theta_i > 0 \) and the rate of impatience is \( \eta \), which is common to all agents. Assume

(A2) \( \eta \) is known to all agents at date 1.

At date 1, agent i maximizes (5.8) subject to (5.2) and (5.4). Assuming that the order of the expectation and differentiation operators can be reversed (as will be verified below), the first order conditions for agent i's demands can be written as in Chapter 3, as

\( \nabla U_{i1} = 0 \)

(5.9) \( D_{12} \exp(-\theta_i c_{i1}) = \eta E_{\xi_2} \exp(-\theta_i y_{i12}) \)

(5.10) \( P_1 \exp(-\theta_i c_{i1}) = \eta E_{\xi_2} \tilde{C}_2 \exp(-\theta_i y_{i12}) \).

Note that the Kuhn-Tucker-Lagrangian first order conditions are necessary and sufficient for an optimum because the objective function is strictly concave and the constraints are linear (cf. Zangwill [1969, ch. 2]).
At this point one may use specific probability distributions for the expectations and compute demand functions (at least in principle) and solve for equilibrium prices with the market clearing relation (5.7).

Alternatively, one can proceed as in Chapter 3 and note that, if all agents have the same beliefs and information, the first order conditions and market clearing conditions are satisfied when

\[
\begin{align*}
  m_{i1} &= \theta/\theta_i \\
  f_{i12} &= \frac{1}{1+D_{12}} (y_{i1} - m_{i1} (C_1 + P_1)) \\
  c_{i1} &= f_{i12} + m_{i1} C_1 \\
  (i &= 1, \ldots, I) \tag{5.11}
\end{align*}
\]

(5.12)

\[
P_1 = nE_{\xi_2} \tilde{C} \exp(-\theta(\tilde{C}_2 - C_1)) \quad \text{and,}
\]

(5.13)

\[
D_{12} = nE_{\xi_2} \exp(-\theta(\tilde{C}_2 - C_1))
\]

where \( \theta^{-1} \equiv \left( \sum_i \theta_i^{-1} \right)^{-1} \).

Multiplying (5.13) by \( C_1 \) and subtracting from (5.12) yields

(5.14)

\[
P_1 - C_1 D_{12} = nE_{\xi_2} (\tilde{C}_2 - C_1) \exp(-\theta(\tilde{C}_2 - C_1))
\]

Agents desire to infer \( \tilde{C}_2 \) (when \( \theta \) is unknown, but \( n, C_1 \) are known) from prices \( P_1 \) and \( D_{12} \) in a rational expectations equilibrium. Recalling that \( \tilde{\Delta}_2 \equiv \tilde{C}_2 - C_1 \), it is apparent from (5.13) and (5.14) that, by observing equilibrium prices, agents observe

\[
f(\theta, \tilde{\Delta}_2) \equiv E_{\xi_2} \exp(-\theta \tilde{\Delta}_2) \quad \text{and,}
\]
assuming, again, that the order of expectation and differentiation can be reversed.

As explained in Chapter 2 and applied in Chapters 3 and 4, the procedure for searching for a fully informing rational expectations equilibrium (FRE) will be to assume \( \theta \) and \( \zeta_2 \) are known to all, and then examine under what conditions the revelation of this information can be sustained in equilibrium.

One-period model: Normally distributed returns

Consider, first, a normally distributed \( \Delta_2 \). Suppose

\[
(5.15) \quad \Delta_2 \sim N(\zeta_2, \sigma_2^2)
\]

where \( \sigma_2^2 \) is assumed known.

Then

\[
D_{12} = \eta f(\theta, \zeta_2) = \eta \exp(-\theta \zeta_2 + \frac{\theta^2 \sigma_2^2}{2})
\]

and

\[
P_1 - C_1 D_{12} = -\eta \frac{\partial}{\partial \theta} f(\theta, \zeta_2) = \eta f(\theta, \zeta_2)(\zeta_2 - \theta \sigma_2^2).
\]

Hence, agents observe

\[
(5.16) \quad \ln \left( \frac{D_{12}}{\eta} \right) = -\theta \zeta_2 + \frac{\theta^2 \sigma_2^2}{2}
\]

and

\[
(5.17) \quad \frac{P_1}{D_{12}} - C_1 = \zeta_2 - \theta \sigma_2^2.
\]

Given the left hand sides, these are two equations in two unknowns, \( \theta \) and \( \zeta_2 \). Substituting for \( \zeta_2 \) from (5.17) into (5.16) yields

\[
(5.18) \quad \ln \left( \frac{\eta}{D_{12}} \right) = (\frac{P_1}{D_{12}} - C_1) \theta + \frac{(\sigma_2^2/2) \theta^2}{2}.
\]
Figure 5.1 Solution of Equation (5.18)
The right hand side of (5.18) is a parabola through the origin that opens upward, as graphed in two possible cases in Figure (5.1). Cases a and b occur as zero is the smaller or larger root of the right hand side of (5.18) respectively. In either case, two roots may occur for most values of \( \ln(n/D_{12}) \), so we must take advantage of the restriction \( e > 0 \). That is, we require that the aggregate investor be risk averse. In either case, it is clear that (5.18) provides a unique solution for \( e > 0 \) iff either \( \ln(n/D_{12}) > 0 \) or \( e = (C_1 - P_1/D_{12})/\sigma_2^2 \) and \( \ln(n/D_{12}) = - (P_1/D_{12} - C_1)^2/(2\sigma_2^2) \). The later is a special case where \( e \) is at the minimum of the parabola, and clearly is not robust, since a minor perturbation would yield two distinct solutions to the problem (at least one must exist, since the economy has an equilibrium).

Since this is a one period economy, uninformed agents are not generally interested in inferring \( \theta \). However, having computed \( \theta \), they can compute \( \xi_2 \) using (5.17). This analysis yields the following:
Theorem 5.1: In the one-period economy of this section, where all agents have constant absolute risk aversion and the same rate of impatience \( n \), and the risky asset has a normally distributed payoff with initially unknown mean return \( \zeta_2 + C_1 \), there exists a FRE whereby all agents can infer \( \zeta_2 \) (and aggregate risk aversion \( \theta \)) from prices, provided \( D_{12} < n \), where \( D_{12} \) is the date 1 price of one unit of certain consumption at date 2.

It is of interest to explore the meaning of the condition \( D_{12} < n \). It says that the one period discount factor is less than the rate of impatience. From (5.16), this is equivalent to \( \zeta_2 - \theta \sigma_2^2/2 > 0 \) or \( E(C_2) - \theta \sigma_2^2/2 > C_1 \). That is, a FRE occurs when the certainty equivalent of date 2 social consumption exceeds the date 1 amount of social consumption. (Here, the certainty equivalent is taken while ignoring the impatience parameter, effectively setting \( n = 1 \).)

One-period models: Gamma distributed returns

Suppose \( \tilde{A}_2 \) has a non-central gamma distribution. That is, suppose the density of \( \tilde{A}_2 \) is, given \( \alpha, \beta \) (\( \alpha > 0 \)),

\[
(5.19) \quad p_{\alpha, \beta}(\Delta_2) = \frac{(\Delta_2 - \beta)^{\alpha-1} \exp(-\Delta_2 + \beta)}{\Gamma(\alpha)} \quad (\Delta_2 \geq \beta)
\]

where \( \Gamma(\cdot) \) is the incomplete gamma function which normalizes the probability to integrate to unity. Both \( \alpha \) and \( \beta \) are candidates for the parameter \( \zeta_2 \). \( \beta \) is a non-centrality parameter specifying location of the distribution. Given \( \alpha \), an increase in \( \beta \) yields an increase in the mean increment \( E(\tilde{A}_2) = E(C_2) - C_1 \) in the social
consumption process. We assume

\[ (A3) \quad \beta \geq - C_1. \]

Since \( \tilde{\Delta}_2 = C_2 - C_1 \geq \beta \geq - C_1 \), this ensures that \( \tilde{\Delta}_2 \geq 0 \). This
overcomes one of the problems associated with normally distributed
returns: they lack limited liability and holders of the risky
asset may be forced to consume arbitrarily negative amounts of the
consumption good -- an incomprehensible task.

The mean of \( \tilde{\Delta}_2 \) is \( \alpha + \beta \) and the variance is \( \alpha \), so \( \alpha \) is
simultaneously a location and scaling parameter.

Evaluating the moment generating function of \( \tilde{\Delta}_2 \) yields.

\[
E_{\xi_2} \exp(-\theta \tilde{\Delta}_2) = \exp((-\theta \beta)(1+\theta)^{-\alpha} (\theta > -1))
\]

and,

\[
- \frac{\partial}{\partial \theta} E_{\xi_2} \exp(-\theta \tilde{\Delta}_2) = (\beta + \alpha/(1+\theta)) \exp(-\theta \beta)(1+\theta)^{-\alpha}.
\]

Here, we set \( \xi_2 \) equal to \( \alpha \) (or \( \beta \)) with \( \beta \) (or \( \alpha \)) fixed and known.

Hence, from (5.13) and (5.14), agents observe

\[
D_{12} = \eta \exp(-\theta \beta)(1+\theta)^{-\alpha} \quad \text{or}
\]

\[ (5.20) \quad \ln(D_{12}/\eta) = -\theta \beta - \alpha \ln(1+\theta) \quad \text{and} \]

\[
P_1 - C_1 = (\beta + \alpha/(1+\theta)) D_{12} \quad \text{or}
\]

\[ (5.21) \quad P_1/D_{12} - C_1 = \beta + \alpha/(1+\theta). \]

**Case 1** \( \xi_2 = \alpha; \beta \) known

Solving (5.21) for \( \alpha \) and substituting into (5.20) yields

\[ (5.22) \quad \ln(\eta/D_{12}) = \theta \beta + (1+\theta)(P_1/D_{12} - C_1 - \beta) \ln(1+\theta). \]
Figure 5.2 Graphs of the right hand side of equation (5.22)

Figure 5.3 Graphs of $F(\theta)$, the right hand side of equation (5.23)
The right side of (5.22) is a convex function through the origin as in Figure 5.2. Note from (5.21) that \( P_1/D_{12} - C_1 - \beta > 0 \). The right hand side of (5.22) is strictly monotone increasing if \( \beta > 0 \), always yielding a FRE (since (5.21) gives \( \alpha \) in terms of \( \theta \)). But assuming \( \beta > 0 \) places strong conditions on the social consumption process: it requires that the social consumption levels never decrease from one period to the next. In terms of endogenous variables it is clear that \( \theta \) can be inferred (since \( \theta > 0 \)) iff \( n > D_{12} \), which is the same condition as for Theorem 5.1. Once again we note from (5.20) or (5.13) that this requires that the expected marginal utility of date 2 consumption be less than that of date 1 consumption disregarding impatience. Since marginal utility is decreasing in wealth, it means that the certainty equivalent of date 2 consumption exceeds date 1 consumption.

We have established:

**Theorem 5.2:** In a one-period economy as in this section (constant absolute risk aversion, known impatience \( n \), non-central gamma distributed social payoff increments with known non-centrality parameter \( \beta \)), there exists a FRE whereby all agents can infer \( \xi_2 = \alpha \) (and \( \theta \)) from prices iff \( D_{12} \ll n \).

**Case 2** \( \xi_2 = \beta \); \( \alpha \) known

Solving (5.21) for \( \beta \) and substituting into (5.20) yields

\[
(5.23) \quad \ln(n/D_{12}) = \theta(P_1/D_{12} - C_1) - \alpha \beta/(1+\theta) + \alpha \ln(1+\theta)
\]

Let the right hand side of (5.23) be \( F(\theta) \). Then \( F(0) = 0 \) and \( F(\theta) \rightarrow +\infty \) or \( -\infty \) as \( \theta \rightarrow \infty \), according as \( P_1/D_{12} - C_1 \) is greater than or less than zero, respectively. Now, \( F'(\theta) = P_1/D_{12} - C_1 - \frac{\alpha}{(1+\theta)^2} \), so \( F \) is asymptotically concave as \( \theta \rightarrow \infty \), and \( F' \) has the same sign at \( \theta = 0 \) and \( \theta = +\infty \) as \( P_1/D_{12} - C_1 \). The zeros of \( F' \) are
\[
\theta = - \left( 1 + \frac{\alpha}{2(P_1/D_{12} - C_1)} \right) \pm \sqrt{\left( 1 + \frac{\alpha}{2(P_1/D_{12} - C_1)} \right)^2 - 1}.
\]

If \( P_1/D_{12} - C_1 > 0 \), there are no positive roots of \( F' \) so \( F \) is monotone increasing. If \( P_1/D_{12} - C_1 < -\alpha/2 \), \( F' \) has no roots, so \( F \) is monotone decreasing. If \( -\alpha/2 \leq P_1/D_{12} - C_1 \), there are two positive roots of \( F \) (perhaps not distinct), so \( F \) is decreasing, then increasing, then decreasing as \( \theta \) increases. These cases are illustrated in Figure 5.3.

The first case has a FRE iff \( \ln(\eta/D_{12}) \geq 0 \) and the second has a FRE iff \( \ln(\eta/D_{12}) < 0 \). In the third case, \( F \) is invertible only for suitably large \( \theta \). That is, for some \( \theta^* > 0 \), and \( k < 0 \), \( F(\theta^*) = k \), \( F \) is monotone on \((\theta^*, \infty)\) and the inverse image of \((-\infty, k)\) under \( F \) is \((\theta^*, \infty)\), so that \( F \) is invertible for \( \theta > \theta^* \), or, equivalently, \( \ln(\eta/D_{12}) < k < 0 \). This yields:

**Theorem 5.3**: In the one-period economy of this section with non-central gamma distributed social payoff increments there exists a FRE whereby all agents can infer the non-centrality parameter \( \xi_2 = \beta \) (and \( \theta \)), from prices, given \( \alpha \), iff either

i) \( n \geq D_{12} \) and \( P_1/D_{12} - C_1 \geq 0 \) or,

ii) \( n < D_{12} \) and \( P_1/D_{12} - C_1 \leq -\frac{\alpha}{2} \) or,

iii) \( n < KD_{12} \) and \( -\frac{\alpha}{2} < P_1/D_{12} - C_1 < 0 \),

where \( 0 < K < 1 \) and \( K \) is a function \( P_1, D_{12}, C_1, C_0, n \) and \( \alpha \).

Note that conditions i), ii) and iii) are non-vacuous, for i) obtains if \( \alpha, \beta, \theta > 0 \), ii) obtains if \( 0 < \alpha \) is large, \( \beta < -5\alpha \) (say) and \( 0 < \theta \) is close to 0, and iii) obtains if \( \beta = -\alpha \) and \( \theta > 0 \). The condition \( n > D_{12} \) has already been shown to be equivalent to the requirement that the certainty equivalent (disregarding impatience) of date 2 consumption exceeds date 1 consumption. The condition \( P_1 - C_1 D_{12} > 0 \), for example, is that the value of the risky asset paying off at date 2 should exceed the value of the certain date 1 consumption, if consumption is postponed.
to date 2. These conditions sound very similar to each other, but are mathematically quite distinct, as seen in equations (5.20) and (5.21).

Conditions ii) and iii) show that, in general, \( n > D_{12} \) is neither necessary nor sufficient for a FRE, which is the case in Theorems (5.1) and (5.2). For general probability distributions and parameters, however, it is reasonable to expect that the relationship between \( n \) and \( D_{12} \) should be important in determining the existence of a FRE. From (5.13), \( D_{12} = n \) when \( \theta = 0 \), and, under reasonable conditions this equality should be obtained in the limit as \( \theta \to 0^+ \). (Note that for \( \theta = 0 \), the utility function is a constant, so prices for \( \theta = 0 \) are not well-defined.) This will still be the case if one solves (5.12) for \( \zeta_2 \) as a function of \( \theta \) (conditional on prices) and substitutes into (5.13) to get an equation in \( \theta \) alone. This latter equation will always be comparing \( \ln(n/D_{12}) \) to a function of \( \theta \) that vanishes at 0. The function may have many shapes, as in Figures 5.1 to 5.3, but if it has at most one stationary point for \( \theta > 0 \), a FRE will exist depending only on the sign of \( \ln(n/D_{12}) \).

One period models: Other distributions and utility classes

The exponential utility class was used in the previous sections because it aggregates (yielding parsimony of risk aversion parameters) and because it often yields closed form solutions to the expectation \( f(\theta, \zeta_2) = \mathbb{E}_{\zeta_2} \exp(-\theta \tilde{\Delta}_2) \) that is used in equations (5.13) and (5.14). For exponential utility, one merely needs to assess the moment generating function of \( \tilde{\Delta}_2 \), as was done in the normal and non-central gamma distribution cases.
Another approach involves using the other linear risk tolerance utility functions that permit aggregation, extended power and log:

\[
U_{i1}(c_{i1}, f_{i2}, m_{i1}) = \gamma^{-1}(\theta_i + c_{i1})^\gamma + \eta E_{\xi_2} (\tilde{\theta}_i + \tilde{y}_{i2})^\gamma
\]

(\(0\leq\gamma<1\))

or

\[
U_{i1}(c_{i1}, f_{i2}, m_{i1}) = \ln(\theta_i + c_{i1}) + \eta E_{\xi_2} \ln(\theta_i + y_{i2})(\gamma=0)
\]

Following the type of development of the previous sections and of Chapter 3, prices satisfy

\[
\begin{align*}
(5.24) & \quad D_{12} (C_1 + \theta_A)^{-1} = \eta E_{\xi_2} (\tilde{C}_2 + \theta_A)^{-1} \\
(5.25) & \quad P_1 (C_1 + \theta_A)^{-1} = \eta E_{\xi_2} (\tilde{C}_2 + \theta_A)^{-1} (\gamma<1)
\end{align*}
\]

where \(\theta_A = \Sigma_1 \theta_i\).

Here again, we have assumed that the order of differentiation and expectation can be reversed. The equations can be combined to yield

\[
(5.26) (C_1 + \theta_A)^{-1}(P_1 + \theta_A D_{12}) = \eta E_{\xi_2} (\tilde{C}_2 + \theta_A)^{-1}
\]

In the case of log utility, \(\gamma=0\) and (5.26) becomes

\[
P_1 + \theta_A D_{12} = \eta (C_1 + \theta_A)
\]

which can be solved for \(\theta_A\) provided \(\eta \neq D_{12}\). Substituting \(\gamma = 0\) into (5.24) yields

\[
E_{\xi_2} (\tilde{C}_2 + \theta_A)^{-1} = \frac{D_{12}}{\eta (C_1 + \theta_A)}
\]

The right side is known at date 1 and the question is whether \(\xi_2\)
can be inferred from this equation. Note that $(C_2 + \theta_A)^{-1}$ is a decreasing convex function of $C_2$ so that if the parameter $\zeta_2$ ranks the distributions in accordance with strict second degree stochastic dominance (see, for example, Hanoch-Levy [1969]) then it can be inferred from prices. Thus, for example, if $\zeta_2$ is a location parameter such as the mean, it can be inferred from prices and a FRE exists with log utility.

To state these results more formally we must explicitly state the assumption of the interchangeability of expectation and differentiation:

(A4) Interchange of differentiation and expectation.

Assume for all $\zeta_2$ that

$$\frac{3}{\partial \mu} \cdot E_{\zeta_2} (mC_2 + f)^\gamma = E_{\zeta_2} \frac{3}{\partial \mu} (mC_2 + f)^\gamma$$

and

$$\frac{3}{\partial \gamma} \cdot E_{\zeta_2} (mC_2 + f)^\gamma = E_{\zeta_2} \frac{3}{\partial \gamma} (mC_2 + f)^\gamma \quad (0 < \gamma < 1)$$

or that

$$\frac{3}{\partial \mu} \cdot E_{\zeta_2} \ln(mC_2 + f) = E_{\zeta_2} \frac{3}{\partial \mu} \ln(mC_2 + f)$$

and

$$\frac{3}{\partial \gamma} \cdot E_{\zeta_2} \ln(mC_2 + f) = E_{\zeta_2} \frac{3}{\partial \gamma} \ln(mC_2 + f) \quad (\gamma = 0) .$$

Verification of (A4) can be done, at least in principle, after making specific assumptions about the distribution (family) and then either explicitly differentiating or using the Lebesque dominated convergence theorem. Closed form solutions for expected log utility and its derivatives for interesting probability families
are rare, so it is difficult to provide numerical examples to establish the validity of (A4) or illustrate the following theorem.

**Theorem 5.4** Consider a one period economy as in this section where the distribution of date 2 social wealth $\tilde{C}_2$ is parameterized by $\xi_2$. Suppose that $\xi_2 < \xi_2'$ iff $\tilde{C}_2$ generated by $\xi_2'$ stochastically dominates (second degree) $\tilde{C}_2$ generated by $\xi_2$. Then if all agents have extended log utility, a FRE exists whereby date 1 prices reveal $\xi_2$ and $\theta_A$, if $D_{12} \neq n$.

**Two period models: Exponential utility**

The solutions to the general model at date 1, as expressed in equations (5.9) to (5.14) can be used to obtain a derived utility for date 1 wealth $y_{11}$, which can then be used at date 0 to obtain date 0 prices. Solving (5.11) for $c_{i1}$ and noting (5.9) yields the following expression for date 1 derived utility for agent $i$ (cf. (5.8)):

$$U_{i1}(y_{i1}) = -(1+D_{12})\exp(-\theta_i c_{i1})$$

$$= -(1+D_{12})\exp\left(-\theta_i \left(y_{i1} - \frac{(\theta_i/\theta_i)(C_1+P_1)}{1+D_{12}} + (\theta_i/\theta_i)C_1\right)\right)$$

$$= -(1+D_{12})\phi \exp\left(-\frac{\theta_i}{1+D_{12}} y_{i1}\right)$$

where

$$\phi = \exp\left(-\frac{\theta_i (P_1-D_{12}C_1)}{1+D_{12}}\right).$$

(5.28)
In general, as viewed at date 0, $\phi$ is random, since $P_1$, $D_{12}$ and $C_1$ are random. However, using assumption (A1), which is that the increments $\Delta_t = \tilde{C}_t - \tilde{C}_{t-1}$ of the social consumption process are independent, equation (5.14) shows that $P_1 - D_{12}C_1$ is non-stochastic at date 0 (if $n$ and $\theta$ are known) since it depends only on the distribution of $\Delta_2$ about which agents have the same information at date 0 as at date 1. Similarly, in (5.13) the value of $D_{12}$ depends only on the distribution of $\Delta_2$ if $n$ and $\theta$ are known, so that it is also non-stochastic at date 0, since we are only interested in FRE's in which date 0 prices reveal $n$ and $\theta$ (and $\xi_1$ or $\xi_2$). Note that the constant absolute risk aversion (with no wealth effects on prices) and the assumption of a Markov social consumption process with independent increments combine to make $D_{12}$ (and future spot interest rates) deterministic. Adding in exponential utility for date 0 consumption and discounting the date 1 derived utility by the factor $n$ yields date 0 utility of

$$U_{i0}(c_{i0}, f_{i01}, f_{i02}, m_{i0})$$

$$= - \exp(-\theta c_{i0}) - n \mathbb{E}_{\xi_1}(\exp(-\theta \tilde{c}_{i1}))$$

$$+ n \mathbb{E}_{\xi_2}(\exp(-\theta \tilde{c}_{i2}))$$

$$= - \exp(-\theta \tilde{c}_{i0}) - n \phi(1 + D_{12})$$

$$\mathbb{E}_{\xi_1}(\exp(-\frac{\theta i}{1 + D} y_{i1}))$$

where $y_{i1}$ satisfies (5.3).
Since $D_{12}$ is deterministic at date 0, the pure expectations version of the term structure of interest rates must hold (in the absence of arbitrage) so that

$$(5.30) \quad D_{02} = D_{01} D_{12}. $$

This allows agents to set $f_{i02} = 0$ without loss of generality, since two-period riskless investments can be obtained by rolling over a one-period bond at date 1.

Assume $C_0$ is exogenously revealed to all agents at date 0. Maximizing (5.29) subject to $f_{i02} = 0$ and (5.1) and (5.2) yields demands for $c_{i0}$, $f_{i01}$, and $m_{i01}$. Using (5.6) to clear the market yields prices. In equilibrium we have:

$$m_{i0} = \frac{\theta}{\theta_i} = \frac{\theta}{\theta_i(1+D_{12})}$$

$$f_{i0} = (D_{01}+(1+D_{12})^{-1})^{-1}(y_{i0}-m_{i0}(C_0+P_0))$$

$$c_{i0} = f_{i01}(1+D_{12})^{-1} + m_{i0}C_0$$

$$P_0 = nE_{\xi_1}[(\tilde{C}_1+\tilde{P}_1)\exp(-\theta(1+D_{12})^{-1}(\tilde{C}_1+\tilde{P}_1$$

$$\quad - (1+D_{12})C_0))]$$

$$D_{01} = nE_{\xi_1}[(\exp(-\theta(1+D_{12})^{-1}(\tilde{C}_1+\tilde{P}_1$$

$$\quad - (1+D_{12})C_0))]$$

Now,

$$(5.31) \quad \tilde{C}_1+\tilde{P}_1-(1+D_{12})C_0 = (\tilde{P}_1-D_{12}\tilde{C}_1) + (1+D_{12})\tilde{A}_1$$
and we have seen that $P_1 - D_{12} - C_1$ is non-stochastic at date 0, as is $D_{12}$. Using the definition (5.28) of $\phi$, the price equations simplify to 4)

\begin{align}
(5.32) \quad P_0 &= nE_{\xi_1} [(C_1 + P_1) \exp(-\theta \Delta_1)] \\
(5.33) \quad D_{01} &= nE_{\xi_1} [\exp(-\theta \Delta_1)] .
\end{align}

Note from (5.13) and (5.33) that, if $\Delta_1$ and $\Delta_2$ are identically distributed (i.e., $\xi_1 \sim \xi_2$) in addition to being independent, $D_{01} = D_{12}$. If this were the case then the date 0 prices $P_0$, $D_{01}$ and $D_{02}$ would only provide two pieces of information since $D_{02} = D_{01}^2$ (by 5.30), which is not enough to fully reveal all of $\theta$, $\eta$ and $\xi_1$ or $\xi_2$. Hence we must assume that $\xi_1 \neq \xi_2$ and infer one or the other probability parameter only, in order to obtain FRE's.

It is convenient to use (5.31) and (5.14) to re-express (5.32):

\begin{align}
P_0 &= nE_{\xi_1} [(-n^2 \frac{\partial}{\partial \theta} f(\theta, \xi_2) + (1 + D_{12}) (C_0 + \Delta_1)) \\
& \quad \exp(-\theta \Delta_1)]
\end{align}

where

\begin{align}
f(\theta, \xi_2) &= E_{\xi_2} \exp(-\theta \Delta_2) .
\end{align}

Letting $g(\theta, \xi_1) = E_{\xi_1} \exp(-\theta \Delta_1)$, this becomes, after interchanging the order of expectation and differentiation,

\begin{align}
(5.34) \quad P_0 &= (-n^2 \frac{\partial}{\partial \theta} f(\theta, \xi_2) + n(1 + D_{12}) C_0) g(\theta, \xi_1) \\
& \quad - n(1 + D_{12}) \frac{\partial}{\partial \theta} \cdot g(\theta, \xi_1) .
\end{align}
Two period models: Normally distributed returns

Suppose that
\[ \tilde{\Delta}_2 \sim N(\xi_2, \sigma_2^2) \]
\[ \tilde{\Delta}_1 \sim N(\xi_1, \sigma_1^2) \]

There are two candidates for the probability parameter that is to be revealed: \( \xi_2 \) (\( \xi_1 \) known) and \( \xi_1 \) (\( \xi_2 \) known). Both cases will be analyzed here. The case where \( \xi_1=\xi_2 \) and both are unknown (a stationary process for \( \tilde{\Delta}_t \)) will not yield a FRE since we have seen that it forces \( D_{01}=D_{12} \), eliminating one dimension from the price information.

We have
\[ g(\theta, \xi_1) = E_{\xi_1} \exp(-\theta \tilde{\Delta}_1) \]
\[ = \exp(-\theta \xi_1 + \theta^2 \sigma_1^2/2) \quad \text{and} \]
\[ f(\theta, \xi_2) = \exp(-\theta \xi_2 + \theta^2 \sigma_2^2/2) \]
so, from (5.33), (5.34) and (5.16),
\[ D_{01} = \eta g(\theta, \xi_1) \quad \text{or} \]
(5.35) \[ \ln(D_{01}/n) = -\theta \xi_1 + \theta^2 \sigma_1^2/2 \quad \text{and} \]
(5.36) \[ P_0 = [-n^2(-\xi_2+\theta \sigma_2^2)f(\theta, \xi_2)+n(1+D_{12})C_0]g(\theta, \xi_1) \]
- \[ n(1+D_{12})(-\xi_1+\theta \sigma_1^2)g(\theta, \xi_1) \]
\[ = D_{01}[D_{12} \xi_2+(1+D_{12})\xi_1-(D_{12} \sigma_2^2+(1+D_{12}) \sigma_1^2) \theta + (1+D_{12})C_0] \]
At date 0, agents observe \( P_0, D_{01} \) and \( D_{02} \), or equivalently, from (5.30), they observe \( P_0, D_{01} \) and \( D_{12} \). These prices are given by (5.36), (5.35) and (5.16). Eliminating \( n \) from (5.35) and (5.16) yields

\[
\ln(D_{12}/D_{01}) = \theta(\zeta_1 - \zeta_2) + \theta^2(\sigma_2^2 - \sigma_1^2)/2.
\]

We assume all agents initially know \( C_0 \) and \( \sigma_1^2 \) and \( \sigma_2^2 \).

**Case 1** \( \zeta_2 \) known; \( \zeta_1, n, \theta \) unknown.

Solve (5.36) for \( \zeta_1 \) in terms of \( \theta \) and substitute into (5.37) to get

\[
\ln(D_{12}/D_{01}) = \theta[(1+D_{12})^{-1}(P_0/D_{01} - D_{12}\zeta_2 - (1+D_{12})C_0) - \zeta_2] + \theta^2[(\sigma_1^2 - \sigma_2^2)/2
+ (1+D_{12})^{-1}(D_{12}\sigma_2^2 + (1+D_{12})\sigma_1^2))]
= \theta[(1+D_{12})^{-1}(P_0/D_{01} - D_{12}\zeta_2 - (1+D_{12})C_0) - \zeta_2]
+ \theta^2[\frac{3}{2}\sigma_1^2 + \sigma_2 + \frac{1}{2}D_{12}^{-1}/(D_{12}+1)^2 \sigma_2^2].
\]

The right hand side of (5.38) is a parabola that opens up or down, according as

\[3\sigma_1^2 + (D_{12}-1)/(D_{12}+1) \sigma_2^2\]

is positive or negative, respectively. As in the analysis of Theorem 5.1 in Figure 5.1, this yields a FRE according as \( \ln(D_{12}/D_{01}) \) is positive or negative, respectively. This yields:

**Theorem 5.5** Consider a two period economy where aggregate consumption \( \hat{C}_t \) follows a Markov process with independent, normally distributed increments, as in this section. Agents have exponential
utility with aggregate risk aversion \( \theta \) and impatience parameter \( \eta \).

If the second period mean increment \( \Delta_2 = \xi_2 \) is known, but the first period increment \( \xi_1 \) is unknown, then a FRE exists whereby date 0 prices of the risky asset and the bond term structure reveal the values of \( \theta, \eta \) and \( \xi_1 \), for \( \theta > 0 \), iff either

i) \( D_{12} > D_{01} \) and \( 3\sigma_1^2 + (D_{12} - 1)/(D_{12} + 1)\sigma_2^2 > 0 \) or

ii) \( D_{12} < D_{01} \) and \( 3\sigma_1^2 + (D_{12} - 1)/(D_{12} + 1)\sigma_2^2 < 0 \).

To check that conditions i) and ii) are not vacuous, note that i) occurs if \( \xi_1 > \xi_2 > 0 \) and \( \sigma_1^2 = \sigma_2^2 \) is appropriately large, and ii) occurs if \( \xi_2 \) is large and \( \sigma_2^2 \) and \( \eta \) are small so that \( D_{12} < 1 \) and \( \xi_1 < \xi_2 \), \( \sigma_1^2 > \sigma_2^2 \), for a given \( \theta > 0 \).

Case 2 \( \xi_1 \) known; \( \xi_2, \eta, \theta \) unknown.

Solve (5.36) for \( \xi_2 \) and substitute into (5.37) to get

(5.39) \[ \ln(D_{12}/D_{01}) = \theta(\xi_1 - P_0/(D_{01}D_{12}) - (1+D_{12}^{-1})(\xi_1+C_0)) \]

- \( \theta^2(\sigma_2^2/2 + (3/2 + D_{12}^{-1})\sigma_1^2) \)

The right hand side is a downward opening parabola from which \( \theta > 0 \) can be inferred iff \( D_{12} < D_{01} \). This yields:

Theorem 5.6 In a two period economy as in Theorem 5.5, except that the first period increment \( \xi_1 \) is known, and \( \xi_2, \eta \) and \( \theta \) are unknown, a FRE exists that reveals these parameters iff \( D_{12} < D_{01} \).

It is of interest to study the condition \( D_{12} < D_{01} \). This says that the one period forward rate of interest should exceed the spot rate of interest, or, equivalently, that the term structure of interest rates should be rising.
Two period models: Gamma distributed returns

In this section, the increments to social wealth, $\tilde{\Delta}_2$ and $\tilde{\Delta}_1$ both follow the non-central gamma distribution given in (5.19). However, we shall suppose that the parameters $\alpha$ and $\beta$ vary from date 1 to date 2. That is, assume the densities of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ are

$$P_{\alpha_1, \beta_1}(\Delta_1) = \frac{(\Delta_1 - \beta_1)^{\alpha_1 - 1} \exp(-\Delta_1 + \beta_1)}{\Gamma(\alpha_1)} \quad (\Delta_1 \geq \beta_1)$$

and

$$P_{\alpha_2, \beta_2}(\Delta_2) = \frac{(\Delta_2 - \beta_2)^{\alpha_2 - 1} \exp(-\Delta_2 + \beta_2)}{\Gamma(\alpha_2)} \quad (\Delta_2 \geq \beta_2).$$

Hence

$$f(\theta, \xi_2) = \mathbb{E}_{\xi_2} \exp(-\theta \tilde{\Delta}_2)$$

$$= \exp(-\theta \beta_2)(1+\theta)^{-\alpha_2}$$

and

$$g(\theta, \xi_1) = \mathbb{E}_{\xi_1} \exp(-\theta \tilde{\Delta}_1)$$

$$= \exp(-\theta \beta_1)(1+\theta)^{-\alpha_1},$$

so that, from (5.33), (5.34) and (5.20),

(5.40) \hspace{1cm} D_{01} = n g(\theta_2 \xi_1) \quad \text{or} \quad \ln(D_{01}/n) = -\theta \beta_1 - \alpha_1 \ln(1+\theta) \quad \text{and}

(5.41) \hspace{1cm} P_0 = [n(\beta_2 + \alpha_2/(1+\theta))f(\theta_1 \xi_2) + (1+D_{12})C_0]\ln g(\theta_1 \xi_1), \quad +n(1+D_{12})[\beta_1 + \alpha_1/(1+\theta)]g(\theta, \xi_1)

= D_{01}[D_{12}(\beta_2 + \alpha_2/(1+\theta)) + (1+D_{12})C_0 + (1+D_{12})(\beta_1 + \alpha_1/(1+\theta))].
Equation (5.20) becomes, in this notation,

\begin{equation}
\ln(D_{12}/\eta) = \ln(E_{\zeta_2} \exp(-\theta \Delta_2))
\end{equation}

\begin{equation}
= -\theta \beta_2 - a_2 \ln(1+\theta) .
\end{equation}

Subtracting (5.40) from (5.42) yields

\begin{equation}
\ln(D_{12}/D_{01}) = \theta (\beta_1 - \beta_2) + \ln(1+\theta)(\alpha_1 - \alpha_2) .
\end{equation}

These are four possible candidates for the unknown probability parameter: \( a_1, a_2, \beta_1 \) and \( \beta_2 \). This yields four cases to analyze where one parameter is unknown (along with \( \theta \) and \( \eta \)) and the other three are known (along with \( C_0 \)). The cases will be analyzed and summarized in one theorem.

**Case 1** \( \zeta_1 = a_1 \) unknown; \( a_2, \beta_1, \beta_2 \), known.

Solving (5.41) for \( a \) and substituting into (5.43) yields

\begin{equation}
\ln(D_{12}/D_{01}) = \theta (\beta_1 - \beta_2) + \ln(1+\theta) \left[ \frac{1+\theta}{1+D_{12}} \left( \frac{P_{\theta}}{D_{01}} - D_{12} \beta_2 - (1+D_{12})(C_0 + \beta_1) \right) \right]
\end{equation}

\begin{equation}
- \frac{1+2D_{12}}{1+D_{12}} a_2 \right] .
\end{equation}

By (5.41), the factor in square brackets is, in equilibrium

\begin{equation}
\frac{D_{12} a_2 + (1+D_{12}) a_1}{1+\theta} > 0 ,
\end{equation}

so that the right hand side is a convex function of \( \theta \) which vanishes at the origin and is increasing for large \( \theta \). However, if \( \beta_1 < \beta_2 \), the function may be decreasing for small \( \theta \). The analysis is the same as for Case 1 in the one-period gamma distributed returns model (Theorem 5.2) as illustrated in Figure 5.2, so that a FRIE exists iff \( \ln(D_{12}/D_{01}) > 0 \).
Figure 5.4  Graphs of $F(\theta)$, the right hand side of (5.44)
Case 2  \( \xi_2 = \alpha_2 \) unknown; \( \alpha_1, \beta_1, \beta_2 \) known

Solving (5.41) for \( \alpha_2 \) and substituting into (5.43) yields

\[
(5.44) \quad \ln \left( \frac{D_{12}}{D_{01}} \right) = \frac{\theta (\beta_1 - \beta_2)}{1 + D_{12}} 
+ \ln \left( 1 + \theta \right) \left\{ \left( \frac{2 + D_{12}^{-1}}{D_{12}} \right) \alpha_1 + \frac{1 + \theta}{D_{12}} \left[ \frac{P_0}{D_{01}} - D_{12} \beta_2 \right] 
- (1 + D_{12}) (C_0 + \beta_1) \right\}.
\]

Again, the term in square brackets is positive, and the right hand side is increasing for large \( \theta \) (and for all \( \theta \) if \( \beta_1 > \beta_2 \)), although it is not necessarily convex for all \( \theta > 0 \). (For example, if \( (2 + D_{12}^{-1}) \alpha_1 \) is large and the factor in square brackets is small, the right side of (5.44) is not convex.) Denoting the right side of (5.44) by \( F(\theta) \), three possible graphs of \( F(\theta) \) are illustrated in Figure (5.4). If graph \( F_1(\theta) \) or \( F_3(\theta) \) occurs, \( \ln(D_{12}/D_{01}) > 0 \) is necessary and sufficient for a FRE. If \( F_2(\theta) \) occurs, then there exists a \( k > 0 \) and \( \theta^*>0 \) such that \( F(\theta^*) = k \), \( F \) is monotone on \([\theta^*, \infty)\) and the inverse image of \([k, \infty)\) under \( F \) is \([\theta^*, \infty)\). That is, a FRE exists iff \( \ln(D_{12}/D_{01}) > k > 0 \). Note that \( k \) depends, in general on the known or observed parameters \( P_0, D_{01}, D_{12}, C_0, \alpha_1, \beta_1, \beta_2 \).

Case 3  \( \xi_1 = \beta_1 \) unknown; \( \alpha_1, \alpha_2, \beta_2 \) known.

Using the techniques of Case 1 and Case 2 yields the equation:

\[
(5.45) \quad \ln \left( \frac{D_{12}}{D_{01}} \right) = \frac{\theta}{1 + D_{12}} \left[ \frac{P_0}{D_{01}} - (1 + D_{12}) C_0 - (1 + 2 D_{12}) \beta_2 \right] 
- \frac{\theta}{1 + \theta} \left\{ \frac{D_{12}}{1 + D_{12}} + \alpha_1 \right\} + \ln \left( 1 + \theta \right) \left( \alpha_1 - \alpha_2 \right).
\]
Figure 5.5 Graphs of $G(\theta)$, the right hand side of (5.45) for $\alpha_1 > \alpha_2$

Figure 5.6 Graphs of $G(\theta)$, the right hand side of (5.45) for $\alpha_1 < \alpha_2$
Letting $Y$ be the factor in square brackets, in equilibrium, by (5.41), it must be that $Y = (1+D_{12})(\beta_1-\beta_2) + (D_{12}a_2 + (1+D_{12})a_1)/(1+\theta)$.

This may be positive or negative, but tends to be positive insofar as $\alpha_1, \alpha_2 > 0$. Thus, if the right side of (5.45) is $G(\theta)$, it may be that $G(\theta) \to \pm\infty$ as $\theta \to \infty$. Also, for $\theta > 0$, the second term of $G(\theta)$ is convex decreasing, and the third term is concave increasing or convex decreasing, according as $\alpha_1 > \alpha_2$ or $\alpha_1 < \alpha_2$, respectively.

Since the second term is bounded, $G(\theta)$ is convex or concave for large $\theta$ according as the third term is convex or concave, although $G$ may have any curvature for small positive $\theta$ if the second and third terms have different curvature. Figures 5.5 and 5.6 illustrate possible graphs of $G(\theta)$. In Figure 5.5 where $\alpha_1 > \alpha_2$, $G$ is asymptotically concave as $\theta \to \infty$ and increasing or decreasing as $Y$ is positive or negative. Thus, if $\alpha_1 > \alpha_2$, a FRE exists if $Y > 0$ and $\ln(D_{12}/D_{01}) > 0$ or if $Y < 0$ and $\ln(D_{12}/D_{01}) < 0$ where $k$ is a function of $P_1, D_0, D_{12}, C_0, a_1, a_2$ and $\beta_2$. In Figure 5.5 where $\alpha_1 < \alpha_2$, $G$ is globally concave (for $\theta > 0$). In this case, a FRE exists iff $Y > 0$ and $\ln(D_{12}/D_{01}) > 0$ or $Y < 0$ and $\ln(D_{12}/D_{01}) < 0$.

Case 4 $\xi_2 = \beta_2$ unknown; $\alpha_1, \alpha_2, \beta_2$ known.

Eliminating $\beta_2$ from (5.41) and (5.43) yields

$$\ln(D_{12}/D_{01}) = \frac{\theta}{D_{12}}\left[\ln(D_{01}/D_0) - (1+D_{12})C_0 - \beta_1 \right] - \frac{\theta}{1+\theta}(a_2 + (1+D_{12}^{-1})a_1)$$

$$+ \ln(1+\theta)(\alpha_1 - \alpha_2)$$
### Table 5.1 Conditions for a FRE in Theorem 5.7

<table>
<thead>
<tr>
<th>Unknown Probability Parameter</th>
<th>Conditions for FRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>$D_{12} &gt; D_{01}$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$D_{12} &gt; K_2 D_{01}$</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>$\alpha_1 &gt; \alpha_2$, $\gamma &gt; 0$ and $D_{12} &gt; D_{01}$  or, $\alpha_1 &lt; \alpha_2$, $\gamma &lt; 0$ and $D_{12} &lt; D_{01}$</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$\alpha_1 &gt; \alpha_2$, $\kappa &gt; 0$ and $D_{12} &gt; D_{01}$  or, $\alpha_1 &lt; \alpha_2$, $\kappa &lt; 0$ and $D_{12} &lt; D_{01}$</td>
</tr>
</tbody>
</table>

Note: $K_2 > 1$; $K_2$ depends upon $\alpha_1$, $\beta_1$, $\beta_2$, $P_0$, $D_{01}$, $D_{12}$, $C_0$

$0 < K_3 < 1$; $K_3$ depends upon $\alpha_1$, $\alpha_2$, $\beta_2$, $P_0$, $D_{01}$, $D_{12}$, $C_0$

$0 < K_4 < 1$; $K_4$ depends upon $\alpha_1$, $\alpha_2$, $\beta_1$, $P_0$, $D_{01}$, $D_{12}$, $C_0$

$$\gamma = \frac{P_0}{D_{01}} - \frac{(1+D_{12})C_0}{(1+2D_{12})} - \frac{\beta_2}{D_{12}}$$

$$\kappa = \frac{P_0}{D_{01}} - \frac{(1+D_{12})C_0}{D_{12}} - \beta_1.$$
Letting $\kappa$ be the factor in square brackets, in equilibrium,

$$\kappa = D_{12}(\beta_2 + \beta_1) + (D_{12} \gamma_2 + (1 + D_{12}) \alpha_1)/(1 + \theta),$$

and this may be positive or negative. The analysis is exactly the same as for Case 3, with $\kappa$ replacing $\gamma$.

These cases may be summarized in the following theorem.

**Theorem 5.7:** Consider the two period exponential utility economy where the social wealth stochastic process has independent non-central gamma distributed increments as in this section. At date 0, agents observe prices $P_0$, $D_{01}$, $D_{02}$ (and hence $D_{12}$), and attempt to infer $\theta$, $\eta$ and one of $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, which are probability parameters. Then FRE's occur under the conditions illustrated in Table 5.1.

**Two period models: Portfolio rollovers**

In all of these models, agents have the same holdings of the risky asset at date 1 as at date 0. That is, $m_{i0} = \theta/\theta_i = m_{i1}$. This suggests that some sort of portfolio rollover technique might make it unnecessary to infer the taste and probability parameters, as in Chapter 3. In this case, however, agents will always desire to know something about the probability parameter, since it affects their utilities for final (and perhaps intermediate) wealth. In the models of Chapter 3, all agents were equally informed at date 0 about the probability parameters and all received the same updated information at date 1, so that the contract curve did not shift from date 0 to date 1. In these models, information about the probability parameter is available at date 0, so that all agents will prefer to use it as soon as it is available. If some agents wait until date 1 to use the information, the contract curve will shift adversely for those agents, and any rollover algorithm must fail.
Conclusion

In this chapter, one and two period exchange economy models were studied where agents had intertemporally additive exponential utility, discounted by an impatience parameter $\eta$. Prices depend on $\eta$, an aggregate risk aversion parameter $\theta$, and a probability parameter. Conditions are derived, that are verifiable because they involve observable prices and known parameters, which ensure the existence of a fully informing rational expectations equilibrium (FRE). In a FRE, agents can infer $\theta$ and a probability parameter from the price of a bond and a risky asset in a one-period model, or $\eta, \theta$ and a probability parameter from the prices in a term structure of bonds and the price of a risky asset.

Conditions for a FRE include, inter alia, the relationship between $\eta$ and the price of a discount bond, in the one-period models, and the slope of the term structure of bond yields, in the two-period models. The models used normally distributed and non-central gamma distributed increments to social wealth.

Another one-period model, was found to have a FRE, in which aggregate risk tolerance and a probability parameter $\xi_2$ could be inferred from prices, if the probability parameter ranked the family of distributions in the same order as would strict second degree stochastic dominance. Agents in that model have extended log utility.
Footnotes to Chapter 5

1. Equilibrium prices are unique because they can be derived by portfolio separation from a complete market structure (with a continuum of markets) in which prices are unique because of the aggregation properties of exponential utility. (Cf., Footnote 2, Chapter 3).

2. To verify the interchange of differentiation and expectation, one could either try to find a measurable bound for the difference quotients and use the Lebesgue dominated convergence theorem (cf., Rudin [1964, p. 246]) or actually perform the integration and differentiation. While the former technique is general and may work well for a variety of utility families, it is not clear that it works with exponential utility.

If $\tilde{\Delta} \sim N(\mu, \sigma^2)$ then

$$E(\exp(\gamma \tilde{\Delta})) = \exp(\gamma^2 \sigma^2 / 2 + \mu \gamma)$$ and

$$\frac{d}{d\gamma} E \exp(\gamma \tilde{\Delta}) = (\gamma \sigma^2 + \mu) \exp(\gamma^2 \sigma^2 / 2 + \mu \gamma) .$$

On the other hand,

$$E \left( \frac{d}{d\gamma} \exp(\gamma \tilde{\Delta}) \right) = E \Delta \exp(\gamma \tilde{\Delta})
= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \Delta \exp(\gamma \Delta - (\Delta - \mu)^2/(2\sigma^2)) \, d\Delta
= \frac{\exp(\gamma \mu + \gamma^2 \sigma^2 / 2)}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \exp(-(\Delta - (\mu + \sigma^2 \gamma))^2/(2\sigma^2)) \, d\Delta
= \left[ \exp(\gamma \mu + \gamma^2 \sigma^2 / 2) \right](\mu + \sigma^2 \gamma) .$$
since the last integration is the same as that performed
to calculate the mean of a normal variate. Thus, the order
of differentiation (w.r.t. coefficients of $\tilde{A}_2$) and expectation
can be reversed. With exponential utility, differentiation
with respect to $f$ in $\exp(-\theta(m^\Delta+f))$ presents no problem since
the factor $\exp(-\theta f)$ may be taken outside the expectation.

3. To verify the validity of the interchange of differentiation
and expectation, suppose $\tilde{A}$ is a central gamma variate with density
$$
\Delta^{-1} \exp(-\Delta)/\Gamma(\alpha) \quad (\Delta>0).
$$
The results when $\Delta$ is a non-central gamma variate follow by
translating a central variate. Then
$$
\frac{\partial}{\partial \gamma} \mathbb{E}\exp(-\gamma \tilde{A}) = \frac{\partial}{\partial \gamma} (1+\gamma)^{-\alpha} = -\alpha(1+\gamma)^{-\alpha-1}
$$
$$
(1+\gamma>0).
$$
On the other hand,
$$
\mathbb{E}\frac{\partial}{\partial \gamma} \exp(-\gamma \tilde{A}) = \mathbb{E}(\tilde{A}\exp(-\gamma \tilde{A}))
$$
$$
= -(\Gamma(\alpha))^{-1} \int_{0}^{\infty} \Delta \exp(-\gamma \Delta) \Delta^{-1} \exp(-\Delta) d\Delta
$$
$$
= -(\Gamma(\alpha))^{-1} (1+\gamma)^{-\alpha-1} \int_{0}^{\infty} ((\gamma+1)\Delta)^{\alpha} \exp(-(\gamma+1)\Delta) d[(\gamma+1)\Delta]
$$
$$
= -(\gamma+1)^{-\alpha-1} \Gamma(\alpha+1) \Gamma(\alpha)
$$
$$
= -\alpha(\gamma+1)^{-\alpha-1}
$$
This justifies the interchange of expectation and differentiation.
4. It is interesting to check (5.32) and (5.33) by analyzing the marginal rates of substitution for the aggregate investor. The marginal rate of substitution between a proportional claim to date 1 and 2 consumption and incremental consumption at date 0 for the aggregate investor is:

\[
\frac{\theta n E_{\zeta_1} \xi_{\zeta_2} \left[ \tilde{C}_1 \exp(-\theta \tilde{C}_1) + n \tilde{C}_2 \exp(-\theta \tilde{C}_2) \right]}{\exp(-\theta C_0)}
\]

\[
= n E_{\zeta_1} \tilde{C}_1 \exp(-\theta \tilde{C}_1) + n E_{\zeta_2} \tilde{C}_2 \exp(-\theta \tilde{C}_2)
\]

\[
= n E_{\zeta_1} \tilde{C}_1 \exp(-\theta \Delta_1) + n E_{\zeta_2} \tilde{C}_2 \exp(-\theta \Delta_2)
\]

This is the same as the expression (5.32) for $P_0$.

The marginal rate of substitution between certain (incremental) consumption at date 1 and (incremental) consumption at date 0 is

\[
\frac{\hat{\theta} n E_{\zeta_1} \exp(-\hat{\theta} \tilde{C}_1)}{\exp(-\hat{\theta} C_0)} = n E_{\zeta_1} \exp(-\hat{\theta} \Delta_1),
\]

which is the same as (5.33) for $D_{01}$. 
Chapter 6 Concluding Remarks

Several questions have been asked, and at least partially answered in this thesis. They include:

What is a rational expectations equilibrium?

Drawing upon definitions already in the economics literature, information is defined, in Chapter 2, in terms of \( \sigma \)-algebras or partitions of the states of nature, upon which agents may form conditional expectations. Information is distinguished from beliefs, the latter being an agents' view of the probability distribution of all possible states of nature. This is perhaps the first paper to stress the difference between information and beliefs in the theory of rational expectations. Macro-economic models tend to stress the importance of agents forming "correct" beliefs in rational expectations models. Micro-economic models, like those in this paper, tend to concentrate on the information aspects in the theory of rational expectations, while merely making convenient assumptions about beliefs.

A rational expectations equilibrium is characterized as a fixed point in a function space of price random variables. Existence of equilibrium is affected by at least two sorts of problems: discontinuities in demands induced by discontinuities in information (the continuity problem), and the requirement that agents' excess demands, which may be measurable with respect to different \( \sigma \)-algebras, must sum to a non-random constant in equilibrium (the measurability problem). Heuristic arguments advanced here,
concerning the measurability problem, suggest that the existence of rational expectations equilibria that are not fully informing (i.e., in which all agents do not have the same information after they observe prices) may be a rare occurrence. An interesting question for future research would be to study the robustness of the existence of equilibrium in various non-fully informing models in the literature, such as Futia [1978], Grossman [1977] and Kreps [1977].

Do prices reveal information about preferences?

When the information is about an aggregate preference parameter in the utility classes that aggregate (extended power and log, and exponential), Chapter 3 shows that the answer is "yes": there exists a fully informing rational expectations equilibrium (FRE). The models there have a two period state preference setting, so the results apply to arbitrary probability distributions. Introducing intermediate period consumption does not prevent prices from revealing the aggregate preference parameter, but it is hard to model the problem in such a way that agents know which prices (or combination of prices) to invert to find aggregate preferences. Introducing random intermediate period labor income, which is independent of which state occurs and is initially revealed to the agent who will earn it, does not prevent prices from revealing the salient part of aggregate preferences, as long as total (social) labor income is non-random. If total labor income is random, an extra dimension of noise is added and a FRE does not exist. However, a non-fully informing rational expectations
equilibrium may exist, and this would be an interesting problem to explore, perhaps numerically by computer.

Do prices reveal information about the distribution of agents' endowments?

This is a multivariate problem, with a vector of prices and a vector of endowments. In a two period, complete market setting where there are at least as many intermediate states (and hence prices) as agents, and probabilities are appropriately non-degenerate, Chapter 4 shows that the answer is generically "yes", provided that one is only interested in the existence of a locally fully informing rational expectations equilibrium (LFRE). Compared to other models in the literature, this model makes weaker dimensionality assumptions and uses somewhat more constructive analytic techniques, for the price of getting LFRE's instead of FRE's.

Do prices reveal and distinguish between preference and probability parameter information?

This leads to a class of multivariate problems that can be studied by more constructive techniques than were used in Chapter 4, since the problems are two and three dimensional.

In order to model information about probability parameters, in Chapter 5, specific families of probability distributions are used -- normal and non-central gamma. Probability parameters that are studied are location and location-scale parameters. One and two period models are considered where markets consist of bonds and a risky asset that pays consumption dividends at initial, intermediate and final dates. The one period markets have two relative prices and conditions are found under which these prices reveal preference and probability information. One condition involves the relationship
between bond prices (discount factors) and agents' common rate of impatience, where the exponential utility is intertemporally additive, discounted by the impatience factor. Equivalently, this condition involves the relationship between current social consumption and the un-discounted certainty equivalent of random next-date social consumption. Another condition is the relationship between the value of next-date social consumption, if deferred one period, as well as various technical conditions that do not seem to be amenable to simple economic interpretation.

In the two period models, a term structure of interest rates adds another (bond) price, so that aggregate risk preference, impatience and probability parameters may be revealed by prices. Conditions for the existence of a FRE involve relationships like those in the one period model, as well as the slope of the term structure of interest rates.

The techniques used in Chapter 5 are quite general and may be extended to other utility classes that aggregate, and other probability families, since the constructive analysis involves assessing moment generating functions (for exponential utility) or non-central moments (for extended log and power) of the probability families. A simple one period extension to extended log utility is developed, where a FRE exists with any probability family, as long as the probability parameter ranks the family in the same order as second degree stochastic dominance.
Conclusion

This thesis uses a variety of constructive and non-constructive techniques to study the existence of a fully informing rational expectations equilibrium (FRE) in one and two period models, where prices convey information about preferences, endowments and probabilities. Some of the techniques may be extended to other settings.
Bibliography


Heinkel, Robert [1978]. "Moral hazard, risky debt and credit rationing," Graduate School of Business Administration, University of California, Berkeley, January.


Journal of Business, 46, 605-615.

——[1974]. "An aggregation theorem for securities markets," 
Journal of Financial Economics, 1, 225-244.


——[1979]. "Do stock prices move too much to be justified by subsequent changes in dividends?," Mimeo., University of Pennsylvania, Dept. of Economics. October.


