# A THEORETICAL STUDY OF THE $(\pi, \pi n)$ KNOCK-OUT REACTION 

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## Abstract

The ( $\pi, \pi n$ ) reaction is studied theoretically, the specific case examined being $0^{16}\left(\pi^{+}, \pi^{+} p\right) N^{15}$. Calculations of the differential cross sections for $1 \mathrm{P}_{1 / 2}$ and $1 \mathrm{P}_{3 / 2}$ protons are made at various incident pion energies. The remaining kinematic variables are specified by a geometry which emphasizes the behavior of the two-body pion nucleon interaction. The aim is to examine the influence of the nucleus on the two-body interaction. In particular, the influences of Pauli exclusion of the nucleon, the off-shell effect, and the effective polarization of the nucleon are examined. The computation is performed using the factorized distorted wave impulse approximation. By evaluating the distorted wave matrix element in coordinate space the localization of the knock-out reaction in the nucleus is determined.

The Pauli exclusion of the nucleon is found to have the largest influence on the differential cross section at incident pion energies of 116 MeV . At higher energies the knock-out reaction occurs at the extreme edge of the nucleus and the effect of Pauli exclusion is minimal. At lower energies, the two-body interaction itself is less sensitive to Pauli exclusion. Off-shell effects were found to be very small. The polarization of the proton was found to have a large influence on the cross sections. Furthermore, it is
noted that comparing the cross sections for $1 P_{1 / 2}$ and $1 P 3 / 2$ protons will indicate the effective polarization of the proton in the nucleus.
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## Introduction

The knock-out reaction, which has been an extremely useful tool in nuclear physics, involves an incident particle striking a nucleus and emerging after having knocked a nucleon out.

Early work in this area involved an examination of the energies of the particles in the final state which allowed the binding energy of the struck nucleon to be calculated. These binding energies were found to cluster around the welldefined values predicted by the nuclear shell model. Furthermore, the angular distributions of the knocked out nucleons made it possible to distinguish between particles with zero and non-zero angular momenta. The resulting assignments were in complete agreement with the predictions of the simple shell model. ${ }^{1}$

These binding energies, coupled with the charge distributions of the protons determined from elastic electron scattering, have allowed single-particle potentials to be determined for these shells. ${ }^{2}$

As computers became more powerful, calculation of the effect of the nucleus on the incident and outgoing particles became feasible and tests of optical model potentials could be performed. The ( $\mathrm{p}, 2 \mathrm{p}$ ) reaction was well-suited to this type of analysis as the symmetries present in the reaction
allowed for much simplification of the calculation. ${ }^{3}$ The $(p, 2 p)$ reaction is also extremely sensitive to the effects of nuclear distortion that the nucleus has on the proton wave functions because there are two outgoing protons and their energies vary with angle. ${ }^{4}$

Modern computers have made calculations of knock-out cross sections for more general cases possible. Studies of the $(p, p n)^{5}$ and $(\pi, \pi n)^{6}$ reactions have been reported. With current studies of the ( $\pi, \pi n$ ) knock-out reaction, that have the outgoing pion and nucleon measured in coincidence, now underway at LAMPF and SIN, detailed calculations of this reaction have become desirable.

As stated earlier, these calculations will provide a good test of optical model potentials. Furthermore, the quasi-free nature of the reaction will allow an examination of the twobody interaction between the pion and the struck nucleon. ${ }^{?}$ As this interaction occurs within the nucleus, it is different from the free-space reaction in two aspects. Firstly, since the nucleon resides in the fermi sea of the nucleus, it is prevented, by the Pauli Exclusion Principle, from having a free range of energies. Secondly, because the nucleon is bound within the nucleus, the energy required to free it causes the two-body reaction to be off-shell. ${ }^{8}$

The purpose of this thesis is, therefore, to predict the behavior of these effects on the differential cross section of the knock-out reaction. The theory of the knockout reaction will be discussed in Section II. Sections III
and IV will give a detailed examination of the components of the calculation, namely, the Two-Body $t$ Matrix and the Distorted Wave Matrix Element. The final sections will focus on the results of the knock-out reaction calculation.

The formal framework used in this analysis of the knockout reaction is the Distorted wave Impulse Approximation (DWIA). This approximation considers the reaction to be a sequential process. The particle incident on a nucleon in the nucleus has its wave function distorted by the nucleus. The incident particle undergoes a two-body interaction with the bound nucleon. These two particles then leave the nucleus, which acts upon them to distort their respective wave functions. The general form of the $T$ matrix describing this process is:

$$
\begin{equation*}
T_{i f}=K\left\langle x_{K_{n}^{\prime}}^{(-)} \quad x_{K_{\pi}^{\prime}}^{(-)}\right| t_{\pi n}\left|x_{K_{\pi}}^{(+)} \psi_{n}\right\rangle \tag{1.1}
\end{equation*}
$$

where $\quad t_{\pi n}$ is the two-body ( $\pi n$ ) t matrix
$x_{K_{n}^{\prime}}^{(-)}$is the final nucleon distorted wave function
$\chi_{K^{\prime} \pi}^{(n)}$ is the final pion distorted wave function
$\chi_{K \pi}^{(+)}$is the incident pion distorted wave function $\psi_{n}$ is the nucleon bound state wave function $K$ is a kinematic factor.

The DWIA has been shown to be the first term in a series. Subsequent terms include nucleus recoil effects and multiple scattering. ${ }^{10}$

Comparisons of the DWIA with exact calculations on light nuclei show that the shape of the DWIA is in quite good
agreement, although the normalization varies widely. ${ }^{11}$ Therefore, one quite often considers only the relative behavior of the DWIA calculation when making comparisons with experimental data.

To further simplify the calculation, the two-body $t$ matrix has been removed from the integration involved with equation (1.1). To illustrate this, consider the momentum space representation of equation (1.1): ${ }^{12}$

$$
\begin{align*}
T_{\text {if }} & =\int d^{3} p_{\pi} d^{3} q_{r} d^{3} q_{\pi} d^{3} q_{n} \delta\left(p_{\pi}-q_{\pi}-q_{r}-q_{n}\right) x_{K_{\pi}}^{(+)}\left(p_{\pi}\right) \psi_{n}\left(q_{r}\right) x_{K_{\pi}^{\prime}}^{(-)}\left(q_{\pi}\right) \\
& \left.\times x_{k_{n}^{\prime}}^{(-)}\left(q_{n}\right)<p_{o n}\left|t\left(\varepsilon_{o n}\right)\right|_{p_{o f f}}\right\rangle \tag{1.2}
\end{align*}
$$

where $K_{x}$ indicates the asymptotic momentum of particle " $x$ "
Poff is the initial relative momentum in the two-body $\pi n$ interaction

Pon is the final relative momentum in this interaction $\mathcal{E}_{\text {on }}$ is the energy of the interaction. ${ }^{8}$

At energies above 50 MeV , the distorted wave functions become strongly peaked about the asymptotic value of their respective particle's momentum ${ }^{12}$ and away from the $(3,3)$ resonance, the $t$ matrix is a slowly varying function of energy. ${ }^{13}$ The $t$ matrix, therefore, has little influence on the integration and can be evaluated at the asymptotic momentum values and factored out. Thus, ${ }^{8}$

$$
\begin{equation*}
\left.T_{\text {if }} \propto<p_{\text {Pon }}\left|t\left(\mathcal{E}_{\text {on }}\right)\right|_{P_{\text {off }}}\right\rangle\left\langle x_{K_{\pi}^{\prime}}^{(-)} x_{K_{n}^{\prime}}^{(-)} \mid x_{K_{\pi}}^{(+)} \psi_{n}\right\rangle \tag{1.3}
\end{equation*}
$$

This approximation can be further improved by making corrections to $p_{o n}, \mathcal{E}_{\text {on }}$, and $p_{\text {off }}$ for the effects of the nucleus on the momentum of the incident and outgoing particles. ${ }^{12}$

Since this adds significantly to the complexity of the calculation, this has not been done here.

As indicated in the ( $p, 2 p$ ) reaction, consideration of the special case where the protons emerge at equal but opposite angles greatly simplifies the calculation, while retaining its sensitivity to different optical model potentials. Similarly, selecting a special geometry will reveal the sensitivity of specific aspects of the knock-out reaction; in this case, selecting a geometry that will emphasize the behavior of the two-body $t$ matrix is desired.

If one makes the Plane Wave Impulse Approximation that the nucleus does not distort the wave functions, the Distorted Wave Matrix Element (DWME) becomes: ${ }^{1}$

$$
\begin{equation*}
\left\langle\chi_{K_{n}^{\prime}}^{(-)} \chi_{K_{\pi}^{\prime}}^{(-)} \mid x_{K_{\pi}}^{(+)} \psi_{n}\right\rangle=\int d^{3} r^{\prime} e^{i\left(\vec{k}_{\pi}-\vec{k}_{n}^{\prime}-\vec{k}_{\pi}^{\prime}\right) \cdot \vec{r}^{\prime}} \psi_{n}\left(\vec{r}^{\prime}\right) \tag{1.4}
\end{equation*}
$$

Note $K_{n}=K_{\pi}^{\prime}+K_{n}^{\prime}-K_{\pi}$
where $\mathrm{K}_{\mathrm{n}}$ is the initial nucleon momentum in the nucleus. Therefore, in this approximation the DWME depends only on the initial nucleon momentum. Furthermore, since the differential cross section is summed over all possible orientations of the bound nucleon, the resulting spherical symmetry causes the cross section to be dependent only on the magnitude of the momentum.

The momentum of the nucleus is assumed not to change in the knock-out reaction, ${ }^{3}$ and since the total nucleus and nucleon initial system is stationary in the lab frame, the initial nucleon momentum is equal to and opposite the final
momentum of the nucleus. For these reasons, it is expected that a geometry which fixes all possible variables except the nuclear recoil angle will emphasize the behavior of the two-body $t$ matrix in the full distorted wave calculation of the differential cross section. As the distorted wave functions destroy the spherical symmetry present in the plane wave approximation, 25 the DWNE is expected to cause some variation in the differential cross section even in such a geometry. However, the factors which produce this variation can be minimized. For a given bound state nucleon, the DWIVE is a function of the incident and outgoing particle momentum only. If the particle energies are held constant, or at most, vary only slowly with the recoil angle of the nucleus, then the variations in the DWME will be due mainly to the changing orientations of the particles. This will simplify the behavior of the DWNE and will allow its influence on the differential cross section to be more predictable.

A geometry that will satisfy these conditions specifies the following parameters: ${ }^{8}$
$E_{i}$ is the incident pion kinetic energy $E_{S}$ is the binding energy of the nucleon $q_{r}$ is the recoil momentum of the nucleus $\theta_{r}$ is the recoil angle of the nucleus

$$
\begin{equation*}
\lambda=\frac{T_{f}}{T_{n}+T_{f}} \tag{1.6}
\end{equation*}
$$

where $T_{f}$ is the final pion kinetic energy
$T_{n}$ is the final nucleon kinetic energy.

By varying $\boldsymbol{\theta}_{R}$ it is expected that the behavior of the two-body $t$ matrix will be the dominant feature of $T_{\text {if }}$ (equation (1.1)). However, selecting this geometry does not mean that information cannot be obtained from the DWME. This matrix element is calculated in coordinate space and involves an integral in $\vec{r}$. As will be shown in Section III, one can determine the region in the nucleus that has the most influence on the reaction by examining this integral. The two-body $t$ matrix is influenced by the nucleus due to the Pauli Exclusion Principle; therefore, knowing the region where the reaction is localized will enable one to choose a suitable value for the fermi momentum.

## Section II

Theory

The knock-out reaction can be described as a three body reaction. The general form of the non-relativistic three body transition operator "U", which describes the reaction when particle "a" is incident on the bound pair "bc" resulting in the bound pair "ac" and the free particle "b" is: 9

$$
\begin{equation*}
U_{a b}(z)=-(1-S a b)\left(H_{0}-z\right)+V-V_{a}-V_{b}+S_{a b} V_{b}-\bar{V}_{a} G(z) \bar{V}_{b} \tag{2.1}
\end{equation*}
$$

where $\bar{V}_{x}$ is the interaction between the two particles other than particle "x"

$$
\bar{V}_{x}=V-V_{x}
$$

$$
V=V_{a}+V_{b}+V_{c}
$$

$\boldsymbol{Z}$ is the total $\because$ energy of the system.

For the case where particle "a" knocks the bound pair "bc" apart, ${ }^{10}$

$$
\begin{equation*}
U_{o a}=\bar{V}_{a}-V G \bar{V}_{a} \tag{2.2}
\end{equation*}
$$

Since the final state is three free particles, $z=H_{0}$. For the specific case of a pion " $\pi$ " knocking a nucleon " n " out of the nucleus " N ":

$$
\begin{equation*}
U_{\pi 0}(z)=\bar{V}_{\pi}-V G(z) \bar{V}_{\pi} \tag{2.3}
\end{equation*}
$$

where

$$
\bar{V}_{\pi}=V_{\pi_{n}}+V_{\pi N}
$$

$$
V=V_{\pi n}+V_{\pi N}+V_{n N}
$$

$V_{\pi n}$ is the pion nucleon potential
$V_{\pi N}$ is the pion nucleus potential
$V_{n N}$ is the nucleon nucleus potential
$z$ is the total energy without interactions

$$
z=E-U-V_{\pi N}-V_{\pi n}-V_{n N}
$$

$E$ is the total energy
$U$ is the sum of the interactions within the residual nucleus.
Following the derivation of Kazaks and Koshel, ${ }^{10}$ it is shown that

$$
\begin{equation*}
U_{\pi 0}(z)=\Omega_{\pi n} t \Omega_{\pi}+\Omega_{\pi n}\left[\tau G_{n}\left(V_{\pi N}+V_{n N}\right) G_{0} t\right] \Omega_{\pi}+\Omega_{\pi n} V_{\pi N} \tag{2.4}
\end{equation*}
$$

The first term in this expansion corresponds to the Distorted Wave Impulse Approximation (DWIA) for the knock-out reaction. $\Omega_{\pi_{n}}$ acts on the final " $\pi$ " and " $n$ " plane waves and distorts them by the potential $V_{\pi N}$ and $V_{n N}$ respectively. $\Omega_{\boldsymbol{\pi}}$ acts on the initial pion plane wave and distorts it by the potential $V_{\pi N^{\prime}}$ " $t$ " is the two-body $t$ matrix describing the $\pi n$ reaction. The second term describes multiple scattering effects and the third term describes nucleus recoil effects. These subsequent terms have been considered and are found to leave the relative shape of the distribution unchanged. ${ }^{12}$ Thus the knock-out reaction can be quite accurately described by the matrix

$$
\begin{equation*}
M=\left\langle\phi_{K_{\pi}^{\prime}}^{(-)} \phi_{K_{n}^{\prime}}^{(-)}\right| \Omega_{\pi n} t \Omega_{\pi}\left|\phi_{K_{\pi}}^{(+)} \psi_{n}\right\rangle \tag{2.5}
\end{equation*}
$$

where $\varnothing_{K_{\pi}}^{(+)}$is the initial pion plane wave
$\psi_{n}$ is the nucleon bound state wave function
$\varnothing_{K_{\pi}}^{(-)}$is the final pion plane wave $\varnothing_{K_{n}}^{(-)}$is the final nucleon plane wave.

Using the operators $\Omega \pi n$ and $\Omega_{\pi}$ results in

$$
\begin{equation*}
M=\left\langle\chi_{K_{\pi}^{\prime}}^{(-)} X_{K_{n}^{\prime}}^{(-)}\right| t\left|X_{K_{\pi}^{(t)}}^{(t)} \Psi_{n}\right\rangle \tag{2.6}
\end{equation*}
$$

where $X_{k_{a}}$ is the plane wave distorted by the potential $V_{a N}$. As stated in the Introduction, the matrix element has been factorized to simplify the calculation. Therefore,

$$
\begin{equation*}
M \cong\left\langle\underset{K_{\pi}^{\prime}}{x_{K_{n}^{\prime}}^{(-)}} \chi_{K_{\pi}^{(-)}} \mid x_{K_{\pi}^{(t)}}^{\left(\psi_{n}\right.}\right\rangle t_{\pi n} \tag{2.7}
\end{equation*}
$$

where $\quad t_{\pi n}$ is the $\pi n$ two-body $t$ matrix, and $\left\langle X_{K_{\pi}^{\prime}}^{(-)} \chi_{K_{n}^{\prime}}^{(-)} \mid X_{K_{\pi}^{(t)}}^{\left(\Psi_{n}\right.}\right\rangle$ is the Distorted Wave Matrix Element (DWME). The two-body $t$ matrix and the effect that the nucleus has on it will be considered in Section III, while the Distorted Wave Matrix Element will be considered in Section IV. The differential cross section for a three particle final state with momenta of particles "1", "2", and "3" in the range " $d^{3} k_{1}$ ", " $d^{3} k_{2}$ ", and " $d^{3} k_{3}$ " respectively is: 14 $d \sigma=\frac{(2 \pi)^{4}}{\left|v_{\text {rel }}\right|} d_{k_{1}}^{3} d_{k_{2}}^{3} d_{k_{3}}^{3} \delta\left(K_{i}-K_{f}\right) \delta\left(E_{i}-E_{f}\right)\left|t_{i f}^{38}\right|^{2}$
For the knock-out of particle "2" from "3" by "1", this has been shown to have the form: 15
$\frac{d \sigma}{d E_{1} d \Omega_{1} d \Omega_{2}}=\frac{k_{1} k_{2}^{2} E_{2} E_{c m}^{2}}{k_{0} E_{2 i}\left[k_{2} E_{3}+E_{2}\left(k_{2}-k_{0} \cos \theta_{2}+k_{2} \cos \left(\theta_{1}-\theta_{2}\right)\right)\right]}|\sigma|^{2} \frac{d \sigma}{d \Omega}$
where $E_{1}, E_{2}, E_{3}$ are the final energies of particles " 1 ", " 2 ", and " 3 " respectively
$E_{2 i}$ is the initial energy of the bound particle $\theta_{1}, \theta_{2}$ are the final angles of the outgoing particles " 1 " and " 2 " with respect to the incident particle "1" $\varnothing$ is the DWME $\left\langle\chi_{K_{\pi}^{\prime}}^{(-)} \chi_{K_{n}^{\prime}}^{(-)} \mid \chi_{K_{\pi}}^{(+)} \Psi_{n}\right\rangle$ $\frac{d \sigma}{d \Omega}$ is the $\pi n$ two-body differential cross section. The cross sections

$$
\frac{d \sigma}{d\left(E_{1}-E_{2}\right) d \Omega_{1} d \Omega_{2}} \quad \text { and } \quad \frac{d \sigma}{d E_{2} d \Omega_{1} d \Omega_{2}}
$$

can be obtained from equation (2.9) by multiplying by the appropriate Jacobian.

Section III
Two-Body t Matrix

The two-body $t$ matrix describing the pion nucleon interaction has the same general form as that describing free space scattering. This is then corrected for the effects of Pauli exclusion in the nuclear medium and for the fact that the nuclear binding energy causes the reaction to become off-shell.

The general form of the free space differential cross section at a center of mass momentum " k " and scattering angle " $\theta$ " is : ${ }^{16}$

$$
\begin{equation*}
\frac{d \sigma_{\pi n}}{d \Omega}=\frac{1}{2} \operatorname{Trace}\left(M_{\rho} M^{+}\right) \tag{3.1}
\end{equation*}
$$

where $\rho$ is the density matrix
$M$ is the scattering matrix element,
$M=f(\theta)+i g(\theta) \vec{\sigma} \cdot \vec{n}$
and $\quad g(\theta)$ is the spin flip scattering amplitude
$f(\theta)$ is the non-spin flip scattering amplitude
$\vec{\sigma} \quad$ is the Pauli spin operator
$\vec{n} \quad$ is a vector perpendicular to the plane of scattering.

For a pion nucleon system, isospin and total angular momentum considerations lead to the following forms for
$f(\theta)$ and $g(\theta)$ (with s- and p-waves): ${ }^{17}$
(a) for $\pi^{+} p \rightarrow \pi^{+} p$

$$
\begin{aligned}
& f(\theta)=F_{3}+\left(2 F_{33}+F_{31}\right) \cos \theta+\text { coulomb corrections } \\
& g(\theta)=\left(F_{31}-F_{33}\right) \sin \theta
\end{aligned}
$$

(b) for $\pi^{-} p \rightarrow \pi^{-} p$
$f(\theta)=\frac{1}{3}\left[F_{3}+2 F_{1}+\left(2 F_{33}+F_{31}+4 F_{13}+2 F_{11}\right) \cos \theta+\right.$ coulomb corrections $]$
$g(\theta)=\frac{1}{3}\left[\left(F_{33}-F_{31}-2 F_{13}-2 F_{11}\right) \sin \theta\right]$
(c) for $\pi^{\circ} p \rightarrow \pi_{n}^{0}$
$f(\theta)=\frac{\sqrt{2}}{3}\left[F_{31}-F_{1}+\left(2 F_{33}-2 F_{13}+F_{31}-F_{11}\right) \cos \theta\right]$
$g(\theta)=\frac{\sqrt{2}}{3}\left[\left(F_{33}-F_{13}+F_{11}-F_{31}\right) \sin \theta\right]$
where $\quad F_{\alpha}=\frac{e^{i \delta \alpha} \sin \delta \alpha}{k}$
and $\quad \alpha=2$ (isospin), 2 (total angular momentum) are phase shifts determined by the fit of Salomon. ${ }^{13}$

To correct for off-shell effects, the separable potentials of Thomas ${ }^{18}$ have been used. Briefly, if one considers expansion of the potential into spherical harmonics,

$$
\begin{equation*}
V=\sum_{l j m} v_{l j}\left(p_{i} p^{\prime}\right) Y_{l j}^{m "}(\hat{p}) Y_{l j}^{m}\left(\hat{p}^{\prime}\right) \tag{3.4}
\end{equation*}
$$

and assumes ${ }^{19}$

$$
\begin{equation*}
v_{i j}\left(p, p^{\prime}\right)=\lambda g_{l j}(p) g_{l j}\left(p^{\prime}\right) \tag{3.5}
\end{equation*}
$$

then, the spherical harmonic expansion of the $t$ matrix becomes: ${ }^{18}$

$$
\begin{equation*}
t_{l j}\left(p^{\prime}, p ; E\right)=g_{l j}\left(p^{\prime}\right) D_{l j}^{-1}(E) g_{l j}(p) \tag{3.6}
\end{equation*}
$$

While the separable potentials were fitted quite accurately to Salmon's phase shift fits, to avoid discrepancies in the on-shell cross sections, the " $g_{1 j}$ " form factors are used to
correct, rather than calculate, the off-shell $t$ matrices:

$$
\begin{equation*}
t_{l_{j}}\left(p^{\prime}, p_{E} ; E\right)=t_{l_{j}}(E) \frac{g_{l j}\left(p^{\prime}\right)}{g_{l_{j}}\left(p_{E}\right)} \tag{3.7}
\end{equation*}
$$

where the fully on-shell $t_{1 j}(E)$ are obtained from Salmon's phase shift fits:

$$
\begin{align*}
& t_{\alpha} k=\frac{-F_{\alpha}}{\pi u(k)}  \tag{3.8}\\
& u(k)=\frac{m_{n}\left(k^{2}+m_{\pi}^{2}\right)^{1 / 2}}{\left[m_{n}+\left(k^{2}+m_{\pi}^{2}\right)^{1 / 2}\right]}
\end{align*}
$$

To correct for the effects of Pauli exclusion, it is illustrative to consider the form of the free scattered t matrix:

$$
\begin{equation*}
t=V+V G_{0} t \tag{3.9}
\end{equation*}
$$

or, pictorially:

$=$
 $+$


The nucleus restricts the nucleon from having a free range of momentum. If the nucleus is considered to be a fermi gas, this has the effect of preventing the nucleon from having a momentum less than $\mathrm{k}_{\mathrm{f}}$, the momentum of the sea. Pictorially, this can be represented by:
 $=$
 $+$

or by:

$$
\begin{equation*}
t=V+V G_{0} Q_{\text {pouli }} t \tag{3.10}
\end{equation*}
$$

where $\quad Q_{\text {pauli }}=0$ when $k<k_{f}$

$$
Q_{\text {pauli }}=1 \quad \text { when } k>k_{f}
$$

Thus, when the $t$ matrix is evaluated between initial and final states, it has the form 20
$\left\langle p^{\prime}\right| t_{l j}|p\rangle=v_{l j}\left(p^{\prime}, p\right)+\int_{0}^{\infty} \frac{d p^{\prime \prime} p^{\prime \prime 2} v_{l j}\left(p^{\prime}, p^{\prime \prime}\right) Q_{p a u l i}\left(p, p^{\prime \prime}\right) t_{l j}\left(p^{\prime \prime}, p ; E\right)}{E-E\left(p^{\prime \prime}\right)}$
Using the separable potentials (equation (3.5)),

$$
\begin{equation*}
t_{l j}=g_{l j}(p) D_{l j}^{-1}(E) g_{i j}\left(p^{\prime}\right) \tag{3.11}
\end{equation*}
$$

where

$$
\begin{align*}
D_{l j}(E) & =\lambda-\int_{0}^{\infty} \frac{p^{\prime \prime 2} d p^{\prime \prime} g_{l j}^{2}\left(p^{\prime \prime}\right)}{E-E\left(p^{\prime \prime}\right)} Q_{\text {pauli }}\left(p, p^{\prime \prime}\right) \\
& =D_{\text {ljnopouli }}(E)+\int_{0}^{\infty} \frac{p^{\prime \prime 2} d p^{\prime \prime 2} g_{l j}\left(p^{\prime \prime}\right)}{E-E\left(p^{\prime \prime}\right)}\left(1-Q_{p a u l i}\left(p, p^{\prime \prime}\right)\right) \tag{3.12}
\end{align*}
$$

To simplify this integral, $Q_{\text {pauli }}$ is expanded in spherical harmonics and only the first term is retained. ${ }^{20}$ To avoid small on-shell differences,

$$
\begin{equation*}
D_{\text {no pauli }} \alpha(E)=\frac{-\pi u(k)}{F_{\alpha}} g^{2} \alpha(E) \tag{3.13}
\end{equation*}
$$

where

$$
u(k)=\frac{m_{n}\left(k^{2}+m_{\pi}^{2}\right)^{1 / 2}}{\left[m_{n}+\left(k^{2}+m_{\pi}^{2}\right)^{1 / 2}\right]}
$$

With these corrections applied to the t matrices, the corrected $F_{\alpha}$ in equation (3.3) are:

$$
\begin{equation*}
F_{\alpha}=-\pi u(k) t_{\alpha} \tag{3.14}
\end{equation*}
$$

## Section IV

## Distorted Wave Matrix Element

With the factorized two-body $t$ matrix previously described, the total knock-out reaction matrix also contains a Distorted Wave Matrix Element (DWME) which must now be evaluated.


This is not easily done in coordinate space. The coordinates used are as follows:
(a) initially,

where $\vec{r}_{0}$ goes to the center of mass of $n$ and $N$. Thus,

$$
\vec{r}_{0}=\vec{r}_{12}+\left(\frac{m_{N}}{m_{N}+m_{n}}\right) \overrightarrow{r_{2}}
$$

(b) Finally,


Since the $\pi n$ interaction is so short-range, the simplification can be made:

$$
\vec{r}_{0}=\left(\frac{m_{N}}{m_{N}+m_{n}}\right) \vec{r}_{2}
$$

and $\quad \vec{r}_{1}=\vec{r}_{2}$

Thus,

$$
\begin{equation*}
\text { DWME }=\int d^{3} r \chi_{k_{1}}^{(-)}(\vec{r}) \chi_{k_{2}}^{(-)}(\vec{r}) \psi_{n}(\vec{r}) X_{k_{0}}^{(+)}(a \vec{r}) \tag{4.2}
\end{equation*}
$$

where

$$
a=\frac{m_{N}}{m_{N}+m_{n}}
$$

To evaluate this integral, consider the distorted wave $\boldsymbol{X}(r)$. A particle incident on a nucleus will find its free space behavior (described by the Hamiltonian $H_{o}$ ) altered by the effect of the nuclear forces. This effect may be described by an effective one-body optical potential $V(r)$, and the behavior of the particle near the nucleus can be described by $H=H_{0}+V(r)$. This Hamiltonian may be separated into radial and angular coordinates yielding the general solution: ${ }^{1}$

$$
\begin{equation*}
X_{k}^{(t)}(r)=4 \pi(k r)^{-1} \sum_{i m} i^{i} e^{i \sigma_{i}} U_{l}(k, r) Y_{i}^{m}\left(\Omega_{r}\right) Y_{i}^{m}\left(\Omega_{k}\right) \tag{4.3}
\end{equation*}
$$

where $\boldsymbol{\Omega}_{r}, \boldsymbol{\Omega}_{k}$ are the directions of $\vec{r}$ and the incident wave vector $\vec{k}$
$Y_{i}^{m}(\Omega)$ is a spherical harmonic $U_{1}(k, r)$ is the solution to the radial partial differential
equation and will be discussed later
$\sigma_{1}$ is the coulomb phase shift.
In the single particle shell model, a bound nucleon may also be described by an effective one-particle Hamiltonian, H. (This is closely related to the optical potential developed above, except that at negative energies, there are no open reaction channels.) The wave function of a nucleon in the
shell model state (l, $m, n$ ) is written ${ }^{1}$

$$
\begin{equation*}
\psi_{n l}^{m}=r^{-1} U_{n l}(r) Y_{q}^{m}(\Omega r) \tag{4.4}
\end{equation*}
$$

Thus, the wave functions in the DWME are described by expansions in spherical harmonics. Using the direction of the incident pion to determine the orientation of the system $\Omega_{\mathrm{k}_{\mathrm{o}}}=0$, equation (4.3) reduces to:

$$
\begin{equation*}
\chi_{k_{0}}^{(+)}=4 \pi\left(k_{0} a r\right)^{-1} \sum_{l_{0} m} i_{e}^{i_{0} i \sigma_{0}} U_{i}\left(k_{0}, a r\right) Y_{i}^{0}\left(\Omega_{r}\right) \sqrt{\frac{21_{0}+1}{4 \pi}} \delta_{m 0} \tag{4.5}
\end{equation*}
$$

and the DWNE has the form:

$$
\begin{align*}
\text { DWME } & =(4 \pi)^{3} \sum_{l_{0} l_{1} l_{2} m_{1} m_{2}} \sqrt{\frac{2 l_{0}+i}{4 \pi}} i_{0}-l_{1}-l_{2} \frac{e^{i\left(\sigma_{l_{0}}+\sigma_{l_{1}}+\sigma_{l_{2}}\right)}}{a k_{0} k_{1} k_{2}} Y_{i_{1}}^{m_{1}}\left(\Omega_{k_{1}}\right) \\
& \times Y_{l_{2}}^{m_{2}}\left(\Omega_{k_{2}}\right) \int_{0}^{\infty} \frac{U_{l_{1}}\left(k_{1}, r\right) U_{l_{2}}\left(k_{2}, r\right) U_{l_{0}}\left(k_{0}, a r\right) U_{n i}(r) r^{2} d r}{r^{4}}  \tag{4.6}\\
& \times \int Y_{i_{1}}^{m_{1}^{*}} Y_{i_{2}}^{m_{2}^{*}} Y_{t}^{m} Y_{i_{0}}^{0}
\end{align*}
$$

Using the spherical harmonic identity, ${ }^{21}$

$$
\begin{align*}
Y_{L_{1}}^{M_{1}} Y_{L_{2}}^{M_{2}} & =\sum_{L} \sqrt{\frac{\left(2 L_{1}+1\right)\left(2 L_{2}+1\right)}{4 \pi(2 L+1)}}\left\langle L_{1} L_{2} O 0 \mid L_{1} L_{2} L 0\right\rangle\left\langle L_{1} L_{2} m_{1} m_{2} \mid L_{1} L_{2} L\left(M_{1}+M_{2}\right)\right\rangle \\
& \times Y_{L}^{M_{1}+M_{2}} \tag{4.7}
\end{align*}
$$

the total angular integral has the form:

$$
\begin{align*}
\int Y_{l_{1}}^{m_{1}^{*}} Y_{l_{2}}^{m_{2}^{*}} Y_{L}^{m} Y_{l_{0}}^{0} & =\sum_{2} \sqrt{\frac{(2 L+1)\left(2 l_{0}+1\right)\left(2 l_{1}+1\right)\left(2 l_{2}+1\right)}{4 \pi\left(2 l_{1}+1\right)}}\left\langle l_{0} 00 \mid L l_{0} 20\right\rangle\left\langle l_{1} l_{2} m_{1} m_{2}\right| l_{1} l_{8}|m\rangle \\
& \times\left\langle l_{1} l_{2} 00 \mid l_{1} l_{2} l_{0}\right\rangle\left\langle l_{1} l_{2} m_{1} m_{2}\right| l_{1} l_{2}|m\rangle \tag{4.8}
\end{align*}
$$

where $m=m_{1}+m_{2}$. (Note that the Clebsh Gordon coefficients are nonzero only when $L+I_{0}+l_{1}+I_{2}$ is even.)

Combining equations (4.6) and (4.8) gives the final
form of the DWME:

# DWME $=\frac{(4 \pi)^{3 / 2}}{a k_{0} k_{1} k_{2}} \sqrt{2 L+1} \sum_{l_{1} l_{1}} i^{\left(l_{0}-l_{1}-l_{2}\right)} e^{i\left(\sigma_{l_{0}}+\sigma_{l_{1}}+\sigma_{l_{2}}\right)} \sqrt{\left(2 l_{1}+1\right)\left(2 l_{2}+1\right)}$ <br> $\times \int_{0}^{\infty} \frac{U_{i_{1}} U_{l_{2}} U_{i_{0}} U_{n L}}{r^{2}} d r\left(2 l_{0}+1\right) \sum_{i m_{1}} \frac{Y_{i_{1}}^{m_{1}}\left(\Omega_{k_{1}}\right) Y_{i_{2}}^{m_{2}}\left(\Omega_{k_{2}}\right)}{(2 l+1)}$ 

$\times\left\langle L l_{0} O O \mid L l_{0} l 0\right\rangle\left\langle l_{0} m O \mid L l_{0} l m\right\rangle\left\langle l_{1} l_{2} O O \mid l_{1} l_{2} l 0\right\rangle\left\langle l_{1} l_{2} m_{1} m_{2} \mid l_{1} l_{2} l m\right\rangle$
where $L+I_{0}+l_{1}+l_{2}$ is even and $m=m_{1}+m_{2}$.
The integral

$$
\begin{equation*}
\int_{0}^{\infty} \frac{U_{l_{1}} U_{l_{2}} U_{l_{0}} U_{n L}}{r^{2}} d r \tag{4.10}
\end{equation*}
$$

involves just the radial parts of the wave functions for the bound state and scattering equations (4.3) and (4.4). In the case of the nucleon, this wave function is the solution of the Schroedinger equation since the mass is so large and the energies involved are relatively small:

$$
\begin{equation*}
\nabla^{2} X(r)+\frac{2 u}{\hbar^{2}}[E-V(r)] X(r)=0 \tag{4.11}
\end{equation*}
$$

The separated radial equation is: ${ }^{1}$

$$
\begin{equation*}
\frac{d^{2} U_{l}}{d r^{2}}+\left[\frac{2 u}{\hbar^{2}}[E-V(r)]-\frac{l(l+1)}{r^{2}}\right] U_{l}=0 \tag{4.12}
\end{equation*}
$$

For the initial bound state, $E$ is the binding energy
which is negative. For the final, unbound state, $E$ is just the asymptotic energy; in this case using the de Broglie wave number,

$$
k=\left[\frac{2 m E}{\hbar^{2}}\right]^{1 / 2}
$$

equation (4.12) may be written as:

$$
\begin{equation*}
\frac{d^{2} U_{k}}{d r^{2}}+\left\{\left[k^{2}-\frac{2 U V(r)}{\hbar^{2}}\right]-\frac{q(l+1)}{r^{2}}\right\} U_{1}=0 \tag{4.13}
\end{equation*}
$$

As usual, the optical potential for the nucleon is taken to have a Wood-Saxon shape with a Thomas spin-orbit term: ${ }^{4}$

$$
\begin{align*}
V(r) & =V_{c}(r)-U_{R} f(r)-i\left[W_{V} g_{v}(r)+W_{D} g_{0}(r)\right]+\left(\frac{\hbar}{m_{\pi} c}\right)^{2}\left(U_{s 0}+i W_{s o}\right)  \tag{4.14}\\
& \times \frac{1}{r} \frac{d f_{S}(r)}{d r} \downarrow \cdot Q^{-}
\end{align*}
$$

where $f(r)=\left(1+e^{\frac{r-R_{R}}{a_{R}}}\right)^{-1} \quad R_{R}$ is the radius of the nucleus

$$
\begin{align*}
& g_{v}(r)=\left(1+e^{\frac{r-R_{I}}{a_{I}}}\right)^{-1} \quad R_{I} \text { is the radius of absorption } \\
& g_{0}(r)=\frac{4 e^{\frac{r-R_{I}}{a_{I}}}}{\left(1+e^{\frac{r-R_{I}}{\alpha_{I}}}\right)^{2}}  \tag{4.15}\\
& f_{S}(r)=\left(1+e^{\left.\frac{r-R_{s}}{\alpha_{S}}\right)^{-1}} \quad \begin{array}{l}
R_{S} \text { is the radius of the surface } \\
\text { effects }
\end{array}\right.
\end{align*}
$$

$$
V_{c}(r)=\frac{Z Z^{\prime} c^{2}}{r} \text { if } r \geqslant R_{c} \quad R_{c} \text { is the charge radius }
$$

$$
\begin{equation*}
V_{c}(r)=\frac{Z Z^{\prime} e^{2}}{2 R_{c}}\left(3-\frac{r^{2}}{R^{2}}\right) \quad \text { if } r<R_{c} \tag{4.16}
\end{equation*}
$$

For the bound nucleon the absorption terms are zero, hence $W_{V}=W_{D}=W_{S O}=0$. The other parameters for the bound state have been fit by Elton and Swift and are given in Table 1. Note that the radius parameters and $U_{R}$ are specially set by the program to fit the RMS radius and separation of the bound nucleon. For the unbound nucleon, the parameters have been determined by Jackson ${ }^{4}$ and the values are given in Table 2.

## Table 1

Wood-Saxon Bound State Potential Parameters for $0^{16}$

$$
\begin{aligned}
R_{c} & =3.15 \mathrm{fm} \\
R_{R} & =3.15 \mathrm{fm} \\
R_{\text {So }} & =3.15 \mathrm{fm} \\
a_{R} & =0.65 \mathrm{fm} \\
a_{\text {So }} & =0.65 \mathrm{fm} \\
\mathrm{~V}_{\text {So }} & =-9 . \mathrm{MeV} \\
1 P_{1 / 2} \text { separation energy } & =12.1 \mathrm{MeV} \\
U_{R} & =56.970 \mathrm{MeV} \\
1 P_{3 / 2} \text { separation energy } & =18.45 \mathrm{MeV} \\
\text { RMS radius } & =2.675 \mathrm{fm} \\
U_{R} & =58.442 \mathrm{MeV} \\
\text { RMS radius } & =2.742 \mathrm{fm}
\end{aligned}
$$

Experimental RMS radius $=2.674 \mathrm{fm}$

## Table 2

Potential Parameters for Proton Scattering from $0^{16}$

$$
\begin{aligned}
& \mathrm{U}_{\mathrm{R}}=24.13\left(1-0.00359 \mathrm{E}_{\mathrm{cm}}\right) \\
& \mathrm{W}_{\mathrm{V}}=27.05 \\
& \mathrm{~W}_{\mathrm{D}}=0.0 \\
& \mathrm{U}_{\mathrm{SO}}=-2.064 \\
& \mathrm{~W}_{\mathrm{SO}}=0.0 \\
& R_{R}=3.664 \mathrm{a}_{\mathrm{R}}=0.554 \\
& R_{I}=2.348 \mathrm{a}_{\mathrm{I}}=0.612 \\
& R_{\mathrm{S}}=2.389 \mathrm{a}_{\mathrm{S}}=0.492 \\
& R_{\mathrm{C}}=3.820
\end{aligned}
$$

These parameters give good fit to data for proton elastic scattering from $0^{16}$ in the energy range 65 to 156 MeV .

In the case of the pion wave function, the appropriate scattering equation is a linearized Klein-Gordon equation, ${ }^{22}$

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}-2 \bar{\omega} U_{o p t}\right) x(r)=0 \tag{4.17}
\end{equation*}
$$

where $k$ is the asymptotic pion momentum and $\bar{\omega}$ is the reduced pion energy in the pion nucleus center of mass. The optical potential used is non-local, ${ }^{22}$ the general form being:

$$
\begin{equation*}
2 \bar{\omega} U_{\text {opt }}=q(r)+\nabla \cdot \alpha(r) \nabla \tag{4.18}
\end{equation*}
$$

Because of the gradient terms, the substitution

$$
x=\frac{\phi}{(1-a)^{1 / 2}}
$$

is required to solve equation (4.17). The resultant form is: $\left(\nabla^{2}+k^{2}\right) \boldsymbol{\phi}^{2}=(1-\alpha)^{-1}\left[q \cdot \alpha k^{2} \cdot\left(\frac{\nabla^{2} \alpha}{2}\right)-\frac{(\nabla \alpha)^{2}}{4(1-\alpha)}\right] \varnothing^{\alpha}$

This equation can be separated into radial and angular wave functions, the equation for the radial wave function defined as:

$$
\begin{equation*}
\frac{d^{2} \tilde{U}_{q}}{d r^{2}}+\left\{k^{2}-\beta(r)-\frac{q(l+1)}{r^{2}}\right\} \tilde{U}_{l}=0 \tag{4.20}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta(r)=(1-\alpha)^{-1}\left[q-\alpha k^{2}-\left(\frac{\nabla^{2} \alpha}{2}\right)-\frac{(\nabla \alpha)^{2}}{4(1-\alpha)}\right] \tag{4.21}
\end{equation*}
$$

and where the radial wave function of the pion is:

$$
\begin{equation*}
u_{l}=\frac{\bar{u}_{l}}{(1-\infty)^{1 / 2}} \tag{4.22}
\end{equation*}
$$

The form of the optical potential used in this calculation was developed by Stricker, McManus, and Carr: ${ }^{23}$

$$
\begin{align*}
2 \bar{\omega} U_{o p t} & =-4 \pi\left(b(r)+\rho_{2} \beta_{0} \rho^{2}(r)+\frac{\left(\rho_{1}-1\right)}{2} \nabla^{2} C(r)+\frac{C_{0}\left(\rho_{\rho}-1\right)}{2 \rho_{2}} \nabla^{2} \rho^{2}(r)\right) \\
& +4 \pi\left(\nabla \cdot L(r) C(r) \nabla+\frac{c_{0}}{\rho_{2}} \nabla \cdot \rho^{2}(r) \nabla\right)+\left(2 \bar{\omega} V_{c}(r)\right) \tag{4.23}
\end{align*}
$$

where $b(r)=p_{1}\left(\bar{b}_{0} p(r)-E_{\pi} b_{1} \delta \rho(r)\right)$

$$
\begin{align*}
& C(r)=\frac{1}{\rho_{1}}\left(C_{0} \rho(r)-E_{\pi} C_{1} \delta_{\rho}(r)\right)  \tag{4.25}\\
& L(r)=\left[1+\frac{4 \pi \lambda}{3}\left(\frac{A-1}{A}\right) C(r)\right]^{-1}
\end{align*}
$$

and $\delta_{\rho}(r)=\rho_{n}(r)-p_{p}(r)$ is the difference between the neutron and proton densities.

Finally, we note that $\boldsymbol{\rho}_{1}$ and $\boldsymbol{\rho}_{2}$ are kinematical factors, $E_{\pi}$ is the charge of the pion, and $V_{c}(r)$ is the coulomb potential. The coefficients $b_{o}$ and $c_{0}$ can be calculated from pion nucleon elastic scattering data, and $C_{0}$ takes into account absorption in the nucleus. The Ericson-Ericson factor, $L(r)$, takes into account higher order effects of multiple scattering. The factor $\boldsymbol{\lambda}$ determines the degree to which p-wave scattering is reduced. The values of these parameters are given in Table $3 .{ }^{2}$

The differential equations for $U_{I}$ (equations (4.12) and (4.20)) are solved numerically using the Numerov technique. 24 This technique determines the value of the radial wave function at selected points, the step size " $h$ " between each point being fixed. The error due to this method is of order $h^{5}$, so that if the step size is halved, the error is reduced by $2^{5}$.

## Table 3

Potential Parameters for Pion Scattering from $0^{16}$

|  | 30 MeV | 40 MeV | 50 MeV | 116 MeV | 180 MeV | 220 MeV |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{b}_{0}(\mathrm{fm})$ | $-0.038+i 0.003$ | $-0.038+i 0.004$ | $-0.038+i 0.006$ | $-0.09+i 0.018$ | $-0.12+i 0.032$ | $-0.13+i 0.042$ |
| $\mathrm{~b}_{1}(\mathrm{fm})$ | $-0.11-i 0.001$ | $-0.11-i 0.001$ | $-0.11-i 0.002$ | $-0.12+i 0.001$ | $-0.12+i 0.008$ | $-0.12+i 0.012$ |
| $\mathrm{~B}_{0}\left(\mathrm{fm}^{4}\right)$ | $-0.18+i 0.18$ | $-0.18+i 0.18$ | $-0.18+i 0.18$ | $0.0+i 0.22$ | $0.0+i 0.26$ | $0.0+i 0.26$ |
| $c_{0}\left(\mathrm{fm}^{3}\right)$ | $0.68+i 0.007$ | $0.68+i 0.015$ | $0.68+i 0.029$ | $0.81+i 0.31$ | $0.12+i 0.70$ | $-0.23+i 0.52$ |
| $c_{1}\left(\mathrm{fm}^{3}\right)$ | $0.62+i 0.004$ | $0.62+i 0.007$ | $0.62+i 0.014$ | $0.44+i 0.16$ | $0.08+i 0.35$ | $-0.11+i 0.26$ |
| $c_{0}\left(\mathrm{fm}^{6}\right)$ | $-0.41+i 0.41$ | $-0.41+i 0.41$ | $-0.41+i 0.41$ | $0.0+i 1.35$ | $0.0+i 1.29$ | $0.0+i 0.65$ |

For energies between those described above, values for the parameters have been linearly interpolated.

The radial wave functions are thus integrated using Simpson's method, which has an error of order $h^{4}$ : 24 Since the smaller the step size, the longer the computation time, there is a compromise between accuracy and efficiency. Consideration of various step sizes resulted in the optimal value $\mathrm{h}=0.2 \mathrm{fm}$ being adopted. Reducing this value by half only resulted in changes in the third significant figure, suggesting that the value chosen is sufficiently accurate.

Another consideration is the upper bound on the integration variable. Past a certain radius, the integrand will be negligible, so that calculation beyond this point will not significantly improve the result. This radius is chosen as being the point at which the amplitude of the bound state radial wave function falls below 0.1 percent of its maximum value. This is generally in the range of 12 fm . All radial wave functions are calculated out to this point and then matched via logarithmic derivatives to the asymptotic coulomb wave functions so they may be properly normalized.

As noted in equation (4.9), the DWME involves a summation of the contributions of each partial wave in the expansion of the respective unbound wave functions. As there are infinite numbers of partial waves in each expansion, and one is limited by computation time as to how many terms may be considered, it is important to determine at which point further terms in the series expansion have a negligible influence on the calculation.

Each partial wave corresponds to a trajectory having an angular momentum " 1 " about the nucleus. For large values of
"l", the partial waves correspond to trajectories which completely miss the nucleus and therefore have little influence on a reaction involving the nucleus. Since the region pertinent to this calculation is fixed by the bound state wave function, which goes rapidly to zero outside the nuclear region, partial waves of high angular momentum can be neglected. The computer program, therefore, evaluates the effect of each successive partial wave on the calculation and when the contribution of the "l"th partial wave is less than 0.5 percent of the running total, the series is terminated.

The discussion to this point has considered a pion incident on a specific nucleon in the nucleus. Experimentally; the only criteria for selecting which nucleon has been struck is the binding energy; one cannot be more precise than stating which nuclear shell the nucleon came from. As there may be more than one nucleon present in this shell, one must consider the effect of this multiplicity. Assuming all other nucleons not in the given shell as being part of an inert core, the total initial wave function is:

$$
\begin{equation*}
\psi=c \chi_{\pi} \psi_{n}\left(\xi_{1}, \xi_{2}, \ldots \xi_{n}, R\right) \tag{4.28}
\end{equation*}
$$

where $\quad c$ is the normalization constant
$\boldsymbol{\chi}_{\pi}$ is the pion wave function
$\boldsymbol{\xi}_{1}, \mathbf{s}_{2}, \ldots . . \boldsymbol{\xi}_{n}$ are the coordinates of the nucleons in the specified shell
$\boldsymbol{R}$ is the coordinate of the remaining nucleus.
Since the nucleons in the shell are identical, we should anti-symmetrize in $\Psi_{n}$ the coordinates $\left(\boldsymbol{\xi}_{1}, \ldots \boldsymbol{\xi}_{n}\right)$. The
initial wave function then takes the form:

$$
\begin{equation*}
\psi_{T}=\frac{c}{N!} \sum_{Q} \varepsilon_{Q} Q \psi \tag{4.29}
\end{equation*}
$$

where $N$ is the total number of nucleons
Q is the permutation operator
$\mathcal{E}_{\mathbf{Q}}$ gives the sign based on fermian statistics.
Note that since the pion is totally distinguishable, it has no effect on the statistics.

Since the "n" nucleons in the specified shell are identifiable by their binding energy, the normalization constant is:

$$
\sqrt{\frac{N!}{m!(N-n)!}}
$$

Similarly, the final state is:

$$
\begin{equation*}
\psi^{\prime}=c^{\prime} x_{\pi} \psi_{n-1}\left(\xi_{1}, \xi_{2}, \ldots . \xi_{n-1}, R\right) \chi_{p} \tag{4.30}
\end{equation*}
$$

where
$\boldsymbol{x}_{\pi}$ is the pion wave function
$x_{p}$ is the unbound nucleon distorted wave
$\mathbf{c}^{\prime}$ is the normalization constant
$\boldsymbol{\xi}_{1}, \boldsymbol{\xi}_{2}, \ldots . \boldsymbol{\xi}_{n-1}$ are the coordinates of the remaining nucleons in the specified shell
$R$ is the generalized coordinate describing all the other nucleons.

If we neglect the exchange term involving the outgoing particle, which as usual, should be small, the anti-symmetrized wave function is:

$$
\begin{equation*}
\psi_{T}^{\prime}=\frac{c^{\prime}}{N!} \sum_{p} \varepsilon_{p} p \psi^{\prime} \tag{4.31}
\end{equation*}
$$

Since there are ( $n-1$ ) nucleons in the specified shell, ( $\mathrm{N}-\mathrm{n}$ ) in the inert core, and one free nucleon, the normal-
ization constant is:

$$
\begin{equation*}
C^{\prime}=\sqrt{\frac{N!}{(n-1)!(N-n)!(1)!}} \tag{4.32}
\end{equation*}
$$

Considering the overlap of the initial and final wave functions,

$$
\begin{align*}
\left\langle\psi_{T}^{\prime} \mid \psi_{T}\right\rangle & =\frac{c c^{\prime}}{(N!)^{2}} \sum_{P} \varepsilon_{P} \sum_{Q} \varepsilon_{Q}\left\langle p \psi_{T}^{\prime} \mid Q \psi_{T}\right\rangle  \tag{4.33}\\
& =\frac{c c^{\prime}}{(N!)^{2}} \sum_{Q} \varepsilon_{Q}\left\langle\psi^{\prime} \mid Q \psi_{T}\right\rangle
\end{align*}
$$

However,

$$
c c^{\prime}=\frac{N!\sqrt{n}}{n!(N-n)!}
$$

and since only permutations which change nucleons in the specified shell and/or nucleons in the inert core among themselves give non-zero overlaps, there are only n!( $N-n$ )! non-zero terms in the series. Hence,

$$
\begin{equation*}
\left\langle\psi_{T}^{\prime} \mid \psi_{T}\right\rangle=\sqrt{n}\left\langle\chi_{\pi}^{\prime} \chi_{p}^{\prime} \mid \chi_{\pi} \psi_{n L}\right\rangle \tag{4.34}
\end{equation*}
$$

where $\quad x_{p}$ is the wave function of the final unbound nucleon $\psi_{n k}$ is the wave function of the struck bound nucleon $\chi_{\pi}$ is the pion wave function.
The magnitude is not the only information that can be gleaned from the DWME. It is noted that to this point, the matrix element is calculated for a specific orientation "m" (in the scattering plane) of the angular momentum " 1 " of the bound state wave function, i.e.,

$$
\begin{equation*}
g_{2}^{m}=\left\langle x_{\pi} x_{p} \mid x_{\pi} \psi_{i}^{m}\right\rangle \tag{4.35}
\end{equation*}
$$

The total magnitude of the DWME influence on the total scattering cross section is simply the weighted sum in "m" of the magnitudes $\left|\mathrm{g}_{1}^{\mathrm{m}}\right|^{2}$. The weighting factors given in

Table 4 are just the probabilities that an unpolarized nucleon will have a given orientation. The effective polarization of the nucleon in the nucleus can be determined by rotating the $g_{l}^{m}$ matrix elements to be perpendicular to the scattering plane, then summing in " $m$ " the magnitudes $\left|g_{I}^{m}\right|^{2}$ with the weight factors for a spin up nucleon and then by summing the weight factors for a spin down nucleon. The effective polarization is just the normalized difference between these two, and it is used in the two-body $t$ matrix calculation. Since the plane wave matrix element is symmetric in spin orientation, analysis of the effective polarization will provide a good indication of the influence of nuclear distortion on the wave functions. 25

Additional information which can be obtained from the calculation of the DWWE is the region of the nucleus in which the reaction is localized. This is useful in determining an appropriate value for the fermi momentum cut-off used in the two-body $t$ matrix calculation. The calculation of the DWNE involves an integration of the radial wave functions. To the computer, this is simply a summation of a sequence of terms evaluated at increasing radii. By considering these terms, one can determine the degree of influence each has on the total matrix element, the degree of influence of the "r"th term being:

$$
\begin{equation*}
A_{r}=\sum_{m} W_{m}\left[\left|g_{m}^{2}\right|-\lg _{m}^{\prime} \mid\right]^{2} \tag{4.36}
\end{equation*}
$$

where $W_{m}$ is the orientation weight factor $g_{m}^{l^{\prime}}$ is the DWME calculated without the "r"th term.

## Table 4

Polarization Weight Factors for $g_{m}^{1}$ When $1=1$

$$
\begin{array}{llll} 
& m=-1 & m=0 & m=1 \\
\text { spin up } & 0.333 & 0.167 & 0.0 \\
1 \mathrm{P}_{1 / 2} \text { spin down } & 0.0 & 0.167 & 0.333 \\
1 \mathrm{P}_{3 / 2} \text { spin up } & 0.083 & 0.167 & 0.250 \\
\text { spin down } & 0.250 & 0.167 & 0.083
\end{array}
$$

Notice that $A_{r}$ will be a maximum for the term that has the most influence on the calculation, thus indicating the location in the nucleus where the reaction predominantly occurs.

## Section V

Results

The knock-out reaction $0^{16}\left(\pi^{+}, \pi_{p}^{+}\right) N^{15}$ was considered and calculations were made of the cross sections

$$
\frac{d \sigma}{d\left(E_{\pi}^{\prime}-E_{n}^{\prime}\right) d \Omega_{\pi} d \Omega_{n}}
$$

(equation (2.9)).
As noted in the Introduction, the geometry selected to specify the kinematics varies the nucleus recoil angle while keeping all other kinematic variables constant. Since this keeps the plane wave matrix element constant, it was expected that the DWNE should be slowly varying. To examine the variation of the DWWE, the ratio of DWIA to PWIA was plotted as a function of nucleus recoil angle for incident pion energies of 60 , 116, 180 , and 220 MeV on protons in the $1 \mathrm{P}_{1 / 2}$ and $1 \mathrm{P}_{3 / 2}$ shells. The cross sections were also plotted as a function of nucleus recoil angle (all other kinematic variables held constant) for the above mentioned energies and proton shells.

To test the off-shell and Pauli exclusion effects on the total cross section, the two-body $t$ matrix was calculated on-shell and off-shell, at five different values of fermi momentum cut-off. The fermi momenta specified correspond to: free space scattering ( $0.0 \mathrm{fm}^{-1}$ ), scattering at the edge of the nucleus ( $0.7 \mathrm{fm}^{-1}$ ), scattering at the RVIS radius ( $1.0 \mathrm{fm}^{-1}$ ), and scattering at the center of the nucleus ( $1.36 \mathrm{fm}^{-1}$ ). Also,
through an examination of the radial character of the DNNE, an effective fermi momentum was calculated. This corresponds to the location in the nucleus where the interaction is most predominant, and should be the cut-off momentum felt by the majority of the two-body interactions.

As noted in Section IV, the DWME can be calculated for spin up and spin down orientations of the proton. The normalized difference of these two results give an effective polarization of the proton. The effect of this polarization was also examined in combination with the off-shell and Pauli exclusion effects by calculating the two-body interaction with and without the effective protion polarization (equation (3.2)).

To facilitate examination of the radial character of the DMME for each angle of nuclear recoil, three dimensional plots of $A_{r}$ values (equation (4.36)) were made for incident pion energies of $60,116,180$, and 220 MeV with protons in the $1 P_{1 / 2}$ and $1 P_{3 / 2}$ shells. As the $1 P_{3 / 2}$ plots were very similar to the $1 P_{1 / 2}$ plots for each energy, only the plots for $1 P_{1 / 2}$ proton scattering have been presented.

The cross sections,

(equation (2.9))
were then calculated using the geometry of Levin and Eisenberg. ${ }^{6}$ This geometry requires the outgoing pion energy and angle to be specified; the cross sections were plotted as a function of the final proton angle, Curves were also plotted for scattering from a $1 P_{1 / 2}$ and a $1 P_{3 / 2}$ proton when
the pions are incident with energy 130 MeV and leave with energies of 51.2 or 61 MeV at an angle of 150 or 120 degrees respectively. For comparison, Levin and Eisenberg's curves of DWIA and factorized DWIA calculations were plotted.

Section VI
Discussion

Comparison of DWIA and PWIA Calculations
As indicated in the Introduction, the geometry used was selected to minimize the variation of the DWME. What is immediately apparent from the ratio of PWIA to DWIA (fig. 1-4) is that this ratio is far from constant; furthermore, the ratio is far from unity, reaching 220 in some cases (fig. 3). The size of the PWIA to DWIA ratio does indicate that absorption by the nucleus is quite significant. Examination of the three dimensional plots (fig. 5-8) reveals a general similarity for scattering at 180 and 220 MeV (fig. 7 and 8 respectively). That the pion is strongly absorbed at energies of 180 and 220 MeV is born out by these plots. The curves peak at radii of 3.6 and 4.0 fm respectively; thus, given that the RMS radius of $0^{16}$ is 2.675 fm , the knock-out reaction occurs at the extreme edge of the nucleus.

Analysis of the angular nature of figures 7 and 8 shows that the DWWE increases with increasing scattering angle. This reveals the effect that absorption has on the cross section. Pions incident on the nucleus at these energies will be so readily absorbed that the number available for a knock-out reaction will be heavily reduced as the pions penetrate the nucleus. Furthermore, pions that are scattered
in a knock-out reaction will have a greater chance of emerging if the path they take does not traverse much nuclear medium. Thus, pions that leave at large angles from interactions at the front of the nucleus will be more plentiful than any others. In this geometry, the angle of the scattered pion changes as the nucleus recoil angle is varied, confirming the behavior illustrated in figures 3 and 4.

For pion scattering at 116 MeV the curves of the PWIA to DWIA ratio (fig. 2) are similar to the curves at 180 and 220 MeV . Given the strong absorption of the pion at $116 \mathrm{MeV},{ }^{26}$ it would be expected that the variation of DWME with recoil angle should follow the same form as scattering at 180 and 220 MeV . An examination of the three dimensional plot (fig. 6) does reveal much the same form as the plots at higher energies.

At 60 MeV (fig. 1), the DWNE peaks at angles where the pion is only slightly deflected, and decreases as the pion scattering angle is increased. The ratio of DWIA to PWIA is much smaller, and the three dimensional plot (fig. 5) of the DWME shows that the matrix element is peaked well within the nucleus (2.8-3.0 fm). As the absorption at low energies is small and the potential is attractive, the pion wave functions will form a focus at the back of the nucleus. ${ }^{1}$ Interactions will therefore occur at this focus as well as at the front of the nucleus. This focusing effect is more pronounced for outgoing pions as they have a lower energy and are absorbed to a lesser degree. Pions leaving with small angles of deflection will originate either at the out-going wave function
focus, at the front of the nucleus, or from the incident wave function focus at the back of the nucleus. As the scattering angle of the pion is increased, the out-going focus will move away from the front of the nucleus, and fewer scattered pions will be detected. This agrees with what is observed in the behavior of the DWME (fig. 1).

Off-Shell Effects
As noted previously, ${ }^{8}$ the off-shell effect does not have much influence on the knock-out cross section. This effect only becomes apparent at 60 MeV (fig. 1), where a change in the magnitude of the cross section of 10 to 15 percent is found.

Pauli Exclusion Effects
As found by Jackson et al., ${ }^{8}$ the Pauli exclusion effect is most dramatic at higher energies (approximately 200 MeV ). However, it is noted that the cross sections obtained with the program-calculated effective fermi momentum lie between the curves for free scattering and scattering at the extreme edge of the nucleus. In fact, for the curves at 220 MeV (fig. 18 - 21), the free scattering curves are almost indistinguishable from the curves using the effective fermi momentum. It is not until the energy falls to 116 MeV that effective momentum curves (fig. 11 - 13) are significantly different from free space scattering. To understand this effect, it is instructive to examine the three dimensional plots (fig. 5-8). What is immediately apparent in the plots for 180 and 220 MeV scattering (fig. 7, 8) is the extreme
radius of the interaction. The DWME has a peak at 3.6 fm and 4.0 fm respectively. This extreme behavior at these energies is caused by the highly absorptive nature of the pion-nucleus interaction. If the imaginary terms in the pion optical potential (equation (4.23)) are removed, the DWME peaks at 2.8 fm , a value slightly higher than the RNS radius of the nucleus ( 2.675 fm ) because of the repulsive nature of the potential at these energies.

For scattering at 116 MeV , the three dimensional plot (fig. 6) reveals a similar structure to figures 7 and 8 although the peak is much more pronounced. The radius of the peak is approximately 3.2 fm which corresponds to the radius of the Wood-Saxon potential of the bound state wave function ( 3.15 fm ) (equation (4.15)). This was expected as the pion-nucleus interaction is less absorptive at this energy ${ }^{26}$ and allows greater penetration of the nucleus. Since the nucleus has a skin thickness of 0.65 fm (Table 1), interactions occuring on the surface of the nucleus will vary quite sharply with radius, a fact indicated by the pronounced peak of the DWME (fig. 6).

Analysis of the cross sections at 60 MeV reveals that the Pauli exclusion effect is minimal, the greatest contribution being of the order of 10 percent. This is unfortunate because the three dimensional plot (fig. 5) for this energy indicates that the interaction occurs within the nucleus and the effective fermi momentum is significant (approximately $0.8 \mathrm{fm}^{-1}$ ).

As noted in Section IV, the cross section can be constructed for spin up and spin down orientation of the proton. The normalized difference between these two results gives the effective polarization of the proton. By varying the angle of nuclear recoil, the polarization was found to vary from -0.5 to +0.5 for scattering from the $1 P_{3 / 2}$ proton and from approximately -1.0 to +1.0 for scattering from the $1 P_{1 / 2}$ proton (fig. 22). This behavior did not change radically as the incident pion energy was varied. The more pronounced influence of spin on the $1 \mathrm{P}_{1 / 2}$ calculation was also noted by Levin and Eisenberg ${ }^{6}$ who ascribe this to absorption effects and the spin orbit coupling of the out-going proton. The curves plotted without the polarization effect (fig. $9,10,12,13,15,17,19,21$ ) show that at a given energy, the cross sections for the $1 \mathrm{P}_{1 / 2}$ and $1 \mathrm{P}_{3 / 2}$ proton scattering are quite similar in shape. This is expected as the outgoing particle energies and angles are almost identical for scattering from these two proton shells, causing the large effects due to pion absorption to be similar. However, when the effective polarization is turned on, this similarity is no longer observed.

Examination of figures 18 and 19 for 220 MeV scattering from the $1 P_{1 / 2}$ proton shows that polarization emphasizes the peak at a nucleus recoil angle of 75 degrees and reduces the structure at lower recoil angles. Figures 20 and 21 for $1 P_{3 / 2}$ proton scattering show that the polarization has the reverse effect, emphasizing the peak at lower angles and
reducing the peak at 30 degrees. To understand this behavior, it is illustrative to consider the behavior of the pol,arization vector for these two cases. Figure 22 shows that the effective polarization for $1 P_{3 / 2}$ proton scattering is almost exactly opposite and approximately a factor of two reduced from the corresponding $1 \mathrm{P}_{1 / 2}$ proton case. This is expected as the polarizing weight factors given in Table 4 show; the spin orientation weight factors for $1 \mathrm{P}_{1 / 2}$ scattering are close to the opposite spin orientation factors for $1 \mathrm{P}_{3 / 2}$ scattering, the $1 P_{3 / 2}$ factors being more mixed.

Analysis of the data at 180 Mev (fig. 14 - 17) reveals much the same behavior as described for data at 220 MeV . For $1 \mathrm{P}_{1 / 2}$ proton scattering (fig. 14, 15), the effective polarization emphasizes the peak at a nuclear recoil angle of 60 degrees, whereas $1 P_{3 / 2}$ proton scattering (fig. 16, 17) shows that the peak at 20 degrees is emphasized.

At 116 MeV (fig. 11, 12), the effect is most noticeable in the cross sections of $1 \mathrm{P}_{1 / 2}$ calculated without fermi momentum corrections. The peak at a nucleus scattering angle of 60 degrees becomes larger than the peak at 20 degrees.

At 60 MeV , the spin flip term (equation (3.2)) is quite small and the effective polarization causes less than 10 percent variation in the cross sections, thus, no curves were plotted.

Comparison with the Results of Levin and Eisenberg ${ }^{6}$
Levin and Eisenberg have calculated the knock-out cross section,

$$
\begin{equation*}
\frac{d \sigma}{d E_{\pi}^{\prime} d \Omega_{\pi} d \Omega_{n}} \tag{2.9}
\end{equation*}
$$

using the momentum space calculation (equation (1.2)). To evaluate the effect of the factorization approximation (equation (2.7)), they evaluated the DWME in momentum space. The geometry which Levin and Eisenberg selected examines the behavior of the cross section as the out-going proton angle is varied. The other kinematic variables are set so the initial proton momentum approaches zero as the out-going proton angle is varied. In the Plane Wave Impulse Approximation the DWNE becomes the momentum space representation of the bound state wave function. ${ }^{1}$ Since the $1 P_{1 / 2}$ and $1 P_{3 / 2}$ bound state wave functions approach zero as the momentum approaches zero, the out-going angle giving minimum initial proton momentum will correspond to a minimum of the differential cross section. As the nucleus causes distortion in the wave functions, the value of this minimum should be altered. This provides a test of the distorted wave functions.

In this study, curves calculated to compare with Levin and Eisenberg's results did not contain the spin orbit term in the proton optical potential (equation (4.14)). In figure 24 , the curve for the $1 \mathrm{P}_{3 / 2}$ cross section peaks at the same height as Levin and Eisenberg's full calculation without spin orbit. The minimum corresponds to Levin and Eisenberg's factorized calculation, indicating agreement with a mix of the two effects. The $1 \mathrm{P}_{1 / 2}$ curve (fig. 23) calculated in this study maintained the same form as the $1 P_{3 / 2}$ and did not show the smoothed-out features of Levin and Eisenberg's results. The peak at minimum initial proton momentum indicated in Levin and Eisenberg's full DWIA result definitely was not
observed. Both $1 \mathrm{P}_{1 / 2}$ and $1 \mathrm{P}_{3 / 2}$ curves fall off more rapidly than Levin and Eisenberg's at higher proton scattering angles. This is caused by the increasing effect of pion absorption and as Levin and Eisenberg's pion optical potential does not include true absorption, agreement was not expected. Figures 25 and 26 show much the same behavior, the $1 \mathrm{P}_{1 / 2}$ curve calculated in this study is not as smooth as the $1 \mathrm{P}_{1 / 2}$ curves from Levin and Eisenberg's study.

The knock-out reaction is rich in information about nuclear processes and, with sufficient data, provides a good test of many nuclear models. As indicated in this study, the knock-out reaction is very sensitive to distortion of the wave functions, specifically, to the effect of pion absorption. Thus, data gathered on the knock-out reaction cross section allows one to evaluate optical model potentials.

The cross sections evaluated in this study proved to be insensitive to off-shell effects. However, the Pauli exclusion effect was noted to be significant, and calculation of the distorted wave matrix element in coordinate space was beneficial in understanding this effect. Examination of the radial behavior of the DWME allows one to determine the region of the nucleus in which the knock-out reaction occurs. The region was noted to move to the extreme edge of the nucleus as the energy of the incident pion was increased. This indicates that data gathered at high energies (around 200 MeV ) will contain little information about the Pauli effect, an unfortunate consequence since the two-body pion nucleon interaction is most sensitive to the fermi momentum cut-off at $200 \mathrm{MeV} .{ }^{8}$ However, at lower energies (around 116 MeV ), the reaction will occur further inside the nucleus (fig. 6),
and scattering data gathered at these energies will provide information on the effect of the Pauli exclusion on the pion nucleon interaction (fig. 11, 13).

The spin orientation of the proton was found to have a major influence on the knock-out cross section because the value of the effective polarization varied significantly. In some cases ( $1 \mathrm{P}_{1 / 2}$ ) this value ranged from almost -1.0 to almost +1.0 . The effective polarization for $1 \mathrm{P}_{3 / 2}$ proton scattering is almost exactly opposite and approximately a factor of two reduced from the corresponding $1 P_{1 / 2}$ proton case. In the geometry used in this study, the $1 P_{1 / 2}$ and $1 P_{3 / 2}$ cross sections calculated without the effective polarization are similar in shape. Thus, consideration of the difference in shape between scattering cross sections for the $1 \mathrm{P}_{1 / 2}$ and $1 P_{3 / 2}$ protons measured under similar kinematic conditions will allow one to describe the effective polarization of the proton in the nucleus, and therefore, the effect of proton spin can be observed.

The cross sections calculated to compare with the results of Levin and Eisenberg ${ }^{6}$ did not contain the spin orbit term in the outgoing proton optical potential. As Levin and Eisenberg ${ }^{6}$ claimed that the spin orbit term should strongly influence the knock-out cross section, a proper comparison between the results was not possible. However, it should be noted that the magnitude and general behavior of the cross sections are in fair agreement.

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## Appendix

Figure Captions

Fig. 1 Ratio of the differential cross section for $0^{16}\left(\pi^{+}, \pi^{+} p\right) N^{15}$ calculated using the PWIA, to the differential cross section calculated using the DWIA. The ratios are plotted as a function of the recoil angle of the nucleus, $\theta_{R}$. The incident pion energy, $\mathrm{E}_{\mathrm{i}}=60 \mathrm{MeV}$, the recoil momentum of the nucleus, $q_{R}=0.3 \mathrm{fm}^{-1}$, and the final energy ratio, $\boldsymbol{\lambda}=0.8$. The solid line gives the ratios for knockout of a $1 \mathrm{P}_{1 / 2}$ proton and the dashed line gives the ratios for knock-out of a $1 P_{3 / 2}$ proton.
Fig. 2 Ratio of PWIA to DWIA differential cross sections for $0^{16}\left(\pi^{+}, \pi^{+} p\right) N^{15}$ as a function of $\theta_{R}$. The remaining kinematic variables are $E_{i}=116 \mathrm{MeV}$, $q_{R}=0.6 \mathrm{fm}^{-1}$, and $\lambda=0.8$. The solid line gives the ratios for the $1 P_{1 / 2}$ proton and the dashed line gives the ratios for a $1 P_{3 / 2}$ proton.
Fig. 3 Ratio of PWIA to DWIA differential cross sections for $0^{16}\left(\pi^{+}, \pi^{+} p\right) N^{15}$ as a function of $\theta_{R}$. The incident pion energy, $\mathrm{E}_{\mathrm{i}}=180 \mathrm{MeV}$. The remaining kinematic variables and explanation of lines are the same as fig. 2.

Fig. 4 Ratio of PWIA to DWIA differential cross sections
for $0^{16}\left(\pi^{+}, \pi^{+} p\right) N^{15}$ as a function of $\theta_{R}$. The incident pion energy, $\mathrm{E}_{\mathrm{i}}=220 \mathrm{MeV}$. The remaining kinematic variables and explanation of lines are the same as fig. 2.

Fig. 5 Three dimensional plot of the radial character of the DWNE for a $1 \mathrm{P}_{1 / 2}$ proton in $0^{16}$. The quantity $A_{r}$ (equation (4.36)) is plotted as a function of distance from the center of the nucleus and $\theta_{R}$. The remaining kinematic variables are the same as indicated in fig. 1.
Fig. 6 Three dimensional plot of the radial character of the DWVE for a $1 \mathrm{P}_{1 / 2}$ proton in $0^{16}$. The quantity $A_{r}$ is plotted as a function of distance from the center of the nucleus and $\theta_{R}$. The remaining kinematic variables are the same as indicated in fig. 2.

Fig. 7 Three dimensional plot of the radial character of the DWME for a $1 P_{1 / 2}$ proton in $0^{16}$. The quantity $A_{r}$ is plotted as a function of distance from the center of the nucleus and $\theta_{R}$. The remaining kinematic variables are the same as indicated in fig. 3.

Fig. 8 Three dimensional plot of the radial character of the DWNE for a $1 P_{1 / 2}$ proton in $0^{16}$. The quantity $A_{r}$ is plotted as a function of distance from the center of the nucleus and $\theta_{R}$. The remaining kinematic variables are the same as indicated in fig. 4.

Fig. 9 Five-fold differential cross section of the knockout of a $1 \mathrm{P}_{1 / 2}$ proton from $0^{16}$ as a function of $\theta_{R}$. The solid line gives $\quad \frac{d \sigma}{d \Omega_{\pi} d \Omega_{n} d\left(E_{\pi}^{\prime}-E_{n}^{\prime}\right)}$ calculated with the off-shell effect. The dashed line is the differential cross section calculated without the off-shell effect. Both curves are calculated with a fermi momentum, $\mathrm{K}_{\mathrm{f}}=0.8 \mathrm{fm}^{-1}$ and exclude the effective polarization of the nucleon. The kinematic situation is the same as indicated in fig. 1.

Fig. 10 Five-fold differential cross section of the knockout of a $1 \mathrm{P}_{3 / 2}$ proton from $0^{16}$ as a function of $\theta_{\mathrm{R}}$. The explanation of the lines is the same as in fig. 9. The kinematic situation is the same as indicated in fig. 1.

Fig. 11 Five-fold differential cross section of the knockout of a $1 P_{1 / 2}$ proton from $0^{16}$ as a function of $\theta_{R}$. The lines give the cross sections calculated for various fermi momenta, and include the off-shell effect and the effective polarization of the nucleon. The solid line gives the cross sections calculated with the fermi momentum determined by the localization of the knock-out reaction. In this case, $K_{f}$ local $\approx$ $0.7 \mathrm{fm}^{-1}$. The dotted line gives the cross sections calculated with $K_{f}=0.0$, the dashed line with $K_{f}=0.7 \mathrm{fm}^{-1}$, the dash-dot line with $K_{f}=1.0 \mathrm{fm}^{-1}$ and the dash-two dot line with $K_{f}=1.36 \mathrm{fm}^{-1}$. The
kinematic situation is the same as indicated for fig. 2.

Fig. 12

Fig. 13

Fig. 14 Five-fold differential cross sections calculated for the knock-out of a $1 P_{1 / 2}$ proton from $0^{16}$. The cross sections are calculated at various fermi momenta and include the off-shell effect and the effective polarization of the nucleon. The solid line gives the cross sections calculated with the fermi momentum determined by the localization of the reaction. In this case, $K_{f \text { local }} \approx 0.4 \mathrm{fm}^{-1}$. The dotted line gives the cross sections with $K_{f}=0.0 \mathrm{fm}^{-1}$, the dashed line gives the cross sections with $K_{f}=0.7 \mathrm{fm}^{-1}$. The kinematic situation is the same as indicated in fig. 3.

Fig. 15 Five-fold differential cross sections calculated without the effective polarization of the nucleon.

Except for the polarization, the explanation of the lines and kinematic situation is the same as fig. 14.

Fig. 16 Five-fold differential cross sections calculated for an effectively polarized $1 P_{3 / 2}$ proton. The kinematic situation is the same as indicated for fig. 3. The explanation of the lines is the same as fig. 14, except that the altered localization of the reaction causes $K_{f}$ local $\simeq 0.5 \mathrm{fm}^{-1}$.

Fig. 17 Five-fold differential cross sections calculated without the effective polarization of the nucleon. Except for the polarization, the explanation of the lines and kinematic situation is the same as fig. 16.

Fig. 18 Five-fold differential cross sections calculated for an effectively polarized $1 \mathrm{P}_{1 / 2}$ proton. The kinematic situation is the same as indicated in fig. 4. The explanation of the lines is the same as fig. 14, except that the increased pion energy alters the localization of the reaction and this changes $K_{f \text { local }} \simeq 0.3 \mathrm{fm}^{-1}$.
Fig. 19 Five-fold differential cross sections calculated without the effective polarization of the nucleon. Except for the polarization of the nucleon, the explanation of the lines and the kinematic situation is the same as fig. 18.

Fig. 20 Five-fold differential cross sections calculated for an effectively polarized $1 P_{3 / 2}$ proton. The
explanation of the kinematic situation is the same as indicated in fig. 4. The explanation of the lines is the same as fig. 16, except that the increased pion energy alters the localization of the reaction and this changes $K_{f \text { local }} \simeq 0.4 \mathrm{fm}^{-1}$. Fig. 21 Five-fold differential cross sections calculated without the effective polarization of the nucleon. Except for the polarization of the nucleon, the explanation of the lines and the kinematic situation is the same as fig. 20.

Fig. 22
Effective polarization of the bound proton as a function of $\boldsymbol{\theta}_{R}$. The solid line is the effective polarization of a $1 P_{1 / 2}$ proton. The dashed line is the effective polarization of a $1 \mathrm{P}_{3 / 2}$ proton. The kinematic situation is the same as indicated in fig. 4.

Fig. 23 Five-fold differential cross sections $\frac{d \sigma}{d \Omega_{\pi} d \Omega_{\rho} d E_{\pi}^{\prime}}$
for a $1 P_{1 / 2}$ proton as a function of out-going proton angle. The kinematic conditions are incident pion energy, $E_{\pi}=130 \mathrm{MeV}$, out-going pion energy, $E^{\prime} \pi=51.2 \mathrm{MeV}$, and out-going pion angle $\theta_{\pi}=150^{\circ}$. For the solid line, the cross section is calculated on-shell and includes the effective polarization of the nucleon. The Pauli exclusion effect and the spin orbit term in the out-going nucleon optical model potential have not been included. Other lines are taken from Levin and Eisenberg's ${ }^{6}$ results and
the dashed line is the cross section calculated without the factorization of the two-body $t$ matrix. The dash-dot line is the cross section calculated with the factorization approximation. The dotted line is the cross section calculated without the factorization approximation and without the spin orbit term in the out-going nucleon optical model potential.

Fig. 24 Five-fold differential cross sections for the knockout of a $1 P_{3 / 2}$ proton from $0^{16}$. The explanation of the lines and the kinematic situation is the same as fig. 23.

Fig. 25 Five-fold differential cross sections for the knockout of a $1 \mathrm{P}_{1 / 2}$ proton from $0^{16}$. The explanation of the lines is the same as fig. 23. The kinematic conditions are $E_{\pi}=130 \mathrm{MeV}, \mathrm{E}_{\boldsymbol{\pi}}^{\prime}=61 \mathrm{MeV}$, and $\boldsymbol{\theta}_{\boldsymbol{\pi}}=120^{\circ}$.
Fig. 26 Five-fold differential cross sections for the knockout of a $1 \mathrm{P}_{3 / 2}$ proton from $0^{16}$. The explanation and kinematic situation is the same as fig. 25.

Figure 1


Figure 2


Figure 3


Figure 4


INCIDENT PION 60 MEV
TARGET NUCLEON P1/2
NUCLEUS RECOIL MOM. 0.30 FM
FINAL MOM. RATIO 0.80
RADIAL SCALE 1 GRID:0.40FM


INCIDENT PION 116 MEV
TRRGET NUCLEON P1/2
NUCLEUS RECOIL MOM. 0.60 FM
FINAL MOM. RATIO 0.80
RADIAL SCALE 1 GRID:0.40FM


INCIDENT PION 180MEV
TARGET NUCLEON P1/2
NUCLEUS RECOIL MOM. 0.60 FM
FINAL MOM. RATIO 0.80
RADIAL SCALE 1 GRID:0.40FM


INCIDENT PION. 220MEV
TARGET NUCLEON P1/2
NUCLEUS RECOIL MOM. 0.60 FM
FINAL MOM. RATIO 0.80
RADIAL SCRLE 1 GRID:0.40FM


Figure 9


Figure 10


Figure 11


Figure 12


Figure 13


Figure 14


Figure 15



Figure 16


Figure 17


Figure 18


Figure 19


Figure 20


Figure 21



Figure 23


Figure 24


Figure 25


Figure 26


