COMPARATIVE STATICS AND THE EVALUATION OF AGRICULTURAL DEVELOPMENT PROGRAMS

by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

(Department of Agricultural Economics)

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

October 1979

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ABSTRACT

Perhaps the most important task of any economic analysis of agricultural policy is to estimate the effects of policy on various economic measures such as income and output. This is usually done by combining economic theory with data. However, the economic theory seldom is fully descriptive of the situation and the empirical knowledge generally is far from complete. Thus, even aside from difficulties in aggregating gains and losses over individuals, economic analyses of policies are often unsatisfactory.

The major purpose of this thesis is to extend economic theory and methods so as to be more descriptive of various agricultural policy situations and to make more appropriate use of available empirical knowledge. This leads us to relax some assumptions in the standard theory of the firm that often seem inappropriate, and to propose a potentially more effective method of incorporating available empirical knowledge of farm structure into economic analysis of policy. In addition, we also attempt to verify the appropriateness of other theoretical constructs of fundamental importance.

First, the static theory of the firm is extended to the case of variable factor prices, i.e., factor prices endogenous to the firm. Under these more general conditions, we establish (among other things) (1) the relation between measures of surplus in factor markets and of consumer plus producer surplus, and

(2) relations between the slope of a firm's derived demands schedule and various properties of its production function. It is shown that (2) provides additional support for the well-known fact that traditional qualitative comparative static methods can seldom be useful in economic policy-making.

Second, we introduce a method of "quantitative comparative statics" that in principle overcomes this defect of established comparative static analysis. This methodology incorporates the available degree of empirical knowledge of the firm's structure without imposing further specification of structure (in contrast to, e.g., the traditional linear and nonlinear programming models of the firm, where a full structure must be specified). This degree of knowledge and its relations to comparative static effects of interest can be expressed as a set of quadratic equalities and inequalities. Then the range of quantitative as well as qualitative effects of policy that are consistent with our degree of knowledge of farm structure and the assumption of static optimizing behavior can in principle be calculated by nonlinear programming methods.

Third, we consider the issue of the appropriateness of constructs of static optimizing behavior in predicting farm response to policy. We demonstrate that, by estimating an equilibrium shadow price for an input rather than (e.g.) supply response, one can reduce the significance of many of the problems associated with studies of supply response via representative farm models and investigate this issue more clearly. In this manner, we derive empirical support for the use of the construct of static optimizing behavior in predicting the effects of agricultural policy.

Thus, by extending economic analysis marginally in the direction of more appropriate theory and more appropriate use of empirical knowledge, we hope to contribute towards the improvement in methodology for evaluating agricultural development programs. Towards this end, the extensions in theory and methods are related to a particular policy situation (evaluation of government funded community pasture programs in British Columbia).

ACKNOWLEDGEMENT

I wish to thank Rick Barichello for the opportunity to assist in the evaluation of B.C. ARDA community pasture programs. This experience increased the author's awareness of the problems in the evaluation of agricultural development programs that are considered here, and hence motivated this study. In addition Professor Barichello has shown considerable patience with the author.

I am also particularly grateful to Chuck Blackorby for agreeing to be the "technical expert" on the committee and for suffering through an essentially unintelligible first draft of this thesis. Moreover, his suggestions as to additional readings helped to round out the thesis in many respects.

In addition I am grateful to Earl Jenson for agreeing to round out the committee and for providing useful comments. Conversations with Ramon Lopez on general matters of theory and his own research have been very helpful. His comments on parts of an earlier draft of this study have also improved the exposition. Roger McNeil has also made useful suggestions.

Many people assisted in the construction of the linear programming model reported in one section of this thesis. Rick Barichello helped considerably in collecting the data and testing the model. John Kidder provided valuable and necessary assistance in the field. Robert Holtby (B.C.D.A., Prince George), Albert Isfeld (B.C.D.A., Fort

St. John), Dale Engstrom (B.C.D.A., Dawson Creek) and Gerry Kirtzinger (president of the grazing association of Sunset Prairie Community Pasture) all provided many valuable hours of consultation. Long and helpful interviews were conducted with seven community pasture users at Sunset Prairie, five at Beatton-Doig and twelve at W.M.

I am grateful to Maryse Ellis, Wendy Lymer and Carol Clark for typing drafts of this manuscript with skill and patience.

Finally, I would like to thank Phyllis Coyle for her encouragement and support.

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CHAPTER 1

INTRODUCTION

CHAPTER 1

INTRODUCTION

1.1 Overview

Perhaps the most important task of any economic analysis of agricultural policy is to estimate the effects of policy on various economic measures such as income and output. This is usually done by combining economic theory with data. However, the economic theory seldom is fully descriptive of the situation and the empirical knowledge generally is far from complete. Thus, even aside from difficulties in aggregating gains and losses over individuals, economic analyses of policies are often unsatisfactory.

The major purpose of this thesis is to extend economic theory and methods so as to be more descriptive of various agricultural policy situations and to make more appropriate use of available empirical knowledge. This leads us to relax some assumptions in the microeconomic theory of the firm that often seem inappropriate, and to propose a potentially more effective method of incorporating available empirical knowledge of farm structure into economic analysis of policy. In addition, we also attempt to verify the appropriateness of other theoretical constructs of fundamental importance.

Thus, by extending economic analysis marginally in the direction of more appropriate theory and more appropriate use of empirical knowledge, we hope to contribute towards the improvement in methodology for evaluating agricultural development programs. Towards this end, extensions in theory and methods are related to a particular policy situation (evaluation of government funded community pasture programs in British Columbia).

1.2 The Problem

The problems to which this thesis is addressed are essentially three-fold:

1. endogenous factor prices apparently are realistic in many cases but have not been introduced (correctly) into the theory of the firm,
2. comparative static methods that are presently available generally make inadequate use of the degree of knowledge about particular policy situations (at least at the firm level), and
3. the usefulness of the construct of static, optimizing behavior in estimating farm response via representative farm models apparently remains a matter of some controversy.

These problems can be elaborated upon as follows.

First, the theory of the firm has been formulated under the assumptions of exogenous factor prices.[1] On the other hand, there appear to be many situations where factor prices are endogenous at the firm level. For example,

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[1] Earlier studies by Ferguson (1969, Chapter 8) and Maurice and Ferguson (1971) have tried to analyze the theory of the firm in the case of variable factor prices. However, a series of fundamental errors in these studies will be pointed out in Chapter 2.

situations characterized by a single employer, collusive monopoly, imperfections in information or in mobility, or internal labor market structuring all imply a positive slope to the labor supply schedule faced by the individual firm.[2] The most common cause of endogenous labor supply prices within U.S. and Canadian Industry may be due to the idiosyncratic nature of many skilled and semi-skilled jobs, which seem to require a degree of on-the-job training and investment by the firm. Since the probability of quitting is inversely related to salary, in effect the supply price of such labor is endogenous to the firm, i.e., the expected length of the period of return on investment in such human capital increases with the level of earnings offered by the firm.[3] In addition, it has been estimated that on average a firm in Canadian agriculture is either employing its own labor off-farm or hiring non-family labor during only 1/6 of the year.[4] Since the marginal utility of leisure presumably varies with the level of leisure, and leisure in one time period will not substitute perfectly for leisure in a different period (when, e.g., labor is bought or sold by the farm), it follows that the supply price of labor to the farm typically is (to at least some extent) endogenous to the farm.

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[2]See Addison and Siebert (1979, Chapter 5) for a summary of the literature concerning endogenous supply prices for labor at the firm level.

[3]See Stoikov and Raimon (1968), Parsons (1972) and Williamson _et al_. (1975).

[4]See Statistics Canada (1976), chapter on Multiple Job Holdings).

Such labor in "developed" countries seems far from the only input that is typically endogenous at the firm level. Endogenous prices at the firm level may well be the rule in most "underdeveloped" countries[5] or wherever markets are weakly developed.[6]

Second, methods of comparative static analysis for the firm that are presently available often appear to make poor use of the degree of knowledge about a particular policy situation. It is well known that qualitative comparative static methods, as embodied in Samuelson (1947) and more recent dual and primal-dual approaches, have led to relatively few deterministic results (signed comparative static effects) under reasonable assumptions.[7] Indeed, it seems to be difficult to incorporate many empirically-based quantitative restrictions on production functions into such methods. Thus, in the absence of such restrictions, relatively few predictions of firm response or testable predictions of firm behavior can be obtained.

Unfortunately, the only alternative methods of comparative static analysis that are presently available are essentially dependent upon complete knowledge of the structure of the problem. For example, in

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[5]This view was implicit in the address by Nerlove at the AJAE meetings (1979).

[6]In the Peace River region there is a (sparse) market for rented hayland, and hay and pasture appear to be close substitutes. However, observations of farm behavior suggested that the supply price of pasture is endogenous to the typical user of community pasture in the region. Moreover, in the other region studied for the B.C. ARDA community pastures evaluation (Prince George), the supply price of pasture is clearly endogenous at the farm level due to the absence of any (market-clearing) rental markets for pasture or any close substitutes. See Barichello (1978).

[7]This point is elaborated upon in Sections 3.2.1 and 3.3.2 of Chapter 3.

order to obtain a solution to a static linear or nonlinear programming model of a firm, the entire structure of the production function must be specified; in fact, our knowledge of the production function is generally far from complete. Usually this problem cannot be handled adequately, at least at the firm level. This can be seen most clearly by thinking in terms of local comparative statics: if a twice differentiable objective function $\pi(x;\alpha)$ for the firm has a maximum at $x^* > 0$ (the equilibrium level of inputs), then the comparative static change in x^* due to the effect (π_α) of an infinitesimal change in a policy parameter α can be calculated as

$$\underset{(N\times 1)}{\frac{\partial x^*}{\partial \alpha}} = - \underset{(N\times N)}{[\pi_{ij}(x^*)]^{-1}} \underset{(N\times 1)}{\pi_\alpha} \quad .$$

where $[\pi_{ij}(x^*)]^{-1}$ denotes the inverse of the Hessian of $\pi(x)$ at x^*. Since the majority of the $\frac{N(N+1)}{2}$ individual elements of $[\pi_{ij}(x^*)]$ are usually largely unknown (in the case of a firm's production function) and the relation between $\frac{\partial x^*}{\partial \alpha}$ and $[\pi_{ij}(x^*)]$ is complex, a sensitivity analysis that depends on direct user alteration of structure $[\pi_{ij}(x^*)]$ or analogous forms is seldom adequate.

Third, there is debate as to the utility of such concerns about comparative static theorems and methods. In particular, lists of possible causes of the apparent failure of representative farm studies of supply response typically have included the static, optimizing nature of these models. Indeed, at least some observers have stated that the decision to model static, optimizing behavior rather than dynamic non-optimizing behavior was the major cause of failure for these studies.[8]

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[8]See Chapter 4.

On the other hand, theory suggests that, given our present state of knowledge, farm response generally can be estimated more effectively from static, optimizing models than from dynamic or non-optimizing models. The essential arguments are that comparative dynamic effects can be differentiated from comparative static effects only on the basis of essentially unavailable knowledge of adjustment cost functions, and that static models are internally consistent and (unlike dynamic models) relatively simple in structure.[9] Given this contrast between opinion and theory and the importance of the issue, there appears to be a need to test the relative utility of the construct of static, optimizing behavior in estimating farm response via representative farm models.

1.3 Statement of Purpose

The purpose of this thesis is essentially three fold.

1. Extension of the traditional qualitative comparative statics of derived demand at the firm level to the case of endogenous factor prices. This will involve the development of theorems concerning: properties of derived demand schedules and comparative static effects of a shift in a factor supply schedule for an individual firm facing variable factor prices.

2. Extension of comparative static methods of analysis at the firm level so as to incorporate more fully our empirical knowledge about

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[9]See Appendix 1.

parameters without specifying more than this knowledge, i.e., to develop a method of analysis that provides a useful "middle ground" between the (generally underdeterminate) traditional qualitative methods as embodied in Samuelson (1947) et al. and the (generally overdeterminate) quantitative methods as embodied in (e.g.) static linear and nonlinear programming models of the firm. This will involve the development of a method of local comparative static analysis that in principle incorporates additional restrictions on potentially observable parameters of the firm's maximization problem, i.e., restrictions that have not been incorporated into traditional methods of local comparative static analysis, and that leaves the degree of specification of structure as optional to the user.

3. Examining the appropriateness (in a particular case) of constructs of static, optimizing behavior in the estimation of farm response. This will involve the use of a static linear programming model of a "representative" farm for a particular community pasture in British Columbia.

Since these extensions are in the direction of making theory more relevant to practice, it is hoped that they will not be "empty" theoretical exercises with zero practical implications. Towards this end, and in addition to (3), an attempt is also made to relate the more theoretical parts (1) and (2) to the problem of predicting farm response to ARDA community pasture programs in British Columbia. However, the major task of obtaining computational and practical experience with the "intermediate" method of comparative statics (part 2 above) will be postponed to a future study.

1.4 Research Procedure

The manner in which these objectives are met can be summarized as follows:

1. The theory of derived demand with variable factor prices is investigated by making explicit use in formal analysis of the following equivalence: a firm's derived demand schedule is equivalent to a schedule of shadow prices for the input. This is simply the "intuitively obvious" equivalence leading to the textbook statement that factor market equilibrium occurs at an intersection of factor demand and supply schedules. Nevertheless, this equivalence has not been incorporated previously into formal analysis of derived demand, and in effect this equivalence was even labelled as incorrect by a paper in a prominent journal.[10] Implications of this theory for the evaluation of ARDA community pasture programs are pointed out.

2. The traditional methodology of local comparative statics for the maximizing firm (e.g., Samuelson, 1947) is generalized by expressing the comparative static implications of the maximiza-

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[10] See Schmalensee (1971).

tion hypothesis and of many potentially observable parameters of the firm's maximization problem as a set of nonlinear constraints. These constraints define the comparative static effect $\frac{\partial x}{\partial \alpha}$ in terms of potentially observable parameters ρ of the firm's maximization problem. Then reasonable restrictions corresponding to our degree of knowledge about the structure of the problem are specified for ρ. By solving for the maximum and minimum value of a scalar-valued function $z\left(\frac{\partial x}{\partial \alpha}\right)$ of $\frac{\partial x}{\partial \alpha}$ over a feasible set defined by all of these constraints, we can calculate the range on the comparative static effects $z\left(\frac{\partial x}{\partial \alpha}\right)$ that are consistent with the maximization hypothesis plus the specified restrictions on ρ. Partial solutions to the major computational difficulties of this method are developed.

3. A static linear programming model of a "representative" multi-product farm using the Sunset Prairie community pasture is presented. Data for the model circa 1976 has been gathered from interviews with local farmers and B.C. Ministry of Agriculture personnel. The resulting estimates of the static equilibrium price of pasture in the region circa 1976 are compared with estimates of the rental price of hayland gathered by Barichello[11] and with results obtained by other models. This comparison provides a rough test of the hypothesis that constructs of static, optimizing

[11]See Barichello (1978).

behavior are appropriate in estimating farm response via representative farm models.

1.6 Organization of the Study

Chapter 1 includes a brief statement of the problems, the objectives and the basic methodology to be followed.

Chapter 2 presents a theoretical study of properties of derived demand schedules and comparative static effects for an individual firm facing variable factor prices, and points out the implications of the theory for a methodology of evaluating community pasture programs.

Chapter 3 presents a method for calculating the comparative static effects of a shift in a firm's factor supply schedule. This method in principle incorporates verifiable restrictions excluded from traditional comparative static methods without at the same time specifying essentially unknown aspects of structure. Simple (two and three input) illustrative models are constructed.

Chapter 4 examines the appropriateness of the construct of static, optimizing behavior in the context of estimating farm response via representative farm models, and summarizes the structure of a static linear programming model of a "representative" beef ranch.

Chapter 5 summarizes the study and provides basic conclusions.

Technical material related to Chapters 1-4 (primarily proofs and details of the method of comparative static analysis and the linear programming model) is presented in the appendices.

CHAPTER 2

QUALITATIVE COMPARATIVE STATICS AND DERIVED DEMAND: AN EXTENSION

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QUALITATIVE COMPARATIVE STATICS AND DERIVED DEMAND: AN EXTENSION

2.1 Introduction

In the previous chapter, we pointed out that variable factor prices are not uncommon at the firm level. Indeed, we noted that the supply price of land and (during most of the year) of labor should generally be endogenous to the firm in agriculture. This was confirmed by observation in the case of pasture in the Peace River and Prince George regions of British Columbia.

Suppose that the construct of static, optimizing behavior is of value in the applied economics of agriculture — this assumption will be verified in Chapter 4. Then it follows that extending the theory of the firm to the case of variable factor prices should be a small positive addition to the tools of the profession. Since the received theory of the firm is embodied in a set of formal propositions and proofs, extensions to this theory also should be made in a rigorous manner.

In this chapter, we shall extend the static theory of the firm to the case of variable factor prices and we shall point out implications for the methodology of evaluating community pasture programs. As we shall see, the farm value of community pasture depends solely on the farm's demand and supply schedules for pasture, and the related comparative static output effect[1] can be decomposed as the product of the related comparative static change in total pasture and the comparative static output effect of an exogenous change in the quantity of total pasture employed by the farm.[2] For these reasons, we shall concentrate on extending the theory of derived demand, i.e., price-quantity relations in input markets, to the case of variable factor prices.

The theory of derived demand with variable factor prices is investigated here by making explicit use in formal analysis of the following equivalence: a firm's derived demand schedule is equivalent to a schedule of shadow prices for the input. This relation is the "intuitively obvious" principle that underlies the textbook statement that factor market equilibrium occurs at an intersection of factor demand and supply schedules; but this principle has not previously been incorporated into formal analysis of the theory of derived demand. This equivalence implies that the area under any section of the firm's derived demand schedule is equal to the general equilibrium benefits to the firm (gross of supply costs of the input) of employing the corresponding levels of that input, i.e., the gross value to the firm of those levels of input. This in turn implies that the user value of programs shifting factor supply schedules (and,

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[1]For simplicity, we shall usually refer to "the" comparative static change in beef output. In fact, we can define "short run," "long run," etc. comparative static changes by making appropriate assumptions about the structure of the "stationary state" and the actual underlying adjustment cost function (Rothschild, 1971).

[2]These two statements seem "obviously" true, but the first statement has in effect been the subject of some controversy.

in the absence of market "distortions," etc., the associated change in consumer plus producer surplus) can be determined directly from knowledge of the appropriate factor market.[3] These results support the approach to evaluation of community pasture programs adopted by Barichello (1978): (in the absence of a commercial market for pasture) the farm value of the program is estimated from observations of a commercial market for an alternative use of improved land (plus a correction for any distortions).

In addition, relations between the slope of the derived demand schedule and several properties of the firm's maximization problem are readily established from this equivalence. For example, the derived demand schedule for pasture is necessarily positively inclined given increasing returns to scale and fixed prices for all other inputs, and the schedule can be positively inclined over large areas of its domain given decreasing returns to scale and non-convex isoquants. Moreover, non-convexity of isoquants and increasing returns to scale cannot be ruled out *a priori* in the case of variable factor prices, and the possibility of non-convexity of isoquants cannot readily be verified or rejected by empirical observation. Therefore, we cannot readily deduce an upper bound on the slope of the derived demand schedule for pasture from this theory of the firm plus empirical observation (except in terms of

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[3]This relation between surpluses in factor and product markets had at one time been declared incorrect (Schmalensee, 1971), and has been the subject of additional papers that have proved the relation under various special conditions (Panzar and Willig (1978) provide the most general treatment). Here we shall prove the relation under general conditions and by methods that are quite different from those employed in previous studies.

the supply schedule for this input). This in turn implies that we cannot readily deduce in this manner an upper bound on the comparative static change in total pasture employed by the farm or on the comparative static change in beef output due to the community pasture programs.

Thus the usual qualitative comparative statics methods, employed in this chapter, permit us to conclude that the farm value of the community pasture program can be estimated directly from knowledge of the market for pasture or of the market for an alternative use of improved land; but these methods plus empirical observation can seldom lead to an adequate measure of the comparative static change in pasture input or beef output. In the next chapter we shall present a "quantitative" method of comparative static analysis that, in principle, incorporates restrictions on many empirically observable parameters of the firm's maximization problem.

2.2 Results of Previous Studies

In this section we summarize the results of two classes of previous studies: studies concerning the relation between surpluses in product and factor markets, and studies concerning the relation between the slope of the individual firm's derived demand schedule and properties of the firm's maximization problem.

The following notation will be used. Define the firm's static (primal) maximization problem as

$$\text{maximize} \quad \pi(x) \equiv R(x) - \sum_{i=1}^{N} w^i x^i \qquad \text{. . . .(P)}$$

where $R(x)$ denotes total revenue as a function of input levels x, and $w^i \equiv w^i(x^i; \alpha^i)$ if w^i is endogenous to the firm. Let x^* be a solution to problem P. The firm's derived demand schedule for input i is obtained by varying the exogenous variable w^i or α^i and recording the relation between x^{i*} and the marginal factor cost. For simplicity, denote the firm's derived demand schedule for input i as $x^i(w^i)$ if the supply price w^i is exogenous to the firm.

2.2.1 Relation between Surpluses in Factor and Product Markets

The relation between surpluses under derived demand schedules and consumer's surplus appear to have been considered too obvious for comment[4] until Schmalensee (1971) argued that the change in surplus between a derived demand and supply schedule for an input, generated by a shift in the supply schedule for that factor, generally exceeds the related change in consumer's surplus. Then a short series of papers verified the equivalence between measures of surplus in product and factor markets under special conditions.[5,6]

This literature has established the following:

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[4]For example, see Prest and Turvey (1965), p. 691.

[5]See Wisecarver (1974), Anderson (1976), Schmalensee (1976) and Panzar and Willig (1978).

[6]It should be noted that the controversy has concerned fundamental properties of derived demand schedules rather than difficulties in aggregating over firms. By assuming that prices are exogenous to the industry, Schmalensee (1971) in effect denied that surpluses in factor and product markets were equivalent even in the case of a shift in a factor supply schedule of a single firm. Likewise, later papers on this relation typically simplified the problem by ignoring firm interactions (the one partial exception is Wisecarver, who outlines an argument that assumes constant elasticity of aggregate factor supply).

1. given perfectly elastic or perfectly inelastic supply schedules of inputs at the firm and industry level, the welfare changes resulting from input price changes can be measured as changes in the surplus under the industry factor demand curve; and

2. given that a change in factor price leads to a change in producer's surplus in that market, the welfare changes (change in consumer plus producer surplus) of course cannot be measured simply in terms of changes in consumer surplus in output markets.[7,8]

2.2.2 <u>Slope of the Firm's Derived Demand Schedule</u>

Suppose that the equilibrium supply price w^i is exogenous to the firm, x^* is an interior solution for problem P, $\pi(x)$ is twice differentiable at x^*, and the Hessian matrix $[\pi_{ij}(x^*)]$ is always negative definite. Then $\frac{\partial x^i}{\partial w^i} < 0$, i.e., the derived demand schedule for input i is always negatively inclined. If instead $[\pi_{ij}(x^*)]$ is negative semi-definite, then $\frac{\partial x^{i*}}{\partial w^i} \leqq 0$, i.e., the derived demand schedule for input i is never positively inclined.[9]

Ferguson (1969) and Maurice and Ferguson (1971) attempt to extend the analysis of Samuelson and others to the case of variable factor

[7]See Panzar and Willig (1978).

[8]This second point favors the measurement of effects of community pasture programs in the pasture market rather than in product markets. Measurement of these effects via product market calculations generally requires considerably more information than does measurement via the pasture market (Carlton, 1978).

[9]For example, see Samuelson (1947) or, for a simpler approach using duality, see McKenzie (1956-7), pp. 188-9 and Karlin (1959), p. 273.

prices; but their manner of doing this is fundamentally incorrect. Their major errors can be summarized as follows:

1. totally differentiating the first order conditions with respect to the endogenous variable w^i rather than with respect to α^i (Ferguson, 1969);[10]
2. defining the firm's derived demand schedule in terms of equilibrium supply price rather than equilibrium marginal factor cost (Ferguson, 1969 and Maurice and Ferguson, 1971).

Given this definition of a derived demand schedule, they conclude that "unique factor demand functions do not exist when factor prices are variable to the firm." (Maurice and Ferguson, 1971, p. 133). On the other hand, suppose that the firm's derived demand schedule $x^i(\alpha^i)$ is defined in terms of equilibrium marginal factor cost. Then the statement quoted above is incorrect provided that $w^i \equiv w^i(x^i; \alpha^i)$ rather than $w^i \equiv w^i(x; \alpha^i)$.[11] Moreover, even overlooking this uniqueness problem, the concept of a derived demand schedule is a much more useful analytical tool when it is defined in terms of equilibrium marginal factor cost.[12]

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[10]Silberberg (1974b), p. 738 has criticized Ferguson and Saving (1969) for a similar error.

[11]See Section 2.4.2.

[12]See Sections 2.4.3 - 2.4.5.

2.3 Difficulties in Extending Results via Usual Methods

The theorems summarized in Section 2.2.2 for the case where w^i is exogenous to the firm can be derived in a straight-forward manner from this restriction. Thus it is not surprising that the slope of the derived demand schedule becomes ambiguous when this restriction is relaxed. Nor should it be surprising that the methods commonly used to sign the slope given fixed factor prices are not appropriate for signing the slope under certain quite different restrictions, e.g., various restrictions on the production function. In Section 2.4.5 we shall employ a slightly different method for this purpose.

Here we shall point out that the methods commonly employed in comparative static analysis of the firm — primal, dual and primal-dual methods — are at best clumsy in signing the slope of the derived demand schedule given various restrictions on the firm's production function.

2.3.1 Primal Methods

The usual restrictions

$$\sum_{j=1}^{N} \pi_{ij}(x^*) \frac{\partial x^{j*}}{\partial \alpha^i} - w^i_{\alpha^i} x^{i*} = 0$$

$$\sum_{j=1}^{N} \pi_{jk}(x^*) \frac{\partial x^{j*}}{\partial \alpha^i} = 0 \qquad \text{all} \quad k \neq i$$

$[\pi_{ij}(x^*)]$ negative definite

for the primal problem P, where $w^i \equiv w^i(x^i; \alpha^i)$, exhaust the implications for $\frac{\partial x^*}{\partial \alpha^i}$ of the maximization hypothesis,[13] and it can easily be shown that the slope of the derived demand schedule for input i is unsigned by these restrictions. Nevertheless the primal approach becomes messy and complex when restrictions such as increasing or decreasing returns to scale and convexity/non-convexity of isoquants are introduced.[14]

2.3.2 Dual and Primal-Dual Methods

For the primal problem

$$\text{maximize } \pi(x;p,\alpha) \equiv pF(x) - \sum_{i=1}^{N} \alpha^i w^i(x^i)x^i \,, \qquad \ldots\ldots(P)$$

where $\pi(x;p,\alpha)$ is linear homogeneous in (p,α), define the dual profit function

$$\pi(p,\alpha) \equiv \left\{\text{all } (\max_x \{\pi(x;p,\alpha) : p,\alpha\varepsilon P^0\right\}$$

where P^0 denotes the domain of (p,α). As in the competitive case, $\pi(p,\alpha)$ is convex and linear homogeneous in (p,α).

It appears that in general a second order approximation (in x) of $\pi(x;p,\alpha)$ at any solution $x^*(p,\alpha)$ to p can be constructed from $\pi(p,\alpha)$[15] as in the competitive case.[16] This suggests that, in principle, restrictions on F(x) can be incorporated into a dual approach to comparative statics when factor prices are endogenous to the firm.

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[13]See Section 2.1 of Appendix 3.

[14]For some idea of the complexity of primal methods in such cases, and of the ease with which serious analytical errors can be introduced into such approaches, see Ferguson (1969), Chapter 8.

[15]See Epstein (1978).

[16]See Blackorby and Diewert (1979), and Section 1 of Appendix 4 for an alternative proof.

However, the dual approach may lose its simplicity even if it is possible to incorporate restrictions on $F(x)$ into our analysis. For example, consider the standard assumption that $F(x)$ is concave (which is not necessarily true when $w^i_{ii}(x^i;\alpha) > 0$). The additional restrictions placed on the dual by this assumption are not obvious — convexity and linear homogeneity of $\pi(p,\alpha)$ hold irrespective of $F(x)$ concave. Thus the dual approach to comparative statics is cumbersome even though such restrictions on $F(x)$ apparently can be incorporated into the analysis.

Similar problems arise with the primal-dual method of comparative statics suggested by Silberberg (1974a). For any problem of the form

$$\text{maximize}_x \ \pi(x;\alpha) \qquad \ldots.(\text{P}')$$

define the "primal-dual" problem

$$\text{minimize}_{x,\alpha} \ L(x,\alpha) \equiv \pi(x^*(\alpha),\alpha) - \pi(x,\alpha) \qquad \ldots.(\text{P-D})$$

where $x^*(\alpha)$ denotes the solution to P' as a function of α. The second order condition for an interior solution to problem P-D is positive semi-definiteness of the Hessian matrix

$$L \equiv \left[\begin{array}{c|c} L^*_{xx} & L^*_{x\alpha} \\ \hline L^*_{x\alpha} & L^*_{\alpha\alpha} \end{array}\right]$$

where L^* is evaluated at a solution (x^*,α^*) for P-D. Silberberg shows that many standard comparative static theorems can be immediately deduced from the positive semi-definiteness of the submatrix $L^*_{\alpha\alpha} = \pi_{x\alpha}\frac{\partial x^*}{\partial \alpha}$. However, $F(x)$ concave implies simple restrictions only on F_{xx}, which appears in submatrices other than $L^*_{\alpha\alpha}$. Thus, for our purposes, methods based solely on $L^*_{\alpha\alpha}$ positive semi-definite also seem unsatisfactory.

2.3.3 Use of Aggregate (Industry) Relations

Hicks-type formulas for the industry elasticity of derived demand[19] are consistent with the assumption of variable factor prices at the industry level; but these formulas are not appropriate for the investigation of relations between elasticity of derived demand and other parameters at the firm level even under the assumption of fixed prices for the product and all other inputs. These formulas are derived independently of the second order conditions for a solution to a firm's maximization problem; whereas, the slope of a firm's derived demand schedule for input i depends entirely upon the Hessian for $\pi(x) + w^i x^i$.[20] Thus relations calculated from these formulas can be more ambiguous than the relations implied by the static maximization hypothesis. This criticism can be verified later by comparing Theorem 2 and a formula due to Andrieu (1974).[21,22]

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[19] In particular, see Hicks (1966), pp. 241-46 and Andrieu (1974).

[20] The Hicks-Andrieu formulas express the industry elasticity of derived demand as a function of parameters that (or course) do imply restrictions on the Hessian of $\pi(x) + w^i x^i$; but these restrictions will satisfy the second order conditions for a maximum only by coincidence.

[21] See Appendix 2. The example (where firm and industry analyses are equivalent) shows that the criticism applies at the industry as well as firm level.

[22] Diewert (1978) includes an analysis of industry derived demand in terms of duality theory. Since the assumptions employed there essentially imply integrability (Epstein, 1978), the analysis can be "collapsed" to the firm level without encountering the criticism levelled here against interpreting Hicks-type formulas at the firm level. However, it has already been noted that a dual approach seems inappropriate for analyzing the implications of restrictions on the production function F(x) in the case of variable factor prices (see Section 2.3.2).

2.4 Extensions to the Theory of Derived Demand

Here we present extensions to the theory of derived demand that was summarized in Section 2.2. These extensions, which relax the assumption of fixed factor prices, are developed from the equivalence between a derived demand schedule and a schedule of shadow prices for the input. This equivalence is obviously true and underlies the textbook statement that factor market equilibrium occurs at an intersection of factor demand and supply schedules. Nevertheless this principle does not appear to have been incorporated previously into formal analysis of the theory of derived demand.

The main points that are established here can be summarized as follows:

1. A derived demand schedule for a firm and a construction relating the exogenous quantity and corresponding shadow price for an input are equivalent under very general conditions (viz., under essentially all conditions where a derived demand schedule can be defined).
2. (Using 1) in the absence of "distortions" in the economy and the presence of a shift in the supply schedule of factor i for a firm or group of firms, the change in surplus in the firm or industry's market for input i is always identical to the corresponding change in producer plus consumer surplus;
3. (Using 1) the firm's derived demand schedule for input i always intersects the firm's marginal factor cost schedule for input i "from above" at a (generally unique) equilibrium level of input i; and
4. (Using 1 and 3) the firm's derived demand schedule for input i can be positively inclined over some $\{x^i\}$ even given decreasing returns to scale (provided that some isoquants are not convex), and is positively inclined

over all $\{x^i\}$ given increasing returns to scale.

Statements 2-3 are "intuitively obvious" applications of statement 1. Nevertheless, statement 3 has not been proved previously and statement 2 has even been the subject of some controversy in the literature. Statement 2 is slightly more general than a theorem presented in Panzar and Willig (1978).[23]

The main implications of these theorems for a methodology of evaluating community pasture programs can be summarized as follows. First, by statements 1-2, the farm value of the community pastures program and — in the absence of market "distortions" — the related change in consumer plus producer surplus can be measured directly in the factor market for pasture.[24] Second, by statements 3-4, in the absence of knowledge about the firm's production function we can only infer that the community pasture program does not lead to a comparative static decrease in the level of pasture employed by the firm, i.e., we cannot infer a finite upper bound for the pasture. Thus, even assuming that the ratio of pasture to beef output does not decrease, we cannot infer a finite upper bound for the comparative static change in beef output.

2.4.1 Notation and Definitions

The following notation, definitions and conditions are used in the theorems presented here.

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[23]See Section 2.2.1.

[24]This statement assumes that community pasture provides the same services as other types of pasture. In fact, a community pasture typically employs a rider to move and watch over cattle. This difference is incorporated into the model that is summarized in Chapter 4 and the related appendix.

2.4.1.1 A Definition of Derived Demand

In most analyses of derived demand, where the factor supply price schedule to be varied is defined as a price exogenous to the firm, there is no need to distinguish between factor supply price and marginal factor cost in defining a derived demand relation. However, an endogenous supply price implies that factor price and marginal factor cost are not necessarily equal. Since the possibility of a divergence between factor supply price and marginal factor cost is to be incorporated into our modelling, we must distinguish between the two in defining a derived demand relation. A firm's derived demand schedule for an input is defined here as the set of pairs of equilibrium quantity and marginal factor cost (for the input) which are obtained by varying the total cost schedule of the input in an otherwise unchanged producer maximization problem.

To be more precise, let

$x \equiv$ $N\times 1$ vector of activity levels for the N inputs of a firm

$c^i(x) \equiv$ total cost schedule to the firm for its i'th input[25]

$c^1(x;\alpha) \equiv$ total cost schedule to the firm for its input 1, as a function of x and a parameter α

$y \equiv$ $M\times 1$ vector of activity levels for the M outputs of the firm

---

[25] If $c^i(x)$ is function only of the level of employment of input i by the firm, then $c^i(x)$ can have the following forms: $w^i x^i$ (supply price exogenous), $s^i(x^i)x^i$ (supply price endogenous, and, in general, supply price does not equal marginal factor cost), and $\int_0^{x^i} s^i(\hat{x}^i)d\hat{x}^i$ (supply price endogenous, and supply price of ith unit equals marginal factor cost).

$y = f(x) \equiv$ production function (vector-valued for $M>1$) for the firm

$b(y) \equiv$ total benefits schedule to the firm as a (scalar-valued) function of its M outputs

$R(x) \equiv b(f(x))$, i.e., total benefits schedule to the firm as a (scalar-valued) function of its N inputs

$x^* \equiv$ vector of the N input levels employed by the firm at a solution to a maximization problem

Then a firm's static maximization problem can be defined as follows.

Definition 1. A producer problem P is defined as

$$\text{maximize } \pi(x) \equiv R(x) - c^1(x;\alpha) - \sum_{i=2}^{N} c^i(x) \qquad \text{. . . . (P)}$$

for a particular value of the exogenous variable α, and the solution set to this problem is denoted as $\{x^{*P}(\alpha)\}$.[26]

Given this definition of a firm's static maximization problem, the firm's derived demand schedule for any input 1 can be defined as follows.

Definition 2. The firm's derived demand schedule for input 1 is defined as

$$\{(x^{1*P}(\alpha),\ MFC^1(\alpha)) \quad \text{for all} \quad \alpha\} \equiv D^P$$

where $MFC^1(\alpha) \equiv \dfrac{\partial c^1(x^{*P}(\alpha);\alpha)}{\partial x^1}$. Denote the relation defined by the pairs in D^P as $p^1 = p^1(x^1)$.

[26] It can be shown that this model of a firm's static maximization problem formally applies to both single-enterprise and multi-enterprise models (e.g., see footnote to Theorem 1).

The derived demand schedule is expressed in the form of the relation $p^1 = p^1(x^1)$, which is the inverse of the usual form, for the following two reasons: in this manner the derived demand relation is defined as a function rather than as correspondence,[27] and this form of the relation emphasizes the shadow price interpretation of a derived demand schedule.[28]

2.4.1.2 A Shadow Price Relation Similar to Derived Demand

Let

$\overline{x^1}$ $\equiv$ an exogenously determined level of input 1 employed by the firm.

In the following problem, a quantity constraint rather than a price constraint is associated with input 1. This device will be useful later in developing properties of derived demand schedules (Definition 2).

Definition 3. A producer problem Q is defined as

$$\text{maximize } \pi(x)^Q \equiv R(x) - \sum_{i=2}^{N} c^i(x)$$

$$\text{subject to } x^1 = \overline{x^1} \qquad \ldots\ldots(Q)$$

for a particular value of the exogenous variable $\overline{x^1}$, and the solution set to this problem is denoted as $\{x^{*Q}(\overline{x^1})\}$.

Then the following relation can be formulated.

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[27]See Corollary 2.

[28]See Theorem 1.

Definition 4. The firm's shadow price schedule for input 1 is defined as

$$\left\{ \left(\overline{x^1},\ \frac{\partial\, \pi(x^{*Q}(\overline{x^1}))^Q}{\partial \overline{x^1}}\right) \text{ for all } \overline{x^1} \in X^1 \right\} \equiv D^Q$$

where

$$X^1 \equiv \left\{ x^{1*P}(\alpha) \text{ for all } \alpha \right\}.$$

The derivative $\frac{\partial\, \pi(x^{*Q})^Q}{\partial \overline{x^1}}$ is simply the change in the solution value of the objective function for a problem Q that results from a small change in the exogenous parameter $\overline{x^1}$.

In addition, we can define "corresponding" problems P and Q as follows.

Definition 5. Any particular problem P

$$\text{maximize } R(x) - c^1(x;\alpha) - \sum_{i=2}^{N} c^i(x)$$

is said to "correspond" with a problem of the form Q

$$\text{maximize } R(x) - \sum_{i=2}^{N} c^i(x)$$

$$\text{subject to } x^1 = \overline{x^{1*P}}\ ,$$

where x^{1*P} is an element of a solution for the problem P.

2.4.1.3 List of Major Assumptions

The following assumptions will be made at various times in the theorems to be presented in this chapter.

Condition 1. For any solution x^* to a problem P: $x^{i*} > 0$, $i = 1, \cdots, N$.

Condition 2. In the neighborhood of any solution to a problem P: $R(x)$ and all $c^i(x)$ are twice differentiable.

Condition 3. $c^1 \equiv c^1(x^1; \alpha)$, i.e., the total cost of input 1 to the firm is independent of the levels of inputs $1, \cdots, N$ employed by the firm.[29]

Condition 4. $\dfrac{\partial^2 c^i(x)}{\partial x^j \partial x^k} \geqq 0$ for all x and $i,j,k = 1, \cdots, N$,

i.e., factor supply prices are non-decreasing in x.

Condition 5. $\dfrac{\partial R(x)}{\partial x^i} \geqq 0$ for all x and $i = 1, \cdots, N$,

i.e., inputs are "freely disposable."

Condition 6. If the set of attainable $\pi(x)^Q$ for a problem Q is bounded from above, then the set is also closed from above.

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[29] For our purposes Condition 3 is the most important of these assumptions and it is "generally" correct. Examples where Condition 3 is likely to be violated include (a) the firm in question is a monopsonist in markets for input 1 and another input, which are supplied by a single industry, and (b) input 1 is an intermediate product of the firm (so that the cost of producing input 1 depends on the level of all inputs employed in producing input 1).

Condition 6 simply rules out the unlikely possibility that maximum $\pi(x)^Q \rightarrow K$ (a real number), i.e., the set of attainable $\pi(x)^Q$ for a problem Q is bounded but not closed from above.

2.4.2 Derived Demand as a Schedule of Shadow Prices

Given that $c^1 \equiv c^1(x^1;\alpha)$, i.e., that the total cost of input 1 to the firm is independent of the levels of inputs $2,\cdots,N$ employed by the firm, the firm's derived demand schedule D^P and schedule of shadow prices D^Q for input 1 are equivalent. On the other hand given that $c^1 \equiv c^1(x;\alpha)$, i.e., that the total cost of input 1 to the firm is not independent of the levels of inputs $2,\cdots,N$ employed by the firm, in general D^P and D^Q for input 1 are not equivalent.

These relations between derived demand schedules and shadow price schedules are stated more precisely as Theorem 1 and Corollary 1, respectively. Theorem 1-A is obviously true, and Theorem 1-B follows from 1-A plus the envelope theorem.[30]

The properties of a derived demand schedule listed in Corollary 2 are deduced from Theorem 1. Note that $p^1 = p^1(x^1)$ is a function rather than a correspondence (Corollary 2-B). In addition, the domain of $p^1 = p^1(x^1)$ is a convex set, and $p^1(x^1)$ is continuous and differentiable within its domain.[31]

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[30]The essential points of the proof can be summarized as follows. $c^1 \equiv c^1(x^1;\alpha)$ (Condition 3) implies that input 1 can be fixed at the equilibrium level(s) x^{1*P} and this factor supply schedule can be removed from the maximization problem P without affecting the solution(s) x^{*P}, which establishes Theorem 1-A. Then, by the envelope theorem (i.e., given an infinitesimal change in an exo genous variable, the change in the value of the objective function when all endogenous variables vary optimally is equal to the change when all endogenous variables remain fixed), Theorem 1-B is established.

[31]Theorem 1 and Corollary 2 will be useful in proving the remaining theorem and corollaries in this chapter.

Theorem 1 and Corollary 1 provide the following rationale for employing Condition 3, i.e., $c^1 \equiv c^1(x^1;\alpha)$, in any study of the properties of derived demand schedules. Since the shadow price schedule D^Q is by definition invariant to changes in the supply schedule for input 1, Theorem 1 implies that the derived demand schedule D^P is invariant to changes in the supply schedule of input 1 given only that $c^1 \equiv c^1(x;\alpha)$. Since there is no economy in defining a derived demand schedule D^P for each possible specification of the supply schedule for the input, we shall restrict our study of derived demand schedules to cases where $c^1 \equiv c^1(x^1;\alpha)$.

Theorem 1. Suppose that conditions 1-3 are satisfied. Then

(A) $\{x^{*P}(\alpha)\} \Longleftrightarrow \{x^{*Q}(x^{1*P}(\alpha))\}$ for all α,

i.e., any problem P and the corresponding problem(s) Q have identical solution sets; and

(B) $\{(x^{1*P}(\alpha),\ MFC^1(\alpha))$ for all $\alpha\} \Longleftrightarrow \{(x^{1*P}(\alpha),$

$\dfrac{\partial \pi(x^{*Q}(x^{1*P}(\alpha)))^Q)}{\partial x^1}$ for all $\alpha\}$,

i.e., $D^P \Longleftrightarrow D^Q$.[32]

---

[32] Formally Theorem 1 only applies to the case where input 1 is employed in a single enterprise, since the cost schedule for input 1 is defined as a function of only one input. However, Theorem 1 generalizes to the firm that employs input 1 in M enterprises. In this case, we can define $c^1 \equiv c^1(\sum_{j=1}^{M} x^{1j};\alpha)$ and the quantity constraint in a corresponding problem Q as $\sum_{j=1}^{M} x^{1j} = \overline{x}^1$. It is easily shown that, with these modifications, Theorem 1 applies to the multi-enterprise firm as well as to the single enterprise firm.

Corollary 1. Suppose that conditions 1-2 are satisfied, and that for all α:

$$\frac{\partial c^1(x^{*P}(\alpha);\alpha)}{\partial x^1} \neq 0 \quad \text{for at least one } i \neq 1. \quad \text{Then}$$

(A) $\{x^{*P}(\alpha)\} \cap \{x^{*Q}(x^{1*P}(\alpha))\}$ = null set for all α

i.e., any problem P and any corresponding problem Q do not have any solutions in common; and

(B) for any α: $\{(x^{1*P}(\alpha), MFC^1(\alpha))\} \cap D^Q \neq$ null set if and only if

$$\frac{\partial R(x^{*P})}{\partial x^1} - \sum_{i=2}^{N} \frac{\partial c^i(x^{*P})}{\partial x^1} = \frac{\partial R(x^{*Q})}{\partial x^1} - \sum_{i=2}^{N} \frac{\partial c^i(x^{*Q})}{\partial x^1}$$

for all (or, equivalently, any) $x^{*P} \in \{x^{*P}(\alpha)\}$

$x^{*Q} \in \{x^{*Q}(x^{1*P}(\alpha))\}$.[33]

Corollary 2. Suppose that conditions 1-3 and 5-6 are satisfied, and denote the domain of $p^1 = p^1(x^1)$ as X^D. Then

(A) if x^{1B} is included in a solution to at least one problem P, then all x^{1A} such that $0 < x^{1A} < x^{1B}$ are in X^D:

(B) p^1 is a function of x^1, i.e., $p^1 = p^1(x^1)$ associates one and only one p^1 with any particular x^1 in X^D; amd

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[33] Corollary 1-B also assumes that conditions 5-6 are satisfied.

(C) $p^1(x^1)$ is differentiable for all x^1 "within" X^D, i.e., for all x^1 such that $0 < x^1 < x^{1A}$ and x^{1A} is an element of X^D.

2.4.3 Relation Between Surpluses in Factor and Product Markets

As was stated in Section 2.2.1, previous literature has established under special conditions the equivalence of measures of surplus in a factor market and measures of producer plus consumer surplus. These analyses have assumed that factor supply schedules at both the firm and industry level are either perfectly elastic or perfectly inelastic.

We shall establish this equivalence under general conditions by direct application of Theorem 1. The analysis has the following implications for a methodology of evaluating various programs that directly shift factor supply schedules: under quite general conditions, the user value and (in the absence of distortions in other markets) change in consumer plus producer surplus can be measured in commercial markets for the input.

Since distortions are common and commercial markets for pasture are uncommon in British Columbia, the preceding comments do not apply to the evaluation of community pasture programs. Nevertheless, the results presented here support the approach adopted by Barichello (1978): estimating the farm value of the program by collecting data from the commercial market for an alternative use of improved land, and arriving at a measure of the change in consumer plus producer surplus by attempting to correct this value for distortions.

We shall now show that the usual conception of the relation between changes in surpluses measured in factor and product markets is correct — provided simply that each firm's total cost schedule for the input in question is independent of the levels of other inputs employed by the firm (condition 3). For the case where the shift in a factor supply schedule is limited to a single firm,[34] this will involve demonstrating that the change in surplus calculated in the firm's input market represents the general equilibrium benefits to the firm resulting from the shift in the factor supply schedule (given condition 3), and then noting the conditions under which these private benefits correspond to social benefits (in the sense of consumer plus producer surplus). For the case where the shift in a factor supply schedule is experienced by all firms in a group (e.g., an industry) we need only note (in addition to the above) that an industry demand schedule for an input is a collection of price and quantity combinations for derived demand schedules of individual firms.

2.4.3.1 Shift in Factor Supply Schedule of Single Firm

Given Theorem 1, it seems intuitively obvious that the change in surplus between a firm's derived demand and supply schedule for an input, due to a shift in the supply schedule of that input, is equal to the associated change in equilibrium net benefits for the firm. Likewise, we can prove Corollary 3 directly from Theorem 1.

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[34]As was stated in Section 2.2.1, this is essentially the case considered by previous studies. In other words, by ignoring all interactions between firms and by assuming the existence of an aggregate production function, the "industry" analyses in previous studies were equivalent to analyses of a single firm.

Corollary 3. Suppose that conditions 1-3 and 5-6 are satisfied. Then

(A) for any solution x^{*A} to a problem P where $\alpha = \alpha^A$,

$$\pi(x^{*A}) = \pi(x^*(0))^Q + \int_0^{x^{1*A}} p(x^1)dx^1 - c^1(x^{1*A};\alpha^A)$$

where

$$\pi(x^*(0))^Q \equiv \max\{R(x) - \sum_{i=2}^{N} c^i(x) : x^1 = 0\}$$

$$p^1(0) \equiv \left.\frac{\partial \pi(x^*(0))^Q}{\partial x^1}\right|_+ ,$$ [36] and

(B) for a solution x^{*A} and a solution x^{*B} to two problems of the form P that differ only in terms of $\alpha = \alpha^A$ and $\alpha = \alpha^B$, respectively,

$$\pi(x^{*B}) - \pi(x^{*A}) = \int_{x^{1*A}}^{x^{1*B}} p^1(x^1)dx^1 - c^1(x^{1*B};\alpha^B)$$

$$- c^1(x^{1*A};\alpha^A) .$$

Suppose that there are no "distortions" in the economy, i.e., that

(a) marginal factor cost is always equal to factor supply price for each firm,

(b) marginal revenue is always equal to product demand price for each firm, and

(c) government taxes and subsidies are non-existent.

In addition suppose that

(d) the marginal utility of income is constant for all consumers;

so that consumer's surplus can be defined in terms of ordinary (non-compensated) product demand schedules.

This correspondence between changes in surplus in a firm's factor market and changes in comsumer plus producer surplus can be deduced from Theorem 1 and Corollary 3 as follows. By Theorem 1,

$$p^1(x^{1A}) = \sum_{h=1}^{M} \frac{\partial b(\cdot y)}{\partial y^h} \frac{\partial y^h(x^*(\overline{x^{1A}}))}{\overline{\partial x^1}} - \sum_{i=2}^{N} \frac{\partial c^i(x^*(\overline{x^{1A}}))}{\overline{\partial x^1}}.^{35} \quad \ldots .(1)$$

By (1) and assumptions (a)-(d), $p^1(x^{1A}) - \frac{\partial c^1(x^{1A};\alpha)}{\partial x^1}$ equals the difference between the dollar-equivalent benefits received by consumers from the production associated with the marginal unit of input 1 minus the supply costs incurred during this production. Since these costs are equal to the surplus foregone by employing these resources in this particular manner (given assumptions (a)-(d)), $p^1(x^{1A}) - \frac{\partial c^1(x^{1A};\alpha)}{\partial x^1}$ is equal to the change in consumer plus producer surplus resulting from the production associated with the marginal unit of input 1, irrespective of

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[35]The notation $x^*(\overline{x^{1A}})$ simply implies that $x^*(\overline{x^{1A}})$ is a solution x^* to the problem Q defined by the constraint $x^1 = \overline{x^{1A}}$, or to a corresponding problem P.

[36]Any right hand side derivative $\lim_{\Delta x^1 \to 0} \frac{f(x^1 + \Delta x^1) - f(x^1)}{\Delta x^1}$ for $\Delta x^1 > 0$ is represented here as

$$\left.\frac{\partial f(x^1)}{\partial x^1}\right|_{+} .$$

the number of outputs and inputs involved in production. Likewise, by Corollary 3,

$$\int_{x^{1A}}^{x^{1B}} [p^1(x^1) - \frac{\partial c^1(x^1;\alpha)}{\partial x^1}]\, dx^1 = \int_{\overline{x^{1A}}}^{\overline{x^{1B}}} [\sum_{h=1}^{M} \frac{\partial b(y)}{\partial y^h} \frac{\partial y^h(x^*(\overline{x^1}))}{\partial \overline{x^1}}$$

$$- \sum_{i=1}^{N} \frac{\partial c^i(x^*(\overline{x^1}))}{\partial \overline{x^1}}]\, \overline{dx^1}. \qquad \ldots .(2)$$

By (2) and assumptions (a)-(d), the surplus over the interval (x^{1A}, x^{1B}) in the firm's market for input 1 is equal to the change in consumer plus producer surplus resulting from the associated production. Therefore, any change in surplus generated in the firm's market for an input corresponds exactly to the resulting change in consumer plus producer surplus given assumptions (a)-(d) (and condition 3).

2.4.3.2 Shift in Industry Factor Supply Schedule

The above analysis can be generalized as follows to the industry case, where shifts in $c^1(x^1)$ for all J firms in the industry typically lead to shifts in other factor supply schedules and product demand schedules faced by the individual firm. Suppose that firms always face identical supply schedules for input 1. Then the set of static general equilibria across the J firms $\{x^{*1}, \cdots, x^{*J})\}$ can be expressed as a corresponsence of a single parameter α rather than of J parameters $(\alpha^1, \cdots, \alpha^J)$. If we make the further assumption that a single static general equilibrium exists for each α, then the industry factor demand schedule can be expressed as

$$\{(X^1(\alpha),\ P^1(\alpha) : \text{all } \alpha\} \qquad \ldots .(3)$$

where

$$X^1(\alpha) \equiv \sum_{j=1}^{J} x^{1*}(\alpha)^j$$

$$P^1(\alpha) \equiv \text{maximum } p^1 \text{ in } \{p^1(x^{1*}(\alpha)^j)^j : j = 1, \cdots, J\}$$

$$\equiv p^1(x^{1*}(\alpha)^M)^M \quad .$$

Thus the change in surplus in the industry market for input 1 resulting

from $\Delta\alpha = \alpha^B - \alpha^A$ can be expressed as

$$\Delta S = \int_{\alpha^A}^{\alpha^B} p^1(x^{1*}(\alpha)^M)^M d\alpha - \sum_{j=1}^{J} c^1(x^{1*}(\alpha^B)^j;\alpha^B)$$

$$+ \sum_{j=1}^{J} c^1(x^{1*}(\alpha^A)^j;\alpha^A) \ . \qquad \ldots .(4)$$

Statement (4) implies that the argument presented in the previous paragraph can be applied to the case of a surplus generated in a factor market by a number of interacting firms. Therefore, statement (4) implies that, given the assumptions of no "distortions" in the economy and of a constant marginal utility of income for all consumers, the change in surplus between the industry's demand and supply schedules for any input 1, resulting from a shift in $c^1(x^1)$ for all firms in the industry, is exactly equal to the associated change in consumer plus producer surplus over all markets.[37]

[37] If firms do not face identical supply schedules for input 1, then statement (3) does not necessarily define the industry demand schedule for input 1. However, the industry demand schedule will still be a collection of price and quantity combinations from the derived demand schedules of individual firms; so a change in surplus (correctly measured) in the factor market due to shifts in the factor supply schedules for each firm will still correspond to the associated change in consumer's surplus.

2.4.4 Slope of the Firm's Derived Demand Schedule

Here we present a corollary and a theorem concerning the slope of a derived demand schedule, and effects of a finite shift in a factor supply schedule, that are independent of any assumption of a fixed price for the input. These statements imply the following: in general, an upper bound for the comparative static increase in the quantity of total pasture employed by a single recipient of community pasture cannot be deduced without incorporating empirical knowledge of the firm's production function into the analysis. The reason for this negative result is that the firm's derived demand schedule can be positively inclined under conditions that are reasonable *a priori*. Thus, even if we assume that the community pasture program does not increase the ratio of beef output to the quantity of pasture employed by the user, we cannot deduce an upper bound for the comparative static change in beef output without incorporating empirical knowledge of the firm's production function into the analysis

2.4.4.1 Relation Between Slopes of Derived Demand and Factor Supply Schedules at Equilibrium

Textbook diagrams typically show that a firm's derived demand schedule intersects the supply schedule for the input 'from above' at an equilibrium in the factor market, and the analogue of this condition in the product market was proved long ago.[38] Nevertheless, this statement

[38] For example, see Samuelson (1947), pp. 76-77.

apparently has not been proved in the past. Given the fundamental nature of this proposition and the controversy that eventually arose over the "obvious" relation between surpluses in factor and product markets, a proof of this statement appears desirable.

Given Theorem 1 and conditions 1-3, it seems intuitively obvious that the following condition

> $p^1(x^1)$ intersects $\frac{\partial c^1(x^1;\alpha)}{\partial x^1}$ from above at x^{1A}, or $p^1(x^1)$ coincides with $\frac{\partial c^1(x^1;\alpha)}{\partial x^1}$ at x^{1A} and some level x^1 in the neighborhood of x^{1A} is necessary for activity level x^{1A} to

be included in a global solution to the firm's maximization problem P and (almost) sufficient for x^{1A} to be included in a local solution to P. Likewise, Corollary 4 can be proved from Theorem 1.

Corollary 4. Suppose that conditions 1-3 are satisfied for a problem P.

(A) If x^A is included in a local solution to P, then

$$p^1(x^{1A}) = \frac{\partial c^1(x^{1A};\alpha)}{\partial x^1}$$

$$\frac{\partial p^1(x^{1A})}{\partial x^1} \leq \frac{\partial^2 c^1(x^{1A};\alpha)}{\partial x^{1\,2}} .$$

(B) If $p^1(x^{1A}) = \frac{\partial c^1(x^{1A};\alpha)}{\partial x^1}$

$$\frac{\partial p^1(x^{1A})}{\partial x^1} < \frac{\partial^2 c^1(x^{1A};\alpha)}{\partial x^{1\,2}} ,$$

then x^{1A} is included in a local solution to P.

2.4.4.2 Relation Between Slope of Derived Demand Schedule and Various Properties of the Production Function

The relations between slopes of derived demand schedules and certain additional properties of the firm's maximization problem (especially returns to scale in production, and convexity or non-convexity of isoquants) can be developed essentially from Theorem 1 and Corollary 4.[39] These relations are summarized as Theorem 2. By statements A and B of the theorem, the firm's derived demand schedule for input 1 is never positively inclined when $R(x)$ is concave, irrespective of the slope of $c^1(x^1)$. By statement D, the firm's derived demand schedule is perfectly elastic when $R(x)$ shows constant returns to scale and the price for each input (other than 1) is exogenous to the firm, irrespective of substitution possibilities (shape of isoquants) and the slope of $c^1(x^1)$.

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[39] By means of Corollary 4, we are able to equate the comparative static problem of determining the direction of *change in equilibrium* level of input 1, resulting from a change in the factor cost schedule $c^1(x^1)$, to a problem of determining the *existence of an equilibrium* for particular specifications of $c^1(x^1)$. Since set-theoretic concepts, such as quasi-concavity and returns to scale, are readily incorporated into analyses of the existence of equilibrium, we are able to relate the direction of slope of an individual firm's derived demand schedule to such properties by these methods. An overview of the method of proof for Theorem 2 (as well as the proof itself) is presented in Appendix 2.

Theorem 2. Suppose that conditions 1-6 are satisfied. Denote the domain of $p^1(x^1)$ as X^D, and denote a wage or rental rate that is exogenous to the firm as $\overline{w^i}$. Then the slope of the firm's derived demand schedule is related to certain properties of $R(x)$ and $c^i(x)$ $(i = 2, \cdots, N)$ as follows:

(A) If $R(x)$ is strictly concave,[40] then $\frac{\partial p^1(x^1)}{\partial x^1} \leqq 0$ and $p^1(x^1) > p^1(x^1+e)$ for all (x^1, x^1+e) in X^D, where $e > 0$.

(B) If $R(x)$ is concave,, then $\frac{\partial p^1(x^1)}{\partial x^1} \leqq 0$ for all x^1 in X^D.

(C) If $R(\lambda x) \leqq \lambda R(x)$ for all $\lambda > 1$ and $x \geqq 0$ but $R(x)$ is not concave, then:

(1) $\frac{\partial p^1(x^1)}{\partial x^1} \leqq 0$ always for at least some x^1 in X^D but

(2) for some $R(x)$ and $\sum_{i=2}^{N} c^i(x)$: $\frac{\partial p^1(x^1)}{\partial x^1} > 0$ for some x^1 in X^D.

(D) If $R(\lambda x) = \lambda R(x) = \lambda R(x)$ for all $(x, \lambda) \geqq 0$ and $c^i \equiv \overline{w^i} x^i$ for $i = 2, \cdots, N$, then $\frac{\partial p^1(x^1)}{\partial x^1} = 0$ for all x^1 in X^D.

(E) If $R(\lambda x) > \lambda R(x)$ for all $\lambda > 1$ and $x > 0$ and $c^i \equiv \overline{w^i} x^i$ for $i = 2, \cdots, N$, then $\frac{\partial p^1(x^1)}{\partial x^1} \geqq 0$ and $p^1(x^1) < p^1(x^1+e)$ for all $(x^1, x^1 + e)$ in X^D, where $e > 0$.[41,42] (on following page)

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[40] The firm's total benefits function $R(x)$, which is simply a total revenue function if the firm maximizes profits, is strictly concave if and only if (1) $R(\lambda x) < \lambda R(x)$ for all $\lambda > 1$ and $x > 0$, and (2) all isoquants of $R(x)$ are strictly convex for $x \geqq 0$. Likewise, $R(x)$ is concave if and only if (1) $R(\lambda x) \leqq \lambda R(x)$ for all $\lambda > 1$ and $x > 0$, and (2) all isoquants of $R(x)$ are convex for $x \geqq 0$.

However, a derived demand schedule may be positively inclined under many possible conditions. By statement C, a derived demand schedule may be positively inclined at some points even if R(x) shows decreasing returns to scale, provided that at least some isoquants are not convex. By statement E, a derived demand schedule is positively inclined over the entire domain when R(x) shows increasing returns to scale and the prices of all other inputs are exogenous, irrespective of substitution possibilities for the firm.

Theorem 2 has important implications for the evaluation of community pasture programs. By Theorem 2: a positively inclined derived demand schedule is consistent with the notion of equilibrium for the firm, provided that R(x) shows increasing returns to scale or non-convex isoquants.

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[41]Overlooking mathematical details (concerning inflection points) that are of no economic significance, conditions in A and E imply that

$$\frac{\partial p^1(x^1)}{\partial x^1} < 0 \quad \text{and} \quad \frac{\partial p^1(x^1)}{\partial x^1} > 0 \quad ,$$

respectively.

[42]Note the asymmetry between statements C and E: $p^1(x^1) \gtrless p^1(x^1+e)$ for decreasing returns to scale and fixed factor prices $(i \neq 1)$ whereas, $p^1(x^1) < p^1(x^1+e)$ for increasing returns to scale and fixed factor prices $(i \neq 1)$, where $e > 0$.

Neither of these conditions can be ruled out *a priori*.[43] Therefore, even if we assume that the firm is at long run equilibrium before and after the introduction of the community pasture programs, we cannot rule out *a priori* the possibility than an inframarginal shift, of magnitude x^{cp} units, in the firm's supply schedule of pasture leads to an increase in the level of pasture employed by the firm that is greater than x^{cp} (see Figure 1).

In addition, we can demonstrate that, for a downward shift in a firm's factor supply schedule $c^1(x^1)$,

$$\frac{dx^{1*}}{x^{1*}} \gtreqless \frac{dR(x^*)}{R(x^*)}$$

depending on properties of the particular maximization problem.[44] In other words, under reasonable assumptions the percentage change in (farm value of) output can be either more or less than the percentage change in total pasture that is due to the community pasture program. This defines a second source of difficulty in calculating a finite upper bound for the comparative static change in beef output associated with community pasture programs (in addition to problems in calculating a finite upper bound for the change in total pasture).

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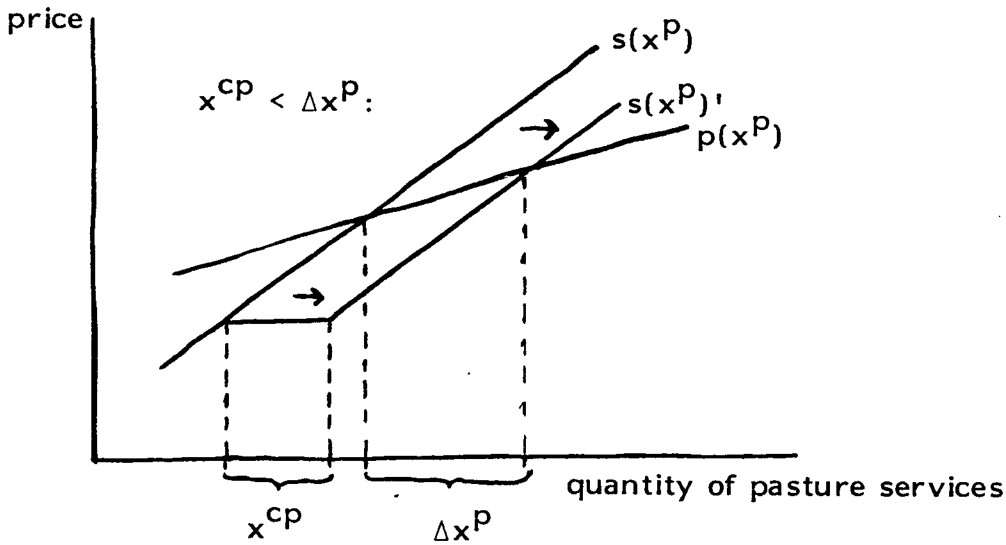
[43]If the supply schedule for any input is positively inclined, then R(x) may show increasing returns to scale and/or non-convexity of isoquants in the neighborhood of an interior long run equilibrium, i.e., these conditions are consistent with the maximization hypothesis. Moreover, increasing returns to scale and non-convexity of isoquants cannot be ruled out *a priori* as "unreasonable" properties of a production process relevant to comparative static modelling. The argument can be summarized as follows Divisibility of the production process would imply decreasing or constant returns to scale, and additivity and divisibility together would imply convexity of isoquants (and also constant returns to scale) (see Malinvaud, 1972, pp. 51-3 for definitions). However, divisibility may be unrealistic, and additivity is reasonable only for models of change in *long run* equilibrium (in any "short run," certain changes in input levels are likely to be, in effect, infeasible due to high adjustment costs).

[44](on the following page)



x^{cp}	$\equiv$	number of units of community pasture alloted to the firm
Δx^p	$\equiv$	change in the number of units of pasture employed by the firm (due to the community pasture program)
$p(x^p)$	$\equiv$	the firm's derived demand schedule for pasture
$s(x^p)$	$\equiv$	the firm's supply schedule for pasture, prior to the community pasture program
$x(x^p)'$	$\equiv$	the firm's supply schedule for pasture, as a result of the community pasture program

Figure 1 Given a Positively-inclined Derived Demand Schedule for Pasture, the (inframarginal) Allotment of Community Pasture is Less than the Resulting Change in the Level of Pasture.

2.5 Summary of Implications for the Evaluation of Community Pasture Programs

In this section, we summarize the major implications of the theory presented in this chapter for evaluations of community pastures programs. The restrictions on comparative static effects of community pastures programs that are implied by this theory have been shown to be extremely weak. Indeed, these restrictions may be considerably weaker than the reader had previously considered possible, or at least reasonable. Thus, the discussion here should help us to avoid errors in our *a priori* theorizing about "likely" comparative static effects of community pastures programs, and underscores the importance of incorporating into our analyses greater knowledge of the producer problem(s) faced by users of community pasture.

2.5.1 Relation Between Surpluses in Factor and Product Markets

The analysis of the relation between surpluses in factor and product markets (Sections 2.2.1 and 2.4.3) implies that, for various programs shifting factor supply schedules, the user value and (overlooking

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[44]Silberberg (1974b) shows that $\frac{dx^{1*}}{x^{1*}} > \frac{dR(x^*)}{R(x^*)}$ for firms that minimize average cost (as in long run competitive equilibrium). Assuming that the number of inputs equals 2 and that $c^2 \equiv \overline{w}^2x^2$, we can also show that $\frac{dx^{1*}}{x^{1*}} > \frac{dR(x^*)}{R(x^*)}$ if $R(x^1,x^2)$ is homogenous of degree less than 1 and all isoquants are convex, and that $\frac{dx^{1*}}{x^{1*}} < \frac{dR(x^*)}{R(x^*)}$ if $R(x^1,x^2)$ is homogeneous of degree greater than 1.

distortions in other markets) change in consumer plus producer surplus could be measured directly in commercial markets for the input (see Figure 2). This analysis, plus the presence of market distortions and absence of commercial markets for pasture in British Columbia, suggests the following approach to the evaluation of community pasture programs. The farm value of the program is estimated from data concerning the commercial market for an alternative use of improved land (e.g., as hay land), and the related change in consumer plus producer surplus is estimated as this farm value plus or minus corrections for distortions.[45]

2.5.2 Slope of Derived Demand Schedule and the Measurement of Distortions

However, the theory presented in this chapter does not provide any useful restrictions concerning the comparative static effects of community pasture programs on distorted markets. In particular, the theory presented in this chapter does not provide any useful restrictions concerning the comparative static change in beef output for the representative farm. Indeed, this theory does not determine an upper bound for the change in total pasture employed by the farm, which in itself precludes the calculation of an upper bound for the change in beef output for the farm. This last statement can be elaborated upon as follows.

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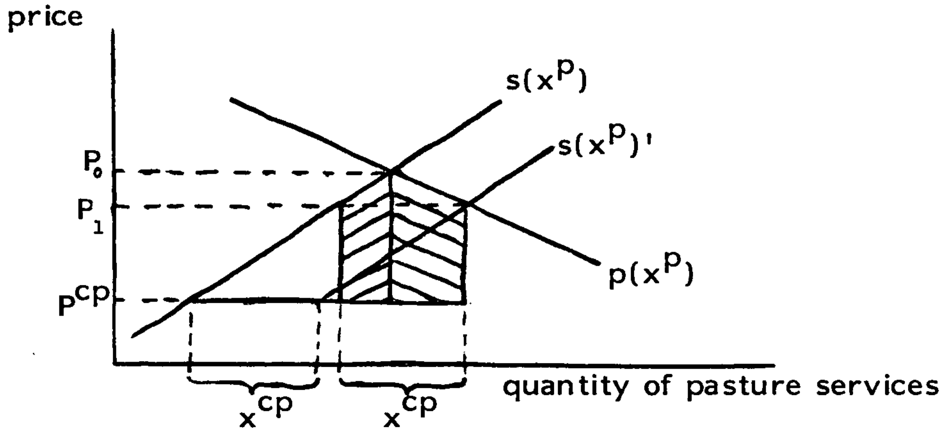
[45]See Barichello (1978).



x^{cp} ≡ number of units of community pasture alloted to the farm

p^{cp} ≡ rental price of community pasture

$p(x^p)$ ≡ firm's derived demand schedule for pasture

$s(x^p)$ ≡ firm's supply schedule for pasture, prior to community pasture program

$s(x^p)'$ ≡ firm's supply schedule for pasture, as a result of community pasture program

p_0 ≡ equilibrium price of pasture, prior to community pasture program

p_1 ≡ equilibrium price of pasture, as a result of community pasture program

 ≡ net user benefits associated with the increase in employment of pasture

 ≡ net user benefits associated with the land freed from use as pasture

Figure 2 Estimating the User Value of Community Pasture Programs in a Market for Pasture.[46]

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[46]This diagram is discussed in Barichello (1978), pp. 28-32.

2.5.2.1 General Case

As shown in Theorem 2, derived demand schedules can be positively inclined under reasonable conditions. In the absence of considerable knowledge of the farm's production function, we can only infer than the derived demand schedule for pasture cuts the supply schedule for pasture "from above" (Corollary 4). However, this condition seldom places any useful restrictions on the change in total pasture employed by the farm.

Assume, for simplicity, that the aggregate cost schedule of pasture to the farm can be written as $c^1 = c^1(x^1;\alpha)$, where x^1 denotes the total quantity of pasture employed by the farm. The community pasture can be represented as a small change in the parameter α that leads to a decrease in the marginal cost of pasture for some levels of x^1, i.e., $\frac{\partial^2 c^1(x^1;\alpha)}{\partial x^1 \partial \alpha} < 0$ for some x^1 and $\frac{\partial^2 c^1(x^1;\alpha)}{\partial x^1 \partial \alpha} = 0$ for all other x^1. If this cost schedule is continuous at the farm's pre-community pasture equilibrium x* and the farm employs pasture at this equilibrium, then the following condition is satisfied:

$$p^1(x^{1*}) - \frac{\partial c^1(x^{1*};\alpha)}{\partial x^1} = 0 \quad . \qquad \ldots\ldots(5)$$

Totally differentiating (5) with respect to α for the change in the level of total pasture employed by the farm:

$$\frac{\partial x^{1*}}{\partial \alpha} = \frac{-\dfrac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha}}{\dfrac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^{1\,2}} - \dfrac{\partial p^1(x^{1*})}{\partial x^1}} \qquad \ldots\ldots(6)$$

Therefore, given (6) and $-\dfrac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} \geqq 0$, the condition

$\dfrac{\partial p^1(x^{1*})}{\partial x^1} < \dfrac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^{1\,2}}$ simply implies the restriction

$$0 \leqq \frac{\partial x^{1*}}{\partial \alpha} \lesseqgtr +\infty \qquad \ldots .(7)$$

which is not helpful. By Theorem 2, a restriction stronger than (7) cannot be obtained without knowledge of the farm's production function.

2.5.2.2 Special Cases

Nevertheless, note that we could obtain meaningful results if the community pasture program did not affect the marginal cost of pasture at equilibrium:

$$\frac{\partial x^{1*}}{\partial \alpha} = 0 \quad \text{given} \quad \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} = 0 \; .$$

This case is accurate when the farm can rent additional pasture or hay land at a constant price at equilibria both before and after introduction of the community pastures program.

However, commercial rental of pasture does not appear to occur in the Peace River and Prince George regions, and appears to be accurate for only a minority of beef ranches in other areas. Moreover, rental of hay land seems uncommon in Prince George, and the rental price of hay land seems essentially endogenous to the firm in the Peace River.

Allowing for discontinuities in the aggregate supply schedule of pasture to the farm implies only one modification of the above conclusions. If the supply curve is vertical at the equilibria established both before and after introduction of the community pastures program, then the resulting change in total pasture is identical to the quantity of community pasture rationed to the farm by the program. Such a supply schedule may be accurate for some short run producer problems P.

However, farms using community pasture in the Peace River and Prince George regions typically allocated own land to both pasture and hay (and often grain as well) prior to and after introduction of the pastures program. This suggests that the assumption of a vertical supply schedule is inappropriate even in the short run for these evaluations.

2.5.3 Further Research

In sum, we have shown that we cannot obtain useful restrictions on the comparative static changes in pasture (and in beef output) in the absence of knowledge about the production function of a farm. However, our knowledge of such production functions is uncertain. Moreover, this knowledge is largely expressed in terms of parameters (e.g., a set of "reasonable" values for a factor substitution effect) that have not been directly incorporated into the traditional methods of qualitative comparative statics.

In the next chapter, we shall develop a technique for incorporating such knowledge into comparative static analysis. In principle, this method will enable us to place many restrictions on the structure of a producer problem P without at the same time specifying the entire structure of the problem.

CHAPTER 3

QUANTITATIVE COMPARATIVE STATICS AND DERIVED DEMAND: A PROPOSED METHODOLOGY

CHAPTER 3

QUANTITATIVE COMPARATIVE STATICS AND DERIVED DEMAND: A PROPOSED METHODOLOGY

3.1 Introduction

In this chapter, we shall introduce a method of "quantitative" comparative statics that is designed to incorporate many empirically-based restrictions into the theory of the firm without specifying essentially unknown parameters. A detailed presentation of the method necessarily includes many equations, and computational experience to date has been minor. For these reasons, details of structure and means of reducing computational problems have been relegated to appendices.

It has been the author's conviction that, in the initial stage, research related to this method of quantitative comparative statics should emphasize clarification of logical structure and means of facilitating computation rather than the collection of numerical results. The theory and methods to be presented in this chapter and related appendices suggest that this methodology will be useful in predicting farm or firm response in various policy situations. Likewise, this methodology may well be useful in generating testable hypotheses of farm or firm behavior.

3.1.1 The Problem

In the previous chapter, we stated that the traditional methods of comparative static analysis, as embodied in Samuelson (1947) and more recent dual and primal-dual approaches, cannot readily incorporate many quantitative restrictions on production functions. We were able to incorporate various properties of production functions into our analysis of derived demand, but the results largely served to emphasize the value of including many empirically based restrictions on production functions in comparative static analysis. These results showed that, in the case of endogenous factor prices, not even the slope of the individual firm's derived demand schedule can be signed unless the analysis incorporates empirically based restrictions that are sufficient to determine convexity or non-convexity of isoquants and decreasing or increasing returns to scale. These latter properties are not easily observed directly.

In addition, we saw that the usual qualitative analysis (supplemented by empirical observation) is seldom able to place any meaningful restrictions on the comparative static effects of community pasture programs. In particular, such an analysis plus empirical observation of the factor supply schedule can seldom lead to a finite upper bound on a comparative static change in beef output (or even pasture) due to a community pasture program.

These results serve to complement previous observations on the difficulties of obtaining many useful results (signed effects) from traditional methods of comparative static analysis. It is well known that relatively few comparative static effects can be signed from the maximization hypothesis plus qualitative knowledge of the elements of the firm's Hessian $[\pi_{ij}(x^*)]$. Although signed results could be obtained by incorporating quantitative restrictions on the elements of $[\pi_{ij}(x^*)]$, there does not appear to be any empirical basis for placing such restrictive conditions directly on the elements of the Hessian $[\pi_{ij}(x^*)]$. Thus such restrictions would have to be derived indirectly from other (more empirically based) restrictions. In the absence of such restrictions, relatively few predictions of farm response or testable predictions of farm behavior can be obtained.

Given our assumption that constructs of static, optimizing behavior have utility in the applied economics of agriculture (an assumption to be verified in the next chapter), alternative methods of comparative static analysis at the firm level seem desirable. The commonly employed alternative has been to specify exactly the structure of the individual firm's problem "maximize $\pi(x)$," to compare the solutions x^* and x^{**} for two different values of the exogenous variable α, and to perform a sensitivity analysis by repeating this procedure for alternative structures $\pi(x)$. However, this second approach also has serious drawbacks in the absence of fairly complete knowledge of the correct structure for $\pi(x)$. Since the number of possible structures is infinite and the relation between structure and comparative static effects $\frac{\Delta x^*}{\Delta \alpha}$ is likely to be complex, any procedure that relies on specifying exactly alternative structures $\pi(x)$ can bound the set of "reasonable" comparative static effects only if the set of "reasonable" structures $\pi(x)$ is quite small.[1]

In general, there appears to be considerably more knowledge of the structure of $\pi(x)$ than has been incorporated into qualitative comparative static methods; but knowledge of $\pi(x)$ is far from complete. Thus there is need for a method that incorporates many restrictions on the structure of the firm's maximization problem into comparative static analysis without specifying an exact structure for $\pi(x)$. Moreover, in order to be most useful as a tool in applied economics, this method should be capable of placing quantitative as well as qualitative bounds on comparative static effects of interest.

3.1.2 A Proposed Methodology

In this chapter, we shall introduce a method for incorporating empirically based quantitative restrictions on $\pi(x)$ into the traditional qualitative comparative static analysis of the firm. Quantitative as well as qualitative bounds on comparative static effects can be calculated by this method. Thus this method can, in principle, calculate a "reasonable" finite upper bound on a comparative static change in beef output from empirically based restrictions on a beef ranch's production function and price schedules.

A detailed discussion of the method and of partial solutions to the important computational problems are presented in accompanying

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[1]See Section 1.2 of Chapter 1.

appendices. In addition, a simple illustrative model for community pasture programs is presented in this chapter. The major task of accumulating computational and practical experience with the method has essentially been postponed to a further study.

We shall express our quantitative restrictions, and the usual restrictions implied by the maximization hypothesis, as (a) a set of equations relating the comparative static change in the firm's activity levels $\frac{\partial x}{\partial \alpha}$ to potentially observable properties ρ of the firm's maximization problem, and (b) a set of empirically derived restrictions on ρ. By calculating the maximum and minimum values of $\frac{\partial x^i}{\partial \alpha}$ (or of a scalar function $z(\frac{\partial x}{\partial \alpha})$) over this feasible set (a and b above), the range of comparative static effects $\frac{\partial x^i}{\partial \alpha}$ (or $z(\frac{\partial x}{\partial \alpha})$) that is consistent with the maximization hypothesis and the specified restrictions on ρ can be determined. These upper and lower values of $z(\frac{\partial x}{\partial \alpha})$ can in principle be calculated as solutions to corresponding (non-linear) programming problems.

The set of variables ρ largely consists of various factor substitution and scale effects defined for various subsets of fixed inputs. The rationale for emphasizing such variables ρ in a comparative static model is essentially as follows: our knowledge (from direct and econometric observation of firm behavior and physical processes) typically is in a form more closely related to such parameters ρ than to the elements of the Hessian $[\pi_{ij}]$.

This method of quantitative comparative statics is not without its drawbacks. In particular, a local solution to the above nonlinear programming problems is not necessarily a global solution, and the number of equations in the set of constraints increases exponentially with the number of inputs included in the firm's maximization problem. However, there appear to be somewhat adequate methods of coping with both problems.

3.2 Previous Methods of Comparative Static Analysis

Methods of comparative static analysis can be classified as either "qualitative" or "quantitative": qualitative methods incorporate restrictions primarily on the sign of parameters, whereas, quantitative methods incorporate many restrictions on magnitudes as well as signs of parameters. Thus qualitative methods can only lead to qualitative restrictions on comparative static effects, whereas, methods classified as quantitative can lead to either quantitative or simply qualitative restrictions on comparative static effects. Qualitative methods have been developed for uses of both minimal and exhaustive restrictions on the signs of parameters; but quantitative methods have been applied essentially only in cases where the structure of the maximization problem (or equilibrium system) is completely specified.

3.2.1 Qualitative Methods

3.2.1.1 Minimal Restrictions

Primal, dual and primal-dual methods of comparative static analysis[2] have been applied to models of the firm or group of firms where only the maximization hypothesis, i.e., the existence of an interior static equilibrium where $\pi(x)$ is twice differentiable, and competitive conditions are assumed.[3] However, any comparative static effect $\frac{\partial x^i}{\partial \alpha^j}$ for a primal problem

$$\text{maximize } \pi(x;\alpha)$$

is signed unambiguously by the maximization hypothesis if and only if $\frac{\partial^2 \pi(x^*)}{\partial x^i \partial \alpha^j}$ is signed and (x^i, α^j) are "conjugate pairs," i.e., $\frac{\partial^2 \pi(x^*)}{\partial x^k \partial \alpha^j} = 0$ for all $k \neq i$. This condition for signing $\frac{\partial x^i}{\partial \alpha^j}$ is not altered by the assumption of perfect competition.[4] Similar comments must apply to dual and primal-dual methods based solely on the maximization hypothesis and competitive conditions.

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[2]See Sections 2.3.1 and 2.3.2.

[3]The assumption $[\pi_{ij}]$ negative definite at x^* actually imposes quantitative as well as qualitative restrictions on $[\pi_{ij}]$; but comparative statics based solely on the maximization hypothesis has nevertheless been defined as a "qualitative" method.

[4]See Samuelson (1947), pp. 30-33 and Archibald (1965).

3.2.1.2 A Calculus of Qualitative Relations

The relation between the signs of the comparative static effect and the signs of the elements of the primal structure $[\pi_{ij}]$ has also been investigated. The model considered by this literature can be expressed as

$$[\pi_{ij}] \frac{\partial x}{\partial \alpha} = \begin{pmatrix} c^1_{1\alpha} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} .^{5}$$

in the case of a shift in the supply schedule of input 1 for a single firm, or more generally as

$$[A]\, dx = b \ .$$

The problem posed by this literature can be expressed as follows: when can the signs of the elements of $\frac{\partial x}{\partial \alpha}$ (or dx) be deduced from knowledge of the signs of the elements of $[\pi_{ij}]$ and $c^1_{1\alpha}$ (or [A] and b)? Thus the central problem considered in this literature is the deducation of the signs of elements of $[\pi_{ij}]^{-1}$ (or $[A]^{-1}$) from knowledge of the signs of elements of $[\pi_{ij}]$ (or [A]).

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[5]In this chapter, partial derivatives will usually be specified in subscript for m with arguments omitted. For example,

$$\frac{\partial^2 \pi(x)}{\partial x^i \partial x^j} \equiv \pi_{ij} \quad \text{and} \quad \frac{\partial^2 c^1(x^1;\alpha)}{\partial x^1 \, \partial \alpha} \equiv c^1_{1\alpha} \ .$$

In addition, the structure of the firm's objective function will be specified as $\pi(x)$ or equivalently $\pi(x;\alpha)$, and the total cost schedule for input 1 will be specified as $c^1(x^1;\alpha)$ (when α is clearly a scalar) or equivalently $c^1(x^1;\alpha)$.

Sameulson (1947, pp. 23-29) pointed out that the sign of an element of $[\pi_{ij}]^{-1}$ can be deduced solely from qualitative knowledge of the elements of $[\pi_{ij}]$ only under unusual conditions, and these conditions have been formulated more precisely by others.[6] Moreover, combining the maximization hypothesis with such qualitative knowledge does not significantly reduce this indeterminacy of comparative static effects.[7]

In addition, Lancaster (1965, 1966) developed a computational procedure for determining the qualitative solution of such systems. Given a system of equations expressed in the form

$$[B]\, y = 0$$

and knowledge of the signs of the elements of [B], the sign pattern of the vector y (where an element is either +, - or indeterminate) can be calculated by a method presented in Lancaster (1966).

3.2.2 Quantitative Methods

Quantitative comparative statics has been employed only in the case where the structure of the primal problem

$$\text{maximize } \pi(x;\alpha)$$

or of the equilibrium system is completely specified. Lancaster (1965) has pointed out that his general approach can incorporate partial quantitative

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[6]See Lancaster (1962) (1964), Gorman (1964) and Bassett et al. (1968).

[7]See Quirk and Ruppert (1968).

information about $[\pi_{ij}]$; but his method does not provide an adequate basis for a quantitative comparative statics of microeconomic units (see Section 3.3.2.2).

3.3 Limitations of Previous Methods of Comparative Static Analysis

In general, there appears to be considerably more knowledge of the individual firm's structure $\pi(x)$ (or, equivalently, $[\pi_{ij}]$) than has been incorporated into qualitative comparative static methods. In particular, there is often considerable knowledge of $\pi(x)$ that is quantitative in nature and difficult to incorporate into established methods of qualitative comparative statics. Moreover, incorporation of quantitative restrictions on $\pi(x)$ into comparative static analysis may permit the calculation of quantitative restrictions on comparative static effects as well as lead to greater qualitative determination of comparative static effects.

On the other hand, knowledge of $\pi(x)$ is far from complete and the relation between structure and comparative static effects is likely to be complex. Thus the established quantitative comparative static methods, which rely on specifying the entire structure of $\pi(x)$, cannot readily bound the set of comparative static effects that is consistent with the set of "reasonable" structures for $\pi(x)$.

3.3.1 The Qualitative Relation Between Comparative Static and Comparative Dynamic Effects

Lancaster (1962, p. 100) presents essentially the following argument for calculating only qualitative properties of comparative static effects:[8] the comparative dynamic effects over time of a unidirectional change in a parameter almost always have the same sign as the corresponding comparative static effect (whereas, the effects obviously have different magnitudes).[9] It is often asserted that this last statement about dynamics follows from the static Le Chatelier principle.

However, this argument is incorrect: comparative dynamic effects of a unidirectional change over time in a parameter can easily vary in sign as well as in magnitude over time. For example, if capital stocks adjust somewhat in the short run, then the short run (impact) effects may differ in sign from the long run (comparative static) effects.[10] The short run and long run effects necessarily have the same sign only if (in the short run) one set of inputs remains fixed at the initial equilibrium levels and all other inputs adjust so as to attain a new full equilibrium given the levels

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[8]Lancaster also states that incorporation of quantitative restrictions greatly complicates the analysis. However, in Section 3.4 we shall outline a comparative static method that can (at least in principle) incorporate quantitative restrictions on $\pi(x)$ at a reasonable cost.

[9]To be more specific, Lancaster states that the signs of the impact (short run) effect and (assuming stability) long run effect of a change in a parameter can always be calculated correctly by comparative static methods, and that intermediate run effects of a (unidirectional) change in a parameter can generally be signed by comparative static methods.

[10]See Nagatani (1976) and Yver (1971).

of the fixed inputs, i.e., these effects necessarily have the same sign only if the short run and long run correspond exactly to the static models considered in the Le Chatelier principle.

Thus comparative statics can in general be employed correctly only in the estimation of long run effects or of essentially static short run or intermediate run effects, which in turn implies that qualitative properties of comparative static effects are only as valid as the quantitative properties of comparative static effects.[11]

3.3.2 Qualitative Methods

Qualitative comparative static analysis is known to be unsatisfactory: $\frac{\partial x^i}{\partial \alpha^j}$ is signed by the maximization hypothesis only if (x^i, α^j) are conjugate pairs, and $\frac{\partial x^i}{\partial \alpha^j}$ is signed by qualitative knowledge of the elements of $[\pi_{ij}(x^*)]$ only under unusual circumstances.[12] Moreover, the elements of $[\pi_{ij}(x^*)]$ are not generally observable; so there is seldom an empirical basis for expressing quantitative restrictions directly on the elements of $[\pi_{ij}(x^*)]$ and incorporating these into the analysis.[13]

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[11] If the restrictions employed in a qualitative analysis are a subset of the restrictions employed in a quantitative analysis, then of course the set of results obtained by the qualitative method are more (or at least not less) ambiguous — and therefore more likely to include measures of comparative dynamic effects within its range — than are quantitative (or qualitative) results obtained by the quantitative method. However, these qualitative results are "superior" to the quantitative results *only* in this trivial sense, i.e., only in the sense that an arbitrary marginal relaxation of quantitative restrictions necessarily leads to "superior" (more ambiguous) results.

[12] See Section 3.2.1.

[13] Note that, if the comparative static problem is $[A]\,dx = b$ where $[A]$ is (e.g.) a matrix of first derivatives of net aggregate supply schedules, then quantitative information about the elements of $[A]$ may be directly available. However, such information is unlikely to be available if $[A]\,dx = b$ describes microeconomic units.

3.3.2.1 Primal, Dual and Primal-Dual Methods

However, quantitative restrictions are not readily incorporated into available methods of qualitative comparative statics that utilize the maximization (or cost minimization) hypothesis. Traditional primal methods are known to be quite messy in the case of such restrictions, and dual and primal-dual methods cannot readily incorporate the restriction of concavity for F(x) in the presence of variable factor prices.[14]

3.3.2.2 A Computational Method of Lancaster

The computational approaches of Lancaster (1965 , 1966) also fail to provide a satisfactory means of incorporating quantitative restrictions on microeconomic units into comparative static methods. His first method (1965) calculates whether or not all elements of a vector y are qualitatively determined by a system of equations

$$[B]y = 0 \qquad \ldots\ldots(1)$$

and particular convex cones as feasible sets for the columns of [B].[15] These convex cones can in principle incorporate quantitative as well as qualitative restrictions. Given the system of equations

$$[\pi_{ij}]\,\frac{\partial x}{\partial \alpha} = \begin{pmatrix} c^1_{1\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \ldots\ldots(2)$$

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[14]See Sections 2.3.1 and 2.3.2 of Chapter 2.

[15]Convex cones are the class of convex sets such that λx is included in the set for all $\lambda \geq 0$ if x is included in the set.

and particular convex cones as feasible sets for the columns of $[\pi_{ij}]$, Lancaster's method can in principle be used to calculate the sign pattern of $\frac{\partial x}{\partial \alpha}$. His second method (1966) is more general in that it directly signs all elements of Y(+, - or indeterminate) from (1); but knowledge of [B] is generally restricted to signs on elements of [B].

However, it appears that Lancaster's approaches cannot incorporate many empirically-based quantitative restrictions at the firm level. The argument for this statement can be sketched as follows. First, we shall later argue that most of our relevant quantitative knowledge is not directly expressed in terms of the elements of $[\pi_{ij}]$. For example, there may be considerable knowledge about the elements of a matrix K that is related to $[\pi_{ij}]$ by the system of equations

$$\left[\begin{array}{c|c} [\pi_{ij}] & c_i^i \\ \hline c_i^{iT} & 0 \end{array}\right] K = I \qquad \ldots .(3)$$

where $c_i^i \equiv \left(\frac{\partial c^1(x^{1*};\alpha}{\partial x^1}, \cdots, \frac{\partial c^N(x^{N*})}{\partial x^N}\right)^T$ and $I \equiv$ an identity matrix.

Second, Equations (2)-(3) cannot be expressed in the form of (1) where y includes all elements of $\frac{\partial x}{\partial \alpha}$ and excludes all elements of $[\pi_{ij}]$. Thus Lancaster's approaches are inappropriate as a method for quantitative comparative statics at the firm level.[16]

[16]Likewise, empirically-based quantitative knowledge of the elements of a matrix B seem unavailable for other microeconomic units. In addition, Lancaster's approaches are not intended to incorporate second order conditions implied by the maximization hypothesis.

3.3.3 Quantitative Methods

The established quantitative approach is to specify $\pi(x)$ precisely and to calculate solutions (x^*, x^{**}) to the problem

maximize $\pi(x; \alpha)$

for two different values of α, and to perform a "sensitivity analysis" by repeating this procedure for a limited number of alternative specifications of $\pi(x)$.

However, knowledge of $\pi(x)$ is far from complete and the relation between structure and comparative static effects is likely to be complex; so an adequate sensitivity analysis seems improbable or at least very costly. Thus this quantitative approach cannot readily bound the set of comparative static effects that is consistent with the set of "reasonable" structures of $\pi(x)$.

In principle, the above procedure can be modified by replacing the fully specified structure $\pi(x)$ with a flexible functional form $\pi(x)^f$ whose structure is not entirely specified.[17] However, this approach also appears unsatisfactory.

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[17] Such an approach of directly calculating $x^*(\alpha)$ and $x^{**}(\alpha + \Delta\alpha)$, if successful, would have the following advantages over the traditional approach of directly calculating $\frac{\partial x^*}{\partial \alpha}$: the social effects of many programs may depend on the levels $x^*(\alpha)$ and $x^*(\alpha + \Delta\alpha)$ rather than simply on the difference $\frac{\partial x^*}{\partial \alpha}$, and some of the parameters of $\pi(x)$ at x^* may depend critically on the level x^* (as in the case of a constant elasticity of factor substitution).

For example, suppose that the procedure is to solve a pair of problems of the form

$$\begin{array}{ll} \text{maximize } \pi(x;\alpha)^f & \text{minimize } \pi(x;\alpha)^f \\ \text{subject to } E(x,\rho) = 0 & \text{subject to } E(x,\rho) = 0 \\ \qquad \rho^L \leqq \rho \leqq \rho^U & \qquad \rho^L \leqq \rho \leqq \rho^U \end{array}$$

for one value of α, and then for a second value of α. The equations $E(x,\rho) = 0$ express restrictions on the structure of $\pi(x)^f$ in terms of a vector ρ of observable parameters, and the inequalities $\rho^L \leqq \rho \leqq \rho^U$ specify "reasonable" restrictions on ρ. However, this approach to quantitative comparative statics has the following serious defects:

(a) the solution values for ρ in any two problems with different values for α will almost always be different, whereas, ρ is (by definition) to be unchanged by changes in α; and

(b) even overlooking (a), this approach cannot bound any comparative static effects other than the change in $\pi(x^*)^f$ or in variables that are in fixed proportion to $\pi(x^*)^f$.

3.4 A Proposed Methodology for Quantitative Comparative Statics

Users of qualitative comparative statics methods have sought to supplement the meagre content of the maximization hypothesis largely by attempting to calculate the signs of elements of $[\pi_{ij}]^{-1}$ from restrictions placed directly on the elements of the Hessian matrix $[\pi_{ij}]$.[18] However,

[18] See Section 3.2.1.

this approach has not been successful in the modelling of the firm or other microeconomic units: such qualitative knowledge of $[\pi_{ij}]$ is seldom sufficient to determine the signs of elements $[\pi_{ij}]^{-1}$, and direct knowledge of the magnitudes of elements of $[\pi_{ij}]$ is in general unavailable since these elements are essentially unobservable. Moreover, these methods of qualitative comparative statics cannot readily incorporate additional quantitative restrictions, and established methods of quantitative comparative statics cannot readily incorporate anything less than a full specification of the structure $\pi(x)$.[19]

Thus, there is need for an additional method of comparative statics analysis that incorporates quantitative restrictions on many potentially observable parameters of the firm's static maximization problem(s) without specifying an exact structure for $\pi(x)$. Since qualitative comparative statics is only as valid as quantitative comparative statics, results obtained with such a method would in principle have the same status as results obtained with methods of qualitative comparative static analysis.[20]

Here we shall propose such a method of comparative static analysis and shall illustrate how this method can be applied in principle to the evaluation of community pasture programs. In contrast to the usual comparative static approaches, which attempt to deduce knowledge

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[19]See Section 3.3.2.

[20]See Section 3.3.1.

of $[\pi_{ij}]^{-1}$ from restrictions placed directly on $[\pi_{ij}]$, we shall place restrictions directly on the *inverse* of matrices that are essentially submatrices of $[\pi_{ij}]$.

In this manner we shall arrive at a system of equations and inequalities which incorporate the restrictions on the comparative static effect $\frac{\partial x}{\partial \alpha}$ that are implied by

(a) the maximization hypothesis, plus

(b) "reasonable" restrictions on potentially observable parameters ρ of the structure of $[\pi_{ij}]$.

Given an interior solution to the producer's static optimization problem "maximize $\pi(x;\alpha)$," the restrictions implied by the maximization hypothesis are the total differential of the first order conditions, i.e.,

$$\underset{(N\times N)}{[\pi_{ij}]} \; \underset{(N\times 1)}{\frac{\partial x}{\partial \alpha}} \; = \; \underset{(N\times 1)}{-\pi_{i\alpha}} \, ,$$ [21]

plus the second order condition

$$\underset{(N\times N)}{[\pi_{ij}]} \quad \text{negative definite.}$$

Thus the range of comparative static effects $z = z(\frac{\partial x}{\partial \alpha})$ that is

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[20]See Section 3.3.1.

[21]In order to make the discussion less abstract, we shall assume that the exogenous change experienced by the firm is a shift in its supply schedule for input 1 $c^1 \equiv c^1(x^1;\alpha)$, i.e.,

$$-\pi_{i\alpha} \equiv \begin{pmatrix} c^1_{1\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{where} \quad c^1_{1\alpha} \equiv \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \, \partial \alpha} .$$

consistent with (a) and (b) can in principle be calculated from the two programming problems

$$\begin{array}{llll} \text{maximize } z\left(\frac{\partial x}{\partial \alpha}\right) & & \text{minimize } z\left(\frac{\partial x}{\partial \alpha}\right) & \\ \text{subject to } [\pi_{ij}]\,\frac{\partial x}{\partial \alpha} = -\pi_{i\alpha} & & \text{subject to } [\pi_{ij}]\,\frac{\partial x}{\partial \alpha} = -\pi_{i\alpha} & \\ \quad [\pi_{ij}] \text{ negative definite} & & \quad [\pi_{ij}] \text{ negative definite} & \\ \quad G([\pi_{ij}], \rho) = 0 & & \quad G([\pi_{ij}], \rho) = 0 & \\ \quad \rho^L \leqq \rho \leqq \rho^U & & \quad \rho^L \leqq \rho \leqq \rho^U & \end{array}$$

where $G([\pi_{ij}], \rho) = 0$ denotes the relations between the Hessian $[\pi_{ij}]$ and the more directly observable parameters ρ, the variables $(\frac{\partial x}{\partial \alpha}, [\pi_{ij}], \rho)$ are endogenous to the problems, and $(\pi_{i\alpha}, \rho^L, \rho^U)$ are exogenous to the problems.[22] The restrictions $\rho^L \leqq \rho \leqq \rho^U$ denote our degree of empirical knowledge about parameters ρ of the structure of $[\pi_{ij}]$. In the case of community pasture programs, the scalar-valued function $z = z(\frac{\partial x}{\partial \alpha})$ may (e.g.) define the comparative static change in producer plus consumer surplus as a function of the comparative static change in the firm's input levels that is induced by a community pasture program.

Note that these two programming problems are essentially analytic rather than simply behavioral in nature. The behavioral implications of the maximization hypothesis are defined there by the relations

[22]The most important types of equations and inequalities for such problems are summarized in Figure 3.

$$[\pi_{ij}] \frac{\partial x}{\partial \alpha} = -\pi_{i\alpha} \quad [\pi_{ij}] \text{ negative definite}$$

where $\frac{\partial x}{\partial \alpha}$ *and* $[\pi_{ij}]$ are treated as endogenous variables. These equations, plus the restrictions

$$G([\pi_{ij}], \rho) = 0 \qquad \rho^L \leqq \rho \leqq \rho^U$$

where ρ is an additional set of endogenous variables, define the analytical relations between the variables ($\frac{\partial x}{\partial \alpha}$, $[\pi_{ij}]$, ρ) that are consistent (feasible) with the behavioral assumptions and the degree of empirical knowledge $\rho^L \leqq \rho \leqq \rho^U$. Thus, solving the two programming problems above is a purely analytical procedure for obtaining the extreme values of the set of values for $z = z(\frac{\partial x}{\partial \alpha})$ that are consistent with the maximization hypothesis and the degree of empirical knowledge $\rho^L \leqq \rho \leqq \rho^U$.

The vector of parameters ρ typically includes measures of the following types of properties of $[\pi_{ij}]$:

(a) possibilities of factor substitution within any subset of inputs, and

(b) scale effects (changes in input levels and "profits") for a given change in output when any subset of inputs is held constant and all other inputs vary optimally in the static sense.

In these cases, the relations $G([\pi_{ij}], \rho) = 0$ in effect decompose the Hessian matrix $[\pi_{ij}]$ into a set of more directly observable parameters ρ. A priori knowledge of a range of "reasonable" values for some of these parameters presumably is available in most cases, in contrast to the essentially unobservable elements of $[\pi_{ij}]$ per se.[23] This knowledge would be derived from observation of physical processes, from econometric estimation of physical processes and "short run" behavior, and from observation of firm behavior that approximates various "short run" comparative static effects.

By formulating these restrictions on ρ as confidence intervals or as Bayes intervals, the corresponding feasible set for $\frac{\partial x}{\partial \alpha}$ and $z(\frac{\partial x}{\partial \alpha})$ can also be interpreted as a confidence-Bayes interval. Thus the values of $z(\frac{\partial x}{\partial \alpha})$ at the solutions to the two programming problems presented above define the confidence-Bayes interval for $z(\frac{\partial x}{\partial \alpha})$ that is implied by the maximization hypothesis and the empirically-based restrictions (confidence-Bayes intervals) $\rho^L \leqq \rho \leqq \rho^U$ in the model.

This method of comparative static analysis is not without its drawbacks. In particular, a local solution to either of the above programming problems is not necessarily a global solution, and the number of equations in these models increases exponentially with the dimension of the input vector x. However, there appear to be somewhat adequate methods of coping with the local-global difficulty and of aggregating inputs and enterprises.[24] Further research on these matters seems desirable.

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[23]Moreover, since the elements of $[\pi_{ij}]$ as well as of ρ are included as endogenous variables in the above programming problems, any direct qualitative or quantitative knowledge of the elements of $[\pi_{ij}]$ can easily be incorporated into our model as restrictions of the form
$\pi_{ij}^L \leq \pi_{ij} \leq \pi_{ij}^U \quad (i,j = 1,\cdots,N)$.

[24]See Appendix 5.

3.4.1 Restrictions Implied by the Maximization Hypothesis

It can be shown that the assumption of maximizing behavior is essentially as realistic as the results of comparative static analysis, and that comparative static methods usually are more appropriate than comparative dynamic techniques for the evaluation of community pasture programs.[25] Thus it is important to incorporate the restrictions implied by the maximization hypothesis, i.e., by the existence of an interior static maximum, into our methodology. However, in order to avoid placing arbitrary restrictions on the structure $\pi(x)$, we should model in this manner only those restrictions that correspond exactly to the comparative static implications of the maximization hypothesis.

The task of determining the precise comparative static implications of the maximization hypothesis has been labelled the "integrability problem" in comparative statics (Silberberg, 1974a), and has been largely solved in the case of the dual approach to comparative statics (Epstein, 1978).

It can also be shown that, for problem P

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x^i) \;, \qquad \ldots\ldots\text{(P)}$$

the usual set of primal restrictions

$$[\pi_{ij}] \frac{\partial x}{\partial \alpha} = \begin{pmatrix} c^1_{1\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

[25]See Appendix 1 and Chapter 4.

$[\pi_{ij}]$ symmetric and negative definite

corresponds exactly to the implications of the maximization hypothesis for primal comparative statics in the case of a shift in a single firm's supply schedule $c^1 \equiv c^1(x^1;\alpha)$ for input 1.[26] Thus the "integrability problem" is solved in this special case. In addition, the restriction $[\pi_{ij}]$ negative definite can be expressed in a form that is more appropriate for our (primal) quantitative comparative statics model:

Theorem. A real symmetric matrix A is negative definite if and only if there exists a real lower triangular matrix H with positive diagonal elements such that $A = -HH^T$.

3.4.2 Major Additional Restrictions

Here we shall outline how the Hessian matrix $[\pi_{ij}]$ at a solution x^* to the producer's static optimization problem "maximize $\pi(x;\alpha)$" can be decomposed into more readily observable factor substitution effects and scale effects within any subset of inputs.[27] These and other relations were denoted in the two programming problems above as $G([\pi_{ij}],\rho) = 0$. In contrast to the usual comparative static methods which place restrictions directly on the elements of $[\pi_{ij}(x^*)]$, these relations shall place restrictions on the inverse of matrices that are essentially submatrices of $[\pi_{ij}(x^*)]$.

[26]Our quantitative comparative statics analysis could be extended easily to the case of a shift in the firm's product demand schedule (see the related section of Appendix 4).

[27]The relations between $[\pi_{ij}(x^*)]$ and the more readily observable properties presented here are fairly obvious, and are detailed within Section 3.1 of Appendix 3.

3.4.2.1 Model with Output Exogenous

Given the firm's static maximization problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \ldots\ldots(P)$$

with solution x^*, we can arbitrarily define the related problem where output is treated as exogenous to the firm

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots(4)$$

$$\text{subject to } R(x) = \overline{R(x^*)} \quad .$$

This problem (4) will enable us to decompose $[\pi_{ij}(x^*)]$ for the producer's static optimization problem P into substitution and scale effects with all inputs variable. Problem (4) can be expressed in Lagrange form as

$$\text{maximize } \pi(x;\alpha) - \lambda(R(x) - \overline{R(x^*)}) \qquad \ldots\ldots(5)$$

where the endogenous variables are (x, λ) and the exogenous variables are $(\alpha, \overline{R})$.

Suppose that the differentials of the interior first order conditions for (5) with respect to each of $(\alpha, \overline{R})$ yield a unique solution for all comparative static effects

$$\left\{\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial \lambda}{\partial \alpha}, \frac{\partial x^{**}}{\partial \overline{R}}, \frac{\partial \lambda}{\partial \overline{R}}\right\} \text{ .}$$ [28]

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[28] It can be shown that this assumption of uniqueness is correct whenever any such comparative static effects exist for a problem (5).

This assumption is equivalent to the restriction that this system of differentials can be expressed in the form

$$[A]\ [K] = I \qquad \ldots\ldots(6)$$

where the matrices [A], [K] and I are as defined in Theorem 3 of Appendix 3. [A] is the Hessian matrix $[\pi_{ij}(x^*)]$ bordered by marginal factor costs

$$c_i^i \equiv (c_1^1(x^{1*}, \alpha^1), \ldots, c_N^N(x^{N*}; \alpha^N))^T \text{ ,}^{29}$$

$$[A] \equiv \left[\begin{array}{c|c} [\pi_{ij}] & c_i^i \\ \hline c_i^{iT} & 0 \end{array}\right], \qquad \ldots\ldots(7)$$

[K] is a matrix of all the comparative static effects $\left(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial \lambda}{\partial \alpha}, \frac{\partial x^{**}}{\partial \bar{R}}, \frac{\partial \lambda}{\partial \bar{R}}\right)$[30] for problem (5), and I is an identity matrix.

Nevertheless, in many situations knowledge of the comparative static substitution and scale effects when all inputs are variable $\left(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial x^{**}}{\partial \bar{R}}\right)$

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[29]The symbol "T" denotes the transpose of a (column) vector.

[30]The "revenue effect" $\frac{\partial x^{i**}}{\partial \bar{R}}$ is related to the corresponding output effect $\frac{\partial x^{i**}}{\partial \bar{F}}$ simply as follows: $\frac{\partial x^{i**}}{\partial \bar{F}} = \frac{\partial x^{i**}}{\partial \bar{R}} \cdot \frac{\partial R(y^*)}{\partial y}$ (by the chain rule where $y \equiv F(x)$ and $R(y) = R(F(x))$. Likewise,

$$\frac{\partial \pi(x^*)}{\partial \bar{F}} = \frac{\partial \pi(x^*)}{\partial \bar{R}} \cdot \frac{\partial R(y^*)}{\partial y}$$

may be almost as scarce as knowledge about the comparative static total effect $\frac{\partial x^*}{\partial \alpha^1}$ itself.[31] Considerably more knowledge about substitution and scale effects may be available for cases where subsets of inputs are fixed for the firm.

3.4.2.2 Model with Output and a Subset of Inputs Exogenous

For many situations where knowledge about $\left(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial x^{**}}{\partial \bar{F}}\right)$ is quite weak, a narrower range of "reasonable" values for substitution and scale effects when some inputs are fixed may be readily available. Such information could be obtained (e.g.) from engineering or field studies of physical processes or econometric estimation of "short run" equilibrium models of the firm.[32] Moreover, this knowledge of substitution and scale

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(Footnote 30 continued)

and so $\frac{\partial^2\pi(x^*)}{\partial \bar{F}} = \frac{\partial(\partial\pi/\partial\bar{F})}{\partial \bar{R}} \frac{\partial R(y^*)}{\partial y}$, which yields $\frac{\partial\lambda}{\partial\bar{F}} = \frac{\partial\lambda}{\partial\bar{R}} \cdot \left(\frac{\partial R(y^*)}{\partial y}\right)^2$.

[31] A potentially important exception to this statement occurs when the econometric estimation of cost minimizing via the dual approach is more appropriate than the estimation of maximizing behavior. (On the advantages of estimating production functions by a dual approach, which involves the estimation of maximizing or cost minimizing factor demand functions, see Varian, 1978, Chapter 4 for an introduction, and Fuss and McFadden, 1979). Even if adjustment costs of varying inputs *per se* are low, observed activities may not correspond to maximizing behavior due to adjustment costs of searching for an optimum (see Appendix 1). Since cost minimization is a weaker condition (involving less search) than maximization and also defines conditional factor demand functions, this assumption is often preferred to maximization. Moreover, to the extent that production approximates constant returns to scale, estimation errors due to endogeneity of output (e.g., when output is adjusted in the short run but not too long run equilibrium levels) can be avoided by estimating conditional factor demand in terms of unit output:

$$x^i(\alpha, F, x^{S+1}, \cdots, x^N) = F \cdot x^i(\alpha, 1, \frac{x^{S+1}}{F}, \cdots, \frac{x^N}{F}).$$

[32] A subset of inputs (S+1, ⋯,N) may be relatively fixed in the short run due to: concavity of adjustment cost functions (Rothschild, 1971), imperfect rental and used capital markets, and indivisibilities. In this case,

effects when various subsets of inputs are fixed may imply strong restrictions on the comparative static effect $\frac{\partial x^*}{\partial \alpha}$ for problem P. This statement can be elaborated upon as follows.

Given the firm's problem P, we can arbitrarily define the related "short run" static maximization problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \text{....(8)}$$

$$\text{subject to } R(x) = \overline{R(x^*)}$$

$$x^j = \overline{x^{j*}} \qquad j = S+1, \cdots, N$$

where output and an arbitrary subset of inputs are "exogenous" to the firm at the equilibrium levels for P. This problem can be expressed in Lagrange form as

$$\text{maximize } \pi(x;\alpha) - \lambda(R(x) - \overline{R(x^*)}) - \sum_{j=S+1}^{N} \gamma^j(x^j - \overline{x^{j*}}) \qquad \text{....(9)}$$

where the endogenous variables are $(x^1, \cdots, x^S, \lambda, \gamma^{S+1}, \cdots, \gamma^N)$ and the exogenous variables are $(\alpha, \bar{R}, \overline{x^{S+1}}, \cdots, \overline{x^N})$.

Suppose that the differentials of the interior first order conditions for (9) with respect to each of $(\alpha, \bar{R})$ yield a unique solution for the comparative static effects $\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial \lambda^S}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \bar{R}}, \frac{\partial \lambda^S}{\partial \bar{R}}\right)$.[33] This assumption

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(Footnote 32 continued)

there is at least a theoretical argument for statistically estimating maximizing or cost-minimizing factor demand functions with inputs (S+1, •••,N) treated as predetermined. Then, in the absence of specification errors, the structure of problems of the form (8) and (12) is being estimated.

[33] There is no loss in generality in assuming that $\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial \gamma^S}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \bar{R}}, \frac{\partial \lambda^S}{\partial \bar{R}}\right)$ is uniquely defined for a given problem (9).

is equivalent to the restriction that this system of differentials can be expressed in the form

$$[\tilde{A}_{11}]\ [L] = I \qquad \ldots .(10)$$

where the matrices $[\tilde{A}_{11}]$, $[L]$ and I are as defined in Corollary 5 of Appendix 3. $[\tilde{A}_{11}]$ consists of (a) the principal submatrix $[\pi_{ij}{}^{A}]$ of $[\pi_{ij}(x^*)]$ that is formed by deleting rows and columns $(S+1,\cdots,N)$ from $[\pi_{ij}(x^*)]$ and (b) the subvector $c_i^{iA} \equiv (c_1^1(x^{1*};\alpha^1),\cdots,c_S^S(x^{S*};\alpha^S))^T$ on the borders of $[\pi_{ij}{}^{A}]$, i.e.,

$$[\tilde{A}_{11}] = \left(\begin{array}{c|c} [\pi_{ij}{}^{A}] & c_i^{iA} \\ \hline c_i^{iA^T} & 0 \end{array}\right) . \qquad \ldots .(11)$$

$[L]$ is a matrix of the comparative static effects $\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial \lambda^S}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \bar{R}}, \frac{\partial \lambda^S}{\partial \bar{R}}\right)$ and I is an identity matrix.

By (10)-(11), knowledge of the elements of $[L]$ and c_i^{iA} places restrictions on $[\pi_{ij}(x^*)]$. Thus knowledge of the comparative static effects $\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \bar{R}}, \frac{\partial \lambda^S}{\partial \bar{R}}\right)$ for problem (9) and of equilibrium marginal factor costs places restrictions on the "long run" comparative static effect $\frac{\partial x^*}{\partial \alpha^1}$ for problem P.[34]

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[34]Since $[L]$ is symmetric and knowledge of $\left(\frac{\partial x^{**S}}{\partial \bar{R}}, c_{1\alpha^1}^1,\cdots,c_{S\alpha^S}^S\right)$ presumably is greater than knowledge of $\frac{\partial \lambda^S}{\partial \alpha}$ *per se*, restrictions on $\frac{\partial \lambda^S}{\partial \alpha}$ seldom would be specified (see Corollary 5-A).

Moreover, the comparative static effect $\frac{\partial x^*}{\partial \alpha^1}$ for problem P can almost always be defined precisely in terms of a set of comparative static effects $\left\{\left(\frac{\partial x^{**S}}{\partial \alpha^i}, \frac{\partial x^{**S}}{\partial \bar{R}}\right)\right\}$ for an appropriate set of problems (9), and these relations are implicit in our standard quantitative comparative comparative statics model. The comparative static effects included in this set will differ in terms of the partition into fixed and variable inputs and the choice of shift parameter α^i. This important relation between $\frac{\partial x^*}{\partial \alpha^1}$ and various sets $\left\{\left(\frac{\partial x^{**S}}{\partial \alpha^i}, \frac{\partial x^{**S}}{\partial \bar{R}}\right)\right\}$ is demonstrated in Appendix 3.

In sum, restrictions on $\left\{\left(\frac{\partial x^{**S}}{\partial \alpha^j}, \frac{\partial x^{**S}}{\partial \bar{R}}, \frac{\partial \lambda^S}{\partial \alpha^j}\right)\right\}$ for various problems (9) and $\{\alpha^j\}$ plus the relations (10) may imply considerable restrictions on $\frac{\partial x^*}{\partial \alpha^1}$ for problem P. Since knowledge of substitution and scale effects will be defined primarily in terms of problems (9) with various subsets of fixed inputs, these restrictions derived from a model with output and a subset of inputs exogenous are a very important aspect of our quantitative comparative statics model.

3.4.2.3 Model With a Subset of Inputs Exogenous

In addition, direct knowledge about the total effects of $d\alpha^i$ when certain subsets of inputs are fixed may be available. Such knowledge can be specified as restrictions on comparative static effects $\left(\frac{\partial x^{*S}}{\partial \alpha^i}\right)$ for various "short run" static problems of the form

$$\begin{aligned} &\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \\ &\text{subject to } \quad x^j = \overline{x^{j*}} \qquad\qquad j = S+1,\cdots,N. \end{aligned} \qquad \ldots .(12)$$

These restrictions plus the following relations can be incorporated into our quantitative comparative statics model:

$$[\pi_{ij}^A]\,[P] = I$$

where $[\pi_{ij}^A]$ is the principal submatrix of $[\pi_{ij}(x^*)]$ obtained by deleting the rows and columns $(S+1,\cdots,N)$ from $[\pi_{ij}(x^*)]$, $[P]$ is symmetric and $P_{ij} = \dfrac{\partial x^{i*S}}{\partial \alpha^j} \Big/ c^j_{j\alpha^j}$. In this manner, knowledge about a "reasonable" range of values for $\dfrac{\partial x^{*S}}{\partial \alpha^i}$ corresponding to any problem (12) places restrictions on $\dfrac{\partial x^*}{\partial \alpha^1}$ for problem P.

3.4.3 Minor Additional Restrictions

Other forms of knowledge about the structure of the firm's static maximization problem P may be available and useful in defining "reasonable" limits on the comparative static effect $\dfrac{\partial x^*}{\partial \alpha^1}$ for problem P. These additional forms of knowledge are of at least two types. First, there may be knowledge of the comparative static effect of a change in the demand schedule for the firm's output or in the firm's production function. Including the corresponding restrictions in our standard quantitative comparative statics model seems likely to lead to a small reduction in the range of feasible values for $\dfrac{\partial x^*}{\partial \alpha^1}$. If such comparative static effects and its "short run" variations with fixed inputs are included in our model, then our model incorporates knowledge of

(First page of Figure 3)

(A) first order conditions for a maximum

$$[\pi_{ij}]\,\frac{\partial x}{\partial \alpha^1} = \overline{\begin{pmatrix} \overline{c^1_{1\alpha^1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}^{38} \qquad \text{N quadratic equations}$$

(B) second order conditions for a maximum

$$-[\pi_{ij}] = [H]\,[H]^T \qquad \frac{N(N+1)}{2} \text{ quadratic equations}$$

$$H_{j,j} > 0 \qquad (j = 1,\cdots,N) \qquad \text{N bounds}$$

(C) long run decomposition (see Theorem 3)

$$\left(\begin{array}{c|c} [\pi_{ij}] & c^i_i \\ \hline c^{iT}_i & 0 \end{array}\right) \underset{(N+1)\times(N+1)}{[K]} = I \qquad \frac{(N+2)(N+1)}{2} \text{ independent quadratic equations}$$

$$\overline{c^{iL}_i} \le c^i_i \le \overline{c^{iU}_i} \qquad (i = 1,\cdots,N)$$

$$\frac{\partial x^{i**}}{\partial \alpha^j}: \overline{K_{i,j}^L} \le \overline{c^j_{j\alpha^j}} \cdot K_{i,j} \le \overline{K_{i,j}^U} \qquad (i,j = 1,\cdots,N)$$

$$\frac{\partial x^{i**}}{\partial \bar{F}}: \overline{K_{i,j}^L} \le R_y \cdot K_{i,j} \le \overline{K_{i,j}^U} \qquad (i = 1,\cdots,N \text{ and } j = N+1) \qquad 2(N+1)^2 \text{ bounds}$$

$$\frac{\partial \lambda}{\partial \bar{F}}: \overline{K_{i,j}^L} \le -R_y^2 \cdot K_{i,j} \le \overline{K_{i,j}^U} \qquad (i,j = N+1)$$

$$\overline{R_y^L} \le R_y \le \overline{R_y^U} \qquad \text{2 bounds}$$

(D) decompositions, given fixed inputs (see Corollary 5); for each decomposition with N-S fixed inputs:

$$\left(\begin{array}{c|c} [\pi_{ij}^A] & c^{iA}_i \\ \hline c^{iAT}_i & 0 \end{array}\right) \underset{(S+1)\times(S+1)}{[L]} = I \qquad \frac{(S+2)(S+1)}{2} \text{ independent quadratic equations}$$

$$\frac{\partial x^{i**S}}{\partial \alpha^j}: \overline{L_{i,j}^L} \le \overline{c^j_{j\alpha^j}} \cdot L_{i,j} \le \overline{L_{i,j}^U} \qquad (i,j = 1,\cdots,S)$$

$$\frac{\partial x^{i**S}}{\partial \bar{F}}: \overline{L_{i,j}^L} \le R_y \cdot L_{i,j} \le \overline{L_{i,j}^U} \qquad (i = 1,\cdots,N \text{ and } j = S+1) \qquad 2(S+1)^2 \text{ bounds}$$

FIGURE 3 Summary of Major Constraints for the Quantitative Comparative Statics Model[39]

[38] The mark " — " is placed above any symbol that refers to a constant rather than an endogenous variable in the model.

[39] For definitions of the symbols used here, see Theorem 3 and Corollary 5. $R_y \equiv \frac{\partial R(y)}{\partial y}$ where $y \equiv F(x)$ and $R(y) = R(F(x))$.

(Second page of Figure 3)

(D) (continued)

$$\frac{\partial \lambda^S}{\partial \bar{F}} : \overline{L_{i,j}^{L}} \le -R_y^{\,2} \cdot L_{i,j} \le \overline{L_{i,j}^{U}} \qquad (i,j = S+1) \Bigg\} \quad 2(S+1)^2 \text{ bounds}$$

(E) non-decompositions (output exogenous), given fixed inputs (see Corollary 6); for each non-decomposition with N-S fixed inputs:

$$\underset{(S\times S)}{[\pi_{ij}^{A}]}\ \underset{(S\times S)}{[P]} = I \qquad \Big\} \ \frac{(S+2)(S+1)}{2} \text{ independent quadratic equations}$$

$$\frac{\partial x^{i*S}}{\partial \alpha^j} : \overline{P_{i,j}^{L}} \le \overline{c_{j\alpha j}^{j}} \cdot P_{i,j} \le \overline{P_{i,j}^{U}} \qquad (i,j = 1,\cdots,S)$$

totals:

$$\frac{(N+2)(N+1)}{2} + \frac{N(N+1)}{2} + N \text{ quadratic equations}$$

$$N(N+1) + \frac{(N+2)(N+1)}{2} + 2N + 1 \text{ variables}$$

$\frac{(S+2)(S+1)}{2}$ additional quadratic equations and variables for each decomposition or non-decomposition (C,D,E) with N-S fixed inputs

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F I G U R E 3 Summary of Major Constraints for the Quantitative Comparative Statics Model [39]

(Footnotes 38 and 39 are the same as the previous page) all types of comparative static effects that can occur realistically at the level of the single firm.[35]

Second, there may be specific knowledge about the functional form of the firm's static maximization problem P. The following examples are considered in Appendix 3: separability of $\pi(x;\alpha)$ in x, linear homogeneity of $\pi(x;\alpha)$ in α, fixed factor proportions for R(x), and homotheticity of $\pi(x;\alpha)$ in x. The first two properties, and presumably many other special properties of $\pi(x;\alpha)$, are easily incorporated into our quantitative comparative statics model. Such restrictions will be useful when (a) observation and/or theory suggests that such a property is closely approximated, or (b) sensitivity of comparative static results to such properties is an important issue.[36] In these circumstances, the imposition of such properties or of limits on the "degree of deviation" from such properties can be useful in our quantitative comparative statics models.

3.4.4 Major Difficulties and Partial Solutions

The two major difficulties with the proposed method of quantitative comparative statics concern the identification of a global solution and the incorporation of a reasonable number of inputs and outputs into the model. Partial solutions for these overlapping problems are suggested here.[37] First,

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[35]See Section 6 of Appendix 3.

[36]For example, calculating the sensitivity of comparative static results to the property of separability may provide a rough estimate of errors due to inappropriate aggregation of inputs in a quantitative comparative statics model (see Sections 3.2 of Appendix 3 and 3.1 of Appendix 5).

[37]See Appendix 5 for details of these partial solutions.

given an algorithm that is reasonably effective in finding local solutions for a quantitative comparative statics model, we can tentatively conclude that there are "relatively few" feasible values for $z(\frac{\partial x}{\partial \alpha_1})$ that are outside of the observed range. This observed range forms an (X-Y) percent confidence-Bayes interval for $z(\frac{\partial x}{\partial \alpha_1})$ when the constraints

$$\rho^L \leqq \rho \leqq \rho^U$$

form an X percent confidence-Bayes interval for the observable parameters ρ, and it becomes approximately an X percent confidence-Bayes interval for $z(\frac{\partial x}{\partial \alpha_1})$ as the search for feasible solutions becomes sufficiently detailed. More precise estimates of confidence-Bayes intervals for observed ranges of feasible $z(\frac{\partial x}{\partial \alpha_1})$ depend largely upon the ability to approximate random sampling of the feasible set.

Second, computational difficulties increase exponentially with the number of inputs included explicitly in a quantitative comparative statics model; so procedures for aggregating such models within and across enterprises are presented here. These aggregation procedures generally lead to some error in characterizing the disaggregate model: correct aggregation of inputs within an enterprise depends on satisfaction of appropriate Leontief separability conditions or fixed factor proportions within the disaggregate enterprise, and correct aggregation across enterprises depends essentially on exogenous marginal factor costs for each enterprise. The aggregation procedures suggested here are shown to have certain optimum properties. In addition, aggregation errors can be crudely estimated by observing the effects of aggregation errors in small models.

3.5 An Illustration of a Quantitative Comparative Statics Model: Initial Models for Estimating Welfare Effects of Community Pasture Programs

In order to illustrate the structure of a quantitative comparative statics model and the potential relevance of this approach to the evaluation of community pasture programs, we shall present a very simple model of the comparative static welfare effects of supplying community pasture to a single farm. First, an objective function denoting the comparative static change in producer plus consumer surplus is derived for this situation. Then the structure of the constraints is illustrated in the case of 2 and 3 input models of the farm. For simplicity, procedures for aggregating over inputs and enterprises are not illustrated here.

3.5.1 Objective Function

Since the change in consumer plus producer surplus due to an input subsidy can be measured correctly in either input or output markets,[40] our analysis can be restricted to the market for pasture and to corrections for "distortions" in other markets. Considerably more information would be required for estimation of the change in surplus directly in output markets.

To a first approximation, the change in producer plus consumer surplus due to the employment of dA units of community pasture by a given farm during a given time period can be expressed in the following form:

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[40] See Section 2.4.3 of Chapter 2.

$$dSW = p^1 \cdot dA + MSB^B \cdot \frac{\partial B}{\partial A} \cdot dA \qquad \ldots .(13)$$

or equivalently

$$\frac{\partial SW}{\partial A} = p^1 + MSB^B \cdot \frac{\partial B}{\partial A} . \qquad \ldots .(14)$$

Here p^1 $\equiv$ farm demand price for an additional unit of community pasture,

$\frac{\partial B}{\partial A}$ $\equiv$ comparative static change in the output of the beef enterprises (measured as total revenue of beef sales), and

MSB^B $\equiv$ marginal social value of beef output minus the marginal value to the farmer of beef output.[41,42]

By the definition of $\frac{\partial B}{\partial A}$, and assuming an interior solution where the farmer's objective function $\pi(x; \alpha)$ is differentiable,

$$\frac{\partial B}{\partial A} = p^1 \cdot \frac{\partial x^1}{\partial A} + \sum_{i=2}^{N} c_i^i \cdot \frac{\partial x^i}{\partial A} . \qquad \ldots .(15)$$

In addition, suppose that the community pasture program simply shifts the farm's pasture supply schedule to the right, as shown in Figure 4. Then

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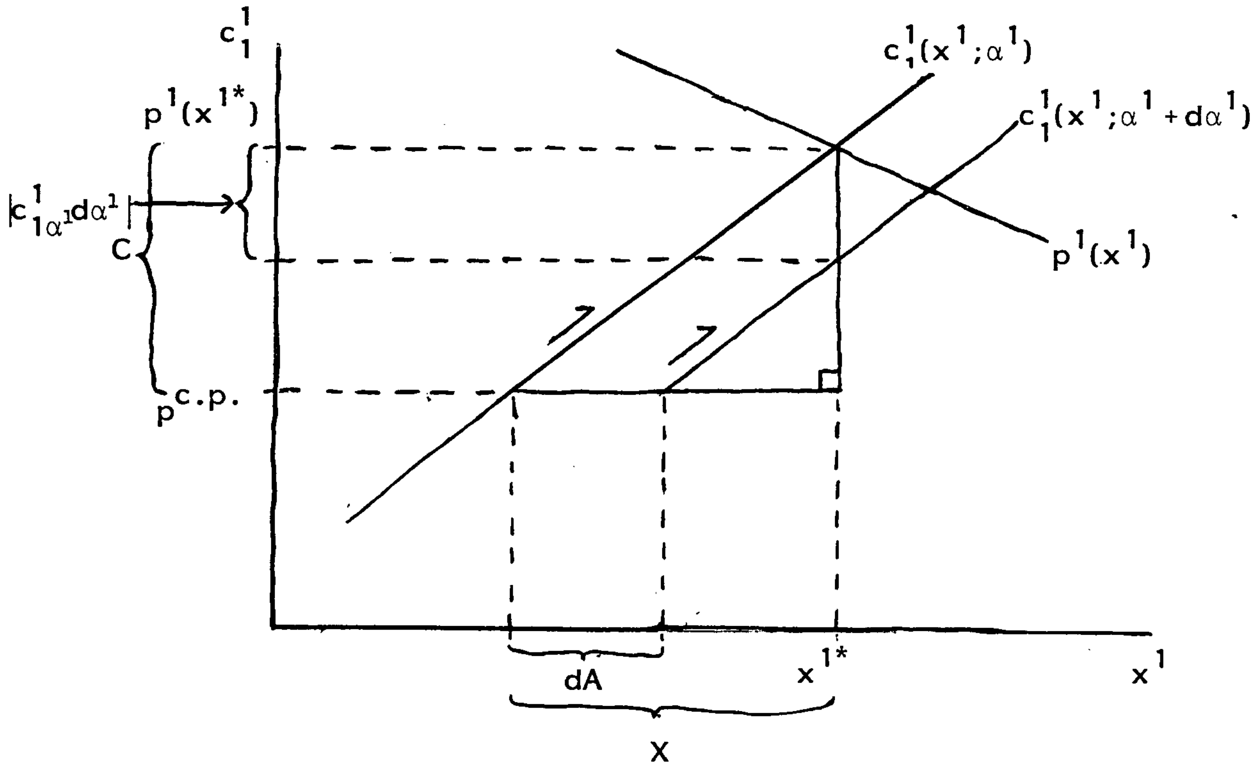
[41]The costs of supplying dA units of community pasture are not considered in (13), since these costs are essentially exogenous to the farms utilizing the pasture. Farm response to these pasture programs is essentially independent of these costs, which are largely borne by the public rather than by the users.

[42]$MSB^B > 0$ due to market "distortions" resulting from import and export taxes or subsidies. Other "distortions" related to the B.C. community pasture programs appear to be minor. In this case, $MSB^B = .13$ circa 1975. See Barichello (1978).



x^{1*} $\equiv$ farm's equilibrium quantity of pasture prior to community pasture program

$p^1(x^1)$ $\equiv$ farm's derived demand schedule for pasture

$c_1^1(x^1;\alpha\,)$ $\equiv$ supply (marginal factor cost) schedule for pasture prior to community pasture program

$c_1^1(x^1;\alpha^1+d\alpha^1)$ $\equiv$ supply (marginal factor cost) schedule for pasture during community pasture program

$p^{c.p.}$ $\equiv$ price at which dA units of community pasture are offered to the farm

Figure 4 Hypothesized Effect of the Community Pastures Program on the Supply Schedule of Pasture to the Parm

$$c^1_{11}(x^{1*};\alpha^1) \;=\; \frac{c}{x} \qquad \ldots.(16)$$

$$dA/x = \left|c^1_{1\alpha^1}(x^{1*};\alpha^1)d\alpha^1\right|/c \;. \qquad \ldots.(17)$$

By (16)-(17),

$$dA \;=\; \frac{\left|c^1_{1\alpha^1}(x^{1*};\alpha^1)\cdot d\alpha^1\right|}{c^1_{11}(x^{1*};\alpha^1)} \quad . \qquad \ldots.(18)$$

Moreover, a shift $dA > 0$ is equivalent (in terms of comparative static effect) to a shift $c^1_{1\alpha^1}(x^{1*};\alpha^1)\cdot d\alpha^1 < 0$, and our convention is to interpret dA and $d\alpha^1$ as +1.[43] Thus, from (18),

$$c^1_{1\alpha^1}(x^{1*};\alpha^1) \;=\; -c^1_{11}(x^{1*};\alpha^1) \quad \text{given} \quad dA \equiv 1,\; d\alpha^1 \equiv 1\,.^{44} \qquad ..(19)$$

In sum, the comparative static change in consumer plus producer surplus due to supplying the farm with one unit of community pasture can be written as

$$\frac{\partial SW}{\partial A} \;=\; p^1 + MSB^B\,(p^1 \cdot \frac{\partial x^1}{\partial \alpha^1} + \sum_{i=2}^{N} c^i_i\,\frac{\partial x^i}{\partial \alpha^1} \qquad \ldots.(20)$$

by (14)-(15) and (18)-(19).

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[43]Since the shift dA is equivalent to a shift $c^1_{1\alpha^1}(x^{1*};\alpha^1)\cdot d\alpha^1$, which is a product of two terms that are independent of the structure of the farm's static maximization problem, the conventions $dA \equiv 1$ and $d\alpha^1 \equiv 1$ are consistent.

[44]From (18), it is obvious that the comparative static effects dx^* are linear homogeneous in dA just as dx^* is linear homogeneous in $c^1_{1\alpha^1}(x^{1*};\alpha^1)\cdot d\alpha^1$. Thus $dA \equiv 1$ would lead to a doubling of comparative static effects for a given $[\pi_{ij}(x^*;\alpha)]$. In other words, $c^1_{1\alpha^1}(x^{1*};\alpha^1) = -2c^1_{11}(x^{1*};\alpha^1)$ given $dA \equiv 2$, $d\alpha^1 \equiv 1$.

Thus the quantitative comparative statics model will have a linear objective function (20) in $\frac{\partial x}{\partial \alpha^1}$ if p^1 and all marginal factor costs c_i^i $(i \neq 1)$ are also treated as exogenous. Presumably, fairly tight bounds on many equilibrium marginal factor costs can often be derived from observation. In addition, the marginal value of community pasture p^1 prior to the pasture program seems to be well estimated by the methods to be described in Chapter 4. Thus, if the quantitative comparative statics model is employed in conjunction with a model similar to the linear programming model to be presented in Chapter 4, the objective function (20) can often be treated as linear in the endogenous variables $\frac{\partial x}{\partial \alpha^1}$.

3.5.2 Constraints (N = 2,3)

Suppose for simplicity that we can construct an "extreme short run" static model with two variable inputs, one enterprise (cow-calf) and a one month time frame (a summer month when pasture is grazed). Of course, such a model is very unrealistic. Let

$x^1 \equiv$ animal unit months of pasture employed by the farm during the month

$x^2 \equiv$ hours of labor supplied to the cow-calf enterprise during the month. . . . . (21)

The comparative static implications of the maximization hypothesis for this model are specified in Parts A and B of Figure 5. Here $c^1_{1\alpha^1}(x^{1*};\alpha^1) \equiv -c^1_{11}(x^{1*};\alpha^1)$, i.e., it is assumed without loss of generality that $dA \equiv 1$.[45]

[45]See the discussion of (19) in the previous section.

(A) $$\pi_{11}\frac{\partial x^1}{\partial \alpha^1} + \pi_{12}\frac{\partial x^2}{\partial \alpha^1} = -c^1_{11}$$

$$\pi_{12}\frac{\partial x^1}{\partial \alpha^1} + \pi_{22}\frac{\partial x^2}{\partial \alpha^1} = 0$$

(B) $$-\pi_{11} = h^2_{11} \qquad -\pi_{12} = h_{11}h_{21} \qquad -\pi_{22} = h^2_{21} + h^2_{22}$$

$$h_{11} \geqq 0.01 \qquad h_{22} \geqq 0.01$$

(C) $$\pi_{11}K_{11} + \pi_{12}K_{12} + p^1K_{13} = 1 \qquad \pi_{11}K_{12} + \pi_{12}K_{22} + p^1K_{23} = 0$$

$$\pi_{12}K_{12} + \pi_{22}K_{22} + c^2_2K_{23} = 1$$

$$\pi_{11}K_{13} + \pi_{12}K_{23} + p^1K_{33} = 0$$

$$\pi_{12}K_{13} + \pi_{22}K_{23} + c^2_2K_{23} = 0$$

$$p^1K_{13} + c^2_2K_{23} = 1$$

(D) $$5 \leqq p^1 \leqq 10, \quad 0 \leqq c^1_{11} \leqq 0.5 \qquad 2 \leqq c^2_2 \leqq 4 \qquad 100 \leqq R_y \leqq 120$$

$$-24 \leqq K_{11} \leqq 0 \quad -80 \leqq K_{22} \leqq 0 \qquad 0 \leqq K_{12} \leqq 32$$

$$0.9 \leqq R_yK_{13} \leqq 1.1 \qquad 0.5 \leqq R_yK_{23} \leqq 2 \qquad 0 \leqq R^2_yK_{33} \leqq 2$$

Figure 5 Comparative Static Constraints for Community Pasture Model (N = 2)

The one decomposition (output exogenous) is specified in Part C. These 6 "upper triangular" equations of the 9 equation matrix system A K = 1 completely describe the comparative static content of this system.[46] Also, p^1 rather than c_1^1 is specified in Part C in order to emphasize the difficulty in obtaining direct market observations of the equilibrium marginal factor cost of pasture.

A set of constraints $\rho^L \leqq \rho \leqq \rho^U$ on p^1, c_{11}^1, c_2^2, R_y and [K] are presented in Part D of Figure 5. Given that these constraints form an X percent confidence-Bayes interval for ρ, the feasible set defines (at least) an X percent confidence-Bayes interval for $\frac{\partial SW}{\partial A}$.[47] The constraints on p^1, c_{11}^1, c_2^2 and R_y have been derived as follows. First, the analysis to be presented in Chapter 4 strongly suggests that the farm value of pasture for the Peace River region circa 1975 typically was between \$5 to \$10 per animal unit month at equilibrium. This estimate defines the bounds on p^1 in the quantitative comparative statics model. Second, suppose that the (inverse) elasticity of factor supply for pasture

$$\frac{dc_1^1}{c_1^1} \Big/ \frac{dx^1}{x^1} \equiv \frac{x^1}{c_1^1} \cdot c_{11}^1 \qquad \ldots . (22)$$

is between 0 and 2 and $x^{1*} \gtrsim 40$, $c_1^1(x^{1*};\alpha^1) \leqq 10$. Then the bounds on the slope of the pasture supply schedule at equilibrium (c_{11}^1) can be specified as 0 and 0.5. Third, interviews with ranchers in the region

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[46]See Section 7 of Appendix 3.

[47]See Section 5 of Appendix 3.

suggested that the opportunity cost of supplying labor to the farm (in terms of foregone leisure or off-farm employment in the case of own labor, or wages for hired labor) typically varied between \$2 and \$4 per hour circa 1975. These are the bounds on c_2^2. Fourth, the total revenue per calf sold by users of British Columbia community pasture circa 1975 typically varied between \$100 and \$120. These define the constraints on R_y.

The constraints on the elements of [K] are related to the static problem

$$\text{maximize } R(y)\,F(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) - \lambda(F(x) - \overline{F(x^*)}) \qquad \ldots .(23)$$

where x* solves the related problem "maximize $\pi(x;\alpha)$." Denote the comparative static effects for (22) as

$$\frac{\partial x^{**}}{\partial \alpha^1},\ \frac{\partial x^{**}}{\partial \alpha^2},\ \frac{\partial x^{**}}{\partial \overline{F}},\ \frac{\partial \lambda}{\partial \overline{F}} \text{ and } \frac{\partial \lambda}{\partial \alpha^1},\ \frac{\partial \lambda}{\partial \alpha^2}.$$

Then (for N = 2)

$$\left.\begin{array}{lll} \dfrac{\partial x^{1**}}{\partial \alpha^1} / c^1_{1\alpha^1} = K_{11} & \dfrac{\partial x^{1**}}{\partial \alpha^2} / c^2_{2\alpha^2} = K_{12} & \dfrac{\partial x^{1**}}{\partial \overline{F}} / R_y = K_{13} \\ \dfrac{\partial x^{2**}}{\partial \alpha^1} / c^1_{1\alpha^1} = K_{21} & \dfrac{\partial x^{2**}}{\partial \alpha^2} / c^2_{2\alpha^2} = K_{22} & \dfrac{\partial x^{2**}}{\partial \overline{F}} / R_y = K_{23} \\ \dfrac{\partial \lambda}{\partial \alpha^1} / c^1_{1\alpha^1} = K_{31} & \dfrac{\partial \lambda}{\partial \alpha^2} / c^2_{2\alpha^2} = K_{32} & \dfrac{\partial \lambda}{\partial \overline{F}} / R_y^{\,2} = -K_{33} \end{array}\right\} \qquad (24)$$

and (by the maximization hypothesis) [K] is symmetric.[48]

[48] See Theorem 3 of Appendix 3.

Thus the direct constraints on [K] are defined by "reasonable" restrictions on comparative static effects for problem (23) and the assumption of symmetry for [K].[49] These constraints have been derived as follows. First, suppose that the own price elasticity of factor substitution for input 1

$$\frac{dx^1}{x^1} \Big/ \frac{dc_1^1}{c^1} \equiv \frac{c_1^1}{x^1} \cdot \frac{1}{c_{1\alpha^1}^1} \cdot \frac{\partial x^{1**}}{\partial \alpha^1} \qquad \ldots\ldots(25)$$

is between -3 and 0 (over a year or, equivalently, over a "typical" one month period). Then, for $x^{1*} \cong 40$ and $c_1^1 > 5$,

$$-24 \leqq K_{11} \leqq 0$$

for a one month model. Likewise, if the own price elasticity of factor substitution for input 2 is between -1 and 0, $x^{2*} \cong 160$ and $c_2^2 \geqq 2$, then

$$-80 \leqq K_{22} \leqq 0 \ .$$

In addition, if the other price elasticity for input 1 is between 0 and 2, and for input 2 is between 0 and 1, then

$$0 \leqq K_{12} \leqq 40 \qquad\qquad 0 \leqq K_{21} \leqq 32 \ .$$

However, $K_{12} = K_{21}$ by the maximization hypothesis. Thus assuming that $0 \leqq K_{12} \leqq 40$ and $0 \leqq K_{21} \leqq 32$ define X percent confidence-Bayes

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[49]Knowledge about $\frac{\partial \lambda}{\partial \alpha^1}, \frac{\partial \lambda}{\partial \alpha^2}$ seems essentially redundant and is therefore ignored. See Section 3.4.2.1.

intervals for these parameters and that the maximization hypothesis is correct, the range

$$0 \leqq K_{12} \leqq 32$$

also defines an X percent confidence-Bayes interval for K_{12}.

Second, suppose that $\frac{\partial x^{1**}}{\partial \bar{F}}$ and $\frac{\partial x^{2**}}{\partial \bar{F}}$, which denote the change in level of inputs 1 and 2 for the month that would be associated with an exogenous increase of one calf of output for the year,[50] are between 0.9 and 1.1, and 0.5 and 2, respectively. Then

$$0.9 \leqq R_y \cdot K_{13} \leqq 1.1 \qquad 0.5 \leqq R_y \cdot K_{23} \leqq 2 .$$

Likewise, if $\frac{\partial^2 \pi(x^*)}{\partial \bar{F}^2} \equiv \frac{\partial \lambda}{\partial \bar{F}}$, which denotes the second derivative of maximum "profits" $\pi(x^*;\alpha)$ for the month with respect to an exogenous increase of one calf of output for the year, is between -2 and 0, then

$$0 \leqq R_y^2 \cdot K_{33} \leqq 2 .$$

Alternatively, suppose that we can construct a static model with three variable inputs that is similar to the above two input model, where

$$x^3 \equiv \text{expenditures on capital services supplied to the enterprise during the month.}^{51} \qquad \ldots .(26)$$

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[50]Since R_y has been defined as the revenue received from the sale of one calf at the end of the year and $\frac{\partial x^{i**}}{\partial \bar{F}} \equiv \frac{\partial x^{i**}}{\partial \bar{R}} \cdot R_y$ (see Section 3.4.2.1), $\frac{\partial x^{i**}}{\partial \bar{F}}$ is to be measured as the change in level of input i for the month that is associated with an exogenous increase of one calf of output for the year.

(Footnote 51 on the following page).

Such a simple model is very unrealistic. Constraints for the three input model are presented in Figure 6, and are constructed in essentially the same manner as are constraints in Figure 5.[52] Knowledge of

$$\frac{\partial x^{i**}}{\partial \alpha^j} \;/\; c^j_{j\alpha j} \;\equiv\; K_{ij} \qquad\qquad (i,j = 1,\cdots,3)$$

often may be relatively scarce; so these restrictions are excluded from Figure 6 in order to emphasize this point.

In addition, note that Figure 6 incorporates

(a) knowledge of comparative static effects for a "short run" decomposition (D), with output and capital fixed, that is equivalent to the decomposition A K = I in Figure 5, and

(b) knowledge of comparative static effects for a "short run" decomposition (E), with capital fixed.[53]

Short run decompositions with pasture and/or labor fixed are excluded from the model in order to emphasize the following: comparative static behavior

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[51]Input x^3 is implicitly defined as an aggregate of various capital inputs. This aggregate model can be derived from knowledge of a more disaggregate comparative static model by aggregation procedures presented within Appendix 5.

[52]The restrictions presented here correspond to the "major" restrictions on comparative static effects (see Section 3.4.2). For a sample of "minor" restrictions that could be included in the model, see Section 3.2 of Appendix 3.

[53]The comparative static effects $\frac{\partial x^{*S}}{\partial \alpha^1}$, $\frac{\partial x^{*S}}{\partial \alpha^2}$ in Part E of Figure 6 refer to a "short run" problem

$$\text{maximize } \pi(x;\alpha)$$
$$\text{subject to } x^3 = \overline{x^{3*}}\,.$$

A. $\pi_{11}\frac{\partial x^1}{\partial \alpha^1} + \pi_{12}\frac{\partial x^2}{\partial \alpha^1} + \pi_{13}\frac{\partial x^3}{\partial \alpha^1} = -c^1_{11}$

$\pi_{12}\frac{\partial x^1}{\partial \alpha^1} + \pi_{22}\frac{\partial x^2}{\partial \alpha^1} + \pi_{23}\frac{\partial x^3}{\partial \alpha^1} = 0$

$\pi_{13}\frac{\partial x^1}{\partial \alpha^1} + \pi_{23}\frac{\partial x^2}{\partial \alpha^1} + \pi_{33}\frac{\partial x^3}{\partial \alpha^1} = 0$

B. $-\pi_{11} = g_{11}{}^2 \qquad -\pi_{12} = g_{11}g_{21} \qquad -\pi_{13} = g_{11}g_{31}$

$-\pi_{22} = g_{21}{}^2 + g_{22}{}^2 \qquad -\pi_{23} = g_{21}g_{31} + g_{22}g_{32} \qquad -\pi_{33} = g_{31}{}^2 + g_{32}{}^2 + g_{33}{}^2$

$g_{11} \geq 0.01 \qquad g_{22} \geq 0.01 \qquad g_{33} \geq 0.01$

C. $\pi_{11}K_{11} + \pi_{12}K_{12} + \pi_{13}K_{13} + p^1K_{14} = 1 \qquad \pi_{11}K_{12} + \pi_{12}K_{22} + \pi_{13}K_{23} + p^1K_{24} = 0$

$\pi_{12}K_{12} + \pi_{22}K_{22} + \pi_{23}K_{23} + c^2_2K_{24} = 1$

$\pi_{11}K_{13} + \pi_{12}K_{23} + \pi_{13}K_{33} + p^1K_{34} = 0 \qquad \pi_{11}K_{14} + \pi_{12}K_{24} + \pi_{13}K_{34} + p^1K_{44} = 0$

$\pi_{12}K_{13} + \pi_{22}K_{23} + \pi_{23}K_{33} + c^2_2K_{34} = 0 \qquad \pi_{12}K_{14} + \pi_{22}K_{24} + \pi_{23}K_{34} + c^2_2K_{44} = 0$

$\pi_{13}K_{13} + \pi_{23}K_{23} + \pi_{33}K_{33} + c^3_3K_{34} = 1 \qquad \pi_{13}K_{14} + \pi_{23}K_{24} + \pi_{33}K_{34} + c^3_3K_{44} = 0$

$p^1K_{14} + c^2_2K_{24} + c^3_3K_{34} = 1$

D. $\pi_{11}L_{11} + \pi_{12}L_{12} + p^1L_{13} = 1 \qquad \pi_{11}L_{12} + \pi_{12}L_{22} + p^1L_{23} = 0$

$\pi_{12}L_{12} + \pi_{22}L_{22} + c^2_2L_{23} = 1$

$\pi_{11}L_{13} + \pi_{12}L_{23} + p^1L_{33} = 0$

$\pi_{12}L_{13} + \pi_{22}L_{23} + c^2_2L_{33} = 0$

$p^1L_{13} + c^2_2L_{23} = 1$

E. $\pi_{11}\frac{\partial x^{1*S}}{\partial \alpha^1} + \pi_{12}\frac{\partial x^{2*S}}{\partial \alpha^1} = c^1_{1\alpha^1} \qquad \pi_{11}\frac{\partial x^{1*S}}{\partial \alpha^2} + \pi_{12}\frac{\partial x^{2*S}}{\partial \alpha^2} = 0$

$\pi_{12}\frac{\partial x^{1*S}}{\partial \alpha^1} + \pi_{22}\frac{\partial x^{2*S}}{\partial \alpha^1} = 0 \qquad \pi_{12}\frac{\partial x^{1*S}}{\partial \alpha^2} + \pi_{22}\frac{\partial x^{2*S}}{\partial \alpha^2} = c^2_{2\alpha^2}$

F. $5 \leq p^1 \leq 10 \qquad 0 \leq c^1_{11} \leq 0.5 \qquad 2 \leq c^2_2 \leq 4 \qquad c^3_3 = 1 \qquad 100 \leq R_y \leq 120$

$0.7 \leq R_yK_{14} \leq 1.05 \qquad 0.3 \leq R_yK_{24} \leq 1.5 \qquad 1 \leq R_yK_{34} \leq 2$

$0 \leq R_y{}^2K_{44} \leq 1.5$

$-24 \leq L_{11} \leq 0 \qquad -80 \leq L_{22} \leq 0 \qquad 0 \leq L_{12} \leq 32$

$0.9 \leq R_yL_{13} \leq 1.1 \qquad 0.5 \leq R_yK_{23} \leq 2 \qquad 0 \leq R_y{}^2L_{33} \leq 2$

$c^1_{1\alpha^1} \equiv 1 \qquad c^2_{2\alpha^2} \equiv 1 \qquad \frac{\partial x^{2*S}}{\partial \alpha^1} \Big/ c^1_{1\alpha^1} = \frac{\partial x^{1*S}}{\partial \alpha^2} \Big/ c^2_{2\alpha^2}$ [53]

$-20 \leq \frac{\partial x^{1*S}}{\partial \alpha^1} \Big/ c^1_{1\alpha^1} \leq 0 \qquad -20 \leq \frac{\partial x^{2*S}}{\partial \alpha^1} \Big/ c^1_{1\alpha^1} \leq 15$

$-90 \leq \frac{\partial x^{2*S}}{\partial \alpha^2} \Big/ c^2_{2\alpha^2} \leq 0$

FIGURE 6 Comparative Static Constraints for Community Pastures Model (N = 3)

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[54] This symmetry condition follows from the maximization hypothesis (see footnote to Theorem 3 in Appendix 3).

is least likely to be observed for decompositions with relatively non-durable inputs defined as fixed.[55,56]

3.6 Summary and Suggestions for Further Research

In this chapter, we have introduced a method of comparative static analysis that is intermediate between traditional qualitative and quantitative methods in terms of the degree of structure imposed on the firm's static maximization problem. Traditional qualitative comparative static methods have incorporated the implications of the maximization hypothesis, signs on elements of $[\pi_{ij}(x^*)]$, and additional elementary restrictions such as fixed factor prices. The method presented here can incorporate these and many other restrictions. We have emphasized restrictions corresponding to various "short run" (fixed input) comparative static effects, especially factor substitution and scale effects. These restrictions can be derived from (e.g.) engineering or field data on physical processes, and from econometric estimation of production processes or of comparative static effects. Since this information can be incorporated into the model in the form of confidence-Bayes intervals, this method avoids the major drawback of traditional quantitative comparative static methods: results obtained by

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[55] Also note that labor can be viewed as a capital asset (rather than as an input whose level can be adjusted quickly without incurring significant adjustment costs) if costs of initial training are borne by the firm; but such costs appear to be minimal for most of the labor employed in a cow-calf enterprise.

[56] Nevertheless, there may be significant knowledge of physical processes that corresponds to comparative static effects for decompositions with pasture or labor fixed.

these determinate methods are dependent upon essentially arbitrary specification of many aspects of the firm's static maximization problem. The manner in which this method can be applied to the estimation of comparative static effects for community pasture programs is illustrated in terms of two simple models.

To the extent that this chapter has been successful in laying the main theoretical foundations for such an intermediate method of comparative static analysis, future research should explore the computational problems and practical significance of this methodology. Computation problems are considered in Appendix 5 of this thesis, and concern difficulties in solving nonlinear programming models that are associated with this method. The material presented there appears to solve these problems in part; but further research along these lines is needed. In particular, the problem of approximating random sampling procedures for obtaining feasible solutions to the nonlinear programming models must be considered more carefully, and experience in solving such models should be accumulated. As a byproduct, such experience should provide some measure of the practical importance of the methodology, i.e., of the extent to which "reasonable" knowledge can define the comparative static effects of interest. It is expected that the degree of success will vary with the comparative static effect of interest and also across "reasonable" degrees of knowledge of the structure of various firms.

Thus, as has been noted previously,[57] this intermediate method and traditional methods of quantitative comparative statics to some extent

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[57]See Section 3.5.

complement each other. Traditional quantitative methods are useful in estimating comparative static effects that depend primarily upon aspects of structure that are known with considerable precision or in estimating comparative static effects that can be compared with alternative measures of the effect. In the next chapter, we shall illustrate both of these points by means of a static (deterministic) linear programming model of a beef ranch. In the process, we shall also gather support for the hypothesis that the comparative static paradigm is useful in the evaluation of community pasture programs. These results will suggest that the essentially theoretical exercises of the last two chapters can make a positive contribution to the applied economics of agriculture.

CHAPTER 4

A STATIC LINEAR PROGRAMMING MODEL OF A REPRESENTATIVE BEEF RANCH

CHAPTER 4

A STATIC LINEAR PROGRAMMING MODEL OF A REPRESENTATIVE BEEF RANCH

4.1 Introduction

The purpose of this chapter and the accompanying Appendix 6 is threefold. First, we shall present a static linear programming model of a "representative" beef ranch for users of community pasture in the Peace River region of British Columbia circa 1975. Such deterministic models can often be useful complements or even substitutes for the method of comparative static analysis that was developed in the previous chapter. In particular, knowledge of the equilibrium shadow price of pasture for the farm is extremely important in the evaluation of community pasture programs, and this equilibrium shadow price depends primarily upon knowledge of parameters that can be specified with some degree of confidence. Moreover, other microeconomic models that have been adapted to the study of derived demand for pasture in Western Canada appear to have been either non-optimizing or partial equilibrium in nature; whereas, the model presented here is explicitly static general equilibrium and optimizing in nature.[1]

Second, solutions to this model are consistent with the assumption that constructs of static, optimizing behavior are

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[1] See Department of Regional Economic Expansion (1976) and Graham (1977), which are mentioned in Section 4.4.2.

often appropriate for microeconomic models designed to estimate farm response. Estimation of aggregate farm supply response via construction and solution of linear programming models of representative farms was a major area of research in the profession in the late 1950's and the 1960's. Unrealistic assumptions of static optimizing behavior have been included on the list of possible explanations for the apparent failure of these studies, and at least some observers have speculated that these have been the primary reasons for failure. However, by attempting to estimate an equilibrium shadow price for an input rather than response, we are able to reduce the significance of many of the problems associated with studies of supply response and focus more clearly on the appropriateness of constructs of static, optimizing behavior.

By comparing solutions to the static optimizing model to be presented here with calculations based on direct observations of hayland rental markets and beef ranch activities, we derive empirical support for the major assumption underlying this thesis: the comparative static paradigm and maximization hypothesis are often useful constructs, and often more useful than alternative constructs, in the empirical estimation of microeconomic behavior. These results are consistent with the theoretical discussion of static, optimizing behavior versus dynamic, nonoptimizing behavior that was summarized earlier in this thesis (Appendix 1).

Third, in this process we demonstrate that the comparative static models and methods of this thesis should be relevant to the problem of predicting farm response to community pasture programs. Thus the theoretical content of this thesis should have constructive (i.e., real world) uses.

4.2 Representative Farm Approaches to Estimating Farm Response

4.2.1 Studies of Supply Response

In the late 1950's and the 1960's, there were many studies attempting to estimate aggregate short run or long run supply response by aggregating estimates of supply response for "representative" farm models. Apparently these studies are considered in large part to have been unsuccessful.[2] Lists of potential causes of this failure have emphasized the following:

(a) use of the unrealistic assumptions of (short run or long run) static equilibrium and optimizing behavior,[3]

(b) poor knowledge of the relevant structure of an individual farm,

(c) poor knowledge of differences in structure among farms, and

(d) poor knowledge of correct procedures for aggregating response over farms, i.e., for modelling interactions of farms.

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[2] For discussion of these studies, see Nerlove and Bachman (1960), L. Day (1963), Carter (1963), Barker and Stanton (1963), and Sharples (1969).

[3] Indeed, the stronger assumptions of static and profit-maximizing behavior were generally employed.

Furthermore, it has been speculated by at least some observers that the primary reason for failure in these studies was the common assumption of static, optimizing behavior.[4]

4.2.2 A Means of Evaluating Constructs of Static Optimizing Behavior

Here we shall formulate a means of investigating the appropriateness of constructs of static, otpimizing behavior in the estimation of farm response via representative farm models. Results will be presented in Section 4.4.

In the previous section, it was pointed out that the apparent failure of microeconomic supply response studies has occasionally been attributed to erroneous assumptions of static and maximizing behavior. On the other hand, theory summarized earlier in this thesis suggests that static optimizing models will often provide the most effective means of estimating supply response. This is primarily because the difference between static and dynamic response presumably depends critically upon essentially unknown parameters of the firm's adjustment cost functions, and static models at least have the virtues of internal consistency and relative simplicity of structure.[5] Thus it seems reasonable to suppose that dynamic and/or non-optimizing representative farm models are even less effective than static, optimizing models in predicting farm response.

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[4]See Carter (1963) especially pp. 1455-64, White (1969) and, in the same spirit, Smith and Martin (1972).

[5]See Appendix 1 for details.

Given this contrast between theory and the opinions of at least some observers on this fundamental issue, it seems desirable to consider empirically the effectiveness of constructs of static, optimizing behavior in predicting farm response. However, an obvious problem here is to separate the effects of such constructs from other potential sources of error in estimating response.

In order to obtain a clearer picture of the effects of such constructs of static optimizing behavior, we shall focus on the equilibrium shadow price of pasture in representative farm models rather than on measures of the change in levels of input (output) for a given change in input (output) price schedules. In this manner, we can significantly reduce the effects of other potential sources of error in the estimation of farm response.

The effects of poor knowledge of the individual farm's static structure $\pi(x;\alpha)$, of the individual farm's adjustment cost functions, of differences in structure among farms, and of interactions of farms all appear to be less important in estimating shadow prices for inputs than in estimating other types of farm response. These points can be elaborated upon as follows. First, the equilibrium shadow price or value of input (e.g., pasture) is more dependent than most other measures of farm response on relatively observable aspects of the structure of an individual farm. For example, in the case of a differentiable function $R(x)$ and a static problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \ldots .(1)$$

with a known interior solution x^{*}[6], the shadow price (demand price for an additional unit) of pasture at equilibrium is defined by

$$p^1(x^{1*}) = R_1(x^*) \qquad \ldots .(2)$$

whereas

$$\frac{\partial x^*}{\partial \alpha^1} = [\pi_{ij}(x^*)]^{-1} \begin{bmatrix} c^1_{1\alpha^1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} . \qquad \ldots .(3)$$

Since knowledge of the first derivatives of $R(x)$ at x^* is likely to exceed the knowledge of the second derivatives of $\pi(x;\alpha)$ at x^*, the shadow price for pasture $p^1(x^{1*})$ can be estimated with more confidence than can effects such as $\frac{\partial x^*}{\partial \alpha^1}$. This statement remains true but in a slightly weaker form when x^* is unknown; to the extent that x^* is estimated with error as x^E and $R^1(x^*) \neq R_1(x^E)$, the accuracy of the estimate for $p^1(x^{1*})$ depends on second (and higher) derivatives of $R(x^E)$ and the degree of knowledge of $R_1(x^E)$; whereas, the accuracy of the estimate for $\frac{\partial x^*}{\partial \alpha}$ depends on third (and higher) derivatives of $\pi(x^E;\alpha)$ and the degree of knowledge of $[\pi_{ij}(x^E)]$. However, unless x^* is estimated with very larger error or second derivatives of $\pi(x)$ are roughly comparable in magnitude to first derivatives, the equilibrium shadow price can still be estimated with more confidence than can effects such as $\frac{\partial x^*}{\partial \alpha^1}$.

In addition, the analysis is essentially unchanged in the case where the model utilized for estimation exhibits a non-differentiable

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[6] x^* is defined as an interior solution if $x^{i*} > 0$, $i = 1, \cdots, N$.

production function. For example, in the linear programming case, the equilibrium price of pasture is calculated from

$$p^1(x^{1*}) \equiv \frac{\partial \pi(x^*)^Q}{\partial x^1} = \frac{\partial R(x^*)}{\partial x^1} - \sum_{i=2}^{N} c_i^i(x^{i*};\alpha^1)\frac{\partial x^{i*}}{\partial x^i} \quad \ldots .(4)$$

rather than from (2) per se, where $\frac{\partial x^*}{\partial x^1}$ defines the fixed factor proportions at x^*. Given that the "true" model is differentiable (or approximately differentiable), we should construct the programming model so that the parameters $\frac{\partial R}{\partial x^1}$, $\frac{\partial x}{\partial x^1}$, c_i^i ($i = 1,\cdots,N$) vary in a piecewise manner over x in rough accordance with our estimates of R_1 and $[\pi_{ij}]$ over x. Thus the analysis presented above for a differentiable model carries over directly to the non-differentiable case.

Second, the equilibrium shadow price is less dependent than most aspect of farm response on the largely unknown structure of the adjustment cost functions of the individual farm. This follows essentially from Equation (2). By definition, $R_1(x^*)$ is independent of the ease in adjusting various inputs. On the other hand, observed changes in input levels resulting from shifts in price schedules are very dependent upon the particular adjustment cost functions for the farm. The argument generalizes to the case of non-differentiable $R(x)$ in roughly the same manner as above.

Third, equilibrium shadow prices of pasture presumably are more uniform across farms than are most measures of response. Suppose that transportation costs of incorporating off-farm improved land into the farm enterprise are negligible, that (equilibrium) rental prices are equal across farms, and that equilibrium exists in the available rental markets for improved land. Then the equilibrium shadow price of pasture will be identical across all farms that trade in these markets and employ improved land as pasture. Although these assumptions may not be realistic, the argument does suggest that market forces tend to reduce the variation in equilibrium shadow price of pasture across farms in the Peace River region of British Columbia. On the other hand, market forces presumably do not tend to reduce the differences between farms in "higher structure" that is analogous to $[\pi_{ij}(x^*)]$ and adjustment cost functions. Moreover, differences in such structure apparently have significant effects on most aspects of farm response.[6'] Thus market forces tend to reduce the differences between $p^1(x^{1*})$ across farms without influencing the variation in response of input and output levels to shifts in price schedules.

Fourth, the farm value of pasture is influenced less by farm inter-relations than are most aspects of farm response. This is because any particular unit of pasture is generally employed only by a single farm, whereas, an exogenous shift in a factor supply or product demand

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[6']This seems obvious from Equations (3) above. In addition, see Day (1963) and Paris and Rausser (1973) for studies that are formulated specifically in the context of linear programming. On the other hand, Day (1969) has also noted that many farmers may tend to imitate the response of managers who are recognized as relatively effective decision-makers. If farm response happens to consist primarily of such behavior, the variation in structure across farms is unlikely to lead to large differences in either shadow prices or other aspects of response within a region dominated by a single manager with recognized decision-making skills. However, results to be presented in Section 4.4 support the hypothesis that such imitation, which is not static and optimizing behavior, is relatively unimportant.

schedule is generally experienced simultaneously by a large number of farms. Thus shifts in factor supply and product demand schedules of farm A due to activities of other farms have less influence on farm A's shadow price for pasture than on farm A's response to an exogenous shift in a price schedule that is also experienced by many other farms.

In sum, there is reason to investigate the assumption that constructs of static, optimizing behavior are relatively useful in farm response studies, and the effects of alternative sources of error (b)-(d) can be controlled more effectively by focussing on equilibrium shadow prices of pasture than on most other types of farm response. Moreover, the marginal value of pasture can also be estimated in the Peace River region from observed rental prices for hay land, given the substitution relationship between hay and pasture observed on farms in the region. Since these estimates are derived from real world data without imposing any significant assumptions about static, optimizing behavior, they provide a criterion for evaluating the appropriateness of constructs of static, optimizing behavior in estimating equilibrium shadow prices for pasture.

In addition, the results of such an examination of shadow prices also provides some information about the appropriateness of the constructs in estimating other types of farm response that are long run or intermediate run in nature. For example, in the case where alternative models have identical differentiable production (revenue) functions R(x), the likelihood of accurately predicting

$R_1(x^*(\alpha_0))$ within a given model presumably increases with the likelihood of accurately predicting $x^*(\alpha_0)$. In addition, the likelihood of predicting the farm responses $x^*(\alpha_1) - x^*(\alpha_0)$ presumably increases with the ability to predict $x^*(\alpha_0)$ and $x^*(\alpha_1)$. Since any actual equilibrium activities $x^*(\alpha)$ are a composite of adjustments over time, it follows that the same model tends to be most appropriate in predicting $R_1(x^*)$ and long run and intermediate run farm response.

4.3 A Static Linear Programming Model of a Representative Beef Ranch

The purpose of this section and the accompanying Appendix 6 is to detail and to explain farm programming models developed for an economic evaluation of British Columbia ARDA community pasture programs in the Peace River region.[7] These models were developed as an alternative to the available non-static or partial equilibrium beef ranch models for Western Canada.

4.3.1 Methodological Problems

Here we outline issues that were considered to be particularly important in choosing a structure for the farm model. In sum, the model (1) specifies static, optimizing behavior rather than dynamic, non-optimizing behavior; (2) generally defines the levels and ratios of various capital

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[7]The material presented in this section and Appendix 6 overlaps considerably with Coyle and Barichello (1978).

stock activities (number of cows on farm and disposition of calves) and enterprise and feeding combinations as endogenous rather than as fixed; and (3) disaggregates labor requirements and supplies over the model year.

4.3.1.1 Static Optimizing Behavior versus Dynamic, Non-optimizing Behavior

First, a model can be static and optimizing or dynamic and/or non-optimizing in nature. In Appendix 1, it has been argued on theoretical grounds that static, optimizing models should be preferable for estimating farm response. In summary, deviations from static, optimizing behavior are due essentially to the existence of "adjustment costs," and our present knowledge of adjustment costs enables us to estimate comparative dynamic and non-optimizing effects only as a series of comparative static and optimizing effects. In addition, static equilibrium models are internally consistent (unlike most dynamic models) and can more easily accommodate a complex structure of production within the unit time period. Thus it was decided that a static programming model was most appropriate. "Short run" and "long run" equilibrium versions were constructed.[8]

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[8]The choice of a one year static model is satisfactory for the purpose of estimating long run equilibrium and response, which are our main concerns. Given the 2.5 year lag in buildup of the beef herd, a static model with a three year time period would be most appropriate for estimating "short run" comparative static effects.

4.3.1.2 Endogenous versus Exogenous Specification of Activities and Combinations of Activities[9]

Second, livestock capital activities, enterprise combinations, etc. can be specified either as endogenous or as exogenous (fixed) in the static model. In the presence of uncertainty about "true" farm structure, i.e., errors in specifying the production function or price schedules, an endogenous specification is not necessarily appropriate. For example, let all such activities be subsumed in the vector x of all farm activities, and let x_B denote any sub-vector of x that may be treated as exogenous to the farm model (all other activities x_A will always be treated as endogenous to the model). Then, in the differentiable case, the problem is to choose between the following estimates of the true equilibrium shadow price $R_1(x^*(\gamma_0);\gamma_0)$:

$$R_1(x^*(\gamma_0 + \Delta^E\gamma);\ \gamma_0 + \Delta^E\gamma) \qquad \ldots.(5)$$

$$R_1(x_A^*(\gamma_0 + \Delta^E\gamma);\ x_B^*(\gamma_0) + \Delta^E x_B,\ \gamma_0 + \Delta^E\gamma) \qquad \ldots.(6)$$

where the true parameters γ_0 (of price schedules and the production function) and true equilibrium levels $x_B^*(\gamma_0)$ are estimated with error $\Delta^E\gamma$ and $\Delta^E x_B$, respectively, and $x_A^*(\gamma_0 + \Delta^E\gamma)$ is the equilibrium when $\gamma = \gamma_0 + \Delta^E\gamma$ and x_B is fixed at its true level $x_B^*(\gamma_0)$. The appropriate choice between (5) and (6) depends essentially on the derivatives π_γ, π_{x_B} (and also R_γ, R_{x_B}) over the relevant region and the prior

[9]This section is the most technical part of this chapter, and can be omitted by the reader without seriously affecting his comprehension of the remainder of this chapter.

distributions of the errors $\Delta^E\gamma$ and $\Delta^E x_B$. Thus an endogenous specification of capital activities, etc. is not necessarily appropriate for estimating the shadow price of pasture in the presence of errors $\Delta^E\gamma$.[10,11]

An endogenous specification of livestock capital activities, enterprise combinations and feeding combinations was selected for the model essentially on the basis of the following extremely crude argument.[12] Suppose that the errors $\Delta^E\gamma$ are "small" relative to the errors $\Delta^E x_B$. Then, in the absence of further information, the expected error in estimating $R_1(x^*(\gamma_0);\gamma_0)$ is less for method (5) than for method (6). Moreover, the parameters seem reasonably specified for the purpose of estimating shadow prices,[13] and current activities x may be quite different from static equilibrium (especially "long run equilibrium") levels. In this case, x should be specified as endogenous.[14,15]

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[10] Similar statements hold when R(x) is non-differentiable and the shadow price is estimated for a discrete change in the level of pasture.

[11] Whether or not an endogenous specification of x_B tends to stabilize equilibrium π (and hence the equilibrium shadow price) in the presence of errors $\Delta^E\gamma$ depends on the (essentially unknown) direction of the errors: $\Delta^E\gamma > 0$ implies that an endogenous specification is destabilizing essentially due to convexity of $\pi(x^*(\gamma);\gamma)$ in γ. For example, if γ^j is the output price for enterprise j, then the firm will maximize the increase in equilibrium π in response to $(\Delta^E\gamma)^j > 0$. Thus, if there are no other errors $\pi(x^*(\gamma_0);\gamma_0) < \pi(x_A^*(\gamma_0+\Delta^E\gamma);x_B^*(\gamma_0),\gamma_0+\Delta^E\gamma) < \pi(x^*(\gamma_0+\Delta^E\gamma);\gamma_0+\Delta^E\gamma)$.

[12] Interviews with farm and B.C.D.A. staff suggested that possibilities for factor substitution could be estimated with at least some accuracy across enterprises and feeding possibilities, but could not be estimated with any accuracy within any particular combination of enterprise and manner of feeding. Thus factor ratios within each of the various enterprises are generally specified as fixed at the observed levels.

(Footnotes 13, 14, 15 on the following page)

4.3.1.3 The Degree of Aggregation and Endogeneity of Labor Demand and Supply

Third, and related to the second issue, labor requirements and supplies can be specified at various levels of disaggregation and endogeneity in the model. It had been suggested that community pasture programs could have various labor-saving effects of considerable value to users, and reasonable point estimates of labor requirements for the various enterprises over the year were obtained. Thus, by the argument for treating capital levels, etc. as endogenous, labor requirements for each enterprise are disaggregated over the model year, and point estimates of the labor-output ratio within each enterprise are specified. Likewise, the supply

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[13] See Section 4.2.2 above.

[14] Even though given errors $\Delta^E\gamma$ have a greater effect on estimates of other aspects of farm response such as $\frac{\partial x^*}{\partial \alpha^i}$, an endogenous specification of x_B also seems preferable for estimating $\frac{\partial x^*}{\partial \alpha^i}$. By the Le Chatelier Principle,

$$\left|\frac{\partial x(x^*(\alpha);\alpha)}{\partial \alpha^i}\right| > \left|\frac{\partial x(x_A{}^*(\alpha);\overline{x_B{}^*(\alpha)},\alpha)}{\partial \alpha^i}\right|$$

where α is a subset of γ. Thus fixing x_B at $x_B{}^*$ leads to errors in estimating $\frac{\partial x^*}{\partial \alpha^i}$ when $\Delta^E\hat{\gamma} = 0$, and on the other hand fixing x_B at $x_B{}^*$ reduces the error in estimating $x_A{}^*(\gamma_0)$ in the presence of any error $(\Delta^E\gamma)^i$. However, this advantage of (correctly) specifying the level of x_B in the model should be less important in estimating the difference $\frac{\partial x^*}{\partial \alpha^i}$ in equilibrium levels.

[15] It should also be noted that the "best" estimates of γ for the model are not the expected values of γ. Since $\pi(x^*(\gamma);\gamma)$ is a convex function of $\hat{\gamma}$ provided only that π_γ does not change sign over $(x^*(\gamma),\gamma)$ (e.g.,

(Footnote 15 continued on the following page)

schedules of family and hired labor to the farm are disaggregated over time and exogenous supply prices are specified.[16] As noted in Chapter 1, the equilibrium supply price of labor appears to be endogenous to the farm during many periods of the year. Nevertheless, the directions of bias on changes in labor use and value of community pasture due to this misspecification are readily determined in this model, and the magnitudes of errors can be estimated simply by varying the supply price of labor in the appropriate directions.[17] Attempts at direct modelling of endogenous supply prices for labor have been avoided here precisely because neither evaluation of direction of bias nor sensitivity analysis could then be done so easily.

4.3.2 Summary of Model Structure

Here we summarize the basic single year linear programming model of a "representative" farm using community pasture in the Peace River region of British Columbia. "Long run" and "short run" variations of the model were constructed. The structure of both versions and sources of data for the models are detailed in Appendix 6.

(Footnote 15 continued)

McFadden, 1978), γ should generally be defined in the model at less than its expected value in order to obtain unbiased estimates of equilibrium π before and after a shift in the supply schedule of pasture. However, we shall ignore this problem on the grounds that the estimated difference in these equilibrium levels of π should be less sensitive to such difficulties.

[16] The exception to this statement is that upper bounds are placed on the supply of family labor available in each time period.

[17] See Section 1.7 of Appendix 6.

The structure of each model can be disaggregated into the following groups of constraints and activities: (1) land, (2) cattle numbers, (3) cattle feeding, (4) labor, (5) income assurance, and (6) income. Each model has the same objective function. The relations between these groups of constraints and activities are summarized in matrix form in Figure 7. In addition, a flow diagram of the model is presented in Figure 8.

Each model farm has 350 acres of improved land that can be used as pasture, or in production (and establishment) of hay, barley or oats, and 150 acres of unimproved land that can only be used as range. Each farm can rent up to 300, 50 and 75 animal unit months of range in summer periods June 1 to September 1, September 1 to September 15 and September 15 to October 7, respectively. Each farm can also rent up to 180, 30 and 45 animal unit months of community pasture in the same summer periods.[18] In addition, a farm can rent up to 50 acres each of hay, barley and oat land during the year. Three-quarters of own acres in hay and in grain are in production during the year. Otherwise, quantities of on-farm and off-farm land in the various uses are free to vary, subject to the supply constraints mentioned above.

However, the structure of the cattle numbers subsection of the models depends on the variant of the model. In a "short run equilibrium"

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[18]In the models, one animal unit month (AUM) is equal to one yearling month of grazing as well as one cow (plus calf) month of grazing. Although one yearling presumably requires less grazing than does one cow (plus calf), it had been suggested that a yearling exhausts about the same quantity of pasture as does a cow plus calf (due to greater trampling of grass by yearlings). In fact cows and yearlings were charged at the same rate on the observed community pastures.

	Own Farm Land	Rented Hay and Grain Land	Cattle on Farm			AUM's Grazing		Hay and Grain				Labor Use			Labor Hired	Roundup Transfer Activities	Cattle Sales		Income		Right Hand Side*
			Opening Stock	Added Cattle	Closing Stock	Own Land	Rented Land	Fed to Cattle	Acres Harvested	Sales	Purchases	Custom	Roundup	Surplus			Total	Subsidized			
I. Land	+ -					+	+													≤ ≤ ≤	b_1 b_3
II. Cattle			+ +	+	- -															= =	
III. Feeding			+	+		-	-	-												≤	
IV. Hay and Grain	-	-						+	+ -	+	-									≤ ≤	
V. Labor	+	+	+	+		+ + -	+ + -		+			+	+ -	+	-	+ -				≤ ≤ ≤	b_9
VI. Income Assurance																	-	+ +		≤ ≤ ≤	b_{12}
VII. Income	+	+	+	+			+			-	+	-			+		-	-	+	≤	
VIII. Objective Function ("MAXIMIZE")														+					+	≤	

*0 unless indicated otherwise

Figure 7. Outline of Matrix for the Peace River Income Assurance Farm Model[1]

[1]For details of this matrix, see Appendix 6.

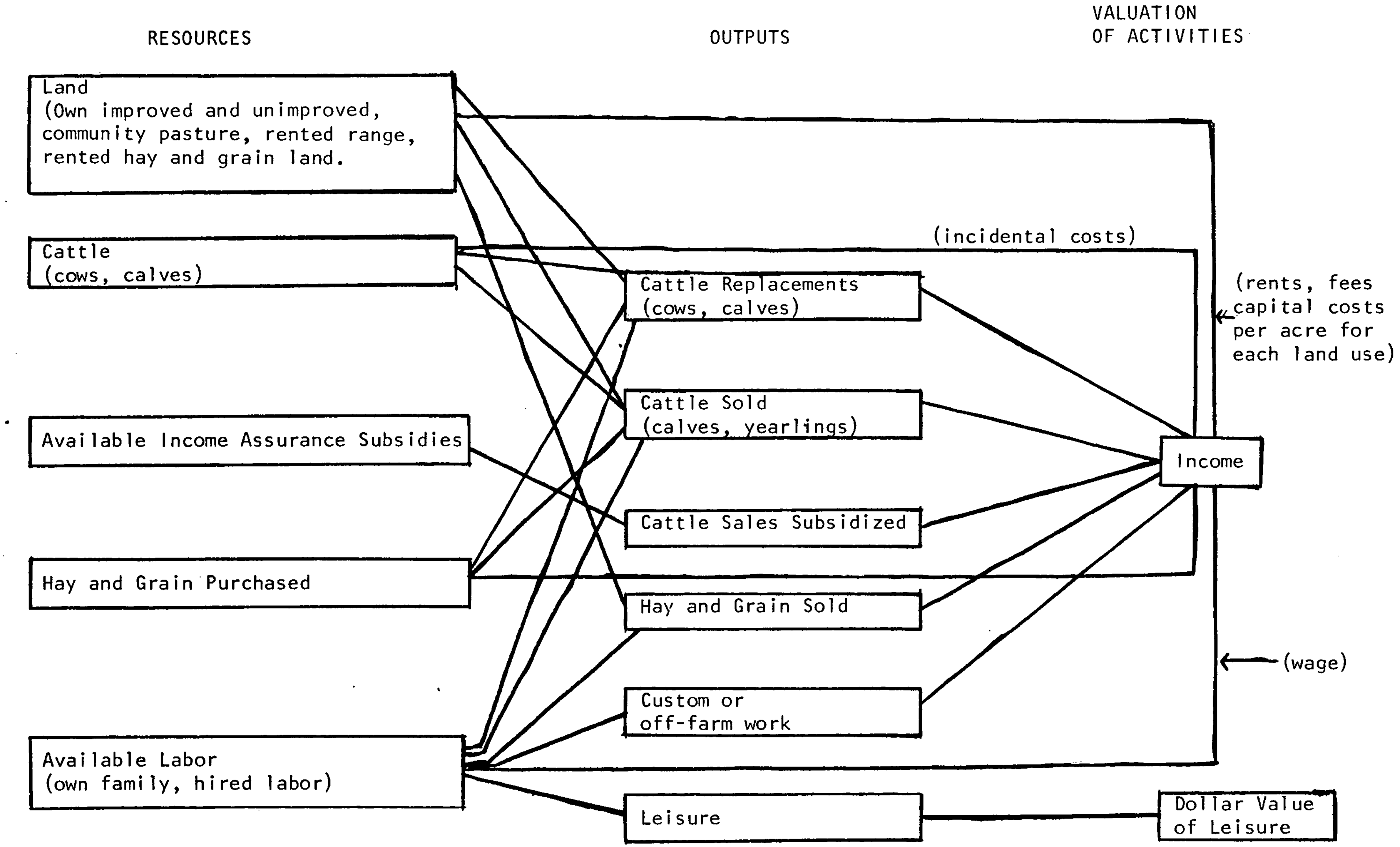

Figure 8. Input-Output Relations for the Farm Model

model, the farm number of cows is defined as greater than or equal to 40, which is approximately the average number of cows on sampled farms. In a "long run equilibrium" model, bounds are never placed on the number of cows. However, both short run and long run models are generally specified such that net investment in cows and calves over the year must be equal to zero. Gross investment in cows then consists solely of replacing cows lost during the year through culling (10% of the opening stock) or through death (2% of the opening stock). Cow replacements must come from the on-farm herd, i.e., cannot be purchased. Likewise, gross investment in calves consists solely of accumulating a stock of calves at the end of the year that is equal to the stock of calves held at the beginning of the year (and sold as yearlings towards the end of the year). In contrast to the level of cows, the opening and closing stock of calves is unbounded in short run as well as in long run models, and the calf replacements (closing stock of calves) can be purchased as well as raised on-farm during the year.[19]

For feeding purposes, the year is divided into six periods, as shown in Table 1.[20] During the first two periods, November 1-June 1,

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[19]Notice that none of the constraints discussed here fixes the levels or ratios of (a) calves sold at the end of the year, (b) calves held over for sale as yearlings in the following year, and (c) calves purchased at the beginning of the year for sale as yearlings towards the end of the year. The levels and ratios of these activities are endogenous to all programming models. A lower bound on cow numbers is generally included in short run models because of their apparent short run fixity (see Section 1.2 of Appendix 6).

[20]The year is defined within the model as beginning and ending November 1. The selection of a starting and terminal date is simply a matter of convenience, provided that the short run or long run equilibrium assumptions on which the model is based are realistic.

Table 1. Feeding and Labor Constraints

A. FEEDING CONSTRAINTS

Feeding Period	Nov. 1 – May 10	May 10 – June 1	June 1 – Sept. 1	Sept. 1 – Sept. 15	Sept. 15 – Oct. 7	Oct. 7 – Nov. 1
Feeding Period No.	1.	2.	3.	4.	5.	6.
Manner of Feeding	Dry fed	Dry fed	Grazed on pasture or range	Grazed on pasture or range	Grazed on pasture, range, hay aftermath or zero-grazed	Grazed on hay aftermath or zero-grazed

B. LABOR CONSTRAINTS

Labour Period	Nov. 1 – Apr. 7	Apr. 7 – Apr. 21	Apr. 21 – May 10	May 10 – June 1	June 1 – July 1	July 1 – Aug. 1	Aug. 1 – Sept. 1	Sept. 1 – Sept. 15	Sept. 15 – Oct. 7	Oct. 7 – Nov. 1
Labour Period No.	1.	2.	1.	3.	4.	5.	6.	7.	8.	9.
Labour Supply (excluding hired labour) hrs./wk.	75	150	75	85	85	120	120	85	85	85
Labour Use	Cattle feeding and management, off-farm or custom work	Cattle feeding and management, off-farm or custom work, calving	Cattle feeding and management, off-farm or custom work	Cattle feeding and management	Cattle feeding and management, hay and grain culture	Cattle feeding and management, hay harvest	Cattle feeding and management, hay harvest	Cattle feeding and management, hay and grain harvest, roundup	Cattle feeding and management, hay and grain harvest, roundup	Cattle feeding and management

all cows and yearlings must be fed hay (produced on-farm or purchased) at a fixed rate; yearlings also receive barley. During the next two periods (June 1-September 15), cows and yearlings must be grazed, either on own pasture, range or community pasture. Weight gains for calves and yearlings are specified as being lower on range than on pasture by 15 pounds per cow (with calf) AUM on range and 21 pounds per yearling AUM on range (in the standard case). Grazing supplies of rented range and pasture cannot be substituted between periods. However, grazing capacities per own acre in pasture or range are defined as fixed aggregates over these two periods, i.e., it is assumed that the total quantity of grazing available on an acre of own pasture or range is invariant with respect to the grazing schedule over these two periods. In the fifth period (September 15-October 7), all cows must be grazed, either on own pasture, range, community pasture, or hay aftermath. Yearlings must be grazed or zero-grazed (fed hay), and also require barley. Weight gains on range are lower than gains on pasture and hay aftermath by 30 pounds per cow (with calf) AUM on range and 42 pounds per yearling AUM on range (in the standard case). Grazing capacity in the fifth period is not transferable to or from other periods. During the sixth feeding period, cows and calves require grazing on hay aftermath, and yearlings must be grazed on hay aftermath or zero grazed (and require barley).

For purposes of labor accounting, the year is divided into nine periods, as also shown in Table 1. A schedule of on-farm labor supply has been estimated (for a work force of one operator, wife, and two school children), and it is assumed that additional labor can be hired at any time at a constant wage rate. During the winter (November 1-May 10) up to 30 hours of an on-farm labor supply of seventy-five hours per week can be allocated to off-farm employment or custom work.

Cows and yearlings require labor at fixed rates within winter and spring periods. In the three labor periods during which community pasture is available (June 1-October 7) cattle demands on farm labor vary with method of feeding (lowest on community pasture) and time of roundup from community pasture and rented range. In the first of these three periods (June 1-July 1), labor also is required for cultivation of hay and grain land. Harvesting of hay can occur within any of the three labor periods from July 1 to September 15, on "appropriate" days (60% of days within the period determined by the vagaries of weather). Thus approximately 60% of the labor available from the farm family in a particular period can be utilized for harvesting. This is the labor constraint on harvesting in the models. Grain can be harvested on appropriate days within the two labor periods from September 1 to October 7. Labor requirements per acre harvested do not vary with the period of harvest; but the yield of hay per acre and grain per acre decreases with the delay in harvesting. The resulting hay and grain can be either sold or fed to animals during the year. In the final period (October 7-November 1), cows, weaned calves and yearlings require labor at fixed rates.[21]

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[21]That component of leisure which is the unemployed surplus of on-farm labor supply is valued in the models. Values are highest during the two week calving period in April and days appropriate for harvesting.

Several Income assurance-related constraints and activities are included in income assurance versions of the "short run" model.[22] This subsection determines the number of beef pounds from calf and yearling sales that qualify for income assurance subsidies to the farm and also determines the level of subsidy. There is an upper bound to the number of qualifying pounds.

Farm income is specified simply as the total revenue for the year from sales of calves, yearlings, hay and grain, plus revenue earned by farm family labor in off-farm or custom work, minus the sum of purchase costs of farm inputs, depreciation and interest costs of capital (excluding land) for the year. In some short run models, costs of maintaining the stock of cows and/or capital in hay and grain enterprises are not specified, i.e., negative net investment is permitted occasionally.

4.3.3 Limitations of the Model

These models of a "representative" farm have many limitations. The most important of these appear to be (1) errors in specifying production functions and extreme difficulties in performing an adequate sensitivity analysis; (2) errors in simulating the effects of adjustment costs (except in long run equilibrium models); (3) errors in specifying expected prices for beef; and (4) failure to incorporate risk into the

[22]The B.C. Farm Income Assurance program subsidizes ranchers in terms of their beef output.

model. The first two points appear to be the most serious weaknesses of these models and of many other models that are designed for estimating farm response.

4.3.3.1 Lack of Knowledge of the Production Function and the Extreme Difficulty in Obtaining an Adequate Sensitivity Analysis

The most serious problem with the model from the viewpoint of estimating "long run" comparative static effects appears to be the difficulty in accurately specifying the farm's production function and in performing an adequate sensitivity analysis for the effects of this uncertainty. Long interviews with farmers and consultations with district agriculturalists led to a rough consensus on current (circa 1975) input-output ratios in various enterprises for the "average" user of community pasture in the Peace River region of British Columbia. However, reliable estimates of possibilities of factor substitution within an enterprise or of returns to scale were not obtained. Moreover, estimates of equilibrium shadow prices should be somewhat sensitive to such uncertainty, and estimates of other comparative static effects of the form $\frac{\Delta x^*}{\Delta \alpha^1}$ should be particularly sensitive to mis-specifications of possibilities for factor substitution and of returns to scale.[23]

Since there is considerable uncertainty about the appropriate structure of the production function and the relation between structure

[23]See Section 4.2.2.

and comparative static effects is generally complex, there should be considerable difficulty in performing an adequate sensitivity analysis for the effects of such uncertainty. This can be seen most clearly in the case of local comparative statics and a twice differentiable production function. Then

$$\frac{\partial x^*}{\partial \alpha^1} = [\pi_{ij}(x^*)]^{-1} \begin{pmatrix} c^1_{1\alpha^1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots .(7)$$

i.e., any comparative static effect $\frac{\partial x^{i*}}{\partial \alpha^1}$ depends on the values of all $\frac{N(N+1)}{2}$ elements (assuming symmetry) of the Hessian $[\pi_{ij}(x^*)]$. For any reasonable number of inputs N and a realistic degree of uncertainty about the structure of the production function, Equations (7) virtually preclude the possibility of doing an adequate sensitivity analysis by varying directly the values of elements of $[\pi_{ij}(x^*)]$. Moreover, a linear programming model cannot even incorporate many reasonable conditions on the production function (increasing returns to scale and non-convex isoquants) and also has difficulties in handling many other reasonable properties (decreasing returns to scale and smooth strictly convex isoquants).

4.3.3.2 Errors in Simulating Adjustment Costs

For the purpose of estimating "short run" comparative static effects, the most serious problem with this model or perhaps any other farm model may be the difficulty in accurately simulating the effects of adjustment costs. As has been noted previously,[24] our poor knowledge of adjustment costs generally implies that comparative static analysis is our most effective means of estimating real response, i.e., real comparative dynamic effects. Nevertheless, even if a static model with a three year time period had been constructed, errors in simulating the effects of adjustment costs presumably would lead to considerable errors in estimating "short run" response of the form $\frac{\partial x^*}{\partial \alpha^1}$.

4.3.3.3 Errors in Specifying Expected Beef Prices

All comparative static effects, including the shadow price of community pasture, will be sensitive to errors in estimating ranchers' expected beef prices. Since there is considerable variation in calf and yearling prices over the ten to eleven year beef cycle and the process of expectations formation for these ranchers has not been quantified, these errors are likely to be significant. On the other hand, a sensitivity analysis for expected calf and yearling prices (two parameters in a one year model), for a given production function, is much more manageable than a sensitivity analysis for the elements of the equilibrium Hessian $[R_{ij}(x^*)]$ of the production function. Thus, in terms of formulating appropriate confidence intervals for comparative static effects, lack

[24] See Appendix 1.

of knowledge about expected prices poses less of a problem than lack of knowledge about second order properties of the production function.

4.3.3.4 The Exclusion of Measures of Risk

Although risk is consistent with the use of static, optimizing models and incorporation of risk would lead to more realistic modelling of behavior, farmer's uncertainty about prices, etc. has not been incorporated directly into the model. The main reason for this is that—in a sensitivity analysis — the effects of risk can be incorporated in terms of variations in expected input and output prices.

4.4 Results and Implications

The equilibrium farm value of community pasture has been estimated under various conditions for the above static linear programming models, and these results shall be summarized here. These results will also be compared with estimates of equilibrium shadow prices for pasture that have been obtained by other methods. Of most importance, the results obtained here are similar to estimates of the marginal value of pasture that were obtained by Barichello (1978) from actual hay market data under essentially independent assumptions. On the other hand, other farm models simulating non-optimizing or partial equilibrium behavior led to quite different results. The conclusion is that the results of these studies are consistent with the argument of Appendix I: assumptions of static, optimizing behavior are more appropriate than alternative constructs in the estimation of supply response or other response via representative farm models.

4.4.1 Results

Solutions obtained from various specifications of the linear programming model strongly suggest that static equilibrium values of pasture in the Peace River region of British Columbia under 1975 conditions typically would be between \$5 and \$10 per AUM. Some of the results supporting this conclusion are presented in Tables 2 to 4. In addition, results presented in Table 5 suggest that (as expected) significantly higher shadow prices for pasture are implied by static equilibrium and extremely high (circa 1979) expected real beef prices.

The results presented in Tables 2-5 illustrate the variation in equilibrium farm value of pasture with respect to expected beef prices, bounds on cow numbers and the possibility of backgrounding (purchase of calves for subsequent sale as yearlings).[25] Best estimates for other parameters of a linear model of a "representative" user of community

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[25]Community pasture differs from on-farm pasture in the following respects: cattle on community pasture are tended by a rider employed by the grazing association, and cattle must be moved to and from the community pasture within specified periods. Model results suggested that the net effect of these differences on income and the dollar-equivalent value of leisure is negligible. Thus we can assume that the marginal products of community pasture and on-farm pasture are equivalent.

pasture in the Peace River region of British Columbia circa 1975 have been employed in obtaining these particular results.[26] These estimates of representative parameters circa 1975 were gathered from interviews with farmers and B.C. Ministry of Agriculture personnel.[27]

Table 2 shows the variation in equilibrium farm value of pasture and several aspects of model solutions with respect to (a) three important combinations of expected beef prices, (b) a lower bound of 40 cows, and (c) the possibility of backgrounding. The following combinations of beef prices are employed: the 1975 market prices of $30 and $36 per cwt. for calves and yearlings; the 1975 market plus Farm Income Assurance subsidy prices of $56 and $50 per cwt. for calves and yearlings; and the average real prices over the previous beef cycle of $50 and $45 per cwt. for calves and yearlings. Since 1975 market prices represent the bottom of the beef cycle and the anticipated subsidies for 1975 were presumably less than the subsidies that were subsequently legislated, the calf and yearling prices for 1975 that were most commonly expected at the beginning of the year should be bounded by the first two combinations.

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[26]The effects of alternative "reasonable" values for some additional parameters (e.g., variable cost and yields of the hay enterprise, hayland rental rates, difference in calf and yearling weight gains on pasture and range, dollar-equivalent value of leisure) have also been considered. These variations do not alter the basic conclusions presented here.

[27]See Section 4 of Appendix 6 for details. In addition, it should be noted that the Income Assurance section of the model is not employed here, and that the expected price for cull cows is varied proportionately with the expected price of calves. For simplicity, these two prices (per cwt.) are equated here (other results show that this assumption does not affect our conclusions).

TABLE II

Farm Value of Community Pasture and Selected Model Activities: 1975 Market Prices, 1975 Market Prices Plus Subsidies, and Long Run Prices for Calves and Yearlings

Calf Price ($/cwt.)	Yearling Price ($/cwt.)	Cow Bound (lower)	Backgrounding Bound (equality)	CATTLE			HAY			AUM's[28] community pasture	Δ Beef pounds per AUM	3.90 + Δ income per AUM	3.90 +[29] Δ OBJ per AUM
				cows	calf sale	yearling sale	own acres	rented acres	tons sold (+) purchased (-)				
30	36	-	-	0	0	123	0	0	-230	-	-	-	-
		-	-	0	0	181	0	0	-339	255	85.8	9.05	7.91
		40	-	40	0	98	43	50	-224	-	-	-	-
		40	-	40	0	122	157	0	-216	255	32.4	7.27	6.21
		-	0	0	0	0	189	0	+177	-	-	-	-
		-	0	0	0	0	189	0	+177	0	-	-	-
		40	0	40	0	28	154	0	-48	-	-	-	-
		40	0	40	0	28	154	0	-48	11	19.6	5.99	6.26
56	50	-	-	0	0	112	0	0	-209	-	-	-	-
		-	-	0	0	170	0	0	-318	255	85.8	8.17	7.03
		40	-	40	0	98	43	50	-232	-	-	-	-
		40	-	40	0	120	43	50	-267	255	37.5	7.22	6.93
		-	0	35	0	24	135	0	-42	-	-	-	-
		-	0	51	0	35	189	0	-66	208	43.0	6.70	4.71
		40	0	40	0	28	111	0	-93	-	-	-	-
		40	0	51	0	35	189	0	-66	208	29.6	6.85	4.79
50	45	-	-	0	0	42	67	0	-9	-	-	-	-
		-	-	0	0	42	67	0	-9	0	-	-	-
		40	-	40	27	1	189	0	+31	-	-	-	-
		40	-	40	2	26	189	0	-11	90	104.4	5.13	4.30
		-	0	20	0	14	77	0	-24	-			
		-	0	20	0	14	77	0	-24	0	-	-	-
		40	0	40	10	19	139	0	-46	-			
		40	0	40	0	28	154	0	-48	68	52.8	5.79	4.64
Average[30]												6.91	5.86

[28] Either 0 ("–") or 255 AUM's of community pasture are supplied to the farm at $3.90 per AUM.

[29] OBJ ≡ income plus dollar-equivalent value of leisure at solution.

[30] These are simple averages of values over all situations where community pasture is utilized.

For our purposes, the most important point to notice about Table 2 is the stability of the equilibrium farm value of pasture (as measured in either of the last two columns) relative to (e.g.) the number of yearlings sold, acres employed in hay, or comparative static change in beef pounds produced on-farm per AUM of community pasture. For these combinations of beef prices, bounds on cow numbers and backgrounding options, the estimated change in farm income (farm income plus dollar-equivalent value of leisure) varies from \$5.13 to \$9.05 (\$4.30 to \$7.91) per AUM of community pasture for farms using community pasture, and has a simple mean value of \$6.91 (\$5.86) per AUM of community pasture. Thus, to the extent that the shadow price of pasture depends on absolute beef prices rather than relative calf and yearling prices, these values should bound the most common equilibrium shadow prices of pasture in the region circa 1975.

Table 3 illustrates the relation between the equilibrium farm value of pasture and calf and yearling prices intermediate between \$30-\$36 per cwt. and \$56-\$50 per cwt., in the absence of bounds on cow numbers or possibilities for backgrounding. These results, together with the results presented in Table 4 (where backgrounding is excluded from solution), suggest that the equilibrium farm value of pasture is highly sensitive to relative calf and yearling prices if and only if backgrounding is defined as feasible in the model. Since backgrounding was observed to be less common than cow-calf or cow-yearling enterprises in the Peace River circa 1975, it seems reasonable to suppose that high prices for yearlings relative to calves were not commonly expected for 1975. In this case, the most common static equilibirum values of pasture in the

TABLE III

Farm Value of Community Pasture: Selected Calf and Yearling Prices Intermediate Between 1975 Market Prices and 1975 Market Prices Plus Subsidies[31]

Calf Price ($/cwt.)	Yearling Price ($/cwt.)	3.90 + Δ Income per AUM	3.90 + Δ OBJ per AUM[32]
35	35	-	-
35	40	8.82	9.05
40	40	6.60	4.96
40	45	12.02	11.32
45	40	-	-
45	45	8.03	6.87
45	50	16.21	15.50
50	40	-	-
50	50	10.52	10 74
55	40	6.90	4.00
55	45	6.90	4.00
Average[33]		9.50	8.30

[31]For all results reported here, cow numbers and numbers of calves purchased for backgrounding were endogenous to the model, i.e., not bounded.

[32]OBJ ≡ income plus dollar-equivalent value of leisure at solution.

[33]These are simple averages of values over all situations where community pasture is utilized.

TABLE IV

Farm Value of Community Pasture: Selected Calf and Yearling Prices Intermediate Between 1975 Market Prices and 1975 Market Prices Plus Subsidies [34]

Calf Price ($/cwt.)	Yearling Price ($/cwt.)	3.90 + Δ Income per AUM	3.90 +[35] Δ OBJ per AUM
40	45	-	-
45	50	6.99	4.63
50	50	6.26	4.24
Average[36]		6.63	4.44

[34] For the results reported here, cow numbers are endogenous (ranging between 0 and 49); but the number of calves purchased for back grounding is defined as 0.

[35] OBJ ≡ income plus dollar-equivalent value of leisure at solution.

[36] These are simple averages of values over all situations where community pasture is utilized.

Peace River region of British Columbia under 1975 conditions should not differ greatly from the averages for Tables 2-3. In sum, static equilibrium values of pasture under 1975 conditions (among users of community pasture in the region) should be between \$5 and \$10 per AUM.

Finally, results presented in Table 5 show that high equilibrium values of pasture can arise from high beef prices irrespective of the relative levels of calf and yearling prices. However, such high expected prices, which are realistic assumptions circa 1979, would have been very unrealistic in 1975 at the low point of the beef cycle.

4.4.2 Results of Related Studies

Here we summarize the results of some other studies that have been designed to calculate the farm value of pasture in British Columbia and other western provinces. For our purposes, the most important of these is a study by Barichello that is based on observations of hayland rental prices.[37] Eleven observations on cash rent paid for hayland were obtained for the Peace River region of British Columbia during 1975-1976. These observations ranged from \$13.50 to \$8.00 per acre with a mean value of \$11.39 and a variance of \$2.77. Moreover, land suitable for the production of hay also was commonly employed as pasture. Thus, given negligible costs of transacting rental agreements and a static equilibrium (with improved land receiving equal rents at the margin in its alternative

[37]See Barichello (1978) for details, especially pp. 30-33. The linear programming model presented here and the study of hayland rental prices were designed as complements in the evaluation of British Columbia ARDA community pasture programs.

TABLE V

Farm Value of Community Pasture for Extremely High Calf and Yearling Prices[38,39]

Calf Price ($/cwt.)	Yearling Price ($/cwt.)	Backgrounding Bound (equality)	3.90 + Δ Income per AUM	3.90 +[40] Δ OBJ per AUM
70	60	-	8.07	9.17
		0	8.24	8.51
80	60	-	9.94	10.27
		0	9.94	10.27
80	70	-	17.18	16.48
		0	12.23	12.19
90	70	-	14.81	14.72
		0	14.82	14.72
90	80	-	25.55	24.85
		0	17.85	13.91
Average[41]			13.86	13.51

[38]These beef prices, in 1975 dollars, correspond to considerably higher prices in 1979 dollars.

[39]For results reported here, cow numbers are endogenous (ranging between 24 and 140).

[40]OBJ ≡ income plus dollar-equivalent value of leisure at solution.

[41]These are simple averages of all situations where community pasture is utilized.

uses as pasture and hayland), the marginal value of pasture on a "representative" farm in the Peace River region during 1975-1976 can be estimated as

$$0.71 \times \$11.39 = \$8.09 \text{ per AUM}.$$ [42]

An earlier set of calculations based directly on observation also yields estimates of the farm value of pasture that are consistent with our static equilibrium beef ranch model. Wiens (1975) calculated a partial budget for a "typical" beef ranch in the Saskatchewan parkland region circa 1975. Given a "typical" set of farm activity levels and observation of a corresponding level of gross farm receipts and all non-grazing costs related to a cow-yearling operation, the value of pasture was estimated as a residual of $8 per AUM of grazing.

On the other hand, a non-optimizing simulation model that was developed for the evaluation of ARDA community pasture programs in the parkland region of Saskatchewan led to quite different results.[43] First a large non-optimizing simulation model was adapted to conditions

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[42] According to the best information obtained from farmers and B.C.D.A. extension staff, three acres of "average" quality pasture are required to summer one animal unit (cow with calf) over the typical grazing season of June 1 to October 7 (4.25 months), i.e. one AUM of grazing capacity corresponds to 0.71 acres of pasture.

[43] See Department of Regional Economic Expansion (1977). This evaluation of Prairie community pasture programs was undertaken simultaneously with the evaluation of British Columbia programs that is reported in Barichello (1978).

in the parkland area of Saskatchewan. Then three sets of farm behavior (resource allocations) in the presence and absence of community pasture were specified for a "representative" farm, and the associated cash flows were generated from the technical coefficients included in the simulation model. For these three sets of base simulations in the absence of community pasture and farm responses to community pasture, the farm value of community pasture (excluding its supply price) varied from $12 to $30 per AUM with a reported "weighted" average of $23 per AUM.

However, when the most important of the data gathered for the Saskatchewan simulation model was incorporated into the British Columbia optimizing model with the assistance of the person responsible for the data, the estimated farm marginal value of community pasture was less than the corresponding values calculated with British Columbia data (see previous section). This result is not too surprising: pasture appears to be a scarcer resource in British Columbia than in Saskatchewan, and one would expect (ceterus paribus) a higher marginal value for pasture in the region where pasture is relatively scarce.

In addition, studies by Graham (1977) and Harrington (1976) have presented estimates of the farm marginal value of pasture within sections of Western Canada. In a preliminary study with a beef farm linear programming model that in effect specifies beef capital activities and many enterprise combinations as exogenous to the model (i.e., as fixed), Graham (1977) obtained estimates of the shadow price of pasture for three British Columbia farms. These estimates varied between $26.00 and $0.62 per AUM for calf-yearling prices between

$40-$43 and $30-$33 per cwt. (backgrounding was excluded.[44,45] In contrast to the above studies, highly aggregated provincial data related to forage, including both pasture and range resources, was included in a large programming model of the Western Canada beef economy by the Economics Branch of Agriculture Canada. The average estimated value of an AUM of forage throughout British Columbia, as reported in Harrington (1976), was between $10 and $11.

4.4.3 Implications

Here we note that the results reported in the previous two sections are consistent with the hypothesis that constructs of static, optimizing behavior are most appropriate in estimating farm response. The close similarity between the value of community pasture for the static linear programming beef ranch model and the results of the hay market study (Barichello, 1978) and the partial budget analysis (Wiens, 1975) suggests that these constructs would be somewhat realistic in the absence of adjustment costs. Since farm adjustment cost functions are essentially unknown at present, it follows that these results lend support to the hypothesis that static, optimizing models are most appropriate in estimating farm response.

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[44] This wide variation in shadow prices presumably can be interpreted in part as empirical support for our decision to specify various capital activities and enterprise combinations as endogenous to the Peace River beef ranch model (see Section 4.3.1.2).

[45] Additional prices were also considered by Graham (1977).

As has been noted previously, the hay market study for the Peace River region of British Columbia and the partial budget analysis for the Saskatchewan parkland both estimated the value of pasture circa 1975 as $8 per AUM. Since there was evidence of substitution between hay and pasture use of land in the Peace River and the variation in market rental prices for hayland was relatively small,[46] the estimates of the hay market study may well be realistic. In addition, given a "typical" set of farm activity levels and observation of corresponding receipts and costs, the estimate of the partial budget analysis would be realistic.

This "realistic" estimate of $8 per AUM for the shadow price of pasture circa 1975 in the Peace River also approximates a "most likely" value in the reasonable range of $5 to $10 per AUM for the static equilibrium beef ranch programming model. Moreover, the results of the linear programming model are essentially independent of the calculations in the hay market study. This independence is demonstrated in Table 6: a $1.00 change in the variable cost of hay production on own land and rented land always leads to considerably less than the corresponding $0.71 change in the shadow price of community pasture.[47] The endogeneity of the equilibrium shadow price reflects the resource constraints and substitution possibilities that are incorporated into the model.[48]

Thus the close similarity between our linear programming results and calculations based directly on observations strongly suggests that

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[46]See Barichello (1978), Chapter 4.

[47]See footnote 42 earlier in this chapter.

[48]See Appendix 6.

TABLE VI

Sensitivity of Shadow Prices for Community Pasture With Respect to Profitability of Hay Enterprises[49]

Calf Price ($/cwt)	Yearling Price ($/cwt)	Variable Cost Own Hayland ($/acre)	Rental Price Hayland ($/acre)	Acres Own Hay		Acres Rented Hay		3.90 + Δ income per AUM c.p.	3.90 +[50] Δ OBJ per AUM c.p.
				Without Com.Pas.	With Com.Pas.	Without Com.Pas.	With Com.Pas.		
30	36	21.25	41.5	304	350	50	50	9.10	8.38
		26.25	46.5	95	122	50	50	9.35	8.00
		31.25	51.5	0	0	0	0	9.05	7.91
		36.25	56.5	0	0	0	0	9.05	7.90
		41.25	61.5	0	0	2	2	9.05	7.91
56	50	21.25	41.5	122	136	50	50	9.01	8.31
		26.25	46.5	75	122	50	50	8.96	7.44
		31.25	51.5	0	0	0	0	8.17	7.03
		36.25	56.5	0	0	2	2	8.23	7.07
		41.25	61.5	0	0	2	2	8.23	7.06
50	45	21.25	41.5	350	350	0	0	7.93	6.57
		26.25	46.5	189	189	0	0	6.73	5.53
		31.25	51.5	67	67	0	0	-	-
		36.25	56.5	3	3	0	0	6.08	4.63
		41.25	61.5	0	0	0	2	6.08	4.63

[49]Cows and calves purchased for backgrounding are unbounded here.

[50]OBJ = income plus dollar-equivalent value of leisure at solution.

(a) models of static, optimizing behavior are appropriate for the estimation of equilibrium shadow prices, and (b) the particular structure of the Peace River linear programming model is adequate for this purpose.

These results also imply that models of static, optimizing behavior are most appropriate in estimating farm supply response given our present state of knowledge of farm adjustment cost functions. As has been noted in Section 4.2.2, adjustment costs (and many other factors) are less important in determining equilibrium shadow prices than in determining supply response. Other aspects of dynamics (price expectations and biologically-determined time lags in beef production) presumably play an important role in determining shadow prices as well as supply response. Since an ability to estimate equilibrium shadow prices with accuracy also suggests an ability to estimate supply response in the absence of significant adjustment costs,[51] our empirical results are consistent with the hypothesis that errors in using constructs of static, optimizing behavior in the estimation of various types of farm response arise essentially from the importance of adjustment costs. Likewise, since farm adjustment cost functions are essentially unknown, our empirical results are consistent with the hypothesis that has been derived from the theory in Appendix 1: constructs of static, optimizing behavior are most appropriate in estimating farm supply response given our present state of knowledge of farm adjustment cost functions.[52] In addition, our results also suggest that, in some respects (e.g., estimation of shadow prices), these seemingly most appropriate constructs can closely approximate real behavior.

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[51] See the last paragraph in Section 4.2.2.

[52] For a first attempt to estimate adjustment cost functions statistically, see Berndt et al. (1979).

4.5 Summary

In this chapter we have (a) outlined a static linear programming model of a "representative" beef ranch for users of community pasture in the Peace River region of British Columbia, (b) formulated a means of examining the appropriateness of constructs of static optimizing behavior in the estimation of farm response, and (c) observed that solutions (farm value of community pasture) to the linear programming model are consistent with these constructs. Thus we have (a) provided an example (estimation of the farm value of community pasture) where such deterministic models are adequate and most appropriate, and (b) in the process gathered empirical support for the hypothesis that the major abstractions from reality that are employed in this thesis, i.e., the assumptions of static, maximizing behavior, are at present most appropriate for the estimation of farm response at the microeconomic level.

CHAPTER 5

SUMMARY AND CONCLUSIONS

CHAPTER 5

SUMMARY AND CONCLUSIONS

5.1 Summary

The main purpose of this thesis has been to extend comparative static theory and methods of the firm so as to be more useful in agricultural policy analysis. The traditional static theory and methods of the firm, which remains largely embodied in Samuelson (1947), has the following defects from the viewpoint of application in agriculture.

1. Endogenous factor prices, i.e., factor prices that are variable to the individual firm, apparently are realistic in many cases but have not been introduced (correctly) into the theory of the firm,

2. Comparative static methods that are presently available generally make inadequate use of the degree of knowledge about particular policy situations. Traditional qualitative methods (e.g., as in Samuelson, 1947) cannot readily incorporate our full degree of knowledge about a firm's production function. In part for this reason, these qualitative methods have led to relatively few useful results. On the other hand, traditional quantitative methods (e.g., use of programming models with a fully specified farm

structure) are usually too restrictive, i.e., they typically derive results that are dependent in unknown ways on a large number of essentially arbitrary assumptions that can only be partially accommodated in a sensitivity analysis.

3. The assumptions of static optimizing behavior that underlie the traditional theory of the firm may not be appropriate.

Thus the main purpose of this thesis has been more specifically three-fold:

1. To extend the traditional qualitative comparative statics of derived demand at the firm level to the case of endogenous factor prices;

2. to extend comparative static methods of analysis at the firm level so as to incorporate more fully our empirical knowledge about parameters without specifying more than this knowledge, i.e., to introduce a method of analysis that provides a useful "middle ground" between the (generally under-determinate) traditional qualitative methods as embodied in Samuelson (1947) et al. and the (generally overdeterminate) quantitative methods as embodied in (e.g.) static linear and nonlinear programming models of the firm; and

3. to examine the appropiateness of constructs of static, optimizing behavior in the estimation of farm response.

These objectives have been pursued in Chapters 2-4, respectively, and in related appendices. In order to make the discussion more concrete and the applications more obvious, the material in each chapter was related to the problem of predicting response to government funded (ARDA) community pasture programs in British Columbia.

In Chapter 2, the theory of derived demand with variable factor prices was investigated by making explicit use of the following "intuitively obvious" equivalence: a firm's derived demand schedule is equivalent (under very general conditions) to a schedule of shadow prices for the input. In Chapter 2 we demonstrated that a failure to recognize the implications of this equivalence has been responsible for a controversy in the American Economic Review during the last ten years concerning the relation between measures of consumer's surplus in product and factor markets, and also in part responsible for the serious errors committed in the previous attempts to incorporate variable factor prices into the theory of the firm (Ferguson, 1969, and Maurice and Ferguson, 1971). Utilizing this equivalence between derived demand and shadow prices, the following statements (among others) were established for the first time.

1. In the absence of market "distortions," the welfare changes (changes in consumer plus producer surplus) of agricultural policy affecting factor supply schedules can always be measured correctly in the related factor market.
2. The derived demand schedule for an input is necessarily positively inclined given increasing returns to scale and fixed prices for all other inputs, and the schedule can be positively inclined over large areas of its domain given decreasing returns to scale and non-convex isoquants.

It was demonstrated that (2) implies that comparative static effects of policies influencing factor supply schedules can seldom be predicted by traditional methods.

In Chapter 3 and accompanying appendices, we introduced a method that in principle overcomes this defect of established comparative static methods by incorporating empirically based quantitative restrictions into the traditional qualitative comparative static analysis of the firm (e.g., Samuelson, 1947). This methodology incorporates the available degree of knowledge of the firm's structure (production function and price schedules) without imposing further specification on the firm's structure (in contrast to, e.g., the traditional linear and nonlinear programming models of the firm, where a full structure must be specified). Then the range of quantitative as well as qualitative predictions of comparative static effects of policy that are consistent with our degree of knowledge can in principle be calculated.

This methodology of "quantitative comparative statics" consists essentially of two nonlinear programming problems each characterized by an identical system of equations and inequalities which incorporate the implications of

(a) the standard assumption that the firm is at a static optimum, plus

(b) "reasonable" restrictions on the firm's production function and price schedules.

The comparative static effects, and the potentially observable parameters of the firm's production function and price schedules on which we have placed "reasonable" restrictions (upper and lower bounds), are all treated as endogenous variables in these two problems. These problems also have the same objective function, which is the comparative static effect of interest, and differ only in the sense that one is a maximization problem and the other is a minimization problem. Thus the solution values of the objective function for these two problems define the range of values for the comparative static effect of interest that are consistent with (a) the assumption of a static optimum and (b) the "reasonable" restrictions on the firm's production function and price schedules.[1]

The empirical knowledge embodied in the restrictions (b) typically would be derived from observation and/or econometric estimation of physical processes and behavior, and would be expressed in the form of confidence-Bayes intervals for these potentially observable parameters of the firm's production function and price schedules. In this case, the range of comparative static effects defined by the solution values to these two problems can also be interpreted as a confidence-Bayes interval for the comparative static effect of interest.

In Chapter 4, we (a) outlined a static linear programming model of a "representative" beef ranch for users of community pasture in the Peace River region of British Columbia, (b) formulated a means of examining the

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[1]A simple schematic model of the methodology of quantitative comparative statics was presented on pages 69-74 in Section 3.4 of Chapter 3.

appropriateness of constructs of static, optimizing behavior in the estimiation of farm response, and (c) observed that solutions (farm value of community pasture) to the linear programming model are consistent with these constructs. We demonstrated that, by estimating an equilibrium shadow price for an input rather than other aspects of farm response, one can reduce the significance of many of the problems associated with studies of supply response (e.g., the effects of poor knowledge of the individual farm's production function) and fucus more clearly on the appropriateness of constructs of static optimizing behavior. By comparing solutions to the static optimizing Peace River model with calculations based on direct observation of hayland rental markets and beef ranch activities, we derived empirical support for the major assumption underlying the theoretical work in Chapters 2 and 3: models of static optimizing behavior are often the most useful constructs in the prediction of microeconomic behavior. In addition, these results showed that models of microeconomic behavior with a fully specified structure, such as the static Peace River programming model, can be useful in estimating some aspects of farm response that are of importance to policy (e.g., farm value of community pasture programs) although they generally seem to be unreliable in estimating changes in input and output levels.

5.2 General Conclusions

The specific conclusions summarized in the previous section support the following broad statements:

1. Assumptions of static optimizing behavior are at present generally more appropriate than alternative constructs in predicting farm response to agricultural policy (Chapter 4 and Appendix 1).

2. Traditional qualitative methods of comparative statics lead to relatively few predictions that are useful in formulating agricultural policy (Chapter 2).

In addition, we have noted that there generally are extreme difficulties in reliably estimating comparative static effects from models of farm behavior with a fully specified structure, e.g., traditional linear and non-linear models of the firm (Section 1.2 of Chapter 1).

The above statements imply that an "intermediate" method of comparative static analysis making full use of our degree of knowledge of structure without being dependent on the specification of more than this degree of knowledge would be very useful in the evaluation of agricultural development programs. Thus the most important conclusions of this study are as follows:

3. Traditional qualitative methods of comparative statics (as in Samuelson, 1947) can be extended to incorporate our degree of knowledge of farm structure (production function and price schedules) without becoming dependent on a specification of more than this degree of knowledge (Chapter 3 and Appendix 3).

4. This method of "quantitative comparative statics" may well be at least somewhat operational now for "small" (or highly aggregated) models of the farm (Chapter 3 and Appendix 5).

Given the promise of this method of quantitative comparative statics, the initial work reported here in Chapter 3 and accompanying appendices should be followed by further studies.

5.3 Suggestions for Further Research

Here we shall point out some areas of future research that are suggested by this study. Since the most important and most experimental part of this thesis concerns the proposed methodology of quantitative comparative statics, we shall limit our comments to that section of the study.

First, there are major unresolved computational difficulties with this proposed method of comparative static analysis. In particular, due to the presence of quadratic equality constraints in the underlying programming models, local solutions are not necessarily global solutions for these models. Thus we cannot estimate with any accuracy the confidence-Bayes interval corresponding to the observed range of local solutions for the maximization and minimization problems unless

(a) there is a procedure for identifying a finite set of points that contains the global solutions to the maximization and minimization problems (so that the global solutions can be calculated from a comparison of these points), or

(b) there is a procedure for obtaining an approximately random sample of the feasible set of the programming problems (so that confidence-Bayes intervals can be estimated for the observed range of comparative static effects).

Thus the first priority for research related to this study should be to reduce computational problems associated with this method by developing somewhat adequate procedures of the form (a) or (b) above. The author speculates that this will be possible in the immediate future for small models involving a few inputs.[2,3]

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[2]For a very preliminary discussion of approaches other than (a) for estimating confidence-Bayes intervals of the observed range of solutions for comparative static effects, see Section 2.2 of Appendix 5.

[3]For an optimal procedure of aggregating large quantitative comparative static models into smaller models when the restrictions on the firm's production function imply that there exists an approximately correct aggregation procedure, see Section 3 of Appendix 5. Unfortunately the conditions for correct aggregation of large quantitative comparative static models to a more manageable size presumably introduces errors into the calculation of the global solutions and feasible set for the comparative static effect of interest. For a procedure that may become somewhat useful in estimating such aggregation biases for particular models, see the end of Section 3.1 of Appendix 5.

Given that these computational problems are handled somewhat adequately, there appear to be many applications in policy research and other empirical work for such a method of quantitative comparative statics. Here we shall simply point out several examples in order to illustrate the diversity of potential applications. First, the methodology could be employed to estimate the range of comparative static effects of community pasture programs that is consistent with our degree of knowledge about the structure of farms receiving this pasture. For example, we could construct quantitative comparative static models roughly similar in type to those presented in Section 3.5 of Chapter 3. Then we could obtain estimates of the range of "reasonable" comparative static changes in producer plus consumer surplus, i.e., of the confidence-Bayes interval for this effect that corresponds to the specified degree of knowledge of farm structure. For the reasons that have been specified, these results would be superior to those obtained by traditional methods of qualitative and quantitative comparative statics.

Second, this method of quantitative comparative comparative statics could be used to investigate the relation between the slope of a firm's derived demand schedule and properties of the firm's production function and price schedules in more detail than was possible with the qualitative methods employed in Chapter 2. Purely qualitative methods do not generally lead to empirically based restrictions on the slopes of derived demand schedules (that are independent of the slope of the factor supply schedule), and knowledge of this slope can be important for policy (see Chapter 2).

Third, it appears that interactions between firms can be incorporated into this methodology. In this case we would obtain a synthesis of maximizing behavior, firm interactions and empirical knowledge for the theory of the firm. With such an extended methodology, we might well be able to predict the effects of, e.g., national or provincial price support programs on farm output and other activities more effectively than in the past.[4,5]

Fourth, this method of quantitative comparative statics may well lead to more effective testing of various theories of firm behavior. Since traditional qualitative and quantitative methods have led to relatively few

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[4]The literature on integrating maximizing behavior and firm interactions seems to be summarized entirely in Silberberg (1974b). Since incorporation of interactions actually increases the ambiguity of results obtained by qualitative methods, there is an even greater need to incorporate empirical information into comparative static methodology for this case than for the standard (no firm interactions) qualitative theory of the firm.

[5]As pointed out previously (Section 2.3.3 of Chapter 2), the standard (Hicks-type) methods of industry analysis, which do not incorporate the implications of the maximization hypothesis, can easily lead to even qualitative errors in predicting comparative static effects.

reliable predictions of comparative static effects, these methods have also led to relatively few reliable tests of hypotheses concerning firm behavior.[6,7] On the other hand, the method of quantitative comparative statics introduced here in principle makes more effective use of the available degree of empirical knowledge than do these traditional methods. Thus this methodology should lead to a greater number of testable hypotheses than do traditional methods of qualitative comparative statics, and these hypotheses should discriminate between various theories of behavior more effectively than hypotheses derived from traditional quantitative methods.

In sum, there is a wide array of potential applications in policy research or other applied work for such a method of quantitative comparative statics. In turn, the potential value of such a methodology in the formulation of agricultural policy justifies further research to make it operational for a wide variety of problems.

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[6]For example, the comparative static predictions of two theories X and Y will generally vary with the structure of the firm's production function. This implies that qualitative methods generally will lead to relatively few predictions that discriminate between the two theories, and that traditional quantitative methods (by erring in their many essentially arbitrary specifications of aspects of the production function) may lead to a rejection of theory X for Y when in fact the reverse is true.

[7]See Archibald (1971) for a discussion of the difficulty in testing Chamberlin's theory of monopolistic competition when empirical knowledge is not incorporated into comparative statics.

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APPENDIX I

WHY COMPARATIVE STATICS AND THE MAXIMIZATION HYPOTHESIS?

Page

APPENDIX I

WHY COMPARATIVE STATICS AND THE MAXIMIZATION HYPOTHESIS?

1.1 Static vs. Dynamic Models

The purpose of this section is to point out the difficulties in modelling stock adjustments to a change in policy, and to argue that the comparative dynamic effect of the community pastures programs apparently can be estimated as accurately by the use of static models as by the use of dynamic models.

An example of a static model is the linear programming model of a beef ranch that will be presented in Chapter 4: this model has a time horizon of one year, and the endogenous opening and closing stocks are restricted to be equal. Therefore, by comparing model solutions in the absence and in the presence of community pasture (ceterus paribus), we can calculate a "comparative static effect" for the community pasture program. By defining an appropriate structure for the model, this effect can be "short-run," "long-run," or whatever.

However, such comparative static calculations can be, at best, only a very rough guide to the comparative dynamic effect of the pastures program. This is because a truly dynamic response primarily results from an effective cost of stock adjustment constraint,[1] and is a function

[1] In the absence of effective cost of stock adjustment constraints and overlooking the inherent lags in the production process, a comparative dynamic effect is simply a series of instantaneous adjustments to changing conditions, i.e., a series of comparative static effects. This series of comparative static effects is defined by the change in the supply schedule

(Footnote 1 continued on following page)

of

(1a) the non-stationary environment (e.g., the different product prices expected over time) perceived by the firm;

(1b) the initial stocks held by the firm; and

(1c) the cost of stock adjustment schedule $C_t = C(I_t)$ faced by the firm where I_t is the level of net investment by the firm at time t.

(1d) inherent lags in production.

Given an effective cost of stock adjustment constraint $C_t = C(I_t)$, none of these influences (1a)-(1d) on a comparative dynamic effect can be modelled correctly by a series of comparative static calculations.

In spite of these weaknesses of comparative static methods, dynamic models do not seem to be much more helpful (and may often be less helpful) than static models in estimating real-world dynamic response of firms to changes in the supply schedule of community pasture, or to changes in exogenous variables in many other situations. This is due to the following:

(Footnote 1 continued)

of community pasture and by changes in the firm's environment over time. In the absence of adjustment costs, the effects of inherent lags in production can be roughly accommodated in static models. For example, given the 2.5 year lag between the change in the beef herd and the resulting production of beef, and overlooking adjustment costs, farm response presumably could be simulated with reasonable accuracy by constructing a static model with a time period of three years. Stocks would be freely variable over the period subject to the constraint of equality at the beginning and end of the three year period.

(2a) the magnitudes of $C(I_t)$ and $C'(I_t)$ for users of community pasture (and in general) seem essentially unknown;

(2b) estimates of initial stocks, by themselves, generally provide little knowledge of comparative dynamic effects; and

(2c) errors in the valuation of the terminal stock occur in non-static models, and lead to errors in the estimation of response.

These points will be elaborated upon in the above order.

The significance of and arguments for statement (2a) are as follows. The difference between the comparative dynamic effect of a change in the supply schedule of pasture and a related series of comparative static effects depends critically upon the cost of adjustment function $C_t = C(I_t)$.[2,3]

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[2]The following generalizations seem correct (Rothschild, 1971). For a highly non-stationary environment, the "average" length of delay in response to pasture depends primarily upon the magnitude of $C(I_t)(>0)$ and the sign and magnitude of $C'(I_t)$, and the degree of fluctuation about this average depends upon the sign and magnitude of $C''(I_t)$. On the other hand, for a highly stationary environment, the average length of delay depends primarily upon the sign and magnitude of $C''(I_t)$ (<0 implies an extremely rapid, non-periodic response).

[3]It should be noted that adjustment costs should also play a role in comparative static calculations: adjustment costs should be incorporated into first and second order conditions for an equilibrium, whether dynamic or static (see Treadway, 1970). However, adjustment costs presumably have considerably more influence on the dynamics of response than on the change in static equilibrium. In this thesis we shall follow the usual procedure of deleting adjustment costs from comparative static calculations.

Hence, any claim for superiority of dynamic models over static models, as predictors of real-world comparative dynamic effects, presumably depends largely upon an ability to estimate the function $C_t = C(I_t)$ with some accuracy. However, an ability to predict the magnitudes of $C(I_t)$ and $C'(I_t)$ seldom seems to exist at present. $C_t = C(I_t)$ appears to be in large part a complex, and so far unidentified, function of such variables as education, elasticities of product and factor supply and demand,[4] and the particular exogenous change. Thus, it is not surprising that we seem to have very little knowledge of these magnitudes for the adjustment-constraining components of $C_t = C(I_t)$, and this in itself suggests that dynamic models seldom will be a significant improvement over static models as estimators of real-world comparative dynamic effects.[5,6]

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[4]See Petzel (1976).

[5]Presumably some components of $C_t = C(I_t)$ are more easily quantified than is indicated here. Perhaps the most obvious example concerns the effect of the firm's debt-equity ratio on its marginal cost of borrowing: $MCB_t = M(D/E_t)$, using obvious notation, and $D/E_t = D(K_t)$, i.e., in the short run the debt-equity ratio increases with the firm's capital stock. However, if the investments being considered by the firm involve only minor changes in techniques and provide relatively quick payoffs, then the firm is likely to face a constant marginal cost of borrowing schedule ($M' = 0$) and cash and credit costs of adjustment $\tilde{C}_t(I_t) \equiv M'D'$ will be zero. This appears to be largely the case for users of B.C. ARDA community pastures. More generally, adjustment will be affected significantly by cash and credit costs $\tilde{C}(I_t)$ presumably only if the firm would be expanding its enterprise in the absence of such costs and $\tilde{C}(I_t)' \equiv M''D'' > 0$.

[6]For a first attempt at the statistical estimation of adjustment cost functions (in manufacturing), see Berndt <u>et al.</u> (1979).

Statement (2b) can be explained briefly as follows. We are interested in the comparative dynamic effect of the pasture program, i.e., the *difference* between the time paths in the presence and in the absence of community pasture. This difference presumably is considerably less dependent on the level of initial stocks, and more dependent on the specified adjustment costs, than are these two time paths. Moreover, time paths are known to be highly sensitive to errors in specifying initial conditions.[7] Hence, even though initial conditions can be incorporated more correctly into dynamic models than into one period models, in general this does not appear to provide dynamic models with a significant advantage over static models as predictors of comparative dynamic effects.

The argument for and significance of statement (2c) is as follows. In dynamic models, capital accumulated at the horizon must be assigned an exogenously-determined per unit value in the objective function, which represents an estimate of the capital's discounted net value in production beyond the horizon. Since the farm value of used capital is in fact endogenous to the farm plan,[8] this procedure inevitably leads to errors in specifying the terminal value of capital. Given such a

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[7]The sensitivity of time paths to initial conditions is documented in growth theory literature, and has been confirmed by simulations with multi-period farm planning models (Boussard, 1971, pp. 475-7).

[8]Due to serious imperfections in markets for used capital (except for the regularly-traded fully depreciated capital, such as cull cows), the farm value of capital at the end of the model year seldom corresponds to the market price. Moreover, even if capital markets accurately reflect current farm value of capital, we would still not be able to compute the farm value of capital that would be consistent with a particular alternative set of expected prices, etc.

mispecification, the time horizon of the dynamic model must be considerably longer than the average life of capital even if the sole intent is to obtain reasonably accurate estimates of activities in the initial time period. Since cows have a productive life of approximately eight years, a dynamic model intended for use in estimating effects of community pasture programs would in general have to be unwieldy, or else extremely simplified within most years,[9] in order to reduce the effects of such a mispecification to insignificant levels. Corresponding problems never occur with static models.[10] Thus, in the presence of very limited knowledge of the magnitude of $C(I_t)$ and $C'(I_t)$ for the firm's cost of stock adjustment function $C_t = C(I_t)$, the solutions of various static models may well provide more information about the comparative dynamic effects of the community pasture programs than will the solutions of dynamic models.[11]

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[9]By defining a highly simplified structure for all but the first year in a dynamic model with a long time horizon (and estimating a dynamic response as the series of first year solutions obtained from recursive runs of the model), we will in general simply be exchanging errors due to a mispecified terminal value of capital for errors due to excessive aggregation.

[10]This statement is justified simply as follows. If the firm's environment and actions in the one year time period of a static model are in effect repeated in all other one year time periods, then the actions that maximize the value of the objective function (flow of farm benefits) in the one year model will also maximize the discounted sum of flows of farm benefits over time.

[11]Since the "flexibility constraint" approach to dynamics (Sahi and Craddock, 1974) incorporates historically observed measures of response over time rather than adjustment cost functions per se, it is not a satisfactory approach to dynamics. In other words, the dynamics of response is not specified as endogenous to the farm in the flexibility approach.

In sum, apparently the best that we can do in estimating the real world responses to community pasture programs is to calculate various comparative static effects for the programs. For simplicity (and also in part due to a lack of confidence in any estimates of the rate of change in the rate of change of the firm's environment at any particular time), these calculations can be limited to "short run" and "long run" comparative static effects. Then the estimated comparative dynamic effect would simply be the straight line connecting the "short run" and "long run" comparative static effects.[12]

1.2 Optimizing vs. Non-Optimizing Models

It is sometimes stated that non-optimizing simulation models are superior to optimizing (or, equivalently, maximizing) models as predictors of farm behavior because "farmers do not optimize." However, we will now argue that this conclusion is incorrect.[13]

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[12]Even this procedure of calculating "short run" and "long run" comparative static effects often may be based on inappropriate assumptions. In particular, a comparative dynamic change at time t, will be similar to a comparative static effect only under certain conditions, e.g., certain properties of adjustment cost functions (Rothschild, 1971), indivisibilities and imperfect capital markets. If these conditions are not sufficiently realistic, then the comparative dynamic change at t, may even have opposite signs from a "short run" comparative static effect calculated for t_1, and the comparative dynamic change over time may bear no resemblance to the time path calculated from the "short run" and "long run" comparative static effects (Nagatani, 1976).

[13]It is known that purposive behavior of microeconomic units can in principle be described by optimization techniques when the decision-making unit's preferences are "consistent," and that inconsistency of preferences can arise when behavior is governed by rules-of-thumb (or is determined collectively). See, e.g., Samuelson (1950). Here we note that

It is a tautology to state that an individual decision-maker always obtains a constrained maximum defined by his preferences, resources and the external environment. In other words, the assumption of purposive behavior implies the existence of a constrained maximum, properly defined. Therefore, such farm behavior can always be described analytically in terms of an appropriate optimization model, and "non-optimal" aspects of such behavior (use of rules-of-thumb in decision-making rather than a "global" search procedure) can always be interpreted as reflections of various types of human capital adjustment costs. However, adjustment costs are by definition zero in a stationary state model, and the influence of human capital adjustment costs on behavior in a non-stationary model cannot at present be predicted with any accuracy.[14]

Therefore,

1. the "non-optimal" aspects of behavior cannot be predicted at present with any degree of accuracy by farm planning models, and

2. actual behavior can be "approximated" by use of a static equilibrium optimization model, with the extent of the approximation depending upon the relevant adjustment cost functions and the rate of change in the firm's environment.

Footnote 13 continued

(a) rules-of-thumb can in principle be incorporated into optimization models as adjustment costs, and (b) static equilibrium optimization models are in practice often superior to static or dynamic non-optimization models as predictors of microeconomic behavior.

[14]See the previous section of this appendix.

Thus there appears to be no point in attempting to incorporate "non-optimal" behavior into models designed to predict farm response to changes in policy. Moreover, non-optimizing models appear to be in principle inferior to optimizing models as predictors of such response precisely because the "optimizing" central tendency of behavior, in contrast to the "non-optimizing" aspects of behavior, can on occasion be modelled with some accuracy.

A comparison of the marginal value of community pasture obtained by the static linear programming model to be presented in Chapter 4 (and Appendix 6) and by other means supports these theoretical arguments, and also suggests that this particular static optimizing model provides reasonably accurate measures of this value. In order to carry out an evaluation of ARDA community pasture programs in Saskatchewan, a large non-optimizing simulation model was adapted to conditions there.[15] This study led to considerably different (generally higher) estimates of the farm marginal value of community pasture than did the British Columbia study to be reported in Chapter 4. However, when the most important of the data gathered for the simulation model was incorporated into the British Columbia optimizing model, the estimated farm marginal value of community pasture was less than the value calculated with British Columbia data. Since pasture appears to be a more scarce resource in British Columbia than in Saskatchewan, one would expect (ceteris paribus) a higher marginal value for pasture in British Columbia than in Saskatchewan.

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[15]See Department of Regional Economic Expansion (1977).

Moreover, the marginal value of community pasture estimated by using British Columbia data in the optimizing model was consistent with an essentially independent measure derived from data in the hay market.

APPENDIX II

QUALITATIVE COMPARATIVE STATICS AND DERIVED DEMAND: PROOFS

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APPENDIX II

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1. Preliminaries

Let

x	$\equiv$	$N \times 1$ vector of activity levels for the inputs of a firm
$c^i(x)$	$\equiv$	total cost schedule to the firm for its i^{th} input ($i \neq 1$)
$c^1(x;\alpha)$	$\equiv$	total cost schedule to the firm for its input 1, as a function of x and a parameter α
y	$\equiv$	$M \times 1$ vector of activity levels for the M outputs of a firm
$y = f(x)$	$\equiv$	production function (vector-valued for $M > 1$) for the firm
$b(y)$	$\equiv$	total benefits schedule to the firm as a (scalar-valued) function of its M outputs
$R(x)$	$\equiv$	$b(f(x))$
x^*	$\equiv$	$N \times 1$ vector of the input levels employed by the firm at a solution to a particular maximization problem
$\overline{x}^1$	$\equiv$	an exogenously determined level of input 1 employed by the firm

Definition 1. A producer problem P is defined as

$$\text{maximize } \pi(x)^P \equiv R(x) - c^1(x;\alpha) - \sum_{i=2}^{N} c^i(x) \qquad \text{....(P)}$$

for a particular value of the exogenous variable α, and the solution set to this problem is denoted as $\{x^{*P}(\alpha)\}$.

Definition 2. The firm's derived demand schedule for input 1 is defined as

$$\{(x^{1*P}(\alpha),\ MFC^1(\alpha)) \text{ for all } \alpha\} \equiv D^P$$

where

$$MFC^1(\alpha) \equiv \frac{\partial c^1(x^{*P}(\alpha);\alpha)}{\partial x^1} .$$

Denote the relation defined by the pairs in D^P as $p^1 = p^1(x^1)$.

Definition 3. A producer problem Q is defined as

$$\begin{aligned} &\text{maximize } \pi(x)^Q \equiv R(x) - \sum_{i=2}^{N} c^i(x) \\ &\text{subject to} \quad x^1 = \overline{x^1} \end{aligned} \qquad \text{....(Q)}$$

for a particular value of the exogenous variable $\overline{x^1}$, and the solution set to this problem is denoted as

$$\{x^{*Q}(\overline{x^1})\} .$$

Definition 4. The firm's shadow price schedule for input 1 is defined as

$$\left\{ (\overline{x^1},\ \frac{\partial \pi\left(x^{*Q}(\overline{x^1})\right)^Q}{\partial \overline{x^1}} \quad \text{for all} \quad \overline{x^1} \in X^1) \right\} \equiv D^Q$$

where

$$X^1 \equiv \{x^{1*P}(\alpha) \quad \text{for all} \quad \alpha\} .$$

Definition 5. Any problem P

$$\text{maximize } R(x) - c^1(x;\alpha) - \sum_{i=2}^{N} c^i(x)$$

is said to "correspond" with the series of problems of the form Q

$$\text{maximize } R(x) - \sum_{i=2}^{N} c^i(x) \qquad \text{maximize } R(x) - \sum_{i=2}^{N} c^i(x)$$

$$\text{subject to } x^1 = \overline{x^{1*A}},\ \cdots\cdots\cdots,\ \text{subject to } x^1 = \overline{x^{1*Z}}$$

where

$$\{x^{1*A}, \cdots, x^{1*Z}\} = \{x^{1*P}\} \text{ for the problem P.}$$

Denote the union of solution sets for this series of problems of the form Q as $\{x^{*Q}\}^c$.

Condition 1. For any solution x^* to a problem P: $x^{i*} > 0$, $i = 1, \cdots, N$.

Condition 2. In the neighbourhood of any solution to a problem P: $R(x)$ and all $c^i(x)$ are twice differentiable.

Condition 3. $c^1(x;\alpha) \Longleftrightarrow c^1(x^1;\alpha)$, i.e., the total cost of input 1 is independent of the levels of inputs $2, \cdots, N$.

Condition 4. $\dfrac{\partial^2 c^i(x)}{\partial x^j \partial x^k} \geqq 0$ for all x and $i, j, k = 1, \cdots, N$, i.e., factor supply prices are non-decreasing in x.

Condition 5. $\dfrac{\partial R(x)}{\partial x^i} \geqq 0$ for all x and $i = 1, \cdots, N$, i.e., input are "freely disposable."

Condition 6. If the set of feasible $\pi(x)^Q$ for a problem Q is bounded from above, then the set is also closed from above.

2. Lemma 1

Lemma 1. Suppose that conditions 1-2 are satisfied for a problem Q. Then

$$\frac{\partial \pi(x^{*Q})^Q}{\partial x^1} = \frac{\partial R(x^*)}{\partial x^1} - \sum_{i=2}^{N} \frac{\partial c^i(x^*)}{\partial x^1}$$

for any

$$x^* \in \{x^{*Q}\} .$$

Proof 1.2. Construct the problem Q

$$\text{maximize } \pi(x)^Q \equiv R(x) - \sum_{i=2}^{N} c^i(x) \qquad \text{....(a)}$$

$$\text{subject to } x^1 = \overline{x^1} \ .$$

By conditions 1-2,

$$R_j(x^*) - \sum_{i=2}^{N} c_j^i(x^*) = 0 \quad \text{for all} \quad j \neq 1 \qquad \text{....(b)}$$

which are first order conditions for a solution. By definition 3, $\frac{\partial \pi(x^*)^Q}{\partial \overline{x^1}}$ for this problem Q can be calculated as

$$\frac{\partial \pi(x^*)^Q}{\partial \overline{x^1}} = R_1(x^*) + \sum_{j\neq 1} R_j(x^*)\frac{\partial x^{j*}}{\partial \overline{x^1}} - \sum_{i\neq 1} c_1^i(x^*)$$

$$- \sum_{i\neq 1} \sum_{j\neq 1} c_j^i(x^*) \frac{\partial x^{j*}}{\partial \overline{x^1}} \qquad \text{....(c)}$$

$$= R_1(x^*) - \sum_{i\neq 1} c_1^i(x^*) + \sum_{j\neq 1} \frac{\partial x^{j*}}{\partial \overline{x^1}} [R_j(x^*) - \sum_{i=1} c_j^i(x^*)] \qquad \text{....(d)}$$

by rearranging (c). Substituting (b) into (d),

$$\frac{\partial \pi(x^*)^Q}{\partial \overline{x^1}} = R_1(x^*) - \sum_{i\neq 1} c_1^i(x^*). \qquad \text{....(e)}$$

$\dfrac{\partial \pi(x^*)^Q}{\partial x^1}$ exists by (e) and $\{x^{*Q}\}$ not null, and $\dfrac{\partial \pi(x^*)^Q}{\partial x^1}$

is unique for Q by Definition 3; so (e) holds for any $x^* \varepsilon \{x^{*Q}\}$. □

1. Lemma 1 can be deduced almost directly from the Viner-Wong envelope theorem of Samuelson, which states that the first order change in the value of the objective function $\pi(x;\alpha)$ as x varies optimally (from an initial interior solution) in response to a change in an exogenous variable α is equal to the change in $\pi(x^*;\alpha)$ for $dx = 0$, i.e.,

$$\frac{\partial \pi(x^*(\alpha),\alpha)}{\partial \alpha} = \frac{\partial \pi(\overline{x^*},\alpha)}{\partial \alpha}$$

where

$$\frac{\partial \overline{x^*}}{\partial \alpha} = 0 \qquad \text{(Samuelson, 1947, p. 34).}$$

2. In proofs, partial derivatives will generally be denoted by subscripts. For example,

$$\frac{\partial R(x)}{\partial x^1} \equiv R_1(x) \quad \text{and} \quad \frac{\partial c^i(x)}{\partial x^j} \equiv c^i_j(x) \quad .$$

3. Theorem 1

<u>Theorem 1.</u> Suppose that conditions 1-3 are satisfied. Then

(A) $\{x^{*P}(\alpha)\} \iff \left\{x^{*Q}(x^{1*P}(\alpha))\right\}^{c}$ for all α

i.e., any problem P and the corresponding problem (s)Q have identical solution sets; and

(B) $\left\{(x^{1*P}(\alpha),\ MFC^{1}(\alpha))\ \text{for all}\ \alpha\right\} \iff \left\{(x^{1*P}(\alpha),\ \dfrac{\partial\pi\left\{x^{*Q}(x^{1*P}(\alpha))\right\}^{Q}}{\partial \overline{x^{1}}})\ \text{for all}\ \alpha\right\}$ i.e., $D^{P} \iff D^{Q}$.[1]

<u>Proof.</u> By condition 1 and definitions 1-2, x^{*P} is a solution to the problem

$$\text{maximize } \pi(x)^{P} \equiv R(x) - c^{1}(x^{1};\alpha) - \sum_{i=2}^{N} c^{i}(x) \text{ .} \qquad \text{. . . . (a)}$$

Construct the related series of problems

$$\begin{array}{lcl} \text{maximize } \pi(x)^{P} & & \text{maximize } \pi(x)^{P} \\ \text{subject to } x^{1} = x^{1*A}, & \cdots\cdots\cdots, & \text{subject to } x^{1} = x^{1*z} \end{array} \qquad \text{. . . . (b)}$$

[1]Formally Theorem 1 only applies to the case where input 1 is employed in a single enterprise, since the cost schedule for input 1 is defined as a function of only one input. However, Theorem 1 readily generalizes to the firm that employs input 1 in M enterprises. In this case, we can define $c^{1} \equiv c^{1}(\sum_{j=1}^{M} x^{1j};\alpha)$ and the quantity constraint in a corresponding Producer Problem Q as $\sum_{j=1}^{M} x^{1j} = \overline{x^{1}}$. It is easily shown that, with these obvious modifications, Theorem 1 applies to the multi-enterprise firm as well as to the single enterprise firm.

where $(x^{1*A},\cdots,x^{1*z}) = \{x^{1*P}\}$ for problem (a), and problems (b) and problem (a) have identical objective functions. By (a) and (b),

$$\{x^{*P}\} = \left\{\{\tilde{x}^{*}\}\right\} \qquad \ldots\ldots(c)$$

where

$\left\{\{x^{*}\}\right\} \equiv$ the set of solutions for series (b).

Since (b) implies that

$\left\{\{\tilde{x}^{1*}\}\right\}$ is exogenous to problems (b), and therefore

$\left\{\{c^{1}(\tilde{x}^{1*};\alpha)\}\right\}$ is exogenous to problems (b),

$\left\{\{\tilde{x}^{*}\}\right\}$ is independent of the specification of $c^{1}(x^{1};\alpha)$ in problems (b), $\qquad \ldots\ldots(d)$

which includes the specification $c^{1}(x^{1};\alpha) = 0$ for all x^{1}. By (c) and (d), for any α

$$\{x^{*P}\} = \{x^{*Q}\}^{c} \qquad \ldots\ldots(***)$$

where

$\{x^{*Q}\}^{c} \equiv$ the set of solutions for the series of problem Q's

corresponding to problem (a) (see Definition 5), which is statement A of the Theorem. By conditions 1-2 and (***), Lemma 1 can be used to calculate

$$\frac{\partial \pi(x^{*P})^Q}{\partial x^1} = R_1(x^{*P}) - \sum_{i=2}^{N} c_1^i(x^{*P}) \qquad \ldots .(e)$$

at any solution x^{*P} for any problem Q corresponding to problem (a). By conditions 1-2,

$$R_1(x^{*P}) - c_1^1(x^{1*P}; \alpha) - \sum_{i=2}^{N} c_1^i(x^{*P}) = 0 \qquad \ldots .(f)$$

which is a first order condition for an interior solution to problem (a). By (e)-(f),

$$\frac{\partial \pi(x^{*P})^Q}{\partial x^1} = c_1^1(x^{1*P};\alpha) \quad . \qquad \ldots .(g)$$

By (***), (g), and Definitions 2 and 4,

$$D^P = D^Q$$

which is statement B of the Theorem. $\square$

4. <u>Corollary 1</u>

<u>Corollary 1.</u> Suppose that conditions 1-2 are satisfied, and that for all α:

$$\frac{\partial c^1(x^{*P}(\alpha);\alpha)}{\partial x^i} \neq 0 \quad \text{for at least one } i \neq 1.$$

Then

(A) $\{x^{*P}(\alpha)\} \cap \{x^{*Q}(x^{1*P}(\alpha))\}^c$ = null set for all α ,

i.e., any problem P and any corresponding problem Q do not have any solutions in common; and

(B) for any α: $\{(x^{1*P}(\alpha),\ MFC^1(\alpha))\} \cap D^Q \neq$ null set

if and only if

$$\frac{\partial R(x^{*P})}{\partial x^1} - \sum_{i=2}^{N} \frac{\partial \epsilon^i(x^{*P})}{\partial x^1} = \frac{\partial R(x^{*Q})}{\partial x^1} - \sum_{i=2}^{N} \frac{\partial c^i(x^{*Q})}{\partial x^1}$$

for all (or, equivalently, any)

$x^{*P} \in \{x^{*P}(\alpha)\}$

$x^{*Q} \in \{x^{*Q}(x^{1*P}(\alpha))\}^c$.[2]

<u>Proof</u>. By conditions 1-2, x^{*P} is any solution to the problem

$$\text{maximize } \pi(x)^P \equiv R(x) - \sum_{i=1}^{N} c^i(x) \qquad \text{. . . . (a)}$$

where

---

[2] Corollary 1-B also assumes that conditions 5-6 are satisfied.

$$x^{i*P} > 0 \qquad i = 1,\cdots,N \quad \ldots.(b)$$

$$R_j(x^{*P}) - \sum_{i=1}^{N} c_j^i(x^{*P}) = 0 \qquad j = 1,\cdots,N. \quad \ldots.(c)$$

By assumption, $c^1(x;\alpha)$ has input 1 and at least one other input (e.g., N) as its arguments, where

$$c_N^1(x^{*P};\alpha) \neq 0 \ . \quad \ldots.(d)$$

Construct the problem Q

$$\text{maximize } R(x) - \sum_{i=2}^{N} c^i(x)$$

$$\text{subject to } x^1 = x^{1*P} \ . \quad \ldots.(e)$$

By (b),

x^{*P} is a solution to problem (e) only if

$$R_j(x^{*P}) - \sum_{i=2}^{N} c_j^i(x^{*P}) = 0 \qquad j = 2,\cdots,N. \quad \ldots.(f)$$

By (c) and (d),

$$R_N(x^{*P}) - \sum_{i=2}^{N} c_N^i(x^{*P}) \neq 0 \ . \quad \ldots.(g)$$

By (f)-(g),

for any P where $c_i^1(x^{*P};\alpha) \neq 0$ for some $i \neq 1$

$$\text{and all } x^{*P}, \qquad \{x^{*P}\} \cap \{x^{*Q}\}^c = \text{null set}, \qquad \ldots\ldots(***)$$

which is statement A of the Theorem. By (c),

$$R_1(x^{*P}) - \sum_{i=2}^{N} c_1^i(x^{*P}) = c_1^1(x^{*P};\alpha) \qquad \ldots\ldots(h)$$

for problem (a). By Lemma 1,

$$\frac{\partial \pi(x^*)^Q}{\partial x^1} = R_1(x^{*Q}) - \sum_{i=2}^{N} c_1^i(x^{*Q}) \qquad \ldots\ldots(i)$$

for problem (e). By statement (f) in the proof of Corollary 2,

$$\frac{\partial \pi(x^*)^Q}{\partial x^1} \text{ is single-valued for a given problem (e).}^3 \qquad \ldots(j)$$

By (h)-(j),

for a given $(x^{1*P}, c_1^1(x^{*P};\alpha))$ on D^P and a related solution x^{*P} to a P, there exists an identical $\left(x^{1*Q}, \frac{\partial \pi(x^*)^Q}{\partial x^1}\right)$ on D^Q if and only if

$$R_1(x^{*P}) - \sum_{i=2}^{N} c_1^i(x^{*P}) = R_1(x^{*Q}) - \sum_{i=2}^{N} c_1^i(x^{*Q})$$

---

[3]Statement (f) in the proof of Corollary 2 depends on conditions 2, 5 and 6 but not on Theorem 1.

for all (or, equivalently, any) solutions x^{*Q} to the corresponding Q's. $\square$

5. Corollary 2

Corollary 2. Suppose that conditions 1-3 and 5-6 are satisfied, and denote the domain of $p^1 = p^1(x^1)$ as X^D. Then

(A) if x^{1B} is included in a solution to at least one problem P, then all x^{1A} such that $0 < x^{1A} < x^{1B}$ are in X^D,

(B) p^1 is a function of x^1, i.e., $p^1(x^1)$ associates one and only one p^1 with any particular x^1 in X^D,

(C) $p^1(x^1)$ is differentiable for all x^1 "within" X^D, i.e., for all x^1 such that $0 < x^1 < x^{1A}$ and x^{1A} is an element of X^D.

Proof. By condition (5),

$$\max \pi(\overline{x^{1A}})^Q \leqq \max \pi(\overline{x^{1B}})^Q \quad \text{if} \quad \overline{x^{1A}} < \overline{x^{1B}} \qquad \ldots .(a)$$

where $\max \pi(\overline{x^1})^Q$ is the maximum attainable value of the objective function $R(x) - \sum_{i \neq 1} c^i(x)$ for the problem Q defined by the constraint $x^1 = \overline{x^1}$. By (a), if Q with the constraint $x^1 = \overline{x^{1B}}$ is bounded,

. . . .(b)

then Q with $x^1 = \overline{x^{1A}}$, where $\overline{x^{1A}} < \overline{x^{1B}}$,

is also bounded

where Q is defined as bounded for $\overline{x^1}$ if and only if

$$\max \pi(\overline{x^1})^Q = k \quad \text{or} \quad \max \pi(\overline{x^1})^Q \to k \quad ,$$

for a real number k. By condition (6) and (b),

if Q with the constraint $x^1 = \overline{x^{1B}}$ has a solution,

then Q with $x^1 = \overline{x^{1A}}$, where $x^{1A} < \overline{x^{1B}}$, ...(c)

also has a solution.

By condition (2) and (c),

if Q with the constraint $x^1 = \overline{x^{1B}}$ has a solution,

then $\dfrac{\partial \pi(x^*(\overline{x^{1A}}))}{\partial \overline{x^1}}$ is defined for all . . . .(d)

$\overline{x^{1A}} < \overline{x^{1B}}$.

By (d) and Theorem 1,

if x^{1B} is included in a solution to at least one P,
then all x^{1A} such that $0 < x^{1A} < x^{1B}$ are in the domain X^D of $p^1(x^1)$ . . . (***)

which is statement A of the Corollary. By Definition 3,

$\max \pi(\overline{x^1})^Q \equiv \pi(x^*(\overline{x^1}))^Q$ exists and is single-valued for each $\overline{x^1}$ where a solution x^* exists for the Q. . . . .(e)

By (d), (e) and statement A(***),

$\dfrac{\partial \pi(x^*(\overline{x^1}))^Q}{\partial \overline{x^1}}$ is single-valued for each $x^{1A} \equiv \overline{x^1}$ such that $0 < x^{1A} < x^{1B}$ and Q has a solution for $\overline{x^1} \equiv x^{1B}$ . . . .(f)

i.e., for each $x^{1A} \equiv \overline{x^1}$ an element of X^D. By (f) and Theorem 1,

$p^1(x^1)$ is a single-valued for all x^1 an element of X^D. . . (***)

which is statement B in the Corollary. By (f), condition (2) and Lemma 1,

$\dfrac{\partial \pi(x^*(\overline{x^1}))^Q}{\partial \overline{x^1}}$ is a differentiable function of $x^1 \equiv \overline{x^1}$ for all x^1 . . . .(g) such that $0 < x^1 < x^{1A}$ and x^{1A} is an element of X^D (g continued)

i.e., for all x^1 within X^D. By (g) and Theorem 1,

$p^1(x^1)$ is differentiable for all x^1 within X^D . . . .(***)

which is statement c of the Corollary. □

6. Corollary 3

Corollary 3. Suppose that conditions 1-3 and 5-6 are satisfied.

Then

(A) for any solution x^{*A} to a problem P where $\alpha = \alpha^A$,

$$\pi(x^{*A})^P = \pi(x^*(0))^Q + \int_0^{x^{1*A}} p^1(x^1)dx^1 - c^1(x^{1*A}; \alpha^A)$$

where

$$\pi(x^*(0))^Q \equiv \max\{R(x) - \sum_{i=2}^{N} c^i(x):x^1 = 0\},$$

$$p^1(0) \equiv \left.\frac{\partial \pi(x^*(0))^Q}{\partial x^1}\right|_{\dagger}, [4]$$

[4] Any right hand side derivative $\lim_{\Delta x^1 \to 0} \frac{f(x^1 + \Delta x^1) - f(x^1)}{\Delta x^1}$ for $\Delta x^1 > 0$ is represented here as $\left.\frac{\partial f(x^1)}{\partial x^1}\right|_{\dagger}$.

and

(B) for a solution x^{*A} and a solution x^{*B} to two problem P's that differ only in terms of $\alpha = \alpha^A$ and $\alpha = \alpha^B$, respectively,

$$\pi(x^{*B})^P - \pi(x^{*A})^P = \int_{x^{1*A}}^{x^{1*B}} p^1(x^1)dx^1 - c^1(x^{1*B};\alpha^B) + c^1(x^{1*A};\alpha^A).$$

Proof. Let x^{*A} be a solution to a P. By conditions 1-2 and Theorem 1-A,

x^{*A} is a solution to the corresponding Q $(x^1 = \overline{x^{1*A}})$. . . . .(a)

By (a), Theorem 1-B, Corollary 2-A and 2-C,

$\dfrac{\partial\, \pi(x^*(\overline{x^1}))^Q}{\partial \overline{x^1}}$ is defined and continuous for all

$0 < \overline{x^1} \leqq x^{1*A}$ . . . . .(b)

By (a), and by (c) in the proof of Corollary 2,

$\pi(x^*(0))^Q$ exists, ...(c)

By (b)-(c),

$\left.\dfrac{\partial\, \pi(x^*(0))^Q}{\partial \overline{x^1}}\right|_+$ exists . . . . .(d) where

$$\left.\frac{\partial\,\pi(x^{*}(0))^{Q}}{\partial\,\overline{x^{1}}}\right|_{+} \equiv \lim_{\overline{\Delta x^{1}}\to 0} \frac{\pi(x^{*}(\overline{\Delta x^{1}}))^{Q} - \pi(x^{*}(0))^{Q}}{\overline{\Delta x^{1}}}$$

for all $\overline{\Delta x^{1}} > 0$.

By (b)-(d) and the definition of

$$\frac{\partial\,\pi(x^{*}(\overline{x^{1}}))^{Q}}{\partial\,\overline{x^{1}}} \;,$$

$$\pi(x^{*A})^{Q} = \pi(x^{*}(0))^{Q} + \int_{0}^{x^{1*A}} \frac{\partial\,\pi(x^{*}(\overline{x^{1}}))^{Q}}{\partial\,\overline{x^{1}}}\,d\overline{x^{1}} \qquad \ldots.(e)$$

where

$$\frac{\partial\,\pi(x^{*}(0))^{Q}}{\partial\,\overline{x^{1}}} \equiv \left.\frac{\partial\,\pi(x^{*}(0))^{Q}}{\partial\,\overline{x^{1}}}\right|_{+} .$$ By (f), Definition

1 and 3,

$$\pi(x^{*A})^{P} = \pi(x^{*}(0))^{Q} + \int_{0}^{x^{1*A}} \frac{\partial\,\pi(x^{*}(\overline{x^{1}}))^{Q}}{\partial\,\overline{x^{1}}}\,d\overline{x^{1}}$$

$$- c^{1}(x^{1*A};\alpha^{A}) \;. \qquad \ldots.(f)$$

By (f),

$$\pi(x^{*B})^{P} - \pi(x^{*A})^{P} = \int_{x^{1*A}}^{x^{1*B}} \frac{\partial\,\pi(x^{*}(\overline{x^{1}}))^{Q}}{\partial\,\overline{x^{1}}}\,d\overline{x^{1}}$$

$$- c^{1}(x^{1*B};\alpha^{B}) + c^{1}(x^{1*A};\alpha^{A}) \;. \qquad \ldots.(g)$$

By (f), (g), condition 3 and Theorem 1,

$$\pi(x^{*A})^P = \pi(x^*(0))^Q + \int_0^{x^{1*A}} p^1(x^1)dx^1 - c^1(x^{1*A};\alpha^A)$$

$$\pi(x^{*B})^P - \pi(x^{*A})^P = \int_{x^{1*A}}^{x^{1*B}} p^1(x^1)dx^1 - c^1(x^{1*B};\alpha^B)$$

$$+ c^1(x^{1*A}: \alpha^A)$$

which are statements A and B of the Corollary. □

7. Corollary 4

Corollary 4. Suppose that conditions 1-3 are satisfied for a problem P.

(A) If x^{1A} is included in a local solution to P, then

$$p^1(x^{1A}) - \frac{\partial c^1(x^{1A};\alpha)}{\partial x^1} = 0$$

$$\frac{\partial p^1(x^{1A})}{\partial x^1} - \frac{\partial^2 c^1(x^{1A};\alpha)}{\partial x^{1\,2}} \leqq 0 \quad .$$

(B) If $p^1(x^{1A}) - \dfrac{\partial c^1(x^{1A};\alpha)}{\partial x^1} = 0$

$$\frac{\partial p^1(x^{1A})}{\partial x^1} - \frac{\partial^2 c^1(x^{1A};\alpha)}{\partial x^{1\,2}} < 0 \quad ,$$

then x^{1A} is included in a local solution to P.

Proof. By conditions 1-2 and Definition 2,

x^A included in a local solution to P =>

$$p^1(x^{1A}) - c^1_1(x^A;\alpha) = 0 \qquad \ldots.(a)$$

$[\pi_{ij}(x^A)^P]$ negative semi-definite, and

$p^1(x^{1A}) - c^1_1(x^A;\alpha) = 0$ and $[\pi_{ij}(x^A)^)]$ negative definite

=> x^A a local solution to P . . . . .(b)

Since

$$\sum_{j=1}^{N} \pi_{ij}(x^*)^P \frac{\partial x^{j*}}{\partial\alpha} = 0 \quad \text{for } i = 2,\cdots,N \quad ,$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \pi_{ij}(x^*)^P \frac{\partial x^{i*}}{\partial\alpha}\frac{\partial x^{j*}}{\partial\alpha} = \sum_{j=1}^{N} \pi_{1j}(x^*)^P \frac{\partial x^{1*}}{\partial\alpha}\frac{\partial x^{j*}}{\partial\alpha}$$

$$= \sum_{j=1}^{N} \pi_{1j}(x^*)^P \frac{\partial x^{1*}}{\partial\alpha}\frac{\partial x^{j*}}{\partial \overline{x^1}}\frac{\partial x^{1*}}{\partial\alpha}$$

. . . . .(c)

by condition 3 and Theorem 1-A. By (c) and Theorem 1-B,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \pi_{ij}(x^*)^P \frac{\partial x^{i*}}{\partial\alpha}\frac{\partial x^{j*}}{\partial\alpha} = (p^1_1(x^{1*}) - c^1_{11}(x^{1*};\alpha))$$

$$\frac{\partial x^{1*2}}{\partial\alpha} . \qquad \ldots.(d)$$

By (a)-(b) and (d),

Corollary 4-A and 4-B are established. □

8. Lemma 2

Lemma 2. Consider a problem Q

$$\text{maximize } \pi(x)^Q \qquad \ldots.(a)$$
$$\text{subject to } x^1 = \overline{x^1}$$

where

$\pi(x)^Q$ is twice differentiable in the neighbourhood of an interior solution x^* (not necessarily unique). Also construct the related problems

$$\text{maximize } \pi(x)^Q \qquad \ldots.(b)$$
$$\text{subject to } x^1 = \overline{x^1} + \overline{\Delta x^1}$$

and

$$\text{maximize } \frac{\Delta^2\pi(x^{*1})^Q}{(\Delta x^1)^2} \qquad \text{maximize } \frac{\Delta^2\pi(x^{*Z})^Q}{(\Delta x^1)^2} \qquad \ldots(c)$$

$$\text{subject to } \Delta x^1 = \overline{\Delta x^1}, \cdots\cdots\cdots\cdots, \text{subject to } \Delta x^1 = \overline{\Delta x^1}$$

where

$$\Delta^2\pi(x^*)^Q \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^*)^Q \, \Delta x^i \, \Delta x^j,$$

$\{x^{*1},\cdots,x^{*Z}\}$ is the solution set to problem (a), and $(\Delta x^2,\cdots,\Delta x^N)$

is the vector of endogenous variables for problems (c).

Then

(A) $$\left\{\frac{x^{*b} - x^{*a}}{\overline{\Delta x^1}}\right\} \rightarrow \left\{\frac{\Delta x^{*c}}{\overline{\Delta x^1}}\right\} \quad \text{as} \quad \overline{\Delta x^1} \rightarrow 0$$

where $x^{*a} \equiv$ a solution to problem (a)

$x^{*b} \equiv$ a solution to problem (b)

$\Delta x^{*c} \equiv$ a solution to a problem (c), and

(B) (even if x^* for problem (a) is not unique)

$$\frac{\partial^2 \pi(x^*)^Q}{\overline{\partial x^1}^{\,2}}$$

for problem (a) is equal to the maximum

$$\frac{\Delta^2 \pi(x^*)^Q}{(\Delta x^1)^2}$$

for any problem (c) $(\overline{\Delta x^1} \neq 0)$.

<u>Proof</u>. Construct the problem Q

$$\begin{aligned} &\text{maximize } \pi(x)^Q \\ &\text{subject to } x^1 = \overline{x^1} \end{aligned} \qquad \ldots .(a)$$

which is assumed to have an interior solution x^* (not necessarily unique). Construct the related problem Q

$$\begin{aligned}&\text{maximize } \pi(x)^Q\\ &\text{subject to } x^1 = \overline{x^1} + t\overline{\Delta x^1}\end{aligned} \qquad \text{....(b)}$$

where t is a given scalar. Problem (b) can be expressed equivalently as

$$\begin{aligned}&\text{maximize } \pi(x^* + t\Delta x)^Q\\ &\text{subject to } \Delta x^1 = \overline{\Delta x^1}\end{aligned} \qquad \text{....(c)}$$

where $(\Delta x^2, \cdots, \Delta x^N)$ are the endogenous variables. Given that $\pi(x)^Q$ is twice differentiable in a neighbourhood of x^* which contains $x^* + t\Delta x$, we can express $\pi(x^* + t\Delta x)^Q$ as a second order Taylor expansion about x^*:

$$\pi(x^* + t\Delta x)^Q = \pi(x^*)^Q + t \sum_{i=1}^{N} \pi_i(x^*)^Q \Delta x^i$$

$$\frac{t^2}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(\hat{x})^Q \Delta x^i \Delta x^j \qquad \text{....(d)}$$

where $\hat{x}$ is some point between x^* and $x^* + t\Delta x$. Substituting the interior first order conditions $\pi_i(x^*)^Q = 0$ $(i = 2, \cdots, N)$ for (a) into (d),

$$\pi(x^* + t\Delta x)^Q = \pi(x^*)^Q + t\pi_1(x^*)^Q \Delta x^1$$

$$+ \frac{t^2}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(\hat{x})^Q \Delta x^i \Delta x^j \ . \qquad \text{....(e)}$$

Construct the related problem

$$\text{maximize} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(\hat{x})^Q \, \Delta x^i \Delta x^j \qquad \text{. . . .(f)}$$

$$\text{subject to } \Delta x^1 = \overline{\Delta x^1}$$

where $(\Delta x^2, \cdots, \Delta x^N)$ again are the endogenous variables. By (e),

problems (c) and (f) have the same set of (primal) solutions. . . . .(g)

By the definition of $\hat{x}$ and the assumption that $\pi(x)$ is twice differentiable at x^*,

$$\pi_{ij}(\hat{x})^Q \rightarrow \pi_{ij}(x^*)^Q \qquad (i, j, = 1, \cdots, N) \text{ as } t \rightarrow 0. \qquad \text{. . . .(h)}$$

By (h),

as $t \rightarrow 0$, the limiting (asymptotic) form of problem (f) is problem A:

$$\text{maximize} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^*)^Q \, \Delta x^i \Delta x^j \qquad \text{. . . .(i)}$$

$$\text{subject to } \Delta x^1 = \overline{\Delta x^1}.$$

Construct the related problem

$$\text{maximize} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^*)^Q \, \Delta x^i \Delta x^j \,/\, (\Delta x^1)^2$$

$$\text{. . . .(j)}$$

$$\text{subject to } \Delta x^1 = \overline{\Delta x^1} \, .$$

Since Δx^1 is exogenous to problems A and (j),

problems A and (j) have the same set of (primal) solutions. . . . . (k)

By (g), (i) and (k),

$$\text{as } t \to 0, \quad \left\{\left(\frac{t\Delta x^{2*}}{t\Delta x^1}, \cdots, \frac{t\Delta x^{N*}}{t\Delta x^1} \right)\right\} \text{ for all (c)}$$

defined by $\{x^*\}$ for (a)

$$\to \left\{\left(\frac{\Delta x^{2*}}{\Delta x^1}, \cdots, \frac{\Delta x^{N*}}{\Delta x^1} \right)\right\} \quad \text{for all (j)} \qquad \ldots\ldots(***)$$

defined by $\{x^*\}$ for (a)

which is statement A of the Lemma. In addition,

$$\frac{1}{(\Delta x^1)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^*)^Q \Delta x^i \Delta x^j = \frac{1}{(\lambda\Delta x^1)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^*)^Q (\lambda\Delta x^i)(\lambda\Delta x^j)$$

for all $(\lambda, \Delta x)$ . . . . . (l)

By (l),

the solution value of the objective function for problem (j) is invariant with respect to the constraint $\Delta x^1 = \overline{\Delta x}^1$ ($\neq 0$). . . . . . (m)

Assuming that a solution or solutions exist for problem (a),

. . . .(n)

$$\frac{\partial^2 \pi(x^*)^Q}{\partial \overline{x^1}^2} \text{ is uniquely defined for problem (a).}$$

By (***), (m)-(n) and the definition of $\frac{\partial^2 \pi(x^*)^Q}{\partial \overline{x^1}^2}$,

$\frac{\partial^2 \pi(x^*)^Q}{\partial \overline{x^1}^2}$ for problem (a) is equal to the solution value

of the objective function for any problem (j) $(\overline{\Delta x^1} \neq 0)$

defined by $\{x^*\}$ for problem (a)

which is statement B of the. Lemma. □

9. Lemma 3

Lemma 3. Suppose that conditions 1-3 are satisfied. Then, for any x^{1A} in the domain of $p^1(x^1)$ and the related α^A and any global solution x^{*A},

$$\frac{\partial p^1(x^{1A})}{\partial x^1} - \frac{\partial^2 c^1(x^{1A};\alpha^A)}{\partial x^{1\,2}} = \text{maximum } \left(\frac{1}{\Delta x^1}\right)^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^{*A})^P \Delta x^i \Delta x^j$$

for all Δx such that $\Delta x^1 \neq 0$.

Proof. Construct the problem P

$$\text{maximize } \pi(x)^P \equiv R(x) - c^1(x^1) - \sum_{i=2}^{N} c^i(x) \qquad \ldots .(a)$$

(for a given α) satisfying conditions 1-2 in the neighbourhood of each global solution x^* (not necessarily unique). Construct the corresponding problem Q

$$\text{maximize } \pi(x)^Q \equiv R(x) - \sum_{i=2}^{N} c^i(x)$$

$$\text{subject to } x^1 = \overline{x^{1A}} \qquad \ldots .(b)$$

where x^{1A} is included in a global solution to problem (a). By (a)-(b) and Lemma 2-B,

$$\frac{\partial \pi(x^{*A})^Q}{\partial \overline{x^1}^{\,2}} - c^1_{11}(x^{1A}) = \text{maximum } \left(\frac{1}{\Delta x^1}\right)^2 \sum_{i=1}^{N} \sum_{j=2}^{N} \pi_{ij}(x^{*A})^P$$

$$\text{for all } \Delta x \text{ such that } \Delta x^1 \neq 0 \qquad \ldots .(c)$$

where $x^{*A} \equiv (x^{1A}, \cdots, x^{NA})$ is a global solution to both problems (a) and (b). By Theorem 1 and Corollary 1-B and 1-C,

$$p^1_1(x^{1A}) = \frac{\partial^2 \pi(x^{*A})^Q}{\partial \overline{x^1}^{\,2}} = \cdots\cdots = \frac{\partial^2 \pi(x^{*Z})^Q}{\partial \overline{x^1}^{\,2}} \qquad \ldots .(d)$$

where

$\{x^{*A}, \cdots, x^{*Z}\}$

denotes the solution set to problem (b). By (c) and (d),

$$p_1^1(x^{1A}) - c_{11}^1(x^{1A}) = \text{maximum } \left(\frac{1}{\Delta x^1}\right)^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^*)^P$$

for all Δx such that $\Delta x^1 \neq 0$ for any global solution x^{*A} to problem (b). □

10. Theorem 2

Since the proof of Theorem 2 consists of several parts corresponding to the statements (A-E) to be proved, it may be useful to precede the proof by a brief statement of the methodology that is common to these parts. As mentioned in Section 2.4.4.2 of Chapter 2, essentially Corollary 4 can be used to transform the comparative statics problem of determining the direction of change in equilibrium level of input 1, resulting from a change in the factor cost schedule $c^1(x^1)$, to a problem of determining the existence of an equilibrium for particular specifications of $c^1(x^1)$. This statement can be elaborated upon as follows.

From Corollary 4 (or, to be exact, Lemmas 2-3) we can deduce the following:

(a) $\dfrac{\partial \pi(x^A)^P}{\partial x^i} = 0$, for all i, is equivalent to $\dfrac{\partial c^1(x^{1A})}{\partial x^1} = p^1(x^{1A})$,

(b) $\pi(x)^P$ concave in the neighbourhood of x^A is equivalent to

$$\frac{\partial^2 c^1(x^1)}{\partial x^{1\,2}} \geq \frac{\partial p^1(x^1)}{\partial x^1}$$

in the neighbourhood of x^{1A}, where the left hand statements in the equivalences a and b are the necessary and sufficient conditions for an interior local solution to the problem P at x^A. Therefore, since a $c^1(x^1) \equiv \overline{w^1} x^1$ ($\overline{w^1}$ exogenous) can always be constructed such that $\overline{w^1} = p^1(x^{1A})$, the slope of the derived demand schedule can be deduced from the answer to the following question: given particular properties of $R(x)$ and $c^i(x)$ for $i \neq 1$, and $c^1(x^1) \equiv \overline{w^1} x^1$ such that $\overline{w^1} = p^1(x^{1A})$ is it

(a') always,

(b') sometimes, or

(c') never

true that $\pi(x)^P$ is concave in the neighbourhood of x^A? Depending on whether a', b' or c' is correct,

$$\frac{\partial p^1(x^{1A})}{\partial x^1} \leq 0 , \qquad \frac{\partial p^1(x^{1A})}{\partial x^1} \gtrless 0$$

or

$$\frac{\partial p^1(x^{1A})}{\partial x^1} \gtrsim 0 , \quad \text{respectively.}$$

Theorem 2. Suppose that conditions 1-6 are satisfied. Denote the the domain of $p^1(x^1)$ as X^D, and denote a wage or rental rate that is exogenous to the firm as $\overline{w^i}$. Then the slope of the firm's derived demand schedule is related to certain properties of $R(x)$ and $c^i(x)$ $(i = 2, \cdots, N)$ as follows.

(A) If $R(x)$ is strictly concave,[5] then $\frac{\partial p^1(x^1)}{\partial x^1} \leqq 0$ and $p^1(x^1) > p^1(x^1 + e)$ for all $(x^1, x^1 + e)$ in X^D, where $e > 0$.

(B) If $R(x)$ is concave, then $\frac{\partial p^1(x^1)}{\partial x^1} \leqq 0$ for all x^1 in X^D.

(C) If $R(\lambda x) \leqq \lambda R(x)$ for all $\lambda > 1$ and $x \geqq 0$ but $R(x)$ is not concave, then

(1) $\frac{\partial p^1(x^1)}{\partial x^1} \leqq 0$ always for at least some x^1 in X^D

but

(2) for some $R(x)$ and $\sum_{i=2}^{N} c^i(x)$: $\frac{\partial p^1(x^1)}{\partial x^1} > 0$ for some x^1 in X^D.

(D) If $R(\lambda x) = \lambda R(x)$ for all $(x, \lambda) \geqq 0$ and $c^i \equiv \overline{w^i} x^i$ for $i = 2, \cdots, N$, then $\frac{\partial p^1(x^1)}{\partial x^1} = 0$ for all x^1 in X^D.

(E) If $R(\lambda x) > \lambda R(x)$ for all $\lambda > 1$ and $x > 0$ and $c^i \equiv \overline{w^i} x^i$ for $i = 2, \cdots, N$, then $\frac{\partial p^1(x^1)}{\partial x^1} \geqq 0$ and $p^1(x^1) < p^1(x^1 + e)$ for all $(x^1, x^1 + e)$ in X^D, where $e > 0$.[6]

Footnotes on the following page (5,6).

Proof. Part A (Introduction)

Construct the problem P

$$P^A : \text{maximize } \pi(x)^A \equiv R(x) - c^1(x^1)^A - \sum_{i=2}^{N} c^i(x). \quad \ldots .(a\text{-}1)$$

Given that $p^1(x^1)$ is constructed from the above R(x) and $\sum_{i=2}^{N} c^i(x)$ and various $c^1(x^1)$ (Definition 2),

if $p^1(x^1)$ is defined for x^{1A} (i.e., $x^{1A} \varepsilon X^D$),
then there exists a $c^1(x^1)^A$ such that P^A has $\quad \ldots .(a\text{-}2)$
a solution $x^A \equiv (x^{1A}, \cdots, x^{NA})$.

Construct the corresponding problem Q

$$\text{maximize } \pi(x)^Q \equiv R(x) - \sum_{i=2}^{N} c^i(x)$$
$$\text{subject to } x^1 = \overline{x^{1A}} \quad \ldots .(a\text{-}3)$$

By (a-2)-(a-3) and Theorem 1-A,

x^A is a global solution to problem (a-3) $\quad \ldots .(a\text{-}4)$

---

[5]The firm's total benefits function R(x), which is simply a total revenue function if the firm maximizes profits, is strictly concave if and only if (1) $R(\lambda x) < \lambda R(x)$ for all $\lambda > 1$ and $x > 0$, and (2) all isoquants of R(x) are strictly convex for $x \geqq 0$. Likewise, R(x) is concave if and only if (1) $R(\lambda x) \leqq \lambda R(x)$ for all $\lambda > 1$ and $x > 0$, and (2) all isoquants of R(x) are convex for $x \geqq 0$.

[6]Note the asymmetry between statements C and E: $p^1(x^1 \gtreqless p^1(x^1 + e)$ for decreasing returns to scale and fixed factor prices $(i \neq 1)$, whereas $p^1(x^1) < p^1(x^1 + e)$ for increasing returns to scale and fixed factor prices $(i \neq 1)$, where $e > 0$.

Replace $c^1(x^1)^A$ in P^A with a $c^1(x^1)^B$ such that

$$c_1^1(x^{1A})^B \quad + \quad c_1^1(x^{1A})^A \qquad \ldots . (a\text{-}5)$$

$$c_{11}^1(x^1)^B \quad = \quad 0 \qquad \text{for all} \quad x^1 \qquad (a\text{-}6)$$

which results in the problem

$$P^B\text{: maximize } \pi(x)^B \equiv R(x) - c^1(x^1)^B - \sum_{i=2}^{N} c^i(x) \ldots . (a\text{-}7)$$

By (a-1)-(a-3), (a-5) and conditions 1-2,

$$\pi_i(x^A)^B \quad = \quad 0 \qquad i = 1, \cdots, N. \qquad \ldots . (a\text{-}8)$$

Given conditions 1-2,

any $\hat{x}$ is a local solution to a P if and only if $\pi_i(\hat{x})^P = 0$ $(i = 1, \cdots, N)$ and $\pi(x)^P$ is concave at $\hat{x}$ ....(a-9)

and

$\pi(x)^P$ is concave at $\hat{x}$ (1) if $[\pi_{ij}{}^P]$ is negative definite at $\hat{x}$, and (2) if and only if $[\pi_{ij}{}^P]$ is negative semi-definite in the neighbourhood of $\hat{x}$[7] ....(a-10)

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[7]See Karlin (1959), p. 406.

where $[\pi_{ij}^{P}]$ denotes the Hessian matrix of

$$\pi(x)^{P} \equiv R(x) - \sum_{i=1}^{N} c^{i}(x)$$

at x.

Part B (proof of Statements A and B)

Since the negative of a convex function is concave, and the sum of a (strictly) concave function and a concave function is (strictly) concave,[8] condition 4 ($c^{i}_{jK} \geqq 0$ for all i, j, K and x) implies that

$$\pi(x) \text{ is (strictly) concave if } R(x) \text{ is (strictly) concave.} \quad \ldots\ldots(b\text{-}1)$$

Given that $\pi(x)$ has a maximum over the convex feasible set of all $x \geqq 0$,

$$\pi(x) \text{ attains a unique local maximum over all } x \geqq 0 \text{ if } \pi(x) \text{ is strictly concave} \quad \ldots\ldots(b\text{-}2)$$

$$\pi(x) \text{ attains either a unique local maximum or a convex set of local maxima (hence every local maximum is a global maximum) over all } x \geqq 0 \text{ if } \pi(x) \text{ is concave.} \quad \ldots\ldots(b\text{-}3)$$

By (a-10) and (b-1),

$$R(x) \text{ concave} \Rightarrow \underset{\Delta x^{1} \neq 0}{\text{maximum}} \left(\frac{1}{\Delta x^{1}}\right)^{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x)^{P} \, \Delta x^{i} \Delta x^{j} \leqq 0 \quad \text{for all } x. \quad \ldots(b\text{-}4)$$

---

[8]Footnote on following page. (8)

By (b-4) and Lemma 3,

$$R(x) \text{ concave} \Rightarrow p_1^1(x^1) \leqq 0 \quad \text{for all} \quad x^1 \in X^D \qquad \ldots .(***)$$

which is statement B of the Theorem. By (***) and Corollary 2-A and 2-C,

$$\begin{aligned} &\text{if } p^1(x^1) = p^1(x^1 + e) \text{ for } R(x) \text{ concave, then} \\ &p^1(x^1) = p^1(x^1 + \lambda e) \text{ for all } 0 \leqq \lambda \leqq 1. \end{aligned} \qquad \ldots .(b\text{-}5)$$

By Theorem 1-B,

$$\begin{aligned} &\text{if } p^1(x^1) = p^1(x^1 + \lambda e) \text{ for all } 0 \leqq \lambda \leqq 1 \text{ where } x^1 \text{ is} \\ &\text{included in a solution to a problem } P^B \text{ (a-5 to a-7),} \\ &\text{then } x^1 + \lambda e \text{ for all } 0 \leqq \lambda \leqq 1 \text{ is included in a local} \\ &\text{solution to } P^B. \end{aligned} \qquad \ldots .(b\text{-}6)$$

By (b-3) and (b-5)-(b-6),

$$\begin{aligned} &\pi(x) \quad \text{strictly concave} \Rightarrow p^1(x^1) \neq p^1(x^1 + e) \text{ for any} \\ &(x^1, x^1 + e) \in X^D. \end{aligned} \qquad \ldots .(b\text{-}7)$$

By (***), (b-7) and Corollary 2-B,

$$\begin{aligned} &R(x) \text{ strictly concave} \Rightarrow p_1^1(x^1) \leqq 0 \text{ and} \\ &p^1(x^1) \neq p^1(x^1 + e) \text{ for all } (x^1, x^1 + e) \in X^D \end{aligned} \qquad \ldots .(***)$$

which is statement A of the Theorem.[9]

---

[8] See Lancaster (1968) for a summary of most of the properties of concave functions and sets that are used here.

[9] Corollary 2-B ($p^1(x^1)$ is single-valued for each $x^1 \in X^D$) implies that the result obtained by statement B and (b-7); i.e., statement A, is independent of the assumption that $c_{11}^1(x^1)^B = 0$ for all x^1.

Part C (proof of statement C)

For a given $R(x) - \sum_{i=2}^{N} c^i(x)$ and $\{\hat{x}\} \equiv$ a particular subset of x that defines all possible factor proportions $\left(\frac{x^2}{x^1}, \cdots, \frac{x^N}{x^1} \right)$, we can construct a problem

$$P^c : \text{maximize } \pi(x)^c \equiv R(x) - c^1(x^1)^c - \sum_{i=2}^{N} c^i(x) \qquad \text{. . . .(c-1)}$$

where

$$c^1_{11}(x^1)^c = 0 \qquad \text{for all } x^1 \qquad \text{. . . .(c-2)}$$

$$\pi(x)^c \leqq 0 \qquad \text{for all } x \in \{\hat{x}\}\ . \qquad \text{. . . .(c-3)}$$

Assume that

$$R(\lambda x) \leqq \lambda R(x) \qquad \text{for all } \lambda > 1 \text{ and all } x \geqq 0\ . \qquad \text{. . . .(c-4)}$$

By (c-3) and condition 4,

$$\text{(c-4)} \Rightarrow \text{(a) } \pi(\gamma x)^c \leqq 0 \text{ for all } \gamma \geqq 1 \text{ and all } x \in \{\hat{x}\}$$

$$\text{(b) } \pi(\alpha x)^c \geqq \pi(x)^c \text{ for all } 0 \leqq \alpha \leqq 1 \text{ and all } x \geqq 0. \qquad \text{. . . .(c-5)}$$

By (c-5),

$$\text{(c-4)} \Rightarrow \{\text{all } x \mid \pi(x)^c \geqq 0 \text{ and } x \geqq 0\} \text{ is closed and bounded.} \qquad \text{. . . .(c-6)}$$

Given that $\{\text{all } x \mid \pi(x)^C \geqq 0 \text{ and } x \geqq 0\}$ is non-empty: (c-6) and Weierstrass's Theorem imply that

$$(c\text{-}4) \Rightarrow \text{problem } P^C \ (c\text{-}1) \text{ has a solution.} \qquad \text{. . . .(c-7)}$$

Given that this solution is interior: (c-2), (c-7), Corollary 4-A and Corollary 2-B imply that

$$R(\lambda x) \leqq \lambda R(x) \quad \text{for all} \quad \lambda > 1 \text{ and } x \geqq 0 \Rightarrow p_1^1(x^1) \leqq 0 \text{ for some } x^1 \in X^D \qquad \text{. . .(***)}$$

which is part 1 of statement C of the Theorem. Given that an interior point $x^A \equiv (x^{1A}, \cdots, x^{NA})$ solves problem P^A (a-1) for an appropriate $c^1(x^1)^A$,

$$x^{1A} \ \varepsilon \ X^D \qquad \text{. . . .(c-8)}$$

by Definition 2, and

$$\pi(x)^A \text{ is concave at } x^A \qquad \text{. . . .(c-9)}$$

by (a-9)-(a-10). Since the sum of non-concave function and concave functions is not necessarily concave,

$$\pi(x)^A \text{ concave at } x^A \not\Rightarrow \pi(x)^B \text{ concave at } x^A \qquad \text{. . . .(c-10)}$$

By the definitions of $\pi(x)^A$ and $\pi(x)^B$ (a-1) and (a-7),

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^A)^A \, \Delta x^i \Delta x^j = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^A)^B \, \Delta x^i \Delta x^j \qquad \ldots\ldots(c\text{-}11)$$

for all Δx such that $\Delta x^1 = 0$.

By (a-10) and (c-11),

for some $R(x)$ and $\sum_{i=2}^{N} c^i(x)$ satisfying (c-4) and condition 4:

$$\underset{\Delta x \cdot \Delta x = 1}{\text{maximum}} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^A)^B \, \Delta x^i \Delta x^j = \underset{\Delta x \cdot \Delta x = 1, \Delta x^1 \neq 0}{\text{maximum}} \qquad \ldots.(c\text{-}12)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^A)^B \, \Delta x^i \Delta x^j > 0 \ .$$

By (a-3)-(a-4) and Lemma 2-B,

$$\frac{\partial^2 \pi(x^A)^Q}{\partial \overline{x^1}^{\,2}} = \underset{\Delta x^1 \neq 0}{\text{maximum}} \left(\frac{1}{\Delta x^1}\right)^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^A)^Q \, \Delta x^i \, \Delta x^j \qquad \ldots.(c\text{-}13)$$

By (a-3)-(a-4), (c-8), (c-12)-(c-13) and Theorem 1-B,

for some $R(x)$ and $\sum_{i=2}^{N} c^i(x)$ satisfying (c-4) and

condition 4: $\ldots.(c\text{-}14)$

$p_1^1(x^1) > 0$ for some $x^1 \varepsilon X^D$

which is part 2 of statement C of the Theorem.

Part D (proof of statement D)

Suppose that, for problem P^A (a-1),

$$R(\lambda x) = \lambda R(x) \quad \text{for all} \quad \lambda > 0 \quad \text{and} \quad x \geqq 0 \qquad \text{....(d-1)}$$

$$c^i(x) \equiv \overline{w}^i x^i \qquad i = 1, \cdots, N \qquad \text{....(d-2)}$$

$$\pi(\lambda\hat{x})^A = 0 \quad \text{for at least one } \hat{x} \neq 0, \text{ all } \lambda > 0 \qquad \text{....(d-3)}$$

$$\pi(\hat{\hat{x}})^A \leqq 0 \quad \text{for all } \hat{\hat{x}} \neq \text{any } \lambda\hat{x} \text{ .}^{10} \qquad \text{....(d-4)}$$

By (d-3) and (d-4),

all $\lambda\hat{x}$ are global solutions to a problem P^A satisfying (d-1)-(d-4) .[11] ....(d-5)

By (d-5), Lemma 3 and Corollary 2-B,

if $R(\lambda x) = \lambda R(x)$ for all $\lambda > 0$ and $x \geqq 0$ and $c^i(x) \equiv \overline{w}^i x^i$ for all $i \neq 1$, then $p_1^1(x^1) = 0$ for all $x^1 \varepsilon X^D$ ....(***)

which is statement D of the Theorem.

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[10] Given (d-1) and (d-2), (d-3) and (d-4) are necessary for the satisfaction of condition 1 (hence are implied by the satisfaction of condition (1). If (d-3) or (d-4) is not satisfied, then either $x^* = 0$ or the problem is unbounded.

[11] Statement (d-5) is in effect Samuelson's substitution theorem (Samuelson, 1951).

Part E (proof of statement E)

Suppose that

$$R(\lambda x) > \lambda R(x) \quad c^i(\lambda x) = \lambda c^i(x) \qquad i = 2,\cdots,N \qquad \text{for all} \quad \lambda > 0,\ x > 0 \ . \qquad \ldots.(e\text{-}1)$$

By the definition of $\pi(x)^B$ (a-7),

$$(e\text{-}1) \Rightarrow \pi(\lambda x)^B > \lambda\pi(x)^B \quad \text{for all } \lambda > 1 \text{ and all } x \ldots.(e\text{-}2)$$

By (e-2),

$$(e\text{-}1) \Rightarrow \pi(x)^B \quad \text{is not concave at any } x. \qquad \ldots.(e\text{-}3)$$

By (a-10), (c-9), (c-11) and (e-3),

$$(e\text{-}1) \Rightarrow \text{maximum} \left(\frac{1}{\Delta x^1}\right)^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^A)^Q \ \Delta x^i \Delta x^j \geqq 0 \ldots..(e\text{-}4)$$

By (c-8), (c-13), (e-4) and Theorem 1-B ,

$$(e\text{-}1) \Rightarrow p_1^1(x^1) \geqq 0 \quad \text{for all} \quad x^1 \varepsilon X^D. \qquad \ldots.(e\text{-}5)$$

By (e-5) and Corollary 2-A,

given (e-1): if $p^1(x^1) = p^1(x^1 + e)$ for a
$(x^1, x^1 + e) \varepsilon X^D$, then $p^1(x^1) = p^1(x^1 + \lambda e)$ ....(e-6)
for all $0 \leqq \lambda \leqq 1$.

By the definition of $\pi(x)^B$ (a-7) and Theorem 1-B,

$$p^1(x^1) = p^1(x^1 + \lambda e) \text{ for an } (x^1, e) \text{ and all } 0 \leqq \lambda \leqq 1 \Rightarrow \pi(x)^B \text{ has a local solution} \quad \ldots .(e\text{-}7)$$

which is presumably interior. By (a-8)-(a-9), (e-3) and (e-6)-(e-7),

$$(e\text{-}1) \Rightarrow p^1(x^1) \neq p^1(x^1 + e) \text{ for any } (x^1, x^1 + e) \varepsilon X^D \quad \ldots .(e\text{-}8)$$

By (e-5) and (e-8),

$$R(\lambda x) > \lambda R(x) \text{ for all } \lambda > 1 \text{ and } x > 0 \text{ and } c^i(x) \equiv \overline{w^i} x^i \text{ for } i = 2, \cdots, N \Rightarrow p_1^1(x^1) \geqq 0 \text{ and } p^1(x^1) \neq p^1(x^1 + e) \text{ for all } (x^1, x^1 + e) \varepsilon X^D \quad \ldots .(***)$$

which is statement E of the Theorem. □

11. On the Hicks-Andrieu Formula for the Elasticity of Derived Demand[12]

Given statement E of Theorem 2, we can easily demonstrate that a solution to the formula for elasticity of industry derived demand developed by Hicks (1963, pp. 241-6) and generalized by Andrieu (1974) is not necessarily consistent with the static maximization hypothesis. From the first order conditions for an interior maximum for competitive firms and assuming an industry production function

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[12]This section of the Appendix supplements section 2.3.3 of Chapter 2.

$F(x^1,x^2)$ homogeneous of degree ρ,[13] Andrieu develops the following formula for the industry elasticity of derived demand for input 1:

$$\lambda_1 \equiv -\frac{\partial x^1}{\partial w^1} \cdot \frac{w^1}{x^1} = \frac{e_2 k_1 + \sigma_{12} - Z\sigma_{12}(1-k_1)e_2}{(1-k_1) - Z(e_2 + k_1\sigma_{12})} \quad \ldots . (1)$$

where

$$\sigma_{12} \equiv \frac{d(x^1/x^2)}{x^1/x^2} \Big/ \frac{d(F_2/F_1)}{F_2/F_1}$$ (industry elasticity of factor substitution)

$$\eta \equiv -\frac{\partial D(p)}{\partial p} \cdot \frac{p}{y}$$ (industry elasticity of product demand)

$$e_2 \equiv \frac{\partial S^2(w^2)}{\partial (w^2)} \cdot \frac{w^2}{x^2}$$ (industry elasticity of supply for input 2)

$$k_1 \equiv \frac{w^1 x^1}{py}$$ (factor share for input 1)

$$Z \equiv (\rho - 1) - \rho/\eta$$

(formula 15, p. 413). For $\rho = 1$, equation 1 reduces to the formula of Hicks. If $\eta \to +\infty$ and $e_2 \to +\infty$, then the numerator and denominator of equation 1 approach $[k_1 - Z\sigma_{12}(1-k_1)]e_2$ and $-Ze_2$, respectively, and $Z \to \rho - 1 > 0$ for $\rho > 1$. So

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[13]For $\rho \neq 1$, these conditions are compatible in the presence of external economies or diseconomies of scale for the individual firm.

$$\text{sign } (\lambda_1) = \text{sign } [Z\sigma_{12}(1-k_1)-k_1] \gtrless 0$$
$$\text{for } \eta \to +\infty,\ e_2 \to +\infty,\ \rho > 1 \qquad \ldots .(2)$$

given only $Z > 0$, $\sigma_{12} > 0$ (convex isoquants for F) and $0 < k_1 < 1$.

Given perfectly elastic product demand schedules and supply schedules for input 2 at both the industry and firm level (so that changes in the level of output produced or input 2 employed do not lead to shifts in price schedules faced by individual firms, the industry derived demand schedule for input 1 would be equivalent to the derived demand schedule for an individual firm facing the production function $F(x^1,x^2)$ and identical price constraints. Therefore, the contrast between statement E of Theorem 2 ($p_1^1 \geqq 0$) and the more ambiguous statement 2 above implies that

(a) a subset of the solutions $\{(\lambda_1, \sigma_{12}, \eta, e_2, k_1, \rho)\}$ to formula 1 is inconsistent with the static maximization hypothesis, and

(b) various qualitative relations calculated by means of formula 1 will be more ambiguous than is warranted by the static maximization hypothesis.[14]

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[14]On the other hand, various qualitative relations implied by the static maximization hypothesis happen to be represented correctly by formula 1. By formula 1: $\lambda_1 = 0$ for $\rho = 1$, $\eta \to +\infty$, $e_2 \to +\infty$ (Hicks, 1963, pp. 373-4), which is equivalent to statement D of Theorem 4. By formula 1: $\lambda_1 > 0$ for $\rho < 1$, $\sigma_{12} > 0$, $\eta \to +\infty$, $e_2 \to +\infty$ and $\lambda_1 \gtrless 0$ for $\rho < 1$, $\sigma_{12} < 0$, $\eta \to +\infty$, $e_2 \to +\infty$, which is in accordance with statements B and C, respectively, of Theorem 2.

These conclusions are not surprising, since Hicks and Andrieu could not incorporate second order conditions for a producer problem P maximum into their formulas.

APPENDIX III

QUANTITATIVE COMPARATIVE STATICS AND DERIVED DEMAND: DETAILS OF THE MODEL

Appendix III. Quantitative Comparative Statics and Derived Demand: Details of the Model

1 Introduction

In this Appendix we shall present a more detailed discussion of the methodology for quantitative comparative statics that was introduced in Chapter 3. Proofs and a discussion of partial solutions to major computational problems will be presented in the next two appendices.

The method of quantitative comparative statics can be schematized as obtaining global solutions to two nonlinear programming problems

maximize	$z(\frac{\partial x}{\partial \alpha})$	maximize	$z(\frac{\partial x}{\partial \alpha})$
subject to	$[\pi_{ij}]\frac{\partial x}{\partial \alpha} = -\pi_{i\alpha}$	subject to	$[\pi_{ij}]\frac{\partial x}{\partial \alpha} = -\pi_{i\alpha}$
	$[\pi_{ij}]$ negative definite		$[\pi_{ij}]$ negative definite
	$G([\pi_{ij}], \rho) = 0$		$G([\pi_{ij}], \rho) = 0$
	$\rho^L \leq \rho \leq \rho^U$		$\rho^L \leq \rho \leq \rho^U$

where $(\frac{\partial x}{\partial \alpha}, [\pi_{ij}], \rho)$ are endogenous variables and the scalar valued function $z = z(\frac{\partial x}{\partial \alpha})$ is the comparative static effect of interest. The restrictions

$$[\pi_{ij}]\frac{\partial x}{\partial \alpha} = -\pi_{i\alpha} \qquad [\pi_{ij}] \text{ negative definite}$$

are the restrictions implied by the assumption of an interior solution to the firm's static maximization problem "maximize $\pi(x;\alpha)$" ($\pi(x;\alpha)$ is twice differentiable), the equations

$$G([\pi_{ij}], \rho) = 0$$

denote the relations between the Hessian matrix $[\pi_{ij}(x^*)]$ and a set of more readily observable parameters ρ, and the restrictions

$$\rho^L \leq \rho \leq \rho^U$$

denote the empirically derived restrictions (confidence-Bayes intervals) for the parameters ρ.

Here we shall discuss primarily

a) the comparative static implications of the maximization hypothesis,
b) various equations $G([\pi_{ij}], \rho) = 0$ relating $[\pi_{ij}(x^*)]$ to more readily observable parameters ρ, and
c) the interpretation of the solution values for $z(\frac{\partial x}{\partial \alpha})$ in the above problems when the restrictions $\rho^L \leq \rho \leq \rho^U$ are formulated as confidence-Bayes intervals.

2. Restrictions Implied by the Maximization Hypothesis

It can be shown that the assumption of maximizing behavior is essentially as realistic as the results of comparative static analysis, and that comparative static methods usually are more appropriate than comparative dynamic techniques for the evaluation of community pasture programs.[1] Thus it is important to incorporate the restrictions implied by the maximization hypothesis, i.e. by the existence of an interior static maximum, into our methodology. However, in order to avoid placing arbitrary restrictions on the structure $\pi(x)$, we should model in this manner only those restrictions that correspond exactly to the comparative static implications of the maximization hypothesis.

The task of determining the precise comparative static implications of the maximization hypothesis has been labelled the "integrability problem" in comparative statics (Silberberg, 1974a), and has been largely solved in the case of the dual approach to comparative statics (Epstein, 1978). In addition, necessary and sufficient conditions for consistency between the competitive firm's factor demand schedules and the maximization hypothesis have been known since Hotelling (1932). Nevertheless, the exact implications of the maximization hypothesis for primal comparative statics apparently has not been demonstrated previously (even in the competitive case) for the problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x^i). \qquad \text{P}$$

In this section we shall show that, for problem P, the usual set of primal restrictions

$$[\pi_{ij}]\,\frac{\partial x}{\partial \alpha} = \begin{bmatrix} c^1_{1\alpha} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$[\pi_{ij}]$ symmetric and negative definite

corresponds exactly to the implications of the maximization hypothesis for primal comparative statics. Thus the "integrability problem" is solved in this

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1. See Appendix 1 and Chapter 4.

special case.[2] In addition, the restriction $[\pi_{ij}]$ negative definite is expressed in a form that is more appropriate for our (primal) quantitative comparative statics model.

2.1 Comparative Static Implications of the Maximization Hypothesis

Given that the primal problem P has an interior global solution x^* where $\pi(x)$ is twice differentiable, the first order conditions for a maximum imply that

$$\left.\begin{aligned} &\sum_{j=1}^{N} \pi_{ij}(x^*)\frac{\partial x^{j*}}{\partial\alpha} - c^1_{1\alpha}(x^{1*};\alpha) = 0 \\ &\sum_{i=1}^{N} \pi_{ij}(x^*)\frac{\partial x^{j*}}{\partial\alpha} = 0 \qquad i = 2,\ldots., N \end{aligned}\right\} \quad (1)$$

for an infinitesimal change $d\alpha$ affecting the cost schedule $c^1(x^1)$ for input 1, and the second order conditions imply that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \pi_{ij}(x^*)dx^i dx^j \leq 0 \qquad \text{for all } dx. \qquad (2)$$

Statement 2 is satisfied if and only if the Hessian matrix $[\pi_{ij}]$ at x^* is either negative definite (implying that the strict inequality relation in 2 holds for all $dx \neq 0$) or negative semi-definite only (implying that the sum on the left hand side of 2 is equal to 0 for some $dx \neq 0$). In addition, Young's theorem implies that

$$[\pi_{ij}(x^*)] \text{ is symmetric.} \qquad (3)$$

Statements (2) and (3) obviously exhaust the restrictions placed on $[\pi_{ij}(x^*)]$ by the assumptions of an interior maximum and twice differentiability of $\pi(x)$.

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2. The "integrability problem" in comparative statics has been described as a "major gap in the theory of comparative statics of maximization models" (Silberberg, 1974a, p. 171); but it is easily solved for the general problem

maximize $\pi(x;\alpha)$
subject to $G(x;\alpha) = 0$

in the context of primal methods of comparative statics in essentially the same manner as for the special case

maximize $\pi(x;\alpha)$

For these reasons, a general discussion of the "integrability problem" in primal comparative statics is included in Appendix 4.

For emphasis, the relation between statements 1-3 and the restrictions on $\frac{\partial x^*}{\partial \alpha}$ implied by the maximization hypothesis for problem P are presented here as Proposition 1. Parts A and B of the Proposition are well known, and follow directly from the fact that a negative definite matrix has full rank and a matrix that is only negative semi-definite does not have full rank. Thus, given that $[\pi_{ij}(x^*)]$ (symmetric) is negative definite and that at least one comparative static effect $\frac{\partial x^*}{\partial \alpha}$ exists for problem P, statement 1 and knowledge of $[\pi_{ij}(x^*)]$ and $c^1_{1\alpha}(x^{1*};\alpha)$ are sufficient to define $\frac{\partial x^*}{\partial \alpha}$ (which is unique). Given that $[\pi_{ij}(x^*)]$ (symmetric) is only negative semi-definite, statement 1 has multiple solutions $\{\frac{\partial x}{\partial \alpha}\}$ for a particular $[\pi_{ij}(x^*)]$ and $c^1_{1\alpha}(x^{1*};\alpha)$. However, by Part C of Proposition 1, $\frac{\partial x^*}{\partial \alpha}$ is in fact undefined by primal comparative static methods when $[\pi_{ij}(x^*)]$ is only negative semi-definite and $d\alpha$ defines a shift in the firm's cost schedule for an input.[3]

The intuitive meaning of Proposition 1 may be clarified somewhat by the following argument. Given that $[\pi_{ij}(x^*)]$ is only negative semi-definite, it can be shown that the derived demand schedule $p^1(x^1)$ and the marginal cost schedule $c^1_1(x^1;\alpha)$ for any input 1 have identical slopes at x^{1*}. Thus, for the purpose of determining the comparative static effect of an infinitesimal change $d\alpha$ (which depends only on the first and second order derivatives of $\pi(x)$ at x^*), the situations shown in Figures 9-A and 9-B are equivalent to the

3 . Proposition 1-C can be proved essentially as follows (an alternative proof is presented in Appendix 4). The first order condition in the product market for an interior solution to problem P can be denoted as

$$MR(y) - MC(y;\alpha) = 0 \qquad \text{(a)}$$

using obvious notation. The total differential of (a) yields

$$(MR_y - MC_y)\frac{\partial y}{\partial \alpha} - MC_\alpha = 0\ ; \qquad \text{(b)}$$

but $[\pi_{ij}(x^*)]$ only negative semi-definite implies (by definition) that

$$MR_y - MC_y = 0 \qquad \text{(c)}$$

Since (b) and (c) are consistent only if $MC_\alpha = 0$, local (primal) comparative statics is meaningless when $[\pi_{ij}(x^*)]$ is only negative semi-definite.

Proposition 1. Suppose that conditions 1-3 are satisfied for a problem

$$\text{maximize } \pi(x) \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x) \qquad \text{(P)}$$

and that this problem has a unique global solution x^*.[4] Denote the set of comparative static effects of $d\alpha$ for this problem as $\{\frac{\partial x^*}{\partial \alpha}\}$, and denote the system of total differentials of the first order conditions for a solution to this problem as

$$[\pi_{ij}]\frac{\partial x}{\partial \alpha} = \begin{bmatrix} c^1_{1\alpha} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (1)$$

where $[\pi_{ij}]$ is defined as the Hessian matrix for $\pi(x)$ at x^*, and $c^1_{1\alpha}$ denotes the exogenous shift in $c^1_1(x^1;\alpha)$ at x^{1*}. Assume that $[\pi_{ij}]$ is negative semi-definite and symmetric. Then

(A) if $[\pi_{ij}]$ is negative definite: equations (1) have a unique solution $\frac{\partial x^*}{\partial \alpha}$;

(B) if $[\pi_{ij}]$ is not negative definite: equations (1) may have multiple solutions $\{\frac{\partial x}{\partial \alpha}\}$; but

(C) if $[\pi_{ij}]$ is not negative definite: $\frac{\partial x^*}{\partial \alpha}$ is undefined ($\{\frac{\partial x^*}{\partial \alpha}\}$ is empty), i.e.

$$\frac{\partial p^1(x^{1*})}{\partial x^1}\frac{\partial x^{1*}}{\partial \alpha} - \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^{1\,2}}\frac{\partial x^{1*}}{\partial \alpha} - \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} = 0$$

by the first equation in (1) ;

$$\frac{\partial p^1(x^{1*})}{\partial x^1} - \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^{1\,2}} = 0$$

by equations 2,...., N in (1), $[\pi_{ij}]$ negative semi-definite (and not

---

4. Assuming other global solutions in the neighborhood of x^* rules out the possibility that $[\pi_{ij}(x^*)]$ is negative definite and does not alter statements B and C.

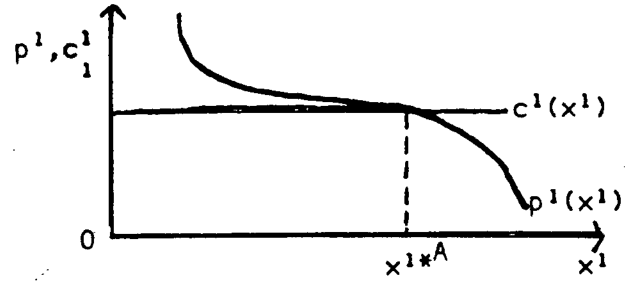
negative definite);

so $\frac{\partial x^{1*}}{\partial \alpha}$ is undefined for $\frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} \neq 0$.

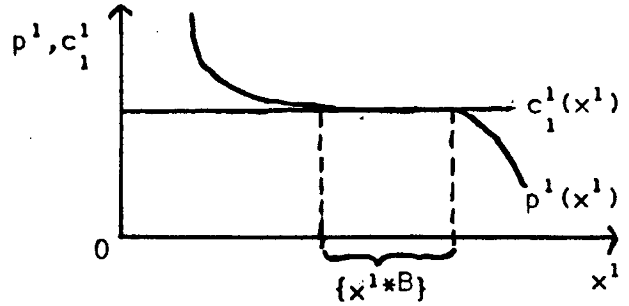
Figure 9. A Discrete Analogue to $[\pi_{ij}(x^*)]$

Only Negative Semi-definite

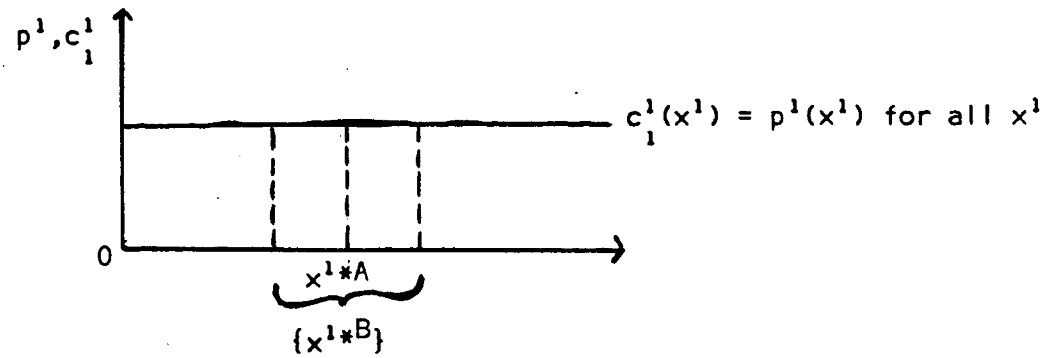
A. $[\pi_{ij}(x^*)]$ only negative semi-definite and x^{1*} unique



B. $[\pi_{ij}(x^*)]$ only negative semi-definite and x^{1*} not unique



C. A discrete analogue to $[\pi_{ij}(x^*)]$ only negative semi-definite



$c_1^1(x^1) \equiv$ the firm's marginal factor cost schedule for input 1

$p^1(x^1) \equiv$ the firm's derived demand schedule for input 1

$x^{1*A} \equiv$ the firm's solution set for input 1 in case A

$x^{1*B} \equiv$ the firm's solution set for input 1 in case B situation shown in 9-C.

In Figure 9-C, the solution set would be undefined after any downward shift in the schedule $c^1_1(x^1)$ for all x^1 [5]; so $\frac{\partial x^{1*}}{\partial \alpha}$ is undefined when $[\pi_{ij}(x^*)]$ is only negative semi-definite and $d\alpha$ defines a change in $c^1_1(x^1)$ at x^{1*}.[6]

In sum, Preposition 1 implies that the set of comparative static effects $\{\frac{\partial x^*}{\partial \alpha}\}$ corresponds to the unique solution for (1) when the given $[\pi_{ij}(x^*)]$ is negative definite, and that $\{\frac{\partial x^*}{\partial \alpha}\}$ is empty when $[\pi_{ij}(x^*)]$ is only negative semi-definite and $c^1_{1\alpha}(x^{1*};\alpha) \neq 0$. Therefore, statement 1 plus the restrictions that $[\pi_{ij}(x^*)]$ is negative definite and symmetric correspond <u>exactly</u> to the restrictions placed on $\{\frac{\partial x^*}{\partial \alpha}\}$ for problem P by the maximization hypothesis.[7]

2.2 <u>Restrictions corresponding to $[\pi_{ij}]$ Negative Definite</u>

In specifying a system of equations that restricts $[\pi_{ij}]$ to be negative definite, we utilize the following theorem:

Theorem. A real symmetric matrix A is positive definite if and only if there exists a real lower triangular matrix H with positive diagonal

---

5. If this relation in Figure 9-C extended throughout the negative orthant for x^1, as is in effect the case in local comparative statics, then the solution set $\{\frac{\Delta x^1}{\Delta \alpha}\}$ also would be undefined for an upward shift in the schedule $c^1(x^1)$ for all x^1.

6. This intuitive explanation of Proposition 1-C suggests that the undefined nature of $\{\frac{\partial x^*}{\partial \alpha}\}$ for $[\pi_{ij}(x^*)]$ only negative semi-definite is fundamental to local comparative static methods rather than a peculiarity of primal methods. In other words, $\{\frac{\partial x^*}{\partial \alpha}\}$ (for problem P) is undefined by any method whenever restrictions employed in the method imply that $[\pi_{ij}(x^*)]$ is only negative semi-definite.

7. For the general problem

maximize $\pi(x;\alpha)$

subject to $G(x;\alpha) = 0$,

the exact implications of the maximization hypothesis for primal comparative statics are analogous to the above restrictions. See the discussion of "integrability" in Appendix 4.

elements such that $A = HH^T$.[8]

Since a negative definite matrix is simply the negative of a positive definite matrix, the following restrictions specify that the NXN real symmetric matrix $[\pi_{ij}]$ is negative definite:

$$\left.\begin{array}{ll} -\pi_{ij} = h_{i,1} \cdot h_{j,1} + h_{i,2} \cdot h_{j,2} + \ldots\ldots + h_{i,j} \cdot h_{j,j} & \begin{array}{l}\text{all } (i,j) \\ \text{such that} \\ j \leq i\end{array} \\ h_{j,j} > 0 & j=1,\ldots,N \end{array}\right\} \quad (4)$$

where all $h_{i,j}$ are also restricted to be real numbers. Restrictions (4) comprise $\frac{N(N+1)}{2}$ quadratic equalities and N bounds.[9]

3 Restrictions Implied by Additional Properties of $[\pi_{ij}(x^*)]$

Given the maximization hypothesis, the comparative static effect $\frac{\partial x^*}{\partial \alpha}$ for the firm's static problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x^i) \qquad (P)$$

is defined by knowledge of the Hessian matrix $[\pi_{ij}(x^*)]$ (negative definite and symmetric) and the exogenous shift $c^1_{1\alpha}(x^{1*};\alpha)$ in the marginal factor cost schedule of input 1.[10] However, qualitative knowledge of the elements of $[\pi_{ij}(x^*)]$ and $c^1_{1\alpha}(x^{1*};\alpha)$ seldom determines $\frac{\partial x^*}{\partial \alpha}$ qualitatively, and direct quantitative knowledge of the elements of $[\pi_{ij}(x^*)]$ is in general very weak.[11]

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8. This theorem can be inferred from Forsyth and Moler (1967), pp. 27-29 and 114-115 plus Murdoch (1970), p. 232.

9. For a more general problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x^i)$$
$$\text{subject to } g(x) = 0\ ,$$

the implications of the maximization hypothesis are not as easily incorporated into our quantitative comparative static methods. However, exclusion of such problems does not seem to limit our analysis significantly (see the discussion of constrained maximization in Appendix 4).

10. See Proposition 1 in the previous section.

11. See section 3.2.1. of Chapter 3.

Thus, even for the purpose of calculating qualitative restrictions on $\frac{\partial x^*}{\partial \alpha}$, there is need for a method of comparative statics that incorporates additional quantitative restrictions on $[\pi_{ij}(x^*)]$.

In this section, we shall show how $[\pi_{ij}(x^*)]$ is related to various potentially observable and quantifiable properties ρ of the structure $\pi(x)$ of the firm's static maximization problem P. In contrast to the usual comparative static approaches, which attempt to deduce knowledge of $[\pi_{ij}(x^*)]^{-1}$ (and hence $\frac{\partial x^*}{\partial \alpha}$) from restrictions placed directly on $[\pi_{ij}(x^*)]$,[12] we shall place restrictions directly on the <u>inverse</u> of matrices that are essentially submatrices of $[\pi_{ij}(x^*)]$.

The vector of parameters ρ typically includes measures of the following types of properties of $[\pi_{ij}(x^*)]$:

(a) possibilities of factor substitution within any subset of inputs,

(b) returns to an exogenous change in output when any subset of inputs is held constant and all other inputs vary optimally in the static sense, and

(c) changes in input levels corresponding to an exogenous change in output when any subset of inputs is held constant and all other inputs vary optimally in the static sense.

A priori knowledge of a range of "reasonable" values for some of these parameters presumably is available in most cases. This knowledge would be derived from observation of physical processes, observation of firm behavior that approximates various short run comparative static effects, and from econometric estimation of physical processes and short run comparative static effects. By formulating these restrictions as confidence intervals or as Bayes intervals, the corresponding feasible set for $\frac{\partial x^*}{\partial \alpha}$ can also be interpreted as a confidence-Bayes interval.[13]

---

12. See section 3.3.2. of Chapter 3.

13. See section 5.

However, the following elementary point should be emphasized: although we can easily formulate conditions that exhaust the comparative static implications of the maximization hypothesis,[14] we cannot formulate conditions that exhaust the relations between $[\pi_{ij}(x^*)]$ and potentially observable data about the structure of the firm's problem P. Thus the relations between $[\pi_{ij}(x^*)]$ and data that are presented here should be viewed only as a subset of all useful relations between comparative static effects and observable structure of the firm's maximization problem.

3.1 Major Restrictions

The restrictions on $[\pi_{ij}(x^*)]$ that are most important in our method of quantitative comparative statics for a shift in a firm's factor supply schedule concern

(a) possibilities of factor substitution within a particular subset of inputs,

(b) returns to an exogenous change in output when a particular subset of inputs is held constant and all other inputs vary optimally in the static sense, and

(c) changes in input levels corresponding to an exogenous change in output when a particular subset of inputs is held constant and all other inputs vary optimally in the static sense.

The relations between $[\pi_{ij}(x^*)]$ and these potentially observable properties of the firm's static maximization problem are detailed in Theorem 3 and Corollary 5.[15,16]

Here we shall explain and elaborate upon these relations between $[\pi_{ij}(x^*)]$

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14. See Proposition 1.

15. Theorem 3 and Corollary 5 owe much to Mundlak (1966, 1968), and in turn to Mosak (1938).

16. Our quantitative comparative statics analysis could be extended easily to the case of a shift in the firm's product demand schedule (see the related section of Appendix 4).

and properties (a)-(c) of the firm's static maximization problem. In contrast to the usual comparative static methods which place restrictions directly on the elements of $[\pi_{ij}(x^*)]$, these relations shall place restrictions on the inverse of matrices that are essentially submatrices of $[\pi_{ij}(x^*)]$.

3.1-1 Model with Output Exogenous

Given the firm's static maximization problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \tag{P}$$

with solution x^*, define the related problem where output is treated as exogenous to the firm

$$\left.\begin{aligned} &\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \\ &\text{subject to } R(x) = \overline{R(x^*)} \end{aligned}\right\} \tag{5}$$

Problem (5) can be expressed in Lagrange form as

$$\text{maximize } \pi(x;\alpha) - \lambda(R(x) - \overline{R(x^*)}) \tag{6}$$

where the endogenous variables are (x,λ) and the exogenous variables are $(\alpha, \overline{R})$.

Suppose that the differentials of the interior first order conditions for (6) with respect to each of $(\alpha, \overline{R})$ yield a unique solution for all comparative static effects $\left(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial \lambda}{\partial \alpha}, \frac{\partial x^{**}}{\partial \overline{R}}, \frac{\partial \lambda}{\partial \overline{R}}\right)$.[17] This assumption is equivalent to the restriction that this system of differentials can be expressed in the form

$$[A]\ [K] = I \tag{7}$$

---

17. Since Proposition 1 can be generalized to the problem

$$\begin{aligned} &\text{maximize } (x;\alpha) \\ &\text{subject to } G(x;\alpha) = 0 \end{aligned}$$

(see the discussion of integrability in Appendix 4), there is no loss in generality in assuming that $\left(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial \lambda}{\partial \alpha}, \frac{\partial x^{**}}{\partial \overline{R}}, \frac{\partial \lambda}{\partial \overline{R}}\right)$ is uniquely defined for a given problem (6). In other words, $\left(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial x^{**}}{\partial \overline{R}}\right)$ is uniquely defined if $[\pi_{ij}(x^*)]$ is negative definite subject to constraint and is undefined if $[\pi_{ij}(x^*)]$ is only negative semi-definite subject to constraint, and $\left(\frac{\partial \lambda}{\partial \alpha}, \frac{\partial \lambda}{\partial \overline{R}}\right)$ is also uniquely defined or undefined (since a maximum or supremum is either uniquely defined or undefined for a given problem).

where the matrices [A], [K] and I are as defined in Theorem 3. [A] is the Hessian matrix $[\pi_{ij}(x^*)]$ bordered by marginal factor costs $c_i^i \equiv (c_1^1(x^{1*},\alpha^1), \ldots, c_N^N(x^{N*};\alpha^N))^T$,

$$[A] \equiv \left[\begin{array}{c|c} [\pi_{ij}] & c_i^i \\ \hline c_i^{iT} & 0 \end{array}\right], \qquad (8.)$$

[K] is a matrix of all the comparative static effects $\left(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial \lambda}{\partial \alpha}, \frac{\partial x^{**}}{\partial \bar{R}}, \frac{\partial \lambda}{\partial \bar{R}}\right)$[18] for problem (6), and I is an identity matrix.[19]

---

18. The "revenue effect" $\frac{\partial x^{i**}}{\partial \bar{R}}$ is related to the corresponding output effect $\frac{\partial x^{i**}}{\partial \bar{F}}$ simply as follows: $\frac{\partial x^{i**}}{\partial \bar{F}} = \frac{\partial x^{i**}}{\partial \bar{R}} \cdot \frac{\partial R(y^*)}{\partial y}$ (by the chain rule) where $y \equiv F(x)$ and $R(y) = R(F(x))$. Likewise, $\frac{\partial \pi(x^*)}{\partial \bar{F}} = \frac{\partial \pi(x^*)}{\partial \bar{R}} \cdot \frac{\partial R(y^*)}{\partial y}$ and so $\frac{\partial^2 \pi(x^*)}{\partial \bar{F}} = \frac{\partial(\partial \pi/\partial \bar{F})}{\partial \bar{R}} \quad \frac{\partial R(y^*)}{\partial y}$, which yields $\frac{\partial \lambda}{\partial \bar{F}} = \frac{\partial \lambda}{\partial \bar{R}} \cdot \left(\frac{\partial R(y^*)}{\partial y}\right)^2$.

19. Knowledge of comparative static effects in the presence of a constraint on expenditure for a particular subset of inputs could be easily incorporated into this approach. For example, consider the problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i)$$

$$\text{subject to } \sum_{i=1}^{S} c^i(x^i;\alpha^i) = \bar{C}$$

or equivalently

$$\text{maximize } \pi(x;\alpha) - \lambda\left(\sum_{i=1}^{S} c^i(x^i;\alpha^i) - \bar{C}\right).$$

Then it can be easily shown that the comparative static effects $\left(\frac{\partial x^E}{\partial \alpha}, \frac{\partial \lambda^E}{\partial \alpha}, \frac{\partial x^E}{\partial \bar{C}}, \frac{\partial \lambda^E}{\partial \bar{C}}\right)$ for this problem are related to [K] as follows:

$$\begin{array}{llll}
\frac{\partial x^{iE}}{\partial \alpha^j} = c^j_{j\alpha j} \cdot K_{i,j} - c^j_{\alpha j} \cdot K_{i,N+1} & & i=1,\ldots,N & j=1,\ldots,S \\
\frac{\partial x^{iE}}{\partial \alpha^j} = c^j_{j\alpha j} \cdot K_{i,j} & = \frac{\partial x^{i**}}{\partial \alpha^j} & i=1,\ldots,N & j=S+1,\ldots,N \\
\frac{\partial x^{iE}}{\partial \bar{C}} = K_{i,N+1} & = \frac{\partial x^{i**}}{\partial \bar{R}} & i=1,\ldots,N & \\
\frac{\partial \lambda^E}{\partial \alpha^j} = -c^j_{j\alpha j} \cdot K_{N+1,j} + c^j_{\alpha j} \cdot K_{N+1,N+1} & & & j=1,\ldots,S \\
\frac{\partial \lambda^E}{\partial \alpha^j} = -c^j_{j\alpha j} \cdot K_{N+1,j} & = \frac{\partial \lambda}{\partial \alpha^j} & & j=S+1,\ldots,N \\
\frac{\partial \lambda}{\partial \bar{C}} = -K_{N+1,N+1} & = \frac{\partial \lambda}{\partial \bar{R}} & &
\end{array}$$

Theorem 3. Suppose that conditions 1-2 are satisfied for a problem P

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \tag{1}$$ [20]

and assume that this problem has a unique global solution x^* where the Hessian matrix for $\pi(x)$ is negative definite. Construct the related problem

$$\text{maximize } \pi(x;\alpha)$$

$$\text{subject to } R(x) = \overline{R(x^*)}$$

which can be expressed in Lagrange form as

$$\text{maximize } \pi(x;\alpha) - \lambda(R(x) - \overline{R(x^*)}). \tag{2}$$

Construct the symmetric matrix

$$\left[\begin{array}{c|c} \pi_{ij} & c_i^i \\ (N \times N) & (N \times 1) \\ \hline c_i^i & 0 \\ (1 \times N) & (1 \times 1) \end{array}\right] \equiv \underset{(N+1)\times(N+1)}{[A]}$$

where $\underset{(N \times N)}{\pi_{ij}}$ denotes the Hessian matrix for $\pi(x;\alpha)$ at x^*, and

$c_i^i \equiv \left(\frac{\partial c^1(x^{1*};\alpha^1)}{\partial x^1}, \ldots\ldots, \frac{\partial c^N(x^{N*};\alpha^N)}{\partial x^N}\right)$. [A] necessarily has full rank, and denote its inverse as [K]:

$$[A]^{-1} \equiv \underset{(N+1)\times(N+1)}{[K]} \quad \text{always exists.}$$

Then,

(A) the comparative static effects for problem 2 are uniquely defined as follows:

$$\frac{\partial x^{i**}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot K_{i,j} \qquad i,j = 1,\ldots\ldots, N$$

$$\frac{\partial x^{i**}}{\partial \overline{R}} = K_{i,N+1} \qquad i = 1,\ldots\ldots, N$$

---

20. This theorem is easily generalized to the case $c^i \equiv c^i(x;\alpha^i)$ $(i=1, \ldots, N)$, but the equations in the generalized theorem are somewhat more detailed than here, and the generalized theorem will not be employed here.

$$\frac{\partial\lambda}{\partial\alpha^j} = -\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial\alpha^j} \cdot K_{N+1,j} \qquad j = 1,\ldots\ldots, N$$

$$\frac{\partial\lambda}{\partial R} = -K_{N+1,N+1}$$

where $K_{i,j} \equiv$ element (i,j) of matrix $[K]$, and $K_{i,j} = K_{j,i}$ $(i,j = 1,\ldots, N+1)$;[21]

and

(B) (a) The comparative static effects $\left(\frac{\partial x^*}{\partial\alpha}\right)$ for problem 1 are unique, and

(b) given that $\sum_{i=1}^{N}\sum_{j=1}^{N} K_{i,N+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$,[22]

$\frac{\partial x^*}{\partial\alpha}$ for problem 1 is uniquely defined in terms of

$\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial\alpha^j}$ and the elements of $[K]$ corresponding to

$\frac{\partial x^{**}}{\partial\alpha^j}$ and $\frac{\partial x^{**}}{\partial R}$ for problem 2, as follows:

$$\frac{\partial x^{i*}}{\partial\alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial\alpha^j} \cdot K_{i,j} + K_{i,N+1} \cdot \frac{\partial R(x^*)}{\partial\alpha^j} \qquad i,j = 1,\ldots\ldots, N$$

$$\frac{\partial R(x^*)}{\partial\alpha^j} = \sum_{i=1}^{N} \frac{\partial c^i(x^{i*};\alpha^j)}{\partial x^i} \cdot \frac{\partial x^{i*}}{\partial\alpha^j} \qquad j = 1,\ldots\ldots, N$$

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21. Thus $\frac{\partial x^{i**}}{\partial\alpha^j} = \left(\frac{\partial^2 c^j(x^{i*};\alpha^j)}{\partial x^j \partial\alpha^j} \Big/ \frac{\partial^2 c^i(x^{i*};\alpha^i)}{\partial x^i \partial\alpha^i}\right)\frac{\partial x^{j**}}{\partial\alpha^i}$ and $\frac{\partial\lambda}{\partial\alpha^j} = -\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial\alpha^j} \cdot \frac{\partial x^{j**}}{\partial R}$ $(i,j = 1,\ldots\ldots, N)$.

22. A sufficient condition for $\sum_{i=1}^{N}\sum_{j=1}^{N} K_{i,N+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$ (a) is that $K_{i,N+1} \geq 0$ $(i=1,\ldots\ldots, N)$, which is equivalent to ruling out the possibility of inferior inputs $\left(\frac{\partial x^{i**}}{\partial R} \geq 0 \iff K_{i,N+1} \geq 0\right)$. Condition (a) would be violated only for a relatively few "appropriate" degrees of inferiority; so condition (a) is not a serious restriction.

Thus equations (7) plus restrictions on the comparative static effects for problem (6) and on equilibrium marginal factor costs imply restrictions on the elements of the matrix $[\pi_{ij}(x^*)]$ for problem P. These relations that are expressed in equations (9) can be summarized as follows:

$$\text{(a)} \quad \sum_{j=1}^{N} \pi_{ik} \frac{\partial x^{i**}}{\partial \alpha^k} - c_k^k \frac{\partial \lambda}{\partial \alpha^k} = c_{k\alpha^k}^k$$

$$\sum_{i=1}^{N} \pi_{ik} \frac{\partial x^{i**}}{\partial \alpha^k} - c_j^j \frac{\partial \lambda}{\partial \alpha^k} = 0 \qquad \text{all } j \neq k \qquad k = 1,\ldots., N$$

$$\text{(b)} \quad \sum_{i=1}^{N} c_i^i \frac{\partial x^{i**}}{\partial \alpha^k} = 0$$

$$\text{(c)} \quad \sum_{i=1}^{N} \pi_{ij} \frac{\partial x^{j**}}{\partial R} - c_j^j \frac{\partial \lambda}{\partial R} = 0 \qquad \text{all } j.$$

$$\text{(d)} \quad \sum_{i=1}^{N} c_i^i \frac{\partial x^{i**}}{\partial R} = 1$$

where all partial derivatives are evaluated at (x^*,α).

Given knowledge of equilibrium marginal factor costs and of N-1 elements of $\frac{\partial x^{**}}{\partial \alpha^k}$ and N-1 elements of $\frac{\partial x^{**}}{\partial R}$, all elements of $\frac{\partial x^{**}}{\partial \alpha^k}$ and $\frac{\partial x^{**}}{\partial R}$ are known (see (b) and (d)). In this case, the comparative static effect $\frac{\partial x^*}{\partial \alpha^k}$ for problem P could be calculated directly from the relations

$$\frac{\partial x^{i*}}{\partial \alpha} = \frac{\partial x^{i**}}{\partial \alpha^k} + \frac{\partial x^{i**}}{\partial R} \cdot \frac{\partial R(x^*)}{\partial \alpha^k} \qquad i = 1,\ldots., N^{23} \qquad (9)$$

$$\frac{\partial R(x^*)}{\partial \alpha^k} = \sum_{i=1}^{N} c_i^i \cdot \frac{\partial x^{i*}}{\partial \alpha^k} \qquad (10)$$

except under unusual circumstances.[24]

23. Statement (9) is essentially Theorem 7-1 of Sakai (1973) (Theorem 7-1 has an obvious typing error).

24. As can be seen from (9)-(10), $\frac{\partial x^{i*}}{\partial \alpha^k}$ is not a simple weighted sum of the pure substitution effect $\frac{\partial x^{i**}}{\partial \alpha^k}$ and scale effect $\frac{\partial x^{i**}}{\partial R}$ in contrast to the Slutsky equation in consumer theory. For the unusual circumstances under which $\frac{\partial x^*}{\partial \alpha^k}$ cannot be calculated from (9)-(10), see Theorem 3-B.

In addition, when knowledge of $(c^1_1,..,c^N_N, \frac{\partial x^{**}}{\partial \alpha^k}, \frac{\partial x^{**}}{\partial \bar{R}})$ is not exact, the restrictions on $\frac{\partial x^*}{\partial \alpha^k}$ are presumably increased by incorporating restrictions on $\frac{\partial x^{**}_j}{\partial \alpha^j}$ as well as on $\frac{\partial x^{**}}{\partial \alpha^k}$ into the quantitative comparative statics model. In this more general case, the quantitative comparative statics model includes the conditions implied by the maximization hypothesis,[25] the equations [A] [K] = I and restrictions on elements of [K]. These restrictions correspond to the "reasonable" range of values for $(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial x^{**}}{\partial \bar{R}}, c^1_{1\alpha^1},.., c^N_{N\alpha^N})$ and also for $\frac{\partial \lambda}{\partial \bar{R}} \equiv \frac{\partial^2 \pi(x^*)}{\partial \bar{R}^2} < 0.$[26,27]

Nevertheless, in many situations knowledge of the comparative static substitution and scale effects when all inputs are variable $(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial x^{**}}{\partial \bar{R}})$ may be almost as scarce as knowledge about the comparative static total effect $\frac{\partial x^*}{\partial \alpha^1}$ itself. Considerably more knowledge about substitution and scale effects may be available for cases where subsets of inputs are fixed for the firm.

3.1.2 Model with Output and a Subset of Inputs Exogenous

For many situations where knowledge about $(\frac{\partial x^{**}}{\partial \alpha}, \frac{\partial x^{**}}{\partial \bar{F}})$ is quite weak, a narrower range of "reasonable" values for substitution and scale effects when some inputs are fixed may be readily available. Moreover, this knowledge of

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25. See Proposition 1.

26. See Theorem 3-A. Since [K] is symmetric and knowledge of $(\frac{\partial x^{**}}{\partial \bar{R}}, c^1_{1\alpha^1},.., c^N_{N\alpha^N})$ is presumably greater than knowledge of $\frac{\partial \lambda}{\partial \alpha}$ <u>per se</u>, restrictions on $\frac{\partial \lambda}{\partial \alpha}$ seldom would be specified.

27. $\frac{\partial \lambda}{\partial \bar{R}} \equiv \frac{\partial^2 \pi(x^*)}{\partial \bar{R}^2} < 0$ for $[\pi_{ij}(x^*)]$ negative definite (since $\frac{\partial \pi(x^*)}{\partial \bar{R}} = 0$ by envelope theorem, and $\frac{\Delta \pi(x^*)}{\Delta \bar{R}} < 0$ for a finite ΔR by $[\pi_{ij}(x^*)]$ negative definite).

substitution and scale effects when various subsets of inputs are fixed may imply strong restrictions on the comparative static effect $\frac{\partial x^*}{\partial \alpha}$ for problem P. This statement can be elaborated upon as follows.

Given the firm's problem P, define the related "short run" static maximization problem

$$\left.\begin{aligned} &\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \\ &\text{subject to } R(x) = \overline{R(x^*)} \\ &\qquad\qquad x^j = \overline{x^{j*}} \qquad\qquad j = S+1,\ldots., N \end{aligned}\right\} \tag{11}$$

where output and an arbitrary subset of inputs are exogenous to the firm at the equilibrium levels for P. This problem can be expressed in Lagrange form as

$$\text{maximize } \pi(x;\alpha) - \lambda(R(x) - \overline{R(x^*)}) - \sum_{j=S+1}^{N} \gamma^j(x^j - \overline{x^{j*}}) \tag{12}$$

where the endogenous variables are $(x^1,.., x^S, \lambda, \gamma^{S+1},.., \gamma^N)$ and the exogenous variables are $(\alpha, \overline{R}, \overline{x^{S+1}},.., \overline{x^N})$.

Suppose that the differentials of the interior first order conditions for (12) with respect to each of $(\alpha, \overline{R})$ yield a unique solution for the comparative static effects $\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial \lambda^S}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \overline{R}}, \frac{\partial \lambda^S}{\partial \overline{R}}\right)$.[28] This assumption is equivalent to the restriction that this system of differentials can be expressed in the form

$$[A_{11}]\,[L] = I \tag{13}$$

where the matrices $[A_{11}]$,$[L]$ and I are as defined in Corollary 5. $[A_{11}]$ consists of (a) the principal submatrix $[\pi_{ij}^A]$ of $[\pi_{ij}(x^*)]$ that is formed by deleting rows and columns (S + 1,.., N) from $[\pi_{ij}(x^*)]$ and (b) the subvector $c_i^{iA} \equiv (c_1^1(x^{1*};\alpha^1),\ldots, c_S^S(x^{S*};\alpha^S))^T$ on the borders of $[\pi_{ij}^A]$, i.e.

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28. There is no loss in generality in assuming that $\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial \gamma^S}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \overline{R}}, \frac{\partial \lambda^S}{\partial \overline{R}}\right)$ is uniquely defined for a given problem (12) (see first footnote in section 3.4.1).

Corollary 5. Construct the problems 1 and 2, and the $(N + 1) \times (N + 1)$ matrices [A] and [K], as in Theorem 3. Partition the Hessian matrix $\underset{(N \times N)}{\pi_{ij}}$ and marginal factor cost vector c_i^i of [A] as follows:

$$\underset{(N \times N)}{[\pi_{ij}]} \equiv \left[\begin{array}{c|c} \underset{(S \times S)}{\pi_{ij}^A} & \underset{(S \times T)}{\pi_{ij}^B} \\ \hline \underset{(T \times S)}{\pi_{ij}^C} & \underset{(T \times T)}{\pi_{ij}^D} \end{array}\right] \qquad \underset{(1 \times N)}{c_i^i} \equiv [\underset{(1 \times S)}{c_i^{iA}} \mid \underset{(1 \times T)}{c_i^{iB}}]$$

where $S + T = N$. Construct the following symmetric matrix

$$\left[\begin{array}{c|c} \underset{(S \times S)}{\pi_{ij}^A} & \underset{(S \times 1)}{c_i^{iA}} \\ \hline \underset{(1 \times S)}{c_i^{iA}} & \underset{(1 \times 1)}{0} \end{array}\right] \equiv \underset{(S + 1) \times (S + 1)}{[\bar{A}_{11}]}.$$

$[\bar{A}_{11}]$ necessarily has full rank, and denote its inverse as [L]:

$[\bar{A}_{11}]^{-1} \equiv [L]$ always exists.

Construct the problem

$$\left.\begin{array}{ll} \text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) & \\ \text{subject to } x^j = \overline{x^{j*}} & j=S + 1, \ldots, N \end{array}\right\} \quad (3)$$

where x^* is the unique global solution to problem 1.

Construct the related problem

$$\text{maximize } \pi(x;\alpha)$$

$$\text{subject to } R(x) = \overline{R(x^*)}$$

$$x^j = \overline{x^{j*}} \qquad j=S + 1, \ldots\ldots, N$$

which can be expressed in Lagrange form as

$$\text{maximize } \pi(x;\alpha) - \lambda(R(x) - \overline{R(x^*)}) - \sum_{j=S+1}^{N} \gamma^j(x^j - \overline{x^{j*}}). \quad (4)$$

Then

(A) the comparative static effects for problem 4 are uniquely defined as follows:

$$\frac{\partial x^{i**S}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot L_{i,j} \qquad i,j=1,\ldots\ldots, S$$

$$\frac{\partial x^{i**S}}{\partial R} = L_{i,S+1} \qquad i=1,\ldots\ldots, S$$

$$\frac{\partial \lambda^S}{\partial \alpha^j} = -\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot L_{S+1,j} \qquad j=1,\ldots\ldots, S$$

$$\frac{\partial \lambda^S}{\partial R} = -L_{S+1,S+1}$$

where $L_{i,j} \equiv$ element (i,j) of $[L]$, and $L_{i,j} = L_{j,i} (i,j=1,\ldots\ldots, S+1)$; and

(B) (a) the comparative static effects $\frac{\partial x^{*S}}{\partial \alpha}$ for problem 3 are unique, and

(b) given that $\sum_{i=1}^{S} \sum_{j=1}^{S} L_{i,S+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$,[29]

$\frac{\partial x^{*S}}{\partial \alpha^j}$ for problem 3 is uniquely defined in terms of

$\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j}$ and the elements of $[L]$ corresponding to

$\frac{\partial x^{**S}}{\partial \alpha^j}$ and $\frac{\partial x^{**S}}{\partial R}$ for problem 4, as follows:

$$\frac{\partial x^{i*S}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot L_{i,j} + L_{i,S+1} \cdot \frac{\partial R(x^*)^S}{\partial \alpha^j} \qquad i,j=1,\ldots\ldots, S$$

$$\frac{\partial R(x^*)^S}{\partial \alpha^j} = \sum_{i=1}^{S} \frac{\partial c^i(x^{i*};\alpha^i)}{\partial x^i} \cdot \frac{\partial x^{i*S}}{\partial \alpha^j} \qquad j=1,\ldots\ldots, S.$$

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29. Assuming that $\sum_{i=1}^{S} \sum_{j=1}^{S} L_{i,S+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$ has implications analogous to those of assuming that $\sum_{i=1}^{N} \sum_{j=1}^{N} K_{i,N+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$ (see footnote to Theorem 3).

$$[A_{11}] = \left[\begin{array}{c|c} [\pi_{ij}{}^{A}] & c_i^{iA} \\ \hline c_i^{iAT} & 0 \end{array}\right] . \qquad (14)$$

[L] is a matrix of the comparative static effects $(\frac{\partial x^{**S}}{\partial\alpha}, \frac{\partial\lambda^S}{\partial\alpha}, \frac{\partial x^{**S}}{\partial R}, \frac{\partial\lambda^S}{\partial R})$ and I is an identity matrix.

By (13)-(14), knowledge of the elements of [L] and c_i^{iA} places restrictions of $[\pi_{ij}(x^*)]$. Thus knowledge of the comparative static effects $(\frac{\partial x^{**S}}{\partial\alpha}, \frac{\partial x^{**S}}{\partial R}, \frac{\partial\lambda^S}{\partial R})$ for problem (12) and of equilibrium marginal factor costs places restrictions on the "long run" comparative static effect $\frac{\partial x^*}{\partial\alpha^i}$ for problem P.[30]

Moreover, the comparative static effect $\frac{\partial x^*}{\partial\alpha^i}$ for problem P can almost always be defined precisely in terms of a set of comparative static effects $\{(\frac{\partial x^{**S}}{\partial\alpha^i}, \frac{\partial x^{**S}}{\partial R})\}$ for an appropriate set of problems (14), and these relations are implicit in our standard quantitative comparative statics model. The comparative static effects included in this set will differ in terms of the partition into fixed and variable inputs and the choice of shift parameter α^i. This important relation between $\frac{\partial x^*}{\partial\alpha^i}$ and various sets $\{(\frac{\partial x^{**S}}{\partial\alpha}, \frac{\partial x^{**S}}{\partial R})\}$ can be demonstrated as follows. Consider a comparative static effect $\frac{\partial x^{*S}}{\partial\alpha^k}$ for the problem

$$\left.\begin{aligned} &\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \\ &\text{subject to} \quad x^j = \overline{x^{j*}} \qquad j=S+1,\ldots, N. \end{aligned}\right\} \quad (15)$$

$\frac{\partial x^{*S}}{\partial\alpha^k}$ is almost always uniquely defined by the relations analogous to (18)-(19)

30. Since [L] is symmetric and knowledge of $(\frac{\partial x^{**S}}{\partial R}, c^1_{1\alpha^1},\ldots, c^S_{S\alpha^S})$ presumably is greater than knowledge of $\frac{\partial\lambda^S}{\partial\alpha}$ *per se*, restrictions on $\frac{\partial\lambda^S}{\partial\alpha}$ seldom would be specified (see Corollary 5-A).

$$\frac{\partial x^{i*S}}{\partial \alpha^k} = \frac{\partial x^{i**S}}{\partial \alpha^k} + \frac{\partial x^{i**}}{\partial R} \cdot \frac{\partial R(x^*)^S}{\partial \alpha^k} \qquad i=1,.., S \tag{16}$$

$$\frac{\partial R(x^*)^S}{\partial \alpha^k} = \sum_{i=1}^{S} c_i^i \frac{\partial x^{i*S}}{\partial \alpha^k} \tag{17}$$

plus a given comparative static effect $(\frac{\partial x^{**S}}{\partial \alpha^k}, \frac{\partial x^{**S}}{\partial R})$ for the corresponding problem (14).[31] In addition, the following relations are satisfied for problem (15):

$$[\pi_{ij}{}^A]\ [P] = I \tag{18}$$

where $[\pi_{ij}{}^A]$ is the principal submatrix of $[\pi_{ij}(x^*)]$ obtained by deleting the rows and columns (S + 1,.., N) from $[\pi_{ij}(x^*)]$, [P] is symmetric and $P_{i,j} = \frac{\partial x^{i*S}}{\partial \alpha^j} \Big/ c^j_{j\alpha^j}$.[32] Thus any principal submatrix of $[\pi_{ij}(x^*)]$ can be uniquely defined by (18) and a suitable partitioning of x into variable and fixed inputs, a suitable choice of shift parameters $\{\alpha^j\}$, and the corresponding given $\{\frac{\partial x^{*S}}{\partial \alpha^j}\}$.[33] By repeating this procedure, $[\pi_{ij}(x^*)]$ and $\frac{\partial x^*}{\partial \alpha^i}$ can be determined. In sum, $\frac{\partial x^*}{\partial \alpha^i}$ can almost always be defined precisely in terms of $(\frac{\partial x^{**S}}{\partial \alpha^j}, \frac{\partial x^{**S}}{\partial R})$ for an appropriate set of problems (14). Moreover, the restrictions

$$\left.\begin{aligned} &[\pi_{ij}(x^*)]\ \frac{\partial x}{\partial \alpha^1} = \begin{bmatrix} c^1_{1\alpha^1} \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ [\pi_{ij}(x^*)] \text{ negative definite} \\ &[\Lambda_{11}]\ [L] = I \end{aligned}\right\} \tag{19}$$

(a) uniquely define the comparative static effects for problem P and a series of problems (14) for a given $[\pi_{ij}(x^*)]$ and a given series of partitionings into variable and fixed inputs, and (b) define $\frac{\partial x^*}{\partial \alpha^i}$ in terms of $[\pi_{ij}(x^*)]$ and a

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31. See Corollary 5-B.

32. See Corollary 6.

33. The symmetry of [P], i.e. $\frac{\partial x^{i*S}}{\partial \alpha^j} \Big/ c^j_{j\alpha^j} = \frac{\partial x^{j*S}}{\partial \alpha^i} \Big/ c^i_{i\alpha^i}$ for all i,j= 1,.. S, implies several degrees of freedom in selecting a $\{\alpha^j\},\{(\frac{\partial x^{**S}}{\partial \alpha^j}, \frac{\partial x^{**S}}{\partial R})\}$ in order to define a given $[\pi_{ij}{}^A]$.

$\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \bar{R}}\right)$ in terms of a subset of elements of $[\pi_{ij}(x^*)]$ and $(c_1^1, \ldots, c_N^N)$. Thus the relation between $\frac{\partial x^*}{\partial \alpha^I}$ and a $\{(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \bar{R}})\}$ is implicit in (19).

In addition, $\frac{\partial x^*}{\partial \alpha^I}$ for problem P apparently can be uniquely defined in terms of $\frac{\partial \lambda^S}{\partial \bar{R}} \equiv \frac{\partial^2 \pi(x^*)^S}{\partial \bar{R}^2}$ for some problems (12) plus a subset of $\{(\frac{\partial x^{**S}}{\partial \alpha^J}, \frac{\partial x^{**S}}{\partial \bar{R}})\}$ specified above. In other words, knowledge of a $\frac{\partial \lambda^S}{\partial \bar{R}}$ for a problem (12) can "substitute" for some knowledge of $\{(\frac{\partial x^{**S}}{\partial \alpha^J}, \frac{\partial x^{**S}}{\partial \bar{R}})\}$ in the determination of $\frac{\partial x^*}{\partial \alpha^I}$ for a problem P.[34]

In sum, restrictions on $\{(\frac{\partial x^{**S}}{\partial \alpha^J}, \frac{\partial x^{**S}}{\partial \bar{R}}, \frac{\partial \lambda^S}{\partial \alpha^J})\}$ for various problems (12) and $\{\alpha^j\}$ plus the relations (19) may imply considerable restrictions on $\frac{\partial x^*}{\partial \alpha^I}$ for problem P. Since knowledge of substitution and scale effects will be defined primarily in terms of problems (12) with various subsets of fixed inputs, these restrictions derived from a model with output and a subset of inputs exogenous are a very important aspect of our quantitative comparative statics model.

3.1.3 Model with a Subset of Inputs Exogenous

In addition, direct knowledge about the total effects of $d\alpha^i$ when certain

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34. Knowledge of a $\frac{\partial \lambda^S}{\partial \bar{R}}$ can "substitute" for some knowledge of $\{(\frac{\partial x^{**S}}{\partial \alpha^J}, \frac{\partial x^{**S}}{\partial \bar{R}})\}$ in the determination of $\frac{\partial x^*}{\partial \alpha^I}$ if $p_1^1(x^{1*})$ is a function of (among other things $\frac{\partial \lambda^S}{\partial \bar{R}}$. Since $\frac{\partial \lambda^S}{\partial \bar{R}} \equiv \frac{\partial \pi(x^*)^S}{\partial \bar{R}^2}$ for a problem (12) and $p_1^1(x^{1*}) \equiv \frac{\partial^2 \pi(x^*)^Q}{\partial x^{1\,2}}$ for a problem Q corresponding to problem P, this relation between $p_1^1(x^{1*})$ and $\frac{\partial \lambda^S}{\partial \bar{R}}$ seems somewhat reasonable. Differentiating the first order condition $p^1 - c_1^1 = 0$ with respect to α^1 yields

$$p_1^1(x^{1*}) \frac{\partial x^{1*}}{\partial \alpha^I} - c_{11}^1 \frac{\partial x^{1*}}{\partial \alpha^I} - c_{1\alpha^1}^1 = 0;$$

so $\frac{\partial \lambda^S}{\partial \bar{R}}$ influences $\frac{\partial x^{1*}}{\partial \alpha^I}$ and hence must "substitute" for some knowledge of $\{(\frac{\partial x^{**S}}{\partial \alpha^j}, \frac{\partial x^{**S}}{\partial \bar{R}})\}$ if $\frac{\partial \lambda^S}{\partial \bar{R}}$ influences $p_1^1(x^{1*})$.

subsets of inputs are fixed may be available. Such knowledge can be specified as restrictions on comparative static effects $\{\frac{\partial x^{*S}}{\partial \alpha^{i}}\}$ for various "short run" static problems of the form (15)

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i)$$

$$\text{subject to} \quad x^j = \overline{x^{j*}} \qquad j=S+1,.., N.$$

These restrictions plus the following relations (18) can be incorporated into our quantitative comparative statics model:

$$[\pi_{ij}^{A}]\ [P] = I$$

where $[\pi_{ij}^{A}]$ is the principal submatrix of $[\pi_{ij}(x^*)]$ obtained by deleting the rows and columns (S + 1,.., N) from $[\pi_{ij}(x^*)]$, [P] is symmetric and $P_{ij} = \frac{\partial x^{i*S}}{\partial \alpha^{j}} \Big/ c^{j}_{j\alpha^j}$.[35] In this manner, knowledge about a "reasonable" range of values for $\frac{\partial x^{*S}}{\partial \alpha^{i}}$ corresponding to any problem (15) places restrictions on $\frac{\partial x^*}{\partial \alpha^{i}}$ for problem P.

3.2 Minor Restrictions

Other forms of knowledge about the structure of the firm's static maximization problem P may be available and useful in defining "reasonable" limits on the comparative static effect $\frac{\partial x^*}{\partial \alpha^{i}}$ for problem P. These additional forms of knowledge are of at least two types. First, there may be knowledge of the comparative static effect of a change in the demand schedule for the firm's output or in the firm's production function. Including the corresponding restrictions in our standard quantitative comparative statics model[36] seems likely to lead to a small reduction in the range of feasible values for $\frac{\partial x^*}{\partial \alpha^{i}}$. If such comparative static effects and its "short run" variations with fixed inputs are included in our model, then our model incorporates knowledge of all types of

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35. See Corollary 6.

36. On the standard model, see Proposition 1-A, Theorem 3 and Corollaries 5-6.

Corollary 6. Construct problems 1 and 3 as above, and partition the (negative definite) Hessian matrix $\underset{(N \times N)}{\pi_{ij}}$ as above. Then the comparative static effects for problem 3 are uniquely defined as follows:

$$\frac{\partial x^{i*S}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot P_{i,j} \qquad i,j=1,\ldots\ldots, S$$

where $P_{i,j} \equiv$ element (i,j) of $[\underset{(S \times S)}{\pi_{ij}{}^A}]^{-1}$ (which always exists), and

$P_{i,j} = P_{j,i}$ $(i,j = 1,\ldots\ldots, S)$.

comparative static effects that can occur realistically at the level of the single firm.[37]

Second, there may be specific knowledge about the functional form of the firm's static maximization problem P. The following examples are considered here: separability of $\pi(x;\alpha)$ in x, linear homogeneity of $\pi(x;\alpha)$ in α, fixed factor proportions for R(x), and homotheticity of $\pi(x;\alpha)$ in x. The first two properties, and presumably many other special properties of $\pi(x;\alpha)$, are easily incorporated into our quantitative comparative statics model. Such restrictions will be useful when (a) observation and/or theory suggests that such a property is closely approximated, or (b) sensitivity of comparative static results to such properties is an important issue.[38] In these circumstances, the imposition of such properties or of limits on the "degree of deviation" from such properties can be useful in our quantitative comparative statics models.

3.2.1 Knowledge of the Comparative Static Effects of a Change in R(x)

Knowledge of the comparative static effects of a shift in the firm's revenue or benefits function also places restrictions on $[\pi_{ij}]$. Define the firm's static maximization problem P as

$$\text{maximize } \pi(x;\alpha) \equiv R(x;\alpha^0) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \text{(P)}$$

i.e. allow for the possibility of shift parameters in the firm's revenue or benefits function as well as in the firm's factor cost schedules.[39] Total

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37. See section 6.

38. For example, calculating the sensitivity of comparative static results to the property of separability may provide a rough estimate of errors due to inappropriate aggregation of inputs in a quantitative comparative statics model (see section 3.2.2 of this Appendix and section 3.1 of Appendix 5).

39. Here we are only interested in knowledge of $\frac{\partial x^*}{\partial \alpha^0}$ as a means of obtaining knowledge about $\frac{\partial x^*}{\partial \alpha^i}$. For several brief remarks on the quantitative comparative statics of a shift in a firm's revenue or benefits schedule, i.e. on the case where our ultimate interest is knowledge of $\frac{\partial x^*}{\partial \alpha^0}$ rather than $\frac{\partial x^*}{\partial \alpha^i}$ $(i \neq 0)$, see Appendix 4.

differentiating the interior first order conditions for problem P with respect to α^0 yields

$$[\pi_{ij}]\ \frac{\partial x^*}{\partial \alpha^0} = \begin{bmatrix} R_{1\alpha^0} \\ \vdots \\ R_{N\alpha^0} \end{bmatrix}$$

where $R_{i\alpha^0} \equiv \frac{\partial^2 R(x^*;\alpha^0)}{\partial x^i \partial \alpha^0}$.[40] Thus restrictions on $\frac{\partial x^*}{\partial \alpha^0}$ and $R_{i\alpha^0}$ $(i=1,.., N)$ imply restrictions on $[\pi_{ij}(x^*)]$.[41]

However, restrictions on $\frac{\partial x^*}{\partial \alpha^0}$ and its decompositions seem considerably less important for our purposes than are the restrictions specified by Proposition 1-A, Theorem 3 and Corollaries 5-6. This statement can be elaborated upon as follows. Prior knowledge of $\frac{\partial x^*}{\partial \alpha^0}$ <u>per</u> <u>se</u> generally appears to be quite weak. Moreover, the decomposition relating $\frac{\partial x^*}{\partial \alpha^0}$ for problem P to $(\frac{\partial x^{**}}{\partial \overline{F}}, \frac{\partial x^{**}}{\partial \alpha^0})$ for the problem

$$\left.\begin{aligned} &\text{maximize } \pi(x;\alpha) \equiv R(x;\alpha^0) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \\ &\text{subject to } F(x) = \overline{F(x^*)} \end{aligned}\right\} \quad (20)$$

only leads to some of the restrictions on $[\pi_{ij}(x^*)]$ already specified by the relations presented in Proposition 1-A and Theorem 3. $\frac{\partial x^{**}}{\partial \overline{F}}$ is simply the scale effect already defined in Theorem 3, and $\frac{\partial x^{**}}{\partial \alpha^0} = 0$ is already implied by the restrictions $[A][K] = I$ specified in Theorem 3.[42] On the other hand,

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40. Given that the firm produces a single output y according to the production function $y = F(x)$ and that α^0 does not enter $F(x)$, $R_{i\alpha^0} = (R_{y\alpha^0}/R_y)c^i_i$ where $R(x;\alpha^0) \equiv R(F(x);\alpha^0)$. If we further assume that $R(y;\alpha^0) \equiv P(y;\alpha^0)y$, then $R_y = P(y^*;\alpha^0) + P_y(y^*;\alpha^0)y^*$ and $R_{y\alpha^0} = P_{\alpha^0}(y^*;\alpha^0) + P_{y\alpha^0}(y^*;\alpha^0)y^*$.

41. In addition, this implies that $\frac{\partial x^*}{\partial \alpha^0}$ is uniquely defined in terms of $([\pi_{ij}], R_{1\alpha^0},.., R_{N\alpha^0})$ given that $[\pi_{ij}(x^*)]$ is negative definite. Of course, $[\pi_{ij}(x^*)]$ is not uniquely defined in terms of $(\frac{\partial x^*}{\partial \alpha^0}, R_{1\alpha^0},.., R_{N\alpha^0})$.

42. Note also that exact knowledge of $(\frac{\partial x^{**}}{\partial \overline{F}}, \frac{\partial x^{**}}{\partial \alpha^0})$ plus the restriction $[\pi_{ij}(x^*)]$ negative definite can only define $\frac{\partial x^*}{\partial \alpha^0}$ up to a positive scalar multiple (see Theorem 4-C and the related discussion in Appendix 4).

knowledge of the comparative static effect $\frac{\partial x^{*S}}{\partial \alpha^{\upsilon}}$ where various inputs are fixed may not be as weak as knowledge of $\frac{\partial x^*}{\partial \alpha^{\upsilon}}$, and may place additional restrictions on $[\pi_{ij}(x^*)]$.[43] Therefore, knowledge of the comparative static effect of a shift in the firm's revenue or benefits function and of such "short run" decompositions may help somewhat in determining "reasonable" upper and lower bounds on $\frac{\partial x^*}{\partial \alpha^1}$.

3.2.2 Several Special Properties of $\pi(x;\alpha)$

The following properties of the functional form of the firm's static maximization problem P are considered here: separability of $\pi(x;\alpha)$ in x, linear homogeneity of $\pi(x;\alpha)$ in α, fixed factor proportions for R(x), and homotheticity of $\pi(x;\alpha)$ in x. The first two properties are easily incorporated into our quantitative comparative statics model. In addition, limits on the "degree of deviation" from these properties are easily included in our model. However, it appears that the last two properties (especially homotheticity) cannot be incorporated into our model.

A twice differentiable function R(x) (with non-zero first derivatives everywhere) is defined as "weakly separable" with respect to the subset $\{1,.., m\}$ of the firm's N inputs when

$R(x) = f(y, x^{m+1}, .., x^N)$ for some (scalar-valued) functions f and $y(x^1, .., x^m)$.

This is equivalent to the "Leontief conditions"

$$\frac{\partial}{\partial x^k}(R_i/R_j) = 0 \quad \text{for all } i,j \in \{1,.., m\} \text{ and } k \in \{m+1,.., N\} \text{ over all } x.^{44,45} \tag{21}$$

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43. See Corollary 7-A in Appendix 4.

44. See Leontief (1947). In addition, $R(x) = f(y(x^1,.., x^m), z(x^{m+1}, .., x^N))$ is equivalent to (21) plus $\frac{\partial}{\partial x^k}(R_i/R_j) = 0$ for all $i,j \in \{m+1,.., N\}$ and $k \in \{1,.., m\}$ over all x. (21′)

45. For discussions of separability and aggregation, see Blackorby et al (1978) and Diewert (1977).

Thus (21) plus the assumption $c^i = c^i(x^i;\alpha)$ $(i = 1,.., N)$ are equivalent to the condition

$$\pi(x;\alpha) = f(y(x^1,.., x^m), x^{m+1},.., x^N) - C(x^1,.., x^m;\alpha) - \sum_{i=m+1}^{N} c^i(x^i;\alpha)$$

$$C(x^1,.., x^m;\alpha) \equiv \sum_{i=1}^{m} c^i(x^i;\alpha)$$

which implies that inputs (1,.., m) can be correctly aggregated in specifying the firm's objective function $\pi(x;\alpha)$ as well as its revenue function $R(x)$. moreover, given $c^i = c^i(x^i;\alpha)$ $(i = 1,.., N)$, inputs (1,.., m) can be correctly aggregated in specifying $\pi(x;\alpha)$ only if inputs (1,.., m) can be correctly aggregated in specifying $R(x)$. Therefore, at an interior solution to the problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha),$$

inputs (1,.., m) and associated comparative static effects can be treated correctly as an aggregate if and only if the following restrictions are satisfied

$$\pi_{ik}c_j^j - \pi_{jk}c_i^i = 0 \quad \text{for all } i,j \in \{1,.., m\} \text{ and } k \in \{m+1,.., N\}. \quad (22)$$

Moreover, limits on the "degree of deviation" from the possibility of correct aggregation can be specified roughly in our model by restrictions of the form

$$\left.\begin{array}{l} \rho^1 \cdot (c_j^j)^2 \leq \pi_{ik}c_j^j - \pi_{jk}c_i^i \leq \rho^2 \cdot (c_j^j)^2 \\ \overline{\rho^{L1}} \leq \rho^1 \leq \overline{\rho^{U1}} \\ \overline{\rho^{L2}} \leq \rho^2 \leq \overline{\rho^{U2}} \text{ 46,47} \end{array}\right\} \quad (23)$$

In addition, on occasion we can reasonably assume that (in the neighborhood of equilibrium)

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46. Under reasonable conditions, approximation to Leontief conditions is equivalent to approximately correct aggregation (Fisher, 1969). This also seems to imply that $R(x)$ approximates Leontief conditions when it approximates fixed factor proportions and is twice differentiable.

47. If Leontief conditions are to be incorporated directly into the comparative static model (as opposed to being used simply as a "justification" for the construction of a model that includes aggregate inputs), then approximations of the form (23) rather than (22) apparently should be used. This statement can be explained as follows: exact separability for a matrix $[\pi_{ij}(x^*)]$ implies that $[\pi_{ij}(x^*)]$ is negative semi-definite only (see Proposition 2 in Appendix 5) and local comparative statics is undefined in this case (by Proposition 1).

$$\pi(x;\alpha) \equiv \alpha^0 R(x) - \sum_{i=1}^{N} \alpha^i c^i(x^i) \tag{24}$$

i.e. $R(x;\alpha^0)$ is linear homogeneous in α^0 and $c^i(x^i;\alpha^i)$ is linear homogeneous in $\alpha^i (i = 1,.., N)$.[48] Then

$$\sum_{j=0}^{N} \frac{\partial x^{i*}}{\partial \alpha^j} \cdot \alpha^j = 0 \qquad i=1,.., N \tag{25}$$

where

$$\left.\begin{array}{ll} \alpha^0 = R_i(x^*;\alpha^0) \,/\, R_{i\alpha^0}(x^*;\alpha^0) & \\ \alpha^i = c^i_i(x^{i*};\alpha^i) \,/\, c^i_{i\alpha^i}(x^{i*};\alpha^i) & i=1,.., N\,^{49} \end{array}\right\} \tag{26}$$

Limits on the "degree of deviation" from linear homogeneity in α (25) can be incorporated into our model in a manner similar to (23).

However, the special property of fixed factor proportions for $R(x)$ is not yet incorporated satisfactorily into our comparative static model for the unconstrained problem "maximize $\pi(x;\alpha)$." As can be seen from (9), fixed proportions between all factors is equivalent to the following:

$$\frac{\partial x^{i**}}{\partial \alpha^j} = 0 \text{ and } \frac{\partial x^{i**}}{\partial \bar{R}} \Big/ \frac{\partial x^{j**}}{\partial \bar{R}} = \frac{x^{i*}}{x^{j*}} \quad i,j=1,.., N. \tag{27}$$

The second statement in (27) is equivalent to the condition that the "isoprofit lines" of $\pi(x)$ for different levels of $\bar{R}$ have identical shapes in a neighborhood of x^*. This condition can be designated as "homotheticity of $\pi(x)$" at x^*. However, imposing such homotheticity on our model may require either exact knowledge of x^* or exact knowledge of the third order partial derivatives of $\pi(x)$ at x^*. Nevertheless, we can at least specify the following consequence of homotheticity of $\pi(x)$:

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48. Note that the effect of the community pasture programs on the firm's pasture supply schedule can be described more accurately as a parallel shift in the schedule rather than as an equiproportional change in the marginal factor cost of pasture at various activity levels (see section 3.5 of Chapter 2). Thus the local effect of the community pasture programs cannot be described accurately in terms of (24).

49. The proof of this statement is quite simple. Condition (24) implies that factor demands are homogeneous of degree zero in α, i.e. $x^*(\alpha) = x^*(\lambda\alpha)$ for all scalar $\lambda > 0$. Then (25) follows directly from Euler's theorem. Equations (26) follow directly from the restrictions $R(x;\alpha^0) \equiv \alpha^0 R(x)$ and $c^i(x^i;\alpha^i) \equiv \alpha^i c^i(x^i)$ $(i=1,.., N)$.

$$\sum_{k=1}^{N} \pi_{ik}(x^*)\frac{\partial x^{k**}}{\partial R} = \sum_{k=1}^{N} \pi_{jk}(x^*)\frac{\partial x^{k**}}{\partial R} \qquad i,j=1,\ldots, N.^{50} \tag{28}$$

Thus, the following restrictions in our model are implied (but not equivalent to) the special property of fixed factor proportions of $\pi(x)$ at x^*:

$$\left.\begin{array}{l} \dfrac{\partial x^{i**}}{\partial \alpha^j} = 0 \text{ and } \displaystyle\sum_{k=1}^{N} \pi_{ik}(x^*)\frac{\partial x^{k**}}{\partial R} = \sum_{k=1}^{N} \pi_{jk}(x^*)\frac{\partial x^{k**}}{\partial R} \\ \qquad\qquad\qquad\qquad\qquad\qquad i,j=1,\ldots, N.^{51} \end{array}\right\} \tag{29}$$

4 A Minor Difficulty in Translating between Local and Observed Comparative Static Effects

Here we note that there can be difficulties in incorporating all knowledge of the form presented in section 3 of this Appendix into a local comparative statics model, but these difficulties will seldom pose serious problems in the use of our quantitative comparative statics model.

The ambiguity relates to the problem of translating between local and global comparative static properties of $\pi(x;\alpha)$. The comparative static effects included in the model presented here are formally defined in terms of local properties of $\pi(x;\alpha)$ and shift parameters ($c^1_{1\alpha^1}$, etc.); whereas the counterparts of these effects that are "observed in reality" depend on more global properties of $\pi(x;\alpha)$. Moreover, local comparative static effects are linear homogeneous in the shift parameters. For example,

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50. Homotheticity of $\pi(x)$ is equivalent to the statement that $\pi_i(\lambda x) = \gamma^{ij}\pi_j(\lambda x)$ for all x and all scalar $\lambda > 0$, and a scalar γ^{ij} for each (i,j); so homotheticity and $\pi_i(x^*) = \pi_j(x^*) = 0$ imply that $\frac{\partial \pi_i(x^*)}{\partial R} = \frac{\partial \pi_j(x^*)}{\partial R}$, which is statement (28). Since (28) can be satisfied by a $\frac{\partial x^{**}}{\partial R}$ that does not preserve initial factor proportions, (28) does not imply homotheticity of $\pi(x)$ at x^*.

51. The assumption of fixed factor proportions "contradicts" the assumption that $\pi(x)$ is continuous at x^*. However, this "contradiction" is trivial: the statement $\frac{\partial x^{i**}}{\partial \alpha^j} = \varepsilon$, where ε is arbitrarily small, does not contradict the assumption that $\pi(x)$ is twice differentiable.

$$\frac{\partial x^*}{\partial \alpha^1} = [\pi_{ij}]^{-1}\begin{bmatrix} c^1_{1\alpha^1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. $\frac{\partial x^*}{\partial \alpha^1}$ is linear homogeneous in $c^1_{1\alpha^1}$ for a given $[\pi_{ij}(x^*)]$. Thus difficulties arise in translating between local and observed comparative static effects if and only if observed comparative static effects are not linear homogeneous in the (parallel) shift of factor supply schedules, etc. Since linear homogeneity will be a special case for observed comparative static effects, there can in principle be difficulties in incorporating all knowledge relevant to estimating comparative static effects into our quantitative comparative statics model.

However, in practice this local comparative statics model should be able to assimilate most of our empirical knowledge concerning comparative static effects. In other words, one can seldom make sharp distinctions between (e.g.) $\frac{\Delta x^{i**}}{\Delta w^j}$ for different ("reasonable") levels of shift in an exogenous wage w^j.[52,53]

5 Restrictions as Confidence Intervals or Bayes Intervals

Here we note that the set of constraints

$$\rho^{Li} \leq \rho^i \leq \rho^{Ui} \qquad i=1,.., M \tag{30}$$

on the potentially observable parameters ρ in our model can be interpreted either as confidence intervals or as Bayes intervals. Thus the corresponding feasible set for the matrix $[\pi_{ij}(x^*)]$ and the vector of comparative static

52. The linear homogeneity property of local comparative statics will generalize to more global comparative statics if the local structure of $\pi(x;\alpha)$ is invariant in a suitable subset of x (the equilibrium path). Thus the assertion that one cannot significantly improve upon linear homogeneity in practice is essentially equivalent to the following: there is considerably more knowledge about the "average" $[\pi_{ij}]$ within this subset than of the differences between $[\pi_{ij}]$ over this subset.

53. Note that linear homogeneity of comparative static effects does rule out a constant elasticity of comparative static effects. However, the assumption of constant elasticity is commonly employed in order to obtain unit-free measures and for other conveniences, and this does not seem to reflect a belief that variations between neighboring $[\pi_{ij}]$ can in effect be measured more accurately than their average.

effects $\frac{\partial x^*}{\partial \alpha^i}$ can be interpreted in a similar manner, and the addition of constraints to (30) leads to a reduction in the size of the confidence-Bayes interval for $\frac{\partial x^*}{\partial \alpha^i}$.

Knowledge of the parameters ρ typically should be expressed in terms of frequency distributions rather than as point estimates. Such distributions can arise in at least four ways. First, the vector ρ may vary significantly across a group of firms, and we may wish to estimate the range in individual response $\frac{\partial x^*}{\partial \alpha^i}$ across these firms. Second, ρ may vary significantly across time for an individual firm, and we may wish to estimate the range of comparative static effects $\frac{\partial x^*}{\partial \alpha^i}$ that could be associated with such a range in ρ. Third, ρ may be observed with error, and this error will generally be stochastic.[54] Fourth, ρ may not be directly observed (with or without error); but there will be a prior distribution summarizing our subjective degree of belief about the unknown values of the elements of ρ.

Knowledge of distributions of the first three types implies particular confidence intervals for ρ. In other words, from a particular set of observations $\{\hat{\rho}^i\}$ and from assumptions about the distribution of ρ^i and of errors in observation, we could construct on X% confidence interval

$$\rho^{Li} \leq \rho^i \leq \rho^{Ui}.$$

If the assumptions about the distributions of ρ^i and of errors in observation are correct, then there is an X% probability that a random observation of the population of (true) ρ^i will be contained in this interval. Likewise, if ρ^i is not observed, an X% Bayes interval

$$\rho^{Li} \leq \rho^i \leq \rho^{Ui}$$

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54. There will be errors in observing ρ in a truly static situation and errors in inferring values for ρ from observations of dynamic situations. Presumably the latter type of error would be more common and more serious and would be systematic (hence not normally distributed with a mean of zero).

can be constructed from our prior distribution for ρ^i.[55] Assuming that $(\rho^1,\ldots,\rho^M)$ are independently distributed, X% confidence-Bayes intervals for these individual parameters together form an X% joint confidence-Bayes interval for ρ (30).[56]

The feasible set $\{(\rho,[\pi_{ij}])\}$ for our model is defined by (30) plus the maximization hypothesis ($[\pi_{ij}]$ negative definite and symmetric) and the equivalence relations between ρ and $[\pi_{ij}]$

$$[A]\,[K] = I,\quad [\widetilde{A}_{11}]\,[L] = I\,,\ldots \tag{31}$$

where all matrices are symmetric (by the symmetry of $[\pi_{ij}]$).[57] Since any vectors ρ that are contradictory or inconsistent with the maximization hypothesis cannot belong to the true population of vectors ρ, the feasible set $\{\rho\}$ for our model and the relations (30) must define identical confidence-Bayes levels for the true population of vectors ρ.[58] Thus the feasible set $\{(\rho,[\pi_{ij}])\}$ for our model forms an X% confidence-Bayes level for the true joint population of $(\rho,[\pi_{ij}])$ and for the true non-joint populations of ρ and $[\pi_{ij}]$.

Since the set of feasible $[\pi_{ij}]$ defines an X% confidence interval and the value of $z(\frac{\partial x}{\partial \alpha^1})$ is determined for a given $[\pi_{ij}]$, it follows that the feasible set of $z(\frac{\partial x}{\partial \alpha^1})$ defines X% of the probability distribution of the true population of $z(\frac{\partial x}{\partial \alpha^1})$. Thus the range of feasible $z(\frac{\partial x}{\partial \alpha^1})$ defines at least an X% confidence-

55. In practice, these constraints on ρ^i often may be derived from a combination of observations and subjective belief.

56. This assumption of independence, which is implicit in (30), can be relaxed by defining constraints that directly limit more than one element of ρ at a time, e.g. $\rho^{Lij} \leq \rho^i + \rho^j \leq \rho^{Uij}$ or $\rho^{Lij} \leq \rho^i \cdot \rho^j \leq \rho^{Uij}$. Moreover, vectors ρ that satisfy (30) but are logically inconsistent (by placing contradictory restrictions on $[\pi_{ij}]$) will be excluded from the feasible set of our model by restrictions of the form (31).

57. Since the inverse of a matrix is unique, (31) defines a one-to-one correspondence between feasible ρ and $[\pi_{ij}]$.

58. The one minor exception to this statement is that $\{[\pi_{ij}]$ negative semidefinite only$\}$ is excluded from the model although this set is consistent with the existence of a maximum; but it can be shown that comparative statics is undefined for this set (see Proposition 1).

Bayes interval for the true population of $z(\frac{\partial x}{\partial \alpha^1})$.[59,60]

Furthermore, we have now justified the following argument for incorporating as many restrictions of the form (30) into our comparative statics model as is possible: as the number of restrictions of the form (30) that define a given X% confidence-Bayes interval is increased, the size of the corresponding confidence-Bayes interval for $z(\frac{\partial x^*}{\partial \alpha^1})$ is decreased (or at least does not increase).

6 The Possibility of Additional Restrictions

Earlier we formulated conditions that exhaust the comparative static implications of the maximization hypothesis.[61] On the other hand, the structure $[\pi_{ij}(x^*)]$ can be described in many ways, i.e. in terms of a large (and perhaps unlimited) number of overlapping parameters and properties. In addition, incorporating observations of additional parameters and properties of $[\pi_{ij}(x^*)]$ into our analysis will lead to a reduction in the size of confidence-Bayes intervals for $\frac{\partial x^*}{\partial \alpha^1}$.[62] Thus the restrictions described in the two previous sections are only a subset of all potentially useful relations between comparative

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59. Whereas ρ and $[\pi_{ij}]$ are uniquely related by (31), z and $[\pi_{ij}]$ are related by

$$[\pi_{ij}] \frac{\partial x}{\partial \alpha^1} = \begin{bmatrix} c^{1}_{1\alpha^{1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$z = z(\frac{\partial x}{\partial \alpha^1}).$$

Thus feasible values for z may be duplicated by elements of the true population of $[\pi_{ij}]$ that are infeasible for our model, i.e. the range of feasible $z(\frac{\partial x}{\partial \alpha^1})$ defines at least an X% confidence-Bayes level. For simplicity, and as a first approximation when X is large, we shall generally assume that this range forms an X% confidence-Bayes level.

60. Note that our arguments do not imply that the probability distribution of (ρ, $[\pi_{ij}]$, z) within this X% interval of the true population and within its corresponding feasible set are equivalent. Indeed, the probability content of the true population of (ρ, $[\pi_{ij}]$, z) seems likely to be much more concentrated around its mean than is the uniformly distributed population of feasible (ρ, $[\pi_{ij}]$, z) that is implicit in the model.

61. See Proposition 1.

62. See the previous section.

static effects and observable structure of the firm's maximization problem.

Here we attempt to assess the importance of the restrictions described in sections 2 and 3 relative to the entire set of relations between comparative static effects and observable structure. We conclude that the most important of the generally applicable relations may well have been specified in these sections. The discussion is highly speculative.

First, consider the set of all special properties that can be imposed directly on $\pi(x;\alpha)$ and incorporated into our comparative static analysis, e.g. properties such as separability. Such properties can be useful in particular cases. However, these properties typically are observed either to hold or not to hold for a particular firm. Thus the specification of limits on the "degree of deviation" from these properties tends to be arbitrary rather than based on observation.[63] In other words, any particular special property is not generally appropriate for our quantitative comparative statics model.

In addition, the set of properties that are always true for the individual firm may be large, i.e. the set of parameters that can be correctly specified as varying numerically over all firms may be large. Nevertheless, the only properties of this type that have come to mind concern comparative static effects.

In the remainder of this section, we note the following: (a) all types of changes in exogenous variables that are observable at the level of the individual firm have been incorporated into our analysis, and (b) all types of comparative static effects that can be derived by the usual methods (primal, primal-dual, dual) for these changes in exogenous variables have been incorporated into our analysis. Consider the following problems

$$\text{maximize } R(x;\alpha^0) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \text{minimize } \sum_{i=1}^{N} c^i(x^i;\alpha^i) \text{ subject to } R(x) = \overline{R}$$

---

63. This is true even when an "average" of firms is to be modelled. For example, it is not clear how to average one firm where $\pi(x;\alpha)$ is separable and another firm where $\pi(x;\alpha)$ is observed to be non-separable.

$$\text{maximize } R(x;\alpha^0) - \sum_{i=1}^{N} c^i(x^i;\alpha^i)$$

$$\text{subject to } \sum_{i=1}^{S} c^i(x^i;\alpha^i) = \overline{C}$$

with exogenous variables $(\alpha^0,.., \alpha^N)$, $(\alpha^1,.., \alpha^N, \overline{R})$ and $(\alpha^0,.., \alpha^N, \overline{C})$ respectively. An arbitrary subset of inputs can also be considered fixed for these problems. These problems appear to accommodate all types of changes in exogenous variables that are observable at the level of the individual firm, and relations defining $[\pi_{ij}(x^*)]$ in terms of comparative static effects of isolated changes in these variables have been incorporated into our model.[64,65]

The form of these relations between $[\pi_{ij}(x^*)]$ and comparative static effects of isolated changes in exogenous variables for the above problems has been derived by primal methods rather than by primal-dual or dual methods. Further relations involving comparative static changes in equilibrium x and λ cannot be derived by primal methods; but it is not immediately obvious that relations between $[\pi_{ij}(x^*)]$ and equivalent comparative static effects expressed in a different and more observable form cannot be derived by primal-dual or dual methods. Nevertheless, it appears that primal-dual and dual methods do not lead to any additional comparative static properties that can be associated with $[\pi_{ij}(x^*)]$.[66]

7 Summary of Major Quantitative Restrictions

In this section, we summarize the most important of the previously established relations between (a) knowledge of various parameters of the producer problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x;\alpha^0) - \sum_{i=1}^{N} c^i(x^i;\alpha^i)$$

64. See section 3.

65. A simultaneous change in two or more exogenous variables would be realistic if (e.g.) the factor supply or product demand schedules (or production function) faced by one farm receiving community pasture are significantly affected by the activities of other farms receiving community pasture. The effects of simultaneous changes in exogenous variables can in principle be incorporated easily into our approach; but such modifications do not appear appropriate for the community pasture programs studied and in general will not be easy to quantify.

66. See the discussion of primal-dual and dual methods that is included in Appendix 4.

(b) the maximization hypothesis, and (c) the comparative static effect $\frac{\partial x^*}{\partial \alpha^1}$ for this problem. These relations are presented in Figure 3.

First, note that the $(N + 1)^2$ equations in the system $[A]\ [K] = I$ can be reduced to $\frac{(N + 2)(N + 1)}{2}$ equations without any loss of content, and the $(S + 1)^2$ equations in a system $[\tilde{A}_{11}]\ [L] = I$ can be reduced to $\frac{(S + 2)(S + 1)}{2}$ equations without any loss in content. For example, the restrictions implied by the $(N + 1)^2$ equations

$$\left[\begin{array}{c|c} [\pi_{ij}] & c_i^i \\ \hline c_i^{iT} & 0 \end{array}\right] [K] \equiv [M] = I$$

in our model are expressed exactly by the $\frac{(N + 2)(N + 1)}{2}$ right hand upper triangular equations of the system $[M] = I$:

$$\underset{M}{[\overline{\diagdown}\!\!\!\!{}^{\,0}]} = \underset{I}{[\overline{\diagdown}\!\!\!\!{}^{\,0}]} \text{ or equivalently } M_{i,j} = I_{i,j} \text{ for all } (i,j) \text{ such that } i = 1,\ldots,$$
$$N + 1 \text{ and } j \geq i. \qquad (32)$$

The argument for this statement can be sketched as follows. The matrix

$$\left[\begin{array}{c|c} [\pi_{ij}] & c_i^i \\ \hline c_i^{iT} & 0 \end{array}\right] \equiv \underset{(N+1)\times(N+1)}{[A]}$$

has full rank by the restriction that $[\pi_{ij}]$ is negative definite.[67] Thus the equations $M_{i,j} = I_{i,j}$ $(i = 1,\ldots, N + 1)$ determine $(K_{1,j},\ldots, K_{N+1,j})$ for all $j = 1,\ldots, N + 1$ and any feasible $[\pi_{ij}]$. Therefore, the equations $M_{i,N+1} = I_{i,N+1}$ $(i = 1,\ldots, N + 1)$ determine $(K_{1,N+1},\ldots, K_{N+1,N+1})$ in our model, and (given the value for $K_{N,N+1} = K_{N+1,N}$) the equations $M_{i,N} = I_{i,N}$ $(i = 1,\ldots, N)$ determine $(K_{1,N},\ldots, K_{N,N})$, etc.[68] Thus the relations between any feasible matrix [A] and any symmetric matrix [K] that are implied

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67. See Theorem 3.

68. The reader can verify that, in contrast to the case of right hand upper triangular equations, necessary and sufficient conditions for determining [K] are not automatically satisfied in the cases of left hand upper triangular equations and of left or right hand lower triangular equations.

by the restriction [A] [K] = I can be incorporated into our model as the $\frac{(N + 2)(N + 1)}{2}$ equations (32). Likewise, the restrictions implied by the $(S + 1)^2$ equations

$$\left[\begin{array}{c|c} [\pi_{ij}{}^{A}] & c_i^{i\,A} \\ \hline c_i^{i\,AT} & o \end{array}\right] [L] \equiv [N] = I$$

in our model are expressed exactly as the $\frac{(S + 2)(S + 1)}{2}$ right hand upper triangular equations of the system [N] = I:

$$N_{i,j} = I_{i,j} \text{ for all } (i,j) \text{ such that } i = 1,.., S + 1 \text{ and } j \geq i. \quad (33)$$

As can be seen from Figure 3, the total number of equations and variables in the set of constraints increases exponentially with the number of inputs N and also with the number of decompositions. For N = 3, 37 quadratic equations and 45 variables are defined when each of the three possible decompositions, given one fixed input, is included in the system. For N = 4, relations A-C involve 29 quadratic equations and 44 variables, and the set of all possible decompositions, given 1 to 2 fixed inputs, adds 76 quadratic equations and 76 variables to the system. For N = 5, relations A-C define 36 quadratic equations and 62 variables, and the set of all possible decompositions with 1 to 3 fixed inputs adds 235 quadratic equations and 235 variables to the system.

Thus the size of the system of constraints is particularly sensitive to the number of decompositions that are included in the system. For N > 3, it is quite tedious to incorporate all such decompositions into the model. However, in general knowledge of the structure of the firm's static maximization problem will relate only (or primarily) to a subset of decompositions.

(First page of Figure 3)

(A) first order conditions for a maximum

$$[\pi_{ij}] \frac{\partial x}{\partial \alpha^1} = \begin{pmatrix} \overline{c^1_{1\alpha^1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{69}$$ N quadratic equations

(B) second order conditions for a maximum

$$-[\pi_{ij}] = [H]\,[H]^T$$ $\frac{N(N+1)}{2}$ quadratic equations

$$H_{j,j} > 0 \qquad (j = 1, \cdots, N)$$ N bounds

(C) long run decomposition (see Theorem 3)

$$\left(\begin{array}{c|c} [\pi_{ij}] & c^i_i \\ \hline c^{iT}_i & 0 \end{array}\right) \underset{(N+1)\times(N+1)}{[K]} = I$$ $\frac{(N+2)(N+1)}{2}$ independent quadratic equations

$$\overline{c^{iL}_i} \leq c^i_i \leq \overline{c^{iU}_i} \qquad (i = 1, \cdots, N)$$

$$\left.\begin{array}{lll} \dfrac{\partial x^{i**}}{\partial \alpha^j} : \overline{K_{i,j}^{\ L}} \leq \overline{c^j_{j\alpha^j}} \cdot K_{i,j} \leq \overline{K_{i,j}^{\ U}} & (i,j = 1, \cdots, N) \\ \dfrac{\partial x^{i**}}{\partial \bar{F}} : \overline{K_{i,j}^{\ L}} \leq R_y \cdot K_{i,j} \leq \overline{K_{i,j}^{\ U}} & (i = 1, \cdots, N \text{ and } j = N+1) \\ \dfrac{\partial \lambda}{\partial \bar{F}} : \overline{K_{i,j}^{\ L}} \leq -R^2_y \cdot K_{i,j} \leq \overline{K_{i,j}^{\ U}} & (i,j = N+1) \end{array}\right\} 2(N+1)^2 \text{ bounds}$$

$$\overline{R_y^{\ L}} \leq R_y \leq \overline{R_y^{\ U}}$$ 2 bounds

(D) decompositions, given fixed inputs (see Corollary 5); for each decomposition with N-S fixed inputs:

$$\left(\begin{array}{c|c} [\pi_{ij}^{\ A}] & c^{iA}_i \\ \hline c^{iAT}_i & 0 \end{array}\right) \underset{(S+1)\times(S+1)}{[L]} = I$$ $\frac{(S+2)(S+1)}{2}$ independent quadratic equations

$$\left.\begin{array}{ll} \dfrac{\partial x^{i**S}}{\partial \alpha^j} : \overline{L_{i,j}^{\ L}} \leq \overline{c^j_{j\alpha^j}} \cdot L_{i,j} \leq \overline{L_{i,j}^{\ U}} & (i,j = 1, \cdots, S) \\ \dfrac{\partial x^{i**S}}{\partial \bar{F}} : \overline{L_{i,j}^{\ L}} \leq R_y \cdot L_{i,j} \leq \overline{L_{i,j}^{\ U}} & (i = 1, \cdots, N \text{ and } j = S+1) \end{array}\right\} 2(S+1)^2 \text{ bounds}$$

F I G U R E 3 Summary of Major Constraints for the Quantitative Comparative Statics Model [70]

[69] The mark " — " is placed above any symbol that refers to a constant rather than an endogenous variable in the model.

[70] For definitions of the symbols used here, see Theorem 3 and Corollary 5. $R_y \equiv \frac{\partial R(y)}{\partial y}$ where $y \equiv F(x)$ and $R(y) = R(F(x))$.

(Second page of Figure 3)

(D) (continued)

$$\frac{\partial \lambda^S}{\partial \bar{F}} : \overline{L_{i,j}^L} \leq -R_y^2 \cdot L_{i,j} \leq \overline{L_{i,j}^U} \qquad (i,j = S+1) \quad \Big\} \quad 2(S+1)^2 \text{ bounds}$$

(E) non-decompositions (output exogenous), given fixed inputs (see Corollary 6); for each non-decomposition with N-S fixed inputs:

$$\underset{(S\times S)}{[\pi_{ij}^A]}\ \underset{(S\times S)}{[P]} = I \qquad \Big\} \quad \frac{(S+2)(S+1)}{2} \text{ independent quadratic equations}$$

$$\frac{\partial x^{i*S}}{\partial \alpha^j} : \overline{P_{i,j}^L} \leq \overline{c_{j\alpha j}^J} \cdot P_{i,j} \leq \overline{P_{i,j}^U} \qquad (i,j = 1,\cdots,S)$$

totals:

$$\frac{(N+2)(N+1)}{2} + \frac{N(N+1)}{2} + N \text{ quadratic equations}$$

$$N(N+1) + \frac{(N+2)(N+1)}{2} + 2N + 1 \text{ variables}$$

$\frac{(S+2)(S+1)}{2}$ additional quadratic equations and variables for each decomposition or non-decomposition (C,D,E) with N-S fixed inputs

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FIGURE 3 Summary of Major Constraints for the Quantitative Comparative Statics Model[70]

(Footnotes 69 and 70 are the same as the previous page)

APPENDIX IV

QUANTITATIVE COMPARATIVE STATICS AND DERIVED DEMAND: PROOFS

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1. On Integrability in Comparative Statics

The problem of determining the precise conditions corresponding to the comparative static implications of the maximization hypothesis has been called one of the major remaining challenges in the theory of comparative statics (Silberberg, 1974a), and has been largely solved in the context of generalized duality theory (Epstein, 1978). Here we shall sketch a solution to this integrability problem in terms of primal methods of comparative statics and point out relations between the primal and dual approaches.

Consider the general primal problem

$$\text{maximize } \pi(x;\alpha) \qquad \ldots.(P)$$

$$\text{subject to } g(x;\alpha) = 0$$

with an interior solution $x^*(\alpha) > 0$. The only conditions that are placed on the Hessian $[\pi_{ij}(x^*)]$ by the assumption of an interior solution and twice differentiability are negative semi-definiteness subject to constraint and symmetry. However, it is well known that the comparative static effect $\frac{\partial x}{\partial \alpha}$ is determined uniquely by $[\pi_{ij}(x^*)]$ and $g_x(x^*)$ whenever

$[\pi_{ij}(x^*)]$ is negative definite subject to constraint. Moreover, it can be shown that the condition $[\pi_{ij}(x^*)]$ only negative semi-definite subject to constraint contradicts the notion of local comparative statics, i.e., the equations of the total differential are consistent with the implications of this condition only in the meaningless case where $\pi_{i\alpha}(x^*) - \lambda g_{i\alpha}(x^*) = 0$ for all i.[1] Thus the conditions of symmetry and negative definiteness subject to constraint for $[\pi_{ij}(x^*)]$ correspond exactly to the comparative static implications of the maximization hypothesis, i.e., are necessary and sufficient conditions for a solution $\frac{\partial x}{\partial \alpha}$ to the total differential of first order conditions for P to be consistent with the existence of a maximum at $x^*(\alpha)$ when $\pi(x;\alpha)$ is twice differentiable.[2,3]

Thus we have the following conditions for economic integrability:

(a) given a primal $\pi(x;\alpha)$ that is twice differentiable, integrability occurs if and only if $[\pi_{ij}(x^*)]$ is negative definite subject to constraint, and

(b) given a dual $\pi(\alpha)$ that is twice differentiable, integrability occurs if $[\pi_{\alpha\alpha}]$ is positive definite (when $\pi(x;\alpha) \equiv R(x) - \sum_i \alpha i \phi(x)$).

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[1]For the unconstrained case where $\pi(x;\alpha) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i)$, see the proof of Proposition 1-C. Proposition 1-C is easily generalized to problem P.

[2]The importance of negative definiteness for integrability in the primal does not seem to have been noticed. In his survey article, Hurwicz (1971) states that semi-definiteness and symmetry of "indirect" (primal) as well as "direct" (dual) demand functions implies economic integrability.

[3]Whereas primal methods appear to attain integrability in the general problem P, dual methods appear to be unsatisfactory for the general case where α enters the constraint function g as well as π (Epstein, 1978).

These two conditions can be related roughly as follows. First,

$$[\pi_{ij}(x^*)] \text{ exists and only negative semi-definite subject to constraint} \Rightarrow x^*(\alpha) \text{ undefined;}^{4} \quad \text{....(c)}$$

so the dual $\pi(\alpha)$ also is undefined. Second, the implications for the primal of the assumption $[\pi_{\alpha\alpha}]$ only positive semi-definite can easily be established in the competitive case

$$\text{maximize } \pi(x;w) \equiv R(x) - \sum_{i=1}^{N} w^i x^i \ .$$

The total differentials of the first order conditions can be expressed in matrix notation as

$$\underset{(N \times N)}{[\pi_{ij}(x^*)]} \ \underset{(N \times N)}{\frac{\partial x}{\partial w}} = I \ ; \quad \text{....(d)}$$

so by Hotelling's Lemma

$$[\pi_{ij}(x^*)] \ [\pi_{ww}] = -I \quad \text{....(e)}$$

only assuming that $[\pi_{ij}(x^*)]$ and $[\pi_{ww}]$ are defined. Thus, by (d) and (e),

$[\pi_{ww}]$ only positive semi-definite $\Rightarrow$

$[\pi_{ij}(x^*)]$ is undefined, i.e., $R(x)$ is not twice differentiable.[5]

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[4]For example, a competitive equilibrium is indeterminate under constant returns to scale.

[5](On the following page).

Footnote 5

Thus knowledge of $[\pi_{ww}(p,w)]$ can be used to define $[\pi_{ij}(x^*(p,w))]$, or vice versa, if and only if $\pi(x;p,w)$ is strictly concave at $x^*(p,w)$ or (equivalently) $[\pi_{ww}(p,w)]^{-1}$ exists. Analogous simple approaches also lead directly to the relation between second-order approximations for the production function/cost function and direct utility/indirect utility cases, except that here the implications of linear homogeneity in prices are avoided by considering the matrix system of comparative static equations that is defined by variations in the exogenous prices plus the exogenous output or income.

2. Lemma 4

Lemma 4. Suppose that conditions 1-3 are satisfied for a problem P

$$\text{maximize } \pi(x)^P \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x)$$

and that this problem has a unique global solution x^*.[1′, 2′] Denote the set of comparative static effects of $d\alpha$ for this problem as

$$\left\{ \frac{\partial x^*}{\partial \alpha} \right\},$$

and denote the set of comparative static effects of $d\overline{x^1}$ for the corresponding problem Q

$$\text{maximize } \pi(x)^Q \equiv R(x) - \sum_{i=2}^{N} c^i(x)$$

$$\text{subject to } x^1 = \overline{x^{1*}}$$

as $\left\{ \frac{\partial x^*}{\partial \overline{x^1}} \right\}$ (these sets are not necessarily non-empty). Then

(A) $$\left\{ \frac{\partial x^*}{\partial \overline{x^1}} \right\} = \left\{ dx^{*A}, \cdots, dx^{*Z} \right\},$$

where

$$\left\{ dx^{*A}, \cdots, dx^{*Z} \right\}$$

is the set of solutions for the problem

$$\text{maximize } \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \pi(x^*)^Q}{\partial x^i \partial x^j} dx^i dx^j$$

Footnotes 1′ & 2′ on following page.

subject to $dx^1 = 1$; and

$$\text{(B)} \quad \left\{ \frac{\partial x^*}{\partial \alpha} \right\} = \left\{ dx^{*A} \cdot \frac{\partial x^{1*}}{\partial \alpha}, \cdots, dx^{*Z} \cdot \frac{\partial x^{1*}}{\partial \alpha} \right\},$$

where $\frac{\partial x^{1*}}{\partial \alpha}$ is uniquely defined by the equation

$$\frac{\partial p^1(x^{1*})}{\partial x^1} \cdot \frac{\partial x^{1*}}{\partial \alpha} - \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^{1\,2}} \cdot \frac{\partial x^{1*}}{\partial \alpha}$$

$$- \frac{\partial^2 c^1(x^{1*};\alpha}{\partial x^1 \partial \alpha} = 0$$

(assuming that $\left\{ \frac{\partial x^*}{\partial \alpha} \right\}$ is non-empty, and

$$\frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} \neq 0 \;) \;.$$

1′ Assuming other global solutions in the neighbourhood of x^* does not change our results substantively.

2′ The analysis is essentially unaffected by relaxing condition 3, i.e., by assuming $c^1 \equiv c^1(x;\alpha)$. Statement A still holds for any problem Q. Given

$$\frac{\partial^2 c^1(x;\alpha)}{\partial x^i \partial \alpha} = 0$$

for all $i \neq 1$ or equivalently $c^1(x;\alpha) \equiv c^{1A}(x) + c^{1B}(x^1;\alpha)$, statement B holds for the problem $\tilde{Q}$ "maximize $\pi(x)^{\tilde{Q}} \equiv R(x)$ $-c^{1A}(x) - \sum_{i=2}^{N} c^i(x)$ subject to $x^1 = \overline{x^{1*}}$," where $\frac{\partial x^{1*}}{\partial \alpha}$ is uniquely defined by $\frac{\partial^2 \pi(x^*)^{\tilde{Q}}}{\partial \overline{x^1}^{\,2}} \cdot \frac{\partial x^{1*}}{\partial \alpha} - \frac{\partial^2 c^{1B}(x^{1*};\alpha)}{\partial x^{1\,2}} \cdot \frac{\partial x^{1*}}{\partial \alpha} - \frac{\partial^2 c^{1B}(x^{1*};\alpha)}{\partial x^1 \partial \alpha} = 0.$

Proof. Define the problem P

$$\text{maximize} \quad \pi(x)^P \equiv R(x) - c^1(x^1;2) - \sum_{i=2}^{N} c^i(x) \quad \ldots .(a)$$

where by assumption

problem (a) has a unique solution x^*. ....(b)

Denote the corresponding problem Q as

$$\text{maximize} \quad \pi(x)^Q \equiv R(x) - \sum_{i=2}^{N} c^i(x)$$

$$\text{subject to } x^1 = \overline{x^{1*}} \quad \ldots .(c)$$

By Theorem 1-A x* is also a unique solution for (c); so by Lemma 2 each $\frac{\partial x^*}{\partial \overline{x^1}}$ for problem (c), and only these $\frac{\partial x^*}{\partial \overline{x^1}}$, is a solution to the problem .....(***)

$$\text{maximize} \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}(x^*)^Q \frac{\partial x^i}{\partial \overline{x^1}} \frac{\partial x^j}{\partial \overline{x^1}}$$

where $\frac{\partial x^{1*}}{\partial \overline{x^1}} \equiv 1$, which is statement A of the Lemma. By Corollary 4,

$$p^1(x^{1*}) - c_1^1(x^{1*}; \alpha) = 0 \quad . \quad \ldots .(d)$$

Assuming that conditions 1-3 hold at the solution(s) to a P both before and after $d\alpha$,

$$p_1^1(x^{1*})\frac{\partial x^{1*}}{\partial \alpha} - c_{11}^1(x^{1*};\alpha)\frac{\partial x^{1*}}{\partial \alpha} - c_{1\alpha}^1(x^{1*};\alpha) = 0 \qquad \ldots.(e)$$

by total differentiating (d). By Corollary 2-B,

$p_1^1(x^{1*})$ is uniquely defined. $\ldots.(f)$

By assumption

$c_{11}^1(x^{1*};\alpha)$ and $c_{1\alpha}^1(x^{1*};2)$ are uniquely defined $\ldots.(g)$

By (f)-(g),

$\frac{\partial x^{1*}}{\partial \alpha}$ is uniquely defined by (e) for problem (a). $\ldots.(h)$

By Theorem 1-A,

if $\overline{dx^1} = \frac{\partial x^{1*}}{\partial \alpha}$ for problems (c) and (a), respectively,

then $\{\frac{\partial x^*}{\partial \overline{x^1}} \cdot \overline{dx^1}\} = \{\frac{\partial x^*}{\partial \overline{x^1}} \cdot \frac{\partial x^{1*}}{\partial \alpha}\} = \{\frac{\partial x^*}{\partial \alpha}\}$. $\ldots.(i)$

By (h)-(i),

$\{\frac{\partial x^*}{\partial \alpha}\} = \{\frac{\partial x^*}{\partial \overline{x^1}} \cdot \frac{\partial x^{1*}}{\partial \alpha}\}$ where $\frac{\partial x^{1*}}{\partial \alpha}$ is uniquely determined

by the equation

$$p_1^1(x^{1*})\frac{\partial x^{1*}}{\partial \alpha} - c_{11}^1(x^{1*};\alpha)\frac{\partial x^{1*}}{\partial \alpha} - c_{1\alpha}^1(x^{1*};\alpha) = 0$$

which is statement B of the Lemma $\square$

3. Proposition 1

Proposition 1. Suppose that conditions 1-3 are satisfied for a problem P

$$\text{maximize } \pi(x)^P \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x)$$

and that this problem has a unique global solution x^*.[3'] Denote the set of comparative static effects of $d\alpha$ for this problem as $\{\frac{\partial x^*}{\partial \alpha}\}$, and denote the system of total differentials of the first order conditions for a solution to this problem as

$$[\pi_{ij}^P] \frac{\partial x}{\partial \alpha} = \begin{bmatrix} c^1_{1\alpha} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad \ldots\ldots(1)$$

where $[\pi_{ij}^P]$ is defined as the Hessian matrix for $\pi(x)^P$ at x^*. Assume that $[\pi_{ij}^P]$ is negative semi-definite.

(A) If $[\pi_{ij}^P]$ is negative definite, then equations (1) have a unique solution $\frac{\partial x^*}{\partial \alpha}$.

(B) If $[\pi_{ij}^P]$ is not negative definite, then equations (1) may have multiple solutions $\{\frac{\partial}{\partial \alpha}\}$.

[3'] Assuming other global solutions in the neighbourhood of x^* rules out the possibility that $\pi(x)^P$ is negative definite at x^*, and does not alter statements B and C.

(C) However, if $[\pi_{ij}^{P}]$ is not negative definite, then $\frac{\partial x^*}{\partial \alpha}$ is undefined ($\{\frac{\partial x^*}{\partial \alpha}\}$ is empty):

$$\frac{\partial p^1(x^{1*})}{\partial x^1}\,\frac{\partial x^{1*}}{\partial \alpha} - \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^{1\,2}}\,\frac{\partial x^{1*}}{\partial \alpha} - \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} = 0$$

by the first equation in (1);

$$\frac{\partial p^1(x^{1*})}{\partial x^1} - \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^{1\,2}} = 0$$

by equations 2,⋯,N in (1), $[\pi_{ij}^{P}]$ negative semi-definite (and not negative definite) and Lemma 3; so $\frac{\partial x^{1*}}{\partial \alpha}$ is undefined for

$$\frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} \neq 0 \text{ [4',5']}$$

[4'] In order to obtain local comparative statics results $\{\frac{\partial x^*}{\partial \alpha}\}$ for $d\alpha$ (where $x^* + dx^*$ is in the neighbourhood of x^*), we must assume that

$$\frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} \neq 0 . \quad \text{If} \quad \frac{\partial^2 c^1(x^{1*};\alpha)}{\partial x^1 \partial \alpha} \equiv 0 , \quad \text{but}$$

$$\frac{\partial^2 c^1(x^1;\alpha)}{\partial x^1 \partial \alpha} \not\equiv 0 \quad \text{for some } x^1 \neq x^{1*}$$ leads to a change in global solution Δx^*, then Δx^* is finite and our methods no longer apply: (in general) equations 1 (and Lemma 4) are correct only for an infinitesimal change in global solution dx^*.

[5'] These results are essentially unaffected by relaxing condition 3, i.e., by assuming

$$c^1 \equiv c^1(x;\alpha).$$

Proof. By definition,

$$[\pi_{ij}^{P}] \text{ negative definite} \iff x'[\pi_{ij}^{P}]\,x < 0 \text{ for all } x \neq 0 \quad \ldots (a)$$

where $[\pi_{ij}^{P}] \equiv N \times N$ Hessian matrix for $\pi(x)^{P}$ at a unique solution x^{*} for a problem P. By (a),

$$[\pi_{ij}^{P}] \text{ negative definite} \Rightarrow [\pi_{ij}^{P}]\,x \neq 0 \text{ for any } x \neq 0 \quad \ldots (b)$$

i.e., $[\pi_{ij}^{P}]$ has full rank N. Since a square matrix [A] has an inverse if and only if [A] has full rank, (b) implies that

$$[\pi_{ij}^{P}] \text{ negative definite} \Rightarrow [\pi_{ij}^{P}]\,\frac{\partial x}{\partial \alpha} = \begin{pmatrix} c^{1}_{1\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \ldots (***)$$

has a unique solution $\frac{\partial x^{*}}{\partial \alpha}$

which is statement A of the Proposition. By definition,

$$[A] \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is negative semi-definite only.}^{6} \quad \ldots (c)$$

By (c),

$$[A]\begin{pmatrix} x^{1} \\ 0 \\ x^{3} \end{pmatrix} = \begin{pmatrix} -x^{1} \\ 0 \\ 0 \end{pmatrix} \text{ is satisfied by all } (x^{1}, x^{3}). \quad \ldots (d)$$

[6] $|A - \lambda I| = (-1-\lambda)^{2}(-\lambda) = 0$ has roots $\lambda = -1, -1, 0$, which implies that [A] is negative semi-definite and is not negative definite.

By (c)-(d),

$$[\pi_{ij}^{P}] \text{ negative semi-definite only } \not\Rightarrow$$

$$[\pi_{ij}^{P}]\,\frac{\partial x}{\partial \alpha} = \begin{pmatrix} c^{1}_{1\alpha} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ has no solution } \frac{\partial x^{*}}{\partial \alpha}. \qquad \ldots.(e)$$

We can show that

given that [A] is an $N \times N$ matrix and C, X are $N \times 1$ vectors: $[A]X = C$ has a unique solution x^{*} if and only if [A] has rank N,[7] ....(f)

[A] negative semi-definite only $\Rightarrow [A]X = 0$ for at least one $X \neq 0$.[8] ....(g)

By (e)-(g),

if $[\pi_{ij}^{P}]$ is negative semi-definite only, the system

$$[\pi_{ij}^{P}]\,\frac{\partial x}{\partial \alpha} = \begin{pmatrix} c^{1}_{1\alpha} \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \text{ may have multiple solutions } \left\{\frac{\partial x^{*}}{\partial \alpha}\right\} \qquad \ldots.(***)$$

which is statement B of the Proposition. By definition,

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[7]See Murdoch (1970), p. 112.

[8][A] negative semi-definite only implies that there exists a scalar $\lambda^{*} = 0$ and a vector $x^{*} \neq 0$ such that $[A]\,x^{*} = \lambda^{*}x^{*}$ (see Hadley, 1961, p. 256); so $[A]x^{*} = 0$ for an $x^{*} \neq 0$.

$$[\pi_{ij}{}^{P}] \text{ negative semi-definite only} \Rightarrow x'[\pi_{ij}{}^{P}]x = 0 \text{ for some } x \neq 0, \quad x'[\pi_{ij}{}^{P}]x \leqq 0 \text{ for all } x. \quad \ldots .(h)$$

By (h) and Lemma 4,

$$[\pi_{ij}{}^{P}] \text{ negative semi-definite only} \Rightarrow \frac{\partial x^{*\prime}}{\partial \alpha}[\pi_{ij}{}^{P}]\frac{\partial x^{*}}{\partial \alpha} = 0.^{9} \quad \ldots .(i)$$

Assuming that conditions 1-3 hold before and after $d\alpha$,

$$\sum_{j=1}^{N} \pi_{1j}(x^{*})^{P} \frac{\partial x^{j*}}{\partial \alpha} - c^{1}_{1\alpha}(x^{1*};\alpha) = 0 \quad \ldots .(j)$$

$$\sum_{j=1}^{N} \pi_{ij}(x^{*})^{P} \frac{\partial x^{j*}}{\partial \alpha} = 0 \quad i = 2,\cdots,N. \quad \ldots .(k)$$

By (i) and (k),

$$[\pi_{ij}{}^{P}] \text{ negative semi-definite only} \Rightarrow \sum_{j=1}^{N} \pi_{1j}(x^{*})^{P} \frac{\partial x^{j*}}{\partial \alpha} = 0 \quad \ldots .(l)$$

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[9] By (h), maximum $z'[\pi_{ij}{}^{P}]z = 0$ where z is $N \times 1$, and $\{z^{1*}\}$ includes $z^{1} = 1$. Therefore, given $\pi(x)^{P} \equiv \pi(x)^{Q} - c^{1}(x^{1};\alpha)$, the $\{z^{*}: z^{1*} = 1\}$ is the solution set for the problem "maximize $z'[\pi_{ij}{}^{Q}]z$ subject to $z' = 1$"; so maximum $z'[\pi_{ij}{}^{Q}]z = c^{1}_{11}(x^{1*};\alpha)$ given $z^{1} \equiv 1$. Thus, by Lemma 4-A,

$$\frac{\partial x^{*\prime}}{\partial \overline{x^{1}}}[\pi_{ij}{}^{Q}]\frac{\partial x^{*}}{\partial \overline{x^{1}}} = c^{1}_{11}(x^{1*};\alpha); \text{ so } \frac{\partial x^{*\prime}}{\partial \alpha}[\pi_{ij}{}^{P}]\frac{\partial x^{*}}{\partial \alpha} = 0 \text{ by Lemma 4-B}$$

and by $\pi(x)^{P} \equiv \pi(x)^{Q} - c^{1}(x^{1};\alpha)$.

which contradicts (j) for $c^1_{1\alpha}(x^{1*};\alpha) \neq 0$. By Theorem 1-A and $\pi(x)^P \equiv \pi(x)^Q - c^1(x^1;\alpha)$,

$$\sum_{j=1}^{N} \pi_{1j}(x^*)^P \frac{\partial x^{j*}}{\partial \alpha} = \sum_{j=1}^{N} \pi_{1j}(x^*)^Q \frac{\partial x^{j*}}{\partial \overline{x^1}} \frac{\partial x^{j*}}{\partial \alpha} - c^1_{11}(x^{1*};\alpha)\frac{\partial x^{1*}}{\partial \alpha} . \qquad \text{....(m)}$$

By Lemma 1,

$$\sum_{j=1}^{N} \pi_{1j}(x^*)^Q \frac{\partial x^{j*}}{\partial \overline{x^1}} = \frac{\partial^2 \pi(x^*)^Q}{\partial \overline{x^1}^2} . \qquad \text{....(n)}$$

By Theorem 1-B,

$$\frac{\partial^2 \pi(x^*)^Q}{\partial \overline{x^1}^2} = p^1_1(x^{1*}) . \qquad \text{....(o)}$$

By (j) and (l)-(o),

if $[\pi_{ij}^P]$ is negative semi-definite only and $c^1_{1\alpha}(x^{1*};\alpha) \neq 0$,

then $\frac{\partial x^*}{\partial \alpha}$ is undefined: $p^1_1(x^{1*})\frac{\partial x^{1*}}{\partial \alpha} - c^1_{11}(x^{1*};\alpha)\frac{\partial x^{1*}}{\partial \alpha}$

$$- c^1_{1\alpha}(x^{1*};\alpha) = 0$$

by (j), whereas $p^1_1(x^{1*}) - c^1_{11}(x^{1*};\alpha) = 0$ by the assumptions $[\pi_{ij}^P]$ negative semi-definite only, (k) and Lemma 3

which is statement C of the Proposition. □

4. Comments on Constrained Maximization

The problem P considered in our comparative static model is of the form "maximize $\pi(x;\alpha)$" rather than of the form

$$\begin{aligned}&\text{maximize } \pi(x;\alpha)\\ &\text{subject to } g^i(x;\alpha) = 0 \qquad i = 1,\cdots,M\end{aligned} \qquad \ldots .(1)$$

which is the general classical problem. Here we shall point out that it seems difficult to extend our method of comparative statics to such a problem. However, we shall also note that this does not appear to be a serious limitation of our approach.

The main (or at least serious) difficulties in incorporating problem 1 into our approach stem from the following: the second order conditions for a solution to 1 do not require that a matrix relating $(\frac{\partial x^*}{\partial \alpha}, \frac{\partial \lambda}{\partial \alpha})$ to shift parameters $\pi_\alpha(x^*;\alpha)$ and $g^i_\alpha(x^*;\alpha)$ be negative definite or semi-definite. Problem 1 can be expressed in Lagrange form as

$$\text{maximize } L(x,\lambda;\alpha) \equiv \pi(x;\alpha) - \sum_{j=1}^{M} \lambda^j g^j(x;\alpha) \ . \qquad \ldots .(2)$$

Total differentiating the first order conditions for 2 yields

$$\begin{aligned}&\sum_{j=1}^{N} L_{x^i x^j}(x^*,\lambda^*;\alpha)\frac{\partial x^{j*}}{\partial \alpha} = -L_{x^i\alpha}(x^*,\lambda^*;\alpha)\\ &\qquad + \sum_{j=1}^{M} \frac{\partial \lambda^j}{\partial \alpha}\cdot g^j_i(x^*;\alpha) \qquad i = 1,\cdots,N\\ &\sum_{j=1}^{N} g^i_j(x^*;\alpha)\frac{\partial x^{j*}}{\partial \alpha} = -g^i_\alpha(x^*;\alpha) \qquad i = 1,\cdots,N.\end{aligned} \qquad \ldots .(3)$$

Equations 3 can be expressed in matrix form as

$$\underset{(M+N)\times(M+N)}{\left[\begin{array}{c|c} \underset{(M\times M)}{0} & \underset{(M\times N)}{G_x} \\ \hline \underset{(N\times M)}{G_x} & \underset{(N\times N)}{L_{xx}} \end{array}\right]} \underset{(M+N)\times 1}{\left[\begin{array}{c} \underset{(M\times 1)}{-\dfrac{\partial\lambda}{\partial\alpha}} \\ \hline \underset{(N\times 1)}{\dfrac{\partial x}{\partial\alpha}} \end{array}\right]} = - \underset{(M+N)\times 1}{\left[\begin{array}{c} \underset{(M\times 1)}{G_\alpha} \\ \hline \underset{(N\times 1)}{L_{x\alpha}} \end{array}\right]} \qquad \ldots\ldots(4)$$

using obvious notation. Denote the $(M + N) \times (M + N)$ matrix in 4 as [L]. This matrix cannot be negative definite (due to the $M \times M$ submatrix of 0's). In addition, [L] has full rank by the usual second order conditions for a constrained maximum;[10] so [L] cannot be negative semi-definite only.

Thus we cannot specify [L] as negative definite, and instead must specify the more clumsy conditions that the determinant of [L] has the sign of $(-1)^N$, the largest principal minor of [L] has a sign opposite to this, and successively smaller principal minors alternate in sign, down to the principal minor of order $M + 1$.

However, apparently we can ignore problems P of the general classical form (1) without restricting our comparative static method in any serious way. The solution set $x^* = x^*(\alpha)$ for a problem 1 can also be obtained as the solution set for a suitably defined unconstrained problem

$$\text{maximize} \quad \pi(x;\alpha)' \qquad \ldots\ldots(5)$$

[10]See Intrilligator (1971), pp. 496-7.

where $\pi(x^{*}(\alpha);\alpha)' = \pi(x^{*}(\alpha);\alpha)$ for all α.[11] The form of $\pi(x;\alpha)'$ implied by the particular problem 1 may not be obvious. However, we shall be interested only in specifying the restrictions on $\frac{\partial x^{*}}{\partial \alpha}$ that are implied by a subset of possible forms $\pi(x;\alpha)$ and $G(x;\alpha) = 0$ for problem 1, and many of these restrictions can be incorporated into a set of equations $G(\frac{\partial x^{*}}{\partial \alpha}, \rho) = 0$.[12] In this case, defining quantitative comparative statics in terms of an unconstrained maximization problem does not lead to a serious loss in generality.

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[11] Assuming that $\pi(x^{*}(\alpha),\alpha) > 0$ for all α, we can "simply" construct $\pi(x;\alpha)'$ such that $\pi(x^{*}(\alpha),\alpha)' = \pi(x^{*}(\alpha);\alpha)$ for all α and $\pi(x;\alpha)' = 0$ for all combinations (x,α) that do not satisfy the relation $x^{*} = x^{*}(\alpha)$.

[12] In some respects, the comparative static effects of fixed factor proportions can be modelled more accurately in terms of (1) than in terms of an unconstrained maximization problem. We can incorporate some — but not all — of the comparative static implications of fixed factor proportions into a set of equations $G(\frac{\partial x^{*}}{\partial \alpha}, \rho) = 0$ (see Chapter 3). On the other hand, any particular example — but not the general case — of fixed factor proportions can easily be expressed as $G(x) = 0$ (e.g., $x^{1} - 2x^{2} = 0$).

5. Theorem 3

Theorem 3. Suppose that conditions 1-2 are satisfied for a problem P

$$\text{maximize } \pi(x) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \ldots\ldots(1)^{13}$$

and assume that this problem has a unique global solution x^* where the Hessian matrix for $\pi(x)$ is negative definite.

Construct the related problem

$$\text{maximize } \pi(x)$$
$$\text{subject to } R(x) = \overline{R(x^*)}.$$

which can be expressed in Lagrange form as

$$\text{maximize } \pi(x) - \lambda(R(x) - \overline{R(x^*)}). \qquad \ldots\ldots(2)$$

Construct the symmetric matrix

$$\left[\begin{array}{c|c} \underset{(N\times N)}{\pi_{ij}} & \underset{(1\times N)}{c_i^i} \\ \hline \underset{(N\times 1)}{c_i^i} & \underset{(1\times 1)}{0} \end{array}\right] \equiv \underset{(N+1)\times(N\times 1)}{[A]}$$

where $\underset{(N\times N)}{\pi_{ij}}$ denotes the Hessian matrix for $\pi(x)$ at x^*,

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13 This theorem is easily generalized to the case $c^i \equiv c^i(x;\alpha^i)$ $(i = 1,\cdots,N)$; but the equations in the generalized theorem are somewhat more detailed than here, and the generalized theorem will not be employed in our research.

and $\underset{(N \times 1)}{c_i^i} \equiv \left(\frac{\partial c^1(x^{1*};\alpha^1)}{\partial x^1}, \ldots\ldots, \frac{\partial c^N(x^{N*};\alpha^N)}{\partial x^N}\right).$

[A] necessarily has full rank, and denote its inverse as [K]:

$$[A]^{-1} \equiv \underset{(N+1)\times(N+1)}{[K]} \quad \text{always exists.}$$

Then

(A) the comparative static effects for problem 2 are uniquely defined as follows:

$$\frac{\partial x^{i**}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot K_{i,j} \qquad i,j = 1,\cdots,N$$

$$\frac{\partial x^{i**}}{\partial \bar{R}} = K_{i,N+1} \qquad i = 1,\cdots,N$$

$$\frac{\partial \lambda}{\partial \alpha^j} = -\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot K_{N+1,j} \qquad j = 1,\cdots,N$$

$$\frac{\partial \lambda}{\partial \bar{R}} = -K_{N+1,N+1}$$

where

$K_{i,j} \equiv$ element (i,j) of matrix [K], and $K_{i,j} = K_{j,i} (i,j = 1,\cdots,N+1)$; and

(B) (a) the comparative static effects $(\frac{\partial x^*}{\partial \alpha})$ for problem 1 are unique, and

(b) given that $\sum_{i=1}^{N} \sum_{j=1}^{N} K_{i,N+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$, [15]

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[14]Thus $\frac{\partial x^{i**}}{\partial \alpha^j} = \left(\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} / \frac{\partial^2 c^i(x^{i*};\alpha^i)}{\partial x^i \partial \alpha^i}\right) \frac{\partial x^{j**}}{\partial \alpha^i}$ and

$$\frac{\partial \lambda}{\partial \alpha^j} = -\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot \frac{\partial x^{j**}}{\partial \bar{R}} \quad (i,j = 1,\cdots,N).$$

[15]A sufficient condition for $\sum_{i=1}^{N} \sum_{j=1}^{N} K_{i,N+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$ (a)

is that $K_{i,N+1} \geqq 0$ $(i = 1,\cdots,N)$, which is equivalent to ruling out the possibility

$\frac{\partial x^*}{\partial \alpha^j}$ for problem 1 is uniquely defined in terms of

$\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j}$ and the elements of [K] corresponding to

$\frac{\partial x^{**}}{\partial \alpha^j}$ and $\frac{\partial x^{**}}{\partial \bar{R}}$ for problem 2, as follows:

$$\frac{\partial x^{i*}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot K_{i,j} + K_{i,N+1} \cdot \frac{\partial R(x^*)}{\partial \alpha^j} \qquad i,j = 1,\cdots,N$$

$$\frac{\partial R(x^*)}{\partial \alpha^j} = \sum_{i=1}^{N} \frac{\partial c^i(x^{i*};\alpha^i)}{\partial x^i} \cdot \frac{\partial x^{i*}}{\partial \alpha^j} \qquad j = 1,\cdots,N.$$

Proof. Suppose that x^* is a unique interior global solution for the problem P

$$\text{maximize } \pi(x) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \ . \qquad \ldots\text{(a)}$$

Construct the related problem

$$\begin{aligned} &\text{maximize } \pi(x) \\ &\text{subject to } R(x) = \overline{R(x^*)} \end{aligned} \qquad \ldots\text{(b)}$$

which can be expressed in Lagrange form as

$$\text{maximize } \pi(x) - \lambda(R(x) - \overline{R(x^*)}) \ . \qquad \ldots\text{(c)}$$

Footnote 15 (continued)
of inferior inputs ($\frac{\partial x^{i**}}{\partial \bar{R}} \geqq 0$, $i = 1,\cdots,N$). Condition (a) would be violated only for a relatively few "appropriate" degrees of inferiority; so condition (a) is not a serious restriction.

By (a)-(b),

$$x^* \text{ is the solution set to problem (c).} \qquad \ldots\ldots(d)$$

By (c)-(d),

$$\begin{aligned} \pi_i(x^*) - \lambda R_i(x^*) &= 0 \qquad i = 1,\cdots,N \\ R(x^*) &= \overline{R(x^*)} \end{aligned} \qquad \ldots\ldots(e)$$

which are the first order conditions for a solution to problem (c).

Total differentiating (e),

$$\sum_{j=1}^{N} \pi_{ij}(x^*)\, dx^j - c^i_{i\alpha^i}\,(x^{i*};\alpha^i) - R_i(x^*)d\lambda$$

$$- \lambda \sum_{j=1}^{N} R_{ij}(x^*)\, dx^j = 0 \qquad i = 1,\cdots,N \qquad \ldots\ldots(f)$$

$$\sum_{i=1}^{N} R_i(x^*)\, dx^i - d\bar{R} = 0 \qquad \ldots\ldots(g)$$

given (a). By (c),

$$\begin{aligned} \lambda &= \frac{\partial \pi(x^*)}{\partial \bar{R}} \\ &= \sum_{i=1}^{N} \pi_i(x^*) \frac{\partial x^{i**}}{\partial \bar{R}} = 0 \end{aligned} \qquad \ldots\ldots(h)$$

by (d) and conditions 1-2. By (h), (f) reduces to

$$\sum_{j=1}^{N} \pi_{ij}(x^*)dx^j - R_i(x^*)d\lambda - c^i_{i\alpha^i}(x^{i*};\alpha^i) = 0 \qquad i = 1,\cdots,N. \ldots\ldots(i)$$

By (a) and conditions 1-2, (g) can be rewritten as

$$\sum_{i=1}^{N} c_i^i(x^{i*};\alpha^i)\, dx^i - d\bar{R} = 0 \quad . \qquad \ldots\ldots(j)$$

Construct the symmetric matrix

$$\left(\begin{array}{c|c} \underset{(N\times N)}{\pi_{ij}} & \underset{(1\times N)}{c_i^i} \\ \hline \underset{(N\times 1)}{c_i^i} & \underset{(1\times 1)}{0} \end{array}\right) \equiv \underset{(N+1)\times(N+1)}{[A]} \qquad \ldots\ldots(k)$$

where

$$\underset{(N\times N)}{\pi_{ij}} \equiv \text{Hessian matrix for } \pi(x) \text{ at } x^*$$

$$\ldots\ldots(l)$$

$$\underset{(1\times N)}{c_i^i} \equiv \left(c_1^1(x^{1*};\alpha^1),\ \cdots,\ c_N^N(x^{N*};\alpha^N)\right)^T .$$

By definition, [A] has less than full rank if and only if

there exists a vector $v \neq 0$ such that $\sum_{j=1}^{N} v^j\, \pi_{ij} - c_i^i = 0$

$$i = 1, \cdots, N$$

$$\sum_{j=1}^{N} v^j\, c_j^j = 0 ;$$

but this statement implies that

there exists a vector $v \neq 0$ such that $\sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij} v^i v^j = 0$

which contradicts the assumption that $[\pi_{ij}]$ is negative definite.
Thus

$$[\pi_{ij}] \text{ negative definite} \Rightarrow [A] \text{ has full rank.} \qquad \ldots .(***)$$

By (i)-(l),

$$\underset{}{[A]} \underset{(N\times 1)}{X} = \underset{(N\times 1)}{C} \qquad \ldots .(m)$$

where

$$\underset{(N\times 1)}{X} \equiv (dx^1, \cdots, dx^N, -d\lambda)^T \text{ for problem (c)}$$

$$\ldots .(n)$$

$$\underset{(N\times 1)}{C} \equiv \left(c^1_{1\alpha^1}(x^{1*};\alpha^1)d\alpha^1, \cdots, c^N_{N\alpha^N}(x^{N*};\alpha^N)d\alpha^N, d\bar{R}\right)^T .$$

By assumption,

$$[A]^{-1} \text{ exists, and } [A]^{-1} \equiv [K] \text{ is symmetric} \qquad \ldots .(o)$$

by the symmetry of [A]. By (o),

$$X = [K]\, C \;. \qquad \ldots .(p)$$

By (n)-(p), for problem (c)

$$\frac{\partial x^{i**}}{\partial \alpha^j} = c^j_{j\alpha^j}(x^{j*};\alpha^j) \cdot K_{i,j} \qquad i,j = 1,\cdots,N$$

$$\frac{\partial x^{i**}}{\partial \bar{R}} = K_{i,N+1} \qquad i = 1,\cdots,N$$

$$\frac{\partial \lambda}{\partial \alpha^j} = -c^j_{j\alpha^j}(x^{j*};\alpha^j) \cdot K_{N+1,j} \qquad j = 1,\cdots,N \qquad \ldots .(***)$$

$$\frac{\partial \lambda}{\partial \bar{R}} = -K_{N+1,N+1}$$

$$K_{i,j} = K_{j,i} \qquad i,j = 1,\cdots,N+1$$

where $K_{ij} \equiv$ element (i,j) of $[K]$, which is statement A of the Theorem. By Proposition1-A and the assumption that the Hessian matrix for $\pi(x)([\pi_{ij}])$ is negative definite at x^*,

$$\left\{\frac{\partial x^*}{\partial \alpha^j}\right\} \text{ for problem (a) contains only one}$$

$$\text{vector } \frac{\partial x^*}{\partial \alpha^j} \qquad j = 1,\cdots,N. \qquad \ldots .(q)$$

By the implicit function theorem and $[\pi_{ij}]$ negative definite at x^*, we can solve the first order conditions of problem (a) for x^* as a function of α:

$$x^{i*} = x^{i*}(\alpha) \qquad i = 1,\cdots,N \text{ for problem (a).} \qquad \ldots .(r)$$

Given $\bar{R} \equiv R(x^*)$, x^* is also the solution set for problem (c); so we can also solve (e) of problem (c) for x^* as a function of $(\alpha, \bar{R})$:

$$x^{i*} = x^{i**}(\alpha, \bar{R}) \qquad i = 1, \cdots, N \quad \text{given } \bar{R} \equiv R(x^*) \qquad \text{for problem (c).} \qquad \ldots .(s)$$

Substituting (r) into (s),

$$x^{i*} = x^{i**}\left(\alpha, \overline{R(x^{1*}(\alpha), \cdots, x^{N*}(\alpha))}\right) \qquad i = 1, \cdots, N. \qquad \ldots .(t)$$

By (t) and the rule for the differential of a composite function,

$$\frac{\partial x^{i*}}{\partial \alpha^j} = \frac{\partial x^{i**}}{\partial \alpha^j} + \frac{\partial x^{i**}}{\partial \bar{R}} \cdot \frac{\partial R(x^*)}{\partial \alpha^j} \qquad i,j = 1, \cdots, N, \quad \ldots .(u)$$

where

$$\frac{\partial R(x^*)}{\partial \alpha^j} = \sum_{i=1}^{N} R_i(x^*) \frac{\partial x^{i*}}{\partial \alpha^j} \qquad j = 1, \cdots, N. \qquad \ldots .(v)$$

By (u) and (***),

$$\frac{\partial x^{i*}}{\partial \alpha^j} = c^j_{j\alpha^j}(x^{j*}; \alpha^j) \cdot K_{i,j} + K_{i,N+1} \cdot \frac{\partial R(x^*)}{\partial \alpha^j} \qquad i,j = 1, \cdots, N. \quad . .(w)$$

By (v) and conditions 1-2,

$$\frac{\partial R(x^*)}{\partial \alpha^j} = \sum_{i=1}^{N} c_i^i(x^{i*};\alpha^i)\,\frac{\partial x^{i*}}{\partial \alpha^j} \qquad j = 1,\cdots,N\,. \qquad \ldots.(x)$$

By (w)-(x),

$$\underset{(N+1)\times(N+1)}{\left[\begin{array}{c|c} \underset{(N\times N)}{I} & \underset{(N\times 1)}{-\tilde{K}_{i,N+1}} \\ \hline \underset{(N\times 1)}{c_i^i} & \underset{(1\times 1)}{-1} \end{array}\right]} \underset{(N+1)\times 1}{\begin{bmatrix} \frac{\partial x^{1*}}{\partial \alpha^j} \\ \vdots \\ \frac{\partial x^{N*}}{\partial \alpha^j} \\ \frac{\partial R(x^*)}{\partial \alpha^j} \end{bmatrix}} = c^j_{j\alpha^j}(x^{j*};\alpha^j) \begin{bmatrix} K_{1,j} \\ \vdots \\ K_{N,j} \\ 0 \end{bmatrix} \qquad j = 1,\cdots,N \qquad \ldots.(y)$$

where $\underset{(N\times N)}{I} \equiv$ identity matrix and $\underset{(N\times 1)}{\tilde{K}_{i,N+1}} \equiv$ N+1'st column of [K].

Denote the $(N+1)\times(N+1)$ matrix in (y) as [L]. By (y) and the definition of a determinant,

$$\Big|\,[L]\,\Big| = -1 - \sum_{i=1}^{N}\sum_{j=1}^{N} K_{i,N+1}\cdot c_j^j(x^{j*};\alpha^j)\,. \qquad \ldots.(z)$$

Since (y) has a unique solution

$$\left(\frac{\partial x^*}{\partial \alpha^j},\ \frac{\partial R(x^*)}{\partial \alpha^j}\right)$$ (j arbitrary) if and only if $[L]^{-1}$ exists

or equivalently $|[L]| \neq 0$, (q) and (y)-(z) imply that

(a) $\frac{\partial x^*}{\partial \alpha^{j}}$ is unique for problem (a) $(j = 1,\cdots,N)$

(b) given that $\sum_{i=1}^{N} \sum_{j=1}^{N} K_{i,N+1} \cdot c^j_j(x^{j*};\alpha^j) \neq -1$,

$\frac{\partial x^*}{\partial \alpha^j}$ for problem (a) is uniquely defined in terms of

$c^j_{j\alpha^j}(x^{j*};\alpha^j)$ and the elements of [K] corresponding to

$\frac{\partial x^*}{\partial \alpha^j}$ and $\frac{\partial x^*}{\partial \bar{R}}$ for problem (c) $(j = 1,\cdots,N)$,

as follows:

$$\frac{\partial x^{i*}}{\partial \alpha^j} = c^j_{j\alpha^j}(x^{j*};\alpha^j) \cdot K_{i,j} + K_{i,N+1} \cdot \frac{\partial R(x^*)}{\partial \alpha^j} \qquad i,j = 1,\cdots,N$$

$$\frac{\partial R(x^*)}{\partial \alpha^j} = \sum_{i=1}^{N} c^i_i(x^{i*};\alpha^i) \cdot \frac{\partial x^{i*}}{\partial \alpha^j} \qquad j = 1,\cdots,N$$

which is statement B of the Theorem. □

6. Corollary 5

Corollary 5. Construct the problems 1 and 2, and the $(N+1)\times(N+1)$ matrices [A] and [K], as in Theorem 9. Partition the Hessian matrix $\underset{(N\times N)}{[\pi_{ij}]}$ and marginal factor cost vector $\underset{(1\times N)}{c^i_i}$ of [A] as follows:

$$\underset{(N\times N)}{[\pi_{ij}]} \equiv \left[\begin{array}{c|c} \underset{(S\times S)}{\pi_{ij}^{A}} & \underset{(S\times T)}{\pi_{ij}^{B}} \\ \hline \underset{(T\times S)}{\pi_{ij}^{C}} & \underset{(T\times T)}{\pi_{ij}^{D}} \end{array}\right] \qquad \underset{(1\times N)}{c^i_i} \equiv \left(\begin{array}{c|c} \underset{(1\times S)}{c^{iA}_i} & \underset{(1\times T)}{c^{iB}_i} \end{array}\right)$$

where S+T = N. Construct the following symmetric matrix

$$\left[\begin{array}{c|c} \pi_{ij}^{A} & c_i^{iA} \\ (S\times S) & (S\times 1) \\ \hline c_i^{iA} & 0 \\ (1\times S) & (1\times 1) \end{array}\right] \equiv \underset{(S+1)\times(S+1)}{[\tilde{A}_{11}]} .$$

$[\tilde{A}_{11}]$ necessarily has full rank, and denote its inverse as [L]:

$$[\tilde{A}_{11}]^{-1} \equiv [L] \quad \text{always exists.}$$

Construct the problem

$$\begin{aligned} &\text{maximize } \pi(x) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \\ &\text{subject to } x^j = x^{j*} \qquad j = S+1,\cdots,N \end{aligned} \qquad \ldots.(3)$$

where x^* is the unique global solution to problem 1.

Construct the related problem

$$\begin{aligned} &\text{maximize } \pi(x) \\ &\text{subject to } R(x) = \overline{R(x^*)} \\ &\qquad x^j = \overline{x^{j*}} \qquad j = S+1,\cdots,N \end{aligned}$$

which can be expressed in Lagrange form as

$$\text{maximize } \pi(x) - \lambda(R(x) - \overline{R(x^*)}) - \sum_{j=S+1}^{N} \gamma^j(x^j - \overline{x^{j*}}). \qquad \ldots.(4)$$

Then

(A) the comparative static effects for problem 4 are uniquely defined as follows:

$$\frac{\partial x^{i**S}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot L_{i,j} \qquad i,j = 1,\cdots,S$$

$$\frac{\partial x^{i**S}}{\partial \bar{R}} = L_{i,S+1} \qquad i = 1,\cdots,S$$

$$\frac{\partial \lambda^S}{\partial \alpha^j} = -\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot L_{S+1,j} \qquad j = 1,\cdots,S$$

$$\frac{\partial \lambda^S}{\partial \bar{R}} = -L_{S+1,S+1}$$

where $L_{i,j} \equiv$ element (i,j) of $[L]$, and $L_{i,j} = L_{j,i} (i,j = 1,\cdots,S+1)$;

and

(B) (a) the comparative static effects $\frac{\partial x^{*S}}{\partial \alpha}$ for problem 3 are unique, and

(b) given that $\sum_{i=1}^{S} \sum_{j=1}^{S} L_{i,S+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$,[16]

---

[16] Assuming that $\sum_{i=1}^{S} \sum_{j=1}^{S} L_{i,S+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$ has implications analogous to those of assuming that $\sum_{i=1}^{N} \sum_{j=1}^{N} K_{i,N+1} \cdot \frac{\partial c^j(x^{j*};\alpha^j)}{\partial x^j} \neq -1$ (see footnote to Theorem 3).

$\frac{\partial x^{*S}}{\partial \alpha^j}$ for problem 3 is uniquely defined in terms of

$\frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j}$ and the elements of [L] corresponding to

$\frac{\partial x^{**S}}{\partial \alpha^j}$ and $\frac{\partial x^{**S}}{\partial \bar{R}}$ for problem 4, as follows:

$$\frac{\partial x^{i*S}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot L_{i,j} + L_{i,S+1} \cdot \frac{\partial R(x^*)^S}{\partial \alpha^j}$$

$$i,j = 1,\cdots,S$$

$$\frac{\partial R(x^*)^S}{\partial \alpha^j} = \sum_{i=1}^{S} \frac{\partial c^i(x^{i*};\alpha^i)}{\partial x^i} \cdot \frac{\partial x^{i*S}}{\partial \alpha^j} \qquad j = 1,\cdots,S .$$

Proof. Suppose that x^* is a unique interior solution for the problem P

$$\text{maximize } \pi(x) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \quad . \qquad \ldots\ldots\text{(a)}$$

Construct the related problems

$$\begin{aligned} &\text{maximize } \pi(x) \\ &\text{subject to } x^j = \overline{x^{j*}} \qquad j = S+1,\cdots,N \end{aligned} \qquad \ldots\ldots\text{(b)}$$

$$\begin{aligned} &\text{maximize } \pi(x) \\ &\text{subject to } x^j = \overline{x^{j*}} \qquad j = S+1,\cdots,N \\ &\qquad\qquad R(x) = \overline{R(x^*)} \quad . \end{aligned} \qquad \ldots\ldots\text{(c)}$$

Problem (c) can be expressed in Lagrange form as

$$\text{maximize } \pi(x) - \lambda(R(x) - \overline{R(x^*)}) - \sum_{j=S+1}^{N} \gamma^j (x^j - \overline{x^{j*}}). \quad \ldots .(d)$$

By (a)-(c),

x^* is the solution set for both problems (b) and (d) . . . .(e)

By (d)-(e),

$$\left.\begin{array}{ll} \pi_i(x^*) - \lambda R_i(x^*) = 0 & i = 1,\cdots,S \\ R(x^*) = \overline{R(x^*)} & \end{array}\right\} \quad \ldots .(f)$$

$$x^j = \overline{x^{j*}} \qquad j = S+1,\cdots,N \quad \ldots .(g)$$

which are the first order conditions for a solution to (d). By arguments identical to (f)-(n) in the proof of Theorem 3,

$$\underset{(S+1)\times(S+1)}{[\tilde{A}_{11}]} \quad \underset{(S\times 1)}{X^S} = \underset{(S\times 1)}{C^S} \qquad \text{given (g)} \quad \ldots .(h)$$

where

$$\underset{(S+1)\times(S+1)}{[\tilde{A}_{11}]} \equiv \left[\begin{array}{c|c} \underset{(S\times S)}{\pi_{ij}^A} & \underset{(S\times 1)}{c_i^{iA}} \\ \hline \underset{(1\times S)}{c_i^{iA}} & \underset{(1\times 1)}{0} \end{array}\right]$$

$$\underset{(S\times S)}{\pi_{ij}^A} \equiv \text{submatrix for inputs } i = 1,\cdots,S \text{ of the Hessian for } \pi(x) \text{ at } x^* \quad \ldots .(i)$$

((i) continued on following page)

$$\underset{(S\times 1)}{c_i^{iA}} \equiv (c_1^1(x^{1*};\alpha^1),\cdots,c_S^S(x^{S*};\alpha^S))^T$$

$$\underset{(S+1)\times 1}{x^S} \equiv (dx^1,\cdots,dx^S, - d\lambda)^T \qquad \ldots\ldots(i)$$

$$\underset{(S+1)\times 1}{c^S} \equiv (c^1_{1\alpha^1}(x^{1*};\alpha^1)d\alpha^1,\cdots,c^S_{S\alpha^S}(x^{S*};\alpha^S)d\alpha^S,d\bar{R})^T .$$

By definition, $[\tilde{A}_{11}]$ has less than full rank if and only if there exists a vector $v \neq 0$ such that

$$\sum_{j=1}^{S} v^j\, \pi_{ij} - c_i^i = 0 \qquad i = 1,\cdots,S$$

$$\sum_{j=1}^{S} v^j\, c_j^j = 0 \;;$$

but this statement implies that

there exists a vector $v \neq 0$ such that $\sum_{i=1}^{S}\sum_{j=1}^{S} \pi_{ij}\, v^i v^j = 0$

which contradicts the assumption that $[\pi_{ij}]$ (hence $[\pi_{ij}^A]$) is negative definite. Thus

$$[\pi_{ij}] \text{ negative definite} \Rightarrow [\tilde{A}_{11}] \text{ has full rank.} \qquad \ldots\ldots(***)$$

Then

$$[\tilde{A}_{11}]^{-1} \equiv \underset{(S+1)\times(S+1)}{[L]} \text{ is symmetric} \qquad \ldots\ldots(j)$$

by $[\tilde{A}_{11}]$ symmetric. By (h) and (j),

$$X^S = [L]\, C^S \quad . \qquad \ldots\ldots(k)$$

By (i)-(k), for problem (d)

the comparative static effects $\left(\frac{\partial x^{**S}}{\partial \alpha}, \frac{\partial \lambda^S}{\partial \alpha}, \frac{\partial x^{**S}}{\partial \bar{R}}, \frac{\partial \lambda^S}{\partial \bar{R}}\right)$

are uniquely defined as follows:

$$\frac{\partial x^{i**S}}{\partial \alpha^j} = c^j_{j\alpha^j}(x^{j*};\alpha^j) \cdot L_{i,j} \qquad i,j = 1,\cdots,S$$

$$\frac{\partial x^{i**S}}{\partial \bar{R}} = L_{i,S+1} \qquad i = 1,\cdots,S$$

$$\ldots\ldots(***)$$

$$\frac{\partial \lambda^S}{\partial \alpha^j} = -\, c^j_{j\alpha^j}\,(x^{j*};\alpha^j) \cdot L_{S+1,j} \qquad j = 1,\cdots,S$$

$$\frac{\partial \lambda^S}{\partial \bar{R}} = -L_{S+1,S+1}$$

where $L_{i,j} \equiv$ element (i,j) of [L], and $L_{i,j} = L_{j,i}$(i,j=1, $\cdots$,S+1)

which is statement A of the Corollary. By (b) and (e),

$$\pi_i(x^*) = 0 \qquad i = 1,\cdots,S$$

$$\ldots\ldots(l)$$

$$x^j = x^{j*} \qquad j = S+1,\cdots,N$$

which are the first order conditions for an interior solution to problem (b). By (i) and (l), differentiating the first S first order conditions with respect to α^j ($j = 1, \cdots, S$) yields

$$\underset{(S\times S)}{[\pi_{ij}^{A}]}\ \underset{(S\times 1)}{\frac{\partial x^{*S}}{\partial \alpha^j}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -c^j_{j\alpha^j}(x^{j*};\alpha^j) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \alpha^j = \alpha^1, \cdots, \alpha^S \quad \ldots.(m)$$

where $\frac{\partial x^{*S}}{\partial \alpha^j} \equiv \left(\frac{\partial x^{1*S}}{\partial \alpha^j}, \cdots, \frac{\partial x^{S*S}}{\partial \alpha^j}\right)^T$. Since $\underset{(N\times N)}{[\pi_{ij}]}$ negative definite implies that $\underset{(S\times S)}{[\pi_{ij}^{A}]}$ is negative definite,

$\left\{\frac{\partial x^{*S}}{\partial \alpha^j}\right\}$ for problem (b) contains one and only one vector

$$\frac{\partial x^{*S}}{\partial \alpha^j} \quad (j = 1, \cdots, S). \qquad \ldots.(n)$$

by (m), $\underset{(N\times N)}{[\pi_{ij}]}$ negative definite and Proposition 1-A. By arguments identical to (v)-(z) in the proof of Theorem 3,

given that $\sum_{i=1}^{S} \sum_{j=1}^{S} L_{i,S+1} \cdot c^j_j(x^{j*};\alpha^j) \neq -1$,

$\frac{\partial x^{*S}}{\partial \alpha^j}$ for problem (b) is uniquely defined in terms of

$c^j_{j\alpha^j}(x^{j*};\alpha^j)$ and the elements of [L] corresponding to $\ldots.(***)$

((***) continued on the following page)

$\frac{\partial x^{**S}}{\partial \alpha^j}$ and $\frac{\partial x^{**S}}{\partial \bar{R}}$ for problem (d) (j = 1,⋯,S), as

follows: $\frac{\partial x^{i*S}}{\partial \alpha^j} = c^j_{j\alpha j}(x^{j*};\alpha^j) \cdot L_{i,j} + L_{i,S+1} \cdot \frac{\partial R(x^*)^S}{\partial \alpha^j}$ . . . .(***)

$i,j = 1,\cdots,S$

$$\frac{\partial R(x^*)^S}{\partial \alpha^j} = \sum_{i=1}^{S} c^i_i(x^{i*};\alpha^i) \cdot \frac{\partial x^{i*S}}{\partial \alpha^j} \qquad j = 1,\cdots,S.$$

Statements (n) and (III) are equivalent to statement B of the Corollary. □

7. Corollary 6

Corollary 6. Construct problems 1 and 3 as above, and partition the (negative definite) Hessian matrix $\underset{(N\times N)}{[\pi_{ij}]}$ as above. Then the comparative static effects for problem 3 are uniquely defined as follows:

$$\frac{\partial x^{i*S}}{\partial \alpha^j} = \frac{\partial^2 c^j(x^{j*};\alpha^j)}{\partial x^j \partial \alpha^j} \cdot P_{i,j} \qquad i,j = 1,\cdots,S$$

where $P_{i,j} \equiv$ element (i,j) of $\underset{(S\times S)}{[\pi_{ij}^A]^{-1}}$ (which always exists),

and $P_{i,j} = P_{j,i}$ (i,j = 1,⋯,S).

<u>Proof</u>. Construct the problems

$$\text{maximize } \pi(x) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \ldots\ldots(a)$$

$$\text{maximize } \pi(x) \equiv R(x) - \sum_{i=1}^{N} c^i(x^i;\alpha^i)$$

$$\text{subject to } x^j = \overline{x^{j*}} \qquad j = S+1, \cdots, N \qquad \ldots\ldots(b)$$

where x^* is a unique global solution to problem (a). Partition the Hessian matrix of $\pi(x)$ at x^* as follows:

$$\underset{(N\times N)}{[\pi_{ij}]} \equiv \left(\begin{array}{c|c} \underset{(S\times S)}{\pi_{ij}^{A}} & \underset{(S\times T)}{\pi_{ij}^{B}} \\ \hline \underset{(T\times S)}{\pi_{ij}^{C}} & \underset{(T\times T)}{\pi_{ij}^{D}} \end{array}\right) . \qquad \ldots\ldots(c)$$

By the assumptions that $\underset{(N\times N)}{[\pi_{ij}]}$ is negative definite and symmetric,

$$\underset{(N\times N)}{[\pi_{ij}]^{-1}} \text{ and } [\pi_{ij}^{A}]^{-1} \text{ exist and are symmetric.} \qquad \ldots\ldots(d)$$

By (a)-(b),

$$x^* \text{ is the solution set to (b) as well as (a).} \qquad \ldots\ldots(e)$$

By conditions 1-2, the first order conditions for a solution to (b) are

$$\pi_i(x) = 0 \qquad i = 1,\cdots,S \qquad \ldots.(f)$$

$$x^j = \overline{x^{j*}} \qquad j = S+1,\cdots,N \; . \qquad \ldots.(g)$$

By (c)-(e) and (g), total differentiating (f) yields

$$\underset{(S\times 1)}{\begin{pmatrix} dx^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ dx^S \end{pmatrix}} = \underset{(S\times S)}{[\pi_{ij}{}^A]^{-1}} \cdot \underset{(S\times 1)}{C} \qquad \ldots.(h)$$

where $C \equiv (c^1_{1\alpha^1}(x^{1*};\alpha^1)d\alpha^1,\cdots,c^S_{S\alpha^S}(x^{S*};\alpha^S)d\alpha^S)^T$. By (d) and (h), for (b)

$$\frac{\partial x^{i*S}}{\partial \alpha^j} = c^j_{j\alpha^j}(x^{j*};\alpha^j) \cdot P_{i,j} \qquad i,j = 1,\cdots,S$$

, where $P_{i,j} \equiv$ element (i,j) of $[\pi_{ij}{}^A]^{-1}$, and $P_{i,j} = P_{j,i}$ (i,j=1,$\cdots$,S)

which is the Corollary. □

8. A Theorem on the Quantitative Comparative Statics of a Shift in a Firm's Product Demand Schedule

Relations between various potentially observable properties of a problem

$$\text{maximize } \pi(x;\alpha) \equiv R(x;\alpha^0) - \sum_{i=1}^{N} c^i(x^i;\alpha^i) \qquad \text{. . . .(P)}$$

and

$\dfrac{\partial x^*}{\partial \alpha^0}$, when $d\alpha^0$ implies a shift in the product demand schedule faced by the firm, are presented in Theorem 4 and Corollary7.[17] These relations differ from those specified in Theorem 3 and Corollaries 5-6 in one particularly important respect, which can be explained as follows.

When α^0 is a parameter in the product demand schedule, $\dfrac{\partial x^*}{\partial \alpha^0}$ can be decomposed as

$$\frac{\partial x^{i*}}{\partial \alpha^0} = \frac{\partial x^{i**}}{\partial \bar{F}} \cdot \frac{\partial F(x^*)}{\partial \alpha^0} \qquad i = 1,\cdots,N \qquad \text{. . . .(1)}$$

where $\dfrac{\partial x^{**}}{\partial \bar{F}}$ is the comparative static effect of $d\bar{F}$ for the problem

$$\text{maximize } \pi(x) \equiv P(F(x);\alpha^0)F(x) - \sum_i c^i(x^i;\alpha^i)$$

$$\text{subject to } F(x) = \overline{F(x^*)}$$

---

[17]The proof of Corollary 7 is not presented here (Corollary 7 can be established in an abvious manner by the methods used in other proofs).

($y = F(x)$ denotes the firm's production function). In addition,

$$\frac{\partial F(x^*)}{\partial \alpha^0} = (1/\hat{R}_y) \sum_{i=1}^{N} \frac{\partial c^i(x^{i*})}{\partial x^i} \cdot \frac{\partial x^{i*}}{\partial \alpha^0} \qquad \ldots .(2)$$

where $\hat{R}(F(x);\alpha^0) = R(x;\alpha^0)$ for all (x,α^0). For a given vector $\frac{\partial x^{**}}{\partial \bar{F}}$, equations 1-2 constitute a homogeneous system of N+1 equations in N+1 unknowns $\left(\frac{\partial x^*}{\partial \alpha^0}, \frac{\partial F(x^*)}{\partial \alpha^0}\right)$. Thus, equations 1-2 can determine the unique $\frac{\partial x^*}{\partial \alpha^0}$ only up to a scalar multiple, i.e., only ratios

$$\left(\frac{\partial x^{1*}}{\partial \alpha^0} / \frac{\partial x^{i*}}{\partial \alpha^0}, \; \ldots, \; \frac{\partial x^{N*}}{\partial \alpha^0} / \frac{\partial x^{i*}}{\partial \alpha^0}\right)$$

can be uniquely defined by 1-2. A similar statement holds for the decomposition of $\frac{\partial x^{*S}}{\partial \alpha^0}$ (when α is a parameter in the product demand schedule). Therefore, knowledge of $\frac{\partial x^{**}}{\partial \bar{F}}$ or $\{\frac{\partial x^{**S}}{\partial \bar{F}}\}$ defined by all possible partitions of x into fixed and variable inputs (and knowledge of $\hat{R}_y$, $c_1^1(x^{1*}), \cdots, c_N^N(x^{N*})$) is insufficient to define the unique $\frac{\partial x^*}{\partial \alpha^0}$. The additional restrictions due to the second order conditions for a maximum only imply that the unique $\frac{\partial x^*}{\partial \alpha^0}$ is determined up to a *positive* scalar multiple.[18]

---

[18]The proof of this statement can be sketched as follows. The first order conditions imply that the "correct" Hessian $[\pi_{ij}{}^*]$ and comparative static effect $\frac{\partial x^*}{\partial \alpha^0}$ satisfy a system of equations of the form

$$[\pi_{ij}{}^*] \; \frac{\partial x^*}{\partial \alpha^0} = \underset{N\times 1}{[K]} \quad . \qquad \ldots .(a)$$

Exact knowledge of $(\frac{\partial x^{**}}{\partial \bar{F}}, \hat{R}_y, c_1^1(x^{1*}), \cdots, c_N^N(x^{N*}))$ implies only the following relation:

Thus restrictions must be placed on other parameters in order to obtain both upper and lower bounds on $\frac{\partial x^*}{\partial \alpha^0}$ by our methods. In particular, knowledge of $\frac{\partial x^*}{\partial \alpha^i}$ $(i \neq 0)$ and its various decompositions seems quite important in the quantitative comparative statics of changes in the firm's revenue or benefits schedule (whereas, prior knowledge of $\frac{\partial x^*}{\partial \alpha^0}$ and its decompositions is relatively unimportant in the quantitative comparative statics of changes in factor supply schedules).

Theorem 4. Suppose that conditions 1-3 are satisfied for a problem P

$$\text{maximize } \pi(x) \equiv R(y;\alpha^0) - \sum_{i=1}^{N} c^i(x^i) \qquad \text{....(1)}$$

where $y \equiv F(x)$ is a scalar function, and assume that problem 1 has a unique global solution x^* where the Hessian $\underset{(N\times N)}{[\pi_{ij}]}$ is negative definite.[19] Construct the related problem

---

(18 continued)

$$[\frac{1}{\gamma} \otimes [\pi_{ij}^*]][\gamma \cdot \frac{\partial x^*}{\partial \alpha^0}] = [K] \qquad \text{....(b)}$$

where γ is an arbitrary scalar. Given that $[\pi_{ij}^*]$ is negative definite: $\frac{1}{\gamma} \otimes [\pi_{ij}^*]$ is negative definite if and only if $\gamma > 0$. Thus, relation (b) plus the second order conditions has the solution set $\{\gamma \cdot \frac{\partial x^*}{\partial \alpha^0} \mid \gamma > 0\}$.

[19]The comparative static effect $\frac{\partial x^*}{\partial \alpha^0}$ is undefined when $[\pi_{ij}]$ is only negative semi-definite (see Proposition 1).

$$\text{maximize } \pi(x) \equiv R(y;\alpha^0) - \sum_{i=1}^{N} c^i(x^i)$$

$$\text{subject to } y = \overline{F(x^*)} \quad . \qquad \ldots .(2)$$

Let $R_y \equiv \dfrac{\partial R(y^*;\alpha^0)}{\partial y}$

$$R_{y\alpha^0} \equiv \frac{\partial^2 R(y^*;\alpha^0)}{\partial y \partial \alpha^0}$$

$$R_{i\alpha^0} \equiv \frac{\partial^2 R(F(x^*);\alpha^0)}{\partial x^i \partial \alpha^0} = (R_{y\alpha^0}/R_y)\, c_i^i(x^{i*}).$$ [20]

Construct matrix [A] as in Theorem 3, so that $[A]^{-1} \equiv [K]$ always exists. Then

(A) $\{\dfrac{\partial x^*}{\partial \alpha^0}\}$ for problem 1 corresponds to the single solution to

the system

$$\underset{(N\times N)}{[\pi_{ij}]} \; \frac{\partial x}{\partial \alpha^0} = \underset{(N\times 1)}{- R_{i\alpha^0}}$$

---

[20] If $R(y;\alpha^0) \equiv P(y;\alpha^0)y$, then $R_y = P(y^*;\alpha^0) + P_y(y^*;\alpha^0)y^*$ and $R_{y\alpha^0} = P_{\alpha^0}(y^*;\alpha^0) + P_{y\alpha^0}(y^*;\alpha^0)y^*$. For the more general case where the firm sells all y units at an identical price and also receives non-pecuniary benefits B(t) from the t'th unit of y, $R(y;\alpha^0) \equiv P(y;\alpha^0)y + \int_0^y B(t)dt$.

where $\underset{(N\times 1)}{R_{i\alpha^0}} \equiv (R_{1\alpha^0}, \cdots, R_{N\alpha^0})^T$;

(B) the comparative static effects for problem 2 are defined in terms of [A] as follows:

$$\frac{\partial x^{i**}}{\partial \bar{F}} = R_y \cdot K_{i,N+1}$$

$$[K] \equiv [A]^{-1} \Rightarrow \frac{\partial x^{i**}}{\partial \alpha^0} = 0 \qquad i = 1, \cdots, N$$

$$\frac{\partial \lambda}{\partial \bar{F}} = -R_y^{\,2} \cdot K_{N+1,N+1} \qquad i = 1, \cdots, N$$

$$[K] \equiv [A]^{-1} \Rightarrow \frac{\partial \lambda}{\partial \alpha^0} = R_{y\alpha^0}$$

where $K_{i,j} \equiv$ element (i,j) of $[A]^{-1}$, and $K_{i,j} = K_{j,i}$

$(i,j = 1, \cdots, N+1)$;

and

(C) given that $\sum_{j=1}^{N} \sum_{i=1}^{N} c_j^j \cdot K_{i,N+1} \neq -1$,

$\frac{\partial x^*}{\partial \alpha^0}$ is determined up to a scalar multiple by R_y, c_i^i and the elements of $[A]^{-1}$ corresponding to $\frac{\partial x^{**}}{\partial \bar{F}}$, i.e., the following system has as solution the $\{\text{all } \gamma(\frac{\partial x^*}{\partial \alpha^0}, \frac{\partial F(x^*)}{\partial \alpha^0})\}$ (γ an arbitrary scalar):

$$\frac{\partial x^{i*}}{\partial \alpha^0} = R_y \cdot K_{i,N+1} \cdot \frac{\partial F}{\partial \alpha^0} \qquad i = 1, \cdots, N$$

$$\frac{\partial F}{\partial \alpha^0} = (1/R_y) \sum_{i=1}^{N} c_i^i \cdot \frac{\partial x^{i*}}{\partial \alpha^0} \quad .$$

Proof. Construct the problem

$$\text{maximize } \pi(x) \equiv R(y;\alpha)y - \sum_{i=1}^{N} c^i(x^i) \qquad \ldots.(a)$$

where $y = F(x)$, or equivalently

$$\text{maximize } \pi(x) \equiv \hat{R}(x;\alpha) - \sum_{i=1}^{N} c^i(x^i) \qquad \ldots.(b)$$

where

$$\hat{R}(x;\alpha) \equiv R(F(x);\alpha) \quad . \qquad \ldots.(c)$$

Total differentiating the first order conditions for an interior solution to (b),

$$\sum_{j=1}^{N} \pi_{ij}(x^*)\frac{\partial x^{i*}}{\partial \alpha} + \hat{R}_{i\alpha}(x^*;\alpha) = 0 \qquad i = 1, \cdots, N. \qquad \ldots.(d)$$

Since a negative definite matrix has an inverse (see a-b in the proof of Proposition 1),

$$\underset{(N\times N)}{[\pi_{ij}]} \text{ negative definite} \Rightarrow \text{equations (d) has a unique solution } \frac{\partial x^*}{\partial \alpha} \qquad \ldots.(***)$$

which is statement A of the Theorem. Construct the problem

$$\text{maximize } \pi(x) \equiv R(y;\alpha) - \sum_{i=1}^{N} c^i(x^i)$$

$$\text{subject to } y = \overline{F(x^*)} \qquad \text{. . . .(e)}$$

where x* is the unique global solution to problem (b). Problem (e) can be expressed in Lagrange form as

$$\text{maximize } R(y;\alpha) - \sum_{i=1}^{N} c^i(x^i) - \lambda(F(x) - \overline{F(x^*)}). \qquad \text{. . . .(f)}$$

By (b) and (e),

x* is the solution set to problem (f) as well as (b). . . . .(g)

By the manner in which α enters into the objective function for problem (f),

$$\frac{\partial x^{**}}{\partial \alpha} = 0 \qquad \text{for problem (f)} \ . \qquad \text{. . . .(h)}$$

By (f)-(g),

$$\pi_i(x^*) - \lambda F_i(x^*) = 0 \qquad i = 1, \cdots, N$$

$$F(x) - \overline{F(x^*)} = 0 \ . \qquad \text{. . . .(i)}$$

Total differentiating these first order conditions (i),

$$\sum_{j=1}^{N} \pi_{ij}(x^*)\, dx^i + \hat{R}_{i\alpha}(x^*;\alpha)d\alpha - F_i(x^*)d\lambda = 0 \qquad i = 1,\cdots,N \qquad \text{....(j)}$$

$$\sum_{i=1}^{N} F_i(x^*)\, dx^i - d\bar{F} = 0$$

since $\lambda = 0$ (see h in the proof of Theorem 3). By conditions 1-2,

$$R_y(y^*;\alpha)\; F_i(x^*) - c_i^i(x^{i*}) = 0 \qquad i = 1,\cdots,N \qquad \text{....(k)}$$

where (x^*, y^*) solves problem (a). By (c),

$$\hat{R}_{i\alpha}(x^*;\alpha) = R_{y\alpha}(y^*;\alpha) F_i(x^*) \qquad i = 1,\cdots,N\,. \qquad \text{....(l)}$$

By (j)-(k),

$$[\hat{A}]\; X = \hat{C} \qquad \text{....(m)}$$

where

$$\left(\begin{array}{c:c} \pi_{ij} & (1/R_y)c_i^i \\ (N\times N) & (1\times N) \\ \hdashline (1/R_y)c_i^i & 0 \\ (N\times 1) & (1\times 1) \end{array}\right) \equiv \underset{(N+1)\times(N+1)}{[\hat{A}]} \qquad \text{....(n)}$$

$$\underset{(N+1)\times 1}{X} \equiv (dx^1,\cdots,dx^N, -d\lambda)^T$$

$$\underset{(N+1)\times 1}{\hat{C}} \equiv (-\hat{R}_{1\alpha}d_\alpha,\cdots,-\hat{R}_{N\alpha}d_\alpha, d\bar{F})^T\,.$$

$|H| = \lambda |G|$ if every element of a row or column of a matrix G is multiplied by a scalar λ to give a new matrix H, and G^{-1} = (adjoint of G) / $|G|$. By these facts and the definitions of $[\hat{A}]$ (n) and [A],

$$
\begin{aligned}
&[A]^{-1} \equiv [K] \text{ exists if and only if } [\hat{A}]^{-1} \equiv [\hat{K}] \text{ exists} \\
&\hat{K}_{ij} = R_y^2 K_{ij} \quad \text{if } i = N+1,\ j = N+1 \\
&\hat{K}_{ij} = R_y K_{ij} \quad \text{if } i = N+1 \text{ or } j = N+1,\ i \neq j \\
&\hat{K}_{ij} = \quad K_{ij} \quad \text{if } i = 1,\cdots,N,\ j = 1,\cdots,N\ .
\end{aligned}
\qquad \ldots .(o)
$$

By (h), (m)-(o), the existence of $[A]^{-1}$ (see Theorem 3) and the symmetry of $[A]^{-1}$ (by the symmetry of [A]), for problem (f)

$$
\begin{aligned}
&\frac{\partial x^{i**}}{\partial \bar{F}} = R_y \cdot K_{i,N+1} && i = 1,\cdots,N \\
&\frac{\partial x^{i**}}{\partial \alpha} = - \sum_{j=1}^{N} \hat{R}_{j\alpha}(x^*;\alpha) K_{ij} && i = 1,\cdots,N \\
&\frac{\partial \lambda}{\partial \bar{F}} = - R_y^{\,2} \cdot K_{N+1,N+1} && \\
&\frac{\partial \lambda}{\partial \alpha} = R_y \sum_{j=1}^{N} \hat{R}_{j\alpha}(x^*;\alpha)\, K_{N+1,j} &&
\end{aligned}
\qquad \ldots .(p)
$$

where (by k-l) $\hat{R}_{i\alpha} = (R_{y\alpha}/R_y)c_i^i$ and $K_{ij} = K_{ji}$ $(i,j = 1,\cdots,N+1)$.

By the definitions of [A] and [K],

$$AK = I \Rightarrow \sum_{j=1}^{N} c_j^j K_{ij} = 0 \qquad i = 1, \cdots, N$$

$$\sum_{j=1}^{N} c_j^j K_{j,N+1} = 1 \qquad \ldots .(q)$$

By (p)-(q),

$$\frac{\partial x^{i**}}{\partial \overline{F}} = R_y K_{i,N+1} \qquad i = 1, \cdots, N$$

$$AK = I \Rightarrow \frac{\partial x^{i**}}{\partial \alpha} = 0 \qquad \ldots .(***)$$

$$\frac{\partial \lambda}{\partial \overline{F}} = - R_y^{\ 2} K_{i,N+1}$$

$$AK = I \Rightarrow \frac{\partial \lambda}{\partial \alpha} = R_{y\alpha}$$

which is statement B of the Theorem. By arguments analogous to (r)-(s) in the proof of Theorem 3,

$$\frac{\partial x^{i*}}{\partial \alpha} = \frac{\partial x^{i**}}{\partial \alpha} + \frac{\partial x^{i**}}{\partial \overline{F}} \cdot \frac{\partial F(x^*)}{\partial \alpha} \qquad i = 1, \cdots, N \qquad \ldots .(r)$$

where

$$\frac{\partial F(x^*)}{\partial \alpha} = \sum_{i=1}^{N} F_i(x^*) \frac{\partial x^{i*}}{\partial \alpha} \quad . \qquad \ldots .(s)$$

By (h) and (r),

$$\frac{\partial x^{i*}}{\partial \alpha} = \frac{\partial x^{i**}}{\partial \bar{F}} \frac{\partial F(x^*)}{\partial \alpha} \qquad i = 1,\cdots,N . \qquad \ldots .(t)$$

By (k), (s)-(t) and statement B of the Theorem (***),

$$\underset{(N+1)\times(N+1)}{\left[\begin{array}{c|c} \underset{(N\times N)}{I} & \underset{(N\times 1)}{-R_y\tilde{K}_{i,N+1}} \\ \hline \underset{(1\times N)}{(1/R_y)c_i^i} & \underset{(1\times 1)}{-1} \end{array}\right]} \underset{(N+1)\times 1}{\begin{bmatrix} \frac{\partial x^{1*}}{\partial \alpha} \\ \vdots \\ \frac{\partial x^{N*}}{\partial \alpha} \\ \frac{\partial F(x^*)}{\partial \alpha} \end{bmatrix}} = 0 \qquad \ldots .(u)$$

where $I \equiv$ identity matrix and $\tilde{K}_{i,N+1} \equiv N + 1$'st column of $[\hat{A}]^{-1}$. Denote the $(N+1)\times(N+1)$ matrix in (u) as $[\hat{L}]$. By (u) and the definition of a determinant,

$$|[\hat{L}]| = -1 - \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{K}_{i,N+1} \cdot c_j^j(x^{j*}) . \qquad \ldots .(v)$$

Since $[\hat{L}]^{-1}$ exists if and only if $|[\hat{L}]| \neq 0$, (w) implies that

given that $\sum_{i=1}^{N} \sum_{j=1}^{N} \hat{K}_{i,N+1} \cdot c_j^j(x^{j*}) \neq -1$,

the solution $(\frac{\partial x}{\partial \alpha}, \frac{\partial F}{\partial \alpha})$ to the linear homogeneous $\qquad \ldots .(***)$

system (u) is unique except for a scalar multiple

which is statement C of the Theorem. □

Corollary 7. Construct problems 1 and 2 as above, and partition the (negative definite) Hessian matrix $\underset{(N\times N)}{[\pi_{ij}]}$ and marginal factor cost vector $\underset{(1\times N)}{c_i^i}$ at x^* as follows:

$$\underset{(N\times N)}{[\pi_{ij}]} \equiv \left[\begin{array}{c|c} \underset{(S\times S)}{\pi_{ij}^{A}} & \underset{(S\times T)}{\pi_{ij}^{B}} \\ \hline \underset{(T\times S)}{\pi_{ij}^{C}} & \underset{(T\times T)}{\pi_{ij}^{D}} \end{array}\right] \qquad \underset{(1\times N)}{c_i^i} \equiv [\underset{(1\times S)}{c_i^{iA}} \mid \underset{(1\times T)}{c_i^{iB}}]$$

where $S+T = N$ and x^* is the solution to problem 1. Construct the matrix $[\tilde{A}_{11}]$ as in Corollary 2, and denote its inverse as $[L]: [\tilde{A}_{11}]^{-1} \equiv [L]$ always exists.

Construct the problem

$$\begin{aligned} &\text{maximize } \pi(x) \equiv R(y;\alpha^0)\, y - \sum_{i=1}^{N} c^i(x^i) \\ &\text{subject to } x^j = \overline{x^{j*}} \qquad\qquad j = S+1, \cdots, N. \end{aligned} \qquad \ldots .(3)$$

Construct the related problem

$$\begin{aligned} &\text{maximize } \pi(x) \\ &\text{subject to } y = \overline{F(x^*)} \\ &\qquad\qquad x^j = \overline{x^{j*}} \qquad\qquad j = S+1, \cdots, N \end{aligned}$$

which can be expressed in Lagrange form as

$$\text{maximize } \pi(x) - \lambda(F(x) - \overline{F(x^*)}) - \sum_{j=S+1}^{N} \gamma^j (x^j - \overline{x^{j*}}). \quad \ldots .(4)$$

Then

(A) the comparative static effects $\frac{\partial x^{*S}}{\partial \alpha}$ for problem 3 are uniquely defined as follows:

$$\frac{\partial x^{i*S}}{\partial \alpha^0} = (R_{y\alpha 0}/R_y) \sum_{j=1}^{S} \frac{\partial c^j(x^{j*})}{\partial x^j} \cdot \hat{P}_{i,j} \qquad i = 1, \cdots, S$$

where $\hat{P}_{i,j} \equiv$ element (i,j) of $[\pi_{ij}^A]^{-1}$ (which always exists),

and $\hat{P}_{i,j} = \hat{P}_{j,i}$ $(i,j = 1, \cdots, S)$, and

(B) the comparative static effects for problem 4 are uniquely defined as follows:

$$\frac{\partial x^{i**S}}{\partial \overline{F}} = R_y \cdot L_{i,S+1} \qquad i = 1, \cdots, S$$

$$[L] \equiv [\tilde{A}_{11}]^{-1} \Rightarrow \frac{\partial x^{i**S}}{\partial \alpha^0} = 0 \qquad i = 1, \cdots, S.$$

$$\frac{\partial \lambda^S}{\partial \overline{F}} = -R_y^{\,2} \cdot L_{S+1,S+1}$$

$$[L] \equiv [\tilde{A}_{11}]^{-1} \Rightarrow \frac{\partial \lambda^S}{\partial \alpha^0} = R_{y\alpha 0}$$

where $L_{ij} \equiv$ element (i,j) of $[\tilde{A}_{11}]^{-1}$, and $L_{i,j} = L_{j,i}$

$(i,j = 1,\cdots,S+1)$; and

(C) given that $\sum_{j=1}^{S} \sum_{i=1}^{S} \frac{\partial c^j(x^{j*})}{\partial x^j} \cdot L_{i,S+1} \neq -1$,

$\frac{\partial x^{*S}}{\partial \alpha^0}$ is determined up to a scalar multiple by R_y, c_i^i and

the elements of [L] corresponding to $\frac{\partial x^{**S}}{\partial \bar{F}}$, i.e., the

following system has as solution

$$\{\text{all } \gamma \; (\frac{\partial x^{*S}}{\partial \alpha^0} , \frac{\partial F(x^*)^S}{\partial \alpha^0})\} \; (\gamma \text{ an arbitrary scalar}):$$

$$\frac{\partial x^i}{\partial \alpha^0} = L_{i,S+1} \cdot \frac{\partial F}{\partial \alpha^0} \qquad i = 1,\cdots,S$$

$$\frac{\partial F}{\partial \alpha^0} = (1/R_y) \sum_{i=1}^{S} \frac{\partial c^i(x^{i*})}{\partial x^i} \cdot \frac{\partial x^i}{\partial \alpha^0} \quad .$$

9. On Primal-Dual and Dual Methods of Quantitative Comparative Statics

The quantitative comparative statics method presented in Chapter 3 is developed directly from the primal problem

$$\text{maximize}_x \quad \pi(x;\alpha) \equiv R(x) - c^1(x^1;\alpha) - \sum_{i=2}^{N} c^i(x^i). \qquad \ldots .(P)$$

In Chapter 2, we noted that "primal-dual" and "dual" problems can be formulated from P and that many standard comparative static theorems can be derived more easily from these problems than from P per se. Here we shall consider the possibility of using primal-dual and dual methods as substitutes or complements to our primal approach to quantitative comparative statics.

Our primal approach exhausts the restrictions placed on $\frac{\partial x^{*P}}{\partial \alpha}$ by the maximization hypothesis[21] but does not exhaust the relations between the parameters relevant to comparative statics $([\pi_{ij}(x^{*P};\alpha)],\ c^1_{1\alpha}(x^{1*P};\alpha))$ and other potentially observable data. Thus the possible advantages of alternative or supplementary approaches to quantitative comparative statics are ease of computation and elucidation of the relations between these restrictions and any a priori knowledge about the structure of P. However, we shall argue that a primal-dual or dual approach to quantitative comparative statics can seldom substitute for a primal approach and can seldom suggest important relations between $\frac{\partial x^{*P}}{\partial \alpha}$ and potentially observable data that are not already incorporated into our primal approach. Indeed, a primal-dual approach can never suggest relations between $\frac{\partial x^{*P}}{\partial \alpha}$ and potentially observable data that are not already incorporated into our primal approach.

[21] See Proposition 1.

The primal-dual problem corresponding to P can be formulated as

$$\text{minimize}_{x,\alpha}\ L(x,\alpha) \equiv \pi(x^*(\alpha),\alpha) - \pi(x,\alpha) \qquad \ldots .(PD)$$

where $x^*(\alpha)$ expresses the solution to P as a function of its exogenous variable α.[22] The solution set to PD is {all $(x^*(\alpha),\alpha)$}. Since all variables (x,α) in P are defined as endogenous in PD, we cannot totally differentiate the first order conditions for a solution to PD. Thus we cannot derive precise quantitative comparative static relations from problem PD, i.e,, there does not exist a set of equations that defines the comparative static effect $\frac{\partial x^*}{\partial \alpha}$ in terms of the structure of problem PD.

In order to obtain a quantitative comparative statics model related to the primal-dual approach, we can construct the following "modified primal-dual" problem

$$\text{minimize}_{x,\gamma}\ L(x,\gamma;\alpha) \equiv \pi(x^*(\gamma,\alpha),\ \gamma;\alpha) - \pi(x,\gamma;\alpha) \qquad \ldots .(PD')$$

where (γ,α) is the set of K+1 variables exogenous to problem P and $x^*(\gamma,\alpha)$ expresses the solution to P as a function of (γ,α). The first order conditions for an interior solution to PD' are

$$L_{xi} = -\pi_{x^i}(x^*,\gamma^*;\alpha) = 0 \qquad i = 1,\cdots,N \qquad \ldots .(a)$$

$$L_{\gamma i} = \sum_{j=1}^{N} \pi_{x^j}(x^*(\gamma^*,\alpha),\gamma^*;\alpha)\ \frac{\partial x^{j*}}{\partial \gamma^i} = 0 \qquad i = 1,\cdots,K \qquad \ldots .(b)$$

and the (necessary) second order conditions are

---

[22]See Silberberg (1974a).

$$[L] \equiv \begin{bmatrix} L_{xx} & L_{x\gamma} \\ L_{x\gamma} & L_{\gamma\gamma} \end{bmatrix} \quad \text{positive semi-definite at } (x^*, \gamma^*; \alpha) \qquad \ldots .(2°)$$

where

$$L_{x^i x^j} = -\pi_{x^i x^j} \qquad i,j = 1, \cdots, N$$

$$L_{x^i \gamma^j} = -\pi_{x^i x^j} \qquad i = 1, \cdots, N \text{ and } j = 1, \cdots, K$$

$$L_{x^i \gamma^j} = \sum_{K=1}^{N} \sum_{l=1}^{N} \pi_{x^K x^l} \frac{\partial x^{K*}}{\partial \gamma^i} \frac{\partial x^{l*}}{\partial \gamma^j} + \sum_{K=1}^{N} \pi_{x^K \gamma^j} \frac{\partial x^{K*}}{\partial \gamma^i} \qquad i,j = 1, \cdots, K.$$

Total differentiating (a)-(b) with respect to the variable α exogenous to PD' yields

$$\sum_{j=1}^{N} \pi_{x^i x^j} \frac{\partial x^{j*}}{\partial \alpha} + \sum_{j=1}^{K} \pi_{x^i \gamma^j} \frac{\partial \gamma^{j*}}{\partial \alpha} + \pi_{x^i \alpha} = 0 \qquad i = 1, \cdots, N. \ldots (c)$$

$$\sum_{i=1}^{N} \frac{\partial x^{i*}}{\partial \gamma^K} \left(\sum_{j=1}^{N} \pi_{x^i x^j} \frac{\partial x^{j*}}{\partial \alpha} + \sum_{j=1}^{N} \pi_{x^i \alpha^j} \frac{\partial \gamma^{j*}}{\partial \alpha} + \pi_{x^i \alpha} \right) + \sum_{i=1}^{N} \pi_{x^i} \left(\sum_{j=1}^{K} \frac{\partial^2 x^{i*}}{\partial \gamma^K \partial \gamma^j} \cdot \frac{\partial \gamma^{j*}}{\partial \alpha} + \frac{\partial^2 x^{i*}}{\partial \gamma^K \partial \alpha} \right) = 0 \qquad k = 1, \cdots, K. \qquad \ldots .(d)$$

Since (b) is implied by (a), and (d) is implied by (a) and (c), the comparative static content of PD' is contained entirely in (c) plus (2°). Combining (c) with the total differential of the first order conditions for the primal only implies that

$$\sum_{j=1}^{N} \pi_{x^i x^j} \frac{\partial x^{j*}}{\partial \alpha} + \pi_{x^i \alpha} = 0 \qquad i = 1, \cdots, N \qquad \ldots .(e)$$

$$\sum_{j=1}^{K} \pi_{x^i \gamma^j} \frac{\partial \gamma^{j*}}{\partial \alpha} = 0 \qquad i = 1, \cdots, N \qquad \ldots .(f)$$

where (e) denotes the total differential of the first order conditions for the primal. Since $\frac{\partial \gamma^*}{\partial \alpha}$ is either undefined or zero (all γ solves PD' for each α), (f) cannot place any restrictions on the structure of PD' or comparative static effects. Thus (a)-(d) plus the total differential of primal first order conditions (e) only implies (e), i.e., the first order conditions for PD' plus their total differentials are redundant given the total differential of primal first order conditions. In sum, any restrictions on $\frac{\partial x^*}{\partial \alpha}$ that are derived from (a)-(d) can be obtained more readily from the primal approach.

In addition, second order conditions for a primal-dual problem do not add to the set of relations between $[\pi_{ij}]$ and potentially observable data that are already formulated in our primal approach. This statement can be elaborated upon as follows. It can be seen from (2°) that the second order conditions for the primal-dual problem

$$\text{minimize}_{x,\gamma,\alpha}\, L(x,\gamma,\alpha) \equiv \pi(x^*(\gamma,\alpha),\gamma,\alpha) - \pi(x,\gamma,\alpha)$$

place restrictions on $[\pi_{ij}]$ in terms of $([\pi_{x\gamma}], \pi_{x\alpha}, [\frac{\partial x^*}{\partial \gamma}], \frac{\partial x^*}{\partial \alpha})$.

However, if (γ,α) are defined solely as shift parameters for factor supply schedules, then the relations between $[\pi_{ij}]$ and $([\pi_{x\gamma}]$, $\pi_{x\alpha}$, $[\frac{\partial x^*}{\partial \gamma}]$, $\frac{\partial x^*}{\partial \alpha})$ are defined exactly by a primal approach incorporating Proposition 1 and Theorem 3. If some element of γ is a shift parameter for the firm's revenue or benefits schedule $R(x)$, then the relations between $[\pi_{ij}]$ and the above set of variables are defined exactly by a primal approach incorporating Proposition 1 and Theorems 3-4. Since the relations between $[\pi_{ij}]$ and the above variables are defined exactly in these cases by our primal approach, and all of the restrictions and relations implied by the maximization hypothesis on the set of parameters $([\pi_{ij}], c^1_{1\alpha}, \frac{\partial x^*}{\partial \alpha})$ directly relevant to comparative statics are incorporated into our primal approach (by Proposition 1), the primal-dual second order condition for any particular γ must already be included in some version of our model.Thus inclusion of the primal-dual second order conditions in our quantitative comparative statics model does not appear to be useful.

In order to formulate a meaningful dual approach we need to impose some regularity conditions on P, e.g., we can define the primal problem

$$\text{maximize}_x \ \pi(x;\alpha)' \equiv \alpha^0 R(x) - \sum_{i=1}^{N} \alpha^i c^i(x^i) \ . \qquad \text{. . . .(P')}$$

The structure of P' indicates the possibility of non-competitive behaviour in the firm's product and factor markets and also assigns a special role to the parameters α similar to that of (p,w), i.e., a change in α^i leads to an equi-proportional change in revenue over all activity levels $(i = 0)$ or to an equi-proportional change in the cost of factor i over all activity levels $(i \neq 0)$. The dual profit function for P' can be defined as

$$\pi(\alpha) \equiv \left\{ \text{all } (\max_x \{\pi(x;\alpha)'\};\alpha) : \alpha \in P^0 \right\}$$

where α is defined over a subset P^0 of R^{N+1} for problem P'. Then $\pi(\alpha)$ has the same properties as in the competitive case: $\pi(\alpha)$ is linear homogeneous and convex in α, non-decreasing in α^0, non-increasing in $\alpha^i (i \neq 0)$, and continuous in α. Linear homogeneity and convexity imply relations between comparative static effects $\frac{\partial x^*}{\partial \alpha}$ and potentially observable data for P'. However, we shall show that these relations already are incorporated into our primal approach.

First, we shall consider the comparative static implications of linear homogeneity given appropriate assumptions about differentiability. By Euler's theorem, $\pi(\alpha)$ linear homogeneous in α implies that

$$\pi(\alpha) = \sum_{i=0}^{N} \pi_{\alpha^i}(\alpha) \cdot \alpha^i . \qquad \ldots .(g)$$

Since (g) is true for all α,

$$\pi_{\alpha^j}(\alpha) = \sum_{i=0}^{N} \pi_{\alpha^i \alpha^j}(\alpha) \cdot \alpha^i + \pi_{\alpha^j}(\alpha) \qquad j = 0, \cdots, N$$

or equivalently

$$\sum_{i=0}^{N} \pi_{\alpha^i \alpha^j}(\alpha) \cdot \alpha^i = 0 \qquad j = 0, \cdots, N \ \ldots (h)$$

By P' and the generalized Hotelling's lemma,[23] (h) is equivalent to

---

[23]By P' and Samuelson's envelope theorem,

$$\pi_{\alpha^0}(\alpha) = R(x^*) \text{ and } \pi_{\alpha^j}(\alpha) = c^j(x^{j^*}) \ (j \neq 0).$$

This can be called a generalized form of Hotelling's lemma.

$$\sum_{j=1}^{N} R_j \Big(\sum_{i=0}^{N} \frac{\partial x^{j*}(\alpha)}{\partial \alpha^i} \alpha^i \Big) = 0$$

$$c_j^j \Big(\sum_{i=0}^{N} \frac{\partial x^{i*}(\alpha)}{\partial \alpha^i} \alpha^i \Big) = 0 \qquad j = 1, \cdots, N .$$

which is already incorporated into our primal approach for a problem P'. For practical purposes, condition (i) exhausts the comparative static implications of linear homogeneity for $\pi(\alpha)$.[24]

Second, we can easily show that the comparative static implications of convexity of $\pi(\alpha)$ also are already incorporated into our primal approach. Convexity of $\pi(\alpha)$ is equivalent to the restriction

$$\pi_{\alpha\alpha} \equiv [\pi_{\alpha^i \alpha^j}] \quad \text{positive semi-definite} \qquad \ldots .(j)$$

where, by P' and the generalized Hotelling's lemma,

$$\pi_{\alpha^0 \alpha^j} = \sum_{i=1}^{N} R_i(x^*) \frac{\partial x^{i*}}{\partial \alpha^j} \qquad j = 0, \cdots, N$$

$$\ldots .(k)$$

$$\pi_{\alpha^i \alpha^j} = c_i^i (x^{i*}) \frac{\partial x^{i*}}{\partial \alpha^j} \qquad i = 1, \cdots, N \quad j = 0, \cdots, N$$

---

[24]Differentiating (i) with respect to α^K yields

$$\frac{\partial x^{j*}(\alpha)}{\partial \alpha^K} + \sum_{i=0}^{N} \frac{\partial^2 x^{j*}(\alpha)}{\partial \alpha^i \partial \alpha^K} \alpha^i = 0 \qquad i = 1, \cdots, N \quad K = 0, \cdots, N.$$

Unless we wish to incorporate relations between $\frac{\partial^2 x^*(\alpha)}{\partial \alpha^i \partial \alpha^j}$ and potentially observable data into our model, these conditions are irrelevant.

and $\pi_{\alpha^i \alpha^j} = \pi_{\alpha^j \alpha^i}$, $\alpha^0 R_i(x^*) = \alpha^i c^i(x^{i*})$. Since the relations between the variables included in (j)-(k) are defined exactly by our general primal approach, these implications of convexity of $\pi(\alpha)$ already are incorporated into our model.

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APPENDIX V

PARTIAL SOLUTIONS FOR THE MAJOR DIFFICULTIES WITH THE PROPOSED METHOD OF QUANTITATIVE COMPARATIVE STATICS

1. Introduction

The two major difficulties with the proposed method of quantitative comparative statics concern the identification of a global solution and the incorporation of a reasonable number of inputs and outputs into the model. Partial solutions for these overlapping problems are suggested here.[1] First, given an algorithm that is reasonably effective in finding local solutions for a quantitative comparative statics model, we can tentatively conclude that there are "relatively few" feasible values for $z(\frac{\partial x}{\partial \alpha^1})$ that are outside of the observed range. This observed range forms an (X-Y)% confidence-Bayes interval for $z(\frac{\partial x}{\partial \alpha^1})$ when the constraints

$$\rho^L \leqq \rho \leqq \rho^U$$

form an X% confidence-Bayes interval for the observable parameters ρ, and it becomes approximately an X% confidence-Bayes interval for $z(\frac{\partial x}{\partial \alpha^1})$ as the search for feasible solutions becomes sufficiently detailed. More precise estimates of confidence-Bayes intervals for observed ranges of feasible $z(\frac{\partial x}{\partial \alpha^1})$ depend largely upon the ability to approximate random sampling of the feasible set.

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[1]The content of this appendix was briefly alluded to in Section 3.4.4 of Chapter 3.

Second, computational difficulties increase exponentially with the number of inputs included explicitly in a quantitative comparative statics model; so procedures for aggregating such models within and across enterprises are presented here. These aggregation procedures generally lead to some error in characterizing the disaggregate model : correct aggregation of inputs within an enterprise depends on satisfaction of appropriate Leontief separability conditions or fixed factor proportions within the disaggregate enterprise, and correct aggregation across enterprises depends essentially on exogenous marginal factor costs for each enterprise. The aggregation procedures suggested here are shown to have certain optimum properties. In addition, aggregation errors can be crudely estimated by observing the effects of aggregation errors in small models.

2. Local versus Global Solutions for the Model

The feasible set for our quantitative comparative statics models

(a) maximize $z(\frac{\partial x}{\partial \alpha^1})$ (b) minimize $z(\frac{\partial x}{\partial \alpha^1})$

subject to $G(\frac{\partial x}{\partial \alpha^1}, \rho) = 0$ subject to $G(\frac{\partial x}{\partial \alpha^1}), \rho) = 0$ . . .(1)

$\rho^L \leqq \rho \leqq \rho^U$ $\rho^L \leqq \rho \leqq \rho^U$

is not convex due to the nonlinear (quadratic) equality constraints $G(\frac{\partial x}{\partial \alpha^1}, \rho) = 0$. Therefore there may not be a guaranteed procedure for obtaining a feasible solution to either of these problems, and many local solutions may not be global solutions to these problems. Moreover, these problems (1) apparently cannot be transformed into concave and convex problems (respectively).[2]

However, this inability to identify global solutions is not in itself a serious problem for our model: the stochastic nature of the constraints on ρ implies that the range of feasible z does not in general span the entire population of "true" values for $z(\frac{\partial x}{\partial \alpha^1})$. Thus the following procedures, if properly implemented, can lead to results that are almost as satisfactory as global solutions: (a) calculation of a large number of local solutions by means of an algorithm that is reasonably effective in finding feasible solutions, and (b) estimation of the confidence-Bayes level for the observed range of feasible values of $z(\frac{\partial x}{\partial \alpha^1})$. These problems in computational methods and statistical inference are considered briefly in the following two sections.

We shall see that (a) and to a lesser extent (b) can presumably be accomplished somewhat satisfactorily at present. First, many local solutions can probably be calculated at reasonable cost for models specifying only a few inputs.[3] Second, an upper bound can easily be

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[2]See a discussion of geometric programming and related methods (Avriel, 1976).

[3]As will be discussed in Section 3, large multi-input comparative static models can be aggregated into models with a small number of inputs.

placed on the confidence-Bayes level of the observed range of feasible $z(\frac{\partial x}{\partial \alpha^1})$, and it can be stated that the confidence-Bayes level approaches this upper bound as the number of observed feasible solutions increases. However, more accurate estimation of this level seems to require further development of procedures for approximating a random sample of the feasible set for (1).

2.1 Algorithms for Calculating Solutions to the Model

In attempting to solve an optimization problem that is subject to constraints, the original problem can either be re-cast as an unconstrained optimization problem (by incorporating constraints into the objective function as penalty functions)[4] or handled directly as a constrained problem. Examples of these two methods are the exact penalty function routine of Fletcher (1973a,b) and the generalized reduced-gradient algorithm of Abadie and Carpentier (1969), respectively. The generalized reduced-gradient method was found by Colville (1970) to be the most effective of the methods tested in handling nonlinear equality constraints, and the more recent exact penalty function approaches seem likely to be more effective than reduced gradient methods.[5]

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[4]For example, from the problem "maximize $g(x)$ subject to $h(x) = 0$" we can construct the unconstrained problem "maximize $g(x) - wh(x)^2$" where w is a positive constant. For an appropriate choice of w, these two problems have identical solution sets.

[5]See Avriel (1976), Chapter 12.4-6.

In order to obtain some idea of the ability of these methods to analyze optimization problems of the type formulated here, a generalized reduced-gradient algorithm[6] was applied to two and three input models of the firm.[7] The results indicated the following:

(a) as expected, the reduced-gradient algorithm was at times unable to locate a feasible solution and in general would not locate a global solution in a small number of runs, and

(b) nevertheless, a large sample of local solutions can be obtained for these simple problems by making a significantly larger number of runs with different starting points.[8]

2.2 Approximations to Confidence-Bayes Levels for the Observed Range of Feasible $z(\frac{\partial x}{\partial \alpha^1})$

The main points of this section can be summarized as

(a) the X% confidence-Bayes level of the constraints on ρ is an upper bound on the confidence-Bayes level for the observed range of feasible z,

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[6]See Wales (1977).

[7]These are similar to models that were presented in Section 3.5 of Chapter 3.

[8]These runs can differ in terms of either the specified starting value of endogenous variables and / or an auxiliary constraint $z(\frac{\partial x}{\partial \alpha^1}) \geq z(\frac{\partial x}{\partial \alpha^1})^* + \varepsilon$ (for a maximization problem 1a), where $z(\frac{\partial x}{\partial \alpha^1})^*$ is the largest of the feasible solutions previously observed ($\varepsilon > 0$).

(b) a Chebyshev lower bound on the confidence-Bayes level for the observed range of feasible $z(\frac{\partial x}{\partial \alpha^1})$ can be estimated from the sample mean and variance,

(c) this confidence-Bayes level can be estimated with considerable accuracy from a random sample of the feasible set for (1), and

(d) random sampling of the feasible set for (1) in a subspace of $\{(\rho, [\pi_{ij}], z)\}$ can often be very crudely approximated, but further research towards devising closer approximations is advisable.

Statements (a)-(d) can be elaborated upon as follows.

As the sample of local solutions for $z(\frac{\partial x}{\partial \alpha^1})$ increases, the confidence-Bayes level of the observed range for $z(\frac{\partial x}{\partial \alpha^1})$ approximates more closely the X% level of the constraints for problems (1).[9] Thus an upper bound of X% can be placed on the confidence-Bayes level of the observed range for $z(\frac{\partial x}{\partial \alpha^1})$, and this upper bound is approached more closely as the size of the sample of local solutions for problems (1a) and (1b) increases.

In addition, given either a random sample or a sufficiently large sample from the feasible set for (1), a likely lower bound on this confidence-Bayes level can be calculated from the Chebyshev inequality:Chebyshev Inequality. If X is a (univariate) random variable with mean U_X and

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[9]See Section 5 of Appendix 3.

standard deviation σ_x, then

$$P(|X-u_x| < k\sigma_x) \geq 1 - \frac{1}{k^2}$$

where

$P(|X-u_x| \geq k\sigma_x)$ can be restated as "the probability that a random observation of X will be within k standard deviations from the mean." This use of the Chebyshev inequality can be elaborated upon as follows. An unbiased and consistent estimator of the Chebyshev lower bound on the probability distribution of the observed range of $z(\frac{\partial x}{\partial \alpha^1})$ within the feasible set $\{(\rho,[\pi_{ij}],z)\}$ for (1) can easily be obtained from a random sample of this feasible set. In the case of non-random sampling, this estimator is consistent but biased.[10] Finally, the probability content of the true population of $(\rho,[\pi_{ij}],z)$ seems likely to be much more concentrated around its mean than is the uniformly distributed population of feasible $(\rho,[\pi_{ij}],z)$ that is implicit in (1); so it seems reasonable to assume that

$$\sigma_z^T - \sigma_z^M >> |u_z^T - u_z^M| \qquad \ldots.(2)$$

---

[10] Since $E(|X-\hat{u}_x| - k\hat{\sigma}_x) = |X-u_x| - k\sigma_x$, where E denotes the expectations operator and $\hat{u}_x$, $\hat{\sigma}_x$ denote the usual estimators of u_x and σ_x which are unbiased under random sampling, the corresponding estimator of the Chebyshev lower bound is unbiased given random sampling. Since a function of consistent estimators is also consistent, and the usual estimators of u_x and σ_x are consistent even for non-random sampling (Goldberger, 1964, pp. 118-19, 128-30 and 142-46), this estimator of the Chebyshev bound is always consistent.

where T and M designate statistics of the true and model populations, respectively. In sum, given either a random sample or a sufficiently large sample from the feasible set for (1), we can easily use the Chebyshev inequality to calculate a number that can be interpreted with considerable confidence as a lower bound on the confidence-Bayes level of the observed range of feasible $z(\frac{\partial x}{\partial \alpha^1})$.

However, in general we cannot determine a sample size yielding an interval of $z(\frac{\partial x}{\partial \alpha^1})$ with a confidence-Bayes level that is likely to approximate the upper bound of X%. Moreover, the Chebyshev lower bound may considerably underestimate the corresponding confidence-Bayes level, and this lower bound does not in general approach the true level as the sample size increases. Thus knowledge of the upper bound of X% and of Chebyshev lower bounds is unlikely to define the confidence-Bayes level of the observed range of feasible $z(\frac{\partial x}{\partial \alpha^1})$ with much precision.

On the other hand, considerably stronger results can be obtained from a random sample of the feasible set for problem (1). First, suppose that knowledge of only the ranks of observations of $z(\frac{\partial x}{\partial \alpha^1})$ within the sample is used in estimating confidence regions. Denote the cumulative probability distribution for z that is implicit in the model (1) as $F(x)^m$, and denote the "true" cumulative probability distribution as $F(z)^t$. Then the probability that the observation z^L in a random sample of n observations exceeds the smallest 95% of the model's probability content for z can be calculated simply as follows:

$$P(F(z^L)^m > .95^n) = 1 - .95^n$$

where $.95^n$ is the probability that all n observations are in the smallest 95% of the probability content. More generally,

$$
\begin{aligned}
&P(F(z^L)^m > q) = 1 - q^n \\
&P(F(z^S)^m < 1 - q) = 1 - q^n
\end{aligned}
\qquad \ldots\ldots(3)
$$

where z^S is the smallest observation in the sample. In addition,

$$P(F(z^L)^m > q\ , \quad F(z^S)^m < 1 - q) = 1 - 2q^n - 2n(1-q)q^{n-1} \qquad \ldots(4)$$

where $n(1-q)q^{n-1}$ is the probability that exactly n-1 observations are in the smallest (largest) 95% of the probability content.[11]

The true population of $(\rho, [\pi_{ij}], z)$ is more likely to be bunched about its mean than is the uniformly distributed population of $(\rho, [\pi_{ij}], z)$ implicit in the model; so Equations (3)-(4) provide estimates of lower bounds on the "true" confidence-Bayes levels for z^L and z^S. To be more precise,

$$
\begin{aligned}
&P(F(z^L)^t > q - \alpha) && \geqq (1-q^n) \\
&P(F(z^S)^t < 1 - q + \alpha) && \geqq (1-q^n) \\
&P(F(z^L)^t > q - \alpha,\ F(z^S)^t < 1 - q + \alpha) && \geqq 1 - 2q^n - 2n(1-q)q^{n-1}
\end{aligned}
\qquad \ldots(5)
$$

q is "large" (e.g., $.9 \leqq q < 1$)

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[11] For further discussion of the use of the binomial distribution in calculating confidence limits for ranks in a random sample, see Bradley (1968), pp. 186-91.

when the constraints $\rho^L \leqq \rho \leqq \rho^U$ define a $100(1-\alpha)\%$ confidence-Bayes interval.

Thus random sampling of the feasible set for (1) is the critical assumption in estimating (with any precision) confidence-Bayes levels for the observed range of feasible $z(\frac{\partial x}{\partial \alpha^1})$. Unfortunately, such random sampling cannot be done directly, and there does not appear to be any method of testing an algorithm for such properties unless the feasible set of the particular problem is known *a priori*. These two statements can be explained as follows. First, direct random sampling of any subinterval of elements $(\rho, [\pi_{ij}], z)$ of the feasible set for (I) would involve random sampling of the set $\{(\rho, [\pi_{ij}], z)\}$ enclosed in the corresponding subspace (of feasible and non-feasible points) and calculating the subset of feasible points in this sample. However, the number of feasible points usually will be an infinitesimal fraction of the elements in the subspace, i.e., a feasible point is unlikely to be found by such a non-directed search procedure. Second, random sampling of the population of $z(\frac{\partial x}{\partial \alpha^1})$ that is implied by the feasible set for a problem (1) consists specifically of

(a) independent draws of the feasible set, where

(b) the probability of a $z(\frac{\partial x}{\partial \alpha^1})$ being drawn such that $a \leqq z \leqq b$ is equal to the frequency of occurrence of this range of z in the feasible set.

Only condition (a) can be tested in the absence of knowledge of the feasible set for the particular problem (1); but it is property (b) of random sampling that is of direct interest in the analysis of a problem (1).

Nevertheless, we can at least devise a method for rejecting the hypothesis (b) and also a crude means of more closely approximating condition (b). First, randomly select starting points for the algorithm and test for condition (a). If an independent sample of starting points does not lead to an independent sample of feasible values for $z(\frac{\partial x}{\partial \alpha^1})$, then the algorithm is in some sense "biased" in its selection of elements from the feasible set. In this manner, condition (b) can be rejected along with condition (a).[12] On the other hand, random sampling of starting points plus independent draws from the feasible set is insufficient to establish condition (b). Condition (b) generally requires that the set of starting points and the set of feasible points be distributed over the subspace in roughly the same manner. However, several procedures can be suggested for improving upon a random sample of starting points. For example, feasible solutions to (1) would generally be obtained by calculating global solutions ($G^2 = 0$) to the problem

$$\text{minimize } G^2 \equiv \sum_{i=1}^{M} g^i\left(\frac{\partial x}{\partial \alpha^1}, \rho\right)^2 \qquad \ldots\ldots(6)$$

$$\text{subject to } \rho^L \leqq \rho \leqq \rho^U$$

and the number of non-global local solutions to (6) ($G^2 > 0$) that are obtained prior to a global solution should be loosely related to the frequency of feasible solutions to (1) in the vicinity of the starting

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[12]Methods of testing for independence in sampling of the feasible set, viz., means of rejecting condition (b), are discussed elsewhere. For a relatively powerful class of tests, see Blum *et al.* (1961). For simpler but less powerful tests, see Bradley (1968), pp. 73-76, 83-84, 87, 91-96.

point. Thus a more appropriate set of starting points may well be obtained by dividing the subspace into various regions and selecting the number of starting points within each region in accordance with the ease of obtaining feasible solutions from various randomly selected starting points.

3. Aggregation Procedures for Quantitative Comparative Static Models

The discussion in Section 2 suggests that there are not any serious computational problems with our quantitative comparative statics model if a reasonably large and approximately random sample of feasible solutions can be identified. However, the size of the model increases exponentially with the specified number of inputs and outputs, and current algorithms apparently cannot obtain feasible solutions at low cost for models with many nonlinear equalities.

In this section, it is shown that the size of comparative static models can often be reduced most effectively by the application of Leontief and Hicks-type aggregation procedures. We present a method for aggregating models (1) over inputs that has certain optimal properties and we suggest a means of crudely estimating the errors that arise from this method. In addition, it is shown that similar procedures can be applied to multiple enterprise models.

3.1 Aggregation of Inputs

Here we outline a procedure for using small models (1) in order to estimate comparative static effects for larger models of a single enterprise firm employing many inputs. This will be the optimal aggregation procedure whenever the firm's disaggregate objective function $\pi(x;\alpha)$ can be exactly or approximately described in separable form. Otherwise, errors in estimating comparative static effects from aggregate models may be significant. Nevertheless, a rough idea of the biases associated with such aggregation procedures presumably can be obtained by comparing comparative static results for disaggregate and aggregate small models.

The essential problems in aggregation of inputs for comparative static models (1) can be illustrated as follows. Suppose that our knowledge of the structure of a firm's static maximization problem and of related comparative static effects enables us to formulate the constraints

$$[\pi_{ij}^D]\,\frac{\partial x^D}{\partial \alpha^1} = \begin{pmatrix} c^1{}_1\alpha^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$[\pi_{ij}^D] \text{ negative definite} \qquad \ldots .(7)$$

$$A^D K^D = I, \quad \tilde{A}_{11}{}^D L_{11}{}^D = I, \ldots$$

$$\rho^{LD} \leq \rho \leq \rho^{UD}$$

for a large disaggregated model (1). If sampling of the feasible set for (7) is very expensive, then it may be wise to transform (7) into a more aggregated set of constraints

$$[\pi_{ij}^{A}] \frac{\partial x^{A}}{\partial \alpha^{1}} = \begin{pmatrix} c^{1}_{1\alpha^{1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$[\pi_{ij}^{A}] \text{ negative definite} \qquad \ldots.(8)$$

$$A^{A}K^{A} = I,\ \tilde{A}^{A}_{11}L^{A}_{11} = I,\ \ldots$$

$$\rho^{LA} \leqq \rho \leqq \rho^{UA}$$

and to calculate feasible solutions for (8) rather than for (7). From the structure of (7) and (8), it can be seen that aggregation of inputs implies

(a) aggregation of elements in

$$([\pi_{ij}^{D}],\ A^{D},\ \tilde{A}^{D}_{11},\ldots) \qquad \ldots.(9)$$

into matrices

$$([\pi_{ij}A],\ A^{A},\ \tilde{A}^{A}_{11},\ldots) \qquad \ldots.(10)$$

and

(b) aggregation of elements in

$$(K^D, L^D_{11}, \cdots, \rho^{LD}, \rho^{UD}) \qquad \ldots .(11)$$

into matrices and vectors

$$(K^A, L^A, \cdots, \rho^{LA}, \rho^{UA}) \ . \qquad \ldots .(12)$$

Thus the central problem in aggregating a comparative statics model (7) is to find methods for calculating (10) from (9) and (12) from (11) that lead to minimum error when comparative static results are obtained from the corresponding aggregate model (8) rather than from the "true" disaggregate model (7). Since possible "true" structures $\pi(x;\alpha)^D$ and aggregation procedures are both infinite in number, this is an impossible task.

However, the problem becomes manageable if we assume that a reasonably effective aggregation procedure exists, i.e., if we restrict our attention to the case of "true" structures $\pi\,(x;\alpha)^D$ that can be expressed approximately in separable form by an appropriate choice of aggretation procedure. In particular, suppose that we identify conditions under which (7) can be exactly or approximately described in a form (9), and suppose that these conditions define (10) from (9) and (12) from (11) in such a way that (10) and (12) are consistent.[13]

---

[13]Since K^A characterizes A^A just as K^D characterizes A^D, etc., the aggregation procedures (a) and (b) must define identical sets of aggregate inputs.

Then an aggregation procedure with the corresponding optimal property has been identified: these conditions minimize or approximately minimize errors in aggregating a given model (7) when there exists an aggregation procedure leading to zero errors or relatively small errors. Here we shall develop such a procedure.

In order to transform the disaggregated matrices (9) into (10), each of various subsets of inputs can be aggregated into a "conditional revenue" function, and these revenue functions plus any remaining disaggregate inputs can be treated as the set of inputs in (8).[14] It will be shown that this procedure is exactly or approximately correct when $\pi(x;\alpha)^D$ can be exactly or approximately specified in separable form.[15]

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[14]Alternatively, expenditure functions $\sum_i c^i(x^i;\alpha^i)$ could be used as aggregators. Neither approach appears to have a significant advantage over the other. Conditional revenue functions are employed here in order to facilitate discussion of the critical assumption in aggregation: separability of $R(x)$.

[15]If $R(x)$ is non-differentiable and exhibits fixed factor proportions within various subsets of inputs, then each of these subsets can be correctly treated as an aggregate input. The corresponding aggregator functions can be defined as "conditional revenue" functions in the manner discussed below, or each can be defined more simply as the activity level of any one input in the particular subset.

The matrices (9) directly describe properties of the firm's objective function $\pi(x;\alpha)$ in the neighbourhood of x^*, i.e., these matrices are combinations of elements of the Hessian matrix $[\pi_{ij}(x^*)]$ and of the associated marginal factor costs. Therefore (9) can be correctly transformed into (10) only if $\pi(x;\alpha)$ satisfies appropriate conditions for all x in a neighbourhood of x^*. These conditions can be developed as follows. First, it has already been noted that, given twice differentiability of R(x), there exist functions f and g such that

$$R(x) = f(g(x^1, \cdots, x^g), x^{g+1}, \cdots, x^N) \qquad \text{....(13)}$$

for all x in a neighbourhood of x^*

if and only if

$$\partial\left(\frac{R_i(x^*)}{R_j(x^*)}\right) / \partial x^k = 0 \qquad \text{....(14)}$$

for all $i,j \in \{1,\cdots,g\}$ and $k \in \{g+1,\cdots,N\}$

where

$$R_i(x^*) \neq 0 \qquad (i = 1,\cdots,N).$$ [16]

In addition, given twice differentiability of R(x), (13) will be closely approximated if and only if (14) is closely approximated.[17] Thus R(x) is approximately separable if and only if the appropriate Leontief

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[16] See Section 3.2.2 of Appendix 3.

[17] This statement is established, with minor qualifications, in Fisher (1969).

conditions are approximately satisfied. Moreover, this aggregator function can be defined as the following "conditional revenue" function

$$g \equiv R(x^1, \cdots, x^g; \; x^{g+1*}, \cdots, x^{N*}) \qquad \ldots .(15)$$

without loss of generality.[18] Thus, denoting $(x^1, \cdots, x^g) \equiv X^g$ and $(x^{g+1}, \cdots, x^N) \equiv X^h$, given twice differentiability of $\pi(x;\alpha)$, there exist functions f, g, $\hat{C}$ and $\hat{g}$ such that

$$\pi(x;\alpha) = f(g(X^g), X^h) - C(\hat{g}(X^g), X^h; \alpha) \qquad \ldots .(16)$$

for all x in a neighbourhood of x^* if and only if (14) is satisfied, where g satisfies (15) and $\hat{g} \equiv g$ without loss of generality.[19] Likewise, (16) is closely approximated if and only if (14) is closely approximated where g satisfies (15) and $g \equiv \hat{g}$.

Given these necessary and sufficient conditions (14) for correct or approximately correct aggregation of (9), we can derive the structure of the matrices (10) and the relation between maximization conditions in the two models. First, elements of (10) are related to (9) as follows:

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[18] See the proof of Theorem 1 in Leontief (1947).

[19] The statement "$\hat{g} \equiv g$ in a neighbourhood of x^* without loss of generality" can be justified as follows.

$$C \equiv \sum_{i=1}^{N} c^i(x^i;\alpha^i)$$

implies that there always exist functions $\hat{C}$ and $\hat{g}$ such that (a) $C = \hat{C}(\hat{g}(X^g), X^h; \alpha)$ for all x. Since $[\hat{C}_{ij}(g(X^{g*}), X^{h*}, \alpha)]$ can be constructed from (a), there always exists a $\hat{C}$ such that $\hat{g}$ can be defined as g in a neighbourhood of x^*.

$$\text{(a)}\quad \pi_{RR}^{A} = \pi_{ij}^{D} / c_i^{iD} \cdot c_j^{jD} \qquad \text{for all } i, j \in \{1, \cdots, g\}$$

$$\text{(b)}\quad \pi_{Rj}^{A} = \pi_{ij}^{D} / c_i^{iD} \qquad \text{for all } i \in \{1, \cdots, g\} \text{ and } j \in \{g+1, \cdots, N\}$$

$$= \pi_{jR}^{A} \qquad \ldots .(17)$$

$$\text{(c)}\quad \pi_{ij}^{A} = \pi_{ij}^{D} \qquad \text{for all } i, j \in \{g+1, \cdots, N\}$$

$$\text{(d)}\quad c_R^{RA} = 1$$

$$\text{(e)}\quad c_i^{iA} = c_i^{iD} \qquad \text{for all } i \in \{g+1, \cdots, N\}$$

where

$$R \equiv R(x^1, \cdots, x^g;\ x^{g+1*}, \cdots, x^{N*}).$$ [20,21]

If direct constraints on elements in $[\pi_{ij}^{D}]$ that correspond to the aggregated inputs $(1, \cdots, g)$ are excluded from the disaggregate model (7), then constraints of the form (17a-b) would have to be included directly in the

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[20] Relations (17 a-c) are derived in the proof of Proposition 2, and (17d) follows from $c_R^{RA} \equiv f_g$ in (10) and $f_g = 1$ given (13) and (15).

[21] Given (17), the aggregate relations

$$[\pi_{ij}^{A}] \frac{\partial x^A}{\partial \alpha^1} = \begin{pmatrix} c^1_{1\alpha^1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{are to be interpreted as, e.g.,}$$

$$\begin{pmatrix} \pi_{11} & \pi_{1R} \\ \pi_{1R} & \pi_{RR} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial \alpha^1} \\ \frac{\partial R}{\partial \alpha^1} \end{pmatrix} = \begin{pmatrix} c^1_{1\alpha^1} \\ 0 \end{pmatrix}, \quad \frac{\partial R}{\partial \alpha^1} \equiv \frac{\partial R(x^{2*}, \cdots, x^{N*};\ \overline{x^{1*}})}{\partial \alpha^1}$$

in the case where a single conditional revenue function $R = R(x^2, \cdots, x^N; \overline{x^{1*}})$ is defined.

aggregate model (8). On the other hand, in the absence of such constraints on $[\pi_{ij}^D]$, aggregate elements of the matrices (10) need not be defined directly in the model in terms of their subaggregates in (9). Several aggregators of the form $R(X^g;X^{h^*})$ probably should be employed in transforming (7) to (8). In general, the subaggregates X^g of these aggregators should not overlap.[22]

Second, including the restriction $[\pi_{ij}^A]$ negative definite in (8) rather than the corresponding disaggregate condition does not lead to any relaxation of the comparative static restrictions implied by the maximization hypothesis, provided that $[\pi_{ij}^D]$ also satisfies the appropriate separability conditions. This matter is stated more precisely in Proposition 2 (see the following page). Since local comparative statics is undefined for a Hessian that is negative semi-definite only,[23] this proposition implies that the restriction $[\pi_{ij}^A]$ negative definite is equivalent to the

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[22]Overlap in the sets of aggregated inputs X^{g1}, X^{g2}, places considerable restrictions on the structure of $[\pi_{ij}^A]$: if $R^1(X^{g1};X^{h1*})$, $R^2(X^{g2};X^{h2*})$, $R^3(X^{g3};X^{h3*})$ denote aggregators for an aggregate function $\pi(R^1,R^2,R^3,X^h;\alpha)^A$ where $X^h \equiv \overline{X^{g1} \cup X^{g2} \cup X^{g3}}$, then $X^{g1} \cap X^{g2} \neq$ null set $\Rightarrow \pi_{R^1R^1}{}^A = \pi_{R^2R^2}{}^A$, $\pi_{R^1j}{}^A = \pi_{R^2j}{}^A$ for all $j \notin X^{g1} \cup X^{g2}$, and $X^{g1} \cap X^{g3} \neq$ null set $\Rightarrow \pi_{R^1R^1}{}^A = \pi_{R^3R^3}{}^A$, $\pi_{R^1j}{}^A = \pi_{R^3j}{}^A$ for all $j \notin X^{g1} \cup X^{g3}$. If two disaggregate matrices $\tilde{A}_{11}{}^D$, $\tilde{A}_{22}{}^D$ treat inputs X^{g1}, X^{g2}, respectively, as fixed and $X^{g1} \subset X^{g2}$, then these matrices can be aggregated without imposing the above restrictions by employing the following aggregators: $R^1 \equiv R(X^{g1}; X^{h1*})$, $R^2 \equiv R(X^{g3}; X^{h3*})$ where $X^{g3} \equiv X^{g2} \cap \overline{X^{g1}}$.

[23]See Section 2.1 of Appendix 3.

implications of the maximization hypothesis for the disaggregate model in the case where $[\pi_{ij}{}^{D}]$ satisfies the intended Leontief conditions.

In sum, the above analysis has demonstrated the following: aggregate matrices $([\pi_{ij}{}^{A}], A^{A}, \widetilde{A}_{11}{}^{A},...)$(10) can be correctly defined or approximated (as above) —and $[\pi_{ij}{}^{A}]$ negative definite is equivalent to or approximates concavity of $\pi(x;\alpha)^{D}$ at x* —if and only if $\pi(x;\alpha)^{D}$ approximates Leontief conditions for separability. Next we shall show that correct aggregation of the matrices and vectors $(K^{D}, L_{11}{}^{D}, \cdots, \rho^{LD}, \rho^{UD})$ (11) is *always* possible, and we shall present such an aggregation procedure.

<u>Proposition 2</u>. Suppose that $\pi(x;\alpha)^{D} = \pi(g(X^{g}), X^{h};\alpha)^{A}$ in a neighbourhood of an interior maximum $x^*(\alpha)$ for some functions $\pi(g, X^{h};\alpha)^{A}$ and g, and that $\pi(x,\alpha)^{D}$ is twice differentiable. Then

(A) $[\pi_{ij}(x^*(\alpha);\alpha)^{D}]$ is negative semi-definite only, i.e.,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}[\pi_{ij}(x^*(\alpha);\alpha)^{D}]dx^{i}dx^{j} \leqq 0 \text{ for all } dx \text{ and } = 0 \text{ for a } dx \neq 0,$$[24] and

(B) $[\pi_{ij}(x^*(\alpha);\alpha)^{D}]$ negative semi-definite only

$\iff [\pi_{ij}(g(X^{g*}), X^{h*};\alpha)^{A}]$ negative semi-definite.

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[24]Statement A of the proposition implies that local comparative statics is undefined at x* for a disaggregate $\pi(x;\alpha)^{D}$ that exactly satisfies Leontief conditions for separability at x* (see Section 2.1 of Appendix 3).

Outline of Proof.

Differentiating (16) with respect to all N inputs under the conditions (14), $\hat{g} \equiv g$ and $C \equiv \sum_{i=1}^{N} c^i(x^i;\alpha^i)$ yields

$$\pi(x;\alpha)^D = f(R(X^g;X^{h*}),X^h) - C(R(X^g;X^{h*}),X^h;\alpha)$$

$$\pi_{ij}{}^D = (f_{RR} - C_{RR})\, R_i R_j + (f_R - C_R)\, R_{ij} \qquad i,j \,\varepsilon\{1,\cdots,g\}$$

$$= \pi_{RR}^A\, c_i^i\, c_j^j \quad \text{since} \quad R_i = c_i^i \quad \text{and} \quad f_R = C_R \text{ at } x^*$$

$$\pi_{ij}{}^D = (f_R - C_{Rj})\, R_i \qquad i\,\varepsilon\{1,\cdots,g\},\ j\,\varepsilon\{g+1,\cdots,N\}$$

$$= \pi_{Rj}^A\, c_i^i$$

$$\pi_{ij}{}^D = \pi_{ij}{}^A \qquad i,j\,\varepsilon\{g+1,\cdots,N\}$$

which in turn yields (17a-c). For simplicity let N = 3 and (without loss of generality) define $g \equiv R(x^2,x^3;x^{1*})$. Then

(a) $S^D \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} \pi_{ij}{}^D v_i v_j \leqq 0$ for all (v_1, v_2, v_3)

at an interior solution x^*. By (17a-b),

(b) $S^D \equiv \pi_{11}{}^A v_1^2 + 2\pi_{1R}{}^A v_1(c_2^2 v_2 + c_3^3 v_3) + \pi_{RR}{}^A (c_2^2 v_2 + c_3^3 v_3)$

at x^*.

Thus $S^D = 0$ for some $v \equiv (0, v_2, v_3) \neq 0$, i.e. $[\pi_{ij}(x^*)^D]$ is negative semi-definite only (Part A of Proposition 2). By (a)-(b) and $(u_1, u_2) \equiv (v_1, c_2^2 v_2 + c_3^3 v_3)$,

$$\pi_{11}{}^A u_1^2 + 2\pi_{1R}{}^A u_1 u_2 + \pi_{RR}{}^A u_2^2 \leqq 0 \quad \text{for all } (u_1, u_2) \text{ at } x^*$$

which is equivalent to $[\pi_{ij}{}^A]$ negative semi-definite at x^*. Thus $[\pi_{ij}(x^*)^D]$ negative semi-definite only <=> $[\pi_{ij}(x^*)^A]$ negative semi-definite (Part B of Proposition 2).□

Since the aggregate matrices $(K^A, L_{11}{}^A, \cdots)$ are to be used in specifying restrictions on $(A^A, \tilde{A}_{11}, \cdots)$ in the same manner as $(K^D, L_{11}{}^D, \cdots)$ specifies restrictions on $(A^D, \tilde{A}_{11}{}^D, \cdots)$, the same aggregator functions must be employed in defining $(K^A, L_{11}{}^A, \cdots)$ as in defining $([\pi_{ij}{}^A], A^A, \tilde{A}_{11}{}^A, \cdots)$. Thus (11) must be aggregated into (12) in terms of the conditional revenue functions employed in aggregating (9). However, for the purposes of aggregation there is one significant difference between (9) and (11): the former matrices directly concern the properties of $R(x)$ and $c^i(x^i; \alpha^i)(i = 1, \cdots, N)$ for all x in the immediate neighbourhood of $x^*(\alpha_0)$, whereas, the latter matrices directly concern only changes in equilibrium x^* in the immediate neighbourhood of $x^*(\alpha_0)$.

The comparative static effects specified in (11) can always be correctly aggregated in a manner that is consistent with (10), i.e., irrespective of any special conditions (such as separability) on the disaggregate structure $\underline{\pi}(x; \alpha)^D$. This statement follows from the Hicks-type aggregation theorems summarized in Proposition 3 (on following page):

Proposition 3 shows that $\pi(x;\alpha)^D$ can always be aggregated in terms of a conditional revenue function $R(X^g;X^{h^*}(\alpha_0)$ over equilibria in the "immediate" neighborhood of $x^*(\alpha_0)$ provided that the factor cost schedules $c^i(x^i;\alpha^i)(i = 1,\cdots,g)$ do not shift or shift equiproportionally. Thus $(K^D, L^D_{11}, \cdots, \rho^{LD}, \rho^{UD})$ (11) can be aggregated without error by weighting the corresponding comparative static effects such that factor cost schedules within any sub-aggregate shift equiproportionally.

<u>Proposition 3.</u> Suppose that $\alpha \equiv (\alpha^1,\cdots,\alpha^g,\alpha^{g+1}(\beta),\cdots,\alpha^N(\beta))$,

$c^i \equiv \alpha^i \hat{c}^i(x^i)$ $(i = g+1,\cdots,h)$, and

$\frac{\partial\alpha^i}{\partial\beta} / \alpha^i = \frac{\partial\alpha^j}{\partial\beta} / \alpha^j$ $(i,j = g+1,\cdots,h)$. Define

$\alpha^G \equiv (\alpha^1,\cdots,\alpha^g)$,

$$\pi(x;\alpha^G,\beta) \equiv R(x) - \sum_{i=1}^{g} c^i(x^i;\alpha^i) - \sum_{i=g+1}^{h} \alpha^i(\beta)\hat{c}^i(x^i) - \sum_{i=h+1}^{N} c^i(x^i;\alpha^i(\beta)).$$

Denote the maximum of $\pi(x;\alpha^G,\beta)$ as $X^* = x^*(\alpha)$,

$\alpha_0 \equiv (\alpha_0^1,\cdots,\alpha_0^N(\beta_0))$, $X^g \equiv (x^1,\cdots,x^g)$, $X^h \equiv (x^{g+1},\cdots,x^h)$,

$X^n \equiv (x^{h+1},\cdots,x^N)$. Then

(A) $\pi(x^*(\alpha_0^G,\beta);\alpha_0^G,\beta)$ has the asymptotic distribution

$f(R^g,R^h,X^{n^*}(\alpha_0^G,\beta);\alpha_0^G,\beta)$ in β for $\beta \to \beta_0$ and α^G fixed at α_0^G

where

$$R^g \equiv R(X^{g*}(\alpha^G,\beta);\ X^{h*}(\alpha_0),\ X^{n*}(\alpha_0))$$

$$R^h \equiv R(X^{h*}(\alpha^G,\beta);\ X^{g*}(\alpha_0),\ X^{n*}(\alpha_0)),\ \text{and}$$

(B) $\pi(x^*(\alpha^G,\beta_0);\ \alpha^G,\beta_0)$ has the asymptotic distribution $\hat{f}(R^h,R^n,X^{g*}(\alpha^G,\beta_0);\alpha^G,\beta_0)$ in $\alpha^G \to \alpha_0^G$ and β fixed at β_0

where

$$R^n \equiv R(X^{n*}(\alpha^G,\beta),\ X^{g*}(\alpha_0),\ X^{h*}(\alpha_0)).^{25}$$

<u>Outline of Proof</u>.

Part A. The cost minimizing x subject to fixed levels of output and various inputs is unaffected by equiproportional shifts in the factor supply schedules for all variable inputs. Thus the set $x^*(\alpha)$ for any $\beta(\alpha^G$ fixed at $\alpha_0^G)$ is exactly determined by knowledge of $R(x^*(\alpha)$, $X^{N*}(\alpha)$ and of the schedules $R(x)$, $c^i(x^i;\alpha_0^i)$ $(i = 1,\cdots,N)$ provided that $\Delta\beta$ leads to equiproportional shifts in supply schedules for inputs $(1,\cdots,h)$. Thus

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[25]Statements A and B generalize in an obvious manner to the cases where $\{x^1,\cdots,x^g\}$ and $\{x^{g+1},\cdots,x^h\}$ are partitioned into subsets so as to define multiple aggregators $(R^{g1},R^{g2},\cdots,R^{h1},R^{h2})$: $f \equiv f(R^{g1},R^{g2},\cdots,X^{N*}(\alpha_0^G,\beta),\ \alpha_0^G,\beta)$ in A and $\hat{f} \equiv \hat{f}(R^{h1},R^{h2},\cdots,X^{g*}(\alpha^G,\beta_0),\ X^{N*}(\alpha^G,\beta_0);\alpha^G,\beta_0)$ in B.

(a) $\pi(x^*(\alpha)) = \hat{\pi}(R(x^*(\alpha)), X^{N*}(\alpha); \alpha)$ for some $\hat{\pi}$ over such $\Delta\beta$.[26]

Since $\frac{\partial R}{\partial \beta} = \sum_{i=1}^{N} R_i \frac{\partial x^i}{\partial \beta}$, in the limit the effects of ΔX^G, ΔX^h and ΔX^N on $R(x)$ can be separated from each other. Thus (a) can be disaggregated to yield Statement A of the proposition.

Part B. Statement B can be established in a similar manner. □

The manner of aggregating (11) can be illustrated in terms of the matrix K^A corresponding to an aggregate matrix A^A of the simple form

$$\begin{pmatrix} \pi_{11} & \pi_{1R^C} & c_1^1 \\ \pi_{1R^C} & \pi_{R^C R^C} & 1 \\ c_1^1 & 1 & 0 \end{pmatrix} \quad \ldots\ldots(18)$$

where $R^C \equiv R(x^2, \cdots, x^N : x^{1*}(\alpha_0))$. Given that $[\pi_{ij}^A]$ is negative definite as well as symmetric, K^A exists and is symmetric. Moreover, the form of (18) implies that

[26]See Pollak (1969) for a similar treatment of the Hicks Aggregation Theorem.

$$K^A \equiv \begin{bmatrix} \dfrac{\partial x^{1**}}{\partial \alpha^1} \Big/ c^1_{1\alpha^1} & \dfrac{\partial x^{1**}}{\partial \beta} \Big/ MFC^R_\beta & \dfrac{\partial x^{1**}}{\partial \bar{R}} \\ \dfrac{\partial R^{c**}}{\partial \alpha^1} \Big/ c^1_{1\alpha^1} & \dfrac{\partial R^{c**}}{\partial \beta} \Big/ MFC^R_\beta & \dfrac{\partial R^{c**}}{\partial \bar{R}} \\ \dfrac{\partial x^{1**}}{\partial \bar{R}} & \dfrac{\partial R^{c**}}{\partial \bar{R}} & \dfrac{\partial \lambda}{\partial \bar{R}} \end{bmatrix} \qquad \ldots\ldots(19)$$

where

$$x^{**} \equiv x^{**}(\alpha, \bar{R}) \quad \text{and}$$

$$\text{(a)} \quad \frac{\partial R^{c**}}{\partial \alpha^1} \equiv \sum_{i=2}^{N} c^i_i \frac{\partial x^{i**}}{\partial \alpha^1}$$

$$\text{(b)} \quad \frac{\partial R^{c**}}{\partial \bar{R}} \equiv \sum_{i=2}^{N} c^i_i \frac{\partial x^{i**}}{\partial \bar{R}} \qquad \ldots\ldots(20)$$

$$\text{(c)}\quad \frac{\partial x^{1**}}{\partial \beta} \Big/ MFC^{R}_{\beta} \;\equiv\; \sum_{i=2}^{N} c^{i}_{i} \frac{\partial x^{1**}}{\partial \alpha^{i}} \Big/ c^{i}_{i\alpha^{i}}$$

$$\text{(d)}\quad \frac{\partial R^{c**}}{\partial \beta} \Big/ MFC^{R}_{\beta} \;\equiv\; \sum_{i=2}^{N} \sum_{j=2}^{N} c^{i}_{i}\, c^{j}_{j} \frac{\partial x^{j**}}{\partial \alpha^{i}} \Big/ c^{i}_{i\alpha^{i}} \,.^{27} \qquad \ldots .(20)$$

The constraints in $\rho^{LA} \leqq \rho \leqq \rho^{UA}$ pertaining to elements of K^{A} are constructed from disaggregate constraints on elements of K^{D} in the manner implied by (19)-(20). For example, if

$$\rho^{L}_{i} \;\leqq\; c^{i}_{i} \;\leqq\; \rho^{U}_{i} \qquad i = 1,\cdots,N$$

$$\rho^{L}_{S+i} \;\leqq\; \frac{\partial x^{i**}}{\partial \alpha^{1}} \;\leqq\; \rho^{U}_{S+i} \qquad i = 1,\cdots,N$$

in the disaggregate model, then

$$\sum_{i=2}^{N} \rho^{L}_{i}\, \rho^{L}_{S+i} \;\leqq\; \frac{\partial R^{c**}}{\partial \alpha^{1}} \;\leqq\; \sum_{i=2}^{N} \rho^{U}_{S+i}\, \rho^{U}_{S+i}$$

constrains element K^{A}_{12} in the aggregate model. Note that $d\alpha^{1}$ and $d\bar{R}$ do not influence the factor supply schedules $c^{i}(x^{i};\alpha^{i})(i = 2,\cdots,N)$ and that we arbitrarily assumed

$$c^{i}_{i\alpha^{i}} \frac{\partial \alpha^{i}}{\partial \beta} \Big/ c^{i}_{i} \;=\; c^{j}_{j\alpha^{j}} \frac{\partial \alpha^{j}}{\partial \beta} \Big/ c^{j}_{j} \qquad i,j = 1,\cdots,N \qquad \ldots .(21)$$

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[27]Equations (20c-d) can be derived as follows. Since the total cost of the aggregate input $R^{c}(x^{A};x^{B})$ is $\sum_{i\varepsilon A} c^{i}(x^{i};\alpha^{i})$, MFC^{R}(marginal

in deriving equations (20c-d). Thus, by Proposition 3, defining K^A by (19)-(20) does not lead to any errors in aggregation under any circumstances.

Finally, it should be re-emphasized that special conditions on the structure $\pi(x;\alpha)^D$ are required for correct aggregation of $([\pi_{ij}{}^D], A^D, \tilde{A}_{11}{}^D \cdot\cdot)$ allbeit not for correct aggregation of $(K^D, L_{11}{}^D, \cdot\cdot)$. To the extent that appropriate Leontief conditions are not implied by the structure of the disaggregate constraints (7) and factor proportions are not fixed, aggregation of inputs will unduly restrict the confidence-Bayes intervals for comparative static effects. For example, suppose that A^D is incorrectly aggregated into the form (8). Since we have shown that the set of feasible comparative static effects

$$\left(\frac{\partial x^{1*}}{\partial \alpha^1}, \frac{\partial R^{c*}}{\partial \alpha^1} \equiv \sum_{i=2}^{N} c_i^i \frac{\partial x^{i*}}{\partial \alpha^1}\right)$$

are identical in the aggregate and disaggregate models when the disaggregate model satisfies appropriate Leontief conditions, adding the implied Leontief conditions to the disaggregate model is equivalent to the incorrect aggregation procedure. By the assumption of incorrect aggregation, these Leontief restrictions would not be redundant additions to the disaggregate model. Thus the feasible set of comparative static effects

27(continued)
factor cost of R^c) is $\sum_{i \varepsilon A} c_i^i \frac{\partial x^{i**}}{\partial \bar{R}}(\alpha, \bar{R}, x^B) = 1$ and

$$MFC^R_\beta \equiv \sum_{i \varepsilon A} c^i_{i\alpha^i} \frac{\partial \alpha^i}{\partial \beta} \frac{\partial x^{i**}}{\partial \bar{R}} = S \cdot MFC^R \quad \text{where}$$

(a) $c^i_{i\alpha^i} \frac{\partial \alpha^i}{\partial \beta} / c_i^i = c^j_{j\alpha^j} \frac{\partial \alpha^j}{\partial \beta} / c_j^j \equiv s$ for all $i, j \varepsilon A$.

$$\left(\frac{\partial x^{1*}}{\partial \alpha^1}, \frac{\partial R^{c*}}{\partial \alpha^1}\right)$$

for the aggregate constraints (8) is more restricted than is implied by the disaggregate constraints (7). These aggregation errors, albeit less than the errors obtained by any other procedure for aggregating these inputs when appropriate Leontief conditions or fixed factor proportions are reasonably approximated, may be significant: adding the implied Leontief conditions to the disaggregate model may significantly restrict the feasible set. Unfortunately, it seems difficult to estimate directly the degree of approximation to Leontief conditions for separability.[28]

However, useful estimates of such aggregation biases for large models presumably can be obtained from judicious use of small models. For example, the definition

$$\frac{\partial R^{c*}}{\partial \alpha^1} = \sum_{i=2}^{N} c_i^i \frac{\partial x^{i*}}{\partial \alpha^1}$$

can be added to a small disaggregate model (7), and a smaller aggregate model (8) can be defined for the aggregator function $R^c \equiv R^c(x^2,\cdots,x^N;x^{1*})$.

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[27] (concluded)
Thus (b) $MFC_\beta^R = s$. Statement (20c) follows from

$\frac{\partial x^{j*}(\alpha(\beta))}{\partial \beta} = \sum_{i \varepsilon A} \frac{\partial x^{j**}}{\partial \alpha^i} \frac{\partial \alpha^i}{\partial \beta}$ for all $j \varepsilon B$ and from (a)-(b). Statement (20d) follows from $\frac{\partial R^c(x^{A*}(\alpha(\beta));x^B)}{\partial \beta} = \sum_{i \varepsilon A} \sum_{j \varepsilon A} c_i^i \frac{\partial x^{i**}}{\partial \alpha^j} \frac{\partial \alpha^j}{\partial \beta}$ and from (a)-(b).

[28]See Denny and Fuss (1977).

The difference between the range of feasible

$$\left(\frac{\partial x^{1*}}{\partial \alpha^1}, \frac{\partial R^{c*}}{\partial \alpha^1}\right)$$

for these two models defines the bias on the X% confidence-Bayes interval for

$$\left(\frac{\partial x^{1*}}{\partial \alpha^1}, \frac{\partial R^{c*}}{\partial \alpha^1}\right)$$

that is implicit in the aggregation of inputs 2,...,N into a single input. By applying such procedures to carefully selected small models, one can presumably obtain crude estimates of the biases in comparative static effects that are implicit in our procedure for aggregating a particular large model (7) down to a model with a small number of inputs.[29,30]

3.2 Aggregation of Enterprises

Thus far our quantitative comparative statics method has been presented in the context of a single enterprise firm (with one output). However, many firms consist of multiple enterprises, and comparative static changes in various outputs may be of interest. For example, many users of community pasture in British Columbia have various beef, hay

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[29]The extent to which separability conditions are consistent with a particular model will vary greatly with the model. Thus the aggregation bias observed for one model or a series of models can be used only with great caution as a guide to the effects of aggregating a different model.

[30]Note that the aggregation bias cannot be estimated by imposing Leontief restrictions directly on disaggregate models (the modelled Hessian $[\pi_{ij}^{D}]$ would not be negative definite in this case). Moreover, imposing approximations to Leontief conditions on a disaggregate model would underestimate the bias in aggregation: aggregation of (9) to (10) implicitly assumes exact separability.

and grain enterprises, and the estimation of comparative static changes in the number of beef pounds sold is an important part of evaluating community pasture programs.

The procedures presented in the last section for aggregating inputs within an enterprise can be generalized to the multi-enterprise firm with only minor modifications. For this reason, these methods will not be detailed here. However, there is one major difference between aggregating inputs within an enterprise and aggregating over enterprises: whereas, correct aggregation within an enterprise typically depends on Leontief conditions for separability that are not easily observed or tested, separation of multi-enterprise models into single-enterprise models depends essentially on exogenous product and factor prices.

The static objective function for a firm employing N inputs over J enterprises can be written as

$$\pi(x;\alpha)^D \equiv R(x) - C(x;\alpha) \qquad \ldots .(22)$$

where

$$x \equiv (x^{11},\cdots,x^{N1},x^{12},\cdots,x^{NJ})$$

and

$$C(x;\alpha) \equiv \sum_{i=1}^{N} c^i(\sum_{j=1}^{N} x^{ij};\alpha^i) . \qquad \ldots .(23)$$

Note that, if the prices of inputs $(1,\cdots,m)$ are exogenous to the firm, then C is weakly separable in inputs $(x^{11},\cdots,x^{m1},x^{12},\cdots,m^{mk})$ for any enterprises $(1,\cdots,k)$.

Suppose that

$$R(x) \equiv \sum_{j=1}^{J} R^j(x^{1j},\cdots,x^{Nj}) \qquad \ldots\ldots(24)$$

i.e., that enterprises do not supply other enterprises in the firm. This implies that R is weakly separable in inputs $(x^{1j},\cdots,x^{Nj})$ for any enterprise j. Thus, given (24), fixed prices for inputs $(1,\cdots,m)$ and weak separability of R^j in $(x^{1j},\cdots,x^{mj})$ for $j = (1,\cdots,k)$, $\pi(x;\alpha)^D$ is weakly separable with respect to inputs $(1,\cdots,m)$ over enterprises $(1,\cdots,k)$, i.e., with respect to inputs $(x^{1j},\cdots,x^{mj})$ for $j = (1,\cdots,k)$. Then the corresponding aggregator functions can be defined in the "immediate" neighbourhood of x* as conditional revenue functions

$$R^j(x^{1j},\cdots,x^{mj};\ x^{m+1,j*},\cdots,x^{Nj*}) \qquad (j = 1,\cdots,k)\ .$$

Thus the aggregation procedures of the previous section can be used to simplify the multi-enterprise comparative static model.

Moreover, suppose that the supply price of each input $(1,\cdots,N)$ is exogenous to the firm and R(x) satisfies (24). Then the multi-enterprise model can be correctly reduced to J single enterprise comparative static models.

On the other hand, one or more enterprises may supply other enterprises in a multi-enterprise firm. Consider the following model:

$$
\begin{aligned}
R(x) &\equiv R^1(y^{1A}) + R^2(y^2) + R^3(y^{3A}) \\
y^{1A} + y^{1B} &= y^1(X^1,\ y^{3B}) \\
y^2 &= y^2(X^2, y^{1B}, y^{3c}) \\
y^{3A} + y^{3B} + y^{3c} &= y^3(X^3)
\end{aligned}
\qquad \ldots . (25)
$$

where

y^{1A} $\equiv$ calf pounds sold

y^{1B} $\equiv$ calf pounds transferred to yearling enterprise

y^2 $\equiv$ yearling pounds sold

y^{3A} $\equiv$ tons of hay sold

y^{3B} $\equiv$ tons of hay transferred to cow-calf enterprise

y^{3c} $\equiv$ tons of hay transferred to yearling enterprise

X^j $\equiv$ vector of levels of inputs $(1, \cdots, N)$ employed in enterprise j.

The constraints for the corresponding quantitative comparative statics model can be outlined as follows:

$$
\text{(a)} \quad \underset{3N \times 3N}{[\pi_{ij}{}^D]} \ \underset{3N \times 1}{\frac{\partial x}{\partial \alpha^1}} = \underset{3N \times 1}{b}
$$

where

$$\pi_{x^{ij}x^{k\ell}}{}^D \equiv 0 \quad \text{for} \quad i \neq k,\ j \neq \ell$$

$$b^i \equiv c^1_{1\alpha^1} \quad \text{for} \quad i = 1,\ \ 1 + J,\ 1 + 2J$$

$$b^i \equiv 0 \quad \text{otherwise}$$

(b) $R^{1}_{1A} = R^{2}_{2A} \cdot y^{2A}_{1B}$

$R^{3}_{3A} = R^{1}_{1A} \cdot y^{1A}_{3B} = R^{2}_{2A} \cdot y^{2A}_{3c}$

(c) total differential of (b) with respect to α^1

(d) $[\pi_{ij}^{D}]$ negative definite

$$\text{(e)} \quad \begin{bmatrix} \underset{(N\times N)}{y^{1}_{ij}} & \underset{(N\times 1)}{y^{1}_{i,3B}} & \underset{(N\times 1)}{y^{1}_{i}} \\ \underset{(1\times N)}{y^{1}_{i,3B}} & \underset{(1\times 1)}{y^{1}_{3B,3B}} & \underset{(1\times 1)}{y^{1}_{3B}} \\ \underset{(1\times N)}{y^{1}_{i}} & \underset{(1\times 1)}{y^{1}_{3B}} & \underset{(1\times 1)}{0} \end{bmatrix} \cdot [K] = I, \cdots \qquad \ldots\ldots(26)$$

Constraints (26) can be reduced by procedures suggested above for the multi-enterprise structure (24). In particular, if all enterprises supplying other enterprises also sell some of their product at an exogenously-determined price, then the multi-enterprise model (25) is formally equivalent to (24). In other words, if the opportunity cost of employing inputs in other enterprises is exogenous to the firm, then a firm with structure (25) in effect solves a problem with structure (24).

APPENDIX VI

Structure of Static Linear Programming Models for Representative Beef Ranches at Peace River Community Pastures

Page

1 Model Structure - Peace River Income Assurance Model

The "standard case" version of the Peace River Income Assurance model (a short run equilibrium model circa 1975) will be discussed here equation by equation. A flow diagram of the mdoel is presented in Figure 1, and the matrix is outlined in Figure 2. In the text, discussion will be followed by presentation of the corresponding row(s) from Figure 2, by definition of any activities in that row not defined previously,[1] and by derivation of any matrix coefficients that are not basic data assumptions for the model.

1.1 Land Constraints and Activities

Own improved land can be allocated to pasture for grazing, or to production (plus establishment) of hay, oats or barley.[2]

1) IMLAND10:
   1(OWNPAS10) + 1(OWNHAY10) + 1(OWNBAR10) + 1(OWNOAT10) ≤ 350 acres
   OWNPAS10 - acres of own improved land allocated to grazing for the year
   OWNHAY10 - acres of own improved land allocated to hay for the year.
   OWNBAR10 - acres of own improved land allocated to barley for the year
   OWNOAT10 - acres of own imrpoved land allocated to oats for the year

The best improved land gives the highest yield of oats and barley.

2) BESIMP10:
   1(BESOAT10) + 1(BESBAR10) ≤ 150 acres
   BESBAR10 - acres of "best" own improved land allocated to barley for the year
   BESOAT10 - acres of "best" own improved land allocated to oats for the year.

---

[1]The one exception to this statement is that activities appearing solely in the section that calculates labour used in rounding up cattle from rented range and pasture before the end of the grazing season (labour constraints 18-37 are not explicitly defined here. All activities are defined in Section 3.

[2]Later equations (land constraint 19, hay and grain constraints 1, 2 and 3) specify equilibrium distributions of acres in production and acres in establishment or summerfallow. This is accruate in the long run but not in the short run. In particular, all home pasture that is reallocated to hay (due to, e.g. increased access to community pasture) can immediately enter production, i.e. establishment is not immediately necessary on any of the new hay land. Since a dollar today is preferred to a dollar tomorrow (due to possibilities for investment and to a positive marginal rate of time preference for consumption), this error tends to create a (presumably slight) overestimation of the benefits of community pasture.

	Own Farm Land	Rented Hay and Grain Land	Cattle on Farm: Opening Stock	Cattle on Farm: Added Cattle	Cattle on Farm: Closing Stock	AUM's Grazing: Own Land	AUM's Grazing: Rented Land	Hay and Grain: Fed to Cattle	Hay and Grain: Acres Harvested	Hay and Grain: Sales	Hay and Grain: Purchases	Labor Use: Custom	Labor Use: Roundup	Labor Use: Surplus	Labor Hired	Roundup Transfer Activities	Cattle Sales: Total	Cattle Sales: Subsidized	Income	Right Hand Side*
I. Land	+ -					+	+													b_1 b_3
II. Cattle			+ +	+	- -															
III. Feeding			+	+		-	-	-												
IV. Hay and Grain	-	-						+	+ -	+	-									
V. Labor	+	+	+	+		+ + -	+ + -		+			+	+ -	+	-	+ -				b_9
VI. Income Assurance																	-	+ +		b_{12}
VII. Income	+	+	+	+			+			-	+	-			+		-	-	+	
VIII. Objective Function ("MAXIMIZE")														+					+	

*0 unless indicated otherwise

Figure 7. Outline of Matrix for the Peace River Income Assurance Farm Model[1]

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[1] For details of this matrix, see Section 3.

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Table 1. Feeding and Labor Constraints

A. FEEDING CONSTRAINTS

Feeding Period	Nov. 1 – May 10	May 10 – June 1	June 1 – Sept. 1	Sept. 1 – Sept. 15	Sept. 15 – Oct. 7	Oct. 7 – Nov. 1
Feeding Period No.	1.	2.	3.	4.	5.	6.
Manner of Feeding	Dry fed	Dry fed	Grazed on pasture or range	Grazed on pasture or range	Grazed on pasture, range, hay aftermath or zero-grazed	Grazed on hay aftermath or zero-grazed

B. LABOR CONSTRAINTS

Labour Period	Nov. 1 – Apr. 7	Apr. 7 – Apr. 21	Apr. 21 – May 10	May 10 – June 1	June 1 – July 1	July 1 – Aug. 1	Aug. 1 – Sept. 1	Sept. 1 – Sept. 15	Sept. 15 – Oct. 7	Oct. 7 – Nov. 1
Labour Period No.	1.	2.	1.	3.	4.	5.	6.	7.	8.	9.
Labour Supply (excluding hired labour) hrs./wk.	75	150	75	85	85	120	120	85	85	85
Labour Use	Cattle feeding and management, off-farm or custom work	Cattle feeding and management, off-farm or custom work, calving	Cattle feeding and management, off-farm or custom work	Cattle feeding and management	Cattle feeding and management, hay and grain culture	Cattle feeding and management, hay harvest	Cattle feeding and management, hay harvest	Cattle feeding and management, hay and grain harvest, roundup	Cattle feeding and management, hay and grain harvest, roundup	Cattle feeding and management

Acres of "best" own improved land in oats and in barley cannot exceed total acres of own improved land in oats and in barley.

3) BESOAT10:
$1(\text{BESOAT10}) - 1(\text{OWNOAT10}) \leq 0$ acres

4) BESBAR10:
$1(\text{BESBAR10}) - 1(\text{OWNBAR10}) \leq 0$ acres

In the third and fourth feeding periods: own pasture can be grazed either by cows or yearlings, and grazing can be substituted between the two periods.

5) OWNPAS10:
$1(\text{OPASTC13}) + 1(\text{OPASTY13}) + 1(\text{OPASTC14}) + 1(\text{OPASTY14}) - 1.15(\text{OWNPAS10}) \leq 0$ AUM's
OPASTC13 - AUM's of own pasture grazed by cows in feeding period 3
OPASTY13 - AUM's of own pasture grazed by yearlings in feeding period 3
OPASTC14 - AUM's of own pasture grazed by cows in feeding period 4
OPASTY14 - AUM's of own pasture grazed by yearlings in feeding period 4

note: $$1.15 = \frac{4\text{ aum/3 acres tame}}{120\text{ day season}} \times \left(\frac{.75(120\text{ day season})}{\text{third feeding period}} + \frac{.125(120\text{ day season})}{\text{fourth feeding period}}\right)^{1}$$

In the fifth feeding period: own pasture can be grazed by cows and yearlings, and grazing capacity cannot be substituted to (or from) other periods.

6) OWNPAS15:
$1(\text{OPASTC15}) + 1(\text{OPASTY15}) - .25(\text{OWNPAS10}) \leq 0$ AUM's
OPASTC15 - AUM's of own pasture grazed by cows in feeding period 5
OPASTY15 - AUM's of own pasture grazed by yearlings in feeding period 5

note: $$.25 = \frac{4\text{ aum/3 acres tame}}{120\text{ day season}} \times \frac{.1875(120\text{ day season})}{\text{fifth feeding period}}$$

Own unimproved land can be used only as range.

7) UNLAND10:
$1(\text{OWNRAN10}) \leq 150$ acres
OWNRAN10 - acres of own unimproved land allocated to grazing for the year

In the third and fourth feeding periods: own range can be grazed by cows or yearlings, and grazing can be substituted between the two periods.

---

[1]All derivations of coefficients in the model are presented as transformations of basic data. For example, the data in this case are (1) 3 acres of pasture (tame) are required to graze one cow (plus calf) or one yearling over 120 days, (2) the third feeding period consists of 90 days, and (3) the fourth feeding period consists of 15 days.

8) OWNRAN10:
$1(ORANGC13) + 1(ORANGC14) + 1(ORANGY13) + 1(ORANGY14) - .29(OWNRAN10) \leq 0$ AUM's

ORANGC13 - AUM's of own range grazed by cows during feeding period 3
ORANGY13 - AUM's of own range grazed by yearlings during feeding period 3
ORANGC14 - AUM's of own range grazed by cows during feeding period 4
ORANGY14 - AUM's of own range grazed by yearlings during feeding period 4

note: $$.29 = \frac{\text{4 aum/12 acres native}}{\text{120 day season}} \times \left(\frac{\text{.75(120 day season)}}{\text{third feeding period}} + \frac{\text{.125(120 day season)}}{\text{fourth feeding period}}\right)$$

In the fifth feeding period: own range can be grazed by cows and yearlings, and grazing capacity cannot be substituted to (or from) other periods.

9) OWNRAN14:
$1(ORANGC15) + 1(ORANGY15) - .0625(OWNRAN10) \leq 0$ AUM's

ORANGC15 - AUM's of own range grazed by cows during feeding period 5
ORANGY15 - AUM's of own range grazed by yearlings during feeding period 5

note: $$.0625 = \frac{\text{4 aum/12 acres tame}}{\text{120 day season}} \times \frac{\text{.1875(120 day season)}}{\text{fifth feeding period}}$$

There are upper limits to the quantities of range that can be rented in feeding periods three through five, but there are no constraints on the manner in which native range is allocated between cows and yearlings.

10) RENRAN13:
$1(RENRAN13) \leq 300$ AUM's

RENRAN13 - AUM's of range rented for feeding period 3

11) RENRAN14:
$1(RENRAN14) \leq 50$ AUM's

RENRAN14 - AUM's of range rented for feeding period 4

12) RENRAN15:
$1(RENRAN15) < 75$ AUM's
RENRAN15 - AUM's of range rented for feeding period 5

13) RENRAD13:
$1(RRANGC13) + 1(RRANGY13) - 1(RENRAN13) \leq 0$ AUM's

RRANGC13 - AUM's of rented range grazed by cows during feeding period 3
RRANGY13 - AUM's of rented range grazed by yearlings during feeding period 4

14) RENRAD14:
$1(RRANGC14) + 1(RRANGY14) - 1(RENRAN14) \leq 0$ AUM's

RRANGC14 - AUM's of rented range grazed by cows during feeding period 4
RRANGY14 - AUM's of rented range grazed by yearlings during feeding period 4

15) RENRAD15:
$1(RRANGC15) + 1(RRANGY15) - 1(RENRAN15) \leq 0$ AUM's

RRANGC15 - AUM's of rented range grazed by cows during feeding period 5
RRANGY15 - AUM's of rented range grazed by yearlings during feeding period 5

There are upper limits to the quantities of community pasture that can be used in feeding periods three through five, but there are no constraints on the manner in which community pasture is allocated between cows and yearlings.

16) RENPAS13:
$1(RPASTC13) + 1(RPASTY13) \leq 180$ AUM's

RPASTC13 - AUM's of rented pasture grazed by cows during feeding period 3
RPASTY13 - AUM's of rented pasture grazed by yearlings during feeding period 3

17) RENPAS14:
$1(RPASTC14) + 1(RPASTY14) \leq 30$ AUM's

RPASTC14 - AUM's of rented pasture grazed by cows during feeding period 4
RRASTY14 - AUM's of rented pasture grazed by yearlings during feeding period 4

18) RENPAS15:
$1(RPASTC15) + 1(RPASTY15) \leq 45$ AUM's

RPASTC15 - AUM's of rented pasture grazed by cows during feeding period 5
RPASTY15 - AUM's of rented pasture grazed by yearlings during feeding period 5

Own and rented acres in hay production provide hay aftermath for grazing by cows and yearlings in feeding periods five and six.

19) HAYAFT10:
$1(HAYAFC15) + 1(HAYAFY15) + 1(HAYAFC16) + 1(HAYAFY16) - .75(OWNHAY10) - 1(RENHAY10) \leq 0$ acres

HAYAFC15 - acres of hay aftermath grazed by cows during feeding period 5
HAYAFY15 - acres of hay aftermath grazed by yearlings during feeding period 5
HAYAFC16 - acres of hay aftermath grazed by cows during feeding period 6
HAYAFY16 - acres of hay aftermath grazed by yearlings during feeding period 6
RENHAY10 - acres of rented land producing hay during the year

note: $.75 = \dfrac{\text{3 acres in hay production}}{\text{4 acres in hay production plus hay establishment}}$

Up to 50 acres in production can be rented for each of hay, oats and barley.

20) RENHAY10:
$1(RENHAY10) \leq 50$ acres

21) RENOAT10:
$1(RENOAT10) \leq 50$ acres

RENOAT10 - acres of rented land producing oats during the year

22) RENBAR10:
1(RENBAR10) $\leq$ 50 acres

RENBAR10 - acres of rented land producing barley during the year

1.2 Cattle Constraints and Activities

For "standard case" short run and long run models, the size of the cow herd, the disposition of calves, and the number of calves purchased for backgrounding are assumed to be the same in the model year as in the previous year and immediately following year. The effects of such a static "equilibrium" on activities for the modelled year are simulated by the following aspects of the model:
(1) only one cow numbers activity (COWSRHEF) is defined, and this activity must be serviced during each feeding and labor period of the model year (see feeding and labor constraints);
(2) the proportion of calves born on-farm that are designated as replacement heifers is fixed at a level that would maintain the cow herd over time, for the assumed levels of culling and mortality (see cattle constraint 3); and
(3) the stock of calves, born on-farm or purchased, for the yearling enterprises at the end of the model year must equal the stock of calves for the yearling enterprises at the beginning of the model year (see cattle constraints 4 and 5).
Notice that none of these constraints (nor cattle constraint 1) fixes the levels or ratios of (a) calves sold at the end of the year, (b) calves held over for sale as yearlings in the following year, and (c) calves purchased for sale as yearlings towards the end of the year. The levels and ratios of these activities are endogenous to all programming models.

For "standard case" short run models used in the simulation of farm behavior circa 1975, an additional constraint is placed on cow-calf and cow-yearling enterprises: a lower bound is placed on the cow numbers activity (see cattle constraint 1). When binding, this constraint implies a disequilibrium between farm demand and supply of cows, so that farm supply of cows in the model year exceeds farm demand in the model year. In this case it may be most appropriate to permit the closing stock of cows to be somewhat less than the opening stock for the model year (as is occasionally done).

The rationale for including such a constraint (cattle constraint 1) in a short run model is as follows. Markets for mature cows (except culls) appear to be sparse (perhaps because productive capacities of cows are quite variable, and potential sellers have more information about their cows than do potential buyers). This implies that, in the short run, the cow herd cannot be decreased as efficiently as it can be increased; retention rates for heifer calves can be varied more profitably than can death rates and culling rates for cows. For this reason, a lower bound is placed on the number of cows in this short run model. The "standard case" bound of 40 cows approximates the average number of cows on farms in the sample of community pasture users for the region circa 1975.[1,2]

---

[1]Since errors in predicting cattle prices and imperfections in markets for mature cows presumably will occur in the future, a lower bound on the cow numbers activity is also included in some models used in simulation of future behavior.

[2]Since cows have a productive life of approximately 8 years, and cow numbers cannot be changed substantially from one year to the next except at considerable cost, the levels of this capital stock over the modelled year presumably will depend on anticipated prices for cattle in future years as well as in the modelled year. Thus the opening and closing cow numbers for the modelled year may be in large part exogenously determined for a one year model and may not be equal. Changes in cow numbers within the year, and related effects, are accommodated in various "non-standard case" (disequilibrium) short run models.

The "standard case" short run (circa 1975) cattle constraints are as follows. The number of cows plus "old replacement heifers" (to calve in year 2) must be greater than or equal to 40.

1) COWSRHEF:
   1(COWSRHEF) $\geq$ 40 cows plus old replacement heifers

   COWSRHEF - number of cows plus "old replacement heifers" (to calve in the next year)

Just before feeding period 5 (Sept. 15) or just after feeding period 6 (Nov. 1), yearlings are sold.

2) OLDYER15:
   - .98 (YEAR0011) + 1(YERSAL15) + 1(YERSAL21) $\leq$ 0 yearlings

   YEAR0011 - number of yearlings at the beginning of the year (to be sold towards end of the year)
   YERSAL15 - number of yearlings sold just before feeding period 5
   YERSAL21 - number of yearlings sold just after feeding period 6

note: .98 = 1 - .02

where .02 = mortality rate for yearlings (mortality is assumed to occur on Sept. 15).

Just after feeding period six (Nov. 1): calves born in the spring are either sold, held over to be sold as yearlings in year 2, or held over as "new replacement heifers" (to calve in year three). For the "standard case" presented here, the ratio of the number of calves designated as new replacement heifers to the number of cows plus "old replacement heifers" (to calve in year 2) is exogenously determined so as to maintain the size of the cow herd over year 1.

3) CAFDIS21:
   - .64(COWSRHEF) + 1.08(CAFSAL21) + 1.08(YEROWN21) $\leq$ 0 calves plus yearlings
   CAFSAL21 - number of calves (born in the spring of the year) sold after feeding period 6
   YEROWN21 - number of calves (born in the spring of the year) held over for sale as yearlings towards the end of the following year

note: $$.64 = \left[\left(\frac{\#\text{ calves}}{\text{cow}}\right) - \left(\frac{\#\text{new repl. heif.}}{\text{cow}}\right)\right] \left(\frac{\#\text{ cows}}{\#\text{ cows + old repl. heif.}}\right)$$

$$= \left[(1-.15-.02) - (.1+.08(.1))\right] \quad (1-.1-.08(.1)) = .712(.892)$$

where .15 = proportion of cows, that calved prior to year one, without calf in year one
.02 = mortality rate for cows
.10 = culling rate for cows
.08 = mortality rate for calves (including "old replacement heifers")

The number of yearlings at the end of year one must equal the number of yearlings at the beginning of year one.

4) YEAR0021:
$1(\text{YEAR0011}) - 1(\text{YEAR0021}) \leq 0$ yearlings

YEAR0021 - number of yearlings at the beginning of the next year

Yearlings at the end of year one consist of calves raised on the farm and held over for sale in year two and calves purchased at the end of the year.

5) NEWYER21:
$- 1(\text{YEROWN21}) - 1(\text{YERPUR21}) + 1(\text{YEAR0021}) \leq 0$ yearlings

YERPUR21 - number of calves purchased at the end of the year (for sale as yearlings towards the end of the next year)

1.3 Feeding Constraints and Activities

In the first feeding period (Nov. 1 to May 10), hay is fed to cows and replacement heifers (2.25 tons per animal over 190 days) and yearlings (1.5 tons per yearling over 190 days).

1) HAYFED11:
$2.25(\text{COWSRHEF}) + 1.5(\text{YEAR0011}) - 1(\text{HAYFED11}) \leq 0$ tons

HAYFED11 - tons of hay fed to cows and yearlings during feeding period 1

In the first feeding period, barley is fed to yearlings (7 bushels per yearling over 190 days).

2) BARFED11:
$7(\text{YEAR0011}) - 1(\text{BARFED11}) \leq 0$ bushels

BARFED11 - bushels of barley fed to yearlings during feeding period 1

In the second feeding period (May 10 to June 1) hay is fed to cows and replacement heifers (.24 tons per animal over 20 days) and yearlings (.16 tons per yearlings over 20 days).

3) HAYFED12:
$.24(\text{COWSRHEF}) + .16(\text{YEAR0011}) - 1(\text{HAYFED12}) \leq 0$ tons

HAYFED12 - tons of hay fed to cows and yearlings during feeding period 2.

In the third feeding period (June 1 to Sept. 1), each cow and yearling requires 3 AUM's of grazing from own range or pasture, rented range or community pasture.

4) COWFED13:
$3(\text{COWSRHEF}) - 1(\text{ORANGC13}) - 1(\text{OPASTC13}) - 1(\text{RRANGC13}) - 1(\text{RPASTC13}) \leq 0$ AUM's

5) YERFED13:
$3(\text{YEAR0011}) - 1(\text{ORANGY13}) - 1(\text{OPASTY13}) - 1(\text{RRANGY13}) - 1(\text{RPASTY13}) \leq 0$ AUM's

In the fourth feeding period (Sept. 1 to Sept. 15), each cow and yearling requires .5 AUM's of grazing from own range or pasture, rented range or community pasture.

6) COWFED14:
$.5(COWSRHEF) - 1(ORANGC14) - 1(OPASTC14) - 1(RRANGC14) - 1(RPASTC14) \leq 0$ AUM's

7) YERFED14:
$.5(YEAR0011) - 1(ORANGY14) - 1(OPASTY14) - 1(RRANGY14) - 1(RPASTY14) \leq 0$ AUM's

In the fifth feeding period (Sept. 15 to Oct. 7), each cow requires .75 AUM's of grazing from own range or pasture, rented range or community pasture or hay aftermath.

8) COWFED15:
$.75(COWSRHEF) - 1(ORANGC15) - 1(OPASTC15) - 1(RRANGC15) - 1(RPASTC15) - .63(HAYAFC15) \leq 0$ AUM's

note: $.63 = \frac{1.25 \text{ tons hay}}{\text{acre in hay production}} \times \frac{1 \text{ ton aftermath}}{6 \text{ tons hay}} \times \frac{1 \text{ AUM}}{.33 \text{ tons aftermath}}$

where it is assumed that .33 tons of hay aftermath provides the equivalent of 1 AUM.

In the fifth feeding period, each yearling requires .75 AUM's of grazing from own range or pasture, rented range or community pasture, hay aftermath, or must be zero-grazed.

9) YERFED15:
$.75(YERSAL21) - 1(ORANGY15) - 1(OPASTY15) - 1(RRANGY15) - 1(RPASTY15) - .63(HAYAFY15) - .75(YEARZG15) \leq 0$ AUM's

YEARZG15 - number of yearlings zero-grazed during feeding period 5

In the fifth feeding period, each yearling also requires 1.375 bushels of barley.

10) BARFED15:
$1.375(YERSAL21) - 1(BARFED15) \leq 0$ bushels

BARFED15 - bushels of barley fed to yearlings during feeding period 5

In the fifth feeding period, each yearling zero-grazed also requires .17 tons of hay.

11) HAYFED15:
$.17(YEARZG15) - 1(HAYFED15) \leq 0$ tons

HAYFED15 - tons of hay fed to yearlings during feeding period 5

In the sixth feeding period (Oct. 7 to Nov. 1), each cow requires .75 AUM's and each (weaned) calf requires .375 AUM's of grazing on hay aftermath.

12) COWFED16:
1.01(COWSRHEF) + .375(CAFSAL21) + .375(YEROWN21) - .63(HAYAFC16) $\leq$ 0 AUM's

note: $$1.01 = \frac{.75\text{ AUM}}{\text{cow \& old rep.hfr.}} + \frac{.375\text{ AUM}}{\text{calf surviving through year 1}}$$

$$\left(\frac{\text{\# calves surviving through year 1}}{\text{calf born}}\right)\left(\frac{\text{\# calves born}}{\text{cow}}\right)$$

$$\left(\frac{\text{\# cows}}{\text{\# cows \& old rep.hfr}}\right)$$

$$= .75 + .375(1-.08)(1-.15-.02)(1-.1-.08(.1))$$
(see cattle constraint 3)

In the sixth feeding period, each yearling requires .75 AUM's of grazing on hay aftermath or zero-grazing, and requires 1.375 bushels of barley. Each yearling zero-grazed requires .17 tons of hay.

13) YERFED16:
.75(YERSAL21) - .63(HAYAFY16) - .75(YEARZG16) $\leq$ 0 AUM's

YEARZG16 - number of yearlings zero-grazed during feeding period 6

14) BARFED16:
1.375(YERSAL21) - 1(BARFED16) $\leq$ 0 bushels

BARFED16 - bushels of barley fed to yearlings during feeding period 6

15) HAYFED16:
.17(YEARZG16) - 1(HAYFED16) $\leq$ 0 tons

HAYFED16 - tons of hay fed to yearlings during feeding period 6

1.4 Hay and Grain Constraints and Activities

Acres in hay production can be harvested in labor periods five, six and seven (July 1 to Sept. 15), and acres in grain production can be harvested in labor periods seven and eight (Sept. 1 to Oct. 7).

1) HAYHAR10:
- .75(OWNHAY10) - 1(RENHAY10) + 1(HAYHAR15) + 1(HAYHAR16) + 1(HAYHAR17) $\leq$ 0 acres

HAYHAR15 - acres of hay harvested during labor period 5
HAYHAR16 - acres of hay harvested during labor period 6
HAYHAR17 - acres of hay harvested during labor period 7

2) OATHAR10:
- .75(OWNOAT10) - 1(RENOAT10) + 1(OATHAR17) + 1(OATHAR18) $\leq$ 0 acres

OATHAR17 - acres of oats harvested during labor period 7
OATHAR18 - acres of oats harvested during labor period 8

3) BARHAR10:
- .75(OWNBAR10) - 1(RENBAR10) + 1(BARHAR17) + 1(BARHAR18) $\leq$ 0 acres

BARHAR17 - acres of barley harvested during labor period 7
BARHAR18 - acres of barley harvested during labor period 8

note: $.75 = \frac{\text{3 acres in hay or grain production}}{\text{4 acres in hay or grain production plus establishment}}$

Yields of hay and grain vary with the period of harvest and (in the case of grain) with the quality of land. Hay yields are, e.g., 1.25, 1.00 and 0.75 tons per acre (in production) in labor periods five, six and seven, respectively. Oat yields on "average" quality own improved land are 30 and 25 bushels per acre (in production) in labor periods seven and eight, respectively. Oat yields are 10 bushels per acre higher on "best" own improved land and rented land. Barley yields on "average" quality own improved land are 23 and 20 bushels per acre (in production) in labor periods seven and eight, respectively. Barley yields are 7 bushels per acre higher on "best" own improved land and rented land. Hay and grain can be purchased as well as produced, and hay and grain supplies are either sold or (in the case of hay and barley) fed to cattle.

4) HAYDO010:
- 1.25(HARHAR15) - 1.00(HAYHAR16) - .75(HAYHAR17) - 1(HAYPUR10)
+ 1(HAYSAL10) + 1(HAYFED11) + 1(HAYFED12) + 1(HAYFED15) + 1(HAYFED16)
$\leq$ 0 tons

HAYPUR10 - tons of hay purchased during the year
HAYSAL10 - tons of hay sold during the year

5) OATDO010:
- 30(OATHAR17) - 25(OATHAR18) - 10(BESOAT10) - 10(RENOAT10) + 1(OATSAL10)
$\leq$ 0 bushels

OATSAL10 - bushels of oats sold during the year

6) BARDO010:
- 23(BARHAR17) - 20(BARHAR18) - 7(BESBAR10) - 7(RENBAR10) - 1(BARPUR10)
+ 1(BARSAL10) + 1(BARFED11) + 1(BARFED15) + 1(BARFED16) $\leq$ 0 bushels

BARPUR10 - bushels of barley purchased during the year
BARSAL10 - bushels of barley sold during the year

I.5 Labor Constraints and Activities

Labor constraints within the various labor periods are specified on a weekly basis. Labor coefficients and right hand sides in general are derived directly from the data on labor requirements for cattle and crops and the data on labor supplies (presented in Section 4). Derivations that may not be obvious are presented in this section. In particular, the equations that calculate cattle roundup and sorting labor after feeding periods three and four are explained in detail here.

In the first labor period (Nov. 1 to April 7, and April 21 to May 10: 25 weeks), feeding and supervision of cows and yearlings requires a fixed amount of labor (15 hours per week), plus .2 hours per week per cow and per "old replacement heifer" and .15 hours per week per yearling. The "variable" supply of family labor (75 total hours per week minus 15 hours per week of fixed labor) can be allocated to feeding and supervision of cattle, custom or off-farm labor, or leisure ("surplus" labor), and can be supplemented by hired labor.[1]

1) LABR0011:
   .2(COWSRHEF) + .15(YEAR0011) + 1(CUSLAB11) + 1(SURLAB11) - 1(HIRLAB11) $\leq$ 60 hours per week

   CUSLAB11 - hrs./wk. of custom or off-farm work during labor period 1
   SURLAB11 - hrs./wk. of surplus labor ("leisure") during labor period 1
   HIRLAB11 - hrs./wk. of hired labor during labor period 1

In the second labor period (April 7 to April 21: 2 weeks), calving of cows also must be supervised (increasing the variable component of cow labor requirements by 1.3 hours per week per cow), and the total quantity of family labor per week available is twice as high as in period one (150 hours per week in period two).

2) LABR0012:
   1.5(COWRHEF) + .15(YEAR0011) + 1(CUSLAB12) + 1(SURLAB12) - 1(HIRLAB12) $\leq$ 135 hours per week

   CUSLAB12 - hrs./wk. of custom or off-farm work during labor period 2
   SURLAB12 - hrs./wk. of surplus labor ("leisure") during labor period 2
   HIRLAB12 - hrs./wk. of hired labor during labor period 2

Custom or off-farm labor by the farm family cannot exceed 30 hours per week (and is available only in winter).

3) CUSLAB11:
   1(CUSLAB11) $\leq$ 30 hours per week

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[1]Cattle labor requirements have been decomposed into fixed and variable components in order to define the marginal labor requirements as less than the average labor requirements of cattle (and to define the marginal requirements as approaching the average requirements as the herd size becomes quite large). However, the specification of a fixed component (which is subtracted from the right hand side) unfortunately implies that the model in effect underestimates the labor saving that would occur in the absence of cattle. This does not impart a bias in favor of cattle to the model if the labor constraint is not binding; but in this case a distinction between marginal and average labor requirements is not worth considering. This "tradeoff" has been accepted here due to, in effect, the difficulty in modelling increasing returns to scale within a linear programming framework.

4) CUSLAB12:
$1(CUSLAB12) \leq 30$ hours per week

In the third labor period (May 10 to June 1: 2 weeks), labor requirements are identical to those in the first period, and the quantity of family labor available is 10 hours per week higher than in period one.

5) LABR0013:
.2(COWSRHEF) + .15(YEAR0011) + 1(SURLAB13) - 1(HIRLAB13) ≤ 70 hours per week.

SURLAB13 - hrs./wk. of surplus labor ('leisure') during labor period 3
HIRLAB13 - hrs./wk. of hired labor during labor period 3

In the fourth labor period (June 1 to July 1: 4.5 weeks), cows and yearlings must be inspected on range and pasture (.1 hours per week per cow or yearling), except on community pasture (where inspection is provided by the pasture rider), and land in hay and grain must be cultured.

6) LABR0014:
.03(ORANGC13) + .03(ORANGY13) + .03(OPASTC13) + .03(OPASTY13)
+ .03(RENRAN13) + .11(OWNHAY10) + .11(RENHAY10) + .18(OWNBAR10)
+ .18(RENBAR10) + .18(OWNOAT10) + .18(RENOAT10) + 1(SURLAB14)
- 1(HIRLAB14) ≤ 85 hours per week

SURLAB14 - hrs./wk. of surplus labor ('leisure') during labor period 4
HIRLAB14 - hrs./wk. of hired labor during labor period 4

note: $.03 = \frac{\text{.1 hours per week}}{\text{cow or yearling}} \times \frac{\text{1 cow or yearling}}{\text{3 AUM's grazing in feeding period 3}}$

$.11 = \frac{\text{.5 hours per acre of hay cultured}}{\text{4.5 weeks in labor period 4}}$

$.18 = \frac{\text{.8 hours per acre of grain cultured}}{\text{4.5 weeks in labor period 4}}$

In the fifth labor period (July 1 to August 1: 4.5 weeks) and in the sixth labor period (August 1 to September 1: 4.5 weeks), cows and yearlings must be inspected on range and pasture, except on community pasture, and hay land may be harvested.

7) LABR0015:
.03(ORANGC13) + .03(ORANGY13) + .03(OPASTC13) + .03(OPASTY13)
+ .03(RENRAN13) + .53(HAYHAR15) + 1(SURLAB15) + 1(SURLABH5) - 1(HIRLAB15)
- 1(HIRLABH5) ≤ 120 hours per week.[1]

[1] In labor periods five through eight: hired labor is disaggregated into labor hired for harvesting (e.g. activity HIRLABH5) and labor hired for other purposes (e.g. activity HIRLAB15), and surplus labor is disaggregated into surplus for harvest constraints (e.g. activity SURLABH5) and other surplus (e.g. activity SURLAB15). Each of these activities represents the average weekly figure (hrs./wk.) for the entire labor period (e.g. July 1 to Aug. 1).

SURLAB15 - hrs./wk. of own labor that is surplus ('leisure') during non-harvesting in labor period 5
SURLABH5 - hrs./wk. of own labor that is surplus ('leisure') during harvesting in labor period 5
HIRLAB15 - hrs./wk. of labor hired for non-harvesting activities during labor period 5
HIRLABH5 - hrs./wk. of labor hired for harvesting activities during labor period 5

note: $.53 = \frac{\text{3 labor days}}{\text{15 acres hay harvested, etc.}} \times \frac{\text{12 hours}}{\text{labor day}}$

$\times \frac{\text{labor period 4}}{\text{4.5 weeks}}$

8) LABR0016:
.03(ORANGC13) + .03(ORANGY13) + .03(OPASTC13) + .03(OPASTY13) + .03(RENRAN13) + .53(HAYHAR16) + 1(SURLAB16) + 1(SURLABH6) - 1(HIRLAB16) - 1(HIRLABH6) $\leq$ 120 hours per week.

SURLAB16 - hrs./wk. of own labor that is surplus ('leisure') during non-harvesting in labor period 6
SURLABH6 - hrs./wk. of own labor that is surplus ('leisure') during harvesting in labor period 6
HIRLAB16 - hrs./wk. of labor hired for non-harvesting activities during labor period 6
HIRLABH6 - hrs./wk. of labor hired for harvesting activities during labor period 6

In the seventh labor period (September 1 to September 15: 2 weeks), cows and yearlings must be inspected on range and pasture (except on community pasture), cows and yearlings may be rounded up (and sorted) from rented range and community pasture,[1] and hay and grain land may be harvested.

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[1]Labor requirements for roundup (and, or course, sorting) are considerably less for cattle on own range or pasture than on rented range or pasture, and time of roundup presumably is more flexible in the former case (hence less likely to compete with harvesting for labor). For these reasons, there seemed to be no point in adding the 24 rows and 32 additional activities needed to calculate roundup requirements from own range or pasture at the end of feeding periods three and four (see the discussion, later in this section, of activities calculating roundup labor from rented range or pasture at the end of these periods); so roundup requirements from own range and pasture are specified only for roundup at the end of feeding period five.

9) LABR0017:
$.2(ORANGC14) + .2(ORANGY14) + .2(OPASTC14) + .2(OPASTY14) + .2(RENRAN14) + 1(LABRRC17) + 1(LABRRY17) + 1(LABRPC17) + 1(LABRPY17) + 1.2(HAYHAR17) + .44(BARHAR17) + .44(OATHAR17) + 1(SURLAB17) + 1(SURLABH7) - 1(HIRLAB17) - 1(HIRLABH7) \leq 85$ hours per week.

SURLAB17 - hrs./wk. of own labor that is surplus ('leisure') during non-harvesting in labor period 7
SURLABH7 - hrs./wk.of own labor that is surplus ('leisure') during harvesting in labor period 7
HIRLAB17 - hrs./wk. of labor hired for non-harvesting activities during labor period 7
HIRLABH7 - hrs./wk. of labor hired for harvesting activities during labor period 7

note: $.2 = \frac{\text{.1 hour per week}}{\text{cow or yearling}} \times \frac{\text{1 cow or yearling}}{\text{.5 AUM's grazing in feeding period 4}}$

$$1.2 = \frac{\text{3 labor days}}{\text{15 acres hay harvested, etc.}} \times \frac{\text{12 hours}}{\text{labour day}} \times \frac{\text{labor period 7}}{\text{2 weeks}}$$

$$.44 = \text{.88 harvest hours per acre grain} \times \frac{\text{labor period 7}}{\text{2 weeks}}$$

In the eighth labor period (September 15 to October 7: 3 weeks), cows and yearlings must be inspected on range and pasture (except on community pasture) and on hay aftermath, yearlings zero-grazed must be inspected, cows and yearlings on community pasture at September 15 may be rounded up, cows and yearlings on rented range at September 15 and on community pasture at October 7 must be rounded up, and grain may be harvested.

10) LABR0018:
$.17(ORANGC15) + .17(ORANGY15) + .15(OPASTC15) + .15(OPASTY15) + .39(RRANGC15) + .43(RRANGY15) + .11(RPASTC15) + .08(RPASTY15) + .1(YEARZG15) + 1(LABRRC18) + 1(LABRRY18) + 1(LABRPC18) + 1(LABRPY18) + .30(BARHAR18) + .30(OATHAR18) + 1(SURLAB18) + 1(SURLABH8) - 1(HIRLAB18) - 1(HIRLABH8) \leq 85$ hours per week.

SURLAB18 - hrs./wk. of own labor that is surplus ('leisure') during non-harvesting in labor period 8
SURLABH8 - hrs./wk. of own labor that is surplus ('leisure') during harvesting in labor period 8
HIRLAB18 - nrs./wk. of labor hired for non-harvesting activities during labor period 8
HIRLABH8 - hrs./wk. of labor hired for harvesting activities during labor period 8

note: $.17 = \left(\frac{\text{.1 hour inspections/week}}{\text{cow or yearling}} + \frac{\text{.083 hours round-up}}{\text{cow or yearling from own range}} \times \frac{\text{labor period 8}}{\text{3 weeks}}\right) \times \frac{\text{1 cow or yearling}}{\text{.75 AUM's grazing in feeding period 5}}$

$$.15 = (.1 + \frac{.03 \text{ hours round-up}}{\text{cow or yearling from own pasture}} \times \frac{1}{3})(\frac{1}{.75})$$

$$.39 = (.1 + \frac{.58 \text{ hours round-up}}{\text{cow from rented range}} \times \frac{1}{3})(\frac{1}{.75})$$

$$.43 = (.1 + \frac{.67 \text{ hours round-up}}{\text{yearling from rented range}} \times \frac{1}{3})(\frac{1}{.75})$$

$$.11 = \frac{.25 \text{ hours round-up}}{\text{cow from community pasture}}(\frac{1}{3})(\frac{1}{.75})$$

$$.08 = \frac{.17 \text{ hours round-up}}{\text{yearling from community pasture}}(\frac{1}{3})(\frac{1}{.75})$$

$$.30 = .88 \text{ harvest hours per acre grain} \times \frac{\text{labor period 8}}{\text{3 weeks}}$$

Labor constraints on harvesting appear to be more binding than are labor constraints in the aggregate periods (or in an "average" week, as modelled above), since weather typically permits harvesting during only approximately 60 percent of the days within a labor period. Suppose that (a) the quantity of labor available (but not necessarily used) at a particular time during labor periods five through eight is independent of the harvesting possibilities, i.e. is independent of the variations in weather that typically influence harvesting possibilities, (b) time and duration of round-up from community pasture and rented range are independent of the harvesting possibilities, and (c) inspection on and round-up from own range and pasture can be scheduled at times when the weather does not permit harvesting. Condition (b) holds if cattle typically are not removed from rented range or community pasture before a closing date (October 7) that is rigidly enforced. Then the labor constraints for harvesting in labor periods five through eight can be represented as follows. Note that the quantities of labor available for harvesting are 60% of the quantities available for the corresponding aggregate labor constraints, and that the cattle labor coefficients in the harvesting constraints are 60% of the round-up requirements for the corresponding aggregate labor constraints.

11) LABHAR15:
.53(HAYHAR15) + 1(SURLABH5) - 1(HIRLABH5) $\leq$ 75 hours per week

12) LABHAR16:
.53(HAYHAR16) + 1(SURLABH6) - 1(HIRLABH6) $\leq$ 75 hours per week

13) LABHAR17:
.6(LABRRC17) + .6(LABRRY17) + .6(LABRPC17) + .6(LABRPY17)
+ 1.2(HAYHAR17) + .44(BARHAR17) + .44(OATHAR17) + 1(SURLABH7)
- 1(HIRLABH7) $\leq$ 51 hours per week

14) LABHAR18:
.6(LABRRC18) + .6(LABRRY18) + .6(LABRPC18) + .6(LABRPY18)
+ .02(ORANGY15) + .01(OPASTC15) + .01(OPASTY15) + .15(RRANGC15)
+ .18(RRANGY15) + .07(RPASTC15) + .05(RPASTY15) + .30(BARHAR18)
+ .30(OATHAR18) + 1(SURLABH8) - 1(HIRLABH8) $\leq$ 51 hours per week.

In the ninth labor period (October 7 to November 1: 35 weeks), cows and yearlings have the same weekly labor requirements as in labor period one, but the available quantity of farm labor is 10 hours per week higher in period nine than in period one.

15) LABR0019:
$.2(COWRHEF) + .15(YERSAL21) + 1(SURLAB19) - 1(HIRLAB19) \leq 70$ hours per week

SURLAB19 - hrs./wk. of surplus labor ('leisure') during labor period 9
HIRLAB19 - hrs./wk. of hired labor during labor period 9

Total hours of hired labor for the year is a weighted (by weeks per labor period) sum of hours hired per week in each of the nine labor periods.

16) HIRLAB10:
$-1(HIRLAB10 + 25(HIRLAB11) + 2(HIRLAB12) + 3(HIRLAB13) + 4.5(HIRLAB14) + 4.5(HIRLAB15) + 4.5(HIRLABH5) + 4.5(HIRLAB16) + 4.5(HIRLABH6) + 2(HIRLAB17) + 2(HIRLABH7) + 3(HIRLAB18) + 3(HIRLABH8) + 3.5(HIRLAB19) \leq 0$ hours per year.

HIRLAB10 - total hours of labor hired during the year.

An upper bound (of 1500 hours) for a total hours of hired labor is often specified.

17) HIRLAB1T:
$1(HIRLAB10) \leq 1500$ hours

Since the seasonal range use patterns on the rented range and community pastures cannot be predicted with confidence, a number of grazing options have been included in the model: rented range and community pasture can be grazed in any combination of periods June 1 to September 1, September 1 to September 15, and September 15 to October 7.[1] Round-up labor at the end of a period is required for all cattle transferred from or between rented range and community pasture at the end of that period, and sorting labor is required for cattle transferred from rented range or community pasture at the end of the period.[2] The round-up requirements for cows and yearlings on rented range and community pasture in the first two periods have been estimated in the following manner.

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[1] This implies 27 grazing combinations for cows and 27 for yearlings. In each of the three periods there are 3 options with respect to off-farm grazing (graze on community pasture, graze on rented range, do not graze off-farm) for both cows and yearlings, and $3^3=27$.

[2] The assumption that cattle transfers between rented range and community pasture require farm labor for round-up is not quite accurate for users of the Sunset Prairie community pasture. This community pasture is the primary source of rented range for these ranchers, and riders hired by the grazing association manage to transfer cattle between rented range and community pasture. However, the services provided by the riders are not a free good; so the use of riders in such a transfer does affect the welfare of members of the association and should in fact be costed directly within the model. For users at Beatton-Doig and W.M. community pasture, such transfers in general do appear to require farm labor for round-up. Sorting labor is not required for such transfers on the assumption that if any cows (yearlings) on rented range or community pasture section A are transferred to rented range or community pasture section B at any time, then all cows (yearlings) on A are transferred to B at this time. This assumption seems somewhat realistic, and in any case labor requirements for sorting are minor in comparison to labor requirements for round-up from rented range (which appears to be the more common transfer).

Since the logic and structure of the equations calculating round-up requirements is more complex and presumably less important than many other aspects of the model, the reader may wish to omit the remainder of this section on labor constraints and activities.

The approach adopted here consists of constructing four sets of five equations to calculate the labor requirements for cows and for yearlings at the end of feeding periods three and four. For example, consider the problem of determining round-up (and sorting) labor requirements for cows on rented range and community pasture towards the end of feeding period three. The situation can be diagrammed as in Figure 3, where activities W1 through Z2 represent the number of cow AUM's on the four types of pasture for feeding periods three and four. Activities H1, H2, ZG1, ZG2, B and S are included here only to illustrate the logic of the calculations in all four sets of equations. If cows could be grazed on hay aftermath or zero-grazed in either period, or cows could be bought or sold after feeding period three (which is not the case), then the number of aftermath acres grazed in period three (H1) and four (H2), and the number of cows zero-grazed after period three (ZG1) or four (ZG2) and bought (B) or sold (S) also should be considered. Then the system of equations for calculating the round-up and sorting labor is as follows:

a) $\frac{15}{60}\left[\frac{1}{3}(W1) - 2(W2) + (T2)\right] - \frac{5}{60}(T1) \leq 2(LABRPC17)$

b) $\frac{35}{60}\left[\frac{1}{3}(X1) - 2(X2) + (T1)\right] - \frac{5}{60}(T2) \leq 2(LABRRC17)$

c) $\frac{1}{3}(W1) = 2(W2) + (T1 - T2) + (T3 - T4)$

d) $\frac{1}{3}(X1) = 2(X2) + (T2 - T1) + (T5 - T6)$

e) $(T3 - T4) + (T5 - T6) = 2\left[(Y2) + (Z2) + .63(H2)\right] + ZG2 + S - \frac{1}{3}\left[(Y1) + (Z1) + .63(H1)\right] - ZG1 - B$

where (T1 - T2) = number of cows transferred from community pasture to rented range at end of period three;
(T2 - T1) = number of cows transferred from rented range to community pasture at the end of period three.[1]

[1] Since benefits from feeding are in no way increased (nor costs reduced) in the model by changes in the feeding schedules of individual animals that do not alter aggregate feeding schedules, transfers from one source of feed to another can be treated as equal to the corresponding net transfers. Thus the transfer from feed source B to feed source A is the negative of the transfer from A to B for the same feeding period. This in turn implies that each transfer must be represented in the model as the difference between two activities, since all activities in a linear programming model are constrained to be greater than or equal to zero.

	during feeding period three				during feeding period four		
	#AUM's	#Acres	#Cows	#Cows	#AUM's	#Acres	#Cows
Source of Feed:							
Community pasture	W1				W2		
Rented range	X1				X2		
Own pasture	Y1				Y2		
Own range	Z1				Z2		
Hay aftermath		H1				H2	
Zero-grazing			ZG1				ZG2
Purchases at end of feeding period three				B			
Sales at end of feeding period three				S			

(1-m)(# cows fed during period three)+(# cows purchased at end of period three)
= (# cows fed during period four)+(# cows sold at end of period three)
where m =mortality rate for cows between feeding periods three and four ("at end of" three)

Note: M=0 for cows between periods three and four and between periods four and five, and for yearlings between periods three and four.

M=.02 for yearlings between periods four and five.

Figure 10. Model of Disposal of Cows during Feeding Periods Three and Four

(T3 - T4) = number of cows on community pasture transferred from the community pasture, and not placed on rented range, at the end of period three.

(T5 - T6) = number of cows on rented range transferred from rented range, and not to community pasture, at the end of period three.

Note: $\frac{15}{60}$ = labor hours round-up and sorting per cow on community pasture.

$\frac{35}{60}$ = labor hours round-up and sorting per cow on rented range.

$\frac{1}{3}$ = number of cows per AUM of grazing in feeding period three.

2 = number of cows per AUM of grazing in feeding period four, or (only in the case of the coefficients for LABRPC17 and LABRRC17) number of weeks in labor period seven.

Inequalities (a) and (b) state round-up hours per week (LABRPC17, LABRRC17) as a function of the number of cows and yearlings transferred from community pasture and rented range, respectively, at the end of period three. Two aspects of these two constraints may need clarification: the coefficients of the transfer activities T1 and T2; and the use of inequalities rather than equalities. If one cow is transferred from community pasture to rented range at the end of feeding period three, i.e., if T1=1 and T2=0,[1] then that cow requires round-up but not sorting labor; so 5/60 of one hour should be subtracted from 15/60[1/3(W1) - 2(W2)]. If one cow is transferred from rented range to community pasture, i.e., if T1=0 and T2=1, then one more cow is transferred from community pasture to some disposal activity other than rented range than is indicated by the calculation 1/3(W1) - 2(W2); so 15/60 of one hour should be added to 15/60[1/3(W1) - 2(W2)].[2]

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[1]Equations (c)-(e) simply constrain (T1-T2), i.e. the difference between the two transfer activities. So the coefficients for T1 and T2 in (a)-(b) imply the following: if cows are rounded-up from either community pasture or rented range, i.e. if either of constraints (a) and (b) is binding, then the opportunity cost of round-up is minimized, for any (T1-T2), at T1=0 or T2=0.

[2]If the coefficients for T1 in (a) and for T2 in (b) were -15/60 and -35/60, respectively, as would be the case if it were realistic to assume that riders transfer cattle between rented range and community pasture and that riders' services are free, then a cow would be transferred from rented range to (e.g.) own pasture by a transfer to community pasture followed immediately by a transfer from the community pasture to own pasture. More precisely, for these coefficients, and at solution, (T1, T2, T3, T4, T5, T6) equals (0, 1, 1, 0, 0, 0) rather than (0, 0, 0, 0, 1, 0).

Second, since all activities in a linear programming model are constrained to be greater than or equal to zero and (a) and (b) are inequalities, a transfer of cows or yearlings to, rather than from, rented range or community pasture (excluding transfers from one to the other) at the end of period three (leading to a net negative value for the left hand sides of (a) or (b)) implies zero round-up labor for that transfer.

Equations (c) and (d) require that the number of cows on community pasture (rented range) during period three equals the number of cows on community pasture (rented range) in period four plus cows transferred from, or minus cows transferred to, community pasture (rented range) at the end of period three. Equation (e) requires that the transfer from rented range and community pasture (excluding transfers from one to the other) at the end of period three equals (a) the number of cows fed by means other than rented range and community pasture in period four or sold at the end of period three minus (b) the number of cows fed by means other than rented range or community pasture in period three or purchased at the end of period three.

The corresponding twenty equations for calculating round-up hours per week for cows on rented range and pasture in feeding period three, yearlings on rented range and pasture in period three, cows on rented range and pasture in period four, and yearlings on rented range and pasture in period four, respectively are as follows[1].

18) CSWIPR13:
$$\frac{15}{60}\left[\frac{1}{3}(\mathrm{RPASTC13}) - 2(\mathrm{RPASTC14}) + 1(\mathrm{CSWPRB13})\right] - \frac{5}{60}(\mathrm{CSWPRA13}) \leq 2(\mathrm{LABRPC17})$$
hours per week.

19) CSWIRP13:
$$\frac{35}{60}\left[\frac{1}{3}(\mathrm{RRANGC13}) - 2(\mathrm{RRANGC14}) + 1(\mathrm{CSWPRA13})\right] - \frac{5}{60}(\mathrm{CSWPRB13}) \leq 2(\mathrm{LABRRC17})$$
hours per week.

20) CSWPAS13:
$$\frac{1}{3}(\mathrm{RPASTC13}) = 2(\mathrm{RPASTC14}) + 1(\mathrm{CSWPRA13}) - 1(\mathrm{CSWPRB13}) + 1(\mathrm{CSWPOA13}) - 1(\mathrm{CSWPOB13})$$
cows.

21) CSWRAN13:
$$\frac{1}{3}(\mathrm{RRANGC13}) = 2(\mathrm{RRANGC14}) + 1(\mathrm{CSWPRB13}) - 1(\mathrm{CSWPRA13}) + 1(\mathrm{CSWROA13}) - 1(\mathrm{CSWROB13})$$
cows.

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[1] By considering three states (graze on rented range, graze on community pasture, do not graze on rented range or community pasture) for each of the three feeding periods, and modelling the resulting twenty-seven combinations directly for both cows and yearlings, 12 rows (2 types of cattle x 2 round-up periods x 3 states = 12) and 54 additional activities ($2\times3^3=54$) would have been necessary to calculate the above round-up requirements (versus 20 rows and 32 additional activities above). As the number of feeding periods requiring round-up calculations increases, the relative efficiency of the method used here presumably increases greatly.

22) COWSWI13:
$1(CSWPOA13) - 1(CSWPOB13) + 1(CSWROA13) - 1(CSWROB13) = 2\ [1(OPASTC14) + 1(ORANGC14)] - \frac{1}{3}\ [1(OPASTC13) + 1(ORANGC13)]$ cows.

23) YSWITPR13:
$\frac{10}{60}\ [\frac{1}{3}(RPASTY13) - 2(RPASTY14) + 1(YSWPRB13)] - \frac{5}{60}(YSWPRA13) \leq 2(LABRPY17)$
hours per week.

24) YSWIRR13:
$\frac{40}{60}\ [\frac{1}{3}(RRANGY13) - 2(RRANGY14) + 1(YSWPRA13)] - \frac{5}{60}(YSWPRB13) \leq 2(LABRRY17)$
hours per week.

25) YSWPAS13:
$\frac{1}{3}(RPASTY13) = 2(RPASTY14) + 1(YSWPRA13) - 1(YSWPRB13) + 1(YSWPOA13) - 1(YSWPOB13)$ yearlings.

26) YSWRAN13:
$\frac{1}{3}(RRANGY13) = 2(RRANGY14) + 1(YSWPRB13) - 1(YSWPRA13) + 1(YSWROA13) - 1(YSWROB13)$ yearlings.

27) YERSWI13:
$1(YSWPOA13) - 1(YSWPOB13) + 1(YSWROA13) - 1(YSWROB13) = 2\ [1(OPASTY14) + 1(ORANGY14)] - \frac{1}{3}\ [1(OPASTY13) + 1(ORANGY13)]$ yearlings.

28) CSWIPR14:
$\frac{15}{60}\ [2(RPASTC14) - \frac{4}{3}(RPASTC15) + 1(CSWPRB14)] - \frac{5}{60}(CSWPRA14) \leq 3(LABRPC18)$
hours per week.

29) CSWINRP14:
$\frac{35}{60}\ [2(RRANGC14) - \frac{4}{3}(RRANGC15) + 1(CSWPRA14)] - \frac{5}{60}(CSWPRB14) \leq 3(LABRRC18)$
hours per week.

30) CSWPAS14:
$2(RPASTC14) = \frac{4}{3}(RPASTC15) + 1(CSWPRA14) - 1(CSWPRB14) + 1(CSWPOA14) - 1(CSWPOB14)$ cows.

31) CSWRAN14:
$2(RRANGC14) = \frac{4}{3}(RRANGC15) + 1(CSWPRB14) - 1(CSWPRA14) + 1(CSWROA14) - 1(CSWROB14)$ cows.

32) COWSWI14:
$1(CSWPOA14) - 1(CSWPOB14) + 1(CSWROA14) - 1(CSWROB14) = \frac{4}{3}[1(OPASTC15) + 1(ORANGC15) + .63(HAYAFC15)] - 2[1(OPASTC14) + 1(OPASTC14)]$ cows.

33) YSWIPR14:
$\frac{10}{60}[.98(2)(RPASTY14) - \frac{4}{3}(RPASTY15) + 1(YSWPRB14)] - \frac{5}{60}(YSWPRA14) \leq 3(LABRPY18)$ hours per week.

34) YSWIRP14:
$\frac{40}{60}[.98(2)(RRANGY14) - \frac{4}{3}(RRANGY15) + 1(YSWPRA14)] - \frac{5}{60}(YSWPRB14) \leq 3(LABNPY18)$ hours per week.

35) YSWPAS14:
$.98(2)(RPASTY14) = \frac{4}{3}(RPASTY15) + 1(YSWPRA14) - 1(YSWPRB14) + 1(YSWPOA14) - 1(YSWPOB14)$ yearlings

36) YSWRAN14:
$.98(2)(RRANGY14) = \frac{4}{3}(RRANGY15) + 1(YSWPRB14) - 1(YSWPRA14) + 1(CSWROA14) - 1(CSWROB14)$ yearlings.

37) YERSWI14:
$1(YSWPOA14) - 1(YSWPOB14) + 1(YSWROA14) - 1(YSWROB14) = \frac{4}{3}[1(OPASTY15) + 1(ORANGY15) + .63(HAYAFY15)] + 1(YEARZG15) + 1(YERSAL15) - .98(2)[1(OPASTY14) + 1(ORANGY14)]$ yearlings

1.6. Income Assurance Constraints and Activities

In the spring of 1975, B.C. beef ranchers were notified that they would subsequently receive subsidies for "qualifying" pounds of beef sold, under the B.C. Farm Income Assurance program. There is a yearly maximum on qualifying pounds of beef per farm equal to the number of pounds of beef sold from the farm in 1974 or 1975 (whichever is higher) that would have qualified for income assurance subsidies. This is true provided that the number of pounds that would have qualified for income assurance subsidies does not exceed the "global limit" of 121,125 qualifying pounds. Typical beef ranches in the Peace River and Prince George areas claim considerably less than 121,125 qualifying pounds. Within this limit, qualifying pounds are equal to the number of pounds sold, with one important exception: for yearlings backgrounded, i.e. for yearlings purchased as calves rather than raised on-farm from birth, qualifying pounds are equal to yearling selling weight minus calf purchase weight. Thus qualifying pounds can be obtained from four sources in the Peace River models: (1) calf pounds sold on November 1, (2) calf pounds for yearlings raised on-farm from birth and sold on September 15 or November 1 of the current model year, (3) yearling pounds (yearling selling weight minus calf pounds) sold on September 15, and (4) yearling pounds sold on November 1. This is incorporated into the model by the following four inequalities, where CSIA0021, YOWNIA21, YSIA0015 and YSIA0021 represent the number of animal-equivalents for which qualifying pounds are claimed, by source in the order listed above.

1) CSIA0021:
$1(\text{CSIA0021}) - 1(\text{CAFSAL21}) \leq 0$ calves.

CSIA0021 - number of calf pounds sold at end of the year that qualify (as calf pounds) for income assurance subsidies.

2) YOWNIA21:
$1(\text{YOWNIA21}) - .98(\text{YEROWN21}) \leq 0$ calves[1]

YOWNIA21 - number of yearling pounds sold during the year that qualify as calf pounds for income assurance subsidies.

note: $.98 = 1 - .02$

where $.02$ = mortality rate for yearlings.

3) YSIA0015:
$1(\text{YSIA0015}) - 1(\text{YERSAL15}) \leq 0$ yearlings

YSIA0015 - number of yearling pounds sold on Nov. 15 of the year that qualify as yearling pounds for income assurance subsidies.

4) YSIA0021:
$1(\text{YSIA0021}) - 1(\text{YERSAL21}) \leq 0$ yearlings

YSIA0021 - number of yearling pounds sold at end of the year that qualify as yearling pounds for income assurance subsidies.

In order to translate these animal-equivalents into qualifying pounds (and also in order to determine total pounds sold), (1) yearly weight gains per animal are specified on the assumption that animals are not grazed on native grass during the year, and (2) weight losses per AUM of native grazing during the year are specified for cows and for yearlings. Weight gains per animal within any feeding period are assumed to be independent of the nutrient source selected (with the exception of native grass), i.e. of choices between community pasture, own pasture, hay aftermath grazing and zero-grazing.[2] The following weights are estimated to be typical for

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[1] To the extent that the equilibrium spirit of the model is accurate for the time period to be modelled, the number of calves held over from year one for sale as yearlings in year two equals the number of calves that were held over from the previous year for sale as yearlings in year one. Under these conditions, YOWNIA21 can be viewed as constrained by YEROWN21 in the same manner as by the number of calves that were held over from the previous year for sale in the current year.

[2] Assuming independence of weight gains with respect to nutrient sources other than native grazing presumably does not lead to serious errors in the estimation of the effects of A.R.D.A. pasture programs. In the "standard case", hay aftermath grazing and zero grazing possibilities overlap with tame grazing possibilities for less than 20% of the summer grazing season, i.e. for the last 3 weeks (September 15 to October 7) of the 4.25 months.

Peace River animals never grazed on range: 450 pounds for calves on November 1, 750 pound for yearlings on September 15, and 825 pounds for yearlings on November 1. Estimates of weight gain differences for tame and native grazing seem less reliable. In the "standard case", it has been assumed that weight gains on own or rented range are less than weight gains on own or rented pasture by the following amounts: (a) 15 pounds per AUM for calves from June 1 to September 15, (b) 30 pounds per AUM for calves from September 15 to October 7, (c) 21 pounds per AUM for yearlings from June 1 to September 15, and (d) 42 pounds per AUM for yearlings from September 15 to October 7. Then the manner in which the allowable number of qualifying pounds of beef constrains qualifying animal-equivalents can be modelled as follows[1].

5) INCASS10:
450(CSIA0021) + 450(YOWNIA21) + 300(YSIA0015) + 375(YSIA0021) - 10.2(ORANGC13) - 10.2(RRANGC13) - 10.2(ORANGC14) - 10.2(RRANGC14) - 20.4(ORANGC15) - 20.4(RRANGC15) - 20.6(ORANGY13) - 20.6(RRANGY13) - 20.6(ORANGY14) - 20.6(RRANGY14) - 41.2(ORANGY15) - 41.2(RRANGY15) $\leq$ 121, 125 qualifying pounds

note: $$10.2 = \frac{\text{15 pounds}}{\text{AUM cow with calf that survives to Nov. 1}}\left(\frac{\text{\# calves surviving yr. 1}}{\text{calf born}}\right)\left(\frac{\text{\# calves born}}{\text{cow}}\right)\left(\frac{\text{\# cows}}{\text{\# cows \& old rep. hef.}}\right)$$
$$= 15(1 - .08)(1 - .15 - .02)(1 - .1 - .08(.1))$$
(see cattle constraint 3)

In addition, the pounds of beef raised on-farm through the year is calculated for calves and for yearlings, respectively.

6) TOTPOUNC:
1(TOTPOUNC) - 450(CAFSAL21) - 450(YEROWN21) + 10.2(ORANGC13) + 10.2 (RRANGC13) + 10.2(ORANGC14) + 10.2(RRANGC14) + 20.4(ORANGC15) + 20.4(RRANGC15) $\leq$ 0 pounds
TOTPOUNC - number of calf pounds raised on-farm through the year

7) TOTPOUNY:
1(TOTPOUNY) - 300(YERSAL15) - 375(YERSAL21) + 20.6(ORANGY13) + 20.6(RRANGY13) + 20.6(ORANGY14) + 20.6(RRANGY14) + 41.2(ORANGY15) + 41.2(RRANGY15) $\leq$ 0 pounds[2]

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[1]Here the right hand side has been set at the global limit on qualifying pounds, but only for illustration. In fact, considerably more constraining limits on qualifying pounds are derived from non-income assurance models and used in this equation.

[2]These two rows are included in the matrix for the sole purpose of providing useful information about the solution, i.e. they are not intended to influence the level of activities other than TOTPOUNC and TOTPOUNY. Note that pounds raised on-farm per year is not equivalent to pounds sold from farm per year: the sum of activities TOTPOUNC and TOTPOUNY excludes the weight-at-purchase of calves for backgrounding, and includes the weight of all on-farm calves held over at the end of the year for sale in the following year as yearlings (two percent of which die before being marketed).

1.7. Income and Value of Surplus Labor Constraints and Activities

In each model, "income" is calculated on an annual basis, and any cash flow problems within the year are ignored.[1] The calculation of "income" implicitly assumes that all revenues are received, and most expenses for the year are incurred and paid, at the end of the modelled year (Nov. 1). The only expenses not dated implicitly for the end of the year are the costs of purchasing or holding over calves for sale as yearlings, which are discounted forward from the beginning of the year (by the addition of one year's real interest charges to the beginning-of-year costs of purchasing or holding over calves). The errors resulting from such simplifications should be minor in comparison to the uncertainty about revenues and costs: even if an expense dated for the end of the modelled year should be dated for the first day of the year, the resulting error is only 4% of the expense, which is the estimated real yearly interest rate times the estimated (real) expense.

Revenue is derived from market sales of calves and yearlings, income assurance subsidies for sale of calves and yearlings, market sales of hay and grain, wages for non-farm employment or custom work by the farm family, and market sales of cull cows. Estimates of 1975 prices and subsidies are used.[2] Expenses in the income equation reflect the variable costs (excluding labor) of hay and grain enterprises, rental rates for hay and grain land, wage rates for hired labor, grazing fees for range and pasture, market prices for hay and grain, market prices for calves purchased for backgrounding, interest charges on the purchase costs of these calves and on the revenues foregone by holding over own calves, and incidental expenses, depreciation and interest on capital for cows and yearlings.

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[1] As pointed out in section II-B, a dynamic model incorporating cash flow problems and other adjustment costs apparently does not lead to more accurate estimation of the comparative dynamic effect of the community pastures programs than do static models. Moreover, at a static equilibrium from year to year, the yearly costs attributable to borrowing will not exceed the real interest charges incurred between the time of negotiating the average yearly loans and the typical time of repayment of these loans during the year. Thus cash flow problems are likely to be of minor influence on the comparative static effects of community pasture programs, and any influence of these problems on comparative static effects could be roughly simulated in the model by slightly increasing the costs for activities that are relatively cash intensive.

[2] It should be emphasized that 1975 prices are included here only to simplify presentation of the model. Since prices circa 1975 were unusually favorable to grain rather than beef production and fluctuated considerably, even short run price expectations held by farmers circa 1975 may have been considerably different from actual 1975 prices.

Calves and yearlings sold receive market prices of 30¢ per pound and 36¢ per pound, respectively. In accordance with the income assurance rules, qualifying "calf pounds" and qualifying "yearling pounds" receive subsidies of 27¢ per pound and 14¢ per pound, respectively. Each qualifying calf-equivalent, or qualifying yearling-equivalent raised on-farm from birth, provides 400 qualifying "calf pounds". Qualifying "yearling pounds" consist of the remaining fifty pounds for each qualifying calf-equivalent or yearling-equivalent raised on-farm from birth, and the difference between selling weight and 400 pounds for each qualifying yearling-equivalent. These rules and prices are incorporated into the income constraint for each model as follows. First, market revenues per calf and yearling are calculated on the assumption that the animal has never been grazed on range, and these figures are used as coefficients for the calf and yearling sale activities in the model. Second, income assurance subsidies are calculated for an animal-equivalent of each of the four types of qualifying pounds on the assumption that the animal is never grazed on range, and used as coefficients for the activities representing the number of qualifying animal-equivalents in each type (CSIA0021, YOWNIA21, YSIA0015, YSIA0021). Third, corrections are made for native grazing as follows. Revenue (market plus subsidy) foregone per AUM of native grazing by cows and by yearlings in each of feeding periods three through five is calculated, for the relative losses in weight per native AUM specified in the "standard case" (income assurance constraint 5). Then these revenues foregone are used as coefficients (in the income constraint) for native grazing activities, so that income is reduced in relation to the quantity of native grazing[1].

Other calculations for the income constraint are straightforward. In the Peace River model, hay and grain can be bought and sold at the following prices: $40 per ton for hay, $2 per bushel for barley, and $1.30 per bushel for oats. Farm family labor can be used for off-farm labor or custom work, from November 1 to May 10, for $5 per hour (and up to 30 hours per week). In the calculation of net expenses per cow, cull cows are assumed to sell for $150 each. Variable costs for grain and hay enterprises, excluding labor

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[1]Due to the structure of income assurance constraint 5, this manner of specifying the income constraint does not overestimate the income costs of native grazing when pounds sold exceeds qualifying pounds. This can be shown as follows. A one unit increase in native grazing directly reduces revenue from subsidies as well as from the market, but has the indirect effect of leading to an offsetting increase in activities CSIA0021, YOWNIA21, YSIA0015, YSIA0021, provided that income assurance constraint 5 is binding by a number of pounds at least equivalent to this increase. Of course, the income constraint also tends to estimate the income costs of native grazing accurately when pounds sold does not exceed qualifying pounds.

costs, are set at \$45 per acre of grain in production[1] and at \$30 to \$40 per acre of hay in production[2]. Rental rates for hay and grain land are often set at \$11.5 per acre and \$17.5 per acre, respectively. Labor can be hired at \$4 per hour, during any season. Range and community pasture can be rented at \$0.53 per AUM and \$3.90 per AUM respectively. Yearly incidentals are estimated at \$25 per cow and per yearling. Yearly depreciation plus interest costs for the capital stock are estimated at approximately \$20 per cow and per yearling.

1) INCOME10:
1(INCOME10) - 135(CAFSAL21) + 135(YERPUR21) - 270(YERSAL15) - 297(YERSAL21)
- 115(CSIA0021) - 115(YOWNIA21) - 42(YSIA0015) - 52.5(YSIA0021)
+ 4.5(ORANGC13) + 4.5(RRANGC13) + 4.5(ORANGC14) + 4.5(RRANGC14)
+ 9.0(ORANGC15) + 9.0(RRANGC15) + 10.3(ORANGY13) + 10.3(RRANGY13)
+ 10.3(ORANGY14) + 10.3(RRANGY14) + 20.6(ORANGY15) + 20.6(RRANGY15)
- 40(HAYSAL10) + 40(HAYPUR10) - 2.00(BARSAL10) + 2.00(BARPUR10)
- 1.30(OATSAL10) - 125(CUSLAB11) - 10(CUSLAB12) + 27.5(OWNHAY10)
+ 46.5(RENHAY10) + 35(OWNBAR10) + 62.5(RENBAR10) + 35(OWNOAT10)
+ 62.5(RENOAT10) + 4(HIRLAB10) + 0.53(RENRAN13) + 0.53(RENRAN14)
+ 0.53(RENRAN15) + 3.90(RPASTC13) + 3.90(RPASTC14) + 3.90(RPASTC15)
+ 3.90(RPASTY13) + 3.90(RPASTY14) + 3.90(RPASTY15) + 29.5(COWSRHEF)
+ 46.4(YEAR0011) $\leq$ 0 dollars

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[1]Discussions with farmers and B.C.D.A. staff did not lead to estimates of non-labor costs for grain enterprises that could be accepted with any confidence, but did suggest that grain acreage for community pasture users would in general be unaffected by access to community pasture (primarily due to a sharply discontinuous supply curve of on-farm land highly suitable for grain). Moreover, it can be argued that the allocative effects of cash and credit constraints are minimal (see section 1.7). Then, provided that acres in grain are adequately simulated, errors in estimating non-labor costs for grain enterprises primarily influence costs that are fixed with respect to the response to the community pasture programs, i.e. these errors lead to quite minor errors in simulating the comparative static effect of community pasture programs.

[2]The costs per acre of hay in production (excluding labor costs and interest on land) appear to be between \$30 and \$40, provided that the capital stock in the enterprise is maintained (over the modelled year) at a fixed ratio to the number of acres in the enterprise. Occasionally in the short run models \$20 to \$30 is specified as the variable cost per acre in order to simulate roughly the effects of not maintaining the capital stock (which is a possible farm response to the holding of an excess capital stock in the enterprise at the beginning of the modelled year).

Note: $$135 = \frac{\text{market price of \$0.30}}{\text{pound of calf sold}} \times \frac{\text{450 pounds}}{\text{calf sold on Nov. 1 (never on native)}}$$

$$270 = \frac{\text{market price of \$0.36}}{\text{pound of yearling sold}} \times \frac{\text{750 pounds}}{\text{yearling sold on Sept. 15 (never on native)}}$$

$$297 = \frac{\text{market price of \$0.36}}{\text{pound of yearling sold}} \times \frac{\text{825 pounds}}{\text{yearling sold on Nov. 1 (never on native)}}$$

$$115 = \frac{\text{400 calf pounds sold}}{\text{calf sold (never on native)}} \times \frac{\text{\$0.27 subsidy}}{\text{calf pound sold}} + \frac{\text{50 yearling pounds sold}}{\text{calf sold (never on native}} \times \frac{\text{\$0.14 subsidy}}{\text{yearling pound sold}}$$

$$42 = \frac{\text{\$0.14 subsidy}}{\text{qualifying yearling pound}} \times \frac{\text{300 qualifying yearling pounds}}{\text{qualifying yearling-equivalent sold on Sept. 15 (never on native)}}$$

$$52.5 = \frac{\text{\$0.14 subsidy}}{\text{qualifying yearling pound}} \times \frac{\text{375 qualifying yearling pounds}}{\text{qualifying yearling-equivalent sold on Nov. 1 (never on native)}}$$

$$4.5 = 10.2\left(\frac{\text{\$0.30 market price}}{\text{calf pound sold}} + \frac{\text{\$0.14 subsidy}}{\text{qualifying yearling pound sold}}\right)^1$$

where 10.2 = relative loss in calf pounds sold, per AUM of native grazing by calves in feeding periods three and four.

$$10.3 = 20.6\left(\frac{\text{\$0.36 market price}}{\text{yearling pound sold}} + \frac{\text{\$0.14 subsidy}}{\text{qualifying yearling pound sold}}\right)$$

where 20.6 = relative loss in yearling pounds sold, per AUM of native grazing by yearlings in feeding periods three and four.

$$27.5 = .75 \times \frac{\text{\$35 variable cost}}{\text{acre in hay production}} + .25 \times \frac{\text{\$5 variable cost}}{\text{acre in hay (re)establishment}}$$

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1 This calculation is accurate provided that (a) native grazing does not reduce calf weights at November 1 to less than 400 pounds, and (b) calves are not held over to be sold as yearlings. When these conditions are not satisfied errors appear to be minor. This can be shown as follows. If every cow with calf is grazed on range for the full 4.25 months of summer grazing (reducing selling weights from 450 to 375 pounds per calf), and (b) is satisfied, then the correct coefficient is

(1) $10.0\,[.30 + (2/3)(.14) + (1/3)(.26)] = 4.8.$

If all calves are held over to be sold as yearlings, and (a) is satisfied, then the correct coefficient is

(2) $.98(10.0)(.36 + .14) = 4.9.$

Discounting of these future income costs (2) in accordance with some positive marginal rate of time preference is inappropriate, when the equilibrium spirit of the model rate of time preference is inappropriate, when the equilibrium spirit of the model is realistic: equilibrium implies that equivalent income costs occur in year one as a result of native grazing in the previous year by cows with calves that are sold in the current year as yearlings. Finally, note that, if every cow with calf is grazed on range for the full 4.25 months and all are held over for sale as yearlings, the correct coefficient is still 4.9.

$$46.5 = \frac{\$35 \text{ variable cost}}{\text{acre in hay production}} + \frac{\$11.5 \text{ rent}}{\text{acre (in hay production)}}$$

$$35.0 = .75 \times \frac{\$45 \text{ variable cost}}{\text{acre in grain production}} + .25 \times \frac{\$5 \text{ variable cost}}{\text{acre in grain summerfallow}}$$

$$62.5 = \frac{\$45 \text{ variable cost}}{\text{acre in grain production}} + \frac{\$17.5 \text{ rent}}{\text{acre (in grain production)}}$$

$$29.5 = \frac{\$25 \text{ incidentals per year}}{\text{cow and replacement heifer}} + \left(\frac{\$20 \text{ depreciation plus interest per year}}{\text{cow}} - \frac{\$150}{\text{cull cow}} \times \frac{.1 \text{ culls per year}}{\text{cow}}\right) \frac{.892 \text{ cows}}{\text{cow and repl. hef.}} \text{ (see cattle constraint 3)}$$

$$46.4 = \frac{\$25 \text{ incidentals per year}}{\text{yearling equiv.}} + \frac{\$2 \text{ depreciation}}{\text{yearling-equiv.}} + (.04)\left(\frac{\$0.30}{\text{lb}}\right)\left(\frac{450 \text{ lbs.}}{\text{calf equiv.}}\right)\left(\frac{1.08 \text{ calf equiv.}}{\text{yearling equiv.}}\right)$$

where 4% = estimated real interest rate
calculation of interest on capital: see Section 4, 17b (calculation assumes that income assurance constraint is binding, and yearlings sold on Nov. 1)

In order to simulate the activities of a firm, and in order to evaluate changes in activities, the supply curve of labor provided by the firm's owner-operator, as well as the supply curve of hired labor, generally should be incorporated into the model. The approach adopted here has been less ambitious: "surplus labor" (slack activities for labor constraints) for each period has been valued at estimates of the net marginal costs at equilibrium of supplying "own labor" to the farm enterprise.[1] This approach leads to a correct static modelling of labor-leisure decisions when the actual farm demand curve for own labor is downward-sloping and correctly modelled, and the net marginal cost at equilibrium of supplying own labor to the farm also is correctly estimated. In comparative statics applications of the model (e.g., comparison of farm activities when the farm does and does not have access to community pasture), errors presumably result. However, the directions of bias on changes in labor use and value of community pasture are readily determined by this approach, and the magnitudes of errors can be estimated simply by varying the value of surplus labor in the appropriate

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[1]Net marginal costs of supplying own labor to the farm enterprise are equal to the benefits of leisure at the margin minus any non-monetary benefits of supplying own labor to the farm at the margin.

directions[1]. Attempts at direct modelling of own labor supply curves have been avoided here precisely because neither evaluation of direction of bias nor sensitivity analysis could then be done so easily.

There does not appear to be any simple means of obtaining reliable estimates of net mariginal costs at equilibrium for supplying own labor to the farm enterprise[2]. However, on-farm interviews strongly suggest that these marginal costs are much higher during calving and harvesting than at other times of the year. For this reason, surplus labor for the labor constraint in labor period two (calving) and for the harvesting labor constraints have been valued at $2.00 per hour, and for other labor constraints at $0.50 per hour, in the "standard case".

2) VALUESUR:
1(VALUESUR) - 12.5(SURLAB11) - 4(SURLAB12) - 4(SURLAB13) - 1.5(SURLAB13) - 2.25(SURLAB14) - 2.25(SURLAB15) - 9(SURLABH5) - 2.25(SURLAB16) - 9(SURLABH6) - 1(SURLAB17) - 4(SURLABH7) - 1.5(SURLAB18) - 6(SURLABH8) - 1.75(SURLAB19)
$\leq$ 0 dollar-equivalents
VALUESUR - dollar-equivalent value of surplus farm labor ('leisure')

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[1]Farmers presumably optimize in selecting among their possiblities for leisure, and the set of these possibilities diminishes as labor supplied increases and income remains constant. This suggests that the supply curve of own labor typically is upward-sloping, rather than perfectly elastic as in the model. Then, if the model is otherwise accurate, directions of bias can be inferred from model results as follows. If the model shows an increase in use of own labor when the farm receives access to community pasture, then there is in fact an increase in use of own labor, and the model overestimates this increase and also overestimates the value of community pasture to the farm. If the model shows a decrease in use of own labor, then there is in fact a decrease in use of own labor, and the model underestimates this decrease and also underestimates the value of the pasture. If use of own labor is unchanged, the model results are unbiased. If the biases are thought to be significant, then they can be corrected for by making "reasonable" (a) increases or (b) decreases in the value of surplus labor, for the model providing the farm with access to community pasture, when own labor use has (a) increased or (b) decreased, respectively, with the introduction of community pasture into an otherwise unchanged model.

[2]Family labor generally is used for off-farm as well as on-farm employment, and labor generally is hired for harvesting. However, this does not imply that the net marginal costs at equilibrium of supplying own labor to the farm equal the off-farm wage rate (even when adjusted for any monetary and non-monetary expenses incurred by working off-farm)during the winter and the wage rate for harvest labor during harvesting. Apparently farmers generally prefer more off-farm employment during winter at existing wage rates minus additional expenses than they are able to obtain; so the net marginal benefits of off-farm employment must be greater than the net marginal costs at equilibrium of supplying own labor to the farm during the winter. Likewise, own labor on harvesting days generally appears to be at the (productive) maximum, i.e., own labor could not substitute further for hired labor. However, if it is also assumed that hired labor is as productive as own labor, then it can at least be said that the wage rate is greater than the net marginal cost of using own labor in harbesting.

note: $12.5 = \frac{\$0.50}{\text{surplus hour}} \times \frac{\text{25 surplus hours per year}}{\text{surplus hour in labor constraint for labor period one}}$
(see labor constraint 16)

etc.

1.8 Objective Function

The objective function to be maximized is essentially equal to income plus the dollar-equivalent index of leisure for the year modelled. Income is not corrected for taxes, nor are changes in the value of capital stock considered. These omissions are justified as follows.

Various estimates of net marginal costs at equilibrium for supplying own labor to the farm will be used in the model, and a "best guess" as to appropriate marginal costs has neither been formed nor is likely to be inferred from model results. For these reasons, there appears to be little point in correcting income for taxes.

When either the short run or long run equilibrium version of these models is realistic, there is no need to incorporate a valuation of capital stock into the model in order to simulate behaviour. In other words, if we can assume that the farmers in question typically behave as if "close" to a static equilibrium in the absence of community pasture, and as if "close" to a static equilibrium in the presence of community pasture, then in each case the capital stock decision (the decision that maximizes net present value of resulting future flows of income plus leisure) will closely approximate the decision that maximizes income plus leisure for the single year. Estimates of the maintenance costs of all capital in cattle and crop enterprises are incorporated into equilibrium models. Estimates of the annual opportunity cost of this capital (excluding land) are incorporated as the annual interest cost (at a real interest rate of 4%) for estimated values of capital stock. Land values are excluded from the objective functions of all models on the assumption that the land will remain in agriculture, and in order to avoid double counting of benefits from the community pasture program.

Moreover, even if farmers do not in fact typically behave as if "close" to a static equilibrium, the comparative dynamic effect of the community pastures programs apparently can be estimated as well by the use of static models as by dynamic models.[1] By specifying a value for terminal capital in a one year (non-stationary) model, we would in effect be modelling part of a dynamic process rather than a static equilibrium. Thus, there does not appear to be any point in including a valuation of capital in the objective function of these models.[2]

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[1]See Appendix 1 for an elaboration of this statement.

[2]As has been mentioned, static disequilibrium is occasionally allowed for in variants of these models. However, this is done by specifying the disequilibrium aspects as exogenous, i.e. by defining the changes in various capital stocks over the model year rather than the terminal value of these capital stocks.

In addition to summing income and the dollar value of leisure, the objective function costs purchases of calves, hay and barley (so that both sales and purchases of such a product does not appear in solution when its marginal value product is zero), and values total pounds of beef produced (so that this will be calculated in the model).

1) MAXIMIZE:
1(INCOME10) + 1(VALUESUR) - 2.31(YERPUR21) - .01(HAYPUR10) - .01(BARPUR10) - .01(HAYAFC15) - .01(HAYAFY15) - .01(HAYAFC16) - .01(HAYAFY16) + .001(TOTPOUNC) + .001(TOTPOUNY)[1]

INCOME10 - income for the farm, plus income from custom and off-farm work during the year.

[1]In the model, calves can be bought and sold at the same market price. Since 2% of calves held over to be sold as yearlings will die before reaching market, in effect calves may receive a higher income assurance payment if sold as calves ($115 per 450 qualifying pounds) then if sold as yearlings ($112.7 per 441 qualifying calf pounds). Thus, in the absence of a charge of 2.31 dollar-equivalents per calf purchased, holding over own calves for sale as yearlings would not necessarily be preferred to selling own calves and purchasing replacement calves to be sold as yearlings. Observation suggests that the first of these two options is more commonly practiced. This may reflect the fact that (contrary to the model) cattle are not homogeneous and farmers do have more information about their own calves than about other calves.

2. Peace River Non-Income Assurance Models

In contrast to income assurance models, non-income assurance models do not provide subsidies for sales of calves and yearlings. These models differ from income assurance models as follows. First, activities representing qualifying pounds are excluded from the income constraint. Thus activities CSIA0021, YOWNIA21, YSIA0015 and YSIA0021 do not enter row INCOME10. Second, the number of beef pounds sold that would be classified as qualifying pounds under Income Assurance regulations is calculated. A new activity INCASS10 is assigned a weighting of +.001 in the objective function, and income assurance constraint 5 is reformulated as follows.

1) INCASS10:
$$+ 1(\text{INCASS10}) - 450(\text{CSIA0021}) - \dots + 41.2(\text{RRANGY15}) \leq 0 \text{ pounds}$$

Non-income assurance models also differ from income assurance models by using many different combinations of market prices for calves and yearlings.

Non-income assurance models are used for (1) estimating the effect of A.R.D.A. community pasture programs on the yearly maximum number of qualifying pounds for users, and (2) estimating future benefits of the community pasture programs (in the absence of income assurance subsidies). Non-income assurance models designed for the first purpose usually differ from income assurance models solely as listed above. On the other hand, both short run and long run equilibrium models can be used for estimating future benefits, since long run equilibrium may or may not be closely approximated at any particular time (or set of prices) in the future. These short run models differ from the other short run non-income assurance models in terms of prices, and the long run equilibrium models differ from these corresponding short run models by leaving cow numbers unbounded (and by always specifying constant capital stocks in all enterprises over the modelled year).

3. Matrix Format and Column Definitions for Peace River Income Assurance Model

In this section we present a description of the standard short run Peace River Income Assurance farm model in matrix format, and provide a list of definitions for all activities (columns) in this model. For convenience, this model is presented in terms of various submatrices. Elements of a submatrix that are equal to plus or minus one are indicated as such (1, -1), while each other non-zero element is simply denoted by its absolute value (+, -).

The reader is cautioned that this section is not a substitute for discussion in Section 1. The simple structure of this model and its companions may be seriously misinterpreted if the reader is unaware of the economic significance of the assumptions employed here.

			Column	Acres Hay and Grain								Acres Own Range	
				1. OWNHAY1Ø	2. RENHAY1Ø	3. OWNOAT1Ø	4. BESOAT1Ø	5. RENOAT1Ø	6. OWNBAR1Ø	7. BESBAR1Ø	8. RENBAR1Ø	9. OWNPAS1Ø	10. OWNRAN1Ø
			Row										
I.	LAND	1.	IMLAND1Ø	1		1			1			1	
		2.	BESIMP1Ø				1			1			
		3.	BESOAT1Ø			-1	1						
		4.	BESBAR1Ø						-1	1			
		5.	OWNPAS1Ø									-	
		6.	OWNPAS15									-	
		7.	UNLAND1Ø										1
		8.	OWNRAN1Ø										-
		9.	OWNRAN14										-
		19.	HAYAFT1Ø	-		-1							
		20.	RENHAY1Ø			1							
		21	RENOAT1Ø					1					
		22.	RENBAR1Ø								1		
IV.	HAY AND GRAIN	1.	HAYHAR1Ø	-	-1								
		2.	OATHAR1Ø			-		-1					
		3.	BARHAR1Ø						-		-1		
		5.	OATDØØ1Ø				-	-					
		6.	BARDØØ1Ø							-	-		
V.	LABOR	6.	LABRØØ14	+	+	+		+	+		+		
VII.	INCOME	1.	INCOME1Ø	+	+	+		+	+		+		

Figure 11. Submatrix "A" of Peace River Income Assurance Model.

				Cows on Farm	Yearlings					
					on Farm				Zero Grazed	
			Column	11. COWSRHEF	12. YEARØØ11	13. YEARØØ21	14. YEROWN21	15. YERPUR21	16. YEARZG15	17. YEARZG16
			Row							
II.	CATTLE	1.	COWSRHEF	1						
		2.	OLDYER15		-					
		3.	CAFDIS21	-			+			
		4.	YEARØØ21		1	-1				
		5.	NEWYER21			1	-1	-1		
III.	FEEDING	1.	HAYFED11	+	+					
		2.	BARFED11		+					
		3.	HAYFED12	+	+					
		4.	COWFED13	+						
		5.	YERFED13		+					
		6.	COWFED14	+						
		7.	YERFED14		+					
		8.	COWFED15	+						
		9.	YERFED15						-	
		11.	HAYFED15						-	
		12.	COWFED16	+			+			
		13.	YERFED16						-	
		15.	HAYFED16						-	
V.	LABOR	1.	LABRØØ11	+	+					
		2.	LABRØØ12	+	+					
		5.	LABRØØ13	+	+					
		10.	LABRØØ18						+	
		15.	LABRØØ19	+						
		37.	YERSWI14						-1	
VI.	INCOME ASSURANCE	2.	YOWNIA21				-			
		6.	TOTPOUNC					-		
VII.	INCOME	1.	INCOME1Ø	+	+			+		
VIII.	OBJECTIVE FUNCTION	1.	MAXIMIZE					-		

Figure 12. Submatrix "B" of Peace River Income Assurance Model

			Column / Row	18. OPASTC13	19. OPASTC14	20. OPASTC15	21. OPASTY13	22. OPASTY14	23. OPASTY15	24. ORANGC13	25. ORANGC14	26. ORANGC15	27. ORANGY13	28. ORANGY14	29. ORANGY15	30. RPASTC13	31. RPASTC14	32. RPASTC15	33. RPASTY13	34. RPASTY14	35. RPASTY15	36. RENRAN13	37. RENRAN14	38. RENRAN15	39. RRANGC13	40. RRANGC14	41. RRANGC15	42. RRANGY13	43. RRANGY14	44. RRANGY15	45. HAYAFC15	46. HAYAFC16	47. HAYAFY15	48. HAYAFY16
				Own Pasture						Own Range						Rented Pasture						Rented Range									Hay Aftermath			
I.	LAND	5.	OWNPAS1Ø	1	1		1	1																										
		6.	OWNPAS15			1			1																									
		8.	OWNRAN1Ø							1	1		1	1																				
		9.	OWNRAN14									1			1																			
		10.	RENRAN13																			1												
		11.	RENRAN14																				1											
		12.	RENRAN15																					1										
		13.	RENRAD13																			-1			1			1						
		14.	RENRAD14																				-1			1			1					
		15.	RENRAD15																					-1			1			1				
		16.	RENPAS13													1			1															
		17.	RENPAS14														1			1														
		18.	RENPAS15															1			1													
		19.	HAYAFT1Ø																												1	1	1	1
III.	FEEDING	4.	COWFED13	-1						-1						-1									-1									
		5.	YERFED13				-1						-1						-1									-1						
		6.	COWFED14		-1						-1						-1									-1								
		7.	YERFED14					-1						-1						-1									-1					
		8.	COWFED15			-1						-1						-1									-1				-			
		9.	YERFED15						-1						-1						-1									-1			-	
		12.	COWFED16																													-		
		13.	YERFED16																															-
V.	LABOR	6.	LABRØØ14	+			+			+			+									+												
		7.	LABRØØ15	+			+			+			+									+												
		8.	LABRØØ16	+			+			+			+									+												
		9.	LABRØØ17		+			+			+			+									+											
		10.	LABRØØ18			+			+			+			+			+			+						+			+				
		14.	LABHAR18			+			+			+			+			+			+						+			+				
		18.	CSWIPR13													+	-																	
		19.	CSWIRP13																						+	-								
		20.	CSWPAS13													-	+																	
		21.	CSWRAN13																						-	+								
		22.	COWSWI13	+	-					+	-																							
		23.	YSWIPR13																+	-														
		24.	YSWIRP13																									+	-					
		25.	YSWPAS13																-	+														
		26.	YSWRAN13																									-	+					
		27.	YERSWI13				+	-					+	-																				
		28.	CSWIPR14														+	-									-							
		29.	CSWIRP14																							+								
		30.	CSWPAS14														-	+									+							
		31.	CSWRAN14																							-								
		32.	COWSWI14		+	-					+	-																			-			
		33.	YSWIPR14																	+	-													
		34.	YSWIRP14																										+	-				
		35.	YSWPAS14																	-	+													
		36.	YSWRAN14																										-	+				
		37.	YERSWI14					+	-					+	-																		-	
VI.	INCOME ASSURANCE	5.	INCASS1Ø							-	-	-	-	-	-										-	-	-	-	-	-				
		6.	TOTPOUNC							+	+	+													+	+	+							
		7.	TOTPOUNY										+	+	+													+	+	+				
VII.	INCOME	1.	INCOME1Ø							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+				
VIII.	OBJECTIVE FUNCTION	1.	MAXIMIZE																												-	-	-	-

Figure 13. Submatrix "C" of Peace River Income Assurance Model

			Column	Hay, Oats and Barley																		
				Fed to Cattle							Harvest							Purchases and Sales				
			Row	49. HAYFED11	50. HAYFED12	51. HAYFED15	52. HAYFED16	53. BARFED11	54. BARFED15	55. BARFED16	56. HAYHAR15	57. HAYHAR16	58. HAYHAR17	59. OATHAR17	60. OATHAR18	61. BARHAR17	62. BARHAR18	63. HAYPUR10	64. HAYSAL10	65. OATSAL10	66. BARPUR10	67. BARSAL10
III.	FEEDING	1.	HAYFED11	-1																		
		2.	BARFED11					-1														
		3.	HAYFED12		-1																	
		10.	BARFED15						-1													
		11.	HAYFED15			-1																
		14.	BARFED16							-1												
		15.	HAYFED16				-1															
IV.	HAY AND GRAIN	1.	HAYHAR10								1	1	1									
		2.	OATHAR10											1	1							
		3.	BARHAR10													1	1					
		4.	HAYD0010								-	-1	-					-1	1			
		5.	OATD0010	1	1	1	1							-	-					1		
		6.	BARD0010					1	1	1						-	-				-1	1
V.	LABOR	7.	LABR0015								+											
		8.	LABR0016									+										
		9.	LABR0017										+	+		+						
		10.	LABR0018												+		+					
		11.	LABHAR15								+											
		12.	LABHAR16									+										
		13.	LABHAR17										+	+		+						
		14.	LABHAR18												+		+					
VII.	INCOME	1.	INCOME10															+	-	-	+	-
VIII.	OBJECTIVE FUNCTION		MAXIMIZE															-			-	

Figure 14. Submatrix "D" of Peace River Income Assurance Model

			Column	Labor Use																								Labor Hired													
				Custom		Roundup								Surplus in Non-Harv, Harv														In Non-Harvest And in Harvest													
			Row	68. CUSLAB11	69. CUSLAB12	70. LABRPC17	71. LABRPC18	72. LABRPY17	73. LABRPY18	74. LABRRC17	75. LABRRC18	76. LABRRY17	77. LABRRY18	78. SURLAB11	79. SURLAB12	80. SURLAB13	81. SURLAB14	82. SURLAB15	83. SURLAB16	84. SURLAB17	85. SURLAB18	86. SURLAB19	87. SURLABH5	88. SURLABH6	89. SURLABH7	90. SURLABH8	91. VALUESUR	92. HIRLAB11	93. HIRLAB12	94. HIRLAB13	95. HIRLAB14	96. HIRLAB15	97. HIRLAB16	98. HIRLAB17	99. HIRLAB18	100. HIRLAB19	101. HIRLABH5	102. HIRLABH6	103. HIRLABH7	104. HIRLABH8	105. HIRLAB10
V.	LABOR	1.	LABR0011	1										1														-1													
		2.	LABR0012		1										1														-1												
		3.	CUSLAB11	1																																					
		4.	CUSLAB12		1																																				
		5.	LABR0013													1														-1											
		6.	LABR0014														1														-1										
		7.	LABR0015															1					1									-1					-1				
		8.	LABR0016																1					1									-1					-1			
		9.	LABR0017			1		1		1		1								1					1									-1					-1		
		10.	LABR0018				1		1		1		1								1					1									-1					-1	
		11.	LABHAR15																				1														-1				
		12.	LABHAR16																					1														-1			
		13.	LABHAR17			+		+		+		+													1														-1		
		14.	LABHAR18				+		+		+		+													1														-1	
		15.	LABR0019																			1														-1					
		16.	HIRLAB10																									+	+	+	+	+	+	+	+	+	+	+	+	+	-1
		17.	HIRLABT																																						1
		18.	CSWIPR13			-																																			
		19.	CSWIRP13							-																															
		23.	YSWIPR13					-																																	
		24.	YSWIRP13									-																													
		28.	CSWIPR14				-																																		
		29.	CSWIRP14								-																														
		33.	YSWIPR14						-																																
		34.	YSWIRP14										-																												
VII.	INCOME AND VALUE	1.	INCOME10																																						+
	OF SURPLUS LABOR	2.	VALUESUR											-	-	-	-	-	-	-1	-	-	-	-	-	-	1														
VIII.	OBJECTIVE FUNCTION	1.	MAXIMIZE																								1														

Figure 15. Submatrix "E" of Peace River Income Assurance Model

Cattle Transferred Between and From Community Pasture and Rented Range

Cows

		Row \ Column		106. CSWPRA13	107. CSWPRB13	108. CSWPOA13	109. CSWPOB13	110. CSWROA13	111. CSWROB13	112. CSWPRA14	113. CSWPRB14	114. CSWPOA14	115. CSWPOB14	116. CSWROA14	117. CSWROB14
V.	LABOR	18.	CSWIPR13	-	+										
		19.	CSWIRP13	+	-										
		20.	CSWPAS13	1	-1	1	-1								
		21.	CSWRAN13	-1	1			1	-1						
		22.	COWSWI13			1	-1	1	-1						
		23.	YSWIPR13												
		24.	YSWIRP13												
		25.	YSWPAS13												
		26.	YSWRAN13												
		27.	YERSWI13												
		28.	CSWIPR14							-	+				
		29.	CSWIRP14							+	-				
		30.	CSWPAS14							1	-1	1	-1		
		31.	CSWRAN14							-1	1			1	-1
		32.	COWSWI14									1	-1	1	-1
		33.	YSWIPR14												
		34.	YSWIRP14												
		35.	YSWPAS14												
		36.	YSWRAN14												
		37.	YERSWI14												

Yearlings

		Row \ Column		118. YSWPRA13	119. YSWPRB13	120. YSWPOA13	121. YSWPOB13	122. YSWROA13	123. YSWROB13	124. YSWPRA14	125. YSWPRB14	126. YSWPOA14	127. YSWPOB14	128. YSWROA14	129. YSWROB14
V.	LABOR	18.	CSWIPR13												
		19.	CSWIRP13												
		20.	CSWPAS13												
		21.	CSWRAN13												
		22.	COWSWI13												
		23.	YSWIPR13	-	+										
		24.	YSWIRP13	+	-										
		25.	YSWPAS13	1	-1	1	-1								
		26.	YSWRAN13	-1	1			1	-1						
		27.	YERSWI13			1	-1	1	-1						
		28.	CSWIPR14												
		29.	CSWIRP14												
		30.	CSWPAS14												
		31.	CSWRAN14												
		32.	COWSWI14												
		33.	YSWIPR14							-	+				
		34.	YSWIRP14							+	-				
		35.	YSWPAS14							1	-1	1	-1		
		36.	YSWRAN14							-1	1			1	-1
		37.	YERSWI14									1	-1	1	-1

Figure 16. Submatrix "F" of Peace River Income Assurance Model

			Row \ Column	130. CAFSAL21	131. YERSAL15	132. YERSAL21	133. CSIAØØ21	134. YOWNIA21	135. YSIAØØ15	136. YSIAØØ21	137. TOTPOUNC	138. TOTPOUNY	139. INCOME1Ø
				Cattle Sales (Total and Subsidized)							Cattle Pounds		Income
II.	CATTLE	2.	OLDYER15		1	1							
		3.	CAFDIS21	+									
III.	FEEDING	9.	YERFED15			+							
		10.	BARFED15			+							
		12.	COWFED16	+									
		13.	YERFED16			+							
		14.	BARFED16			+							
V.	LABOR	15.	LABRØØ19			+							
		36.	YSWRAN14		-1								
		37.	YERSWI14			-1							
VI.	INCOME ASSURANCE	1.	CSIAØØ21	-1			1						
		2.	YOWNIA21			-1		1					
		3.	YSIAØØ15		-1				1				
		4.	YSIAØØ21			-1				1			
		5.	INCASS1Ø				+	+	+	+			
		6.	TOTPOUNC	-							1		
		7.	TOTPOUNY		-	-						1	
VII.	INCOME	1.	INCOME1Ø	-	-	-	-	-	-	-			1
VIII.	OBJECTIVE FUNCTION	1.	MAXIMIZE								+	+	1

Figure 17. Submatrix "G" of Peace River Income Assurance Model

		Row		Right Hand Side *
I.	LAND	1.	IMLAND1Ø	350 acres
		2.	BESIMP1Ø	150 acres
		3.	BESOAT1Ø	acres
		4.	BESBAR1Ø	acres
		5.	OWNPAS1Ø	AUM's
		6.	OWNPAS15	AUM's
		7.	UNLAND1Ø	150 acres
		8.	OWNRAN1Ø	AUM's
		9.	OWNRAN14	AUM's
		10.	RENRAN13	300 AUM's
		11.	RENRAN14	50 AUM's
		12.	RENRAN15	75 AUM's
		13.	RENRAD13	AUM's
		14.	RENRAD14	AUM's
		15.	RENRAD15	AUM's
		16.	RENPAS13	180 AUM's
		17.	RENPAS14	30 AUM's
		18.	RENPAS15	45 AUM's
		19.	HAYAFT1Ø	acres
		20.	RENHAY1Ø	50 acres
		21.	RENOAT1Ø	50 acres
		22.	RENBAR1Ø	50 acres
II.	CATTLE	1.	COWSRHEF	$\geq$ 40 cows
		2.	OLDYER15	yearlings
		3.	CAFDIS21	calves plus (beginning) yearlings
		4.	YEARØØ21	yearlings
		5.	NEWYER21	yearlings
III.	FEEDING	1.	HAYFED11	tons
		2.	BARFED11	bushels
		3.	HAYFED12	tons
		4.	COWFED13	AUM's
		5.	YERFED13	AUM's
		6.	COWFED14	AUM's
		7.	YERFED14	AUM's
		8.	COWFED15	AUM's
		9.	YERFED15	AUM's
		10.	BARFED15	bushels
		11.	HAYFED15	tons
		12.	COWFED16	AUM's
		13.	YERFED16	AUM's
		14.	BARFED16	bushels
		15.	HAYFED16	tons
IV.	HAY & GRAIN	1.	HAYHAR1Ø	acres
		2.	OATHAR1Ø	acres
		3.	BARHAR1Ø	acres
		4.	HAYDØØ1Ø	tons
		5.	OATDØØ1Ø	bushels
		6.	BARDØØ1Ø	bushels

*Right hand side is $\leq$ 0 unless indicated otherwise.

Figure 18. Right Hand Side of Peace River Income Assurance Model

			Row		Right Hand Side *
V.	LABOR	1.	LABRØØ11		60 hours per week
		2.	LABRØØ12		135 hours per week
		3.	CUSLAB11		30 hours per week
		4.	CUSLAB12		30 hours per week
		5.	LABRØØ13		70 hours per week
		6.	LABRØØ14		75 hours per week
		7.	LABRØØ15		120 hours per week
		8.	LABRØØ16		120 hours per week
		9.	LABRØØ17		85 hours per week
		10.	LABRØØ18		85 hours per week
		11.	LABHAR15		75 hours per week
		12.	LABHAR16		75 hours per week
		13.	LABHAR17		51 hours per week
		14.	LABHAR18		51 hours per week
		15.	LABRØØ19		70 hours per week
		16.	HIRLAB1Ø		hours per week
		17.	HIRLAB1T		1500 hours
		18.	CSWIPR13		hours
		19.	CSWIRP13		hours
		20.	CSWPAS13	=	cows
		21.	CSWRAN13	=	cows
		22.	COWSWI13	=	cows
		23.	YSWIPR13		hours
		24.	YSWIRP13		hours
		25.	YSWPAS13	=	yearlings
		26.	YSWRAN13	=	yearlings
		27.	YERSWI13	=	yearlings
		28.	CSWIPR14		hours
		29.	CSWIRP14		hours
		30.	CSWPAS14	=	cows
		31.	CSWRAN14	=	cows
		32.	COWSWI14	=	cows
		33.	YSWIPR14		hours
		34.	YSWIRP14		hours
		35.	YSWPAS14	=	yearlings
		36.	YSWRAN14	=	yearlings
		37.	YERSWI14	=	yearlings
VI.	INCOME ASSURANCE	1.	CSIAØØ21		calves
		2.	YOWNIA21		calves
		3.	YSIAØØ15		yearlings
		4.	YSIAØØ21		yearlings
		5.	INCASS1Ø		121125 qualifying pounds
		6.	TOTPOUNC	=	pounds
		7.	TOTPOUNY		pounds
VII.	INCOME AND VALUE OF SURPLUS LABOR	1.	INCOME1Ø		dollar
		2.	VALUESUR		dollar equivalents

* Right hand side is ≤ 0 unless indicated otherwise.

Table VII. Definitions of Activities (Columns) of Peace River Income Assurance Model

1.	OWNHAY1Ø	acres of own improved land allocated to hay for the year
2.	RENHAY1Ø	acres of rented land producing hay in the year
3.	OWNOAT1Ø	acres of own improved land allocated to oats for the year
4.	BESOAT1Ø	acres of "best" own improved land allocated to oats for the year
5.	RENOAT1Ø	acres of rented land producing oats in the year
6.	OWNBAR1Ø	acres of own improved land allocated to barley for the year
7.	BESBAR1Ø	acres of "best" own improved land allocated to barley for the year
8.	RENBAR1Ø	acres of rented land producing barley in the year
9.	OWNPAS1Ø	acres of own improved land allocated to pasture for the year
10.	OWNRAN1Ø	acres of own unimproved land allocated to grazing for the year
11.	COWSRHEF	number of cows plus "old" replacement heifers (to calve in the next year) for the year
12.	YEARØØ11	number of yearlings at the beginning of the year (to be sold towards end of year)
13.	YEARØØ21	number of yearlings at the beginning of the next year
14.	YEROWN21	number of calves (born in spring of year) held over for sale as yearlings towards the end of the following year
15.	YERPUR21	number of calves purchased at end of year (for sale as yearlings towards end of next year)
16.	YEARZG15	number of yearlings zero-grazed during feeding period 5
17.	YEARZG16	number of yearlings zero-grazed during feeding period 6
18.	OPASTC13	AUM's of own pasture grazed by cows in feeding period 3
19.	OPASTC14	AUM's of own pasture grazed by cows in feeding period 4
20.	OPASTC15	AUM's of own pasture grazed by cows in feeding period 5
21.	OPASTY13	AUM's of own pasture grazed by yearlings in feeding period 3
22.	OPASTY14	AUM's of own pasture grazed by yearlings in feeding period 4
23.	OPASTY15	AUM's of own pasture grazed by yearlings in feeding period 5
24.	ORANGC13	AUM's of own range grazed by cows in feeding period 3
25.	ORANGC14	AUM's of own range grazed by cows in feeding period 4
26.	ORANGC15	AUM's of own range grazed by cows in feeding period 5
27.	ORANGY13	AUM's of own range grazed by yearlings in feeding period 3
28.	ORANGY14	AUM's of own range grazed by yearlings in feeding period 4
29.	ORANGY15	AUM's of own range grazed by yearlings in feeding period 5
30.	RPASTC13	AUM's of rented pasture grazed by cows in feeding period 3
31.	RPASTC14	AUM's of rented pasture grazed by cows in feeding period 4
32.	RPASTC15	AUM's of rented pasture grazed by cows in feeding period 5
33.	RPASTY13	AUM's of rented pasture grazed by yearlings in feeding period 3
34.	RPASTY14	AUM's of rented pasture grazed by yearlings in feeding period 4
35.	RPASTY15	AUM's of rented pasture grazed by yearlings in feeding period 5
36.	RENRAN13	AUM's of rented range grazed by cattle in feeding period 3
37.	RENRAN14	AUM's of rented range grazed by cattle in feeding period 4
38.	RENRAN15	AUM's of rented range grazed by cattle in feeding period 5

.../Cont'd.

Table VII (Cont'd.)

39. RRANGC13 AUM's of rented range grazed by cows in feeding period 3
40. RRANGC14 AUM's of rented range grazed by cows in feeding period 4
41. RRANGC15 AUM's of rented range grazed by cows in feeding period 5
42. RRANGY13 AUM's of rented range grazed by yearlings in feeding period 3
43. RRANGY14 AUM's of rented range grazed by yearlings in feeding period 4
44. RRANGY15 AUM's of rented range grazed by yearlings in feeding period 5
45. HAYAFC15 acres of hay aftermath grazed by cows in feeding period 5
46. HAYAFC16 acres of hay aftermath grazed by yearlings in feeding period 5
47. HAYAFY15 acres of hay aftermath grazed by cows in feeding period 6
48. HAYAFY16 acres of hay aftermath grazed by yearlings in feeding period 6
49. HAYFED11 tons of hay fed to cows and yearlings during feeding period 1
50. HAYFED12 tons of hay fed to cows and yearlings during feeding period 2
51. HAYFED15 tons of hay fed to cows and yearlings during feeding period 5
52. HAYFED16 tons of hay fed to cows and yearlings during feeding period 6
53. BARFED11 bushels of barley fed to yearlings during feeding period 1
54. BARFED15 bushels of barley fed to yearlings during feeding period 5
55. BARFED16 bushels of barley fed to yearlings during feeding period 6
56. HAYHAR15 acres of hay harvested during labor period 5
57. HAYHAR16 acres of hay harvested during labor period 6
58. HAYHAR17 acres of hay harvested during labor period 7
59. OATHAR17 acres of oats harvested during labor period 7
60. OATHAR18 acres of oats harvested during labor period 8
61. BARHAR17 acres of barley harvested during labor period 7
62. BARHAR18 acres of barley harvested during labor period 8
63. HAYPUR1Ø tons of hay purchased during the year
64. HAYSAL1Ø tons of hay sold during the year
65. OATSAL1Ø bushels of oats sold during the year
66. BARPUR1Ø bushels of barley purchased during the year
67. BARSAL1Ø bushels of barley sold during the year
68. CUSLAB11 hrs./wk. of custom or off-farm work during labor period 1
69. CUSLAB12 hrs./wk. of custom or off-farm work during labor period 2
70. LABRPC17 roundup hrs./wk. in labor period 7 for cows on rented pasture in feeding period 3
71. LABRPC18 roundup hrs./wk. in labor period 8 for cows on rented pasture in feeding period 4
72. LABRPY17 roundup hrs./wk. in labor period 7 for yearlings on rented pasture in feeding period 3
73. LABRPY18 roundup hrs./wk. in labor period 8 for yearlings on rented pasture in feeding period 4
74. LABRRC17 roundup hrs./wk. in labor period 7 for cows on rented range in feeding period 3
75. LABRRC18 roundup hrs./wk. in labor period 8 for cows on rented range in feeding period 4
76. LABRRY17 roundup hrs./wk. in labor period 7 for yearlings on rented range in feeding period 3
77. LABRRY18 roundup hrs./wk. in labor period 8 for yearlings on rented range in feeding period 4
78. SURLAB11 hrs./wk. of own labor that is surplus (leisure) in labor period 1
79. SURLAB12 hrs./wk. of own labor that is surplus (leisure) in labor period 2
80. SURLAB13 hrs./wk. of own labor that is surplus (leisure) in labor period 3
81. SURLAB14 hrs./wk. of own labor that is surplus (leisure) in labor period 4

.../Cont'd.

Table VII (Cont'd.)

82.	SURLAB15	hrs./wk. of own labor that is surplus (leisure) during non-harvesting in labor period 5
83.	SURLAB16	hrs./wk. of own labor that is surplus (leisure) during non-harvesting in labor period 6
84.	SURLAB17	hrs./wk. of own labor that is surplus (leisure) during non-harvesting in labor period 7
85.	SURLAB18	hrs./wk. of own labor that is surplus (leisure) during non-harvesting in labor period 8
86.	SURLAB19	hrs./wk. of own labor that is surplus (leisure) in labor period 9
87.	SURLABH5	hrs./wk.of own labor that is surplus (leisure) during harvesting in labor period 5
88.	SURLABH6	hrs./wk. of own labor that is surplus (leisure) during harvesting in labor period 6
89.	SURLABH7	hrs./wk. of own labor that is surplus (leisure) during harvesting in labor period 7
90.	SURLABH8	hrs./wk. of own labor that is surplus (leisure) during harvesting in labor period 8
91.	VALUESUR	dollar equivalent value of surplus farm labor (leisure) for the year
92.	HIRLAB11	hrs./wk. of labor hired in labor period 1
93.	HIRLAB12	hrs./wk. of labor hired in labor period 2
94.	HIRLAB13	hrs./wk. of labor hired in labor period 3
95.	HIRLAB14	hrs./wk. of labor hired in labor period 4
96.	HIRLAB15	hrs./wk. of labor hired for non-harvesting activities in labor period 5
97.	HIRLAB16	hrs./wk. of labor hired for non-harvesting activities in labor period 6
98.	HRILAB17	hrs./wk. of labor hired for non-harvesting activities in labor period 7
99.	HIRLAB18	hrs./wk. of labor hired for non-harvesting activities in labor period 8
100.	HIRLAB19	hrs./wk. of labor hired in labor period 9
101.	HIRLABH5	hrs./wk. of labor hired for harvesting activities in labor period 5
102.	HIRLABH6	hrs./wk. of labor hired for harvesting activities in labor period 6
103.	HIRLABH7	hrs./wk. of labor hired for harvesting activities in labor period 7
104.	HIRLABH8	hrs./wk. of labor hired for harvesting activities in labor period 8
105.	HIRLAB1Ø	total hours of labor hired during the year
106.	CSWPRA13	difference (A-B) equals number of cows transferred from community pasture to rented range at end of feeding period 3
107.	CSWPRB13	
108.	CSWPOA13	difference (A-B) equals number of cows transferred from community pasture, but not to rented range, at end of feeding period 3
109.	CSWPOB13	
110.	CSWROA13	difference (A-B) equals number of cows transferred from rented range, but not to community pasture, at end of feeding period 3
111.	CSWROB13	
112.	CSWPRA14	difference (A-B) equals number of cows transferred from community pasture to rented range at the end of feeding period 4
113.	CSWPRB14	
114.	CSWPOA14	difference (A-B) equals number of cows transferred from community pasture, but not to rented range, at end of feeding period 4
115.	CSWPOB14	

.../Cont'd.

Table VII (Cont'd.)

116.	CSWROA14	difference (A-B) equals number of cows transferred from rented
117.	CSWROB14	range, but not to community pasture, at end of feeding period 4
118.	YSWPRA13	difference (A-B) equals number of yearlings transferred from
119.	YSWPRB13	community pasture to rented range at end of feeding period 3
120.	YSWPOA13	difference (A-B) equals number of yearlings transferred from
121.	YSWPOB13	community pasture, but not to rented range,at end of feeding period 3
122.	YSWROA13	difference (A-B) equals number of yearlings transferred from
123.	YSWROB13	rented range, but not to community pasture,at end of feeding period 3
124.	YSWPRA14	difference (A-B) equals number of yearlings transferred from
125.	YSWPRB14	community pasture to rented range at end of feeding period 4
126.	YSWPOA14	difference (A-B) equals number of yearlings transferred from
127.	YSWPOB14	community pasture, but not to rented range, at end of feeding period 4
128.	YSWROA14	difference (A-B) equals number of yearlings transferred from
129.	YSWROB14	rented range, but not to community pasture,at end of feeding period 4
130.	CAFSAL21	number of calves born in spring of year sold after feeding period 6
131.	YERSAL15	number of yearlings sold just before feeding period 5
132.	YERSAL21	number of yearlings sold just after feeding period 6
133.	CSIA0021	number of calf pounds sold at end of the year that qualify (as calf pounds) for income assurance subsidies
134.	YOWNIA21	number of yearling pounds sold during the year that qualify (as calf pounds) for income assurance subsidies
135.	YSIA0015	number of yearling pounds sold on Nov. 15 that qualify (as yearling pounds) for income assurance subsidies
136.	YSIA0021	number of yearling pounds sold at end of the year that qualify (as yearling pounds) for income assurance subsidies
137.	TOTPOUNC	number of calf pounds raised on-farm during the year
138.	TOTPOUNY	number of yearling pounds raised on-farm during the year
139.	INCOME10	farm family income for the year

4. Peace River Models: Basic Data

Most of the data collected for and ultimately employed in Peace River models is presented here. When parameters are considered relatively uncertain and important in the evaluation of the A.R.D.A. community pastures programs, several estimates are included in programming models in order to estimate the sensitivity of model results to likely errors in data. In several cases, the range within which the "correct" value of the parameter seems likely to lie is presented in parentheses after the "best guess".

1. Farm land supply

   a. 150 acres native grass
   b. 350 acres suitable for tame grass, hay or grain
   c. of these 350 acres, 150 acres are particularly suitable for grain
   d. up to 50 acres each of hay, oats, and barley land can be rented per year
   e. access to community pasture and community range or forestry range.

	Tame AUM's	Native AUM's
June 1 - Sept. 1	180	300
Sept. 1 - Sept. 15	30	50
Sept. 15 - Oct. 1	45	75

Source: B.C.D.A. staff in Dawson Creek, Fort St. John and Prince George, farm interviews.

2. Farm labor supply (= labor hours typically worked by the farm family, per week)

   a. Nov. 1 - Apr. 7, Apr. 21 - May 10 (winter, excluding calving): 75 hrs./wk., including up to 30 hrs./wk. of custom or off-farm work
   b. Apr. 7 - Apr. 21 (calving season): 150 hrs./wk. including up to 30 hrs./wk. of custom or off-farm work
   c. May 10 - July 1: 85 hrs./wk.
   d. July 1 - Sept. 1 (primary harvesting season): 120 hrs./wk.
   e. Sept. 1 - Nov. 1: 85 hrs./wk.

Source: B.C.D.A. staff in Dawson Creek and Fort St. John, farm interviews.

Note: the following estimates were used in arriving at the aggregate farm family labor figures above (excluding calving period)

	Labour hours per week supplied to the farm (excluding custom or off-farm work during winter)				
	Winter	Spring	May-June	July-Aug.	Fall
farm owner-operator	30	60	60	60	60
wife	10	15	15	20	15
older child	5	10	10	40	10

3. Weather constraint on harvesting:
   weather permits harvesting on 60% of days from July 1 to Oct. 7.

Source: farm interviews, B.C.D.A. staff at Fort St. John

4. Feeding requirements

   a. Nov. 1 - May 10 (190 days):
      2.25 t hay/cow, 1.7 t hay/yearling, 7 bu. barley/yearling
   b. May 10 - June 1 (20 days):
      .24 t hay/cow, .16 t hay/yearling
   c. June 1 - Sept. 1 (90 days):
      3 AUM grazing on pasture/cow and yearling
   d. Sept. 1 - Sept. 15 (15 days):
      .5 AUM grazing on pasture/cow and yearling
   e. Sept. 15 - Oct. 7 (22 days):
      .75 AUM grazing on pasture or hay aftermath/cow and grazed yearling,
      .17 t hay/yearling zero grazed, 1.375 bu. barley/yearling
   f. Oct. 7 - Nov. 1 (22 days):
      .75 AUM grazing on hay aftermath/cow and grazed yearling,
      .375 AUM grazing on hay aftermath/weaned calf
      .17 t hay/yearling zero-grazed, 1.375 bu. barley/yearling

   Source: B.C.D.A. staff in Dawson Creek, Fort St. John and Prince George, J. Kidder.

5. Calf and yearling weight gain differences for pasture and range
   (= increase in selling weight per AUM of tame grazing during the period
   minus
   increase in selling weight per AUM of native grazing during the period)

	Calves	Yearlings
June 1-Sept. 15	+15 lbs./AUM grazing by cow (10to20)	+ 21 lbs./AUM
Sept. 15-Oct. 7	+30 lbs./AUM grazing by cow (20to45)	+ 42 lbs./AUM

   Source: R. Beames, B.C.D.A. staff in Fort St. John, G. Kirtzinger

6. Animal selling and purchase dates and selling and purchase weights

   a. Calves sold, and purchased (for backgrounding), on Nov. 1, at 450 lbs. if never grazed on native
   b. Yearlings sold (1) on Sept. 15, at 750 lbs. if never grazed on native (as yearlings or as calves on the farm), or (2) on Nov. 1, at 825 lbs. if never grazed on native (as yearlings or as calves on the farm)

   Source: B.C.D.A. staff in Dawson Creek and Fort St. John

7. Grazing productivity <u>on farm</u>

   a. 3 acres pasture required for 4 summer months of grazing per cow or yearling (3 to 4 acres)
   b. 12 acres range required for 4 summer months of grazing per cow or yearling
   c. hay aftermath grazing capacity:

$$\frac{1.25 \text{ t hay yield}}{\text{ac. in hay production}} \times \frac{1 \text{ t aftermath yield}}{6 \text{ t hay yield}}$$

$$\times \frac{1 \text{ AUM}}{.33 \text{ t aftermath required}} = .63 \text{ AUM per acre of hay aftermath grazed}$$

Source: B.C.D.A. staff in Fort St. John, Prince George consensus report on hay, J. Kidder.

8. Crop rotation: in any year, 3/4 of own acres in hay, in oats and in barley are in production

   Source: B.C.D.A. staff in Dawson Creek and Fort St. John

9. Crop yields per acre harvested (as a function of time of harvest)

	hay	"ave." oatland	"poor" oatland	"ave." barley	"poor" barley
harvested July 1-Aug.1	1.25t/ac(1.0to1.5)	--	--	--	--
harvested Aug.1-Sept.1	1.00t/ac	--	--	--	--
harvested Sept.1-Sept.15	.75t/ac	40bu/ac	30bu/ac	30bu/ac	23bu/ac
harvested Sept.15-Oct.7		35bu/ac	25bu/ac	27bu/ac	20bu/ac

Source: B.C.D.A. staff in Fort St. John, farm interviews, Dawson Creek Rolla concensus report on barley, etc.

10. "Variable costs" per acre of hay and grain (excludes cost of labor and interest on land, includes cost of maintaining capital stock)

hay	\$35/ac in production (30-40)	\$5/ac in establishment
oats & barley	\$45/ac in production	\$5/ac in summerfallow

Source: farm interviews, B.C.D.A. staff in Dawson Creek and Fort St. John, Dawson Creek-Rolla concensus report on barley, etc., Prince George concensus report on hay.

Note: 1) Data in the Prince George concensus report was used only in the estimation of differences between short and long run variable costs, and interest on capital is calculated for an interest rate of 4% (an estimate of the real interest rate for borrowing funds).

2) These estimates of reasonable range of variable costs per acre of hay in production are also consistent with other data for the Peace River model. If aftermath is grazed and own labor (at \$2 /hr.) is used for harvesting, then a hay variable cost of \$40/acre is consistent with a rental rate of \$10/acre (see 12):

TR/ac = 1.25t/ac x \$40/t + .63AUM's aftermath/ac x \$8/AUM
≈ \$55/ac

TC/ac = \$40 VC/ac + 2.5 hrs./ac x \$2/hr. ≈ \$45/ac

On the other hand, if aftermath is not grazed and hired labor (at $4/hr.) is used for harvesting, then a hay variable cost of $30/acre is consistent with a rental rate of $10/acre.

11. Labor requirements for culture and harvest of hay and grain

	Cultivation	Harvesting
hay	.5 hrs/ac cultivated	2.4 hrs/ac harvested = $\frac{\text{3 labor days}}{\text{15 ac harvested, etc.}} \times \frac{\text{12 hrs.}}{\text{labor day}}$
oats & barley	.8 hrs/ac cultivated	.88 hrs/ac harvested

Source: B.C.D.A. staff in Fort St. John, Dawson Creek-Rolla concensus report on barley, etc.

12. Rental rates per acre hay and grain (in current production) of "average" quality

hay: $11.5/ac (standard deviation = 1.7)
oats & barley: $17.5/ac (15-20)

Source: farm interviews, B.C.D.A. in Dawson Creek and Fort St. John.

Note: all acres are cultivated.

13. Pasture grazing fees

a. community pasture: $3.90/AUM
b. community range or forestry range: $0.53/AUM

Source: G. Kirtzinger, B.C.D.A. staff in Dawson Creek and Fort St. John

14. 1975 market prices (Sept.-Nov.)

a. hay: $40/ton
b. oats: $1.60/bu.
c. barley: $2.40/bu.
d. calves: (450 lbs.): 30¢/lb.
e. yearlings: (750 or 825 lbs.): 36¢/lb.
f. cull cows: $150/cow (at 1,000 lbs./cow)

Source: B.C.D.A. staff in Dawson Creek and Fort St. John, Canadian Livestock and Meat Trade Report (June 1975), J. Kidder

Note: As pointed out in the text, 1975 market prices are highly unrepresentative of price in other years. Hence alternative price combinations were employed in the application of the model.

15. 1975 B.C. Income Assurance subsidies for beef producers

    a. 27¢/"qualifying calf pound"
    b. 14¢/"qualifying yearling pound"

    Source: B.C.D.A. staff in Prince George.

16. Incidental costs (vet., etc.)

    a. $25/cow (including calf)
    b. $25/yearling

    Source: B.C.D.A. staff in Dawson Creek and Fort St. John.

    Note: This estimate of $25 incidental expenses per cow is consistent with a more detailed estimate provided by J. Sinclair (1975, p. 262)[1]:
    $13.75/cow for straw, salt, minerals, vet. and medicine
    $10.00/cow for breeding.

17. Depreciation and Interest Costs for Cattle Enterprise

    a. cow-calf

depreciation (on buildings and equipment, fences)	$7/cow
interest on capital excluding land, at 4% (buildings and equipment, fences, cows)	$13/cow

    b. yearling phase

depreciation (on buildings and equipment, fences)	$2/yearling
interest on capital excluding land, at 4% (buildings and equipment, fences, calves)	$A/yearling

$A \equiv$ (revenue from sale of 450 lb. calf) x .04 x B/365
$B \equiv$ # days between potential sale as calf and actual sale as yearling

    Source: article by J. Sinclair (1975)

    Note: Interest on capital is calculated from Sinclair's data for an interest rate of 4% (an estimate of the real interest rate for borrowing funds).

18. Average size of cow herd (1975): 40 cows

    Source: farm interviews.

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[1] Farm Management Specialist, Manitoba Department of Agriculture, Brandon, Manitoba.

19. Calving, mortality and culling rates (%)

	Calving Rate	Mortality Rate	Culling Rate
a. cows more than two years old, i.e. cows that have calved before	85	2	10
b. cows that have not calved before	80	2	10
c. calves	--	8	--
d. yearlings	--	2	--

Source: farm interviews

20. Labor prices

a. wage for hired labor: \$4/hr. at any time during year
b. wage for custom work or off-farm work by the farm family: \$5/hr.

Source: B.C.D.A. staff in Dawson Creek and Fort St. John, farm interviews.

21. Cattle labor requirements

a. during summer (Junel-Oct. 7: 18.5 weeks)

1) inspection of cows and calves by farm labor:
0 hrs./wk. on community pasture
6 hrs./wk. for 60 cows on grazing other than community pasture

ii) inspection of yearlings by farm labor:
0 hrs./wk. on community pasture
6 hrs./wk. for 60 yearlings on grazing other than community pasture.

iii) roundup (and sorting) requirements (total hours for 60 animals)

	com. past.	rented native	own tame	own native
roundup of 60 cows	10	30	2	5
sorting of 60 cows	5	5	0	0
roundup of 60 yearlings	5	35	2	5
sorting of 60 yearlings	5	5	0	0

iv) inspection, roundup and sorting requirements on a given type of pasture or range are assumed to be proportional to the number of cows and to the number of yearlings.

b. rest of year, excluding calving requirements (33.5 weeks)

i) 60 cows plus 60 yearlings: 36 hrs./wk.

ii) these 36 hrs./wk. are assumed to have fixed and proportional components:
36 hrs. = 15 hrs. + .2 hrs./cow x 60 cows + .15 hrs./yearling x 60 yearlings

c. calving requirements (2 weeks)

i) 60 cows: 180 hours total for 2 weeks

ii) calving requiremtns are assumed to be proportional to the number of cows

Source: G. Kirtzinger, J. Kidder.

Note: Sum of a, b, and c (hrs./yr.) equals

$$33.5(15) + 18.5\left(\frac{0 \text{ or } .1}{\text{cow}}\right)(60 \text{ cows}) + 18.5\left(\frac{0 \text{ or } .1}{\text{yearling}}\right)$$

$$(60 \text{ yearlings}) + \frac{.03 \text{ to } .58}{\text{cow}}(60 \text{ cows}) + \frac{.03 \text{ to } .67}{\text{yearling}}$$

$$(60 \text{ yearlings}) + 33.5\left(\frac{.2}{\text{cow}}\right)(60 \text{ cows}) + 33.5\left(\frac{.15}{\text{yearling}}\right)$$

$$(60 \text{ yearlings}) + \frac{3}{\text{cow}}(60 \text{ cows})$$

$$= 502.5 + \frac{9.7 \text{ to } 12.1}{\text{cow}}(60 \text{ cows}) + \frac{5.0 \text{ to } 7.5}{\text{yearling}}(60 \text{ yearlings})$$

So 60 cows, 0 yearlings implies 1084.5 hrs./yr. to 1228 hrs./yr., and 40 cows, 0 yearlings implies 890 hrs./yr. to 986 hrs./yr. This is relatively close to results obtained in a regression analysis of 1971 Alberta Peace River cow-calf enterprises:

labor hrs./yr. for 60 cows $600 + \frac{6}{\text{cow}}(60 \text{ cows}) = 960$

labor hrs./yr. for 40 cows $500 + \frac{8}{\text{cow}}(40 \text{ cows}) = 820.$

[1] Kirk (1972).