PRODUCTIVITY ANALYSIS AND FUNCTIONAL SPECIFICATION: A PARAMETRIC APPROACH

by

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This dissertation proposes to develop a functional specification of the cost function which can be used to discriminate among the alternative flexible functional forms and to analyze productivity growth. The problem of choosing among alternative functional forms is tackled by developing a nested representation which takes as special cases several well known forms. Productivity growth is analyzed by considering input price effects, non-neutral scale effects and biased technical change.

Total factor productivity (TFP) is used as an index of technical change. Traditionally, the rate of TFP growth has been computed, assuming constant returns to scale, as the residual of the rate of growth of real output minus the rate of growth of aggregate real input. The parametric approach to productivity analysis adopted here allows estimation of the TFP growth rate without requiring constant returns to scale.

The stochastic specification of the model incorporates errors arising out of imperfect cost minimizing behaviour. The resulting likelihood function is continuous in all the parameters.

A four input version of the model has been used to estimate the total U.S. Manufacturing Technology, 1947-71. The inputs are capital, labour, energy and other intermediate materials. The parameters of the model have been estimated by utilizing the maximum likelihood method.
The main hypotheses tested are neutrality of technical change and homotheticity.

This investigation suggests that total U.S. Manufacturing, 1947-71 may be characterized by non-neutral scale effects and biased technical change. The scale effects have been capital, labour and energy saving while technical progress has been capital and energy using and labour neutral. The elasticity results indicate that capital and labour are substitutes as are labour and energy while capital and energy are complements. Capital and energy are more own price elastic than labour.

Scale economies seem to have contributed considerably to productivity growth in total U.S. Manufacturing, 1947-71. The contribution of total factor productivity is, however, uncertain and perhaps small.
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CHAPTER I: INTRODUCTION

Since the introduction of the Cobb-Douglas (CD) [1928] production form considerable progress has occurred in the development of functional forms for production and cost. Historically, attention was focussed primarily on the elasticity of substitution\(^1\) between inputs. The CES generalization (Arrow-Chenery-Minhas-Solow [1961]) of the CD form allowed arbitrary constant elasticity of substitution different from unity but in the multifactor context all partial elasticities of substitution have to equal the same constant.

Functional forms yielding partial elasticities of substitution that can differ pairwise and also along isoquants have been proposed by V. Mukerji [1963], W. M. Gorman [1965] and G. Hanoch [1971]. However, in these functions the elasticities of substitution stand in constant ratios.\(^2\)

Moreover, all the above functional forms are not flexible in that they do not have enough free parameters to provide second order approximations to an arbitrary function of the relevant variables.

Growing realization of the implications of duality theory has generated renewed interest in functional form research at the multifactor context. This has resulted in the development of several very useful flexible functional forms e.g., the Generalized Leontief (GL) (W. E. Diewert [1971]), the transcendental logarithmic (TLOG) (L. R. Christensen, D. W. Jorgenson and L. J. Lau [1971] and the Quadratic Square Rooted (QSR) (Diewert [1974a]) representations which do not constrain the elasticities of substitution or their ratios to be constant.

The presence of several alternative flexible forms, however, raises the
question of how to discriminate among them. This choice is important because key estimates like elasticities of substitution may not be robust to alternative functional specifications. Furthermore, results of hypothesis tests may depend crucially on the functional representation. Consequently, which function is employed can have important empirical and policy implications.

Discrimination among such functional forms on theoretical grounds is difficult because each flexible form can be interpreted as a second order approximation to an arbitrary twice differentiable function of the variables involved. Choice could be made on the criterion of ease of estimation and/or appropriate stochastic specification. For example, in the context of cost functions, when share equations are to be estimated the TLOG form is convenient in that it yields a share system linear in parameters. On the other hand, for estimating an input-output coefficient system it is the GL form which yields a system of input-output equations linear in parameters. Another criterion for discriminating among flexible functional forms is in terms of goodness of fit; such a Bayesian framework has been utilized by E. R. Berndt, M. N. Darrough and W. E. Diewert [1977]. By utilizing the Box-Cox [1964] transformation, N. Kiefer [1975] has recently developed a very attractive general representation which can take on a variety of functional forms including the GL and TLOG as special or limiting cases.

The main purpose of this research is to modify and extend Kiefer's formulation to develop a flexible functional form for the production sector which yields most of the currently available flexible forms as special or limiting parametric cases. Such a nested framework allows classical test
3.

procedures to be employed to discriminate among the different forms. Since the variables appearing in this specification are transformed in the method originally used by G. E. P. Box and D. R. Cox we call the formulation presented here the Generalized Box Cox (GBC) functional form.

The GBC form allows parametric tests of homotheticity, homogeneity and symmetry. R. W. Parks (1971) and A. D. Woodland (1975) have provided a simple non-homothetic extension of the GL form. Although this formulation is econometrically more convenient than ours, it is not possible to impose homotheticity without also imposing homogeneity. An attractive feature of the GBC representation in this thesis is that it permits separate tests of homotheticity and homogeneity.

We have also incorporated non-neutral technical change in our formulation. P. Diamond and D. McFadden (1965) have shown that in the absence of sufficient structure on the nature of technical change it is not possible, by utilizing time series data alone, to simultaneously identify the bias of technical change and the elasticities of substitution. Consequently we have specified technical change and returns to scale in our formulation in such a way that simultaneous identification of substitution elasticities, bias of technical change and scale economies is possible. This formulation also allows parametric estimation of total factor productivity. In this method the productivity measure is free of errors of cost minimizing behaviour, while in our context the traditional residual measure of total factor productivity computed as the rate of growth of real output minus the rate of growth of real input is not free of such errors. Moreover, although a theoretical justification for the traditional measure exists in the case of constant returns to scale technology, this
measure cannot be interpreted as rate of total factor productivity (i.e. an index of technical change) when technology is characterized by nonconstant returns to scale. The residual measure would in this case include both the effects of scale economies and technical change. However, in the GBC representation total factor productivity can be separately estimated even in case of a nonconstant returns to scale technology.

Finally, the GBC representation provides a useful framework to study the interrelationships among the various functional forms. The richness of the GBC parameter base allows consideration of a multiplicity of scale and technical change combinations.

The remainder of this thesis is organized as follows. In chapter II we introduce the GBC formulation. We first look at the special cases of our simplest general form. This formulation is then extended to take into account technical change and scale economies. Next we compare the GBC representation with other general specifications. The various elasticity expressions are then derived. An examination of the various properties of the GBC formulation is followed by a discussion of the returns to scale and total factor productivity measures that are used. Some interesting comparative static expressions are presented next. In the final section of chapter II we consider the impact of imposing weak separability restrictions on the GBC formulation.

In chapter III we discuss the issues of empirical implementation. The estimating equations are derived first. In the next section we specify a stochastic framework for our share equations and the cost equation. This stochastic specification has the advantage of yielding a likelihood function that is continuous over the entire range of values of the Box-Cox transformation parameter ($\lambda$) including $\lambda = 0$ (the TLOG limiting case).
So, not only are the different flexible functional forms nested in the GBC function but also their likelihood functions (corresponding to our stochastic framework) are nested in the GBC likelihood function. In section 3 of this chapter we discuss why the GBC function is not treated as an approximation in estimation. In the final two sections we consider the test criteria, the measure of fit and the testing design that we use in chapter IV.

In chapter IV we apply the GBC specification to formulate and estimate a four input—capital, labour, energy and materials—cost function model of the U.S. Manufacturing sector, 1947-71. For estimation we use the KLEM data originally utilized by E. R. Berndt and D. O. Wood [1975a]. First, a model of the technology of U.S. Manufacturing is developed. We then discuss the important features of the data and some relevant empirical evidence. A notable feature of the data body is that while the prices of capital services and energy grew at a slower rate than that of labour services, the quantity demanded of capital services and energy increased at a faster rate than the demand for labour services. Our results indicate that this trend is essentially due to relatively large own price elasticities of demand for capital and energy services and considerable capital-energy complementarity on the one hand and significantly capital and energy using bias of technical change and relatively large labour saving bias of scale economies on the other in U.S. Manufacturing, 1947-71. In presenting the results we first focus on the choice of the appropriate technology and functional form. We then examine the properties of the estimated cost functions. Next we turn to a discussion of the various elasticity estimates. In the final three sections we obtain the measures of returns
to scale and total factor productivity and also examine the response of these estimates to changes in the explanatory variables viz., input prices, output quantity and time. It is intriguing that we have considerable evidence for increasing returns to scale in the aggregate U.S. manufacturing technology over 1947-71 while there is hardly any evidence for comparable total factor productivity gains. As is well known, if the assumption of competition in the output market is maintained then such scale economies in the aggregate manufacturing industry can arise only due to productivity augmenting external economies that arise with the growth of the entire industry. Our measure of total factor productivity is parametric. We also compare it with the traditional 'residual measure' of total factor productivity utilizing the 'mean of order r' index.  

In chapter V we present a summary of the important results and our conclusions. We also make an assessment of the GBC formulation and indicate possible further research.

There are three appendices in this thesis: list of equations used in the approximation theorem (Appendix 1), proofs of theorems and corollaries (Appendix 2) and data (Appendix 3).
Different definitions of elasticity of substitution have been proposed in the context of more than two inputs:

(a) Allen [1938]-Uzawa [1962] partial elasticity of substitution, $\sigma_{ij}$ measures the normalized (normalized so that $\sigma_{ij} = \sigma_{ji}$) response of demand for input $i$ when price of input $j$ changes, output and all other prices being held constant.

(b) Direct Elasticity of Substitution (Hicks [1963]-Allen [1938]), $\sigma^{*}_{ij}$ between inputs $i$ and $j$ gives the response of the ratio of the two factor demands when their price ratio changes while output and all other input quantities are held fixed.

(c) Shadow Elasticity of Substitution (D. McFadden [1962]). $\sigma^{*\prime}_{ij}$ describes the response of input price ratio to a change in their quantity ratio when output, shadow cost and all other prices are held fixed. In this research we concentrate only on the Allen-Uzawa Partial elasticity of Substitution $\sigma_{ij}$. The other two elasticities can readily be obtained utilizing the $\sigma_{ij}$'s.


3. Among the flexible functional forms only the QSR function has a dual which also has the QSR form. See W. E. Diewert [1974a].
4. The Box-Cox transformation of a variable \( x \) is \( x(\lambda) = \frac{x^\lambda - 1}{\lambda} \) where \( \lambda \) is a transformation parameter. Defining \( x(0) = \ln x \) it can be seen that \( x(\lambda) \) is a continuous transformation.

5. Further comparison of our formulation with Kiefer's has been done in chapter II Section 5.

6. It should, however, be noted that the classical method can compare a general form with its special cases but may not be able to compare the special cases themselves e.g. if neither the GL nor the TLOG forms can be rejected, then the problem of choice between them remains. In this case a Bayesian method as in Berndt, Darrough and Diewert seems necessary.

7. For a separate test of homogeneity in a homothetic CES function see C. A. K. Lovell [1973].

8. For a nonhomothetic and non-neutral (technical change) generalization of the Leontief cost function see L. J. Lau and S. Tamura [1972]. For a similar generalization of the GL cost function see R. W. Parks [1971] and A. D. Woodland [1975].


10. See W. E. Diewert [1976].
Chapter II
THE GENERALISED BOX-COX FUNCTIONAL FORM

It is assumed that there is a production function,

\[ Y = f(X) \]

where \( f \) is a continuous twice differentiable nondecreasing and quasiconcave function of a vector of inputs \( X \geq 0 \). As is well known, if the producer competitively minimizes the cost of production subject to producing a given amount of output then the technology (2.1) is completely describable by the dual cost function,

\[ C = g(P, Y) \]

where \( g \) is a nondecreasing, linearly homogeneous and concave function of prices, \( P > 0 \).

In this chapter we introduce the GBC function to represent a cost function,

\[ C = \left\{ 1 + \lambda G(P) \right\}^{\frac{1}{\lambda}} Y \]

where,

\[ G(P) = \alpha_0 + \sum_i \alpha_i P_i(\lambda) + \frac{1}{2} \sum_i \sum_j \gamma_{ij} P_i(\lambda) P_j(\lambda) \]

and,

\[ P_i(\lambda) = \frac{\lambda}{\lambda^2 - 1} , \]

\( P_i = \) price of input \( i \), \( C = \) total cost and \( Y = \) output. Imposing symmetry,
\[(2.6) \quad \gamma_{ij} = \gamma_{ji}, \quad \forall i, j\]

and linear homogeneity in prices,

\[(2.7) \quad \sum_i \alpha_i = 1 + \lambda \alpha_0 \]
\[\sum_j \gamma_{ij} = \frac{\lambda}{2} \alpha_i, \quad \forall i \]

we write the GBC cost function \((2.3)\) as,

\[(2.8) \quad C = \left( \frac{2}{\lambda} \sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}} \right)^{\frac{1}{\lambda}} \]

II.1 Special and limiting cases of GBC cost function.

The GBC cost function \((2.3)\) yields several special and limiting cases. We can write \((2.3)\) as,

\[(2.9) \quad \frac{\hat{C}^\lambda - 1}{\lambda} = \hat{G}(P)\]

where \(\hat{C} = \frac{C}{Y}\) is unit cost. Taking the limit of \((2.9)\) as \(\lambda \to 0\) we have the TLOG unit cost function,

\[(2.10) \quad \ln \hat{C} = \alpha_0 + \sum_i \alpha_i \ln P_i + \frac{\lambda}{2} \sum_i \sum_j \gamma_{ij} \ln P_i \ln P_j .\]

The function \((2.10)\) is linearly homogeneous in prices, given symmetry \((2.6)\), if

\[(2.11) \quad \sum_i \alpha_i = 1 \]
\[\sum_j \gamma_{ij} = 0, \quad \forall i \]

Note that these restrictions are implied by \((2.7)\) as \(\lambda \to 0\) .
When $\gamma_{ij} = 0$, $\psi_{i,j}$ in the TLOG unit cost function we have the Cobb-Douglas (C-D) unit cost function,

\[(2.12) \quad \ln \hat{C} = \alpha_0 + \sum_i \alpha_i \ln P_i .\]

The GL and the Quadratic Square Rooted (QSR) cost functions are obtained by setting $\lambda = 1$ and $\lambda = 2$ respectively in (2.8). The resulting GL unit cost function is,

\[(2.13) \quad \hat{C} = \sum_i \sum_j 2 \gamma_{ij} (P_i P_j)^{1/2} ;\]

When, in addition, $\gamma^+_{ij} = 0$, $i \neq j$ we obtain the corresponding Leontief cost (LC) function,

\[(2.14) \quad \hat{C} = \sum_i 2 \gamma_{ii} P_i .\]

The QSR unit cost function is obtained when $\lambda = 2$ in (2.8),

\[(2.15) \quad \hat{C} = \left\{ \sum_i \sum_j \gamma_{ij} P_i P_j \right\}^{1/2} .\]

To obtain the CES form we set, in 2.8, $\gamma_{ij} = 0$, $i \neq j$ ,

\[(2.16)^5 \quad \hat{C} = \left\{ \sum_i \frac{2}{\lambda} \gamma_{ii} P_i^\lambda \right\}^{1/\lambda} .\]

The mean of order two (MOT) cost function$^6$ results when $\lambda = 2$ in (2.16),

\[(2.17) \quad \hat{C} = \left\{ \sum_i \gamma_{ii} P_i^2 \right\}^{1/2} .\]

To highlight the interrelationship between the different functional forms we can use figure 1.

In the above we have considered only positive values of $\lambda$. However, two other interesting forms are obtained if we set $\lambda = -1$ and -2.
Figure 1

SPECIAL AND LIMITING CASES OF THE GBC FORM

GBC

\[ \lambda \to 0 \]

\[ \gamma_{ij} = 0, \quad i \neq j \]

TLOG

\[ \gamma_{ij} = 0, \quad i \neq j \]

CES

\[ \lambda = 1 \]

\[ \lambda = 2 \]

GL

\[ \gamma_{ij} = 0, \quad i \neq j \]

QSR

\[ \gamma_{ij} = 0, \quad i \neq j \]

CD

LC

MOT
For $\lambda = -1$ in (2.8) we obtain,

\[(2.18) \quad \frac{Y}{C} = \sum \sum \gamma^*_{ij}/(P_i P_j)^{\lambda/2}\]

where $\gamma^*_{ij} = -2 \gamma_{ij}$. This equation gives inverse of unit cost. For $\lambda = -2$ in (2.8),

\[(2.19) \quad \frac{Y}{C} = \{\sum \sum \gamma^*_{ij}/(P_i P_j)^{\lambda/2}\}

where $\gamma^*_{ij} = -\gamma_{ij}$. 

**II.2 Scale Effects**

We now extend the GBC form (2.3) to take account of scale effects,

\[(2.20) \quad C = \{1 + \lambda G(P)\}^{1/\lambda} \beta(Y,P)\]

where $G(P)$ has been defined in (2.4) and

\[(2.21) \quad \beta(Y,P) = \beta + \frac{\theta}{2} \ln Y + \sum \phi_i \ln P_i .

Linear homogeneity in prices now requires the extra restriction,

\[(2.22) \quad \sum \phi_i = 0 .

Production is homothetic if the cost function factors into a function of prices and a function of output. This occurs in (2.20) when

\[(2.23) \quad \phi_i = 0, \quad \psi_i .

If in addition,

\[(2.24) \quad \theta = 0 ,

then the underlying production function is homogeneous of degree $\beta^{-1}$ in the
inputs. Constant returns to scale is obtained by further restricting

\[(2.25) \quad \beta = 1.\]

II.3 GBC with Technical Change

We will now introduce technical change into the GBC form,

\[(2.26) \quad C = \left(1 + \lambda G(P)\right)^{\frac{1}{\lambda}} \beta(Y, P) e^{T(t, P, Y)}\]

where \(G(P)\) and \(\beta(Y, P)\) are as defined in (2.4) and (2.21) respectively and

\[(2.27) \quad T(t, P, Y) = (\gamma + \frac{\delta}{2} t + \sum_i \tau_i \ln P_i + n \ln Y) t.\]

Linear homogeneity of \(C\) in prices now requires, in addition to (2.7) and (2.22),

\[(2.28) \quad \sum_i \tau_i = 0.\]

With these restrictions imposed (2.26) becomes,

\[(2.29) \quad C = \left\{\frac{2}{\lambda} \sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{1}{\lambda}} \beta(Y, P) e^{T(t, P, Y)}\right\}\]

The \(i^{th}\) share equation corresponding to (2.29) is, utilizing Shephard's Lemma,

\[(2.30) \quad s_i = \frac{\sum_j \gamma_{ij} (P_i P_j)^{\frac{1}{\lambda}}}{\sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{1}{\lambda}}} + \phi_i \ln Y + \tau_i t.\]

Technological progress is Hicks neutral if

\[(2.31) \quad \tau_i = 0, \quad \Psi_i.\]
The shares are in this case independent of $t$. There would be no interaction\(^{10}\) between scale and time if,

$$
\hat{n} = 0 .
$$

When, in addition to (2.31) and (2.32),

$$
\delta = 0
$$
technological change is Hicks neutral and of constant exponential form.

The non-neutrality parameters can be interpreted as,

$$
\frac{\partial s_i}{\partial t} = \tau_i
$$

where $s_i$ is share of input $i$ in total cost and prices and output are held fixed. If $\tau_i < 0$ then technological change causes share of $i$ to decline. Given prices and output this means that the amount required of input $i$ per unit of output declines by a proportionately greater amount than that of other inputs.$^{11}$ So depending on the sign of $\tau_i$ we can talk of $i$-saving, $i$-neutral or $i$-using bias of technical progress.$^{12}$

An analogous interpretation can be given to the nonhomotheticity parameters,

$$
\frac{\partial s_i}{\partial \ln Y} = \phi_i
$$

where input prices and $t$ are unchanged. The $\phi_i$ parameters indicate how the economies of scale are distributed over the various inputs. Scale economies, if evenly distributed (i.e. the homothetic case), would result in the same proportionate decline in all the input-output coefficients. However, in the nonhomothetic case, when $\phi_i < 0$, an increase in output results in larger economies in the $i^{th}$ input, i.e., there occurs a larger
proportionate decline in the per unit requirements of input i.\(^{13}\)

Because sufficient structure has been imposed on both the scale and time effects (2.21 and 2.27) in the GBC specification, unique identification of elasticities of substitution, bias of technical change and returns to scale are possible.

To obtain the TLOG case in this extended GBC representation, we can rewrite (2.26) as,

\[
(2.36) \quad \frac{\frac{\partial}{\partial \lambda} C}{\lambda} - \frac{1}{\lambda} = G(P)
\]

where,

\[
(2.37) \quad \tilde{C} = C/(\gamma^\beta(Y,P) e^{T(t,P,Y)}).
\]

Taking the limit of (2.36) as \(\lambda \rightarrow 0\), gives

\[
(2.38) \quad \ln C = \alpha_0 + \sum_i \alpha_i \ln P_i + \frac{1}{2} \sum_i \sum_j \gamma_{ij} \ln P_i \ln P_j \\
+ \beta(Y,P) \ln Y + T(t,P,Y)
\]

which is a TLOG formulation. Other special cases can be obtained by substituting in the relevant values of \(\lambda\) in (2.29).

II.4 An Alternative Specification of Nonhomotheticity and Nonneutrality in the GBC Representation

It is possible to formulate the bias of scale economies and technical change in the GBC specification in a number of ways. One representation is the following,

\[
(2.39) \quad C = (1 + \lambda G(P) + \lambda \phi(P) \ln Y + \lambda T(P) t)^{\frac{1}{\lambda}} \gamma^\beta(Y) e^{T(t,Y)}
\]
where \( G(P) \) is as in (2.4), \( \beta(Y) = \beta + \frac{\gamma}{2} \ln Y \), \( T(t,Y) = (\tau + \frac{\delta}{2} t + \eta \ln y)t \) and,

\[
(2.40) \quad \phi(P) = \sum_i \phi_i \frac{P_i^\lambda - 1}{\lambda}
\]

and

\[
(2.41) \quad T(P) = \sum_i \tau_i \frac{P_i^\lambda - 1}{\lambda}
\]

The restrictions of linear homogeneity in prices are the same as before. When these conditions are imposed we can write (2.39) as,

\[
(2.42) \quad C = \frac{2}{\lambda} \left\{ \sum_i \sum_j \gamma_{ij} (p_i^\lambda p_j^\lambda)^{\frac{\lambda}{2}} + \sum_i \phi_i p_i^\lambda \ln Y + \sum_i \tau_i p_i^\lambda t \right\} \frac{1}{\lambda} \beta(Y) e^{T(t,Y)}
\]

The TLOG limiting case can be obtained from (2.39) by rewriting it as

\[
(2.43) \quad \frac{\gamma_{ij}^\lambda - 1}{\lambda} = G(P) + \phi(P) \ln Y + T(P)t
\]

where

\[
(2.44) \quad C = C/(Y^\beta(Y) e^{T(t,Y)})
\]

and then taking limit of (2.43) as \( \lambda \to 0 \). Special cases of other flexible forms can be obtained from (2.42) by substituting in appropriate values of \( \lambda \).

However, in this thesis we adopt the specification in sections 2 and 3 because of its more straightforward interpretation of the nonhomotheticity and nonneutrality parameters. The interpretation of these parameters in the present specification can be obtained from the 'generalized share equations'. Utilizing Shephard's Lemma we obtain from (2.42) the following generalized share equations,

\[
(2.45) \quad s^*_i = \frac{2}{\lambda} \sum_j \gamma_{ij} (p_j/p_i)^{\frac{\lambda}{2}} + \phi_i \ln y + \tau_i t
\]
where

\[ (2.46) \quad s^*_i = s_i \left( \frac{\tilde{C}}{P_i} \right)^\lambda \]

and \( \tilde{C} \) is as defined in (2.44). From (2.45) we see that

\[ (2.47) \quad \frac{\partial s^*_i}{\partial \ln y} = \phi_i \]

and,

\[ (2.48) \quad \frac{\partial s^*_i}{\partial t} = \tau_i \]

However, \( s^*_i \) gives the \( i \)th input cost share only when \( \lambda \to 0 \). The output elasticity of cost and rate of cost diminution are respectively given by,

\[ (2.49) \quad \frac{\partial \ln C}{\partial \ln y} = \beta + \theta \ln y + \eta t + \sum_i \phi_i \frac{p_i^\lambda}{\lambda} \]

and,

\[ (2.50) \quad \frac{\partial \ln C}{\partial t} = \tau + \delta t + \eta \ln y + \sum_i \tau_i \frac{p_i^\lambda}{\lambda} \]

II.5 The GBC Form in Relation to Other General Forms

Although the GBC form is attractive in that it takes as special cases a number of well known flexible functional forms, recently several other highly general functional forms have been formulated. We now examine the GBC form in relation to these other highly general representations.

M. K. Denny [1974] and G. Hasenkamp [1976] have introduced the Generalized Quadratic (GQ) formulation,

\[ (2.51) \quad \frac{C}{Y} = \left\{ \frac{\lambda}{\sum_i \sum_j Y_{ij} p_i^2 p_j^2} \right\} \]

which has been used as the marginal cost function by R. Russell and M. Norman [1976] in their Gorman Polar Form cost function,
where output enters linearly. Russell and Norman call \( \sum \alpha_i P_i \) 'committed expenditure'. In the context of the cost function it can be interpreted as fixed cost (i.e., independent of output level).

W. E. Diewert [1975] has shown that the homogeneous TLOG form appears as a limiting case of (2.51). This is done by redefining the parameters in (2.51), for all \( \lambda \neq 0 \), as

\begin{equation}
(2.53) \quad \gamma_{ij} = \frac{2}{\lambda} \gamma^*_{ij},
\end{equation}

where \( \gamma^*_{ij} \) are the parameters of the TLOG form which results when we let \( \lambda \to 0 \).

It is of interest to compare the GBC and the GQ forms. In particular the coefficients in the GBC formulation (2.8) and the GQ representation (2.51) differ by the multiplicative factor \( \frac{2}{\lambda} \) which is precisely the factor brought in by the redefinition (2.53). The presence of the factor \( \frac{2}{\lambda} \) in (2.8), however, allows it to yield the TLOG form as a limiting case without any redefinition of parameters.

A general form for an indirect utility function that gives the TLOG and GL share equation as special cases has been developed by N. Kiefer [1975] using the Box-Cox transformation on the right hand variables,

\begin{equation}
(2.54) \quad U = \sum_i a_i V_i(\lambda) + \frac{1}{2} \sum_i \sum_j b_{ij} V_i(\lambda) V_j(\lambda)
\end{equation}

where, \( V_i(\lambda) = \frac{V_i}{\lambda} \) and \( V_i = P_i/m \) is a normalized price. The same method has been utilized by E. Appelbaum [1976] to obtain a general form for production and reciprocal indirect (utility) production functions that yield the TLOG and GL expenditure ratio and share equations as special cases. Appelbaum's functions have essentially the same form as (2.54).
Since the Box-Cox transformation has been used only on the right hand variables neither Kiefer's nor Appelbaum's general forms provide the TLOG form itself as a limiting case. This is of no consequence for Kiefer's estimating equations because the utility maximizing demand equations are invariant to a monotonic transformation of the utility function. It also doesn't matter for Appelbaum's estimating equations for production functions because he uses expenditure ratio equations which are again independent of a monotonic transformation of the production function. But the issue is important in the case of production theory, if cost minimizing demand or share equations are being used for estimation.

Another difficulty with general forms like (2.54) is that in some cases, e.g., Appelbaum's production function,

\[
F(X) = \sum_{i} \alpha_i X_i(\lambda) + \frac{1}{2} \sum_{i} \sum_{j} b_{ij} X_i(\lambda) X_j(\lambda),
\]

they cannot be made linearly homogeneous. The problem arises because of two reasons: first, the dependent variable is not transformed and, second, the independent variables are transformed with the parameter \( \lambda \) rather than \( \frac{\lambda}{2} \). All these problems are avoided in the GBC formulation which applies appropriate Box-Cox transformation on both the dependent and independent variables as can be seen in (2.9). Another important new feature of our GBC representation is the incorporation of nonneutral technical change and nonhomothetic scale economies.

II.6 The Elasticities

The own price elasticity is defined as \( \varepsilon_{ii} = \frac{p_i \frac{\partial X_i}{\partial p_i}}{X_i} \) which, by using Shephard's Lemma, can also be written as \( \varepsilon_{ii} = \frac{C_i}{C_{ii}} \) where \( C_i \) and \( C_{ii} \) are
first and second order partial derivatives of \( C \) with respect to \( P_i \). For the GBC cost function (2.29),

\[
(2.56) \quad \varepsilon_{ii} = (1-\lambda)s_i + \gamma_{ii} \frac{p_i^{\lambda}}{s_i} \frac{C^{-\lambda}}{C} + \lambda f_i(Y,t) + \lambda (s_i - F_i(Y,t)) \frac{f_i(Y,t)}{s_i} + \frac{\lambda}{2} \left( 1 - \frac{F_i(Y,t)}{s_i} \right) - 1
\]

where,

\[
(2.57) \quad F_i(Y,t) = \phi_i \ln Y + \tau_i t, \quad \forall_i
\]

and \( \hat{C} \) is as defined in (2.37). In the case of the homothetic GBC cost function with neutral technical progress,

\[
(2.58) \quad \varepsilon_{ii} = (1-\lambda)s_i + \gamma_{ii} \frac{p_i^{\lambda}}{s_i} \frac{C^{-\lambda}}{C} + \frac{\lambda}{2} - 1
\]

The cross price elasticity is \( \varepsilon_{ij} = \frac{p_j}{p_i} \frac{\partial x_i}{\partial P_j} = \frac{p_j}{C_i} C_{ij} \). For the GBC cost function (2.29),

\[
(2.59) \quad \varepsilon_{ij} = (1-\lambda)s_j + \gamma_{ij} \frac{(P_i P_j)^{\lambda/2}}{s_i} \frac{C^{-\lambda}}{C} + \lambda f_j(Y,t) + \lambda (s_j - F_j(Y,t)) \frac{f_j(Y,t)}{s_i}
\]

from which we obtain, for the homothetic GBC cost function with neutral technical change,

\[
(2.60) \quad \varepsilon_{ij} = (1-\lambda)s_j + \gamma_{ij} \frac{(P_i P_j)^{\lambda/2}}{s_i} \frac{C^{-\lambda}}{C}
\]

The Allen-Uzawa partial elasticity of substitution can be obtained by using the relation,

\[
(2.61) \quad \varepsilon_{ij} = \sigma_{ij} s_j
\]
The elasticities for the different functional forms result from (2.56 - 2.60) by substituting in the respective parametric restrictions on \( \lambda \). The special cases of 2.58, 2.60 and correspondingly of 2.61 have been presented in tables II.1 and II.2.

II.7 The Cost Function Properties of the GBC Form

The GBC cost function,

\[
(2.62) \quad C = \frac{1}{\lambda} \sum_{i} \sum_{j} \gamma_{ij} (P_i P_j) \frac{\lambda Y_{P}^2}{\lambda} \exp \left( \frac{t P_i P_j}{\lambda} \right)
\]

is linearly homogeneous in prices. Also since \( \gamma_{ij} = \gamma_{ji} \) in (2.62) the elasticities of substitution are symmetric. One set of sufficient conditions for positivity of \( C \) is that \( \lambda > 0 \) and \( \Gamma = [\gamma_{ij}] \) be positive definite. It can be seen that \( C \) can be positive even with \( \lambda < 0 \). However, we would then require \( \Gamma \) to be negative definite. A still simpler set of sufficient conditions for positivity of \( C \) is that \( 1 > \lambda > 0 \) and \( \gamma_{ij} > 0 \) or \( \lambda < 0 \) and \( \gamma_{ij} < 0 \) for all \( i, j \).

The cost function would be nondecreasing in prices if \( s_i \geq 0 \), where,

\[
(2.63) \quad s_i = \frac{\sum_{j} \gamma_{ij} (P_i P_j) \frac{\lambda Y_{P}^2}{\lambda}}{\sum_{i} \sum_{j} \gamma_{ij} (P_i P_j) \frac{\lambda Y_{P}^2}{\lambda}} + \phi_i \ln Y + \tau_i t
\]

It is sufficient for \( s_i \) to be nonnegative that \( \gamma_{ij} \geq 0, \phi_i \geq 0, \tau_i \geq 0, V_{i, j} \) or \( \gamma_{ij} \leq 0, \phi_i \geq 0 \) and \( \tau_i \geq 0, V_{i, j} \).

The GBC cost function would be concave in prices if its Hessian \( H = [C_{ij}] \) is negative semidefinite. The sufficient parametric conditions for \( H \) to be so are not clear. The definiteness of \( H \) has to be checked at every point in the domain of \( C \). In particular, if \( C \) is concave in prices
Table II.1
THE PRICE ELASTICITIES

<table>
<thead>
<tr>
<th></th>
<th>Own(^{14})</th>
<th>Cross(^{15})</th>
</tr>
</thead>
<tbody>
<tr>
<td>TLOG((\lambda \to 0))</td>
<td>( s_i - 1 + \frac{\gamma_{ii}}{s_i} )</td>
<td>( s_j + \frac{\gamma_{ij}}{s_j} )</td>
</tr>
<tr>
<td>CD((\gamma_{ij} = 0 \text{ in TLOG}))</td>
<td>( s_i - 1 )</td>
<td>( s_j )</td>
</tr>
<tr>
<td>CES((\gamma_{ij} = 0, \text{i} \neq \text{j} \text{ in GBC}))</td>
<td>((s_i-1)(1-\lambda))</td>
<td>(s_j(1-\lambda))</td>
</tr>
<tr>
<td>LC((\lambda = 1 \text{ in CES}))</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>GL((\lambda = 1 \text{ in GBC}))</td>
<td>( \frac{1}{2} \left( \sum_{i} \frac{\gamma_{ii}p_i^{\lambda_2}}{\sum_{j} \gamma_{ij}p_j^{\lambda_2}} - 1 \right) )</td>
<td>( \frac{1}{2} \left( \frac{\sum_{i} \gamma_{ii}p_i^{\lambda_2}}{\sum_{j} \gamma_{ij}p_j^{\lambda_2}} \right) )</td>
</tr>
<tr>
<td>QSR((\lambda = 2 \text{ in GBC}))</td>
<td>( \frac{1}{2} \left( \frac{\gamma_{ii}p_i}{\sum_{j} \gamma_{ij}p_j} - s_i \right) )</td>
<td>( \frac{1}{2} \left( \frac{\sum_{j} \gamma_{ij}p_j}{\sum_{j} \gamma_{ij}p_j} - s_j \right) )</td>
</tr>
</tbody>
</table>
### TABLE II.2

**The Allen Partial Elasticities**

<table>
<thead>
<tr>
<th></th>
<th>Own</th>
<th>Cross</th>
</tr>
</thead>
<tbody>
<tr>
<td>TLOG</td>
<td>$1 - \frac{1}{s_i} + \frac{\gamma_{ii}}{s_i^2}$</td>
<td>$1 + \frac{\gamma_{ij}}{s_is_j}$</td>
</tr>
<tr>
<td>CD</td>
<td>$1 - \frac{1}{s_i}$</td>
<td>1</td>
</tr>
<tr>
<td>CES</td>
<td>$(1 - \frac{1}{s_i})(1 - \lambda)$</td>
<td>$1 - \lambda$</td>
</tr>
<tr>
<td>LC</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>GL</td>
<td>$\frac{1}{s_i} \left{ \frac{\gamma_{ii}}{s_i} \frac{p_i^2}{s_i} - 1 \right}$</td>
<td>$\frac{1}{s_j} \left{ \frac{\sum_j \gamma_{ij} p_j^2}{s_j^2} - 1 \right}$</td>
</tr>
<tr>
<td>QSR</td>
<td>$\frac{\gamma_{ii} p_i}{s_i} - 1$</td>
<td>$\frac{\sum_j \gamma_{ij} p_j}{s_j} - 1$</td>
</tr>
</tbody>
</table>
then the own price effect $C_{jj}$ would be negative. The GBC cost function is uniformly increasing in output $Y$ if the scale elasticity, $\frac{\partial \ln C}{\partial \ln Y} > 0$, where

$$\frac{\partial \ln C}{\partial \ln Y} = \beta + \Theta \ln Y + \sum_{i} \phi_i \ln P_i + \eta t \quad (2.64)$$

A set of sufficient conditions is that $\beta > 0$, $\Theta > 0$, $\phi_i > 0$, $\Psi_i$, and $\eta > 0$. Since,

$$\frac{\partial^2 \ln C}{\partial \ln Y^2} = \Theta \quad (2.65)$$

the condition $\Theta > 0$ would ensure that the minimum point of the average cost curve is reached as output increases, i.e., this condition gives the unit cost curve its traditional U shape.

II.8 The GBC Form as an Approximation

We have introduced the GBC form as a true cost function. However, given $\lambda$, it can be shown that the GBC cost function has enough free parameters to approximate, in the sense of W. E. Diewert [1974b], an arbitrary twice differentiable cost function $C^* = C^*(P,Y,t)$ at given $P^*$, $Y^*$, $t^*$. In the discussion below we use Diewert's method of proof.

Given $\lambda$, symmetry of $r$ and linear homogeneity in prices, the GBC cost function has $(N+2)(N+3)/2$ free parameters. These can be chosen so that the following $(N+2)(N+3)/2$ equations are satisfied:

$$C_i^*(P^*,Y^*,t^*) = C_i(P^*,Y^*,t^*) , \quad i = 1,\ldots,N \quad (2.66)$$

$$C_{ij}^*(P^*,Y^*,t^*) = C_{ij}(P^*,Y^*,t^*) , \quad 1 \leq i < j \leq N \quad (2.67)$$

$$C_{iy}^*(P^*,Y^*,t^*) = C_{iy}(P^*,Y^*,t^*) , \quad i = 1,\ldots,N \quad (2.68)$$

$$C_{it}^*(P^*,Y^*,t^*) = C_{it}(P^*,Y^*,t^*) , \quad i = 1,\ldots,N \quad (2.69)$$
\[(2.70) \quad C_{YY}(P^*, Y^*, t^*) = C_{YY}(P^*, Y^*, t^*)\]

\[(2.71) \quad C_{tt}(P^*, Y^*, t^*) = C_{tt}(P^*, Y^*, t^*)\]

\[(2.72) \quad C_{tt}(P^*, Y^*, t^*) = C_{tt}(P^*, Y^*, t^*),\]

where \(C_{i}^*, \ldots \) are the derivatives of the given cost function \(C^*\) and \(C_{i}, C_{ij}, C_{ij}, C_{iY}, C_{it}, C_{yy}, C_{tt}\) and \(C_{yt}\) are the derivatives of the GBC cost function.

It can be shown that \(C_{i}^*, C_{ij}^*, C_{ij}^*\) for \(1 \leq j < i \leq N\), \(C_{Y}^*, C_{Yt}^*\) are all determined by \(C_{i}^*, C_{ij}^*\) for \(1 \leq i < j \leq n, C_{Y}^*, C_{Yt}^*\). If the parameters of the GBC cost function are chosen to satisfy 2.66 - 2.72 then it will be true that \(C^* = C, C_{i}^* = C_{i}, C_{ij}^* = C_{ij}, C_{ij}^* = C_{ij}, C_{ij}^* = C_{ij}, C_{iY}^* = C_{iY}, C_{it}^* = C_{it}, C_{yy}^* = C_{yy}, C_{tt}^* = C_{tt}\) and \(C_{yt}^* = C_{yt}\) at \(P^*, Y^*, t^*\) and hence satisfying the requirements for second order differential approximation.

II.9 Returns to Scale and Total Factor Productivity

The measures of total factor productivity and returns to scale in the context of a nonconstant returns to scale technology have been discussed by M. Ohta [1974]. Here we use his definition to obtain the relevant measures for the GBC formulation.

The rate of total cost diminution for the GBC representation is, with input prices and output fixed,

\[(2.73) \quad d = -\frac{3\ln C}{3t} = -(\tau + \delta t + \sum_{i} \tau_{i} \ln P_{i} + \eta \ln Y)\]

and the elasticity of cost with respect to output is

\[(2.74) \quad \epsilon_{CY} = \frac{3\ln C}{3\ln Y} = \beta + \phi \ln Y + \sum_{i} \phi_{i} \ln P_{i} + \eta t \]

Ohta defines the rate of returns to scale as
where input prices are fixed. The rate of total factor productivity is then,

\[(2.76) \quad \Pi = r \cdot d \quad .\]

Note that this measure of total factor productivity is parametric as compared with the traditional residual measure. In chapter IV we compare total factor productivity measures by both these methods.

II.10 More Comparative Statics

A variety of interesting sensitivity analyses are possible in the GBC representation. As we have seen the response of shares to change in output or time provide an interesting interpretation of nonhomotheticity and nonneutrality parameters,

\[(2.77) \quad \frac{\partial s_i}{\partial \ln Y} = \phi_i \]

\[(2.78) \quad \frac{\partial s_i}{\partial t} = \tau_i \quad .\]

The response of shares to prices is given by,

\[(2.79) \quad \frac{\partial s_i}{\partial \ln p_i} = (\sigma_{ii} - 1)s_i^2 + s_i \]

and,

\[(2.80) \quad \frac{\partial s_i}{\partial \ln p_j} = (\sigma_{ij} - 1)s_is_j \]

where \(\sigma_{ii}\) and \(\sigma_{ij}\) are own and cross elasticities of substitution.
It is also of interest to consider the effects of scale, time and price changes on the rates of total cost diminution, returns to scale and total factor productivity. The effects of change in output are,

\[
\frac{\partial d}{\partial \ln Y} = -n
\]

(2.81)

\[
\frac{\partial r}{\partial \ln Y} = -\theta r^2
\]

(2.82)

and,

\[
\frac{\partial \pi}{\partial \ln Y} = -(\eta + \Theta \pi)r
\]

(2.83)

If \( n < 0 \) then an increase in output would increase the rate of total cost diminution. This might be possible if larger scale of operation facilitated the realization of benefits of technical change. If \( \theta > 0 \) then returns to scale would decline as output increased and the minimum point of the average cost curve would be reached. However, if \( n < 0, \theta > 0 \) and \( \Pi > 0 \), then the impact of increase in output on total factor productivity (\( \Pi \)) is uncertain.

The consequence of change in time are the following,

\[
\frac{\partial d}{\partial t} = -\delta
\]

(2.84)

\[
\frac{\partial r}{\partial t} = -n r^2
\]

(2.85)

and,

\[
\frac{\partial \pi}{\partial t} = -(\delta + \eta \Pi)r
\]

(2.86)

If \( \delta < 0 \) then the rate of cost diminution increases over time. If \( n < 0 \) then technical change would increase returns to scale. The rate of total factor productivity would increase over time if \( \delta < 0, n < 0 \) and \( \Pi > 0 \).
More important for policy purposes are the price effects.

\[ (2.87) \quad \frac{\partial d}{\partial \ln P_i} = -\tau_i, \]
\[ (2.88) \quad \frac{\partial r}{\partial \ln P_i} = -\phi_i r^2, \]

and,

\[ (2.89) \quad \frac{\partial \Pi}{\partial \ln P_i} = -(\tau_i + \phi_i \Pi) r. \]

If \( \tau_i < 0 \), i.e., if technical change is input \( i \) saving, then an increase in the price \( i \) increases the rate of total cost diminution. On the other hand if \( \phi_i < 0 \), i.e., if input \( i \) has economies of scale in it then a rise in \( P_i \) would raise the returns to scale. In the case when \( \tau_i < 0, \phi_i < 0 \) and \( \Pi > 0 \) then the rate of total factor productivity would increase with growth in \( P_i \).

Besides the total factor productivity index other commonly used productivity indexes are the average products, \( \frac{Y}{X_i} \) of the various inputs. These are the reciprocals of the input-output coefficients. Unlike our total productivity measure the partial productivity indices, i.e., the average products would be affected by changes in relative prices even when scale economies are homothetic and technological change is neutral.

In the nonhomothetic nonneutral representation the comparative statics of the partial productivity indexes are the following,

\[ (2.90) \quad \frac{\partial \ln Y}{\partial \ln y} = 1 - \rho_{CY} - \frac{\phi_i}{s_i}, \quad \text{(input prices fixed)} \]
\[ (2.91) \quad \frac{\partial \ln Y}{\partial t} = \rho_Y - \frac{\tau_i}{s_i}, \quad \text{(input prices and output fixed)} \]

and,
The effects of an increase in scale on the average products depend on the respective input shares and the nonhomothetic bias. Similarly, the time effects also depend on the input shares and the nonneutral bias. The effects of the input prices on the average products are simply the negative of the price elasticities.

II.11 Separability in the GBC Functional Form

For estimation we will regard the GBC form as the true function. Recently C. Blackorby, D. Primont and R. Russell [1977] have shown that when the flexible forms are treated as exact the separability restrictions on these forms are stronger than intended. Not only is separability imposed but the aggregators are necessarily restricted as well. Unfortunately this is also true of the GBC formulation; we show this in the two theorems below which are, of course, direct consequences of Blackorby, Primont and Russell's more general theorems. However, in the discussion below we present these theorems in a simple and nested framework.

The conditions for separability of the $i^{th}$ and $j^{th}$ variables from the $k^{th}$ variable in the homothetic GBC cost function with neutral technical change,

\begin{equation}
C = (1 + \lambda G(P))Y^{\beta(Y)} e^{T(t)}
\end{equation}

where $\beta(Y) = \beta + \frac{\sigma}{2} \ln Y$ and $T(t) = (\tau + \frac{\delta}{2} t)t$, are,

\begin{equation}
\alpha_i Y_j k - \alpha_j Y_i k = 0, \quad \text{and}
\end{equation}

\begin{equation}
\gamma_{ih} Y_j k - \gamma_{jh} Y_i k = 0, \quad \forall h.
\end{equation}
For weak separability $i, j \in$ one subset and $k \in$ another subset of inputs.
For strong separability $i \in$ one subset, $j \in$ another subset and $k \in$ yet another subset of inputs.

The conditions (2.94) can be satisfied in two ways,

\begin{align}
(2.95) \quad & \gamma_{ij} = 0 = \gamma_{jk} \quad \text{(linear or additive separability) or,} \\
(2.96) \quad & \frac{\alpha_i}{\alpha_j} = \frac{\gamma_{ik}}{\gamma_{jk}} = \frac{\gamma_{ih}}{\gamma_{jh}} \quad \text{(nonlinear or nonadditive separability).}
\end{align}

Theorem I below and the corollaries thereof relate to linear separability (L.S.). Theorem II and its corollaries give the implications of nonlinear separability (NLS).

Let the set of $N$ inputs be partitioned into $S$ disjoint subsets $N_1, \ldots, N_S$ so that $N_1 \cup N_2 \cup \ldots \cup N_S = N$.

**Theorem I.** If each subset is linearly separable from its complement in the set then the homothetic GBC function can be written as a homothetic 'CES type' aggregate of GBC aggregators.

(Proofs of theorems and corollaries can be found in Appendix 2).

The following corollaries can be obtained as parametric special or limiting cases of theorem I.

**Corollary I.1.** The homothetic GBC cost function is a homothetic CES aggregate of GBC aggregator functions linearly homogeneous in prices.

**Corollary I.2.** The homothetic TLOG cost function is a homothetic CD aggregate of TLOG aggregator functions linearly homogeneous in prices.

**Corollary I.3.** The homothetic GL cost function is a homothetic linear aggregate of GL aggregator functions.
Theorem II. If each subset is nonlinearly separable from its complement in the set then the homothetic GBC function is a homothetic GBC aggregate of the 'CES type' aggregators.

The corollaries below are parametric special or limiting cases of theorem II.

Corollary II.1. The homothetic GBC cost function is a homothetic GBC aggregate function linearly homogeneous in CES aggregators.

Corollary II.2. The homothetic TLOG cost function is a homothetic TLOG aggregate function linearly homogeneous in CD aggregators.

Corollary II.3. The homothetic GL cost function is a homothetic GL aggregate function linearly homogeneous in mean of order half aggregators.

Corollary II.4. The homothetic QSR cost function is a homothetic QSR aggregate function linearly homogeneous in linear aggregators.

It is clear from the two theorems and their corollaries that the GBC generalization is also subject to Blackorby, Primont and Russell's result that in testing weak separability with flexible functional forms, we also necessarily test the functional form of the aggregate or the aggregators.
Footnotes to Chapter II

1. This result is known as the Shephard duality theorem. See. R. W. Shephard [1953] and W. E. Diewert [1974b].

2. The GBC production function is,

\[ Y = \left(1 + \lambda G(X)\right)^{\frac{1}{\lambda}} \]

where, \( G(X) = a_0 + \sum \frac{\lambda}{i} a_i X_i(\lambda) + \frac{\lambda}{2} \sum \sum \frac{a_{ij} X_i(\lambda) X_j(\lambda)}{\lambda} \)

\[ X_i(\lambda) = \frac{X_i^2 - 1}{\frac{\lambda}{2}} \] ,

and \( Y = \) output, \( X_i = \) quantity of \( i\)th input. Depending on the value of \( \lambda \) a variety of different functional forms of production are possible.

3. The GBC unit cost function (2.8) is a slight generalization of M. Denny's [1974] 'mean of order \( \lambda \)' cost function. This is discussed further in section 3.

4. We utilize L' Hospital's Rule by which

\[ \lim_{\lambda \to 0} X(\lambda) = \ln X \text{ where } X(\lambda) = \frac{X^{\frac{\lambda}{2}} - 1}{\frac{\lambda}{2}} . \]

5. The CD form can be obtained as a special case of the CES form (2.16) when the parameters are redefined, for \( \lambda \neq 0 \), as \( \gamma_{ii} = \frac{\lambda}{2} \gamma_{ii}^* \). If \( \alpha_0 = 0 \) we have the relation \( \sum \frac{2}{\lambda} \gamma_{ii} = \sum \gamma_{ii}^* = 1 \). Using this and letting \( \lambda \to 0 \) we obtain the CD form,

\[ C = \Pi_{i} \gamma_{ii}^* . \]

As we shall see later, this method has been used by W. E. Diewert [1975] in obtaining the homogeneous TLOG as a special case of M. K. Denny's GQ form.
6. For this case and also for the mean of order r, see G. H. Hardy, J. E. Littlewood and G. Polya [1934].

7. As we shall see in chapter IV, it is important that negative values of \( \lambda \) be considered. In several models the maximum likelihood estimate of \( \lambda \) is close to \(-1\). For negative estimates of \( \lambda \) also see H. Chang [1977].

8. With restriction \( \sum \phi_i = 0 \) imposed we only have to set \( \phi_i = 0 \) for \( i = 1, \ldots, N-1 \) to obtain homotheticity.

9. This definition of Hicks neutrality along an expansion path corresponds to C. Blackorby, C. A. K. Lovell and M. Thursby's [1976] definition of Extended Hicks neutrality. When \( \tau_i = 0 \), \( \psi_i \) the GBC cost function factors into a function of prices and output and a function of time and output. The input cost shares are in this case independent of time.

10. Interaction might be present for the following reasons: (a) Technological change may shift the minimum point of the average cost curve to the right increasing the range of output over which scale economies are present. To emphasize such interdependence T. G. Cowing [1974] calls it 'scale augmenting' technical change. (b) On the other hand increase in scale of operation may enable adoption of technological innovations not worthwhile at a lower scale. For an interpretation of interaction between scale effects and technical change also see B. Gold [1974].

11. This can be seen from the following,

\[
\dot{s}_i = s_i \left( \frac{\dot{a}_i}{a_i} - \frac{AC}{\dot{AC}} \right) = \tau_i
\]

where \( s_i \) is the share of \( i \), \( a_i \) is the \( i^{th} \) input-output coefficient and \( AC \) is unit cost. The dot over a variable denotes time derivative.
If technological progress occurs, $\hat{A}C < 0$. So, when $\tau_i < 0$ it follows that $\dot{a}_i < 0$ and also that $\frac{\dot{a}_i}{a_i} < \frac{\hat{A}C}{AC}$, i.e. the $i^{th}$ input-output coefficient declines faster than unit cost. If $\tau_i = 0$ then $\dot{a}_i < 0$ but $\frac{\dot{a}_i}{a_i} = \frac{AC}{AC}$. When $\tau_i > 0$, $\frac{\dot{a}_i}{a_i} > \frac{AC}{AC}$. So $\dot{a}_i$ can be $< 0$.

12. This definition of nonneutrality is along an expansion path. K. Sato [1965] has similarly defined bias of technical invention along a ray.

13. To see this we can write,

$$\frac{\partial s_i}{\partial \ln Y} = s_i \left( \frac{\partial \ln X_i}{\partial \ln Y} - \frac{\partial \ln C}{\partial \ln Y} \right) = \phi_i$$

where $X_i$ is quantity of input $i$ and $C$ is total cost. If scale economies are present $\frac{\partial \ln C}{\partial \ln Y} < 1$. Now if $\phi_i < 0$ then $\frac{\partial \ln X_i}{\partial \ln Y} < 1$ so that input-output coefficient $a_i$ declines as $Y$ is increased.

14. The special cases of own price elasticities for CES, GL and QSR forms result from writing (2.58) as,

$$\varepsilon_{ii} = (1-\lambda)s_i + \frac{\gamma_{ii} \lambda}{2} + \frac{\lambda}{2} - 1$$

15. The special cases of cross price elasticities for GL and QSR forms are obtained by writing (2.60) as,

$$\varepsilon_{ij} = (1 - \lambda)s_j + \frac{\gamma_{ij} \lambda}{\lambda \sum_j \gamma_{ij} p_{ij}^2}$$

16. These are obtained by dividing the own and cross elasticities in table II.1 by $s_i$ and $s_j$ respectively.
17. These conditions also ensure that

\[ \left\{ \frac{2}{\lambda} \sum_{i} \sum_{j} \gamma_{ij} \left( \frac{p_i}{p_j} \right)^{2 \lambda} \right\}^{\frac{1}{\lambda}} \]

is concave.

18-22. These equations are presented in Appendix 1.

23. See W. E. Diewert [1974b]. Linear homogeneity of \( C^* \) in prices implies, applying Euler's theorem,

\[ C^* = \sum_{i} C^*_{i} p^*_i \]

\[ 0 = \sum_{j} C^*_{ij} p^*_j \]

and Young's theorem implies that

\[ C^*_{ij} = C^*_{ji} \]

To the above set of equation we add the following equations, which are also consequences of Euler's Theorem,

\[ C^*_{yi} = \sum_{i} C^*_{yi} p^*_i \]

\[ C^*_t = \sum_{i} C^*_{ti} p^*_i \]

Through all the above equations \( C^*, C^*_{ii}, C^*_{ij} \) for \( i \leq j < i \leq N \), \( C^*_{yi} \) and \( C^*_t \) are determined by \( C^*_{yi}, C^*_{ij} \) for \( 1 \leq i < j \leq N \), \( C^*_{yi} \) and \( C^*_{ti} \).
24. This measure of returns to scale is along the expansion path, i.e., costs are minimized at every level of output given input prices. G. Hanoch [1975] has argued that it is more appropriate to measure scale economies along the expansion path in a cost minimization framework. L. Christensen and H. Greene [1976] define scale economies as 
\[
1 - \frac{\partial \ln C}{\partial \ln \mathbf{Y}}.
\]
This number is positive for scale economies and negative for scale diseconomies. By Ohta's measure, \( r > 1 \) for scale economies and \( r < 1 \) for scale diseconomies.


26. The effects of price changes on \( d \), \( r \) and \( \Pi \) and the comparative statics of average factor productivity have been derived in E. R. Berndt and M. S. Khaled [1977].

27. Separability cannot be tested in the nonhomothetic and nonneutral case in the GBC formulation of section 2 and 3. However, the GBC representation in section 4 permits test of separability even in the nonhomothetic and nonneutral case.

28. For derivation of these conditions in a TLOG production function see E. R. Berndt and L. Christensen [1973].

29. We can obtain as special cases of corollaries I.1 and II.1 a CES aggregate of CES aggregator functions. This special case can in turn be treated as a special case of Sato's [1967] two level CES function. For other two level functions see D. McFadden [1962] and H. Uzawa [1962].

30. Recently, however, A. D. Woodland [1976] has suggested a profit maximization framework to test for weak separability of a single group without requiring the aggregator function to be less than flexible.
31. For simplicity our proof is in terms of four inputs only.

32. Throughout appendix 2 we use the following notation:

\[ X(\frac{\lambda}{2}) = \frac{\lambda}{2} \left( \frac{X^2}{\frac{\lambda}{2}} - 1 \right), \quad X(\lambda) = \frac{X^\lambda - 1}{\lambda} \] where \( X \) is any variable.
Chapter III
TOWARDS EMPIRICAL IMPLEMENTATION

In this chapter we discuss the issues of estimating a cost function model using the GBC specification. In section 1 we derive the system of estimating equations. In section 2 the estimating equations are set in an appropriate stochastic framework. In section 3 we discuss some other econometric issues. In the final two sections we consider the testing procedure and the measure of fit used in chapter IV.

III.1 The Share Equations

An econometric problem in the direct estimation of the cost function is multi-collinearity. Moreover, the notion of optimizing behaviour is not fully incorporated by simply estimating the cost function; the cost function is only one equation in a complete model.

Optimizing behaviour can be modelled more fully by considering, along with the cost function, the factor demand equations, easily obtained by differentiating the cost function. Our cost function is,

\[ C = \left(1 + \lambda G(P)\right)^{\lambda} \beta(Y, P) e^{T(t, P)} \]

where,

\[ G(P) = \alpha_0 + \sum_i \alpha_i P_i(\lambda) + \frac{1}{2} \sum_i \sum_j \gamma_{ij} P_i(\lambda) P_j(\lambda) \]

\[ P_i(\lambda) = \frac{P_i^2 - 1}{(\lambda^2)} \]

\[ \beta(Y, P) = \beta + \frac{\Theta}{2} \ln Y + \sum_i \phi_i \ln P_i \]

and
Differentiating (3.1) with respect to $P_i$ and by rearranging terms we obtain the following system of equations,

$$s_i = \frac{\sum_j \gamma_{ij}(P_j)^{\frac{\lambda}{2}}}{1 + \lambda G(P)} + \phi_i \ln Y + \tau_i t$$

where $s_i$ is the share of the $i$th input in total cost.

Imposing the restrictions of symmetry (2.6) and linear homogeneity in prices (2.7), (2.22) and (2.28) in (3.6) we obtain the GBC share equations,

$$s_i = \frac{\sum_j \gamma_{ij}(P_j)^{\frac{\lambda}{2}}}{\sum_i \sum_j \gamma_{ij}(P_i P_j)^{\frac{\lambda}{2}}} + \phi_i \ln Y + \tau_i t$$

The share systems for GL and QSR cost functions can be obtained by setting $\lambda = 1$ and $\lambda = 2$ respectively in (3.7). Taking the limit of (3.6) as $\lambda \to 0$ we obtain the TLOG share equations,

$$s_i = \alpha_i + \sum_j \gamma_{ij} \ln P_j + \phi_i \ln Y + \tau_i t$$

The tests of symmetry and linear homogeneity in prices and of homotheticity, homogeneity or linear homogeneity in output and neutrality of technological change can all be done separately and parametrically in our system of equations (3.1) and (3.6) and in their special or limiting cases.

In the absence of any restrictions the complete system has $N^2 + 3N + 5$ free parameters. In table III.1 we show how the number of free parameters is reduced when restrictions are imposed.
Table III.1  
**THE NUMBER OF FREE PARAMETERS IN THE GBC FORM**

<table>
<thead>
<tr>
<th>Number of Parameters in the unrestricted model</th>
<th>N Factor Model</th>
<th>4 Factor Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N^2 + 3N + 5$</td>
<td>33</td>
</tr>
<tr>
<td>Number of Parameters in the restricted Models:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Symmetry and linear homogeneity in prices</td>
<td>$\frac{N^2 + 5N}{2} + 2$</td>
<td>20</td>
</tr>
<tr>
<td>2. Homotheticity and 1</td>
<td>$\frac{N^2 + 3N}{2} + 3$</td>
<td>17</td>
</tr>
<tr>
<td>3. Neutrality and 1</td>
<td>$\frac{N^2 + 3N}{2} + 3$</td>
<td>17</td>
</tr>
<tr>
<td>4. Homotheticity, Neutrality and 1</td>
<td>$\frac{N^2 + N}{2} + 4$</td>
<td>14</td>
</tr>
<tr>
<td>5. Homogeneity in Output and 3</td>
<td>$\frac{N^2 + N}{2} + 3$</td>
<td>13</td>
</tr>
<tr>
<td>6. CRTS and 3</td>
<td>$\frac{N^2 + N}{2} + 2$</td>
<td>12</td>
</tr>
</tbody>
</table>
II.2 Stochastic Specification and the Likelihood Function

For estimation it is necessary that the system of equations be set up in an appropriate stochastic framework.

In estimating the flexible forms researchers have often specified a deterministic production, cost or utility function and additive normally distributed errors in the demand or share equation to represent deviation from optimal demand decisions.

In this thesis we adopt the assumption of additive errors of optimizing behaviour in the share equations,

\[
(3.9) \quad s_i = \alpha_i P_i^2 + \sum_j \gamma_{ij} P_j (\lambda P_i^2) + \phi_i \ln Y + \tau_i t + \varepsilon_i
\]

and an additive error in the following cost equation,

\[
(3.10) \quad \frac{C^\lambda}{\lambda} - 1 = G(P) + u
\]

where \( \frac{C}{\beta(Y, P)} e^{T(t, P)} \) is cost adjusted for effects of scale and technical change.

Since shares add up to unity, one of the errors in (3.9) is not independent. This problem is handled by dropping any one of the share equations from the estimating system. It is known that this is of no consequence on the parameter estimates if we use a maximum likelihood method of estimation.

We assume that the error \( u \) and the \((N-1)\) errors \( \varepsilon_i \) have a joint normal distribution with a mean vector zero and a non-singular covariance matrix \( \Omega \). The variance covariance matrix is assumed constant because
scale effects on error variances and covariances have hopefully been taken into account in the share equations. Furthermore, one purpose of the transformation of \( C \) in (3.10) is to make the error term \( u \) homoscedastic.

The observed endogenous variables in our model (3.9 and 3.10) are the shares \( s_i \) and total cost \( C \). However, it can be seen from (3.10) that \( C \) has been transformed. So the likelihood function for the observed endogenous variables will have a Jacobian in it. The Jacobian is,

\[
(3.11) \quad J_t = \begin{vmatrix}
\frac{\partial u}{\partial C} & \frac{\partial u}{\partial s_i} \\
\frac{\partial \varepsilon_i}{\partial C} & \frac{\partial \varepsilon_i}{\partial s_j}
\end{vmatrix}
= \begin{vmatrix}
\frac{C_t^\lambda}{C_t} & 0 \\
0 & 1
\end{vmatrix}
= \frac{C_t^\lambda}{C_t}.
\]

The log of the concentrated likelihood function for the observed total cost \( C_t \) and the \( (n-1) \) shares \( s_{it}, t = 1, \ldots, T \) is then,

\[
(3.12) \quad L(b) = B - \frac{T}{2} \ln |\hat{\Omega}| + \sum_{t=1}^{T} \ln \{\text{abs}(J_t)\}
= B - \frac{T}{2} \ln |\hat{\Omega}| + \lambda \sum_{t=1}^{T} \ln \hat{C}_t - \sum_{t=1}^{T} \ln C_t
\]

where \( b \) is the vector of all the parameters, \( B \) is a constant term, \( \hat{\Omega} \) is the maximum likelihood estimate of \( \Omega \) and \( \text{abs}(J_t) \) is the absolute value of the Jacobian.

When we impose symmetry and linear homogeneity in prices, the log of the concentrated likelihood for the GBC cost function is (3.12) with,

\[
\hat{\Omega} = \frac{1}{T} \begin{vmatrix}
\hat{u}' \hat{\varepsilon} & \hat{u}' \varepsilon_1 & \ldots & \hat{u}' \varepsilon_{n-1} \\
\hat{\varepsilon}_1' \hat{\varepsilon} & \hat{\varepsilon}_1' \varepsilon_1 & \ldots & \hat{\varepsilon}_1' \varepsilon_{n-1} \\
\vdots & & & \\
\hat{u}' \varepsilon_{n-1} & \ldots & \hat{\varepsilon}_{n-1}' \varepsilon_{n-1}
\end{vmatrix}
\]
where,

\[
(3.13) \quad u = \frac{C^\lambda}{\lambda} - \frac{2}{\lambda^2} \sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}}
\]

and,

\[
(3.14) \quad e_i = s_i - \frac{\sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}}}{\sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}}} - \phi_i \ln Y - \tau \bar{t}, \quad i = 1, \ldots, N-1.
\]

The special cases of the likelihood function for GL and QSR functions can be obtained by setting \(\lambda = 1\) and \(\lambda = 2\) respectively in the above equations. The limiting case of (3.12) as \(\lambda \to 0\) gives the likelihood function for a TLOG system of equations. This limiting form is,

\[
(3.12a) \quad L_0(b) = B - \frac{T}{2} \ln |\hat{\beta}_0| - \sum_{t=1}^T \ln C_t
\]

where,

\[
\hat{\beta}_0 = \frac{1}{T} \begin{bmatrix}
\hat{u}_0 & \hat{v}_0 & \hat{e}_{1,0} & \cdots & \hat{u}_0 & \hat{e}_{n-1,0} \\
\hat{u}_0 & \hat{e}_{1,0} & \hat{v}_1 & \hat{e}_{1,1,0} & \cdots & \hat{v}_1 & \hat{e}_{1,0} & \cdots & \hat{v}_1 & \hat{e}_{1,0} & \hat{e}_{1,1,0} \\
\vdots & \hat{e}_{1,0} & \hat{v}_1 & \hat{e}_{1,1,0} & \cdots & \hat{v}_1 & \hat{e}_{1,0} & \cdots & \hat{v}_1 & \hat{e}_{1,0} & \hat{e}_{1,1,0} \\
\hat{u}_n & \hat{v}_n & \hat{e}_{n-1,0} & \cdots & \hat{v}_n & \hat{e}_{n-1,0} & \hat{e}_{n-1,0} & \cdots & \hat{v}_n & \hat{e}_{n-1,0} & \hat{e}_{n-1,0}
\end{bmatrix}
\]

\[
(3.15) \quad u_0 = \lim_{\lambda \to 0} u
\]

\[
= \lim_{\lambda \to 0} \left[ \frac{C^\lambda}{\lambda} - \frac{1}{\lambda} - G(P) \right], \quad \text{using (3.10)}
\]

\[
= \ln C - \alpha_0 - \sum_i \alpha_i \ln P_i - \frac{1}{2} \sum_{i} \sum_{j} \gamma_{ij} \ln P_i \ln P_j
\]

and,
The likelihood function (3.12) is then continuous everywhere including \( \lambda = 0 \). Therefore, given data on prices and quantities of input services and quantity of output, the parameters of the GBC cost function and of its special or limiting cases can be estimated by maximizing the sample value of (3.12) or of its corresponding special or limiting cases. It is also notable that the likelihood function (3.12) and its special or limiting cases allow a variety of hypotheses to be tested concerning the different types of scale effects and technological change.

The likelihood function (3.12) reaches a maximum when the gradients,

\[
\frac{\partial L(b)}{\partial b} = 0
\]

and the Hessian,

\[
\frac{\partial^2 L(b)}{\partial b \partial b'}
\]

is negative definite. If the maximum of (3.13) obtained as above is global then the corresponding parameter estimates have the properties of consistency, asymptotic efficiency and asymptotic normality.

A consistent estimate of the asymptotic variance covariance matrix of the maximum likelihood parameter estimates \( \hat{b} \) is given by
(3.19) \( V(b) = - \left[ \frac{\partial^2 L(\hat{b})}{\partial b \partial b'} \right]^{-1} \)

which is the negative of the inverse of the Hessian matrix at the maximum likelihood parameter estimates.

III.3 Other Econometric Issues

In estimating the flexible forms some researchers (e.g. D. F. Burgess [1975], M. Denny and M. Fuss [1975]) have treated these forms as approximations. This approach raises the possibility of approximation induced autocorrelation and heteroscedasticity of an unknown magnitude in the residuals. Approximation also leads to a conflict between desirable closeness of data points to the point of approximation and desirable dispersion of data for efficient parameter estimates. Finally, as argued by L. J. Lau [1974] many properties of the true functions are not preserved in the approximating functions. Consequently testing a property of the true function on the approximating function may be misleading. In this thesis we take the approach of treating the GBC equation as exact rather than as an approximation.

III.4 The Test Criterion and Measure of Fit

For testing hypotheses we utilize the likelihood ratio test. The test statistic is \( 2(L_u - L_c) \), where \( L_u \) is the log of likelihood of the more general model and \( L_c \) is the log of likelihood of the constrained model. This statistic is of course distributed asymptotically as \( \chi^2_q \) where \( q \) is the number of restrictions in \( L_c \) as compared to \( L_u \).
To evaluate the precision of the parameter estimates we also make use of the 'asymptotic t ratios' defined as the ratio of the parameter estimate to its asymptotic standard error. We compare this ratio with critical t values to test the null hypothesis that the relevant parameter is zero. The square of this ratio is called 'Wald Statistic' and is also asymptotically distributed as $\chi^2$ with 1 degree of freedom.

For measuring the goodness of fit of the complete system we compute our $R^2$ measure following the method of N. D. Baxter and J. G. Cragg [1970],

$$R^2 = 1 - \exp\{2(L_0 - L_1)/N\}$$

where $L_0$ is the sample maximum of log of likelihood when all the 'slope coefficients' (i.e., all parameters other than the intercept $\alpha_0$ and the transformation parameter $\lambda$) are zero, $L_1$ is the sample maximum of log of likelihood when some or all of these slope coefficients are not constrained and $N$ is the number of observations in the entire system.

III.5 The Testing Design

We impose symmetry and linear homogeneity in prices as our maintained hypothesis. We test for homotheticity, homogeneity (in output) and constant returns to scale. Corresponding to each of the different forms of returns to scale we test for neutrality or absence of technological change.

Having chosen the form of technological change and returns to scale we test for the different functional forms by testing the different $\lambda$ values.

Positivity of the cost function, its monotonicity in input prices and output and concavity in prices will be checked at each observation for the chosen models.
Footnotes to Chapter III


2. For an alternative stochastic specification of the GBC system see E. R. Berndt and M. S. Khaled [1977].

3. E. R. Berndt [1977] has related this $R^2$ measure to the likelihood ratio test statistic for all coefficients other than the constant being zero.

4. In this most restrictive case we let $C = \frac{C}{Y}$ so that the 'slope coefficients' that are set equal to zero are $\alpha_i, \gamma_{ij}, \phi_i, \tau_i$ for all $i,j$ and $\beta^* (= \beta - 1)$, and $\tau$. When all these parameters are zero the likelihood function (3.12) has $\hat{\Omega} = \frac{1}{\lambda} \hat{w}' \hat{w}$ where,

$$
\hat{w} = \begin{bmatrix}
\hat{u} \\
\hat{\epsilon}_1 \\
\hat{\epsilon}_2 \\
\vdots \\
\hat{\epsilon}_{n-1}
\end{bmatrix}
= \begin{bmatrix}
\frac{\chi^2}{\lambda} - \frac{1}{\lambda} - \alpha_0 \\
\lambda \cdot s_1 \\
\lambda \cdot s_2 \\
\vdots \\
\lambda \cdot s_{n-1}
\end{bmatrix}
$$
Chapter IV

THE GBC FUNCTIONAL FORM: AN APPLICATION
WITH U.S. MANUFACTURING DATA, 1947-71

In this chapter we utilize the GBC functional form to estimate a four factor--capital (K), labour (L), energy (E) and other intermediate materials (M)--model of U.S. Manufacturing, 1947-71.

IV.1 A Model

It is assumed that the production technology of the U.S. Manufacturing sector is characterized by a continuous, twice differentiable and concave function in the four inputs and that the input levels are chosen to minimize the total cost of producing a specified quantity of output subject to the given prices and knowledge. We let the effect of time, t represent the effect of technological progress. We can then obtain the minimum total cost function as,

\[(4.1) \quad C = C(P_K, P_L, P_E, P_M, Y, t)\]

which, by duality, would have the properties of linear homogeneity and concavity in prices.

In this thesis we use the following GBC specification to represent (4.1),

\[(4.2) \quad \frac{C^\lambda - 1}{\lambda} = G(P)\]

where,

\[(4.3) \quad G(P) = \alpha_0 + \sum_i \alpha_i \lambda P_i(\lambda) + \sum_i \sum_j \gamma_{ij} P_i(\lambda) P_j(\lambda), \quad i, j = K, L, E, M\]
\[ P_i(\lambda) = \frac{\lambda}{P_i^2 - 1} \quad , \quad i = K, L, E, M \]

\[ \dot{C} = \frac{C}{\gamma Y P} e^{T(t,P)} \]

\[ \beta(Y,P) = \beta + \frac{\theta}{2} \ln Y + \sum_i \phi_i \ln P_i \quad , \quad i = K, L, E, M \]

and,

\[ T(t,P) = (\tau + \sum_i \tau_i \ln P_i)t \quad , \quad i = K, L, E, M \]

If \( \lambda \) is free but we impose symmetry (2.7) and the linear homogeneity restrictions (2.7), (2.22) and (2.28) on (4.2) we obtain the following GBC cost function,

\[ C = \left\{ \frac{2}{\lambda} \sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}} \right\} \frac{1}{\lambda} \gamma Y P e^{T(t,P)} \quad , \quad i, j = K, L, E, M . \]

On the other hand, if \( \lambda \to 0 \) in (4.2) we obtain the translog cost function,

\[ \ln C = \alpha_0 + \sum_i \alpha_i \ln P_i + \frac{\theta}{2} \sum_i \sum_j \gamma_{ij} \ln P_i \ln P_j + \beta(Y,P) \ln Y + T(t,P) \quad , \quad i, j = K, L, E, M \]

where the restrictions of symmetry and linear homogeneity in prices are the same as those in (4.8) but with \( \lambda = 0 \).

By utilizing Shephard's Lemma we obtain from (4.8) and (4.9) the respective share equations, which after adding errors of optimizing behaviour, are,

\[ s_i = \frac{\sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}}}{\sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}}} \phi_i \ln Y + \tau_i t + \varepsilon_i \quad i = K, L, E, M \]
(4.11) \[ s_i = \alpha_i + \sum_j \gamma_{ij} \ln P_j + \phi_i \ln Y + \tau_i t + \varepsilon_i , \quad i = K, L, E, M. \]

However, since shares add up to unity one of the errors \( \varepsilon_K, \varepsilon_L, \varepsilon_E \) and \( \varepsilon_M \) is not independent. Since maximum likelihood parameter estimates are invariant to whichever equation is dropped we arbitrarily choose to delete the materials share equation. The lost degrees of freedom in the systems (4.10) or (4.11) can be regained by including in the respective systems the following cost equations which are, with additive errors,

\[ \hat{\gamma}^T \hat{\gamma} = \frac{2}{\lambda} \sum_i \sum_j \gamma_{ij} (P_i P_j)^{\frac{\lambda}{2}} + u \]

and,

\[ \ln \hat{C} = \alpha_0 + \sum_i \alpha_i \ln P_i + \frac{1}{2} \sum_i \sum_j \gamma_{ij} \ln P_i \ln P_j + u. \]

Inclusion of the cost equations also permits estimation of parameters that do not appear in the share equations.

We assume that the vector of errors, \( w = \{u, \varepsilon_K, \varepsilon_L, \varepsilon_E\} \) is multivariate normally distributed with mean vector 0 and nonsingular variance-covariance matrix \( \Omega \) at every observation. The log of the concentrated likelihood function of observed total cost and shares is then,

\[ \sum_{i=1}^T \ln |\hat{\Omega}| + \sum_{t=1}^T \ln \text{abs}(J_t) \]

where \( b \) is the vector of all the parameters to be estimated, \( Q \) is a constant, \( T \) is the number of observations in each of the four estimating equations in the system, \( \hat{\alpha} \) is the maximum likelihood estimate of the variance-covariance matrix \( \Omega \),
and \( \text{abs} \left( J_t \right) \) is the absolute value of the Jacobian of the transformation from the errors to the observed endogenous variables,

\[
(4.16) \quad J_t = \frac{\gamma^\lambda}{C_t} \quad t = 1, \ldots, T
\]

The parameter estimates are chosen so that the sample value of the log of likelihood (4.14) is maximized. It is known that the maximum likelihood estimators are consistent, asymptotically efficient and asymptotically normally distributed. The asymptotic variance-covariance matrix of the parameter estimates is consistently estimated as

\[
(4.17) \quad V(b) = - \left[ \frac{\partial^2 L(\hat{b})}{\partial b \partial b'} \right]^{-1}
\]

When a maximum has been obtained, i.e., the gradients \( \frac{\partial L(\hat{b})}{\partial b} = 0 \) and, the Hessian \( \frac{\partial^2 L(\hat{b})}{\partial b \partial b'} \) is negative definite, the estimated variance-covariance matrix (4.17) is positive definite.

We utilize the Fletcher Algorithm as incorporated in the UBC Computing Centre write-up UBC:NLMON [1975] to minimize the negative of log of the likelihood. The computation were carried out on an IBM 360, Model 168 computer at the UBC Computer Centre.
IV.2 The Data

The required data for this study are the prices and quantities of the four input services and the output quantities in U.S. Manufacturing 1947-71. In this study we use the annual KLEM data on the U.S. Manufacturing sector, 1947-71 utilized originally by E. R. Berndt and D. O. Wood [1975a]. The data on the price and quantity indexes of the four inputs have been presented in their table 1. The output data is from Jack Faucett Associates [1973]. A notable feature of this body of data is that while the quantities demanded of K and E grew at a faster rate than that of L, the prices $P_K$ and $P_E$ increased at a slower rate than $P_L$. Compared to 1947 the average productivity of L has increased considerably in 1971 while that of K has fallen slightly. The average productivity of E has approximately remained unchanged over 1947-71.

The uneven growth in input demand can be due to a variety of reasons the most important ones being the own price effect, the substitution effects, nonhomothetic returns to scale and nonneutral technical change. An important purpose of this study is to disentangle these different effects with a view to evaluating their importance.

Simultaneous estimation of bias in technical change and substitution elasticities in the absence of sufficient structure on the nature of technical change has been hindered by an impossibility succinctly summarized by P. A. Diamond and D. McFadden [1965]. However, we have imposed sufficient structure in specifying both technical change and returns to scale in the GBC formulation so that not only the elasticities of substitution and bias of technical change but also returns to scale can be simultaneously identified.
Previous studies in U.S. Manufacturing have demonstrated that technical change may be nonneutral. For example, P. David and T. Van de Klundert [1965] have found that technical progress in the private domestic sector of the U.S. Economy, 1900-1960 has been more labour augmenting than capital augmenting. R. Sato [1970] estimates that labour efficiency tends to rise faster than capital efficiency in the U.S. private nonfarm sector.

There is also evidence that the U.S. Manufacturing sector is characterized by considerable economies of scale. According to E. F. Denison [1974], economies of scale added 13.3 to 16.4 percent to the contribution of other determinants of output growth in the U.S. nonresidential business, 1947-69. In an earlier study he suggested that this contribution might be as high as 24 percent.

The results of this study confirm the findings of nonneutral technical change and nonconstant returns to scale in U.S. Manufacturing. These results are presented below in detail. We give particular attention to the following issues: (1) choice among alternative models of technology, (2) choice among alternative functional forms, (3) the bias of technical change and scale effects, (4) properties of the estimated cost functions, (5) elasticities of substitution and price elasticities, (6) measures of returns to scale and total factor productivity and (7) the response of returns to scale and factor productivity measures to changes in prices, scale and time.

IV.3 Choice Among Alternative Models of Technology

The richness of the GBC specification enables us to estimate a variety of models depending on the nature of returns to scale and technical change. We consider three states of technical change--no technical
change, neutral technical change and non-neutral technical change—and four types of returns to scale—constant returns to scale, homogeneous returns to scale, nonhomogeneous but homothetic returns to scale and nonhomothetic returns to scale. So, altogether twelve different models are possible. The model classification has been presented in table IV.1 below. CRTS-N etc. are the names given to the different models.

In figure 2 below we present the sample maximum of log of likelihood values for the twelve models. Our most general model is the NHT-NN specification in which we have twenty free parameters. The least general model is the CRTS-NO with eleven free parameters.

The chi square test statistics for the different scale and technological restrictions have been presented in table IV.2 The restriction of homotheticity is rejected irrespective of the form of technical change assumed. The 1% chi square critical value for 3 degrees of freedom is 11.3449 while the chi-square test statistics defined as $2(L(\text{unconstrained}) - L(\text{constrained}))$ are 52.1620, 36.1436 and 36.295 for the nonneutral, neutral and no technical change specifications respectively. However, if one erroneously imposes homotheticity then whatever the type of technical change the additional restriction of homogeneity does not result in any significant loss of fit when compared to the wrong alternative hypothesis of homotheticity at the 1% level. The test statistics are .0410, .0258 and .4180 respectively for the different types of technical change as compared to the 1% critical chi-square value of 6.6349. Indeed even the restriction of constant returns to scale doesn't result in any significant loss of fit under the rejected alternative hypothesis of homotheticity except in the case of no technical change. The 1% critical value of chi-square for 2 degrees of freedom is 9.21 as compared to the test
### Table IV.1
**MODEL CLASSIFICATION**

<table>
<thead>
<tr>
<th>Nature of Tech. Change</th>
<th>Type of returns to scale</th>
<th>No Technical Change</th>
<th>Neutral Technical Change</th>
<th>Non-neutral technical change</th>
</tr>
</thead>
<tbody>
<tr>
<td>CRTS-NO</td>
<td>CRTS-N</td>
<td>CRTS-NN</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HG-NO</td>
<td>HG-N</td>
<td>HG-NN</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HT-NO</td>
<td>HT-N</td>
<td>HT-NN</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NHT-NO</td>
<td>NHT-N</td>
<td>NHT-NN</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 2
MAXIMUM SAMPLE LOG OF LIKELIHOOD VALUES
(Number of restrictions in circles)
Table IV.2

$\chi^2$ TEST STATISTICS FOR ALTERNATIVE MODELS OF TECHNOLOGY

Tests of Returns to Scale

<table>
<thead>
<tr>
<th></th>
<th>Technical Change</th>
<th>No Technical Change</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nonneutral</td>
<td>Neutral</td>
</tr>
<tr>
<td>Homotheticity</td>
<td>52.1620*</td>
<td>36.1436*</td>
</tr>
<tr>
<td>Homogeneity given</td>
<td>.0410</td>
<td>.0258</td>
</tr>
<tr>
<td>homotheticity CRTS</td>
<td>2.4996</td>
<td>2.4188</td>
</tr>
<tr>
<td>CRTS given</td>
<td>2.4996</td>
<td>2.4188</td>
</tr>
<tr>
<td>homotheticity CRTS</td>
<td>2.4996</td>
<td>2.4188</td>
</tr>
</tbody>
</table>

Tests of Technical Change

<table>
<thead>
<tr>
<th></th>
<th>Nonhomothetic</th>
<th>Homothetic but Nonhomogeneous</th>
<th>Homogeneous</th>
<th>CRTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutrality</td>
<td>18.2304*</td>
<td>2.2120</td>
<td>2.1968</td>
<td>2.1312</td>
</tr>
<tr>
<td>Absence of technical</td>
<td>.0082</td>
<td>.1596</td>
<td>.5518</td>
<td>55.1856*</td>
</tr>
<tr>
<td>change given neutrality</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Indicates that the relevant null hypothesis is rejected at 1% level of significance.
statistics 2.4996, 2.4188 and 57.4448 for the nonneutral, neutral and no technical change representation respectively.

In the context of technical change neutrality is rejected only in the nonhomothetic specification. The test statistic is 18.2304 as against the 1% critical value 11.3449. On the other hand once neutrality is mistakenly imposed on the nonhomothetic model the extra restriction of absence of technical change fails to be rejected, the test statistic being 0.0082 as compared to the 1% critical chi-square value 6.6349. In the homothetic specifications (HT, HG and CRTS) where neutrality cannot be rejected the extra restriction of no technical change is rejected under the alternative hypothesis of neutrality only in the case of CRTS where the test statistic is 55.1856 while the 1% chi-square critical value is 6.6349.

Our basic results in the above are then that U.S. Manufacturing 1947-71 has been characterized by significant nonhomothetic returns to scale and nonneutral technical change (NHT-NN). However, once homotheticity and neutrality are wrongly imposed the simplest specifications that we cannot reject as compared to the wrong alternative hypothesis are the constant returns to scale with neutral technical change (CRTS-N) and the homogeneous returns to scale with no technical change (HG-NO) representations. In the CRTS-N model all the growth in productivity due to factors other than input deepening is attributed to technical progress while in the HG-NO model it is attributed to scale economies. It is not possible to compare these models themselves by classical methods since one is not nested in the other. Each is of course nested in the homogeneous with neutral technical change (HG-N) model. However, compared to the latter both of them result in almost equally insignificant loss of fit, the chi-square test statistics with one degree of freedom are 2.3930 and
for the CRTS-N and HG-NO models respectively. The HG-N model, therefore, fails to resolve the indecisiveness as to whether scale economies or technological change accounts for the growth in productivity in U.S. Manufacturing 1947-71 that is not due to increase in input quantities alone. This problem is, however, irrelevant in our context because the HG-N model is itself decisively rejected as compared to the NHT-NN representation. The alternative sources of productivity growth should more properly be evaluated in the NHT-NN framework. However, for purposes of comparison we report below results based on the NHT-NN, HG-N, HG-NO and CRTS-N specifications.

IV.4 Choice Among Alternative Functional Forms

The sample maximum of log of likelihood values corresponding to different given values of λ for the selected models have been presented in table IV.3. At the top of the table we also present the maximum value of log of likelihood when λ is freely estimated with the asymptotic t-ratio in parenthesis. The results embodied in table IV.3 have been summarized in table IV.4 where we present the χ² test statistics for H₀: λ = i for i = -2, -1, 0, 1, 2 versus H₁: λ ≠ i in all the selected models.

In our accepted model (NHT-NN) the Generalized Leontief (λ = 1) is the only functional form that cannot be rejected at 1% level of significance (critical chi-square value is 6.6349). In the other models (HG-N, CRTS-N and HG-NO) the Generalized Leontief (λ = 1), Translog (λ = 0) and λ = -1 functional forms cannot be rejected at 1% level while at 5% level (critical chi-square value is 3.841) only the translog and λ = -1 functional forms cannot be rejected.
Table IV.3
SAMPLE MAXIMUM OF LOG OF LIKELIHOOD VALUES
FOR SELECTED MODELS

<table>
<thead>
<tr>
<th>$\lambda$ free</th>
<th>NHT-NN</th>
<th>HG-N</th>
<th>CRTS-N</th>
<th>HG-NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\lambda}$</td>
<td>.79</td>
<td>-.93</td>
<td>-.89</td>
<td>-.92</td>
</tr>
<tr>
<td>$t$</td>
<td>(5.181)</td>
<td>(-2.319)</td>
<td>(-2.014)</td>
<td>(-2.341)</td>
</tr>
<tr>
<td>$\ln L$</td>
<td>446.4521</td>
<td>419.2522</td>
<td>418.0557</td>
<td>418.9763</td>
</tr>
<tr>
<td>$\lambda$ fixed</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>438.3638</td>
<td>416.0754</td>
<td>412.3850</td>
<td>415.4885</td>
</tr>
<tr>
<td>1</td>
<td>445.6995</td>
<td>416.2612</td>
<td>414.8760</td>
<td>416.2512</td>
</tr>
<tr>
<td>0</td>
<td>439.8987</td>
<td>417.8804</td>
<td>416.8538</td>
<td>417.6677</td>
</tr>
<tr>
<td>-1</td>
<td>436.1582</td>
<td>419.2380</td>
<td>418.0237</td>
<td>418.9543</td>
</tr>
<tr>
<td>-2</td>
<td>430.0864</td>
<td>413.1089</td>
<td>412.5105</td>
<td>411.8989</td>
</tr>
</tbody>
</table>
Table IV.4

$\chi^2$ TEST STATISTICS FOR ALTERNATIVE FUNCTIONAL FORMS IN THE SELECTED MODELS

<table>
<thead>
<tr>
<th>$\lambda$ fixed at</th>
<th>NHT-NN</th>
<th>HG-N</th>
<th>CRTS-N</th>
<th>HG-NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16.1766</td>
<td>6.3536*</td>
<td>11.3414</td>
<td>6.9756</td>
</tr>
<tr>
<td>1</td>
<td>1.5052*</td>
<td>5.9820*</td>
<td>6.3594*</td>
<td>5.4502*</td>
</tr>
<tr>
<td>0</td>
<td>13.1068</td>
<td>2.7436*</td>
<td>2.4038*</td>
<td>2.6172*</td>
</tr>
<tr>
<td>-1</td>
<td>20.5878</td>
<td>.0284*</td>
<td>.0640*</td>
<td>.0440*</td>
</tr>
<tr>
<td>-2</td>
<td>32.7314</td>
<td>12.2866</td>
<td>11.0904</td>
<td>14.1548</td>
</tr>
</tbody>
</table>

*Denotes that the relevant null hypothesis cannot be rejected at 1% level of significance.
From the above results we conclude that the NHT-NN specification of U.S. Manufacturing 1947-71 can be represented by a GBC functional form with $\lambda = 0.79$. It can also be represented by the Generalized Leontief functional form without any significant loss of fit at 1% level. However, once the rejected hypotheses of homotheticity and neutrality are imposed the alternatives to the GBC functional form that do not result in any significant loss of fit at the 1% level of significance are $\lambda = -1$, translog and Generalized Leontief functional forms in the HG-N, CRTS-N and HG-NO representation. Indeed in these models the $\lambda = -1$ functional form virtually does not result in any loss of fit as compared to the GBC functional form, the chi-square test statistics being 0.0284, 0.0640 and 0.0440.

IV.5 The Nature of Scale Economies and Technological Change

The maximum likelihood estimates of the parameters along with asymptotic t-ratios have been presented in Table IV.5 below. The NHT-NN-R and NHT-NN-R (GL) are two variants of the NHT-NN model to be explained below.

Our results in the NHT-NN representation indicate that significant nonhomothetic scale economies were present in U.S. Manufacturing over this period. The nonhomothetic coefficients $\phi_K$, $\phi_L$ and $\phi_E$ are significantly negative indicating substantial scale economies in these inputs as output is increased. In other words, using the relation,

$$ (2.33) \quad \frac{\partial \theta_i}{\partial \ln Y} = \phi_i $$
### Table IV.5
MAXIMUM LIKELIHOOD PARAMETER ESTIMATES IN SELECTED GBC MODELS
(asymptotic t-ratios in parentheses)

<table>
<thead>
<tr>
<th></th>
<th>NHT-NN</th>
<th>NHT-NN-R</th>
<th>NHT-NN-R(GL)</th>
<th>HG-N</th>
<th>HG-NO</th>
<th>CRTS-N</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{KK}$</td>
<td>-0.059</td>
<td>-0.032</td>
<td>-0.080</td>
<td>0.027</td>
<td>0.020</td>
<td>0.049</td>
</tr>
<tr>
<td></td>
<td>(-0.883)</td>
<td>(-1.117)</td>
<td>(-7.231)</td>
<td>(2.253)</td>
<td>(3.598)</td>
<td>(6.081)</td>
</tr>
<tr>
<td>$\gamma_{KL}$</td>
<td>0.431</td>
<td>0.236</td>
<td>0.316</td>
<td>-0.009</td>
<td>-0.007</td>
<td>-0.016</td>
</tr>
<tr>
<td></td>
<td>(1.404)</td>
<td>(3.306)</td>
<td>(4.493)</td>
<td>(-2.000)</td>
<td>(-3.775)</td>
<td>(-1.776)</td>
</tr>
<tr>
<td>$\gamma_{KE}$</td>
<td>0.128</td>
<td>0.068</td>
<td>0.100</td>
<td>-0.006</td>
<td>-0.005</td>
<td>-0.010</td>
</tr>
<tr>
<td></td>
<td>(1.342)</td>
<td>(2.878)</td>
<td>(5.677)</td>
<td>(-2.160)</td>
<td>(-2.316)</td>
<td>(-3.419)</td>
</tr>
<tr>
<td>$\gamma_{KM}$</td>
<td>-0.141</td>
<td>-0.081</td>
<td>-0.054</td>
<td>-0.027</td>
<td>-0.020</td>
<td>-0.051</td>
</tr>
<tr>
<td></td>
<td>(-1.142)</td>
<td>(-1.658)</td>
<td>(-0.828)</td>
<td>(-2.134)</td>
<td>(-3.883)</td>
<td>(-3.463)</td>
</tr>
<tr>
<td>$\gamma_{LL}$</td>
<td>0.527</td>
<td>0.293</td>
<td>0.300</td>
<td>0.095</td>
<td>0.071</td>
<td>0.168</td>
</tr>
<tr>
<td></td>
<td>(1.454)</td>
<td>(3.300)</td>
<td>(2.835)</td>
<td>(2.551)</td>
<td>(9.504)</td>
<td>(2.990)</td>
</tr>
<tr>
<td>$\gamma_{LE}$</td>
<td>0.307</td>
<td>0.176</td>
<td>0.221</td>
<td>-0.011</td>
<td>-0.008</td>
<td>-0.019</td>
</tr>
<tr>
<td></td>
<td>(1.460)</td>
<td>(3.692)</td>
<td>(5.059)</td>
<td>(-2.333)</td>
<td>(-6.264)</td>
<td>(-2.169)</td>
</tr>
<tr>
<td>$\gamma_{LM}$</td>
<td>-0.635</td>
<td>-0.349</td>
<td>-0.323</td>
<td>-1.144</td>
<td>-0.108</td>
<td>-2.252</td>
</tr>
<tr>
<td></td>
<td>(-1.461)</td>
<td>(-3.363)</td>
<td>(-2.708)</td>
<td>(-2.491)</td>
<td>(-14.801)</td>
<td>(-2.469)</td>
</tr>
<tr>
<td>$\gamma_{EE}$</td>
<td>-0.111</td>
<td>-0.062</td>
<td>-0.095</td>
<td>0.021</td>
<td>0.015</td>
<td>0.040</td>
</tr>
<tr>
<td></td>
<td>(-1.302)</td>
<td>(-2.556)</td>
<td>(-5.587)</td>
<td>(2.147)</td>
<td>(3.950)</td>
<td>(3.602)</td>
</tr>
<tr>
<td>$\gamma_{EM}$</td>
<td>-0.069</td>
<td>-0.042</td>
<td>-0.019</td>
<td>-0.017</td>
<td>-0.012</td>
<td>-0.033</td>
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<tr>
<td></td>
<td>(-0.975)</td>
<td>(-1.259)</td>
<td>(-4.49)</td>
<td>(-1.720)</td>
<td>(-2.741)</td>
<td>(-2.206)</td>
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<td>$\gamma_{MM}$</td>
<td>0.916</td>
<td>0.502</td>
<td>0.504</td>
<td>0.011</td>
<td>0.007</td>
<td>0.028</td>
</tr>
<tr>
<td></td>
<td>(1.487)</td>
<td>(4.020)</td>
<td>(3.596)</td>
<td>(4.04)</td>
<td>(3.49)</td>
<td>(6.30)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.604</td>
<td>0.332</td>
<td>0.832</td>
<td>0.876</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.236)</td>
<td>(78.632)</td>
<td>(79.327)</td>
<td>(11.053)</td>
<td>(102.325)</td>
<td></td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.001</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.002</td>
<td>0.0</td>
<td>-0.007</td>
</tr>
<tr>
<td></td>
<td>(0.832)</td>
<td>(-0.742)</td>
<td>(16.739)</td>
<td>(16.739)</td>
<td>(16.739)</td>
<td>(16.739)</td>
</tr>
<tr>
<td>$\phi_{K}$</td>
<td>-0.042</td>
<td>-0.040</td>
<td>-0.038</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(-7.098)</td>
<td>(-7.067)</td>
<td>(-7.514)</td>
<td>(0.0)</td>
<td>(0.0)</td>
<td>(0.0)</td>
</tr>
<tr>
<td>$\phi_{L}$</td>
<td>-0.043</td>
<td>-0.046</td>
<td>-0.040</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(-2.614)</td>
<td>(-2.895)</td>
<td>(-2.591)</td>
<td>(0.0)</td>
<td>(0.0)</td>
<td>(0.0)</td>
</tr>
<tr>
<td>$\phi_{E}$</td>
<td>-0.029</td>
<td>-0.029</td>
<td>-0.027</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(-10.333)</td>
<td>(-10.489)</td>
<td>(-10.542)</td>
<td>(0.0)</td>
<td>(0.0)</td>
<td>(0.0)</td>
</tr>
<tr>
<td>$\phi_{M}$</td>
<td>0.113</td>
<td>0.115</td>
<td>0.105</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(5.756)</td>
<td>(5.938)</td>
<td>(5.936)</td>
<td>(0.0)</td>
<td>(0.0)</td>
<td>(0.0)</td>
</tr>
<tr>
<td>$\tau_{K}$</td>
<td>0.0013</td>
<td>0.0012</td>
<td>0.0012</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(4.164)</td>
<td>(4.088)</td>
<td>(4.187)</td>
<td>(0.0)</td>
<td>(0.0)</td>
<td>(0.0)</td>
</tr>
<tr>
<td>$\tau_{L}$</td>
<td>-0.0004</td>
<td>-0.0001</td>
<td>-0.0002</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(-0.333)</td>
<td>(-0.077)</td>
<td>(-0.228)</td>
<td>(0.0)</td>
<td>(0.0)</td>
<td>(0.0)</td>
</tr>
<tr>
<td></td>
<td>NHT-NN</td>
<td>NHT-NN-R</td>
<td>NHT-NN-R(GL)</td>
<td>HG-N</td>
<td>HG-NO</td>
<td>CRTS-N</td>
</tr>
<tr>
<td>-------</td>
<td>--------</td>
<td>----------</td>
<td>--------------</td>
<td>------</td>
<td>-------</td>
<td>--------</td>
</tr>
<tr>
<td>(\tau_E)</td>
<td>.0007</td>
<td>.0007</td>
<td>.0008</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(3.004)</td>
<td>(3.068)</td>
<td>(3.047)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\tau_M)</td>
<td>-.0016</td>
<td>-.0019</td>
<td>-.0017</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>(-1.271)</td>
<td>(-1.490)</td>
<td>(-1.456)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\lambda)</td>
<td>.788</td>
<td>.775</td>
<td>1.0</td>
<td>-.932</td>
<td>-.918</td>
<td>-.890</td>
</tr>
<tr>
<td></td>
<td>(5.181)</td>
<td>(5.117)</td>
<td></td>
<td>(-2.319)</td>
<td>(-2.341)</td>
<td>(-2.014)</td>
</tr>
<tr>
<td>(\ln L)</td>
<td>446.4521</td>
<td>445.9758</td>
<td>445.0581</td>
<td>419.2522</td>
<td>418.9763</td>
<td>418.0557</td>
</tr>
<tr>
<td>(\hat{R}^2)</td>
<td>.9857</td>
<td>.9855</td>
<td>.9853</td>
<td>.9753</td>
<td>.9752</td>
<td>.9747</td>
</tr>
</tbody>
</table>
the scale effect in U.S. Manufacturing has been share saving in case of capital, labour and energy. This is, however, not true of intermediate materials. The presence of scale economies is highly significant in the case of capital and energy.

Our results in the NHT-NN specification also indicate that the effects of technological change in U.S. Manufacturing, 1947-71 have been significantly capital and energy using and approximately labour neutral. This can be seen from the $\tau_K$, $\tau_L$ and $\tau_E$ parameters which have the interpretation.

\[
(2.32) \quad \frac{\partial s_i}{\partial t} = \tau_i
\]

The sign of the labour technological change coefficient ($\tau_L$) is negative indicating labour saving bias. However, this estimate does not differ significantly from zero as reflected by the small t ratio. This result is not inconsistent with the finding of labour saving bias in U.S. Manufacturing by others (e.g. David and Klundert [1965] and R. Sato [1970]) because the labour variable in the Berndt-Wood data has already been adjusted for embodied quality change using an educational attainment index.

A simpler NHT-NN representation is suggested by the following results. The estimate of $\theta$ ($0.034$) is positive indicating that the minimum point of the average cost curve is reached as output is increased, i.e. the unit cost curve has the traditional U shape. This can be seen from the relation,

\[
(2.56) \quad \frac{\partial^2 \ln C}{\partial \ln Y^2} = \theta
\]

However, the above estimate of $\theta$ is virtually zero, its t ratio being ($0.724$). The point estimate of $\tau$ ($0.001$) is also not significantly different from zero. Its t ratio is $0.832$. In the absence of bias in technological
change \( \tau \) represents the neutral rate of total cost diminution. We can, therefore, let \( \tau = 0 \) without any loss of generality with respect to biased technological change. We have estimated a simpler NHT-NN model (denoted NHT-NN-R) with \( \theta = 0 = \tau \). These restrictions do not result in any significant loss of fit. The chi-square test statistic is .9526 as compared to the 1% critical value (9.21) with two degrees of freedom. We have also estimated another simple NHT-NN model (called NHT-NN-R(GL)) with \( \theta = 0 = \tau, \lambda = 1 \). This representation also doesn't result in any significant loss of fit. When compared to the NHT-NN-R, the chi-square test statistic is only 1.8354. The results in the two simpler NHT-NN models show that our conclusions concerning scale effects and technological change remain unchanged. It is, however, noteworthy that other parameter estimates in the NHT-NN-R and NHT-NN-R(GL) representations are much more precise when compared to the NHT-NN model; this is indicated by the generally smaller standard errors of parameter estimates in the simpler models.

We now turn, for the sake of comparison, to a discussion of the still simpler (but rejected) representations of U.S. Manufacturing 1947-71. In the CRTS-N specification there can be no scale economies. All growth in productivity not explained by increase in quantities of inputs is attributed to neutral technological progress. The estimated rate of total cost diminution (.7%) is significantly different from zero. It has a t ratio of (16.739). At the other extreme is the HG-NO representation where all increase in productivity not due to input deepening is imputed to scale economies. The scale parameter (\( \beta \)) is significantly different from 1. The t ratio for \( H_0 : \beta = 1 \) is -22.90. The estimate .817 of \( \beta \) implies that the estimated degree of homogeneity of the underlying
production function \((\frac{1}{\beta})\) in input quantities is 1.224, i.e., scale economies contribute an extra 22% to output growth. In the HG-N model it is notable that the rate of cost diminution (\(\tau\)) is not significantly different from zero (t ratio is -.748) nor is the scale parameter (\(\beta\)) significantly different from 1 (t ratio is -1.565) although taken together they explain productivity at least as good as the CRTS-N and HG-NO representations. This may be due to high correlation (.979) between \(\ln Y\) and \(t^5\).

Finally, we note that our models explain the data quite well. The system measures of \(R^2\) obtained by using (3.20) are all above .97.

IV.6 Properties of the Estimated Cost Functions

We imposed linear homogeneity in prices and symmetry of substitution effects as our maintained hypotheses. The estimated cost functions also satisfy the properties of positivity and monotonicity in prices. The cost and share estimates are positive at every observation. The cost functions are all increasing in output. This is indicated by estimated output elasticities of cost that are positive at every point of observation. To check the concavity of the cost functions in prices we use the matrix of the Allen partial elasticities of substitution. For concavity of the cost function we require this matrix to be negative semi-definite. In our four input model this condition would be satisfied if the own elasticities of substitution \((\sigma_{ii})\) are nonpositive, all the minors of order two are nonnegative and all the minors of order three are nonpositive. The determinant of the full matrix would be zero because of linear homogeneity.

The above requirements are all satisfied at every observation except 1949 and 1971 for the three restrictive models CRTS-N, HG-NO and HG-N.
In the NHT-NN-R representation the estimated own elasticities, $\sigma_{ii}$ are negative at every observation for $i = K, L, E$ but $\sigma_{MM}$ is negative only over the latter half of the sample. The requirements on minors of order two and three are satisfied for the period 1962-1971. Similar results have been obtained in the NHT-NN-R(GL) representation.

The concavity violation in the general representations (NHT-NN-R and NHT-NN-R(GL)) in the first half of the sample is, however, not significant. The estimates of $\gamma_{MM}$ (.01 to .03) in the restrictive models (CRTS-N, HG-NO, HG-N), in which concavity is in general satisfied, are much smaller than those (.5) in the NHT-NN-R and NHT-NN-R(GL) specifications. To discover the sensitivity of concavity to the estimates of $\gamma_{MM}$ we have estimated the NHT-NN-R model with different $\gamma_{MM}$ values. For example, $\gamma_{MM} = .4$ does not result in any significant loss of fit at 1% level (the chi-square test statistic is only .7) but all the concavity requirements are satisfied in the last 15 years.

IV.7 Substitution and Price Elasticities

We now turn to a discussion of the Allen partial elasticities of substitution based on the maximum likelihood parameter estimates in table IV.5 and the relations (2.56), (2.59) and (2.61). In table IV.6 we report the elasticities of substitution estimates for the year 1959, the midpoint of our sample. The elasticity estimates for this year are typical among the estimates for all years.

The most important results in table IV.6 are the following. Irrespective of the model specification capital and energy display substantial complementarity. The estimated $\sigma_{KE}$ range from -1.96 to -2.69 in the different models. On the other hand there is considerable
Table IV.6
ALLEN PARTIAL ELASTICITIES OF SUBSTITUTION IN
U.S. MANUFACTURING, 1959 IN SELECTED GBC MODELS

<table>
<thead>
<tr>
<th></th>
<th>NHT-NN-R</th>
<th>NHT-NN-R (GL)</th>
<th>HG-N</th>
<th>HG-NO</th>
<th>CRTS-N</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_{KK} )</td>
<td>-6.453</td>
<td>-6.944</td>
<td>-8.773</td>
<td>-9.033</td>
<td>-8.316</td>
</tr>
<tr>
<td>( \sigma_{KL} )</td>
<td>1.738</td>
<td>1.667</td>
<td>1.011</td>
<td>1.012</td>
<td>.987</td>
</tr>
<tr>
<td>( \sigma_{KE} )</td>
<td>-2.369</td>
<td>-2.418</td>
<td>-2.359</td>
<td>-2.688</td>
<td>-1.962</td>
</tr>
<tr>
<td>( \sigma_{KM} )</td>
<td>.007</td>
<td>.088</td>
<td>.514</td>
<td>.558</td>
<td>.458</td>
</tr>
<tr>
<td>( \sigma_{LL} )</td>
<td>-.559</td>
<td>-.631</td>
<td>-1.635</td>
<td>-1.638</td>
<td>-1.636</td>
</tr>
<tr>
<td>( \sigma_{LE} )</td>
<td>1.880</td>
<td>1.413</td>
<td>.545</td>
<td>.534</td>
<td>.551</td>
</tr>
<tr>
<td>( \sigma_{LM} )</td>
<td>-.052</td>
<td>.021</td>
<td>.591</td>
<td>.593</td>
<td>.593</td>
</tr>
<tr>
<td>( \sigma_{EE} )</td>
<td>-13.160</td>
<td>-11.250</td>
<td>-12.227</td>
<td>-12.816</td>
<td>-11.223</td>
</tr>
<tr>
<td>( \sigma_{EM} )</td>
<td>.351</td>
<td>.425</td>
<td>.836</td>
<td>.910</td>
<td>.730</td>
</tr>
<tr>
<td>( \sigma_{MM} )</td>
<td>-.003</td>
<td>-.049</td>
<td>-.366</td>
<td>-.376</td>
<td>-.355</td>
</tr>
</tbody>
</table>
evidence of capital-labour, labour-energy and energy-material substitutability which persists even in different model specifications. The $\sigma_{KL}$ estimates range from .99 to 1.74 and $\sigma_{LE}$ from .53 to 1.88. The estimates of $\sigma_{EM}$ range from .35 to .91. The capital-material and labour-material relations are, however, not robust to alternative representations. In general, in our accepted models (NHT-NN-R and NHT-NN-R (GL)) capital-material and labour-material complementarity holds in the first half of the sample while in the second half the relation is one of substitutability. It thus appears that in the more recent years only capital and energy are the complementary inputs.

The price elasticity estimates for 1959 have been presented in table IV.7. Energy demand is quite sensitive to its own price; the estimated own price elasticity is -.6 in the NHT-NN-R specification. The own price elasticities of K, L and M are respectively -.38, -.16 and -.002.

The cross price elasticity estimates indicate some interesting results. An increase in the price of energy or the price of capital would increase the demand for labour; the estimated price elasticities $\varepsilon_{LE}$ and $\varepsilon_{LK}$ are .09 and .10 respectively. However, an increase in the price of labour would substantially raise the demand for energy and capital. The $\varepsilon_{EL}$ and $\varepsilon_{KL}$ estimates are respectively .52 and .48. Capital-energy complementarity is reflected in the cross price elasticity estimates $-0.11$ ($\varepsilon_{KE}$) and $-0.14$ ($\varepsilon_{EK}$). A 1% fall in the price of either of the two inputs tends to increase the quantity demanded of the other by about .1%. Similar results, at least qualitatively, are obtained in all the different specifications.
Table IV.7

PRICE ELASTICITIES OF INPUT DEMAND IN U.S. MANUFACTURING,
1959 IN SELECTED GBC MODELS

<table>
<thead>
<tr>
<th></th>
<th>NHT-NN-R</th>
<th>NHT-NN-R (GL)</th>
<th>HG-N</th>
<th>HG-NO</th>
<th>CRTS-N</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_{KK}$</td>
<td>-0.379</td>
<td>-0.406</td>
<td>-0.495</td>
<td>-0.509</td>
<td>-0.471</td>
</tr>
<tr>
<td>$\epsilon_{KL}$</td>
<td>0.483</td>
<td>0.462</td>
<td>0.279</td>
<td>0.279</td>
<td>0.272</td>
</tr>
<tr>
<td>$\epsilon_{KE}$</td>
<td>-0.108</td>
<td>-0.110</td>
<td>-0.104</td>
<td>-0.118</td>
<td>-0.087</td>
</tr>
<tr>
<td>$\epsilon_{KM}$</td>
<td>0.004</td>
<td>0.055</td>
<td>0.321</td>
<td>0.349</td>
<td>0.286</td>
</tr>
<tr>
<td>$\epsilon_{LK}$</td>
<td>0.102</td>
<td>0.098</td>
<td>0.057</td>
<td>0.057</td>
<td>0.056</td>
</tr>
<tr>
<td>$\epsilon_{LL}$</td>
<td>-0.155</td>
<td>-0.175</td>
<td>-0.450</td>
<td>-0.451</td>
<td>-0.450</td>
</tr>
<tr>
<td>$\epsilon_{LE}$</td>
<td>0.086</td>
<td>0.064</td>
<td>0.024</td>
<td>0.023</td>
<td>0.024</td>
</tr>
<tr>
<td>$\epsilon_{LM}$</td>
<td>-0.032</td>
<td>0.013</td>
<td>0.369</td>
<td>0.370</td>
<td>0.370</td>
</tr>
<tr>
<td>$\epsilon_{EK}$</td>
<td>-0.139</td>
<td>-0.141</td>
<td>-0.133</td>
<td>-0.151</td>
<td>-0.111</td>
</tr>
<tr>
<td>$\epsilon_{EL}$</td>
<td>0.522</td>
<td>0.391</td>
<td>0.150</td>
<td>0.147</td>
<td>0.152</td>
</tr>
<tr>
<td>$\epsilon_{EE}$</td>
<td>-0.600</td>
<td>-0.513</td>
<td>-0.539</td>
<td>-0.564</td>
<td>-0.496</td>
</tr>
<tr>
<td>$\epsilon_{EM}$</td>
<td>0.217</td>
<td>0.263</td>
<td>0.522</td>
<td>0.568</td>
<td>0.456</td>
</tr>
<tr>
<td>$\epsilon_{MK}$</td>
<td>0.0004</td>
<td>0.005</td>
<td>0.029</td>
<td>0.031</td>
<td>0.026</td>
</tr>
<tr>
<td>$\epsilon_{ML}$</td>
<td>-0.015</td>
<td>0.006</td>
<td>0.163</td>
<td>0.163</td>
<td>0.163</td>
</tr>
<tr>
<td>$\epsilon_{ME}$</td>
<td>0.016</td>
<td>0.019</td>
<td>0.037</td>
<td>0.040</td>
<td>0.032</td>
</tr>
<tr>
<td>$\epsilon_{MM}$</td>
<td>-0.002</td>
<td>-0.030</td>
<td>-0.229</td>
<td>-0.235</td>
<td>-0.222</td>
</tr>
</tbody>
</table>
To give an idea of the various elasticity estimates in the recent years we present in Table IV.8 the elasticities in 1971 based on the NHT-NN-R parameter estimates. Capital energy complementarity again comes out quite strongly. The estimated $\sigma_{KE}$ is -5.40. The cross price elasticity estimates $\varepsilon_{KE}$ and $\varepsilon_{EK}$ indicate that if the price of capital or energy rises by 1% the quantity demanded of the other input would fall by about .25%. The other elasticity results are also similar to those obtained before.

IV.8 Measures of Returns to Scale and Total Factor Productivity

Estimates of scale effects and total factor productivity in 1971 in some selected models have been presented in table IV.9. These results are based on (2.73) - (2.76). There is evidence of significant increasing returns to scale in our preferred NHT-NN-R specification. Estimated growth of output due to scale economies is about 22% in the NHT-NN-R and HG-N representations and 14% in the HG-N representation. The null hypothesis of constant returns to scale is decisively rejected in the NHT-NN-R and HG-NO models; the respective asymptotic t ratios are 11.25 and 18.71. We will come back to a discussion of this result. It is, however, worth noting at this point that our estimates of scale economies are in the neighbourhood of E. F. Denison's estimate of 15% in Denison [1974] and 24% as implied in Denison [1967].

In contrast to the scale economies, growth in total factor productivity (i.e., technological progress) seems to have contributed very little to growth in output. The estimated rate of growth of total factor productivity is only .04% in 1971 in the NHT-NN-R model. Furthermore, this estimate is not significantly different from zero. The relatively
Table IV.8
SUBSTITUTION AND PRICE ELASTICITIES IN U.S. MANUFACTURING,
1971 BASED ON NHT-NN-R REPRESENTATION

<table>
<thead>
<tr>
<th>Allen Partial Elasticities of Substitution</th>
<th>Price Elasticities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{KK}$</td>
<td>$\epsilon_{KK}$</td>
</tr>
<tr>
<td>-3.763</td>
<td>-0.179</td>
</tr>
<tr>
<td>$\sigma_{KL}$</td>
<td>$\epsilon_{KL}$</td>
</tr>
<tr>
<td>0.870</td>
<td>0.263</td>
</tr>
<tr>
<td>$\sigma_{KE}$</td>
<td>$\epsilon_{KE}$</td>
</tr>
<tr>
<td>-5.401</td>
<td>-0.252</td>
</tr>
<tr>
<td>$\sigma_{KM}$</td>
<td>$\epsilon_{KM}$</td>
</tr>
<tr>
<td>0.279</td>
<td>0.168</td>
</tr>
<tr>
<td>$\sigma_{LL}$</td>
<td>$\epsilon_{LK}$</td>
</tr>
<tr>
<td>-0.522</td>
<td>0.042</td>
</tr>
<tr>
<td>$\sigma_{LE}$</td>
<td>$\epsilon_{LL}$</td>
</tr>
<tr>
<td>1.385</td>
<td>-0.158</td>
</tr>
<tr>
<td>$\sigma_{LM}$</td>
<td>$\epsilon_{LE}$</td>
</tr>
<tr>
<td>0.086</td>
<td>0.065</td>
</tr>
<tr>
<td>$\sigma_{EE}$</td>
<td>$\epsilon_{LM}$</td>
</tr>
<tr>
<td>-11.934</td>
<td>0.052</td>
</tr>
<tr>
<td>$\sigma_{EM}$</td>
<td></td>
</tr>
<tr>
<td>0.655</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{MM}$</td>
<td></td>
</tr>
<tr>
<td>-0.115</td>
<td></td>
</tr>
</tbody>
</table>
Table IV.9
MEASURES OF SCALE, TOTAL COST DIMINUTION AND
TOTAL FACTOR PRODUCTIVITY IN SELECTED GBC MODELS
U.S. MANUFACTURING, 1971
(Asymptotic standard errors in parentheses*)

<table>
<thead>
<tr>
<th>Returns to scale (r)</th>
<th>HG-N</th>
<th>HG-NO</th>
<th>CRTS-N</th>
</tr>
</thead>
<tbody>
<tr>
<td>NHT-NN-R</td>
<td>1.22912</td>
<td>1.14138</td>
<td>1.22377</td>
</tr>
<tr>
<td></td>
<td>(.02036)</td>
<td>(.10327)</td>
<td>(.01196)</td>
</tr>
<tr>
<td>Rate of total cost</td>
<td>.00032</td>
<td>.00227</td>
<td>0.0</td>
</tr>
<tr>
<td>diminution (d)</td>
<td>(.00060)</td>
<td>(.00304)</td>
<td></td>
</tr>
<tr>
<td>Total factor</td>
<td>.00039</td>
<td>.00259</td>
<td>0.0</td>
</tr>
<tr>
<td>Productivity</td>
<td>(.00073)</td>
<td>(.00323)</td>
<td></td>
</tr>
<tr>
<td>(Π = r·d)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*The calculated asymptotic standard errors are conditional on the input prices and output quantity. The formula used is the first order approximation discussed in Klein [1953, pp. 258].
large asymptotic standard error indicates that a 95% confidence interval around the estimate would include zero and even some negative values. On the other hand, in the (rejected) CRTS-N representation the estimated rate of growth of total factor productivity is .7% per annum; this estimate is significantly different from zero as indicated by the relatively small standard error. Furthermore, this estimate is close to the estimates of .59% and .60% obtained by S. Star [1974] and E. R. Berndt and D. Wood [1975b] respectively in the context of a CRTS-N specification. In our (rejected) CRTS-NN representation the estimated rate of growth of total factor productivity is .74% in 1971. Our results, however, indicate that when the CRTS restriction is erroneously imposed in the presence of increasing returns to scale the estimate of total factor productivity has an upward bias. This is because even the contribution of scale economies is attributed to total factor productivity.

In our preferred NHT-NN-R representation where both returns to scale and total factor productivity are freely estimated, total factor productivity appears to have contributed very little to growth in productivity while the contribution of scale economies is very impressive. Our finding on total factor productivity is consistent with the D. Jorgenson and Z. Griliches [1967] conclusion that the contribution of total factor productivity to growth is small.

The above results, which are based on aggregate time series data should, however, be handled cautiously. As is well known, over time both output and input quantities grow so that in practice the contribution of scale economies and total factor productivity are mixed. Furthermore, if technological progress is mainly of 'learning by doing' type and if growth of output is a better proxy for 'doing' than time then the resulting estimate of scale economies in our kind of model would be
upward biased. As an indirect test of this possibility we have also considered the following specification of technological change in the NHT-NN-R model,

\[
\frac{\partial \ln C}{\partial t} = \tau + \sum_i \tau_i \ln p_i + \eta \ln Y
\]

where the rate of cost diminishing technical change also depends on output quantity. However, the estimate of \( \eta (0.000286) \) is not significantly different from zero; its asymptotic standard error is 0.000289. The chi-square test statistic for the restriction (\( \eta = 0 \)) is only 0.9644. The interaction between scale effects and technical change is, however, much more complicated than that in (4.18).

To abstract from technical change attempts have in the past been made to measure returns to scale from cross section data alone. However, as is well known, if the industry is competitive then in the long run equilibrium all the different firms tend to produce at the same unit cost. Consequently any cross section estimate of increasing returns to scale is expected to be low. The typical cross section estimate of scale economies is 7\%.\(^6\)

The cross section estimate fails to take account of external economies (resulting from, for example, improved market organization, transportation etc.) that are obtained as the entire industry grows. This is the familiar Marshallian case of a 'decreasing cost industry'. Although each firm can have the traditional U-shaped long run average cost curve the competitive industry as a whole would have a downward sloping long run supply curve. In this case the aggregate production function for the whole industry would show increasing returns to scale.\(^7\) This may explain our finding of considerable scale economies in U.S. Manufacturing
(1947-71) if the assumption of competition is maintained. If, however, the manufacturing sector is not competitive then the aggregate time series data can exhibit increasing returns to scale due to the presence of excess capacity in each firm.

Whatever the reasons behind it the finding of increasing returns to scale in the manufacturing industry seems to be quite robust. Using a Cobb-Douglas specification for U.S. Manufacturing 1900-1948, G. J. Stigler [1961] has estimated the scale economies to be 36%. He has reported a similar estimate (34%) by comparing U.S. and U.K. cross section data in 1947 and 1948. In commenting on these results, R. M. Solow [1961, pp. 67] admits "I am not trying to debunk increasing returns to scale. As an economic theorist I would gladly pay tithes to a Society for the Preservation of Constant Returns to Scale, and consider the money well spent. But analytical convenience aside, the way of the world is surely quite different."

IV.9 Comparison with Residual Measures of Total Factor Productivity

Consider the following linear homogeneous production function,

\[ Y = A(t)F(K, L, E, M) \]

In this framework, \( A(t) = Y/F \) is an index of pure technical change or total factor productivity if (a) the CRTS restriction is true, (b) the inputs have been properly measured and (c) \( F \) has been correctly specified. The rate of change of total factor productivity, \( \frac{\Delta A}{A} = \tau \) if \( A(t) = e^{\tau t} \). This rate also gives the rate of neutral technical progress. It can be measured either by estimating the production function or by computing the residual,
where \( X \) is an aggregate of the four inputs and the aggregation formula is assumed to be known. \( X \) is commonly obtained as a Divisia Index.\(^9\)

Total factor productivity is measured also from the point of view of the dual. We can utilize the unit cost function,

\[
C = a(t)f(P_K, P_L, P_E, P_M)
\]

where the negative of \( \frac{\dot{a}}{a} \) gives the rate of unit cost diminution due to neutral technical progress. This rate can again be estimated (as in our CRTS-N model) or computed as a residual,

\[
\frac{\dot{a}}{a} = \frac{\dot{P}}{P} - \frac{\dot{C}}{C}
\]

where \( P \) is an aggregate of the input prices. The use of a Törnqvist price index to obtain \( P \) is justifiable if \( f \) is a TLOG unit cost function.\(^10\)

If the assumption of competition is made then output price \( q \) equals unit cost. Consequently, a residual measure of total factor productivity can also be obtained as,

\[
\frac{\dot{a}}{a} = \frac{\dot{P}}{P} - \frac{\dot{q}}{q}
\]

where \( q \) is the output price index. This measure would, however, be different from that in (4.22) if \( q \) differs from unit cost nonsystematically.

The index numbers that are exact for mean of order \( \lambda \) aggregator functions are the mean of order \( \lambda \) indexes introduced by W. E. Diewert [1976]. Since the GBC unit cost function is a mean of order \( \lambda \) aggregator we compute the residual measures also by utilizing the mean of order \( \lambda \)
indexes. The price index is,

\[
P = \left\{ \sum_i s_{i,t-1} \left( \frac{P_{i,t}}{P_{i,t-1}} \right)^{\frac{\lambda}{2}} \right\} \frac{1}{\lambda} \sum_i s_{it} \left( \frac{P_{i,t-1}}{P_{it}} \right)^{\frac{\lambda}{2}}
\]

where \( s_i \)'s are the shares. The quantity index has the same form as (4.24) with the prices replaced by the corresponding quantities.

To compute indexes like (4.24) we need an estimate of \( \lambda \). In the total factor productivity measures below we use \( \lambda = -.89 \), the estimate obtained in our CRTS-N model. We present the residual measures of total factor productivity in Table IV.10 below. We also include our parametric measures of total factor productivity for purposes of comparison.

The residual measures by the two kinds of indexes are virtually the same. Among the residual measures, the primal \( \left( \frac{A}{A} = .63\% \right) \) and the unit cost dual \( \left( - \frac{a}{a} = .63\% \right) \) measures are the same while the dual measure using output price \( \left( - \frac{A}{A} = .65\% \right) \) is slightly different. This discrepancy is due to a mark up of output price over unit cost.

All the residual measures are, however, close to our parametric measure of total factor productivity in the CRTS models \( (.70\% \) and \( .71\% \)). This is not surprising because both kinds of measures are based on the CRTS assumption. Any difference is due to errors of cost minimizing behaviour which are present in the residual measures. To see this we can look at (4.24) where observed shares are used. In the context of our stochastic specification the observed shares have errors in them; in particular

\[
(4.25) \ s_{it} = E(s_{it}) + \varepsilon_{it}, \quad \Psi_{i,t}
\]
Table IV.10
MEAN AND STANDARD DEVIATION OF TOTAL FACTOR PRODUCTIVITY MEASURES,
U.S. MANUFACTURING, 1947-71

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. Residual Measures</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Törnqvist Divisia ($\lambda=0$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) Primal ($\dot{A}/A$)</td>
<td>.006309</td>
<td>.01284</td>
</tr>
<tr>
<td>(b) Unit cost dual ($-\dot{A}/a$)</td>
<td>.006310</td>
<td>.01284</td>
</tr>
<tr>
<td>(c) Competitive dual ($-\dot{a}/a$)</td>
<td>.006475</td>
<td>.01268</td>
</tr>
<tr>
<td>2. Mean of order $\lambda$ Divisia ($\lambda = -.89$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) Primal</td>
<td>.006310</td>
<td>.01284</td>
</tr>
<tr>
<td>(b) Unit cost dual</td>
<td>.006310</td>
<td>.01284</td>
</tr>
<tr>
<td>(c) Competitive dual</td>
<td>.006470</td>
<td>.01269</td>
</tr>
<tr>
<td><strong>B. Parametric Measures</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. CRTS-N</td>
<td>.00701</td>
<td>.00015</td>
</tr>
<tr>
<td>2. CRTS-NN</td>
<td>.00713</td>
<td>.00015</td>
</tr>
<tr>
<td>3. HG-N</td>
<td>.00259</td>
<td>.00015</td>
</tr>
<tr>
<td>4. NHT-NN-R</td>
<td>.00015</td>
<td>.00014</td>
</tr>
</tbody>
</table>
where the errors $\varepsilon_{it}$ result from imperfect cost minimizing behaviour. In our parametric estimation these errors have been taken into account. It is also notable that the residual measures have a relatively larger standard deviation than the parametric measures. We note further that the CRTS restriction has been decisively rejected in our model. In our accepted NHT-NN-R representation the estimate of average total factor productivity is rather small (.015%). So, in our context both the CRTS parametric estimates and the traditional residual measures attribute the contribution of scale economies to total factor productivity. Residual measures comparable to our nonconstant returns to scale parametric measures could be developed if somehow the contribution of scale economies could also be taken out of total output growth.

IV.10 Some Comparative Static Results

It is of interest to consider the effect of technological change, change in scale of production and in input prices on our key measures. The effects of such changes on the estimated shares in 1971 based on NHT-NN-R parameter estimates and the relations (2.77)-(2.80) have been presented in table IV.11.

As can be seen from the last row of Table IV.11 technological change tends to increase capital and energy cost shares and decrease the shares of materials and labour. This reflects the relatively capital and energy using and labour and materials saving bias of technological change. On the other hand due to the presence of significant scale economies, growth in output decreases the shares of capital, labour and energy. The economies are, however, not evenly distributed; in fact the materials share increases with a rise in output displaying scale diseconomies in
Table IV.11
EFFECTS OF CHANGE IN OUTPUT, TIME AND INPUT PRICES ON ESTIMATED SHARE IN NHT-NN-R MODEL OF U.S. MANUFACTURING, 1971

<table>
<thead>
<tr>
<th>$\frac{\partial s_K}{\partial Z}$</th>
<th>$\frac{\partial s_L}{\partial Z}$</th>
<th>$\frac{\partial s_E}{\partial Z}$</th>
<th>$\frac{\partial s_M}{\partial Z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z = $ln P_K$</td>
<td>.037</td>
<td>-.002</td>
<td>-.014</td>
</tr>
<tr>
<td>= $ln P_L$</td>
<td>-.002</td>
<td>.163</td>
<td>.005</td>
</tr>
<tr>
<td>- $ln P_E$</td>
<td>-.014</td>
<td>.005</td>
<td>.019</td>
</tr>
<tr>
<td>= $ln P_M$</td>
<td>-.021</td>
<td>-.167</td>
<td>-.010</td>
</tr>
<tr>
<td>= $ln Y$</td>
<td>-.040</td>
<td>-.046</td>
<td>-.029</td>
</tr>
<tr>
<td>= t</td>
<td>.0012</td>
<td>-.0001</td>
<td>.0007</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-.0018</td>
</tr>
</tbody>
</table>
this input. These scale effects are listed in the fifth row of table IV.11.

Since the inputs are all own price inelastic the effects of own price increases is to raise the shares of all inputs. These effects can be seen in the diagonal elements of the upper 4 x 4 matrix in table IV.11. The cross price effects on the shares are symmetric. An increase in the price of capital services lowers all the other shares. This reflects the fact that the cross elasticities of substitution between capital and all other inputs are less than unity. A rise in materials price has similar effects. However, a growth in price of labour services or of energy tends to raise each other's share while lowering shares of capital and materials. This is due to the greater than unity elasticity of substitution between labour and energy.

We will now turn to a discussion of the effects of input price changes on the measures of total factor productivity and returns to scale. These responses computed by using (2.87)-(2.89), have been presented in table IV.12. Because of the presence of significant non-homothetic scale economies in K, L and E growth in the prices of these inputs increases the returns to scale measure (r). A rise in the price of capital and energy would reduce the rate of total cost diminution (d) while an increase in labour and materials prices would raise the rate of total cost diminution. These effects are also reflected in the response of the total factor productivity measure (\( \Pi \)) to change in the input prices. Here again increases in \( P_K \) and \( P_E \) have dampening effects on total factor productivity.

The response of the average factor productivities to changes in the input prices, output quantity and time are reported in table IV.13. These measures are based on (2.90)-(2.92). An increase in the price of
Table IV.12
EFFECTS OF INPUT PRICE CHANGES ON RATE OF
TOTAL COST DIMINUTION, RETURNS TO SCALE AND
TOTAL FACTOR PRODUCTIVITY: U.S. MANUFACTURING
NHT-NN-R MODEL, 1971

<table>
<thead>
<tr>
<th></th>
<th>$\frac{\partial d}{\partial \ln P_i}$</th>
<th>$\frac{\partial r}{\partial \ln P_i}$</th>
<th>$\frac{\partial H}{\partial \ln P_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = K$</td>
<td>0.0012</td>
<td>0.0608</td>
<td>0.0015</td>
</tr>
<tr>
<td>$i = L$</td>
<td>0.0001</td>
<td>0.0697</td>
<td>0.0001</td>
</tr>
<tr>
<td>$i = E$</td>
<td>-0.0007</td>
<td>0.0438</td>
<td>-0.0008</td>
</tr>
<tr>
<td>$i = M$</td>
<td>0.0018</td>
<td>-0.1743</td>
<td>0.0022</td>
</tr>
</tbody>
</table>
Table IV.13
EFFECT OF INPUT PRICES, OUTPUT QUANTITY AND TIME ON AVERAGE FACTOR PRODUCTIVITY: U.S. MANUFACTURING, 1971
NHT-NN-R REPRESENTATION

\[
\begin{align*}
\frac{\partial \ln Y}{\partial Z} & \quad \frac{\partial \ln Y}{\partial Z} & \quad \frac{\partial \ln Y}{\partial Z} & \quad \frac{\partial \ln Y}{\partial Z} \\
Z = \ln P_K & \quad .179 & \quad -.042 & \quad .258 & \quad -.013 \\
= \ln P_L & \quad -.263 & \quad .158 & \quad -.418 & \quad -.026 \\
= \ln P_E & \quad .252 & \quad -.065 & \quad .556 & \quad -.031 \\
= \ln P_M & \quad -.168 & \quad -.052 & \quad -.395 & \quad .070 \\
= \ln Y & \quad 1.031 & \quad .339 & \quad .808 & \quad -.005 \\
= t & \quad -.0257 & \quad .0006 & \quad -.0147 & \quad .0034
\end{align*}
\]
capital and energy would have the effects of increasing the average productivities of K and E while dampening the average productivity of labour. An increase in the price of labour would have the opposite effects. These results reflect capital-labour and labour-energy substitutability and capital-energy complementarity. The presence of substantial scale economies is exhibited in the response of the average productivities to change in scale. With output growth the average productivities of capital, labour and energy inputs tend to increase. This response is considerable in case of K and E because of their larger scale economies and smaller shares. The average productivity of materials tends to decline due mainly to strong scale diseconomies in this input. On the other hand, because of the capital and energy using bias of technical change the average productivities of these inputs tend to decline with time while the average productivities of labour and materials have a tendency to rise due primarily to labour and energy saving bias of technical progress.

In conclusion we note that the faster growth over 1947-71 in the demand for capital and energy is mainly explainable by the following reasons. First, these two inputs have substantial own price effects and are also complementary with each other while being substitutable with labour. Consequently, a slower growth in the prices of these two inputs due to various tax concessions as compared to a faster growth in the price of labour have reinforced each other in causing a faster increase in the demand for capital and energy. Second, capital and energy are characterized by a using bias of technical change. Finally, scale economies are relatively more labour saving.
Footnotes to Chapter IV

1. The data on the input prices and quantities and the quantity of gross output have been reproduced in Appendix 3.

2. See E. F. Denison [1967].

3. The 1% critical t value for 80 degrees of freedom is 2.64. The 5% critical t is 1.99.


5. We use as time variable the integers from 1 to 25.

6. For example, this is the estimate reported by Z. Griliches and V. Ringstad [1971] for Norwegian Manufacturing. Similar estimates for U.S. Manufacturing have been reported by Hildebrand and Liu [1965].

7. For a more recent discussion of this argument see G. J. Stigler [1961] and R. M. Solow [1961].


9. The most important works in the residual approach are those of R. M. Solow [1957], E. F. Denison [1974] and Jorgenson and Griliches [1967]. The latter writers have utilized the 'Törnqvist Divisia quantity index'. A theoretical justification for this procedure has been provided by W. E. Diewert [1977].

10. See W. E. Diewert [1977].

11. Diewert [1977] has shown that the Törnqvist indexes are limiting cases of the corresponding mean of order \( \lambda \) indexes as \( \lambda \to 0 \). A quick proof is given below. We can write (4.24) as

\[
    p_\lambda = \frac{\sum_i s_{i,t-1} \left\{ \frac{p_{it}}{p_{i,t-1}} \right\}^{\frac{\lambda}{2}}}{\sum_i s_{it} \left\{ \frac{p_{i,t-1}}{p_{it}} \right\}^{\frac{\lambda}{2}}}
\]
By subtracting 1 from both sides and dividing by \( \lambda \), we obtain

\[
\frac{p_{t+1} - p_t}{\lambda} = \left[ \frac{\sum_i s_{i,t} (\frac{p_{i,t}}{p_{i,t-1}})^{\frac{\lambda}{2}} - \sum_i s_{i,t} (\frac{p_{i,t-1}}{p_{i,t}})^{\frac{\lambda}{2}}}{\lambda} \right]
\]

Taking the limit of the above as \( \lambda \to 0 \) and utilizing L'Hospital's rule, we obtain,

\[
\ln p = \frac{1}{2} \left[ \sum_i s_{i,t} (\ln p_{i,t} - \ln p_{i,t-1}) - \sum_i s_{i,t} (\ln \frac{p_{i,t}}{p_{i,t-1}}) \right]
\]

which gives the Törnqvist Divisia index.

12. The estimate of \( \lambda \) in the CRTS-NN model is -.81. So use of this estimate of \( \lambda \) would not really make any difference to the resulting index numbers.

13. We also computed the measure \( \frac{A}{\lambda} \) using the implicit quantity indexes corresponding to the price indexes. The resulting measures are the same as those obtained from the direct quantity indexes (\( \frac{A}{\lambda} = .00631 \) by the mean of order \( \lambda \) implicit quantity index and \( \frac{A}{\lambda} = .00631 \) by the Törnqvist implicit quantity index.)
Chapter V
SUMMARY AND CONCLUSIONS

In this final chapter we present a summary of the study and the main conclusions. We also indicate some areas of further research.

The main purpose of this research has been to develop an appropriate functional representation in order to analyze productivity in U.S. Manufacturing. This goal is elaborated in chapter I. In order to place the different flexible forms of cost function in a nested framework we have modified and extended N. Kiefer's [1975] Box-Cox formulation of an indirect utility function. This nested structure allows classical test procedures (e.g. likelihood ratio tests) to be employed in order to discriminate among the various flexible forms proposed in the literature. We also set as part of our goal in chapter I the incorporation of nonhomothetic returns to scale and nonneutral technical change in the production technology in such a way that simultaneous unique identification of substitution elasticities, scale economies and bias of technical change is possible.

In chapter II we developed the theoretical foundations of our cost function models. Our approach is the familiar method of modelling the production technology via the dual cost function. We start with a simple generalized Box-Cox (GBC) unit cost function and derive the various well known functional forms as parametric special or limiting cases. The possibility of negative values of $\lambda$, the Box-Cox transformation parameter, suggests two new interesting functional forms corresponding to $\lambda = -1$ and $-2$ respectively. We then extend the simple GBC formulation to incorporate nonhomothetic scale economies and nonneutral technical change.
The resulting model is parametrically quite rich and allows a total of twelve scale and technical change combinations to be considered. In particular it is possible to test separately for homotheticity, homogeneity or constancy of returns to scale and neutrality or absence of technical change. The scale economies and technical change have been sufficiently structured in order to allow separate identification of bias of technical change and scale economies and the elasticities of substitution. The homotheticity and the nonneutrality parameters have a convenient interpretation of share using or share saving bias. We have also indicated an alternative specification of scale effects and technical change in the GBC formulation. Estimation of this alternative representation is a topic for further research.

We have compared the extended GBC formulation with other highly general representations proposed in the literature, e.g., those of M. Denny [1974] and N. Kiefer [1975]. The various elasticity formulas corresponding to the extended GBC form were then derived. It is shown that the elasticity formulas for the other functional forms are themselves nested in the GBC elasticity formulas. We then examined the various cost function properties of the GBC form. It was shown that the GBC formulation has attractive approximation properties as well.

The measures of returns to scale and total factor productivity in the context of a nonhomothetic and nonneutral (technical change) production technology for the GBC model were developed following M. Ohta [1974]. We also developed various comparative static expressions relating our key measures such as estimated shares, returns to scale, total factor productivity and average factor productivity to changes in input prices, scale and time. These qualitative results provide interesting alternative interpretations of the scale and technical change parameters.
Chapter II was concluded by a discussion of the consequences of imposing weak separability on the GBC representation. It was shown that the GBC representation does not escape the problem of failing to provide a test for weak separability alone, a result recently proved by C. Blackorby, D. Primont and R. Russell [1977] in the context of the available flexible functional forms.

Chapter III was devoted to formulating a stochastic representation in order to estimate the GBC cost function. Our estimating equations are the cost function and the associated system of share equations. It is shown that the system of estimating equation for other flexible functional forms are nested in the GBC system of estimating equations.

Our error specification consists of the classical additive multivariate normally distributed errors in the share equations and the transformed cost equation. The errors in the share equations arise due to imperfect cost minimizing behaviour. However, a careful and detailed analysis of the error in the cost equation deserves extensive research. Our error structure allows correlation between the error in the cost equation and those in the share equations. But an analytical relationship between these errors is yet to be worked out.\(^2\)

Our error assumption, however, is not unreasonable. In the cost equation we have adjusted cost for scale effects and technical change. This adjusted cost has then been transformed in the method of G.E.P. Box and D. R. Cox [1964]. In our remaining estimating equations shares are the dependent variables. Consequently, it is reasonable to assume that the errors in our estimating system are multivariate normally distributed with homoscedastic variances and covariances.
An attractive feature of our error specification is that it yields a likelihood function that takes as parametric special or limiting cases the likelihood functions for other functional forms with a corresponding error structure.

A possible extension of our error specification is to allow for serial correlation. The GBC model can also be extended to incorporate a dynamic adjustment process in which actual shares adjust to the optimal shares with a lag.

The main null hypotheses tested are homotheticity, homogeneity and constancy of returns to scale and neutrality or absence of technical change. We employ likelihood ratio tests to determine the validity of these hypotheses. For computing an $R^2$ measure for the whole system we follow the method of N. D. Baxter and J. G. Cragg [1970]. This chapter was concluded with a brief discussion of the testing design that we followed.

In chapter IV we applied the GBC formulation to estimate the production technology of the U.S. Manufacturing sector. Our methodology is to postulate the existence of a well-behaved aggregate production function in the U.S. Manufacturing sector and then estimate it via the dual cost function utilizing a GBC functional form and the stochastic formulation in chapter III. Our data on U.S. Manufacturing is the capital (K)-Labour (L)-Energy (E)-Materials (M) (KLEM) data on U.S. Manufacturing 1947-71 originally used by E. R. Berndt and D. O. Wood [1975a]. We utilize the Fletcher Algorithm as incorporated in the UBC:NLMON (1975) to obtain maximum likelihood parameter estimates.

The most important trend in the KLEM data is that demand for capital and energy grew at a faster rate than that of labour while their prices grew at a smaller pace than that of labour. Our goal in this chapter
was to isolate the contribution to the above trend of own and cross price effects, scale effects and technical change.

We first turned our attention to a choice among alternative models of technology. The hypothesis of homotheticity was decisively rejected irrespective of the nature of technical change. Given a nonhomothetic production structure the null hypothesis of neutrality of technical change was also firmly rejected. We, therefore, accept that U.S. Manufacturing 1947-71 was characterized by nonhomothetic scale effects and nonneutral technical change. We called this representation NHT-NN. However, once we wrongly imposed homotheticity and neutrality we could not reject the simpler specifications of constant returns to scale with neutral technical change (CRTS-N) and homogeneous returns to scale with no technical change (HG-NO). In the former representation all growth in productivity not due to input deepening is attributed to technical change while in the latter it is attributed to scale economies alone. A model allowing both homogeneous scale economies and neutral technical change (HG-N) is another simple representation that we could not reject as compared to the rejected alternative hypothesis of homotheticity and neutrality. However, given the HG-N representation we could reject neither the CRTS-N nor the HG-NO specifications. For the sake of comparison though we reported our empirical results based on the four representations—NHT-NN, HG-N, HG-NO and CRTS-N.

In the GBC framework the Box-Cox transformation parameter $\lambda$ was used to discriminate among alternative flexible functional forms. For the Translog (TLOG) form $\lambda = 0$, for the Generalized Leontief (GL) form $\lambda = 1$ and for the Quadratic Square Root (QSR) form $\lambda = 2$. The maximum likelihood estimate of $\lambda$ in the NHT-NN representation is .8 while in HG-N,
HG-NO and CRTS-N models it is -.9. However, in the NHT-NN representation the GL form does not result in any significant loss of fit at the 1% level as measured by likelihood ratio test. The functional forms that are not rejected in the other simpler models at 1% level are the $\lambda = -1$, TLOG and GL forms. In these latter models the $\lambda = -1$ form does not result in virtually any loss of fit.

Our results in this chapter indicate that in our accepted NHT-NN representation of U.S. Manufacturing technical change has been significantly capital and energy using. The estimated nonneutrality parameters associated with labour and materials are negative indicating saving bias in these inputs. But these two parameters are not significantly different from zero. This result is, however, consistent with the finding of labour saving bias by other researchers (e.g., P. David and T. Van de Klundert [1965]) because the labour variable in the KLEM data that we are using has been preadjusted for educational attainment. Our results also imply the presence of substantial scale economies distributed unevenly over the various inputs. The scale effects are capital, labour and energy saving but materials using. The scale economies are, however, largest in the case of labour.

The maximum likelihood parameter estimates suggest a simpler version of NHT-NN representation which we call NHT-NN-R. This results from a slight simplification of the scale and technical change specifications. Our subsequent results are mainly based on this model.

The estimated cost functions are in general well behaved. Linear homogeneity in prices and symmetry of substitution effects are our maintained hypotheses. The monotonicity (in prices and output) and positivity properties of the cost functions are satisfied at every point
in the sample. In the NHT-NN-R representation the curvature conditions are satisfied over approximately the latter half of the sample while in the simpler HG-N, HG-NO and CRTS-N specifications the curvature conditions are satisfied over the entire sample except two years.

The elasticity results indicate that capital and energy display substantial complementarity while they are each substitutable with labour. The price elasticity estimates show that energy and capital are more own price elastic than labour. The cross price elasticity estimates in 1971 in the NHT-NN-R representation imply that if the price of labour rises by 1% then the quantity demanded of energy and capital services would grow by .4% and .3% respectively. On the other hand between capital and energy a fall in the price of energy or capital by 1% would lead to a .25% rise in the quantity demanded of the other input.

In our NHT-NN-R model where both scale effects and technical change are freely estimated, the measure of returns to scale is 1.23 in 1971. This estimate is significantly different from unity (CRTS). This measure of scale economies is close to the 24% measure implied in E. F. Denison [1967]. In a later study Denison [1974] estimates scale economies to have contributed an extra 15% to the output growth in the U.S. nonresidential business sector, 1947-69. We obtain a similar measure of 14% in the HG-N specification. This estimate (1.14), however, does not differ significantly from 1.

In contrast to the large and significant scale economies the estimate of the rate of growth of total factor productivity in 1971 is only .04% in our preferred NHT-NN-R representation. This measure, however, has a large standard error and so it should be interpreted cautiously. Since output and input quantities all grow with time, the effects of scale and technical change are in practice mixed. Technical change is a very
complicated phenomenon. Discovering its true contribution calls for a more careful modelling. At a general level a combined time series and cross section analysis may be helpful. Another alternative is to attempt to obtain some independent measure of economic progress. At a micro level this may be accomplished by using an engineering approach in deriving economic production functions. This approach is based on a knowledge of the physical technology of a production process. T. G. Cowing [1974] has applied this method in modelling the technology of the U.S. Steam electric power.

In the context of aggregate U.S. Manufacturing, however, our impression is that scale economies have contributed substantially to growth in output while the effect of total factor productivity is uncertain and perhaps small. If the output market is assumed to be competitive then the source of such scale economies must be productivity augmenting external economies. In an imperfect output market scale economies may arise due to the utilization of excess capacity in each firm.

It is interesting to note that when we impose the (rejected) CRTS restriction the estimated rate of growth of total factor productivity is .7% which is significantly different from zero and is similar to the estimate obtained in the literature (e.g. S. Star [1974]) based on CRTS representations and traditional productivity calculations.

A comparison of the 'residual measures' of total factor productivity with our 'parametric measures' indicates that the residual measures closely approximate the parametric measures when the CRTS restriction is imposed. The remaining discrepancy can even be analytically represented if the error portion in the Divisia index implied by the stochastic specification in the parametric approach can be identified. For example, from the
parametric approach the estimated share equation is, say $s_{it} = \hat{s}_{it} + \varepsilon_{it}$.

In the Törnqvist index then,

$$\ln P = \sum_{i} \frac{1}{2} (s_{it} + s_{i,t-1}) \ln \frac{P_{it}}{p_{i,t-1}}$$

and

$$\ln P = \sum_{i} \frac{1}{2} (\hat{s}_{it} + \hat{s}_{i,t-1}) \ln \frac{P_{it}}{p_{i,t-1}} + \sum_{i} \frac{1}{2} (\varepsilon_{it} + \varepsilon_{i,t-1}) \ln \frac{P_{it}}{p_{i,t-1}}$$

$$= \ln \hat{P} + w, \text{ say.}$$

$\hat{P}$ may be called an estimated Divisia index. If $\hat{P}$ is obtained by utilizing the residuals in the parametric approach to take out $w$ from $P$, then the residual measure based on $\hat{P}$ is likely to be the same as the parametric measure.

If however, the CRTS restriction is not true and hence not imposed in parametric estimation then the residual measures (which are based on the CRTS assumption) would differ widely from the parametric measures of total factor productivity. Residual measures comparable to the parametric measures in the case of a nonconstant returns to scale technology can be developed only if independent information on scale effects are available. Otherwise, the residual measures would be misleading.

We have also examined the response of our productivity and scale measures to changes in input prices, output quantity and time. Due to the presence of significant scale economies in capital, labor and energy an increase in the prices of these inputs would raise the returns to scale. On the other hand, due mainly to capital and energy using bias of technical change a growth in the price of these inputs would have a dampening effect on total factor productivity.
Finally, we note that the GBC representation explains productivity in U.S. Manufacturing reasonably well. Our models generate results which are consistent with economic theory and most of the available empirical evidence. On the theoretical side the GBC formulation provides a nested framework to discriminate among alternative functional forms and to test a variety of hypotheses concerning scale effects and technical change.
Footnotes to Chapter V

1. Indeed our results in chapter IV indicate that the $\lambda = -1$ functional form may be quite useful. This functional form virtually doesn't result in any loss of fit as compared to the GBC form in modelling the homogeneous and neutral technical change representations of U.S. Manufacturing, 1947-71.

2. Stochastic specifications of CD and CES production functions with multiplicative log normal errors or additive normal errors have been discussed widely in the literature. Here a distinction is often made between a 'technical error' in the production function and 'economic errors' in the first order optimizing equations. See, for example, R. Bodkin and L. Klein [1967], A. S. Goldberger [1968], J. Kmenta [1963, 1967], Y. Mundlak [1973], Y. Mundlak and I. Hoch [1965], M. Nerlove [1965] and A. Zellner, J. Kmenta and J. Drèze [1966]. Some researchers have recently turned their attention to modelling production frontier models. They explicitly take account of a one-sided error component in the production function. See, for example, D. J. Aigner, C. A. K. Lovell and P. Schmidt [1976] and J. Richmond [1974]. The component of the error in the cost function arising out of cost minimizing behaviour is one sided. To model this component a one-sided error distribution can be used. In any case, analyzing the nature of the error in the cost equation deserves special attention.

3. It should be noted that Denison's returns to scale measure is based on a value added specification while ours is a gross output measure. The value added measure of returns to scale is likely to underestimate the gross output measure.
4. T. G. Cowing [1974] states that "one of the features of the engineering approach is a more explicit analysis of technical change, including the precise ways in which technical change comes about and the specific impact of technical change on the basic production process."

5. It is, of course, assumed that the monopolistic firms are not replicable.
REFERENCES


UBC NLMON, 1975, The University of British Columbia Computing Centre.


Appendix 1: List of Equations in the Approximation Theorem.

(2.66)  \[ C^*_i = \left(\sum_j \gamma_{ij} p_i^2 p_j^2 / \left(\sum_i \gamma_{ij} p_i^2 p_j^2\right) + \phi_i \ln Y + \tau_i t\right) \frac{C}{p_i} \]

(2.67)  \[ C^*_{ij} = \epsilon_{ij} \frac{C_i}{p_j} \]

(2.68)  \[ C^*_{iY} = \frac{C_i}{Y} (\beta + \phi \ln Y + \sum_i \phi_i \ln p_i + \eta t + \phi_i \frac{C}{p_i C_i}) \]

(2.69)  \[ C^*_t = c_i (\tau + \delta t + \sum i \tau_i \ln p_i + \eta \ln Y + \tau_i \frac{C}{p_i C_i}) \]

(2.70)  \[ C^*_{YY} = \frac{C^2_Y}{C} + \phi \frac{C}{Y^2} - \frac{C_Y}{C} \]

(2.71)  \[ C^*_tt = \frac{C^2_t}{C} + \delta C \]

(2.72)  \[ C^*_Yt = \frac{C_Y C_t}{C} + \eta \frac{C}{Y} \]

In the equations above C is the GBC cost function (2.62) and \( \epsilon_{ii} \) and \( \epsilon_{ij} \) are as in (2.56) and (2.59).
Appendix 2: Proofs of Theorems and Corollaries

(We follow the method of M. Denny and M. Fuss [1977])

Proof of Theorem I. 1, 2 Consider the set of inputs. \( I = \{X_1, X_2, X_3, X_4\} \)

\[ = (X_1, X_2) \cup (X_3, X_4) = I_1 \cup I_2. \]

If \( I_1 \) is linearly separable from \( I_2 \) then

\[ \gamma_{13} = 0 = \gamma_{23} \text{ and } \gamma_{14} = 0 = \gamma_{24}. \]

With symmetry \( I_2 \) is also linearly separable from \( I_1 \). Consequently in (2.4)

\[ (A2.1) \quad G(P) = \alpha_0 + \alpha_K G_K(P) + \alpha_L G_L(P), \] where,

\[ (A2.2) \quad G_K(P) = \sum_{i \in I_1} \beta_i P_i \left( \frac{\lambda}{2} \right) + \frac{1}{2} \sum_{i, j \in I_1} \beta_{ij} P_i \left( \frac{\lambda}{2} \right) P_j \left( \frac{\lambda}{2} \right) \]

\[ (A2.3) \quad G_L(P) = \sum_{i \in I_2} \beta_i P_i \left( \frac{\lambda}{2} \right) + \frac{1}{2} \sum_{i, j \in I_2} \beta_{ij} P_i \left( \frac{\lambda}{2} \right) P_j \left( \frac{\lambda}{2} \right) \]

\[ (A2.4) \quad \alpha_i = \alpha_K \beta_i, \quad i \in I_1 \]

\[ = \alpha_L \beta_i, \quad i \in I_2 \]

\[ (A2.5) \quad \gamma_{ij} = \alpha_K \beta_{ij}, \quad i, j \in I_1 \]

\[ = \alpha_L \beta_{ij}, \quad i, j \in I_2 \]

Also defining,

\[ (A2.6) \quad C_K(\lambda) = G_K(P) \]

\[ (A2.7) \quad C_L(\lambda) = G_L(P) \]

The homothetic GBC cost function (2.93) can be written,

\[ (A2.8) \quad C = \left\{ 1 + \lambda \alpha_0 + \lambda \alpha_K C_K(\lambda) + \lambda \alpha_L C_L(\lambda) \right\} \frac{1}{\lambda} \gamma^\beta(Y) e^{T(t)} \]

where,
(A2.9) \[ \beta(y) = \beta + \frac{\theta}{2} \ln Y \]

(A2.10) \[ T(t) = (\tau + \frac{\delta}{2} t) t \]

It can be seen that (A2.8) is a homothetic 'CES type' aggregate of GBC aggregators \( C_K(\gamma) \) and \( C_L(\gamma) \).

Q.E.D.

Proof of Theorem II. If \( I_1 \) is nonlinearly separable from \( I_2 \) and conversely then,

(A2.11) \[ \frac{\alpha_1}{\alpha_2} = \frac{\gamma_{11}}{\gamma_{21}} = \frac{\gamma_{12}}{\gamma_{22}} = \frac{\gamma_{13}}{\gamma_{23}} = \frac{\gamma_{14}}{\gamma_{24}} = \frac{\theta_1}{\theta_2}, \quad \text{say, and} \]

(A2.12) \[ \frac{\alpha_3}{\alpha_4} = \frac{\gamma_{31}}{\gamma_{41}} = \frac{\gamma_{32}}{\gamma_{42}} = \frac{\gamma_{33}}{\gamma_{43}} = \frac{\gamma_{34}}{\gamma_{44}} = \frac{\theta_3}{\theta_4}, \quad \text{say} \]

Defining,

(A2.13) \[ \alpha_K = \alpha_1/\theta_1, \quad \alpha_L = \alpha_3/\theta_3, \]

(A2.14) \[ P_K(\frac{\lambda}{2}) = \theta_1 P_1(\frac{\lambda}{2}) + \theta_2 P_2(\frac{\lambda}{2}) \]

(A2.15) \[ P_L(\frac{\lambda}{2}) = \theta_3 P_3(\frac{\lambda}{2}) + \theta_4 P_4(\frac{\lambda}{2}) \]

(A2.16) \[ \gamma_{KK} = \gamma_{11}/\theta_1^2, \quad \gamma_{LL} = \gamma_{33}/\theta_3^2, \quad \gamma_{KL} = \gamma_{13}/\theta_1 \theta_3, \]

the homothetic GBC function (2.93) can be written as,

(A2.17) \[ C = \{1 + \lambda \alpha_0 + \sum_{i=K,L} \lambda \alpha_i P_i(\frac{\lambda}{2}) + \frac{\lambda}{2} \sum_{i,j=K,L} \lambda \gamma_{ij} P_i(\frac{\lambda}{2}) P_j(\frac{\lambda}{2}) \} \left[ \frac{1}{\gamma^2(Y)} e^{T(t)} \right] \]

which is a homothetic GBC aggregate of 'CES type' aggregators \( P_K(\frac{\lambda}{2}) \) and \( P_L(\frac{\lambda}{2}) \).

Q.E.D.

Proof of Corollary I.1. With linear homogeneity in prices in the GBC cost function \( 1 + \lambda \alpha_0 = \sum_{i=K,L} \alpha_i \) \( \alpha_K(\beta_1 + \beta_2) + \alpha_L(\beta_3 + \beta_4) \) using (A2.4).

If \( \beta_1 + \beta_2 = 1 = \beta_3 + \beta_4 \) then \( 1 + \lambda \alpha_0 = \alpha_K + \alpha_L \). (A2.8) can then be written
as,

\[(A2.18) \quad C = \{\alpha KC + \alpha LC\}^\lambda \frac{1}{Y^\beta(Y)} e^{T(t)}\]

The other condition for linear homogeneity in prices of the GBC cost function is \(\sum_{j=1}^{4} y_{ij} = \frac{\lambda}{2} \alpha_i, \quad i = 1, \ldots, 4.\) This implies \(\sum_{j \in I_1} \beta_{ij} = \frac{\lambda}{2} \beta_i, \quad i \in I_1\) and \(\sum_{j \in I_2} \beta_{ij} = \frac{\lambda}{2} \beta_i, \quad i \in I_2.\) Also if \(\beta_1 + \beta_2 = 1 = \beta_3 + \beta_4,\) we have,

\[(A2.19) \quad C_K = \left(\frac{\lambda}{2} \sum_{i, j \in I_1} \beta_{ij} (P_i P_j)^{\lambda/2} \right) \frac{1}{Y^{\beta(Y)}}\]

and

\[(A2.20) \quad C_L = \left(\frac{\lambda}{2} \sum_{i, j \in I_2} \beta_{ij} (P_i P_j)^{\lambda/2} \right) \frac{1}{Y^{\beta(Y)}}\]

Q.E.D.

Proof of Corollary 1.2. We can write (A2.8) as

\[(A2.21) \quad \frac{\lambda}{\lambda - 1} C^\lambda - 1 = \alpha_0 + \alpha_K C_K(\lambda) + \alpha_L C_L(\lambda)\]

where,

\[(A2.22) \quad C^\lambda = C/\{Y^\beta(Y) e^{T(t)}\}\]

Now taking the limit of (A2.21), (A2.6) and (A2.7) as \(\lambda \to 0\) we obtain,

\[(A2.23) \quad \ln C = \alpha_0 + \alpha_K \ln C_K + \alpha_L \ln C_L + \beta \ln Y + \frac{\theta}{2} (\ln Y)^2 + \tau t + \frac{\delta}{2} t^2\]

\[(A2.24) \quad \ln C_K = \sum_{i \in I_1} \beta_i \ln P_i + \frac{\lambda}{2} \sum_{i, j \in I_1} \beta_{ij} \ln P_i \ln P_j\]

and,

\[(A2.25) \quad \ln C_L = \sum_{i \in I_2} \beta_i \ln P_i + \frac{\lambda}{2} \sum_{i, j \in I_2} \beta_{ij} \ln P_i \ln P_j\].
Now linear homogeneity in prices in the TLOG cost function implies
\[ 1 = \frac{4}{\lambda} \sum_{i=1}^{4} \alpha_i = \alpha_K (\beta_1 + \beta_2) + \alpha_L (\beta_3 + \beta_4) = \alpha_K + \alpha_L \text{ only if } \beta_1 + \beta_2 = 1 = \beta_3 + \beta_4. \]
Also \( 0 = \sum_{j=1}^{4} \gamma_{ij}, i = 1, \ldots, 4 \) implies \( \sum_{j \in I_1} \beta_{ij} = 0, i \in I_1 \) and \( \sum_{j \in I_2} \beta_{ij} = 0, i \in I_2. \)

Q.E.D.

Proof of Corollary 1.3. Set \( \lambda = 1 \) in (A2.18), (A2.19) and (A2.20). Q.E.D.

Proof of Corollary 1.4. Set \( \lambda = 2 \) in (A2.18), (A2.19) and (A2.20). Q.E.D.

Proof of Corollary II.1. If \( \phi_1 + \phi_2 = 1 = \phi_3 + \phi_4 \) then by linear homogeneity of the GBC cost function in prices \( 1 + \lambda a_0 = \sum_{i=1}^{4} \alpha_i = \alpha_K + \alpha_L \)
and \( \sum_{j=1}^{4} \gamma_{ij} = \frac{\lambda}{2} \alpha_i, i = 1, \ldots, 4 \) implies \( \sum_{j=K,L} \gamma_{ij} = \frac{\lambda}{2} \alpha_i, i = K,L. \) We can then write (A2.17) as,
\[
(A2.26) \quad C = \left\{ \frac{2}{\lambda} \sum_{i,j=K,L} \gamma_{ij} (P_i P_j)^\frac{\lambda}{2} \right\} \frac{1}{\lambda} \gamma^\beta (Y) e^T (t)
\]
with,
\[
(A2.27) \quad P_K = \left\{ \frac{\lambda}{2} P_1^\frac{\lambda}{2} + \frac{\lambda}{2} P_2^\frac{\lambda}{2} \right\} \frac{2}{\lambda}
\]
and
\[
(A2.28) \quad P_L = \left\{ \frac{\lambda}{2} P_3^\frac{\lambda}{2} + \frac{\lambda}{2} P_4^\frac{\lambda}{2} \right\} \frac{2}{\lambda}. \quad \text{Q.E.D.}
\]

Proof of Corollary II.2. We can write (A2.17) as,
\[
(A2.29) \quad \frac{C^{\lambda} - 1}{\lambda} = a_0 + \sum_{i=K,L} \alpha_i P_i (\frac{\lambda}{2}) + \frac{\lambda}{2} \sum_{i,j=K,L} \gamma_{ij} P_i (\frac{\lambda}{2}) P_j (\frac{\lambda}{2})
\]
where \( C \) is as defined in (A2.22). Now taking the limit of (A2.29), (A2.14) and (A2.15) as \( \lambda \to 0 \), we obtain,
(A2.30) \[ \ln C = \alpha_0 + \sum_{i=K,L} \alpha_i \ln P_i + \frac{1}{2} \sum_{i,j=K,L} \gamma_{ij} \ln P_i \ln P_j \]
\[ + \beta \ln Y + \frac{\theta}{2} (\ln Y)^2 + \tau t + \frac{\delta}{2} t^2 \]

where,

(A2.31) \[ \ln P_K = \theta_1 \ln P_1 + \theta_2 \ln P_2 \]

and,

(A2.32) \[ \ln P_L = \theta_3 \ln P_3 + \theta_4 \ln P_4 \]

If \( \theta_1 + \theta_2 = 1 = \theta_3 + \theta_4 \) then linear homogeneity in prices of the TLOG cost function implies \( 1 = \sum_{i=1}^{4} \alpha_i = \alpha_K + \alpha_L \). Also, then \( 0 = \sum_{j=1}^{4} \gamma_{ij} \), \( i = 1, \ldots, 4 \) implies \( 0 = \sum_{j=K,L} \gamma_{ij}, i = K,L \). Q.E.D.

Proof of Corollary II.3. Set \( \lambda = 1 \) in (A2.26), (A2.27) and (A2.28). Q.E.D.

Proof of Corollary II.4. Set \( \lambda = 2 \) in (A2.26), (A2.27) and (A2.28). Q.E.D.
Footnotes to Appendix 2

1. For simplicity our proof is in terms of four inputs only.

2. Throughout appendix 2 we use the following notation:

\[ X(\frac{\lambda}{2}) = \frac{X^2 - 1}{\frac{\lambda}{2}}, \quad X(\lambda) = \frac{X^\lambda - 1}{\lambda} \]

where \( X \) is any variable.

3. This proof is strictly valid only if \( \gamma_{ij} \)'s are all non-zero and are either all positive or all negative. For a general proof see Blackorby, Primont and Russell [1977].
Appendix 3

QUANTITIES OF INPUT SERVICES AND OUTPUT IN
U.S. MANUFACTURING, 1947-71

(Values in billions of 1947 dollars)

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Source: E. R. Berndt and D. O. Wood [1975a]

K: capital
L: labour
E: energy
M: other intermediate materials
Y: output
PRICE INDEXES OF INPUTS IN U.S. MANUFACTURING, 1947-71

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Source: E. R. Berndt and D. O. Wood [1975a]

PK: Price of capital services
PL: Price of labour services
PE: Price of energy
PM: Price of other intermediate materials.