TESTS OF THE BLACK SCHOLES OPTION PRICING MODEL

by

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To my Mother

ABSTRACT

Black and Scholes developed the first Option Pricing Model based on observable variables. This model was subsequently extended by Merton, Cox and Ross, Schwartz and others. In the past, empirical studies using the Black-Scholes option pricing model have obtained fairly satisfactory results. However, these tests have either assumed that discrete hedging will not significantly affect the results in any way or that it causes uncertain returns which could be diversified away. This paper shows that the use of discrete hedging will result in a significant bias in the excess returns. As such a bias is shown to be a function of the distribution of the rate of return on the stock, there is a possibility that the covariance between the excess return on a hedged position and that of the market are not zero. This implies the existence of systematic risk which could not be diversified away.

Tests of the Montreal Stock Exchange's option market were also carried out. These tests were subjected to certain statistical problems as assumptions of the regression model used were violated. Despite these violations, the results indicate that profit opportunities do exist in the market. However, it is doubtful that such profit opportunities would still exist if transaction costs etc. are taken into consideration.

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1. INTRODUCTION

An option is a contract to buy (call options) or sell (put option) a specified number of shares in a given time period at an agreed price. An 'American' option is one that can be exercised at any time up to the date the option expires. An 'European' option on the other hand is one with the exercise right limited to a specified future date. As put and call options are the simplest forms of contingent claims, it is generally believed that a study of these will provide insights into more complex contingent claims situations.

In 1973, Black and Scholes [2] published a paper which represented a major breakthrough in this area. Since then, a substantial amount of research has been done in this field. Most of this research has been of a theoretical nature. The few studies dealing with empirical tests have all used American Data.

The purpose of this paper is twofold; First, to test the performance of the Black-Scholes option pricing model using data from the Montreal Option Market and second to test the efficiency of that market. Efficiency refers to the possibilities of earning higher than normal returns after taking into consideration the risk taken.

Section 2 introduces the Black-Scholes model and discusses some subsequent criticisms and extensions. Section 3 presents some of the major empirical tests. Section 4 analyses the effect of discrete hedging on excess returns. Section 5 describes the main features of the Montreal Option Market. A description of the data used is given in Section 6. Section 7 discusses the tests performed and the results and Section 8 presents some conclusions and suggestions for further research.

2. THEORY

2.1 The Black-Scholes Model:

The earliest option[1] valuation formula was derived by Bachelier (1900). Since then, there have been papers by Sprenkle (1962), Boness (1964) and Sameulson (1965) and others. A good survey of these earlier results outlining their strengths and limitations is given in Smith [11].

Black and Scholes [2] present the first complete option pricing model which depends only on observable variables. Their model is developed on the principle that in equilibrium conditions assets which are similar would yield the same rate of return. In deriving their formula for the value at time t, of a European option which matures at time t*, the following assumptions were made:

(1) The stock price follows a Geometric Brownian motion through time, i.e.

$$\frac{dS}{S} = \mu dt + \sigma dz \qquad (2.1)$$

where S is the value of the stock, μdt is the drift or deterministic term and σdz is the stochastic term with dz following a normal distribution with mean zero and variance dt.

(2) The short term risk free interest rate is known and is constant through time.

(3) The stock pays no dividends or other distributions during the life of the options.

(4) The option is European, i.e. it can only be exercised at date of maturity.

(5) There are no transaction costs or taxes. Borrowing and lending interest rates are equal and that there are no

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(1) Options referred to in this paper are call options.

restrictions on short-selling.

With these assumptions, the value of the options can be expressed as a function of S, the stock price and T, the time to maturity. ($T = t^* - t$). It is then possible to create a long (short) position in the stock and a short (long) position in the option in such a way that profits (losses) as a result of a rise (fall) in stock prices will be offset by losses (profits) as a result of the short (long) position in options. Consequently, the value of this hedged portfolio will not depend on the price of the stock but on T, the time to maturity and other constants. If we write the value of the option as $C(S,T)$, then changes in the stock price dS will change the value of the option by approximately

$$C_S dS$$

where C_S is the partial derivative of $C(S.T)$ with respect to S. If we were to establish a hedged portfolio such that for one share of stock long, we sell $\frac{1}{C_S}$ number of options short then the change in the value of the hedged portfolio as a result of a change in stock price of dS will be

$$dS - \frac{1}{C_S} C_S \, dS \doteq 0 \qquad (2.3)$$

By adjusting the hedged portfolio continuously, the return on the hedged portfolio will then be non-stochastic and therefore riskless. Consequently the portfolio must earn the riskless rate of return.

In general, if the value of a hedged portfolio consisting of one share of stock long and $\frac{1}{C_S}$ options short is

$$V = S - \frac{1}{C_S} C \qquad (2.4)$$

where C is the value of one option. The change in the value of the portfolio in a short interval dt is

$$dV = dS - \frac{1}{C_S} dC \qquad (2.5)$$

In equilibrium conditions, the rate of return on V should be equal to the riskfree rate. Therefore

$$\frac{dV}{V} = rdt \tag{2.6}$$

Substituting (2.4) and (2.5) into (2.6) we have

$$dS - \frac{1}{C_S} dC = (S - \frac{1}{C_S} C) \; rdt \tag{2.7}$$

Using stochastic calculus to expand dC and simplifying, Black and Scholes found that

$$C_T = rC - rSC_S - \tfrac{1}{2} C_{SS} \sigma^2 S^2 \tag{2.8}$$

which defines a differential equation for the values of the option subject to the boundary condition that

$$C(S,0) = \text{Max} \; [0, S-E] \tag{2.9}$$

This condition states that at maturity date the option price must be equal to the maximum of either the difference between the stock price and the exercise price or zero.

Transforming the differential equation (2.8) into the heat exchange equation from physics and solving, Black and Scholes arrived at the option valuation formula:

$$C(S,t) = S.N(d_1) - Xe^{-rT} N(d_2)$$

where $N(d_1) = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$ (2.10)

$d_2 \;=\; d_1 - \sigma\sqrt{T}$

C = price of the option for a single share of stock

S = the current price of the stock

X = the striking or exercise price of the option

r = short term interest rate (riskless)

σ^2 = variance of the rate of return on stock

T = time to maturity

$N(d_1)$ = cumulative nominal density function

Equation (2.8) is a function of the variables, r, S, T, σ^2, X. Cox and Ross [5] have noted that in deriving it, no assumptions were made with regard to the preferences of the investors except that two assets which are perfect substitutes should earn the same rate of return. This implies that the solution obtained by solving for equation (2.8) will hold for any preference structure. Therefore, if we assume that the market is composed of risk-neutral investors, then the equilibrium rate of return for the stock price and the option should be equal to the riskless rate of return, r. The option price will then be the discounted expected option price at the date of maturity. The expected option value at maturity can be obtained by integrating over the distribution of stock prices at maturity to obtain

$$Se^{rT} N(d_1) - XN(d_2) \tag{2.11}$$

Multiplying (2.10) by the discount factor of e^{-rT}, we obtain equation (2.9).

2.2 Extension of the Black-Scholes Model

The basic Black-Scholes model has been extended in a number of ways by modifying the underlying assumptions.

With no dividend payment, Merton [8](2) has shown that prior to expiration, the holder of an American option will rather sell than exercise the option. This implies that an American call will never be

(2) Refer to Merton [8] for a more detailed discussion.

exercised before its maturity date and thus will be valued at the same price as a European call. Therefore, the Black-Scholes model could be used to value American options on non-dividend paying stocks.

When dividends are paid and assuming that D is the continuous dividend per share per unit time, Merton [8] arrived at the following differential equation for the value of an option:

$$C_T = \tfrac{1}{2}\sigma^2 S^2 \; C_{SS} - (rS - D)\; C_S - rC \tag{2.12}$$

where subscripts denote partial derivatives.

If the option is European, equation (2.12) is subject to the boundary condition

$$C(S,0) = \mathrm{Max}[0, S - E] \tag{2.9}$$

A general closed formed solution for equation (2.12) subject to boundary conditions (2.9) has not been found. Only in one particular case has a solution been found for the value of a European call with a finite time to expiration. This is the case when underlying stock is assumed to pay a constant dividend yield of ρ. i.e. $D = \rho S$.

Substituting $D = \rho S$ in equation (2.12) and solving Merton arrived at the following formula for the European option or dividend paying stock;

$$\begin{aligned} C(S_{div}, T) = e^{-\rho T} S.N \left\{ \frac{\ln(S/X) + (r - \rho + \sigma^2/2)T}{\sigma\sqrt{T}} \right\} - \\ Xe^{-rT}\, N \left\{ \frac{\ln(S/X) + (r - \rho - \frac{\sigma^2}{2})T}{\rho\sqrt{T}} \right\} \end{aligned} \tag{2.13}$$

However, with dividend payments, the value of the American option is no longer equal to the value of the European option because of the possibility of premature exercise. Therefore equation (2.13) is not applicable to the American option.

To arrive at the equation for the value of the American option, it is necessary to account for the possibility of premature exercise. Equation (2.12) will be subject to the following additional boundary condition:

Assuming that the stock price will drop by the amount of the dividend immediately after the stock goes ex-dividend, the value of the option for the ex-dividend stock price must be greater than the exercise price cum-dividend, because, otherwise it will be exercised.

No closed form solution has been found for this case. However, Schwartz [10] successfully employed numerical integration to solve this equation and arrived at an optimal exercise strategy.

Boyle [4] adopted a different approach whereby the distribution of the stock price at maturity date can be generated using Monto Carlo simulation techniques. The advantage of such a technique is the ease at which dividends can be taken in consideration. Furthermore, a change in the underlying distribution of stock prices can be effected by changing a different process of generating the random variables employed. The method was used to evaluate a hypothetical options using 5,000 trials. These results were compared to that which were arrived at using numerical integration and in all cases the 95% confidence interval contains the true answer.

So far, stock prices are assumed to follow a log-normal diffusion process. However, Cox and Ross [5] argued that new information tends to arrive in the market in discrete lumps rather than a smooth flow. Therefore, stock prices can be more accurately described as following a jump rather than a diffusion process. Assuming that the percentage change in the stock price from t to t + dt is

$$\frac{dS}{S} = \mu dt + \begin{cases} \xrightarrow{\lambda dt} \; k-1 \\ \xrightarrow[1-\lambda dt]{} \; 0 \end{cases} \tag{2.14}$$

where k-1 = the jump amplitude

μ = drift term

λdt = probability of a jump in the time interval dt

they were able to form a hedged position with the stock, the option and the riskfree bond and arrive at a complicated formula for the option price independent of λ.

3. EMPIRICAL TESTS

To date, the only major empirical tests of the Black-Scholes option pricing model have been performed by Black and Scholes in 1971 and Galai in 1975. As it is intended to replicate some of these tests, the procedures employed and the results obtained are discussed in considerable detail.

Black-Scholes Tests

The first empirical tests of the model were performed by Black and Scholes [1], who obtained their data from the diaries of an option broker from 1966 to 1969. At the writing date of the call option, the model was used to compute its theoretical value as well as the number of shares to balance against each option, i.e. $N(d_1)$.(3) If the market price was greater (less) than the model price the option was considered overvalued (undervalued). Four portfolios were constructed based on the following strategies.

a) Buying all calls at model prices

b) Buying all calls at market prices

c) Buying undervalued calls and selling overvalued calls at model prices

d) Buying undervalued calls and selling overvalued calls at market prices.

On every purchase (sale) of a call option a hedge was established by selling (buying) $N(d_1)$ number of shares. Each portfolio consisted of a number of hedges. However, as the proportion $N(d_1)$ was not adjusted continuously, the hedge will generate an uncertain return. Black

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(3) $N(d_1)$ is the derivative of the Black Scholes option pricing formula with respect to the stock price.

and Scholes [2] however have shown that this return is theoretically uncorrelated with the market and could therefore be diversified away by holding a portfolio of hedges. As the market will not compensate the investors for diversifiable risk, the return on the hedge should be equal to the riskless rate r. Therefore

$$dC - C_S dS = (C - C_S S)\ r\Delta t \tag{3.1}$$

The realized excess dollar return is therefore defined as

$$dC - C_S dS - (C - C_S S)\ r\Delta t \tag{3.2}$$

The period from May 1966 to July 1969 was divided into ten sub-periods. For each portfolio the realized excess dollar returns were aggregated daily to form a total daily portfolio return. A regression model of the form

$$\tilde{R}_{pt} = \alpha_p + \beta_p \tilde{R}_{mt} + \varepsilon_t \tag{3.3}$$

where

$\tilde{R}_{mt}$ = Return on Standard and Poor Composite Index on day t

$\tilde{R}_{pt}$ = Total portfolio return on day t

α = The intercept α. Its significance is taken to be a measure of the performance of the model for the respective portfolios.

β = Slope coefficient

$\varepsilon_{\tilde{t}}$ = Residual.

was used for each of the ten sub periods and the total period for each portfolio. The results obtained could be summarized as follows:

i) As expected the $\hat{\beta}$ coefficients were insiginficantly different from zero

ii) For the first two strategies where the contracts were purchased at model and again at market prices, the excess portfolio returns were insignificantly different from zero

iii) When undervalued contracts were bought and overvalued contracts were sold at model prices, the $\hat{\alpha}$'s were found to be significantly negative. However, when this strategy was performed with market prices, the $\hat{\alpha}$'s were significantly positive. In other words, buying and selling at model prices will incur significant losses while buying and selling at market prices results in significant gains.

Black and Scholes explained the findings in terms of the measurement errors of past variances. Estimates of true variances using past variances result in a wider spread in the distribution of the estimated variance than that of the true variance. Consequently, high variances will be over-estimated while low variances will be under-estimated. Therefore the model tends to overprice options on high variance stock and underprice options on low variance stock. However, the reverse is true in the case of the options trader, i.e. the estimation of the spread of the distribution of variances is smaller than what it should be. Therefore, the market will tend to underprice options of high variance stock and overprice options on low variance stock. If the true option price is between the model and the market price, then buying and selling at model prices results in buying high and selling low thus explaining the significantly negative returns. Using market prices results in the opposite effect. To test their hypothesis Black and Scholes divided all the stocks into 4 portfolios. The first portfolio consisted of stocks which have the lowest 25% of the variance and so on. Contracts were bought and sold as before using model and market prices. Realized excess portfolio returns were calculated and regressed against the market. The results clearly showed that the model over-estimates the values of options on high variance securities and under-estimates the values of options on low variance securities. The reverse was true for the market. Thus it would be possible to make money by following a strategy of selling contracts on low variance security and by buying contracts on high variance securities. However, Black-Scholes found that this profit would be eliminated as a result of transaction costs of trading in the current option market. To demonstrate that the model could price correctly if it has the proper variance, Black and Scholes repeated the above tests using variances calculated with the actual stock prices over the life of the option. The results for the portfolio using the first two strategies were as before insignificant. However, in contrast to previous tests, the results when contracts were bought and sold at model prices were insignificant while trading at market prices resulted in a very significant postive return. These tests are in a way biased against the market but nevertheless show that the model will perform well given the correct variance.

3.2 Galai's Tests

The second set of empirical tests on the Black-Scholes option pricing model was performed by Galai [6] on the Chicago Board Options Exchange. Using data from April 20, 1973 to November 30, 1973, Galai divided his tests into ex post and ex ante tests. Ex post tests assume that it is possible to trade on a day's closing prices after the trading rule has been based on that price. Ex ante tests on the other hand assume that transactions can only take place on the next day's closing prices.

Ex Post Tests: Initially Galai duplicated the Black-Scholes tests by maintaining the initial option position and adjusting the hedge through changes in stock position. Returns on the hedge are calculated in two ways. The first method is calculated as follows; on the day after the hedged position was established

$$R_{Ht} = (C_t^M - C_{t-1}^A) - N(d_1)_{t-1} (S_t - S_{t-1}) \tag{3.4}$$

where R_{Ht} = Return on the hedge in period t

C_t^M = Model option price on period t

C_{t-1}^A = Actual option price on period t-1, i.e. the day the option was first traded

S_t = Stock price at period t.

For subsequent days, the model prices are assumed to be the actual option prices and

$$R_{Ht} = \{C_t^M - C_{t-1}^M\} - N(d_1)_{t-1} (S_t - S_{t-1}) \tag{3.5}$$

The second method uses actual option prices for the first as well as subsequent days to calculate R_{Ht}, so

$$R_{Ht} = (C_t^A - C_{t-1}^A) - N(d_1)_{t-1} (S_t - S_{t-1}) \tag{3.6}$$

The returns on the hedged positions were tested and Galai reported that they appeared to fit a normal distribution. The average daily return of each hedged position was then tested for significance using a t test.

The above equations for calculating returns differ from those used by Black and Scholes (equation (3.2)) by a term $(C - C_S S)\, r\Delta t$. This interest rate factor was omitted because the effect was found by Galai to be minimal.[4]

Using the above test procedure, the average returns for almost all of the hedged positions using the second method of computation were not significantly different from zero. This is in contrast with results obtained using the first method where almost all the hedged positions were found to be significantly different from zero. Galai explained these results as follows: Actual option prices differ from the true model prices by an error term. This can be written as:

$$C_A = C_M + \varepsilon \tag{3.7}$$

C_A = Actual option price

C_M = Model price

ε = Error term where $E(\varepsilon) = 0$

The variance of returns calculated using the actual prices will therefore be greater than those using the model price. Consequently, most of the t statistics of the coefficients using the model prices will be divided by a smaller variance thus resulting in their significance.

The original B-S tests were performed on the over-the-counter options for which there were no secondary markets. Therefore, it is necessary to

i) assume model prices to be actual prices after the first day

ii) maintain the hedge position by adjusting the stock

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(4) A 10% riskfree rate compounded daily is 0.00027401 if the equity position is \$10. This interest rate factor is only \$0.0027401.

position daily

However, Galai's tests were done on the Chicago Board Option Exchange in which there is a very active secondary market. The adjustment of the hedge by changing stock positions will ignore the information available in the daily deviations of actual option prices from model prices. Therefore the second ex post test was performed by adjusting the hedge positions through changes in the option position.

For each day, the model price C_t^M was compared to the actual option price C_t^A. If $C_t^M > C_t^A$, then the option would be purchased at the day's actual closing prices and immediately an amount $N(d_1)$ of underlying stock sold. If $C_t^A \geq C_t^M$, then the option would be sold and $N(d_1)$ stocks purchased. This hedged position at t is assumed to be liquidated at t+1 and the return is

$$R_{Ht} = (C_{t+1}^A - C_t^A) - N(d_1)_t \; (S_{t+1} - S_t) \tag{3.8}$$

When the hedge was liquidated at t+1, a new hedge was immediately set up based on the relationship between C_{t+1}^M and C_{t+1}^A. This procedure was repeated and the average returns at the end of the period was calculated. Comparing the results with the first test, shows that the model is able to differentiate over and underpriced options very well. The average returns was \$10 per option per day versus about \$1 in the case of the first tests.

As a variation of the above tests, the hedge was not liquidated until the second, third or fifth days, i.e. if a position was taken at day t, it will not be liquidated until t=2. Meanwhile at t=1 another position was taken using t+1 prices. The average returns for maintaining the hedge for two days is still significant though the amount was less than that derived for the 1 day holding period. No attempts were made by

Galai to adjust for the correct transaction costs in each case. Instead, an arbitrary percentage of 1% was selected and its inclusion eliminated the significant returns arrived at in earlier tests.

The Black-Scholes option pricing model assumes zero dividend payment during the life of the options and Galai in performing the above tests ignored the dividend payments. He however tested the effect of dividend payments in the following ways:

i) Four portfolios were constructed with the first portfolio consisting of all options on stocks with the lowest dividend yield and the last portfolio consisting of all options on stocks with the highest dividend yield. The daily average returns on each portfolio were regressed against the return on Standard and Poor's Index. As the daily average return on portfolio is based on a different number of options traded, a problem of heteroskedasticity was suspected. To overcome this problem a weighted least square was performed. As in the B-S case, the intercept $\hat{\alpha}_p$ indicates the average profits on a portfolio with zero systematic risk.
The $\hat{\alpha}_p$ for the four portfolios were found to be significantly different from zero. However as he moved from the highest dividends portfolio to the lowest, the $\hat{\alpha}_p$ increases.

ii) The returns for hedged positions using equation (3.8) were calculated for all hedges where no ex-dividend dates were expected during the period remaining to expiration. The overall daily average returns was \$10.7 per contract and the majority of the hedged positions had significant average returns.

iii) Solutions of the dividend adjusted differential equation using numerical methods are expensive with more advanced computers. Galai therefore used a simple adjustment to the Black-Scholes formula. The adjustment is carried out by subtracting the present value of the expected dividends from the price of the underlying security. The option pricing equations become

$$C_t\ (V, D_t, \quad \ldots.\ D_{tn}) = (V - \sum_{i=1}^{n} e^{-rT} D_{ti})\ N(d_1) - X_e^{-rt}\ N(d_2) \qquad (3.9)$$

where

$$d_1 = \{\ln[(V - \sum_{i=1}^{n} e^{-rt} D_{ti})/X] + (r + \frac{\sigma^2}{2})T\}/\sigma\sqrt{T}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Unpublished simulated analysis by Black shows that this adjustment produces very similar results to that arrived at using numerical integration.

Model prices and $N(d_1)$ calculated using the above equations were then used in equation (3.8) to arrive at the average returns for each hedged position. The results showed that the overall daily average returns were higher than those arrived at using the model without adjusting for dividends. Furthermore, the number of hedged positions with significant average returns were greater.

Generally the results of the tests support the hypothesis that the model performs better in situations in which the assumptions are better met.

Ex Ante Tests: Instead of using the closing prices of the same day for determining and establishing the hedge, it is assumed in these tests that the trader had to wait 24 hours before executing the desired transaction. On day t the Black-Scholes option pricing formula was used, as before, to determine if an option is over or under priced. However, the hedge is established based on t+1 closing prices and liquidated at t+2 closing prices. The return on the hedge is calculated as

$$R_{Ht+2} = (C^A_{t+2} - C^A_{t+1}) - N(d_1)_t \; (V_{t+2} - V_{t+1}) \qquad (3.10)$$

The returns for the ex ante tests were significant. However, the averages were lower than those obtained on the ex post tests. Repeating the tests with a holding period of 2 days results in substantially lower returns.

Effects of dividends were tested by repeating the procedures in the ex post tests. Dividing the options into 4 portfolios ranged according to the dividend yields of the underlying stock, provided results similar to that of the ex post tests. However, if a dividend adjustment was made to the Black-Scholes formula, the result was significantly different. In the ex post test adjustment of B-S formula results in significantly higher average returns, but with the ex ante tests, the average returns were similar to those when no adjustments were made. Galai explained that "the added accuracy in determining the hedge ratio and the position is washed out by delaying execution by one day".

Options spreading is a technique commonly used by traders. It refers to the position of being long and short on different options of the same underlying stock. Galai tested the performance of the model based on the spreading strategy by using a spreading ratio derived from Black-Scholes Formula. The ratio is derived as follows: If we buy 1 share of stock i, hedging is done by selling $1/N(d_{1k})$ of option k. However, we could also establish another hedge by selling 1 share of stock i and buying $1/N(d_{1j})$ of option j. Note that option k and j have the same underlying stock but differ in striking price or maturity date. As the one share of stock i is common in both hedges, it is possible for us to arrive at the same riskless position by selling $1/N(d_{1k})$ of option k and buying $1/N(d_{1j})$ of options j or by buying one option j and selling $N(d_{1j})/N(d_{1k})$ of option k.

On day t, the decision as to whether to buy or sell option k (or option j) will depend on the ratio of C^M_{jt}/C^M_{kt}. If $C^M_{jt}/C^M_{kt} > C^A_{jt}/C^A_{kt}$, then we will buy 1 option j and sell $N(d_{1j})/N(d_{1k})$ of option k. At t+1 this position is liquidated and a new position based on closing prices at t+1 will be

$$R_{st+1} = [C_{jt+1} - C_{jt}] - \frac{N(d_{1jt})}{N(d_{1kt})} [C_{kt+1} - C_{kt}] \qquad (3.11)$$

In cases where $C^M_{jt}/C^M_{kt} < C^A_{jt}/C^A_{kt}$, 1 option j will be sold and $N(d_{1j})/N(d_{1k})$ of option k purchased. The return will then be

$$R_{st+1} = \frac{N(d_{1jt})}{N(d_{1kt})} [C_{kt+1} - C_{kt}] - [C_{jt+1} - C_{jt}] \qquad (3.12)$$

The results showed that:

i) the average returns for the spreading strategy were greater than the average returns for hedging strategy

ii) the options with high average returns were usually on high priced stock

The above spreading strategy is not operational as it is assumed that closing prices at which the decision rule is based upon is known before hand so that trading can be done at that closing price. To make it operational, Galai used day t closing prices to determine the position to take. Transactions were done using t+1 prices and the hedged position liquidated with t+2 closing prices (as in the ex ante tests). The average returns were calculated and found to be less than half of those obtained using the ex post spreading strategy. However, they were still significant and above normal profits could still be made.

4. EFFECT OF DISCRETE HEDGING ON EXCESS RETURN

The empirical tests of the Black Scholes option pricing model performed so far produced fairly satisfactory results. However, these tests were carried out on the assumption that the use of discrete instead of continuous hedging will not in any way bias the excess returns. Black and Scholes [2, pp. 642] wrote:

"Note also that the direction of the change in the equity value is independent of the direction of change in the stock price. This means that.......the covariance between the return on the equity and the return on the stock will be zero. If the stock price and the value of the "market portfolio" follow a joint continuous random walk with constant covariance rate, it means that the covariance between the return on the equity and the return on the market will be zero."

Since the covariance between the return on the equity and the return on the market will be zero they justified the use of discrete hedging on the grounds that "if the position is not adjusted continuously, the risk is small and.......can be diversified away by forming a portfolio."

In this section we will attempt to show that with discrete hedging, although the direction of the change in the equity value is independent of the direction of change in the stock price, it is however not independent of the magnitude of the change in the stock price. Consequently it is possible that the covariance between the return on the market and the return on the equity will not be zero resulting in systematic risk that is not diversifiable by the formation of the portfolios.

Figure I illustrates the relationship between the option value and the stock price at a point in time. The curves W and W_1 represent the value of the option at T and T-1 days to maturity respectively. On day

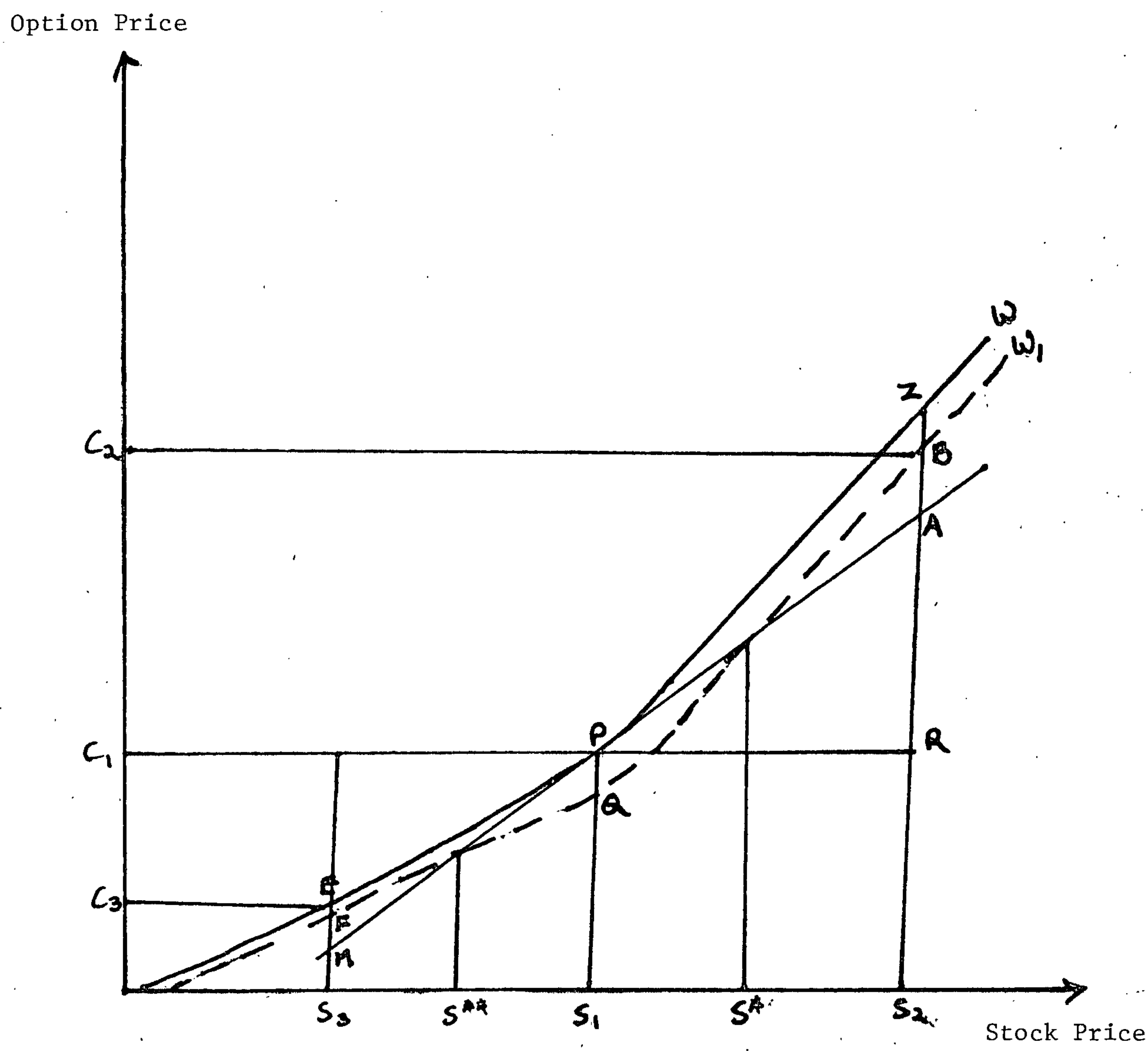

Figure I: Relationship Between Stock and Option Price

T, assuming the stock price is S_1, which gives an option value of C_1 a hedged position is established by buying one option at C_1 and selling $N(d_1)$ stocks at S_1. Since $N(d_1)$ is the derivative of the option pricing formula with respect to the stock price, it is therefore the slope of the curve W at point P. This slope can be better represented by drawing a line AM tangent to the curve W at point P.

Assuming that the hedged position is adjusted daily, the effects on the hedged position is two-fold. First, the $N(d_1)$ used will be represented by the slope of the line AM instead of the slope at various points of the curve W. Second, the value of the option will reduce by a discrete amount represented by a downward shift in the curve W to W_1 as we approach maturity.

As we move from T to T-1 days to maturity, let us assume that the stock price moves from S_1 to S_2. The option price will move from C_1 to C_2. The excess return on the hedged position is calculated as:

$$(C_2 - C_1) - N(d_1)\ (S_2 - S_1) \quad (5) \qquad (4.1)$$

With the stock price at S_2, the slope of the line AM has a lower gradient than the slope of W at the point Z. This implies that the $N(d_1)$ used in establishing the hedged position is lower than what it should be if continuous hedging is possible. The use of a lower $N(d_1)$ will bias the excess return upward. This bias is represented by AZ.

The downward shift of the curve from W to W_1 will result in a negative bias (represented by ZB) as C_2 is lower than what it would be if Δt is very small. Note that this bias is a function of Δt and not the

(5) To simplify the analysis the interest rate adjustment factor of $(C_1 - N(d_1)S_1)r\Delta t$ is ignored at this stage. It will be considered subsequently.

change in stock price. Since we assume Δt to be 1 day, this effect causes a small negative bias independent of the change in the stock price.

The net result of these two effects (represented by AB) is positive as ZA is greater than ZB. AB is also the excess return as calculated by equation (4-1). This can be shown as follows:

The term $(C_2 - C_1)$ in equation (4-1) is represented by BR and $N(d_1)$ $(S_2 - S_1)$ is the distance AR. The excess return is therefore BR - AR = AB.(6)

Using the same analysis, if the stock price moves downward from S_1 to S_3, the $N(d_1)$ used in the establishment of the hedged position is higher than what it should be. This will still cause a positive bias as the excess return is now calculated as

$$N(d_1)\ (S_1 - S_3) - (C_1 - C_3) \tag{4-2}$$

For any movement of stock prices from S_1 the excess return is represented by the vertical distance between the curve W_1 and the line AM. It can be clearly seen that if the change in stock price is 'small' (stock price falling within S* and S** on the next day) the excess return will be negative. For a 'large' change in stock price (stock price falling beyond S* or S** on the next day) the excess return is positive.

The effect of using yesterdays $N(d_1)$ causes a positive bias which increases with the size of the stock price changes but decreases to zero if the stock price remains unchanged. On the other hand the downward shift in the curve from W to W_1 causes a small negative bias independent of the stock price movement. The net result will depend on the size of each bias. Hence, for 'large' (small) changes in stock price the positive

(6) This was pointed out to me by Phelim Boyle.

(negative) bias outweighs the negative (positive) thus incurring a net positive (negative) excess return. Note, however, that the net negative excess return cannot exceed PQ whereas the amount of the positive excess return increases rapidly with the size of stock price changes beyond S* or S**. Therefore, we will expect that the distribution of the excess return to be skewed rather than normal.

Having shown that excess return is indeed a function of the magnitude of changes in the stock price, the interest rate adjustment factor will be introduced.

Interest rate adjustment is calculated as

$$(C_1 - N(d_1)S_1)r\Delta t \tag{4-3}$$

with r as the riskfree rate. As $N(d_1)S_1$ is always greater than C_1 the addition of this positive constant to the excess returns will tend to inflate the amount of positive and reduce the amount of negative excess return.

Theoretically, this interest rate adjustment factor is supposed to offset the reduction in the option values as we move from T to T-1. In figure 1, the reduction in the option value is the vertical distance between the curves W and W_1, the effect of the interest rate adjustment factor can be seen as moving the W_1 curve to the same position as the W curve thus eliminating the negative bias discussed above. When this occurs, the effect of the discontinuous hedging on excess return is a positive bias which gets larger the further the stock price on the next day is away from S_1.

If the interest rate adjustment factor is insufficient, then the curve W_1 will not move back to the same position as W but will be some-where between the present W and W_1. If this occurs, the phenomenon of obtaining negative excess returns for small changes in stock prices and positive excess returns for large changes in stock prices will occur.

As a validation and also to determine if the analysis is applicable in cases when the rates of return of stock prices do not follow a log-normal distribution a simple simulation study was done. 100 stock prices were obtained by simulating the rates of return firstly from a log-normal and subsequently from a normal and t distribution with two degrees of freedom. The hedged positions were formed and the excess return calculated as before. A sort of the excess return in ascending order of stock price changes clearly shows that in every case, a 'large' jump in stock price results in a positive returns while a 'small' jump results in a negative returns. The phenomenon is present whether the stock price returns follow a log-normal, normal or t distribution. This result tends to suggest that the interest rate factor described in the previous paragraph is insufficient.(7) Histograms of the excess returns were plotted for the three distributions. As expected they show a skewness to the left. As an illustration, the histogram of excess return using a log-normal distribution is presented in figure II.

In summary, the interest rate adjustment factor tends to reduce the range within which stock price movements will cause a negative excess return. But it is, however, not big enough to eliminate the negative return altogether. Therefore the effect of using discrete instead of continuous hedging will cause the excess return for a particular day to be a function of the stock price changes. For the period over which the option exists, the distribution of excess returns will be a function of the distribution of rates of return of stock prices of that period. If the distribution of

(7) In Appendix III, it is shown that the interest rate adjustment factor is insufficient by $Xe^{-rT}[Z(d_2)\frac{\sigma}{2\sqrt{T}}]\Delta t$ where Z is the normal density function.

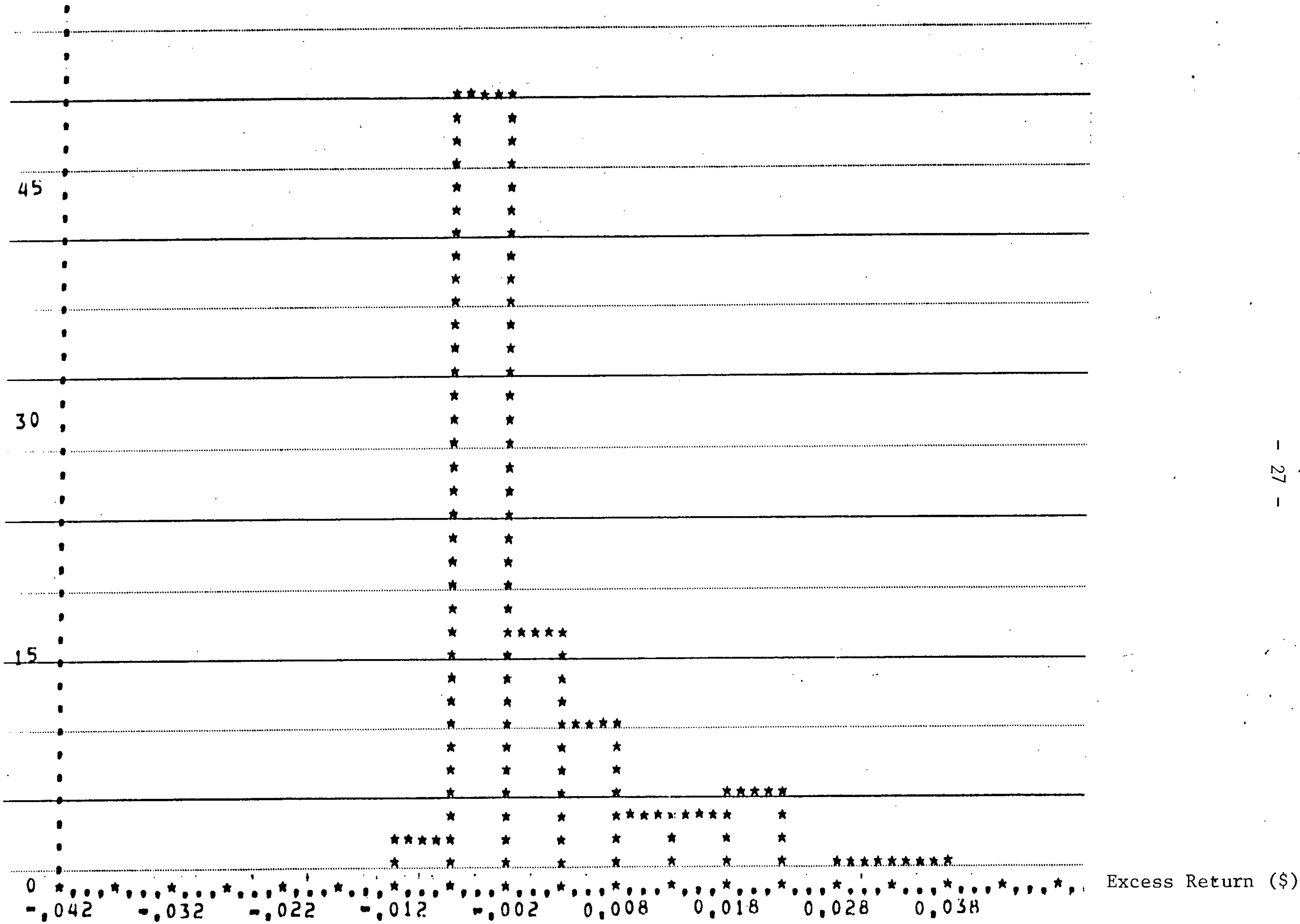

FIGURE II: Histogram of Excess Returns for a Log-Normal Distribution rates of return of stock prices during the period is almost uniform, then the distribution of the excess return will probably be skewed to the left. On the other hand, if the distribution is concentrated around the mean then we will expect the reverse to be true.

5. THE CANADIAN OPTION MARKET [8]

Introduction:

Call options were first traded on the Montreal Stock Exchange on September 15, 1975 through the Montreal Option Clearing Corporation. On September 29, 1975, the Canadian Option Clearing Corporation (COCC) was set up as a successor to the Montreal Option Clearing Corporation. The function of COCC was described as ".....issuing and acting as the primary obligor on the Exchange Traded Options and for clearing transactions in Options". The shares of the COCC were divided equally between the Montreal Stock Exchange and the Toronto Stock Exchange. Its name was subsequently amended to Trans Canada Options Inc. As of July 1977, options on 23 stocks are being traded on both exchanges. (A listing of these stocks is given in Appendix II.)

Descriptions of Options

Options on underlying securities are permitted only if the underlying securities meet the following conditions:

i) The underlying security is listed on one of the Exchanges.

ii) The issuer of the underlying security is incorporated, organized or continued under the laws of Canada or a province or territory thereof.

iii) The issuer and its consolidated subsidiaries have had a net income, after taxes but before extraordinary items net of tax effect, of at least $500,000 for each of the last 3 fiscal years.

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[8] This section draws heavily from the prospectus of Trans Canadian Option Inc., May 6, 1977.

iv) The issuer and its consolidated subsidaries have not defaulted during the past 3 fiscal years in the payment of any dividends, or sinking fund instalment, interest and principal of any borrowed money or in payments of rental under long-term leases.

v) The issuer earned in three of the last five fiscal years, any dividends, including the fair market value of any stock dividends, paid in each such year on all classes of securities.

There may be exceptions to requirements (iii) to (v) of the above.

Besides the above requirements, the Corporations also established certain guidelines for the selection of the underlying securities. They are:

i) The issuer of the underlying securities shall have outstanding a minimum of 5,000,000 shares.

ii) The issuer shall have a minimum of 5,000 registered shareholders.

iii) Total trading value in the underlying security on all stock exchanges on which it is listed shall have been at least 800,000 shares per year in each of the past two years.

iv) The market price of the underlying security shall be at least $5 per share.

Maturity dates of options on underlying securities traded on both exchanges or only on the Montreal Stock Exchange are restricted to February, May, August and November. At any point in time only options with the next three maturity dates will be opened for trading, e.g. in June 1977, options with maturity at August, November and February will be opened for trading. Trading on options once opened, subjected to certain restrictions, will remain opened until the third Friday of the expiring month.

As prices of the underlying stock fluctuate, new options with higher or lower exercise price will be introduced by the exchange. The new exercise price will differ from the old price in multiple of \$2.50 for shares trading below \$25.00, \$5.00 for shares trading between \$25.00 and \$50.00, \$10.00 for shares trading between \$50.00 and \$100.00, \$20,00 for shares trading between \$100.00 and \$200.00 and lastly \$25.00 for shares trading above \$200.00 Departure from this general practice is permitted if the result would be to provide better liquidity for options covering a particular underlying security. Options are traded in multiples of 100 and are exercisable any time after issuance until expiration except in the following circumstances:

i) The number of options that can be exercised covering the same underlying securities is restricted to 1,000 contracts or 100,000 shares.

ii) The Exchange can restrict the exercise of any options if it feels that such action is advisable in the interest of maintaining a fair and orderly market.

Adjustment of Options

Adjustment to options are necessary as a result of dividends, stock splits or reorganisation. In the case of dividends, no adjustment is made to any of the terms of the exchange traded options. The rule is that if the holder of a call option files an effective exercise notice prior to the ex-dividend date, then he or she is entitled to that dividend even though the writer to whom the exercise is assigned may not receive the assignment notice until after the ex-dividend date.

Stock splits, stock dividends or other stock distribution which increase the number of outstanding shares of the issuer have the effect of proportionately increasing the number of shares of the underlying stock covered by the options and proportionately decreasing the exercise price as of the ex-date. In cases where stock split or stock distribution results in the distribution of one or more whole shares for each share outstanding then an adjustment will be made to the number of options contract. Adjustment as a result of reorganization will be made with respect to exercise price or unit of trading, if the corporation considered such adjustments to be fair to the holders and writers of such options.

Limitations on Trading

The number of options of the same underlying security regardless of maturity dates which may be held by a single investor or group of investors is restricted to 500 contracts or 50,000 shares. The limit on the number of options of the same underlying security on any maturity date is 1,000 contracts or 100,000 shares. These limits are applicable to both buying and selling options. The exchange is empowered to order liquidation of any position exceeding these limits or to order any other sanctions.

Besides these limits, margin requirements are imposed on the option writer. These requirements differ between Exchange members. However, as an indication of these requirements, we take a look at the margin requirement the Exchange imposed on its various members.

Each Exchange member is required either

i) to deposit with the Corporation the underlying security, represented by the option. In this instance, an escrow receipt issued by approved institution

or

ii) to maintain with the Corporation, each security of the government of Canada or a province, or bank of credit, equalled to 30% of the market price of the underlying securities, increased (reduced) by the difference between the market price of the securities and the exercise price.

The amount of deposits required are adjusted daily.

6. THE DATA

Daily option prices traded on the Montreal Stock Exchange were collected from the Montreal Gazette for the period 15th September 1975 to 31st December 1976. They were sorted into eight maturity dates starting with those maturing on November 1975 and ending with those maturing on August 1977. This gives a total of options on 18 underlying stocks. (A listing of the stocks are given in Appendix II.) The closing stock prices traded on the Montreal Stock Exchange for the period 15th September 1974 to 31st December 1976 were collected from the Toronto Globe and Mail. Dividends information was obtained from the official monthly publications of the Montreal Stock Exchange for the same period.

The prices were checked for accuracy by picking various sample and tracing these back to the original source document. Furthermore, prices with a percentage change of 10% or more were verified.

Besides the above, estimates of the riskless rate of interest and the variance of the stock's rate of return distribution were required. For the riskless interest rate, the average weekly interest rate of the 90-day treasury bills over the life of the option was used. These rates were obtained from the Bank of Canada Review for the period concerned.

In the case of the variance of the stock's rate of return distribution, two estimates were used, namely a daily adjusted past variance and a constant variance.

On any day, the past variance is calculated using the rates of return of the stock for the past 12 months. For subsequent days, this past variance is updated by dropping the earliest observation and taking into consideration the latest rate of return. For purposes of calculating the variances the rates of return were adjusted for dividends. The constant variance is arrived at by taking the average of the daily adjusted past variance over the life of the option.

7. TEST AND RESULTS

7.1 Tests of the Black-Scholes Model

The analysis in Section 4 suggests the possibility of a bias in using discrete instead of continuous hedging. Such a bias is dependent on the distribution of the rates of return of the stock price for the period concerned. Hence, prior to using the Black-Scholes model for testing market efficiency, it will be necessary to determine if such a bias exists for the test period, and if it exists, the extent of the bias. Therefore, a test similar to the 'buy all options at model prices' (Buy all) strategy performed by Black and Scholes [1] was carried out whereby (i) a constant variance rate and actual closing stock prices from the Montreal Stock Exchange were used, and (ii) the hedge position was adjusted on a daily basis.

For each day, the theoretical model price and $N(d_1)$ were calculated using equation (2.9). The actual closing stock price for that day was used. A hedged position was then set up by buying one option at the model price and selling $N(d_1)$ stock at the market price. On the next day this position was liquidated and a new one established. The excess return for holding the hedged position for one day is calculated as follows:

$$(\Delta C - N(d_1)\Delta S) - (C - N(d_1)S)r \qquad (7.1)$$

This procedure was repeated for each option traded on the Montreal Stock Exchange from the day trading on the option commenced to the expiration date. These excess returns were examined and the following points noted:

(i) options with the same underlying stocks tend to have positive returns on the same day

(ii) changes in stock prices of a large (small) amount give rise to positive (negative) excess returns for the hedged positions

(iii) almost 95% of the excess returns on the hedged position for the first six maturity dates are negative. Furthermore, the number of negative returns reduces significantly for the last two maturity dates, i.e. May and August, 1977

Points (i) and (ii) above indicate that the excess returns are a function of the stock price changes. Point (iii) suggests that the distributions of the rates of return of stock prices for the first six maturity dates tend to be 'peaked' (i.e. concentrated around the mean) while the distribution of the rates of return of stock prices for the last two maturity dates tend to be 'fat-tailed' (i.e. greater probability of having a big rate of returns of the stock prices). To test the above hypothesis, histograms of the rates of return of stock prices for each maturity date of the 18 stocks were plotted. For every stock, a comparison of the histogram of the eight maturity dates shows the above hypothesis to be true. Four histograms (for maturity date; February, August, 1977 and 1978) of a stock are presented in Appendix I for purposes of illustration.

The above observations indicate the existence of a bias along the lines indicated in Section 4. To examine the direction and extent of the bias these excess returns for each option were individually regressed against the excess return on the market. The regression model was used:

$$\tilde{R}_{Ht} = \alpha + \beta \, \tilde{R}_{Mt} + \varepsilon_t \tag{7.2}$$

Maturity Date	No. of Options	No. of α significantly positive	No. of α significantly negative	No. of α insignificant	No. of β significant	Average α
Nov. 75	31	-	23	8	-	-0.009715
Feb. 76	38	-	35	3	2	-0.006341
May 76	51	-	49	2	4	-0.004757
Aug. 76	55	-	55	-	2	-0.004271
Nov. 76	55	-	50	5	15	-0.003995
Feb. 77	61	-	42	19	6	-0.003367
May 77	60	-	12	48	3	-0.002492
Aug. 77	27	-	4	23	-	-0.007971

Table I: Returns on the Hedged Positions*
Using a Buyall Strategy (Constant Variance)

Maturity Date	α	t-α	β	t-β
Nov. 75	-0.2526	-4.7879	1.1795	0.1516
Feb. 76	-0.2296	-6.5931	1.8871	0.3638
May 76	-0.2369	-7.8078	5.3385	1.1374
Aug. 76	-0.2347	-8.1321	7.7844	1.5848
Nov. 76	-0.2046	-6.8352	10.2870	2.2230
Feb. 77	-0.1680	-4.0229	3.0571	0.5090
May 77	-0.0928	-1.7387	1.6400	0.2538
Aug. 77	-0.0367	-0.9175	3.5821	0.7388

Table III: Returns on the Portfolios*
Using a Buyall Strategy (Constant Variance)

* Serial correlations of the residuals are not significant.

where

$\tilde{R}_{Ht}$ = excess return on the hedged position on day t

$\tilde{R}_{Mt}$ = excess return on the Montreal Stock Exchange Composite Index on day t

ε_t = error term

The slope coefficient β is interpreted as the measure of risk while α, the intercept is the amount of excess returns which can be earned after taking into consideration any risk which the hedged position might have. The α is therefore an indication of the amount and direction of the bias. The results of the regression were sorted according to the maturity date and the summarized results are presented in Table I. These results show that

(i) although most of the $\hat{\beta}$'s are insignificantly different from zero, the $\hat{\alpha}$'s in most cases are significantly negative

(ii) the percentage of $\hat{\alpha}$'s which are significantly negative is relatively low for the last two maturity dates

The above results should be taken as indicative rather than conclusive proof of the direction and extent of the bias. As explained in Section 4, we expect the distribution of excess returns to be skewed rather than normal depending on the distribution of the rates of return of the stock price.(9) With the excess returns not being normally distributed, the $\hat{\alpha}$'s and $\hat{\beta}$'s obtained above are unbiased but not the most

(9) To be certain, histograms of the excess returns of a few options were plotted and they were found to be skewed.

efficient estimators of the true α and β. Therefore interpretation of the results should be made with caution.(10)

In an attempt to overcome the above problems as well as to diversify away errors in variables that can affect each position individually, portfolios are formed by summing daily all the individual excess returns on the hedged position of the same maturity date. These portfolio excess returns were tested for normality using a Kolmogorov-Smirnoff goodness of fit test and the results are presented in Table II. Of the eight portfolios we can conclude that only three of them (indicated by '*') have excess returns that are normal.

Maturity Date	K-S D Statistics	0.05 Critical Level	0.01 Critical Level	Maturity Date	K-S D Statistics	0.05 Critical Level	0.01 Critical Level
Nov. 75	0.1757	0.1984	0.2378*	Nov. 76	0.2654	0.1000	0.1198
Feb. 76	0.2637	0.1303	0.1561	Feb. 77	0.2374	0.1107	0.1327
May 76	0.2296	0.1037	0.1243	May 77	0.1149	0.1442	0.1728*
Aug. 76	0.1891	0.1003	0.1202	Aug. 77	0.2011	0.2570	0.3130*

Table II Kolmogorov-Smirnoff goodness of fit test on excess return

The daily portfolio excess returns were then regressed against the market returns using equation (7.2).(11) The regression results are presented in Table III. The $\hat{\alpha}$'s were significantly negative for the first six maturity dates and insignificant for the last two maturity dates. It is interesting to note that the portfolio of options maturing on November 1976 has a significant postive $\hat{\beta}$.

(10) For ten hedged positions, the residuals obtained from the regression were plotted against the predicted R_{Ht}, the independent $\tilde{R}_{Mt}$ and tested for normality. There appears to be no further violation of the other assumptions of the regression model used.

(11) As the number of outstanding contracts each day is constant for each maturity date, no problem of heteroskedasticity is encountered.

Maturity Date	No. of Options	No. of α significantly positive	No. of α significantly negative	No. of α insignificant	No. of β significant	Average α
Nov. 75	31	-	25	6	5	-0.011330
Feb. 76	38	-	35	3	5	-0.008664
May 76	51	-	48	3	16	-0.006837
Aug. 76	55	-	51	4	17	-0.005955
Nov. 76	55	-	47	8	-	-0.004899
Feb. 77	61	-	33	28	-	-0.004105
May 77	60	-	3	57	2	-0.003994
Aug. 77	27	-	-	27	-	-

Table IV: Returns on the Hedged Positions*
Using a Buyall Strategy (Daily Adjusted Variances)

Maturity Date	α	t-α	β	t-β
Nov. 75	-0.3064	-5.7953	12.3820	1.5891
Feb. 76	-0.3135	-7.6482	9.6116	1.5745
May 76	-0.3280	-8.9023	15.3030	2.6855
Aug. 76	-0.3126	-9.1490	15.2220	5.8153
Nov. 76	-0.2486	-7.8127	3.8960	0.7924
Feb. 77	-0.1749	-3.3982	-4.0397	-0.5474
May 77	-0.0126	-0.1606	-0.6514	-0.0689
Aug. 77	0.0639	1.0535	4.0845	0.5560

Table V: Returns on the Portfolios*
Using a Buyall Strategy (Daily Adjusted Variance)

* Serial correlations of the residuals are not significant.

To determine the sensitivity of the result obtained, the test was repeated using the daily adjusted past variance. The results are presented in Table IV for individual hedges and Table V for portfolios.

The use of daily adjusted variance did not change the results significantly. The observations made with regard to the result of the test using constant variance are still valid. However, there are two portfolios which have significantly positive $\hat{\beta}$'s. The fact that significantly positive $\hat{\beta}$'s are consistently obtained supports the point made in Section 4 that formation of portfolio will not necessarily eliminate all the risk due to discrete hedging.

7.2 Efficiency of the Montreal Option Market

We will attempt to test the efficiency of the Montreal Option Market by using a simple strategy. Efficiency refers to the possibility of earning higher than normal returns after taking into consideration the risk taken. The strategy used is similar to the 'buy (sell) all undervalued (overvalued)[12] options at market value' strategy performed by Black and Scholes [1].

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(12) An option whose model price is greater (less) than its market price is considered undervalued (overvalued).

On the first day the option is traded, equation (2.10) is used to calculate the theoretical option price as well as $N(d_1)$. This price is compared to the option's closing market price for the day. If the option is undervalued (overvalued), a hedge position is established by buying (selling) one option at the market price and selling (buying) $N(d_1)$ amount of shares. The hedge position is liquidated the next day and the excess return is calculated as

$$R_{Ht} = (C_t^M - C_{t-1}^A) - N(d_1)_{t-1}(S_t - S_{t-1}) - (C_{t-1}^A - N(d_1)_{t-1}S_{t-1})r \qquad (7.3)$$

where

R_{Ht} = Return on the hedge at day t

C_t^M = Model option price at day t

C_t^A = Actual (market) option price at day t

S_t = Closed stock price at day t

On second and subsequent days until maturity, the model price is assumed to be the market price and the excess return is calculated using

$$R_{Ht} = (C_t^M - C_{t-1}^M) - N(d_1)_{t-1}(S_t - S_{t-1}) - (C_{t-1}^M - N(d_1)_{t-1}S_{t-1})r \qquad (7.4)$$

The assumption of the model price being the market price is necessary as the secondary activity on the Montreal Option Market does not provide a daily option price for the majority of the options traded. However, we will attempt to simulate trading at the market price by adding to the daily excess return a positive amount. This amount is calculated as an annuity payment over the life of the option equivalent to the difference between the initial market price and the model price. The rationale behind this adjustment is that as the option approaches maturity, the market price will tend towards the option. At the maturity date, the market price will be equal to the option price. Therefore the total of all the differences between excess returns of two hedged positions (one using model prices and the other using market prices) over the life of the option is approximately equal to the difference between initial market and model prices.

The above procedure is used to arrive at the adjusted daily excess returns for every option traded on the Montreal Option Market for the period 15th September 1975 to 31st December 1976. The individual excess returns were regressed against the market using equation (7.2). The test was carried out using the daily adjusted past variances and the summarized results are presented in Table VI.

For comparison purposes, portfolios are formed according to the eight maturity dates. Each portfolio consists of all the contracts outstanding on a particular day. A series of daily excess returns on the portfolio is obtained by aggregating the excess returns each day over all contracts outstanding.

Since the individual options start trading on different days, the number of contracts outstanding vary between days. Hence, the portfolio return on day t (R_{pt}) is divided by the number of hedged positions outstanding that day. The average portfolio excess returns are then regressed

Maturity Date	No. of Options	No. of α significantly positive	No. of α significantly negative	No. of α insignificant	No. of β significant	Average α
Nov. 75	31	13	5	13	5	0.007924
Feb. 76	38	12	10	16	5	0.003779
May 76	51	23	13	15	11	0.007453
Aug. 76	55	21	18	16	4	0.002296
Nov. 76	55	39	5	11	6	0.009672
Feb. 77	61	17	6	38	1	0.008062
May 77	60	12	-	48	2	0.011929
Aug. 77	27	1	-	26	1	0.004688

Table VI: Returns on the Hedged Positions*
Using a Buy and Sell Strategy (Daily Adusted Variance)

Maturity Date	α	t-α	β	t-β
Nov. 75	0.009515	34.9260	-0.0045	-0.1141
Feb. 76	0.001566	8.2921	0.06204	2.2099
May 76	0.001715	9.5967	-0.02135	-0.7419
Aug. 76	0.002144	16.7830	-0.05281	-2.1135
Nov. 76	0.009377	30.1830	-0.00382	-0.9349
Feb. 77	0.002570	20.661	0.05067	3.4217
May 77	0.004841	36.057	0.05333	3.7263
Aug. 77	0.006291	35.341	-0.01647	-0.8518

Table VII: Returns on the Portfolios*
Using a Buy and Sell Strategy (Daily Adjusted Variance)

* Serial correlations of the residuals are not significant.

against the market and the regression model used is

$$\tilde{\tilde{R}}_{pt}\sqrt{\frac{1}{N_t}} = \alpha\sqrt{N_t} + \beta(\tilde{R}_{Mt}\sqrt{N_t}) + \mu_t \quad ^{(13)} \tag{7.4}$$

where

$\tilde{R}_{pt}$ = portfolio excess returns on day t

$\tilde{R}_{Mt}$ = excess return on the market on day t

N_t = number of hedged positions outstanding on day t

μ_t = error term

The regression line was forced to go through the origin with $\tilde{R}_{pt}\sqrt{\frac{1}{N_t}}$ as the dependent variable and $\sqrt{N_t}$ and $R_{Mt}\sqrt{N_t}$ as independent variables. The α's and β's can be interpreted as before. The results are presented in Table VII.

The following points can be noted from the results presented in Tables VI and VII.

(i) Although the average of the significant α's for the individual hedged positions are positive there are a large number of α's which are significantly negative. However, it is not known whether these negative α's are caused by market inefficiency or the bias as a result of using discrete hedging.

(ii) The very significant α's in the case of portfolios will tend to suggest that it will be possible to make excess returns on the market by following a simple buy and sell strategy. e.g. If a portfolio consisting of all the hedged positions maturing on November

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(13) To overcome the problem of heteroskedasticity that arises, we multiply the standard regression equation throughout by $\sqrt{N_t}$.

(14) A problem of multicollinearity is encountered here. This will tend to inflate the standard error of the coefficients $\hat{\alpha}$'a and $\hat{\beta}$'s thus reducing their reliability and understating their significance.

1975 were held and adjusted daily then (from Table VII) a daily average return of \$0.95 will be made with zero investment.(12)

(iii) Despite the negative bias caused by discrete hedging, it is still possible to obtain a significantly positive excess return on all the portfolios. This will indicate that profit opportunities do exist in the market. However it is doubtful that when transaction costs, taxes, different lending and borrowing rates are taken into consideration, such profit opportunities will still exist in the market.

(iv) Four of the eight portfolios have significant $\hat{\beta}$'s. The significant $\hat{\beta}$'s show that the risk on the portfolios is not zero implying the existence of systematic risk.

To test if the model can perform better when dividends are taken into consideration we divide all the options of each maturity date into four portfolios. The first portfolio consists of options on stocks with the lowest dividend yield and the last portfolio consisting of all options on stock with the highest dividend yield. The excess returns on the portfolio were calculated in the same manner and regressed against the market using equation (7.4).

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(12) The α on Table VII shows the dollar amount to be made on a portfolio consisting of hedged positions on one option. However, options are only traded in multiples of a hundred, therefore the return will be \$0.0095 x 100 = \$0.95.

Although we would expect the model to perform better thus resulting in higher α's in those cases when the assumptions of the model are better met, the results show that the α's obtained are independent of the amount of dividend payment.

In summary, this section shows that the use of the model for this period will result in a significant negative bias for excess returns in six of the eight maturity dates. Despite this bias, we are able to make significant positive profits by following a simple buy and sell strategy implying that the market is inefficient. However such profit opportunities are unlikely to exist in the market when transaction costs, different lending and borrowing rates are taken into consideration.

8. Summary and Conclusion

Past empirical studies using the Black-Scholes option pricing model have either assumed that the discrete hedging will not significantly affect the results in any way or that it causes uncertain returns, but such uncertainty could be diversified away by the formation of portfolio. However the analyses and tests in Sections 4 and 7.1 have shown that the use of discrete hedging has resulted in a significant bias which is a function of the distribution of the rates of return on the stock prices. Consequently, there is a possibility that the hedge contains systematic risks which cannot be diversified away by the formation of portfolios.

Despite this bias, we are able to obtain significantly positive excess returns when a simple buy and sell strategy is followed. This indicates that the Montreal Option Market is inefficient. Note that if the model is adjusted to take into consideration discrete hedging, we will expect to obtain even higher positive excess returns as the model will be able to better differentiate profit opportunities. However, it is unlikely that such profit opportunities will exist if transaction costs, etc. are considered.

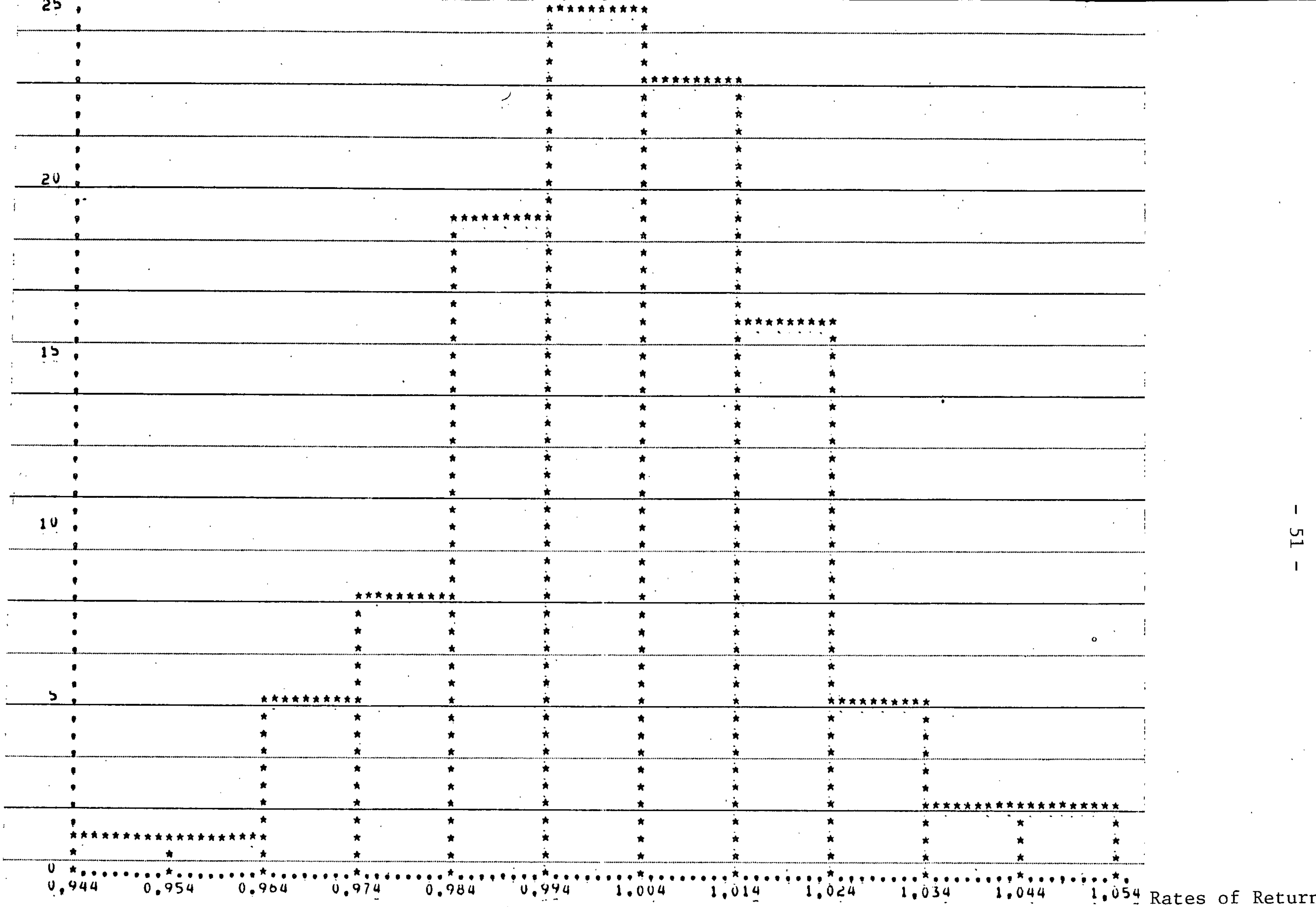
This paper does not provide all the answers as to the effects of discrete hedging on a model which assumes continuous time. However, it does show that violation of this assumption is of sufficient importance to warrant further investigation.

Further research can be done in the following areas

(i) in deriving the Option Pricing formula, higher order terms in the expansion of dC in equation (2.7) were ignored on the basis that dt is very small. This is true only in continuous time. However, as hedging can only be done in discrete time, adjustment can be made to the Black-Scholes model to take into consideration such higher order terms.

(ii) in using the regression model we encounter various problems as the underlying assumptions are violated. New techniques should therefore be derived for the purpose of testing market efficiency.

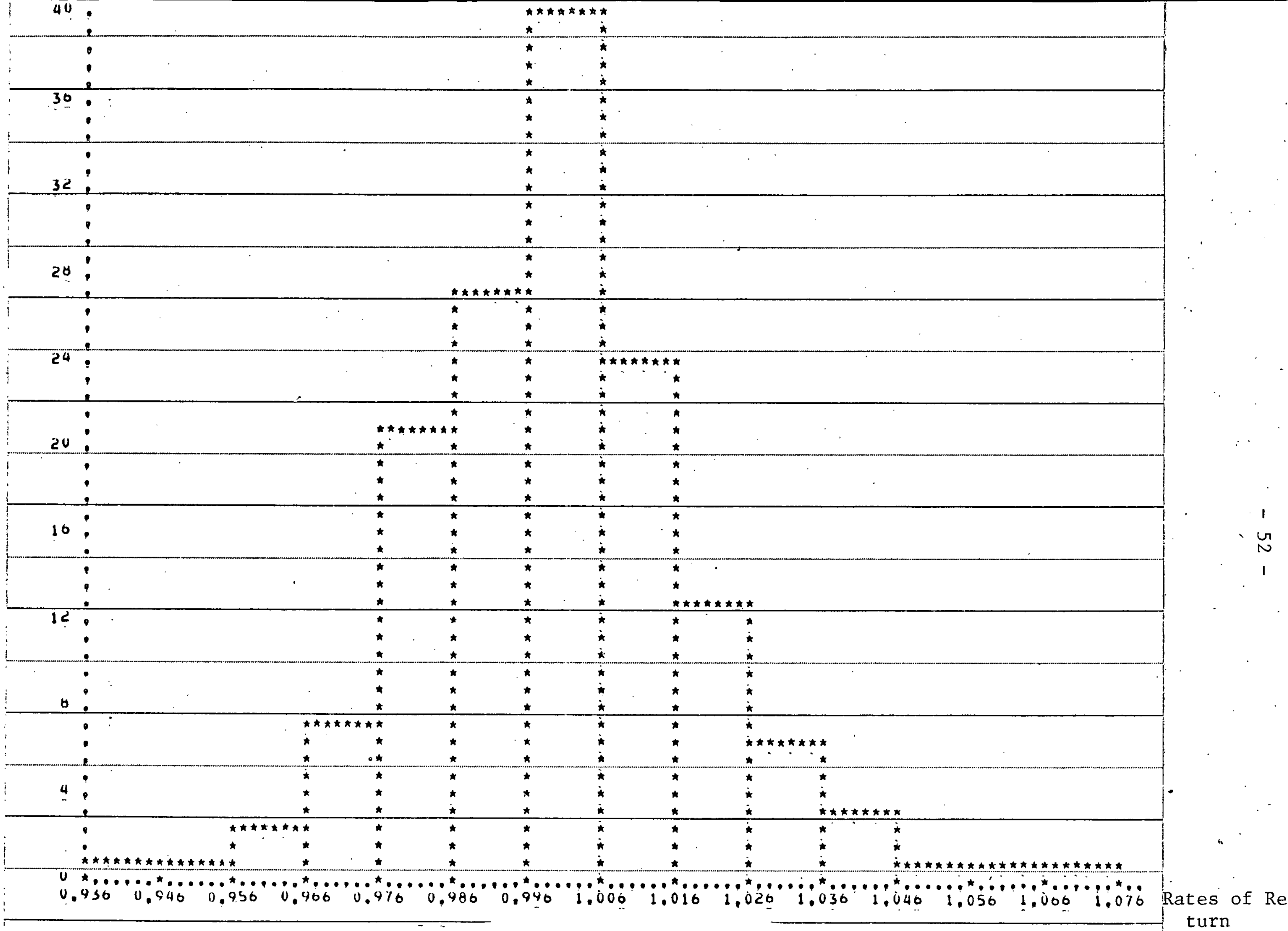
(iii) deriving an option pricing model in discrete time.



Appendix IA

Rates of Return for Shell Canada

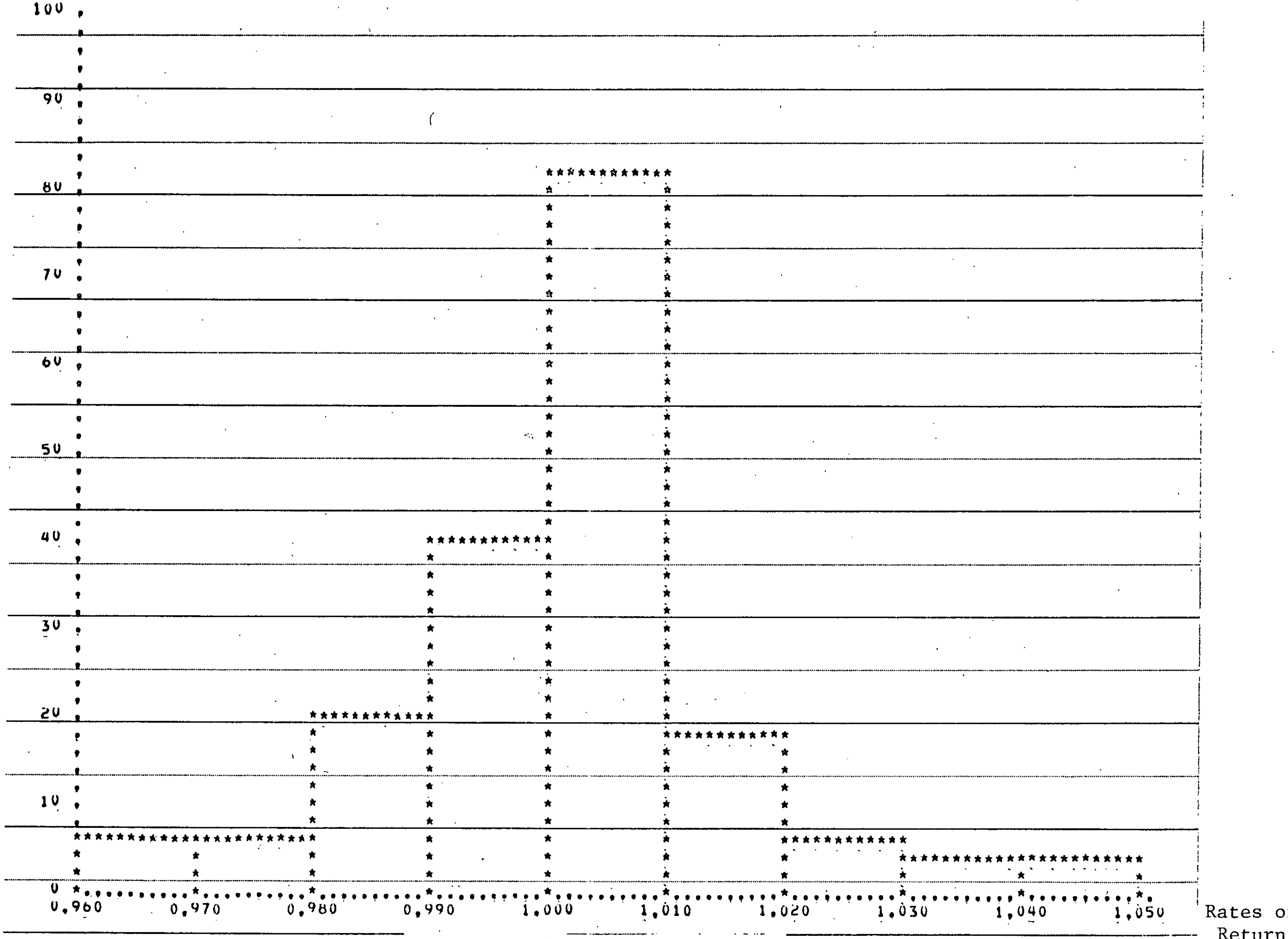
From 16th Sepember to 20th February 1976



Appendix IB

Rates of Returns for Shell Canada

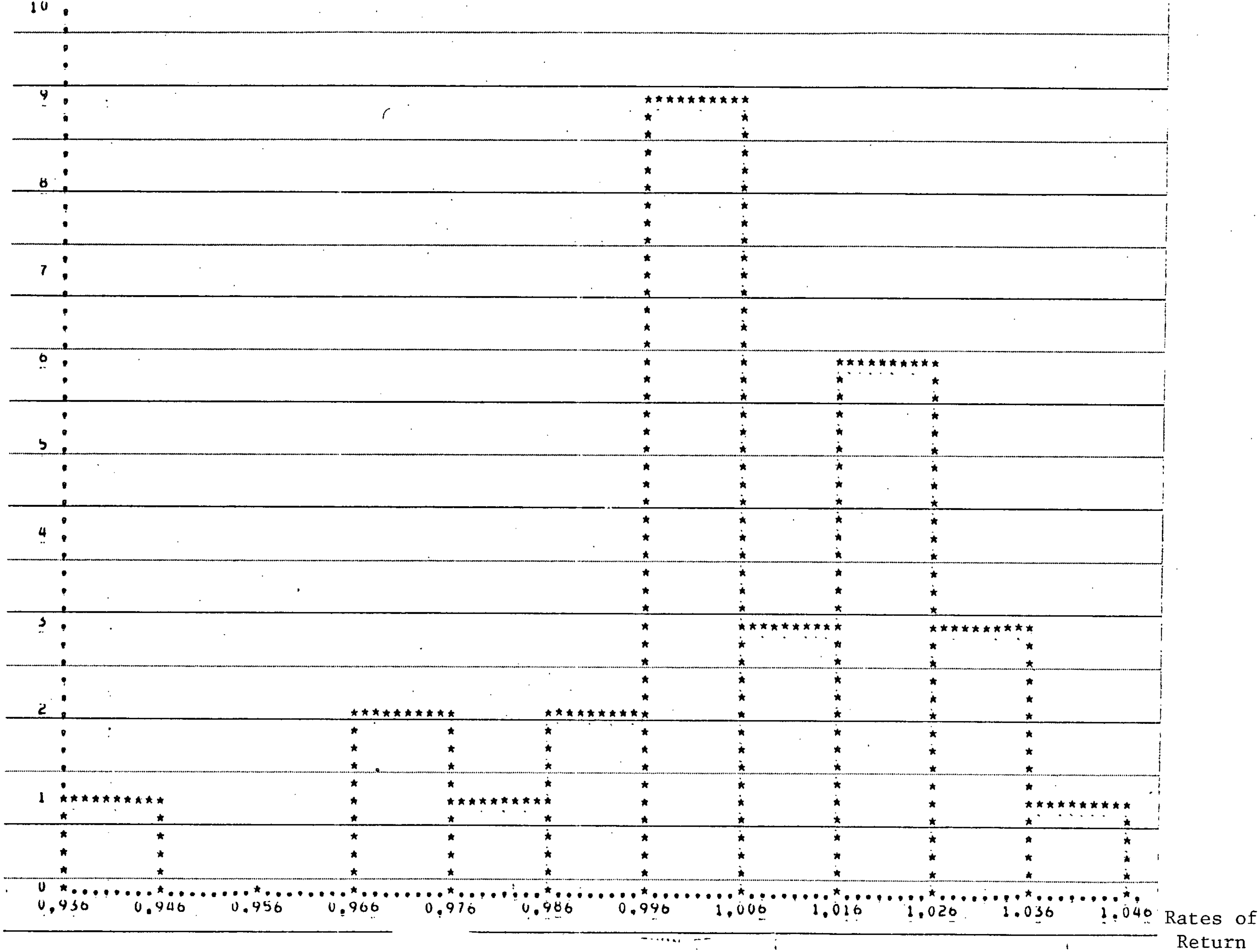
From 25th November to 20th August 1976



Appendix IC

Rates of Returns of Shell Canada

From 26th May to 31st December 1976



Appendix ID

Rates of Returns for Shell Canada

From 23rd November to 31st December

Appendix II

Listing of Underlying Stocks Whose Options are Traded on the Montreal Options Market as at 31st December 1976

1. Abitibi Paper Company
2. Alcan Aluminium
3. Bank of Montreal
4. Bell Canada
5. Brascan
6. Canadian Pacific
7. Gulf Oil
8. Imperial Oil
9. Inter-Provincial Pipelines
10. International Nickel Company
11. MacMillan Bloedal
12. Massey Ferguson
13. Moore Corporation
14. Noranda Mines
15. Pacific Petroleum
16. Shell Canada
17. Steel Company of Canada
18. Trans Canada Pipelines

APPENDIX III

Inadequacy of the Interest Rate Adjustment Factor

The interest rate adjustment factor is

$$-[C_1 - N(d_1)S_1]\ r\Delta t \qquad \text{(i)}$$

Substituting equation (2.10) into (i) we have

$$-[-Xe^{-rT}\ N(d_2)]\ r\Delta t \qquad \text{(ii)}$$

As the option price is a function of both S and T, a change in the option price dC can be expressed as

$$dC = C_T\Delta t + C_S\Delta S \qquad \text{(iii)}$$

where subscripts denote partial derivatives. Since we are only interested in changes in the option price as a result of a change in T, ΔS is assumed to be zero. Substituting

$$C_T = \{Xe^{-rT}[Z(d_2)\frac{\sigma}{2\sqrt{T}} + rN(d_2)]\}\Delta t \qquad \text{(iv)}$$

with Z as the normal density function, into equation (iii) we have

$$dC = \{Xe^{-rT}[Z(d_2)\frac{\sigma}{2\sqrt{T}} + rN(d_2)]\}\Delta t \qquad \text{(v)}$$

The difference between dC and the interest rate adjustment factor given in equation (ii) is therefore

$$Xe^{-rT}[Z(d_2)\frac{\sigma}{2\sqrt{T}} + rN(d_2)]\Delta t-[Xe^{-rT}\ N(d_2)]r\Delta t$$

$$= \quad Xe^{-rT}[Z(d_2)\frac{\sigma}{2\sqrt{T}}\ ]\Delta t \qquad \text{(vi)}$$

Equation (vi) will be approximately zero as the hedged position is adjusted continuously. However if discrete hedging is used, the interest rate adjustment factor will not offset the reduction in the option price by this amount.

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