VALUATIONS FOR THE QUANTUM PROPOSITIONAL STRUCTURES AND HIDDEN VARIABLES FOR QUANTUM MECHANICS

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Abstract

The thesis investigates the possibility of a classical semantics for quantum propositional structures. A classical semantics is defined as a set of mappings each of which is (i) bivalent, i.e., the value 1 (true) or 0 (false) is assigned to each proposition, and (ii) truth-functional, i.e., the logical operations are preserved. In addition, this set must be "full", i.e., any pair of distinct propositions is assigned different values by some mapping in the set. When the propositions make assertions about the properties of classical or of quantum systems, the mappings should also be (iii) "state-induced", i.e., values assigned by the semantics should accord with values assigned by classical or by quantum mechanics.

In classical propositional logic, (equivalence classes of) propositions form a Boolean algebra, and each semantic mapping assigns the value 1 to the members of a certain subset of the algebra, namely, an ultrafilter, and assigns 0 to the members of the dual ultraideal, where the union of these two subsets is the entire algebra. The propositional structures of classical mechanics are likewise Boolean algebras, so one can straightforwardly provide a classical semantics, which also satisfies (iii). However, quantum propositional structures are <u>non</u>-Boolean, so it is an open question whether a semantics satisfying (i), (ii) and (iii) can be provided.

Von Neumann first proposed (1932) that the algebraic structures of the subspaces (or projectors) of Hilbert space be regarded as the propositional structures P_{QM} of quantum mechanics. These structures have been formalized in two ways: as orthomodular lattices which have the binary operations "and", "or", defined among all elements, compatible ⫰ and incompatible ⫰̸ ; and as partial-Boolean algebras which have the

binary operations defined among only compatible elements. In the thesis, two basic senses in which these structures are non-Boolean are discriminated. And two notions of truth-functionality are distinguished: truth-functionality (δ, $\not\delta$) applicable to the P_{QM} lattices; and truth-functionality (δ) applicable to both the P_{QM} lattices and partial-Boolean algebras. Then it is shown how the lattice definitions of "and", "or", among <u>in</u>compatibles rule out a bivalent, truth-functional (δ, $\not\delta$) semantics for any P_{QM} lattice containing incompatible elements. In contrast, the Gleason and Kochen-Specker proofs of the impossibility of hidden-variables for quantum mechanics show the impossibility of a bivalent, truth-functional (δ) semantics for three-or-higher dimensional Hilbert space P_{QM} structures; and the presence of incompatible elements is necessary but is not sufficient to rule out such a semantics.

As for (iii), each quantum state-induced expectation-function on a P_{QM} truth-functionally assigns 1 and 0 values to the elements in a ultrafilter and dual ultraideal of P_{QM}, where in general the union of an ultrafilter and its dual ultraideal is smaller than the entire structure. Thus it is argued that each expectation-function is the quantum analog of a classical semantic mapping, even though the domain where each expectation-function is bivalent and truth-functional is usually a <u>non</u>-Boolean <u>sub</u>structure of P_{QM}.

The final portion of the thesis surveys proposals for the introduction of hidden variables into quantum mechanics, proofs of the impossibility of such hidden-variable proposals, and criticisms of these impossibility proofs. And arguments in favour of the partial-Boolean algebra, rather than the orthomodular lattice, formalization of the quantum propositional structures are reviewed.

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CHAPTER 0

INTRODUCTION

In 1932, von Neumann published the first proofs of the completeness of quantum mechanics and the impossibility of introducing hidden variables into quantum theory. Thirty years later in 1963, Jauch and Piron published another proof of the impossibility of hidden variables which they regarded as a strengthening of von Neumann's impossibility result. Although von Neumann's proofs were later challenged, the completeness of quantum mechanics and the impossibility of hidden variables were proven anew by Gleason in 1957. And ten years later, Kochen and Specker published their version of Gleason's impossibility proof.

The proofs by Jauch-Piron and Kochen-Specker are especially interesting because they connect the enterprise of introducing hidden variables into quantum theory with the enterprise of assigning 0, 1 values to the algebraic structures of the subspaces (or projectors) of Hilbert space. Such algebraic structures have been regarded as the propositional or logical structures of quantum mechanics ever since von Neumann's proposals, in 1932 and 1936, to consider the subspaces (or projectors) as the mathematical representatives of quantum propositions and to consider the operations and relations among the subspaces as the mathematical representatives of logical operations and relations. So these proofs of the impossibility of assigning 0, 1 values to the algebraic structures of quantum propositions in a manner which preserves the logical operations and relations among the propositions are proofs--or at least suggestive of a proof--of the impossibility of a classical, that is, a bivalent and truth-functional, semantics for the quantum propositions.

In Chapter I, the algebraic notions employed throughout this thesis are presented. In Chapter II, the Boolean Lindenbaum algebra L of a set of well-formed formulae of classical propositional logic is introduced, and the notion of a bivalent, truth-functional semantics for an L is defined as a complete collection of algebraic valuations on L. We shall see in Chapter V how such a semantics fails to work for the algebraic structures of quantum propositions. And we shall see in Chapter VI how such a semantics does work for certain substructures of the quantum propositional structures.

The quantum propositional structures, labeled P_{QM}, have been formalized in two ways: as orthomodular lattices P_{QML} which have the operations $\wedge$ (and), $\vee$ (or) defined among all elements, compatible $\circlearrowleft$ and incompatible $\not\circlearrowleft$; and as partial-Boolean algebras P_{QMA} which have $\wedge$, $\vee$ defined among only compatible elements. In Chapter IV, some differences between the P_{QML} and the P_{QMA} formalizations are described. Also, two notions of truth-functionality are distinguished: truth-functionality $(\circlearrowleft, \not\circlearrowleft)$ appropriate to a P_{QML} and truth-functionality $(\circlearrowleft)$ appropriate to a P_{QMA}. And two basic senses in which the quantum propositional structures may be said to be non-Boolean are elaborated. Then in Chapter V, it is shown how the lattice definitions of $\wedge$, $\vee$ among incompatibles cause truth-functionality problems which rule out a bivalent, truth-functional $(\circlearrowleft, \not\circlearrowleft)$ semantics for any quantum P_{QML} containing incompatible elements. In contrast, the Kochen-Specker 1967 impossibility proof, which semantically interpreted is a proof of the impossibility of a bivalent, truth-functional $(\circlearrowleft)$ semantics for any three-or-higher dimensional Hilbert space $P_{QMA}^{n\geq 3}$ structure, rests upon truth-functionality problems caused by the presence of overlapping maximal Boolean substructures in $P_{QMA}^{n\geq 3}$; the presence of incompatible elements is necessary but not sufficient to rule out such a classical semantics for a $P_{QMA}^{n\geq 3}$.

In Chapter VI, another semantic proposal is considered for the quantum propositional structures. This is the proposal of a state-induced semantics, which is partly motivated by the fact that, as described in Chapter III, the state-induced semantics for a Boolean propositional structure P_{CM} determined by classical mechanics is exactly analogous to the classical (algebraic) semantics for a Boolean Lindenbaum algebra L. So the notion of a state-induced semantics for a P_{QM} , with the quantum state-induced expectation-functions Exp_{ψ} as semantic mappings, is investigated. Each Exp_{ψ} on a P_{QM} truth-functionally assigns 0, 1 values to the elements in a <u>sub</u>structure of P_{QM} in a manner exactly analogous to the way algebraic valuations on an L truth-functionally assign 0, 1 values to the elements in L. Thus Exp_{ψ} may be regarded as the quantum analog of the standard valuation of classical propositional logic, even though the domain where each Exp_{ψ} is bivalent and truth-functional is only a <u>sub</u>-structure of P_{QM} rather than the entire P_{QM} , and even though that substructure may be larger than any Boolean substructure of P_{QM} and so may be a <u>non</u>-Boolean substructure of P_{QM} . In short, the basic methodology of the quantum state-induced semantics for a P_{QM} is exactly like the methodology of the classical (algebraic) semantics for an L. Thus, when the classical semantic method is applied to a non-Boolean quantum P_{QM} , the result is a semantics (which happens to also be state-induced and) which is <u>non</u>-classical in the sense that the domain of each semantic mapping Exp_{ψ} is a <u>non</u>-Boolean <u>sub</u>structure of

$\mathcal{P}_{QM}$. In contrast, in the case of classical mechanics, the domain of each state-induced semantic mapping is the entire Boolean $\mathcal{P}_{CM}$; and likewise, in the case of classical propositional logic, the domain of each algebraic valuation is the entire Boolean L.

Chapter VII surveys hidden variable (HV) proposals, proofs of their impossibility, and criticisms of these HV impossibility proofs.[1] Kochen-Specker present the clearest notion of the goal of proposed HV theories: to give a classical, Boolean reconstruction of quantum mechanics, whereby the statistical results of quantum mechanics are reproduced by classical probability measures on a proposed Boolean structure $\mathcal{P}_{HV}$ of (subsets of) a proposed classical phase space of hidden variables. Kochen-Specker require that such a classical HV reconstruction of quantum mechanics preserve the functional relations among quantum magnitudes and the logical operations among compatible quantum propositions; in other words, an HV reconstruction must preserve the partial-Boolean structural features of the quantum $\mathcal{P}_{QM}$. Such a requirement may be called a structural condition. Von Newmann and Jauch-Piron each impose an additional structural condition requiring the preservation of an operation among incompatibles. That is, according to von Newmann and Jauch-Piron, an HV theory must preserve some of the lattice features of the quantum $\mathcal{P}_{QM}$; this view is criticized in three notes at the end of Chapter VII. Now Kochen-Specker show that their notion of an HV reconstruction of quantum mechanics is possible IFF there exists what in this thesis is called a complete collection of bivalent, truth-functional ($\flat$) mappings on $\mathcal{P}_{QM}$. In this way, the problem of hidden variables for quantum mechanics is connected with the problem of a classical semantics for the quantum propositional structures. And as mentioned above, Kochen-Specker prove that for $\mathcal{P}^{n\ 3}_{QM}$ structures, bivalent, truth-functional (ϕ) mappings are impossible, and so a classical HV reconstruction is impossible for the quantum mechanics of three-or-higher dimensional Hilbert space. The other HV impossibility proofs similarly involve showing the impossibility of proposed bivalent, operation-preserving HV mappings on the $\mathcal{P}_{QM}$ structures.

Critics of these HV impossibility proofs argue that the proofs rest upon contradictions caused by requiring the proposed HV mappings to satisfy the various structural conditions imposed by the authors of the HV impossibility proofs. So whether or not the proofs are accepted depends upon whether or not the structural conditions are accepted as justifiably imposed requirements. And as Bub makes clear, the latter depends upon how quantum mechanics is interpreted. In particular, we have the following dichotomy articulated by Bub: Either quantum mechanics is taken to be a (principle) theory which posits a non-Boolean logical-property-event structure for quantum phenomena, as given by the $\mathcal{P}_{QMA}$ structure abstracted from the fundamental postulates of quantum mechanics; in this case, the quantum $\mathcal{P}_{QMA}$ must be preserved, and as shown by Gleason and Kochen-Specker, quantum mechanics is a complete theory of quantum phenomena and an HV reconstruction of quantum mechanics is impossible. Or the enterprise of providing a classical HV reconstruction of quantum mechanics is treated as paramount, with respect to which the quantum $\mathcal{P}_{QMA}$ need not be preserved; in this case, as proved by Kochen-Specker, the quantum $\mathcal{P}_{QMA}$ cannot be preserved, and as exemplified by the so-called contextual HV theories, a classical HV reconstruction which does not preserve $\mathcal{P}_{QMA}$ is possible and quantum mechanics is incomplete relative to such an HV reconstruction. Bub argues that behind each of these two positions there is a presupposition about logic: The latter is motivated by the presupposition that the logical structure of quantum phenomena and quantum theory must be a Boolean structure like the Boolean $\mathcal{P}_{CM}$ structure of classical phenomena and classical mechanics. The former is motivated by an open acceptance of the non-Boolean character of the logical structure of quantum phenomena and quantum theory, as manifested in the non-Boolean $\mathcal{P}_{QMA}$ structure (which is abstracted from the quantum formalism by the same way that the Boolean $\mathcal{P}_{CM}$ structure is abstracted from the formalism of classical mechanics). Thus one's views on logic may colour one's interpretation of quantum mechanics.

But regardless of the above logical point, since 1967 it has been clear that a classical HV reconstruction of quantum mechanics which preserves the partial-Boolean structural features of the quantum $\mathcal{P}_{QM}$ is impossible. And it is arguable that because the contextual HV theories do not preserve the quantum $\mathcal{P}_{QMA}$, such theories are not really reconstructions of quantum mechanics but rather are entirely separate theories of quantum phenomena which, as Bub puts it, will have to stand on their own feet. Their feet are shaky since so far, experiments have falsified the deviations from quantum mechanics predicted by the contextual HV theories. Thus quantum mechanics, whose state-induced Exp_ψ mappings do preserve the partial-Boolean structural features of $\mathcal{P}_{QM}$ and do successfully predict the results of experiments, marks a radical departure from classical physical theories and may also mark a radical departure from classical logic.

[1] In this thesis, two types of HV theories are investigated, namely, what are called by Belinfante HV theories of the "zeroth kind" (proved impossible by von Neumann, Jauch-Piron, Gleason, Kochen-Specker) and HV theories of the "first kind" (also called contextual HV theories). What Belinfante calls HV theories of the "second kind," that is, the so-called local HV theories, are not discussed in this thesis. And in particular, the celebrated paper by Einstein, Podolsky, and Rosen, in which the non-locality of quantum phenomena is highlighted, is not discussed in this thesis.

Bernard d'Espagnat, in his paper "The Quantum Theory and Reality" in a recent Scientific American (Vol. 241, No. 5, November, 1979) presents a lucid and accessible description of the non-locality of quantum phenomena and of the proposal of a local HV theory. Though d'Espagnat does not say so, his explanation of the derivation of Bell's Inequality in a local HV theory makes it clear that the derivation depends upon a set-theoretic, i.e., Boolean, manipulation of the properties of correlated quantum systems. Bub makes a similar point in his book (Bub, 1974, pp. 79, 83); he argues that the crucial assumption in the derivation of Bell's Inequality in a local HV theory is not the assumption of locality but rather the assumption that certain quantum probabilities are to be computed as though they were classical conditional probabilities on a classical, i.e., Boolean, probability space. Thus the problems and issues raised by HV theories of the "second kind" may in fact be no different from the problems and issues raised by HV theories of the "first kind" which hinge upon attempted Boolean treatments of quantum properties and propositions. A full explication of these points is left for future work.

CHAPTER I

ALGEBRAIC PRELIMINARIES

Section A. Group and Ring Structures

Consider an arbitrary, nonempty collection of elements $E = \{a, b, c, d, e, . . .\}$ with a binary (univalent) operation plus $+$ defined from $E \times E$ to E such that E is closed with respect to $+$; i.e., for any $b, c \in E$, $b + c \in E$, and the following conditions obtain for any elements in E:

(1) $+$ is associative, i.e., $b + (c + d) = (b + c) + d$.

(2) There exists a distinguished element $0 \in E$ such that $b + 0 = 0 + b = 0$, for any $b \in E$.

(3) For any $b \in E$, there exists a $c \in E$ such that $b + c = 0$. It can be proven that c is unique; it is designated as "$-b$" (the additive inverse of b) and satisfies $b + (-b) = 0 = (-b) + b$. For any $b, c \in E$, $b + (-c)$ is also written as $b - c$.

The ordered triple $\langle E, +, 0\rangle$ satisfying closure and (1), (2), (3) is an additive group. For example, $\langle S, \Delta, \emptyset\rangle$ is a set-theoretic realization of an additive group, where S is a set of subsets of some set, Δ is symmetric difference, and $\emptyset$ is the empty set.

If an additive group $\langle E, +, 0\rangle$ is such that:

(4) $+$ is commutative, i.e., $b + c = c + b$, then $\langle E, +, 0\rangle$ is an abelian or commutative additive group.

Now let a second binary (univalent) operation dot $\cdot$ be defined from $E \times E$ to E such that E is closed with respect to $\cdot$, i.e., for any $b, c \in E$, $b \cdot c \in E$ (by convention, $b \cdot c$ is also written bc), and these two conditions obtain:

(5) $\cdot$ is associative.

(6) $\cdot$ is distributive with respect to $+$, i.e., $b \cdot (c + d) = bc + bd$ and $(c + d) \cdot b = cb + db$.

The ordered quadruple $\langle E, +, \cdot, 0\rangle$ satisfying the two closure conditions and (1)-(6) is a ring. For example, $\langle S, \Delta, \cap, \emptyset\rangle$ is a set-theoretic realization of a ring, where $\cap$ is the intersect operation.

If a ring is such that:

(7) $\cdot$ is commutative,

then the ring is a commutative ring.

Consider also this condition:

(8) There exists a distinguished element $1 \in E$ such that $b \cdot 1 = 1 \cdot b = b$, for any $b \in E$.

The ordered five-tuple $\langle E, +, \cdot, 0, 1\rangle$ satisfying closure, (1)-(6), and (8) is a ring-with-unit; and a ring-with-unit which satisfies (7) is a commutative ring-with-unit.

Consider such a ring which also satisfies:

(9) $b \cdot b = b$, for any $b \in E$, that is, each element in E is idempotent. (By convention, $b \cdot b$ is also written b^2.)

Two conditions follow from (9):

(10) Each element $b \in E$ is its own additive inverse.

Proof: For any $b \in E$, $(b + b)^2 = b^2 + b^2 + b^2 + b^2$. And by (9), $(b + b)^2 = b + b = b + b + b + b$. So by (2), $b + b = 0$, and so by (3), $b = -b$. Q.E.D. Thus for any $b, c \in E$, $b + c = b + (-c) = b - c$.

(7) $\cdot$ is commutative.

Proof: For any $b, c \in E$, $(b + c)^2 = b^2 + bc + cb + c^2$. And by (9), $(b + c)^2 = b + c = b + bc + cb + c$. So by (2), $bc + cb = 0$, and so by (3), $bc = -(cb)$, and hence by (10), $bc = cb$. Q.E.D. (Halmos, 1963, p. 2)

The ordered quadruple $\langle E, +, \cdot, 0\rangle$ satisfying closure, (1)-(6), (9), and hence (10) and (7) is a <u>Boolean ring</u>. And the ordered five-tuple $\langle E, +, \cdot, 0, 1\rangle$ satisfying closure, (1)-(10) is a <u>Boolean ring-with-unit</u>. Or in other words, the idempotent elements of a commutative ring form a Boolean ring, and the idempotent elements of a ring-with-unit or a commutative ring-with-unit form a Boolean ring-with-unit. For example, $\langle S, \Delta, \cap, \emptyset, X\rangle$ is a set-theoretic realization of a Boolean ring-with-unit, where S is the set of subsets of a given set X.

<u>Section B.</u> <u>The Boolean Algebra and the Boolean Lattice</u>

In a Boolean ring-with-unit, two binary operations <u>meet</u> $\wedge$ and <u>join</u> $\vee$ are defined from $E \times E$ to E and a unary operation <u>complementation</u> $'$ is defined from E to E in terms of the ring operations $+, \cdot$ as follows: for any $b, c \in E$, $b \wedge c = b \cdot c$, $b \vee c = b + c - (b \cdot c)$, $b' = 1 - b$. The resulting sextuple $\langle E, \wedge, \vee, ', 0, 1\rangle$ is a <u>Boolean algebra</u>. For example, $\langle S, \cap, \cup, ', \emptyset, X\rangle$ is a set-theoretic realization of a Boolean algebra, where S is the set of all subsets of a given set X, $\emptyset$ is the empty set, and $\cap, \cup, '$ are the set operations intersect, union, complementation, respectively.

From the above list of conditions (1)-(10) which a Boolean ring-with-unit satisfies with respect to $+$ and $\cdot$, we can derive a lengthy list of conditions which a Boolean algebra satisfies with respect to its operations. However, in addition to the closure of E w.r.t. $\wedge, \vee, '$, and the existence of the distinguished 0 and 1-elements in E, the following five conditions are necessary and sufficient to characterize a Boolean algebra: for any $b, c, d \in E$,

(B1) Commutativity: $b \wedge c = c \wedge b$ and $b \vee c = c \vee b$, by (4) and (7).

(B2) Associativity: $(b \wedge c) \wedge d = b \wedge (c \wedge d)$ and $(b \vee c) \vee d = b \vee (c \vee d)$, by (1), (4), (5), (6), (10).

(B3) Absorption: $b \wedge (b \vee c) = b$ and $(b \wedge c) \vee c = c$, by (2), (3), (4), (6), (9).

(B4) Complementation: $b \wedge b' = 0$ and $b \vee b' = 1$, by (1), (3), (6), (8), (9), (10).

(B5) Distributivity: $b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$ and $b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$, by (3), (6), (9).

Among the many other identities and conditions which can be derived from (1)-(10) we note the following:

Idempotence: $b \wedge b = b$ and $b \vee b = b$.

Distinguished elements: $b \wedge 0 = 0$, $b \vee 0 = b$, $b \wedge 1 = b$, and $b \vee 1 = 1$.

Involution of complementation: $(b')' = b$, by (1), (2), (10).

Moreover, in a Boolean algebra we may define a binary relation $\leq$ in terms of the meet or join operations as: for any $b, c \in E$, $b \leq c$ IFF $b \wedge c = b$, and $b \leq c$ IFF $b \vee c = c$. It follows that $0 \leq b \leq 1$, for every $b \in E$.

Then by (2), (3), (6), (8), and (1), the $'$ operation also satisfies the condition: for any $b, c \in E$, $b \leq c$ IFF $c' \leq b'$. This condition together with (B4) and the involution condition define $'$ as <u>orthocomplementation</u> $\perp$. Since in a Boolean algebra, complementation is orthocomplementation, I hereafter substitute $\perp$ for $'$ in the ordered sextuple designation of a Boolean algebra $\langle E, \wedge, \vee, ^{\perp}, 0, 1\rangle$.

Any Boolean algebra also satisfies the following conditions, for any $b, c \in E$:

De Morgan's laws: $(b \wedge c)^{\perp} = b^{\perp} \vee c^{\perp}$ and $(b \vee c)^{\perp} = b^{\perp} \wedge c^{\perp}$.

Compatibility: $(b \wedge c\ ) \vee c = (c \wedge b\ ) \vee b$, proven as follows. By (B5), $(b \wedge c^{\perp}) \vee c = (b \vee c) \wedge (c^{\perp} \vee c)$, which by (B4) and by the distinguished character of the 1-element equals $(b \vee c)$. And by the same conditions, $(c \wedge b^{\perp}) \vee b = (c \vee b) \wedge 1 = (c \vee b)$. And by (B1), $b \vee c = c \vee b$. Q.E.D.

Modularity: If $b \leq c$ then $b \vee (e \wedge c) = (b \vee e) \wedge c$, for any $e \in E$. Modularity follows from (B5).

Orthomodularity: If $b \leq c$ then $b = (b \vee c^{\perp}) \wedge c$ and $c = (c \wedge b^{\perp}) \vee b$, which again follows from (B5).

Any elements $b, c \in E$ are said to be disjoint or orthogonal IFF $b \leq c^{\perp}$, where $b \leq c^{\perp}$ IFF $c \leq b^{\perp}$. Moreover, $b \leq c^{\perp}$ IFF $b \wedge c = 0$, proven as follows. Assume $b \leq c^{\perp}$, then $b \wedge c \leq c^{\perp} \wedge c$, and so by (B4), $b \wedge c \leq 0$, i.e., $b \wedge c = 0$ since $0 \leq e$, for every $e \in E$. Assume $b \wedge c = 0$. Then since $b = b \wedge 1 = b \wedge (c \vee c^{\perp}) = (b \wedge c) \vee (b \wedge c^{\perp})$ $= 0 \vee (b \wedge c^{\perp}) = b \wedge c^{\perp}$, we have $b \leq c^{\perp}$. Q.E.D.

The compatibility, modularity, and orthomodularity conditions and the relation of disjointedness or orthogonality are mentioned here because they are important for the quantum structures described in Sections D and E.

With the binary relation $\leq$ defined as above in a Boolean algebra, it follows from the five conditions (B1)-(B5) that w.r.t. $\leq$ a Boolean algebra is a Boolean lattice, as will be shown below. A Boolean. i.e., an orthocomplemented and distributive, lattice is defined as follows.

Consider an arbitrary, nonempty collection of elements

$E = \{a,b,c,d,e, \ldots\}$ with a binary relation $\le \subseteq E \times E$ which has the following properties, for any $b,c,d \in E$:

($\le$ a) Reflexivity: $b \le b$.

($\le$ b) Anti-symmetry: If $b \le c$ and $c \le b$, then $b = c$.

($\le$ c) Transitivity: If $b \le c$ and $c \le d$, then $b \le d$.

The ordered pair $\langle E,\le\rangle$ is a <u>partially ordered set</u>, also called a <u>poset</u>.

With respect to $\le$, define the greatest lower bound (g.l.b.) and the least upper bound (l.u.b.) of any subset $F \subseteq E$ as follows. The g.l.b. of F is that element $b \in E$ such that, for every $f \in F$, $b \le f$, and for any $e \in E$, if $e \le f$ for every $f \in F$ then $e \le b$. The l.u.b. is <u>defined dually</u>, i.e., substitute $\ge$ for $\le$. (The dual of any condition is the result of interchanging $\le$ and $\ge$, $\wedge$ and $\vee$, and 0 and 1 (Halmos, 1963, pp. 7-8, 22).) The uniqueness of the g.l.b. and l.u.b. of any $F \subseteq E$ follows from ($\le$ b). A <u>lattice</u> $\langle E,\le,\wedge,\vee\rangle$ is a poset any two of whose elements $b,c \in E$ have a g.l.b., written $b \wedge c$ and called the <u>meet</u> of b, c, and have a l.u.b., written $b \vee c$ and called the <u>join</u> of b, c. For example, $\langle S,\subseteq,\cap,\cup\rangle$ is a set-theoretic realization of a lattice, where $\subseteq$ is the set-inclusion relation.

If a lattice has a g.l.b., 0, and a l.u.b., 1, and if, for any $b \in E$, there exists at least one $c \in E$ such that $b \wedge c = 0$ and $b \vee c = 1$ (such a c will be called the <u>complement</u> of b and be denoted "b'"), then the lattice is a <u>complemented lattice</u> $\langle E,\le,\wedge,\vee,',0,1\rangle$.

If a lattice has a g.l.b., 0, and a l.u.b., 1, and if, for any $b \in E$, there exists a unique orthocomplement $b^{\perp} \in E$ satisfying: $b \wedge b^{\perp} = 0$, $b \vee b^{\perp} = 1$, $(b^{\perp})^{\perp} = b$, and $b \le c$ IFF $c^{\perp} \le b^{\perp}$, then the lattice is an <u>orthocomplemented lattice</u> $\langle E,\le,\wedge,\vee,^{\perp},0,1\rangle$.

If the meet and join operations are distributive then the lattice is _distributive_. If a lattice is complemented and distributive, then complementation is unique and is orthocomplementation (Birkhoff, 1948, p. 152). An orthocomplemented, distributive lattice is called a _Boolean lattice_.

It is easy to prove that a Boolean algebra is a Boolean lattice with respect to the partial-ordering relation $\leq$ defined in a Boolean algebra as above. For (B1), (B2), and (B3) ensure that $\leq$ satisfies ($\leq$a), ($\leq$b), and ($\leq$c). And (B1), (B2), (B3) ensure that the element $b \wedge c$ is a lower bound for the subset $\{b,c\}$ because $b \wedge (b \wedge c) = (b \wedge b) \wedge c = b \wedge c$, thus $b \wedge c \leq b$, and $c \wedge (c \wedge b) = (c \wedge c) \wedge b = c \wedge b$, thus $b \wedge c \leq c$. And if d is any lower bound for $\{b,c\}$, i.e., $d \wedge b = d$ and $d \wedge c = d$, then $(b \wedge c) \wedge d = b \wedge (c \wedge d) = b \wedge d = d$, and hence $d \leq (b \wedge c)$; thus $b \wedge c$ is the greatest lower bound of $\{b,c\}$. Dually, the element $b \vee c$ is the least upper bound of the subset $\{b,c\}$. Moreover, $b \wedge c$ and $b \vee c$ are each unique because $\wedge$, $\vee$ in a Boolean algebra are operations, i.e., they are univalent. Hence, a Boolean algebra is a lattice with respect to the $\leq$ relation defined in a Boolean algebra as above. And in other words, the (B1), (B2), (B3) conditions completely characterize a lattice (Birkhoff, 1948, p. 18).

It follows from (B4) that the distinguished 0 and 1 elements of a Boolean algebra are the greatest lower bound of E and the least upper bound of E, respectively, as shown next. For any $b \in E$, $b \wedge 1 = b \wedge (b \vee b^{\perp}) = b$, so $b \leq 1$, and $b \wedge 0 = b \wedge (b \wedge b^{\perp}) = (b \wedge b) \wedge b^{\perp} = b \wedge b^{\perp} = 0$, so $0 \leq b$; that is, 1 is an upper bound of E, and 0 is a lower bound of E. If there is an $e \in E$ such that $1 \leq e$, i.e., $1 \wedge e = 1$, then either $e = 1$ or (B3) is violated, and dually, if there is an $e \in E$ such that $e \leq 0$, i.e., $e \vee 0 = 0$, then either $e = 0$ or (B3) is violated; thus 1 is the least upper bound of E and 0 is the greatest lower bound of E. Q.E.D. So a Boolean algebra is a complemented lattice. And by (B5), a Boolean algebra is a distributive lattice whose complementation is unique and is orthocomplementation.

Thus a Boolean algebra is a Boolean lattice with respect to the $\leq$ relation defined in a Boolean algebra as above. Conversely, it is easy to prove that a Boolean lattice is a Boolean algebra.(Bell and Slomson, 1969, pp. 9-11). Hereafter, I use the phrase <u>Boolean structure</u> and the sextuple $\mathcal{B} = \langle E,\wedge,\vee,^{\perp},\leq,0,1\rangle$ to refer to both a Boolean algebra and a Boolean lattice indiscriminately.

The Boolean structures determined by classical mechanics, which I label $\mathcal{P}_{CM}$ and describe in Chapter III(B), are σ-complete and atomic. The Boolean structures determined by classical logic, which I label $\mathcal{L}$ and describe in Chapter II(B), are complete and atomic <u>if</u> they are finite. These additional conditions are defined as follows.

Completeness: $\mathcal{B}$ is <u>complete</u> if every subset of elements in $\mathcal{B}$ has a g.l.b. and a l.u.b. $\mathcal{B}$ is <u>σ-complete</u> if every denumerable subset of elements in $\mathcal{B}$ has a g.l.b. and a l.u.b..

Atomicity: $\mathcal{B}$ is <u>atomic</u> if, for every non-zero element $b \in \mathcal{B}$, there is an atom $a \in \mathcal{B}$ such that $b \geq a$, where an <u>atom</u> is an element which <u>covers</u> the 0-element, i.e., $a > 0$ and $a > e > 0$ is not satisfied by any $e \in \mathcal{B}$. (For any $b,c \in \mathcal{B}$, $b > c$ IFF $b > c$ and $b \neq c$.) $\mathcal{B}$ is <u>non-atomic</u> if it has no atoms (Halmos, 1963, p. 69).

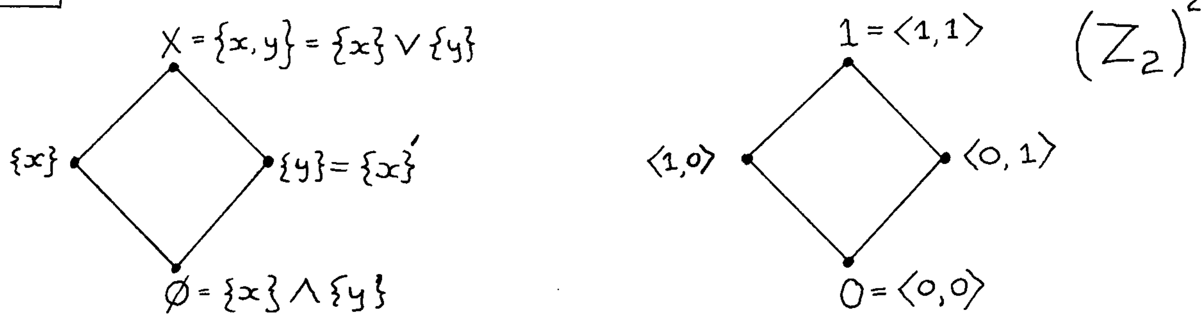
It follows that in an atomic $\mathcal{B}$, every element is the l.u.b. of the atoms it dominates (Halmos, 1963, p. 70). And in an atomic $\mathcal{B}$, two elements are equal IFF they dominate the same atoms (Rutherford, 1965, p. 83). Thus for any distinct elements $b \neq c$ in an atomic $\mathcal{B}$, there is an atom $a \in \mathcal{B}$ such that $a \leq b$ but $a \nleq c$, or $a \leq c$ but $a \nleq b$.

Every finite $\mathcal{B}$ is atomic and complete. Every finite $\mathcal{B}$ is isomorphic to the cartesian product $(Z_2)^n$, where n is the number of atoms in $\mathcal{B}$ and Z_2 is the two-element Boolean structure $\langle E = \{0,1\}, \wedge, \vee, ^{\perp}, \leq, 0, 1\rangle$. Every finite $\mathcal{B}$ is isomorphic to the power set Boolean structure $\langle E =$ the set of all subsets of a given set $X, \cap, \cup, ', \leq, \emptyset, X\rangle$, where the number of elements in X is the same as the number of atoms in $\mathcal{B}$.

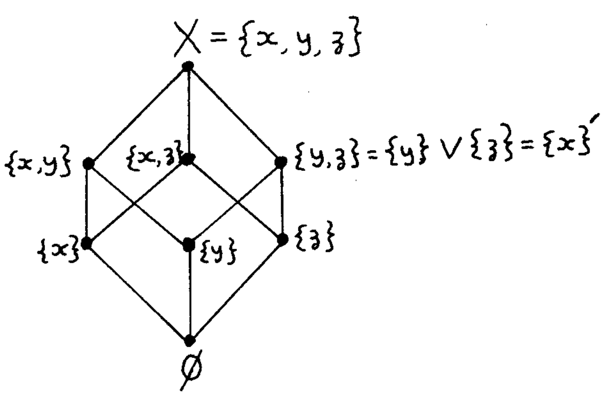
The diagram of a finite $\mathcal{B}$ looks like a two-dimensional representation of an n dimensional cube, where n is the number of atoms in $\mathcal{B}$. For example:

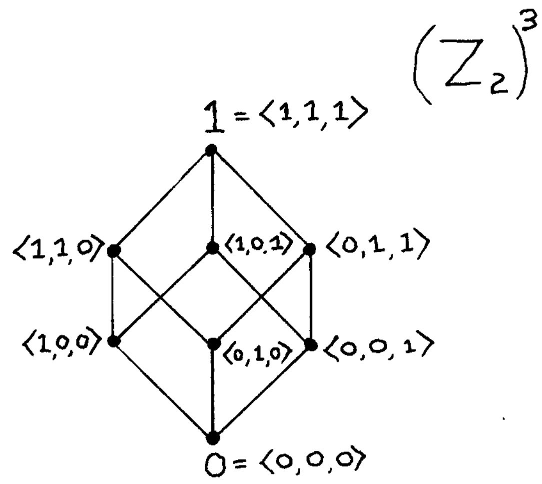
n = 1

n = 2

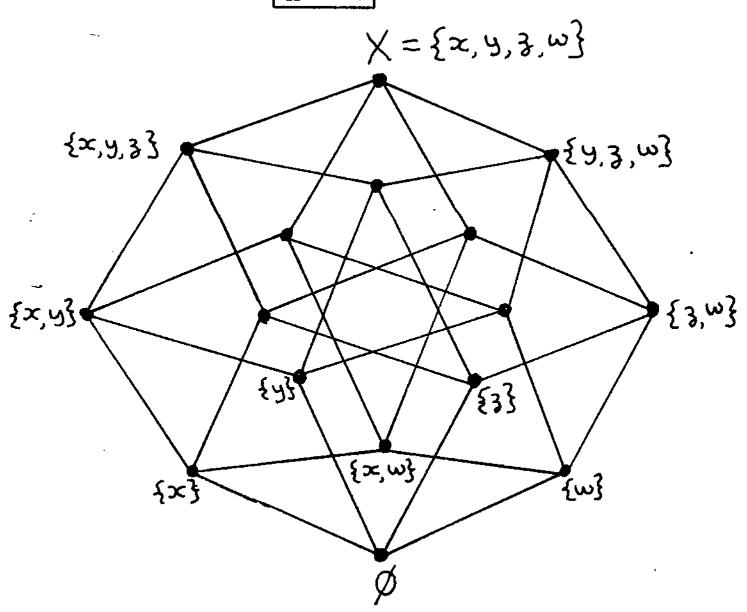


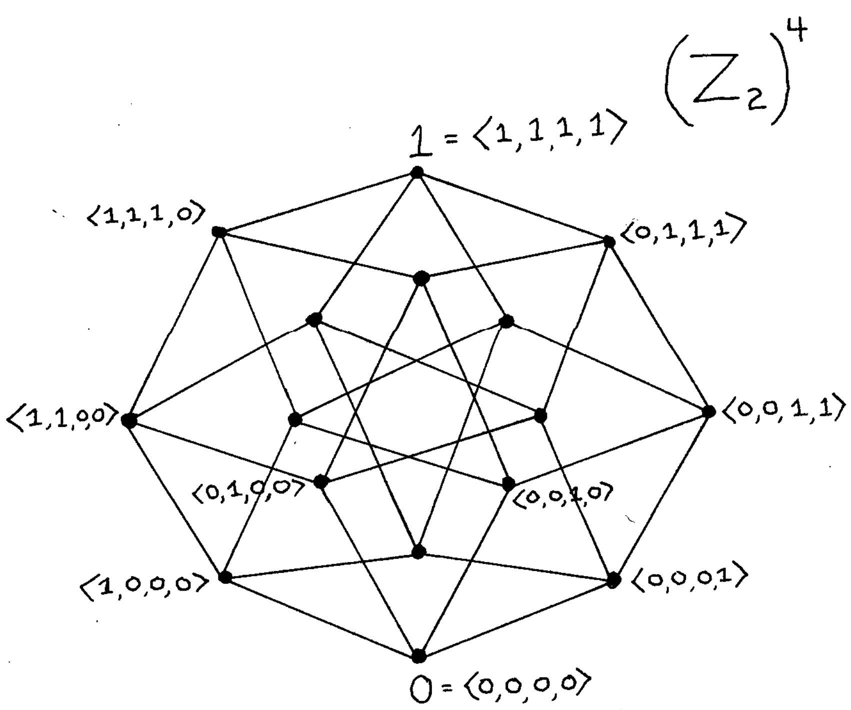
$\boxed{n = 3}$



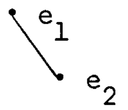


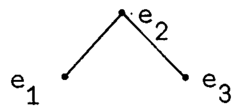
$\boxed{n = 4}$



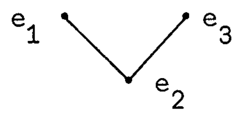


In these diagrams, the dots represent the elements of the structure and the lines connecting the dots represent the operations and relations among the elements, e.g.,



represents $e_2 \le e_1$,


represents $e_2 = e_1 \vee e_3$,

and


represents $e_2 = e_1 \wedge e_3$.

Section C. Subsets of a Boolean Structure

In conformity with standard mathematical parlance, I do not distinguish between the structure $\mathcal{B}$ and its set of elements E. So members or subsets of the set E are also more simply referred to as members or subsets of $\mathcal{B}$.

A subalgebra or sublattice of $\mathcal{B}$ is a non-empty subset of $\mathcal{B}$ which is closed with respect to the operations $\wedge, \vee, ^{\perp}$ of $\mathcal{B}$. A non-empty subset of $\mathcal{B}$, when closed with respect to the operations of $\mathcal{B}$, is said to generate a subalgebra or sublattice of $\mathcal{B}$.

A (proper) filter in $\mathcal{B}$ is a non-empty (proper) subset F of $\mathcal{B}$ which satisfies:

(a) For any $b, c \in F$, $b \wedge c \in F$.

(b) For any $b \in F$ and for any $e \in \mathcal{B}$, if $b \le e$ then $e \in F$.

A (proper) ideal in $\mathcal{B}$ is defined dually, that is, a (proper) ideal is a non-empty (proper) subset I of $\mathcal{B}$ which satisfies: (a')

(a') For any $b, c \in I$, $b \vee c \in I$.

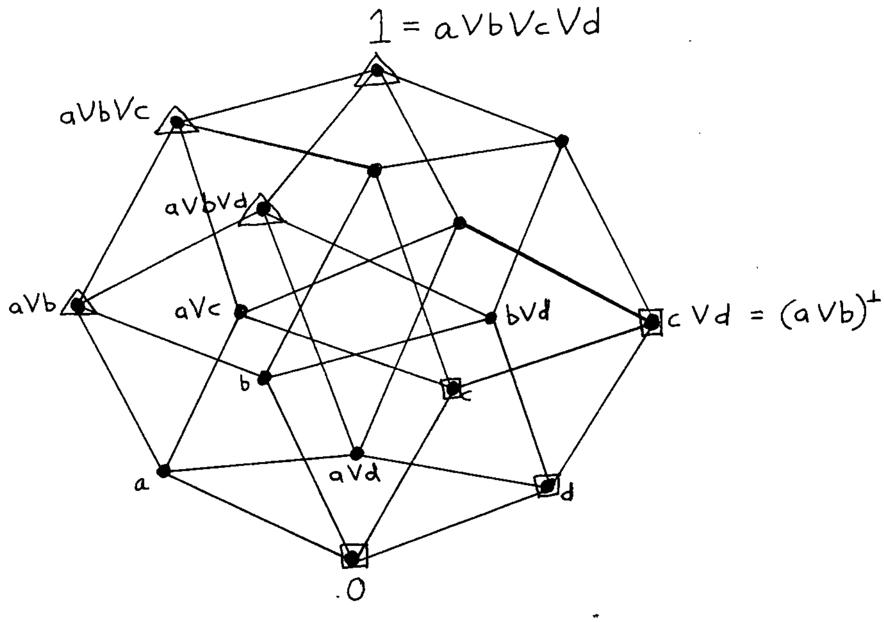
(b') For any $b \in I$ and for any $e \in \mathcal{B}$, if $b \ge e$ then $e \in I$.

The distinguished 1-element of $\mathcal{B}$ is a member of every filter in $\mathcal{B}$ and is itself a filter in $\mathcal{B}$. Dually, the 0-element is a member of every ideal in $\mathcal{B}$ and is itself an ideal in $\mathcal{B}$. Moreover, it follows from de Morgan's laws and from one of the conditions defining $^{\perp}$ (if $b \leq c$ then $c^{\perp} \leq b^{\perp}$) that, for any filter F in $\mathcal{B}$, the set of elements $\{b \in \mathcal{B} : b^{\perp} \in F\}$ is an ideal in $\mathcal{B}$; and dually, for any ideal I in $\mathcal{B}$, the set of elements $\{b \in \mathcal{B} : b^{\perp} \in I\}$ is a filter in $\mathcal{B}$ (Sikorski, 1960, pp. 9, 11). Thus for any F and dual I in $\mathcal{B}$ we have:

(c) For any $b \in \mathcal{B}$, $b^{\perp} \in F$ IFF $b \in I$.

(c') For any $b \in \mathcal{B}$, $b^{\perp} \in I$ IFF $b \in F$.

And if $b \in F$ then $b \notin I$, and if $b \in I$ then $b \notin F$. For example, a filter and dual ideal are designated in the Boolean structure diagrammed below by triangles around the elements in the filter and squares around the elements in the dual ideal:



The union of any filter and its dual ideal in $\mathcal{B}$ is a subalgebra or a sublattice of $\mathcal{B}$ (Bell and Slomson, 1969, p. 17).

An <u>ultrafilter</u> UF in $\mathcal{B}$ is a proper filter which is not the proper subset of any proper filter in $\mathcal{B}$. An <u>ultraideal</u> UI in $\mathcal{B}$ is a proper ideal which is not the proper subset of any proper ideal in $\mathcal{B}$. Every filter in $\mathcal{B}$ is contained in an ultrafilter; every ideal in $\mathcal{B}$ is contained in an ultraideal (Sikorski, 1960, pp. 15, 17). Moreover, every ultrafilter and every ultraideal in any $\mathcal{B}$ is <u>prime</u>, that is, for any $b,c \in \mathcal{B}$:

(d) If $b \vee c \in UF$, then either $b \in UF$ or $c \in UF$.

(d') If $b \wedge c \in UI$, then either $b \in UI$ or $c \in UI$.

And equivalently, if $b \notin UF$ then $b^{\perp} \in UF$, and if $b \notin UI$ then $b^{\perp} \in UI$. (Bell and Slomson, 1969, p. 20). It follows that for any UF and its dual UI in any $\mathcal{B}$ and for any $b \in \mathcal{B}$, $b \in UF$ or $b \in UI$, and thus $\mathcal{B} = UF \cup UI$. Proof: For any UF in any $\mathcal{B}$ and for any $b \in \mathcal{B}$, $b \in UF$ or $b \notin UF$. If $b \notin UF$ then $b^{\perp} \in UF$, and so by (c), $b \in UI$. Q.E.D.

Each ultrafilter and its dual ultraideal in a finite atomic $\mathcal{B}$ is a <u>principal</u> ultrafilter and a <u>principal</u> ultraideal defined with respect to an atom $a \in \mathcal{B}$ as follows: $UF_a = \{b \in \mathcal{B} : b \geq a\}$ and $UI_a = \{b \in \mathcal{B} : b \leq a\}$. And in an atomic $\mathcal{B}$, there is a one-to-one correspondence between atoms and ultrafilters (and dual ultraideals) (Bell and Slomson, 1969, p. 27).

For any pair of distinct elements $b \neq c$ in any $\mathcal{B}$, there is an ultrafilter UF in $\mathcal{B}$ containing one but not the other. Also, each non-zero element in a $\mathcal{B}$ is contained in some ultrafilter in $\mathcal{B}$. (Bell and Slomson, 1969, p. 16).

<u>Section D.</u> <u>The Quantum Partial-Boolean Algebra</u>

Kochen and Specker define a partial-Boolean algebra by first defining a partial-algebra over a field. In short, a partial-algebra over a field is a set of elements E with the usual ring operations + and •

defined from $\between$ to E, where $\between \subseteq E \times E$ is called the compatibility relation. A commutative algebra is a special case of a partial-algebra, namely the case where $\between = E \times E$. The idempotent elements of a partial-algebra form a partial-Boolean algebra $\langle E, \between, \wedge, \vee, ^{\perp}, 0, 1 \rangle$ which has the Boolean operations $\wedge, \vee, ^{\perp}$, defined in terms of the ring operations $+, \cdot$, as usual, but the binary operations $\wedge$, $\vee$ are again defined from only $\between$ to E. A Boolean algebra is a special case of a partial-Boolean algebra, namely the case where $\between = E \times E$. (Kochen-Specker, 1965, pp. 180, 183; 1967, pp. 64-65). Using the terminology and style of Sections (A) and (B), these structures are described as follows.

A <u>partial-ring-with-unit</u> $\langle E, \between, +, \cdot, 0, 1 \rangle$ is a non-empty set of elements $E = \{a, b, c, d, e, \ldots,\}$ including the distinguished 0 and 1, with a binary relation of <u>compatibility</u> $\between \subseteq E \times E$ and two binary operations $+$ and $\cdot$ defined from $\between$ to E such that:

($\between$a) $\between$ is <u>reflexive</u>, i.e., for any $b \in E$, $b \between b$, <u>symmetric</u>, i.e., for any b, $c \in E$, if $b \between c$ then $c \between b$, and <u>non-transitive</u>, i.e., for any $b, c, d \in E$, if $b \between c$ and $c \between d$, it does <u>not</u> follow that $b \between d$.

($\between$b) For every $b \in E$, $b \between 1$ and $b \between 0$.

($\between$c) $\between$ is <u>closed</u> under $+$ and $\cdot$, i.e., for any $b, c, d \in E$, if b, c, d are pairwise compatible then $(b+c) \between d$, $(b \cdot c) \between d$, etc. for all combinations.

And for any subset $F \subseteq E$, if all the elements in F are pairwise compatible, then by closure they generate a commutative-ring-with-unit. (Kochen-Specker, 1965, p. 180; 1967, p. 64). Finally, since $+, \cdot$, are defined from only $\between$ to E, rather than from $E \times E$ to E, they are called partial-operations or partial-functions by Kochen-Specker (1965, pp. 177, 178).

Kochen and Specker do not state any other conditions which the 0 and 1 elements and the $+, \cdot$ operations must satisfy. However, since any partial-ring-with-unit which has $\between = E \times E$ is a commutative-ring-with-unit and since any subset of mutually compatible elements in a partial-ring-with-unit form a commutative-ring-with-unit, the 0 and 1 elements and the $+, \cdot$ operations of a partial-ring-with-unit presumably must satisfy all the conditions (1)-(8) which define the $0, 1, +, \cdot$, of a commutative-ring-with-unit.

The idempotent elements of a partial-ring-with-unit form a partial-Boolean-ring-with-unit which is a partial-Boolean algebra $A = \langle E, \between, \wedge, \vee, {}^{\perp}, 0, 1 \rangle$ when the $\wedge, \vee, {}^{\perp}$ operations are defined in terms of $+$ and $\cdot$ as usual and $\between$, 0, 1 are defined as above (Kochen-Specker, 1965, p. 183). In particular, E is non-empty; $0, 1$ are the distinguished elements; $\between$ is reflexive, symmetric, and non-transitive; for every $b \in E$, $b \between 1$ and $b \between 0$; $\between$ is closed under $\wedge, \vee, {}^{\perp}$; and finally, for any subset $F \subseteq E$, if all the elements in F are pairwise compatible, then by closure they generate a Boolean (sub)structure. Moreover, since all the elements in a partial-Boolean-ring-with-unit are idempotent, not only conditions (1)-(8) but also condition (9) and hence condition (10) are all satisfied. Thus the binary operations $\wedge$, $\vee$ defined from $\between$ to E in terms of $+$ and $\cdot$ as usual satisfy the commutativity, associativity, absorption, and distributivity conditions which follow from the conditions (1)-(7), (9), (10) satisfied by $+$ and $\cdot$. And since $\wedge$, $\vee$ are defined from only $\between$ to E, rather than from $E \times E$ to E, they are in fact partial-operations. The unary operation $^{\perp}$ defined from E to E in terms of + and the 1-element as usual satisfies the complementation and involution conditions and (assuming $\leq$ is defined in terms of $\wedge$ or $\vee$ as usual) the condition: $b \leq c$ IFF $c^{\perp} \leq b^{\perp}$, which follow from the conditions (1)-(3), (6), (8)-(10). Thus $^{\perp}$ is orthocomplementation.

The partial-ordering relation $\leq$ is defined in a partial-Boolean algebra in terms of $\wedge$ or $\vee$ as usual, i.e., $b \leq c$ IFF $b \wedge c = b$, and $b \leq c$ IFF $b \vee c = c$. Since the meet $b \wedge c$ and the join $b \vee c$ are defined in A IFF $b \between c$, we can be sure that, for any $b,c \in A$, if $b \leq c$ then $b \between c$. The partial-ordering relation so defined in A is reflexive and anti-symmetric as usual. However, if $b \leq c$ and $c \leq d$ but $b \not\between d$, then $b \wedge d$, $b \vee d$ are not defined in A and so it does not follow that $b \leq d$. So $\leq$ may not be transitive, in which case $\leq$ is not a partial-ordering. But $\leq$ is transitive in the partial-Boolean algebras considered in this thesis, namely the partial-Boolean algebras determined by quantum mechanics, which shall be labeled $P_{QMA} = \langle E, \between, \wedge, \vee, ^{\perp}, \leq, 0, 1\rangle$.

The P_{QMA} structures are associative, transitive,[1] and atomic. A partial-Boolean algebra A is associative IFF, for any $b,c,d \in A$ such that $b \between c$ and $c \between d$: $b \between (c \wedge d)$ IFF $(b \wedge c) \between d$; and $b \between (c \wedge d)$ implies $b \wedge (c \wedge d) = (b \wedge c) \wedge d$. A transitive partial-Boolean algebra A satisfies the condition: For any $b,c,d \in A$, if $b \leq c$, and $c \leq d$, then $b \between d$ and $b \leq d$. And an atomic partial-Boolean algebra satisfies the same atomicity condition as an atomic B. (An additional condition on P_{QMA} structures is introduced in Chapter VI(D); nothing before Chapter VI(D) is affected by this additional condition.)

The notion of a partial-Boolean algebra is further elucidated by the following construction due to Kochen-Specker. Consider a nonempty family of

Boolean algebras $\{\mathcal{B}_i\}_{i \in \mathrm{Index}}$ such that the intersection of two algebras of the family is itself an algebra of the family; so all the $\mathcal{B}_i$ share the same distinguished 0, 1 elements. And if $\{e_1, e_2, \ldots\}$ are elements of the union $E = \bigcup_i \mathcal{B}_i$ such that every pair of them lie in some common algebra $\mathcal{B}_i$, then there is a $\mathcal{B}_k$, $k \in$ Index such that $\{e_1, e_2, \ldots\} \in \mathcal{B}_k$. Then a partial-Boolean algebra $\mathcal{A}$ is defined on the union E as follows. For any $b, c, d \in E$, $b \between c$ in $\mathcal{A}$ IFF there exists a $\mathcal{B}_i$ such that $b, c \in \mathcal{B}_i$; $b \wedge c = d$ in $\mathcal{A}$ IFF there exists a $\mathcal{B}_i$ such that $b \wedge c = d$ in $\mathcal{B}_i$; $b \vee c = d$ in $\mathcal{A}$ IFF there exists a $\mathcal{B}_i$ such that $b \vee c = d$ in $\mathcal{B}_i$; $b = c$ in $\mathcal{A}$ IFF there exists a $\mathcal{B}_i$ such that $b = c$ in $\mathcal{B}_i$; 1 and 0 in $\mathcal{A}$ are the common distinguished elements of all the $\mathcal{B}_i$. Kochen-Specker state and Hughes proves that every $\mathcal{A}$ is isomorphic to an $\mathcal{A}$ constructed on a family of Boolean algebras as above. (Kochen-Specker, 1965, pp. 183-184; Hughes, 1978, pp. 113-114).

Section E. The Quantum Orthomodular Lattice

Jauch's definition of the lattice structures determined by quantum mechanics, which I label $\mathcal{P}_{QML}$, starts with an orthocomplemented lattice $\langle E, \leq, \wedge, \vee, {}^{\perp}, 0, 1 \rangle$ which is complete in the usual sense that every subset of E has a g.l.b. and a l.u.b. Then Jauch defines the compatibility relation $\between$ in this lattice as follows: A subset $F \subseteq E$ is a <u>compatible set</u> if the lattice generated by F is a Boolean sublattice of the original lattice. (Let $\{SL_i\}_{i \in \mathrm{Index}}$ be the family of all the sublattices which contain F; the sublattice $SL_0 = \bigcap_i SL_i$ is the lattice <u>generated</u> by F. (Jauch, 1968, pp. 74-77, 80-81).) As a binary relation, compatibility $\between \subseteq E \times E$ is reflexive, symmetric, and non-transitive. And it is easy to show that, for any $b \in E$, $b \between 0$, $b \between 1$, and $b \between b$.

In order to define P_{QML}, Jauch furthermore postulates the conditions:

(P) If $b \leq c$ then $b \mathrel{\acute{\circ}} c$, for any $b, c \in E$. Jauch calls this condition <u>weak modularity</u>.

(A1) Atomicity (as usual).

(A2) If a is an atom and $a \wedge e = 0$, then $a \vee e$ covers e, for any $e \in E$ (Jauch, 1968, pp. 86-87). And it follows that if a is any atom, then $(a \vee e) \wedge e^{\perp}$ is also an atom, for any $e \in E$ (Piron, 1976, p. 24).

Thus $P_{QML} = \langle E, \leq, \wedge, \vee, {}^{\perp}, \acute{\circ}, 0, 1 \rangle$ is a complete, orthocomplemented, weakly modular, atomic lattice.

It is easy to show that in such a lattice, for any $b, c \in E$: $b \mathrel{\acute{\circ}} c$ IFF $(b \wedge c^{\perp}) \vee c = (c \wedge b^{\perp}) \vee b = b \vee c$; $b \mathrel{\acute{\circ}} c$ IFF $(b \wedge c) \vee (b \wedge c^{\perp}) = b$; and $b \mathrel{\acute{\circ}} c$ IFF the elements $b, b^{\perp}, c, c^{\perp}$, satisfy the distributive law for any combination (Jauch, 1968, p. 87; Piron, 1976, p. 26). Moreover, since $b \leq c$ IFF $b \wedge c = b$, it follows from weak modularity that, for any $b, c \in E$, if $b \leq c$ then $b = (b \vee c^{\perp}) \wedge c$, and if $b \leq c$ then $c = (c \wedge b^{\perp}) \vee b$. This is the <u>orthomodularity</u> condition, according to Rose (1964, p. 331) and according to Piron (1976, p. 24). The phrase "orthomodular" subsumes the two conditions of orthocomplementation and weak modularity; thus P_{QML} is a complete, atomic, orthomodular lattice.

Piron develops his definition of the complete, atomic, orthomodular lattice P_{QML} in a different manner which reveals the fact that each element $b \in P_{QML}$ may have non-unique complements defined in P_{QML} besides the unique orthocomplement $b^{\perp}$. Piron starts with a lattice which is complete in the usual sense. Since completeness ensures that the entire lattice has a g.l.b. which is the distinguished 0-element and a l.u.b. which is the distinguished 1-element, a complete lattice is an ordered sextuple $\langle E,\leq,\wedge,\vee,0,1\rangle$.

In order to define $\mathcal{P}_{QML}$, Piron furthermore postulates the conditions:

(A1) and (A2), as in Jauch.

(C) For each element $b \in E$, there is at least one *compatible complement* $b^{\perp} \in E$, where $b, b^{\perp}$ are *complements* satisfying the usual complementation condition ($b \wedge b^{\perp} = 0$ and $b \vee b^{\perp} = 1$), and $b, b^{\perp}$ are *compatible* in a sense which Piron defines independently of the $\wedge$, $\vee$, operations and $\leq$ relation. Most simply, any b, c are compatible in Piron's sense if they are associated with simultaneously measurable quantum propositions.

(P) For any $b,c \in E$, if $b \leq c$ then the sublattice generated by $b, b^{\perp}, c, c^{\perp}$ is distributive. Piron calls this condition *weak modularity* (Piron, 1976, pp. 21-23).

Two results follow from Piron's weak modularity. First, if $b \leq c$, then by (P), the elements $b, b^{\perp}, c$ are distributive and so $b \vee (b^{\perp} \wedge c) = (b \vee b^{\perp}) \wedge (b \vee c) = 1 \wedge (b \vee c) = c$; similarly, if $b \leq c$ then $c \wedge (c^{\perp} \vee b) = b$. This result will be mentioned again shortly. Secondly, according to Piron, it follows immediately from (P) that, for any $b,c \in E$, if $b \leq c$ then $c^{\perp} \leq b^{\perp}$, and it follows that the compatible complement of each element is unique. Thus the association of an element b with its unique $b^{\perp}$ is *orthocomplementation* satisfying: $b \wedge b^{\perp} = 0$, $b \vee b^{\perp} = 1$, $(b^{\perp})^{\perp} = b$, and if $b \leq c$ then $c^{\perp} \leq b^{\perp}$ (Piron, 1976, pp. 23-24).[2]

Substituting $^{\perp}$ for $^{\spadesuit}$, the first result following from (P) becomes the orthomodularity condition. And Piron proves that if the orthocomplement is interpreted as a compatible complement, then any orthomodular lattice satisfies his conditions (C) and (P).

Moreover, Piron's weak modularity can be shown to be equivalent to Jauch's weak modularity. Piron later defines the compatibility relation $\delta \subseteq E \times E$ in a complete lattice satisfying (C) and (P) as follows: $b \, \delta \, c$ IFF the sublattice generated by $b, b^{\spadesuit}, c, c^{\spadesuit}$ is distributive. With this definition of compatibility, Jauch's (P) is equivalent to Piron's (P) with $^{\perp}$ substituted for $^{\spadesuit}$. Jauch also says that his weak modularity is equivalent to the postulate that the compatible complement is unique, that is, the second result which Piron derives from his weak modularity (Jauch, 1968, p. 87).

So like Jauch, Piron defines P_{QML} as a complete, atomic, orthomodular lattice. Moreover, Piron makes it clear that an element in P_{QML} may have non-unique complements which satisfy the complementation condition but which are not compatible complements and are not orthocomplements. Thus there arises in P_{QML} the problem of a complementation which is not unique (and hence is not an operation), as will be discussed in Chapter IV(F). The Boolean L and P_{CM} structures and the partial-Boolean algebra P_{QMA} each have only one complementation, namely, the orthocomplementation, which is unique.

Finally, as with P_{QMA} , a Boolean structure is a special case of an orthomodular lattice P_{QML} , namely, the case where $\delta = E \times E$. Moreover, any quantum P_{QMA} can be extended to an orthomodular lattice P_{QML} by defining the $\wedge$, $\vee$ operations among incompatible elements. The two structures P_{QMA} and P_{QML} will be further compared in Chapter IV(E) and (F).

Section F. Subsets of P_{QMA} and P_{QML}

The notion of a filter, ideal, ultrafilter, ultraideal, principal ultrafilter, and principal ultraideal are defined in any lattice, e.g., in the quantum P_{QML}, exactly as they are defined in a Boolean lattice (Birkhoff, 1967, pp. 25, 28). Bub mentions that a filter and an ultrafilter in the quantum P_{QMA} (and dually an ideal and an ultraideal in P_{QMA}) are defined as in a Boolean algebra, i.e., any filter satisfies (a), (b), any any ideal satisfies (a'), (b') (Bub, 1974, p. 120). However, R. Hughes modifies condition (a) (and dually, (a')). The modification is motivated by the fact that, for any b, c in any filter $F \subseteq P_{QMA}$ if $b \not\leftrightarrow c$ then $b \wedge c$ is not defined in P_{QMA}. Hughes's modified definition is: A filter in a P_{QMA} is a non-empty subset F of P_{QMA} such that, for any $b,c,d \in P_{QMA}$.

(a_H) If $b,c \in F$, then there is a $d \in F$ such that $d \leq b$ any $d \leq c$.

And Hughes adds as a proviso the condition:

(c_H) $0 \notin F$.

Condition (b) is left as before; that is, (b), (a_H), and (c_H) define a filter in a P_{QMA}.

According to the definition of a filter in a Boolean structure $\mathcal{B}$, the entire $\mathcal{B}$ is a filter, albeit an improper filter. But according to Hughes's definition of a filter in P_{QMA}, the entire P_{QMA} is not a filter since $0 \in P_{QMA}$ but according to condition (c_H), 0 is not a member of any filter. Conditions (b), (a_H), (c_H), actually define a proper filter in $\mathcal{P}_{QMA}$. So we may drop condition (c_H) and define a <u>filter</u> in a $\mathcal{P}_{QMA}$ as a non-empty subset F of $\mathcal{P}_{QMA}$ which satisfies (a_H) and (b).

The difference between (a) and (a_H) may be characterized as follows. For any $b,c \in F$, according to (a) and assuming that $b \wedge c$ is defined in $\mathcal{P}_{QMA}$ (i.e., $b \overset{|}{\circ} c$), the element $b \wedge c$ is a member of F, where $b \wedge c$ is the greatest lower bound of $\{b,c\}$, as shown in Section (D); while according to (a_H) and regardless of whether or not $b \wedge c$ is defined in $\mathcal{P}_{QMA}$, any one of the lower bounds of $\{b,c\}$ is a member of F. Now if $b \wedge c$ is defined in $\mathcal{P}_{QMA}$, then (a_H) and (b) do ensure that $b \wedge c \in F$ if $b,c \in F$. For by (a_H), some lower bound of $\{b,c\}$ is a member of F if $b,c \in F$, and so by (b), the g.l.b. $\{b,c\} = b \wedge c$ is a member of F if $b,c \in F$. That is, though a filter in a $\mathcal{P}_{QMA}$ is defined by condition (a_H) rather than (a), nevertheless a filter in $\mathcal{P}_{QMA}$ does satisfy condition (a) for those $b,c \in F$ such that $b \overset{|}{\circ} c$.

The dual modified condition (a'_H) which, together with the unmodified (b'), defines an <u>ideal</u> I in a $\mathcal{P}_{QMA}$ is, of course, for any $b,c,d \in \mathcal{P}_{QMA}$:

(a'_H) If $b,c \in I$, then there is a $d \in I$ such that $d \geq b$ and $d \geq c$.

And as above, an ideal in a $\mathcal{P}_{QMA}$ does satisfy the unmodified condition (a') for those $b,c \in I$ such that $b \overset{|}{\circ} c$.

As in the Boolean case, we define an <u>ultrafilter</u> (<u>ultraideal</u>) in a $\mathcal{P}_{QMA}$ as a proper filter (ideal) which is not the proper subset of any proper filter (ideal) in $\mathcal{P}_{QMA}$. And a <u>principal</u> ultrafilter and a <u>principal</u> ultraideal are defined with respect to an atom of $\mathcal{P}_{QMA}$ as in Section C.

Hereafter, $\mathcal{P}_{QM}$ refers to both $\mathcal{P}_{QMA}$ and $\mathcal{P}_{QML}$ indiscriminately.

A <u>substructure</u> of $\mathcal{P}_{QM}$ is a non-empty subset of elements of $\mathcal{P}_{QM}$ which is closed with respect to the $\wedge, \vee, {}^{\perp}$ operations of $\mathcal{P}_{QM}$ (where the $\wedge, \vee$ operations of $\mathcal{P}_{QM}$ are partial-operations, as described in Section (D).) Any non-empty subset of elements of $\mathcal{P}_{QM}$ <u>generates</u> a substructure of $\mathcal{P}_{QM}$ when closed with respect to the operations of $\mathcal{P}_{QM}$. A substructure of $\mathcal{P}_{QM}$ is <u>Boolean</u> IFF its elements are mutually (i.e., pairwise) compatible. Any non-empty subset of mutually compatible elements in $\mathcal{P}_{QM}$ generates a Boolean substructure of $\mathcal{P}_{QM}$ when closed with respect to the operations of $\mathcal{P}_{QM}$. And for any $P_1 \not\leftrightarrow P_2$ in $\mathcal{P}_{QM}$, there is no Boolean substructure in $\mathcal{P}_{QM}$ which contains both P_1, P_2. Any element $P \in \mathcal{P}_{QM}$ is a member of some Boolean substructure in $\mathcal{P}_{QM}$, at least the Boolean substructure consisting of just the elements $\{P, P^{\perp}, 0, 1\}$. A <u>maximal Boolean substructure</u> mBS of $\mathcal{P}_{QM}$ is a Boolean substructure which is not the proper subset of any other Boolean substructure of $\mathcal{P}_{QM}$. And by Zorn's lemma, any Boolean substructure of $\mathcal{P}_{QM}$ is contained in a maximal one (Varadarajan, 1962, p. 204).

The <u>centre</u> of a $\mathcal{P}_{QM}$ is the subset of elements in $\mathcal{P}_{QM}$ which are compatible with every element in $\mathcal{P}_{QM}$. This subset is in fact a closed substructure of $\mathcal{P}_{QM}$, and moreover, it is a Boolean substructure. The centre of any $\mathcal{P}_{QM}$ contains at least the 0, 1 elements of $\mathcal{P}_{QM}$ since the 0, 1 elements are compatible with every other element in $\mathcal{P}_{QM}$.[3]

<u>Section G</u>. <u>Mappings on a Structure</u>

Let $\mathcal{X}, \mathcal{Y}$ be any algebraic and/or lattice-theoretic structures which have $\wedge, \vee, {}^{\perp}$ operations defined on a set of elements including the distinguished 0-element and 1-element. Any <u>mapping</u> $m : \mathcal{X} \to \mathcal{Y}$ from any structure X to any structure Y assigns values as follows:

Ma For any $b,c,d \in X$, $m(b)$ is unique, that is, if $b = c$ in X then $m(b) = m(c)$ in Y. For example, if $b \wedge c = d$ in X then $m(b \wedge c) = m(d)$ in Y, if $b\ = c$ in X then $m(b\ ) = m(c)$ in Y.

Mb $m(0) = 0$ in Y.

Moreover, any non-trivial mapping $m : X \to Y$ also assigns:

Mc $m(1) = 1$ in Y.

If Y is the two-element Boolean structure Z_2, then m is a <u>bivalent</u> mapping designated as $m : X \to \{0,1\}$. A <u>homomorphic</u> mapping $h : X \to Y$ <u>preserves the operations</u> defined in X, i.e., for any $b,c \in X$,

H1 $h(b \wedge c) = h(b) \wedge h(c)$.

H2 $h(b \vee c) = h(b) \vee h(c)$.

H3 $h(b^{\perp}) = (h(b))^{\perp}$.

A mapping $m : X \to Y$ is said to be <u>injective</u> IFF, for any $b,c \in X$, if $b \neq c$ then $m(b) \neq m(c)$. Clearly, an injective mapping is one-to-one into Y. A mapping $m : X \to Y$ is said to be <u>surjective</u> IFF $m(X) = Y$, i.e., the image of X under m is the entire Y. An <u>isomorphic</u> mapping $m : X \to Y$ is an injective and surjective mapping, i.e., a one-to-one mapping, which preserves the operations of X (Lang, 1971, pp. 87, 90, 106; Birkhoff, 1948, p. vii). An <u>imbedding</u> of one structure into another is a homomorphic mapping which is injective (Bub, 1974, p. 68).

<u>Notes</u>

[1] Hughes discusses the problem of the transitivity of $\leq$ in a partial-Boolean algebra and proves that a quantum partial-Boolean algebra

of subspaces of a Hilbert space is associative and transitive (Hughes, 1978, p. VI.18).

[2] As described in note 5 of chapter IV(E), orthocomplementation is defined as a type of mapping, namely, a dual automorphism.

[3] Piron defines the centre of a $\mathcal{P}_{QML}$; the centre of a $\mathcal{P}_{QMA}$ can be defined in exactly the same way. (Piron, 1976, p. 29).

CHAPTER II

THE CLASSICAL PRECEDENT FOR A BIVALENT TRUTH-FUNCTIONAL SEMANTICS

Section A. The Standard Semantics of Classical Propositional Logic

Classical propositional logic assigns truth values to a set $L = \{f_1, f_2, \ldots\}$ of well-formed formulae by semantic mappings, called valuations, which are bivalent and truth-functional. A valuation v on an L initially assigns the value 0 (false) or 1 (true) to each of the atomic (sub)formulae in L. And then the valuation assigns 0, 1 values to every other formula in L in the following recursive manner: for any $f_1, f_2, f \in L$,

TF1 $v(f_1 \wedge f_2) = 1$ IFF $v(f_1) = v(f_2) = 1$

TF2 $v(f_1 \vee f_2) = 1$ IFF $v(f_1) = 1$ or $v(f_2) = 1$

TF3 $v(f^{\perp}) = 1$ IFF $v(f) = 0$, where "$\wedge$" designates "and," "$\vee$" designates "or," and "$\perp$" designates "not." This (redundant) list of biconditionals characterizes the truth-functionality condition on the valuations. The bivalency condition requires that every formula in L be assigned a 0 or 1 value.

According to the truth-table method of schematizing valuations, there are as many valuations for a set L of formulae as there are rows in the truth-table for L, where each row in the truth-table is specified by a different initial assignment of 0, 1 values to the atomic (sub)formulae occurring in L. And if n is the number of atomic (sub)formulae in L, then there are exactly 2^n valuations for L. Such a collection of valuations can be regarded as a bivalent truth-functional semantics for L.

This notion of a bivalent, truth-functional semantics for an L will be restated in algebraic terms in Section (D).

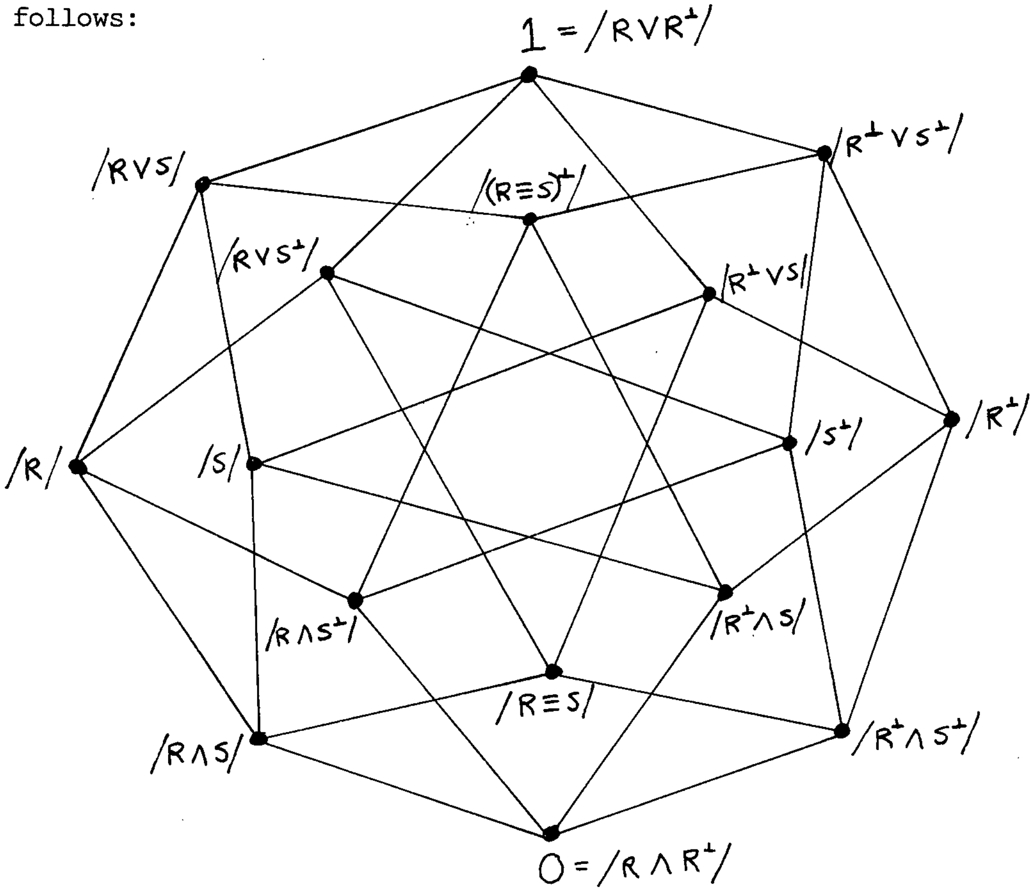
Section B. The Boolean Structure Determined by Classical Propositional Logic

In the algebraic approach to classical propositional logic, we start with a set L of formulae which is closed with respect to the $\wedge$, $\vee$, $^{\perp}$ operations. Such a closed L is partitioned into equivalence classes with respect to the standard, classical proof theoretic equivalence relation: for any $f_1, f_2 \in L$, $f_1 \sim f_2$ IFF $\vdash f_1 \supset f_2$ and $\vdash f_2 \supset f_1$, where $\vdash$ is classical derivability. The resulting set of equivalence classes form a Boolean structure, often called the Lindenbaum algebra, which shall be labeled $\mathcal{L} = \langle E = \{/f_1/, /f_2/, \ldots\}, \wedge, \vee, ^{\perp}, \leq, 0, 1\rangle$. (The equivalence class containing f_1 is designated "$/f_1/$".) For any $f_1, f_2 \in L$, $/f_1/ \wedge /f_2/ = /f_1 \wedge f_2/$; $/f_1/ \vee /f_2/ = /f_1 \vee f_2/$; $/f_1/^{\perp} = /f^{\perp}/$; and $/f_1/ \leq /f_2/$ IFF $f_1 \vdash f_2$. The 0-element of $\mathcal{L}$ is the equivalence class of anti-theorems or contradictions, while the 1-element is the equivalence class of theorems or tautologies. When the number n of atomic (sub)formulae in L is finite, then the $\mathcal{L}$ structure of L is finite and atomic, with exactly 2^{2^n} elements and 2^n atoms. But when the number of atomic (sub)formulae in L is infinite, then the $\mathcal{L}$ structure of L is infinite and atomless.

For example, the closed set L_2 of propositional formulae in just two propositional variables, say R and S, is partitioned into exactly 16 equivalence classes:

	$/R/$	$/S/$	$/R \wedge S/$	$/R \wedge S^{\perp}/$	$/R^{\perp} \wedge S/$	$/R^{\perp} \wedge S^{\perp}/$	$/R \equiv S/$	$/(R \equiv S)^{\perp}/$	$/S^{\perp}/$	$/R^{\perp}/$	$/R \vee S/$	$/R \vee S^{\perp}/$	$/R^{\perp} \vee S/$	$/R^{\perp} \vee S^{\perp}/$	$/R \wedge R^{\perp}/$	$/R \vee R^{\perp}/$
υ^1	1	1	1	0	0	0	1	0	0	0	1	1	1	0	0	1
υ^2	1	0	0	1	0	0	0	1	1	0	1	1	0	1	0	1
υ^3	0	1	0	0	1	0	0	1	0	1	1	0	1	1	0	1
υ^4	0	0	0	0	0	1	1	0	1	1	0	1	1	1	0	1

These equivalence classes form the Lindenbaum algebra L_2 diagrammed as follows:



Notice that the four atoms in this Lindenbaum algebra are not the equivalence classes of the atomic formulae R, S, but rather are the following: $/R \wedge S/$, $/R \wedge S^{\perp}/$, $/R^{\perp} \wedge S/$, $/R^{\perp} \wedge S^{\perp}/$.

Every Lindenbaum algebra of (equivalence classes of) formulae of classical propositional logic is a Boolean structure. The simplest (non-trivial) Boolean structure has just the two elements 0 and 1 and is often called Z_2 ; it shall also be labeled $\{0,1\}$. Any Boolean structure can be homomorphically mapped onto this simplest Boolean structure, as described next.

Section C. Bivalent Homomorphic Mappings on Any Boolean Structure

Given any Boolean structure $\mathcal{B}$, there exist bivalent, homomorphic mappings $h : \mathcal{B} \rightarrow \{0,1\}$ which can be defined with respect to the ultrafilters in $\mathcal{B}$ since there is a one-to-one correspondence between ultrafilters in $\mathcal{B}$ and bivalent homomorphisms on $\mathcal{B}$. Sikorski defines each bivalent homomorphism h with respect to an ultrafilter UF as follows: for any element $b \in \mathcal{B}$, $h(b) = 1$ if $b \in$ UF and $h(b) = 0$ if $b \notin$ UF (Sikorski, 1960, p. 16). However, each h on a $\mathcal{B}$ may be equivalently defined with respect to UF as: for any $b \in \mathcal{B}$, $h(b) = 1$ if $b \in$ UF and $h(b) = 0$ if $b \in$ UI, where UI is the unique ultraideal dual to UF. In this thesis, the latter is taken as my usual definition of a bivalent homomorphism. For any $\mathcal{B}$, my definition and Sikorski's definition are equivalent because, for any UF and dual UI in $\mathcal{B}$ and for any $b \in \mathcal{B}$, $b \notin$ UF IFF $b \in$ UI, as shown in Chapter I(C). But when we consider the non-Boolean propositional structures determined by quantum mechanics, it is not always the case that if $b \notin$ UF then $b \in$ UI. So the two definitions differ and it is argued in Chapter VI(B) that my definition is more useful.

Each mapping $h : \mathcal{B} \rightarrow \{0,1\}$ is clearly bivalent. And by definition, a homomorphism satisfies the conditions H1, H1, H3 listed in

Chapter I(G), where $1 \wedge 1 = 1 \vee 1 = 1$, $0 \wedge 0 = 0 \vee 0 = 0$, $1 \wedge 0 = 0 \wedge 1 = 0$, $1 \vee 0 = 0 \vee 1 = 1$, and $1^{\perp} = 0$, $0^{\perp} = 1$. It is easy to show that a bivalent mapping on an algebraic structure is homomorphic qua H1, H2, H3, IFF it is truth-functional qua TF1, TF2, TF3 (Bub, 1974, p. 99). Thus each bivalent homomorphism on a Lindenbaum algebra is bivalent and truth-functional.

Alternately, the truth-functional character of every bivalent homomorphism on a $\mathcal{B}$ can be shown as follows. As mentioned in Chapter I(C), every ultrafilter and dual ultraideal in any $\mathcal{B}$ is prime, and each ultrafilter together with its dual ultraideal completely exhaust $\mathcal{B}$, i.e., $\mathcal{B} = \text{UF} \cup \text{UI}$. Moreover, it follows from the eight conditions (a)-(d), (a')-(d'), listed in Chapter I(C), which define prime UF and prime UI, that each bivalent homomorphism defined with respect to UF and UI is truth-functional. For the eight conditions yield the following biconditionals, for any $b, b_1, b_2 \in \mathcal{B}$:

U1 $b_1 \wedge b_2 \in \text{UF}$ IFF $b_1 \in \text{UF}$ and $b_2 \in \text{UF}$, by (a) and (b).

$b_1 \wedge b_2 \in \text{UI}$ IFF $b_1 \in \text{UI}$ or $b_2 \in \text{UI}$, by (b') and (d').

U2 $b_1 \vee b_2 \in \text{UF}$ IFF $b_1 \in \text{UF}$ or $b_2 \in \text{UF}$, by (b) and (d).

$b_1 \vee b_2 \in \text{UI}$ IFF $b_1 \in \text{UI}$ and $b_2 \in \text{UI}$, by (a') and (b').

U3 $b^{\perp} \in \text{UF}$ IFF $b \in \text{UI}$, by (c).

$b^{\perp} \in \text{UI}$ IFF $b \in \text{UF}$, by (c').

So by the definition of $h : \mathcal{B} \to \{0,1\}$ with respect to UF and UI:

TF1 $h(b_1 \wedge b_2) = 1$ IFF $h(b_1) = h(b_2) = 1$

$h(b_1 \wedge b_2) = 0$ IFF $h(b_1) = 0$ or $h(b_2) = 0$

TF2 $h(b_1 \vee b_2) = 1$ IFF $h(b_1) = 1$ or $h(b_2) = 1$

$h(b_1 \vee b_2) = 0$ IFF $h(b_1) = h(b_2) = 0$

TF3 $h(b^{\perp}) = 1$ IFF $h(b) = 0$

$h(b^{\perp}) = 0$ IFF $h(b) = 1$.

Thus each bivalent homomorphism on a $\mathcal{B}$ is truth-functional.

Furthermore, any $\mathcal{B}$ admits many bivalent homomorphisms. If $b_1 \neq b_2$ are any pair of distinct elements in $\mathcal{B}$, then as mentioned in Chapter I(C), there is some ultrafilter in $\mathcal{B}$ which contains one element but not the other. Hence there is some bivalent homomorphism on $\mathcal{B}$ which assigns the value 1 to one element and 0 to therother. In other words, for any pair of distinct elements $b_1 \neq b_2$ in a $\mathcal{B}$, there is some h such that $h(b_1) \neq h(b_2)$; this has been called the <u>semi-simplicity property of</u> $\mathcal{B}$ (Kochen-Specker, 1967, p. 67). And in particular, as Halmos shows, for any nonzero $b \neq 0$ in a $\mathcal{B}$, there is some h such that $h(b) \neq h(0) = 0$, i.e., such that $h(b) = 1$ since every h assigns the value 0 to the 0-element (Halmos, 1963, p. 77). The former notion shall be taken to define a <u>complete collection</u> of bivalent homomorphisms on an algebraic structure X, that is, a collection of bivalent homomorphisms on an X is <u>complete</u> IFF, for any distinct $b \neq c$ in X, there is an h such that $h(b) \neq h(c)$.[1] Clearly, the completeness of the collection of bivalent homomorphisms on a Boolean structure $\mathcal{B}$ is ensured by the semi-simplicity property of $\mathcal{B}$.

When $\mathcal{B}$ is atomic, then besides the above-mentioned one-to-one correspondence between bivalent homomorphisms and ultrafilters (and dual ultraideals) there is also a one-to-one correspondence between ultrafilters and atoms. Each atom $a \in \mathcal{B}$ is a member of exactly one ultrafilter in $\mathcal{B}$, namely $UF_a = \{b \in \mathcal{B} : b \geq a\}$. And each atom a is assigned the value 1 by exactly one bivalent homomorphism on $\mathcal{B}$, namely the h_a defined with respect to UF_a and its dual UI_a. It is easy to show that a collection of bivalent homomorphisms on an atomic $\mathcal{B}$ is complete IFF it is as large as the number of atoms in $\mathcal{B}$. Proof: By definition, a complete collection is large enough so that every atom $a \neq 0$ is assigned the value $1 = h(a) \neq h(0) = 0$ by some h on $\mathcal{B}$. Since each atom is assigned the value 1 by exactly one bivalent homomorphism, the complete collection is as large as the number of atoms. Conversely, consider the collection of bivalent homomorphisms on an atomic $\mathcal{B}$ which is as large as the number of atoms in $\mathcal{B}$. By definition, each bivalent homomorphism in this collection is an h_a defined (via UF_a and UI_a) with respect to an atom $a \in \mathcal{B}$. Now by a theorem due to Rutherford (Chapter I(B)), for any $b \neq c$ in $\mathcal{B}$, there is an atom $a \in \mathcal{B}$ such that $a \leq b$ but $a \nleq c$, or $a \leq c$ but $a \nleq b$. If $a \leq b$ but $a \nleq c$, $b \in UF_a$ and $c \notin UF_a$, and so $h_a(b) = 1 \neq h_a(c)$. Similarly, if $a \leq c$ but $a \nleq b$, then $c \in UF_a$ and $b \notin UF_a$, and so $h_a(c) = 1 \neq h_a(b)$. Thus for any $b \neq c$ in $\mathcal{B}$, there is an h_a on $\mathcal{B}$ such that $h_a(b) \neq h_a(c)$. Q.E.D.

Section D. The Algebraic Semantics for the Lindenbaum Algebra

These facts about bivalent homomorphisms on a Boolean structure are relevant for the concept of a bivalent truth-functional semantics for the Lindenbaum algebras of classical propositional logic.

Each ultrafilter in the $\mathcal{L}$ structure of a (closed) set of formulae L is itself a subset of (equivalence classes of) formulae in L which is deductively complete in the sense that, for any UF in the $\mathcal{L}$ of an L, and for any formulae $f_1, f_2 \in L$, if $/f_1/ \in UF$ and $f_1 \vdash f_2$, then $/f_2/ \in UF$. And each ultrafilter in $\mathcal{L}$ is maximally consistent in the sense that the meet of all the elements in any UF is never the 0-element of $\mathcal{L}$, i.e., the conjunction of all the (equivalence classes) of formulae in UF is never a contradiction; but if any element in $\mathcal{L}$ which is outside a given UF were added to that UF, then the meet of all the elements in UF would be the 0-element of L.

As described in the previous section, each bivalent homomorphism on a Lindenbaum algebra is bivalent and truth-functional. Moreover, for any element $/f/ \in L$ and any $UF \subset L$, either $/f/ \in UF$ or $/f^{\perp}/ \in UF$ but not both; hence, no bivalent homomorphism on L assigns the value 1 to both $/f/$ and $/f^{\perp}/$ since every bivalent homomorphism assigns the value 1 to an ultrafilter of elements in L. And if a bivalent homomorphism were to assign the value 1 to any other element in L besides those elements in the ultrafilter which defines h, then h would assign the value 1 to the 0-element of L. So each bivalent homomorphism can be said to be a maximally consistent mapping on L.

Moreover, each bivalent homomorphism on the Lindenbaum algebra of an L is the algebraic version of one of the standard valuations for L. That is, for any given valuation v_0 on an L, there is a corresponding bivalent homomorphism h_0 on the L of that L such that, for every formula $f \in L$, $v_0(f) = h_0(/f/)$ (Bub, 1974, p. 102). And finally, in this thesis the complete collection of bivalent homomorphisms on a Lindenbaum algebra is regarded as a bivalent, truth-functional semantics.

The analogy between the complete collection of bivalent homomorphisms on an L and the truth table collection of valuations for an L may be elaborated as follows. If we assume that the number n of atomic (sub)formulae in L is finite, then the L structure of L is finite and atomic, with exactly 2^n atoms. Thus the complete collection of bivalent homomorphisms on L contains 2^n bivalent homomorphisms, just as the truth-table collection of valuations for that L contains 2^n valuations.

Each valuation for L is specified by its initial assignment of 0, 1 values to the n atomic (sub)formulae in L, and likewise each bivalent homomorphism on the L structure of L is specified by its initial assignment of 0, 1 values to the n equivalence classes of atomic formulae in L. For example, an initial assignment of the values 0 to R and 1 to S specifies the valuation v_3 in the truth table for L_2 given in Section (B). Similarly, the initial assignment of the values 0 to /R/ and 1 to /S/ specifies the unique atom $/R^{\perp}/\wedge S/$ in the Lindenbaum algebra L_2 of L_2; this atom in turn specifies the unique ultrafilter $UF_{/R^{\perp}\wedge S/} = \{/R^{\perp} \wedge S/, /S/, /R^{\perp}/, /R \vee S/, /R^{\perp} \vee S/, /R^{\perp} \vee S^{\perp}/, /(R \equiv S)^{\perp}/, /R \vee R^{\perp}/\}$; and this ultrafilter specifies a unique bivalent homomorphism $h_{/R^{\perp}\wedge S/}$ on L_2, where $h_{/R^{\perp}\wedge S/}(/f/) = v_3(f)$ for every formula $f \in L_2$.

The concept of a bivalent, truth-functional semantics for a Boolean Lindenbaum algebra described in this chapter will be treated in this thesis as a precedent for any proposed bivalent, truth-functional semantics for the Boolean propositional structures determined by classical mechanics and the non-Boolean propositional structures determined by quantum mechanics. In particular, subsequent chapters make use of the following:

For any propositional structure P, a mapping which assigns the value 1 to an ultrafilter UF of elements in P and assigns the value 0 to the dual ultraideal UI of elements in P is not only bivalent but also truth-functional with respect to the elements in UF ∪ UI. Such a bivalent, truth-functional mapping defined with respect to an UF and dual UI may be called an <u>ultravaluation</u> because, on a Lindenbaum algebra of classical propositional logic, such a mapping is the algebraic version of a standard valuation, which is regarded in this thesis as the paradigm semantic mapping.

The 0, 1 values assigned by an ultravaluation on a propositional structure may be interpreted as the truth-values true and false, again because, on a Lindenbaum algebra, an ultravaluation is the algebraic version of a standard valuation.

And use is especially made of the notion that a bivalent truth-functional semantics for a $\mathcal{P}$ is a complete collection of bivalent, truth-functional mappings. So it is clear that the existence of only one or several bivalent, truth-functional mappings on a $\mathcal{P}$ does not yet constitute a bivalent, truth-functional semantics for $\mathcal{P}$. But in order to show the impossibility of such a semantics, it obviously suffices to show that there is not even one bivalent, truth-functional mapping on $\mathcal{P}$.

Notes

[1] This notion of a complete collection of bivalent homomorphisms was suggested to me by Kochen and Specker. In their 1967 Theorem 0, Kochen-Specker prove that a partial-Boolean algebra of quantum propositions, labeled $\mathcal{P}_{QMA}$, can be imbedded into a Boolean algebra $\mathcal{B}$ IFF there exists what in this thesis is called a complete collection of bivalent homomorphisms on $\mathcal{P}_{QMA}$. Kochen-Specker also define a weak imbedding of a $\mathcal{P}_{QMA}$ into a $\mathcal{B}$; such an imbedding exists IFF there exists a large enough collection of bivalent homomorphisms on $\mathcal{P}_{QMA}$ so that, for every non-zero element $P \neq 0$ in $\mathcal{P}_{QMA}$, there is some $h : \mathcal{P}_{QMA} \to \{0,1\}$ such that $h(P) \neq h(0)$, i.e., $h(P) = 1$ since every h assigns the value 0 to the 0-element (Kochen-Specker, 1967, pp. 67,884). Such a collection may be called <u>weakly complete</u>. The notion of a weakly complete collection of bivalent homomorphisms on a propositional structure is mentioned in Chapters V and VI.

CHAPTER III

THE CLASSICAL PRECEDENT FOR A STATE-INDUCED SEMANTICS

Preface

We consider propositions[1] which make assertions about the real-number values of the <u>magnitudes</u>, i.e., measurable properties, of a classical physical system, for example:

The kinetic energy of a 1 kg swinging pendulum is between 19-20 kg m^2/sec^2.[2]
(magnitude) (-------system-------) (value)

As will be described in this chapter, such propositions and the logical operations "and," "or," "not" among such propositions can be associated with various mathematical machinery in the formalism of classical mechanics. These associations determine the structure of a set of such propositions. This structure is a σ-complete, atomic Boolean structure $\mathcal{P}_{CM}$.

Moreover, the formalism of classical mechanics includes state-induced bivalent homomorphisms, or equivalently, state-induced dispersion-free probability measures, which can be regarded as performing the semantic task of assigning truth-values to the elements of $\mathcal{P}_{CM}$. For each bivalent homomorphism or dispersion-free probability measure induced by the state of a classical system is an ultravaluation on the $\mathcal{P}_{CM}$ structure of propositions describing the system, just as each of the standard valuations for a set L of formulae of classical logic is an ultravaluation on the $\mathcal{L}$ structure of L. This straightforward analogy is a strong motivation for seriously considering the notion of a state-induced semantics for the propositional structures determined by classical mechanics and also considering the notion of a state-induced semantics for the propositional structures determined by quantum mechanics, as shall be proposed in Chapter VI.

Section A. The States of a Classical System Determine the Real Values of That System's Magnitudes

According to the Hamiltonian formalization of classical mechanics, a physical system is associated with an abstract phase space Ω which is parameterized by position and momentum coordinates and whose dimensionality reflects the degrees of freedom of the system. For example, a physical system with only one degree of freedom, such as a ball falling in a straight line, is associated with the simplest phase space which is two dimensional and has one position coordinate and one momentum coordinate. Each point $w \in \Omega$ represents a pure state of the system associated with Ω, for a pure state is a specification of the system's position and momentum values. According to classical mechanics, the values of every other (mechanical) magnitude of the system can be calculated once the system's state is specified. In particular, the classical formalism represents each magnitude A by a real-valued, measurable[3] function $f_A : \Omega \rightarrow R$ on the phase space associated with the system such that the image of any point $w \in \Omega$ under the function f_A is the real-number value $a \in R$ (the real-number line) of the magnitude A when the system is in the state w.

The real-valued functions $f_A, f_B, \ldots$ representing the classical magnitudes $A, B, \ldots$ have the ring operations $+$ and $\cdot$ defined among them as the usual sum and product of functions: for any f_A, f_B on Ω and for every $w \in \Omega$, $(f_A + f_B)(w) = f_A(w) + f_B(w)$, and $(f_A \cdot f_B)(w) = f_A(w) \cdot f_B(w)$. (Here $+$ and $\cdot$ work like the addition and multiplication of real numbers.)

For example, consider as a system a 1 kg pendulum swinging so that its maximum height is 2 m above its minimum height. Let w_1 and w_2 be the following states.

w_1: At the top of its swing, the pendulum's height position is 2 m and its momentum is 0 kg m/sec.

w_2: Near the bottom of its swing, the pendulum's height position is nearly 0 m and its momentum is nearly maximal, say 6.2 kg m/sec.

The magnitude kinetic energy, K, is represented in the classical formalism by the real-valued function $f_K = \frac{1}{2 \cdot \text{mass}} \cdot (\text{momentum})^2$. So when the pendulum's state is w_1, the real-number value of K is 0 kg m^2/sec^2. And when the pendulum's state is w_1, the value of K is 19.2 kg m^2/sec^2.

So the fact that the real-number values of a classical system's magnitudes depend upon the system's state has been formalized by representing each magnitude A by a real-valued, measurable function $f_A : \Omega \to \mathcal{R}$ on a classical phase space whose points represent the system's states.

Alternately, each state $w \in \Omega$ can itself be regarded as a mapping from a (closed) set $\mathcal{F}_{CM}$ of functions representing classical magnitudes to the real-number line, i.e., $w : \mathcal{F}_{CM} \to \mathcal{R}$, such that, for any point $w \in \Omega$ and for any function $f_A : \Omega \to \mathcal{R}$, $w(f_A) = f_A(w)$. The mapping $w : \mathcal{F}_{CM} \to \mathcal{R}$ may be called the <u>state-induced mapping</u>. It follows that each state-induced mapping preserves the $+$ and $\cdot$ operations defined among the functions: for any given, fixed $w \in \Omega$ and for any functions f_A, f_B on Ω, $w(f_A + f_B) = (f_A + f_B)(w) = f_A(w) + f_B(w)$ $= w(f_A) + w(f_B)$; and $w(f_A \cdot f_B) = (f_A \cdot f_B)(w) = f_A(w) \cdot f_B(w) = w(f_A) \cdot w(f_B)$.

This mathematical machinery of real-valued functions and state-induced mappings not only formalizes the procedure by which real-number values are assigned to the magnitudes of a classical system, but also implicitly formalizes a procedure by which truth values can be assigned to the propositions which make assertions about the real-number values of a classical system's magnitudes, as will be made explicit in Section (C).

Section B. The Propositional Structure Determined by Classical Mechanics

When a set of real-valued, measurable functions on a Ω is a closed set with respect to the $+, \cdot$ operations, then the set forms a commutative-ring-with-unit, labeled $\mathcal{F}_{CM} = \langle\{f_A, f_B, \ldots\}, +, \cdot, 0, 1\rangle$. The 0-element is the constant function f_0 which assigns the real-number 0 to every $w \in \Omega$, and the 1-element is the constant function f_1 which assigns the real-number 1 to every point in Ω.

Some of the functions in $\mathcal{F}_{CM}$ are idempotent functions f_P satisfying: $f_P \cdot f_P = f_P$, i.e., for every $w \in \Omega$, $(f_P \cdot f_P)(w) = f_P(w)$. Since the product $f_P \cdot f_P$ is defined as, for every $w \in \Omega$, $(f_P \cdot f_P)(w) = f_P(w) \cdot f_P(w)$, it follows that the real-number value $r = f_P(w)$ of an idempotent function is either 0 or 1. In other words, each f_P is a function from Ω to $\{0,1\}$. A set of idempotent functions which is closed with respect to the $+, \cdot$ operations forms a Boolean-ring-with-unit, or in other words, the idempotent elements of $\mathcal{F}_{CM}$ form a Boolean-ring-with-unit, as defined in Chapter I(A). And in this Boolean-ring-with-unit, the Boolean operations $\wedge, \vee, ^{\perp}$, and the lattice partial-ordering relation $\leq$ can be defined in terms of the ring operations $+$ and $\cdot$ as usual, yielding a Boolean structure of idempotent functions on a Ω.

Each idempotent function on a classical phase space is a <u>characteristic function</u> defined with respect to a unique subset $W_P \subseteq \Omega$ as follows: for any point $w \in \Omega$, $f_P(w) = 1$ if $w \in W_P$ and $f_P(w) = 0$ if $w \notin W_P$, i.e., $w \in W_P^{\perp}$. Each W_P is a measurable (i.e., Borel) subset of Ω and $W_P = f_P^{-1}(\{1\}) = \{w \in \Omega : f(w) = 1\}$; and each $W_P^{\perp}$ is the set-theoretic (ortho)complement of W_P, i.e., $W_P^{\perp} = \Omega - W_P$. Thus the idempotent functions on a Ω are in a one-to-one correspondence with the Borel subsets of Ω; each Borel subset uniquely defines an idempotent function (<u>qua</u> characteristic function) and each idempotent function uniquely specifies a Borel subset (via its inverse image $f_P^{-1}(\{1\})$). The Borel subsets of a Ω form a Boolean-ring-with-unit (with $+$, $\cdot$, 0, 1, interpreted as symmetric difference, set-intersection, the empty set, and the entire space Ω, respectively), which is isomorphic to the Boolean-ring-with-unit of idempotent functions on Ω. And the Boolean-ring-with-unit of Borel subsets of a Ω is also a <u>Boolean structure</u> (with $\wedge$, $\vee$, $^{\perp}$, $\leq$, interpreted as set-intersection, set-union, set-(ortho)complementation, and set-inclusion, respectively), which is isomorphic to the Boolean structure of idempotent functions on Ω (Bub, 1974, p. 105).

The Boolean structure of idempotent functions on a classical phase space, or isomorphically, the Boolean structure of Borel subsets of the phase space, have each been regarded as a propositional structure determined by classical mechanics, labeled $\mathcal{P}_{CM}$. For in one way or another, propositions which make assertions about the real-number values of a classical system's magnitudes have been associated with either the idempotent functions on the system's phase space or the uniquely corresponding Borel subsets of the system's phase space. For example, in his 1932 book, von Neumann argues that propositions which make assertions about the values of a system's magnitudes can themselves be regarded as idempotent magnitudes whose 0, 1 values can be interpreted as the "verification" and the non-verification of the propositions. Mentioning von Neumann's argument, Kochen-Specker likewise regard propositions as idempotent magnitudes whose 0, 1 values are interpreted as falsity and truth. There is a better reason, given in Section (C), why the 0, 1 values exhibited by the idempotent magnitudes may be interpreted as the truth-values of propositions. Nevertheless, in the classical formalism, idempotent magnitudes are represented by the above-described idempotent functions on a phase space. On the other hand, in their 1936 paper, von Neumann and Birkhoff associate propositions which make assertions about a classical system's magnitudes with the subsets of the system's phase space.[4] Similarly, Jauch associates such propositions with the Borel subsets of the system's phase space. Either association yields the Boolean propositional structure $\mathcal{P}_{CM} = \langle E = \{P_1, P_2, \ldots\}, \wedge, \vee, ^{\perp}, \leq, 0, 1\rangle$. The elements of $\mathcal{P}_{CM}$ may be thought of either as idempotent functions or as Borel subsets of the phase space; the elements of $\mathcal{P}_{CM}$ represent or are associated with propositions. The $\mathcal{P}_{CM}$ structure of any Ω is a σ-complete atomic, Boolean structure. And each atom P_w in a $\mathcal{P}_{CM}$ is a one-point idempotent function f_w uniquely corresponding with the singleton Borel subset $\{w\}$.

Section C. The Bivalent, Truth-Functional, State-Induced Semantics for the Boolean $\mathcal{P}_{CM}$ Structures

Just as the real-number values of a system's magnitudes depend upon the system's state (i.e., upon the values of the system's position and momentum), likewise the truth values of propositions which make assertions about the real-number values of a system's magnitudes depend upon that system's state. For example, when the pendulum described in Section A is in the state w_1, the truth value of the following proposition is false: The kinetic energy of the pendulum is between 19-20 kg m^2/sec^2. And when the pendulum is in the state w_2, the truth value of that proposition is true. The fact that a system's state determines the true values of propositions which make assertions about the real-number values of the system's magnitudes may be formalized by defining state-induced ultravaluations on the P_{CM} structure of these propositions, and such ultravaluations may be described in two ways. Both ways shall be elaborated, even though each yields the same notion of state-induced ultravaluations on a P_{CM}. For one way makes use of concepts introduced in Section A and thus shows the continuity between the state's determining the real-number values of magnitudes and the state's determining the truth values of propositions. And the other way makes use of the concept of a dispersion-free probability measure, which recurs in Chapters V, VI and VII.

As described in Section A, each state-induced mapping $w : F_{CM} \to R$ preserves + and $\cdot$, i.e., each state-induced mapping on an F_{CM} is real-valued and homomorphic. It follows that each state-induced mapping on the Boolean structure P_{CM} of idempotent elements of an F_{CM} is bivalent and homomorphic. For by the definition of the mapping w, for any $w \in \Omega$ and for any f_P on Ω, $w(f_P) = f_P(w) = 1$ if $w \in W_P$, and $w(f_P) = f_P(w) = 0$ if $w \in W_P^{\perp}$, where $W_P \cup W_P^{\perp} = \Omega$. Thus $w : P_{CM} \to \{0,1\}$, and in other words, each pure state of a classical system induces a bivalent, truth-functional mapping $w : P_{CM} \to \{0,1\}$ on the propositional structure P_{CM} of the phase space associated with the system.

In fact, as we would expect, each state-induced mapping on a $\mathcal{P}_{CM}$ is an ultravaluation which assigns the value 1 to an ultrafilter of elements in $\mathcal{P}_{CM}$ and assigns the value 0 to the dual ultraideal of elements in $\mathcal{P}_{CM}$, as shown next. Each point $w \in \Omega$ specifies a unique atom P_w in the $\mathcal{P}_{CM}$ structure of Ω, namely, the one-point idempotent function f_w or the corresponding singleton Borel subset $\{w\}$. And the set $\{P \in \mathcal{P}_{CM} : P \geq P_w\}$ is an ultrafilter in $\mathcal{P}_{CM}$, namely the unique ultrafilter UF_w defined by the atom P_w; dually, the set $\{P \in \mathcal{P}_{CM} : P \leq P_w^{\perp}\}$ is the unique ultraideal UI_w dual to UF_w. Now for any point $w \in \Omega$ and for any Borel subset $W_P \subset \Omega$, $w \in W_P$ IFF $P_w \leq P$, that is, $f_w \leq f_P$ or $\{w\} \subset W_P$, and also $w \in W_P^{\perp}$ IFF $P_w \leq P^{\perp}$, i.e., IFF $P \leq P_w^{\perp}$. So by substitution into the above definition of the mapping w, for any element $P \in \mathcal{P}_{CM}$, $w(P) = 1$ if $P \in UF_w$ and $w(P) = 0$ if $P \in UI_w$. Hence, each state-induced mapping on a $\mathcal{P}_{CM}$ is an ultravaluation and so is the classical-mechanical analogue of a standard valuation of classical-propositional logic.[5] And thus the 0, 1 values assigned by the state-induced ultravaluations to the elements of $\mathcal{P}_{CM}$ can be interpreted as the truth values false and true.

This notion of the state-induced ultravaluations on a $\mathcal{P}_{CM}$ structure may also be developed as follows.

According to the mathematical formalism of classical mechanics and classical statistical mechanics, each pure state w of a system can be regarded as inducing a dispersion-free probability measure on the $\mathcal{P}_{CM}$ structure of the system's phase space.[6] A <u>measure</u> μ is a real-valued function on a Boolean algebra, e.g., on $\mathcal{P}_{CM}$, which satisfies the following conditions:

(μa) For any countable set $\{P_i\}_{i \in \text{Index}}$ of disjoint elements of $\mathcal{P}_{CM}$ $\mu(\bigvee_i P_i) = \sum_i \mu(P_i)$. This is the additivity condition.

(μb) $0 \leq \mu(P) \leq \infty$, for every $P \in \mathcal{P}_{CM}$.

(μc) $\mu(0) = 0$.

It follows that μ is isotone, i.e.,

(μi) If $P_1 \leq P_2$, then $\mu(P_1) \leq \mu(P_2)$, for any $P_1, P_2 \in \mathcal{P}_{CM}$ (Sikorski, 1960, p. 10).

A probability measure is a normed measure satisfying:

(μn) $\mu(1) = 1$.

And hence, for every element $P \in \mathcal{P}_{CM}$, $0 \leq \mu(P) \leq 1$, that is, $\mu : \mathcal{P}_{CM} \to [0,1]$, where $[0,1]$ is the closed interval from 0 to 1 on the real-number line. And finally, a dispersion-free probability measure satisfies the condition:

(μd) $\mu(P^2) - (\mu(P))^2 = 0$, for every $P \in \mathcal{P}_{CM}$.

A dispersion-free probability measure on a $\mathcal{P}_{CM}$ is bivalent. Proof: Since every element $P \in \mathcal{P}_{CM}$ is idempotent, i.e., $P^2 = P$, condition (μd) yields the equation: $\mu(P) = (\mu(P))^2$, for every $P \in \mathcal{P}_{CM}$. Thus $\hat{\mu}(P) = 1$ or 0. Q.E.D. (Bub, 1974, p. 60). So each dispersion-free probability measure, hereafter labeled μ_w, is a bivalent mapping $\mu_w : \mathcal{P}_{CM} \to \{0,1\}$. Moreover, each dispersion-free probability measure on a $\mathcal{P}_{CM}$ is also a homomorphic mapping, as shown by the following proof due to Gudder (though Gudder does not refer to a Boolean structure like $\mathcal{P}_{CM}$).

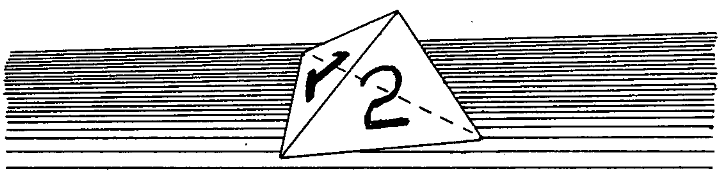
First, it is easy to show that, for any $P_1, P_2 \in \mathcal{P}_{CM}$, $\mu_w(P_1 \vee P_2) = \mu_w(P_1) + \mu_w(P_2) - \mu_w(P_1 \wedge P_2)$. Proof: The join $P_1 \vee P_2$ of any $P_1, P_2 \in \mathcal{P}_{CM}$ can be written as the join of three mutually disjoint elements, e.g., $P_1 \vee P_2 = P_3 \vee P_4 \vee P_5$, where $P_3 = P_1 \wedge P_2^{\perp}$; $P_4 = P_2 \wedge P_1^{\perp}$; and $P_5 = P_1 \wedge P_2$. Then by additivity, $\mu_W(P_1 \vee P_2) = \mu_W(P_3) + \mu_W(P_4) + \mu_W(P_5)$. And by substitution and additivity: $\mu_W(P_1) + \mu_W(P_2) = \mu_W(P_3 \vee P_5) + \mu_W(P_4 \vee P_5) = \mu_W(P_3) + \mu_W(P_5) + \mu_W(P_4) + \mu_W(P_5) = \mu_W(P_1 \vee P_2) + \mu_W(P_5) = \mu_W(P_1 \vee P_2) + \mu_W(P_1 \wedge P_2)$. Thus $\mu_W(P_1 \vee P_2) = \mu_W(P_1) + \mu_W(P_2) - \mu_W(P_1 \wedge P_2)$. Q.E.D. With this result, it is easy to prove that any dispersion-free probability measure $\mu_W : \mathcal{P}_{CM} \to \{0,1\}$ is homomorphic, i.e., for any P, $P_1, P_2 \in \mathcal{P}_{CM}$, $\mu_W(P^{\perp}) = (\mu_W(P))^{\perp}$ and $\mu_W(P_1 \vee P_2) = \mu_W(P_1) \vee \mu_W(P_2)$. Proof: For any $P \in \mathcal{P}_{CM}$, $\mu_W(P \vee P^{\perp}) = \mu_W(1) = 1$; and by additivity, $\mu_W(P \vee P^{\perp}) = \mu_W(P) + \mu_W(P^{\perp})$. Hence $1 = \mu_W(P) + \mu_W(P^{\perp})$, and so $\mu_W(P^{\perp}) = 1 - \mu_W(P) = (\mu_W(P))^{\perp}$. Now $\mu_W(P) = 0$ or 1, for every $P \in \mathcal{P}_{CM}$ so in the next part of this proof, there are two cases, one of which has two subcases. For Case 1, assume $\mu_W(P_1 \vee P_2) = 1$, and in addition, for Subcase 1a, assume $\mu_W(P_1 \wedge P_2) = 1$. Then since $P_1 \wedge P_2 \leq P_1$ and $P_1 \wedge P_2 \leq P_2$, by condition (μi) we have $\mu_W(P_1) = 1$ and also $\mu_W(P_2) = 1$. Hence $\mu_W(P_1 \vee P_2) = \mu_W(P_1) \vee \mu_W(P_2)$. For Subcase 1b, assume $\mu_W(P_1 \wedge P_2) = 0$. Then since $\mu_W(P_1 \vee P_2) = \mu_W(P_1) + \mu_W(P_2) - \mu_W(P_1 \wedge P_2)$, either $\mu_W(P_1) = 1$ and $\mu_W(P_2) = 0$ or else $\mu_W(P_1) = 0$ and $\mu_W(P_2) = 1$. Hence, $\mu_W(P_1 \vee P_2) = \mu_W(P_1) \vee \mu_W(P_2) = 1$. For case 2, assume $\mu_W(P_1 \vee P_2) = 0$. Then since $P_1 \leq P_1 \vee P_2$ and $P_2 \leq P_1 \vee P_2$, by condition (μi) we have $\mu_W(P_1) = \mu_W(P_2) = 0$. Hence $\mu_W(P_1 \vee P_2) = \mu_W(P_1) \vee \mu_W(P_2) = 0$. Q.E.D. (based on Gudder, 1970, pp. 433-434).

Thus each pure state of a classical system induces a dispersion-free probability measure $\mu_W : \mathcal{P}_{CM} \to \{0,1\}$ which is a bivalent homomorphism on the $\mathcal{P}_{CM}$ structure of the phase space associated with the system.

Moreover, each $\mu_w : P_{CM} \to \{0,1\}$ is an ultravaluation on P_{CM} and is in fact the ultravaluation $w : P_{CM} \to \{0,1\}$ described above, as shown next. A dispersion-free probability measure on a Boolean algebra of Borel subsets of a Ω is an atomic measure concentrated on a single point in Ω (Bub, 1974, p. 47), namely the point w representing the state which is said to induce the measure. That is, each μ_w on the P_{CM} structure of a Ω assigns probability 1 to the singleton subset $\{w\}$ (which is the atom P_w in P_{CM}) and assigns probability 0 to every other singleton subset of points in Ω. Now since $\mu_w(P_w) = 1$ and since μ_w preserves the $^\perp$ operation as shown above, it follows that $\mu_w(P_w^\perp) = 1^\perp = 0$. Then since μ_w is isotone, we have, for any $P \in P_{CM}$, if $P \geq P_w$ then $\mu_w(P) = 1$, and if $P \leq P_w^\perp$ then $\mu_w(P) = 0$. Thus each state-induced, dispersion-free probability measure on a P_{CM} assigns values as follows: for any $P \in P_{CM}$, $\mu_w(P) = 1$ if $P \in UF_w = \{P \in P_{CM} : P \geq P_w\}$ and $\mu_w(P) = 0$ if $P \in UI_w = \{P \in P_{CM} : P \leq P_w^\perp\}$. So each μ_w is an ultravaluation on P_{CM}. And clearly, each μ_w is the very mapping $w : P_{CM} \to \{0,1\}$ described above; conversely, each mapping $w : P_{CM} \to \{0,1\}$ is a dispersion-free probability measure on P_{CM}. Also, since each μ_w is an ultravaluation, the 0, 1 values assigned by μ_w to the elements of P_{CM} can be interpreted as the truth values false and true.

So the fact that a system's state determines the truth values of the propositions which make assertions about the real-number values of the system's magnitude is formalized via the notion of state-induced ultravaluations on the P_{CM} structure of the phase space associated with the system. And this state-induced procedure of assigning truth values to the elements of a propositional structure P_{CM} works exactly like the procedure by which truth values are assigned to the elements of an L structure determined by classical propositional logic.

The straightforward analogy between the state-induced ultravaluations on a P_{CM} and the ultravaluations on an L suggests, for example, that we may postulate a physical system with an associated phase space underlying the L_2 structure diagrammed in Chapter II(B) so that each ultravaluation on L_2 is induced by a state of the postulated system. Consider a tetrahedral die with the numbers 1, 2, 3, 4 marked on each side, respectively, and with the convention that we read the bottom face of the die as the outcome of a throw and thus as the <u>state</u> of the die.



The phase space associated with the die consists of four points $\Omega_0 = \{w_1, w_2, w_3, w_4\}$, each representing one of the four discrete states of the die. In order that L_2 be the propositional structure of this Ω_0, we may interpret the element $/R/ \in L_2$ as the proposition: "A number less than three appears (on the bottom face of the die)."[7] This proposition is associated with the idempotent function $f_R : \Omega_0 \to \{0,1\}$ defined as follows: for any $w_i \in \Omega_0$, $f_R(w_i) = 1$ if $w_i \in \{w_1, w_2\}$ and $f_R(w_i) = 0$ if $w_i \in \{w_1, w_2\}^{\perp} = \{w_3, w_4\}$. And we may interpret the element $/S/ \in L_2$ as the proposition: "An odd number appears." This proposition is associated with the idempotent function f_S defined as follows: for any $w_i \in \Omega_0$, $f_S(w_i) = 1$ if $w_i \in \{w_1, w_3\}$ and $f_S(w_i) = 0$ if $w_i \in \{w_2, w_4\}$. Each of the four ultravaluations on L_2 is state-induced because it is the state of the die which specifies an atom in L_2 which in turn specifies an ultrafilter and dual ultraideal defining an ultravaluation on L_2. Thus each state of the postulated system is the classical-mechanical analogue of the initial assignment of 0, 1 values to R and S which specifies an atom in L_2, as described in Chapter II(D).

Finally, by the semi-simplicity of the Boolean structure P_{CM}, the collection of state-induced ultravaluations on a P_{CM} is complete. Thus the complete collection of state-induced ultravaluations on a P_{CM} can be regarded as a state-induced, bivalent, truth-functional semantics for P_{CM}. This state-induced semantics for P_{CM} shall be regarded as the precedent for a proposed state-induced semantics for the quantum propositional structures, as developed in Chapter VI.

Notes

[1] I use the term "proposition" in a philosophically unsophisticated way; "sentence" or "statement" could serve as well.

[2] As suggested by R. E. Robinson, the units kg m^2/sec^2, which help make sense of the real-number values, may be considered to be part of the magnitude.

[3] The measurability condition on the functions representing classical magnitudes requires that, for any measurable (i.e., Borel) subset $R \subseteq \mathcal{R}$, the set W of all points $w \in \Omega$ such that $f_A(w) \in R$ is itself a Borel subset of Ω. (This set W is the inverse image of R under f_A.) The measurability restriction on the subsets $R \subseteq \mathcal{R}$ and $W \subseteq \Omega$ rules out sets such as the set of irrational numbers between 0 and 1, which is a non-denumerable infinity of disjoint points so that the measure of this set cannot be expressed as a countable union or sum of the measures of each of the set's elements. A singleton, one-point set is a Borel set of measure 0.

[4] Birkhoff and von Neumann actually specify a more restricted class of measurable subsets of Ω than the class of Borel subsets, see (Jauch, 1968, p. 79).

[5] Bub describes this connection between classical states, ultrafilters, and bivalent homomorphisms, see (Bub, 1974, pp. 97-106). However, Bub defines a bivalent homomorphism by the Sikorski definition, as discussed in Chapter II(C).

[6] The domain of a classical probability measure is usually specified to be a Boolean ring, field, or algebra of sets, in particular, the Boolean algebra of Borel subsets of classical phase space. However, M. Strauss, I. Segal, and others argue that the (isomorphic) Boolean algebra of idempotent random variables (i.e., idempotent, real-valued, measurable functions) is preferable as the domain of the measures of probability theory (Strauss, 1973, p. 268; Segal, 1954, p. 721). Similarly, Gleason proposes that we may regard his quantum measures as being defined on the set of idempotent operators on a Hilbert space rather than the set of subspaces of Hilbert space (Gleason, 1957, p. 885).

[7] It may seem initially more plausible to interpret the propositional variables R, S as propositions associated with idempotent functions on Ω_0. Thus any propositions P which makes assertions about what appears after a throw of the die is a molecular combination of R, S. Let L_2 label the closed, denumerable set of all molecular combinations of R, S. We have no equivalence relation with which to partition L_2 in order to get the equivalence classes which are the elements of $\mathcal{L}_2$. Strictly speaking, it is the elements of $\mathcal{L}_2$ which I want to interpret as propositions associated with idempotent functions on Ω_0. However, let every $P \in L_2$ be directly associated with an idempotent function f_P (where $f_P(w)$ is the truth value of P given w), and say that, for any $P, Q \in L_2$, $P \sim Q$ IFF $f_P(w) = f_Q(w)$ for every $w \in \Omega_0$, where if $f_P(w) = f_Q(w)$ for every $w \in \Omega_0$, then $f_P = f_Q$. Thus all the members of the equivalence class $/P/$ are associated with a single idempotent function f_P, as we want. And in other words, $P \sim Q$ IFF P, Q have the same truth table, which is the semantic counterpart of the proof-theoretic equivalence relation stated in Chapter II(B).

CHAPTER IV

THE NON-BOOLEAN PROPOSITIONAL STRUCTURES DETERMINED BY QUANTUM MECHANICS

Section A. The Fundamental Postulates of Quantum Mechanics

What follows is an extremely simplified exposition of some of the mathematical formalism of quantum mechanics. It is postulated that a physical system is associated with a Hilbert space $\mathcal{H}$ whose dimensionality reflects the degrees of freedom of the system. Each magnitude A of the system is represented by a self-adjoint operator A on the system's $\mathcal{H}$. The operator A has a spectral representation (for the case of a discrete spectrum):

$$\text{(I)} \qquad \hat{A} = \sum_i a_i \hat{P}_{\psi_i}, \text{ where for each } i \in \text{Index}, \ \hat{P}_{\psi_i} = |\psi_i\rangle\langle\psi_i| \text{ and } \hat{A}|\psi_i\rangle = a_i|\psi_i\rangle.$$

The real numbers $\{a_i\}_{i \in \text{Index}}$ are called the eigenvalues of $\hat{A}$ and of A. They are the real-number values and the only real-number values exhibited by the magnitude A.[1] A pure state ψ of a quantum system is represented by a vector $|\psi\rangle$ in the system's $\mathcal{H}$ or by a density operator $\hat{P}_\psi = |\psi\rangle\langle\psi|$ on $\mathcal{H}$. The operator $\hat{P}_\psi$ is self-adjoint and idempotent, that is, $\hat{P}_\psi$ is a projection operator which is also called a projector, more generally designated $\hat{P}$, $\hat{P}_1$, $\hat{P}_2$, etc. Each projector $\hat{P}$ on an $\mathcal{H}$ corresponds uniquely to a subspace H of $\mathcal{H}$, where a subspace is a set of vectors which form a closed linear manifold (see Bub, 1974, pp. 10, 12). The projectors $\{\hat{P}_{\psi_i}\}_{i \in \text{Index}}$ and the vectors $\{|\psi_i\rangle\}_{i \in \text{Index}}$ appearing in the spectral representation of any operator $\hat{A}$ represent the (pure) eigenstates of $\hat{A}$ and of A. The set of eigenstates of any A are mutually orthogonal (as defined in Section C) and satisfy $\bigvee_i |\psi_i\rangle = \mathcal{H}$ and $\sum_i \hat{P}_{\psi_i} = \hat{I}$. ($\hat{I}$ is the identity operator which satisfies $\hat{I}|\psi\rangle = 1|\psi\rangle$ for every $|\psi\rangle \in \mathcal{H}$.)

The state of a quantum system determines the real-number values of the system's magnitudes via the following formalism. When a system is in an eigenstate $|\psi_j>$, for some $j \in$ Index, of the magnitude A, then the real-number value of A is the eigenvalue a_j affiliated with that eigenstate $|\psi_j>$ by the equation $\hat{A}|\psi_j> = a_j|\psi_j>$. But when a system is in an arbitrary pure state ψ which is not an eigenstate of A, then upon measurement the magnitude A may exhibit any of its real-number eigenvalues; the quantum formalism does not specify which eigenvalue A will exhibit. However, for any pure state ψ, the probability that the real-number value of A is the eigenvalue a_j, for some $j \in$ Index, is determined by the quantum formalism:

(II) $$\rho_{\psi,A}(a_j) = |<\psi_j|\psi>|^2 = <\psi|\psi_j><\psi_j|\psi> = <\psi|\hat{P}_{\psi_j}|\psi> .$$

This probability is a real-number in the closed interval [0,1] of the real-number line. The probability equals 1 (certainty) IFF the system is in the eigenstate ψ_j affiliated with the eigenvalue a_j, i.e., $\rho_{\psi_j,A}(a_j) = |<\psi_j|\psi_j>|^2 = 1$. And this probability equals 0 (impossibility) IFF the system is in any one of the other eigenstates of A.

The average value, i.e., the expectation value, of A when the system is in an arbitrary pure state ψ is defined as the following weighted sum of eigenvalues of A:

(III) $$\begin{aligned} \mathrm{Exp}_\psi(A) &= \sum_i a_i \rho_{\psi_i,A}(a_i) = \sum_i (a_i)<\psi|\psi_i><\psi_i|\psi> \\ &= <\psi\Big|\sum_i (a_i)\,|\psi_i><\psi_i|\Big|\psi> \\ &= <\psi|\hat{A}|\psi>. \end{aligned}$$

And when the system is in an eigenstate ψ_j of A, then the expectation value of A is the eigenvalue a_j .

Clearly, the expression for the probability (II) is equal to the expectation value of a projector according to (III), i.e., for any pure state ψ and for any magnitude A, $\rho_{\psi,A}(a_j) = \mathrm{Exp}_{\psi}(\hat{P}_{\psi_j})$, where $\hat{P}_{\psi_j}$ is the projector representing the eigenstate affiliated with the eigenvalue a_j . In fact, instead of moving from the probability expression (II) to the expectation value expression (III), the former expression (II) can be derived from (III), as is done, for example, by Messiah (1966, pp. 176-179). In other words, (I) together with either (II) or (III) are regarded as the foundational postulates of quantum mechanics. For example, von Neumann considers (II) to be the more general probability expression but he regards (III) to be preferable as a fundamental postulate (von Neumann, 1932, pp. pp. 200-206).

Section B. Incompatibility

In both classical and quantum mechanics, a sufficient condition for the simultaneous measurability of any set of magnitudes $\{A_i\}_{i\in \mathrm{Index}}$ is that each magnitude is equal to a (Borel) function of some common magnitude, say B; that is, for each $i \in$ Index, $A_i = g_i(B)$ for some Borel function g_i (Kochen-Specker, 1967, p. 64). Now for any magnitude B and any Borel function g, the magnitude g(B) is by definition that magnitude which exhibits the value g(b) when B exhibits the value b. So when the real-number value of the common magnitude B is b, then the real-number value of each $A_i = g_i(B)$ is $g_i(b)$. Hence a single measurement of B suffices to determine the real-number values of all the magnitudes

$\{A_i\}_{i \in Index}$. For example, as mentioned in Chapter III(A), every classical magnitude is a (Borel) function of the position and/or momentum magnitudes, and so all classical magnitudes are simultaneously measurable. But it is not the case that every quantum magnitude is a function of the position and/or momentum magnitudes. Moreover, the quantum position and momentum magnitudes are themselves not simultaneously measurable. And in general, the set of magnitudes describing a quantum system includes magnitudes which are not simultaneously measurable.

With respect to the (self-adjoint) operator representation of the quantum magnitudes, a necessary and sufficient condition for the simultaneous measurability of any magnitudes is the commutativity of their representative operators. Any operators $\hat{A}$, $\hat{B}$ commute IFF $\hat{A}$, $\hat{B}$ have all their eigenstates in common. Moreover, any set $\{\hat{A}_i\}_{i \in Index}$ of operators is mutually commutative IFF there is an operator $\hat{B}$ and Borel functions $\{g_i\}_{i \in Index}$ such that $\hat{A}_i = \widehat{g_i(B)} = g_i(\hat{B})$, for every $i \in$ Index (von Neumann, 1932, p. 173). Now for any magnitude B and for any Borel function g, if B has the operator $\hat{B}$, then g(B) has the operator $\widehat{g(B)} = g(\hat{B})$. (von Neumann, 1932, p. 204; Fano, 1971, p. 394). Thus it follows that any quantum magnitudes are simultaneously measurable IF their representative operators are mutually commutative; the converse is also shown by von Neumann (1932, pp. 223-228).

Commuting operators and simultaneously measurable magnitudes are said to be compatible; such operators or magnitudes have all their eigenstates in common. Operators which do not commute and magnitudes which are not simultaneously measurable are said to be incompatible; such operators or magnitudes may nevertheless have one or several eigenstates in common so that one or several of their eigenvalues may be simultaneously determined by measurement.

When we talk of a propositional structure determined by quantum mechanics, the propositions we consider are propositions which make assertions about the real-number eigenvalues of quantum magnitudes. Propositions which make assertions about the eigenvalues of compatible magnitudes are said to be compatible. Propositions which make assertions about the eigenvalues of incompatible magnitudes are said to be incompatible with the following exception. If the eigenvalues happen to be associated with eigenstates which are shared in common by the incompatible magnitudes, then propositions which make assertions about such eigenvalues of incompatible magnitudes are said to be compatible.[2] The attempt to assign truth-values to incompatible quantum propositions is a problematic enterprise, as will be shown in Chapter V(A).

Section C. The Propositional Structure Determined by Quantum Mechanics

As in the classical case described in Chapter III, the self-adjoint operators representing quantum magnitudes have the binary ring operations $+$ and $\cdot$ defined among them as follows: for any $\hat{A}$, $\hat{B}$ on H and for every $|\psi\rangle \in H$, $(\hat{A}+\hat{B})|\psi\rangle = \hat{A}|\psi\rangle + \hat{B}|\psi\rangle$, and $(\hat{A}\cdot\hat{B})|\psi\rangle = \hat{A}(\hat{B}|\psi\rangle)$. The $+$ operation so defined is associative and commutative, as usual. And the $\cdot$ operation so defined is associative and distributive with respect to $+$, as usual. But $\cdot$ is not commutative, i.e., $\hat{A}(\hat{B}|\psi\rangle)$ need not equal $\hat{B}(\hat{A}|\psi\rangle)$ for every $\hat{A}, \hat{B}, |\psi\rangle$. In particular, if $\hat{A}(\hat{B}|\psi\rangle) = \hat{B}(\hat{A}|\psi\rangle)$ for every $|\psi\rangle \in H$, then $\hat{A}$, $\hat{B}$ are said to commute or to be compatible. This suggests that a closed set of self-adjoint operators on a Hilbert space has the structure of a non-commutative ring-with-unit whose 0-element is the constant operator $\hat{0}$ satisfying $\hat{0}|\psi\rangle = 0$, for every $|\psi\rangle \in H$, and whose 1-element is the constant operator $\hat{I}$ satisfying $\hat{I}|\psi\rangle = 1$, for every $|\psi\rangle \in H$. However, a set of self-adjoint operators is not closed with respect to $\cdot$ unless $\cdot$ is restricted to commuting, i.e., compatible, operators. For although the sum of any two self-adjoint operators is itself a self-adjoint operator, the product of two self-adjoint operators is not itself a self-adjoint operator unless the two commute (von Neumann, 1932, p. 98). So rather than a non-commutative ring-with-unit, a set of self-adjoint operators representing quantum magnitudes which is closed with respect to $+$ and $\cdot$ form a structure which may be called a <u>partial-dot-ring-with-unit</u> $\langle E = \{\hat{A},\hat{B}, \ldots\},+,\eth,\cdot,0,1\rangle$, where $\eth \subseteq E \times E$, $+$ is defined from $E \times E$ to E, and $\cdot$ is defined from only $\eth$ to E.

Taking this notion of restricting the binary $\cdot$ operation to a partial-operation defined from only $\eth$ to E one step further, we may define the structure of the self-adjoint operators to be a partial-ring-with-unit which has both the $+$ and the $\cdot$ operations defined from only $\eth$ to E, where again, $\eth \subseteq E \times E$. As mentioned in Chapter I(D), Kochen-Specker call such a structure a partial algebra. And they define the structure of a quantum system's magnitudes, which are represented by and presumably reflect the structure of self-adjoint operators on the system's Hilbert space, as a partial-algebra; i.e., as a partial-ring-with-unit in my terminology.

But regardless of the exact structuring of the self-adjoint operators, it is clear that the structure of the operators representing the magnitudes of quantum mechanics is different from the structure of the real-valued functions representing the magnitudes of classical mechanics.

Thus it is reasonable to expect that the structure of the quantum propositions which make assertions about the real-number eigenvalues of the quantum magnitudes is different from the Boolean P_{CM} structure of classical propositions.

Nevertheless, the procedure by which a quantum propositional structure is abstracted from the mathematical formalism of quantum mechanics is exactly analogous to the procedure by which a P_{CM} structure is abstracted from the classical formalism, as described in Chapter III(B). For quantum propositions have historically been associated either with the projectors (i.e., idempotent, self-adjoint operators) on an H or with the uniquely corresponding subspaces of H. And the logical operations "and," "or," "not," either have been indirectly defined in terms of the projector $+$ and $\cdot$ operations or have been directly associated with the subspace intersect $\wedge$, span $\vee$, and orthocomplementation $^{\perp}$ operations, as will be described shortly. These associations determine the structure of a set of quantum propositions, or in von Neumann's terms, these associations determine "a sort of logical calculus" or a "propositional calculus" for quantum mechanics (von Neumann, 1932, p. 253).

In his 1932 book, von Neumann discusses classical and quantum propositions under the categorical label: properties of the state of the system. That is, von Neumann's properties are in fact propositions which make assertions about the real-number (eigen)values of a system's magnitudes (von Neumann, 1932, p. 249). For example: The spin_x of an electron is $+\frac{1}{2}\hbar$. Von Neumann argues that each such proposition can be associated with a magnitude which is defined such that its value is 1 if the proposition is verified and 0 if the proposition is not verified. In other words, each proposition which makes assertions about the real-number eigenvalues of a quantum system's magnitudes can itself be regarded as or associated with an idempotent magnitude of the system. Since an idempotent magnitude is represented by a projector on the system's Hilbert space and each projector in turn corresponds uniquely to a subspace of that Hilbert space, namely, to the subspace onto which the projector projects every vector in Hilbert space, von Neumann concludes that quantum propositions can be associated either with projectors on a Hilbert space or equally well with subspaces of a Hilbert space.

For example, consider a proposition which asserts that the value of the magnitude A is in some Borel subset R of the real-number line. Such propositions are regarded by most authors as the paradigm quantum (or classical) propositions. As described above, the only values exhibited by A are its eigenvalues, and each eigenvalue a_i is uniquely associated with a projector $\hat{P}_{\psi_i} = |\psi_i\rangle\langle\psi_i|$. So depending upon how many eigenvalues of A are in the Borel subset R, the above paradigm proposition specifies either the unique projector $\hat{P}_{\psi_i}$ and its corresponding subspace H_{ψ_i} when only one $a_i \in R$, or the unique projector $\sum_{i=1}^{n} \hat{P}_{\psi_i}$ and its corresponding subspace $\bigvee_{i=1}^{n} H_{\psi_i}$ when several $a_1, \ldots, a_n \in R$.

All other authors who discuss a quantum propositional structure or a quantum logical calculus also somehow or other associate quantum propositions with either the projectors on a Hilbert space or the subspaces of a Hilbert space. So the structure of the projectors on a Hilbert space, or isomorphically, the structure of the subspaces of that Hilbert space, is regarded as the propositional structure determined by quantum mechanics, labeled $\mathcal{P}_{QM} = \langle E = \{P, P_1, P_2, \ldots\}, \mathrel{\text{⅄}}, \leq, \wedge, \vee, {}^{\perp}, 0, 1\rangle$. The elements of $\mathcal{P}_{QM}$ may be thought of either as projectors or as subspaces of a Hilbert space;

the elements of $\mathcal{P}_{QM}$ represent or are associated with propositions $p, p_1, p_2, \ldots$. The $\mathcal{P}_{QM}$ structures have been formalized in two different ways. But before describing these two ways in the next section, the features of $\mathcal{P}_{QM}$ which are common to both formalizations are first described, as follows.

A $\mathcal{P}_{QM}$ is an atomic structure whose atoms, written P_ψ or sometimes P_a , are the one-dimensional projectors on $\mathcal{H}$, e.g., $\hat{P}_\psi = |\psi\rangle\langle\psi|$, or the corresponding one-dimensional subspaces of $\mathcal{H}$, e.g., the subspace H_ψ which is the range of $\hat{P}_\psi$. The distinguished 0-element of a $\mathcal{P}_{QM}$ is the null projector $\hat{0}$ or the corresponding zero-subspace of $\mathcal{H}$; the distinguished 1-element is the identity projector $\hat{I}$ or the corresponding entire $\mathcal{H}$. As with the $\mathcal{L}$ and the $\mathcal{P}_{CM}$ structures described in Chapters II and III, the 0-element of a $\mathcal{P}_{QM}$ is associated with impossible or contradictory quantum propositions, and the 1-element is associated with certain or tautological quantum propositions.

The <u>compatibility</u> relation $\mathrel{\text{ᐝ}}$ of $\mathcal{P}_{QM}$ is reflexive, symmetric, and non-transitive, and is defined in terms of the $\wedge$, $\vee$, $^\perp$ operations as follows. For any $P_1, P_2 \in \mathcal{P}_{QM}$ $P_1 \mathrel{\text{ᐝ}} P_2$ IFF there exist three mutually disjoint (i.e., orthogonal) elements P_{11}, P_{22}, P_3 such that $P_1 = P_{11} \vee P_3$ and $P_2 = P_{22} \vee P_3$. And assuming that $P_1 \mathrel{\text{ᐝ}} P_2$, it can be shown that $P_{11} = P_1 \wedge P_2^\perp$, $P_{22} = P_2 \wedge P_1^\perp$, $P_3 = P_1 \wedge P_2$ (Jauch, 1968, pp. 28, 97; Kochen-Specker, 1967, p. 65). Any $P_1, P_2 \in \mathcal{P}_{QM}$ are <u>disjoint</u> or <u>orthogonal</u> written $P_1 \perp P_2$ IFF $P_1 \leq P_2^\perp$ (Piron, 1976, p. 29). It follows that, for any $P_1, P_2 \in \mathcal{P}_{QM}$, if P_1, P_2 are disjoint then $P_1 \mathrel{\text{ᐝ}} P_2$; and $P_1 \mathrel{\text{ᐝ}} P_2$ IFF $P_1 \mathrel{\text{ᐝ}} P_2^\perp$.

The binary relation $\leq$ of $\mathcal{P}_{QM}$, defined in terms of $\wedge$ or $\vee$ as usual (i.e., $P_1 \leq P_2$ IFF $PP_1 \wedge P_2 = P_1$ and $P_1 \leq P_2$ IFF $P_1 \vee P_2 = P_2$), is a partial-ordering (i.e., it is reflexive, anti-symmetric, and transitive). Moreover, the compatibility of any $P_1, P_2 \in \mathcal{P}_{QM}$ is a necessary condition for their being related to the partial-ordering $\leq$, that is, if $P_1 \leq P_2$ then $P_1 \smile P_2$, for any $P_1, P_2 \in \mathcal{P}_{QM}$.[3]

And finally, the operations $\wedge, \vee, ^{\perp}$ of $\mathcal{P}_{QM}$ are defined and discussed in the next section.

Section D. The Partial-Boolean Algebra and the Orthomodular Lattice Quantum Propositional Structures

The $\mathcal{P}_{QM}$ structure has been formalized in two ways: as a transitive, atomic, partial-Boolean algebra $\mathcal{P}_{QMA}$ and as a complete, atomic, orthomodular lattice $\mathcal{P}_{QML}$. These structures are defined in Chapter I(D) and (E). I retain the label $\mathcal{P}_{QM}$ to refer to a $\mathcal{P}_{QMA}$ or a $\mathcal{P}_{QML}$ indiscriminately. The basic difference between a $\mathcal{P}_{QMA}$ and a $\mathcal{P}_{QML}$ is that the former has the binary operations $\wedge, \vee$ defined among only compatible elements while the latter has $\wedge, \vee$ defined among all elements, compatible and incompatible. The two formalizations do not differ with respect to any of the other entries in the ordered octuple $\mathcal{P}_{QM}$.

That the quantum propositional structures have been formalized in these two ways is at least partly due to differences between the projectors and the subspaces of $\mathcal{H}$. For despite the one-to-one correspondence between the projectors and the subspaces, the association of quantum propositions with projectors naturally yields a $\mathcal{P}_{QMA}$ while the association of quantum propositions with subspaces suggests a $\mathcal{P}_{QML}$, as will be shown in this section.

In his 1932 book, von Neumann proposes a logical calculus of quantum propositions which has "and" and "or" restricted to compatible propositions.[4] First von Neumann defines "not." For any quantum proposition p associated with the projector $\hat{P}$ whose corresponding subspace is H, the proposition "not p" is associated with the projector $\hat{I} - \hat{P} = \hat{P}^{\perp}$ whose corresponding subspace is $H^{\perp}$. Next, for any compatible propositions p_1, p_2, the proposition "p_1 and p_2" is associated with the projector $\hat{P}_1 \cdot \hat{P}_2$ whose corresponding subspace is $H_1 \wedge H_2$, where $\wedge$ is interpreted among subspaces as the set-theoretic intersect operation. Classically "p_1 or p_2" is equivalent to "not ((not p_1) and (not p_2))"; analogously, von Neumann associates "p_1 or p_2," for any compatible p_1, p_2, with the projector $\hat{I} - ((\hat{I}-\hat{P}_1) \cdot (\hat{I}-\hat{P}_2)) = \hat{P}_1 + \hat{P}_2 - (\hat{P}_1 \cdot \hat{P}_2)$ whose corresponding subspace is the closed linear sum of H_1, H_2, i.e., $H_1 \vee H_2$, where $\vee$ is interpreted as the subspace span operation. Thus von Neumann's 1932 logical calculus has the "and," "or," "not" operations among propositions defined in terms of the $+$, $\cdot$ operations among projectors in the usual way that the Boolean operations $\wedge$, $\vee$, $^{\perp}$, are defined in terms of the ring operations $+$, $\cdot$. But the binary "and," "or" operations are defined among only compatible propositions. A similar calculus of quantum propositions is developed and discussed by Strauss under the appellation "complementary logic" (Strauss, 1936, p. 196) and later by Kochen-Specker under the appellation "partial-Boolean algebra."

A lattice structure of calculus of quantum propositions was first proposed by Birkhoff and von Neumann in their celebrated 1936 paper. There, in a discussion of their initial association of experimental propositions with the subsets of a phase space, Birkhoff and von Neumann are especially concerned to preserve the relation of logical implication among the propositions. Logical implication is reflexive, anti-symmetric, and transitive, and so can be regarded as a partial-ordering. So Birkhoff and von Neumann postulate that a propositional calculus, determined by either classical mechanics or quantum mechanics, is a partially ordered set. They then assume that a propositional calculus has a distinguished 0-element, interpreted as the "identically false" or "absurd" proposition, and a distinguished 1-element, interpreted as the "identically true" or "self-evident" proposition. Next Birkhoff and von Neumann claim that: "In any calculus of propositions, it is natural to imagine that there is a weakest proposition implying, and a strongest proposition implied by, a given pair of propositions" (Birkhoff and von Neumann, 1936, pp. 828-829). In other words, with respect to the partial-ordering of logical implication, Birkhoff and von Neumann assume that any given pair of propositions p_1, p_2, in a propositional structure has a g.l.b. (the meet $p_1 \wedge p_2$) and a l.u.b. (the join $p_1 \vee p_2$), which they interpret as logical conjunction and disjunction, respectively. Hence, Birkhoff and von Neumann postulate that a propositional structure is a lattice which has $\wedge$, $\vee$ defined for every pair of propositions.

But Birkhoff and von Neumann immediately mention the problematic character of the meets and joins of incompatible propositions. They say that the meet or the join of incompatible experimental propositions cannot itself be defined as an experimental proposition but rather must be expressed as a class of logically equivalent experimental propositions which they call a physical quality. Nevertheless, Birkhoff and von Neumann go on to associate quantum propositions with the subspaces of a Hilbert space, and they associate "not," "and," "or," among compatible and incompatible propositions _qua_ subspaces with the subspace $\perp$, $\wedge$, $\vee$, as defined by von Neumann in 1932.

It is noteworthy that the orthocomplement $H^{\perp}$ of any subspace H of a Hilbert space is itself a subspace, and likewise the set-theoretic intersect $H_1 \wedge H_2$ and the closed linear sum $H_1 \vee H_2$ of any pair of subspaces H_1 , H_2 of a Hilbert space are themselves subspaces. So it is clear that the meets and joins of incompatible propositions _qua_ subspaces are at least sure to exist, whether as experimental propositions or as "physical qualities."

Birkhoff and von Neumann conclude that the orthocomplemented, modular, non-distributive lattice of subspaces of a Hilbert space may be regarded as the logical structure or propositional calculus of quantum mechanics. Later, Jauch shows that the subspaces of an infinite dimensional Hilbert space are not modular, and so Jauch weakens the modularity condition on the quantum lattice of subspaces to weak modularity (see Chapter I(E)). Consequently, authors who favour the lattice formalization of quantum propositions initiated by Birkhoff and von Neumann consider the propositional structure or calculus of quantum mechanics to be a complete, atomic, orthomodular (i.e., orthocomplemented and weakly modular) lattice.

However, when quantum propositions are associated with the projectors on a Hilbert space rather than the subspaces, then the existence of the meets and joins of incompatible propositions _qua_ projectors is more problematic. As mentioned in Section (B), the operators and projectors on a Hilbert space have $+$ and $\cdot$ interpreted as addition and multiplication defined among them. But a theorem states that the product of any two projectors is itself a projector IFF the two are compatible; the sum of any two projectors is itself a projector IFF the two are orthogonal (von Neumann, 1932, p. 81). In addition, any $\hat{P}$ is a projector IFF $\hat{I}-\hat{P}$ is a projector (von Neumann, 1932, p. 79). So a set of projectors is closed with respect to $+$ and $\cdot$ only if the $+$ operation is restricted to orthogonal projectors and the $\cdot$ operation is restricted to compatible projectors resulting in a sort of partial-Boolean ring-with-unit $\langle E = \{\hat{P}_1, \hat{P}_2, \ldots\}, \perp, \mathrel{\circ\!\!\mid}, +, \cdot, 0, 1\rangle$, where $\perp \subseteq \mathrel{\circ\!\!\mid} \subseteq E \times E$, $+$ is defined from $\perp$ to E, and $\cdot$ is defined from $\mathrel{\circ\!\!\mid}$ to E. I write $\perp \subseteq \mathrel{\circ\!\!\mid}$ because, for any $\hat{P}_1$, $\hat{P}_2$, if $\hat{P}_1 \perp \hat{P}_2$ then $\hat{P}_1 \mathrel{\circ\!\!\mid} \hat{P}_2$, but not the converse.

Now although $\hat{P}_1 + \hat{P}_2$ is a projector IFF $\hat{P}_1 \perp \hat{P}_2$, it is easy to show that the sum less the product: $\hat{P}_1 + \hat{P}_2 - \hat{P}_1\hat{P}_2$, of any $\hat{P}_1$, $\hat{P}_2$, is a projector IFF $\hat{P}_1 \mathrel{\circ\!\!\mid} \hat{P}_2$. For $\hat{P}_1 + \hat{P}_2 - \hat{P}_1\hat{P}_2 = \hat{I} - \hat{I} + \hat{P}_1 + \hat{P}_2 - \hat{P}_1\hat{P}_2$ $= \hat{I} - ((\hat{I}-\hat{P}_1) - (\hat{P}_2-\hat{P}_1\hat{P}_2)) = \hat{I} - ((\hat{I}-\hat{P}_1)\hat{I} - (\hat{I}-\hat{P}_1)\hat{P}_2)) = \hat{I} - ((\hat{I}-\hat{P}_1)\cdot(\hat{I}-\hat{P}_2))$. And by the theorem and additional result stated in the previous paragraph, for any $\hat{P}_1$, $\hat{P}_2$, $(\hat{I}-\hat{P}_1)$ and $(\hat{I}-\hat{P}_2)$ are each projectors; and $\hat{I} - ((\hat{I}-\hat{P}_1)\cdot(\hat{I}-\hat{P}_2))$ is a projector IFF $((\hat{I}-\hat{P}_1)\cdot(\hat{I}-\hat{P}_2))$ is a projector; the latter is a projector IFF $(\hat{I}-\hat{P}_1) \mathrel{\circ\!\!\mid} (\hat{I}-\hat{P}_2)$, which is the case IFF $\hat{P}_1 \mathrel{\circ\!\!\mid} \hat{P}_2$. Q.E.D. So when the $\wedge$, $\vee$, $^{\perp}$ operations are defined among projectors in terms of $+$ and $\cdot$ as usual, then a set of projectors is closed with respect to $\wedge$, $\vee$, $^{\perp}$ only if $\wedge$ and $\vee$ are restricted to compatible projectors resulting in a partial-Boolean algebra of quantum propositions <u>qua</u> projectors.

The subspace representation of quantum propositions easily lends itself to a partial-Boolean algebra structuring, as first suggested by von Neumann in 1932. Merely restrict the above defined $\wedge$, $\vee$ operations among subspaces to compatible subspaces (e.g., see Kochen-Specker, 1967,

p. 65). On the other hand, the projector representation of quantum propositions may be structured as an orthomodular lattice, but the $\wedge$, $\vee$ operations can be defined in terms of projector addition $+$ and multiplication $\cdot$ in the usual way among only compatibles. Among incompatible propositions qua projectors, the $\wedge$, $\vee$ operations are defined by Jauch as follows: $P_1 \wedge P_2 = \lim_{n\to\infty}(P_1 \cdot P_2)^n$ and $P_1 \vee P_2 = (P_1^\perp \wedge P_2^\perp)^\perp$ $= I - \lim_{n\to\infty}((I-P_1)\cdot(I-P_2))^n$ (Jauch, 1968, pp. 38, 219). These definitions of $\wedge$, $\vee$ reduce to the usual definitions of $\wedge$, $\vee$ in terms of $+$, $\cdot$ when $P_1 \overset{\mathsf{c}}{\leftrightarrow} P_2$. So an orthomodular lattice $\mathcal{P}_{QML}$ of quantum propositions qua projectors is also defined.

Thus regardless of whether quantum propositions are associated with the projectors or the subspaces of a Hilbert space, both alternatives have been structured as a $\mathcal{P}_{QMA}$ and both have been structured as a $\mathcal{P}_{QML}$. I have described how the alternatives have been formalized as a $\mathcal{P}_{QMA}$ and as a $\mathcal{P}_{QML}$ in order to highlight the problematic character of the meets and joins of incompatibles defined in $\mathcal{P}_{QML}$. In summary, when quantum propositions are associated with the subspaces of a Hilbert space, then the meets and joins of incompatibles are at least sure to exist and the propositions qua subspaces can be structured as a $\mathcal{P}_{QML}$. However, Birkhoff and von Neumann, for example, do not regard the meets and joins of incompatible propositions as propositions but rather as "physical qualities." When quantum propositions are associated with the projectors on a Hilbert space and the $\wedge$, $\vee$, $^\perp$ operations are defined in terms of projector addition $+$ and multiplication $\cdot$ as usual, then the resulting structure is a $\mathcal{P}_{QMA}$ rather than a $\mathcal{P}_{QML}$. In order to define a $\mathcal{P}_{QML}$ of quantum propositions qua projectors, Jauch must introduce definitions of $\wedge$ and $\vee$ which involve the limits of infinite products.

Section E. Ramifications of the Basic Difference between $\mathcal{P}_{QMA}$ and $\mathcal{P}_{QML}$

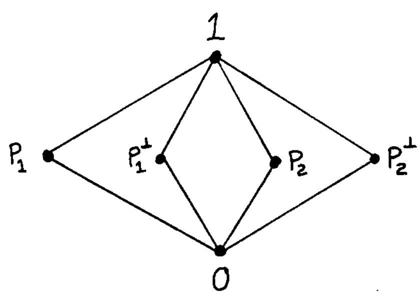
The fact that a $\mathcal{P}_{QML}$ has $\wedge, \vee$ defined among incompatible elements while a $\mathcal{P}_{QMA}$ does not have $\wedge, \vee$ defined among incompatible elements may suggest that a $\mathcal{P}_{QMA}$ is in some sense missing elements compared to a $\mathcal{P}_{QML}$. For example, given an initial set of one-dimensional subspaces of a Hilbert space, both a $\mathcal{P}_{QMA}$ and a $\mathcal{P}_{QML}$ can be generated by closing the initial set with respect to the $\wedge, \vee, ^{\perp}$ operations as defined in each structure. When the initial set is finite, the $\mathcal{P}_{QMA}$ generated by closing the initial set is also finite. In contrast, the lattice definitions of $\wedge, \vee$ among incompatibles often results in a proliferation of lattice elements so that the $\mathcal{P}_{QML}$ generated by closing a finite initial set may be denumerably infinite. An example of this proliferation of elements is given in Chapter VI(C). This proliferation of lattice elements does not occur in the $\mathcal{P}_{QML}$ structures of subspaces of two-dimensional Hilbert space. And it does not occur in higher dimensional Hilbert space structures when there are certain angular relations among the subspaces in the initial set. An example is given in note 8 below. In these cases when the proliferation of lattice elements does not occur, both the $\mathcal{P}_{QML}$ and the $\mathcal{P}_{QMA}$ generated by closing an initial set have exactly the same elements.

And in any case, it is not correct to consider a $\mathcal{P}_{QMA}$ to be missing elements compared with a $\mathcal{P}_{QML}$. For given any finite or infinite $\mathcal{P}_{QML}$, there is a corresponding finite or infinite $\mathcal{P}_{QMA}$ which has exactly the same elements as $\mathcal{P}_{QML}$ but is missing some of the lattice relations among these same elements. Specifically, an element $P \in \mathcal{P}_{QML}$ may be the meet or join of two incompatible elements in $\mathcal{P}_{QML}$, e.g., $P = P_1 \wedge P_2$, with $P_1 \not\leftrightarrow P_2$, but the same three elements P, P_1, P_2 , in the $\mathcal{P}_{QMA}$ which corresponds to that $\mathcal{P}_{QML}$ will not be so related because $\wedge$ and $\vee$ are not defined among incompatibles in a $\mathcal{P}_{QMA}$.

Strauss makes a similar point when he argues that the lattice interpretation of an element P as the meet of two incompatible elements $P_1 \wedge P_2$ is a misinterpretation because $P \neq P_1 \cdot P_2$. In other words, the $\wedge$ operation cannot be defined in terms of the $\cdot$ operation as usual when $P_1 \not\leftrightarrow P_2$. Strauss concludes that, compared with a (orthomodular) lattice, a partial-Boolean algebra does not omit any elements but rather prevents the misinterpretation of elements (Strauss, 1936, p. 203). Of course, authors who favour the lattice structure can argue that the interpretation of an element $P \in \mathcal{P}_{QML}$ as the meet of two incompatible elements $P_1 \wedge P_2$ is not a misinterpretation, in spite of the fact that $P \neq P_1 \cdot P_2$, since Jauch has created the infinite-product definition of the meet of two incompatible elements in $\mathcal{P}_{QML}$.

Regardless of whether or not the lattice definitions of $\wedge$ and $\vee$ among incompatibles results in misinterpretations, the lattice meets and joins of incompatibles do cause truth-functionality problems which are peculiar to the $\mathcal{P}_{QML}$ structures but are avoided in the $\mathcal{P}_{QMA}$ structures. For a truth-functional mapping on a $\mathcal{P}_{QMA}$ must preserve the unary $^\perp$ operation and the binary $\wedge, \vee$ operations among only compatibles; while a truth-functional mapping on a $\mathcal{P}_{QML}$ must preserve the unary $^\perp$ operation and the binary $\wedge, \vee$ operations among compatibles and incompatibles. Hereafter, let truth-functional $(\leftrightarrow)$ refer to the former condition and let truth-functional $(\leftrightarrow, \not\leftrightarrow)$ refer to the latter condition. The latter condition is not applicable to a mapping on a $\mathcal{P}_{QMA}$ since a $\mathcal{P}_{QMA}$ has no operations defined among incompatibles. However, both conditions can be applied to a mapping on a $\mathcal{P}_{QML}$, though a truth-functional (ó) mapping on a $\mathcal{P}_{QML}$ ignores the lattice meets and joins of incompatibles and thus preserves only the partial-Boolean structural features of $\mathcal{P}_{QML}$. In Chapter V(A), it is shown how the lattice meets and joins of incompatibles cause truth-functionality (ó,$\not{\acute{o}}$) problems which rule out a bivalent, truth-functional (ó,$\not{\acute{o}}$) semantics for any $\mathcal{P}_{QML}$ which contains incompatible elements.

The fact that an orthomodular lattice $\mathcal{P}_{QML}$ has $\wedge$, $\vee$ defined among incompatibles also affects the notion of a complement in $\mathcal{P}_{QML}$. For as mentioned in Chapter I(E), any element P in a $\mathcal{P}_{QML}$ containing incompatible elements may have non-unique, incompatible complements. That is, for any $P_1 \in \mathcal{P}_{QML}$, there may be an element $P_2 \not{\acute{o}}\, P_1$ such that $P_1 \wedge P_2 = 0$ and $P_1 \vee P_2 = 1$; so P_1, P_2 are complements, but P_1, P_2 are not compatible and are not orthocomplements. For example, consider the orthomodular lattice diagrammed as follows, with $P_1 \not{\acute{o}}\, P_2$ (and hence $P_1 \not{\acute{o}}\, P_2^{\perp}$):



In this lattice, P_1, $P_1^{\perp}$ are compatible and are orthocomplements; likewise, P_2, $P_2^{\perp}$ are compatible and are orthocomplements. But moreover, as is clear from the diagram: $P_1 \wedge P_2 = 0$, $P_1 \vee P_2 = 1$, and $P_1 \wedge P_2^{\perp} = 0$, $P_1 \vee P_2^{\perp} = 1$. So besides its unique <u>ortho</u>complement P_1, the element P_1 also has two other complements, namely, the element P_2 and the element

$P_2^{\perp}$, which are not compatible with P_1 and are not orthocomplements of P_1.

However, when we consider the corresponding partial-Boolean algebra which has exactly the same elements as the above orthomodular lattice but does not have $\wedge$, $\vee$ defined among incompatibles, these same elements P_1, P_2, $P_2^{\perp}$ are not related via the $\wedge$, $\vee$ operations and so they are not complements. The only complements in a partial-Boolean algebra are the orthocomplements which are compatible and unique, just as the only complements in a Boolean structure such as the classical L and $\mathcal{P}_{CM}$ are orthocomplements which are compatible and unique. In contrast, the elements in an orthomodular lattice may have other complements. The presence of these other complements in a $\mathcal{P}_{QML}$ contributes to the lattice truth-functionality (ط,ط) problems, as shall be shown in Chapter V(A). And the presence of these other complements in a $\mathcal{P}_{QML}$ raises the question of whether the logical "not" operation should be associated with orthocomplementation or with complementation. The fact that the "not" of classical logic is an operation, that is, is a function which is univalent, provides a precedent for associating "not" with orthocomplementation rather than the other non-unique complementation.[5]

It is also worth noting that in a partial-Boolean algebra $\mathcal{P}_{QMA}$, the material conditional $\supset$ of (classical) formal logic can be defined in terms of $\vee$ "or" and $^{\perp}$ "not" as usual; moreover, as so defined, the material conditional in $\mathcal{P}_{QMA}$ is transitive as usual. But in a $\mathcal{P}_{QML}$, the material conditional cannot be defined as usual, which raises the question of how to define $\supset$ in $\mathcal{P}_{QML}$.

In classical logic, the material conditional is defined as, for any formulae $f_1, f_2 \in L$, $f_1 \supset f_2 = \text{df. } f_1^{\perp} \vee f_2$. And the material conditional is transitive, i.e., for any $f_1, f_2, f_3 \in L$, if $\vdash f_1 \supset f_2$ and $\vdash f_2 \supset f_3$, then $\vdash f_1 \supset f_3$, or equivalently, if $\models f_1 \supset f_2$ and $\models f_2 \supset f_3$, then $\models f_1 \supset f_3$. Algebraically, for any elements $/f_1/$, $/f_2/$ in the $\mathcal{L}$ structure of L, $/f_1/ \supset /f_2/ = /f_1 \supset f_2/$ is an element in $\mathcal{L}$, namely the element $/f_1/^{\perp} \vee /f_2/$. And the relations of logical implication $\vdash$ or semantic entailment $\models$ are interpreted as the partial-ordering relation $\leq$, where for any $/f/ \in \mathcal{L}$, $\leq /f/$ IFF $/f/ =$ the 1-element. Then the above transitivity condition can be restated algebraically as follows: For any $/f_1/, /f_2/, /f_3/ \in \mathcal{L}$, if $/f_1/^{\perp} \vee /f_2/ = 1$ and $/f_2/^{\perp} \vee /f_3/ = 1$, then $/f_1/^{\perp} \vee /f_3/ = 1$.

With respect to a quantum $\mathcal{P}_{QMA}$, the material conditional defined in terms of $^{\perp}$ and $\vee$ as above does satisfy this transitivity condition, i.e., for any $P_1, P_2, P_3 \in \mathcal{P}_{QMA}$, if $P_1^{\perp} \vee P_2 = 1$ and $P_2^{\perp} \vee P_3 = 1$, then $P_1^{\perp} \vee P_3 = 1$. But with respect to a quantum $\mathcal{P}_{QML}$, if the material conditional is defined in terms of $^{\perp}$ and $\vee$ as usual, then the material conditional is transitive IFF the lattice is Boolean, as shown by Fay (1967, p. 267). According to Jauch and others who worry about how to define the material conditional in a non-Boolean quantum $\mathcal{P}_{QML}$, the transitivity of the material conditional is necessary for a logic. And so these lattice theoreticians conclude that $\supset$ cannot be defined in terms of $^{\perp}$, $\vee$ as usual in a quantum $\mathcal{P}_{QML}$ (Jauch-Piron, 1970, p. 174). So the correct definition of the material conditional and even the possibility of a rule like modus ponens have been controversial issues among lattice-theoreticians.

Yet another ramification of the basic difference between $\mathcal{P}_{QMA}$

and $\mathcal{P}_{QML}$ is described in the next section.

Section F. The Two Basic Senses in Which the Quantum Propositional Structures Are Non-Boolean

In contrast to the Boolean propositional or logical structures determined by classical mechanics and classical propositional logic, the quantum propositional structures are said to be non-Boolean. However, both an orthomodular lattice $\mathcal{P}_{QML}$ and a partial-Boolean algebra $\mathcal{P}_{QMA}$ can be non-Boolean in various senses. In this section four senses are described, three of which are equivalent.

The most celebrated sense is the failure of distributivity. If an algebra or lattice is Boolean, then its binary $\wedge$, $\vee$ operations are distributive. So if the $\wedge$, $\vee$ operations in an algebra or lattice are not distributive, then the structure is non-Boolean. In particular, any quantum $\mathcal{P}_{QML}$ which contains incompatible elements exhibits at least one instance of the failure of distributivity. For as mentioned in Chapter I(E), for any $P_1, P_2 \in \mathcal{P}_{QML}$, the four elements P_1, $P_1^{\perp}$, P_2, $P_2^{\perp}$, satisfy the distributive identity for any combinations of these elements IFF $P_1 \leftrightarrow P_2$. It follows that distributivity fails in $\mathcal{P}_{QML}$ if $P_1 \not\leftrightarrow P_2$, for any $P_1, P_2 \in \mathcal{P}_{QML}$. Most authors who favour the lattice formalization of the quantum propositional structures, e.g., von Neumann and Birkhoff (1936, p. 831), Jauch (1963, p. 831), Putnam (1969, p. 226), Friedman and Glymour (1972, pp. 18, 20), focus upon the failure of distributivity as the peculiarly non-Boolean feature of the quantum propositional structures which distinguishes them from the Boolean propositional structures determined by classical mechanics. Moreover, it is a theorem that a lattice is distributive

IFF every pair of elements in it is compatible (Jauch-Piron, 1963, p. 831). It follows that a P_{QML} is non-Boolean in the failure of distributivity sense IFF it contains incompatible elements. And hence, we can be sure that instances of the failure of distributivity in a P_{QML} always involve the meets and joins of incompatibles.

Since a P_{QMA} has $\wedge$, $\vee$ defined among only compatibles these operations are distributive in a P_{QMA}. Thus the failure of distributivity can neither capture the sense in which a P_{QMA} is non-Boolean nor distinguish a P_{QMA} from the Boolean propositional structures determined by classical mechanics.

However, Piron defines another sense of non-Boolean for the P_{QML} structures which is equivalent to the failure of distributivity sense and which can also be applied to the P_{QMA} structures. Piron defines the centre of a lattice as stated in Chapter I(F). And it is a theorem that a lattice is Boolean IFF its centre is the entire lattice (Piron, 1976, p. 29). So if the centre of a lattice is smaller than the entire lattice, i.e., if there is an element in the lattice which is not compatible with all other elements, then the lattice is non-Boolean. Any quantum P_{QML} containing incompatible elements is non-Boolean in this sense. And Piron takes this fact to be the peculiarly non-Boolean feature of the quantum P_{QML} structures. By the definition of the centre, a P_{QML} is non-Boolean in the Piron sense IFF it contains incompatible elements. So we expect that a P_{QML} is non-Boolean in the Piron sense IFF it is non-Boolean in the failure of distributivity sense, as it is easy to show. If distributivity fails in a P_{QML}, then as mentioned above, not all pairs of elements in P_{QML} are compatible. And so by the definition of centre, the centre of P_{QML} is smaller than the entire P_{QML}. Conversely, if the centre of P_{QML} is smaller than the entire $\mathcal{P}_{QML}$, that is, if there is an element $P \in \mathcal{P}_{QML}$ which is not in the centre of $\mathcal{P}_{QML}$, then that P is incompatible with at least one other element in $\mathcal{P}_{QML}$. Hence not all pairs of elements in $\mathcal{P}_{QML}$ are compatible, and so distributivity fails in $\mathcal{P}_{QML}$. Q.E.D.

But unlike the failure of distributivity sense of non-Boolean, the Piron sense of non-Boolean does not involve the meets and joins of incompatibles. So the Piron sense of non-Boolean can be applied to a $\mathcal{P}_{QMA}$, with the centre of a $\mathcal{P}_{QMA}$ defined exactly as the centre of a $\mathcal{P}_{QML}$. And as defined in Chapter I(D), a partial-Boolean algebra is in fact a Boolean algebra IFF its elements are all mutually compatible, i.e., IFF its centre is the entire algebra. Thus a $\mathcal{P}_{QMA}$ is non-Boolean in the Piron sense if its centre is smaller than the entire $\mathcal{P}_{QMA}$. And as before, a $\mathcal{P}_{QMA}$ is non-Boolean in this Piron sense IFF it contains incompatible elements.

Similarly, the mere presence of incompatible elements in a $\mathcal{P}_{QML}$ or a $\mathcal{P}_{QMA}$ is a necessary and sufficient condition for the ultrafilters (and dual ultrideals) in $\mathcal{P}_{QML}$ or $\mathcal{P}_{QMA}$ to be not prime; this provides us with a third sense of non-Boolean. For as mentioned in Chapter I(C), the ultrafilters (and dual ultraideals) in a Boolean structure are all prime. So if the ultrafilters in a $\mathcal{P}_{QML}$ or a $\mathcal{P}_{QMA}$ are not all prime, then that structure can be said to be non-Boolean. As shown in Chapter VI(B), if a $\mathcal{P}_{QM}$ contains incompatible elements, then there is at least one ultrafilter in $\mathcal{P}_{QM}$ which is not prime, where a prime ultrafilter satisfies the condition (d) stated in Chapter I(C). Hence the presence of incompatible elements in a $\mathcal{P}_{QM}$ is a sufficient condition for $\mathcal{P}_{QM}$ to be non-Boolean in the sense that its ultrafilters are not all prime. Moreover, this condition is also necessary. For if all the elements of a $\mathcal{P}_{QM}$ are mutually compatible, then that $\mathcal{P}_{QM}$ is in fact a Boolean structure whose ultrafilters are all prime. So a $\mathcal{P}_{QM}$ is non-Boolean in the sense that its ultrafilters are not all prime IFF $\mathcal{P}_{QM}$ contains incompatible elements.

In summary, with respect to a $\mathcal{P}_{QML}$, the failure of distributivity sense, the Piron sense, and the not-prime ultrafilter sense of non-Boolean are all equivalent. And with respect to a $\mathcal{P}_{QMA}$, the Piron sense and the not-prime ultrafilter sense of non-Boolean are equivalent. For these senses of non-Boolean are each biconditionally connected with the mere presence of incompatible elements in a $\mathcal{P}_{QML}$ or a $\mathcal{P}_{QMA}$.[6]

However, there is an entirely different sense of non-Boolean which is not biconditionally connected with the mere presence of incompatible elements. This sense is suggested by Kochen-Specker, who refer specifically to $\mathcal{P}_{QMA}$ structures although their results also apply to $\mathcal{P}_{QML}$ structures. According to Kochen-Specker, a $\mathcal{P}_{QMA}$ is distinguished from the Boolean propositional structures determined by classical mechanics if the $\mathcal{P}_{QMA}$ cannot be imbedded into a Boolean algebra. And in their Theorem 0, Kochen-Specker prove that a $\mathcal{P}_{QMA}$ can be imbedded into a Boolean algebra $\mathcal{B}$ IFF there exists a sufficiently large collection of bivalent homomorphisms on $\mathcal{P}_{QMA}$ so that, for any pair of distinct elements $P_1 \neq P_2$ in $\mathcal{P}_{QMA}$, there is at least one bivalent homomorphism $h : \mathcal{P}_{QMA} \to \{0,1\}$ such that $h(P_1) \neq h(P_2)$. Next Kochen-Specker produce a finite, three-dimensional Hilbert space $\mathcal{P}_{QMA}$ which is shown in their Theorem 1 to admit no bivalent homomorphisms. Kochen-Specker conclude that the three-or-higher dimensional Hilbert space $\mathcal{P}_{QMA}$ structures of quantum mechanics likewise admit no bivalent homomorphisms, and thus by Theorem 0, these structures cannot be imbedded into a $\mathcal{B}$. Theorem 1 will be discussed in Chapter V; the Kochen-Specker proof of Theorem 0 is discussed here.

The "only if" half of the biconditional statement of Theorem 0 follows immediately from the semi-simplicity property of Boolean structures. Let % be the proposed imbedding. An imbedding is by definition injective, i.e., for any elements $P_1 \neq P_2$ in $\mathcal{P}_{QMA}$, $\%(P_1) \neq \%(P_2)$. And assuming that the imbedding $\% : \mathcal{P}_{QMA} \to \mathcal{B}$ exists, the semi-simplicity property of $\mathcal{B}$ guarantees that there is a bivalent homomorphism $h : \mathcal{B} \to \{0,1\}$ such that $h(\%(P_1)) \neq h(\%(P_2))$ for any $P_1 \neq P_2$ in $\mathcal{P}_{QMA}$. Thus the composition $h \circ \% : \mathcal{P}_{QMA} \to \{0,1\}$ is the desired homomorphism of $\mathcal{P}_{QMA}$ onto $\{0,1\}$ which separates $P_1 \neq P_2$, for any $P_1, P_2 \in \mathcal{P}_{QMA}$.

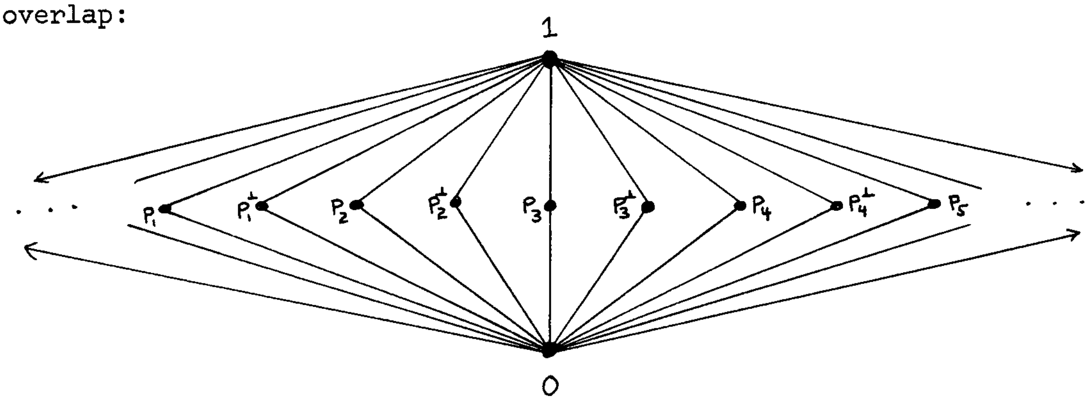
Kochen-Specker's proof of the converse half of Theorem 0 is also worth restating here because it suggests how to construct a Cartesian product Boolean structure into which a $\mathcal{P}_{QMA}$ can be imbedded _if_ the requisite set of bivalent homomorphisms on $\mathcal{P}_{QMA}$ exist. Assume that this set exists: let $\{h_i\}_{i \in \text{Index}}$ be the set and let s be the cardinality of this set. Then an imbedding of $\mathcal{P}_{QMA}$ into the Cartesian product Boolean structure $(Z_2)^s$, i.e., $\% : \mathcal{P}_{QMA} \to (Z_2)^s$, is given by the association of each element $P \in \mathcal{P}_{QMA}$ with the function $g_P : \{h_i\}_{i \in \text{Index}} \to \{0,1\}$ defined so that $g_P(h_i) = h_i(P)$ for every $i \in$ Index, where of course $h_i(P) \in \{0,1\}$ for every $i \in$ Index. So for example, the image of any given $P \in \mathcal{P}_{QMA}$ under the imbedding is $\%(P) = \langle h_1(P), h_2(P), \ldots, h_s(P) \rangle \in (Z_2)^s$ (Kochen-Specker, 1967, p. 67). This construction will be referred to again shortly.

The Kochen-Specker imbeddings and homomorphisms preserve the operations and relations of a $\mathcal{P}_{QMA}$ structure. More exactly, a homomorphism

$h : X \to Y$ between any partial-Boolean algebras X, Y, satisfies, for any compatible elements $b \,\text{ȯ}\, c \in X$: $h(b) \,\text{ȯ}\, h(c)$, $h(b \wedge c) = h(b) \wedge h(c)$, $h(b^{\perp}) = (h(b))^{\perp}$, $h(1) = 1$ (Kochen-Specker, 1967, pp. 66-67). In my terminology, an h satisfying the above is a <u>homomorphism(ȯ)</u> from X to Y, an injective h satisfying the above is an <u>imbedding(ȯ)</u> of X into Y, and when Y is $\{0,1\}$, i.e., when h is bivalent, an h satisfying the above is truth-functional(ȯ). Thus Kochen-Specker's Theorem 0 biconditionally connects the possibility of imbedding(ȯ) a P_{QMA} into a Boolean structure with the existence of what I call a complete collection of bivalent, truth-functional(ȯ) mappings on P_{QMA}, or in other words, a bivalent, truth-functional(ȯ) semantics for P_{QMA}. And it is the impossibility of imbedding(ȯ) P_{QMA} into a Boolean structure, or equivalently, the impossibility of a bivalent, truth-functional(ȯ) semantics for P_{QMA}, which Kochen-Specker focus upon as the distinguishing non-Boolean feature of the quantum P_{QMA} structures. Of course, this sense of non-Boolean can also be applied to the quantum P_{QML} structures, although an imbedding(ȯ) of a P_{QML} into a Boolean structure or a bivalent, truth-functional(ȯ) semantics for a P_{QML} ignore the lattice meets and joins of incompatibles and preserve only the partial-Boolean structural features of P_{QML}.

Like the other senses of non-Boolean described above, the presence of incompatible elements in a P_{QMA} or P_{QML} is a necessary condition for that structure to be non-Boolean in the Kochen-Specker sense. For as mentioned in Chapter I(D) and (E), if all the elements of a P_{QMA} or a P_{QML} are mutually compatible, then that P_{QM} is a Boolean structure as defined in Chapter I(B). And any Boolean structure admits a complete collection of bivalent, homomorphic(ȯ) mappings, i.e., any Boolean structure can be imbedded(b) into another Boolean structure. (The suffix (b) is redundant and harmless since all elements in a Boolean structure are mutually compatible.) So if a $\mathcal{P}_{QM}$ is non-Boolean in the Kochen-Specker sense, then the elements of that structure cannot be mutually compatible, that is, the structure must contain incompatible elements. However, unlike the other senses of non-Boolean described above, the mere presence of incompatible elements in a $\mathcal{P}_{QM}$ is *not* a *sufficient* condition for the structure's being non-Boolean in the Kochen-Specker sense. In particular, regardless of the presence of incompatible elements, a $\mathcal{P}_{QM}$ structure of *two*-dimensional Hilbert space does admit a complete collection of bivalent, homomorphic(b) mappings, i.e., a $\mathcal{P}_{QM}$ structure of two-dimensional Hilbert space can be imbedded(b) into a Boolean structure, as shown below.

The peculiar structural feature of three-or-higher dimensional Hilbert space $\mathcal{P}^{n\geq 3}_{QM}$ structures which makes them non-Boolean in the Kochen-Specker sense is the presence of *overlapping maximal Boolean substructures*. Any Boolean structure has only one maximal Boolean substructure, namely, itself. And the two-dimensional Hilbert space $\mathcal{P}^{2}_{QM}$ structure diagrammed below has many maximal Boolean substructures, but they do not overlap:



Except for the trivial Boolean substructure containing just the 0, 1 elements of P^2_{QM}, every other Boolean substructure of P^2_{QM} contains the four elements P_i, $P_i^{\perp}$, 0, 1, for some $i = 1, 2, \ldots,$ and is a maximal Boolean substructure mBS_i. The mBS's of P^2_{QM} do share the 0, 1 elements but do not share any other elements and so are <u>non-overlapping</u>. As shown next, any two-dimensional Hilbert space P^2_{QM} can be imbedded(b) into the Cartesian product Boolean structure $(Z_2)^{2 \cdot r}$, where r is the cardinality of the set of mBS's in P^2_{QM} and 2 is the dimensionality of the Hilbert space.

Each mBS_i in P^2_{QM} is isomorphic to the Cartesian product Boolean structure $(Z_2)^2$ diagrammed in Chapter I(B). And by semi-simplicity each mBS_i has exactly two bivalent homomorphisms, for example, the h_a, h_b on mBS_1 and the h_c, h_d on mBS_2 listed in the following table:

	P_1	P_1	P_2	P_2	0	1
h_a	1	0			0	1
h_b	0	1			0	1
h_c			1	0	0	1
h_d			0	1	0	1

Since the mBS's of P^2_{QM} do not overlap, it is possible to define at least $2 \cdot r$ bivalent homomorphisms(b) on the entire P^2_{QM} by simply adding together the bivalent homomorphisms on each mBS_i. For example, assume that r is just 2, i.e., consider the six-element fragment of P^2_{QM} consisting of just mBS_1 and mBS_2 together. The above four bivalent homomorphisms h_a , h_b , h_c , h_d on mBS_1 , mBS_2 , respectively, can be added together as follows to yield $2 \cdot 2$ bivalent homomorphisms(ð) on this six-element fragment of P^2_{QM} :

	P_1	P_1	P_2	P_2	0	1
$h_a + h_c = h_1$	1	0	1	0	0	1
$h_b + h_d = h_2$	0	1	0	1	0	1
$h_b + h_c = h_3$	0	1	1	0	0	1
$h_a + h_d = h_4$	1	0	0	1	0	1

Similarly, for any P^2_{QM} with r mBS's, it is possible to define $2 \cdot r$ bivalent homomorphisms(ð) on the entire P^2_{QM} . And thus, as Kochen-Specker show in their proof of Theorem 0, for each element $P \in P^2_{QM}$, the mapping $\%(P) = \langle h_1(P), h_2(P), \ldots, h_{2 \cdot r}(P) \rangle$ defines an imbedding(ð) of P^2_{QM} into the Cartesian product Boolean structure $(Z_2)^{2 \cdot r}$. The latter is also written: $\Pi^r_{i=1}(Z_2)^2_i$. For example, the six-element fragment of P^2_{QM} consisting of just mBS_1 and mBS_2 can be imbedded(ð) into the Cartesian product Boolean structure $(Z_2)^{2 \cdot 2} = (Z_2)^4$ diagrammed in Chapter I(B) as follows: $\%(P_1) = \langle h_1(P_1), h_2(P_1), h_3(P_1), h_4(P_1) \rangle = \langle 1,0,0,1 \rangle$; $\%(P_1^\perp) = \langle 0,1,1,0 \rangle$; $\%(P_2) = \langle 1,0,1,0 \rangle$; $\%(P_2^\perp) = \langle 0,1,0,1 \rangle$; $\%(1) = \langle 1,1,1,1 \rangle$; $\%(0) = \langle 0,0,0,0 \rangle$.

If the maximal Boolean substructures of any three-or-higher dimensional Hilbert space $P^{n \cdot 3}_{QM}$ structure did not overlap, then it would similarly be possible to imbed(ð) that structure into the Cartesian product Boolean structure $(Z_2)^{n \cdot r}$, where again r is the cardinality of the set of mBS's in the structure and n is the dimensionality of the Hilbert space. For each mBS_i in a $P_{QM}^{n\geq 3}$ structure is isomorphic to the Boolean structure $(Z_2)^n$ and by semi-simplicity has exactly n bivalent homomorphisms. So if the mBS's of $P_{QM}^{n\geq 3}$ did not overlap, then it would be possible to simply add together these bivalent homomorphisms on each mBS_i yielding at least $n\cdot r$ bivalent homomorphisms(♁) on the entire $P_{QM}^{n\geq 3}$ structure. And thus by the Kochen-Specker Theorem 0, the $P_{QM}^{n\geq 3}$ could be imbedded(♁) into the Cartesian product Boolean structure $(Z_2)^{n\cdot r}$, which is also written: $\Pi_{i=1}^{r}(Z_2)_i^n$.[7]

However, the mBS's of a three-or-higher dimensional Hilbert space $P_{QM}^{n\geq 3}$ may overlap and do overlap in quantum mechanically relevant $P_{QM}^{n\geq 3}$. Consequently, the attempt to define bivalent homomorphisms(♁) on a $P_{QM}^{n\geq 3}$, by simply adding together the separate bivalent homomorphisms existing on each mBS of $P_{QM}^{n\geq 3}$, is problematic and in fact is impossible. Kochen-Specker prove this impossibility; their proof is discussed in Chapter V(B). An example of a trivial exception to this impossibility is given in the note below; such exceptional $P_{QM}^{n\geq 3}$ structures are not quantum mechanically relevant.[8]

In summary, there are two basic senses in which the quantum propositional structures may be said to be non-Boolean and may be distinguished from the Boolean propositional structures determined by classical mechanics and classical logic. One basic sense subsumes the failure of distributivity, the Piron, and the not-prime ultrafilter senses of non-Boolean; the presence of incompatible elements in a P_{QM} is necessary and sufficient for the structure to be non-Boolean in this basic sense. The other basic sense is suggested by Kochen-Specker's work; the mere presence of incompatible elements in a P_{QM} is necessary but is <u>not</u> sufficient for the structure to be non-Boolean in this second basic sense.

Notes

[1] This fact is actually derived from one or the other of the fundamental postulates (II) or (III) which define $\rho_{\psi,A}$ and $\mathrm{Exp}_{\psi}(A)$ (Messiah, 1966, pp. 178, 297).

[2] According to the terminology of his 1932 book, von Neumann calls such propositions <u>simultaneously decidable</u>. Von Neumann's notion of the <u>simultaneous decidability of propositions</u> is a refinement of his notion of the simultaneous measurability of magnitudes. The latter requires that the self-adjoint operators representing the magnitudes commute. The former requires that only the projectors representing the propositions commute, but the magnitudes mentioned in the propositions need not be simultaneously measurable, i.e., their operators need not commute. So while the operators representing simultaneously measurable magnitudes share all their eigenstates, the operators representing the magnitudes mentioned in simultaneously decidable propositions need share only the eigenstate(s) specified by the propositions. Von Neumann has his own unusual use of the terms compatible and incompatible. Nevertheless, simultaneously decidable propositions are compatible in the usual sense that their representative projectors commute (von Neumann, 1932, pp. 251, 253).

[3] With respect to an orthomodular lattice $\mathcal{P}_{QML}$, this condition is weak modularity, which characterizes the quantum $\mathcal{P}_{QML}$ structures. With respect to a partial-Boolean algebra $\mathcal{P}_{QMA}$, this condition holds because by definition, $P_1 \leq P_2$ IFF $P_1 \wedge P_2 = P_1$, but $P_1 \wedge P_2$ is defined in $\mathcal{P}_{QMA}$ IFF $P_1 \mathrel{\overset{\downarrow}{\circ}} P_2$.

[4] Von Neumann restricts "and" and "or" to what he calls simultaneously decidable propositions. As mentioned in Note 2 above, such propositions are compatible in the usual sense that their representative projectors commute.

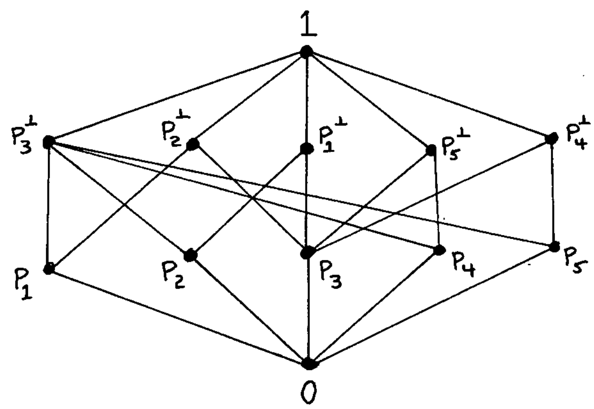
[5] This point was suggested by Dr. R. E. Robinson.

In her discussion of Birkhoff and von Neumann's quantum lattice structures, S. Haack incorrectly claims that an element in such a structure may have more than one orthocomplement (Haack, 1974, p. 161). Though it is true that an element may have more than one complement, the orthocomplement of an element is by definition unique. For according to Birkhoff, the association of an element b with its orthocomplement b is a type of mapping (namely, a dual automorphism $h : \mathcal{X} \to \mathcal{X}$ which is an isomorphism of a structure with itself satisfying, for every $b, c \in \mathcal{X}$, $b \leq c$ IFF $h(b) \geq h(c)$) (Birkhoff, 1967, p. 3). And as stated in Chapter I(G), condition Ma, the image of any element $b \in \mathcal{X}$ under any mapping $h : \mathcal{X} \to \mathcal{Y}$ is unique, e.g., $h(b) = b^{\perp}$ is unique.

[6] Likewise the failure of bivalence sense of non-Boolean proposed by van Fraassen is biconditionally connected with the mere presence of incompatible elements in a quantum propositional structure (van Fraassen, 1973, p. 89).

[7] Bub makes a similar point (1974, pp. 144-146).

[8] $\mathcal{P}^{n\geq 3}_{QMA}$ structures whose mBS's do not overlap and $\mathcal{P}^{n\geq 3}_{QM}$ structures which admit bivalent homomorphisms(ȸ) even though their mBS's do overlap, may be generated by closing certain limited sets of one-dimensional subspaces (or projectors) of $\mathcal{H}^{n\geq 3}$ with respect to the $\wedge$, $\vee$, $^{\perp}$ operations of $\mathcal{P}_{QMA}$ or $\mathcal{P}_{QM}$. For an example of the latter, consider the following twelve-element $\mathcal{P}^{3}_{QM}$ structure generated by closing an initial set of five one-dimensional subspaces of $\mathcal{H}^{3}$ with respect to the $\wedge$, $\vee$, $^{\perp}$ operations of $\mathcal{P}_{QM}$, where $\{P_1, P_2, P_3\}$ are mutually compatible and likewise $\{P_3, P_4, P_5\}$ are mutually compatible.



The following five bivalent homomorphisms(ȸ) constitute a complete collection of bivalent homomorphisms(ȸ) on this twelve-element $\mathcal{P}^{3}_{QM}$:

	P_1	P_1	P_2	P_2	P_3	P_3	P_4	P_4	P_5	P_5	0	1
h_1	1	0	0	1	0	1	1	0	0	1	0	1
h_2	0	1	1	0	0	1	0	1	1	0	0	1
h_3	0	1	0	1	1	0	0	1	0	1	0	1
h_4	0	1	1	0	0	1	1	0	0	1	0	1
h_5	1	0	0	1	0	1	0	1	1	0	0	1

(Just the first three bivalent, homomorphisms(ȸ) h_1, h_2, h_3, constitute a weakly complete collection.) This twelve-element $\mathcal{P}^{3}_{QM}$ is also an example of a phenomenon mentioned in Section (E) above, namely, an example of how the proliferation of lattice elements due to the lattice definitions of

$\wedge$, $\vee$ among incompatibles does not occur in $\mathcal{P}^{n\geq 3}_{QML}$ structures generated by closing a finite initial set of one-dimensional subspaces of $\mathcal{H}^{n\geq 3}$ when there are certain angular relations among the subspaces in the initial set; most simply, in this case, P_1, P_2, P_4, P_5 are all in the same two-dimensional subspace P_3. And in this case, the $\mathcal{P}^3_{QML}$ and the $\mathcal{P}^3_{QMA}$ generated by closing the initial set of five one-dimensional suspaces of $\mathcal{H}^3$ with respect to the $\wedge$, $\vee$, $^\perp$ operations of $\mathcal{P}_{QML}$ and $\mathcal{P}_{QMA}$ respectively, each have exactly the same twelve elements, as diagrammed above.

Nevertheless, as exemplified by Kochen-Specker, for $\mathcal{H}^{n\geq 3}$, the sets of one-dimensional subspaces representing quantum propositions which describe actual quantum mechanical systems and situations yield, upon closure, $\mathcal{P}^{n\geq 3}_{QM}$ structures whose mBS's do overlap and overlap in such a way that bivalent homomorphisms(ƅ) on $\mathcal{P}^{n\geq 3}_{QM}$ are ruled out. (Kochen-Specker, 1967, Section 4). In other words, quantum mechanically relevant $\mathcal{P}^{n\geq 3}_{QM}$ structures have overlapping mBS's which rule out bivalent homomorphisms(ƅ).

CHAPTER V

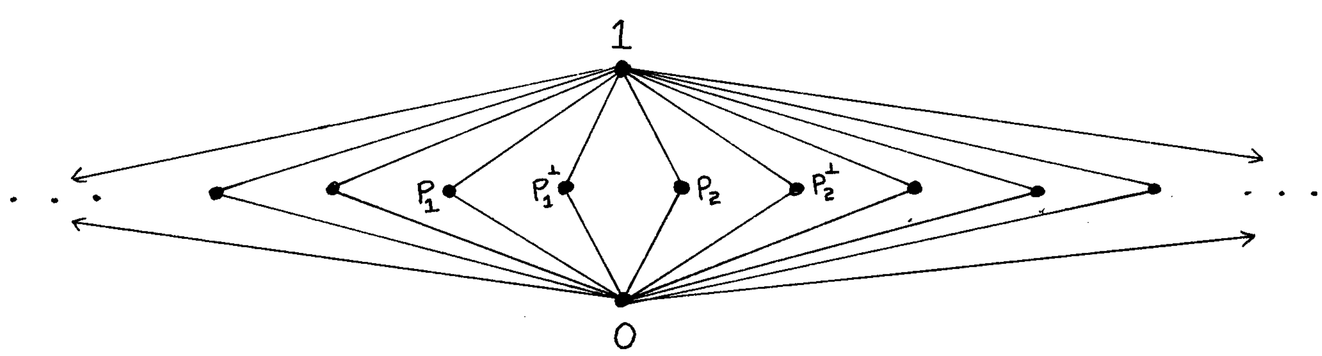
THE IMPOSSIBILITY OF A BIVALENT, TRUTH-FUNCTIONAL SEMANTICS FOR THE NON-BOOLEAN PROPOSITIONAL STRUCTURES DETERMINED BY QUANTUM MECHANICS

Preface

As described in Chapter IV(F), there are two basic senses in which the quantum propositional structures may be said to be non-Boolean. And as mentioned in Chapter II(C), any Boolean structure admits a complete collection of bivalent homomorphisms, and this collection is a bivalent, truth-functional semantics when the Boolean structure is a logical or propositional structure. But if a propositional structure is in some sense non-Boolean, then whether or not it admits such a semantics is an open question. With respect to the non-Boolean quantum propositional structures, answers to this question have already been given or at least suggested by von Neumann, Jauch-Piron, Gleason and Kochen-Specker in their proofs and arguments against the possibility of hidden variables in quantum mechanics. For as shall be described in Section (D), when interpreted semantically Gleason's impossibility proof and Kochen-Specker's Theorem 1 show the impossibility of a bivalent, truth-functional($\mathring{\shortmid}$) semantics for three-or-higher dimensional Hilbert space $\mathcal{P}_{QM}^{n\geq 3}$ structures, whether $\mathcal{P}_{QMA}$ or $\mathcal{P}_{QML}$. And interpreted semantically, von Neumann's proof of the impossibility of dispersion-free quantum ensembles and Jauch-Piron's Corollary 1 suggest the impossibility of a bivalent, truth-functional($\mathring{\shortmid}$,$\not\mathring{\shortmid}$) semantics for two-or-higher dimensional Hilbert space $P^{n\geq 2}_{QML}$ structures. The proofs by von Neumann and Jauch-Piron must be interpreted as referring to the orthomodular lattice structures and the truth-functionality(δ,$\not\delta$) condition because von Neumann and Jauch-Piron do define operations among incompatibles and do require the preservation of these operations. In the next section, von Neumann, Jauch-Piron suggestion is pursued.

Section A. The Impossibility of a Bivalent, Truth-Functional(δ,$\not\delta$) Semantics for any P_{QML} Which Contains Incompatible Elements

Consider the fragment of the P^2_{QM} structure of two-dimensional Hilbert space diagrammed below, with $P_1 \not\delta P_2$:



As mentioned in Chapter IV(E), P^2_{QML} and P^2_{QMA} have the same elements. In both structures, the 0, 1 elements are equal to the following meets and joins of compatibles: $P_1 \wedge P_1^{\perp} = 0$, $P_2 \wedge P_2^{\perp} = 0$, $P_1 \vee P_1^{\perp} = 1$, $P_2 \vee P_2^{\perp} = 1$. In addition, the 0, 1 elements are equal to the following meets and joins of incompatibles in P^2_{QML} : $P_1 \wedge P_2 = 0$, $P_1 \wedge P_2^{\perp} = 0$, $P_1 \vee P_2 = 1$, $P_1 \vee P_2^{\perp} = 1$.

Any non-trivial mapping m on P^2_{QML} assigns the value 0 to each meet which equals the 0-element, e.g., $m(P_1 \wedge P_2) = m(0) = 0$.

Likewise, m assigns the value 1 to each join which equals the 1-element.[1] And now it is easy to prove the following:

<u>Theorem 0.</u> A bivalent, truth-functional($⫰$,$\not⫰$) mapping on the fragment of $\mathcal{P}^2_{QML}$ diagrammed above is impossible.

Proof: Any bivalent m assigns either the value 1 or the value 0 to the element P_1 , so there are two cases. Case 1: Assume $m(P_1) = 1$. We have $m(P_1 \wedge P_2) = m(0) = 0$, so by truth-functionality($⫰$,$\not⫰$), $m(P_1 \wedge P_2) = m(P_1) \wedge m(P_2) = 0$. Hence $m(P_2) \neq 1$ since $1 \wedge 1 = 1$, thus $m(P_2) = 0$. So by truth-functionality($⫰$,$\not⫰$) $m(P_2^{\perp}) = (m(P_2))^{\perp} = 0^{\perp} = 1$. Also we have $m(P_1 \wedge P_2^{\perp}) = m(0) = 0$, so by truth-functionality($⫰$,$\not⫰$), $m(P_1 \wedge P_2^{\perp}) = m(P_1) \wedge m(P_2^{\perp}) = 0$. Hence $m(P_2^{\perp}) = 0$. So we have a contradiction. Case 2: Assume $m(P_1) = 0$. Then as in Case 1, $m(P_1 \vee P_2) = m(1) = 1$ and $m(P_1 \vee P_2^{\perp}) = m(1) = 1$ yield contradictory assignments of values to the element $P_2^{\perp}$. Q.E.D.

Hence a bivalent, truth-functional($⫰$,$\not⫰$) semantics for this fragment of $\mathcal{P}^2_{QML}$ is impossible. This proof can be generalized to include any two-or-higher dimensional Hilbert space $\mathcal{P}^{n\geq 2}_{QML}$ structure which contains incompatible elements. (The trivial case of a one-dimensional Hilbert space structure is excluded because that structure contains just a 0-element and a 1-element which are compatible.)

The generalization makes use of the following lemmas:

<u>Lemma A.</u> For any atom P_a in any $\mathcal{P}_{QM}$ and for any element $P \in \mathcal{P}_{QM}$, $P_a ⫰ P$ IFF $P_a \leq P$ or $P_a \leq P^{\perp}$.

Assume $P_a ⫰ P$. By definition of $⫰$, there exist three mutually disjoint elements $P_1, P_2, P_3 \in \mathcal{P}_{QM}$ such that $P_a = P_1 \vee P_3$

and $P = P_2 \vee P_3$. Since $P_a = P_1 \vee P_3$ is an atom, $P_1 \vee P_3 > 0$ and there is no element $P_x \in \mathcal{P}_{QM}$ such that $P_1 \vee P_3 > P_x > 0$. Since $P_1 \vee P_3 \geq P_1$ and $P_1 \vee P_3 \geq P_3$, either $P_1 = 0$ and $P_a = P_3$, or $P_3 = 0$ and $P_a = P_1$. If the former, then $P = P_2 \vee P_a \geq P_a$. If the latter, then $P = P_2 \vee 0 = P_2$; and since P_1 , P_2 are disjoint, $P_a = P_1$ and $P = P_2$ are disjoint.

Assume $P_a \leq P$, then $P_a \eth P$. (See note 3 of Chapter IV.) Likewise, if $P_a \leq P^{\perp}$, then $P_a \eth P^{\perp}$, where $P_a \eth P^{\perp}$ IFF $P_a \eth P$. Q.E.D.

<u>Lemma B</u>. For any atom P_a in any $\mathcal{P}_{QML}$ and for any element $P \in \mathcal{P}_{QML}$, if $P_a \not\leq P$ then $P_a \wedge P = 0$.

By assumption, $P_a > 0$ and there is no element $P_x \in \mathcal{P}_{QML}$ such that $P_a > P_x > 0$. But $P_a \geq P_a \wedge P \geq 0$. So either $P_a = P_a \wedge P$ or $P_a \wedge P = 0$. The former is ruled out because $P_a = P_a \wedge P$ IFF $P_a \leq P$, which contradicts the antecedent of Lemma B. Hence $P_a \wedge P = 0$. Q.E.D.

<u>Lemma C</u>. Every element $P \neq 0$ in $\mathcal{P}_{QML}$ is the join of the atoms it dominates.

Let P_i be any atom in $\mathcal{P}_{QML}$ such that $P_i \leq P$, and let $\bigvee_i P_i$ be the (finite or infinite) join of all such atoms. (This join exists because $\mathcal{P}_{QML}$ is complete.) And let $P_x = \bigvee_i P_i$. We want to show that $P = P_x$. Clearly, $P_x \leq P$, and so $P_x \eth P$. Now if $P_x^{\perp} \wedge P = 0$, then $P = 1 \wedge P = (P_x \vee P_x^{\perp}) \wedge P = (P_x \wedge P) \vee (P_x^{\perp} \wedge P) = (P_x \wedge P) \vee 0 = P_x \wedge P$, i.e., $P \leq P_x$ and thus $P_x = P$. Assume on the contrary that $P_x^{\perp} \wedge P \neq 0$. Then since $\mathcal{P}_{QML}$ is atomic, there is an atom P_a in $\mathcal{P}_{QML}$ such that $P_a \leq P_x^{\perp} \wedge P$, so $P_a \leq P_x^{\perp}$ and $P_a \leq P$. Since $P_a \leq P$, $P_a = P_i$, for some i, and so $P_a \leq P_x$, i.e., $P_a \wedge P_x = P_a$. And since $P_a \leq P_x^{\perp}$, $P_a = P_a \wedge P_x = P_x^{\perp} \wedge P_x = 0$, a contradiction. Q.E.D.

Lemma D. The join of all the atoms in any $\mathcal{P}_{QML}$ is equal to the 1-element.

Let P_i be any atom in $\mathcal{P}_{QML}$ and let $\bigvee_i P_i$ be the (finite or infinite) join of all the atoms in $\mathcal{P}_{QML}$.

Assume on the contrary that $\bigvee_i P_i \neq 1$. Then $(\bigvee_i P_i)^{\perp} \neq 0$, and so $(\bigvee_i P_i)^{\perp} \geq P_j$, for some atom P_j. Clearly, $\bigvee_i P_i \geq P_j$. It follows that $0 = (\bigvee_i P_i) \wedge (\bigvee_i P_i)^{\perp} \geq (\bigvee_i P_i) \wedge P_j = P_j$, which is impossible. Hence $(\bigvee_i P_i) = 1$. Q.E.D.

Lemma E. Any proposed bivalent, truth-functional($\circ$,$\not\circ$) mapping on any $\mathcal{P}_{QML}$ must assign the value 1 to at least one of the atoms in $\mathcal{P}_{QML}$.

Again, let P_i be any atom in $\mathcal{P}_{QML}$ and let $\bigvee_i P_i$ be the join of all the atoms in $\mathcal{P}_{QML}$. I assume that the truth-functionality($\circ$,$\not\circ$) condition includes the preservation of infinite meets and joins.

By Lemma D, $\bigvee_i P_i = 1$, and so any (non-trivial) bivalent, truth-functional($\circ$,$\not\circ$) mapping m on $\mathcal{P}_{QML}$ assigns the value $\bigvee_i m(P_i) = m(\bigvee_i P_i) = m(1) = 1$. And for every P_i, $m(P_i) = 0$ or 1, since m is bivalent. If $m(P_i) = 0$ for every P_i, then $\bigvee_i m(P_i) = 0 \neq 1$. Thus at least one of the atoms in $\mathcal{P}_{QML}$

is assigned the value 1 by m. Q.E.D.[2]

Besides these lemmas, the generalization makes use of the distinction between irreducible and reducible $\mathcal{P}_{QML}$ structures, defined as follows. As defined in Chapter I(F), the centre of any $\mathcal{P}_{QML}$ contains at least the 0 and 1 elements of $\mathcal{P}_{QML}$. A $\mathcal{P}_{QML}$ whose centre contains just the 0, 1 elements is <u>irreducible</u>. A $\mathcal{P}_{QML}$ whose centre contains other elements besides the 0, 1 elements is <u>reducible</u>. As described in Chapter IV(F), any $\mathcal{P}_{QML}$ contains incompatible elements IFF its centre is less than the entire structure. Clearly, any $\mathcal{P}_{QML}$ containing incompatible elements is either irreducible or reducible. And any irreducible $\mathcal{P}_{QML}$ contains incompatible elements. A reducible $\mathcal{P}_{QML}$ may have all its elements mutually-compatible, but such a reducible $\mathcal{P}_{QML}$ is in fact a Boolean structure.

If the centre of a reducible $\mathcal{P}_{QML}$ contains any atoms of $\mathcal{P}_{QML}$, then the structure does admit some bivalent, truth-functional(ὁ,ὃ) mappings, as shall be described in a brief digression. Each central atom P_c of such a reducible $\mathcal{P}_{QML}$ specifies an ultrafilter UF_c and dual ultraideal UI_c by the usual definitions: $UF_c = \{P \in \mathcal{P}_{QML} : P \geq P_c\}$ and $UI_c = \{P \in \mathcal{P}_{QML} : P \leq P_c^{\perp}\}$. And since each central atom is by definition compatible with every other element in $\mathcal{P}_{QML}$, it follows by Lemma A that, for every element $P \in \mathcal{P}_{QML}$, and for any given central atom P_c, either $P \geq P_c$ or $P^{\perp} \geq P_c$. Since by definition of $^{\perp}$, $P^{\perp} \geq P_c$ IFF $P \leq P_c^{\perp}$, either $P \geq P_c$ or $P \leq P_c^{\perp}$. So every element in $\mathcal{P}_{QML}$ is either a member of UF_c or a member of UI_c; thus $\mathcal{P}_{QML} = UF_c \cup UI_c$. Then as will be shown in Chapter VI(B), it follows by the conditions defining an ultrafilter and dual ultraideal that the bivalent homomorphism $h_c : \mathcal{P}_{QML} \to \{0,1\}$, defined with respect to UF_c and UI_c as usual, truth-functionally(ὁ,ὃ) assigns 0, 1 values to every element in $\mathcal{P}_{QML}$. In particular, each h_c assigns the value 1 to its affiliated central atom P_c and assigns the value 0 to every other atom $P_a \neq P_c$ in $\mathcal{P}_{QML}$. For every other atom P_a is compatible with P_c and so by Lemma A, $P_a \leq P_c^{\perp}$ (the alternative $P_a \leq P_c$ is ruled out since P_c is an atom); thus $P_a \in UI_c$. There are as many such bivalent, truth-functional(δ,$\not\delta$) mappings on a reducible $\mathcal{P}_{QML}$ as there are central atoms in $\mathcal{P}_{QML}$. This ends the digression.

Now the previous Theorem 0 is generalized as follows.

> Theorem A. A bivalent, truth-functional(δ,$\not\delta$) semantics is impossible for any (two-or-higher dimensional Hilbert space) $\mathcal{P}_{QML}$ which contains incompatible elements.

Case 1: Irreducible $\mathcal{P}_{QML}$. By Lemma E, any proposed bivalent, truth-functional(δ,$\not\delta$) mapping on any irreducible $\mathcal{P}_{QML}$ assigns the value 1 to at least one atom in $\mathcal{P}_{QML}$, say $m(P_a) = 1$. Since P_a is not in the centre of the irreducible $\mathcal{P}_{QML}$, there is some element $P \in \mathcal{P}_{QML}$ such that $P_a \not\delta P$. Then by Lemma A, $P_a \not\leq P$ and $P_a \not\leq P^{\perp}$. So by Lemma B, $P_a \wedge P = 0$ and $P_a \wedge P^{\perp} = 0$. Then it follows by the reasoning given in the case 1 of Theorem 0 that m assigns contradictory values to the element $P^{\perp}$. So a proposed bivalent truth-functional(δ,$\not\delta$) mapping on an irreducible $\mathcal{P}_{QML}$ is impossible. Hence a bivalent, truth-functional(δ,$\not\delta$) semantics for an irreducible $\mathcal{P}_{QML}$ is impossible.

Case 2: Reducible $\mathcal{P}_{QML}$ (containing incompatible elements). Any reducible $\mathcal{P}_{QML}$ contains at least one non-central atom. For if every atom in $\mathcal{P}_{QML}$ were central, then since the centre is a sublattice closed with respect to the join operation it follows by Lemma C that every element in $\mathcal{P}_{QML}$ would be central, i.e., there would be no incompatible elements in $\mathcal{P}_{QML}$. A non-central atom in $\mathcal{P}_{QML}$ is clearly distinct from the 0-element, and so a complete collection of bivalent, truth-functional(δ,$\not\delta$) mappings on $\mathcal{P}_{QML}$ must include a mapping which assigns the value 1 to the non-central atom. But by the same reasoning given in case 1 of this proof, any proposed bivalent, truth-functional(δ,$\not\delta$) mapping which assigns the value 1 to a non-central atom in $\mathcal{P}_{QML}$ will assign contradictory values to some other element in $\mathcal{P}_{QML}$ which is incompatible with that atom. So although a reducible $\mathcal{P}_{QML}$ may admit some bivalent, truth-functional(δ,$\not\delta$) mappings, as shown in the digression above, a reducible $\mathcal{P}_{QML}$ does not admit enough such mappings to constitute a bivalent, truth-functional(δ,$\not\delta$) semantics for $\mathcal{P}_{QML}$. Q.E.D.[3]

One way of avoiding the contradictions which yield this impossibility proof is to weaken the truth-functionality(δ,$\not\delta$) condition to just truth-functionality(δ), thus allowing the semantic mappings on a $\mathcal{P}_{QML}$ to ignore the lattice meets and joins of incompatibles. Such bivalent, truth-functional(δ) mappings which preserve the partial-Boolean structural features of a $\mathcal{P}_{QML}$ or a $\mathcal{P}_{QMA}$ are bivalent homomorphisms(δ) and are considered by Kochen-Specker

<u>Section B.</u> <u>The Kochen-Specker Proof of the Impossibility of Bivalent Homomorphisms(δ) on a Three-Dimensional Hilbert Space $\mathcal{P}^3_{QMA}$</u>

As described in Chapter IV(F), two-dimensional Hilbert space $\mathcal{P}^2_{QM}$ structures do admit a complete collection of bivalent homomorphisms(δ), i.e., they do admit a bivalent, truth-functional(δ) semantics, in spite of the fact that they contain incompatible elements. But three-or-higher dimensional Hilbert space $\mathcal{P}^{n\geq 3}_{QM}$ structures do not admit bivalent homomorphisms($\flat$). The peculiar structural feature of $\mathcal{P}^{n\geq 3}_{QM}$ which rules out bivalent homomorphisms($\flat$) is not just the presence of incompatible elements but rather the presence of overlapping maximal Boolean substructures (for which the presence of incompatibles is a necessary condition).

In their Theorem 1, Kochen-Specker consider a particular finite $\mathcal{P}^3_{QMA}$ and show that this structure does not admit even a single bivalent homomorphism($\flat$). By definition, a proposed bivalent homomorphism($\flat$) h on $\mathcal{P}^3_{QMA}$ satisfies, for any three mutually orthogonal atoms $P_1, P_2, P_3 \in \mathcal{P}^3_{QMA}$, $h(P_1) \vee h(P_2) \vee h(P_3) = h(P_1 \vee P_2 \vee P_3) = h(1) = 1$, and $h(P_i) \wedge h(P_j) = h(P_i \wedge P_j) = h(0) = 0$ for $1 \leq i \neq j \leq 3$. Thus exactly one of every three mutually orthogonal atoms in $\mathcal{P}^3_{QMA}$ is assigned the value 1 by a bivalent homomorphism($\flat$) on $\mathcal{P}^3_{QMA}$ (Kochen-Specker, 1967, p. 67). More generally, a bivalent homomorphism($\flat$) h on any n dimensional Hilbert space $\mathcal{P}^n_{QM}$ satisfies:

(KS1) For any n mutually orthogonal atoms $P_1, P_2, \ldots, P_n \in \mathcal{P}^n_{QM}$, $h(P_1) \vee h(P_2) \vee \ldots \vee h(P_n) = h(P_1 \vee P_2 \vee \ldots \vee P_n) = h(1) = 1$, and $h(P_i) \wedge h(P_j) = h(P_i \wedge P_j) = h(0) = 0$ for $1 \leq i \neq j \leq n$.

By the Lemma A of the previous section, any two atoms in a $\mathcal{P}_{QM}$ are orthogonal IFF they are compatible. By closure with respect to the $\wedge$, $\vee$, $^\perp$ operations of $\mathcal{P}_{QM}$, a set of n mutually orthogonal (i.e., compatible) atoms in a $\mathcal{P}^n_{QM}$ generates a Boolean substructure of $\mathcal{P}^n_{QM}$. In particular, such a set generates a maximal Boolean substructure of $\mathcal{P}^n_{QM}$ since the maximum number of mutually compatible atoms in a $\mathcal{P}^n_{QM}$ structure of n dimensional Hilbert space is n.[4] Thus condition (KS1) refers to the (maximal) Boolean substructures of a $\mathcal{P}^n_{QM}$ and ensures that their Boolean structural features are preserved by a bivalent homomorphism(ḃ). And just the usual definition of a bivalent homomorphism on a Boolean structure ensures that any bivalent homomorphism $h : mBS \rightarrow \{0,1\}$ satisfies (KS1). So the fact that a bivalent homomorphism(ḃ) on any P^n_{QM} by definition satisfies (KS1) does not focus attention upon that peculiarly non-Boolean structural feature of $P^{n\geq 3}_{QM}$ structure, namely, the presence of overlapping mBS's in $P^{n\geq 3}_{QM}$.

Bivalent homomorphisms(ḃ) on a $P^{n\geq 3}_{QM}$ preserve not only the Boolean structural features of every (maximal) Boolean substructure but also the partial-Boolean structural features of the entire $P^{n\geq 3}_{QM}$, in particular, the overlap patterns of the mBS's in $P^{n\geq 3}_{QM}$. One way in which the overlap patterns can be violated is by allowing different values to be assigned to a given element P which is a member of two or more overlapping mBS's; the value assigned to P in the context of one mBS may be different from the value assigned to P in the context of another mBS. Such proposed violations of the overlap patterns are further discussed in Chapter VII. A bivalent homomorphism(ḃ) on a $P^{n\geq 3}_{QM}$ does not violate the overlap patterns in this way (or in any way). For a bivalent homomorphism(ḃ) is a mapping, i.e., $h(P)$ is unique, as stated in Chapter I(G), and in particular, $h(P)$ does not depend upon which mBS is being considered. Kochen-Specker do not explicitly state this aspect of a bivalent homomorphism(ḃ); here and in Chapter VII it shall be referred to as:

(KS2) The values assigned by a bivalent homomorphism(ḃ) $h : P_{QM} \rightarrow \{0,1\}$ are unique and do not vary with or depend upon the mBS's of P_{QM}.

This aspect of the notion of a bivalent homomorphism(ḃ) is articulated, though in different terms, by Belinfante (1973, p. 41).

The particular P^3_{QMA} considered by Kochen-Specker contains 192 atoms and 118 mBS's.[5] Since each mBS in a P^3_{QMA} contains three orthogonal atoms, clearly the mBS's in Kochen-Specker's P^3_{QMA} share atoms, that is, they overlap each other in complex patterns. As mentioned above, the overlap patterns make bivalent homomorphisms(♭) on a $P^{n\geq 3}_{QM}$ impossible. I shall restate Kochen-Specker's proof of this impossibility for their P^3_{QMA} in a manner which elucidates the effect of the overlap patterns and which explicitly refers to (KS1) and (KS2).

By (KS1), any bivalent homomorphism(♭) h on P^3_{QMA} must assign the value 1 to exactly one atom in the mBS_0 specified by the three atoms Kochen-Specker label p_0, q_0, r_0. Let us initially assume that $h(p_0) = 1$ and thus $h(q_0) = h(r_0) = 0$. This initial assignment of values to the atoms in mBS_0 of course determines the values assigned to all the other non-atomic elements of mBS_0. But in addition, this initial assignment of values to the atoms in mBS_0 places restrictions upon the values assigned by h to the atoms in every mBS which overlaps with mBS_0. For example, consider an mBS which contains p_0 and two other atoms; by assumption and by (KS2), $h(p_0) = 1$. And so by (KS1), the other two atoms in every mmBS containing p_0 are each assigned the value 0 by h. These value assignments in turn determine the values assigned by h to the atoms in every mBS which overlaps with any of the mBS which overlap with mBS_0. And this process continues through the P^3_{QMA} structure until we get $h(q_0) = 1$, which contradicts the statement that $h(q_0) = 0$ (which follows by (KS1) from the initial assumption that $h(p_0) = 1$). A similar contradiction results if we instead initially assume that

$h(q_0) = 1$ and $h(p_0) = h(r_0) = 0$. And likewise a contradiction results if we initially assume that $h(r_0) = 1$ and $h(p_0) = h(q_0) = 0$. Thus, since any bivalent homomorphism(♭) on this $\mathcal{P}^3_{QMA}$ must assign the value 1 to one of the three atoms p_0, q_0, r_0 in mBS_0 yet all three attempts lead to contradiction, a bivalent homomorphism(♭) on this $\mathcal{P}^3_{QMA}$ considered by Kochen-Specker is impossible.

In other words, a bivalent, truth-functional(♭) mapping on this $\mathcal{P}^3_{QMA}$ is impossible, and so a bivalent, truth-functional(♭) semantics for this $\mathcal{P}^3_{QMA}$ is impossible.

Kochen-Specker also consider a much smaller $\mathcal{P}^3_{QMA}$ which contains 27 atoms and 16 overlapping mBS's. This $\mathcal{P}^3_{QMA}$ does admit some bivalent homomorphisms(♭), but as Kochen-Specker point out, there are two distinct atoms in this structure such that no bivalent homomorphism(♭) assigns different values to these two distinct elements. That is, the collection of bivalent homomorphisms(♭) which do exist on this $\mathcal{P}^3_{QMA}$ is not complete. So like the reducible orthomodular lattice $\mathcal{P}_{QML}$ structures discussed in the digression in the previous section, this $\mathcal{P}^3_{QMA}$ does admit some bivalent, truth-functional(♭) mappings, but it does not admit a bivalent, truth-functional(♭) semantics[6] (Kochen-Specker, 1967, p. 67).

The Kochen-Specker result is further discussed in Section (D) and in Chapter VII.

Section C. Avoiding These Impossibility Proofs

There are at least two ways of avoiding not only Theorem A but also the Kochen-Specker impossibility proof. One way is to further weaken the truth-functionality(♭) condition; another way is to restrict the domains of proposed semantic mappings on $\mathcal{P}_{QM}$ to certain <u>sub</u>structures of $\mathcal{P}_{QM}$.

With regard to the latter, for example, if the domain of the mapping m on the $\mathcal{P}^2_{QML}$ discussed in the beginning of Section (A) were restricted such that $m(P_2)$, $m(P_2^{\perp})$ are not defined when $m(P_1)$ is defined, then m would not assign contradictory values.[7] In other words, if the domain of each proposed semantic mapping on a $\mathcal{P}_{QM}$ were restricted to an mBS of $\mathcal{P}_{QM}$, then both impossibility proofs would clearly be avoided. More interestingly, semantic mappings which avoid both impossibility proofs yet whose domains are substructures of $\mathcal{P}_{QM}$ which include overlapping mBS's are described in Chapter VI; they are the quantum state-induced expectation-functions.

With regard to the first mentioned way of avoiding both impossibility proofs, Friedman and Glymour propose, for quantum $\mathcal{P}_{QML}$ structures, semantic mappings which are required to preserve the $^{\perp}$ operation and the $\leq$ relation of $\mathcal{P}_{QML}$ but are allowed to ignore the meets and joins of both compatibles and incompatibles (Friedman-Glymour, 1972). However, as shown in Chapter VI(B), the Friedman-Glymour semantic mappings are in fact bivalent and truth-functional(⫰,⫰̸) on substructures of $\mathcal{P}_{QML}$ which include overlapping mBS's; in this respect, the Friedman-Glymour mappings are exactly like the quantum state-induced expectation-functions mentioned above. So a weakening of the truth-functionality(⫰,⫰̸) condition to just $\perp$, $\leq$ preservation nevertheless ensures the preservation of the meets and joins of compatible and incompatible elements in certain substructures of $\mathcal{P}_{QML}$.

More extremely, the so-called contextual hidden-variable theories propose bivalent mappings for $\mathcal{P}_{QM}$ which are not required to preserve even the $\perp$ operation and the $\leq$ relation of P_{QM} and which avoid both impossibility proofs. This proposal is discussed in Chapter VII.

Section D. The Meaning of the Hidden-Variable Impossibility Proofs for the Issue of a Classical Semantics for the Quantum Propositional Structures

In his 1957 proof of the completeness of quantum mechanics, Gleason refers to the infinite set of all subspaces (or projectors) of a three-or-higher dimensional Hilbert space, but Gleason does not explicitly state whether the structure of such a set is an orthomodular lattice or a partial-Boolean algebra. As mentioned in Chapter IV(E), such infinite P_{QML} and corresponding infinite P_{QMA} structures have exactly the same elements, but the P_{QML} has its $\wedge$, $\vee$ operations defined among compatible and incompatible elements while the P_{QMA} has its $\wedge$, $\vee$ operations defined among only compatible elements. Nevertheless, Gleason is effectively committed to partial-Boolean algebra structures because the mapping μ which he defines on the subspaces must satisfy his additivity condition:

(Ga) For any denumerable collection $\{P_i\}_{i \in \text{Index}}$ of mutually orthogonal subspaces, $\mu(\bigvee_i P_i) = \sum_i \mu(P_i)$; for example, $\mu(P_1 \vee P_2) = \mu(P_1) + \mu(P_2)$ (Gleason, 1957, p. 885).

This additivity condition ensures that when Gleason's mappings are dispersion-free, i.e., bivalent, then the mappings preserve the unary operation and binary $\wedge$, $\vee$ operations among compatibles.[8] But the mappings do not preserve the $\wedge$, $\vee$ operations among incompatibles. In other words, dispersion-free hidden-variable mappings which satisfy (Ga) are bivalent homomorphisms(δ), and vice versa. Viewed semantically, such mappings are bivalent and truth-functional(δ).

Such mappings preserve the operations and relations of a P_{QMA}. But such mappings on a P_{QML} ignore the lattice meets and joins of incompatibles. So Gleason is effectively referring to $P^{n\geq 3}_{QMA}$ structures of subspaces, although his results also do apply to $P^{n\geq 3}_{QML}$ structures. Clearly, since dispersion-free Gleason mappings ignore the meets and joins of incompatibles, they do not run into the truth-functionality(δ,$\not\delta$) problems which are the basis of Theorem A. However, the mappings do run into truth-functionality(δ) problems. For a corollary to Gleason's completeness proof shows that proposed dispersion-free hidden-variable mappings satisfying (Ga) are impossible on the infinite set of all subspaces of a three-or-higher dimensional Hilbert space. This corollary is known as Gleason's proof of the impossibility of hidden variables.

The Kochen-Specker 1967 Theorem 1, described in Section B is a finite version of Gleason's impossibility proof which makes explicit the fact that Gleason's proof considers bivalent homomorphisms(δ) on P_{QMA} structures (although Gleason's result also applies to P_{QML} structures). Moreover, with their orthohelium example, Kochen-Specker provide a concrete, quantum mechanical realization of their finite P^{3}_{QMA} (1967, pp. 71-74). Thus Gleason's proof, which refers to all subspaces or projectors of a Hilbert space, is protected from critics who argue that only some finite set of operators in fact represent quantum magnitudes or argue that only some "essential" magnitudes need be assigned values by proposed dispersion-free hidden-variable mappings (Belinfante, 1973, pp. 48-49; Ballentine, 1970, p. 376).

In contrast to the Gleason, Kochen-Specker proofs, both the von Neumann and the Jauch-Piron impossibility proofs consider mappings which are required to preserve an operation among incompatibles, and both proofs include the case of two-dimensional Hilbert space.

In his 1932 proofs of the completeness of quantum mechanics and the impossibility of dispersion-free hidden-variable ensembles in quantum mechanics, von Neumann does not explicitly refer to bivalent, operation-preserving mappings on either P_{QMA} or P_{QML} structures of subspaces or projectors of Hilbert space. Rather, von Neumann considers expectation-functions whose domain is the (infinite) set of quantum magnitudes represented by self-adjoint operators on a Hilbert space of any dimension, and he requires that expectation-functions preserve the $+$ operation defined among the magnitudes represented by operators. However, dispersion-free expectation-functions which satisfy von Neumann's requirements can be shown to be bivalent, operation-preserving mappings on quantum propositional structures as follows.

Consider only the idempotent quantum magnitudes represented by projectors on a Hilbert space, and let Exp_W be a dispersion-free expectation-function. As described in Chapter IV(D), the structure of the projectors on a Hilbert space is a P_{QM}, with the $\wedge, \vee, ^{\perp}$ operations of P_{QM} defined in terms of the ring operations $+, \cdot,$ as usual for compatible projectors and by means of Jauch's definitions for incompatible projectors. Moreover, with respect to the idempotent magnitudes represented by projectors, the dispersion-free condition with which von Neumann characterizes his hidden-variable Exp_W mappings ensures that the mappings are bivalent, as shown by a simple proof.[9] Thus von Neumann's Exp_W mappings are bivalent mappings on $\mathcal{P}_{QM}$ structures. Furthermore, von Neumann requires any expectation-function Exp to satisfy his additivity condition, which may be split into two parts:

(vNó) For any compatible magnitudes A,B, . . . ,

$\mathrm{Exp}(A + B + \ldots) = \mathrm{Exp}(A) + \mathrm{Exp}(B) + \ldots .$

(vN$\not{ó}$) For any incompatible magnitudes A,B, . . .

$\mathrm{Exp}(A + B + \ldots) = \mathrm{Exp}(A) + \mathrm{Exp}(B) + \ldots .$[10]

In particular, an Exp must preserve the + operation among compatible and incompatible idempotent magnitudes represented by projectors. Now like condition (Ga), the condition (vNó) ensures that the bivalent Exp_w mappings preserve the unary $^{\perp}$ operation and the binary $\wedge$, $\vee$ operations among compatible projectors.[11] Thus the von Neumann Exp_w mappings are bivalent and truth-functional(ó) mappings on $\mathcal{P}_{QM}$ structures, as are the Gleason, Kochen-Specker mappings. However, von Neumann's mappings are also required to preserve the + operation among incompatibles. So considering just the idempotent magnitudes represented by projectors, von Neumann is effectively committed to something like a $\mathcal{P}_{QML}$ structure as the domain of his Exp_w mappings, because he requires his mappings to preserve a binary operation among <u>in</u>compatibles.

In their 1963 proof of the impossibility of hidden variables in quantum mechanics, Jauch-Piron do explicitly refer to dispersion-free, i.e., bivalent, mappings w on $\mathcal{P}_{QML}$ structures. The mappings are required to satisfy certain conditions, especially:

(JPó) For any elements $P_1, P_2 \in \mathcal{P}_{QML}$, if P_1 ó P_2 then

$w(P_1) + w(P_2) = w(P_1 \vee P_2) + w(P_1 \wedge P_2)$.

(JPბ̸) For any subset $\{P_i\}_{i \in \text{Index}}$ of elements in a P_{QML}, if $w(P_i) = 1$ for every $i \in$ Index, then $w(\bigwedge_i P_i) = 1$; for example, if $w(P_1) = w(P_2) = 1$, then $w(P_1 \wedge P_2) = 1$ (Jauch-Piron, 1963, p. 833).

Like (Ga) and (vNბ), the condition (JPბ) ensures that the bivalent mappings preserve the unary $\perp$ operation and the binary $\wedge$, $\vee$ operations among compatibles.[12] So the Jauch-Piron mappings are bivalent and truth-functional(ბ). But the mappings are in addition required to satisfy (JPბ̸) which involves preserving the $\wedge$ operation among compatible and incompatible elements of a P_{QML}. So Gleason, Kochen-Specker, von Neumann, and Jauch-Piron all require their proposed hidden-variable mappings to be truth-functional(ბ), but in addition, von Neumann and Jauch-Piron require their mappings to preserve an operation among incompatibles. And it is precisely these additional conditions (vNბ̸) and (JPბ̸) which allow the von Neumann and the Jauch-Piron proofs to work at all and which allow their proofs to include the two-dimensional Hilbert space case which is excluded from the Gleason, Kochen-Specker proofs.

Specifically, using the trace-formalism developed in his completeness proof, von Neumann shows that dispersion-free expectation-functions which satisfy his conditions are impossible on the (infinite) set of operators on a Hilbert space of any dimension (von Neumann, 1932, pp. 320-321). And Jauch-Piron prove in their Corollary 1 that, with respect to any irreducible P_{QML}, bivalent mappings which satisfy their conditions are impossible.[13]

Semantically interpreted, since the truth-functionality(ბ,ბ̸) condition is even stronger than the conditions imposed by either von Neumann or Jauch-Piron, their impossibility proofs suggest that in general and including the two-dimensional Hilbert space case, quantum $\mathcal{P}_{QML}$ structures do not admit bivalent, truth-functional(δ,$\not\delta$) mappings and hence do not admit a bivalent, truth-functional(δ,$\not\delta$) semantics; this is proven in Section A as Theorem A.

There is an impossibility proof by Zierler and Schlessinger involving a condition which is as strong as my truth-functionality(δ,$\not\delta$) condition.[14] In their Theorem 3.1, Zierler-Schlessinger show that if there is a strongly additive embedding m of an orthomodular partially ordered set P into a Boolean algebra, then the join $P_1 \vee P_2$ exists in P only when P_1 commutes with P_2 (i.e., $P_1 \mathrel{\delta} P_2$). A <u>strongly additive embedding</u> preserves $\leq, {}^{\perp}, \vee$, and moreover is monomorphic, i.e., if $m(P_1) \leq m(P_2)$ then $P_1 \leq P_2$ (Zierler-Schlessinger, 1964, pp. 254-255, 260).

It is easy to prove that a monomorphic mapping $m : P \rightarrow \mathcal{B}$ which preserves $\leq$ is <u>injective</u>, i.e., for any $P_1 \neq P_2$ in P, $m(P_1) \neq m(P_2)$. Proof: Assume on the contrary that $m(P_1) = m(P_2)$ in $\mathcal{B}$. Then since $\leq$ is reflexive, $m(P_1) \leq m(P_2)$ and also $m(P_2) \leq m(P_1)$. Since m is monomorphic, it follows that $P_1 \leq P_2$ and $P_2 \leq P_1$, thus $P_1 = P_2$ which contradicts $P_1 \neq P_2$. Q.E.D. And since an <u>imbedding</u> is an injective homomorphism, it follows that a strongly additive embedding of P into a $\mathcal{B}$ is in fact an imbedding (δ,$\not\delta$) of P into $\mathcal{B}$. So the contrapositive of Theorem 3.1 says: If the join $P_1 \vee P_2$ exists in P and $P_1 \mathrel{\not\delta} P_2$, then an imbedding ($\delta$,$\not\delta$) of P into a $\mathcal{B}$ is impossible. Or in other words, with respect to an orthomodular lattice $\mathcal{P}_{QML}$, which has $\vee$ defined for any $P_1, P_2 \in \mathcal{P}_{QML}$, Theorem 3.1 yields: If a $\mathcal{P}_{QML}$ contains incompatible elements, that is, if the join $P_1 \vee P_2$ of incompatible P_1, P_2 exists in $\mathcal{P}_{QML}$, then an imbedding (δ,$\not\delta$) of $\mathcal{P}_{QML}$ into a $\mathcal{B}$ is impossible.

Then assuming that there is a theorem for imbedding($\circ\!\!\!\mid$, $\not\circ\!\!\!\mid$) like the Kochen-Specker Theorem 0 for imbedding($\circ\!\!\!\mid$), i.e., an imbedding($\circ\!\!\!\mid$, $\not\circ\!\!\!\mid$) of a P_{QML} into a B exists IFF a complete collection of bivalent homomorphisms($\circ\!\!\!\mid$, $\not\circ\!\!\!\mid$) exists on P_{QML}, the above restatement of the contrapositive of Zierler-Schlessinger's Theorem 3.1 is equivalent to my Theorem A: If a P_{QML} contains incompatible elements, then a bivalent, truth-functional($\circ\!\!\!\mid$, $\not\circ\!\!\!\mid$) semantics for P_{QML} is impossible.

Summary

The general fact of the impossibility of a bivalent, truth-functional semantics for the propositional structures determined by quantum mechanics should be more subtly demarcated according to whether the structures are taken to be orthomodular lattices P_{QML} or partial-Boolean algebras P_{QMA}; according to whether the semantic mappings are required to be truth-functional($\circ\!\!\!\mid$, $\not\circ\!\!\!\mid$) or truth-functional($\circ\!\!\!\mid$); and according to whether two-or-higher dimensional Hilbert space P_{QM} structures or three-or-higher dimensional Hilbert space P_{QM} structures are being considered.[15]

If the quantum P_{QM} structures are taken to be orthomodular lattices, then bivalent mappings which preserve the operations and relations of a P_{QML} must be truth-functional($\circ\!\!\!\mid$, $\not\circ\!\!\!\mid$). Then as suggested by von Neumann and Jauch-Piron and as proven in Section A, the mere presence of incompatible elements in a P_{QML} is sufficient to rule out any semantical or hidden-variable proposal which imposes this strong condition, for any <u>two</u>-or-higher dimensional Hilbert space $P^{n\geq 2}_{\text{QML}}$ structure. Thus from the orthomodular lattice perspective, the peculiarly non-classical feature of quantum mechanics and the peculiarly non-Boolean feature of the quantum propositional structures is the existence of incompatible magnitudes and propositions.

However, the weaker truth-functionality(b) condition can instead be imposed upon the semantic or hidden-variable mappings on the $\mathcal{P}_{QML}$ structures, although such mappings ignore the lattice meets and joins of incompatibles and preserve only the partial-Boolean algebra structural features of the $\mathcal{P}_{QML}$ structures. Or alternatively, the quantum propositional structures can be taken to be partial-Boolean algebras, where bivalent mappings which preserve the operations and relations of a $\mathcal{P}_{QMA}$ need only be truth-functional(b). In either case, the Gleason, Kochen-Specker proofs show that any semantical or hidden variable proposal which imposes this truth-functionality(b) condition is impossible for any <u>three</u>-or-higher dimensional Hilbert space $\mathcal{P}^{n\geq 3}_{QMA}$ or $\mathcal{P}^{n\geq 3}_{QML}$ structures. But such semantical or hidden-variable proposals are possible for any two-dimensional Hilbert space $\mathcal{P}^{2}_{QMA}$ or $\mathcal{P}^{2}_{QML}$ structures, in spite of the presence of incompatibles, and in spite of the fact that these structures are non-Boolean in the Piron sense and in the not-prime ultrafilter sense.[16]

Notes

1 It is worth noting that these value assignments would be acceptable to the lattice theoreticians Jauch (1968, p. 76), Putnam (1969, p. 222), van Fraassen (1973, p. 90), Friedman and Glymour (1972, p. 18). For these authors do associate the 0 element of a $\mathcal{P}_{QML}$ with contradictory propositions and the 1 element with tautological propositions. So even though some of these authors do not discuss semantic proposals for $\mathcal{P}_{QML}$, all would accept the value assignments $m(P_1 \wedge P_2) = m(0) = 0$ and $m(P_1 \vee P_2) = m(1) = 1$, for any proposed semantic mapping m on a $\mathcal{P}_{QML}$. For example, Putnam explicitly discusses the conjunction of two quantum propositions associated with two incompatible, one-dimensional subspaces

whose intersect is the 0 subspace, e.g., our $P_1 \wedge P_2 = 0$ and $P_1 \wedge P_2^{\perp} = 0$. Such a conjunction is logically false, according to Putnam, and so he is committed to the value assignments $m(P_1 \wedge P_2) = 0$ and $m(P_1 \wedge P_2^{\perp}) = 0$.

[2] Thanks to Edwin Levy, L. Peter Belluce, and Richard E. Robinson for auditing these proofs. Dr. Belluce especially helped with Lemmas A and B, and he proved Lemma C, adding that it is a standard proof in Boolean lattice theory. Dr. Robinson suggested a more economical restatement of the proofs.

[3] This impossibility holds whether a semantics for a $\mathcal{P}_{QML}$ is taken to be a complete collection or a weakly complete collection of bivalent, truth-functional(δ,$\not\delta$) mappings. The notion of a weakly complete collection is defined in note 1 of Chapter II.

[4] The dimension of a Hilbert space $\mathcal{H}$ is the maximum number of linearly independent vectors in the Hilbert space (Jauch, 1968, p. 20), and is designated by the superscript n = 1,2, ... So any $\mathcal{H}^n$ has n linearly independent vectors; this set of vectors are a <u>basis</u> for $\mathcal{H}^n$ (Lande, 1972, p. 47). A basis may be orthogonalized (by the Gram-Schmidt process) and normalized (by dividing each vector by its length) yielding an <u>orthonormal</u> <u>basis</u>. Thus the maximum number of mutually orthogonal vectors in any $\mathcal{H}^n$ is n. Since each vector $|\psi\rangle$ corresponds uniquely with the one-dimensional projector $\hat{P}_\psi = |\psi\rangle\langle\psi|$ and the one-dimensional subspace H_ψ which is the range of $\hat{P}_\psi$, the maximum number of mutually orthogonal, one-dimensional projectors or subspaces of any $\mathcal{H}^n$ is n. Each one-dimensional projector or subspace is an atom in the $\mathcal{P}_{QM}$ structure of the Hilbert space, and by Lemma A, any two atoms in a $\mathcal{P}_{QM}$ are orthogonal IFF they are compatible. Thus the maximum number of mutually compatible atoms in the $\mathcal{P}^n_{QM}$ structure of any $\mathcal{H}^n$ is n. And so I claim without proof that when a set of n mutually compatible or orthogonal atoms in a $\mathcal{P}^n_{QM}$ is closed with respect to the $\wedge$, $\vee$, $^{\perp}$ operations of $\mathcal{P}_{QM}$ we obtain an mBS of $\mathcal{P}^n_{QM}$.

[5] The three orthogonal atoms in each mBS of the $\mathcal{P}^3_{QMA}$ which Kochen-Specker consider in their Theorem 1 are represented by a triangle in the completion of the graph Kochen-Specker label Γ_2 (Kochen-Specker, 1967, pp. 68-69). Each identical subportion of this graph, which Kochen-Specker draw separately as Γ_1, contains 13 points (atoms) and eight overlapping triangles (mBS's) upon completion. There are 15 such subportions in Γ_2, so the completion of Γ_2 contains 195 points and 120 triangles. However, Kochen-Specker further identify the points $p_o = a$, $q_o = b$, and $r_o = c$, so that three points and two triangles are redundant. Thus the $\mathcal{P}^3_{QMA}$ considered by Kochen-Specker contains 192 atoms and 118 mBS's.

[6] This impossibility holds whether a semantics for a P_{QMA} is taken to be a complete or a weakly complete collection of bivalent, truth-functional($\flat$) mappings.

[7] In precisely this manner, the semantic mappings proposed by I. Hacking for the quantum P_{QML} structures, namely, the evaluations, side-step the Theorem A impossibility proof. This proposal was made in an unpublished, 1974 paper which has since been rescinded.

[8] A dispersion-free Gleason mapping is bivalent, as mentioned by Gudder; a proof is given by Bub and restated in note 9 below. And Gudder proves that dispersion-free mappings satisfying Gleason's additivity condition are bivalent homomorphisms($\flat$) (Gudder, 1970, pp. 433-434). A version of this proof is restated in Chapter III(C).

[9] By definition, an expectation-function is dispersion-free IFF, for any quantum mechanical magnitude A, $Exp(A^2) = (Exp(A))^2$. So with respect to any idempotent magnitude P which by definition satisfies $P^2 = P$, $Exp(P) = (Exp(P))^2$. That is, $Exp(P) = 0$ or 1. So an Exp on a P_{QM} dispersion-free IFF, for any element $P \in P_{QM}$, $Exp(P) = 0$ or 1 (Bub, 1974, p. 60).

[10] Condition (vN$\flat$) alone is labeled (D) by von Neumann in his book. And conditions (vN$\flat$), (vN$\not\flat$) together are subsumed by one condition von Neumann labels (B') (von Neumann, 1932, pp. 309, 311).

[11] Kochen-Specker prove that dispersion-free expectation-functions which preserve the $+$ operation among compatible operators or projectors also preserve the $\cdot$ operation among compatibles (Kochen-Specker, 1967, p. 81). Since the $\wedge, \vee, ^{\perp}$ operations of a P_{QM} structure can be defined in terms of the ring operations $+, \cdot$, as usual among compatible projectors, mappings on a P_{QM} which preserve the $+, \cdot$ operations among compatibles also preserve the $\wedge, \vee, ^{\perp}$ operations among compatibles.

[12] The proof by Gudder cited in note 8 above works with condition (JP$\flat$) as well as with condition (Ga).

[13] Jauch-Piron's Corollary 1 speaks of coherent proposition systems; coherency is irreducibility and a proposition system is an orthomodular lattice (Jauch-Piron, 1963, pp. 831, 834).

[14] Zierler and Schlessinger's work was called to my attention by Prof. W. Demopoulos.

[15] Some authors do not make these distinctions. For example, M. Gardner claims that, "Kochen and Specker have proven that there is <u>no</u> homomorphism of P [i.e., P_{QMA}] into Z_2" (Gardner, 1971, p. 519). Gardner does clarify that the homomorphisms considered by Kochen-Specker are homomorphisms($\flat$). But Gardner does not mention that two-dimensional Hilbert space P^2_{QMA} structures are exempt from Kochen-Specker's proof;

that Kochen-Specker give an example of a $\mathcal{P}^3_{QMA}$ which does admit some homomorphisms(σ) into Z_2 (i.e., bivalent homomorphisms(σ)) but does not admit a complete collection of such mappings; and that the Kochen-Specker proof also applies to $\mathcal{P}_{QML}$ structures, as explained in Sections B and D.

[16] Most of this chapter is to be published as "The impossibility of a classical semantics for the quantum propositional structures," in a forthcoming issue of Philosophia.

CHAPTER VI

A STATE-INDUCED SEMANTICS FOR THE NON-BOOLEAN PROPOSITIONAL STRUCTURES DETERMINED BY QUANTUM MECHANICS

Section A. The Quantum State-induced Expectation-Functions

As described in Chapter IV(A), the quantum formalism associates a physical system with a Hilbert space $\mathcal{H}$, represents each pure state ψ of the system by a one-dimensional projector on $\mathcal{H}$, and represents each magnitude A of the system by a self-adjoint operator on $\mathcal{H}$. The expectation value of each of the system's magnitudes A,B, , . . is determined by the state ψ of the system according to the expression $\mathrm{Exp}_\psi(A) = \langle\psi|\hat{A}|\psi\rangle$, which is one of the real-number eigenvalues $\{a_i\}_{i\in \mathrm{Index}}$ of A when the state of the system is one of the eigenstates $\{\psi_i\}_{i\in \mathrm{Index}}$ of A, e.g., $\mathrm{Exp}_{\psi_j}(A) = a_j$. Thus, when the state of a system is an eigenstate of any of the system's magnitudes, then the state of the system determines the exact values of those magnitudes via the expectation-function. When the state ψ of the system is not an eigenstate of a given magnitude A, then the state determines the probabilities of that magnitude A exhibiting any one of its eigenvalues according to the expression $\rho_{\psi,A}(a_i) = \mathrm{Exp}(\hat{P}_{\psi_i})$, where $\hat{P}_{\psi_i}$ is the projector representing the eigenstate ψ_i associated with the eigenvalue a_i . So for any of the system's magnitudes, each pure state ψ of the system determines, via the expectation-function Exp_ψ , either the exact real-number value of the magnitude or the probabilities of the magnitude exhibiting any one of its exact (eigen)values. And for any pure state ψ, the expectation-function Exp_ψ is unique to ψ, and conversely, Exp_ψ unambiguously defines the state ψ (Fano, 1971, p. 399).

In fact, for any pure state ψ, the expectation-function Exp_ψ acts as a mapping from the set of magnitudes represented by operators to the real-number line, i.e., $\mathrm{Exp}_\psi : \{\hat{A},\hat{B}, \ldots\} \to \mathcal{R}$, which satisfies:

(Ea) $\mathrm{Exp}_\psi(\hat{A} + \hat{B} + \ldots) = \mathrm{Exp}_\psi(\hat{A}) + \mathrm{Exp}_\psi(\hat{B}) + \ldots$

(Eb) If $\hat{A} \geq \hat{0}$ then $\mathrm{Exp}_\psi(\hat{A}) \geq 0$.

(Ec) $\mathrm{Exp}_\psi(\hat{I}) = 1$. (Fano, 1971, p. 398; von Neumann, 1932, p. 308)

For any pure state ψ, the uniquely associated mapping $\mathrm{Exp}_\psi : \{A,B, \ldots\} \to \mathcal{R}$ may be called the quantum state-induced mapping, just as, for any pure state w of a classical system, the uniquely associated mapping $w : \{f_A,f_B, \ldots\} \to \mathcal{R}$ is called the state-induced mapping in Chapter III.

As will be shown in this section, conditions (Ea), (Eb), (Ec), ensure that, with respect to the idempotent elements of $\{\hat{A},\hat{B}, \ldots\}$ which form a $\mathcal{P}_{QM}$ structure, each Exp_ψ is a probability measure $\mathrm{Exp}_\psi : \mathcal{P}_{QM} \to [0,1]$. Classically, the analogous result is that each classical state-induced mapping $w : \{f_A,f_B, \ldots\} \to \mathcal{R}$ is a dispersion-free probability measure $\mu_w : \mathcal{P}_{CM} \to \{0,1\}$ with respect to the $\mathcal{P}_{CM}$ structure of idempotent elements of $\{f_A,f_B, \ldots\}$.

Moreover, as in the classical case described in Chapter III(C), this mathematical machinery of quantum state-induced mappings on the set of operators on a Hilbert space not only formalizes the procedure by which real-number values and probabilities are assigned to the magnitudes of a quantum system, but also implicitly formalizes a procedure by which truth-values and probabilities may be assigned to the propositions which make assertions about the real-number values of a quantum system's magnitudes, as shall be shown in this chapter.

As described in Chapter IV(C), propositions which make assertions about the values of a quantum system's magnitudes have been associated with the projectors or subspaces of the system's Hilbert space, and the logical operations among the propositions have been interpreted as or defined in terms of operations among the projectors or subspaces, yielding a propositional structure $\mathcal{P}_{QM}$. In order to describe how the state-induced mappings Exp_ψ act with respect to a $\mathcal{P}_{QM}$, we focus temporarily upon the elements of $\mathcal{P}_{QM}$ as projectors, which are by definition idempotent, self-adjoint, bounded operators whose only eigenvalues are the real-numbers 0 and 1. With respect to a $\mathcal{P}_{QM}$ of propositions qua projectors, each state-induced Exp_ψ on $\mathcal{P}_{QM}$ satisfies the five conditions which define a probability measure μ, as listed in Chapter III(C). For any Exp_ψ on a $\mathcal{P}_{QM}$ and for any $\hat{P}_1, \hat{P}_2 \in \mathcal{P}_{QM}$:

(μa) As stated in Chapter IV(D), if $\hat{P}_1 \mathrel{\acute{o}} \hat{P}_2$, then P_1 $\hat{P}_1 \vee \hat{P}_2 = \hat{P}_1 + \hat{P}_2 - \hat{P}_1 \cdot \hat{P}_2$ and $\hat{P}_1 \wedge \hat{P}_2 = \hat{P}_1 \cdot \hat{P}_2$. And if $\hat{P}_1$, $\hat{P}_2$ are disjoint, i.e., $\hat{P}_1 \leq \hat{P}_2^{\perp}$, then $\hat{P}_1 \mathrel{\acute{o}} \hat{P}_2$ and $\hat{P}_1 \wedge \hat{P}_2 = \hat{0}$ (since $\hat{P}_1 \wedge \hat{P}_2 \leq \hat{P}_2 \wedge \hat{P}_2^{\perp} = \hat{0}$ and $\hat{P} \geq \hat{0}$ for every $\hat{P} \in \mathcal{P}_{QM}$). So if $\hat{P}_1$, $\hat{P}_2$ are disjoint, then $\hat{P}_1 \vee \hat{P}_2 = \hat{P}_1 + \hat{P}_2$. Thus by (Ea), for any disjoint $\hat{P}_1$, $\hat{P}_2$, $\mathrm{Exp}_\psi(\hat{P}_1 \vee \hat{P}_2)$ $= \mathrm{Exp}_\psi(\hat{P}_1 + \hat{P}_2) = \mathrm{Exp}_\psi(\hat{P}_1) + \mathrm{Exp}_\psi(\hat{P}_2)$. And for any countable set

$\{\hat{P}_i\}_{i \in \text{Index}}$ of pairwise disjoint elements of $\mathcal{P}_{QM}$, $\text{Exp}_\psi(\vee_i P_i) = \Sigma_i \text{Exp}_\psi(P_i)$. Thus ($\mu$a) is satisfied.

(μb) Every element $\hat{P} \in \mathcal{P}_{QM}$ is by definition idempotent, i.e., $\hat{P} = \hat{P}^2$; it follows that every element is nonnegative, i.e., $\hat{P} \geq \hat{0}$ (von Neumann, 1932, p. 308). Hence by (Eb), for every $\hat{P} \in \mathcal{P}_{QM}$, $\text{Exp}_\psi(\hat{P}) \geq 0$. Moreover, since a projector is by definition a bounded operator (Fano, 1971, p. 288), $\text{Exp}_\psi(\hat{P}) \leq \infty$, for every $\hat{P} \in \mathcal{P}_{QM}$ (Fano, 1971, p. 396). So we have: $0 \leq \text{Exp}_\psi(\hat{P}) \leq \infty$, for every element $\hat{P} \in \mathcal{P}_{QM}$. Thus ($\mu$b) is satisfied.

(μc) By (Eb), if $\hat{P} = \hat{0}$ then $\text{Exp}_\psi(\hat{P}) = 0$, i.e., $\text{Exp}_\psi(\hat{0}) = 0$. Thus ($\mu$c) is satisfied.

(μi) In Halmos's discussion of the probability measure μ, he says that the isotone character of μ follows from the non-negative character of μ (Halmos, 1950, p. 37). This result for any Exp_ψ on a $\mathcal{P}_{QM}$ is shown as follows. For any $\hat{P}_1, \hat{P}_2 \in \mathcal{P}_{QM}$, $\text{Exp}_\psi(\hat{P}_1) \leq \text{Exp}_\psi(\hat{P}_1) + \text{Exp}_\psi(\hat{P}_2 \wedge \hat{P}_1^\perp)$ because $\text{Exp}_\psi(\hat{P}) \geq 0$ for any $\hat{P} \in \mathcal{P}_{QM}$, in particular, for $\hat{P} = \hat{P}_2 \wedge \hat{P}_1^\perp$. And as stated in Chapter IV(D), if $\hat{P}_1 \leq \hat{P}_2$, then $\hat{P}_1 \,\breve{\circ}\, \hat{P}_2$ (where $\hat{P}_1 \,\breve{\circ}\, \hat{P}_2$ IFF $\hat{P}_1^\perp \,\breve{\circ}\, \hat{P}_2$) and $\hat{P}_1 \wedge \hat{P}_2 = \hat{P}_1$; and so by the mutual compatibility of $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_1^\perp$ and by the definition of $\hat{I}$, $\hat{P}_2 = \hat{P}_2 \wedge \hat{I} = \hat{P}_2 \wedge (\hat{P}_1 \vee \hat{P}_1^\perp) = (\hat{P}_2 \wedge \hat{P}_1) \vee (\hat{P}_2 \wedge \hat{P}_1^\perp)$ $= \hat{P} \vee (\hat{P}_2 \wedge \hat{P}_1^\perp)$. So if $\hat{P}_1 \leq \hat{P}_2$, then $\text{Exp}_\psi(\hat{P}_2)$ $= \text{Exp}_\psi(\hat{P}_1 \vee (\hat{P}_2 \wedge \hat{P}_1^\perp))$. Moreover, since $\hat{P}_2 \wedge \hat{P}_1^\perp \leq \hat{P}_1^\perp$, $\hat{P}_1$ and $\hat{P}_2 \wedge \hat{P}_1^\perp$ are disjoint, and so by (μa): $\text{Exp}_\psi(\hat{P}_1 \vee (\hat{P}_2 \wedge \hat{P}_1^\perp))$ $= \text{Exp}_\psi(\hat{P}_1) + \text{Exp}_\psi(\hat{P}_2 \wedge \hat{P}_1^\perp)$. Hence, for any $\hat{P}_1, \hat{P}_2 \in \mathcal{P}_{QM}$, if

$\hat{P}_1 \leq \hat{P}_2$ then $\mathrm{Exp}_\psi(\hat{P}_1) \leq \mathrm{Exp}_\psi(\hat{P}_1) + \mathrm{Exp}_\psi(\hat{P}_2 \wedge \hat{P}_1^\perp) = \mathrm{Exp}_\psi(\hat{P}_2)$. So Exp_ψ is an isotone mapping, that is, (μi) is satisfied.

(μn) By (Ec), $\mathrm{Exp}_\psi(\hat{I}) = 1$, thus ($\mu$i) is satisfied. And so for every $\hat{P} \in \mathcal{P}_{QM}$, $0 \leq \mathrm{Exp}_\psi(\hat{P}) \leq 1$, that is, $\mathrm{Exp}_\psi : \mathcal{P}_{QM} \rightarrow [0,1]$.

So conditions (Ea), (Eb), (Ec), ensure that an Exp_ψ on a $\mathcal{P}_{QM}$ satisfies the five conditions which define a probability measure μ.

However, this classical probability measure μ is defined on a Boolean structure, e.g., on a $\mathcal{P}_{CM}$, while the quantum Exp_ψ is defined on a non-Boolean $\mathcal{P}_{QM}$ structure. So the quantum expectation-functions on a $\mathcal{P}_{QM}$ can be regarded as <u>generalized probability measures</u> which satisfy all the usual defining conditions of a classical probability measure but which are defined on a non-Boolean $\mathcal{P}_{QM}$ structure rather than on a Boolean structure.

The notion of a generalized probability measure on a $\mathcal{P}_{QMA}$ is defined by Bub, and a different notion of a generalized probability measure on a $\mathcal{P}_{QML}$ is defined by Jauch-Piron (Bub, 1974, p. 89; Jauch-Piron, 1963, p. 833). Bub and Jauch-Piron agree that the classical notion of a probability measure on a Boolean structure must be generalized for $\mathcal{P}_{QMA}$, $\mathcal{P}_{QML}$, in such a way that on every (maximal) Boolean substructure of $\mathcal{P}_{QMA}$, $\mathcal{P}_{QML}$, the generalized probability measure reduces to the classical probability measure μ. In addition, with respect to the entire non-Boolean $\mathcal{P}_{QMA}$, $\mathcal{P}_{QML}$, both Bub and Jauch-Piron require that a generalized probability measure be additive with respect to orthogonal elements; this additivity condition is Gleason's (Ga) stated in Chapter V(D) Bub does not state this requirement explicitly, but it is clear that he wants a generalized probability measure to satisfy (Ga). In their 1963 paper, Jauch-Piron do explicitly require their generalized probability measures to satisfy an additivity condition which amounts to (Ga), namely, the condition (JPb) stated in Chapter V(D). And elsewhere, Jauch explicitly imposes (Ga) rather than (JPb) (1976, p. 135). Any quantum Exp_ψ on a P_{QM} does satisfy (Ga). For as shown above, any Exp_ψ on a P_{QM} satisfies (μa) which is equivalent to (Ga) since disjointedness and orthogonality are equivalent notions, as stated in Chapter IV(D).[1] Besides (Ga), Jauch-Piron require their generalized probability measures on a P_{QML} to satisfy the condition (JP$\not b$) stated in Chapter V(D), and Jauch-Piron claim that the quantum Exp_ψ mappings do satisfy (JP$\not b$) (1963, p. 833). Bub does not impose this condition. The notion of a generalized probability measure is further discussed in Chapter VII; in particular, Jauch-Piron's imposition of (JP$\not b$) as part of the conditions defining a generalized probability measure is criticized.

Nevertheless, each state-induced mapping $\mathrm{Exp}_\psi : P_{QM} \to [0,1]$ is a generalized probability measure (as defined by either Bub or Jauch-Piron) on P_{QM}, just as each classical state-induced mapping $w : P_{CM} \to \{0,1\}$ is a classical probability measure $\mu_w : P_{CM} \to \{0,1\}$, as discussed in Chapter III(C). But the classical measures μ_w are dispersion-free, where a dispersion-free measure satisfies the condition (μd) which ensures bivalency, while the quantum measures Exp_ψ assign dispersive, probability values <u>between</u> 0 and 1 to some elements of P_{QM}. Moreover, unlike the classical measures which are truth-functional mappings on P_{CM}, the quantum Exp_ψ measures are not truth-functional ((b) or (b,$\not b$)) mappings on P_{QM}. Conditions (Ea) and (Ec) do ensure that the quantum measures preserve the $\perp$ operation of P_{QM}, i.e., for any Exp_ψ on a P_{QM} and for any $\hat{P} \in P_{QM}$, $\mathrm{Exp}(\hat{P}^\perp) =$, by substitution, $\mathrm{Exp}(\hat{I} - \hat{P}) =$, by (Ea), $\mathrm{Exp}(\hat{I}) - \mathrm{Exp}(\hat{P}) =$, by (Ec) and substitution, $1 - \mathrm{Exp}(\hat{P}) =$, by definition of $^{\perp}$, $(\mathrm{Exp}(\hat{P}))^{\perp}$. But the quantum measures do not always preserve the $\wedge$, $\vee$ operations of $\mathcal{P}_{QM}$. For example, consider the projectors $\hat{P}_1 = |\psi_1\rangle\langle\psi_1|$ and $\hat{P}_2 = |\psi_2\rangle\langle\psi_2|$ such that $\hat{P}_1 \between \hat{P}_2$ and $\hat{P}_1 \wedge \hat{P}_2 = \hat{0}$; and consider the pure state ψ represented by the projector $\hat{P}_\psi$ such that $\hat{P}_\psi \not\between \hat{P}_1$ and $\hat{P}_\psi \not\between \hat{P}_2$. The Exp_ψ induced by this state ψ assigns values as follows: $\mathrm{Exp}_\psi(\hat{P}_1 \wedge \hat{P}_2) = \mathrm{Exp}_\psi(\hat{0}) = 0$, but $\mathrm{Exp}_\psi(\hat{P}_1) \wedge \mathrm{Exp}_\psi(\hat{P}_2) \neq 0$ because, for each $i = 1,2$, $\mathrm{Exp}_\psi(\hat{P}_i) = \| |\psi\rangle\langle\psi_i| \|^2 \neq 0$.

However, each quantum expectation-function $\mathrm{Exp}_\psi : \mathcal{P}_{QM} \to [0,1]$ induced by the pure state ψ of a quantum system is bivalent with respect to a certain subset of elements of the $\mathcal{P}_{QM}$ structure of the system's $\mathcal{H}$, namely, the subset of elements in $\mathcal{P}_{QM}$ which are compatible with the atom P_ψ which qua projector $\hat{P}_\psi = |\psi\rangle\langle\psi|$ represents the pure state ψ which induced Exp_ψ. In particular, according to the quantum formalism, for any Exp_ψ on a $\mathcal{P}_{QM}$, $\mathrm{Exp}_\psi(\hat{P}_\psi) = \| |\psi\rangle\langle\psi| \|^2 = 1$, and $\mathrm{Exp}_\psi(\hat{P}_\psi^{\perp}) = 1 - \mathrm{Exp}_\psi(\hat{P}_\psi) = 0$. In addition, for any element $\hat{P} \in \mathcal{P}_{QM}$, if $\hat{P} \not\between \hat{P}_\psi$, then $\mathrm{Exp}_\psi(\hat{P}) \in (0,1)$; if $\hat{P} \geq \hat{P}_\psi$, then by ($\mu$i), $\mathrm{Exp}_\psi(\hat{P}) = 1$; and if $\hat{P} \leq \hat{P}_\psi^{\perp}$, then by ($\mu$i), $\mathrm{Exp}_\psi(\hat{P}) = 0$. Rewritten: For any Exp_ψ on a $\mathcal{P}_{QM}$ and for any element $P \in \mathcal{P}_{QM}$,

$$\mathrm{Exp}_\psi(P) = \left\{\begin{array}{ll} 1 & \text{if } P \geq P_\psi \\ 0 & \text{if } P \leq P_\psi^{\perp} \\ \in(0,1) & \text{if } P \not\between P_\psi \end{array}\right\}.$$

So each quantum expectation-function $\mathrm{Exp}_\psi : \mathcal{P}_{QM} \to [0,1]$ is bivalent with respect to the subset $\{P \in \mathcal{P}_{QM} : P \geq P_\psi \text{ or } P \leq P_\psi^{\perp}\}$; by Lemma A of Chapter V(A), this is the subset of elements in $\mathcal{P}_{QM}$ which are compatible with P_ψ. And each quantum Exp_ψ assigns probability values between 0 and 1 to all other elements in $\mathcal{P}_{QM}$, i.e., to all elements in $\mathcal{P}_{QM}$ which are incompatible with P_ψ, the atom which qua projector $\hat{P}_\psi$

represents the state ψ which induced Exp_ψ .

In the next section, I shall show that, for any atom $P_\psi \in \mathcal{P}_{QM}$, the subset $\{P \in \mathcal{P}_{QM} : P \geq P_\psi \text{ or } P \leq P_\psi^\perp\}$ is a closed substructure of $\mathcal{P}_{QM}$, and the expectation-function Exp_ψ is not only bivalent but also truth-functional ((⫰) or (⫰,⫰̸)) on that substructure of $\mathcal{P}_{QM}$.

Section B. The Quantum Expectation-Function As an Ultravaluation on an Ultrasubstructure of $\mathcal{P}_{QM}$

As described in Chapter I(F), the notions of a filter and dual ideal are defined in a $\mathcal{P}_{QML}$ by the conditions (a), (b), and the dual conditions (a'), (b'), listed in Chapter I(C). To define these notions in a $\mathcal{P}_{QMA}$ which has the $\wedge$, $\vee$ operations defined among only compatibles, conditions (a) and (a') are modified to the conditions (a_H) and (a'_H) given in Chapter I(F); nevertheless, any filter in a $\mathcal{P}_{QMA}$ still satisfies the unmodified conditions (a) and (b), and any ideal in a $\mathcal{P}_{QMA}$ still satisfies (a') and (b'). As in the case of a Boolean structure, an ultrafilter (ultraideal) in a quantum $\mathcal{P}_{QM}$ is a proper filter (ideal) which is not the proper subset of any proper filter (ideal) in $\mathcal{P}_{QM}$.

Making use of Lemma B of Chapter V(A), it is easy to prove that the subset of elements $\{P \in \mathcal{P}_{QML} : P \geq P_\psi\}$, for any given atom $P_\psi \in \mathcal{P}_{QML}$, is an ultrafilter UF_ψ in $\mathcal{P}_{QML}$. For any $P_1, P_2 \in \mathcal{P}_{QML}$, if P_1 , P_2 are members of the set $S = \{P \in \mathcal{P}_{QML} : P \geq P_\psi\}$, i.e., $P_1 \geq P_\psi$ and $P_2 \geq P_\psi$, then $P_1 \wedge P_2 \geq P_\psi \wedge P_2 = P_\psi$ and so $P_1 \wedge P_2 \in S$; thus S satisfies (a). For any $P_1 \in \mathcal{P}_{QML}$ and for any $P_2 \in S$ (i.e., $P_2 \geq P_\psi$), if $P_1 \geq P_2$, then since $P_2 \geq P_\psi$ we have $P_1 \geq P_\psi$, and so $P_1 \in S$; thus S satisfies (b). So S is a filter in $\mathcal{P}_{QML}$. Moreover, S is a proper filter, that is, $S \neq \mathcal{P}_{QML}$, e.g., $0 \notin S$ since $0 \ngeq P_\psi$. And finally, S is not the proper subset of any proper filter in $\mathcal{P}_{QML}$. For assume on the contrary that there is a proper filter F in $\mathcal{P}_{QML}$ such that $S \subset F$. Then there is an element $P \in \mathcal{P}_{QML}$ such that $P \in F$ but $P \notin S$, i.e., $P \not\geq P_\psi$. Since $P_\psi \in S \subset F$, both P_ψ, P are members of F and so by (a), $P \wedge P_\psi \in F$. But since $P \not\geq P_\psi$, by Lemma B, $P \wedge P_\psi = 0$. Thus, $0 \in F$, and so by (b), $F = \mathcal{P}_{QML}$, which contradicts the assumption that F is a proper filter in $\mathcal{P}_{QML}$. Q.E.D.

The proof that the subset of elements $\{P \in \mathcal{P}_{QML} : P \leq P_\psi^\perp\}$ is an ultraideal UI_ψ in $\mathcal{P}_{QML}$ proceeds dually.

Similarly, the subset of elements $\{P \in \mathcal{P}_{QMA} : P \geq P_\psi\}$, for any given atom $P_\psi \in \mathcal{P}_{QMA}$, is an ultrafilter UF_ψ in $\mathcal{P}_{QMA}$, as shown next. For any $P_1, P_2 \in \mathcal{P}_{QMA}$, if P_1, P_2 are members of the set $S = \{P \in \mathcal{P}_{QMA} : P \geq P_\psi\}$, i.e., $P_1 \geq P_\psi$ and $P_2 \geq P_\psi$, then there is a $d \in S$ such that $d \leq P_1$ and $d \leq P_2$, namely, $d = P_\psi$; thus S satisfies (a_H). For any $P_1 \in \mathcal{P}_{QMA}$ and for any $P_2 \in S$ (i.e., $P_2 \geq P_\psi$), if $P_1 \geq P_2$, then since $P_2 \geq P_\psi$ we have $P_1 \geq P_\psi$, and so $P_1 \in S$; thus S satisfies (b). So S is a filter in $\mathcal{P}_{QMA}$. Moreover, S is a proper filter, that is, $S \neq \mathcal{P}_{QMA}$, e.g., $0 \notin S$ since $0 \not\geq P_\psi$. And finally, S is not the proper subset of any proper filter in $\mathcal{P}_{QMA}$. For assume on the contrary that there is a proper filter F in $\mathcal{P}_{QMA}$ such that $S \subset F$. Then there is an element $P \in \mathcal{P}_{QMA}$ such that $P \in F$ but $P \notin S$, i.e., $P \not\geq P_\psi$. Since $P_\psi \in S \subset F$, both P_ψ, P are members of F and so by (a_H), there is a $d \in F$ such that $d \leq P_\psi$ and $d \leq P$. Since P_ψ is an atom, only $0 \leq P_\psi$ and $P_\psi \leq P_\psi$. However, $P_\psi \not\leq P$, and so by (a_H), $0 \in F$. But then by (b), $F = \mathcal{P}_{QMA}$, which contradicts the assumption that F is a proper filter in $\mathcal{P}_{QMA}$. Q.E.D.

The proof that the subset of elements $\{P \in \mathcal{P}_{QMA} : P \leq P_\psi^\perp\}$ is an ultraideal UI_ψ in $\mathcal{P}_{QMA}$ proceeds dually.

Any such ultrafilter UF_ψ and dual ultraideal UI_ψ defined with respect to an atom of $\mathcal{P}_{QM}$ is called a principal ultrafilter and a principal ultraideal, respectively, as mentioned in Chapter I(C) and (F). In the case of an infinite dimensional Hilbert space $\mathcal{P}_{QM}$, not every ultrafilter and not every dual ultraideal is principle.[2] Nevertheless, since a quantum pure state, as represented by a vector in Hilbert space, is an atom in the $\mathcal{P}_{QM}$ structure of Hilbert space, each (pure) state-induced mapping <u>is</u> defined with respect to a principle ultrafilter and dual principle ultraideal in $\mathcal{P}_{QM}$. So we need only consider principle ultrafilters, labeled UF_ψ, and principle ultraideals, labeled UF_ψ, in this discussion of a state-induced semantics for a $\mathcal{P}_{QM}$.

As mentioned above, any filter in a $\mathcal{P}_{QML}$ by definition satisfies (a), (b), and any ideal in a $\mathcal{P}_{QML}$ by definition satisfies (a'), (b'). Any filter in a $\mathcal{P}_{QMA}$ by definition satisfies (b) and also satisfies (a), as shown in Chapter I(F), and any ideal in a $\mathcal{P}_{QMA}$ by definition satisfies (b') and also satisfies (a'). Moreover, it is easy to show that any ultrafilter UF_ψ and dual ultraideal UI_ψ in a $\mathcal{P}_{QM}$ satisfy the conditions (c) and (c') stated in Chapter I(C):

(c) For any $P \in \mathcal{P}_{QM}$, $P^\perp \in UF_\psi$ IFF $P \in UI_\psi$.

(c') For any $P \in \mathcal{P}_{QM}$, $P^\perp \in UI_\psi$ IFF $P \in UF_\psi$.

Proof: For any $P \in \mathcal{P}_{QM}$, $P^\perp \in UF_\psi$ IFF $P^\perp \geq P_\psi$ IFF $P_\psi^\perp \geq P$ IFF $P \in UI_\psi$. And for any $P \in \mathcal{P}_{QM}$, $P^\perp \in UI_\psi$ IFF $P^\perp \leq P_\psi^\perp$ IFF $P_\psi \leq P$ IFF $P \in UF_\psi$. Q.E.D.

These conditions (a), (a'), (b), (b'), (c), (c'), ensure that, for any atom $P_\psi \in \mathcal{P}_{QM}$, the union $UF_\psi \cup UI_\psi = \{P \in \mathcal{P}_{QM} : P \geq P_\psi \text{ or } P \leq P_\psi^\perp\}$

is closed with respect to the $\wedge$, $\vee$, $^{\perp}$ operations of $\mathcal{P}_{QM}$, as shown next.[3] For any elements $P_1, P_2 \in \mathcal{P}_{QM}$, if both $P_1, P_2 \in UF_{\psi}$, then by (a) $P_1 \wedge P_2 \in UF_{\psi}$, by (b) $P_1 \vee P_2 \in UF_{\psi}$ (since $P_1 \leq P_1 \vee P_2$), and by (c') $P_1^{\perp} \in UI_{\psi}$ and $P_2^{\perp} \in UI_{\psi}$. If both $P_1, P_2 \in UI_{\psi}$, then by (a') $P_1 \vee P_2 \in UI_{\psi}$, by (b') $P_1 \wedge P_2 \in UI_{\psi}$ (since $P_1 \wedge P_2 \leq P_1$), and by (c) $P_1^{\perp} \in UF_{\psi}$ and $P_2^{\perp} \in UF_{\psi}$. If $P_1 \in UF_{\psi}$ and $P_2 \in UI_{\psi}$, then by (b) $P_1 \vee P_2 \in UF_{\psi}$, by (b') $P_1 \wedge P_2 \in UI_{\psi}$, by (c') $P_1^{\perp} \in UI_{\psi}$, and by (c) $P_2^{\perp} \in UF_{\psi}$. Since a filter F and an ideal I are each by definition nonempty and since, for any $P \in F$, $P \leq 1$, and for any $P \in I$, $P \geq 0$, it follows by (b) that the 1 element of $\mathcal{P}_{QM}$ is a member of UF_{ψ}, and it follows by (b') that the 0 element of $\mathcal{P}_{QM}$ is a member of UI_{ψ}. In other words, letting US_{ψ} label the union $UF_{\psi} \cup UI_{\psi}$, we have $0 \in US_{\psi}$, $1 \in US_{\psi}$, and for any elements $P_1, P_2 \in \mathcal{P}_{QM}$, if $P_1, P_2 \in US_{\psi}$, then $P_1 \wedge P_2 \in US_{\psi}$, $P_1 \vee P_2 \in US_{\psi}$, and $P_1^{\perp}, P_2^{\perp} \in US_{\psi}$. Thus, for any atom $P_{\psi} \in \mathcal{P}_{QM}$, the subset $US_{\psi} = UF_{\psi} \cup UI_{\psi}$ is a closed substructure of $\mathcal{P}_{QM}$ which may be called an <u>ultrasubstructure</u>.[4] Specifically, US_{ψ} is a <u>subalgebra</u> of $\mathcal{P}_{QMA}$, and US_{ψ} is a <u>sublattice</u> of $\mathcal{P}_{QML}$. This result is analogous to the result: In any Boolean structure $\mathcal{B}$ (algebra or lattice), the union of a filter and dual ideal form a substructure (subalgebra or sublattice) of $\mathcal{B}$ (Bell and Slomson, 1969, p. 17).

However, it is important to note that this closure of $US_{\psi} = UF_{\psi} \cup UI_{\psi}$ with respect to the $\wedge$, $\vee$, $^{\perp}$ operations of $\mathcal{P}_{QM}$ guarantees for any elements $P_1, P_2 \in \mathcal{P}_{QM}$, <u>neither</u> that if $P_1 \vee P_2 \in US_{\psi}$ then $P_1 \in US_{\psi}$ or $P_2 \in US_{\psi}$, <u>nor</u> that if $P_1 \wedge P_2 \in US_{\psi}$ then $P_1 \in US_{\psi}$ or $P_2 \in US_{\psi}$. For any US_{ψ} in a $\mathcal{P}_{QM}$, such meets and joins which are themselves members of US_{ψ} but whose constituent elements P_1, P_2 are not both members of US_{ψ} are hereafter called <u>US_{ψ}-extra meets and joins</u>.

It is also worth noting that, for any atom P_ψ in a $\mathcal{P}_{QM}$, the ultrastructure US_ψ is the union of all the Boolean substructures in $\mathcal{P}_{QM}$ which contain P_ψ, and in particular, US_ψ is the union of all the overlapping mBS's in $\mathcal{P}_{QM}^{n\geq 3}$ which contain P_ψ. As mentioned in the previous section, by Lemma A, $\{P \in \mathcal{P}_{QM} : P \geq P_\psi \text{ or } P \leq P_\psi^\perp\}$ $= \{P \in \mathcal{P}_{QM} : P \,\check{\circ}\, P_\psi\}$, that is, US_ψ is the (unique) subset of all elements in $\mathcal{P}_{QM}$ which are compatible with P_ψ. Let $mBS_{\psi,i}$ be any mBS in $\mathcal{P}_{QM}$ which contains P_ψ, and let $\bigcup_i mBS_{\psi,i}$ be the union of all such $mBS_{\psi,i}$'s in $\mathcal{P}_{QM}$. It is easy to show that, for any given atom $P_\psi \in \mathcal{P}_{QM}$ and for every element $P \in \mathcal{P}_{QM}$, $P \in US_\psi$ IFF $P \in \bigcup_i mBS_{\psi,i}$. If $P \in US_\psi$, then $P \,\check{\circ}\, P_\psi$ and so the set of elements $\{P, P^\perp, P_\psi, P_\psi^\perp, 0, 1\}$ form a Boolean substructure in $\mathcal{P}_{QM}$ which contains P_ψ and which, by Zorn's lemma, is itself contained in some maximal Boolean substructure $mBS_{\psi,i}$ which contains P_ψ; thus $P \in \bigcup_i mBS_{\psi,i}$. Conversely, if $P \in \bigcup_i mBS_{\psi,i}$, then $P \,\check{\circ}\, P_\psi$, and so $P \in US_\psi$. Q.E.D. So for any given atom $P_\psi \in \mathcal{P}_{QM}$ and for every $mBS_{\psi,i}$ containing P_ψ, $mBS_{\psi,i} \subseteq US_\psi \subseteq \mathcal{P}_{QM}$. In particular, all the elements in an $mBS_{\psi,i}$ are compatible with P_ψ and are also mutually compatible, while all the elements in US_ψ are compatible with P_ψ but need not be mutually compatible.[5] Since, as described in Chapter IV(F), the mBS's in a two-dimensional Hilbert space $\mathcal{P}_{QM}^2$ do not overlap, e.g., any atom $P_\psi \in \mathcal{P}_{QM}^2$ is a member of only one mBS in $\mathcal{P}_{QM}^2$, $US_\psi = \bigcup_i mBS_{\psi,i} = mBS_\psi$. That is, an ultrasubstructure in a $\mathcal{P}_{QM}^2$ is always just a maximal Boolean substructure of $\mathcal{P}_{QM}^2$. But since the mBS's in a three- or higher-dimensional Hilbert space $\mathcal{P}_{QM}^{n\geq 3}$ may overlap, e.g., any atom $P_\psi \in \mathcal{P}_{QM}^{n\geq 3}$ may be a member of many mBS's in $\mathcal{P}_{QM}^{n\geq 3}$, $US_\psi = \bigcup_i mBS_{\psi,i}$ may be larger than any $mBS_{\psi,i}$. That is, an ultrasubstructure in a

$P_{QM}^{n\geq 3}$ may contain incompatible elements and thus may in some sense be a <u>non</u>-Boolean substructure of $P_{QM}^{n\geq 3}$.

As stated in the previous section, any Exp_{ψ} on a P_{QM} assigns values as follows: For any $P \in P_{QM}$, $Exp_{\psi}(P) = \begin{cases} 1 & \text{if } P \geq P_{\psi} \\ 0 & \text{if } P \leq P_{\psi}^{\perp} \\ \in(0,1) & \text{if } P \not\circlearrowright P_{\psi} \end{cases}$.

Since $UF_{\psi} = \{P \in P_{QM} : P \geq P_{\psi}\}$, $UI_{\psi} = \{P \in P_{QM} : P \leq P_{\psi}^{\perp}\}$, and $US_{\psi} = \{P \in P_{QM} : P \circlearrowright P_{\psi}\}$, it follows that any Exp_{ψ} on a P_{QM} assigns values as follows: For any $P \in P_{QM}$, $Exp_{\psi}(P) = \begin{cases} 1 & \text{if } P \in UF_{\psi} \\ 0 & \text{if } P \in UI_{\psi} \\ \in(0,1) & \text{if } P \notin US_{\psi} = UF_{\psi} \cup UI_{\psi} \end{cases}$.

This result suggests that each Exp_{ψ} on a P_{QM} is an ultravaluation on the ultrasubstructure US_{ψ}. (Hereafter, an Exp_{ψ} and its US_{ψ} may be said to be <u>affiliated</u>.) Of course, an Exp_{ψ} is bivalent with respect to the elements in US_{ψ}. Moreover, it shall be shown below that an Exp_{ψ} is truth-functional (($\circlearrowright$) or ($\circlearrowright$,$\not\circlearrowright$)) with respect to the elements in US_{ψ}. Thus an Exp_{ψ} is a bivalent, truth-functional (($\circlearrowright$) or ($\circlearrowright$,$\not\circlearrowright$)) mapping on US_{ψ} defined with respect to the ultrafilter UF_{ψ} and the dual ultraideal UI_{ψ}, that is, an Exp_{ψ} is an ultravaluation on the affiliated ultrasubstructure US_{ψ}.

The conditions (a), (a'), (b), (b'), (c), (c'), satisfied by any UF_{ψ} and dual UI_{ψ} in a P_{QM} yield the following biconditionals and conditionals. For any UF_{ψ} and dual UI_{ψ} in a P_{QM}, for any $P \in P_{QM}$, and for any $P_1,P_2 \in P_{QM}$ (<u>qua</u> P_{QML}), for any $P_1 \circlearrowright P_2 \in P_{QM}$ (<u>qua</u> P_{QMA}):

U1 $\quad P_1 \wedge P_2 \in UF_{\psi}$ IFF $P_1 \in UF_{\psi}$ and $P_2 \in UF_{\psi}$, by (a) and (b);

$P_1 \wedge P_2 \in UI_\psi$ IF $P_1 \in UI_\psi$ or $P_2 \in UI_\psi$, by (b');

U2 $P_1 \vee P_2 \in UF_\psi$ IF $P_1 \in UF_\psi$ or $P_2 \in UF_\psi$, by (b);

$P_1 \vee P_2 \in UI_\psi$ IFF $P_1 \in UI_\psi$ and $P_2 \in UI_\psi$, by (a') and (b');

U3 $P^\perp \in UF_\psi$ IFF $P \in UI_\psi$, by (c);

$P^\perp \in UI_\psi$ IFF $P \in UF_\psi$, by (c').

It clearly follows that the Exp_ψ on $\mathcal{P}_{QM}$ which assigns the value 1 to the elements in UF_ψ and assigns the value 0 to the elements in UI_ψ satisfies all of the conditions TF1, TF2, TF3, which define a truth-functional mapping and are listed in Chapter II(C), except the following two: If $Exp_\psi(P_1 \vee P_2) = 1$, then $Exp_\psi(P_1) = 1$ or $Exp_\psi(P_2) = 1$; if $Exp_\psi(P_1 \wedge P_2) = 0$, then $Exp_\psi(P_1) = 0$ or $Exp_\psi(P_2) = 0$. These two conditionals are missing from the list of conditions satisfied by Exp_ψ because the following two conditionals are missing from the list of conditions U1, U2, U3, satisfied by UF_ψ , UI_ψ : If $P_1 \vee P_2 \in UF_\psi$, then $P_1 \in UF_\psi$ or $P_2 \in UF_\psi$. If $P_1 \wedge P_2 \in UI_\psi$, then $P_1 \in UI_\psi$ or $P_2 \in UI_\psi$. These two conditionals in fact characterize a prime ultrafilter and a prime ultraideal, respectively, as shall be discussed next.

Using the definition stated in Chapter I(C), we shall say that an ultrafilter UF_ψ in a $\mathcal{P}_{QM}$ is <u>prime</u> IFF, for any $P_1, P_2 \in \mathcal{P}_{QM}$,

(d) If $P_1 \vee P_2 \in UF_\psi$, then $P_1 \in UF_\psi$ or $P_2 \in UF_\psi$.

If we take $\mathcal{P}_{QM}$ to be a $\mathcal{P}_{QMA}$ and if $P_1 \not\!\!\!\not\circ P_2$, then $P_1 \vee P_2$ is not defined and so trivially, the antecedent of (d) does not obtain, So no special provision is made for $\mathcal{P}_{QMA}$. Dually, an ultraideal UI_ψ in a $\mathcal{P}_{QM}$ is <u>prime</u> IFF, for any $P_1, P_2 \in \mathcal{P}_{QM}$, (d') If $P_1 \wedge P_2 \in UI_\psi$, then $P_1 \in UI_\psi$ or $P_2 \in UI_\psi$.

Every ultrafilter (ultraideal) in a Boolean structure is prime. But as stated without proof in Chapter IV(F), if a $\mathcal{P}_{QM}$ contains incompatible elements, then there is some ultrafilter in $\mathcal{P}_{QM}$ which is not prime; i.e., if a $\mathcal{P}_{QM}$ contains incompatible elements, then not every ultrafilter in $\mathcal{P}_{QM}$ is prime. This claim shall be proven with the help of the following propositions.

<u>Proposition A</u>: For any UF_ψ in a $\mathcal{P}_{QM}$, if UF_ψ is prime, then $UF_\psi \cup UI_\psi = \mathcal{P}_{QM}$. In other words, for any UF_ψ in a $\mathcal{P}_{QM}$, if UF_ψ satisfies (d), then, for any $P \in \mathcal{P}_{QM}$, either $P \in UF_\psi$ or $P \in UI_\psi$ (where UI_ψ is the ultraideal dual to UF_ψ).

For any UF_ψ in a $\mathcal{P}_{QM}$, and for any $P \in \mathcal{P}_{QM}$, either $P \in UF_\psi$ or $P \notin UF_\psi$. Assuming that UF_ψ satisfies (d), $P \notin UF$ implies $P^\perp \in UF$. For $P \vee P^\perp = 1 \in UF_\psi$, and so by (d), either $P \in UF_\psi$ or $P^\perp \in UF_\psi$; so if $P \notin UF$ then $P^\perp \in UF_\psi$. And by (c), $P^\perp \in UF_\psi$ implies $P \in UI_\psi$. So for any $P \in \mathcal{P}_{QM}$, either $P \in UF_\psi$ or $P \in UI_\psi$. Q.E.D.

<u>Proposition B</u>: If all the atoms in a $\mathcal{P}_{QMA}$ are mutually compatible, then every element $P \neq 0$ in $\mathcal{P}_{QMA}$ is the join of the atoms it dominates.

Let P_i be any atom in $\mathcal{P}_{QMA}$ such that $P_i \leq P$, and let $\bigvee_i P_i$ be the (finite or infinite) join of all such atoms. (This join is defined because by assumption, all the atoms in $\mathcal{P}_{QMA}$ are mutually compatible.) The rest of the proof proceeds exactly as the proof of Lemma C in Chapter V(A), with P_{QMA} substituted for P_{QML} .

Now the claim stated above may be proven as follows.

Theorem B: If a P_{QM} contains incompatible elements, then not every ultrafilter UF_ψ in P_{QM} is prime.

Proof: Assume on the contrary that P_{QM} contains incompatible elements and every ultrafilter UF_ψ in P_{QM} is prime. Then by Proposition A, for every UF_ψ in P_{QM} , $UF_\psi \cup UI_\psi = P_{QM}$, where $UF_\psi \cup UI_\psi = US_\psi$ $= \{P \in P_{QM} : P \mathrel{\rotatebox[origin=c]{180}{$\text{\textperthousand}$}} P_\psi\}$ for some atom $P_\psi \in P_{QM}$. Thus each atom P_ψ in P_{QM} is compatible with every element in P_{QM} ; in particular, each atom is compatible with every other atom in P_{QM} , that is, the atoms in P_{QM} are mutually compatible. It follows that the set of atoms in P_{QM} generates a Boolean substructure when closed with respect to the $\wedge, \vee, {}^\perp$ operations of P_{QM} , for as stated in Chapter I(D), (E), (F), any set of mutually compatible elements in a P_{QM} generate a Boolean substructure when closed with respect to the operations of P_{QM} . Moreover, for P_{QM} qua P_{QML} , by Lemma C of Chapter V(A), every element $P \neq 0$ in P_{QML} is the join of the atoms it dominates. And similarly, for P_{QM} qua P_{QMA} , by Proposition B, every element $P \neq 0$ in P_{QMA} is the join of the atoms it dominates, where all the atoms in P_{QMA} are mutually compatible. Thus every element P in P_{QM} is a member of the Boolean substructure generated by closing the set of atoms in P_{QM} with respect to the $\wedge, \vee, {}^\perp$ operations of P_{QM} . And so all elements in P_{QM} are mutually compatible, which contradicts the assumption that P_{QM} contains incompatible elements. Q.E.D.

Proposition A and Theorem B, with "ultraideal UI_ψ" interchanged with "ultrafilter UF_ψ," can also be proven. In short, any P_{QM} which contains incompatible elements contains an ultrafilter which does not satisfy (d) and contains an ultraideal which does not satisfy (d'), and thus contains an ultrasubstructure $US_\psi = UF_\psi \cup UI_\psi$ which is a proper subset of P_{QM}.

However, for any $US_\psi \subset P_{QM}$, if we restrict our attention to the elements of P_{QM} which are in US_ψ, then we do have, for any $P_1, P_2 \in US_\psi = UF_\psi \cup UI_\psi$:

(d) If $P_1 \vee P_2 \in UF_\psi$, then $P_1 \in UF_\psi$ or $P_2 \in UF_\psi$.

(d') If $P_1 \wedge P_2 \in UI_\psi$, then $P_1 \in UI_\psi$ or $P_2 \in UI_\psi$.

Proof: Assume on the contrary that $P_1 \vee P_2$ is a member of the ultrafilter UF_ψ but $P_1 \notin UF_\psi$ and $P_2 \notin UF_\psi$. Then since $P_1, P_2 \in US_\psi = UF_\psi \cup UI_\psi$, $P_1, P_2 \in UI_\psi$. So by (c), $P_1^\perp, P_2^\perp \in UF_\psi$, and so by (a), $P_1^\perp \wedge P_2^\perp = (P_1 \vee P_2)^\perp \in UF_\psi$. (If $P_1 \vee P_2$ is defined in P_{QMA}, then $P_1 \,\dot{\circ}\, P_2$, and it follows that P_1, $P_1^\perp$, P_2, $P_2^\perp$, are mutually compatible and so their meets and joins are all defined in P_{QMA}.) Then by (a) again, $0 = (P_1 \vee P_2)^\perp \wedge (P_1 \vee P_2) \in UF_\psi$. And so by (b), $UF_\psi = P_{QM}$, which contradicts the assumption that UF_ψ is an ultrafilter, which is a proper filter, in P_{QM}. The proof of (d') proceeds dually. Q.E.D.

It is noteworthy that if we take US_ψ to be an improper substructure of P_{QM}, i.e., $US_\psi = P_{QM}$ rather than $US_\psi \subset P_{QM}$, then the above works as a proof of the converse of Proposition A: For any UF_ψ in a P_{QM}, if $UF_\psi \cup UI_\psi = P_{QM}$, then UF_ψ is prime. Proof: Assume on the contrary that, $UF_\psi \cup UI_\psi = P_{QM}$, and for any $P_1, P_2 \in P_{QM}$, $P_1 \vee P_2 \in UF_\psi$

but $P_1 \notin UF_\psi$ and $P_2 \notin UF_\psi$. Then since $\mathcal{P}_{QM} = UF_\psi \cup UI_\psi$, $P_1, P_2 \in UI_\psi$. The rest of the proof continues as above to the end of the penultimate sentence. Q.E.D. Thus we have: For any UF_ψ in a $\mathcal{P}_{QM}$, UF_ψ is prime IFF $UF_\psi \cup UI_\psi = \mathcal{P}_{QM}$. And equivalently, for any UF_ψ in a $\mathcal{P}_{QM}$, UF_ψ is not prime IFF $UF_\psi \cup UI_\psi \neq \mathcal{P}_{QM}$, i.e., IFF $UF_\psi \cup UI_\psi$ is a proper substructure of $\mathcal{P}_{QM}$.

Nevertheless, the point of the proof given in the paragraph preceding the previous paragraph is to show that, even when $UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$, i.e., even when UF_ψ and UI_ψ are each not prime in $\mathcal{P}_{QM}$, with respect to the elements in the ultrasubstructure $US_\psi = UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$, UF_ψ does satisfy (d) and UI_ψ does satisfy (d'), and so UF_ψ and UI_ψ may each be said to be prime with respect to US_ψ . Concordantly, for any atom $P_\psi \in \mathcal{P}_{QM}$, even when the state-induced expectation function Exp_ψ on $\mathcal{P}_{QM}$ does not satisfy all of the conditions listed as TF1, TF2, TF3, nevertheless, with respect to the ultrasubstructure $US_\psi \subset \mathcal{P}_{QM}$, Exp_ψ does satisfy all the conditions: For any $P \in US \subset \mathcal{P}_{QM}$, for any $P_1, P_2 \in US_\psi \subset \mathcal{P}_{QM}$ (<u>qua</u> $\mathcal{P}_{QML}$), for any $P_1 \text{\d{o}} P_2 \in US_\psi \subset \mathcal{P}_{QM}$ (<u>qua</u> $\mathcal{P}_{QMA}$):

TF1 $\quad Exp_\psi(P_1 \vee P_2) = 1$ IFF $Exp_\psi(P_1) = Exp_\psi(P_2) = 1$

$\quad\quad Exp_\psi(P_1 \vee P_2) = 0$ IFF $Exp_\psi(P_1) = 0$ or $Exp_\psi(P_2) = 0$

TF2 $\quad Exp_\psi(P_1 \wedge P_2) = 1$ IFF $Exp_\psi(P_1) = 1$ or $Exp_\psi(P_2) = 1$

$\quad\quad Exp_\psi(P_1 \wedge P_2) = 0$ IFF $Exp_\psi(P_1) = Exp_\psi(P_2) = 0$

TF3 $\quad Exp_\psi(P^\perp) = 1$ IFF $Exp_\psi(P) = 0$

$\quad\quad Exp_\psi(P^\perp) = 0$ IFF $Exp_\psi(P) = 1$

Thus Exp_ψ is an ultravaluation on the ultrasubstructure US_ψ ; that is, Exp_ψ , as defined with respect to UF_ψ and the dual UI_ψ , is a bivalent, truth-functional ((δ) or $(\delta,\not\delta)$) mapping on $US_\psi = UF_\psi \cup UI_\psi$. More exactly, each quantum state-induced Exp_ψ on a $\mathcal{P}_{QMA}$ truth-functionally (δ) assigns 0, 1 values to the elements in its affiliated US_ψ, which is a subalgebra of $\mathcal{P}_{QMA}$. And each quantum state-induced Exp_ψ on a $\mathcal{P}_{QML}$ truth-functionally $(\delta,\not\delta)$ assigns 0, 1 values to the elements in its affiliated ultrasubstructure US_ψ, which is a sublattice of $\mathcal{P}_{QML}$.

The truth-functional $(\delta,\not\delta)$ character of Exp_ψ on the domain $US_\psi \subseteq \mathcal{P}_{QML}$ may seem surprising in the light of the Chapter V(A) description of the truth-functionality $(\delta,\not\delta)$ problems caused by the meets and joins of incompatible elements in $\mathcal{P}_{QML}$. Yet Exp_ψ satisfies TF1, TF2, TF3, for <u>any</u> elements P_1, P_2 in $US_\psi \subseteq \mathcal{P}_{QML}$. Thus Exp_ψ preserves the $\wedge$, $\vee$ operations of $\mathcal{P}_{QML}$ among any compatible and incompatible elements in US_ψ; in other words, Exp_ψ is truth-functional $(\delta,\not\delta)$ on $US_\psi \subseteq \mathcal{P}_{QML}$.

As mentioned in Chapter V(C), Friedman and Glymour propose, for the quantum $\mathcal{P}_{QML}$ structures, semantic mappings which are required to preserve the $^\perp$ operation and the $\leq$ relation of $\mathcal{P}_{QML}$ but are not required to preserve the $\wedge$, $\vee$ operations among either compatible or incompatible elements of $\mathcal{P}_{QML}$. However, it is easy to show that a Friedman-Glymour mapping is in fact bivalent and truth-functional $(\delta,\not\delta)$ on an ultrasubstructure of $\mathcal{P}_{QML}$, just like the quantum state-induced Exp_ψ mapping. The Friedman-Glymour semantic mappings are called S3-valuations $v : \mathcal{P}_{QML} \to \{0,1\}$ and need only satisfy the following two conditions: For any $P_1, P_2 \in \mathcal{P}_{QML}$,

(i) $v(P_1) = 1$ IFF $v(P_1^\perp) = 0$

(ii) If $v(P_1) = 1$ and $P_1 \leq P_2$, then $v(P_2) = 1$.

It follows from (i), (ii), that, for any S3-valuation v on a $\mathcal{P}_{QML}$ and for any given element $P_0 \in \mathcal{P}_{QML}$, if $v(P_0) = 1$, then for any $P \in \mathcal{P}_{QML}$, $v(P) = 1$ if $P \geq P_0$ and $v(P) = 0$ if $P \leq P_0^{\perp}$. For if $v(P_0) = 1$ and $P \geq P_0$, then by (ii), $v(P) = 1$. And since $P \leq P_0^{\perp}$ IFF $P_0 \leq P^{\perp}$, if $v(P_0) = 1$ and $P \leq P_0^{\perp}$, then $P_0 \leq P^{\perp}$ and so by (ii), $v(P^{\perp}) = 1$, and then by (i), $v(P) = v((P^{\perp})^{\perp}) = 0$. When P_0 is an atom P_{ψ} in $\mathcal{P}_{QML}$, then as shown in this section, the set $\{P \in \mathcal{P}_{QML} : P \geq P_{\psi}\}$ is an ultrafilter UF_{ψ} in $\mathcal{P}_{QML}$ and the set $\{P \in \mathcal{P}_{QML} : P \leq P_{\psi}^{\perp}\}$ is the dual ultraideal UI_{ψ} in $\mathcal{P}_{QML}$. And it follows from the conditions satisfied by UF_{ψ} and UI_{ψ} that a mapping like the S3-valuation which assigns the value 1 to the elements in UF_{ψ} and assigns the value 0 to the elements in UI_{ψ} is not only bivalent but also truth-functional($\eth, \not\eth$) on the ultrasubstructure $UF_{\psi} \cup UI_{\psi}$ of $\mathcal{P}_{QML}$. So besides being bivalent and $\perp$, $\leq$ preserving with respect to the entire $\mathcal{P}_{QML}$, the S3-valuations are also bivalent, truth-functional($\eth, \not\eth$) ultravaluations on the ultrasubstructures of $\mathcal{P}_{QML}$, as are the quantum state-induced expectation-functions.

Of course, for any atom $P_{\psi} \in \mathcal{P}_{QM}$, if the ultrasubstructure $US_{\psi} = UF_{\psi} \cup UI_{\psi}$ is an improper substructure of $\mathcal{P}_{QM}$, i.e., if $US_{\psi} = \mathcal{P}_{QM}$, then the quantum expectation-function Exp_{ψ}, which is induced by the pure state represented by P_{ψ}, is a bivalent, truth-functional (($\eth$) or ($\eth, \not\eth$)) ultravaluation on the entire $\mathcal{P}_{QM}$ structure. In particular, as described in the digression prior to the proof of Theorem A in Chapter V(A), if $\mathcal{P}_{QM}$ has a nontrivial centre which includes an atom P_{ψ} (labeled P_c in the digression) of $\mathcal{P}_{QM}$, so this P_{ψ} is compatible with every $P \in \mathcal{P}_{QM}$, then the ultrasubstrucutre $US_{\psi} = \{P \in \mathcal{P}_{QM} : P \eth P_{\psi}\} = \mathcal{P}_{QM}$. And so the mapping (labeled h_c in the digression) which assigns the value 1 to the elements in UF_{ψ} and assigns the value 0 to the elements in UI_{ψ}, namely, the state-induced Exp_ψ , truth-functionally $((\eth)$ or $(\eth, \not\eth))$ assigns 0, 1 values to <u>every</u> element in $P_{QM} = \mathrm{US}_\psi = \mathrm{UF}_\psi \cup \mathrm{UI}_\psi$.

However, if P_{QM} contains incompatible elements, then as shown by Theorem B, there is some ultrafilter UF_ψ in P_{QM} which is not prime, and so by the converse of Proposition A, $\mathrm{UF}_\psi \cup \mathrm{UI}_\psi \neq P_{QM}$, i.e., $\mathrm{UF}_\psi \cup \mathrm{UI}_\psi \subset P_{QM}$. It is precisely because every quantum P_{QM} containing incompatible elements has at least one ultrasubstructure which is smaller than the entire P_{QM} that I have chosen to assign 0, 1 truth-values to the elements of any propositional or logical structure P according to the definition: For any element $P \in P$, $v(P) = 1$ if $P \in \mathrm{UF}$ and $v(P) = 0$ if $P \in \mathrm{UI}$, rather than according to Sikorski's definition of a bivalent homomorphism: for any element $P \in P$, $v(P) = 1$ if $P \in \mathrm{UF}$ and $v(P) = 0$ if $P \notin \mathrm{UF}$. With respect to a Boolean propositional or logical structure B, e.g., L or P_{CM} , the two definitions are equivalent because $\mathrm{UF} \cup \mathrm{UI} = B$ for every UF and dual UI in B, since every UF (and dual UI) in a Boolean structure is prime. So each may be regarded as the definition of an ultravaluation on a Boolean propositional or logical structure B. That is, each definition defines a bivalent, truth-functional mapping on a B with respect to an UF and dual UI in B; such a mapping is called an ultravaluation because, with respect to the Lindenbaum algebra L of classical propositional logic, such a mapping is the algebraic version of a standard valuation. But the two definitions are not equivalent whenever $\mathrm{UF} \cup \mathrm{UI} \subset P$. In particular, the two definitions are not equivalent with respect to a quantum P_{QM} which contains incompatible elements and thus contains at least one ultrasubstructure $\mathrm{UF}_\psi \cup \mathrm{UI}_\psi \subset P_{QM}$.

According to both definitions, any $P \in P_{QM}$ such that

$P_\psi \in UF_\psi \cup UI_\psi$ is assigned the value 1 if $P \in UF_\psi$ and is assigned the value 0 if $P \in UI_\psi$, because for any such $P \in UF_\psi \cup UI_\psi$, $P \in UI_\psi$ IFF $P \notin UF_\psi$. So with respect to a given ultrasubstructure $UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$, both definitions are equivalent. In particular, a mapping which assigns 0, 1 values according to either definition is bivalent and truth-functional ((⌽) or (⌽,⍉)) on the ultrasubstructure $UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$. But the two definitions differ with respect to the elements of $\mathcal{P}_{QM}$ which are outside of a given $UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$. Every $P \in \mathcal{P}_{QM}$ such that $P \notin UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$ is assigned the value 0 according to the Sikorski definition since every such P is not a member of UF_ψ. However, the assignment of the value 0 to every $P \notin UF_\psi \cup UI_\psi$ according to the Sikorski definition is not a truth-functional ((⌽) or (⌽,⍉)) assignment, as shown by the following example. For any $P \in \mathcal{P}_{QM}$, if $P \notin UF_\psi \cup UI_\psi$ then $P^\perp \notin UF_\psi \cup UI_\psi$. For assume on the contrary that $P \notin UF_\psi \cup UI_\psi$, i.e., $P \notin UF_\psi$ and $P \notin UI_\psi$, and $P^\perp \in UF_\psi \cup UI_\psi$, i.e., $P^\perp \in UF_\psi$ or $P^\perp \in UI_\psi$. If $P^\perp \in UF_\psi$, then by (c), $P \in UI_\psi$, which contradicts the assumption $P \notin UI_\psi$. And if $P^\perp \in UI_\psi$, then by (c'), $P \in UF_\psi$, which contradicts the assumption $P \notin UF_\psi$. Thus if $P \notin UF_\psi \cup UI_\psi$, then also $P^\perp \notin UF_\psi \cup UI_\psi$. In particular, both P, $P^\perp \notin UF_\psi$, and so according to the Sikorski definition, $v(P) = v(P^\perp) = 0$. But $P \vee P^\perp = 1 \in UF_\psi$, and so $v(P \vee P^\perp) = 1$. Hence, for any $P \notin UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$, $v(P \vee P^\perp) = 1 \neq 0 = v(P) \vee v(P^\perp)$. So although a mapping which assigns values according to the Sikorski definition is bivalent on the entire $\mathcal{P}_{QM}$, it is not truth-functional ((⌽) or (⌽,⍉)) on the entire $\mathcal{P}_{QM}$. In contrast, the other definition which uses the condition "if $P \in UI$" rather than the condition "if $P \notin UF$" leaves open the questions of how and what values are to be assigned to such elements

$P \notin UF_\psi \cup UI_\psi \subset P_{QM}$. So the only difference between the two definitions is that one leaves these questions open while the other assigns the value 0 to the elements outside a given ultrasubstructure. Since these 0 value assignments are not truth-functional ((⫰) or (⫰,⫰̸)), including them as part of the definition of an ultravaluation on a P_{QM} actually adds little beyond satisfying in a trivial way the bivalency desideratum. Thus we have taken the definition which uses the condition "if $P \in UI$" as the definition of an ultravaluation on a P_{QM} .

As described in the preceding sections, a state-induced ultravaluation Exp_ψ assigns values between 0 and 1 to the elements outside $UF_\psi \cup UI_\psi \subset P_{QM}$. And Exp_ψ does preserve the $\perp$ operation and the $\leq$ relation of P_{QM} as it assigns these intermediate values, but the $\wedge$, $\vee$ operations of P_{QM} are not preserved. So an Exp_ψ is neither bivalent nor truth-functional ((⫰) or (⫰,⫰̸)) on the entire P_{QM} .

Friedman and Glymour propose that their S3-valuations on a P_{QML} , which have been shown to be ultravaluations on the ultrasubstructures of P_{QML} , also assign 0, 1 values to the elements outside their affiliated ultrasubstructures. Most simply, the value 0 may be assigned to every atom (one-dimensional subspace) and the value 1 may be assigned to the orthocomplement of every atom (two-dimensional subspace) outside a given ultrasubstructure of a three-dimensional Hilbert space P^3_{QML} (Friedman-Glymour, 1972, p. 27). Again, the $\perp$ operation and the $\leq$ relation of P_{QML} are preserved by such 0, 1 value assignments to the elements outside an ultrasubstructure. And this proposal avoids at least some of the truth-functionality(⫰,⫰̸) problems of the more simple proposal that the value 0 be assigned to every element outside an ultrasubstructure. But Friedman-Glymour do not describe how 0, 1 values may be assigned for, say, a four-dimensional Hilbert space P^4_{QML} which has not only one- and two-dimensional subspaces but also three-dimensional subspaces outside any given ultrasubstructure of P^4_{QML}. And of course, this Friedman-Glymour proposal, and any other proposal of a bivalent semantics for the quantum P_{QML} structures, inevitably runs into truth-functionality($\leftrightarrow$,$\not\leftrightarrow$) problems, as shown in Chapter V(A), and also truth-functionality($\leftrightarrow$) problems, as shown by Kochen-Specker.

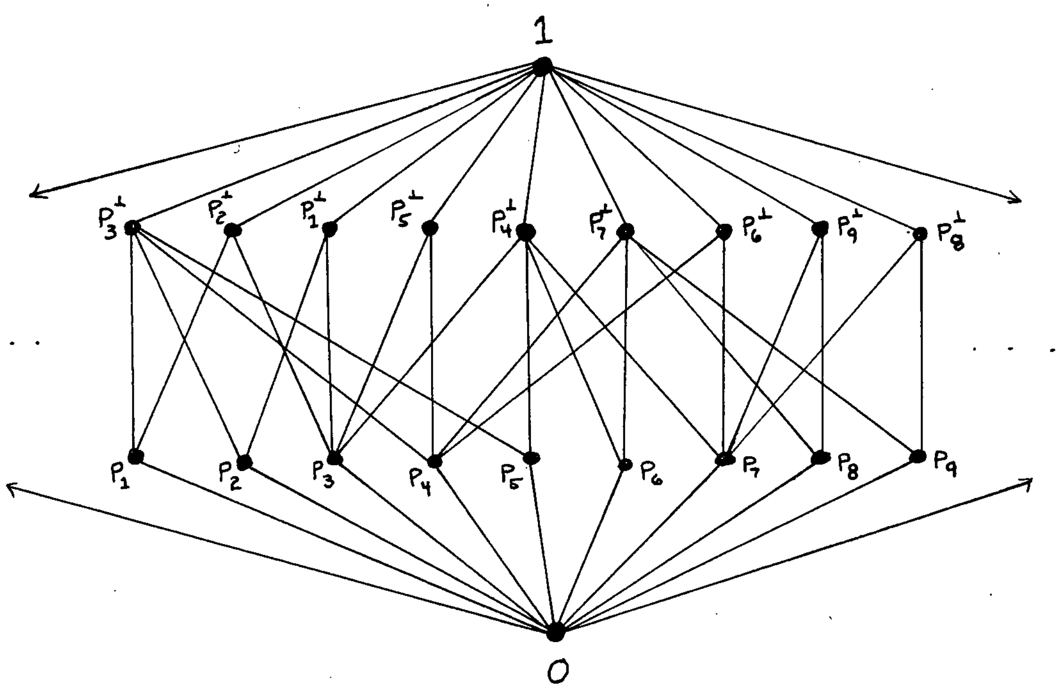
While addressing the issue of a predicate calculus for a Kochen-Specker P_{QMA} type of quantum propositional logic, Levy proposes that, besides the 0, 1 values assigned by a state-induced ultravaluation to the elements in an ultrasubstructure of P_{QMA}, a third truth value, <u>inappropriate</u>, labeled N, be assigned to the elements outside a given ultrasubstructure. Such a three-valued semantics for a quantum P_{QMA} or P_{QML} is, of course, not bivalent and is also not truth-functional ($\leftrightarrow$) or ($\leftrightarrow$,$\not\leftrightarrow$)), as Levy mentions.[6] An example of a violation of truth-functionality (($\leftrightarrow$) or ($\leftrightarrow$,$\not\leftrightarrow$)) is given at the end of the next section.

This Levy proposal of three-valued semantic mappings for P_{QM} structures is different from previous proposals of a three-valued semantics for quantum propositions. For example, Reichenbach assigns his third truth value I (Indeterminate) to quantum propositions which are meaningless according to the Bohr-Heisenberg interpretation of quantum mechanics. In particular, if $P_1 \not\leftrightarrow P_2$ then at most one of P_1, P_2 is meaningful while the other is meaningless, and also $P_1 \wedge P_2$ and $P_1 \vee P_2$ are each meaningless (Reichenback, 1965, pp. 143-145). However, even though $P_1 \not\leftrightarrow P_2$, they may both be together in some ultrasubstructure of P_{QM}, in which case both of them, and their meet and their join are all assigned the usual 0, 1 truth values by the state-induced ultravaluation affiliated with that ultrasubstructure.

In short, although semantic mappings on a $\mathcal{P}_{QM}$ which assign values between 0 and 1 or which assign a third truth-value like N to the elements outside a given ultrasubstructure of $\mathcal{P}_{QM}$ are <u>not</u> bivalent semantic mappings on the entire $\mathcal{P}_{QM}$ when $UF_\psi \cup UI_\psi \subset \mathcal{P}_{QM}$, nevertheless such mappings <u>are</u> truth-functional $((\flat))$ or $((\flat, \not\flat))$ wherever they are bivalent, namely, on $UF_\psi \cup UI_\psi$. Thus the proposal of such semantic mappings for $\mathcal{P}_{QM}$ has the virtue of clearly demarcating the substructures of $\mathcal{P}_{QM}$ with respect to which bivalent, truth-functional $((\flat))$ or $((\flat, \not\flat))$ value assignments are possible, namely, the ultrasubstructures $UF_\psi \cup UI_\psi$, for any atom $P_\psi \in \mathcal{P}_{QM}$.

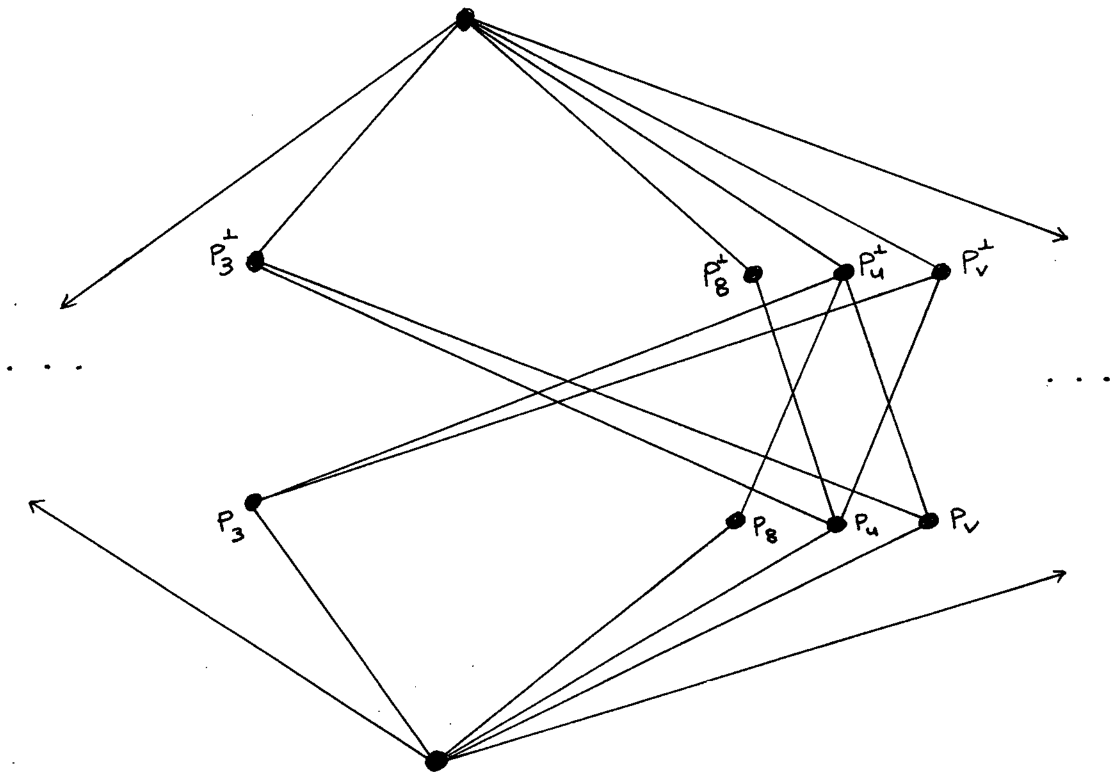
<u>Section C</u>. <u>An Example</u>

Consider the fragment of the $\mathcal{P}^3_{QM}$ structure of subspaces (or projectors) of three-dimensional Hilbert space diagrammed below:



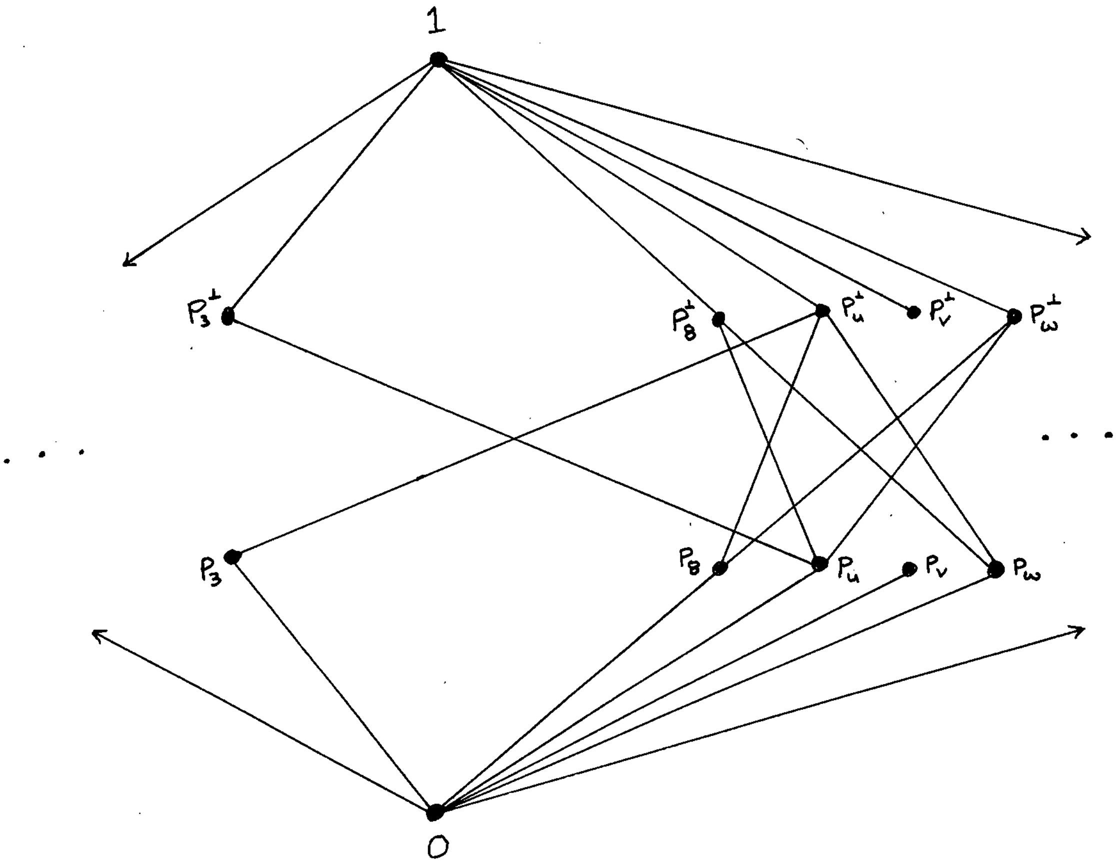
This fragment contains four maximal Boolean substructures: mBS_2 generated by the atoms $\{P_1,P_2,P_3\}$, mBS_5 generated by the atoms $\{P_3,P_4,P_5\}$, mBS_6 generated by the atoms $\{P_4,P_6,P_7\}$, and mBS_9 generated by the atoms $\{P_7,P_8,P_9\}$. Clearly, these four mBS's overlap since they share atoms.

If we had started with the initial set $S = \{P_1,P_2, \ldots ,P_9\}$ of these nine one-dimensional subspaces of $\mathcal{H}^3$, then the partial-Boolean algebra generated by closing S with respect to the $\wedge$, $\vee$, $^\perp$ operations of $\mathcal{P}_{QMA}$ is the finite fragment of 20 elements diagrammed above. However, the orthomodular lattice generated by closing S with respect to the $\wedge$, $\vee$, $^\perp$ operations of $\mathcal{P}_{QML}$ is denumerably infinite and so exemplifies the proliferation of lattice elements due to the lattice definitions of $\wedge$, $\vee$ among incompatible elements, as mentioned in Chapter IV(E). Let us focus on the element P_3 which is compatible with P_1, P_2, P_4, P_5. Consider the incompatible elements $P_3 \not\mathrel{\text{ó}} P_6$, their join $P_3 \vee P_6 = P_4^\perp$. This $P_4^\perp$ is also equal to $P_3 \vee P_5$ and to $P_6 \vee P_7$, where $P_3 \mathrel{\text{ó}} P_5$ and $P_6 \mathrel{\text{ó}} P_7$. So the join $P_3 \vee P_6$ does not introduce any new element. And the join $P_3 \vee P_6$ is an example of what Strauss would call the lattice <u>mis</u>interpretation of the element $P_4^\perp$. Similarly, consider the incompatible elements $P_3 \not\mathrel{\text{ó}} P_7$. Again their join $P_3 \vee P_7 = P_4^\perp$, so no new element is introduced by having the $\vee$ operation defined among these two incompatible elements. But now consider the two incompatible elements $P_3 \not\mathrel{\text{ó}} P_8$; their join $P_3 \vee P_8$ and the meet of their orthocomplements $P_3^\perp \wedge P_8^\perp$ are each not equal to any of the twenty elements in the above diagram. Let $P_3^\perp \wedge P_8^\perp = P_u$ and so $P_3 \vee P_8 = (P_3^\perp \wedge P_8^\perp)^\perp = P_u^\perp$. Clearly, $P_3 \leq P_u^\perp$, and so $P_3 \mathrel{\text{ó}} P_u$ and also $P_3^\perp \mathrel{\text{ó}} P_u^\perp$. Let $P_3^\perp \wedge P_u^\perp = P_v$ and so $P_3 \vee P_u = (P_3^\perp \wedge P_u^\perp)^\perp = P_v^\perp$. Clearly, $P_v^\perp \geq P_3$ and $P_v^\perp \geq P_u$, and so $P_v \mathrel{\text{ó}} P_3$ and $P_v \mathrel{\text{ó}} P_u$. Thus $\{P_3,P_u,P_v\}$ are three mutually compatible atoms which generate another maximal Boolean substructure, say mBS_v . The relations among these elements are diagrammed below; for clarity, all the elements of the first diagram have been omitted except the P_3 , $P_3^\perp$, P_8 , $P_8^\perp$, 0, 1 elements:



But besides $P_3 \leq P_u^\perp$, when we let $P_3^\perp \wedge P_8^\perp = P_u$ and so $P_3 \vee P_8 = P_u^\perp$, we also have: $P_8 \leq P_u^\perp$, and so $P_8 \,\delta\, P_u$ and also $P_8^\perp \,\delta\, P_u^\perp$. Let $P_8^\perp \wedge P_u^\perp = P_w$ and so $P_8 \vee P_u = (P_8^\perp \wedge P_u^\perp)^\perp = P_w^\perp$. Clearly, $P_w^\perp \geq P_8$ and $P_w^\perp \geq P_u$, and so $P_w \,\delta\, P_8$ and $P_w \,\delta\, P_u$. Thus $\{P_8, P_u, P_w\}$ are three mutually compatible atoms which generate yet another maximal Boolean substructure, say mBS_w . Moreover: $P_8 \vee P_w = P_8 \vee (P_8^\perp \wedge P_u^\perp)$ $= (P_8 \vee P_8^\perp) \wedge (P_8 \vee P_u^\perp) = 1 \wedge P_u^\perp = P_u^\perp$. So $P_u^\perp \wedge P_w^\perp = (P_8 \vee P_w) \wedge P_w^\perp$ $(P_8 \wedge P_w^\perp) \vee (P_w \wedge P_w^\perp) = P_8 \vee 1 = P_8$, and thus

$P_8^{\perp} = (P_u^{\perp} \wedge P_w^{\perp})^{\perp} = P_u \vee P_w$. All these relations are diagrammed below; for clarity, all the elements of the first diagram have been omitted except the P_3 , $P_3^{\perp}$, P_8 , $P_8^{\perp}$, 0, 1 elements:



Similarly, consider the two incompatible elements $P_3 \not\leftrightarrow P_9$; their join $P_3 \vee P_9$ and the meet of their orthocomplements $P_3^{\perp} \wedge P_9^{\perp}$ are each not equal to any of the 26 elements in the above (combined) diagrams. Let $P_3^{\perp} \wedge P_9^{\perp} = P_x$ and so $P_3 \vee P_9 = (P_3^{\perp} \wedge P_9^{\perp})^{\perp} = P_x^{\perp}$. Thus two more elements have been introduced, and as described above, by closure, four more elements $P_y = P_3^{\perp} \wedge P_x^{\perp}$, $P_y^{\perp} = P_3 \vee P_x$, $P_z = P_9^{\perp} \wedge P_x^{\perp}$, and $P_z^{\perp} = P_9 \vee P_x$ will be introduced, where $\{P_3, P_x, P_y\}$ and $\{P_9, P_x, P_z\}$ will be two sets of mutually compatible atoms, each generating two more mBS's, mBS_y and mBS_z , in the P^3_{QML} generated by closing the initial set S with respect to the $\wedge$, $\vee$ operations of P_{QM} . Likewise, the incompatible pairs $P_3 \not\leftrightarrow P_w$, $P_3 \not\leftrightarrow P_z$ may be joined and meeted to introduce even more elements. And, of course, when we focus upon another element besides P_3 , say P_2 , which is incompatible with P_4 , P_5 , P_6 , P_7 , P_8 , P_9 , P_u , P_v , P_w , P_x , P_z , the joins of P_2 with each of these elements will introduce even more elements, etc. Thus the P^3_{QML} generated by closing the initial finite set S with respect to the $\wedge, \vee, ^\perp$ operations of P_{QML} will contain a denumerable infinity of elements. Nevertheless, the infinite P^3_{QML} and the corresponding infinite P^3_{QMA} of all subspaces of H^3 each contain the same elements, so it is not correct to consider a partial-Boolean algebra of subspaces to be missing elements compared with an orthomodular lattice of subspaces. The point of the above example is to show how, when an orthomodular lattice of subspaces is <u>generated</u> from an initial set of subspaces by closing the initial set with respect to the $\wedge, \vee, ^\perp$ operations of P_{QML} , the lattice definitions of $\wedge$, $\vee$ among incompatibles may result in a proliferation of elements which does not occur when a partial-Boolean algebra of subspaces is generated from the same initial set by closing the set with respect to the $\wedge, \vee, ^\perp$ operations of P_{QMA} .

Let us assume that the quantum system, which is associated with H^3 and which $P_1, P_2, \ldots, P_9$, represent propositions about, is in the pure state ψ_3 represented by the projector $\hat{P}_3$ which is the atom P_3 in the (combined) diagram, which is a fragment of the system's propositional structure P^3_{QM} . So we focus on the state-induced expectation-function Exp_3 and its affiliated ultrasubstructure $US_3 = UF_3 \cup UI_3 \in P^3_{QM}$. With respect to the twenty element P^3_{QMA} generated by the initial set S, we have:

$UF_3 = \{1, P_3, P_3 \vee P_1 = P_2^{\perp}, P_3 \vee P_2 = P_1^{\perp}, P_3 \vee P_4 = P_5^{\perp}, P_3 \vee P_5 = P_4^{\perp}\}$ and $UI_3 = \{0, P_3^{\perp}, P_2, P_1, P_5, P_4\}$. With respect to the denumerably infinite $\mathcal{P}^3_{QML}$ generated by the initial set S, we have:

$UF_3 = \{1, P_3, P_3 \vee P_1 = P_2^{\perp}, P_3 \vee P_2 = P_1^{\perp}, P_3 \vee P_4 = P_5^{\perp}, P_3 \vee P_5 = P_4^{\perp} =$
$= P_3 \vee P_6 = P_3 \vee P_7$, $P_3 \vee P_8 = P_u^{\perp} = P_3 \vee P_v$, $P_3 \vee P_u = P_v^{\perp}$,
$P_3 \vee P_9 = P_x^{\perp} = P_3 \vee P_y$, $P_3 \vee P_x = P_y^{\perp}$, etc., . . . a denumerable infinity of two-dimensional subspaces of $\mathcal{H}^3$, each containing $P_3\}$.

$UI_3 = \{0, P_3^{\perp}, P_2, P_1, P_5, P_4, P_u, P_v, P_x, P_y$, etc., . . . a denumerable infinity of one-dimensional subspaces of $\mathcal{H}^3$, each contained in $P_3^{\perp}\}$.

And with respect to the infinite $\mathcal{P}^3_{QMA}$ and $\mathcal{P}^3_{QML}$ of all subspaces of $\mathcal{H}^3$, UF_3 in both structures includes the 1 element, P_3 , and the nondenumerable infinity of all two-dimensional subspaces of $\mathcal{H}^3$ containing P_3 . And UI_3 in both structures includes the 0 element, $P_3^{\perp}$, and the nondenumerable infinity of all one-dimensional subspaces of $\mathcal{H}^3$ contained in $P_3^{\perp}$. Hereafter, let us just focus on the twenty element $\mathcal{P}^3_{QMA}$ generated by S and the denumerably infinite $\mathcal{P}^3_{QML}$ generated by S.

Clearly, $US_3 = UF_3 \cup UI_3$ is larger than the two maximal Boolean substructures mBS_2 and mBS_5 which contain P_3 in the twenty element $\mathcal{P}^3_{QMA}$. And likewise US_3 is larger than any of the maximal Boolean substructures mBS_2 , mBS_5 , mBS_v , mBS_y , etc., which contain P_3 in the denumerably infinite fragment $\mathcal{P}^3_{QML}$. Moreover, by inspection it is clear that in the finite $\mathcal{P}^3_{QMA}$, $US_3 = mBS_2 \cup mBS_5$; and by inspection it is clear that in the denumerable $\mathcal{P}^3_{QML}$, considering just the explicitly listed elements in US_3 and just the explicitly listed mBS's containing P_3 , the listed elements in $US_3 = mBS_2 \cup mBS_5 \cup mBS_v \cup mBS_y$. That is,

US_3 equals the union of all the mBS's containing P_3, as proven in Section B.

It is also worth noticing how, if we had used conditions (a), (b), rather than conditions (a_H), (b), to define a filter in a $\mathcal{P}_{QMA}$, then the set $S' = UF_3 \cup \{P_7^\perp\}$ would be a proper filter in the twenty element $\mathcal{P}^3_{QMA}$ diagrammed above. Using (a_H), it is easy to show that S' is <u>not</u> a filter in this $\mathcal{P}^3_{QMA}$. If S' is a filter, then since $P_2^\perp, P_7^\perp \in S'$, by (a_H), there is an element $d \in S'$ such that $d \leq P_2^\perp$ and $d \leq P_7^\perp$. In the twenty element $\mathcal{P}^3_{QMA}$, $P_2^\perp \leq P_2^\perp$, $0 \leq P_2^\perp$, $P_1 \leq P_2^\perp$, $P_3 \leq P_2^\perp$, $P_7^\perp \leq P_7^\perp$, $0 \leq P_7^\perp$, $P_4 \leq P_7^\perp$, $P_6 \leq P_7^\perp$, $P_8 \leq P_7^\perp$, and $P_9 \leq P_7^\perp$; so only $0 \leq P_2^\perp$ and $0 \leq P_7^\perp$. But $0 \notin S'$, and so S' is not a filter. Q.E.D. But using (a), it turns out that S' is a proper filter in the twenty element $\mathcal{P}^3_{QMA}$. If S' is a filter, then since $P_2^\perp, P_7^\perp \in S'$, by (a), the meet of $P_2^\perp$, $P_7^\perp$ is a member of S', but this meet is not defined in the twenty element $\mathcal{P}^3_{QMA}$ since $P_2^\perp \not\leftrightarrow P_7^\perp$. (If the meet were defined, as in a $\mathcal{P}^3_{QML}$ containing $P_2^\perp$ and $P_7^\perp$, then the meet $P_2^\perp \wedge P_7^\perp = 0$; thus since $0 \notin S'$, S' would not be a filter.) Moreover, except for the 1 element, every other element in S' is incompatible with $P_7^\perp$ and so the meets of $P_7^\perp$ with every other element in S' are not defined in the twenty element $\mathcal{P}^3_{QMA}$. And $1 \wedge P_7^\perp = P_7^\perp \in S'$. Thus S' satisfies (a). Also $S' = UF_3 \cup \{P_7^\perp\}$ satisfies (b); for UF_3 satisfies (b), and only $1 \geq P_7^\perp$, $P_7^\perp \geq P_7^\perp$, and $1, P_7^\perp \in S'$. So S' is a filter in the twenty element $\mathcal{P}^3_{QMA}$. Q.E.D. Moreover, since $0 \notin S'$, S' is a proper filter in this $\mathcal{P}^3_{QMA}$. So UF_3 is the proper subset of a proper filter in this $\mathcal{P}^3_{QMA}$. Thus UF_3 is not an ultrafilter in this $\mathcal{P}^3_{QMA}$, a very undesirable result of using (a) rather than (a_H) to define a filter in a $\mathcal{P}_{QMA}$.

Returning to the state-induced Exp_3, which assigns the value 1 to elements in UF_3 and assigns the value 0 to elements in UI_3, it is easy to find examples of how Exp_3 is <u>not</u> a truth-functional($\flat$) mapping on the entire twenty element P^3_{QMA}. Consider the compatible elements $P_6^{\perp}, P_7^{\perp} \in P^3_{QMA}$: $P_6^{\perp} \wedge P_7^{\perp} = P_4 \in UI_3$, so $Exp_3(P_6^{\perp} \wedge P_7^{\perp}) = 0$. But $P_6^{\perp}, P_7^{\perp} \notin UF_3 \cup UI_3$, so $Exp_3(P_6^{\perp}) \neq 0$ (and $\neq 1$) and $Exp_3(P_7^{\perp}) \neq 0$ (and $\neq 1$). Thus $Exp_3(P_6^{\perp}) \wedge Exp_3(P_7^{\perp}) \neq 0 = Exp_3(P_6^{\perp} \wedge P_7^{\perp})$. Similarly, it is easy to find examples of how Exp_3 is <u>not</u> a truth-functional($\flat,\not\flat$) mapping on the entire denumerable P^3_{QML}. Consider the incompatible elements $P_3^{\perp}, P_8 \in P^3_{QML}$: $P_3^{\perp} \vee P_8 = 1 \in UF_3$, so $Exp_3(P_3^{\perp} \vee P_8) = 1$. But $P_3^{\perp} \in UI_3$, so $Exp_3(P_3^{\perp}) = 0$, and $P_8 \notin UF_3 \cup UI_3$, so $Exp_3(P_8) \neq 1$ (and $\neq 0$). Thus $Exp_3(P_3^{\perp}) \vee Exp_3(P_8) = 0 \vee Exp_3(P_8) = Exp_3(P_8) \neq 1 = Exp_3(P_3^{\perp} \vee P_8)$.

Since the elements $P_6^{\perp}, P_7^{\perp}, P_8 \notin UF_3 \cup UI_3$, the meet $P_6^{\perp} \wedge P_7^{\perp}$ and the join $P_3 \vee P_8^{\perp}$ are examples of what were called US_ψ-extra meets and joins in Section B, where here, US_ψ is US_3. These are the meets and joins which cause truth-functionality (($\flat$) or ($\flat,\not\flat$)) problems for Exp_3. Whether they are the meets and joins of compatible elements or of incompatible elements is irrelevent. What makes such meets and joins problematic for Exp_3 is that one or the other or both of their subformulae are elements of P_{QM} which are <u>not</u> members of US_3. Moreover, every violation of truth-functionality (($\flat$) or ($\flat,\not\flat$)) by an Exp_ψ on a P_{QM} involves such US_ψ-extra meets and joins. For as has been shown in Section B, any Exp_ψ is truth-functional (($\flat$) or ($\flat,\not\flat$)) on the domain US_ψ, that is, Exp_ψ does preserve the meets and joins of the elements of P_{QM} which <u>are</u> members of US_ψ.

As mentioned in Section B, the truth-functional($\flat,\not\flat$) character of an Exp_ψ on $US_\psi \subset P_{QML}$ may seem surprising in the light of the Chapter V (A) description of the truth-functionality($\flat,\not\flat$) problems caused by the meets and joins of incompatible elements in $\mathcal{P}_{QML}$. However, we can find many examples of the truth-functionality($\circlearrowleft,\not\circlearrowleft$) of Exp_3 on the ultrasubstructure US_3 of the denumerable $\mathcal{P}_{QML}$ considered in this section. Consider the incompatible pairs $P_2^{\perp} \not\circlearrowleft P_5$, $P_u^{\perp} \not\circlearrowleft P_y$, $P_1 \not\circlearrowleft P_x$, and the following meets and joins of these incompatible pairs: $P_2^{\perp} \wedge P_5$, $P_u^{\perp} \vee P_y$, $P_1 \wedge P_x$. Clearly, $P_2^{\perp} \in UF_3$, $P_5 \in UI_3$, and $P_2^{\perp} \wedge P_5 = 0 \in UI_3$; thus $Exp_3(P_2^{\perp}) \wedge Exp_3(P_5) = 1 \wedge 0 = 0 = Exp_3(P_2^{\perp} \wedge P_5)$. Clearly, $P_u^{\perp} \in UF_3$, $P_y \in UI_3$, and $P_u^{\perp} \vee P_y = 1 \in UF_3$; thus $Exp_3(P_u^{\perp}) \vee Exp_3(P_y) = 1 \vee 0 = 1 = Exp_3(P_u^{\perp} \vee P_y)$. Clearly, $P_1 \in UI_3$, $P_x \in UI_3$, and $P_1 \wedge P_x = 0 \in UI_3$; thus $Exp_3(P_1) \wedge Exp_3(P_x) = 0 \wedge 0 = 0 = Exp_3(P_1 \wedge P_x)$.

It is also easy to find examples of violations of truth-functionality($\circlearrowleft,\not\circlearrowleft$) by a semantic mapping v which assigns 0, 1 values to the elements in US_3 and in addition assigns 0, 1 values to the elements outside of US_3 according to the Friedman-Glymour proposal mentioned in Section B. Consider the three mutually compatible elements $P_9^{\perp}, P_8^{\perp}, P_7 \notin US_3$ in the denumerable $\mathcal{P}_{QML}$. According to the Friedman-Glymour proposal, $v(P_9^{\perp}) = v(P_8^{\perp}) = 1$ and $v(P_7) = 0$. However, $P_9^{\perp} \wedge P_8^{\perp} = P_7$ in $\mathcal{P}_{QML}$, and so $v(P_9^{\perp} \wedge P_8^{\perp}) = v(P_7) = 0 \neq 1 \wedge 1 = v(P_9^{\perp}) \wedge v(P_8^{\perp})$.

Finally, as an example of a violation of truth-functionality($\circlearrowleft$) by a semantic mapping v which assigns 0, 1 values to the elements in US_3 and in addition assigns the value N to the elements outside of US_3 according to the Levy proposal mentioned in Section B, consider these two joins of compatible elements in the twenty element $\mathcal{P}_{QMA}$: $P_6 \vee P_6^{\perp}$ and $P_6 \vee P_7$. Since $P_6, P_6^{\perp}, P_7 \notin US_3 \subset \mathcal{P}_{QMA}$, $v(P_6) = v(P_6^{\perp}) = v(P_7) = N$. Similarly, since $P_6 \vee P_7 = P_8^{\perp} \notin US_3$, $v(P_6 \vee P_7) = N$. But $P_6 \vee P_6^{\perp} = 1 \in UF_3$, so $v(P_6 \vee P_6^{\perp}) = 1$. In order to show that v is not truth-functional($\circlearrowleft$), assume on the contrary that it is truth-functional($\circlearrowleft$).

Then $1 = v(P_6 \vee P_6^{\perp}) = v(P_6) \vee v(P_6^{\perp}) = N \vee N = v(P_6) \vee v(P_7) = v(P_6 \vee P_7)$ $= N$, i.e., $1 = N$, which contradicts the presupposition that $N \neq 1$.

Section D. A State-induced Semantics for the $\mathcal{P}_{QM}$ Structures

As described in Chapter II, a bivalent, truth-functional semantics for a Lindenbaum Boolean algebra of classical propositional logic is a complete collection of ultravaluations on the Lindenbaum algebra. And as described in Chapter III, a state-induced, bivalent, truth-functional semantics for a Boolean $\mathcal{P}_{CM}$ of classical mechanics is a complete collection of state-induced ultravaluations on the $\mathcal{P}_{CM}$. With these classical precedents in mind, in order to fully elaborate the notion of a state-induced semantics for a quantum $\mathcal{P}_{QM}$, it remains to be shown that the collection of state-induced ultravaluations on the ultrasubstructures of a $\mathcal{P}_{QM}$ is complete.

We can establish completeness in the required sense if we can show that, for any given pair of distinct elements $P_1 \neq P_2$ in a $\mathcal{P}_{QM}$, the set of atoms dominated by P_1 is not the same as the set of atoms dominated by P_2. For clearly, if P_ψ is an atom dominated by P_1, i.e., $P_\psi \leq P_1$, but not dominated by P_2, i.e. $P_\psi \nleq P_2$, then the state-induced mapping Exp_ψ by definition assigns the values $Exp(P_1) = 1 \neq Exp(P_2)$. Now as pointed out by van Fraassen,[7] if the elements of a $\mathcal{P}_{QM}$ are regarded as subspaces of a Hilbert space, it is easy to show that, for any $P_1 \neq P_2$ in a $\mathcal{P}_{QM}$, the set of atoms dominated by P_1 differs from the set of atoms dominated by P_2. For as stated in Chapter IV(A), a subspace of a Hilbert space is a set of vectors (which forms a closed linear manifold). Thus any two subspaces of a Hilbert space are distinct IFF the two subspaces do not contain exactly the same vectors, where a vector in a Hilbert space is uniquely associated with an atom in the $\mathcal{P}_{QM}$ structure of the Hilbert space.

However, we may also consider supporting the completeness result by an algebraic proof which does not invoke the subspace character of the elements of a $\mathcal{P}_{QM}$. For the case of a $\mathcal{P}_{QML}$, an algebraic proof of the completeness result can easily be shown to follow from Lemma C of Chapter V(A). An algebraic proof of the completeness result for a $\mathcal{P}_{QMA}$ is more difficult. Nevertheless, the <u>weak</u> completeness of the collection of state-induced ultravaluations on a $\mathcal{P}_{QMA}$ or a $\mathcal{P}_{QML}$ is easily proved as follows:

> <u>Proposition C</u>: For any $\mathcal{P}_{QM}$, the collection of state-induced ultravaluations on the ultrasubstructures of $\mathcal{P}_{QM}$ is weakly complete, i.e., for any element $P \neq 0$ in $\mathcal{P}_{QM}$, there is an Exp_ψ such that $Exp_\psi(P) \neq Exp_\psi(0)$.
>
> By the atomicity of $\mathcal{P}_{QM}$, for any $P \neq 0$ in $\mathcal{P}_{QM}$ there is an atom $P_\psi \in \mathcal{P}_{QM}$ such that $P_\psi \leq P$, and so the ultrafilter $UF_\psi = \{P \in \mathcal{P}_{QM} : P \geq P_\psi\}$ contains P, while the dual ultraideal $UI_\psi = \{P \in \mathcal{P}_{QM} : P \leq P_\psi^\perp\}$ contains 0 since $0 \leq P_\psi^\perp$. Thus the state-induced ultravaluation Exp_ψ which assigns the value 1 to the members of UF_ψ and assigns the value 0 to the members of UI_ψ satisfies: $Exp_\psi(P) = 1 \neq 0 = Exp_\psi(0)$. Q.E.D.

For the case of a $\mathcal{P}_{QML}$, the completeness result is an immediate consequence of the following Proposition D which follows from Lemma C.

> <u>Proposition D</u>: For any $\mathcal{P}_{QML}$ and for any $P_1, P_2 \in \mathcal{P}_{QML}$, if $P_1 \neq P_2$, then there is an atom $P_\psi \in \mathcal{P}_{QML}$ such that either $P_\psi \leq P_1$ and $P_\psi \nleq P_2$, or $P_\psi \leq P_2$ and $P_\psi \nleq P_1$.
>
> Assume on the contrary that $P_1 \neq P_2$ and for every atom $P_\psi \in \mathcal{P}_{QML}$,

$P_\psi \le P_1$ IFF $P_\psi \le P_2$.

Let $\{P_i\}_{i \in \text{Index}}$ be the set of atoms of $\mathcal{P}_{QML}$ which are dominated by P_1 and let $\bigvee_i P_i$ be the join of all such atoms. Let $\{P_k\}_{k \in \text{Index}}$ be the set of atoms $\mathcal{P}_{QML}$ which are dominated by P_2 and let $\bigvee_k P_k$ be the join of all such atoms. By assumption, for every atom $P_\psi \in \mathcal{P}_{QML}$, $P_\psi \in \{P_i\}_{i \in \text{Index}}$ IFF $P_\psi \in \{P_k\}_{k \in \text{Index}}$, thus $\{P_i\}_{i \in \text{Index}} = \{P_k\}_{k \in \text{Index}}$, and so $\bigvee_i P_i = \bigvee_k P_k$. But by Lemma C, $P_1 = \bigvee_i P_i$ and $P_2 = \bigvee_k P_k$; thus $P_1 = P_2$, which contradicts the assumption $P_1 \ne P_2$. Q.E.D.

Now the desired result follows as an immediate

<u>Corollary to Proposition D</u>: For any $\mathcal{P}_{QML}$, the collection of state-induced ultravaluations on the ultrasubstructures of $\mathcal{P}_{QML}$ is complete, i.e., for any $P_1 \ne P_2$ in $\mathcal{P}_{QML}$, there is an Exp_ψ such that $\text{Exp}_\psi(P_1) \ne \text{Exp}_\psi(P_2)$.

If $P_1 \ne P_2$, then by Proposition D, there is an atom $P_\psi \in \mathcal{P}_{QML}$ such that either $P_\psi \le P_1$ and $P_\psi \not\le P_2$, or $P_\psi \le P_2$ and $P_\psi \not\le P_1$. If $P_\psi \le P_1$ and $P_\psi \not\le P_2$, then $P_1 \in UF = \{P \in \mathcal{P}_{QML} : P \ge P_\psi\}$ but $P_2 \notin UF_\psi$. And so $\text{Exp}_\psi(P_1) = 1$ but $\text{Exp}_\psi(P_2) \ne 1$; thus $\text{Exp}_\psi(P_1) \ne \text{Exp}_\psi(P_2)$. Similarly, if $P_\psi \le P_2$ and $P_\psi \not\le P_1$, then $P_2 \in UF_\psi$ but $P_1 \notin UF_\psi$. And so $\text{Exp}_\psi(P_2) = 1$ but $\text{Exp}_\psi(P_1) \ne 1$; thus $\text{Exp}_\psi(P_2) \ne \text{Exp}_\psi(P_1)$. Q.E.D.

For the case of $\mathcal{P}_{QMA}$, we now assume that $\mathcal{P}_{QMA}$ structures are not only associative, transitive, and atomic partial-Boolean algebras (as defined in Chapter I(D)), but also satisfy the following

<u>Condition A</u>: Every maximal Boolean substructure of a $\mathcal{P}_{QMA}$ is atomic.

And we make use of the following two lemmas proved by Edwin Levy:

<u>Lemma F</u>: For any P_1, P_2 in a $\mathcal{P}_{QM}$, if $P_1 \mathrel{\delta} P_2$, then there are these four non-exclusive but jointly exhaustive possibilities: $P_1 \le P_2$, or $P_2 \le P_1$, or $P_1 \perp P_2$, or there are non-zero disjoint elements $P_{11}, P_{22} \in \mathcal{P}_{QM}$ such that $P_{11} < P_1$ and $P_{22} < P_2$.

Proof: If $P_1 \mathrel{\delta} P_2$, then by the definition of compatibility (Chapter IV(C)) there exist three mutually orthogonal elements $P_{11}, P_{22}, P_3 \in \mathcal{P}_{QM}$ such that $P_1 = P_{11} \vee P_3$ and $P_2 = P_{22} \vee P_3$. We have eight cases depending upon which of P_{11}, P_{22}, P_3 are or are not equal to the 0 element. (1) If $P_{11} = 0$ then $P_1 = 0 \vee P_3 = P_3$ and $P_2 = P_{22} \vee P_3 = P_{22} \vee P_1$; thus $P_1 \le P_2$. (2) If $P_{22} = 0$, then $P_2 = 0 \vee P_3 = P_3$ and $P_1 = P_{11} \vee P_3 = P_{11} \vee P_2$; thus $P_2 \le P_1$. (3) If $P_{11} = P_{22} = 0$, then $P_1 = P_2 = P_3$. (4) If $P_{11} = P_3 = 0$, then $P_1 = 0$ (and so $P_1 \le P_2$). (5) If $P_{22} = P_3 = 0$, then $P_2 = 0$ (and so $P_2 \le P_1$). (6) If $P_{11} = P_{22} = P_3 = 0$, then $P_1 = P_2 = 0$. (Clearly, the results of cases (3), (4), (5), (6), are subsumed by the results of cases (1) and (2).) (7) If $P_3 = 0$, then $P_1 = P_{11} \vee 0 = P_{11}$ and $P_2 = P_{22} \vee 0 = P_{22}$; thus $P_1 \perp P_2$ since $P_{11} \perp P_{22}$. (8) If $P_{11} \ne 0$ and $P_{22} \ne 0$ and $P_3 \ne 0$, then since $P_1 = P_{11} \vee P_3$, $P_{11} \le P_1$. However, $P_{11} = P_1$ is ruled out as follows, leaving just $P_{11} < P_1$. Since $P_{11} \ne 0$, $P_{11} = P_1$ IFF $P_3 \le P_{11}$. And since by assumption $P_3 \perp P_{11}$, i.e., $P_3 \le P_{11}^{\perp}$, we have $P_3 \le P_{11}$ only if $P_3 = 0$. But for this case (8), by assumption $P_3 \ne 0$. Thus $P_{11} < P_1$. <u>Mutatis</u> <u>mutandis</u> for $P_{22} < P_2$.
Q.E.D.

Lemma G: All the atoms of a maximal Boolean substructure of a $\mathcal{P}_{QM}$ are also atoms of $\mathcal{P}_{QM}$.

Proof: Assume on the contrary that there is a maximal Boolean substructure mBS_0 in $\mathcal{P}_{QM}$ and an element $P_0 \in \mathcal{P}_{QM}$ such that P_0 is an atom of mBS_0 but P_0 is not an atom of $\mathcal{P}_{QM}$. Since $\mathcal{P}_{QM}$ is an atomic structure, there is an atom $P_a \in \mathcal{P}_{QM}$ such that $P_a < P_0$ ($P_a \neq P_0$ since by assumption, P_0 is not an atom of $\mathcal{P}_{QM}$; and $P_a \notin mBS_0$ since by assumption P_0 is an atom of mBS_0 .) Now since all elements in a maximal Boolean substructure are mutually compatible, for every element $P \in mBS_0$, $P \mathrel{\overset{|}{\circ}} P_0$. It follows by Lemma F that, for every element $P \in mBS_0$ such that $P \neq 0$ and $P \neq P_0$, either (1) $P < P_0$, or (2) $P_0 < P$, or (3) $P_0 \perp P$, or (4) there are nonzero elements $P', P_0' \in mBS_0$ such that $P' < P$ and $P_0' < P_0$. Since by assumption, P_0 is an atom of mBS_0 and $P \neq 0$, possibility (1) $P < P_0$ is ruled out. Similarly, since by assumption, P_0 is an atom of mBS_0 , possibility (4) is ruled out. Now considering possibility (2), if $P_0 < P$, then since $P_a < P_0$ we have $P_a < P$, and so $P_a \mathrel{\overset{|}{\circ}} P$. Similarly, considering possibility (3), if $P_0 \perp P$, i.e., $P_0 \leq P^{\perp}$, then since $P_a < P_0$ we have $P_a < P^{\perp}$, and so $P_a \mathrel{\overset{|}{\circ}} P^{\perp}$, hence $P_a \mathrel{\overset{|}{\circ}} P$. So for every element $P \in mBS_0$ such that $P \neq 0$ and $P \neq P_0$, we have $P_a \mathrel{\overset{|}{\circ}} P$. Moreover, for $P = 0$, since $P_a \mathrel{\overset{|}{\circ}} 0$ we likewise have $P_a \mathrel{\overset{|}{\circ}} P$. And for $P = P_0$, since $P_a \mathrel{\overset{|}{\circ}} P_0$ we likewise have $P_a \mathrel{\overset{|}{\circ}} P$. That is, for every element $P \in mBS_0$, $P \mathrel{\overset{|}{\circ}} P_0$ and $P \mathrel{\overset{|}{\circ}} P_a$. So the set of mutually compatible elements $mBS_0 \cup \{P_a\}$ generate a Boolean substructure of $\mathcal{P}_{QM}$ which contains all the elements of mBS_0 plus P_a (and perhaps others). Thus mBS_0 is the proper subset of a Boolean substructure of $\mathcal{P}_{QM}$, which contradicts the definition of mBS_0 as a maximal Boolean substructure. Q.E.D.

Furthermore, given the conjecture that every mBS of a P_{QM} is atomic, it is a trivial point that all atoms of P_{QM} which are in an mBS of P_{QM} are also atoms of the mBS. For the only way an atom P_a of P_{QM} which is in an mBS of P_{QM} could <u>not</u> be an atom of mBS is if some other element $P \in$ mBS were between P_a and the 0-element in mBS but not in P_{QM}. But since mBS is a <u>sub</u>structure of P_{QM}, if some $P \in$ mBS were such that $0 \leq P \leq P_a$ in mBS then also $0 \leq P \leq P_a$ in P_{QM}, and so P_a would not be an atom of P_{QM}.

We also make use of the following results. As mentioned in Chapter I(D), Hughes has proven that any partial-Boolean algebra is isomorphic to a partial-Boolean algebra constructed on a family of Boolean algebras $\{B_i\}_{i \in \text{Index}}$, as described by Kochen-Specker. Among other conditions, the constructed partial-Boolean algebra A satisfies, for any elements $b,c,d \in A$, $b \vee c = d$ in A IFF there is a B_i such that $b \vee c = d$ in B_i. Now as part of his proof, Hughes shows that any partial-Boolean algebra can be constructed on the family of its own Boolean subalgebras. So in particular, any P_{QMA} can be constructed on the family of its own Boolean subalgebras. Thus we have the following

<u>Proposition E</u>: For any P_{QMA} and for any $P, P_1, P_2 \in P_{QMA}$, $P = P_1 \vee P_2$ in P_{QMA} IFF there is a Boolean substructure BS of P_{QMA} such that $P = P_1 \vee P_2$ in BS.

We also make use of these two lemmas.

<u>Lemma H</u>: For any $P_1, P_2 \in P_{QMA}$, if $P_1 \vee P_2$ is defined in P_{QMA}, i.e., if $P_1 \circ P_2$, then $P_1 \vee P_2$ is the <u>least upper bound</u> of $\{P_1, P_2\}$ in P_{QMA}.

Proof: Clearly, $P_1 \vee P_2 \geq P_1$ and $P_1 \vee P_2 \geq P_2$; thus $P_1 \vee P_2$ is an upper bound of $\{P_1,P_2\}$. And for any $P \in \mathcal{P}_{QMA}$, if $P \geq P_1$, i.e., $P \vee P_1 = P$ (and $P \mathrel{\stackrel{|}{\circ}} P_1$), and $P \geq P_2$, i.e., $P \vee P_2 = P$, then because $\mathcal{P}_{QMA}$ is an associative partial-Boolean algebra which satisfies: $P_1 \mathrel{\stackrel{|}{\circ}} (P_2 \vee P)$ IFF $(P_1 \vee P_2) \mathrel{\stackrel{|}{\circ}} P$, we have $(P_1 \vee P_2) \mathrel{\stackrel{|}{\circ}} P$ since $P_1 \mathrel{\stackrel{|}{\circ}} P$ and $P = P \vee P_2 = P_2 \vee P$ (i.e., $P_1 \mathrel{\stackrel{|}{\circ}} (P_2 \vee P)$). So $P \mathrel{\stackrel{|}{\circ}} (P_1 \vee P_2)$ and moreover, $P = P \vee P_2 = (P \vee P_1) \vee P_2 = P \vee (P_1 \vee P_2)$, i.e., $P \geq P_1 \vee P_2$. Q.E.D.

<u>Halmos' Lemma</u>: In an atomic Boolean algebra, every element is the join (least upper bound) of the atoms it dominates (Halmos, 1963, p. 70).

Now we may prove the following Theorem C for $\mathcal{P}_{QMA}$, which corresponds to the above Proposition D for $\mathcal{P}_{QML}$. (The proof is due to Levy, Robinson, Chernavska.)

<u>Theorem C</u>: For any $\mathcal{P}_{QMA}$ and for any $P_1,P_2 \in \mathcal{P}_{QMA}$, if $P_1 \neq P_2$, then the set of atoms dominated by P_1 is not equal to the set of atoms dominated by P_2 (i.e., there is an atom $P_\psi \in \mathcal{P}_{QMA}$ such that either $P_\psi \leq P_1$ and $P_\psi \nleq P_2$, or $P_\psi \leq P_2$ and $P_\psi \nleq P_1$).

Proof: Let A_1 be the set of atoms of $\mathcal{P}_{QMA}$ which are dominated by P_1, i.e., $A_1 = \{P_\psi \in \mathcal{P}_{QMA} : P_\psi \leq P_1\}$; and let A_2 be the set of atoms of $\mathcal{P}_{QMA}$ which are dominated by P_2. Assume $A_1 = A_2$. Clearly, if $A_1 = A_2 = \emptyset$ (the empty set), then $P_1 = P_2 = 0$. Assume then that $A_1 = A_2 \neq \emptyset$. Since $A_1 = A_2 \neq \emptyset$, there is a nonempty set $A_{\stackrel{|}{\circ}}$ of mutually compatible atoms of $\mathcal{P}_{QMA}$, each of which is dominated by P_1 and by P_2. (1) Since P_1 dominates each member of $A_{\stackrel{|}{\circ}}$, P_1 is compatible with each member of A_2 . Thus, $A_2 \cup \{P_1\}$ is a set of mutually compatible elements of $\mathcal{P}_{QMA}$. Hence there is a Boolean subalgebra of $\mathcal{P}_{QMA}$ containing P_1 and also containing all members of A_2 ; and this Boolean subalgebra is contained in a maximal Boolean subalgebra mBS' of $\mathcal{P}_{QMA}$. By Condition A, mBS' is atomic, and by Lemma G, all of its atoms are atoms of $\mathcal{P}_{QMA}$. Let $A' = \{P'_{\psi_i}\}_{i \in \text{Index}}$ be the set of all atoms of mBS' which are dominated by P_1 ; clearly, $A' \subseteq A_1$. Now by Halmos's Lemma, P_1 is the least upper bound of A' in mBS', i.e., $P_1 = \bigvee_i P'_{\psi_i}$ in mBS'. Then by Proposition E $P_1 = \bigvee_i P'_{\psi_i}$ in $\mathcal{P}_{QMA}$. And by Lemma H, $\bigvee_i P'_i$ is the least upper bound of A' in $\mathcal{P}_{QMA}$. Now P_2 dominates every member of $A_1 = A_2$, and $A' \subseteq A_1$, so P_2 dominates every member of A', and hence P_2 dominates the least upper bound of A', namely, P_1 .

(2) By a similar argument it can be shown thet P_1 dominates P_2 . Thus, $P_1 = P_2$. So if $P_1 \neq P_2$, then $A_1 \neq A_2$. Q.E.D.

And as in the $\mathcal{P}_{QML}$ case, the desired completeness result follows as an immediate

<u>Corollary to Theorem C</u>: For any $\mathcal{P}_{QMA}$, the collection of state-induced ultravaluations on the ultrasubstructures of $\mathcal{P}_{QMA}$ is complete, i.e., for any $P_1 \neq P_2$ in $\mathcal{P}_{QMA}$, there is an Exp_ψ such that $\text{Exp}_\psi(P_1) \neq \text{Exp}_\psi(P_2)$.

The proof of the corollary proceeds as in the $\mathcal{P}_{QML}$ case.

<u>Summary</u>

As described in Chapters II and III, a bivalent, truth-functional semantics for a Lindenbaum Boolean algebra L of classical propositional logic is a complete collection of ultravaluations on L, and a state-induced, bivalent, truth-functional semantics for a Boolean P_{CM} of classical mechanics is a complete collection of state-induced ultravaluations on P_{CM}. In both cases, an ultravaluation is a mapping which assigns the value 1 to the elements in an ultrafilter UF and assigns the value 0 to the elements in the dual ultraideal UI; thus an ultravaluation is said to be defined with respect to an UF and dual UI. Clearly, an ultravaluation is a bivalent mapping on UF $\cup$ UI, i.e., every element in UF $\cup$ UI is assigned a 0 or a 1 value. And it follows from the conditions satisfied by any UF and dual UI in a Boolean structure that an ultravaluation is a truth-functional mapping on UF $\cup$ UI. Moreover, because the L, P_{CM} structures are Boolean, for any UF and dual UI in an L or a P_{CM}, UF $\cup$ UI = L and UF $\cup$ UI = P_{CM}, thus the domain of each ultravaluation is the entire L, P_{CM} structure. And the completeness of the collection of ultravaluations on an L or a P_{CM} is ensured by the semi-simplicity property of Boolean structures. Furthermore, for the case of classical propositional logic, each ultravaluation on the L structure of equivalence classes of well-formed formulae in a (closed) set L is an algebraic version of one of the standard valuations for L, which is part of the reason ultravaluations are so called and is the main reason why ultravaluations on any other propositional or logical structure are regarded in this thesis as semantic mappings. And for the case of classical mechanics, ultravaluations are said to be state-induced because in fact it is the states of a classical mechanical system which induce mappings, namely, dispersion-free classical probability measures, each of which μ_w is an ultravaluation on the $UF_w \cup UI_w = P_{CM}$ structure of propositions which make assertions about the values of the system's magnitudes.

When we consider a P_{QM} structure of propositions which make assertions about the values of a quantum mechanical system's magnitudes, the states of the system similarly induce mappings, namely, dispersive generalized probability measures, each of which Exp_{ψ} is an ultravaluation on $UF_{\psi} \cup UI_{\psi} \subseteq P_{QM}$. And as in the classical cases, each state-induced ultravaluation Exp_{ψ} is a <u>bivalent</u> mapping on $UF_{\psi} \cup UI_{\psi}$; and it follows from the conditions satisfied by any UF_{ψ} and dual UI_{ψ} in a P_{QM} that each state-induced ultravaluation is a <u>truth-functional ((⫰) or (⫰,⫰̸))</u> mapping on $UF_{\psi} \cup UI_{\psi}$. But unlike the classical cases in which $UF \cup UI = L$ and $UF_{w} \cup UI_{w} = P_{CM}$ for every ultrafilter and dual ultraideal in L, P_{CM} , for the quantum case, if P_{QM} contains incompatible elements, then not every ultrafilter and dual ultraideal in P_{QM} is such that $UF_{\psi} \cup UI_{\psi} = P_{QM}$, rather, for some UF_{ψ} and dual UI_{ψ} , $UF_{\psi} \cup UI_{\psi} \subset P_{QM}$. When $UF_{\psi} \cup UI_{\psi}$ is less than the entire P_{QM} , we can at least be sure that $UF_{\psi} \cup UI_{\psi}$ is a closed substructure of P_{QM} , which may be called an ultrasubstructure. However, the affiliated state-induced ultravaluation Exp_{ψ} is a bivalent, truth-functional ((⫰) or (⫰,⫰̸)) mapping on just that ultrasubstructure US_{ψ} of P_{QM} . Thus while every ultravaluation on an L and every state-induced ultravaluation on a P_{CM} is a bivalent, truth-functional mapping on the entire structure, at least some of the state-induced ultravaluations on a P_{QM} containing incompatible elements are bivalent, truth-functional ((⫰) or (⫰,⫰̸)) mappings on <u>just ultrasubstructures</u> of P_{QM} rather than on the entire P_{QM} . Moreover, the <u>completeness</u> of the collection of state-induced ultravaluations on the ultrasubstructures of a P_{QM} must be proven, as done in Section D.

However, the fact that $UF_{\psi} \cup UI_{\psi} \subset P_{QM}$ for some UF_{ψ} and dual

UI_ψ in a P_{QM} containing incompatible elements <u>need</u> <u>not</u> <u>be</u> a problematic feature and <u>is</u> <u>not</u> <u>the</u> <u>only</u> problematic feature of the quantum P_{QM} structures. As described in Chapter V(B), if we ignore the lattice meets and joins of incompatibles and consider the proposal of a bivalent, truth-functional(♁) semantics for a P_{QM}, the presence of incompatible elements in P_{QM} is necessary but not sufficient to rule out a bivalent, truth-functional(♁) semantics for P_{QM}. For a two-dimensional Hilbert space P^2_{QM} does admit a bivalent, truth-functional(♁) semantics in spite of the presence of incompatible elements. The peculiar structural feature of three-or-higher dimensional Hilbert space $P^{n\geq 3}_{QM}$ structures which does rule out a bivalent, truth-functional(♁) semantics is the presence of overlapping maximal Boolean substructures in $P^{n\geq 3}_{QM}$, for which the presence of incompatible elements is a necessary (but not a sufficient) condition. The following similar remarks apply here in the Chapter VI discussion of the proposal of a semantics for P_{QM} consisting of a complete collection of (state-induced) ultravaluations on the ultrasubstructures of P_{QM}:

If we ignore the lattice meets and joins of incompatibles and consider the proposal of a bivalent, truth-functional(♁) semantics for P_{QM} consisting of ultravaluations, then the fact that $UF_\psi \cup UI_\psi \subset P_{QM}$ (rather than $UF_\psi \cup UI_\psi = P_{QM}$) for some UF_ψ and dual UI_ψ in any P_{QM} which contains incompatible elements would by itself be harmlessly unproblematic <u>if</u> $UF_\psi \cup UI_\psi$ were equal to just a mBS of P_{QM} and <u>if</u> the mBS's of P_{QM} were non-overlapping. For example, both these "if's" obtain in a P^2_{QM} and as described in Chapter IV(F), a P^2_{QM} does admit a complete collection of bivalent, truth-functional(♁) mappings, where by inspection it is clear that each of the four bivalent, truth-functional(♁) mappings on the six-element P^2_{QM} explicitly considered in that Chapter IV(F) is in fact a sum of two ultravaluations, each defined on one of the two ultrasubstructures of the six-element P^2_{QM} . That is, $UF_1 = \{P \in P^2_{QM} : P \geq P_1\} = \{P_1,1\}$, $UI_1 = \{P \in P^2_{QM} : P \leq P_1^{\perp}\} = \{P_1^{\perp},0\}$, and $US_1 = UF_1 \cup UI_1 = mBS_1$; $UF_2 = \{P_2,1\}$, $UI_2 = \{P_2^{\perp},0\}$, and $US_2 = UF_2 \cup UI_2 = mBS_2$. So each US_i equals an mBS_i in P^2_{QM} , and as described earlier in that section, the mBS's of P^2_{QM} do not overlap. (And the six-element P^2_{QM} equals the union of the two ultrasubstructures $US_1 \cup US_2$.) Moreover, the two mappings h_a , h_b , are ultravaluations on US_1 , the two mappings h_c , h_d are ultravaluations on US_2 , and each of the four bivalent, truth-functional(♁) mappings h_1 , h_2 , h_3 , h_4 , on the entire six-element P^2_{QM} is the sum of an ultravaluation on US_1 plus an ultravaluation on US_2 . Thus, the six-element P^2_{QM} , and more generally, any P^2_{QM} does admit a bivalent, truth-functional(♁) semantics consisting of a complete collection of bivalent, truth-functional(♁) mappings on the entire P^2_{QM} , each of which is a sum of ultravaluations on the ultrasubstructures of P^2_{QM} . So the fact that $UF_\psi \cup UI_\psi \subset P_{QM}$ for some UF_ψ and dual UI_ψ in any P_{QM} containing incompatible elements <u>need not be</u> a problematic feature. In particular, the ultravaluations on the ultrasubstructures of a P^2_{QM} containing incompatible elements may be added together to yield a complete collection of bivalent, truth-functional(♁) mappings on the entire P^2_{QM} , and thus a bivalent, truth-functional(♁) semantics for P^2_{QM} , in spite of that fact.

However, neither of the above, underlined "if's" obtain in a three-or-higher dimensional Hilbert space $P^{n\geq 3}_{QM}$. That is, the mBS's of a $P^{n\geq 3}_{QM}$ may overlap, and the ultrasubstructures in a $P^{n\geq 3}_{QM}$ may be larger than any mBS, for each ultrasubstructure US_ψ in a $P^{n\geq 3}_{QM}$ is equal to the union of all the overlapping mBS's in $P^{n\geq 3}_{QM}$ which share the atom P_ψ , as shown in Section B. So the fact that $UF_\psi \cup UI_\psi \subset P^{n\geq 3}_{QM}$ for some UF_ψ and dual

UI_ψ in any $P_{QM}^{n\geq 3}$ containing incompatible elements is problematic. In particular, the best we can do for such a $P_{QM}^{n\geq 3}$ is to define bivalent, truth-functional ((⫰) or (⫰,⫰̸)) mappings on its ultrasubstructures. We cannot add together these ultravaluations on the ultrasubstructures of a $P_{QM}^{n\geq 3}$ to get bivalent, truth-functional(⫰) mappings on the entire $P_{QM}^{n\geq 3}$. But the fact that $UF_\psi \cup UI_\psi \subset P_{QM}^{n\geq 3}$ <u>is not the only reason</u> why we cannot add the ultravaluations in the suggested manner; the other reason is that an ultrasubstructure $UF_\psi \cup UI_\psi$ in a $P_{QM}^{n\geq 3}$ is a union of overlapping mBS's in $P_{QM}^{n\geq 3}$.

Other sorts of semantic mappings may be and have been proposed for the quantum P_{QM} structures. But in this thesis, only two semantic proposals have been seriously considered: the proposal of a bivalent, truth-functional semantics for P_{QM} and the proposal of a state-induced semantics for P_{QM}. The former is motivated by the success and usefulness of such a semantics for classical logical and propositional structures such as L, P_{CM}. The latter is motivated by the fact that the state-induced semantics for a P_{CM}, consisting of state-induced μ_w mappings already present in the formalism of classical mechanics, works exactly like the algebraic version of the standard, bivalent, truth-functional semantics of classical propositional logic. And the proposal of a state-induced semantics for P_{QM} is motivated by the fact that the quantum formalism, like the classical formalism, includes state-induced mappings which assign 0, 1 values to representatives of quantum propositions, i.e., to the projectors or subspaces of a Hilbert space. So for a P_{QM}, it is worth considering the notion of a state-induced semantics consisting of the state-induced Exp_ψ mappings already present in the quantum formalism. Like the classical semantic mappings on an L, like the classical state-induced μ_w mappings on a $\mathcal{P}_{CM}$, and like the Friedman-Glymour S3-valuations proposed for a $\mathcal{P}_{QML}$, the state-induced Exp_ψ mappings are ultravaluations on the ultrasubstructures of $\mathcal{P}_{QM}$. So the basic semantic method in all these cases is the same. The crucial difference between the classical and the quantum cases is that, for some UF_ψ and dual UI_ψ in any $\mathcal{P}_{QM}$ containing incompatible elements, $UF_\psi \cup UI_\psi$ is <u>smaller</u> than $\mathcal{P}_{QM}$ rather than being equal to the entire $\mathcal{P}_{QM}$, and moreover, $UF_\psi \cup UI_\psi$ is <u>larger</u> than any mBS in a $\mathcal{P}_{QM}^{n\geq 3}$ because $UF_\psi \cup UI_\psi$ is the union of all overlapping mBS's in $\mathcal{P}_{QM}^{n\geq 3}$ which contain the atom P_ψ .

<u>Notes</u>:

[1]Though other conditions are sometimes taken as defining the orthogonality relation, e.g., P_1 , P_2 are orthogonal IFF $P_1 \cdot P_2 = 0$, these conditions are satisfied by any $P_1, P_2 \in \mathcal{P}_{QM}$ IFF $P_1 \leq P_2^{\perp}$, i.e., IFF P_1 , P_2 are disjoint. And, for example, Piron takes the $P_1 \leq P_2^{\perp}$ condition as defining the orthogonality relation (Piron, 1976, p. 29).

[2]This was pointed out to me independently by Dr. L. P. Belluce and Dr. J. V. Whittaker.

[3]The fact that $P_1 \wedge P_2$ and $P_1 \vee P_2$ are not defined in $\mathcal{P}_{QMA}$ when $P_1 \not\leftrightarrow P_2$ does not mean that the union $UF \cup UI$ is in any way <u>not</u> closed with respect to the $\wedge$, $\vee$ operations of $\mathcal{P}_{QMA}$. The $\wedge$, $\vee$ operations of $\mathcal{P}_{QMA} = \langle E, \leftrightarrow, \leq, \wedge, \vee, ^{\perp}, 0, 1 \rangle$ are defined from $\leftrightarrow \subseteq E \times E$ to E rather than from $E \times E$ to E. Thus Kochen-Specker call them partial-operations or partial-functions (1965, pp. 177, 178). By closure with respect to the $\wedge$, $\vee$ operations of $\mathcal{P}_{QMA}$, I mean closure with respect to these operations <u>qua</u> partial-operations.

[4]Thanks to Dr. Edwin Levy for suggesting the "ultra" terminology.

[5]In an earlier draft, I claimed that each quantum expectation-function Exp_ψ on a $\mathcal{P}_{QM}$ is bivalent and truth-functional with respect to a Boolean substructure of mutually compatible elements in $\mathcal{P}_{QM}$. Thanks to Jeffrey Bub and Edwin Levy for helping clarify that in fact, the subset of elements in a $\mathcal{P}_{QM}$ which are assigned 0, 1 values by an Exp_ψ on $\mathcal{P}_{QM}$ may include incompatible elements and so may be larger than

any Boolean substructure of $\mathcal{P}_{QM}$.

[6]Page 1 of a manuscript by Edwin Levy circulated in December, 1977.

[7]As the external examiner, van Fraassen pointed out this alternate proof of the completeness result in his report on the thesis.

CHAPTER VII

HIDDEN-VARIABLES RECONSIDERED

Preface

In classical mechanics, a pure state w specifies an exact value for any magnitude. But in quantum mechanics, a pure state ψ specifies an exact (eigen)value for only those magnitudes whose eigenstates are compatible with ψ. Any magnitude A whose eigenstates are incompatible with ψ may, upon measurement, exhibit any of its (eigen)values. In the quantum formalism, for the given state ψ the average value of A is determined by $\mathrm{Exp}_{\psi}(A) = \Sigma_i \, a_i \| \psi_i >< \psi \|^2$, and the probability that A will exhibit any one of its (eigen)values, say a_j, is determined by $\mathrm{Exp}_{\psi}(\hat{P}_{\psi_j}) = \| \psi_j >< \psi \|^2$, where $\hat{P}_{\psi_j}$ represents the eigenstate of A associated with the (eigen)value a_j. But the quantum formalism does not determine which exact (eigen)value A will exhibit. In other words, quantum systems characterized by the same quantum state ψ exhibit, upon measurement, different values for the same magnitude A, yet the quantum formalism does not determine which of the different values of A will be exhibited. For this reason, it has been argued that quantum mechanics is incomplete and should be supplemented by a hidden variable theory which reflects the different possible outcomes of a measurement of A for a given ψ.

In terms of the quantum propositions, this problem is connected with the fact, described in Chapter VI, that any P_{QM} which contains incompatible elements has at least one ultrasubstructure $US_{\psi} = UF_{\psi} \cup UI_{\psi}$

which is smaller than the entire $\mathcal{P}_{QM}$, and each element of $\mathcal{P}_{QM}$ which is outside US_ψ is assigned a value between 0 and 1 by the affiliated state-induced Exp_ψ rather than being assigned an exact 0 or 1 value by Exp_ψ . At the very least, such a value between 0 and 1 is interpreted as the probability that an element $P \notin US_\psi$, qua idempotent magnitude, will upon measurement exhibit its (eigen)value 1, that is, as the probability that P, qua proposition, is true of a system or ensemble of systems whose state is ψ. So again, quantum systems characterized by the same quantum state ψ exhibit upon measurement, sometimes the truth-value 0 and sometimes the truth-value 1 for the same proposition $P \notin US_\psi$, but which of these truth-values will be the outcome of a measurement is not determined by the quantum formalism.

Now if we presume that the physical theory of quantum phenomena should include a formalism which does determine, given the state of a quantum system, exactly whether any $P \in \mathcal{P}_{QM}$ is true or false, then quantum mechanics is indeed an incomplete theory and we must seek a supplementary formalism. The proposals of such supplementary formalisms have been called hidden-variable theories. Hidden-variable (HV) theories are extensions or reconstructions of quantum mechanics which introduce further specifications of the state of a quantum system so that the so-called hidden state determines the exact values of magnitudes and propositions which are assigned dispersive values by a quantum state-induced Exp_ψ . So while a quantum state ψ induces the generalized probability measure $Exp_\psi : \mathcal{P}_{QM} \to [0,1]$ which is dispersive with respect to every $P \notin US_\psi$, a hidden state induces or is associated with a dispersion-free probability measure which somehow assigns an exact 0 or 1 value to such $P \notin US_\psi$. And so in an HV reconstruction of quantum mechanics, the presumed incompleteness of quantum mechanics is reflected by the fact that the set of quantum Exp_{ψ} measures is a proper subset of a larger set of measures which includes the dispersion-free HV measures.

The dispersion-free measures added by an HV theory may be classical probability measures on some Boolean structure proposed by the HV theory, or they may be some sort of generalized probability measures defined on the quantum P_{QM} (or on substructures of P_{QM}). Von Neumann, Jauch-Piron and Gleason-Kochen-Specker prove the impossibility of three kinds of generalized dispersion-free measures on the quantum P_{QM} structures, as described in Chapter V(D); they thus rule out three kinds of HV theories, as shall be elaborated below. But besides these three proposed but impossible kinds of HV theories, contextual HV theories whose dispersion-free measures avoid the above impossibility proofs have also been proposed. In all four cases, each quantum Exp_{ψ} measure is represented in the proposed HV theory as a mixture or complex, e.g., a convex sum or weighted integral, of dispersion-free HV measures. And all four kinds of HV proposals impose a statistical condition requiring that the complexes which represent the quantum Exp_{ψ} measures in the HV theory must yield statistical results which reproduce the results given by the quantum Exp_{ψ} measures (and so far observed by experiment) (Kochen-Specker, 1967, p. 59; Belinfante, 1973, p. 9).

However, as Kochen-Specker argue, the imposition of this statistical condition alone does not yet take into consideration the structural and functional relations among the quantum magnitudes (and propositions). These relations are embodied in the algebraic structure of the quantum magnitudes, and concordantly, in the P_{QM} structure of the quantum propositions. Von Neumann, Jauch-Piron, Gleason, and Kochen-Specker do take this consideration into account by requiring that some or all of the operations and relations of $\mathcal{P}_{QM}$ must be preserved in an HV reconstruction of quantum mechanics. Such requirements may be called *structural conditions*. As shown at length in Chapter V(D), each of these authors imposes a structural condition which boils down to the requirement that dispersion-free HV measures, *qua* generalized probability measures on the quantum $\mathcal{P}_{QM}$, must *preserve the partial-Boolean structural features of $\mathcal{P}_{QM}$* (i.e., $\mathcal{P}_{QMA}$-preservation), or in other words, proposed dispersion-free HV measures must be bivalent homomorphisms($\flat$) on $\mathcal{P}_{QM}$. In addition, von Neumann and Jauch-Piron each impose a structural condition, labeled (vN$\not\flat$) and (JP$\not\flat$) in Chapter V(D), which requires that proposed dispersion-free HV measures preserve an operation among *in*compatibles. So von Neumann's notion and Jauch-Piron's notion of what is a generalized probability measure on $\mathcal{P}_{QM}$ is clearly different from Gleason's and Kochen-Specker's notion. Now all of these structural conditions are satisfied by the quantum Exp_ψ measures on $\mathcal{P}_{QM}$. The contentious issue is whether or not the proposed dispersion-free HV measures introduced by a proposed HV extension or reconstruction of quantum mechanics must also be required to satisfy these structural conditions.

The three kinds of HV proposals which require their dispersion-free HV measures to satisfy the three different sets of structural conditions imposed by von Neumann, Jauch-Piron, and Gleason-Kochen-Specker have been shown by these authors to be impossible; either the dispersion-free HV measures are themselves impossible or else complexes of the dispersion-free HV measures cannot reproduce the statistical results of the quantum Exp_ψ

measures, as required by the statistical condition. However, critics of the above HV impossibility proofs and advocates of the contextual HV proposals have brought forth the following three sorts of arguments against the imposition of the structural conditions upon proposed dispersion-free HV measures:

(i) The structural conditions are inconsistent with the other conditions which are imposed upon the proposed HV measures, and so the structural conditions immediately rule out a HV theory. But rather than concluding that an HV theory is impossible, we should reject the structural conditions. (ii) The structural conditions relate the results of different measurements in ways which are not justified if we take into account the interaction between measuring instruments and quantum phenomena. Thus the imposition of the structural conditions begs the question, and these conditions should be rejected. (iii) The imposition of the structural conditions and the development of the impossibility proofs beg the HV question in other ways. So the structural conditions should be rejected and the von Neumann, Jauch-Piron, Gleason, and Kochen-Specker proofs do not in fact show the impossibility of an HV reconstruction of quantum mechanics.

In Section A, these criticisms are described in detail. Then in Section B, another perspective on quantum mechanics and the problem of hidden-variables is introduced, according to which the structural conditions (vN∅) and (JP∅) (and thus the von Neumann and the Jauch-Piron proofs) succumb to the above criticisms, but the structural condition of P_{QMA}-preservation (and thus the Gleason and Kochen-Specker proofs) are rescued from these criticisms.

Section A. Criticisms of the Hidden-Variable Impossibility Proofs

Von Neumann poses the question of whether the dispersive ensembles of quantum systems can be resolved into sub-ensembles which are dispersion-free for any quantum magnitude; in his view, an HV reconstruction of quantum mechanics involves such a resolution. Ensembles of quantum systems are characterized by expectation-functions, and so the question is whether the dispersive quantum Exp_ψ functions can be represented as mixtures or weighted sums of different dispersion-free HV Exp_w functions (von Neumann, 1932, pp. 305-307, 324). Von Neumann defines an expectation-function by a list of conditions, one of which subsumes the two conditions labeled $(vN\flat)$ and $(vN\not\flat)$ in Chapter V(D). The domain of an expectation-function is the set of quantum magnitudes as represented by operators on a Hilbert space. And the functional relations among the magnitudes are given by the functional relations among the operators, that is, by the algebraic structure of the operators. So in terms of the quantum propositions qua idempotent magnitudes, a necessary condition for an HV reconstruction of quantum mechanics is, in von Neumann's view, the existence of dispersion-free Exp_w functions on the quantum $\mathcal{P}_{QM}$ structures.[1]

As mentioned in Chapter V(D), using his trace-formalism, von Neumann proves that no such dispersion-free Exp_w exist; I referred to this result as von Neumann's impossibility proof. In addition, von Neumann proves that homogeneous expectation-functions do exist and in fact correspond to the quantum Exp_ψ functions induced by the pure quantum ψ states. So the quantum Exp_ψ cannot be represented as mixtures of dispersion-free Exp_w, first because the quantum Exp_ψ are themselves homogeneous (where by definition a homogeneous Exp cannot be represented as a weighted sum of different Exp-functions), and second because the dispersion-free Exp_w do not exist (von Neumann, 1932, p. 324). It is thus that an HV reconstruction of quantum mechanics is impossible, according to von Neumann.

In 1966, Bell discredited von Neumann's impossibility proof by arguing that it rests upon an inconsistency between the requirement that HV expectation-functions satisfy (vN$\not{b}$) and the requirement that HV expectation-functions be dispersion-free. For (vN$\not{b}$) requires the additivity of the expectation values of incompatible magnitudes and incompatible propositions qua idempotent magnitudes, and the dispersion-free expectation value of any magnitude or proposition is an eigenvalue of the magnitude or proposition. But since the eigenvalues of incompatible magnitudes or propositions are not additive, an HV expectation-function which satisfies (vN$\not{b}$) and is dispersion-free is impossible (Bell, 1966, p. 449). The Kochen-Specker version of von Neumann's impossibility proof shows clearly how (vN$\not{b}$) is the culprit in the proof and so further substantiates Bell's criticism (Kochen-Specker, 1967, pp. 81-82). Such HV proposals whose imposed conditions are inconsistent with each other are called HV theories of the zero-th kind by Balinfante; their impossibility is not surprising.

Bell also appeals to the problem of measurement interaction in order to argue that HV measures (or expectation-functions) need not satisfy (vn$\not{b}$). The result of a measurement of the sum $P_1 + P_2$ of two incompatible propositions cannot be calculated by simply adding together the results of separate measurements for P_1, P_2. For as exemplified by von Neumann (1932, p. 310), a measurement of a sum $P_1 + P_2$ of incompatibles involves an experimental arrangement which is entirely different from the arrangements by which P_1 and P_2 are each measured separately. Now although the expectation-value assigned to $P_1 + P_2$ by any quantum Exp_ψ always does equal the sum of the expectation-values assigned by Exp_ψ to each of the P_1 , P_2 separately, this is not a trivial or necessary feature of the quantum Exp_ψ measures. Rather, it is a very peculiar feature of the quantum Exp_ψ measures, especially when, as Bell suggests, one remembers with Bohr "the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with measuring instruments which serve to define the conditions under which the [quantum] phenomena appear" (Bohr quoted by Bell, 1966, p. 447). Bell concludes that there is no reason to demand that proposed dispersion-free HV measures must be additive with respect to incompatible magnitudes and propositions, as (vN$\not{b}$) requires. So when von Neumann imposes his condition (vN$\not{b}$) and then proves that dispersion-free HV measures are impossible and thus proves that an HV reconstruction of quantum mechanics is impossible, he is open to the charge of begging the HV question since (vN$\not{b}$) is unjustified.

Furthermore, von Neumann's imposition of (vN$\not{b}$) on proposed dispersion-free HV measures begs the HV question in another way. One of the conditions which von Neumann incorporates as part of his list of conditions defining an expectation-function Exp in general is the following, which he labels (E):

(E) If $A, B, \ldots$ are arbitrary magnitudes, then there is an additional magnitude $A + B + \cdots$ (which does not depend on the choice of the expectation-function), such that
$\mathrm{Exp}(A + B + \cdots) = \mathrm{Exp}(A) + \mathrm{Exp}(B) + \cdots$

(von Neumann, pp. 309, 311). With this condition (E), von Neumann lets the expectation-functions define the sum of incompatible magnitudes, e.g., the sum of A, B is that magnitude which satisfies (E) for all expectation-functions. Von Neumann motivates this definition by two facts: The sum of the operators $\hat{A}$, $\hat{B}$ (representing the magnitudes A, B) is itself a self-adjoint operator which can represent a quantum magnitude; and for all quantum Exp_{ψ} expectation-functions, $Exp_{\psi}(\hat{A} + \hat{B}) = Exp_{\psi}(\hat{A}) + Exp_{\psi}(\hat{B})$. Now if we assume that dispersion-free HV Exp_{W} expectation-functions do exist, then the sum of A, B as defined by all the quantum Exp_{ψ} and HV Exp_{W} may be different from the sum of A, B as defined by just all the quantum Exp_{ψ}. And for example, although the operator $\hat{A} + \hat{B}$ does represent the magnitude which is the sum of A, B as defined by all the quantum Exp_{ψ}, the operator $\hat{A} + \hat{B}$ may not represent the sum of A, B as defined by all the quantum Exp_{ψ} and HV Exp_{W}, in which case $Exp_{W}(\hat{A} + \hat{B}) \neq Exp_{W}(\hat{A}) + Exp_{W}(\hat{B})$, contrary to von Neumann's (vN$\not{\phi}$) condition. Of course, if the dispersion-free HV Exp_{W} are impossible, then the two sums are the same. However, von Neumann imposes (vN$\not{\phi}$) which presumes that the two sums are the same (and so presumes that dispersion-free HV Exp_{W} do not exist) and which requires proposed dispersion-free HV Exp_{W} to satisfy $Exp_{W}(\hat{A} + \hat{B}) = Exp_{W}(\hat{A}) + Exp_{W}(\hat{B})$, and then von Neumann proves that the proposed dispersion-free HV Exp_{W} are impossible. Thus von Neumann is begging the HV question because the imposition of condition (vN$\not{b}$) presumes what is being proved, namely, the impossibility or non-existence of dispersion-free HV Exp_{W} functions.[2]

As mentioned in Chapter V(D), using the structural condition (JP$\not{b}$), Jauch-Piron prove in their Corollary 1 that dispersion-free measures are impossible on any irreducible orthomodular lattice. This, they say, is von Neumann's old result, i.e., von Neumann's proof of the impossibility of dispersion-free measures, proven without the contentious condition (vN$\not{b}$). However, Jauch-Piron argue that the quantum superselection rules ensure that the quantum orthomodular lattice $\mathcal{P}_{QML}$ structures are not irreducible but rather are reducible lattices with non-trivial centres. So Corollary 1 does not rule out dispersion-free measures on the quantum $\mathcal{P}_{QML}$.

Now according to Jauch-Piron, a quantum $\mathcal{P}_{QML}$ which does admit hidden-variables is characterized by the following property: Every measure on a $\mathcal{P}_{QML}$ which admits hidden-variables can be represented as a weighted sum of dispersion-free measures on $\mathcal{P}_{QML}$, in particular, every quantum Exp_ψ measure on $\mathcal{P}_{QML}$ can be so represented. Then in their Corollary 3 and again in their Theorem 2, Jauch-Piron prove that an orthomodular lattice admits hidden-variables only if all its elements are mutually compatible, i.e., only if the lattice is Boolean. So any quantum $\mathcal{P}_{QML}$ which contains incompatible elements does not admit hidden-variables, and hence hidden-variables are impossible in quantum mechanics (Jauch-Piron, 1963, pp. 835-837).

Bub's elucidation of Jauch-Piron's work shows clearly how condition (JP$\not{b}$) is the culprit in their impossibility proof(s). For Bub shows how the quantum Exp_ψ measures on a $\mathcal{P}_{QML}$ cannot be represented as weighted sums of dispersion-free measures on $\mathcal{P}_{QML}$ when the dispersion-free HV measures are required to satisfy (JP$\not{b}$) (Bub, 1974, pp. 61-62). For example, consider a quantum Exp_ψ which assigns values to two incompatible atoms P_ψ, P_φ of a $\mathcal{P}_{QML}$ as follows: $\mathrm{Exp}_\psi(P_\psi) = 1$, $\mathrm{Exp}_\psi(P_\varphi) = \||\varphi\rangle\langle\psi|\|^2 \in (0,1)$, and since $P_\psi \wedge P_\varphi = 0$, $\mathrm{Exp}_\psi(P_\psi \wedge P_\varphi) = \mathrm{Exp}_\psi(0) = 0$. According to the Jauch-Piron characterization of a hidden-variables proposal, if $\mathcal{P}_{QML}$ admits hidden-variables then this Exp_ψ measure on $\mathcal{P}_{QML}$ can be represented as a weighted sum $\Sigma_i \lambda_i w_i$, where $\Sigma_i \lambda_i = 1$ and each w_i is a dispersion-free (HV) measure on $\mathcal{P}_{QML}$. Now in order to reproduce the assignment $\mathrm{Exp}_\psi(P_\psi) = 1$, each w_i must assign the value 1 to P_ψ, i.e., for every w_i in the sum representing Exp_ψ, $w_i(P_\psi) = 1$ so that $\Sigma_i \lambda_i w_i(P_\psi) = 1 = \mathrm{Exp}_\psi(P_\psi)$. And since $P_\psi \wedge P_\varphi = 0$, $w_i(P_\psi \wedge P_\varphi) = w_i(0) = 0$, for every w_i in the sum representing Exp_ψ. Moreover, none of the w_i can assign the value 1 to P_ψ because by (JPϕ), $w_i(P_\psi) = 1$ and $w_i(P_\varphi) = 1$ yields $w_i(P_\psi \wedge P_\varphi) = 1$, which contradicts $w_i(P_\psi \wedge P_\varphi) = 0$; so $w_i(P_\varphi) = 0$ for every w_i in the sum representing Exp_ψ. Thus the <u>non</u>zero value assigned by Exp_ψ to P_φ cannot be reproduced by any weighted sum which reproduces the value assignment $\mathrm{Exp}_\psi(P_\psi) = 1$. That is, a weighted sum of dispersion-free (HV) measures satisfying (JPϕ) cannot reproduce the value assignments of this quantum Exp_ψ measure.

So we can view the impossibility of a Jauch-Piron type of HV proposal as being due to an inconsistency between three conditions imposed on proposed HV measures: the structural condition (JPϕ), the dispersion-free condition, and the statistical condition, which requires that the value assignments of the quantum Exp_ψ measures be reproduced by, e.g., a weighted sum of dispersion-free HV measures. Thus, as Belinfante says, rather than proving the impossibility of hidden-variables, Jauch-Piron have merely shown that their type of HV proposal is of the zero-th kind (Belinfante, 1973, p. 59).

Bell's objection to the structural condition (JPϕ) is similar to his objection to (vNϕ). When P_ψ, P_φ are incompatible, a measurement of their meet $P_\psi \wedge P_\varphi$ involves an experimental arrangement which differs from the arrangements by which P_ψ and P_φ are each measured separately.

yet (JP$\not\psi$) requires a proposed dispersion-free HV measure to assign the value 1 to $P_\psi \wedge P_\varphi$ if it assigns the value 1 to each P_ψ, P_φ, separately. In spite of the different experimental arrangements, the quantum Exp_ψ measures do satisfy (JP$\not\psi$). And so it is reasonable to require that the weighted sums of dispersion-free HV measures which represent the quantum Exp_ψ measures in an HV reconstruction likewise satisfy (JP$\not\psi$). But it is not reasonable to require that each dispersion-free HV measure must itself satisfy (JP$\not\psi$), especially when we recall the problem of measurement interaction. So when Jauch-Piron impose their structural condition (JP$\not\psi$) on proposed dispersion-free HV measures and then show that hidden-variables are impossible, they are open to the charge of begging the HV question since their imposition of (JP$\not\psi$) is not justified.

Bub also argues that the Jauch-Piron impossibility proof(s) beg the HV question in the following manner. Jauch-Piron prove the impossibility of representing the quantum Exp_ψ measures on a $\mathcal{P}_{\mathrm{QML}}$ as mixtures of dispersion-free HV measures on $\mathcal{P}_{\mathrm{QML}}$. That is, the HV measures considered by Jauch-Piron are a sort of <u>generalized</u> probability measure defined on the quantum $\mathcal{P}_{\mathrm{QML}}$. But then the Jauch-Piron proof does not rule out the further possibility of representing the quantum Exp_ψ measures as mixtures of dispersion-free HV measures which are <u>classical</u> probability measures defined on a Boolean structure (Bub, 1974, p. 63).

The same criticism can be directed against the proofs and arguments by which von Neumann purports to show the impossibility of hidden-variables. For von Neumann refers to dispersion-free HV expectation-functions defined on the set of quantum propositions, <u>qua</u> idempotent magnitudes represented by projectors, whose structure is a quantum $\mathcal{P}_{\mathrm{QM}}$. Similarly, Gleason's impossibility proof and the

Kochen-Specker Theorem 1 version of Gleason's proof are also subject to this criticism. For Gleason's proof shows that his sort of generalized dispersion-free HV measures (which satisfy (Ga) and thus are P_{QMA}-preserving) are impossible on the quantum $P_{QM}^{n\geq 3}$ structures, that is, in Kochen-Specker's version, bivalent homomorphisms(δ) are impossible on $P_{QM}^{n\geq 3}$. But Gleason's proof and Kochen-Specker's Theorem 1 do not address the above-mentioned further possibility of representing the quantum Exp_ψ measures as mixtures of <u>classical</u> dispersion-free HV measures defined on a Boolean structure.

Moreover, Bub argues that this further possibility precisely captures the HV enterprise which Kochen-Specker <u>do</u> correctly formulate and address yet which their Theorem 1 alone does <u>not</u> rule out. Correctly formulated, the HV enterprise can be said to be the attempt to reconstruct the statistical results given by $\langle H, P_{QM}, \mathrm{Exp}_\psi \rangle$, i.e., the quantum generalized Exp_ψ probability measures on the non-Boolean P_{QM} structure of the quantum phase space H (Hilbert space), in terms of a classical measure space $\langle \Omega, P_{HV}, \mu \rangle$, i.e., classical μ probability measures on the Boolean P_{HV} structure of a postulated HV classical phase space Ω (Kochen-Specker, 1967, pp. 62, 75). Thus an HV theory may be said to be a <u>Boolean reconstruction of quantum mechanics</u>.[3]

More explicitly, as described by Kochen-Specker, a Boolean HV reconstruction of quantum mechanics can be formulated as follows. Like the formalism of classical mechanics described in Chapter III, an HV theory posits a classical phase space Ω; each point $w \in \Omega$ represents a pure hidden state, and each real-valued (Borel) function $f_A : \Omega \to R$ represents a magnitude in the HV theory. The idempotent functions on Ω, or equivalently, the Borel subsets of Ω, form a Boolean structure which may be labeled P_{HV}. Like the P_{CM} structure, this P_{HV} is regarded as the propositional structure of the HV theory. That is, an idempotent function $f_P : \Omega \to \{0,1\}$, or corresponding Borel subset $W_P \subset \Omega$, represents a proposition in the HV theory. Each pure hidden state w induces a dispersion-free classical probability measure $\mu_w : \mathcal{P}_{HV} \to \{0,1\}$ which is a bivalent homomorphism on $\mathcal{P}_{HV}$.

The HV reconstruction of quantum mechanics proceeds by representing or associating each of the quantum magnitudes A,B,... with a real-valued function $f_A, f_B, \ldots$ on the HV phase space Ω. Each quantum proposition P, qua idempotent magnitude, is likewise associated with an idempotent function f_P on Ω or corresponding Borel subset W_P of Ω. That is, quantum propositions are associated with the elements of $\mathcal{P}_{HV}$; and let % label this association. Kochen-Specker take the structure of the quantum propositions to be a partial-Boolean algebra $\mathcal{P}_{QMA}$; this fact is further discussed below.

Next, each quantum pure state ψ is represented in the HV reconstruction as a mixed state which induces a dispersive classical probability measure $\mu_\psi : \mathcal{P}_{HV} \to [0,1]$ on the Boolean $\mathcal{P}_{HV}$ structure. In the HV theory, these dispersive μ_ψ measures represent the quantum Exp_ψ measures. And these μ_ψ measures are required to satisfy the statistical condition, which Kochen-Specker give as follows: For any quantum ψ and for any quantum P, $\int_\Omega f_P(w)\, d\mu_\psi(\{w\}) = \mathrm{Exp}_\psi(P)$ (Kochen-Specker, 1967, pp. 61, 75). Now by definition, for any f_P on Ω and for any hidden state $w \in \Omega$, $f_P(w) = 1$ if $w \in W_P$ and $f_P(w) = 0$ if $w \in W_P^\perp$. So by substitution, the statistical condition reduces to: $\mathrm{Exp}_\psi(P) =$
$= \int_{W_P} 1\, d\mu_\psi(\{w\}) + \int_{W_P} 0\, d\mu_\psi(\{w\}) = \mu_\psi(W_P)$, where $W_P = f_P^{-1}(\{1\})$. Thus for a quantum system (or ensemble of quantum systems) whose state is given

by ψ in quantum mechanics, the probability that the quantum proposition P is true is equal to the probability that the pure hidden state w of the quantum system is a member of that subset $W_P \subseteq \Omega$ of hidden states with respect to which the HV representative of P, namely, f_P, has the value 1.

Besides the statistical condition, Kochen-Specker also impose the following <u>structural condition</u>: The association % of the quantum propositions with the elements of $\mathcal{P}_{HV}$ must be an imbedding(δ) which preserves the $\mathcal{P}_{QMA}$ structure of the quantum propositions. That is, an imbedding(δ) % : $\mathcal{P}_{QMA} \to \mathcal{P}_{HV}$ is a necessary condition for an HV reconstruction of quantum mechanics, according to Kochen-Specker. The arguments by which Kochen-Specker motivate this imbedding(δ) condition are further discussed below.

Next, Kochen-Specker prove in their Theorem 0, discussed in Chapter IV(F), that an imbedding(δ) % : $\mathcal{P}_{QMA} \to \mathcal{P}_{HV}$ exists IFF a complete collection of bivalent homomorphisms(δ) h : $\mathcal{P}_{QMA} \to \{0,1\}$ exists. We can better understand the "if" half of this biconditional by noting that the <u>classical</u> probability measures $\mu_w : \mathcal{P}_{HV} \to \{0,1\}$ and $\mu_\psi : \mathcal{P}_{HV} \to [0,1]$ of the HV reconstruction can also be regarded, via the imbedding(δ) % : $\mathcal{P}_{QMA} \to \mathcal{P}_{HV}$, as <u>generalized</u> probability measures on the quantum $\mathcal{P}_{QMA}$. The relationships among these mappings can be schematized as follows:

$$
\begin{tikzcd}[column sep=small]
\mathcal{P}_{QMA} \arrow[dd, bend right, "\%"'] \arrow[dr, bend left, "\mu_\omega \circ \%"] \arrow[drrr, bend left=40, "Exp_\psi \equiv \mu_\psi \circ \%"] & & & \\
& \{0,1\} & \subset & [0,1] \\
\mathcal{P}_{HV} \arrow[ur, bend right, "\mu_\omega"'] \arrow[urrr, bend right=40, "\mu_\psi"'] & & &
\end{tikzcd}
$$

The equivalence between the quantum $\mathrm{Exp}_\psi : \mathcal{P}_{QMA} \to [0,1]$ and the composition $\mu_\psi \circ \% : \mathcal{P}_{QMA} \to [0,1]$ is ensured by the statistical condition. And for every pure hidden state w, the composition $\mu_w \circ \% : \mathcal{P}_{QMA} \to \{0,1\}$ is a generalized dispersion-free HV measure on $\mathcal{P}_{QMA}$ which preserves the partial-Boolean structural features of $\mathcal{P}_{QMA}$, or in other words, each composition $\mu_w \circ \%$ is a bivalent homomorphism(ð) on $\mathcal{P}_{QMA}$. Moreover, as described in Chapter IV(F), an imbedding is by definition an injective mapping, i.e., for any $P_1 \neq P_2$ in $\mathcal{P}_{QMA}$, $\%(P_1) \neq \%(P_2)$. And by the semi-simplicity property of the Boolean structure $\mathcal{P}_{HV}$, for any $f_{P_1} \neq f_{P_2}$ in $\mathcal{P}_{HV}$, there is a bivalent homomorphism on $\mathcal{P}_{HV}$, namely, a classical dispersion-free probability measure $\mu_w : \mathcal{P}_{HV} \to \{0,1\}$ for some w, such that $\mu_w(f_{P_1}) \neq \mu_w(f_{P_2})$. So if the imbedding(ð) $\% : \mathcal{P}_{QMA} \to \mathcal{P}_{HV}$ exists, then for every pure hidden state w, the composition $\mu_w \circ \%$ is a bivalent homomorphism(ð) on $\mathcal{P}_{QMA}$. And for any $P_1 \neq P_2$ in $\mathcal{P}_{QMA}$, we can be sure that $f_{P_1} = \%(P_1) \neq \%(P_2) = f_{P_2}$ in $\mathcal{P}_{HV}$, and we can be sure that for some w, $\mu_w(\%(P_1)) \neq \mu_w(\%(P_2))$, that is, we can be sure that the collection of bivalent homomorphisms(ꝺ) on P_{QMA} is complete.

Conversely, if a complete collection of bivalent homomorphisms(ꝺ) exist on P_{QMA}, then as described in Chapter IV(F), P_{QMA} can be imbedded(ꝺ) into a Cartesian product Boolean structure. For example, any two-dimensional Hilbert space P^2_{QMA} (or P^2_{QML}) can be imbedded(ꝺ) into the Cartesian product Boolean structure $(Z_2)^{2\cdot r}$, or equivalently, $\Pi^r_i (Z_2)^2_i$, where r is the cardinality of the set of maximal Boolean substructures of the P^2_{QM}. This Cartesian product Boolean structure can be taken to be the Boolean P_{HV} structure of a proposed HV reconstruction of quantum mechanics, e.g., $(Z_2)^{2\cdot r}$ can be regarded as the P_{HV} of a proposed HV reconstruction of the quantum mechanics of $\langle H^2, P^2_{QM}, \mathrm{Exp}_\psi\rangle$. Or in other words, as described by Bub, the classical measure space $\langle \Omega, P_{HV}, \mu\rangle = X$ which provides a Boolean HV reconstruction of the quantum mechanical statistical results given by $\langle H^2, P^2_{QM}, \mathrm{Exp}_\psi\rangle$ can be regarded as a Cartesian product measure space $X = \Pi_i X_i$ where i ranges over the set of maximal Boolean substructures of P^2_{QM} and each $X_i = \langle \Omega_i, P_{HV}, \mu_i\rangle$ is a classical measure space introduced for each maximal Boolean substructure mBS_i of P^2_{QM} (Bub, 1974, p. 145). Since each mBS_i of P^2_{QM} is isomorphic to $(Z_2)^2$, each P_{HV_i} is isomorphic to $(Z_2)^2$, and so $P_{HV} = \Pi_i P_{HV_i}$ is the Cartesian product $\Pi^r_i (Z_2)^2_i$ mentioned above.

Now the Kochen-Specker proof of the impossibility of such a proposed $\langle \Omega, P_{HV}, \mu\rangle$ reconstruction of the quantum mechanics of $\langle H^{n\geq 3}, P^{n\geq 3}_{QM}, \mathrm{Exp}_\psi\rangle$ proceeds in two stages. First in Theorem 1, which is their version of Gleason's impossibility proof, Kochen-Specker show that bivalent homomorphisms(ꝺ), i.e., generalized, dispersion-free Gleason measures, are impossible on any $P^{n\geq 3}_{QMA}$ (and this result also applies to $P^{n\geq 3}_{QML}$). Then it follows by Kochen-Specker's Theorem 0 that an imbedding(δ) of any $P^{n\geq 3}_{QMA}$ into any proposed Boolean P_{HV} structure of an HV reconstruction is impossible, and hence, since such an imbedding(δ) is a necessary condition for an HV reconstruction, an HV reconstruction of the quantum mechanical statistical results of $<H^{n\geq 3}, P^{n\geq 3}_{QM}, \mathrm{Exp}_{\psi}>$ in terms of some classical HV measure space $<\Omega, P_{HV}, \mu>$ is impossible; this is the second stage of the Kochen-Specker proof of the impossibility of an HV reconstruction of quantum mechanics.

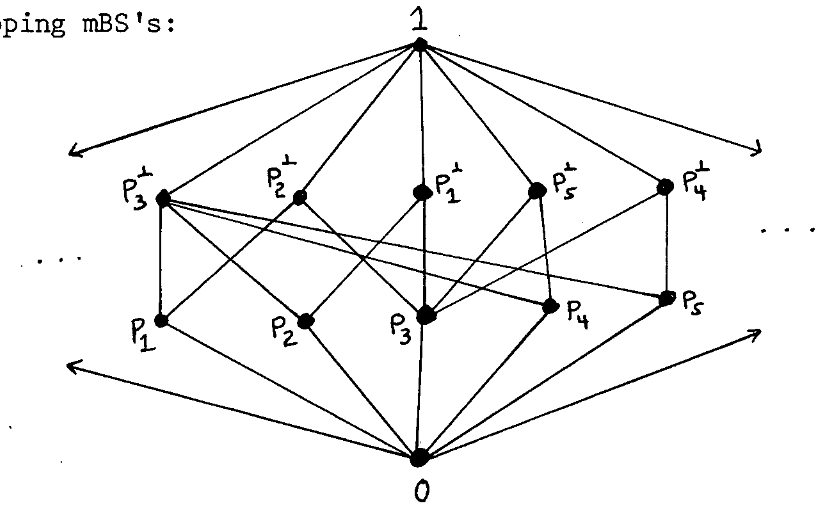
So while Gleason's impossibility proof and Kochen-Specker's Theorem 1 just show the impossibility of bivalent homomorphisms(δ), i.e., generalized, dispersion-free Gleason measures, on $P^{n\geq 3}_{QM}$, Kochen-Specker's Theorem 0 and imbedding(δ) condition connect this result with the further question of the possibility of a $<\Omega, P_{HV}, \mu>$ type of HV reconstruction of quantum mechanics. For the imbedding(δ) condition, according to which an imbedding(δ) $\% : P_{QMA} \to P_{HV}$ is a necessary condition for such an HV reconstruction, ensures that proposed classical dispersion-free HV measures $\mu_w : P_{HV} \to \{0,1\}$ preserve the P_{QMA} structure of the quantum propositions so that, for each hidden state w, the composition $\mu_w \circ \%$ is a generalized, dispersion-free Gleason measure on P_{QMA}. And Theorem 0, which biconditionally connects the existence of a complete collection of such measures on a P_{QMA} with the existence of an imbedding(δ) $\% : P_{QMA} \to P_{HV}$, thus entails that the existence of a complete collection of generalized, dispersion-free Gleason measures is a necessary condition for a $<\Omega, P_{HV}, \mu>$ type of HV reconstruction. In this way, Kochen-Specker apply Gleason's result to the correctly formulated HV question; while in contrast, the von Neumann and the Jauch-Piron proofs do not even address the HV question as so formulated.

Like the structural conditions (vN$\not\flat$) and (JP$\not\flat$), the structural condition P_{QMA}-preservation, whose imposition upon proposed dispersion-free HV measures is entailed by the imposition of the (vN$\flat$) condition, or the (JP$\flat$) condition, or the (Ga) condition, or the Kochen-Specker imbedding($\flat$) condition, has also been subject to the three sorts of criticisms listed in the Preface above. These criticisms will be elaborated next. But in Section B, another perspective on the problem of hidden-variables is introduced, according to which the structural condition of P_{QMA}-preservation emerges unscathed by these criticisms.

In Belinfante's view, the type of HV theory proved impossible by Gleason and Kochen-Specker is like the types proved impossible by von Neumann and Jauch-Piron; they are all HV theories of the zeroth kind whose impossibility is due to an inconsistency between the conditions which the proposed dispersion-free HV measures are required to satisfy (Belinfante, 1973, p. 17). However, the structural condition of P_{QMA}-preservation is not simply inconsistent with the dispersion-free condition, in the way that (vN$\not\flat$) is. Nor is P_{QMA}-preservation inconsistent with the dispersion-free condition together with the statistical condition, in the way that (JP$\not\flat$) is. On the contrary, the structural condition of P_{QMA}-preservation follows from the dispersion-free condition together with (Ga), or the dispersion-free condition together with (vN$\flat$), or the dispersion-free condition together with (JP$\flat$), as described in Chapter V(D). The trouble with P_{QMA}-preservation is more subtle than the troubles with (vN$\not\flat$) and (JP$\not\flat$). In fact, the trouble with P_{QMA}-preservation has to do with the overlap patterns among the mBS's of a $P_{QM}^{n\geq 3}$ structure, and as Belinfante points out, the trouble with P_{QMA}-preservation has to do with the assumption that HV measures are noncontextual, as shall be described below.

Bell's criticism of Gleason's impossibility proof (and thus of Kochen-Specker's Theorem 1) hinges upon the difference between what are sometimes called contextual and noncontextual HV theories, though Bell does not use these terms. Bell presents a version of Gleason's proof which focuses upon two structural conditions which Bell derives from Gleason's additivity (Ga). Both conditions are subsumed by the structural condition of P_{QMA}-preservation, which likewise follows from (Ga). Bell shows how the second condition which he derives from (Ga) rules out (generalized) dispersion-free HV measures on the set of all projectors or subspaces of a three-or-higher dimensional Hilbert space; and as shall be described shortly, this second condition ensures that (generalized) dispersion-free HV measures are in fact noncontextual. Bell criticizes the imposition of this second condition upon HV measures because the second condition relates in a nontrivial and unjustified way the results of measurements which cannot be performed simultaneously.

Although Gleason's proof refers to an infinite set of subspaces (or projectors) of three-or-higher dimensional Hilbert space, in order to understand Bell's explication and critique of Gleason's proof we need only consider the following twelve-element fragment of P^3_{QM} which includes two overlapping mBS's:



One maximal Boolean substructure mBS_1 is generated by the three mutually orthogonal (i.e., compatible) atoms $\{P_1, P_2, P_3\}$ and the other mBS_4 is generated by $\{P_3, P_4, P_5\}$. A generalized measure μ on $\mathcal{P}^3_{QM}$ which satisfies (Ga) assigns values to these five atoms as follows:
$\mu(P_1) + \mu(P_2) + \mu(P_3) = \mu(P_1 \vee P_2 \vee P_3) = \mu(1) = 1$, and
$\mu(P_3) + \mu(P_4) + \mu(P_5) = \mu(P_3 \vee P_4 \vee P_5) = \mu(1) = 1$. It follows that if $\mu_w(P_3) = 1$ then $\mu_w(P_1) = \mu_w(P_2) = 0$; and similarly, if $\mu_w(P_3) = 1$ then $\mu_w(P_4) = \mu_w(P_5) = 0$. (The subscript w is added because a measure which assigns 0, 1 values is dispersion-free.) These two conditionals are instances of the first condition which Bell derives from (Ga) and which he labels (A) (Bell, 1966, p. 450).

Belinfante argues that because Bell's (A) refers to only one triad of mutually orthogonal atoms at a time, we cannot yet conclude that, if $\mu_w(P_3) = 1$ then $\mu_w(P_1) = \mu_w(P_2) = \mu_w(P_4) = \mu_w(P_5) = 0$ (Belinfante, 1973, p. 65). But such a conclusion <u>is</u> guaranteed by the second condition which Bell derives from (Ga) and which he labels (B). An instance of (B) is: If $\mu_w(P_1) = \mu_w(P_2) = 0$ then, for any other $P \leq P_1 \vee P_2$, $\mu_w(P) = 0$. Thus if $\mu_w(P_3) = 1$, then by (A), $\mu_w(P_1) = \mu_w(P_2) = 0$, and then, since $P_4 \leq P_1 \vee P_2$ and $P_5 \leq P_1 \vee P_2$, by (B) $\mu_w(P_4) = \mu_w(P_5) = 0$.

These two conditions (A) and (B) which Bell derives from (Ga) correspond to the two conditions (KS1) and (KS2) stated in Chapter V(B) and to the two conditions labeled (61b) and (64) in Belinfante's description of Kochen-Specker's work (Belinfante, 1973, pp. 39, 41). The first condition of each pair, namely, (A), (KS1), (61b), ensure that the assignment of 0, 1 values to the atoms in a given mBS of a $\mathcal{P}_{QM}$ preserve the Boolean operations and relations, i.e., the Boolean structural features, of the mBS. And the second condition of each pair, namely, (B), (KS2), (64), ensure that the assignment of 0, 1 values to the atoms in any overlapping mBS's in a $\mathcal{P}_{QM}^{n\geq 3}$ preserve the overlap patterns among the mBS's. Both sorts of conditions are subsumed by the structural conditions of $\mathcal{P}_{QMA}$-preservation, which itself has two aspects: First, it ensures that the Boolean structural features of each mBS in a $\mathcal{P}_{QM}$ are preserved; second, it ensures that the partial-Boolean structural features of the entire $\mathcal{P}_{QM}$ are preserved; in particular, it ensures that the overlap patterns among the mBS's in a $\mathcal{P}_{QM}^{n\geq 3}$ are preserved. So Bell rightly points to the second condition (B), which he derives from (Ga), as the crucial part of Gleason's impossibility proof. For as described in Chapter V(B), it is the preservation of the overlap patterns which makes bivalent homomorphisms(δ) impossible in Kochen-Specker's Theorem 1 version of Gleason's proof.

Bell argues that proposed dispersion-free HV measures need not be required to satisfy (B). For any proposition P which is less than or equal to $P_1 \vee P_2$ is incompatible with each of P_1, P_2, unless $P = P_1$ or $P = P_2$. And if $P \not\delta P_1$ and $P \not\delta P_2$ then a measurement of P cannot be made simultaneously with a measurement of P_1 and P_2. Bell also poses the question of how this condition (B), which in fact refers to the values assigned to incompatibles, could follow from condition (Ga) which explicitly refers to only orthogonal elements which are compatible. Bell answers that it was "tacitly assumed" that a measurement of, say, P_3 must yield the same value regardless of whether P_3 is measured together with P_1, P_2 or together with P_4, P_5. But since P_1, P_2 are each incompatible with each of P_4, P_5, a measurement of P_1, P_2, P_3 requires an experimental arrangement different from the arrangement by which P_3, P_4, P_5 are measured, so there is no reason to believe that the result of a measurement of P_3 together with P_1, P_2, should be the same as the result of a measurement of P_3 together with P_4, P_5 (Bell, 1966, p. 451).

An HV theory which allows its dispersion-free HV measures to assign different 0, 1 values to a given element $P \in \mathcal{P}_{QM}$ depending upon which other elements are measured together with P have been called contextual HV theories. And the tacit assumption mentioned by Bell is the assumption that an HV theory is non-contextual; i.e., its dispersion-free HV measures assign a unique 0 or 1 value to a given element $P \in \mathcal{P}_{QM}$ regardless of which other elements are measured together with P.[4]

Now in quantum mechanics, the outcome of a measurement of any magnitude A (which is always one of A's eigenvalues) or of any idempotent magnitude P (which is always one of P's 0 or 1 eigenvalues) is determined by the quantum state ψ, though if ψ is incompatible with any of A's or P's eigenstates, then as described in the Preface, the quantum formalism at best determines the probability of any one of A's or P's eigenvalues being the outcome of a measurement and determines the average value (i.e., expectation-value) of A or P for a large number of the same measurements of A or P on many quantum systems whose state is described by ψ. In a contextual HV theory, the outcome of a measurement of A or P is determined by the hidden state and the context of measurement. A hidden state, labeled w above, is specified in a contextual HV theory by the quantum state ψ together with the hidden variable(s) ξ; so hereafter, a hidden state of a contextual HV theory shall be designated by ψ, ξ. And the context is taken to be the set of all possible outcomes of the measurement as specified by a complete, orthogonal set of eigenstates of the measured magnitude. As mentioned in Chapter IV(A), the eigenstates of any magnitude, as represented by projectors $\{\hat{P}_i\}_{i \in \text{Index}}$ on a Hilbert space, are orthogonal and satisfy $\sum_i \hat{P}_i = \hat{I}$. In order that the set of eigenstates of a magnitude be complete, it suffices that each $\hat{P}_i$ is a one-dimensional projector on $\mathcal{H}$; i.e., an atom in the $\mathcal{P}_{QM}$ structure of $\mathcal{H}$.[5] Thus the context of a measurement of a magnitude represented by an operator on an n dimensional Hilbert space $\mathcal{H}^n$ is specified by a set of n orthogonal one-dimensional projectors on $\mathcal{H}^n$, i.e., by a set of n mutually orthogonal atoms in the $\mathcal{P}^n_{QM}$ structure of $\mathcal{H}^n$. And since a set of n mutually orthogonal atoms in $\mathcal{P}^n_{QM}$ generates a unique maximal Boolean substructure of $\mathcal{P}^n_{QM}$,[6] the context of a measurement of a magnitude represented by an operator on $\mathcal{H}^n$ can equally well be specified by an mBS in the $\mathcal{P}^n_{QM}$ structure of $\mathcal{H}^n$, as suggested by Gudder (1970, p. 432). In particular, when we consider any idempotent magnitude P, which is represented by the projector $\hat{P}$ on $\mathcal{H}^n$ and so is an element in the $\mathcal{P}^n_{QM}$ structure of $\mathcal{H}^n$, P, qua element of $\mathcal{P}^n_{QM}$, is itself a member of any of the mBS's in $\mathcal{P}^n_{QM}$ which specify possible contexts of measurement of P. For $\hat{P}$ is itself a member (or a sum of members) of any set of n orthogonal, one-dimensional projectors on $\mathcal{H}^n$ representing a complete, orthogonal set of eigenstates of the idempotent magnitude P and so specifying the context of a measurement of P; thus P, qua element of $\mathcal{P}^n_{QM}$, is itself a member (or a join of members) of any set of n mutually orthogonal atoms in $\mathcal{P}^n_{QM}$ specifying the context of a measurement of P; and so P, qua element of $\mathcal{P}^n_{QM}$, is itself a member of any mBS in $\mathcal{P}^n_{QM}$ specifying the context of a measurement of P. In short, in a contextual HV theory, the outcome of a measurement of any $P \in \mathcal{P}_{QM}$ is determined by the hidden state ψ, ξ and the context of measurement, specified by an mBS in $\mathcal{P}_{QM}$ with $P \in$ mBS.

The fact stated in the last sentence can be and has been formalized in any number of ways. Most abstractly, since the outcome of a measurement of P is always one of P's 0 or 1 eigenvalues, we may talk of a contextual HV theory proposing contextually-dependent 0, 1 value assignments to the elements of $\mathcal{P}_{QM}$. For example, Belinfante talks of a contextual HV theory, which he refers to as a "realistic" HV theory, introducing, for a given hidden state ψ, ξ, a bivalent mapping v whose arguments are quantum propositions and which depend not only upon ψ, ξ, but also upon the context of measurement (Belinfante, 1973, pp. 40-42). Less abstractly, since in this chapter and in Chapters III and VI we have described how in classical mechanics, quantum mechanics, and proposed HV theories, 0, 1 value assignments to the elements in $\mathcal{P}_{CM}$, $\mathcal{P}_{QM}$, $\mathcal{P}_{HV}$ structures are preformed by various kinds of state-induced dispersion-free probability measures, we can in a similar vein say that the hidden states of a contextual HV theory induce dispersion-free HV measures which assign 0, 1 values to elements of $\mathcal{P}_{QM}$ in a contextually dependent manner. For example, Bub talks in this way (1974, pp. 146-147; 1973, p. 51). While according to Gudder's way of formalizing the contextual HV proposal, a hidden state of a contextual HV theory induces a dispersion-free HV measure on only an mBS of $\mathcal{P}_{QM}$ so that the contextual dependence of the measure is at least partly handled by restricting its domain to one context, i.e., one mBS (Gudder, 1970, p. 433).

We shall focus upon the notion of the hidden states of a contextual HV theory inducing dispersion-free HV measures which assign 0, 1 values to the elements of $\mathcal{P}_{QM}$ in a contextually dependent manner. The contextual-dependence of the dispersion-free measures may be and has been formulated in two equivalent ways. One way involves contextualizing proposed generalized, dispersion-free HV measures $\mu_{\psi,\xi}$ on $\mathcal{P}_{QM}$ by having the domain of each $\mu_{\psi,\xi}$ be the cross-product of P_{QM} and the set of mBS's in P_{QM} so that the value which $\mu_{\psi,\xi}$ assigns to an element $P \in P_{QM}$ depends upon which mBS containing P is being considered (i.e., depends upon the context in which P is being measured). Thus a hidden state ψ, ξ induces a contextualized, generalized, dispersion-free HV measure $\mu_{\psi,\xi} : P_{QM} \times \{mBS_i\}_{i \in Index} \longrightarrow \{0,1\}$ such that, for example, $\mu_{\psi,\xi}(\langle P_3, mBS_1 \rangle)$ need not equal $\mu_{\psi,\xi}(\langle P_3, mBS_4 \rangle)$. According to Bub, the Bohm 1952 HV proposal is such a contextual HV theory. However, one would be hard pressed to find anything like this $\mu_{\psi,\xi} : P_{QM} \times \{mBS_i\}_{i \in Index} \longrightarrow \{0,1\}$ in Bohm's work or even in Bub's description of Bohm's work (Bub, 1973, p. 51). For again, the above notion of an $\mu_{\psi,\xi}$ measure is an abstraction which helps make sense of Bub's description of Bohm's work and which was suggested to me by Belinfante's method of contextualizing his bivalent v mappings (with R. E. Robinson suggesting the cross-product formulation). Now the alternative way involves proposing that a Boolean P_{HV} structure be the domain of proposed classical dispersion-free HV measures $\mu_{\psi,\xi}: P_{HV} \rightarrow \{0,1\}$ induced by the hidden states, with a contextualized association of the elements of P_{QM} with the elements of the P_{HV}. Thus we have a contextualized association $\% : P_{QM} \times \{mBS_i\}_{i \in Index} \longrightarrow P_{HV}$ such that, for example, $\%(\langle P_3, mBS_1 \rangle)$ need not equal $\%(\langle P_3, mBS_4 \rangle)$, and so $\mu_{\psi,\xi}(\%(\langle P_3, mBS_1 \rangle))$ need not equal $\mu_{\psi,\xi}(\%(\langle P_3, mBS_4 \rangle))$. The Bohm-Bub 1966 HV proposal is such a contextual HV theory. According to Bub, both ways of formulating the contextual HV proposal, either in terms of contextualized measures on P_{QM} or in terms of a contextualized association of P_{QM} with P_{HV}, are formally equivalent (Bub, 1973, p. 51). Clearly, both have the same effect, namely, the proposed dispersion-free HV measure induced by a hidden state in a contextual HV theory does not assign a unique 0 or 1 value to a given element $P \in \mathcal{P}_{QM}$ when P is a member of more than one mBS in $\mathcal{P}_{QM}$, i.e., when P is a member of two or more overlapping mBS's in $\mathcal{P}_{QM}$.

Thus the dispersion-free HV measures induced by the hidden states of a contextual HV theory especially break up the overlap patterns among the mBS's of any $\mathcal{P}^{n\geq 3}_{QM}$ in the manner suggested in Chapter V(B), namely, by assigning different values to a single element which is in more than one mBS of $\mathcal{P}^{n\geq 3}_{QM}$. So, for example, although $P_3 = P_3$ in the twelve-element fragment of $\mathcal{P}^{3}_{QM}$ diagrammed above, and although $\mathrm{Exp}_\psi(P_3) = \mathrm{Exp}_\psi(P_3)$ for every quantum Exp_ψ on $\mathcal{P}^{3}_{QM}$, nevertheless, in a contextual HV theory, P_3 may be assigned different values, as exemplified in the previous paragraph. In this sense, the dispersion-free HV measures induced by the hidden states of a contextual HV theory do not preserve the relation $P_3 = P_3$. That is, they do not preserve the $=$ relation of $\mathcal{P}_{QM}$, and so it clearly follows that with respect to elements in overlapping mBS's in $\mathcal{P}_{QM}$, the dispersion-free HV measures induced by the hidden states of a contextual HV theory do not preserve any of the operations and relations of $\mathcal{P}_{QM}$. A contextual HV theory and its dispersion-free HV measures thereby avoid HV impossibility proofs. Or as Bub puts it, in terms of the second formulation of the contextual HV proposal which includes a Boolean $\mathcal{P}_{HV}$ and classical dispersion-free HV measures on $\mathcal{P}_{HV}$, a contextual HV theory is a type of <u>Boolean reconstruction of quantum mechanics</u> (Bub, 1974, p. 146) which avoids the Kochen-Specker impossibility proof by letting the association of the elements of $\mathcal{P}_{QM}$ with the elements of $\mathcal{P}_{HV}$ be a contextualized mapping which breaks up the overlap patterns among the mBS's of $\mathcal{P}^{n\geq 3}_{QM}$ rather than demanding, as Kochen-Specker do, that this association be an imbedding(b) which preserves $\mathcal{P}_{QMA}$, i.e., preserves all the partial-Boolean structural features of $P_{QM}^{n\geq 3}$, including the $\leq$ relation (and thus the $=$ relation) and including the overlap patterns among the mBS's.

Now to continue with the third sort of criticism, labeled (iii) in the Preface, of the imposition of the structural condition of P_{QMA}-preservation and of the development of the Gleason, Kochen-Specker HV impossibility proofs. If, as Bell argues, there is no reason why the partial-Boolean structural features of P_{QM}, in particular, the overlap patterns among the mBS's of P_{QM}, must be preserved by non-contextually assigning the same unique 0 or 1 value to, say, P_3 regardless of whether P_3 is measured in the context mBS_1 or in the context mBS_4, then the Gleason and Kochen-Specker HV impossibility proofs beg the HV question. For these proofs rest upon contradictions caused by requiring that 0, 1 values be assigned to the elements of a P_{QM} in a non-contextual, P_{QMA}-preserving manner which is not justified. Moreover, these proofs do not rule out a contextual HV reconstruction of quantum mechanics, and so they do not rule out hidden variables, as they purport to do.

Kochen-Specker's work is especially vulnerable to the above criticism because of the following ambiguity, pointed out by Bub, in the manner in which Kochen-Specker ground the partial-Boolean algebra of quantum propositions which they require an HV theory to preserve:

(a) On the one hand, Kochen-Specker regard the quantum propositional structure as simply given by the partial-Boolean algebra of projectors or subspaces of Hilbert space, which has been labeled P_{QMA}. For according to Kochen-Specker, it is a "basic tenet" of quantum mechanics that quantum magnitudes are represented by operators on a Hilbert space and similarly, quantum propositions, qua idempotent magnitudes, are represented by projectors on a Hilbert space (Kochen-Specker, 1967, p. 65). That is, the quantum propositional structure is a $\mathcal{P}_{QM}$ structure, in particular, a $\mathcal{P}_{QMA}$ structure, of projectors or subspaces of a Hilbert space. And for example, two propositions are equivalent in $\mathcal{P}_{QMA}$ if they are represented by the same projector (or subspace).

(b) But on the other hand, Kochen-Specker define a partial-Boolean algebra of quantum propositions with respect to a set of states and measures; the defined structure shall be labeled $p\mathcal{B}\mathcal{A}$ to distinguish it from the above $\mathcal{P}_{QMA}$. The definition of $p\mathcal{B}\mathcal{A}$ may yield, for example, that two propositions are equivalent in $p\mathcal{B}\mathcal{A}$ if their expectation-values are equal for all quantum Exp_{ψ} measures.

If the quantum propositional structure which Kochen-Specker require an HV theory to preserve is such a $p\mathcal{B}\mathcal{A}$ defined with respect to the quantum measures, then Kochen-Specker's impossibility proof, which rests upon the requirement that the quantum propositional structure be preserved, begs the HV question. For there is no reason why proposed HV measures must preserve such a $p\mathcal{B}\mathcal{A}$, and in particular preserve the equivalence $P_3 = P_3'$ in $p\mathcal{B}\mathcal{A}$, if $P_3 = P_3'$ in $p\mathcal{B}\mathcal{A}$ only because $Exp_{\psi}(P_3) = Exp_{\psi}(P_3')$ for all quantum Exp_{ψ} measures. Moreover, if dispersion-free HV measures do exist, then the $p\mathcal{B}\mathcal{A}$ defined with respect to the quantum measures and the HV measures may be different from the $p\mathcal{B}\mathcal{A}$ defined with respect to just the quantum Exp_{ψ} measures. These criticisms of Kochen-Specker's defined $p\mathcal{B}\mathcal{A}$ are similar to the criticisms of von Neumann's use of his condition (E) to define the sums of incompatibles.

This ambiguity between (a) and (b), and the way in which (b) leads to a misunderstanding of Kochen-Specker's work and makes the Kochen-Specker impossibility result appear to be especially vulnerable to Bell's criticism, are described by Bub (1974, pp. 84-88). Bub concludes that Kochen-Specker are best understood referring to P_{QMA} rather than pBA, that is, Kochen-Specker should have used just the (a) notion and not discussed the (b) notion at all. Moreover, as shall be described in Section B, from Bub's perspective on the problem of hidden-variables, the ambiguity between (a) and (b) is not substantially important, though it is confusing and leads to a misunderstanding of Kochen-Specker's work, and so the ambiguity is worth clarifying. In the rest of this section, Kochen-Specker's (b) definition of pBA is elaborated, and a reason why Kochen-Specker may have been motivated to develop this (b) definition is given.

According to Kochen-Specker, a physical theory like classical mechanics or quantum mechanics or a proposed HV theory consists of a set of magnitudes $\{A, \ldots\}$, a set of states $\{\psi, \ldots\}$, and a set of (classical) probability measures $\{\rho_{\psi,A}, \ldots\}$ on the real-number line $\mathcal{R}$, or more exactly, on the Boolean structure $\mathcal{B}_{\mathcal{R}}$ of Borel subsets of $\mathcal{R}$. For any Borel subset $R \subseteq \mathcal{R}$, for any magnitude A, and for any state ψ, $\rho_{\psi,A}(R) \in [0,1]$ is the probability that the real-value of A is a member of R. These $\rho_{\psi,A}$ measures on $\mathcal{B}_{\mathcal{R}}$ are related to the more familiar expectation-functions Exp and are related to the HV measures μ of a Kochen-Specker type of HV reconstruction of quantum mechanics, by equations given below.

Now Kochen-Specker argue that the magnitudes of a physical theory are not independent of each other but rather are functionally related, e.g., the magnitude A^2 is clearly a function of A. And the function A^2 of the magnitude A can be measured by simply measuring A and squaring the resulting value. That is, the real value of any (Borel) function $g(A)$ of any magnitude A is calculated by simply applying that function g to the real value of A. The last sentence is a statement of what may be regarded as an <u>uncontentious general principle</u> which applies to the magnitudes of any physical theory.

Kochen-Specker also assume that the magnitudes of a physical theory are determined by the $\rho_{\psi,A}$ measures in the following sense:

> (*) For any magnitudes A, B, if $\rho_{\psi,A}(R) = \rho_{\psi,B}(R)$ for every state ψ and any Borel subset $R \subseteq \mathcal{R}$, then $A = B$.

With (*), the above general principle suggests the following definition, which Kochen-Specker label (3), for a function $g(A)$ of any magnitude A:

> (3) For any A and any Borel function g,
>
> $\rho_{\psi,g(A)}(R) = \rho_{\psi,A}(g^{-1}(R))$ for any state ψ and any $R \subseteq \mathcal{R}$

(Kochen-Specker, 1967, pp. 61, 63).

In fact, (3) can be regarded as a restatement of the uncontentious general principle. For if the real value of A is a member of some Borel subset $R \subseteq \mathcal{R}$, then by the general principle, the real value of $g(A)$ is a member of the Borel subset $g(R) \subseteq \mathcal{R}$. Likewise, if the real value of $g(A)$ is a member of some $R \subseteq \mathcal{R}$, then by the general principle, the real value of A is a member of the Borel subset $g^{-1}(R) \subseteq \mathcal{R}$. So assuming that $\rho_{\psi,g(A)}(R)$ is the probability that the real value of $g(A)$ is in R, and assuming that the $\rho_{\psi,A}$ measures determine the magnitude of a physical theory in the above (*) sense, then by the general principle we can be sure that $\rho_{\psi,g(A)}(R) = \rho_{\psi,A}(g^{-1}(R))$.

Moreover, with respect to a Kochen-Specker type of HV reconstruction of quantum mechanics, in which each quantum magnitude A is represented by a function $f_A : \Omega \to R$ on the HV phase space Ω and the real value of A for any hidden state $w \in \Omega$ is $f_A(w)$, the general principle yields the identity: for any $w \in \Omega$, $f_{g(A)}(w) = g(f_A(w))$. So in a Kochen-Specker type of HV reconstruction, the functions $\{f_A, \ldots\}$ representing the quantum magnitudes in the proposed HV theory must satisfy the following structural condition labeled (4) by Kochen-Specker:

(4) For any quantum magnitude A and any Borel function g,

$$f_{g(A)} = g(f_A).$$

Kochen-Specker aim to show that an HV reconstruction of quantum mechanics which satisfies (4) is impossible. But first Kochen-Specker replace (4) by a more tractable structural condition as follows.

Using (*) and (3), Kochen-Specker define the relation of commeasurability, i.e., compatibility, among the magnitudes of a physical theory as stated in Chapter IV(B). Then using (*) and (3) again, Kochen-Specker define the ring operations + and $\cdot$ among commeasurable magnitudes as follows: For any magnitudes A_1, A_2, if A_1, A_2 are commeasurable, then for some magnitude B and Borel functions g_1, g_2, $A_1 = g_1(B)$ and $A_2 = g_2(B)$, and then

$$(5) \quad A_1 + A_2 = (g_1 + g_2)(B),$$
$$A_1 \cdot A_2 = (g_1 \cdot g_2)(B).$$

With + and $\cdot$ so defined among compatible magnitudes, the set of magnitudes of a physical theory acquires the structure of a partial-algebra, or in the terminology of Chapter I(D), a partial-ring-with-unit. And thus the set of propositions of a physical theory, _qua_ idempotent magnitudes, i.e., _qua_ idempotent elements of a partial-ring-with-unit, has the structure of a partial-Boolean algebra. In particular, by (*), (3), and (5), the mutually compatible magnitudes of classical mechanics form a commutative-ring-with-unit, which is a special case of a partial-ring-with-unit, namely, the case where all elements are mutually compatible, as described in Chapters III(B) and I(D). And the propositions of classical mechanics form a Boolean algebra, which again is the special case of a partial-Boolean algebra where all elements are mutually compatible. Likewise, the magnitudes of a proposed Kochen-Specker type of HV theory form a commutative-ring-with-unit, and the propositions of such an HV theory form a Boolean algebra. And finally, by (*), (3), and (5) the magnitudes of quantum mechanics form a partial-ring-with-unit, and the propositions of quantum mechanics form a partial-Boolean algebra, labeled $p\mathcal{BA}$. This completes the Kochen-Specker definition of a $p\mathcal{BA}$ of quantum propositions, which shall be further discussed shortly.

Kochen-Specker then note that their condition (4) implies that the partial-operations $+$ and $\cdot$, which are defined among (just) compatible quantum propositions $\{P_1, P_2, \ldots\}$ and among the compatible HV representatives $\{f_{P_1}, f_{P_2}, \ldots\}$ of quantum propositions by the condition (5), are preserved by the mapping % which associates the quantum propositions with their HV representatives, in this case $\% : p\mathcal{BA} \to \mathcal{P}_{HV}$. For example, as elaborated by Bub, for any compatible P_1, P_2, which are by the definition of compatibility Borel functions of some common P, say $P_1 = g_1(P)$ and $P_2 = g_2(P)$, we have: $f_{P_1+P_2} = f_{g_1(P)+g_2(P)} =$ (by (5)) $f_{(g_1+g_2)(P)} =$ (by (4)) $(g_1+g_2)(f_P) =$ (by (5)) $g_1(f_P)+g_2(f_P) =$ (by (4)) $f_{g_1(P)} + f_{g_2(P)} = f_{P_1} + f_{P_2}$ (Bub, 1974, p. 87). So, for example, if $\%(P_1) = f_{P_1}$ and $\%(P_2) = f_{P_2}$ and $\%(P_1 + P_2) = f_{P_1+P_2}$, then by (4) and

(5) we have: $\%(P_1 + P_2) = \%(P_1) + \%(P_2)$. Thus the mapping % which associates quantum propositions with their HV representatives preserves the partial-operation + among compatible quantum propositions. Similarly, it can be shown that, by (4) and (5), % preserves the partial-operation $\cdot$ among compatible quantum propositions. And so with the operation $^{\perp}$ and the partial-operations $\wedge$, $\vee$ defined in terms of $+$, $\cdot$ as usual, the mapping $\% : pBA \rightarrow P_{HV}$ preserves these $\wedge$, $\vee$, $^{\perp}$ operations since it preserves the $+$, $\cdot$ operations. In other words, % is an imbedding($\overset{\cdot}{\cup}$) (Kochen-Specker, 1967, pp. 63-66).

However, as pointed out by Bub, it is clear that in this (b) definition of pBA, Kochen-Specker rely upon the $\rho_{\psi,A}$ measures to define, by (*), the <u>equivalence</u> of quantum propositions, and to define, with (*) and (3), the <u>functional relations</u> and the <u>compatibility relations</u> among the quantum propositions. That is, the pBA structure of quantum propositions which Kochen-Specker require an HV reconstruction to preserve is defined with respect to the $\rho_{\psi,A}$ measures. These measures on B_{R} are related to expectation-functions Exp by the equation: For any magnitude A and any state ψ, $\mathrm{Exp}_{\psi}(A) = \int_{-\infty}^{+\infty} r \, d\rho_{\psi,A}(\{r\})$, $r \in R$. And the $\rho_{\psi,A}$ measures are related to the μ measures of a Kochen-Specker type of HV reconstruction by the equation: For any magnitude A, any state ψ, and any Borel subset $R \subseteq R$, $\rho_{\psi,A}(R) = \mu_{\psi}(f_A^{-1}(R))$ (Kochen-Specker, 1967, p. 61). Now so far, A and ψ designate any magnitude and any state in any physical theory. So with respect to the issue of a proposed HV reconstruction of quantum mechanics, it is not clear whether the set of ψ states, which via the $\rho_{\psi,A}$ measures defines pBA, includes just the <u>quantum</u> states, which are usually designated by ψ, or includes both the quantum states <u>and</u> the hidden states proposed by an HV reconstruction. And as suggested above, the $p\mathcal{B}A$ defined with respect to just the quantum states may be different from the $p\mathcal{B}A$ defined with respect to both the quantum and the hidden states; in particular, while the former is isomorphic to P_{QMA}, the latter might not be.

If Kochen-Specker mean the set of states which define, via the $\rho_{\psi,A}$ measures, their $p\mathcal{B}A$ to include both quantum and hidden states, then they are presuming that hidden states exist and they thus beg the HV question in a trivial way. If it does not matter whether the set of states includes just the quantum states or includes both quantum and hidden states, then Kochen-Specker beg the HV question in the sense that they presume that the $p\mathcal{B}A$ defined with respect to the quantum ψ states is the same as the $p\mathcal{B}A$ defined with respect to both quantum ψ and hidden w states; in particular, they presume that $\mu_w(P_3) = \mu_w(P_3)$ just as $\mathrm{Exp}_\psi(P_3) = \mathrm{Exp}_\psi(P_3)$. But in a contextual HV theory, an element P_3 which is a member of two or more overlapping mBS's in $p\mathcal{B}A$ is not assigned a unique value for a given hidden state w specified by ψ, ξ, e.g., $\mu_{\psi,\xi}(\langle P_3, mBS_1\rangle)$ may not equal $\mu_{\psi,\xi}(\langle P_3, mBS_4\rangle)$. And finally, if Kochen-Specker mean the set of states which define their $p\mathcal{B}A$ to include just the quantum states, then they beg the HV question in the manner described on page 190. For then by (*), quantum states determine the identity of the quantum magnitudes and quantum propositions; i.e., for any quantum propositions P_1, P_2, $P_1 = P_2$ if $\rho_{\psi,P_1}(R) = \rho_{\psi,P_2}(R)$ for every quantum state ψ and any Borel subset $R \subseteq \mathcal{R}$. Or in other words, by the above equation connecting Exp_ψ with $\rho_{\psi,A}$ we have: For any quantum propositions P_1, P_2, $P_1 = P_2$ if $\mathrm{Exp}_\psi(P_1) = \mathrm{Exp}_\psi(P_2)$ for every quantum state. But there is no reason why proposed dispersion-free HV measures induced by the hidden states of a proposed HV theory must preserve this equivalence which is defined with respect to the quantum states and measures. Thus contextual HV measures which do not preserve the equivalences in pBA may be proposed especially in order to avoid the Kochen-Specker HV impossibility proof.

So if Kochen-Specker had only the (b) definition of pBA, then their crucial imbedding($\flat$) condition would unjustifiably demand the preservation of a structure defined with respect to maybe just the quantum states and measures. However, Kochen-Specker have not only the defined pBA but also the (a) P_{QMA} given by the basic tenets of quantum mechanics. And as Bub argues, both the Kochen-Specker HV impossibility proof and the contextual HV counter-proposal are best understood if we give Kochen-Specker the benefit of the doubt and resolve their ambiguity between (a) and (b) in favour of the (a) P_{QMA}. The very fact that Kochen-Specker require that the quantum propositional structure be preserved in an HV reconstruction suggests that they regard it as something more than a merely statistical structure defined with respect to the dispersive quantum states and measures. Moreover, Kochen-Specker specifically declare their Theorem 1 to be a finite version of Gleason's impossibility proof, which refers to the projectors or subspaces of Hilbert space. Thus Kochen-Specker's finite version of Gleason's proof may likewise be understood as referring to the P_{QMA} structure of projectors or subspaces of Hilbert space rather than referring to the pBA structure.

Kochen-Specker may have been motivated to develop their (b) definition of pBA in order that their contentious imbedding($\flat$) condition should follow from the uncontentious general principle as described above. But then Kochen-Specker should have used the general principle only to support, via (4), their imbedding($\flat$) condition rather than to help define, via (*), (3), (4), (5), a *pBA* of quantum propositions. For example, we may take the quantum propositional structure to be a P_{QMA} of projectors or subspaces of Hilbert space; so $P_1 = P_2$ if $\hat{P}_1 = \hat{P}_2$, and the partial-operations $+$, $\cdot$ are defined among compatible propositions as projector addition and multiplication. Then we may still argue that in a proposed HV reconstruction of quantum mechanics, where any quantum proposition P is represented by an idempotent function $f_P : \Omega \to \{0,1\}$ on the HV phase space Ω and any Borel function $g(P)$ is correspondingly represented by the idempotent function $f_{g(P)}$, the uncontentious general principle requires that the 0, 1 values issued by $f_{g(P)}$ must be g-functions of the 0, 1 values issued by f_P. And the fulfillment of this requirement is best ensured by making $f_{g(P)} = g(f_P)$, for any P and any Borel function g. Thus we have condition (4), from which the imbedding($\flat$) condition follows as described above. In other words, the crucial Kochen-Specker imbedding($\flat$) condition, which requires that P_{QMA} be preserved in any proposed HV reconstruction of quantum mechanics, is supported by the uncontentious general principle which it seems no critic of Kochen-Specker's HV impossibility proof could reasonably object to.

However, without realizing or disregarding the above elaborated connection between the general principle and the imbedding($\flat$) condition, critics of the Kochen-Specker proof may argue that even if Kochen-Specker are understood as referring to P_{QMA} rather than *pBA*, their proof begs the HV question because their imbedding($\flat$) condition, which requires P_{QMA}-preservation and which rules out hidden-variables, is not justified. In other words, critics may argue that there is no reason why a proposed HV reconstruction must preserve even this P_{QMA} given by the fundamental postulates of quantum mechanics. In fact, Bell must be understood as making this further argument, for he addresses himself to the Gleason impossibility proof and thus to an (a) type of structure rather than a (b) type of structure.

Bub rescues the Gleason, Kochen-Specker proofs from this criticism, as described in the next section.

Section B. Either P_{QMA}-preservation or Boolean Reconstruction

Bub argues that the concept of an HV reconstruction of quantum mechanics does not make sense unless the quantum propositional structure is preserved. For according to Bub, quantum mechanics is a principle theory rather than a constructive theory. The distinction is due to Einstein and is described by Bub as follows. Constructive theories "aim to reduce a wide class of diverse systems to component systems of a particular kind (e.g., the molecular hypothesis of the kinetic theory of gases)." In contrast, principle theories "introduce abstract structural constraints that events are held to satisfy," e.g., special and general relativity can be viewed as principle theories of space-time structure (Bub, 1974, pp. vii, 142). Bub regards quantum mechanics and classical mechanics as principle theories of logical structure because,

> . . . they introduce constraints on the way in which the properties of a physical system are structured. The logical structure of a physical system is understood as imposing the most general kind of constraint on the occurrence and non-occurrence of events. (Bub, 1974, p. 149)

The logical-property-event structure of a physical system is given by the propositional structure as determined by the mathematical formalism of the physical theory describing the system, namely, the classical $\mathcal{P}_{CM}$ and the quantum $\mathcal{P}_{QM}$. So at the very core of quantum mechanics is the non-Boolean $\mathcal{P}_{QM}$ structure, which Bub and Kochen-Specker explicitly and Gleason implicitly take(s) to be a $\mathcal{P}_{QMA}$. And according to Bub, (i) the question of the completeness of quantum mechanics must be posed with respect to $\mathcal{P}_{QMA}$, (ii) the quantum probability measures are defined on $\mathcal{P}_{QMA}$ and the statistical results of quantum mechanics make sense with respect to $\mathcal{P}_{QMA}$, and (iii) any HV reconstruction or extension of quantum mechanics must preserve the quantum $\mathcal{P}_{QMA}$.

Now as shown by Kochen-Specker and by Gudder, a Boolean HV reconstruction of quantum mechanics which does <u>not</u> preserve the $\mathcal{P}_{QMA}$ structure is always possible. By a trivial construction, Kochen-Specker show that is is always possible to introduce a classical measure space $\langle\Omega,\mathcal{P}_{HV},\mu\rangle = X$ which reproduces the quantum statistics but does not preserve $\mathcal{P}_{QMA}$ (Kochen-Specker, 1967, p. 63). And Gudder proves that it is always possible to introduce a contextual HV Boolean reconstruction which reproduces the quantum statistics and preserves the Boolean structural features of the mBS's of $\mathcal{P}_{QMA}$ but which breaks up the overlap patterns among the mBS's and so does not preserve $\mathcal{P}_{QMA}$ (Gudder, 1970, pp. 434-436).

However, as Bub argues:

> The contribution of Kochen-Specker lies in showing that the problem of hidden variables is not that of fitting a theory--i.e., a class of event structures--to a statistics. This can always be done in an infinite number of ways; in particular, a Boolean representation is always possible. Rather, the problem concerns the kind of statistics definable on a given class of event structures. (Bub, 1974, p. 88).

The event structures given by the fundamental postulates of quantum mechanics are the non-Boolean P_{QM} structures, in particular, the P_{QMA} structures. So the problem of the completeness of quantum mechanics and the concordant problem of hidden variables is correctly addressed with respect to the quantum P_{QMA}, as done by Gleason and Kochen-Specker. In Bub's view, Gleason's completeness proof shows that the quantum formalism generates all possible (generalized) probability measures on the $P_{QMA}^{n\geq 3}$ structures of three-or-higher dimensional Hilbert space. That is, with respect to P_{QMA}, the quantum mechanics of three-or-higher dimensional Hilbert space is complete. And it follows as a corollary that, for $P_{QMA}^{n\geq 3}$, dispersion-free (generalized) probability measures which preserve the partial-Boolean structural features of $P_{QMA}^{n\geq 3}$ are impossible. And so by Kochen-Specker's Theorem 0, an imbedding(δ) of P_{QMA} into a Boolean structure is impossible. Thus a Boolean HV reconstruction of quantum mechanics which preserves P_{QMA} is impossible. That is, with respect to P_{QMA}, an HV reconstruction of the quantum mechanics of three-or-higher dimensional Hilbert space is impossible.

The above interpretation of Gleason and Kochen-Specker's work actually depends upon our acknowledging the priority of the P_{QMA} structure as the core, or at least part of the core, of quantum mechanics which *must be preserved*. For dispersion-free HV measures and a Boolean HV reconstruction which do not preserve P_{QMA} are always possible. So if P_{QMA} were not required to be preserved, then in spite of Gleason's completeness proof, the fact that all the measures generated by the quantum formalism are dispersive would signal the *in*completeness of quantum mechanics relative to a possible Boolean HV reconstruction which included dispersion-free HV measures.

Now as Bub mentions, the completeness of quantum mechanics with respect to P_{QMA}, i.e., the fact that the quantum formalism generates all possible (generalized) probability measures on any $P_{QMA}^{n\geq 3}$, guarantees that the P_{QMA} structure given by the fundamental quantum postulates and the *pBA* structure defined with respect to the quantum measures are isomorphic (Bub, 1974, p. 45). So the ambiguity, described in Section (A), in Kochen-Specker's notion of the quantum propositional structure as a (a) given P_{QMA} and a (b) defined *pBA* is not harmful but merely confusing. In particular, we can be sure that if $\mathrm{Exp}_\psi(P_1) = \mathrm{Exp}_\psi(P_2)$ for all quantum Exp_ψ, then $P_1 = P_2$ in P_{QMA}.

Moreover, if we acknowledge the priority of the P_{QMA} structure in quantum mechanics, then the structural condition of P_{QMA} preservation and the Gleason, Kochen-Specker HV impossibility proofs emerge unscathed by the three sorts of criticisms described in the previous section. In particular, P_{QMA}-preservation must still be required of a proposed HV theory in spite of the fact that this condition leads to contradictions which make the HV theory impossible and of the zeroth kind, in Belinfante's terminology. And P_{QMA}-preservation must be required of the proposed dispersion-free measures of an HV theory in spite of the considerations of measurement interaction which Bell raises in order to dissuade our imposing this condition. And finally, the Gleason, Kochen-Specker proofs cannot be charged with begging the HV question because they impose the P_{QMA}-preservation condition, for the question of an HV reconstruction of quantum mechanics does not even make sense except with respect to the quantum P_{QMA} structure, which must be preserved.

In contrast, an HV advocate may choose to regard to quantum P_{QM}

structure, whether P_{QMA} or P_{QML}, as not worthy of preservation when considered with respect to the larger enterprise of providing a classical, Boolean reconstruction or re-interpretation of quantum mechanics, especially because such a reconstruction is possible _if_ the quantum P_{QM} is not preserved. So rather than affirming the priority of the quantum P_{QM} structure in the interpretation of quantum mechanics, an HV advocate may instead affirm that (i') the problem of the completeness of any physical theory only makes sense when posed or framed with respect to a Boolean logical-property-event structure, (ii') the probability measures of any statistical theory like quantum mechanics are to be defined on a Boolean structure, and (iii') a Boolean HV reconstruction of quantum mechanics need not preserve the quantum P_{QM} structure.

As described by Bub, if we acknowledge the priority of a Boolean HV reconstruction of quantum mechanics by affirming these three primed conditions, then quantum mechanics is incomplete and an HV reconstruction is possible and completes quantum mechanics. Most simply, a Boolean structure always admits dispersion-free measures, yet quantum mechanics lacks dispersion-free measures. So with respect to a Boolean logical-property-event structure, quantum mechanics is incomplete; and quantum mechanics is completed when reconstructed as a Boolean HV theory which includes dispersion-free measures. Moreover, if we acknowledge the priority of a Boolean HV reconstruction of quantum mechanics, then the ambiguity in the Kochen-Specker notion of the quantum propositional structure is again not harmful but merely confusing; for neither the (a) given P_{QMA} nor the (b) defined $\mathcal{p}\mathcal{B}\mathcal{A}$ need be preserved. It also follows from the above acknowledgement that the structural condition of P_{QMA}-preservation succumbs to the three sorts of criticisms described in the previous section, as do the structural conditions (vn$\not{b}$) and (JP$\not{b}$). In particular, since there is no reason why an HV reconstruction must satisfy any of these structural conditions, the von Neumann, the Jauch-Piron, the Gleason and the Kochen-Specker impossibility proofs do beg the HV question since each rests upon contradictions caused by the imposition of an unjustified condition. Bell's considerations of measurement interaction lend further support to the rejection of the structural conditions as unjustified. And since the structural conditions lead to contradictions, in other words, since HV theories which include these structural conditions are of the zeroth kind and are impossible, we can be sure that the structural conditions are precisely what a proposed HV reconstruction of quantum mechanics must <u>not</u> be required to satisfy.

So there are these two ways of interpreting quantum mechanics: <u>Either</u> the $\mathcal{P}_{QMA}$ structure is regarded as the core of quantum mechanics which must be preserved, in which case quantum mechanics is complete (as proved by Gleason) and a Boolean HV reconstruction of quantum mechanics is impossible (as proved by Kochen-Specker). <u>Or</u> the possibility of a Boolean reconstruction of quantum mechanics is regarded as the most important consideration in the interpretation of quantum mechanics, in which case a contextual Boolean HV reconstruction which does not preserve $\mathcal{P}_{QMA}$ is possible and quantum mechanics is incomplete relative to this reconstruction. The articulation of this dichotomy is Bub's decisive contribution to the interpretation of quantum mechanics and the problem of hidden-variables (see, e.g., Bub, 1973, p. 48). And notice that this dichotomy undercuts the three sorts of arguments described in Section A . For regardless of the inconsistency and question begging claims, and regardless of Bell's considerations of measurement interaction, the structural conditions and the HV impossibility proofs either stand or fall depending upon which side of the dichotomy one favours. In fact, which side of the dichotomy one favours also determines whether the inconsistency and question begging claims stand or fall.

In the rest of this section, some arguments in favour of the P_{QMA}-preservation side of this dichotomy are described. One might also consider regarding the orthomodular lattice P_{QML} rather than the partial-Boolean algebras P_{QMA} as the core of quantum mechanics which must be preserved; some arguments against regarding P_{QML} as the core of quantum mechanics are suggested by various points made throughout this thesis.[7]

Both sides of the dichotomy imply the imposition of structural conditions on a proposed Boolean HV reconstruction of quantum mechanics. Clearly, on the P_{QMA}-preservation side, the Boolean structural features of each mBS in a P_{QM} and the overlap patterns among the mBS's in a $P_{QM}^{n\geq 3}$ must be preserved. And on the Boolean reconstruction side, the Boolean structural features of each mBS in a P_{QM} may be preserved but, by virtue of the Gleason, Kochen-Specker results, the overlap patterns among the mBS's in a $P_{QM}^{n\geq 3}$ can<u>not</u> be preserved. So it is not the case that one side of the dichotomy imposes stringent structural conditions while the other side does not. Rather, both sides impose equally stringent conditions: either the overlap patterns among the mBS's <u>must be preserved</u>, or the overlap patterns <u>must be violated</u>.

The simple proposal that in a proposed Boolean reconstruction, the operations and relations among compatibles ought to be preserved while the operations and relations among incompatibles ought to be ignored, does not help decide between the two sides of the dichotomy. All the elements in an mBS of a $\mathcal{P}_{QM}$ are mutually compatible; and for any non-overlapping mBS_i , mBS_j of $\mathcal{P}_{QM}$, every element $P_i \in mBS_i$ (except the distinguished 0, 1 elements) is incompatible with every element $P_j \in mBS_j$ (except the distinguished 0, 1 elements). But the elements in any overlapping mBS's of $\mathcal{P}_{QM}^{n\geq 3}$ are inextricably compatible and incompatible with each other in the following sense. On the one hand, if the operations and relations among compatibles are preserved, then the overlap patterns are preserved, and then it follows, as Bell rightly argues, that some relations among incompatibles are also preserved. On the other hand, if the overlap patterns are not preserved, then these relations among incompatibles are not preserved, but also some relations among compatibles are not preserved. For example, consider the relation $P_1 \leq P_4 \vee P_5$ among the elements P_1 , P_4 , P_5 in the two overlapping mBS's of the twelve-element $\mathcal{P}_{QM}^3$ diagrammed in Section A . If the overlap pattern between mBS_1 and mBS_4 is preserved, then this relation is preserved even though $P_1 \not\unicode{x2AF0} P_4$ and $P_1 \not\unicode{x2AF0} P_5$, as Bell criticizes. But if the overlap pattern is not preserved, then even though $P_1 \unicode{x2AF0} P_4 \vee P_5$, this relation is not preserved in the sense that, for example, in a contextual HV theory, for a given hidden state ψ, ξ, P_1 may be assigned a value which is not less-than-or-equal-to the value assigned to $P_4 \vee P_5$, i.e., $\mu_{\psi,\xi}(\langle P_1, mBS_1\rangle) \not\leq \mu_{\psi,\xi}(\langle P_4 \vee P_5, mBS_4\rangle)$. In short, relations among compatible elements in overlapping mBS's cannot be preserved without also preserving relations among incompatibles, and relations among incompatible elements in overlapping mBS's cannot be ignored without also ignoring relations among compatibles.

The contextual HV proposals are the serious contenders on the Boolean reconstruction side of the dichotomy. As Gudder makes clear, contextual HV theories preserve the Boolean structural features of the mBS's in a P_{QM} structure (Gudder, 1970, p. 435). But as described in Section A , the dispersion-free HV measures induced by the hidden states of a contextual HV theory violate the overlap patterns among the mBS's by assigning different 0 or 1 values to a given element $P \in P_{QM}$ when P is a member of overlapping mBS's. That is, the value assigned to P when considered in the context of one mBS may be different from the value assigned to P when considered in the context of another mBS. In this sense, the identity P = P is violated; so clearly, any other operation or relation among elements in overlapping mBS's may be violated. And, for example, the dispersion-free HV measures of a contextual HV theory cannot even preserve just the $^{\perp}$ operation and the $\leq$ relation of P_{QM} because, as shown in Chapter V(C), just $^{\perp}$, $\leq$ preservation is sufficient to ensure that all operations and relations among compatible and incompatible elements in the ultrasubstructures of P_{QM} are preserved, where an ultrasubstructure in a $P_{QM}^{n\geq 3}$ is a union of overlapping mBS's; thus the overlap patterns among mBS's in the ultrasubstructures of $P_{QM}^{n\geq 3}$ are preserved if $^{\perp}$, $\leq$ are preserved.

Quantum mechanics itself is the serious contender on the P_{QMA}-preservation side of the dichotomy; HV proposals which preserve P_{QMA} are impossible. The quantum Exp_{ψ} measures on a P_{QM} do preserve the $\leq$, = relations and the $^{\perp}$ operation of P_{QM} , and they do preserve the $\wedge$, $\vee$ operations among any compatible pairs of elements in P_{QM} (even though Exp_{ψ} may not be bivalent with respect to every element in P_{QM}); thus the Exp_{ψ} measures do preserve P_{QMA} and do preserve the overlap patterns among the mBS's. And in particular, with respect to the domain US_ψ where each Exp_ψ is bivalent, Exp_ψ preserves all the operations and relations among all (compatible and incompatible) elements in the overlapping mBS's in US_ψ, as shown in Chapter VI(B).

Tutsch gives an example of how the quantum mechanical ordering of propositions, i.e., the $\leq$ relation of $\mathcal{P}_{QM}$ and thus the $=$ relation, is not preserved in the Bohm-Bub contextual HV theory. In this theory, once the hidden state ψ, ξ and the context of measurement are specified, the outcome of a measurement of a magnitude A (which is an eigenvalue of A) is determined by the so-called polychotomic algorithm, hereafter called the HV algorithm. Tutsch's example shows how according to the HV algorithm, for a given hidden state ψ, ξ, the outcome of a measurement of the magnitude $/S_z/$, the absolute value of spin-1 in the z direction, is the eigenvalue 0, while the outcome of a measurement of the magnitude S_z, spin-1 in the z direction, is the eigenvalue -1. Clearly, $/S_z/$ is a function of S_z; each magnitude is represented by an operator on three-dimensional Hilbert space; and both magnitudes share the eigenstate ψ_0 associated with their 0 eigenvalues and represented by the one-dimensional projector $\hat{P}_0 = |\psi_0\rangle\langle\psi_0|$. In both quantum mechanics and a contextual HV theory, the outcome 0 for a measurement of $/S_z/$ or for a measurement of S_z is connected with the assignment of the value 1 to the element $P_0 \in \mathcal{P}_{QM}$ which qua projector represents the eigenstate ψ_0. That is, in quantum mechanics, for a given quantum state ψ, the outcome of a measurement of $/S_z/$ is the eigenvalue 0 (i.e., $Exp\ (/S_z/) = 0$) IFF $Exp_\psi(P_0) = 1$; and likewise, the outcome of a measurement of S_z is the eigenvalue 0 (i.e., $Exp_\psi(S_z) = 0$) IFF $Exp_\psi(P_0) = 1$. Similarly, in a contextual HV theory, for a given hidden state $\psi, \xi,$ and for any context mBS, the outcome of a measurement of $/S_z/$ is the eigenvalue 0 IFF $\mu_{\psi,\xi}(\langle P_0, mBS\rangle) = 1$; and likewise, the outcome of a measurement of S_z is the eigenvalue 0 IFF $\mu_{\psi,\xi}(\langle P_0, mBS\rangle) = 1$. Furthermore, in the quantum propositional structure P^3_{QM} of three-dimensional Hilbert space, the element P_0, which qua projector represents the eigenstate ψ_0, represents both of the following propositions: "The eigenvalue of $/S_z/$ is 0." "The eigenvalue of S_z is 0." Since both propositions are represented by the same element $P_0 = P_0$ in P^3_{QM}, each proposition implies the other in the sense that $P_0 \leq P_0$ and $P_0 \geq P_0$ (where the $\leq$ of P_{QM} is interpreted as logical implication). And since, for every quantum state ψ, $Exp_\psi(/S_z/) = 0$ IFF $Exp_\psi(P_0) = 1$ IFF $Exp_\psi(S_z) = 0$, each proposition implies the other in the sense that, for any quantum state ψ, if the outcome of a measurement of $/S_z/$ is the eigenvalue 0, then the outcome of a measurement of S_z is the eigenvalue 0, and conversely.[8] But in spite of the fact that quantum mechanically, each of the above propositions implies the other (in both senses of implies), Tutsch gives an example of how in a contextual HV theory, the proposition "The eigenvalue of $/S_z/$ is 0." need not imply (in either sense) the proposition "The eigenvalue of S_z is 0."

According to the description of contextual HV theories given in Section A, such a deviation from quantum mechanics is possible because P_0 is a member of (at least) two overlapping mBS's specifying possible contexts of measurements of $/S_z/$, S_z, described as follows. Besides the 0 eigenvalue, the magnitude S_z has two other eigenvalues, 1, -1, each associated with an eigenstate represented by a one-dimensional projector

$\hat{P}_1 = |\psi_1\rangle\langle\psi_1|$ and $\hat{P}_{-1} = |\psi_{-1}\rangle\langle\psi_{-1}|$; but the magnitude $/S_z/$ has only one other eigenvalue besides 0, namely, the eigenvalue 1 associated with an eigenstate represented by a two-dimensional projector, say $\hat{P}_{1,-1} = \hat{P}_1 \vee \hat{P}_{-1}$ (i.e., the eigenvalue 1 of $/S_z/$ is degenerate). Though $P_{1,-1} = P_1 \vee P_{-1}$, it is equally true that $\hat{P}_{1,-1} = \hat{P}_a \vee \hat{P}_b$ for any orthogonal $\hat{P}_a, \hat{P}_b$ which satisfy $\hat{P}_a \vee \hat{P}_b = \hat{P}_1 \vee \hat{P}_{-1}$. So the set $\{P_0, P_1, P_{-1}\}$ (i.e., the mBS_1 generated by that set) may be the context of a measurement of $/S_z/$ as well as the set $\{P_0, P_a, P_b\}$ (i.e., the mBS_a generated by that set); but only the set $\{P_0, P_1, P_{-1}\}$ (i.e., mBS_1) may be the context of a measurement of S_z. And clearly, mBS_1 overlaps with mBS_a since both share the element P_0. Now as described in Section A, for a unique hidden state ψ,ξ, it is possible that $\mu_{\psi,\xi}(\langle P_0, mBS_a\rangle) \neq \mu_{\psi,\xi}(\langle P_0, mBS_1\rangle)$. So given the connection between the outcome 0 for a measurement of $/S_z/$ or S_z and the assignment of the value 1 to the element P_0 described above, this possibility: $\mu_{\psi,\xi}(\langle P_0, mBS_a\rangle) \neq \mu_{\psi,\xi}(\langle P_0, mBS_1\rangle)$ means that it is possible that if $/S_z/$ is measured in the context mBS_a and S_z is measured in the context mBS_1, then for a unique hidden state ψ,ξ, the outcome of the measurement of $/S_z/$ is the eigenvalue 0 (which occurs IFF $\mu_{\psi,\xi}(\langle P_0, mBS_a\rangle) = 1$), while the outcome of the measurement of S_z is one of S_z's other eigenvalues not equal to 0 (which occurs IFF $\mu_{\psi,\xi}(\langle P_0, mBS_1\rangle) \neq 1$). In his example, Tutsch gives a hidden state which, according to the HV algorithm, assigns values which exemplify this possibility. In particular, his hidden state yields the outcome 0 for $/S_z/$ but the outcome -1 for S_z. And although Tutsch does not explicitly state that in his example, $/S_z/$ is measured in a context different from the context in which S_z is measured, Tutsch does conclude that his example could mean that the two propositions: "The eigenvalue of $/S_z/$ is 0." and "The eigenvalue of S_z is 0." refer to "properties of the system plus apparatus and hence, different apparatus may produce different results." This conclusion suggests that in his example, $/S_z/$ is measured in a context different from the context in which S_z is measured (Tutsch, 1969, pp. 1118-1119).

Belinfante speaks of Tutsch's example as an example of a paradox which is derived from the HV algorithm[9] and which is related to but in fact worse than the Kochen-Specker troubles (which motivate the contextual HV proposals) in that such paradoxes are "much less (if at all) justifiable as a 'result of the influence of the measuring arangement'" (Belinfante, 1973, p. 135). Assuming that the above analysis of Tutsch's example is correct, the example is not an example of a paradox. For $/S_z/$ and S_z are measured in different contexts (involving different experimental arrangements), and as Bub makes clear, we must expect the dispersion-free measures induced by the hidden states of a contextual HV theory to assign different values even to the same magnitude when measured in different contexts. Gudder, who was in contact with Tutsch at the time of the publication of each of their papers, likewise understands Tutsch's example as involving measurements in different contexts (Gudder, 1970, p. 436). Moreover, the "paradoxes" exemplified by Tutsch's example are related to the Kochen-Specker troubles only in the sense that such "paradoxes" are features of a contextual HV theory which are necessary in order to avoid the Kochen-Specker HV impossibility proof. For if the dispersion-free HV measures induced by the hidden states of a contextual HV theory did not assign different values to the same magnitude when measured in different contexts and instead assigned unique values to an element P_0 even though P_0 is a member of overlapping mBS's, then such measures would be ruled out by the Kochen-Specker proof.

Gudder suggests that the sort of contextual HV deviations from quantum mechanics which are exemplified by Tutsch's example may be candidates for experimental verification or falsification. In addition to Tutsch's sort of deviations, there is also the following sort of contextual HV deviations from quantum mechanics which has been the subject of experimental test.

In a contextual HV theory, a pure quantum state ψ is treated as a mixed state with respect to the possible hidden states (each represented by ψ together with some ξ), and the mixed state ψ describes an ensemble of hidden states with a so-called equilibrium distribution of hidden variables. In order that the statistical condition, mentioned in the Preface to this chapter, be satisfied, this equilibrium distribution of hidden variables together with ψ must reproduce, via the HV algorithm, the statistical results of quantum mechanics given by the Exp_ψ measures (Belinfante, 1973, p. 136); or in other words, the statistical results of quantum mechanics are derived from the HV algorithm by assuming that the hidden variables are in an equilibrium distribution. For example, as described by Belinfante, consider a large number of quantum systems whose quantum state is ψ on which we perform measurements of a magnitude A and, whenever the outcome is a particular eigenvalue a_j , we follow up by measuring a different magnitude B. In quantum mechanics, the average value of B is determined not by $Exp_\psi(B)$ but rather by $Exp_{\psi_j}(B)$, where ψ_j is the eigenstate of A associated with the eigenvalue a_j ; that is, the first measurement of A is assumed to have reduced the initial state

ψ to the eigenstate ψ_j of A. In a contextual HV theory, in order that the HV algorithm reproduce this quantum mechanical result $\text{Exp}_{\psi_j}(B)$, besides the reduction of ψ to ψ_j, it must also be assumed that the hidden variables, which together with ψ_j describe the hidden states of the ensemble of quantum systems after the measurement of A, are in an equilibrium distribution before the measurement of B occurs (Belinfante, 1973, pp. 139-140). However, after the measurement of A, the HV algorithm calculations yield a non-equilibrium or biased distribution of hidden-variables. It is assumed that such biased distributions of hidden-variables very rapidly relax to the equilibrium distribution which reproduces the $\text{Exp}_{\psi_j}(B)$ result. But if the measurement of B is performed before the biased distribution resulting from the measurement of A has relaxed to the equilibrium distribution, then the biased distribution predicts via the HV algorithm statistical results for B which differ from what quantum mechanics predicts via its $\text{Exp}_{\psi_j}(B)$ formalism (Belinfante, 1973, p. 163).

These sorts of deviations from quantum mechanics which are connected with non-equilibrium distributions of hidden variables after measurement are different from the Tutsch sort of deviations which are connected with different contexts of measurement. For as described by Bohm-Bub (1966, p. 466), the non-equilibrium sort of deviations occur for measurements of magnitudes represented by operators on two-dimensional Hilbert space. But clearly, the contextual sort of deviations can occur for measurements only of magnitudes represented by operators on a three-or-higher dimensional Hilbert space since only $P_{QM}^{n\geq 3}$ structures have overlapping mBS's.

The existence of these deviations make it at least in principle possible to experimentally verify or falsify the predictions of the HV algorithm and thus to decide between quantum mechanics and the proposed contextual, Boolean HV reconstructions of quantum mechanics. Experiments testing for the non-equilibrium distribution sort of deviations with a time interval of less than 10^{-13} seconds between the measurements of different magnitudes (like the measurements of A and B described above) have so far found no deviations from the predictions of quantum mechanics and have thus falsified the predictions of the HV algorithm. However, HV advocates may argue that the time it takes a biased distribution of hidden variables to relax to the equilibrium distribution is less than the 10^{-13} second interval between the measurements of the experiments mentioned above. For as described by Belinfante, "it has not yet been established how fast one may theoretically expect biased hidden-variable distributions to relax. . . ." So even if the HV algorithm is falsified at an even shorter time interval in some future experiment, HV advocates may nevertheless continue to argue that the shorter time interval is not yet short enough to capture the biased distribution of hidden-variables before it relaxes to the equilibrium distribution which reproduces the quantum mechanical predictions. Thus while experiments have so far falsified and may continue to falsify the HV algorithm, it may be that no experiment will ever conclusively decide between quantum mechanics and the proposed contextual HV theories (Belinfante, 1973, pp. 88, 100).

Nevertheless, quantum mechanics is so far supported by experimental evidence. And as pointed out by Belinfante, the formalism of quantum mechanics is simpler than the formalism of the contextual HV theories. So by the usual criteria of experimental evidence and formal simplicity, quantum mechanics is a better theory of quantum phenomena than is a contextual HV theory. So why is quantum mechanics still challenged by the contextual HV proposals? Four reasons are described, to the end of this section.

1. One reason quantum mechanics is vulnerable to a contextual HV proposal is because even if it is granted that the classical notion of a probability measure defined on a Boolean structure may be generalized so as to be defined on the non-Boolean quantum P_{QM} structures, the notion of a generalized measure on P_{QM} is open with regard to the issue of which operations and relations of P_{QM} ought to be required to be preserved by the generalized measures. As described in Chapter IV(A), Bub and Jauch-Piron each define two different sorts of generalized measures on P_{QM}. The contextual HV measures can be regarded as a third sort of generalized probability measure on P_{QM} (even though the domain of a contextual HV measure is $P_{QM} \times \{mBS_i\}_{i \in Index}$). All three sorts of generalized measures preserve the Boolean structural features of the (maximal) Boolean substructures of P_{QM}. In addition, Bub, Gleason, Kochen-Specker, and Jauch-Piron require that a generalized probability measure satisfy Gleason's additivity condition (Ga) which ensures that dispersion-free generalized probability measures preserve the partial-Boolean structural features of P_{QM}, in particular, preserve the overlap patterns among the mBS's of $P_{QM}^{n \geq 3}$. Jauch-Piron further require their generalized measures to satisfy (JPϕ). An argument against the inclusion of (JPϕ) as part of the conditions defining a generalized probability measure is given in the note below.[10] Here we consider whether or not (Ga), which entails P_{QMA}-preservation, ought to be included.

The dispersion-free HV measures induced by the hidden states of a contextual HV theory do not and cannot satisfy (Ga) because together with the dispersion-free condition,(Ga) yields $\mathcal{P}_{QMA}$-preservation and (Ga) also yields the condition labeled (B) by Bell, neither of which is satisfied by contextual HV dispersion-free measures. However, the inclusion of at least Gleason's additivity condition as part of the conditions defining a generalized probability measure on a $\mathcal{P}_{QM}$ is strongly supported by the precedent that in classical probability theory, condition (μa) is included among the conditions defining a classical probability measure on a Boolean structure, as stated in Chapter III(C). (see for example, Kolmogorov, 1933, p. 2). Since elements in a $\mathcal{P}_{QM}$ are disjoint IFF they are orthogonal, or in other words, orthogonality is the quantum analogue of disjointedness, condition (Ga), which requires that a generalized probability measure on a $\mathcal{P}_{QM}$ be additive with respect to orthogonal elements of $\mathcal{P}_{QM}$, is the quantum analogue of condition (μa), which requires that a classical probability measure on a Boolean structure be additive with respect to disjoint elements. Or in other words, (Ga) is simply the condition (μa) as applied to the quantum $\mathcal{P}_{QM}$ structures. So it is arguable that because (μa) is one of the conditions defining a classical probability measure, (Ga) ought to be one of the conditions defining a generalized probability measure.

Moreover, as elaborated at the end of Section A, the condition of $\mathcal{P}_{QMA}$-preservation which follows from (Ga) is independently supported by the uncontentious general principle according to which the real value of say Borel function of any magnitude in any physical theory is calculated or determined by simply applying that Borel function to the real value of the magnitude. Since any magnitude is compatible with any Borel function of itself, the general principle refers to the preservation of functional relations among compatible magnitudes (or propositions). So in a contextual

HV theory, while the functional relations among compatible elements in any mBS of $\mathcal{P}_{QM}$ are preserved, the functional relations among compatible elements in overlapping mBS's of $\mathcal{P}_{QM}$ are not preserved since the $\mathcal{P}_{QMA}$ structure is not preserved and thus the general principle which entails $\mathcal{P}_{QMA}$-preservation is not satisfied in a contextual HV theory. For example, as suggested by Gudder, if one considered two different mBS's containing P and g(P) respectively, then one would get <u>independence</u> of the representing functions f_P, $f_{g(P)}$, rather than the <u>functional relation</u> $f_{g(P)} = g(f_P)$. And the excuse given by contextual HV advocates for this violation of the general principle is that the consideration of two different mBS's involves two separate measurements with different experimental arrangements, and so in such cases one would expect to obtain <u>independent</u> results for P and g(P) (Gudder, 1970, p. 435). This excuse ignores or makes light of the fact that, as determined by quantum mechanics and as (so far) experimentally observed, the results of any measurements of P, g(P), are not independent but rather are functionally related in accordance with the general principle.

2. As suggested again in the previous paragraph, quantum theory is vulnerable to the contextual HV proposals if measurement interaction or measurement disturbance is regarded as the cause or basis of the non-classical peculiarities of quantum mechanics and as (at least part of) the reason why the von Neumann, Jauch-Piron, and Kochen-Specker type of HV proposals are impossible. For example, according to Heisenberg's version of the Copenhagen interpretation of quantum mechanics, one reason why quantum ensembles cannot be resolved into subensembles which are dispersion-free (as required in von Neumann's HV proposal) is because quantum systems are disturbed by measurement. And for an example of how measurement considerations

support contextual HV proposals, we have of course Bell's argument, from the perspective of Bohr's version of the Copenhagen interpretation, that structural conditions like P_{QMA}-preservation which refer even indirectly to measurements of incompatible magnitudes must not be imposed upon the proposed dispersion-free measures of an HV theory because of the inextricable wholeness of quantum phenomena and measuring devices.

Now the outcome of a measurement at best determines an assignment of 0, 1 values to a maximal Boolean substructure of elements in a P_{QM}. For a measurement of any magnitude A can at best be a measurement of what is called a complete set of commuting magnitudes including A (and including <u>just</u> A if none of A's eigenvalues are degenerate) whose eigenstates $\{\psi_i\}_{i\in \text{Index}}$, as represented by (n) orthogonal atoms $\{\hat{P}_i\}_{i\in \text{Index}}$ in a $P^{(n)}_{QM}$ structure which generate a unique maximal Boolean substructure mBS_A of $P^{(n)}_{QM}$, specify the context of the measurement of A. And the outcome of the measurement, which is an eigenvalue a_j of A associated with an eigenstate ψ_j in the set $\{\psi_i\}_{i\in \text{Index}}$, determines via Exp_{ψ_j} in quantum mechanics and via $\mu_{\psi_j,\xi}$ in a contextual HV theory, an assignment of 0, 1 values to the elements in that mBS_A. The contextual HV measure does no more without changing its 0, 1 value assignments to the members of mBS_A. However, without changing its value assignments to the members of mBS_A, the quantum measure $Exp_{\psi_j} : US_{\psi_j} \to \{0,1\}$ in addition assigns 0, 1 values to every element in the ultrasubstructure $US_{\psi_j} = \{P \in P_{QM} : P \,\delta\, P_j\} \supseteq mBS_A$, where (unless US_{ψ_j} happens to equal mBS_A) US_{ψ_j} is a union of overlapping mBS's including mBS_A.

These additional 0, 1 value assignments by the quantum measure Exp_{ψ_j} mean the following. Let B be any magnitude which shares the eigenstate ψ_j with A even though $B \not\delta A$ (i.e., B and A do not share all their eigenstates). Either alone (if none of B's eigenvalues are degenerate) or as part of a complete set of commuting magnitudes, B specifies a unique maximal Boolean substructure mBS_B of P_{QM} which clearly overlaps with mBS_A since the atom P_j , which qua projector represents the eigenstate ψ_j , is a member of both mBS_B and mBS_A . And since every element in mBS_B is compatible with P_j , clearly $mBS_B \subset US_{\psi_j}$. Thus Exp_{ψ_j} assigns 0, 1 values to every element in mBS_B . And these value assignments by Exp_{ψ_j} mean that if B is measured after A is measured, or if B, instead of A, had been measured with the outcome b_j , then the outcome of the measurement of B, namely, the eigenvalue b_j associated with the eigenstate ψ_j , determines an assignment of 0, 1 values to the elements in mBS_B such that the values assigned to the common elements in $mBS_A \cap mBS_B$ match the value assignments determined by the outcome of the (first) measurement of A.

Thus the 0, 1 value assignments by the quantum measure Exp_{ψ_j} to every element in both $mBS_A \subset US_{\psi_j}$ and $mBS_B \subset US_{\psi_j}$ are determined by the outcome of one measurement yet refer to the outcomes of more than one measurement. For A and B cannot be measured simultaneously, i.e., $A \not\delta B$. (And if $A \,\delta\, B$, then $mBS_A = mBS_B$ in P_{QM} .) In other words, the fact that a quantum Exp_ψ measure assigns 0, 1 values to overlapping mBS's of elements in a manner which preserves the overlap patterns says something about different measurements of incompatible magnitudes. Similarly, if proposed dispersion-free HV measures are required to assign 0, 1 values to overlapping mBS's of elements in a manner which preserves the overlap patterns, then this requirement does refer to different measurements of incompatible magnitudes, as the contextual HV advocates argue. For example, the 192 atoms contained in 118 overlapping mBS's in the $\mathcal{P}^3_{\mathrm{QMA}}$ considered by Kochen-Specker in the Theorem 1 part of their HV impossibility proof represent the eigenstates of 118 incompatible magnitudes which cannot all be measured together, yet Kochen-Specker require proposed dispersion-free HV measures to preserve the overlap patterns among the eigenstates of these magnitudes. This requirement, which is part of the $\mathcal{P}_{\mathrm{QMA}}$-preservation condition, is very hard to motivate if measurement interaction, as described by Bohr and Bell, or measurement disturbance, as described by Heisenberg with his Uncertainty Principle, are treated as central in the interpretation of quantum mechanics, as the cause of the non-classical peculiarities of quantum mechanics, and as the reason why hidden-variables are either impossible or else dependent upon the context of measurement.

In contrast, if the non-Boolean $\mathcal{P}_{\mathrm{QMA}}$ structure abstracted from the fundamental postulates of quantum mechanics is treated as central in the interpretation of quantum mechanics, then the non-classical peculiarities of quantum mechanics are regarded as due to the non-Boolean character of the $\mathcal{P}_{\mathrm{QMA}}$ structure rather than due to measurement interaction or disturbance.[11] And as Kochen-Specker and Bub make clear, consideration of measurement interaction or disturbance are beside the point if the problem of hidden-variables is correctly understood as posing the question of whether the statistical results of $\langle \mathcal{H}, \mathcal{P}_{\mathrm{QMA}}, \mathrm{Exp}_\psi \rangle$ can be reconstructed in terms of a classical measure space $\langle \Omega, \mathcal{P}_{\mathrm{HV}}, \mu \rangle$ in a manner which preserves the core $\mathcal{P}_{\mathrm{QMA}}$ structure of quantum mechanics. For example, <u>in spite of</u> Heisenberg's Uncertainty Principle, the statistical results of $\langle \mathcal{H}^2, \mathcal{P}^2_{\mathrm{QMA}}, \mathrm{Exp}_\psi \rangle$ can be classically reconstructed, as Kochen-Specker demonstrate by producing an HV theory for that part of quantum mechanics which involves just two-dimensional Hilbert space (Kochen-Specker, 1967, pp. 75-80, 86).

3. Metaphysical prejudices, like the "religious belief that 'nature must be deterministic' . . ." mentioned by Belinfante (1973, p. 18) make quantum mechanics especially vulnerable to the contextual HV proposals. As described by Bub, the main reason why quantum mechanics is vulnerable to contextual HV proposals is because of the presupposition that the logical-property-event structure of reality and of any physical theory about any portion of reality <u>is</u> and can only be a <u>Boolean</u> structure. Bub argues that behind the affirmation of the three primed conditions (i'), (ii'), (iii') describing the Boolean reconstruction side of the dichotomy in the interpretation of quantum mechanics is the (metaphysical) presupposition that the logical-property-event structure of quantum phenomena must be a Boolean structure, like the Boolean logical-property-event structure of classical phenomena and classical mechanics. In contrast, behind the affirmation of the three un-primed conditions (i), (ii), (iii) describing the $\mathcal{P}_{QMA}$-preservation side of the dichotomy, there is an open acceptance of the notion of a <u>non</u>-Boolean logical-property-event structure of quantum phenomena and quantum mechanics (Bub, 1973, p. 54; 1974, p. 144). This acceptance is motivated by the following analogy. The logical-property-event structure of classical phenomena as described by classical mechanics is identified with (or is considered to be isomorphic with) the Boolean propositional structure $\mathcal{P}_{CM}$ abstracted from the formalism of classical mechanics. The non-Boolean propositional structure $\mathcal{P}_{QM}$, in particular, the $\mathcal{P}_{QMA}$ structure, is abstracted from the formalism of quantum mechanics

in a manner exactly analogous to the way in which P_{CM} is abstracted from the classical formalism. So the logical-property-event structure of quantum phenomena, as successfully described by quantum mechanics, may be and ought to be identified with (or considered to be isomorphic with) the P_{QMA} structure rather than any proposed Boolean P_{HV} structure.

Now if the quantum P_{QMA} could be imbedded($\flat$) into a Boolean structure, then the statistical results of quantum mechanics could be reconstructed in terms of a classical measure space $<\Omega, P_{HV}, \mu>$ with a Boolean structure at its core, and thus quantum mechanics could be regarded as a rather baroque elaboration of what is essentially a classical statistical theory. For example, P^2_{QMA} can be imbedded($\flat$) into a Boolean structure, and the $<H^2, P^2_{QMA}, \mathrm{Exp}_\psi>$ statistical results can be reconstructed in terms of a classical measure space, as demonstrated by Kochen-Specker. So if quantum mechanics made use of just two-dimensional Hilbert space rather than any higher dimensional Hilbert spaces, then quantum mechanics would in fact be a classical statistical theory since all of its statistical results could be classically reconstructed. However, the quantum $P^{n\geq 3}_{QMA}$ structures cannot be imbedded($\flat$) into a Boolean structure, and the $< H^{n\geq 3}, P^{n\geq 3}_{QMA}, \mathrm{Exp}_\psi>$ statistical results cannot be reconstructed in terms of a classical measure space. For Kochen-Specker and for Bub, this fact demarcates quantum mechanics from classical mechanics.[12] As Bub says:

> I have argued that the transition from classical to quantum mechanics is to be understood as a generalization of the Boolean propositional structures of classical mechanics to a particular class of non-Boolean structures. (1974, pp. 149-150)

So the fact that a $P^{n\geq 3}_{QMA}$ is not imbeddable($\flat$) into a Boolean structure signals the separate but equal status of the P_{QMA} structure and the P_{CM} structure; each structure is theoretically located at the core of quantum mechanics and classical mechanics, respectively. And the very fact that it is the preservation of the partial-Boolean structural features of $P^{n\geq 3}_{QMA}$, in particular, the preservation of the overlap patterns among the mBS's in $P^{n\geq 3}_{QMA}$, which must be given up in order to make a Boolean HV reconstruction of quantum mechanics, in particular, a contextual HV theory, possible lends further support to locating the P_{QMA} structure at the core of quantum mechanics which must be preserved.

As suggested above, the preservation of the P_{QMA} structure is further motivated by regarding P_{QMA} as the logical structure of quantum mechanics and as the logical space, in a Wittgensteinian sense, of micro-events, as Bub does (Bub, 1973, p. 52; Wittgenstein, 1921, p. 35). Now whether or not the P_{QMA} (or the P_{QML}) structure is accepted as a new quantum logic depends upon one's views of what logic is and of what role logic plays in a physical theory. Bub argues that the structure of logical space is not "parasitic on the syntactic properties of a formalized language," is not conventional, and is not "a priori in the sense that the laws of logic characterize necessary features of any linguistic framework suitable for the description and communication of experience." Rather, "logic is about the world, not about language" (Bub, 1973, pp. 52-53). And: "The logical structure of a physical system is understood as imposing the most general kind of constraint on the occurrence and non-occurrence of events" (Bub, 1974, p. 149). Moreover, as first suggested by Putnam, just as geometry plays an explanatory role in relativistic mechanics, e.g., the curved geometry of space-time "explains" gravity, similarly, quantum logic plays an explanatory role in quantum mechanics, e.g., the fact that the logical core of quantum mechanics is the non-Boolean P_{QMA} structure "explains" the non-classical peculiarities of quantum mechanics (Bub, 1973, p. 52; Putnam, 1969).

4. Finally, as suggested at the end of point (2) above, an inadequate or incorrect view of the problem of hidden-variables and the problem of the completeness of quantum mechanics makes quantum mechanics vulnerable to the contextual HV proposals. As Bub argues, the notion of a completion or extension of a physical theory only makes sense with respect to the underlying logical-property-event structure as given by the propositional structure determined by the theory's formalism. So a contextual HV theory which does not preserve the quantum P_{QMA} structure abstracted from the fundamental postulates of quantum mechanics is not a completion of quantum mechanics but rather is an entirely separate theory of quantum phenomena which will have to stand on its own feet (Bub, 1974, p. 147). Considering the experimental falsifications of the contextual HV deviations from quantum mechanics, we may conclude with Stapp that quantum mechanics is complete, in at least the pragmatic sense that

> . . . no theoretical construction can yield [or has so far yielded] experimentally verifiable predictions about atomic phenomena that cannot be extracted from the quantum theoretic description. (Stapp, 1972, p. 1108)[13]

Notes:

[1]This statement is corroborated by Kochen-Specker (1967, p. 81) and Gudder (1970, p. 432).

[2]Belinfante makes a similar point (1973, pp. 25-26).

[3]Bub uses this phrase in reference to contextual HV theories, as will be discussed below (Bub, 1974, p. 146).

[4]As mentioned above, Bell claims that Gleason's impossibility proof rests upon the tacit assumption that dispersion-free HV measures are non-contextual. Belinfante substantiates Bell's claim by elaborating how the condition Bell labels (B) follows from the condition Bell labels (A) together with the non-contextual assumption (Belinfante, 1973, p. 65).

[5]If any of the $\hat{P}_i$ representing the eigenstates of A are two-or-higher dimensional projectors, which obtains if any of A's eigenvalues are degenerate, then contextual HV theories have a procedure, e.g., Tutsch's rule, for augmenting the set of eigenstates so that the set is complete and unique, and so specifies a context (Belinfante, 1973, pp. 132-133).

[6]See note 4 of Chapter V, and the discussion of measurement under point 2. at the end of the next Section B.

[7]Various points made throughout this thesis suggest the problematic character of the lattice definitions of $\wedge$, $\vee$ among incompatibles and so favour the partial-Boolean algebra P_{QMA} formalization of the quantum propositional structures.

As described in Chapter IV, von Neumann first developed, in 1932, something like a partial-Boolean algebra of quantum propositions. A lattice of quantum propositions was developed by von Neumann four years later with the collaboration of Birkhoff, who had just founded lattice theory and so no doubt had the idea of a lattice, with $\leq$ interpreted as logical implication, strongly in mind. But Birkhoff and von Neumann immediately recognized the problematic character of the meets and joins of incompatible propositions, which they said could not be interpreted as experimental propositions. Moreover, the definition of $\wedge$, $\vee$ among incompatible propositions <u>qua</u> projectors cannot be given simply in terms of $+$ and $\cdot$ as usual, rather, Jauch had to create definitions involving the limits of infinite products. It has been said that the lattice definitions of $\wedge$, $\vee$ among incompatibles results in misinterpretations of the elements of P_{QM}. And the lattice definitions of $\wedge$, $\vee$ among incompatibles can cause a proliferation of lattice elements, as exemplified in Chapter VI(C), and do cause truth-functionality(δ,$\not\delta$) problems which are peculiar to the P_{QML} and are avoided by the P_{QMA} structures, as described in Chapter V.

Also see notes 10 and 12 below for further criticisms of the orthomodular lattice P_{QML} formalization of the quantum P_{QM}.

[8]When Tutsch talks of the "ordinary heuristic sense" of implication, I understand him to be referring to the latter sense of implication which involves the outcomes of two measurements. The two measurements may be successive measurements, or the two measurements may be alternate measurements performed on two systems in the same prepared state. In the former case, it is assumed that the first measurement is "reproducible," which loosely speaking means that the measurement can be followed up by another measurement (e.g., the measured system has not been annihilated) and which more strictly speaking means that the measurement can serve as

what has been called a <u>state preparation</u> (Belinfante, 1973, p. 6; Ballentine, 1970, p. 366).

[9]Belinfante points to the HV algorithm and the so-called Tutsch's rule as the basis of the "paradox" exemplified by Tutsch's example (Belinfante, 1973, pp. 142, 217). As mentioned in note 5 above, Tutsch's rule is a rule by which a unique and complete context of measurement is determined for a measurement of a magnitude which has degenerate eigenvalues. This rule may determine, for example, the context $\{P_0, P_a, P_b\}$ (i.e., mBS_a) for a measurement of the magnitude $/S_z/$ whose eigenvalue 1 is degenerate.

[10]In classical probability theory, a classical probability measure is defined as a function $\mu : \mathcal{B} \rightarrow [0,1]$ satisfying:

(i) Gleason's additivity condition (Ga), where orthogonality is equivalent to disjointedness

(ii) $\mu(0) = 0$ and $\mu(1) = 1$ (see Chapter III(C)).

And it is easy to show that μ satisfies condition (JP$\not\!\!\!\circ$), i.e., for any $b,c \in \mathcal{B}$, if $\mu(b) = \mu(c) = 1$ then $\mu(b \wedge c) = 1$ (Jauch, 1976, pp. 136-137). Now Jauch defines a generalized probability measure as a function $\mu : \mathcal{P}_{QML} \rightarrow [0,1]$ satisfying:

(i) (Ga)

(ii) $\mu(0) = 0$ and $\mu(1) = 1$

(iii) (JP$\not\!\!\!\circ$)

And Jauch remarks that property (iii) "must be postulated since it cannot be derived from the other two as in the classical probability calculus" (Jauch, 1976, pp. 135, 136).

In order to help motivate the inclusion of (JP$\not\!\!\!\circ$) as part of the conditions defining a generalized probability measure, Jauch mentions his passive filter interpretation of quantum propositions, according to which the conjunction $P_1 \wedge P_2$, for $P_1 \not\!\!\!\circ\, P_2$, is interpreted as an infinite, alternating sequence of filters representing P_1, P_2. (This passive filter interpretation, described in greater detail in Jauch's (1968, pp. 74-76), has always seemed suspect to me; Jauch proposes it in order to make sense of the lattice definition of $\wedge$ among incompatibles.) And using Gleason's Completeness result, Jauch gives a derivation of (iii), i.e., (JP$\not\!\!\!\circ$), from (i) and (ii) for the case of $\mathcal{P}^{n\geq 3}_{QML}$. So although Gleason does <u>not</u> include (JP$\not\!\!\!\circ$) as part of his definition of a generalized probability measure, Jauch uses Gleason's result to help make the inclusion of (JP$\not\!\!\!\circ$) as part of Jauch's definition of a generalized probability measure more palatable. However, Jauch adds:

> The only example known to me of a probability measure on a lattice which does not satisfy (iii) is in a lattice with a maximal chain of three elements $\left[\text{i.e., } P^2_{QML}\right]$. This is of course precisely the case that is excluded by the hypothesis of Gleason's theorem that dim $\mathcal{H} \geq 3$. In view of this fact it would be of considerable interest to prove property (iii) in the lattice-theoretic setting. No such proof is known to me.' (Jauch, 1976, p. 139)

Thus, since there is no derivation of (iii), i.e., (JP$\not\phi$), from (i) and (iii) for the quantum P_{QML} of arbitrary dimension, Jauch chooses to make his (JP$\not\phi$) <u>part of the definition</u> of a generalized probability measure.

In contrast, I would conclude that the notion of a generalized probability measure is better defined by just (i) and (ii), <u>rejecting</u> (JP$\not\phi$) altogether. It follows then that a generalized probability measure on a P_{QML} preserves only the partial-Boolean structural features of P_{QML}, which calls into question the whole enterprise of an orthomodular lattice formalization of the quantum P_{QM} structure. Why bother formalizing P_{QM} as an orthomodular lattice P_{QML} when the measures and mappings defined on P_{QM} preserve only the partial-Boolean features of P_{QM} and ignore the lattice definitions of $\wedge$ and $\vee$ among incompatibles? Even Jauch's difficult to motivate inclusion of (JP$\not\phi$) as part of the conditions defining a generalized probability measure μ on a P_{QML} only preserves the $\wedge$ operation among incompatible P_1, P_2, when $\mu(P_1) = \mu(P_2) = 1$; otherwise, the $\wedge$ operation among incompatibles is ignored, as is the $\vee$ operation among incompatibles. In fact, <u>nothing less than</u> the truth-functionality(∂,$\not\phi$) condition considered in Chapters IV and V ensures the preservation of the features of P_{QML} which distinguish P_{QML} from P_{QMA}, namely, the meets and joins of incompatibles. But as far as I know, no author who has considered the problem of how best to define the notion of a generalized probability measure on the quantum P_{QM} structure has advocated the inclusion of a condition as strong as truth-functionality(∂,$\not\phi$).

[11] Some examples of how the non-Boolean character of the quantum P_{QMA} structure is the basis of the non-classical peculiarities of the quantum statistical results are described by Bub (1974, pp. 149, 120-122, 125-127).

[12] Bub's point about the demarcation between classical mechanics and quantum mechanics suggests the following criticism of the orthomodular lattice P_{QML} formalization of the quantum P_{QM} structure.

As described in Chapters IV(F) and V, from the P_{QML} perspective, the peculiarly non-classical feature which distinguishes the quantum propositional structures from the classical ones and which, for example, makes a classical HV reconstruction of quantum mechanics impossible, is the presence of incompatible elements. This is made clear in the statement of Theorem A (Chapter V(A)) and, for example, in Jauch-Piron's concluding remark about their HV impossibility proof: "To rule out hidden variables it suffices to exhibit two propositions of a physical system which are not compatible" (Jauch-Piron, 1963, p. 837). In contrast, from the P_{QMA} perspective, it is not the presence of incompatibles but rather the presence of overlapping mBS's (for which the presence of incompatibles is necessary but not sufficient) which distinguishes the quantum propositional structures from the classical ones and makes a classical HV reconstruction of quantum mechanics impossible. However, as described in Chapter IV(F), any P_{QM} whose mBS's happen to not overlap can be imbedded($\dot{\circ}$) into a Boolean structure and so is classical, in a Kochen-Specker sense. And in particular, the mBS's of a P^2_{QM} structure do not overlap, P^2_{QM} can be imbedded($\dot{\circ}$) into a Boolean structure, and the quantum mechanics of two-dimensional Hilbert space does admit a classical HV reconstruction. Thus from the P_{QMA} perspective, there is a classical/quantum demarcation between P_{QM} structures with non-overlapping mBS's and P_{QM} structures with overlapping mBS's. And in particular, from the P_{QMA} perspective, there is a classical/quantum demarcation between P^2_{QM} and $P^{n\geq 3}_{QM}$ structures.

These classical/quantum demarcations, which are not recognized from the P_{QML} perspective, are in fact corroborated by the lattice-theoretician Jauch as follows. Jauch argues that although P_{QML} structures may be irreducible or may be reducible, the quantum superselection rules ensure that every quantum mechanically relevant P_{QML} is reducible (Jauch, 1968, p. 109). Thus Jauch makes a non-quantum/quantum demarcation between irreducible P_{QML} and reducible P_{QML}. Now it is intuitively obvious that if a P_{QML} is reducible, then P_{QML} contains overlapping mBS's. For in a reducible P_{QML}, there is at least one element $P_0 \neq 0,1$, such that $P_0 \,\dot{\circ}\, P$ for every $P \in P_{QML}$; thus P_0 must be a member of more than one mBS of P_{QML}. Contrapositively, if none of the mBS's in a P_{QML} overlap, then the only elements shared by any mBS's are the $0, 1$ elements, and so P_{QML} must have a trivial centre, i.e., P_{QML} is irreducible. So if we consider a reducible P_{QML} (which falls on the quantum-side of Jauch's demarcation), since a reducible P_{QML} contains overlapping mBS's, a reducible P_{QML} also falls on the quantum-side of the P_{QMA} demarcation. And if we consider a two-dimensional Hilbert space P^2_{QML} (which falls on the classical-side of the P_{QMA} demarcation), since the mBS's in a P^2_{QML} do not overlap, a P^2_{QML} is irreducible and so falls on the non-quantum-side of Jauch's demarcation. This suggests, in broad outline, a correlation between Jauch's demarcation and the P_{QMA} demarcations, even though the latter are demarcations between classical/quantum while the former is a demarcation between non-quantum/quantum. So although the P_{QMA} demarcations between P_{QM} with overlapping mBS's and P_{QM} with non-overlapping mBS's and between P^{2}_{QM} and $P^{n\geq 3}_{QM}$ are not recognized from the P_{QML} perspective, they are reflected in Jauch's demarcation between irreducible and reducible structures, which suggests that the P_{QMA} demarcations are worth recognizing.

[13]Bits of this chapter were included as my contribution to a co-authored (with Edwin Levy and R. I. G. Hughes) review of Bub's The Interpretation of Quantum Mechanics. The review appears in the June, 1977, issue of Philosophy of Science.

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