THE DRESSING TRANSFORMATION AND ITS APPLICATION TO A FERMION-BOSON TRILINEAR INTERACTION

by

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Abstract

In this thesis, various fermion-boson strong interaction potentials are determined as functions of the basic fermion-boson trilinear vertex function.

Working in Fock space, we note that the fermion-boson trilinear interaction does not explicitly involve physical particles. We develop a transformation, called the dressing transformation, which acts on the fundamental particle creators and annihilators. They are transformed into physical particle operators, and the invariance properties and commutation relations of the theory are preserved. A precise technique for perturbatively determining the dressing transformation is formulated, and is applied to some simple models in field theory.

The dressing transformation makes explicit the physical particle interactions implicit in the original trilinear interaction. When applied to the nucleon-pion trilinear interaction, we find a nucleon mass renormalization, a nucleon-pion scattering term, and a nucleon-nucleon scattering term present in the second-order dressed Hamiltonian. Using the $NN\pi$ vertex function derived from the Cloudy Bag Model, the nucleon-nucleon coordinate space potential can be calculated. We discover that providing the two nucleons are separated by a distance greater than twice the bag radius, the potential between them is given by the one pion exchange potential modified in strength by a function of the bag radius.

TABLE OF CONTENTS

List of Tables

List of Figures

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Chapter 1 Introduction

The problem of finding a useful and accurate theory of the strong interactions of nucleons and pions is an ongoing one. Certainly one important advance in this area was the concept of one boson exchange potentials, implying that the underlying fermion-boson strong interaction is a trilinear one. These one boson exchange potentials provide the basis for the functional form of the phenomenological nucleon-nucleon potentials. In this thesis, we develop a technique for finding potentials for direct fermion-fermion, boson-boson, and fermion-boson interactions, as well as boson production on two fermions. These interaction potentials are determined in terms of the fermion-boson trilinear vertex function. They can serve as the basis for a phenomenological strong interaction Hamiltonian for systems of pions and nucleons at intermediate energies.

Using an approach which takes the fundamental dynamical variables of the theory to be the elementary fermion and boson creators $F^\dagger$ and $B^\dagger$, discussed in Chapter 2, the trilinear interaction involves the integral $\int hF^\dagger FB$, where the vertex function h depends on the momenta involved. We show in Chapter 3 that requiring this interaction to be invariant under certain space-time transformations greatly restricts the vertex function h. The Cloudy Bag Model is then used to determine a specific form for the strong interaction vertex function.

One difficulty with this fermion-boson trilinear interaction is that it does not explicitly involve physical particles since $F^\dagger|0\rangle$ is not an eigenket of the Hamiltonian. In

Chapter 4 we formulate a technique, called the dressing transformation, for transforming the elementary particle creators into physical particle creators. This transformation alters none of the invariance properties or commutation relations of the theory. The initial work on the dressing transformation was done by Greenberg and Schweber (1958), who considered simple, soluble theories such as the scalar field model and the Lee model. In this thesis we have generalized the concept of the dressing transformation in order to apply it to more realistic theories. We give a detailed prescription for determining the dressing transformation and also for determining the Hamiltonian as a function of the physical particle operators. Both are calculated in a perturbation series in the strong interaction coupling constant. (A different perturbation series for the dressing transformation has been given by Faddeev (1964)).

In Sections 4.2 and 4.3, the dressing transformation is applied to two simple theories - the scalar field model and the Lee model. This application illustrates many features of the transformation, the physical particle creators, and the dressed Hamiltonian which are present in more complicated theories. The nucleon-pion trilinear interaction is dressed to second order in Chapter 5. We consider the resulting physical nucleon-pion and nucleon-nucleon interactions. Using the Cloudy Bag Model vertex function, we calculate the second-order nucleon-nucleon potential. We discover that providing the two nucleons are not touching, this potential is simply a one pion exchange potential that has been slightly modified in strength.

Many of the Appendices provide useful mathematical formulae and techniques used in the thesis. For example, they discuss the rotation matrices, spherical harmonics, angular momentum coupling, and Bessel functions. Other Appendices provide some background to concepts used in the thesis, discussing such things as two-body potentials or one pion exchange potentials.

In Appendix H we have generalized the trilinear interaction to include fermions and bosons of arbitrary spin and isospin. Thus our technique can be applied not only to interactions of nucleons and pions, but to interactions of other fermions and bosons as well.

Finally, in Appendix K, we discuss a dressing transformation for a Poincaré invariant system of interacting fermions and bosons.

Chapter 2 Fermion and Boson Fundamental Dynamical Variables and Their Properties

This Chapter will provide the background information necessary for an understanding of the rest of the thesis. We will introduce particle creators and annihilators, and we will give their commutation relations and space-time transformation properties.

2.1 The Fundamental Dynamical Variables

Our system of fundamental fermions and bosons with arbitrary spin and isospin is described by a Hilbert space which is a direct product of fermion Fock space and boson Fock space. The fundamental dynamical variables, in terms of which all observables and operators can be expressed, are the particle creation and annihilation operators. These are defined as follows:

$F^{si\dagger}_{m\mu}(\underline{x})$, when acting on the vacuum state, yields a one-fermion ket corresponding to an elementary fermion at position $\underline{x}$, having spin s, z-axis projection m, isospin i, and isospin z-axis projection μ. The adjoint operator, $F^{si}_{m\mu}(\underline{x})$, when acting on this one-fermion ket, gives the vacuum state.

Equivalently, we can define the momentum space fermion creators and annihilators, $F^{si\dagger}_{m\mu}(\underline{p})$ and $F^{si}_{m\mu}(\underline{p})$. These create, or destroy, an elementary fermion with momentum $\underline{p}$. The two creators are related by a Fourier transform:

$$F^{si\,+}_{m\mu}(\underline{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3x \; e^{i\underline{p}\cdot\underline{x}/\hbar} \, F^{si\,+}_{m\mu}(\underline{x}) \tag{2.1.1}$$

Note that

$$\langle 0 | F^{si\,+}_{m\mu}(\underline{p}) = \langle 0 | F^{si\,+}_{m\mu}(\underline{x}) = 0 \tag{2.1.2}$$

and

$$F^{si}_{m\mu}(\underline{p}) | 0 \rangle = F^{si}_{m\mu}(\underline{x}) | 0 \rangle = 0 \tag{2.1.3}$$

Similarly, $B^{si\,+}_{m\mu}(\underline{x})$, when acting on the vacuum state, gives a position ket corresponding to an elementary boson at $\underline{x}$ having spin s with projection m and isospin i with projection μ. The operators $B^{si}_{m\mu}(\underline{x})$, $B^{si\,+}_{m\mu}(\underline{p})$, and $B^{si}_{m\mu}(\underline{p})$ are defined analogously to the fermion case. The position and momentum creators are related by

$$B^{si\,+}_{m\mu}(\underline{x}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3p \; e^{-i\underline{p}\cdot\underline{x}/\hbar} \, B^{si\,+}_{m\mu}(\underline{p}) \tag{2.1.4}$$

The following commutation relations are satisfied by the fundamental dynamical variables:

$$\{ F^{si}_{m\mu}(\underline{\xi}), F^{s'i'\,+}_{m'\mu'}(\underline{\xi}') \} = \delta(\underline{\xi}-\underline{\xi}') \, \delta_{ss'} \, \delta_{mm'} \, \delta_{ii'} \, \delta_{\mu\mu'} \tag{2.1.5}$$

$$\{ F^{si}_{m\mu}(\underline{\xi}), F^{s'i'}_{m'\mu'}(\underline{\xi}') \} = \{ F^{si\,+}_{m\mu}(\underline{\xi}), F^{s'i'\,+}_{m'\mu'}(\underline{\xi}') \} = 0 \tag{2.1.6}$$

$$[B^{si}_{m\mu}(\underline{\xi}), B^{s'i'\,+}_{m'\mu'}(\underline{\xi}')] = \delta(\underline{\xi}-\underline{\xi}') \, \delta_{ss'} \, \delta_{mm'} \, \delta_{ii'} \, \delta_{\mu\mu'} \tag{2.1.7}$$

$$[B^{si}_{m\mu}(\underline{\xi}), B^{s'i'}_{m'\mu'}(\underline{\xi}')] = [B^{si\,+}_{m\mu}(\underline{\xi}), B^{s'i'\,+}_{m'\mu'}(\underline{\xi}')] = 0 \tag{2.1.8}$$

$$[B^{si\,+}_{m\mu}(\underline{\xi}), F^{s'i'}_{m'\mu'}(\underline{\xi}')] = [B^{si\,+}_{m\mu}(\underline{\xi}), F^{s'i'\,+}_{m'\mu'}(\underline{\xi}')] = 0 \tag{2.1.9}$$

where [ ] denotes a commutator and { } an anticommutator; $\underline{\xi}$ represents either $\underline{x}$ or $\underline{p}$.

2.2 Space-Time Transformation Properties of the Fundamental Dynamical Variables

In this Section we discuss the effect of displacements, rotations, space inversion and time reversal on the particle creators and annihilators. These results are a consequence of the requirement of general Poincaré invariance for a physical system (Kalyniak (1978)). We will use them in Section 3.1 to determine the form which invariant particle interactions must have.

To a spatial displacement of the system by amount $\underline{a}$ there corresponds a linear, unitary operator $D(\underline{a})$ such that

$$D(\underline{a})\, F^{si\,+}_{m\mu}(\underline{x})\, D^{+}(\underline{a}) = F^{si\,+}_{m\mu}(\underline{x}+\underline{a}) \tag{2.2.1}$$

$$D(\underline{a})\, B^{si\,+}_{m\mu}(\underline{x})\, D^{+}(\underline{a}) = B^{si\,+}_{m\mu}(\underline{x}+\underline{a}) \tag{2.2.2}$$

The resulting creation operator creates a particle at position $\underline{x} + \underline{a}$ rather than $\underline{x}$. Using (2.1.1) and (2.1.4) we have

$$D(a)\, F^{si\,+}_{m\mu}(\underline{p})\, D^{+}(\underline{a}) = F^{si\,+}_{m\mu}(\underline{p})\, e^{-i\underline{a}\cdot\underline{p}/\hbar} \tag{2.2.3}$$

$$D(a)\, B^{si}_{m\mu}(\underline{p})\, D^{+}(\underline{a}) = B^{si}_{m\mu}(\underline{p})\, e^{i\underline{a}\cdot\underline{p}/\hbar} \tag{2.2.4}$$

To a spatial rotation of the system through Euler angles α, β, γ there corresponds a linear, unitary operator $\mathcal{R}(\alpha\beta\gamma)$, such that

$$\mathcal{R}(\alpha\beta\gamma)\, F^{si\,+}_{m\mu}(\underline{x})\, \mathcal{R}^{+}(\alpha\beta\gamma) = \sum_{m'=-s}^{s} D^{s}_{m'm}(\alpha\beta\gamma)\, F^{si\,+}_{m'\mu}(\underline{x}_{-R}) \tag{2.2.5}$$

$$\mathcal{R}(\alpha\beta\gamma)\, B^{si}_{m\mu}(\underline{x})\, \mathcal{R}^{+}(\alpha\beta\gamma) = \sum_{m'=-s}^{s} D^{s\,*}_{m'm}(\alpha\beta\gamma)\, B^{si}_{m'\mu}(\underline{x}_{-R}) \tag{2.2.6}$$

The conventions for the Euler angles and rotation matrices $D^{s}_{mm'}(\alpha\beta\gamma)$ are as given in Rose (1957). Note that

$$\underline{x}_{R} = M\,\underline{x} \tag{2.2.7a}$$

$$\underline{x}_{-R} = M^{-1}\underline{x} \tag{2.2.7b}$$

where $\underline{x}$ is a column vector and M is the matrix defined in Rose (1957, p.65).

From (2.1.1) and (2.1.4) it follows that

$$\mathcal{R}(\alpha\beta\gamma)\, F^{si\,+}_{m\mu}(\underline{p})\, \mathcal{R}^{+}(\alpha\beta\gamma) = \sum_{m'=-s}^{s} D^{s}_{m'm}(\alpha\beta\gamma)\, F^{si\,+}_{m'\mu}(\underline{p}_{-R}) \tag{2.2.8}$$

$$\mathcal{R}(\alpha\beta\gamma)\, B^{si}_{m\mu}(\underline{p})\, \mathcal{R}^{+}(\alpha\beta\gamma) = \sum_{m'=-s}^{s} D^{s\,*}_{m'm}(\alpha\beta\gamma)\, B^{si}_{m'\mu}(\underline{p}_{-R}) \tag{2.2.9}$$

Both the momentum and spatial coordinates, as well as the spin, are rotated.

To a space inversion transformation there corresponds a linear unitary operator $\mathcal{P}$ such that

$$\mathcal{P}\, F^{si\,+}_{m\mu}(\underline{\xi})\, \mathcal{P}^{+} = \pm F^{si\,+}_{m\mu}(-\underline{\xi}) \tag{2.2.10}$$

$$\mathcal{P}\, B^{si}_{m\mu}(\underline{\xi})\, \mathcal{P}^{+} = \pm B^{si}_{m\mu}(-\underline{\xi}) \tag{2.2.11}$$

where the plus sign applies to positive parity particles, and the minus sign to negative parity particles. $\underline{\xi}$ represents either $\underline{x}$ or $\underline{p}$. Both the position and momentum vectors are inverted by this transformation.

To a time reversal transformation on the system there corresponds an antilinear, antiunitary operator $\mathcal{J}$ such that

$$\mathcal{J} F^{si\,+}_{m\mu}(\underline{x}) \mathcal{J}^{+} = \eta_F \sum_{m'=-s}^{s} D^{s}_{m'm}(0\pi 0)\; F^{si\,+}_{m'\mu}(\underline{x}) \tag{2.2.12}$$

$$\mathcal{J} B^{si}_{m\mu}(\underline{x}) \mathcal{J}^{+} = \eta_B \sum_{m'=-s}^{s} D^{s\,*}_{m'm}(0\pi 0)\; B^{si}_{m'\mu}(\underline{x}) \tag{2.2.13}$$

$$\mathcal{J} f(\underline{x}) \mathcal{J}^{+} = f^{*}(\underline{x})\, \mathcal{J}\mathcal{J}^{+} = f^{*}(\underline{x}) \tag{2.2.14}$$

where $f(\underline{x})$ is an arbitrary complex function, and $|\eta_F|^2 = \eta_B{}^2 = 1$. η is the time reversal parity of the particle. It is determined from the space inversion parity and charge conjugation parity by the requirement of overall TCP invariance (see Schweber (1961, p.268)). For nucleons, $\eta_F = +1$; for pions, $\eta_B = -1$.

The time reversal transformation does not change the position coordinates, but does reverse both the momentum and spin vectors. Reversing the spin is equivalent to rotating the observer through 180 degrees about an arbitrary axis. We have chosen this axis to be the y-axis, so the spin rotation has Euler angles $0, \pi, 0$. Using

$$D^{s}_{m'm}(0\pi 0) = d^{s}_{m'm}(\pi) = (-)^{s+m'}\, \delta_{m',-m} \tag{2.2.15}$$

as well as (2.1.1) and (2.1.4), it follows that

$$\mathcal{J} F^{si\,+}_{m\mu}(\underline{p}) \mathcal{J}^{+} = \eta_F\, (-)^{s-m}\, F^{si\,+}_{-m\mu}(-\underline{p}) \tag{2.2.16}$$

$$\mathcal{J} B^{si}_{m\mu}(\underline{p}) \mathcal{J}^{+} = \eta_B\, (-)^{s-m}\, B^{si}_{-m\mu}(-\underline{p}) \tag{2.2.17}$$

Finally, to a rotation in isospin space through Euler angles α, β, γ there corresponds a linear, unitary operator $\mathcal{R}_I(\alpha\beta\gamma)$ such that

$$\mathcal{R}_I(\alpha\beta\gamma)\, F^{si\,\dagger}_{m\mu}(\underline{\xi})\,\mathcal{R}_I^{\dagger}(\alpha\beta\gamma) = \sum_{\mu'=-i}^{i} D^{i}_{\mu'\mu}(\alpha\beta\gamma)\, F^{si\,\dagger}_{m\mu'}(\underline{\xi}) \tag{2.2.18}$$

$$\mathcal{R}_I(\alpha\beta\gamma)\, B^{si}_{m\mu}(\underline{\xi})\,\mathcal{R}_I^{\dagger}(\alpha\beta\gamma) = \sum_{\mu'=-i}^{i} D^{i\,*}_{\mu'\mu}(\alpha\beta\gamma)\, B^{si}_{m\mu'}(\underline{\xi}) \tag{2.2.19}$$

The space-time transformations of $F^{si}_{m\mu}(\underline{\xi})$ and $B^{si\,\dagger}_{m\mu}(\underline{\xi})$ are easily obtained by taking the adjoint of the transformation equations for $F^{si\,\dagger}_{m\mu}(\underline{\xi})$ and $B^{si}_{m\mu}(\underline{\xi})$ given in this Section.

Chapter 3 The Trilinear Fermion-Boson Interaction

In this Chapter, we introduce and discuss a Hamiltonian for strongly interacting nucleons and pions. (A more general trilinear Hamiltonian involving other fermions and bosons is considered in Appendix H.) We will show, in the first Section, that when this interaction is required to be invariant under certain space-time transformations, it can only then depend on a single function of one variable. In the next Section, we specify this vertex function for the Cloudy Bag Model. Finally, we discuss the concepts of elementary particle and physical particle, within the context of the trilinear interaction.

3.1 Restrictions Due to Certain Space-Time Invariances

Using the fundamental dynamical variables defined in Section 2.1, we construct the following Hamiltonian involving interacting spin one-half, isospin one-half, positive parity fermions, i.e. nucleons, and spin zero, isospin one, negative parity bosons, i.e. pions:

$$H = H_0 + \lambda H_1 \tag{3.1.1a}$$

$$H_0 = \sum_{m\mu\mu'} \int d^3p \left[\mathcal{E}_{F0}(\underline{p}) F_{m\mu}^{+}(\underline{p}) F_{m\mu}(\underline{p}) + \mathcal{E}_{B0}(\underline{p}) B_{\mu'}^{+}(\underline{p}) B_{\mu'}(\underline{p}) \right] \tag{3.1.1b}$$

$$H_1 = \sum_{\substack{m_1 m_2 \\ \mu_1\mu_2\mu_3}} \int d^3p \, d^3q \; h^{\mu_1\mu_2\mu_3}_{m_1 m_2}(\underline{q}) \, F_{m_1\mu_1}^{+}(\underline{p}) \, F_{m_2\mu_2}(\underline{p}-\underline{q}) \, B_{\mu_3}(\underline{q}) + \text{adj.} \tag{3.1.1c}$$

The operator $F_{m\mu}^{\dagger}(\underline{p})$ corresponds, in the notation of Section 2.1, to the operator $F_{m\mu}^{si\dagger}(\underline{p})$ with s=1/2 and i=1/2; m and μ can take the values ±1/2. We drop the labels s and i for simplicity. Similarly, $B_{\mu'}(\underline{p})$ corresponds to $B_{m\mu'}^{si}(\underline{p})$ with s=m=0, i=1, μ'= ±1,0. Also,

$$\mathcal{E}_{F_0}(\underline{p}) = \left[p^2c^2 + m_{F_0}{}^2c^4\right]^{1/2} \tag{3.1.2}$$

and

$$\mathcal{E}_{B_0}(\underline{p}) = \left[p^2c^2 + m_{B_0}{}^2c^4\right]^{1/2} \tag{3.1.3}$$

are the energies of the elementary fermion and boson, respectively. Note that the particles are treated 'semi-relativistically' by including relativistic kinematics. Note also that the 'vertex function' $h_{m_1m_2}^{\mu_1\mu_2\mu_3}(\underline{q})$ is chosen to be a function only of $\underline{q}$.

The trilinear interaction H_1 may be pictured as follows:

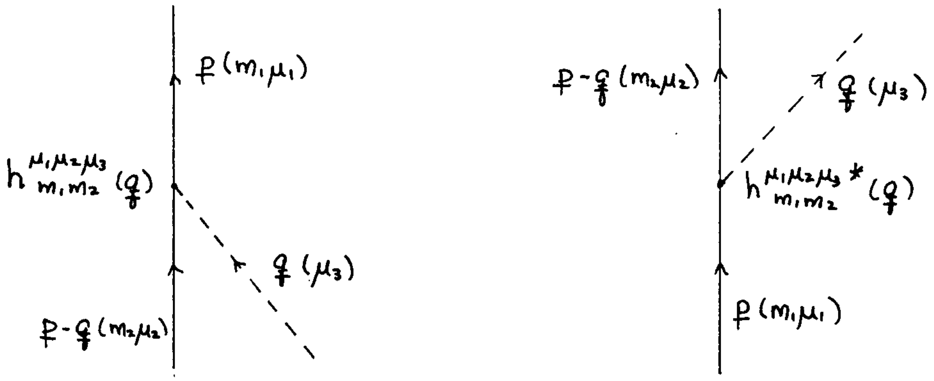

Fig. 1 The Trilinear F-B Interaction
Solid lines are fermions; dashed lines are bosons

The Hamiltonian (3.1.1) is translationally invariant and conserves the total number of fermions. We can see this by noting that the total momentum operator

$$\underline{P} = \sum_{m\mu\mu'} \int d^3p \; \underline{p} \left[F_{m\mu}^{\dagger}(\underline{p}) F_{m\mu}(\underline{p}) + B_{\mu'}^{\dagger}(\underline{p}) B_{\mu'}(\underline{p}) \right] \tag{3.1.4}$$

and the fermion number operator

$$N = \sum_{m\mu} \int d^3p \; F_{m\mu}^{+}(\underset{\sim}{p})\, F_{m\mu}(\underset{\sim}{p}) \tag{3.1.5}$$

both commute with H.

We now demand that H also be invariant under rotations in isospin space, rotations in ordinary space, space inversion, and time reversal. H_0 is already invariant under all these transformations, but in order that H_1 also be invariant, the form of the function $h^{\mu_1\mu_2\mu_3}_{m_1m_2}(\underset{\sim}{q})$ appearing in (3.1.1c) must be restricted (see eq. (3.1.35)).

First, let us calculate the effect on H_1 of a rotation in isospin space. Using the results (2.2.18) and (2.2.19), we see that

$$\mathcal{R}_I(\alpha\beta\gamma)\, H_1\, \mathcal{R}_I^{+}(\alpha\beta\gamma) = \sum_{\substack{m_1m_2\\ \mu_1\mu_2\mu_3,\\ \mu_1'\mu_2'\mu_3'}} \int d^3p\, d^3q\; h^{\mu_1\mu_2\mu_3}_{m_1m_2}(\underset{\sim}{q})\, D^{1/2}_{\mu_1'\mu_1}(\alpha\beta\gamma)\, D^{1/2\,*}_{\mu_2'\mu_2}(\alpha\beta\gamma)\cdot$$

$$\cdot D^{1\,*}_{\mu_3'\mu_3}(\alpha\beta\gamma)\; F^{+}_{m_1\mu_1'}(\underset{\sim}{p})\, F_{m_2\mu_2'}(\underset{\sim}{p}-\underset{\sim}{q})\, B_{\mu_3'}(\underset{\sim}{q}) \;+ \text{adj}. \tag{3.1.6}$$

Using (A.1) and (A.8) to combine the three rotation matrices into a single one, we find

$$\mathcal{R}_I\, H_1\, \mathcal{R}_I^{+} = \sum_{\substack{m_1m_2\mu_1\mu_2\mu_3\\ \mu_1'\mu_2'\mu_3'\sigma\sigma'}} \int d^3p\, d^3q\; h^{\mu_1\mu_2\mu_3}_{m_1m_2}(\underset{\sim}{q})\, (-)^{\mu'-\mu}\, D^{\sigma'}_{\lambda'\lambda}(\alpha\beta\gamma)\cdot$$

$$\cdot(\tfrac{1}{2}1\,\mu_2'\mu_3'|\sigma\mu')\,(\tfrac{1}{2}1\,\mu_2\mu_3|\sigma\mu)\,(\tfrac{1}{2}\sigma\,\mu_1'\,{-\mu'}|\sigma'\lambda')\,(\tfrac{1}{2}\sigma\mu_1\,{-\mu}|\sigma'\lambda')\cdot$$

$$\cdot\, F^{+}_{m_1\mu_1'}(\underset{\sim}{p})\, F_{m_2\mu_2'}(\underset{\sim}{p}-\underset{\sim}{q})\, B_{\mu_3'}(\underset{\sim}{q}) \;+ \text{adj}. \tag{3.1.7}$$

where $(j_1j_2m_1m_2|jm)$ is a Clebsch-Gordan coefficient. (See Appendix D.) In order that (3.1.7) be equal to H_1, it can have no dependence on the angles α, β, γ. Therefore, taking

$\sigma' = \lambda' = \lambda = 0$ and using the property (D.12) of the Clebsch-Gordan coefficients in Appendix D, we have

$$\mathcal{R}_I H_1 \mathcal{R}_I^+ = \sum_{\substack{\mu_2\mu_3 \\ m_1 m_2 \mu_2'\mu_3'}} \int d^3p\, d^3q\; h^{\mu_1\mu_2\mu_3}_{m_1 m_2}(q)\, \tfrac{1}{2}\, (-)^{1-2\mu_2-2\mu_3} \cdot (\tfrac{1}{2} \mid \mu_2'\mu_3' \mid \tfrac{1}{2}\mu_1')\,(\tfrac{1}{2}\mid \mu_2\mu_3 \mid \tfrac{1}{2}\mu_1) \cdot F^+_{m_1\mu_1'}(p)\, F_{m_2\mu_2'}(p-q)\, B_{\mu_3'}(q) + \text{adj}. \tag{3.1.8}$$

Comparing with (3.1.1c), we see that

$$\mathcal{R}_I H_1 \mathcal{R}_I^+ = H_1 \tag{3.1.9}$$

if

$$h^{\mu_1'\mu_2'\mu_3'}_{m_1 m_2}(q) = \sum_{\mu_2\mu_3} h^{\mu_1\mu_2\mu_3}_{m_1 m_2}(q)\,(\tfrac{1}{2})\,(-)^{1-2\mu_2-2\mu_3} \cdot (\tfrac{1}{2}\mid \mu_2'\mu_3' \mid \tfrac{1}{2}\mu_1')\,(\tfrac{1}{2}\mid \mu_2\mu_3 \mid \tfrac{1}{2}\mu_1) \tag{3.1.10}$$

From this equation, we note that all of the μ_1, μ_2, μ_3 dependence of $h^{\mu_1\mu_2\mu_3}_{m_1 m_2}(q)$ is contained in the coefficient $(\tfrac{1}{2}\mid \mu_2\mu_3 \mid \tfrac{1}{2}\mu_1)$. Therefore, we may write

$$h^{\mu_1\mu_2\mu_3}_{m_1 m_2}(q) = (\tfrac{1}{2}\mid \mu_2\mu_3 \mid \tfrac{1}{2}\mu_1)\, h_{m_1 m_2}(q) \tag{3.1.11}$$

Secondly, we determine the effect on H_1 of a spatial rotation. From (2.2.8) and (2.2.9), we have

$$\mathcal{R}(\alpha\beta\gamma) H_1 \mathcal{R}^+(\alpha\beta\gamma) = \sum_{\mu_1\mu_2\mu_3} (\tfrac{1}{2}\mid \mu_2\mu_3 \mid \tfrac{1}{2}\mu_1) \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \int d^3p\, d^3q\; h_{m_1 m_2}(q) \cdot D^{1/2}_{m_1' m_1}(\alpha\beta\gamma)\, D^{1/2\,*}_{m_2' m_2}(\alpha\beta\gamma)\, F^+_{m_1'\mu_1}(p_{-R})\, F_{m_2'\mu_2}(p_{-R} - q_{-R})\, B_{\mu_3}(q_{-R}) + \text{adj}. \tag{3.1.12}$$

This will equal H_1 if

$$h_{m_1 m_2}(q) = \sum_{m_1' m_2'} h_{m_1' m_2'}(q_R)\, D^{1/2}_{m_1 m_1'}(\alpha\beta\gamma)\, D^{1/2\,*}_{m_2 m_2'}(\alpha\beta\gamma) \tag{3.1.13}$$

We write

$$h_{m_1 m_2}(q) = i \sum_{\ell m} Y^*_{\ell m}(q)\; h_{\ell m m_1 m_2}(q) \tag{3.1.14}$$

where $Y_{\ell m}(q)$ is a spherical harmonic of order ℓ, whose properties are described in Appendix B.

Substituting (3.1.14) into (3.1.13), we obtain

$$\sum_{\ell m} Y^*_{\ell m}(q)\; h_{\ell m m_1 m_2}(q) = \sum_{\substack{m_1' m_2' \\ \ell' m'}} Y^*_{\ell' m'}(q_R)\; h_{\ell' m' m_1' m_2'}(q)\cdot$$
$$\cdot\, D^{1/2}_{m_1 m_1'}(\alpha\beta\gamma)\; D^{1/2\,*}_{m_2 m_2'}(\alpha\beta\gamma) \tag{3.1.15}$$

Using (B.2),

$$Y^*_{\ell' m'}(q_R) = \sum_{m''} D^{\ell'\,*}_{m'' m'}(\alpha\beta\gamma)\; Y^*_{\ell' m''}(q) \tag{3.1.16}$$

we act on both sides of (3.1.15) with $\int Y_{\ell m}(q)\, d\Omega_q$ to obtain

$$h_{\ell m m_1 m_2}(q) = \sum_{m_1' m_2' m'} h_{\ell m' m_1' m_2'}(q)\; D^{1/2}_{m_1 m_1'}(\alpha\beta\gamma)\, D^{1/2\,*}_{m_2 m_2'}(\alpha\beta\gamma)\, D^{\ell\,*}_{m m'}(\alpha\beta\gamma) \tag{3.1.17}$$

Using (A.8) to combine the first two rotation matrices this becomes

$$h_{\ell m m_1 m_2}(q) = \sum_{\substack{m_1' m_2' m' \\ \ell m \lambda}} \left(\frac{2\ell+1}{3}\right) h_{\ell m' m_1' m_2'}(q)\,(\tfrac{1}{2}\,\ell\, m_2\, m \,|\, \tfrac{1}{2} m_1)(\tfrac{1}{2}\,\ell\, m_2'\, \lambda \,|\, \tfrac{1}{2} m_1')\cdot$$
$$\cdot\, D^{\ell}_{m\lambda}(\alpha\beta\gamma)\; D^{\ell\,*}_{m m'}(\alpha\beta\gamma) \tag{3.1.18}$$

Now we integrate on both sides with respect to the Euler angles α, β, γ. Using (A.9), we find

$$h_{\ell m m_1 m_2}(q) = \frac{1}{3}\sum_{m_1' m_2' m'} h_{\ell m' m_1' m_2'}(q)\,(\tfrac{1}{2}\,\ell\, m_2\, m \,|\, \tfrac{1}{2} m_1)(\tfrac{1}{2}\,\ell\, m_2'\, m' \,|\, \tfrac{1}{2} m_1') \tag{3.1.19}$$

This equation determines the m_1, m_2, m dependence of $h_{\ell m m_1 m_2}(q)$:

$$h_{\ell m m_1 m_2}(\underset{\sim}{q}) = (\tfrac{1}{2}\,\ell\, m_2\, m \,|\, \tfrac{1}{2} m_1)\, h_\ell(\underset{\sim}{q}) \tag{3.1.20}$$

Referring to (3.1.14), we write

$$h_{m_1 m_2}(\underset{\sim}{q}) = i \sum_{\ell} Y^*_{\ell m}(\underset{\sim}{q}) \; (\tfrac{1}{2}\,\ell\, m_2\, m \,|\, \tfrac{1}{2} m_1)\, h_\ell(\underset{\sim}{q}) \tag{3.1.21}$$

Thirdly, we require that H_1 be invariant under space inversion. From (2.2.10) and (2.2.11) one has

$$\begin{aligned} \mathcal{P} H_1 \mathcal{P}^\dagger = - \sum_{\substack{m_1 m_2 \\ \mu_1 \mu_2 \mu_3}} \int d^3p\, d^3q\; h_{m_1 m_2}(\underset{\sim}{q}) \; (\tfrac{1}{2}\, 1\, \mu_2 \mu_3 \,|\, \tfrac{1}{2} \mu_1) \cdot \\ \cdot F^{+}_{m_1 \mu_1}(-\underset{\sim}{p})\, F_{m_2 \mu_2}(-\underset{\sim}{p}+\underset{\sim}{q})\, B_{\mu_3}(-\underset{\sim}{q}) + \mathrm{adj}. \end{aligned} \tag{3.1.22}$$

This will equal H_1 if

$$h_{m_1 m_2}(\underset{\sim}{q}) = - h_{m_1 m_2}(-\underset{\sim}{q}) \tag{3.1.23}$$

(One deals with positive parity bosons in a similar manner. For such bosons we would have instead

$$h_{m_1 m_2}(\underset{\sim}{q}) = h_{m_1 m_2}(-\underset{\sim}{q})).$$

From (3.1.14), we have

$$\begin{aligned} h_{m_1 m_2}(-\underset{\sim}{q}) &= i \sum_{\ell m} Y^*_{\ell m}(-\underset{\sim}{q})\, h_{\ell m m_1 m_2}(\underset{\sim}{q}) \\ &= i \sum_{\ell m} (-)^\ell\, Y^*_{\ell m}(\underset{\sim}{q})\, h_{\ell m m_1 m_2}(\underset{\sim}{q}) \end{aligned} \tag{3.1.24}$$

where the second step follows from (B.4). Therefore, (3.1.23) can only be satisfied if

$$(-)^\ell = -1$$

i.e., if ℓ is odd. Referring to (3.1.20) for $h_{\ell m m_1 m_2}(\underset{\sim}{q})$ and to Figure 1, we see that ℓ may be interpreted as the angular momentum of the boson in the trilinear interaction. The Clebsch-Gordan coefficient $(\frac{1}{2}\, \ell\, m_2\, m \,|\, \frac{1}{2} m_1)$ implies that the only possible odd value of ℓ is

$$\ell = 1 \tag{3.1.25}$$

That is, only p-wave pions are allowed. We write

$$h_1(\underset{\sim}{q}) = h(\underset{\sim}{q}) \tag{3.1.26}$$

Finally, we demand that H_1 be invariant under a time reversal transformation. From (2.2.14), (2.2.16) and (2.2.17), we see that

$$J H_1 J^{+} = -\sum_{\substack{m_1 m_2 \\ \mu_1 \mu_2 \mu_3}} \int d^3p\, d^3q\, (\tfrac{1}{2}\, 1\, \mu_2\, \mu_3 \,|\, \tfrac{1}{2}\mu_1)\ h^{*}_{m_1 m_2}(\underset{\sim}{q})\ (-)^{1-m_1-m_2} \cdot$$

$$\cdot F^{+}_{-m_1 \mu_1}(-\underset{\sim}{p})\ F_{-m_2 \mu_2}(-\underset{\sim}{p}+\underset{\sim}{q})\ B_{\mu_3}(-\underset{\sim}{q}) + \mathrm{adj}. \tag{3.1.27}$$

This equals H_1 if

$$h_{m_1 m_2}(\underset{\sim}{q}) = (-)^{m_1+m_2}\, h^{*}_{-m_1\,-m_2}(-\underset{\sim}{q}) \tag{3.1.28}$$

or, from (3.1.23),

$$h_{m_1 m_2}(\underset{\sim}{q}) = (-)^{1+m_1+m_2}\, h^{*}_{-m_1,-m_2}(\underset{\sim}{q}) \tag{3.1.29}$$

Now, from (3.1.21) and (3.1.25), we have

$$h^{*}_{-m_1,-m_2}(\underset{\sim}{q}) = -i\, Y_{1m}(\underset{\sim}{q})\, (\tfrac{1}{2}\, 1\, -m_2\, m \,|\, \tfrac{1}{2}\, -m_1)\, h^{*}(\underset{\sim}{q}) \tag{3.1.30}$$

Using (B.5) and (D.9b), equation (3.1.30) becomes

$$h^{*}_{-m_1,-m_2}(\underset{\sim}{q}) = i\, (-)^{m}\, Y^{*}_{1-m}(\underset{\sim}{q})\, (\tfrac{1}{2}\, 1\, m_2\, -m \,|\, \tfrac{1}{2}\, m_1)\, h^{*}(\underset{\sim}{q}) \tag{3.1.31}$$

Since m_2 is half odd integral,

$$(-)^{-m_2} = -(-)^{m_2} \tag{3.1.32}$$

It follows that

$$h^*_{-m_1-m_2}(\underset{\sim}{q}) = i(-)^{-m_1-m_2+1} Y^*_{1m}(\underset{\sim}{q})(\tfrac{1}{2}1m_2m|\tfrac{1}{2}m_1)h^*(q) \tag{3.1.33}$$

Therefore, (3.1.29) is satisfied if

$$h(q) = h^*(q) \tag{3.1.34}$$

i.e., if h(q) is a real function.

In summary, the requirements of invariance under displacements, isospin rotations, spatial rotations, space inversion and time reversal have determined the vertex function for the trilinear interaction (3.1.1c) to be

$$h^{\mu_1\mu_2\mu_3}_{m_1m_2}(\underset{\sim}{q}) = i(\tfrac{1}{2}1\mu_2\mu_3|\tfrac{1}{2}\mu_1)(\tfrac{1}{2}1m_2m|\tfrac{1}{2}m_1)Y^*_{1m}(\underset{\sim}{q})h(q) \tag{3.1.35}$$

where h(q) is an arbitrary, real function. Thus the interaction (3.1.1c) can be written

$$H_1 = i\sum_{\substack{m_1m_2\\ \mu_1\mu_2\mu_3}}\int d^3p\, d^3q\ (\tfrac{1}{2}1\mu_2\mu_3|\tfrac{1}{2}\mu_1)(\tfrac{1}{2}1m_2m|\tfrac{1}{2}m_1)Y^*_{1m}(\underset{\sim}{q})h(q)\cdot$$
$$\cdot F^+_{m_1\mu_1}(\underset{\sim}{p})F_{m_2\mu_2}(\underset{\sim}{p}-\underset{\sim}{q})B_{\mu_3}(\underset{\sim}{q}) + \text{adj}. \tag{3.1.36}$$

At this point it is illustrative to show that the above expression for H_1 can be put into a manifestly invariant form. First, we define the following operator:

$$S_{\lambda\mu}(\underset{\sim}{q}) = \sqrt{\frac{3}{4\pi}}\sum_{\substack{m_1m_2\\ \mu_1\mu_2}}(\tfrac{1}{2}1m_2\lambda|\tfrac{1}{2}m_1)(\tfrac{1}{2}1\mu_2\mu|\tfrac{1}{2}\mu_1)\cdot$$
$$\cdot\int d^3p\ F^+_{m_1\mu_1}(\underset{\sim}{p})F_{m_2\mu_2}(\underset{\sim}{p}-\underset{\sim}{q}) \tag{3.1.37}$$

Using the properties (2.2.5) and (2.2.18) of the fermion operators, we find that

$$\mathcal{R}(\alpha\beta\gamma)\, S_{\lambda\mu}(\underset{\sim}{q})\, \mathcal{R}^{+}(\alpha\beta\gamma) = \sum_{\lambda'=-1}^{1} D^{1}_{\lambda'\lambda}(\alpha\beta\gamma)\, S_{\lambda'\mu}(\underset{\sim}{q}_{-R}) \tag{3.1.38}$$

and

$$\mathcal{R}_{I}(\alpha\beta\gamma)\, S_{\lambda\mu}(\underset{\sim}{q})\, \mathcal{R}_{I}^{+}(\alpha\beta\gamma) = \sum_{\mu'=-1}^{1} D^{1}_{\mu'\mu}(\alpha\beta\gamma)\, S_{\lambda\mu'}(\underset{\sim}{q}) \tag{3.1.39}$$

i.e., $S_{\lambda\mu}(\underset{\sim}{q})$ transforms as a vector under rotations in both spin and isospin space. Indeed, we may interpret this operator as a fermion spin-isospin transfer operator. For example, $S_{11}(\underset{\sim}{q})$ acting on a one neutron state with spin projection m=-1/2, turns it into a one proton state with spin up. The spin z-axis projection is increased one unit (λ=+1), and the isospin z-axis projection is also increased (μ=+1).

The trilinear interaction H_1 may be expressed in terms of $S_{\lambda\mu}(\underset{\sim}{q})$ as follows. We write $Y_{1\lambda}^{*}(\underset{\sim}{q})$ in terms of the spherical components of the unit vector $\underset{\sim}{q}$ (see Appendix C) as

$$Y_{1\lambda}^{*}(\underset{\sim}{q}) = \sqrt{\frac{3}{4\pi}}\, (-)^{\lambda}\, \hat{q}_{-\lambda} \tag{3.1.40}$$

and substitute this into (3.1.36). The result is

$$H_1 = i\int d^3q\; h(q)\; \hat{\underset{\sim}{q}} \cdot \underset{\approx}{S}(\underset{\sim}{q}) \cdot \underset{\sim}{B}(\underset{\sim}{q}) + \text{adj}. \tag{3.1.41a}$$

where

$$\hat{\underset{\sim}{q}} \cdot \underset{\approx}{S}(\underset{\sim}{q}) \cdot \underset{\sim}{B}(\underset{\sim}{q}) = \sum_{\lambda\mu} (-)^{\lambda}\, \hat{q}_{-\lambda}\, S_{\lambda\mu}(\underset{\sim}{q})\, B_{\mu}(\underset{\sim}{q}) \tag{3.1.41b}$$

Using (A.2), (B.2), and (A.6), we find

$$\mathcal{R}(\alpha\beta\gamma)\, \hat{\underset{\sim}{q}} \cdot \underset{\approx}{S}(\underset{\sim}{q}) \cdot \underset{\sim}{B}(\underset{\sim}{q})\, \mathcal{R}^{+}(\alpha\beta\gamma) = \hat{\underset{\sim}{q}}_{-R} \cdot \underset{\approx}{S}(\underset{\sim}{q}_{-R}) \cdot \underset{\sim}{B}(\underset{\sim}{q}_{-R}) \tag{3.1.42}$$

and

$$\mathcal{R}_I(\alpha\beta\gamma)\, \hat{\underset{\sim}{q}} \cdot \underset{=}{S}(\underset{\sim}{q}) \cdot \underline{B}(\underset{\sim}{q})\, \mathcal{R}_I^{+}(\alpha\beta\gamma) = \hat{\underset{\sim}{q}} \cdot \underset{=}{S}(\underset{\sim}{q}) \cdot \underline{B}(\underset{\sim}{q}) \qquad (3.1.43)$$

The expression (3.1.41) for H_1 is manifestly invariant under isospin space and spatial rotations. Note its similarity to the more conventional $\underline{\sigma} \cdot \underset{\sim}{q}\, \tau_\mu$ form for the nucleon-pion trilinear interaction, such as the Chew-Low interaction discussed in Schweber (1961, pp.376,377). Indeed, the invariance requirements considered in this Section determine all but the arbitrary real function h(q) for the nucleon-pion trilinear interaction.

3.2 The Trilinear Vertex Function in the Cloudy Bag Model

The Cloudy Bag Model of Théberge, Thomas, and Miller (1980) involves a massive pion field in interaction with massless quark fields. The pion field couples to the quark fields only on a spherical surface of radius R ('the bag'). In this model the bare nucleon and delta particles are composed of three massless up and down quarks confined to the bag. Using the known, lowest order bag model quark wave functions, and assuming that the pion field is small, Théberge *et al.* have re-expressed the pion-quark interaction in terms of pion-baryon trilinear interactions. The resulting Hamiltonian is a combination of the Lee model Hamiltonian (see Section 4.3) and the trilinear Hamiltonian discussed in Section 3.1. For further details on the derivation and consequences of the Cloudy Bag Model, we refer the reader to the paper of Thebérge *et al.* (1980), and references therein.

In Appendix F we relate the NNπ piece of the Cloudy Bag Model trilinear interaction to our equation (3.1.36). We find the trilinear vertex function, which we write as $h_{CBM}(q)$, is given by

$$h_{CBM}(q) = \sqrt{\frac{3c}{\pi}} \, \frac{f_0}{m_\pi} \, q \, \frac{u_N(q)}{\sqrt{\mathcal{E}_\pi(q)}} \tag{3.2.1}$$

f_0 is the NNπ coupling constant and

$$u_N(q) = \frac{3 j_1(qR/\hbar)}{qR/\hbar} \tag{3.2.2}$$

where $j_1(qR/\hbar)$ is a spherical Bessel function of order one. The 'form factor' $u_N(q)$ takes into account the finite extent of the bare nucleon; it has the property that

$$\lim_{R\to 0} u_N(q) = 1 \tag{3.2.3}$$

Therefore,

$$\lim_{R\to 0} h_{CBM}(q) = \sqrt{\frac{3c}{\pi}} \, \frac{f_0}{m_\pi} \, q \, \frac{1}{\sqrt{\mathcal{E}_\pi(q)}} \tag{3.2.4}$$

In the limit R$\to$0, the Cloudy Bag Model vertex function becomes exactly the vertex function for the Chew-Low interaction discussed in Schweber (1961, p.374).

3.3 Physical Bosons and Fermions

The Hamiltonian (3.1.1) has the following feature:

$$H\, B_\mu^+(\underline{p})|0\rangle = \mathcal{E}_{Bo}(\underline{p})\; B_\mu^+(\underline{p})|0\rangle \tag{3.3.1}$$

$$H\, F_{m\mu}^+(\underline{p})|0\rangle \neq \mathcal{E}_F(\underline{p})\; F_{m\mu}^+(\underline{p})|0\rangle \tag{3.3.2}$$

for $\mathcal{E}_F(\underline{p})$ an arbitrary function. $B_\mu^+(\underline{p})|0\rangle$ is an eigenket of the Hamiltonian and therefore corresponds to a 'physical boson', i.e., the elementary boson of the theory is a physical particle with mass $m_B = m_{Bo}$. However, $F_{m\mu}^+(\underline{p})|0\rangle$ is not an eigenket of the Hamiltonian. The elementary fermion of the theory is not a physical fermion; we say that $F^\dagger$ creates a 'bare' fermion. The trilinear interaction is thus not explicitly a physical particle interaction. In the next Chapter we shall develop a technique, the dressing transformation, for obtaining physical particle creators and a Hamiltonian which explicitly involves physical particle interactions.

Chapter 4 The Dressing Transformation and Some Simple Applications

As we saw in the previous Chapter, the trilinear interaction does not explicitly involve physical fermions and bosons. We seek a transformation on the fundamental dynamical variables of the theory which will lead to a Hamiltonian expressed in terms of physical particle operators. This transformation must be unitary and possess certain invariance properties in order to preserve the basic commutation relations and transformation properties of the particle creators and annihilators. We will see that the bare particles acquire a composite structure via the transformation; thus it is called a dressing transformation.

In the following Sections, we will formulate the dressing transformation explicitly and apply it to two simple, soluble models - the scalar field model and the Lee model.

4.1 The General Dressing Transformation

Consider the unitary operator

$$\mathcal{U} = e^{D} \qquad \text{where} \qquad D^{\dagger} = -D \tag{4.1.1}$$

and where D is invariant under translations, spatial rotations, space inversion, time reversal, and rotations in isospin space. The operator D will be specified further below; it will be a function of the fundamental dynamical variables, so we write

$$D = D(F,B) \tag{4.1.2}$$

Let

$$\tilde{F}^{si+}_{m\mu}(p) = \mathcal{U}\, F^{si+}_{m\mu}(p)\, \mathcal{U}^{+} \tag{4.1.3}$$

and

$$\tilde{B}^{si}_{m\mu}(p) = \mathcal{U}\, B^{si}_{m\mu}(p)\, \mathcal{U}^{+} \tag{4.1.4}$$

where F and B are the fundamental fermion and boson destruction operators, respectively, defined in Section 2.1. Throughout the rest of this Section, we will omit the spin and isospin labels on the particle operators, as they only complicate the notation and change none of the results. We use the symbol $\sim$ to denote all transformed operators.

Because U is a unitary operator, $\tilde{F}$ and $\tilde{B}$ obey the same commutation relations as do F and B. (See equations (2.1.5) - (2.1.9)). Moreover, because of the invariance properties of U, $\tilde{F}$ and $\tilde{B}$ will also transform under translations, spatial rotations, space inversion, time reversal and rotations in isospin space according to the transformation laws in Section 2.2.

For any operator A = A(F,B) in the Fock space, we have

$$\mathcal{U}\, A(F,B)\, \mathcal{U}^{+} = A(\tilde{F},\tilde{B}) \tag{4.1.5}$$

In particular,

$$D(\tilde{F},\tilde{B}) = \mathcal{U}\, D(F,B)\, \mathcal{U}^{+} = D(F,B) \tag{4.1.6}$$

and using (4.1.5), (4.1.1), and (4.1.6), we have

$$H(F,B) = \mathcal{U}^{+} H(\tilde{F},\tilde{B})\, \mathcal{U} = e^{-D(\tilde{F},\tilde{B})}\, H(\tilde{F},\tilde{B})\, e^{D(\tilde{F},\hat{B})} \equiv \tilde{H}(\tilde{F},\tilde{B}) \tag{4.1.7}$$

Given D, equation (4.1.7) gives the Hamiltonian as some new function $\tilde{H}(\tilde{F},\tilde{B})$ of the new, independent fundamental dynamical variables $\tilde{F}$ and $\tilde{B}$. Indeed, we can determine the functional form of $\tilde{H}$, expressed below as a function of the dummy variables F and B, using

$$\tilde{H}(F,B) = e^{-D(F,B)} \; H(F,B) \; e^{D(F,B)}$$

$$= H + [H,D] + \frac{1}{2!}[[H,D],D] + \ldots \qquad (4.1.8)$$

Our strategy will be to calculate $\tilde{H}$ from this equation; from (4.1.7) we know that the Hamiltonian H(F,B) is equivalent to $\tilde{H}(\tilde{F},\tilde{B})$ where $\tilde{F}$ and $\tilde{B}$ are the transformed operators.

The total momentum operator is

$$\underline{P}(F,B) = \tilde{\underline{P}}(\tilde{F},\tilde{B}) \qquad (4.1.9)$$

where

$$\tilde{\underline{P}}(F,B) = e^{-D(F,B)} \; \underline{P}(F,B) \; e^{D(F,B)} \qquad (4.1.10)$$

Since D(F,B) is translationally invariant, i.e.,

$$[\underline{P}, D] = 0 \qquad (4.1.11)$$

equations (4.1.9) and (4.1.10) imply

$$\tilde{\underline{P}}(F,B) = \underline{P}(F,B) \qquad (4.1.12)$$

Now suppose that we can write the Hamiltonian as

$$H = H_0 + \lambda H_1 \qquad (4.1.13)$$

and that D can be expanded in a perturbation series in λ:

$$D = \sum_{n=1}^{\infty} \lambda^n D_n \qquad (4.1.14)$$

Equation (4.1.8) then becomes the following perturbation series for $\tilde{H}$:

$$\tilde{H} = H_0 + \lambda \{ H_1 + [H_0, D_1] \}$$
$$+ \lambda^2 \{ [H_1, D_1] + \tfrac{1}{2} [[H_0, D_1], D_1] + [H_0, D_2] \} + \ldots \qquad (4.1.15)$$

We now demand that the transformed operators $\tilde{F}^{\dagger}$ and $\tilde{B}^{\dagger}$ create physical particles. That is, we require $\tilde{F}^{\dagger}|0\rangle$ and $\tilde{B}^{\dagger}|0\rangle$ to be eigenkets of the Hamiltonian $H(F,B)$, or equivalently of $\tilde{H}(\tilde{F},\tilde{B})$. That is,

$$\tilde{H}(\tilde{F},\tilde{B})\, \tilde{F}^{\dagger}(\underline{p})|0\rangle = \mathcal{E}_F(\underline{p})\, \tilde{F}^{\dagger}(\underline{p})|0\rangle \qquad (4.1.16)$$

$$\tilde{H}(\tilde{F},\tilde{B})\, \tilde{B}^{\dagger}(\underline{p})|0\rangle = \mathcal{E}_B(\underline{p})\, \tilde{B}^{\dagger}(\underline{p})|0\rangle \qquad (4.1.17)$$

where $\mathcal{E}_F(\underline{p})$ and $\mathcal{E}_B(\underline{p})$ are some functions of $\underline{p}$. Equations (4.1.16) and (4.1.17) will hold if, apart from the terms $F^{\dagger}F$ and $B^{\dagger}B$, there are no terms in $\tilde{H}(F,B)$, where again F and B are dummy variables, which contain only one fermion or boson annihilator, i.e., terms of the form

$$F^{\dagger}FB^{\dagger}, \quad F^{\dagger}FB^{\dagger}B^{\dagger}, \quad \text{etc.} \qquad (4.1.18)$$

$\tilde{H}(F,B)$ may, however, contain terms of the form

$$F^{\dagger}F^{\dagger}FF, \quad B^{\dagger}B^{\dagger}BB, \quad F^{\dagger}B^{\dagger}FB, \quad F^{\dagger}F^{\dagger}FFB^{\dagger}, \quad \text{etc.} \qquad (4.1.19)$$

which correspond, respectively, to direct fermion-fermion, boson-boson, and fermion-boson interactions, and boson production on two fermions. These terms appear in the phenomenological Hamiltonian discussed in Hsieh (1978, p.31).

We thus choose the D_n to eliminate terms of the form (4.1.18) from $\tilde{H}(F,B)$ given by (4.1.15). This may be done order by order in λ. For the case that H_1 is trilinear, we choose D_1 such that

$$[H_0, D_1] = -H_1 \qquad (4.1.20)$$

The expression (4.1.15) then becomes

$$\tilde{H} = H_o + \lambda^2 \left\{ \tfrac{1}{2} [H_1, D_1] + [H_o, D_2] \right\}$$
$$+ \lambda^3 \left\{ \tfrac{1}{3} [[H_1, D_1], D_1] + \tfrac{1}{2} [[H_o, D_2], D_1] + \tfrac{1}{2} [H_1, D_2] + [H_o, D_3] \right\} + \ldots \qquad (4.1.21)$$

Next, knowing D_1, D_2 is chosen to eliminate terms of the form (4.1.18) to second order in λ, and so on.

The operator D constructed to satisfy the above conditions is called a <u>dressing operator</u>. The corresponding unitary transformation is a <u>dressing transformation</u>; the transformed operators $\tilde{F}^{\dagger}$ and $\tilde{B}^{\dagger}$, which create physical particles, are called dressed creators.

We now go on to illustrate the dressing transformation with some simple examples.

<u>4.2 The Scalar Field Model</u>

The scalar field Hamiltonian is a trilinear one involving fermions and bosons. It is very similar to the Hamiltonian discussed in Chapter 3, except that spin and isospin are not included in the scalar field theory. The Hamiltonian is

$$H = H_o + \lambda H_1 \qquad (4.2.1a)$$

$$H_o = \int d^3p \, [\varepsilon_{Fo}(\underline{p}) F^{\dagger}(\underline{p}) F(\underline{p}) + \varepsilon_{Bo}(\underline{p}) B^{\dagger}(\underline{p}) B(\underline{p})] \qquad (4.2.1b)$$

$$H_1 = \int d^3p \, d^3q \, h(\underline{q}) \, [F^{\dagger}(\underline{p}) F(\underline{p}-\underline{q}) B(\underline{q}) + adj.] \qquad (4.2.1c)$$

$$\mathcal{E}_{F_0}(\underset{\sim}{p}) = [p^2c^2 + m_{F_0}^2c^4]^{1/2} \tag{4.2.2}$$

$$\mathcal{E}_{B_0}(\underset{\sim}{p}) = [p^2c^2 + m_{B_0}^2c^4]^{1/2} \tag{4.2.3}$$

where m_{F_0} and m_{B_0} are the masses of the elementary fermion and boson, respectively.

The operators F and B obey the commutation rules (2.1.5) - (2.1.9), omitting all reference to spin and isospin; h(q) is chosen to be a real function independent of the fermion momentum.

We may picture the trilinear interaction (4.2.1c) as in Figure 2:

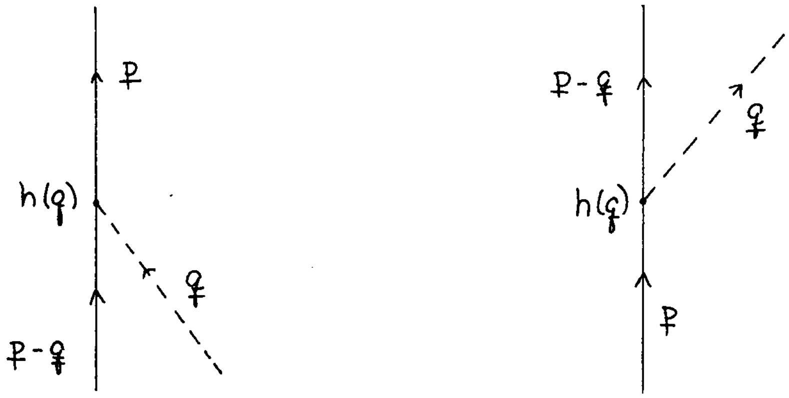

Fig. 2 The Scalar Field Model Interaction

The total momentum operator for the system is

$$\underline{P} = \int d^3p\ \underset{\sim}{p}\ [F^{+}(\underset{\sim}{p})F(\underset{\sim}{p}) + B^{+}(\underset{\sim}{p})B(\underset{\sim}{p})] \tag{4.2.4}$$

The fermion number operator is

$$N = \int d^3p\ F^{+}(\underset{\sim}{p})\ F(\underset{\sim}{p}) \tag{4.2.5}$$

Both $\underline{P}$ and N commute with the Hamiltonian (4.2.1).

We note that $B^\dagger(\underline{p})|0\rangle$ is an eigenket of the Hamiltonian, with eigenvalue $\mathcal{E}_{B_0}(\underline{p})$, while $F^\dagger(\underline{p})|0\rangle$ is not an eigenket. Therefore we may take $m_{B_0} = m_B$, the mass of the physical boson. We seek a dressing transformation as outlined in Section 4.1. The following D_1 satisfies the required invariance properties, and equation (4.1.20):

$$D_1 = \int d^3p\, d^3q\;\; d_1(\underline{p},\underline{q})\left[F^\dagger(\underline{p})\,F(\underline{p}-\underline{q})\,B(\underline{q}) - \text{adj.}\right] \tag{4.2.6a}$$

where

$$d_1(\underline{p},\underline{q}) = \frac{h(\underline{q})}{\Delta(\underline{p},\underline{q})} \tag{4.2.6b}$$

and

$$\Delta(\underline{p},\underline{q}) = \mathcal{E}_{F_0}(\underline{p}-\underline{q}) + \mathcal{E}_B(\underline{q}) - \mathcal{E}_{F_0}(\underline{p}) \tag{4.2.6c}$$

Note that this operator has a structure identical to H_1, the term it must eliminate. We will find this to be a general property of the dressing transformation.

We now can compute $[H_1, D_1]$ and find

$$\begin{aligned}
[H_1,D_1] = &- \int d^3p\, d^3q\, \frac{h^2(\underline{q})}{\Delta(\underline{p},\underline{q})}\, F^\dagger(\underline{p})\, F(\underline{p}) \\
&+ \int d^3p\, d^3q\, d^3q' \left\{ \frac{h^2(\underline{q})}{\Delta(\underline{q}',\underline{q})}\, F^\dagger(\underline{p})\, F^\dagger(\underline{q}'-\underline{q})\, F(\underline{p}-\underline{q})\, F(\underline{q}') \right. \\
&+ h(\underline{q})h(\underline{q}')\left[\frac{1}{\Delta(\underline{p},\underline{q}')} - \frac{1}{\Delta(\underline{p}-\underline{q},\underline{q}')}\right] F^\dagger(\underline{p}-\underline{q}')\, B^\dagger(\underline{q}')\, B(\underline{q})\, F(\underline{p}-\underline{q}) \\
&\left. + h(\underline{q})\, h(\underline{q}')\left[\frac{1}{\Delta(\underline{p},\underline{q}')} - \frac{1}{\Delta(\underline{p}+\underline{q},\underline{q}')}\right] F^\dagger(\underline{p}+\underline{q})\, F(\underline{p}-\underline{q}')\, B(\underline{q})\, B(\underline{q}') \right\} \\
&+ \text{adj.}
\end{aligned} \tag{4.2.7}$$

The last term in $[H_1, D_1]$, being of the form (4.1.18), must be eliminated from the dressed Hamiltonian given by (4.1.21). D_2 can be constructed to accomplish this.

The momentum $\underline{P}$ is given by (4.1.9) and (4.1.12) to be

$$\underline{P} = \int d^3p\, \underline{p} \left[\tilde{F}^+(\underline{p})\, \tilde{F}(\underline{p}) + \tilde{B}^+(\underline{p})\, \tilde{B}(\underline{p}) \right] \tag{4.2.8}$$

The Hamiltonian, dressed to second order, is given by (4.1.7) and (4.1.21) as

$$H(F,B) = \tilde{H}(\tilde{F},\tilde{B}) = T + V_{FF} + V_{FB} \tag{4.2.9}$$

$$T = \int d^3p \left[\mathcal{E}_F(\underline{p})\, \tilde{F}^+(\underline{p})\, \tilde{F}(\underline{p}) + \mathcal{E}_B(\underline{p})\, \tilde{B}^+(\underline{p})\, \tilde{B}(\underline{p}) \right] \tag{4.2.10}$$

$$V_{FF} = \frac{1}{2}\int d^3k\, d^3k'\, d^3K\, \mathcal{V}_{FF}(\underline{k},\underline{k}',\underline{K})\, \tilde{F}^+(\tfrac{1}{2}\underline{K}+\underline{k})\, \tilde{F}^+(\tfrac{1}{2}\underline{K}-\underline{k})\, \tilde{F}(\tfrac{1}{2}\underline{K}-\underline{k}')\, \tilde{F}(\tfrac{1}{2}\underline{K}+\underline{k}') \tag{4.2.11}$$

$$V_{FB} = \frac{1}{2}\int d^3k\, d^3k'\, d^3K\, \mathcal{V}_{FB}(\underline{k},\underline{k}',\underline{K})\, \tilde{F}^+(\tfrac{1}{2}\underline{K}+\underline{k})\, \tilde{B}^+(\tfrac{1}{2}\underline{K}-\underline{k})\, \tilde{B}(\tfrac{1}{2}\underline{K}-\underline{k}')\, \tilde{F}(\tfrac{1}{2}\underline{K}+\underline{k}') \tag{4.2.12}$$

where

$$\mathcal{E}_B(\underline{p}) = \mathcal{E}_{B0}(\underline{p}) \tag{4.2.13}$$

$$\mathcal{E}_F(\underline{p}) = \mathcal{E}_{F0}(\underline{p}) - \lambda^2 \int d^3q\, \frac{h^2(\underline{q})}{\mathcal{E}_{F0}(\underline{p}-\underline{q}) + \mathcal{E}_B(\underline{q}) - \mathcal{E}_{F0}(\underline{p})} \tag{4.2.14}$$

$$\mathcal{V}_{FF}(\underline{k},\underline{k}',\underline{K}) = \frac{-\lambda^2 h^2(|\underline{k}-\underline{k}'|)}{\mathcal{E}_{F0}(\frac{1}{2}\underline{K}-\underline{k}) + \mathcal{E}_B(\underline{k}-\underline{k}') - \mathcal{E}_{F0}(\frac{1}{2}\underline{K}-\underline{k}')} + \underline{k} \leftrightarrow \underline{k}' \tag{4.2.15}$$

$$\mathcal{V}_{FB}(\underline{k},\underline{k}',\underline{K}) = -\lambda^2\, h(|\tfrac{1}{2}\underline{K}-\underline{k}'|)\, h(|\tfrac{1}{2}\underline{K}-\underline{k}|)\, \cdot$$

$$\cdot \left[\frac{1}{\mathcal{E}_{F0}(\underline{k}+\underline{k}') + \mathcal{E}_B(\frac{1}{2}\underline{K}-\underline{k}) - \mathcal{E}_{F0}(\frac{1}{2}\underline{K}+\underline{k}')} - \frac{1}{\mathcal{E}_{F0}(\frac{1}{2}\underline{K}+\underline{k}) + \mathcal{E}_B(\frac{1}{2}\underline{K}-\underline{k}) - \mathcal{E}_{F0}(\underline{K})} \right] + \underline{k} \leftrightarrow \underline{k}' \tag{4.2.16}$$

Referring to the discussion in Appendix E (eq. (E.7)), we recognize $\mathcal{V}_{FF}$ and $\mathcal{V}_{FB}$ as the matrix elements of the fermion-fermion and fermion-boson momentum space potentials. These functions are pictured in Figures 3 and 4:

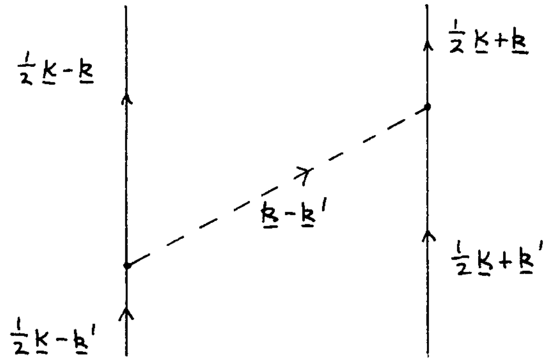

Fig. 3 The Fermion-Fermion Potential $\mathcal{V}_{FF}(\underline{k},\underline{k}',\underline{K})$

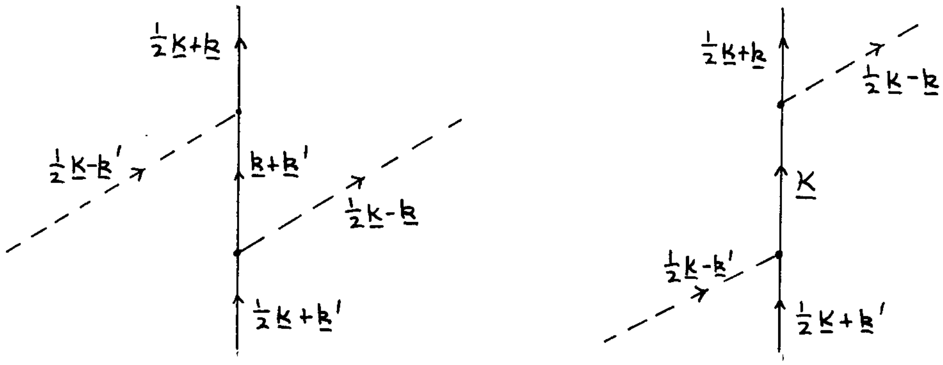

Fig. 4 The Fermion-Boson Potential $\mathcal{V}_{FB}(\underline{k},\underline{k}',\underline{K})$

Note that, to second order, there are two different mechanisms (Fig. 4) by which FB scattering can take place; these correspond to the two terms in $\mathcal{V}_{FB}$.

Note also that

$$\tilde{H}(\tilde{F},\tilde{B})\,\tilde{F}^{\dagger}(\underline{p})|0\rangle = \varepsilon_F(\underline{p})\,\tilde{F}^{\dagger}(\underline{p})|0\rangle \qquad (4.2.17)$$

and

$$\tilde{H}(\tilde{F},\tilde{B})\,\tilde{B}^{\dagger}(\underline{p})|0\rangle = \varepsilon_B(\underline{p})\,\tilde{B}^{\dagger}(\underline{p})|0\rangle \qquad (4.2.18)$$

where $\mathcal{E}_F(\underline{p})$ is the 'renormalized' fermion energy, given to second order by (4.2.14). Thus $\tilde{F}^\dagger(\underline{p})$ now creates a physical fermion. From (4.1.3) and (4.1.4) we see

$$\tilde{F}^\dagger(\underline{p}) = F^\dagger(\underline{p}) - [F^\dagger(\underline{p}), D] + \frac{1}{2!}[[F^\dagger(\underline{p}), D], D] + \ldots \tag{4.2.19}$$

$$\tilde{B}^\dagger(\underline{p}) = B^\dagger(\underline{p}) - [B^\dagger(\underline{p}), D] + \frac{1}{2!}[[B^\dagger(\underline{p}), D], D] + \ldots \tag{4.2.20}$$

Calculation of these dressed creators shows that $\tilde{F}^\dagger(\underline{p})$ is given by an infinite series of terms; the first term is the bare fermion creator $F^\dagger$, the second term involves $F^\dagger B^\dagger$, the third term involves $F^\dagger B^\dagger B^\dagger$, and so on. Thus one says that the physical fermion consists of a bare fermion surrounded by a 'cloud' of bosons. Also, we find that $\tilde{B}^\dagger$ equals $B^\dagger$ plus a series of terms which involve fermion operators, all of which give zero when acting on the vacuum state. Thus $\tilde{B}^\dagger(\underline{p})|0\rangle = B^\dagger(\underline{p})|0\rangle$, although $\tilde{B}^\dagger(\underline{p}) \neq B^\dagger(\underline{p})$. The dressing transformation has not changed the physical one boson state.

If we make the simplifying assumption that

$$\mathcal{E}_{F_0}(\underline{p}) = m_{F_0} c^2 \tag{4.2.21}$$

i.e., the fermion energy is independent of its momentum, we discover several interesting results. In this case, (4.2.6c) simplifies to

$$\Delta(\underline{p}, \underline{q}) = \mathcal{E}_B(\underline{q}) \tag{4.2.22}$$

and as a result the final two terms in eq. (4.2.7) vanish. Moreover, now

$$[[H_1, D_1], D_1] = 0 \tag{4.2.23}$$

Thus $[H_1, D_1]$ now contains no terms of the form (4.1.18), so we may take $D_2 = 0$. In turn (4.2.23) implies that $D_3 = 0$. Indeed, $D_n = 0$ for $n>1$. We may write the dressing transformation exactly as $D = D_1$. The model is thus said to be soluble.

Note that when (4.2.21) holds, the dressed Hamiltonian no longer contains a fermion-boson scattering term. It does, however, have a fermion mass renormalization and a fermion-fermion scattering term. We find equations (4.2.14) and (4.2.15) become, respectively,

$$m_F c^2 = m_{F_0} c^2 - \lambda^2 \int d^3q \, \frac{h^2(q)}{\varepsilon_B(q)} \tag{4.2.24}$$

and

$$\mathcal{V}_{FF}(q) = \frac{-2\lambda^2 h^2(q)}{\varepsilon_B(q)} \tag{4.2.25}$$

The corresponding coordinate space fermion-fermion potential may be obtained by from equation (E.11). It is

$$\mathcal{V}_{FF}(\underline{r}) = \int d^3q \; e^{i q \cdot r/\hbar} \, \mathcal{V}_{FF}(q) \tag{4.2.26}$$

For the specific choice

$$h(q) = \frac{1}{\sqrt{\varepsilon_B(q)}} \tag{4.2.27}$$

for the vertex function, the integral in (4.2.26) can be evaluated. We obtain a coordinate space Yukawa potential, namely

$$\mathcal{V}_{FF}(r) = -4\pi^2\lambda^2 \hbar/c \; \frac{e^{-m_B c r/\hbar}}{r} \tag{4.2.28}$$

The integral (4.2.24) is mathematically divergent; equation (4.2.24) is taken as a definition of the bare mass m_{F_0}, in terms of the physical mass m_F and the (infinite) integral.

In conclusion, the scalar field model has illustrated some interesting features of the trilinear interaction and the dressing transformation. The physical particle interactions which were implicit in the trilinear Hamiltonian (4.2.1) have been made explicit through the dressing transformation. Physical particle potentials can be determined from the trilinear vertex functions. The physical particle creators can be calculated and we find that the physical fermion corresponds to a bare fermion surrounded by a cloud of bosons. Finally, we have shown that the scalar field model is soluble only when the approximation (4.2.21) is made. For the general case that (4.2.2) holds, the dressing operator is given by an infinite series of terms and thus can only be determined by a perturbative procedure such as we have developed.

4.3 The Lee Model

The Lee model describes a trilinear interaction involving two distinguishable fermions and one boson. Like the scalar field model, the spin and isospin of the particles is neglected. The Lee Hamiltonian is:

$$H = H_0 + \lambda H_1 \tag{4.3.1a}$$

$$H_0 = \int d^3p \left[\sum_{\alpha=1}^{2} \mathcal{E}_{\alpha 0}(\underline{p}) F_\alpha^\dagger(\underline{p}) F_\alpha(\underline{p}) + \mathcal{E}_{B0}(\underline{p}) B^\dagger(\underline{p}) B(\underline{p}) \right] \tag{4.3.1b}$$

$$H_1 = \int d^3p \, d^3q \; h(\underline{q}) \left[F_2^\dagger(\underline{p}) F_1(\underline{p}-\underline{q}) B(\underline{q}) + \mathrm{adj.} \right] \tag{4.3.1c}$$

The momentum operator is

$$\underline{P} = \int d^3p \; \underline{p} \left[\sum_{\alpha=1}^{2} F_\alpha^\dagger(\underline{p}) F_\alpha(\underline{p}) + B^\dagger(\underline{p}) B^\dagger(\underline{p}) \right] \tag{4.3.2}$$

$F_1^\dagger$, $F_2^\dagger$, $B^\dagger$ are the creation operators for particles traditionally called N, V, and θ, respectively.

$$\mathcal{E}_{\alpha o}(\underline{p}) = \left[p^2c^2 + m_{\alpha o}^2 c^4 \right]^{1/2} \qquad \alpha = 1,2 \tag{4.3.3}$$

is the energy of the elementary fermion, and

$$\mathcal{E}_{Bo}(\underline{p}) = \left[p^2c^2 + m_{Bo}^2 c^4 \right]^{1/2} \tag{4.3.4}$$

is the energy of the elementary boson.

$h(\underline{q})$ is chosen to be a real function. The Lee model is also solvable when $h = h(\underline{p},\underline{q})$ but this only complicates the notation and changes none of the essential results.

Note that momentum is conserved by the interaction, as is the total number of N and V particles and the difference between the number of N and θ particles.

The interaction H_I may be pictured as in Figure 5:

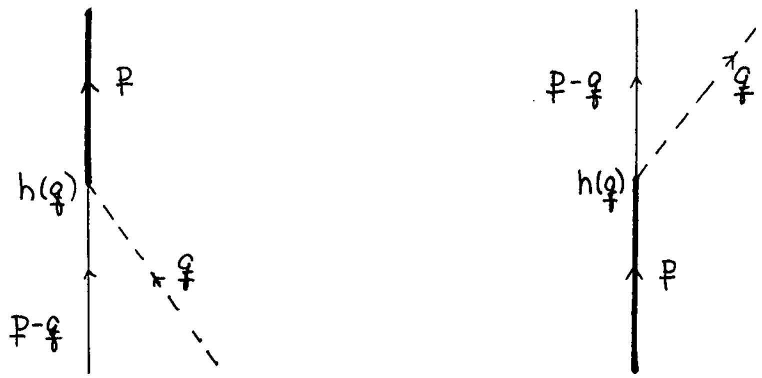

Fig. 5 The Lee Model Interaction
Thick solid lines are V particles, solid lines are N particles, and dashed lines are θ particles.

The creation and destruction operators obey the familiar fermion and boson commutation rules:

$$\{F_\alpha(\underset{\sim}{p}), F_{\alpha'}^{\dagger}(\underset{\sim}{p}')\} = \delta(\underset{\sim}{p}-\underset{\sim}{p}')\,\delta_{\alpha\alpha'} \tag{4.3.5}$$

$$\{F_\alpha(\underset{\sim}{p}), F_{\alpha'}(\underset{\sim}{p}')\} = 0 \tag{4.3.6}$$

$$[B(\underset{\sim}{p}), B^{\dagger}(\underset{\sim}{p}')] = \delta(\underset{\sim}{p}-\underset{\sim}{p}') \tag{4.3.7}$$

$$[B(\underset{\sim}{p}), B(\underset{\sim}{p}')] = 0 \tag{4.3.8}$$

$$[F_\alpha(\underset{\sim}{p}), B(\underset{\sim}{p}')] = [F_\alpha(\underset{\sim}{p}), B^{\dagger}(\underset{\sim}{p}')] = 0 \tag{4.3.9}$$

Because

$$H F_1^{\dagger}(\underset{\sim}{p})|0\rangle = \mathcal{E}_{10}(\underset{\sim}{p})\, F_1^{\dagger}(\underset{\sim}{p})|0\rangle \tag{4.3.10}$$

and

$$H B^{\dagger}(\underset{\sim}{p})|0\rangle = \mathcal{E}_{B0}(\underset{\sim}{p})\, B^{\dagger}(\underset{\sim}{p})|0\rangle \tag{4.3.11}$$

$F_1^{\dagger}$ and $B^{\dagger}$ create physical particles. Therefore, $m_{10} = m_1$, the mass of the physical N particle, and $m_{B0} = m_B$, the mass of the physical boson.

However, $F_2^{\dagger}(\underset{\sim}{p})|0\rangle$ is not an eigenket of the Hamiltonian H so $F_2^{\dagger}$ does not create physical particles. We now perform a dressing transformation according to the prescription in Section 4.1.

Equation (4.1.20) is satisfied by

$$D_1 = \int d^3p\, d^3q\;\; d_1(\underset{\sim}{p},\underset{\sim}{q})\,[F_2^{\dagger}(\underset{\sim}{p})\, F_1(\underset{\sim}{p}-\underset{\sim}{q})\, B(\underset{\sim}{q}) - \text{adj.}] \tag{4.3.12a}$$

where

$$d_1(\underset{\sim}{p},\underset{\sim}{q}) = \frac{h(\underset{\sim}{q})}{\Delta(\underset{\sim}{p},\underset{\sim}{q})} \tag{4.3.12b}$$

and

$$\Delta(\underset{\sim}{p},\underset{\sim}{q}) = \mathcal{E}_1(\underset{\sim}{p}-\underset{\sim}{q}) + \mathcal{E}_B(\underset{\sim}{q}) - \mathcal{E}_{20}(\underset{\sim}{p}) \qquad (4.3.12c)$$

Note that this is not the same $\Delta(\underset{\sim}{p},\underset{\sim}{q})$ as is used in the previous Section. To ensure that $\Delta(\underset{\sim}{p},\underset{\sim}{q})$ does not vanish, we require that $m_{20}<m_1+m_B$, i.e., the V particle is stable against decay into N + θ.

A simple computation now shows that

$$\begin{aligned}[H_1,D_1] = &- \int d^3p\, d^3q\, \frac{h^2(\underset{\sim}{q})}{\Delta(\underset{\sim}{p},\underset{\sim}{q})} F_2^+(\underset{\sim}{p}) F_2(\underset{\sim}{p}) \\
&+ \int d^3p\, d^3q\, d^3q' \Big\{ \frac{h^2(\underset{\sim}{q})}{\Delta(\underset{\sim}{p},\underset{\sim}{q})} F_2^+(\underset{\sim}{p}) F_1^+(\underset{\sim}{q}'-\underset{\sim}{q}) F_1(\underset{\sim}{p}-\underset{\sim}{q}) F_2(\underset{\sim}{q}') \\
&- \frac{h(\underset{\sim}{q})h(\underset{\sim}{q}')}{\Delta(\underset{\sim}{p},\underset{\sim}{q})} F_2^+(\underset{\sim}{p}) B^+(\underset{\sim}{q}') B(\underset{\sim}{q}) F_2(\underset{\sim}{q}'-\underset{\sim}{q}+\underset{\sim}{p}) \\
&+ \frac{h(\underset{\sim}{q})\, h(\underset{\sim}{q}')}{\Delta(\underset{\sim}{p},\underset{\sim}{q})} F_1^+(\underset{\sim}{p}-\underset{\sim}{q}') B^+(\underset{\sim}{q}') B(\underset{\sim}{q})\, F_1(\underset{\sim}{p}-\underset{\sim}{q}) \Big\} \\
&+ \text{adj.}\end{aligned} \qquad (4.3.13)$$

Since $[H_1,D_1]$ contains no terms of the form (4.1.18), we take

$$D_2 = 0 \qquad (4.3.14)$$

A computation of $[[H_1,D_1],D_1]$ indicates that we must choose

$$D_3 = \int d^3p\, d^3q\;\; d_3(\underset{\sim}{p},\underset{\sim}{q}) \left[F_2^+(\underset{\sim}{p}) F_1(\underset{\sim}{p}-\underset{\sim}{q}) B(\underset{\sim}{q}) - \text{adj.}\right] \qquad (4.3.15a)$$

where

$$d_3(\underset{\sim}{p},\underset{\sim}{q}) = - \int d^3q'\, \frac{h^2(\underset{\sim}{q}')\, h(\underset{\sim}{q})}{\Delta(\underset{\sim}{p},\underset{\sim}{q})\,\Delta(\underset{\sim}{p},\underset{\sim}{q}')} \left[\frac{1}{\Delta(\underset{\sim}{p},\underset{\sim}{q})} + \frac{1}{3}\, \frac{1}{\Delta(\underset{\sim}{p},\underset{\sim}{q}')} \right] \qquad (4.3.15b)$$

This eliminates all unsuitable terms of the form (4.1.18) from the dressed Hamiltonian, to order λ^3.

Note that D_1 and D_3 have an identical trilinear structure. Indeed it has been previously shown (Greenberg and Schweber (1958), Piskunov (1974)) that for the Lee model D can be determined exactly and has a trilinear structure. We give such a calculation in Appendix G, and determine there the physical V particle creator $\tilde{F}_2{}^{\dagger}$.

The dressed Hamiltonian, to second order, is given by (4.1.21) and (4.3.13):

$$\tilde{H}(\tilde{F},\tilde{B}) = T + V_{12} + V_{2B} + V_{1B} \tag{4.3.16}$$

$$T = \int d^3p \left[\sum_{\alpha=1}^{2} \mathcal{E}_\alpha(\underline{p})\, \tilde{F}_\alpha{}^{\dagger}(\underline{p})\, \tilde{F}_\alpha(\underline{p}) + \mathcal{E}_B(\underline{p})\, \tilde{B}^{\dagger}(\underline{p})\, \tilde{B}(\underline{p}) \right] \tag{4.3.17}$$

$$V_{12} = \tfrac{1}{2}\int d^3k\, d^3k'\, d^3K\, \mathcal{V}_{12}(\underline{k},\underline{k}',\underline{K})\, \tilde{F}_2^{\dagger}(\tfrac{1}{2}\underline{K}+\underline{k})\, \tilde{F}_1^{\dagger}(\tfrac{1}{2}\underline{K}-\underline{k})\, \tilde{F}_1(\tfrac{1}{2}\underline{K}-\underline{k}')\, \tilde{F}_2(\tfrac{1}{2}\underline{K}+\underline{k}') \tag{4.3.18}$$

$$V_{2B} = \tfrac{1}{2}\int d^3k\, d^3k'\, d^3K\, \mathcal{V}_{2B}(\underline{k},\underline{k}',\underline{K})\, \tilde{F}_2^{\dagger}(\tfrac{1}{2}\underline{K}+\underline{k})\, \tilde{B}^{\dagger}(\tfrac{1}{2}\underline{K}-\underline{k})\, \tilde{B}(\tfrac{1}{2}\underline{K}-\underline{k}')\, \tilde{F}_2(\tfrac{1}{2}\underline{K}+\underline{k}') \tag{4.3.19}$$

$$V_{1B} = \tfrac{1}{2}\int d^3k\, d^3k'\, d^3K\, \mathcal{V}_{1B}(\underline{k},\underline{k}',\underline{K})\, \tilde{F}_1^{\dagger}(\tfrac{1}{2}\underline{K}+\underline{k})\, \tilde{B}^{\dagger}(\tfrac{1}{2}\underline{K}-\underline{k})\, \tilde{B}(\tfrac{1}{2}\underline{K}-\underline{k}')\, \tilde{F}_1(\tfrac{1}{2}\underline{K}+\underline{k}') \tag{4.3.20}$$

where

$$\mathcal{E}_1(\underline{p}) = \left[p^2c^2 + m_1^2c^4\right]^{1/2} \tag{4.3.21}$$

$$\mathcal{E}_2(\underline{p}) = \mathcal{E}_{20}(\underline{p}) + \lambda^2 \int d^3q\, \frac{h^2(\underline{q})}{\mathcal{E}_{20}(\underline{p}) - \mathcal{E}_1(\underline{p}-\underline{q}) - \mathcal{E}_B(\underline{q})} \tag{4.3.22}$$

$$\mathcal{V}_{12}(\underline{k},\underline{k}',\underline{K}) = \frac{\lambda^2 h^2(\underline{k}+\underline{k}')}{\mathcal{E}_1(\tfrac{1}{2}\underline{K}-\underline{k}') + \mathcal{E}_B(\underline{k}+\underline{k}') - \mathcal{E}_{20}(\tfrac{1}{2}\underline{K}+\underline{k})} \quad + \quad \underline{k} \leftrightarrow \underline{k}' \tag{4.3.23}$$

$$v_{2B}(\underline{k},\underline{k}',\underline{K}) = \frac{-\lambda^2 h(\frac{1}{2}\underline{K}-\underline{k}')\, h(\frac{1}{2}\underline{K}-\underline{k})}{\mathcal{E}_1(\underline{k}+\underline{k}') + \mathcal{E}_B(\frac{1}{2}\underline{K}-\underline{k}') - \mathcal{E}_{20}(\frac{1}{2}\underline{K}+\underline{k})} + \underline{k} \leftrightarrow \underline{k}' \qquad (4.3.24)$$

$$v_{1B}(\underline{k},\underline{k}',\underline{K}) = \frac{\lambda^2 h(\frac{1}{2}\underline{K}-\underline{k}')\, h(\frac{1}{2}\underline{K}-\underline{k})}{\mathcal{E}_1(\frac{1}{2}\underline{K}+\underline{k}') + \mathcal{E}_B(\frac{1}{2}\underline{K}-\underline{k}') - \mathcal{E}_{20}(\underline{K})} + \underline{k} \leftrightarrow \underline{k}' \qquad (4.3.25)$$

Note that the energy of the V particle has been renormalized (eq. (4.3.22)).

The functions v_{12}, v_{2B}, and v_{1B} correspond to Figures 6, 7, and 8:

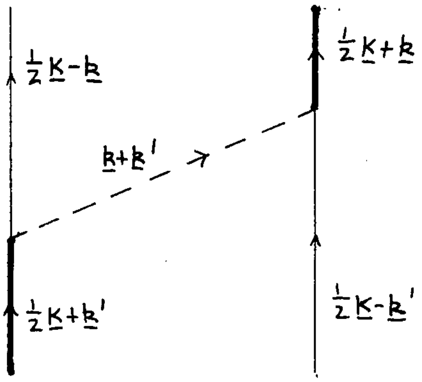

Fig. 6 The N-V Potential v_{12}

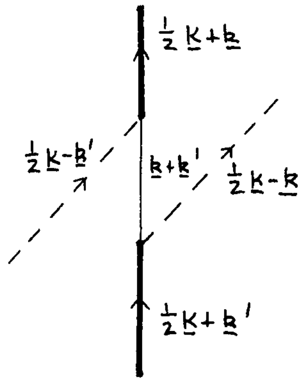

Fig. 7 The V-θ Potential v_{2B}

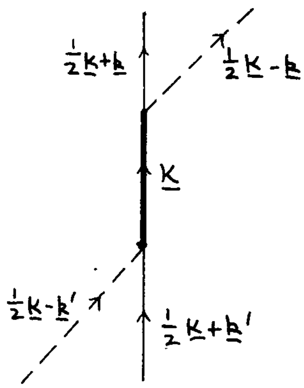

Fig. 8 The N-θ Potential v_{1B}

There are no N-N or θ-θ scattering terms present in the Hamiltonian to second order.

Again we note that the dressing transformation has made explicit the physical particle interactions described by the original trilinear interaction. Matrix elements of the physical particle potentials, $\mathcal{V}_{12}$, $\mathcal{V}_{2B}$ and $\mathcal{V}_{1B}$, have been determined in terms of the trilinear vertex function $h(\underline{q})$. The physical particles (see eq. (G.16)) become composites of elementary ones.

Chapter 5 Dressing the Trilinear Fermion-Boson Interaction: The Fermion-Fermion Potential

We saw in Chapter 3 that the nucleon-pion trilinear interaction discussed there does not explicitly involve physical fermions. In this Chapter, we use the dressing transformation developed in Section 4.1 to transform the elementary particle creators into physical particle creators. Under this transformation, we find that to second order the Hamiltonian contains physical nucleon-nucleon and nucleon-pion interactions. Concentrating on the nucleon-nucleon term, we use the vertex function derived from the Cloudy Bag Model to calculate the second-order nucleon-nucleon potential.

5.1 Dressing the Trilinear Interaction

Consider the nucleon-pion Hamiltonian (3.1.1), with vertex function h given by (3.1.35). $B_{\mu}^{\dagger}(\underline{p})|0\rangle$ is an eigenket of this Hamiltonian so we take $\mathcal{E}_{B_0}(\underline{p}) = \mathcal{E}_B(\underline{p})$, the energy of the physical pion. $F_{m\mu}^{\dagger}(\underline{p})|0\rangle$ is not an eigenket of H. We seek a dressing transformation as in Section 4.1 which will determine a dressed Hamiltonian $\tilde{H}(\tilde{F},\tilde{B})$ expressed explicitly in terms of physical particle operators.

A first-order dressing operator, which satisfies the required invariance properties and (4.1.20) is:

$$D_1 = \sum_{\substack{m_1 m_2 \\ \mu_1\mu_2\mu_3}} \int d^3p\, d^3q \; \frac{h^{\mu_1\mu_2\mu_3}_{m_1 m_2}(q)}{\varepsilon_\pi(q)} F^+_{m_1\mu_1}(p) F_{m_2\mu_2}(p-q) B_{\mu_3}(q) - adj. \tag{5.1.1}$$

In determining this operator we have taken

$$\varepsilon_{No}(p) = m_{No}\, c^2 \tag{5.1.2}$$

We are now able to determine $[H_1, D_1]$. We find

$$[H_1, D_1] = \sum_{\substack{m_1 m \\ \mu_1 \mu}} \int d^3p \;\; f_1{}^{\mu_1\mu}_{m_1 m} \; F^+_{m_1\mu_1}(p) F_{m\mu}(p)$$

$$+ \sum_{\substack{\mu_1\mu_2\mu\mu'\mu'' \\ m_1 m_2 m}} \int d^3p\, d^3q\, d^3q' \Big\{ \sum_{m'} f_2{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m m'}(q) F^+_{m\mu_1}(p+q) F^+_{m'\mu'}(q') F_{m_2\mu_2}(p) F_{m'\mu'}(q'+q)$$

$$+ f_3{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m}(q,q') F^+_{m_1\mu_1}(p+q) B^+_{\mu''}(q') B_{\mu_2}(q) F_{m\mu}(p+q')$$

$$+ f_4{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m}(q,q') F^+_{m\mu}(p+q') F_{m_2\mu_2}(p-q) B_{\mu_1}(q) B_{\mu''}(q') \Big\}$$

$$+ adj. \tag{5.1.3}$$

where

$$f_1{}^{\mu_1\mu}_{m_1 m} = - \sum_{m_2\mu_2\mu''} \int d^3q \; \frac{h^{\mu_1\mu_2\mu''}_{m_1 m_2}(q)\, h^{\mu\mu_2\mu''\,*}_{m m_2}(q)}{\varepsilon_\pi(q)} \tag{5.1.4}$$

$$f_2{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m m'}(q) = \frac{h^{\mu_1\mu_2\mu''}_{m_1 m_2}(q)\;\; h^{\mu\mu'\mu''\,*}_{m m'}(q)}{\varepsilon_\pi(q)} \tag{5.1.5}$$

$$f_3{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m}(q,q') = \frac{1}{\varepsilon_\pi(q')} \Big[h^{\mu'\mu\mu_2}_{m_2 m}(q)\, h^{\mu'\mu_1\mu''\,*}_{m_2 m_1}(q') - h^{\mu_1\mu'\mu_2}_{m_1 m_2}(q)\, h^{\mu\mu'\mu''\,*}_{m m_2}(q') \Big] \tag{5.1.6}$$

$$f_4{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m}(q,q') = h^{\mu\mu'\mu''}_{m m_1}(q')\, h^{\mu'\mu_2\mu_1}_{m_2 m_1}(q) \Big[\frac{1}{\varepsilon_\pi(q)} - \frac{1}{\varepsilon_\pi(q')} \Big] \tag{5.1.7}$$

The first term in $[H_1, D_1]$ can be simplified using the form (3.1.35) for h. We have

$$f_{1\,m_1 m}^{\,\mu_1 \mu} = - \sum_{m_2 \mu_2 \mu''} \int d^3q \, (\tfrac{1}{2} 1 \mu_2 \mu'' | \tfrac{1}{2} \mu_1)(\tfrac{1}{2} 1 \mu_2 \mu'' | \tfrac{1}{2} \mu) \cdot$$

$$\cdot (\tfrac{1}{2} 1 m_2 m' | \tfrac{1}{2} m_1)(\tfrac{1}{2} 1 m_2 m'' | \tfrac{1}{2} m) \, Y_{1m'}^{*}(\hat{q}) \, Y_{1m''}(\hat{q}) \, \frac{h^2(q)}{\mathcal{E}_\pi(q)} \quad (5.1.8)$$

First, (D.5) is used to sum over μ_2 and μ'', leaving $\delta_{\mu_1,\mu}$. Next, (B.8) is used to integrate over $d\Omega_q$, to give $\delta_{m',m''}$. Finally, we use (D.5) to sum over m_2 and obtain $\delta_{m_1,m}$. Thus

$$f_{1\,m_1 m}^{\,\mu_1 \mu} = - \int q^2 dq \, \frac{h^2(q)}{\mathcal{E}_\pi(q)} \, \delta_{\mu_1,\mu} \, \delta_{m_1,m} \quad (5.1.9)$$

The last term in $[H_1, D_1]$ is of the form (4.1.18) and must therefore be eliminated from the dressed Hamiltonian given by (4.1.21). We choose

$$D_2 = -\frac{1}{2} \sum_{\substack{\mu_1 \mu_2 \mu \mu' \mu'' \\ m_1 m_2 m}} \int d^3p \, d^3q \; f_4{}_{\,m_1 m_2 m}^{\,\mu_1 \mu_2 \mu \mu' \mu''}(q, q') \cdot$$

$$\cdot F_{m\mu}^{+}(p+q') \, F_{m_2\mu_2}(p-q) \, B_{\mu_1}(q) \, B_{\mu''}(q') - \mathrm{adj}. \quad (5.1.10)$$

so that $[H_0, D_2]$ exactly equals minus one-half of the last term in $[H_1, D_1]$.

The resulting Hamiltonian, to order λ^2, is:

$$\tilde{H}(\tilde{F}, \tilde{B}) = T + V_{\pi N} + V_{NN} \quad (5.1.11)$$

$$T = \sum_{m\mu\mu'} \int d^3p \, [m_N c^2 \, \tilde{F}_{m\mu}^{+}(p) \, \tilde{F}_{m\mu}(p) + \mathcal{E}_\pi(p) \, \tilde{B}_{\mu'}^{+}(p) \, \tilde{B}_{\mu'}(p)] \quad (5.1.12)$$

$$V_{\pi N} = \frac{\lambda^2}{2} \sum_{\substack{\mu_1\mu_2\mu\mu'\mu'' \\ m_1 m_2 m}} \int d^3k\, d^3k'\, d^3K \;\Big[v_{\pi N}{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m}(\underline{k},\underline{k}',\underline{K}) + v_{\pi N}{}^{\mu\mu''\mu_1\mu'\mu_2}_{m m_2 m_1}(\underline{k}',\underline{k},\underline{K})\Big] \cdot$$

$$\cdot \tilde{F}^{+}_{m_1\mu_1}(\tfrac{1}{2}\underline{K}+\underline{k})\,\tilde{B}^{+}_{\mu''}(\tfrac{1}{2}\underline{K}-\underline{k})\,\tilde{B}_{\mu_2}(\tfrac{1}{2}\underline{K}-\underline{k}')\,\tilde{F}_{m\mu}(\tfrac{1}{2}\underline{K}+\underline{k}') \qquad (5.1.13)$$

$$V_{NN} = \frac{\lambda^2}{2} \sum_{\substack{\mu_1\mu_2\mu\mu'\mu'' \\ m_1 m_2 m m'}} \int d^3k\; d^3k'\; d^3K \;\; v_{NN}{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m m'}(\underline{k}-\underline{k}') \cdot$$

$$\cdot \tilde{F}^{+}_{m_1\mu_1}(\tfrac{1}{2}\underline{K}+\underline{k})\;\tilde{F}^{+}_{m'\mu'}(\tfrac{1}{2}\underline{K}-\underline{k})\,\tilde{F}_{m\mu}(\tfrac{1}{2}\underline{K}-\underline{k}')\,\tilde{F}_{m_2\mu_2}(\tfrac{1}{2}\underline{K}+\underline{k}') \qquad (5.1.14)$$

where

$$m_N c^2 = m_{N_0} c^2 - \lambda^2 \int q^2\, dq\, \frac{h^2(q)}{\varepsilon_\pi(q)} \qquad (5.1.15)$$

$$v_{\pi N}{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m}(\underline{k},\underline{k}',\underline{K}) = \frac{h(|\tfrac{1}{2}\underline{K}-\underline{k}'|)\,h(|\tfrac{1}{2}\underline{K}-\underline{k}|)}{\varepsilon_\pi(\tfrac{1}{2}\underline{K}-\underline{k})}\; Y^{*}_{1\lambda}(\tfrac{1}{2}\underline{K}-\underline{k}')\,Y_{1\lambda'}(\tfrac{1}{2}\underline{K}-\underline{k}) \cdot$$

$$\cdot\Big[(\tfrac{1}{2}\,1\,\mu\,\mu_2\,|\,\tfrac{1}{2}\mu')(\tfrac{1}{2}\,1\,m\,\lambda\,|\,\tfrac{1}{2}m_2)(\tfrac{1}{2}\,1\,\mu_1\mu''\,|\,\tfrac{1}{2}\mu')(\tfrac{1}{2}\,1\,m_1\lambda'\,|\,\tfrac{1}{2}m_2)$$

$$-(\tfrac{1}{2}\,1\,\mu'\mu_2\,|\,\tfrac{1}{2}\mu_1)(\tfrac{1}{2}\,1\,m_2\lambda\,|\,\tfrac{1}{2}m_1)(\tfrac{1}{2}\,1\,\mu'\,\mu''\,|\,\tfrac{1}{2}\mu)(\tfrac{1}{2}\,1\,m_2\lambda'\,|\,\tfrac{1}{2}m)\Big] \qquad (5.1.16)$$

$$v_{NN}{}^{\mu_1\mu_2\mu\mu'\mu''}_{m_1 m_2 m m'}(\underline{q}) = \frac{-2h^2(q)}{\varepsilon_\pi(q)}\; Y^{*}_{1\lambda}(\underline{q})\,Y_{1\lambda'}(\underline{q}) \cdot$$

$$\cdot\Big[(\tfrac{1}{2}\,1\,\mu_2\mu''\,|\,\tfrac{1}{2}\mu_1)(\tfrac{1}{2}\,1\,m_2\lambda\,|\,\tfrac{1}{2}m_1)(\tfrac{1}{2}\,1\,\mu'\,\mu''\,|\,\tfrac{1}{2}\mu)(\tfrac{1}{2}\,1\,m'\lambda'\,|\,\tfrac{1}{2}m)\Big] \qquad (5.1.17)$$

We can see from its structure that $V_{\pi N} = V_{\pi N}{}^{\dagger}$. Using (B.4) and (D.9), one can easily show that $V_{NN} = V_{NN}{}^{\dagger}$ as well.

The nucleon mass has been renormalized (eq. (5.1.15)). The functions (5.1.16) and (5.1.17) may be pictured as in Figures 9 and 10:

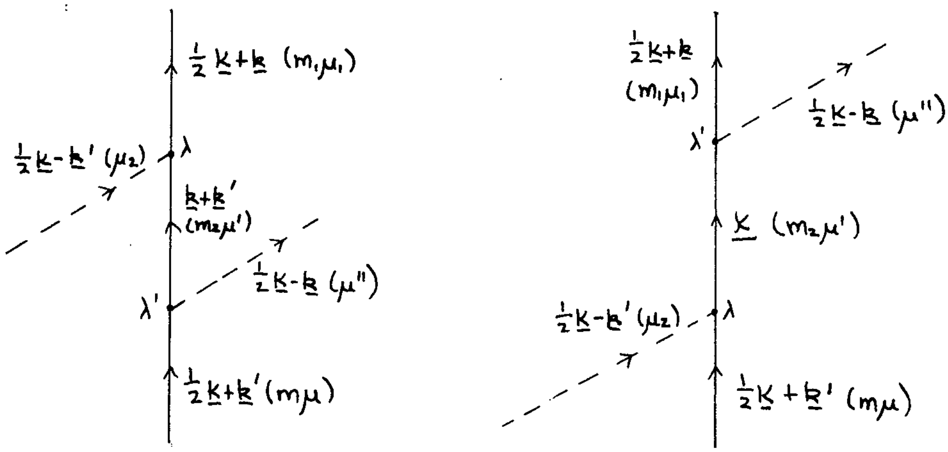

Fig. 9 The π -N Potential $v_{\pi N}{}^{\mu_1\mu_2\,\mu\mu'\mu''}_{m_1 m_2 m}(\underline{k},\underline{k}',\underline{K})$

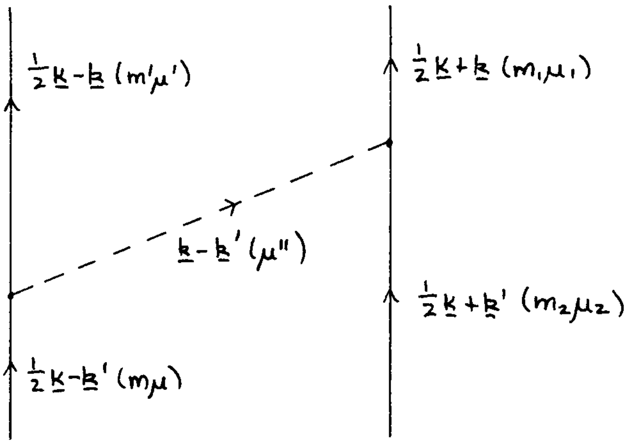

Fig. 10 The N-N Potential $v_{NN}{}^{\mu_1\mu_2\,\mu\mu'\mu''}_{m_1 m_2 m m'}(\underline{k}-\underline{k}')$

From the Figures, we see clearly that the Clebsch-Gordan coefficients appearing in (5.1.16) and (5.1.17) serve to impose angular momentum and isospin conservation at each vertex. The two terms in $v_{\pi N}$ correspond to the two different diagrams for π-N scattering shown in Figure 9.

The physical one-nucleon creator may be calculated from

$$\tilde{F}_{m\mu}^{+}(\underline{p}) = e^{D} F_{m\mu}^{+}(\underline{p})\, e^{-D} \qquad (5.1.18)$$

One finds that the physical nucleon is a composite particle consisting of a bare nucleon surrounded by a cloud of pions.

We have now seen several examples of dressing transformations and may infer certain properties of these transformations. To order n, D_n is constructed to have the same structure as the unsuitable term which must be cancelled. Because of the basic commutation relations, $[H_0, D_n]$ will always cancel exactly and only those terms required. The physical particles become composites of elementary ones via the transformation. Physical particle interactions are expressed explicitly in the dressed Hamiltonian. The series for the dressing operator and the Hamiltonian will terminate or be summable only for very simple models.

5.2 The Second-Order Nucleon-Nucleon Potential

In this Section we use the trilinear fermion-boson interaction of Chapter 3 and results of dressing it to determine a coordinate space nucleon-nucleon potential in terms of the vertex function h(q). It is a second-order potential in the sense that it is obtained from the second-order term V_{NN} in $\tilde{H}(\tilde{F},\tilde{B})$. First, we rewrite V_{NN} in eq. (5.1.14) in terms of the two-nucleon operators $\tilde{A}^{s\sigma\dagger}_{m\mu}$ defined by (E.12) to find

$$V_{NN} = \lambda^2 \sum_{\substack{sM\sigma\beta \\ s'M'\sigma'\beta'}} \int d^3k\, d^3k'\, d^3K \; v^{\sigma\beta\sigma'\beta'}_{sMs'M'}(\underline{k}-\underline{k}')\, \tilde{A}^{s\sigma\dagger}_{m\beta}(\underline{k},\underline{K})\, \tilde{A}^{s'\sigma'}_{M'\beta'}(\underline{k}',\underline{K}) \tag{5.2.1a}$$

where

$$v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(\underset{\sim}{q}) = -2 \sum_{\substack{\mu_1\mu_2\mu\mu'\mu'' \\ m_1 m_2 m m' \ell}} \frac{h^2(q)}{\mathcal{E}_\pi(q)} \sqrt{\frac{3\cdot 3}{4\pi(2\ell+1)}}\, Y_{\ell\alpha}(\underset{\sim}{q})\,(1100|\ell 0)\cdot$$

$$\cdot\left[(\tfrac{1}{2}1\mu_2\mu''|\tfrac{1}{2}\mu_1)(\tfrac{1}{2}\tfrac{1}{2}\mu_1\mu'|\sigma\beta)(\tfrac{1}{2}1\mu'\mu''|\tfrac{1}{2}\mu)(\tfrac{1}{2}\tfrac{1}{2}\mu_2\mu|\sigma'\beta')\right]\cdot$$

$$\cdot\left\{(-)^\lambda(11-\lambda\lambda'|\ell\alpha)(\tfrac{1}{2}\tfrac{1}{2}m_1m'|SM)(\tfrac{1}{2}\tfrac{1}{2}m_2m|S'M')(\tfrac{1}{2}1m_2\lambda|\tfrac{1}{2}m_1)(\tfrac{1}{2}1m'\lambda'|\tfrac{1}{2}m)\right\} \tag{5.2.1b}$$

To arrive at this form for $v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(\underset{\sim}{q})$ we used (B.6) to combine the two spherical harmonics in (5.1.17) into a single one.

Based on the discussion in Appendix E, we recognize $v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(\underset{\sim}{q})$ as the matrix elements of the momentum space two-nucleon potential. Many of the sums over spin and isospin projection quantum numbers in (5.2.1) can be evaluated, as we now show.

We apply (D.18) to the first three isospin Clebsch-Gordan coefficients appearing in (5.2.1) to find

$$\sum_{\mu_1\mu''\mu'} (\tfrac{1}{2}1\mu_2\mu''|\tfrac{1}{2}\mu_1)\,(\tfrac{1}{2}\tfrac{1}{2}\mu_1\mu'|\sigma\beta)\,(-)^1\,(1\tfrac{1}{2}\mu''\mu'|\tfrac{1}{2}\mu)$$

$$= (-)^{\sigma+1}\sqrt{2(2)}\begin{Bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & \sigma & 1/2 \end{Bmatrix}(\tfrac{1}{2}\tfrac{1}{2}\mu_2\mu|\sigma\beta) \tag{5.2.2}$$

where $\begin{Bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & \sigma & 1/2 \end{Bmatrix}$ denotes a 6j symbol (see Appendix D). Using (D.5) to sum over μ_2 and μ, we find that the expression in square brackets reduces to

$$\sum_{\substack{\mu_1\mu''\mu' \\ \mu_2\mu}} [\ \ ] = 2(-)^{\sigma+1}\begin{Bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & \sigma & 1/2 \end{Bmatrix}\delta_{\sigma\sigma'}\,\delta_{\beta\beta'} \tag{5.2.3}$$

We now consider the five Clebsch-Gordan coefficients inside the curly brackets in (5.2.1b). Rearranging the fourth coefficient using (D.9d), we find that these five coefficients are related exactly by (D.25). Thus

$$\sum_{\substack{m_1 m' \\ m_2 m \\ \lambda\lambda'}} \{\ \} = \sum_{\substack{m_1 m' \\ m_2 m \\ \lambda\lambda'}} \left[(-)^{2\lambda}=1\right] \left(\tfrac{1}{2}\tfrac{1}{2}m_1 m' \,|\, SM\right)\left(1\,1\,-\lambda\,\lambda' \,|\, \ell\alpha\right) \cdot$$

$$\cdot \left(\tfrac{1}{2}\,1\,m_1\,-\lambda \,|\, \tfrac{1}{2}m_2\right)\left(\tfrac{1}{2}\,1\,m'\lambda' \,|\, \tfrac{1}{2}m\right)\left(\tfrac{1}{2}\tfrac{1}{2}m_2 m \,|\, S'M'\right)$$

$$= \sqrt{(2S+1)(2\ell+1)(2)(2)}\;\left(S\ell M\alpha \,|\, S'M'\right)\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ 1 & 1 & \ell \\ \frac{1}{2} & \frac{1}{2} & S' \end{Bmatrix} \tag{5.2.4}$$

where $\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ 1 & 1 & \ell \\ \frac{1}{2} & \frac{1}{2} & S' \end{Bmatrix}$ is a 9j symbol (see Appendix D).

We can combine the results (5.2.3) and (5.2.4) to write

$$v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(\underset{\sim}{q}) = \sum_{\ell}\left(S\ell M\alpha \,|\, S'M'\right) Y_{\ell\alpha}(\underset{\sim}{q})\; v^{\sigma\beta\sigma'\beta'}_{SS'}(\underset{\sim}{q}) \tag{5.2.5a}$$

where

$$v^{\sigma\beta\sigma'\beta'}_{SS'}(\underset{\sim}{q}) = \delta_{\sigma\sigma'}\,\delta_{\beta\beta'}\,\frac{24}{\sqrt{4\pi}}\,(-)^{\sigma}\begin{Bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \sigma & \frac{1}{2} \end{Bmatrix}\frac{h^2(\underset{\sim}{q})}{\varepsilon_\pi(\underset{\sim}{q})}\,\cdot$$

$$\cdot\sqrt{2S+1}\;\left(1\,1\,0\,0 \,|\, \ell 0\right)\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S \\ 1 & 1 & \ell \\ \frac{1}{2} & \frac{1}{2} & S' \end{Bmatrix} \tag{5.2.5b}$$

Before continuing let us consider the various factors in this potential. First, we note that the factor $\delta_{\sigma\sigma'}\,\delta_{\beta\beta'}$ makes the conservation of isospin and its z-axis projection manifest. Secondly, it can easily be shown that the requirement of rotational invariance alone means that $v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(\underset{\sim}{q})$ must be of the form (5.2.5a). The specific form (5.2.5b) is a consequence of the trilinear interaction with which we began. Thirdly, note that the Clebsch-Gordan coefficient $(1\,1\,0\,0|\ell\,0)$ is non-zero only for $\ell = 0,2$, corresponding to the scalar and tensor parts of the strong interaction, respectively. The conservation of spin is implicit in the 9j symbol, which is zero unless the three angular momenta in any row or column form a triad. Thus, for $\ell = 0,2$, $S = S'$ necessarily. The quantum numbers M and M' are not required to be equal.

Based on the above discussion, we may write the nucleon-nucleon interaction (5.1.14) as

$$V_{NN} = \lambda^2 \int d^3k\, d^3k'\, d^3K \sum_{\substack{SMM' \\ \sigma\beta}} v^{S\sigma}_{MM'}(\underline{k}-\underline{k}')\, \tilde{A}^{S\sigma\dagger}_{M\beta}(\underline{k},\underline{K})\, \tilde{A}^{S\sigma}_{M'\beta}(\underline{k}',\underline{K}) \quad (5.2.6)$$

where

$$v^{S\sigma}_{MM'}(\underline{q}) = \sum_{\ell=0,2} a^{S\sigma\ell}_{MM'} \frac{h^2(q)}{\mathcal{E}_\pi(q)} Y_{\ell\alpha}(\underline{\hat{q}}) \quad (5.2.7)$$

$$a^{S\sigma\ell}_{MM'} = \frac{24}{\sqrt{4\pi}} (-)^{\sigma} \begin{Bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & \sigma & 1/2 \end{Bmatrix} \sqrt{2S+1}\, (1\,1\,0\,0|\ell\,0) \cdot$$
$$\cdot\, (S\,\ell\,M\,\alpha|S\,M') \begin{Bmatrix} 1/2 & 1/2 & S \\ 1 & 1 & \ell \\ 1/2 & 1/2 & S \end{Bmatrix} \quad (5.2.8)$$

This constant specifies the spin and isospin dependence of the potential. We have determined its numerical values using Messiah (1958, pp. 1065 - 68) to evaluate the 6j and 9j symbols and values are given in Table I for various total spins and isospins.

We may use (E.11) to relate the momentum space nucleon-

nucleon potential $v^{S\sigma}_{MM'}(\underline{q})$ to the corresponding coordinate space potential:

$$v^{S\sigma}_{MM'}(\underline{r}) = \int d^3q \; e^{i\underline{q}\cdot\underline{r}/\hbar} \; v^{S\sigma}_{MM'}(\underline{q}) \tag{5.2.9}$$

Using (5.2.7), (B.7), and (B.8), we obtain

$$v^{S\sigma}_{MM'}(\underline{r}) = \sum_{\ell=0,2} Y_{\ell d}(\underline{r})\, v^{S\sigma\ell}_{MM'}(r) \tag{5.2.10}$$

where

$$v^{S\sigma\ell}_{MM'}(r) = 4\pi\, i^{\ell}\, a^{S\sigma\ell}_{MM'} \int_0^{\infty} \frac{q^2\, dq}{\varepsilon_\pi(q)}\, j_\ell(qr/\hbar)\, h^2(q) \tag{5.2.11}$$

Thus, given a trilinear vertex function h(q) we can compute the corresponding coordinate space second-order N-N potential by performing the integration in equation (5.2.11).

	$\ell=0$, $S=0$	$\ell=0$, $S=1$	$\ell=2$, $S=1$
$\sigma=0$	$\frac{-1}{\sqrt{\pi}}\,\delta_{MM'}\,\delta_{M0}$	$\frac{1}{3\sqrt{\pi}}\,\delta_{MM'}$	$\frac{2}{3}\sqrt{\frac{2}{\pi}}\,(1\,2\,M\,M'-M\,\vert\,1\,M')$
$\sigma=1$	$\frac{1}{3\sqrt{\pi}}\,\delta_{MM'}\,\delta_{M0}$	$\frac{-1}{9\sqrt{\pi}}\,\delta_{MM'}$	$\frac{-\sqrt{2}}{9\sqrt{\pi}}\,(1\,2\,M\,M'-M\,\vert\,1\,M')$

Table I Values of the Constant $a^{S\sigma\ell}_{MM'}$

5.3 The Nucleon-Nucleon Potential in the Cloudy Bag Model

The coordinate space nucleon-nucleon potential functions $\mathcal{V}_{MM'}^{S\sigma,\ell}(r)$, given by (5.2.11), are the matrix elements of the scalar ($\ell=0$) and tensor ($\ell=2$) parts of the second-order nucleon-nucleon potential taken between two-nucleon states having spin S and isospin σ. We now calculate these matrix elements by substituting the Cloudy Bag Model vertex function given by (3.2.1) into (5.2.11). We have

$$\left[\mathcal{V}_{MM'}^{S\sigma,\ell}(r)\right]_{CBM} = \frac{108}{R^2}\;\frac{f_0^2\,\hbar^2\,i^\ell}{m_\pi^2\,c}\;a_{MM'}^{S\sigma,\ell}\;N_\ell(r,R) \tag{5.3.1}$$

where

$$N_\ell(r,R) = \int_0^\infty dq\,\frac{q^2}{q^2+m_\pi^2c^2}\;j_1^2\left(\frac{qR}{\hbar}\right)\;j_\ell\left(qR/\hbar\right) \qquad \ell=0,2 \tag{5.3.2}$$

We show in Appendix I that for the case r > 2R, i.e., when the two bags do not overlap, the integrals $N_\ell(r,R)$ are easily evaluated using a result given in Watson (1966), involving a contour integration over products of Bessel functions. Remarkably, the integrals $N_\ell(r,R)$ factor into the product of a simple function of the bag radius R and a function of the coordinate r. Indeed, taking $k = m_\pi$, $b = R$, and $a = r$ in equations (I.13) and (I.14), we discover

$$\left[\mathcal{V}_{MM'}^{S\sigma,\ell}(\underline{r})\right]_{CBM} = \left[\mathcal{V}_{MM'}^{S\sigma,\ell}(\underline{r})\right]_{OPEP}\;g(R) \tag{5.3.3}$$

where

$$g(R) = \frac{9}{(\mu R)^4}\left[\cosh(\mu R) - \frac{\sinh(\mu R)}{\mu R}\right]^2 \tag{5.3.4}$$

$[\mathcal{V}_{MM'}^{S\sigma\ell}(\underline{r})]_{OPEP}$ are the matrix elements of the central and tensor one pion exchange potential, which is discussed in Appendix J, and

$$\mu = \frac{m_\pi c}{\hbar} \tag{5.3.5}$$

Specifically, the Cloudy Bag Model potentials are

$$[v_{MM'}^{s\sigma 0}(\underline{r})]_{CBM} = Y_{00}(\underline{r})\,(-6\pi f_0^2 \hbar c)\, a_{MM'}^{s\sigma 0}\, \frac{e^{-\mu r}}{r}\, g(R) \qquad r>2R \tag{5.3.6}$$

and

$$[v_{MM'}^{s\sigma 2}(\underline{r})]_{CBM} = Y_{2,M'-M}(\underline{r})\,(-6\pi f_0^2 \hbar c)\, a_{MM'}^{s\sigma 2}\,\cdot$$

$$\cdot\left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)\frac{e^{-\mu r}}{r}\, g(R) \qquad r>2R \tag{5.3.7}$$

Thus, when the two nucleon bags are not touching ($r>2R$), the Cloudy Bag Model nucleon-nucleon potential is <u>exactly</u> the same function of the nucleon separation $\underline{r}$ as is the potential calculated from one pion exchange. The spin and isospin dependence of the two potentials is also identical. In light of the discussion in Appendix J, we recognize the constants $a_{MM'}^{s\sigma \ell}$ as being proportional to the spin-isospin matrix elements of the nucleon operators $\underline{\sigma}_1 \cdot \underline{\sigma}_2$ ($\ell=0$) and S_{12} ($\ell=2$), which appear in V_{OPEP}.

The two potentials V_{CBM} and V_{OPEP} differ in overall strength by the function $g(R)$. Note that

$$\lim_{R\to 0} g(R) = 1 \tag{5.3.8}$$

Thus, for $r > 0$, the Cloudy Bag Model potential becomes exactly the one pion exchange potential as $R \to 0$. Recall (eq. (3.2.4)) that in this limit we obtain the Chew-Low interaction. We see that for this interaction, the integral (5.2.12) diverges when $\ell = 0$. This divergence gives rise to the $\delta(\underline{r})$ term in the one pion exchange potential. There are no such divergences in the Cloudy Bag Model potential. The function $u_N(q)$, which accounts for the finite size of the bare nucleon, acts as a physically meaningful cutoff function which keeps all integrals in the theory finite.

It is interesting to note that for the value R = 0.72 fm. predicted by Théberge et al. we find

$$g\,(0.72\text{ fm.}) = 1.05 \tag{5.3.9}$$

The Cloudy Bag Model potential differs only slightly from the one pion exchange potential, by the factor 1.05.

So far we have considered the Cloudy Bag Model potential for the case r > 2R. The integrals $N_\ell(r,R)$ can also be evaluated for the case r < 2R, although this region is physically less clear because it implies an overlap of the nucleon bags. For completeness, however, we give the result for $\ell = 0$, obtained from a contour integration:

$$\left[v_{MM'}^{S\sigma\,0}(r) \right]_{CBM} = 27\pi f_0^2 \, a_{MM'}^{S\sigma\,0} \, \frac{1}{(\mu R)^6} \hbar c \;\cdot$$

$$\cdot \left\{ (1-\mu^2 R^2) \frac{e^{-\mu r}}{r} + \frac{1}{2}(1+\mu R)^2 \, e^{-2\mu R} \left(\frac{e^{\mu r}}{r} - \frac{e^{-\mu r}}{r} \right) \right.$$

$$\left. - \frac{1}{r}\left(1 + \frac{1}{2}\mu^2 r^2 - \mu^2 R^2\right) \right\} \qquad r < 2R \tag{5.3.10}$$

This solution matches smoothly with (5.3.6) in the limit $r \to 2R$.

Unlike the solution for r > 2R, however, it does not factor into a function of R and a function of r. Finally, in the limit $r \to 0$, it goes to

$$\left[v_{MM'}^{s\sigma,0}(r) \right]_{CBM} = 27\pi f_0^2 \hbar c \; a_{MM'}^{s\sigma,0} \frac{\mu}{(\mu R)^6} \cdot$$

$$\cdot \left\{ (1+\mu R)^2 e^{-2\mu R} - (1-\mu^2 R^2) \right\} \tag{5.3.11}$$

which becomes infinite for $R \to 0$.

Chapter 6 Summary and Conclusions

Our study of strongly interacting fermions and bosons began with the nucleon-pion trilinear interaction of Chapter 3. This interaction was extended to include fermions and bosons of arbitrary spin and isospin in Appendix H. We found that the requirement of invariance under translations, spatial rotations, space inversion, time reversal, and rotations in isospin space greatly restricted the form of the trilinear vertex function. We were able to use the Cloudy Bag Model to obtain a specific expression for the $NN\pi$ trilinear vertex function.

The fermion-boson trilinear interaction does not explicitly involve physical particles, because $F^{\dagger}|0\rangle$ is not an eigenket of the Hamiltonian. In Chapter 4, we developed a dressing transformation which acted on the bare particle creators and annihilators to transform them into physical particle operators. We derived an expression (eq. (4.1.20)) for the Hamiltonian as a function of the physical particle creators and annihilators, and gave a precise method for determining the dressing transformation to any desired order in perturbation theory. Application of this dressing transformation to the scalar field model, the Lee model, the nucleon-pion trilinear interaction, and to the generalized fermion-boson trilinear interaction illustrated several common features of the transformation. First, we discovered that the perturbation series for the dressing operator only terminates for very simple interactions, and that for realistic theories a technique such as we have developed must be used to find the operator D. Secondly, to any given order in perturbation theory, D_n is constructed to have the same structure as the unsuitable term of the form (4.1.18) which must be eliminated from the dressed Hamiltonian. $[H_0, D_n]$ will then cancel exactly and only the term required. Thirdly, the dressing transformation gives a specific expression for the physical particle creators as composites of the elementary particle creators. We found a fermion mass renormalization in the second-order calculations; to higher orders we expect an analogous vertex renormalization. Finally, the physical particle interactions implicit in the trilinear Hamiltonian are made explicit by the dressing transformation, leading to interaction potentials that can be calculated from the original trilinear vertex function.

A second-order nucleon-nucleon potential for the Cloudy Bag Model was determined in Chapter 5. We found that when the two nucleon bags were not touching, this potential was simply the one pion exchange potential multiplied by a function of the bag radius. As the bag radius goes to zero, the Cloudy Bag Model N-N potential goes to the one pion exchange potential.

The nucleon-pion Hamiltonian, when dressed to second order, was found to contain not only a nucleon-nucleon interaction but also a direct nucleon-pion interaction. When the theory is extended to include the Δ, we obtain N-Δ, Δ-Δ, and Δ-π interactions as well. Potentials for these interactions can be calculated analogously to our determination of the nucleon-nucleon potential. When the fermion-boson trilinear Hamiltonian is dressed to third order, we find a term describing boson production on two fermions, i.e. $\int \upsilon\, \tilde{F}^\dagger \tilde{F}^\dagger \tilde{F}\tilde{F}\tilde{B}^\dagger$. The dressing transformation approach allows us to calculate $\mathcal{V}$ in terms of an integral involving the basic trilinear vertex function. This contrasts with the Hamiltonian considered by Hsieh (1978), where this function $\mathcal{V}$ is determined only phenomenologically. Thus our work on the dressing transformation can be used to find such interaction potentials. In turn, they can serve as a basis for the functional form of a long-range, strong interaction Hamiltonian for systems of pions and nucleons at intermediate energies.

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Appendix A The Rotation Matrices $D^{\ell}_{m'm}(\alpha\beta\gamma)$

This Appendix lists some useful properties and formulae involving the $D^{\ell}_{mm'}(\alpha\beta\gamma)$, the irreducible matrix representations of the rotation group.

We take the conventions for these matrices to be those used in Rose (1957) and Messiah (1958). Note that these differ from the conventions used in Edmonds (1960).

From Rose (1957, p.54), we have

$$D^{\ell\,*}_{m'm}(\alpha\beta\gamma) = (-)^{m'-m}\, D^{\ell}_{-m'm}(\alpha\beta\gamma) \tag{A.1}$$

$$D^{\ell\,*}_{m'm}(\alpha\beta\gamma) = D^{\ell}_{mm'}(-\gamma,-\beta,-\alpha) \tag{A.2}$$

When $\ell = 0$,

$$D^{0}_{00}(\alpha\beta\gamma) = 1 \tag{A.3}$$

When $\alpha = \gamma = 0$, the real matrices $d^{j}_{m'm}$ are defined by

$$d^{j}_{m'm}(\beta) = D^{j}_{m'm}(0\,\beta\,0) \tag{A.4}$$

Note that

$$d^{j}_{m'm}(\pi) = (-)^{j+m'}\,\delta_{m',-m} \tag{A.5}$$

The orthogonality properties of the rotation matrices are, from Rose (1957, p.73):

$$\sum_{m} D^{\ell\,*}_{m'm}(\alpha\beta\gamma)\, D^{\ell}_{m''m}(\alpha\beta\gamma) = \delta_{m'm''} \tag{A.6}$$

$$\sum_{m} D^{\ell\,*}_{mm'}(\alpha\beta\gamma)\, D^{\ell}_{mm''}(\alpha\beta\gamma) = \delta_{m'm''} \tag{A.7}$$

The rule for combining two rotation matrices into one is (Rose (1956, p.58)):

$$D^{j_1}_{\mu_1 m_1}(\alpha\beta\gamma)\, D^{j_2}_{\mu_2 m_2}(\alpha\beta\gamma) = \sum_j (j_1 j_2 m_1 m_2 | jm)(j_1 j_2 \mu_1 \mu_2 | j\mu)\, D^{j}_{\mu m}(\alpha\beta\gamma) \tag{A.8}$$

Finally, from Rose (1957, p.75), we have

$$\frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} D^{j_1 *}_{m_1' m_1}(\alpha\beta\gamma)\, D^{j_2}_{m_2' m_2}(\alpha\beta\gamma)\, d\alpha \, \sin\beta \, d\beta \, d\gamma$$

$$= \delta_{m_1' m_2'}\, \delta_{m_1 m_2}\, \delta_{j_1 j_2} \left(\frac{1}{2j_1+1} \right) \tag{A.9}$$

Appendix B The Spherical Harmonics $Y_{\ell m}(q)$

This Appendix lists some useful properties and formulae involving the spherical harmonics. We write

$$Y_{\ell m}(\mathbf{q}) = Y_{\ell m}(\theta, \varphi) \tag{B.1}$$

where θ and φ are the polar and azimuthal angles, respectively, used to specify the direction of the vector $\mathbf{q}$.

$Y_{\ell m}$ transforms under rotations according to (Rose (1957, p.60)):

$$Y_{\ell m}(\mathbf{q}_R) = \sum_{m'=-\ell}^{\ell} D^{\ell}_{m'm}(\alpha\beta\gamma)\, Y_{\ell m'}(\mathbf{q}) \tag{B.2}$$

where

$$\mathbf{q}_R = M\mathbf{q} \tag{B.3}$$

In (B.3), $\mathbf{q}$ is a column vector in its three Cartesian coordinates and M is the matrix given in Rose (1957, p.65).

From Rose (1957, p.61) we have

$$Y_{\ell m}(-\mathbf{q}) = (-)^{\ell}\, Y_{\ell m}(\mathbf{q}) \tag{B.4}$$

$$Y^{*}_{\ell m}(\mathbf{q}) = (-)^{m}\, Y_{\ell -m}(\mathbf{q}) \tag{B.5}$$

$$Y^{*}_{\ell m}(\mathbf{q})\, Y_{\ell' m'}(\mathbf{q}) = (-)^{m}\, Y_{\ell -m}(\mathbf{q})\, Y_{\ell' m'}(\mathbf{q})$$

$$= (-)^{m} \sum_{L\alpha} \sqrt{\frac{(2\ell+1)(2\ell'+1)}{4\pi(2L+1)}}\, (\ell\ell' 00|L0)(\ell\ell' -m m'|L\alpha)\, Y_{L\alpha}(\mathbf{q}) \tag{B.6}$$

where $(j_1 j_2 m_1 m_2|jm)$ is a Clebsch-Gordan coefficient (see Appendix D).

From Rose (1957, p.81),

$$e^{i\mathbf{q}\cdot\mathbf{r}/\hbar} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^{\ell}\, j_{\ell}(qr/\hbar)\, Y^{*}_{\ell m}(\mathbf{q})\, Y_{\ell m}(\mathbf{r}) \tag{B.7}$$

Also, from Rose (1957, p.75), we have

$$\int d\Omega_q \; Y_{\ell m}(q)\, Y^*_{\ell' m'}(q) = \int_0^{2\pi} \int_0^{\pi} Y_{\ell m}(\theta,\varphi)\, Y_{\ell' m'}(\theta,\varphi) \sin\theta \, d\theta \, d\varphi$$

$$= \delta_{\ell\ell'}\, \delta_{mm'} \tag{B.8}$$

Appendix C Irreducible Tensor Operators and the Wigner-Eckart Theorem

An irreducible tensor operator of rank L is a set of 2L+1 quantities T_{LM} $(-L \le M \le L)$ which transform under rotations according to (Rose (1957, p.77)):

$$R\, T_{LM}\, R^{-1} = \sum_{M'=-L}^{L} D^{L}_{M'M}(\alpha\beta\gamma)\, T_{LM'} \tag{C.1}$$

(Note that the above operator R is not the Fock space operator $\mathcal{R}(\alpha\beta\gamma)$).

Any vector $\underline{k} = (k_x, k_y, k_z)$ is an irreducible tensor of rank one, with spherical components

$$k_{\pm 1} = \mp \frac{1}{\sqrt{2}} (k_x \pm i k_y)$$

$$k_0 = k_z \tag{C.2}$$

Alternatively, the spherical components of a vector can be written in terms of the spherical harmonics:

$$\hat{k}_\alpha = \frac{k_\alpha}{k} = \sqrt{\frac{4\pi}{3}}\, Y_{1\alpha}(\underline{k}) \qquad \alpha = \pm 1, 0 \tag{C.3}$$

The rotationally invariant scalar product of two vectors can be written in terms of their spherical components:

$$\underline{k}\cdot\underline{q} = \sum_{i=1}^{3} k_i q_i$$

$$= \sum_{\alpha=\pm 1, 0} (-)^{\alpha}\, k_\alpha\, q_{-\alpha} \tag{C.4}$$

Consider the matrix element of the irreducible tensor operator $T_{L\alpha}$ between angular momentum states. The Wigner-Eckart theorem gives the dependence of this matrix element on the projection quantum numbers. It states (Rose (1957, p.85)):

$$\langle j'm' | T_{L\alpha} | jm \rangle = (j\, L\, m\, \alpha | j'm') \langle j' \| T_L \| j \rangle \tag{C.5}$$

where $\langle j' \| T_L \| j \rangle$ is called the reduced matrix element of the set of tensor operators T_{LM}.

For the case that the T_{LM} are the angular momentum operators $\underline{J}$, this reduced matrix element is (Rose (1957, p.89)):

$$\langle j' \| \underline{J} \| j \rangle = \sqrt{j(j+1)}\ \delta_{jj'} \tag{C.6}$$

Appendix D Angular Momentum Coupling

In this Appendix, we consider the addition of various numbers of angular momenta to form a total angular momentum j(m). Because such a coupling can take place via different intermediate representations, coefficients to connect these representations must be defined. These are the Clebsch-Gordan coefficients, 3j, 6j, 9j and 12j symbols. After defining these coupling coefficients, we list some useful formulae involving them.

(a) Addition of Two Angular Momenta: The Clebsch-Gordan Coefficient and the 3j Symbol

Consider a system involving two angular momenta $\underline{J}_1$ and $\underline{J}_2$ coupled to form a total angular momentum $\underline{J}$:

$$\underline{J}_1 + \underline{J}_2 = \underline{J} \tag{D.1}$$

This system can have four simultaneously diagonalizable operators. Let $|j_1 j_2 m_1 m_2\rangle$ denote the basis state in which $\underline{J}_1$, $\underline{J}_2$, J_{1z}, J_{2z} are diagonal, and $|j_1 j_2 jm\rangle$ the state in which $\underline{J}_1$, $\underline{J}_2$, $\underline{J}$, J_z are diagonal. These two representations are related by a unitary transformation:

$$|j_1 j_2 jm\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm\rangle \tag{D.2}$$

and

$$|j_1 j_2 m_1 m_2\rangle = \sum_{jm} |j_1 j_2 jm\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm\rangle^* \tag{D.3}$$

The elements of this transformation, $\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm\rangle$, are the Clebsch-Gordan coefficients, which we denote by

$(j_1 j_2 m_1 m_2 | j m)$. It is possible to define a phase convention for these coefficients such that they are real:

$$(j_1 j_2 m_1 m_2 | j m) = (j_1 j_2 m_1 m_2 | j m)^* \tag{D.4}$$

Their orthogonality properties are derived from the requirement that the two representations be normalized:

$$\sum_{m_1 m_2} (j_1 j_2 m_1 m_2 | j m)(j_1 j_2 m_1 m_2 | j' m') = \delta_{jj'} \delta_{mm'} \tag{D.5}$$

$$\sum_{jm} (j_1 j_2 m_1 m_2 | j m)(j_1 j_2 m_1' m_2' | j m) = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{D.6}$$

The coefficients have the property that

$$(j_1 j_2 m_1 m_2 | j m) = 0 \tag{D.7}$$

unless $m_1 + m_2 = m$, and (j_1, j_2, j) form a triad, i.e.,

$$|j_1 - j_2| \leq j \leq |j_1 + j_2| \tag{D.8}$$

The symmetry relations of the Clebsch-Gordan coefficients are given in Rose (1957, pp. 38-39):

$$(j_1 j_2 m_1 m_2 | j_3 m_3) = (-)^{j_1 + j_2 - j_3} (j_2 j_1 m_2 m_1 | j_3 m_3) \tag{D.9a}$$

$$= (-)^{j_1 + j_2 - j_3} (j_1 j_2 \; -m_1 \; -m_2 | j_3 \; -m_3) \tag{D.9b}$$

$$= (-)^{j_1 - m_1} \sqrt{\frac{2j_3+1}{2j_2+1}} (j_1 j_3 m_1 \; -m_3 | j_2 \; -m_2) \tag{D.9c}$$

$$= (-)^{j_2 + m_2} \sqrt{\frac{2j_3+1}{2j_1+1}} (j_3 j_2 \; -m_3 \; m_2 | j_1 \; -m_1) \tag{D.9d}$$

$$= (-)^{j_1 - m_1} \sqrt{\frac{2j_3+1}{2j_2+1}} (j_3 j_1 m_3 \; -m_1 | j_2 m_2) \tag{D.9e}$$

$$= (-)^{j_2 + m_2} \sqrt{\frac{2j_3+1}{2j_1+1}} (j_2 j_3 \; -m_2 \; m_3 | j_1 m_1) \tag{D.9f}$$

From Rose (1957, p.42) and (D.9), we have

$$(j_1\, 0\, m_1\, 0 \,|\, j_3\, m_3) = \delta_{j_1 j_3}\, \delta_{m_1 m_3} \tag{D.10}$$

$$(0\, j_2\, 0\, m_2 \,|\, j_3 m_3) = \delta_{j_2 j_3}\, \delta_{m_2 m_3} \tag{D.11}$$

$$(j_1\, j_2\, m_1 m_2 \,|\, 0\, 0) = \frac{(-)^{j_2+m_2}}{\sqrt{2j_2+1}}\, \delta_{j_1 j_2}\, \delta_{m_1,-m_2} \tag{D.12}$$

One can define a 3j symbol which is related to the Clebsch-Gordan coefficients by (Edmonds (1960, p.46)):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}}\, (j_1\, j_2\, m_1\, m_2 \,|\, j_3\, -m_3) \tag{D.13}$$

(b) Addition of Three Angular Momenta: The 6j Symbol

Consider the following two schemes for combining three angular momenta $\underline{J}_1$, $\underline{J}_2$, $\underline{J}_3$ to form a total $\underline{J}$:

$$\underline{J}_1 + \underline{J}_2 = \underline{J}_{12} \qquad \underline{J}_{12} + \underline{J}_3 = \underline{J} \tag{D.14}$$

and

$$\underline{J}_2 + \underline{J}_3 = \underline{J}_{23} \qquad \underline{J}_1 + \underline{J}_{23} = \underline{J} \tag{D.15}$$

In the first case, the operators $\underline{J}_1$, $\underline{J}_2$, $\underline{J}_3$, $\underline{J}_{12}$, $\underline{J}$ can be made diagonal; we denote the corresponding state by $|(j_1 j_2) j_{12}, j_3, j\rangle$. In the second case, the operators $\underline{J}_1$, $\underline{J}_2$, $\underline{J}_3$, $\underline{J}_{23}$, $\underline{J}$ can be diagonalized; the state is $|j_1, (j_2 j_3) j_{23}, j\rangle$. The 6j symbol, which relates these two independent representations is defined by (Edmonds (1960, p.92)):

$$|j_1,(j_2j_3)j_{23},j\rangle$$

$$= \sum_{j_{12}} |(j_1j_2)j_{12},j_3,j\rangle \frac{(-)^{j_1+j_2+j_3+j}}{\sqrt{(2j_{12}+1)(2j_{23}+1)}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \tag{D.16}$$

The two different representations (D.14) and (D.15) can be built up from Clebsch-Gordan coefficients. For example,

$$|j_1,(j_2j_3)j_{23},j\rangle = \sum_{\substack{m_1m_2m_3\\ m_{23}}} (j_2\, j_3\, m_2\, m_3\, |j_{23}\, m_{23})\,(j_1\, j_{23}\, m_1\, m_{23}\, |jm)\cdot$$

$$\cdot |j_1m_1\rangle\, |j_2m_2\rangle\, |j_3m_3\rangle \tag{D.17}$$

Substituting such expansions into (D.16) and using (D.5), one finds

$$\sum_{\substack{m_2m_3\\ m_{12}}} (j_1 j_2\, m_1 m_2\, |j_{12}\, m_{12})\,(j_{12}\, j_3\, m_{12}\, m_3\, |jm)\,(j_2\, j_3\, m_2\, m_3\, |j_{23}\, m_{23})$$

$$= (-)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} (j_1\, j_{23}\, m_1\, m_{23}|jm) \tag{D.18}$$

The 6j symbol is invariant under a permutation of its columns and under interchange of the upper and lower variables in each of any two columns.

In certain cases the 6j symbol can easily be evaluated numerically. From Edmonds (1960, p.95):

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 0 & j_3 & j_2 \end{Bmatrix} = (-)^{j_1+j_2+j_3} \left[(2j_2+1)\,(2j_3+1)\right]^{-1/2} \tag{D.19}$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 1/2 & j_3-1/2 & j_2+1/2 \end{Bmatrix} = (-)^{j_1+j_2+j_3} \left[\frac{(j_1+j_3-j_2)\,(j_1+j_2-j_3+1)}{(2j_2+1)\,(2j_2+2)(2j_3)\,(2j_3+1)}\right]^{1/2} \tag{D.20}$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 1/2 & j_3-1/2 & j_2-1/2 \end{Bmatrix} = (-)^{j_1+j_2+j_3} \left[\frac{(j_1+j_2+j_3+1)\,(j_2+j_3-j_1)}{2j_2\,(2j_2+1)\,(2j_3)\,(2j_3+1)}\right]^{1/2} \tag{D.21}$$

(c) Addition of Four Angular Momenta: The 9j Symbol

Consider the following two schemes for combining four angular momenta to make a total $\underline{J}$:

$$\underline{J}_1+\underline{J}_2=\underline{J}_{12} \quad \underline{J}_3+\underline{J}_4=\underline{J}_{34} \quad \underline{J}_{12}+\underline{J}_{34}=\underline{J} \tag{D.22}$$

and

$$\underline{J}_1+\underline{J}_3=\underline{J}_{13} \quad \underline{J}_2+\underline{J}_4=\underline{J}_{24} \quad \underline{J}_{13}+\underline{J}_{24}=\underline{J} \tag{D.23}$$

The 9j symbol, which relates these two independent representations, is defined by (Edmonds (1960, p.101)):

$$|(j_1j_3)j_{13},(j_2j_4)j_{24},j\rangle = \sum_{j_{12}j_{34}} |(j_1j_2)j_{12},(j_3j_4)j_{34},j\rangle \cdot$$

$$\cdot\left[(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)\right]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \tag{D.24}$$

Expanding the states in (D.24) as in (D.17) and using (D.5) twice, one finds

$$\sum_{\substack{m_1m_2\\m_3m_4\\m_{13}m_{24}}} (j_1j_2m_1m_2|j_{12}m_{12})(j_3j_4m_3m_4|j_{34}m_{34})(j_1j_3m_1m_3|j_{13}m_{13})\cdot$$

$$\cdot(j_2j_4m_2m_4|j_{24}m_{24})(j_{13}j_{24}m_{13}m_{24}|jm)$$

$$=(j_{12}j_{34}m_{12}m_{34}|jm)\left[(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)\right]^{1/2}\cdot$$

$$\cdot\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \tag{D.25}$$

Acting on both sides of (D.25) with $\sum\limits_{\substack{m_{12}\\m_{34}}} (j_{12}j_{34}m_{12}m_{34}|j'm')$ and using (D.5) gives

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} = [(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)]^{-1/2} \cdot$$

$$\cdot \sum_{\substack{m_1 m_2 m_3 m_4 \\ m_{12} m_{13} m_{24} \\ m_{34}}} (j_1 j_2 m_1 m_2 | j_{12} m_{12})(j_3 j_4 m_3 m_4 | j_{34} m_{34}) \cdot$$

$$\cdot (j_1 j_3 m_1 m_3 | j_{13} m_{13})(j_2 j_4 m_2 m_4 | j_{24} m_{24}) \cdot$$

$$\cdot (j_{13} j_{24} m_{13} m_{24} | j m)(j_{12} j_{34} m_{12} m_{34} | j m) \qquad \text{(D.26)}$$

From (D.26) and the properties of the Clebsch-Gordan coefficients, we see that the 9j symbol is zero unless the three angular momenta in any row or column form a triad. The 9j symbol is invariant under a transposition, or an even permutation of rows or columns (i.e., an even number of exchanges of adjacent rows or columns). Under an odd permutation, the phase changes by $(-)^{j_1+j_2+j_3+j_4+j_{12}+j_{34}+j_{13}+j_{24}+j}$

In certain cases, the 9j symbol can be easily evaluated numerically. From (Edmonds (1960, pp. 101,105,106)):

$$\begin{Bmatrix} a & b & e \\ c & d & e \\ f & f & 0 \end{Bmatrix} = \frac{(-)^{b+c+e+f}}{\sqrt{(2e+1)(2f+1)}} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} \qquad \text{(D.27)}$$

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} = \sum_K (-)^{2K} (2K+1) \begin{Bmatrix} j_1 & j_3 & j_{13} \\ j_{24} & j & K \end{Bmatrix} \cdot$$

$$\cdot \begin{Bmatrix} j_2 & j_4 & j_{24} \\ j_3 & K & j_{34} \end{Bmatrix} \begin{Bmatrix} j_{12} & j_{34} & j \\ K & j_1 & j_2 \end{Bmatrix} \qquad \text{(D.28)}$$

(d) Addition of Five Angular Momenta: The 12j Symbol

Consider the following two schemes for combining five angular momenta to make a total $\underline{J}$:

$$\underline{J}_1 + \underline{J}_2 = \underline{J}_{12} \qquad \underline{J}_5 + \underline{J}_{12} = \underline{J}' \qquad \underline{J}_3 + \underline{J}_4 = \underline{J}_{34} \qquad \underline{J}' + \underline{J}_{34} = \underline{J} \tag{D.29}$$

and

$$\underline{J}_1 + \underline{J}_3 = \underline{J}_{13} \qquad \underline{J}_5 + \underline{J}_{13} = \underline{J}'' \qquad \underline{J}_2 + \underline{J}_4 = \underline{J}_{24} \qquad \underline{J}'' + \underline{J}_{24} = \underline{J} \tag{D.30}$$

The 12j symbol, which relates these two independent representations, is defined by (Ord-Smith (1954)):

$$\begin{aligned} &|\{j_5, (j_1 j_3) j_{13}\} j'', (j_2 j_4) j_{24}, j\rangle \\ &= \sum_{j_{12} j_{34} j} |\{j_5, (j_1 j_2) j_{12}\} j', (j_3 j_4) j_{34}, j\rangle \cdot \\ &\qquad \cdot \left[(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)(2j'+1)(2j''+1)\right]^{1/2} \cdot \\ &\qquad \cdot \begin{Bmatrix} j_1 & j_2 & j_{12} & j' \\ j_3 & j_4 & j_{34} & j'' \\ j_{13} & j_{24} & j_5 & j \end{Bmatrix} \end{aligned} \tag{D.31}$$

Expanding the states in (D.31) in terms of Clebsch-Gordan coefficients, and using (D.5) three times, we find

$$\begin{aligned} &\sum_{\substack{m_1 m_2 m_3 \\ m_4 m_5 m_{12} \\ m_{13} m_{24} m''}} (j_1 j_2 m_1 m_2 | j_{12} m_{12})(j_3 j_4 m_3 m_4 | j_{34} m_{34})(j_1 j_3 m_1 m_3 | j_{13} m_{13}) \cdot \\ &\qquad \cdot (j_5 j_{13} m_5 m_{13} | j'' m'')(j_2 j_4 m_2 m_4 | j_{24} m_{24}) \cdot \\ &\qquad \cdot (j_5 j_{12} m_5 m_{12} | j' m')(j'' j_{24} m'' m_{24} | j m) \\ &= \left[(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)(2j'+1)(2j''+1)\right]^{1/2} \cdot \\ &\qquad \cdot (j' j_{34} m' m_{34} | j m) \begin{Bmatrix} j_1 & j_2 & j_{12} & j' \\ j_3 & j_4 & j_{34} & j'' \\ j_{13} & j_{24} & j_5 & j \end{Bmatrix} \end{aligned} \tag{D.32}$$

The symmetry relations of the 12j symbol are given in Ord-Smith (1954). When one element is zero, the 12j symbol simplifies to

$$\begin{Bmatrix} a & b & e & e \\ c & d & f & g \\ g & h & 0 & s \end{Bmatrix} = \left[(2e+1)(2g+1)\right]^{1/2} \begin{Bmatrix} a & b & e \\ c & d & f \\ g & h & s \end{Bmatrix} \qquad \text{(D.33)}$$

Appendix E Two-Particle Operators in Fock Space

This Appendix provides some background to the various two-body potentials and two-fermion operators that are used in the thesis.

Two-particle operators in Fock space are constructed from the corresponding two-body operators in the Hilbert space of n particles. In this Appendix we deal only with fermion Fock space; the extension to include bosons is easily made. For a complete derivation of the form of two-particle operators in Fock space see Schweber (1961,pp.140-2).

Suppose that the total potential V_n of a system of n identical physical fermions is due to two-particle interactions. Then in the n-fermion Hilbert space, we have

$$V_n = \sum_{\alpha=1}^{n} \sum_{\substack{\beta=2 \\ \beta>\alpha}}^{n} V(\xi_\alpha, \xi_\beta) = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \alpha\neq\beta}}^{n} V(\xi_\alpha, \xi_\beta) \tag{E.1}$$

providing that

$$V(\xi_\alpha, \xi_\beta) = V(\xi_\beta, \xi_\alpha) \tag{E.2}$$

ξ_α denotes the coordinates, momentum, spin, and isotopic spin of fermion α. $V(\xi_\alpha, \xi_\beta)$ is the potential between fermions α and β.

The corresponding Hermitian operator in Fock space is:

$$V = \frac{1}{2} \sum_{\substack{s_j\, i_j \\ m_j \mu_j}} \int d^3x\, d^3y\, d^3x'\, d^3y'\; v^{s_j\, i_j}_{m_j \mu_j}(\underline{x}, y, \underline{x}', y') \cdot$$

$$\cdot\, \tilde{F}^{s_1 i_1 \dagger}_{m_1 \mu_1}(\underline{x})\, \tilde{F}^{s_2 i_2 \dagger}_{m_2 \mu_2}(y)\, \tilde{F}^{s_3 i_3}_{m_3 \mu_3}(y')\, \tilde{F}^{s_4 i_4}_{m_4 \mu_4}(\underline{x}') \tag{E.3}$$

where the index j runs over 1,2,3,4. $\tilde{F}^{\dagger}$ is the creation operator for a physical fermion and

$$v^{s_j i_j}_{m_j \mu_j}(\underline{x}, \underline{y}, \underline{x}', \underline{y}')$$

$$= {}_{\alpha}\langle \underline{x}, s_1 m_1 i_1 \mu_1 |_{\beta} \langle \underline{y}, s_2 m_2 i_2 \mu_2 | V(\xi_\alpha, \xi_\beta) | \underline{x}', s_4 m_4 i_4 \mu_4 \rangle_{\alpha} | \underline{y}', s_3 m_3 i_3 \mu_3 \rangle_{\beta} \tag{E.4}$$

That is, the $v^{s_j i_j}_{m_j \mu_j}(\underline{x}, \underline{y}, \underline{x}', \underline{y}')$ are the coordinate space matrix elements of the two-fermion potential $V(\xi_\alpha, \xi_\beta)$ taken between two-fermion states.

Making the change of coordinates

$$\begin{aligned} \underline{x} &= \underline{R} + \tfrac{1}{2}\underline{r} & \underline{y} &= \underline{R} - \tfrac{1}{2}\underline{r} \\ \underline{x}' &= \underline{R}' + \tfrac{1}{2}\underline{r}' & \underline{y}' &= \underline{R}' - \tfrac{1}{2}\underline{r}' \end{aligned} \tag{E.5}$$

to center of mass and relative coordinates, and imposing the condition of displacement invariance, this two-particle operator becomes

$$V = \frac{1}{2} \sum_{\substack{s_j i_j \\ m_j \mu_j}} \int d^3r\, d^3r'\, d^3R\, d^3R' \; v^{s_j i_j}_{m_j \mu_j}(\underline{r}, \underline{r}', \underline{R} - \underline{R}') \cdot$$
$$\cdot\, \tilde{F}^{s_1 i_1 \dagger}_{m_1 \mu_1}(\underline{R} + \tfrac{1}{2}\underline{r})\, \tilde{F}^{s_2 i_2 \dagger}_{m_2 \mu_2}(\underline{R} - \tfrac{1}{2}\underline{r})\, \tilde{F}^{s_3 i_3}_{m_3 \mu_3}(\underline{R}' - \tfrac{1}{2}\underline{r}')\, \tilde{F}^{s_4 i_4}_{m_4 \mu_4}(\underline{R}' + \tfrac{1}{2}\underline{r}') \tag{E.6}$$

Note that v can only depend on the difference $\underline{R}-\underline{R}'$, if V is to be translationally invariant.

Using (2.1.1), V can also be expressed as

$$V = \frac{1}{2} \sum_{\substack{s_j i_j \\ m_j \mu_j}} \int d^3k\, d^3k'\, d^3K \; v^{s_j i_j}_{m_j \mu_j}(\underline{k}, \underline{k}', \underline{K}) \cdot$$
$$\cdot\, \tilde{F}^{s_1 i_1 \dagger}_{m_1 \mu_1}(\tfrac{1}{2}\underline{K} + \underline{k})\, \tilde{F}^{s_2 i_2 \dagger}_{m_2 \mu_2}(\tfrac{1}{2}\underline{K} - \underline{k})\, \tilde{F}^{s_3 i_3}_{m_3 \mu_3}(\tfrac{1}{2}\underline{K} - \underline{k}')\, \tilde{F}^{s_4 i_4}_{m_4 \mu_4}(\tfrac{1}{2}\underline{K} + \underline{k}') \tag{E.7}$$

where

$$\mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{k},\underline{k}',\underline{K}) = \frac{1}{(2\pi\hbar)^3}\int d^3r\ d^3r'\ d^3R\ \ \mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{r},\underline{r}',\underline{R})\cdot$$

$$\cdot\, e^{-i\underline{K}\cdot\underline{R}/\hbar}\ e^{-i\underline{k}\cdot\underline{r}/\hbar}\ e^{i\underline{k}'\cdot\underline{r}'/\hbar} \tag{E.8}$$

For the special case that

$$\mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{k},\underline{k}',\underline{K}) = \mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{k}-\underline{k}') \tag{E.9}$$

we find that the inverse of equation (E.8) is

$$\mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{r},\underline{r}',\underline{R}) = \mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{r})\ \delta(\underline{R})\ \delta(\underline{r}-\underline{r}') \tag{E.10}$$

where

$$\mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{r}) = \int d^3q\ \ e^{i\underline{q}\cdot\underline{r}/\hbar}\ \ \mathcal{V}^{s_j i_j}_{m_j \mu_j}(\underline{q}) \tag{E.11}$$

We now define a two-fermion operator as follows:

$$\tilde{A}^{s\sigma\,+}_{m\mu}(\underline{k},\underline{K}) = \sum_{\substack{s_1 s_2 i_1 i_2\\ m_1 m_2 \mu_1 \mu_2}} \frac{1}{\sqrt{2}}\ (s_1 s_2 m_1 m_2 | s m)\ (i_1 i_2 \mu_1 \mu_2 | \sigma\mu)\cdot$$

$$\cdot\ \tilde{F}^{s_1 i_1 +}_{m_1 \mu_1}(\tfrac{1}{2}\underline{K}+\underline{k})\ \tilde{F}^{s_2 i_2 +}_{m_2 \mu_2}(\tfrac{1}{2}\underline{K}-\underline{k}) \tag{E.12}$$

As a function of the coordinates $\underline{r}$ and $\underline{R}$ this becomes

$$\tilde{A}^{s\sigma\,+}_{m\mu}(\underline{r},\underline{R}) = \sum_{\substack{s_1 s_2 i_1 i_2\\ m_1 m_2 \mu_1 \mu_2}} \frac{1}{\sqrt{2}}\ (s_1 s_2 m_1 m_2 | s m)\ (i_1 i_2 \mu_1 \mu_2 | \sigma\mu)\cdot$$

$$\cdot\ \tilde{F}^{s_1 i_1 +}_{m_1 \mu_1}(\underline{R}+\tfrac{1}{2}\underline{r})\ \tilde{F}^{s_2 i_2 +}_{m_2 \mu_2}(\underline{R}-\tfrac{1}{2}\underline{r}) \tag{E.13}$$

The two operators $\tilde{A}^{s\sigma\,+}_{m\mu}(\underline{k},\underline{K})$ and $\tilde{A}^{s\sigma\,+}_{m\mu}(\underline{r},\underline{R})$, when acting on the vacuum state, create a two-fermion state having total spin s, with projection m, and total isospin σ, with projection μ. They are related, using (2.1.1), by

$$\tilde{A}^{s\sigma\,+}_{m\mu}(\underline{k},\underline{K}) = \frac{1}{(2\pi\hbar)^3}\int d^3r\, d^3R\; e^{i\underline{K}\cdot\underline{R}/\hbar}\; e^{i\underline{k}\cdot\underline{r}/\hbar}\;\tilde{A}^{s\sigma\,+}_{m\mu}(\underline{r},\underline{R}) \qquad \text{(E.14)}$$

Using the properties of the Clebsch-Gordan coefficients in Appendix D, the two-particle interaction V can be written as

$$V = \sum_{\substack{s_j i_j m_j \mu_j \\ s\sigma s'\sigma'}} \int d^3k\, d^3k'\, d^3K\; v^{s\sigma s'\sigma'}_{s_j i_j m_j \mu_j}(\underline{k},\underline{k}',\underline{K})\;\tilde{A}^{s\sigma\,+}_{m\mu}(\underline{k},\underline{K})\;\tilde{A}^{s'\sigma'}_{m'\mu'}(\underline{k}',\underline{K}) \qquad \text{(E.15)}$$

where

$$v^{s\sigma s'\sigma'}_{s_j i_j m_j \mu_j}(\underline{k},\underline{k}',\underline{K}) = (s_1 s_2\, m_1 m_2 | sm)(i_1 i_2\, \mu_1\mu_2 | \sigma\mu)(s_4 s_3\, m_4 m_3 | s'm')\cdot$$

$$\cdot(i_4 i_3\, \mu_4\mu_3 | \sigma'\mu')\quad v^{s_j i_j}_{m_j \mu_j}(\underline{k},\underline{k}',\underline{K}) \qquad \text{(E.16)}$$

The potential functions $v^{s\sigma s'\sigma'}_{s_j i_j m_j \mu_j}(\underline{k},\underline{k}',\underline{K})$ are the momentum space matrix elements of the two-fermion potential $V(\xi_\alpha, \xi_\beta)$ between two-fermion states of total spin s, isospin σ and total spin s', isospin σ'. These functions may be pictured as in Figure 11:

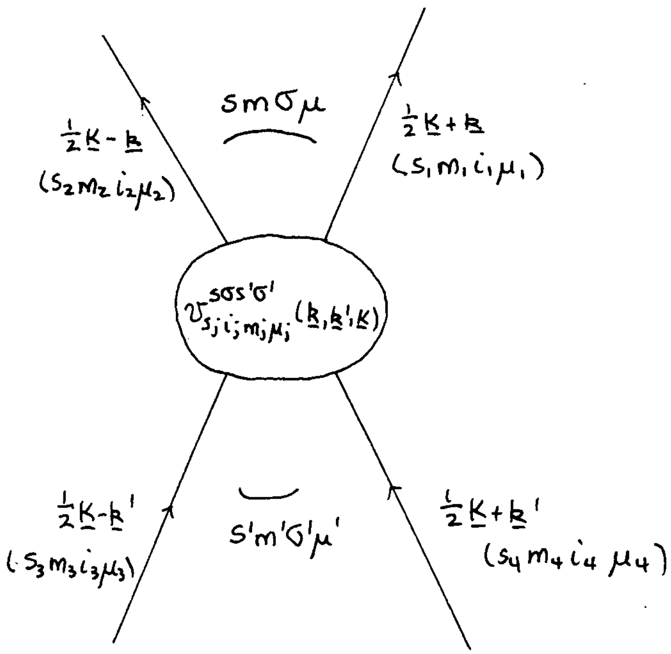

Fig. 11 The Two-Fermion Potential $v^{s\sigma s'\sigma'}_{s_j i_j m_j \mu_j}(\underline{k},\underline{k}',\underline{K})$

Appendix F Calculation of the Trilinear Vertex Function for the Cloudy Bag Model

In this Appendix, we determine the NNπ trilinear vertex function h(q) for the Cloudy Bag Model.

The one-nucleon matrix elements of the nucleon-pion interaction in the Cloudy Bag Model are (Thebérge et al. (1980, eq. (2.25))):

$$\langle N | H_{CBM}^{NN\pi} | N \rangle = \frac{1}{(2\pi)^{3/2}} \sum_{\mu} \int \frac{d^3k}{[2\omega(k)]^{1/2}} \, v_{\mu}^{NN}(\underline{k})\, B_{\mu}(\underline{k}) + \text{adj}. \tag{F.1a}$$

$$v_{\mu}^{NN}(\underline{k}) = i\sqrt{4\pi}\, \frac{f_0}{m_\pi}\, u_N(k) \langle \tfrac{1}{2}m_1, \tfrac{1}{2}\mu_1 | \underline{\sigma}\cdot\underline{k}\, \tau_\mu | \tfrac{1}{2}m_2, \tfrac{1}{2}\mu_2 \rangle \tag{F.1b}$$

where f_0 is the NNπ coupling constant, m_π is the pion mass, and

$$\omega(k) = (k^2 + m_\pi^2)^{1/2} \tag{F.2}$$

$$u_N(k) = 3\, j_1(kR)/kR \tag{F.3}$$

where $j_1(kR)$ is a spherical Bessel function of order one and R is the bag radius. Further, $|\tfrac{1}{2}m, \tfrac{1}{2}\mu\rangle$ is the spin-isospin state describing a nucleon with spin projection m and isospin projection μ; $\underline{S} = \underline{\sigma}/2$ is the nucleon spin operator, where $\underline{\sigma}$ has as its components the Pauli spin matrices; $I_\mu = \tau_\mu/2$ are the spherical components of the nucleon isospin operator. The above is written in units where $\hbar = c = 1$.

Note the similarity between the interaction (F.1) and the form (3.1.41) for the nucleon-pion trilinear interaction. We determine the vertex function h(q) for the Cloudy Bag Model by evaluating the spin-isospin matrix elements in (F.1b) and comparing the result with the one-nucleon matrix elements of H_1 given by (3.1.36).

First, we separate the spin and isospin parts of the matrix element. Expanding $\underset{\sim}{S}\cdot\underset{\sim}{k}$ in spherical components using (C.4), we have

$$\langle \tfrac{1}{2}m_1, \tfrac{1}{2}\mu_1 |\ \underset{\sim}{S}\cdot\underset{\sim}{k}\ I_\mu\ | \tfrac{1}{2}m_2, \tfrac{1}{2}\mu_2 \rangle$$

$$= \sum_{\alpha=\pm 1,0} (-)^\alpha k_{-\alpha} \langle \tfrac{1}{2}m_1 | S_\alpha | \tfrac{1}{2}m_2 \rangle \langle \tfrac{1}{2}\mu_1 | I_\mu | \tfrac{1}{2}\mu_2 \rangle \tag{F.4}$$

The Wigner-Eckart theorem (C.5) gives

$$\langle \tfrac{1}{2}m_1 | S_\alpha | \tfrac{1}{2}m_2 \rangle = (\tfrac{1}{2}\ |\ m_2\alpha\ |\ \tfrac{1}{2}m_1) \langle \tfrac{1}{2} \| \underset{\sim}{S} \| \tfrac{1}{2} \rangle \tag{F.5}$$

$$\langle \tfrac{1}{2}\mu_1 | I_\mu | \tfrac{1}{2}\mu_2 \rangle = (\tfrac{1}{2}\ |\ \mu_2\mu\ |\ \tfrac{1}{2}\mu_1) \langle \tfrac{1}{2} \| \underset{\sim}{I} \| \tfrac{1}{2} \rangle \tag{F.6}$$

where

$$\langle \tfrac{1}{2} \| \underset{\sim}{S} \| \tfrac{1}{2} \rangle = \langle \tfrac{1}{2} \| \underset{\sim}{I} \| \tfrac{1}{2} \rangle = \sqrt{3}/2 \tag{F.7}$$

Substituting (F.5), (F.6), and (F.7) into (F.1), and using (C.3) to express $q_{-\alpha}$ in terms of q and a spherical harmonic, we find

$$\langle N | H_{CBM}^{NN\pi} | N \rangle = i\sqrt{\frac{3}{\pi}}\ \frac{f_0}{m_\pi} \sum_{\alpha\mu} \int \frac{d^3k}{\sqrt{\omega(k)}}\ k\ U_N(k)\ Y_{1\alpha}^*(\underset{\sim}{k}) \cdot$$

$$\cdot\ (\tfrac{1}{2}\ |\ \mu_2\mu\ |\ \tfrac{1}{2}\mu_1)\ (\tfrac{1}{2}\ |\ m_2\alpha\ |\ \tfrac{1}{2}m_1)\ B_\mu(\underset{\sim}{k}) + \text{adj}. \tag{F.8}$$

Now consider the matrix elements of the interaction H_1 in (3.1.36) between one nucleon states. Using the commutation relations for the fermion operators, we find

$$\langle 0 | F_{m_1\mu_1}(\underset{\sim}{p})\, H_1\, F^{+}_{m_2\mu_2}(\underset{\sim}{p}-\underset{\sim}{q})\, | 0 \rangle = i \sum_{\mu} \int d^3q\; h(q)\; Y^{*}_{1m}(\underset{\sim}{q}) \cdot$$

$$\cdot \left(\tfrac{1}{2}\, 1\; m_2\, m \,\middle|\, \tfrac{1}{2}\, m_1\right)\left(\tfrac{1}{2}\, 1\; \mu_2\, \mu \,\middle|\, \tfrac{1}{2}\, \mu_1\right) B_{\mu}(\underset{\sim}{q}) + \text{adj}. \tag{F.9}$$

Comparing (F.8) and (F.9), and putting in $\hbar$ and c as required to give the correct dimensions to the vertex function which we write as $h_{CBM}(q)$, we see that

$$h_{CBM}(q) = \sqrt{\frac{3c}{\pi}}\; \frac{f_0}{m_\pi}\; q\; \frac{U_N(q/\hbar)}{\mathcal{E}_\pi(q)} \tag{F.10}$$

where $\mathcal{E}_\pi(q)$ is given by equation (3.1.3). The two different formalisms can be made identical by the choice (F.10) for the vertex function. We are able to do this because both trilinear interactions, (F.1) and (3.1.36), are required to be invariant under translations, rotations, space inversion, and time reversal. Rotational invariance determines that both interactions will have the same dependence on spin and isospin projection quantum numbers and on the angles $\underset{\sim}{q}$. Space inversion invariance requires that the orbital angular momentum of the pion be one, and this is manifested in (F.1) by the vector character of the operator $\underset{\sim}{S}$ (eq. (F.5)).

Appendix G An Exact Dressing Operator for the Lee Model

In this Appendix we follow the work of Piskunov (1974) in determining an exact dressing operator for the Lee model discussed in Section 4.3. (The method of solution for the case $h = h(\underline{p},\underline{q})$ is analogous to the procedure given here.) As equations (4.3.12a) and (4.3.15a) suggest, we write

$$D = \int d^3p\, d^3q\; d(\underline{p},\underline{q})\,[F_2^+(\underline{p})\, F_1(\underline{p}-\underline{q})\, B(\underline{q}) - adj.] \tag{G.1}$$

and attempt to find the function $d(\underline{p},\underline{q})$. We do this by determining the creator $\tilde{F}_2^+(\underline{p})$ in terms of $d(\underline{p},\underline{q})$ and then requiring that $\tilde{F}_2^+(\underline{p})|0>$ be an eigenket of the Hamiltonian.

Recall from (4.1.3) that

$$\begin{aligned} \tilde{F}_2^+(\underline{p}) &= e^{D}\, F_2^+(\underline{p})\, e^{-D} \\ &= F_2^+(\underline{p}) - [F_2^+(\underline{p}),D] + \frac{1}{2!}\,[[F_2^+(\underline{p}),D],D] + \ldots \end{aligned} \tag{G.2}$$

Using (G.1) for D, and the commutators (4.3.5) - (4.3.9), we find

$$[F_2^+(\underline{p}),D] = \int d^3q\; d(\underline{p},\underline{q})\; F_1^+(\underline{p},\underline{q})\, B^+(\underline{q}) \tag{G.3}$$

and

$$[[F_2^+(\underline{p}),D],D] = -\,d^2(\underline{p})\; F_2^+(\underline{p}) \tag{G.4}$$

where

$$d^2(\underline{p}) = \int d^3q\; d(\underline{p},\underline{q}) \tag{G.5}$$

The series in (G.2) thus separates into two series, each of which can be summed exactly. We find

$$\tilde{F}_2^+(\underset{\sim}{p}) = \cos d(\underset{\sim}{p})\ F_2^+(\underset{\sim}{p}) - \int d^3\underset{\sim}{q}\ f(\underset{\sim}{p},\underset{\sim}{q})\, F_1^+(\underset{\sim}{p}-\underset{\sim}{q}) B^+(\underset{\sim}{q}) \tag{G.6}$$

where

$$f(\underset{\sim}{p},\underset{\sim}{q}) = d(\underset{\sim}{p},\underset{\sim}{q})\ \frac{\sin d(\underset{\sim}{p})}{d(\underset{\sim}{p})} \tag{G.7}$$

This equation gives $\tilde{F}_2^+(\underset{\sim}{p})$ in terms of $d(\underset{\sim}{p},\underset{\sim}{q})$. We now substitute $\tilde{F}_2^+(\underset{\sim}{p})$ into the equation

$$H\,\tilde{F}_2^+(\underset{\sim}{p})|0\rangle = \mathcal{E}_2(\underset{\sim}{p})\,\tilde{F}_2^+(\underset{\sim}{p})|0\rangle \tag{G.8}$$

where $\mathcal{E}_2(\underset{\sim}{p})$ is the energy of the physical V-particle state.

Using the commutators (4.3.5)-(4.3.9) and the expression (4.3.1) for H, equation (G.8) becomes

$$\Big\{\cos d(\underset{\sim}{p})\,[\mathcal{E}_{20}(\underset{\sim}{p}) - \mathcal{E}_2(\underset{\sim}{p})] - \lambda \int d^3\underset{\sim}{q}\ h(\underset{\sim}{q})\, f(\underset{\sim}{p},\underset{\sim}{q})\Big\}\, F_2^+(\underset{\sim}{p})|0\rangle$$

$$+ \int d^3\underset{\sim}{q}\ \Big\{\lambda \cos d(\underset{\sim}{p})\, h(\underset{\sim}{q}) - f(\underset{\sim}{p},\underset{\sim}{q})\,[\mathcal{E}_1(\underset{\sim}{p}-\underset{\sim}{q}) + \mathcal{E}_B(\underset{\sim}{q}) - \mathcal{E}_2(\underset{\sim}{p})]\Big\}\cdot$$

$$\cdot\, F_1^+(\underset{\sim}{p}-\underset{\sim}{q})\, B^+(\underset{\sim}{q})\,|0\rangle = 0 \tag{G.9}$$

Taking the scalar product of this equation from the left with $\langle 0|F_2(\underset{\sim}{p}')$ and then with $\langle 0|F_1(\underset{\sim}{p}'-\underset{\sim}{q}')B(\underset{\sim}{q}')$ gives

$$\mathcal{E}_{20}(\underset{\sim}{p}') - \mathcal{E}_2(\underset{\sim}{p}') = \lambda \int d^3\underset{\sim}{q}\ \frac{h(\underset{\sim}{q})\, f(\underset{\sim}{p}',\underset{\sim}{q})}{\cos d(\underset{\sim}{p}')} \tag{G.10}$$

$$\frac{f(\underset{\sim}{p}',\underset{\sim}{q}')}{\cos d(\underset{\sim}{p}')} = \frac{\lambda\, h(\underset{\sim}{q}')}{\mathcal{E}_1(\underset{\sim}{p}'-\underset{\sim}{q}') + \mathcal{E}_B(\underset{\sim}{q}') - \mathcal{E}_2(\underset{\sim}{p}')} \tag{G.11}$$

Substituting (G.11) into (G.10) gives the result

$$\mathcal{E}_2(\underset{\sim}{p}) = \mathcal{E}_{20}(\underset{\sim}{p}) + \lambda^2 \int d^3\underset{\sim}{q}\ \frac{h^2(\underset{\sim}{q})}{\mathcal{E}_2(\underset{\sim}{p}) - \mathcal{E}_1(\underset{\sim}{p}-\underset{\sim}{q}) - \mathcal{E}_B(\underset{\sim}{q})} \tag{G.12}$$

We now solve for $d(\underline{p},\underline{q})$ by substituting $f(\underline{p},\underline{q})$ from (G.7) into (G.11), squaring the resulting equation, and integrating both sides with respect to $\underline{q}$. We find

$$\tan^2 g(\underline{p}) = \lambda^2 g(\underline{p}) \tag{G.13}$$

where

$$g^2(\underline{p}) = \int d^3q \; \frac{h^2(\underline{q})}{[\mathcal{E}_2(\underline{p}) - \mathcal{E}_1(\underline{p}-\underline{q}) - \mathcal{E}_B(\underline{q})]^2} \tag{G.14}$$

Finally, from (G.11), (G.7), and (G.13), we have

$$d(\underline{p},\underline{q}) = \frac{h(\underline{q}) \; \arctan\,(\lambda g(\underline{p}))}{g(\underline{p})\,[\mathcal{E}_2(\underline{p}) - \mathcal{E}_1(\underline{p}-\underline{q}) - \mathcal{E}_B(\underline{q})]} \tag{G.15}$$

Thus, for the Lee model, we find that one can write the dressing transformation exactly, as in (G.1), with $d(\underline{p},\underline{q})$ given by (G.15). Indeed, if this D is expanded in a power series in λ, one obtains exactly the same dressing transformation to first, second, and third orders as is given by equations (4.3.12), (4.3.14), and (4.3.15). Furthermore, we note that if we iteratively solve (G.12) for $\mathcal{E}_2(\underline{p})$, the solution to order λ^2 is that given by equation (4.3.22).

Using (G.15), (G.7) and (G.6) we obtain the physical V-particle creator:

$$\tilde{F}_2^{+}(\underline{p}) = \cos d(\underline{p}) \left[F_2^{+}(\underline{p}) - \lambda \int d^3q \; \frac{h(\underline{q})\, F_1^{+}(\underline{p}-\underline{q})\, B^{+}(\underline{q})}{[\mathcal{E}_1(\underline{p}-\underline{q}) + \mathcal{E}_B(\underline{q}) - \mathcal{E}_2(\underline{p})]}\right] \tag{G.16}$$

The physical V-particle state is a superposition of a state containing an elementary V particle with a state containing one N and one θ particle.

Appendix H A Dressing Transformation For A Generalized Fermion-Boson Trilinear Interaction

In this Appendix we generalize the trilinear interaction of Chapter 3 to include fermions and bosons of arbitrary spin and isospin. In doing so, we not only are able to consider interactions of nucleons and pions, but also interactions of other types of fermions and bosons. First, we examine the restrictions placed on this generalized interaction by the requirement that it be invariant under certain space-time transformations. Secondly, we perform a dressing transformation on the fundamental dynamical variables as in Section 5.1. Finally, we consider the various terms in the second-order dressed Hamiltonian, particularly the fermion-fermion scattering term.

Although the formulae become more complicated with this generalized interaction, the theory of the dressing transformation can be applied as successfully to this interaction as to the simpler theories we have previously studied.

(a) The Generalized Fermion-Boson Trilinear Interaction

Consider the following generalized trilinear Hamiltonian constructed from the fundamental dynamical variables defined in Section 2.1:

$$H = H_0 + \lambda H_1 \qquad \text{(H.1a)}$$

$$H_0 = \sum_{\substack{sim\mu \\ s'i'm'\mu'}} \int d^3p \left[\mathcal{E}_{Fo\,\mu}^{\;si}(\underline{p}) F_{m\mu}^{si+}(\underline{p})\; F_{m\mu}^{si}(\underline{p}) + \mathcal{E}_{Bo\,\mu'}^{\;s'i'}(\underline{p})\; B_{m'\mu'}^{s'i'+}(\underline{p})\, B_{m'\mu'}^{s'i'}(\underline{p}) \right] \tag{H.1b}$$

$$H_1 = \sum_{\substack{s_1s_2s_3i_1i_2i_3 \\ m_1m_2m_3\mu_1\mu_2\mu_3}} \int d^3p\; d^3q\;\; h_{m_1m_2m_3\mu_1\mu_2\mu_3}^{s_1s_2s_3i_1i_2i_3}(\underline{q})\; F_{m_1\mu_1}^{s_1i_1+}(\underline{p})\, F_{m_2\mu_2}^{s_2i_2}(\underline{p}-\underline{q})\, B_{m_3\mu_3}^{s_3i_3}(\underline{q}) + \text{adj.} \tag{H.1c}$$

where

$$\mathcal{E}_{o\;\mu}^{\;si}(\underline{p}) = \left[p^2c^2 + m_o^2c^4\right]^{1/2} \tag{H.2}$$

is the energy of the elementary fermion or boson having spin s, isospin i and isospin z-axis projection μ.

The fermions and bosons are treated 'semi-relativistically' and the vertex function is chosen to be a function of $\underline{q}$ only.

The interaction H_1 may be pictured as in Figure 12:

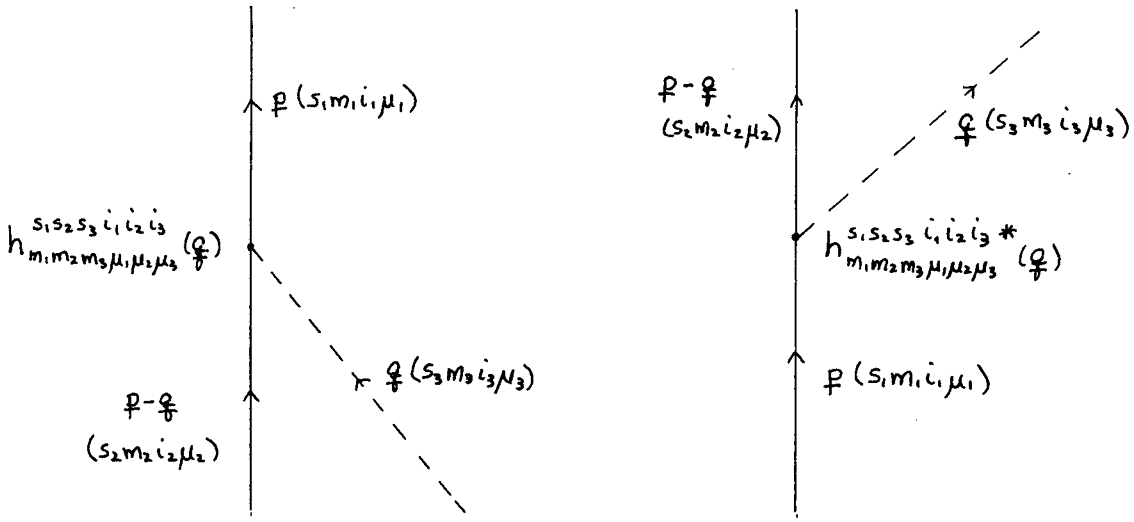

Fig. 12 The Generalized Trilinear F-B Interaction

The total momentum operator for the system is

$$\underline{P} = \sum_{\substack{sim\mu \\ s'i'm'\mu'}} \int d^3p\; \underline{p}\left[F_{m\mu}^{si+}(\underline{p})\, F_{m\mu}^{si}(\underline{p}) + B_{m'\mu'}^{s'i'+}(\underline{p})\; B_{m'\mu'}^{s'i'}(\underline{p}) \right] \tag{H.3}$$

The Hamiltonian H is translationally invariant. We also require that it be invariant under rotations in isospin space, spatial rotations, space inversion, and time reversal. H_0 already satisfies these requirements, but H_1 does not without further restriction.

First, we consider a rotation in isospin space. We must have

$$\mathcal{R}_I(\alpha\beta\gamma)\ H_1\ \mathcal{R}_I^{\dagger}(\alpha\beta\gamma) = H_1 \tag{H.4}$$

Using (2.2.18) and (2.2.19), we proceed similarly to steps (3.1.6) through (3.1.10). We discover that isospin rotational invariance implies

$$h^{s_1 s_2 s_3\, i_1 i_2 i_3}_{m_1 m_2 m_3\, \mu_1 \mu_2 \mu_3}(q) = (i_2 i_3 \mu_2 \mu_3 | i_1 \mu_1)\ h^{s_1 s_2 s_3\, i_1 i_2 i_3}_{m_1 m_2 m_3}(q) \tag{H.5}$$

Secondly, we consider a spatial rotation, demanding

$$\mathcal{R}(\alpha\beta\gamma)\ H_1\ \mathcal{R}^{\dagger}(\alpha\beta\gamma) = H_1 \tag{H.6}$$

Analogously to (3.1.13), using (2.2.8) and (2.2.9), we find

$$h^{s_1 s_2 s_3\, i_1 i_2 i_3}_{m_1 m_2 m_3}(q) = \sum_{m_1' m_2' m_3'} h^{s_1 s_2 s_3\, i_1 i_2 i_3}_{m_1' m_2' m_3'}(q_R)\cdot$$

$$\cdot\, D^{s_1}_{m_1 m_1'}(\alpha\beta\gamma)\ D^{s_2\,*}_{m_2 m_2'}(\alpha\beta\gamma)\ D^{s_3\,*}_{m_3 m_3'}(\alpha\beta\gamma) \tag{H.7}$$

As in (3.1.14), we write

$$h^{s_1 s_2 s_3\, i_1 i_2 i_3}_{m_1 m_2 m_3}(q) = i \sum_{\ell m} Y^{*}_{\ell m}(q)\ h^{s_1 s_2 s_3\, i_1 i_2 i_3}_{\ell m\, m_1 m_2 m_3}(q) \tag{H.8}$$

Substituting this expression into (H.7), we obtain an expression comparable to (3.1.17), namely

$$h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{LM \, m_1 m_2 m_3}(q) = \sum_{m_1' m_2' m_3' m} h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{Lm \, m_1' m_2' m_3'}(q) \cdot$$

$$\cdot D^{L\,*}_{Mm}(\alpha\beta\gamma) \, D^{s_1}_{m_1 m_1'}(\alpha\beta\gamma) \, D^{s_2\,*}_{m_2 m_2'}(\alpha\beta\gamma) \, D^{s_3\,*}_{m_3 m_3'}(\alpha\beta\gamma) \tag{H.9}$$

Using (A.1) and (A.8) to combine the first pair of rotation matrices into one, and then to combine the second pair into one, we proceed as in (3.1.18) and (3.1.19). We discover that the m_1, m_2, m_3, M dependence of the vertex function is given by

$$h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{LM \, m_1 m_2 m_3}(q) = \sum_{S} (-)^{M} \, (s_2 s_3 m_2 m_3 | S\alpha) \, (L s_1 -M m_1 | S\alpha) \cdot$$
$$\cdot \, h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{LS}(q) \tag{H.10}$$

Therefore, from (B.5) and (H.8), we can write

$$h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{m_1 m_2 m_3}(q) = i \sum_{\ell m S} Y_{\ell m}(q) \, h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{\ell S}(q) \cdot$$

$$\cdot (s_2 s_3 m_2 m_3 | S\alpha) \, (\ell s_1 m m_1 | S\alpha) \tag{H.11}$$

Thirdly, we require that H_1 be invariant under space inversion:

$$\mathcal{P} H_1 \mathcal{P}^{\dagger} = H_1 \tag{H.12}$$

This implies, as in (3.1.22) and (3.1.23), that

$$h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{m_1 m_2 m_3}(q) = - h^{s_1 s_2 s_3 \, i_1 i_2 i_3}_{m_1 m_2 m_3}(-q) \tag{H.13}$$

for positive parity fermions and negative parity bosons. Substituting for the vertex function from (H.11), we see that this implies

$$(-)^{\ell} = -1 \tag{H.14}$$

That is, ℓ must be odd in order to have space inversion invariance.

Finally, we consider a time reversal transformation. We require

$$\mathcal{J}\, H_1\, \mathcal{J}^\dagger = H_1 \tag{H.15}$$

Using (2.2.14), (2.2.16) and (2.2.17), this leads to

$$h^{s_1 s_2 s_3\; i_1 i_2 i_3}_{-m_1 -m_2 -m_3}(\underset{\sim}{q}) = (-)^{s_1+s_2+s_3-m_1-m_2-m_3}\, h^{s_1 s_2 s_3\; i_1 i_2 i_3 *}_{m_1 m_2 m_3}(\underset{\sim}{q}) \tag{H.16}$$

where we have taken $\eta_F = 1$ and $\eta_B = -1$. Following steps similar to (3.1.29) through (3.1.34), we conclude

$$h^{s_1 s_2 s_3\; i_1 i_2 i_3 *}_{\ell s}(q) = h^{s_1 s_2 s_3\; i_1 i_2 i_3}_{\ell s}(q) \tag{H.17}$$

Thus the interaction (H.1c) can be written

$$H_1 = i \sum_{\substack{s_1 s_2 s_3 i_1 i_2 i_3 \\ m_1 m_2 m_3 \mu_1 \mu_2 \\ \mu_3\; \ell s}} \int d^3p\, d^3q\; (i_2 i_3 \mu_2 \mu_3 | i_1 \mu_1)(s_2 s_3 m_2 m_3 | s\alpha) \cdot$$
$$\cdot (\ell s_1 m m_1 | s\alpha)\; Y_{\ell m}(\underset{\sim}{q})\; h^{s_1 s_2 s_3\; i_1 i_2 i_3}_{\ell s}(q) \cdot$$
$$\cdot F^{s_1 i_1 \dagger}_{m_1 \mu_1}(\underset{\sim}{p})\, F^{s_2 i_2}_{m_2 \mu_2}(\underset{\sim}{p}-\underset{\sim}{q})\, B^{s_3 i_3}_{m_3 \mu_3}(\underset{\sim}{q}) + \text{adj.} \tag{H.18}$$

where ℓ is odd and $h^{s_1 s_2 s_3\; i_1 i_2 i_3}_{\ell s}(q)$ is a real function. Referring to (H.18) and Figures 12, 13, and 14, we interpret the quantum number s in $h^{s_1 s_2 s_3\; i_1 i_2 i_3}_{\ell s}(q)$ as the total spin resulting from coupling fermion spin s_2 to boson spin s_3. ℓ is the orbital angular momentum of the boson (s_3, i_3) with respect to the fermion (s_2, i_2). We see that $\underset{\sim}{\ell} + \underline{s} = \underline{s}_1$, in order to conserve angular momentum.

For the particular case that $s_1 = s_2 = i_1 = i_2 = 1/2$, $s_3 = m_3 = 0$, $i_3 = 1$, we recover the interaction (3.1.36). From the properties of the Clebsch-Gordan coefficients in (H.18), we see that $s = 1/2$ and $\ell = 1$ for this case. Therefore

$$h^{\frac{1}{2}\frac{1}{2}0\;\frac{1}{2}\frac{1}{2}1}_{m_1 m_2 0\;\mu_1\mu_2\mu_3}(q) = i\,(\tfrac{1}{2}\,1\,\mu_2\mu_3|\tfrac{1}{2}\mu_1)(1\,\tfrac{1}{2}\,m\,m_1|\tfrac{1}{2}\,m_2)\cdot$$

$$\cdot\, Y_{1m}(q)\; h^{\frac{1}{2}\frac{1}{2}0\;\frac{1}{2}\frac{1}{2}1}_{1\frac{1}{2}}(q) \tag{H.19}$$

Noting that

$$Y_{1m}(q)\,(1\,\tfrac{1}{2}\,m\,m_1|\tfrac{1}{2}m_2) = (-)^1\, Y^*_{1m}(q)\,(\tfrac{1}{2}\,1\,m_2\,-m|\tfrac{1}{2}m_1) \tag{H.20}$$

and comparing (H.19) with (3.1.35), we see

$$h(q) = -\,h^{\frac{1}{2}\frac{1}{2}0\;\frac{1}{2}\frac{1}{2}1}_{1\frac{1}{2}}(q) \tag{H.21}$$

where $h(q)$ is the vertex function introduced in Chapter 3 for the nucleon-pion interaction.

(b) Dressing the Generalized Interaction

Just as we have noted for the Hamiltonian (3.1.1), the one boson ket $B^{si\dagger}_{m\mu}(\underline{p})|0\rangle$ is an eigenket of the Hamiltonian (H.1), while the one fermion ket $F^{si\dagger}_{m\mu}(\underline{p})|0\rangle$ is not. We require a dressing transformation as developed in Chapter 4. To first order, a suitably invariant dressing operator that satisfies (4.1.20) is

$$D_1 = \sum_{\substack{s_1s_2s_3\,i_1i_2i_3\\ m_1m_2m_3\mu_1\mu_2\mu_3}} \int d^3p\, d^3q\; \frac{h^{s_1s_2s_3\;i_1i_2i_3}_{m_1m_2m_3\;\mu_1\mu_2\mu_3}(q)}{\mathcal{E}_B{}^{s_3 i_3}_{\mu_3}(q)}\;.$$

$$\cdot\, F^{s_1i_1\dagger}_{m_1\mu_1}(\underline{p})\, F^{s_2i_2}_{m_2\mu_2}(\underline{p}-\underline{q})\, B^{s_3i_3}_{m_3\mu_3}(\underline{q}) - \text{adj}. \tag{H.22}$$

where we have taken

$$\mathcal{E}_{Fo\mu}^{si}(p) = m_{Fo\,\mu}^{\,si}\, c^2 \tag{H.23}$$

The commutator $[H_1, D_1]$ can now be computed. It is found to contain an unsuitable term of the form $F^{\dagger}FBB$ + adj., which can be eliminated from the second-order dressed Hamiltonian through a suitable choice of the dressing operator D_2. This operator will be similar to D_2 given by (5.1.10). From (4.1.21), the resulting second-order Hamiltonian is:

$$\tilde{H}(\tilde{F},\tilde{B}) = T + V_{FB} + V_{FF} \tag{H.24}$$

$$T = \sum_{\substack{sim\mu \\ s'i'm'\mu'}} \int d^3p \left[m_{F\,\mu}^{\,si}\, c^2\, \tilde{F}_{m\mu}^{si\,\dagger}(p)\, \tilde{F}_{m\mu}^{si}(p) + \mathcal{E}_{B\,\mu'}^{\,s'i'}(p)\, \tilde{B}_{m'\mu'}^{s'i'\dagger}(p)\, \tilde{B}_{m'\mu'}^{s'i'}(p) \right] \tag{H.25}$$

$$\begin{aligned} V_{FB} = \frac{\lambda^2}{2} \sum_{\substack{s_1s_2s_3\, i_1i_2i_3 \\ m_1m_2m_3\,\mu_1\mu_2\mu_3 \\ s_2's_3'\, i_2'i_3'\, m_2' \\ m_3'\,\mu_2'\mu_3'}} & \int d^3k\, d^3k'\, d^3K\;\; \upsilon_{FB}(k,k',K)\,\cdot \\ & \cdot\, \tilde{F}_{m_2'\mu_2'}^{s_2'i_2'\dagger}(\tfrac{1}{2}K+k)\, \tilde{B}_{m_3'\mu_3'}^{s_3'i_3'\dagger}(\tfrac{1}{2}K-k)\, \tilde{B}_{m_3\mu_3}^{s_3i_3}(\tfrac{1}{2}K-k')\, \tilde{F}_{m_2\mu_2}^{s_2i_2}(\tfrac{1}{2}K+k') \\ & + \text{adj}. \end{aligned} \tag{H.26}$$

$$\begin{aligned} V_{FF} = \frac{\lambda^2}{2} \sum_{\substack{s_1s_2s_3\, i_1i_2i_3 \\ m_1m_2m_3\,\mu_1\mu_2\mu_3 \\ s_1's_2'\, i_1'i_2' \\ m_1'm_2'\,\mu_1'\mu_2'}} & \int d^3k\, d^3k'\, d^3K\;\; \upsilon_{FF}(k-k')\,\cdot \\ & \cdot\, \tilde{F}_{m_1\mu_1}^{s_1i_1\dagger}(\tfrac{1}{2}K+k)\, \tilde{F}_{m_2'\mu_2'}^{s_2'i_2'\dagger}(\tfrac{1}{2}K-k)\, \tilde{F}_{m_1'\mu_1'}^{s_1'i_1'}(\tfrac{1}{2}K-k')\, \tilde{F}_{m_2\mu_2}^{s_2i_2}(\tfrac{1}{2}K+k') \\ & + \text{adj}. \end{aligned} \tag{H.27}$$

where

$$m_{F\,\mu}^{\,si}\, c^2 = m_{Fo\,\mu}^{\,si}\, c^2 - \lambda^2 \sum_{\substack{s_1s_2i_1i_2 \\ \ell\alpha}} \int q^2 dq \left(\frac{2\alpha+1}{2s+1}\right) \frac{\left(h_{\ell\alpha}^{ss_1s_2\, ii_1i_2}(q)\right)^2}{\mathcal{E}_{B\,\mu_2}^{\,s_2i_2}(q)} \tag{H.28}$$

$$\mathcal{V}_{FB}(\underline{k},\underline{k}',\underline{K}) = \sum_{\ell\ell' ss'} Y_{\ell m}(\tfrac{1}{2}\underline{K}-\underline{k}')\, Y_{\ell' m'}^{*}(\tfrac{1}{2}\underline{K}-\underline{k}) \frac{1}{\mathcal{E}_{B\,\mu_3'}^{\;s_3' i_3'}(\tfrac{1}{2}\underline{K}-\underline{k})} \cdot$$

$$\cdot \Big[(i_2 i_3 \mu_2 \mu_3 | i_1 \mu_1)(i_2' i_3' \mu_2' \mu_3' | i_1 \mu_1)(s_2 s_3 m_2 m_3 | s\alpha) \cdot$$

$$\cdot (\ell s_1 m m_1 | s\alpha)(s_2' s_3' m_2' m_3' | s'\alpha')(\ell' s_1 m' m_1 | s'\alpha') \cdot$$

$$\cdot h_{\ell s}^{s_1 s_2 s_3\, i_1 i_2 i_3}(|\tfrac{1}{2}\underline{K}-\underline{k}'|)\, h_{\ell' s'}^{s_1 s_2' s_3'\, i_1 i_2' i_3'}(|\tfrac{1}{2}\underline{K}-\underline{k}|)$$

$$-(i_1 i_3 \mu_1 \mu_3 | i_2' \mu_2')(i_1 i_3' \mu_1 \mu_3' | i_2 \mu_2)(s_1 s_3 m_1 m_3 | s\alpha) \cdot$$

$$\cdot (\ell s_2' m m_2' | s\alpha)(s_1 s_3' m_1 m_3' | s'\alpha')(\ell' s_2 m' m_2 | s'\alpha') \cdot$$

$$\cdot h_{\ell s}^{s_2' s_1 s_3\, i_2' i_1 i_3}(|\tfrac{1}{2}\underline{K}-\underline{k}'|)\, h_{\ell' s'}^{s_2 s_1 s_3'\, i_2 i_1 i_3'}(|\tfrac{1}{2}\underline{K}-\underline{k}|)\Big] \tag{H.29}$$

$$\mathcal{V}_{FF}(\underline{k}-\underline{k}') = - Y_{\ell m}(\underline{k}-\underline{k}')\, Y_{\ell' m'}^{*}(\underline{k}-\underline{k}') \frac{1}{\mathcal{E}_{B\,\mu_3}^{\;s_3 i_3}(\underline{k}-\underline{k}')} \cdot$$

$$\cdot (i_2 i_3 \mu_2 \mu_3 | i_1 \mu_1)(i_2' i_3 \mu_2' \mu_3 | i_1' \mu_1')(s_2 s_3 m_2 m_3 | s\alpha) \cdot$$

$$\cdot (\ell s_1 m m_1 | s\alpha)(s_2' s_3 m_2' m_3 | s'\alpha')(\ell' s_1' m' m_1' | s'\alpha') \cdot$$

$$\cdot h_{\ell s}^{s_1 s_2 s_3\, i_1 i_2 i_3}(|\underline{k}-\underline{k}'|)\, h_{\ell' s'}^{s_1' s_2' s_3\, i_1' i_2' i_3}(|\underline{k}-\underline{k}'|) \tag{H.30}$$

Equation (H.28) gives the fermion mass renormalization, to order λ^2.

The functions $\mathcal{V}_{FB}$ and $\mathcal{V}_{FF}$ are pictured in Figures 13 and 14:

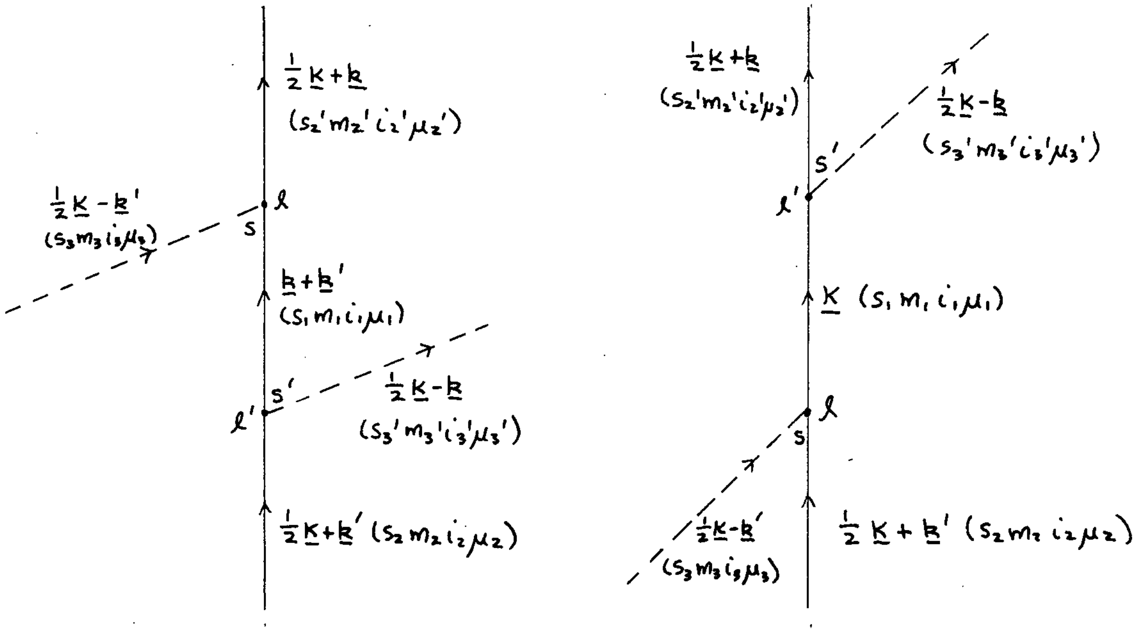

Fig. 13 The Fermion-Boson Potential $v_{FB}(\underline{k},\underline{k}',\underline{K})$

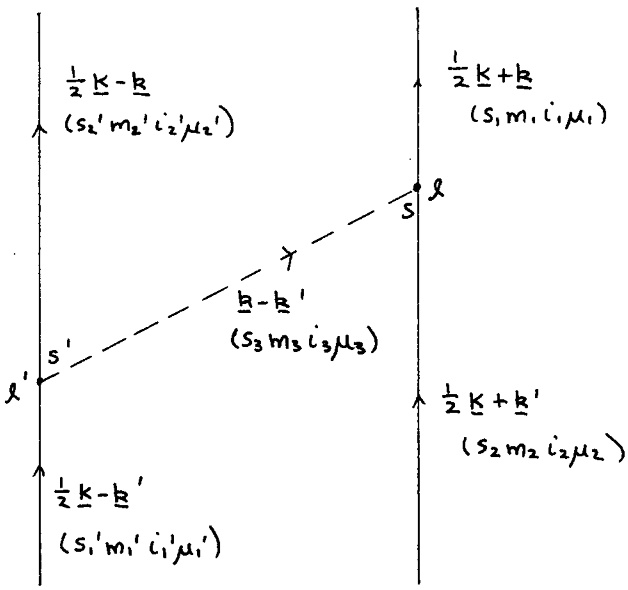

Fig. 14 The Fermion-Fermion Potential $v_{FF}(\underline{k}-\underline{k}')$

The fermion-fermion potential can be simplified by evaluating many of the sums over projection quantum numbers in (H.30). First, we rewrite the fermion-fermion interaction in terms of the two-fermion operators $\tilde{A}^{s\sigma}_{m\mu}$ defined in Appendix E:

$$V_{FF} = \lambda^2 \sum_{\substack{s\sigma s'\sigma' \\ MM'\beta\beta'}} \int d^3k\, d^3k'\, d^3K\; v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(\underline{k}-\underline{k}')\, \tilde{A}^{s\sigma\dagger}_{M\beta}(\underline{k},\underline{K})\, \tilde{A}^{s'\sigma'}_{M'\beta'}(\underline{k}',\underline{K}) + \text{adj}. \qquad \text{(H.31)}$$

where

$$v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(\underline{q}) = - \sum_{\substack{s_1 s_2 s_3\, i_1 i_2 i_3 \\ m_1 m_2 m_3 \mu_1 \mu_2 \mu_3 \\ s_1' i_1' s_2' i_2' m_1' m_2' \\ \mu_1' \mu_2'\, l s\, l' s'\, L\alpha}} \frac{h^{s_1 s_2 s_3\; i_1 i_2 i_3}_{l s}(q)\; h^{s_1' s_2' s_3\; i_1' i_2' i_3}_{l' s'}(q)}{\varepsilon_B{}^{s_3 i_3}_{\mu_3}(q)} \cdot \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2L+1)}}\, (l' l\, 0 0 | L 0)\, Y_{L\lambda}(\underline{q}) \cdot$$

$$\cdot \left[(i_2 i_3 \mu_2 \mu_3 | i_1 \mu_1)(i_2' i_3 \mu_2' \mu_3 | i_1' \mu_1')(i_1 i_2' \mu_1 \mu_2' | \sigma\beta)(i_2 i_1' \mu_2 \mu_1' | \sigma'\beta') \right] \cdot$$

$$\cdot \left\{ (-)^{m'} (s_1 s_2' m_1 m_2' | SM)(s_2 s_1' m_2 m_1' | S'M')(l' l\, m' m | L\lambda)(s_2 s_3 m_2 m_3 | s\alpha) \cdot \right.$$

$$\left. \cdot (l s_1 m m_1 | s\alpha)(s_2' s_3 m_2' m_3 | s'\alpha')(l' s_1' m' m_1' | s'\alpha') \right\} \qquad \text{(H.32)}$$

As in (5.2.3), we use (D.18) and (D.5) to show

$$\sum_{\substack{\mu_1 \mu_2 \mu_3 \\ \mu_1' \mu_2'}} [\;\;] = (-)^{i_1' + i_2 + \sigma} \sqrt{(2i_1+1)(2i_1'+1)} \begin{Bmatrix} i_2' & i_3 & i_1' \\ i_2 & \sigma & i_1 \end{Bmatrix} \delta_{\sigma\sigma'}\, \delta_{\beta\beta'} \qquad \text{(H.33)}$$

Next, we rearrange certain of the coefficients in the curly brackets in equation (H.32) using (D.9), and then apply (D.32) to obtain

$$\sum_{\substack{m_1 m_2 m_3 \\ m_1' m_2' m m' \\ \alpha\alpha'}} \{\;\;\} = \sum_{\substack{m_1 m_2 m_3 \\ m_1' m_2' m \\ m' \alpha\alpha'}} (-)^{s_1' + s_2 + s_3 + 2s_2' - 2S + S' - s + s' + l + l' + M}$$

$$\cdot (s_3 s' m_3 \alpha' | s_2' m_2')(s_2 s_1' m_2 m_1' | S' -M')(s_3 s_2 m_3 m_2 | s\alpha) \cdot$$

$$(s_1 s\, m_1 \alpha | l m)(s' s_1' \alpha' m_1' | l' m')(s_1 s_2' m_1 m_2' | SM)(l l' m m' | L -\lambda)$$

$$= (-)^{s_1' + s_2 - s_3 - S + S' - s - s' + l + l'} (S L M \lambda | S' M') \begin{Bmatrix} s_3 & s' & s_2' & S \\ s_2 & s_1' & S' & l \\ s & l' & s_1 & L \end{Bmatrix} \cdot$$

$$\cdot \left[(2s_2'+1)(2s+1)(2l'+1)(2l+1)(2L+1)(2S+1) \right]^{1/2} \qquad \text{(H.34)}$$

where $\begin{Bmatrix} s_3 & \Delta' & s_2' & S \\ s_2 & s_1' & s' & \lambda \\ \Delta & \ell' & s_1 & L \end{Bmatrix}$ is a 12j symbol, whose properties are described in Appendix D.

Thus we may write the fermion-fermion potential as

$$v^{\sigma\beta\sigma'\beta'}_{SMS'M'}(q) = -\sum_{\substack{s_1 s_2 s_3 i_1 i_2 i_3 \\ s_1' s_2' i_1' i_2' \\ \ell\ell' \Delta\Delta' L}} \delta_{\sigma\sigma'}\, \delta_{\beta\beta'}\, (-)^{i_1'+i_2+\sigma} \sqrt{(2i_1+1)(2i_1'+1)}\, \cdot \begin{Bmatrix} i_2' & i_3 & i_1' \\ i_2 & \sigma & i_1 \end{Bmatrix} \cdot$$

$$\cdot\, (-)^{s_1'+s_2+s_3-S+s'-\Delta-\Delta'+\ell+\ell'} (2\Delta'+1)(2\Delta+1) \sqrt{\frac{(2S+1)(2\ell+1)(2\ell'+1)}{4\pi}}\, \cdot$$

$$\cdot\, h^{s_1 s_2 s_3 i_1 i_2 i_3}_{\ell\Delta}(q)\, h^{s_1' s_2' s_3 i_1' i_2' i_3}_{\ell'\Delta'}(q)\, Y_{L\lambda}(\hat{q})\, (\ell'\ell\, 0\, 0 | L\, 0) \cdot$$

$$\cdot\, (S L M \lambda | S' M') \begin{Bmatrix} s_3 & \Delta' & s_2' & S \\ s_2 & s_1' & s' & \ell \\ \Delta & \ell' & s_1 & L \end{Bmatrix} \qquad \text{(H.35)}$$

The form of this potential is very similar to the nucleon-nucleon potential (5.2.5). Both have the structure required by rotational invariance, and in both the conservation of isospin and isospin z-axis projection is manifest. Indeed, in the case that $s_1 = s_2 = s_1' = s_2' = i_1 = i_2 = i_1' = i_2' = 1/2$, $s_3 = m_3 = 0$, and $i_3 = 1$, i.e., we are considering two nucleons interacting through pion exchange, the interaction V_{FF} can be shown to equal V_{NN} given by (5.2.6), using (H.21) and (D.33).

Equation (H.35) gives a fermion-fermion potential in terms of a trilinear vertex function. If we wished to calculate a nucleon-delta potential, for example, we could use the $N\Delta\pi$ piece of the Cloudy Bag Model Hamiltonian, pictured in Figure 15, to obtain $h^{\frac{1}{2}\frac{3}{2}0\,\frac{1}{2}\frac{3}{2}1}_{\ell\,3/2}(q)$, in the same way as we found

$h_{CBM}(q)$ in Appendix F. We would then be able to use this $N\Delta\pi$ vertex function to integrate as in Section 5.3, giving the desired N-Δ potential.

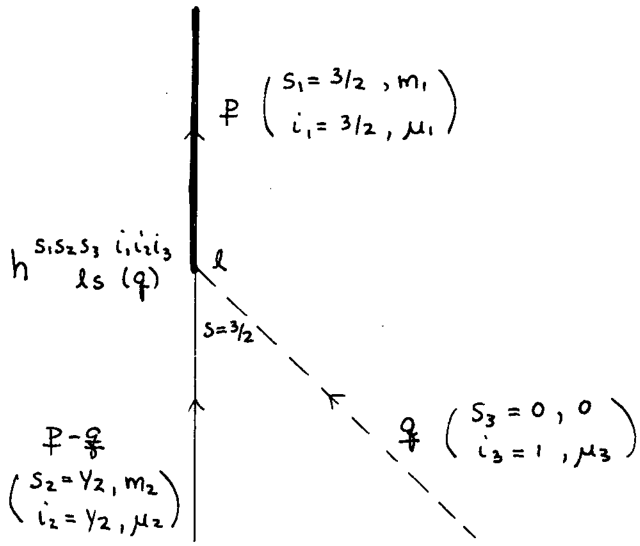

Fig. 15 The $N\Delta\pi$ Trilinear Vertex
Thick solid lines are Δ particles, solid lines are nucleons, dashed lines are pions.

Appendix I Some Properties of Bessel Functions

In this Appendix, we introduce and list some Bessel functions encountered in the discussion in Chapter 5. We then consider certain integrals involving these Bessel functions.

For a complete discussion of Bessel functions, we refer to the book by Watson (1966).

$J_\nu(z)$ denotes an ordinary Bessel function of the first kind, of order ν and argument z. Both ν and z are unrestricted, complex variables. The Bessel functions of half-integral order are called spherical Bessel functions. $j_n(z)$ is a spherical Bessel function of the first kind. It is related to the ordinary Bessel functions by

$$j_n(z) = \sqrt{\frac{\pi}{2z}}\; J_{n+1/2}(z) \qquad n=0,\pm 1,\ldots \tag{I.1}$$

The spherical Bessel functions of the second kind are, from Abramowitz and Stegun (1965, p.433):

$$y_n(z) = (-)^{n+1}\, j_{-n-1}(z) \qquad n=0,\pm 1,\ldots \tag{I.2}$$

We will also require the modified spherical Bessel functions of the first, second, and third kinds, namely $i_n(z)$, $i_{-n}(z)$, and $k_n(z)$, for $n = 0,\pm 1,\ldots$. They are defined in terms of the spherical Bessel functions. From Abramowitz and Stegun (1965, pp. 443-4), we have

$$i_n(z) = \sqrt{\frac{\pi}{2z}}\, I_{n+1/2}(z) = e^{-n\pi i/2}\, j_n(iz) \qquad (-\pi < \arg z \le \tfrac{\pi}{2})$$

$$= e^{3n\pi i/2}\, j_n(iz) \qquad (\tfrac{\pi}{2} < \arg z \le \pi) \tag{I.3}$$

$$i_{-n}(z) = \sqrt{\frac{\pi}{2z}}\, I_{-n-1/2}(z) = e^{3(n+1)\pi i/2}\, y_n(iz) \qquad (-\pi < \arg z \le \tfrac{\pi}{2})$$

$$= e^{-(n+1)\pi i/2}\, y_n(iz) \qquad (\tfrac{\pi}{2} < \arg z \le \pi) \tag{I.4}$$

$$k_n(z) = \sqrt{\frac{\pi}{2z}}\, K_{n+1/2}(z) = (-)^n \frac{\pi}{2}\left[i_{-n}(z) - i_n(z)\right] \tag{I.5}$$

We now list some Bessel functions of small order, taken from Abramowitz and Stegun (1965, pp. 433-4, 443-4).

$$j_0(z) = \frac{\sin z}{z} \qquad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z \tag{I.6}$$

$$y_0(z) = -j_{-1}(z) = \frac{-\cos z}{z}$$

$$y_1(z) = j_{-2}(z) = \frac{-\cos z}{z^2} - \frac{\sin z}{z}$$

$$y_2(z) = -j_{-3}(z) = \left(\frac{-3}{z^3} + \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z \tag{I.7}$$

$$i_0(z) = \frac{\sinh z}{z}$$

$$i_1(z) = \frac{\cosh z}{z} - \frac{\sinh z}{z^2}$$

$$i_2(z) = \left(\frac{3}{z^3} + \frac{1}{z}\right)\sinh z - \frac{3}{z^2}\cosh z \tag{I.8}$$

$$k_0(z) = \frac{\pi}{2z} e^{-z}$$

$$k_1(z) = \frac{\pi}{2z} e^{-z}\left(1 + \frac{1}{z}\right)$$

$$k_3(z) = \frac{\pi}{2z} e^{-z}\left(1 + \frac{3}{z} + \frac{3}{z^2}\right) \tag{I.9}$$

The small argument limit of the spherical Bessel functions is:

$$j_n(z) \xrightarrow[z\to 0]{} \frac{z^n}{(2n+1)!!}$$

$$y_n(z) \xrightarrow[z\to 0]{} -(2n-1)!!\left(\frac{1}{z}\right)^{n+1} \tag{I.10}$$

where (2n+1)!! = (2n+1)(2n-1)...(3)(1)

We now consider certain contour integrations involving spherical Bessel functions. From Watson (1966, eq. (9), p.430), choosing $\rho=3/2$, $\mu_1=3/2=\mu_2$, $b_1=b_2=b$, and $\nu=\ell+1/2$ for ℓ an even integer, we have

$$\int_0^\infty \frac{x^{1/2}}{x^2+k^2}\left[J_{3/2}(bx)\right]^2 (-)^{\ell/2}\left[J_{\ell+1/2}(ax)\right]dx$$

$$= -\left[I_{3/2}(bk)\right]^2 K_{\ell+1/2}(ak)\, k^{-1/2} \tag{I.11}$$

providing a>2b and $\ell<4$.

Writing the ordinary Bessel functions in terms of spherical ones using (I.1), (I.3), and (I.5), this becomes

$$\int_0^\infty \frac{x^2\,dx}{x^2+k^2}\left[j_1(bx)\right]^2 j_\ell(ax)$$

$$= (-)^{1+\ell/2}\, k\left[i_1(bk)\right]^2 k_\ell(ak) \qquad \begin{matrix} a>2b \\ \ell<4 \end{matrix} \tag{I.12}$$

When $\ell=0$, using (I.8) and (I.9), we find

$$\int_0^\infty \frac{x^2 dx}{x^2+k^2} \left[j_1(bx) \right]^2 j_0(ax)$$

$$= -\frac{\pi}{2} \frac{e^{-ak}}{ab^2k^2} \left[\cosh(bk) - \frac{\sinh(bk)}{bk} \right]^2$$

$a>2b$ (I.13)

When $\ell=2$, using (I.8) and (I.9), the integral becomes

$$\int_0^\infty \frac{x^2 dx}{x^2+k^2} \left[j_1(bx) \right]^2 j_2(ax)$$

$$= -\frac{\pi}{2} \frac{e^{-ak}}{ab^2k} \left[1 + \frac{3}{ak} + \frac{3}{(ak)^2} \right] \left[\cosh(bk) - \frac{\sinh(bk)}{bk} \right]^2$$

$a>2b$ (I.14)

Appendix J One Pion Exchange Potentials (OPEP)

In this Appendix, we summarize the results of calculations of nucleon-nucleon potentials from one pion exchange. These potentials are discussed in most texts on nuclear physics. See, for example, the book by Moravcsik (1963).

Beginning with a suitable non-relativistic limit of the Lagrangian interaction

$$\mathcal{L}^{int} = \sqrt{4\pi}\ (2m_N)\ \frac{f}{m_\pi}\ \bar{\psi}\gamma_5\psi\,\phi_\pi \tag{J.1}$$

and considering one pion exchange in a perturbative theory, the following second-order nucleon-nucleon potential is obtained:

$$V_{OPEP}(\underline{r}) = \frac{1}{3} f^2 (\underline{\tau}_1\cdot\underline{\tau}_2)\cdot\Big\{\underline{\sigma}_1\cdot\underline{\sigma}_2\,\frac{e^{-m_\pi r}}{r}$$

$$+\left(1+\frac{3}{m_\pi r}+\frac{3}{m_\pi^2 r^2}\right)\frac{e^{-m_\pi r}}{r}S_{12} - \frac{4\pi}{m_\pi^2}\underline{\sigma}_1\cdot\underline{\sigma}_2\,\delta(\underline{r})\Big\} \tag{J.2a}$$

where

$$S_{12} = 3\,\underline{\sigma}_1\cdot\hat{\underline{r}}\ \underline{\sigma}_2\cdot\hat{\underline{r}} - \underline{\sigma}_1\cdot\underline{\sigma}_2 \tag{J.2b}$$

$\underline{\sigma}_1$ and $\underline{\sigma}_2$ are the Pauli spin operators in the Hilbert spaces of nucleons 1 and 2, respectively; τ_1 and τ_2 are the analogous isospin operators; f is the NNπ coupling constant ($f^2 \cong .08$). The relative separation of the nucleons is $\underline{r} = \underline{r}_2 - \underline{r}_1$; m_π is the pion mass. The above is written in units where $\hbar = c = 1$.

Note that this interaction has a scalar part which is proportional to $\underline{\sigma}_1\cdot\underline{\sigma}_2$, and a tensor part, proportional to the operator S_{12}.

Consider the scalar OPEP, for $\underline{r} \neq 0$. Taking matrix elements between two-nucleon states of spin S and isospin σ , we obtain

$$[v^{S\sigma \ell=0}_{MM'}(\underline{r})]_{OPEP} = \langle SM ; \sigma\beta \mid V^{\ell=0}_{OPEP}(\underline{r}) \mid S'M' ; \sigma'\beta' \rangle$$

$$= \frac{1}{3} f^2 \frac{e^{-m_\pi r}}{r} \langle SM \mid \underline{\sigma}_1 \cdot \underline{\sigma}_2 \mid S'M' \rangle \langle \sigma\beta \mid \underline{\tau}_1 \cdot \underline{\tau}_2 \mid \sigma'\beta' \rangle \qquad (J.3)$$

A simple calculation shows

$$\langle SM \mid \underline{\sigma}_1 \cdot \underline{\sigma}_2 \mid S'M' \rangle = \delta_{SS'} \delta_{MM'} \begin{cases} -3 & S = S' = 0 \\ 1 & S = S' = 1 \end{cases} \qquad (J.4)$$

Similarly

$$\langle \sigma\beta \mid \underline{\tau}_1 \cdot \underline{\tau}_2 \mid \sigma'\beta' \rangle = \delta_{\sigma\sigma'} \delta_{\beta\beta'} \begin{cases} -3 & \sigma = \sigma' = 0 \\ 1 & \sigma = \sigma' = 1 \end{cases} \qquad (J.5)$$

The resulting matrix elements of the potential are given in Table II.

	$\ell = 0$ $S = 0$	$\ell = 0$ $S = 1$
$\sigma = 0$	$3f^2 \delta_{MM'} \delta_{M0} \frac{e^{-m_\pi r}}{r}$	$-f^2 \delta_{MM'} \frac{e^{-m_\pi r}}{r}$
$\sigma = 1$	$-f^2 \delta_{MM'} \delta_{M0} \frac{e^{-m_\pi r}}{r}$	$\frac{1}{3} f^2 \delta_{MM'} \frac{e^{-m_\pi r}}{r}$

Table II Matrix Elements of the Scalar OPEP $[v^{S\sigma 0}_{MM'}(\underline{r})]_{OPEP}$

Appendix K Dressing a Poincaré Invariant System

In this Appendix we consider the consequences of applying a dressing transformation to a system which is invariant under a Poincaré transformation, i.e., one which is not only invariant under translations, spatial rotations, space inversion and time reversal, but also under homogeneous Lorentz boosts.

To describe a system which is Poincaré invariant, one must construct from the fundamental dynamical variables of the system ten Hermitian operators P^j, J^j, H, K^j (j=1,2,3) satisfying the Poincaré algebra (see, for example, Kalyniak (1978, p.23):

$$[P^j,P^k]=0 \qquad [J^j,P^k]=i\hbar\,\varepsilon_{jk\ell}\,P^\ell \qquad [J^j,J^k]=i\hbar\,\varepsilon_{jk\ell}\,J^\ell \tag{K.1a)-(K.1c}$$

$$[P^j,H]=0 \qquad [J^j,H]=0 \qquad [K^j,J^k]=i\hbar\,\varepsilon_{jk\ell}\,K^\ell$$

$$[K^j,P^k]=-i\hbar\,\delta_{jk}\,H/c^2 \tag{K.2a)-(K.2d}$$

$$[K^j,H]=-i\hbar\,P^j \qquad [K^j,K^k]=-i\hbar\,\varepsilon_{jk\ell}\,J^\ell/c^2 \tag{K.3a)-(K.3b}$$

$$j,k,\ell=1,2,3$$

The momentum operator $\underline{P}$ is the generator of spatial translations; the angular momentum $\underline{J}$ is the generator of spatial rotations; the Hamiltonian H is the generator of time translations; the operator $\underline{K}$ is the generator of Lorentz boosts.

For a system of free fermions and bosons the momentum operator $\underline{P}$ and the Hamiltonian H are given in Fock space by equations such as (4.2.4) and (4.2.1b). $\underline{J}_0$ and $\underline{K}_0$ will also involve $F^\dagger F$ and $B^\dagger B$ terms and expressions for these free particle operators in Fock space can be obtained from the corresponding n-particle operators. For example, see

Kalyniak (1978, pp.76,77) for the n-particle angular momentum and Lorentz boost operators.

To describe a Poincaré invariant system of interacting fermions and bosons, we introduce interactions using the instant form of Dirac (1949), i.e., we let

$$\underline{P} = \underline{P}_o$$

$$\underline{J} = \underline{J}_o$$

$$H = H_o + \lambda H_1$$

$$\underline{K} = \underline{K}_o + \lambda \underline{K}_1 \qquad \text{(K.4)}$$

Thus we have a system of interacting fermions and bosons with generators $\underline{P}_o$, $\underline{J}_o$, H, $\underline{K}$ which satisfy the Poincaré algebra (K.1) - (K.3).

We choose the interactions H_1 and $\underline{K}_1$ to be trilinear. The vertex function h must be a function of the fermion momentum $\underline{p}$ as well as the boson momentum $\underline{q}$ in order to satisfy the commutation relations involving $\underline{K}$ and H. The systems which we have considered previously in this thesis could not be made Poincaré invariant because we chose the vertex function to depend only on $\underline{q}$. Taking h = h($\underline{p}$,$\underline{q}$), the functional dependence of the vertex function on $\underline{p}$ and $\underline{q}$ will be restricted by the Poincaré algebra, just as its dependence on spin was determined in Chapter 3 by space-time invariance requirements.

Now consider a dressing transformation on the fundamental dynamical variables of this system. The transformation is given by equations (4.1.1) - (4.1.4). Since the dressing transformation is unitary, the operators $\tilde{\underline{P}}_o$, $\tilde{\underline{J}}_o$, $\tilde{H}$, $\tilde{\underline{K}}$ must still obey the Poincaré algebra. D is invariant under translations and rotations, i.e.,

$$[\underline{P}_0, D] = [\underline{J}_0, D] = 0 \tag{K.5}$$

so the functional form of $\underline{P}_0$ and $\underline{J}_0$ will not be changed by the dressing transformation:

$$\tilde{\underline{P}}_0(F,B) = \underline{P}_0(F,B) \tag{K.6}$$

$$\tilde{\underline{J}}_0(F,B) = \underline{J}_0(F,B) \tag{K.7}$$

As is Section 4.1 we write

$$D = \sum_{n=1}^{\infty} \lambda^n D_n \tag{K.8}$$

The dressed Hamiltonian $\tilde{H}$ will be given by (4.1.15); $\tilde{\underline{K}}$ will be given by a similar expression.

We now choose D_n to eliminate unsuitable terms of the form (4.1.18) from $\tilde{\underline{K}}$ to order n. Thus

$$\tilde{\underline{K}} = \underline{K}_0 + \lambda^2 \left\{ \tfrac{1}{2} [\underline{K}_1, D_1] + [\underline{K}_0, D_2] \right\} + \cdots \tag{K.9}$$

That is,

$$\tilde{\underline{K}} = \underline{K}_0 + \sum_{n=2}^{\infty} \lambda^n \tilde{\underline{K}}_n \tag{K.10}$$

where $\tilde{\underline{K}}_n$ contains no terms with a single fermion or boson annihilator (other than $F^\dagger F$ and $B^\dagger B$). This implies that equation (4.1.20) for $\tilde{H}$ holds, as we now show.

The dressed operators satisfy the Poincaré algebra; therefore

$$-i\hbar\, \delta_{jk}\, \tilde{H}/c^2 = [\tilde{K}^j, P_0^k] \tag{K.11}$$

We also have, from (K.2d),

$$-i\hbar\,\delta_{jk}\,H_0/c^2 = [K_0^j, P_0^k] \tag{K.12}$$

$$-i\hbar\,\delta_{jk}\,H_1/c^2 = [K_1^j, P_0^k] \tag{K.13}$$

Taking the commutator of equation (K.9) with P_0^k, we find

$$[\tilde{K}^j, P_0^k] = [K_0^j, P_0^k] + \lambda^2 \left\{ \tfrac{1}{2} [[K_1^j, D_1], P_0^k] + [[K_0^j, D_2], P_0^k] \right\} + \ldots \tag{K.14}$$

Note that we can use (K.5), (K.13) and a Jacobi identity to show, for example,

$$[[K_1^j, D_1], P_0^k] = [[K_1^j, P_0^k], D_1] + [[\cancelto{0}{P_0^k, D_1}], K_1^j]$$

$$= -i\hbar/c^2\ \delta_{jk}\,[H_1, D_1] \tag{K.15}$$

Therefore, using (K.11), (K.12), (K.13) and (K.5) equation (K.14) becomes

$$\tilde{H} = H_0 + \lambda^2 \left\{ [H_1, D_1] + [H_0, D_2] \right\} + \ldots \tag{K.16}$$

or

$$\tilde{H} = H_0 + \sum_{n=2}^{\infty} \lambda^n \tilde{H}_n \tag{K.17}$$

where $\tilde{H}_n$ also contains no unsuitable terms of the form (4.1.18). This is the same series for $\tilde{H}$ that we obtained in equation (4.1.20). Thus if we choose D to eliminate all unsuitable terms from $\underline{\tilde{K}}$, we automatically eliminate all such terms from $\tilde{H}$ as well.

Is the Poincaré invariance of the resulting theory satisfied order by order in λ? Certainly (K.1) is still satisfied by $\tilde{\underline{P}}_0$ and $\tilde{\underline{J}}_0$. Also, in light of (K.5), we have

$$[\tilde{P}_0 , \tilde{H}_n] = [\tilde{\underline{J}}_0 , \tilde{H}_n] = 0 \tag{K.18}$$

From (K.11), (K.12) and (K.5), it must hold that

$$[\tilde{K}_n , \tilde{P}_0^k] = -i\hbar\, \delta_{jk}\, \tilde{H}_n^k \tag{K.19}$$

and

$$[\tilde{K}_n^j , J_0^k] = i\hbar\, \varepsilon_{jkl}\, \tilde{K}_n^l \tag{K.20}$$

Thus (K.2) is satisfied by the dressed generators to order n. Consider (K.3a):

$$[\tilde{K}^j , \tilde{H}] = -i\hbar P_0^j \tag{K.21}$$

Since

$$[K_0^j , H_0] = -i\hbar P_0^j \tag{K.22}$$

this implies

$$\sum_{n=2}^{\infty} \lambda^n \{ [K_0^j, \tilde{H}_n] + [\tilde{K}_n^j, H_0] \} + \sum_{n,m=2}^{\infty} \lambda^{n+m} [\tilde{K}_n^j , \tilde{H}_m] = 0 \tag{K.23}$$

Thus (K.3) cannot be satisfied order by order in λ. If the series for $\tilde{\underline{K}}$ and $\tilde{H}$ are truncated, (K.23) will not hold, and the Poincaré invariance of the theory is destroyed.

In summary, we have shown that it is possible to apply a dressing transformation to a Poincaré invariant system by constructing the dressing operator D_n to eliminate the unsuitable terms of the form (4.1.18) from $\tilde{\underline{K}}$, the generator for

Lorentz boosts. Procedures similar to those given in Chapter 4 can be used to carry out this dressing transformation. As a consequence, the dressed Hamiltonian contains no unsuitable terms and thus $\tilde{F}^\dagger$ and $\tilde{B}^\dagger$ create physical particles. One cannot truncate the series for $\tilde{H}$ or $\underline{\tilde{K}}$ and maintain Poincaré invariance.

An alternative perturbative approach to describing a Poincaré invariant system is given in Glöckle and Müller (1981).