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CONSEQUENCES OF SPACE-TIME INVARIANCES

IN QUANTUM MECHANICS

AND DIRECT INTERACTION THEORIES

by

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Abstract

The problem of describing a quantum mechanical system is considered. For a system which is invariant under the Poincaré (inhomogeneous Lorentz) transformations this description is provided by the generators of those transformations, which satisfy the usual commutation relations. General expressions for the centre of mass position and internal angular momentum operators, in terms of these generators, are obtained. The generators of the Poincaré transformations are written in terms of the fundamental dynamical variables for several systems. The systems considered are those consisting of a single free spinless particle, a single free particle with spin, a single free Dirac particle, n noninteracting particles, and n interacting particles. In each case, the centre of mass position and internal angular momentum are given in terms of the fundamental dynamical variables of the system. For the first two systems listed above, these two operators are found to be equal to the Cartesian coordinates and spin of the particle, respectively. In the case of the Dirac particle, these operators are seen to be related to the Cartesian coordinates and spin of the particle via the Pryce-Foldy-Wouthuysen transformation. Following Bakamjian and Thomas, interaction is introduced to the n particle system via a single operator which depends only on internal variables. The condition of "asymptotic covariance" of the scattering operator is discussed for two particle scattering. The scattering operator for a two particle system with no bound states and with Poincaré generators of the Bakamjian-Thomas form is seen to be asymptotically covariant.

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Chapter 1 Introduction

We are concerned here with the construction of a description of a physical quantum mechanical system. By requiring that a system be invariant under certain space-time transformations, we are led to a set of operators which provide a description of that system. First, however, the meaning of "invariance under a space-time transformation" must be clarified.

Since any information about a quantum mechanical system is the result of the measurement of observables, represented by Hermitian operators in the quantum theory, our basic postulate is a condition of the results of such measurements. Consider a system which has been prepared by some apparatus in a state $|\psi\rangle$ and an observable A which is to be measured by another apparatus. Now if both the preparation and measuring apparatus undergo the same transformation in space and time, then the state $|\psi'\rangle$ is prepared and the observable A' is measured. The result of the measurement of the observable A in the state $|\psi\rangle$ must be equal, on the average, to the result of the measurement of the observable A' in the state $|\psi'\rangle$. Symbolically, we write

$$\langle\psi|A|\psi\rangle = \langle\psi'|A'|\psi'\rangle$$

This postulate must hold for all states $|\psi\rangle$ and all observables A. The system is then said to be invariant under the space-time transformation.

The transformations of concern here are, for the most part, the Poincaré (inhomogeneous Lorentz) transformations. These include time translations, space displacements, rotations, and Lorentz boosts. They are given explicitly in Section 2.1. We require that any realistic system be invariant under the Poincaré transformations. This requirement implies, as we show, that to each one-parameter transformation there corresponds a unitary linear operator $\mathcal{U}_\alpha(\tau)$ which may be expressed as

$$\mathcal{U}_\alpha(\tau) = e^{-iC_\alpha \tau/\hbar}$$

τ is the parameter of the transformation. These unitary operators provide the relations between the states $|\Psi\rangle$ and $|\Psi'\rangle$ and between the observables A and A'. The Hermitian operators C_α are called the generators of the transformation and provide the characterization of the system which we seek. (The labels α identify the one-parameter transformation.) There are ten such operators corresponding to the Poincaré transformations.

A system may also be invariant under two transformations which are not Poincaré transformations. These are the operations of space inversion and time reversal which we introduce in Sections 2.6 and 2.7, respectively. As for the Poincaré transformations, there exists a unitary linear operator corresponding to the operation of space inversion. An antiunitary antilinear operator corresponds to the time reversal operation.

The ten generators of the Poincaré transformations obey a set of commutation relations, which we derive in Section 2.2. Thus, to construct a description of a physical system which is invariant under the Poincaré transformations, one must find expressions for the ten generators, in terms of the fundamental dynamical variables of the system, which satisfy the proper commutation relations.

Six of the generators may also be expressed, in a general form, in terms of two operators, $\underset{\sim}{X}$ and $\underset{\sim}{S}$, which we introduce in Section 2.8. $\underset{\sim}{X}$ is the centre of mass position operator and $\underset{\sim}{S}$ is the total internal angular momentum operator. They obey a set of commutation relations which ensures that the generators satisfy the proper set of commutation relations. We write the six generators in terms of these operators and, inversely, give general expressions for $\underline{X}$ and $\underline{S}$ in terms of all the generators of the Poincaré transformations.

In the third chapter, we consider several quantum mechanical systems. Their fundamental dynamical variables are introduced and the generators of the Poincaré transformations are given in terms of these variables. The first two systems considered are those of a single free spinless particle and a single free particle with spin, in Sections 3.1 and 3.2, respectively. The operators $\underline{X}$ and $\underline{S}$ are expressed particularly simply in terms of the fundamental dynamical variables for these systems. Also, the unitary operators corresponding to the Poincaré transformations and to space inversion as well as the antiunitary time reversal operator are expressed in integral form for these systems.

We also give the generators of the Poincaré transformations for a single Dirac particle in Section 3.3 and obtain expressions for $\underline{X}$ and $\underline{S}$ which are in agreement with several authors.

In Section 3.4, for a system of a fixed number of noninteracting particles, we give the generators in terms of the single particle generators. As a final example, we consider a fixed number of interacting particles in Section 3.5. The interaction is completely specified by an operator M, the mass operator, when the Hamiltonian takes the Bakamjian-Thomas form.[1] Certain restrictions are imposed on M and, in order to satisfy these restrictions, internal variables are defined. We do not consider the question of the separability of the n particle Hamiltonian.

In Appendix I, we consider a system of n interacting nonrelativistic particles wherein the interaction is again specified by a single operator. We find that the generators of the Poincaré transformations for this system may be obtained by taking the limit $c \to \infty$ in the expressions for the relativistic system.

In a consideration of two particle scattering in Chapter 4, we find that several conditions are imposed on the Moller operators $\Omega_{\pm}$ and the scattering operator S as a result of the requirement of Poincaré invariance. The further condition of "asymptotic covariance" of the scattering operator is introduced and discussed. Equivalent conditions are written in terms of the generators of the Poincaré transformations and the Bakamjian-Thomas expressions given in Section 3.5 for the generators for a system of two interacting particles are seen to satisfy these conditions.

Chapter 2 Consequences of Space-time Invariances in Quantum Mechanics

This chapter provides a general discussion of the description of a quantum mechanical system in terms of the generators of the transformations under which the system is invariant. In Section 2.1, the Poincaré transformations are introduced and the concept of Lorentz invariance of a physical system is discussed in some detail. The ten generators of the Poincaré transformations, H, $P^{(j)}$, $J^{(j)}$, $K^{(j)}$ $(j=1,2,3)$, are identified and, in Section 2.2, their commutation relations are derived. In Section 2.3, the operators $\mathcal{U}_\beta C_\alpha \mathcal{U}_\beta^\dagger$ are derived, where $\mathcal{U}_\beta$ is a unitary operator corresponding to a one-parameter Poincaré transformation and C_α is the generator of a one-parameter Poincaré transformation. If C_α is measured by some apparatus, then $\mathcal{U}_\beta C_\alpha \mathcal{U}_\beta^\dagger$ is the observable measured by that apparatus after it has undergone the transformation to which $\mathcal{U}_\beta$ corresponds, while $\mathcal{U}_\beta^\dagger C_\alpha \mathcal{U}_\beta$ is the observable measured when the preparation apparatus has undergone the transformation.

Two Poincaré invariants M^2 and $\mathcal{W}^2$ are defined in Section 2.4; such operators are invariant under all the Poincaré transformations and, thus, as we show, commute with all the generators of the Poincaré transformations. Some of the results of the first three sections are put into four-vector notation in Section 2.5.

The operations of space inversion and time reversal, to which correspond operators $\mathcal{P}$ and $\mathcal{T}$, are introduced in Sections 2.6 and 2.7, respectively. Their effect, in the sense of Section 2.3, on the generators of the Poincaré transformations is investigated. That is, we obtain $\mathcal{P} C_\alpha \mathcal{P}^\dagger$ and $\mathcal{T} C_\alpha \mathcal{T}^\dagger$.

We introduce the centre of mass position $\underline{X}$, the total internal angular momentum $\underline{S}$, and the centre of mass velocity $\underline{V}$ in Section 2.8 and write $J^{(j)}$ and $K^{(j)}$ $(j=1,2,3)$ in terms of $\underline{X}$ and $\underline{S}$. The commutation relations obeyed by

$\underline{\mathcal{X}}$ and $\underline{\mathcal{S}}$ are given and general expressions for these operators, in terms of the generators of the Poincaré transformations, are obtained. Their properties under the Poincaré transformations, in the sense of Section 2.3, are given in Section 2.9. In that section, we see that $\underline{\mathcal{X}}$ transforms in a complicated manner under Lorentz boosts. Therefore, we introduce, in Section 2.10, operators $\mathcal{T}$ and $\underline{\mathcal{X}}(\mathcal{T})$ which transform more simply.

2.1 Lorentz Invariance of a Physical System

An apparatus used to prepare a state $|\psi\rangle$ of some physical system is placed in a fixed space-time reference frame in a laboratory. (We shall except when noted always use Schrodinger's picture of quantum mechanics.) The placement of the apparatus in the laboratory is specified by ten real numbers which may be chosen to be the time it was switched on (one number), the position of a fixed point on the apparatus (three numbers), the angles axes fixed on the apparatus make with the fixed laboratory frame (three numbers), and the velocity of its centre of mass in the laboratory (three numbers). The placement of the apparatus in the laboratory is now changed without changing its intrinsic structure. The changed apparatus prepares the state $|\psi'\rangle$. The change involves changing one or more of the above ten numbers and can be characterized by the equations

$$x^{(\mu)} \longrightarrow x^{(\mu)'} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{(\nu)} + a^{(\mu)} \qquad \mu = 0,1,2,3 \qquad (2.1.1)$$

where the $x^{(\mu)} = (ct, \underline{x})$ are the space-time coordinates of a point on the apparatus before the change (t is the time the apparatus was switched on), $x^{(\mu)'}$ are the coordinates of the point on the apparatus after the change, and $\Lambda^{\mu}{}_{\nu}$ and $a^{(\mu)}$ are quantities independent of $x^{(\mu)}$ and $x^{(\mu)'}$ which characterize the change in placement. More specifically,

i) if the apparatus is switched on ϵ seconds earlier:

$$(\underline{x}, t) \longrightarrow (\underline{x}', t') = (\underline{x}, t-\epsilon) \qquad (2.1.2)$$

ii) if it is displaced along the 1-axis by a distance a:

$$(\underline{x}, t) \longrightarrow (\underline{x}', t') = (x^{(1)}+a, x^{(2)}, x^{(3)}, t) \qquad (2.1.3)$$

iii) if it is rotated about the 1-axis by an angle θ:

$$(\underline{x},t) \longrightarrow (\underline{x}',t') = (x^{(1)}, x^{(2)}\cos\theta - x^{(3)}\sin\theta, x^{(2)}\sin\theta + x^{(3)}\cos\theta, t) \qquad (2.1.4)$$

iv) if it is boosted along the 1-axis by speed v:

$$(\underline{x},t) \longrightarrow (\underline{x}',t') = (\gamma\,(x^{(1)} + vt), x^{(2)}, x^{(3)}; \gamma\,(t + v/c^2\, x^{(1)})) \qquad (2.1.5a)$$

or

$$(x^{(0)},\underline{x}) \longrightarrow (x^{(0)\prime},\underline{x}') = (x^{(0)}\cosh u + x^{(1)}\sinh u, x^{(1)}\cosh u + x^{(0)}\sinh u, x^{(2)}, x^{(3)}) \qquad (2.1.5b)$$

where c is the speed of light,

$$\gamma = (1 - v^2/c^2)^{-1/2} \qquad (2.1.6)$$

and

$$\tanh u = v/c \qquad (2.1.7)$$

Similar considerations can be given to a second apparatus which measures the value of some observable of the system. We denote by A the observable measured in the first placement and by A' the observable measured in the second placement.

A physical system is said to be Lorentz invariant if for every Poincaré transformation, i.e., every orthochronous inhomogeneous Lorentz transformation (2.1.1) one obtains the same number on the average when one transforms both the preparation and the measuring apparatus. That is,

$$\langle \psi' | A' | \psi' \rangle = \langle \psi | A | \psi \rangle \qquad (2.1.8)$$

Moreover, equation (2.1.8) must hold for every preparation and measuring apparatus; that is, it must be true for every state $|\psi\rangle$ and every observable A. We show in Appendix A that equation (2.1.8) implies that corresponding probabilities are equal. That is,

$$|\langle\varphi'|\psi'\rangle|^2 = |\langle\varphi|\psi\rangle|^2 \qquad \forall\, |\varphi\rangle, |\psi\rangle \tag{2.1.9}$$

Wigner has shown that equation (2.1.9) implies that

$$|\psi'\rangle = \mathcal{U}\,|\psi\rangle \tag{2.1.10}$$

where $\mathcal{U}$ is either a linear unitary operator or an antilinear antiunitary operator.[2] Equation (2.1.8) then follows for all $|\psi\rangle$ and A provided

$$A' = \mathcal{U} A\, \mathcal{U}^{\dagger} \tag{2.1.11}$$

where $\mathcal{U}^{\dagger}$ is the adjoint of $\mathcal{U}$.

Further, it follows from the group properties of the Poincaré transformations that $\mathcal{U}$ can be written in terms of products of one or more of ten one-parameter operators $\mathcal{U}_\alpha(\tau)$, $\alpha=1,2,...,10$, which have the properties

$$\mathcal{U}_\alpha(0) = 1 \qquad \forall\, \alpha \tag{2.1.12}$$

$$\mathcal{U}_\alpha(\tau_1)\,\mathcal{U}_\alpha(\tau_2) = \mathcal{U}_\alpha(\tau_1 + \tau_2) \qquad \forall\, \alpha \tag{2.1.13}$$

Since the square of a unitary linear operator or of an antiunitary antilinear operator is unitary, it follows from equation (2.1.13) that the $\mathcal{U}_\alpha$ are linear unitary operators.

The unitary operators $\mathcal{U}_\alpha(\tau)$ can be specified further. It follows from Stone's Theorem that one can write

$$\mathcal{U}_\alpha(\tau) = e^{-i\tau C_\alpha} \tag{2.1.14}$$

where C_α is a Hermitian operator.[3] The operators C_α are called the generators of the Poincaré group. It is convenient to label the ten generators as follows: H generates time translations; $P^{(1)}, P^{(2)}, P^{(3)}$ generate space translations along the 1-,2-,3-axis, respectively; $J^{(1)}, J^{(2)}, J^{(3)}$ generate rotations about the 1-,2-,3-axes, respectively; $K^{(1)}, K^{(2)}, K^{(3)}$ generate Lorentz boosts along the 1-,2-,3-axes, respectively. The corresponding unitary operators will be written as

$$U(t) = e^{-iHt/\hbar} \qquad \text{:time translation or evolution operator} \tag{2.1.15}$$

$$D^{(j)}(a) = e^{-iP^{(j)}a/\hbar} \qquad \text{:displacement operator along j-axis} \tag{2.1.16}$$

$$\mathcal{R}^{(j)}(\theta) = e^{-iJ^{(j)}\theta/\hbar} \qquad \text{:rotation operator about j-axis} \tag{2.1.17}$$

$$L^{(j)}(u) = e^{-icK^{(j)}u/\hbar} \qquad \text{:Lorentz boost operator along j-axis} \tag{2.1.18}$$

Thus $U(t)$ corresponds to transformation (2.1.2); $D^{(1)}(a)$ corresponds to transformation (2.1.3); $\mathcal{R}^{(1)}(\theta)$ corresponds to transformation (2.1.4); $L^{(1)}(u)$ corresponds to transformation (2.1.5).

The operator H is called the Hamiltonian of the system; $P^{(1)}, P^{(2)}, P^{(3)}$ are called the components of the total linear momentum of the system; $J^{(1)}, J^{(2)}, J^{(3)}$ are the components of the total angular momentum of the system.

2.2 Commutation Relations for the Generators of the Poincaré Group

The generators of the Poincaré group obey a set of commutation relations which we now derive. Two infinitesmal transformations generated by C_α and C_β with parameters τ and σ, respectively, are applied to the preparation apparatus of the system. ($\alpha,\beta=1,2,\ldots,10$) They are succeeded by their inverses. The corresponding unitary operators can be expanded in a power series as follows:

$$e^{i\sigma C_\beta} e^{i\tau C_\alpha} e^{-i\sigma C_\beta} e^{-i\tau C_\alpha} = 1 + \tau\sigma [C_\alpha, C_\beta] + \cdots \tag{2.2.1}$$

where the commutator

$$[C_\alpha, C_\beta] = C_\alpha C_\beta - C_\beta C_\alpha \tag{2.2.2}$$

The result of the same set of four successive infinitesmal transformations can be exhibited upon application to a space-time point ($\underline{x}$,t) on the preparation apparatus. This result is some net transformation with the corresponding unitary operator

$$e^{i\tau\sigma \mathcal{C}_{\alpha\beta}} = 1 + i\tau\sigma\, \mathcal{C}_{\alpha\beta} + \ldots \tag{2.2.3}$$

Comparison of the terms which are first order in $\tau\sigma$ in equations (2.2.1) and (2.2.3) yields the commutator (2.2.2). However, the unitary operators (2.2.1) and (2.2.3) may differ by a phase factor. That is, we may write

$$e^{i\sigma C_\beta} e^{i\tau C_\alpha} e^{-i\sigma C_\beta} e^{-i\tau C_\alpha} = e^{i\tau\sigma b_{\alpha\beta} I}\, e^{i\tau\sigma \mathcal{C}_{\alpha\beta}}$$

$$= e^{i\tau\sigma(b_{\alpha\beta} I + \mathcal{C}_{\alpha\beta})}$$

where $b_{\alpha\beta}$ is a real number and I is the identity operator. Thus it follows that

$$[C_\alpha, C_\beta] = i(\mathcal{C}_{\alpha\beta} + b_{\alpha\beta} I) \tag{2.2.4}$$

The multiples of the identity are included for now because of the possibility of a phase factor. We will show later how they are eliminated from the commutation relations of the Poincaré group.

Case 1

The preparation apparatus of the system is displaced along the 1-axis by a, then along the 2-axis by b. These are followed by the inverse displacements.

$$e^{iP^{(2)}b/\hbar}\, e^{iP^{(1)}a/\hbar}\, e^{-iP^{(2)}b/\hbar}\, e^{-iP^{(1)}a/\hbar} = 1 + \frac{ab}{\hbar^2}[P^{(1)}, P^{(2)}] + \cdots$$

$$\begin{aligned}(x^{(1)}, x^{(2)}, x^{(3)}, t) &\to (x^{(1)} + a, x^{(2)}, x^{(3)}, t)\\ &\to (x^{(1)} + a, x^{(2)} + b, x^{(3)}, t)\\ &\to (x^{(1)} + a - a, x^{(2)} + b, x^{(3)}, t)\\ &\to (x^{(1)}, x^{(2)} + b - b, x^{(3)}, t)\end{aligned}$$

Thus

$$(x^{(1)\prime}, x^{(2)\prime}, x^{(3)\prime}, t) = (x^{(1)}, x^{(2)}, x^{(3)}, t)$$

The initial and final placements of the preparation apparatus are the same; therefore we can take

$$[P^{(1)}, P^{(2)}] = i\hbar a_{12} I$$

Similarly,

$$[P^{(j)}, P^{(k)}] = i\hbar a_{jk} I \qquad j,k = 1,2,3 \qquad (2.2.5)$$

Case 2

The preparation apparatus is now displaced along the 1-axis by a and then a time translation by σ is performed. The inverse transformations succeed these.

$$e^{iH\sigma/\hbar} e^{iP^{(1)}a/\hbar} e^{-iH\sigma/\hbar} e^{-iP^{(1)}a/\hbar} = 1 + \frac{a\sigma}{\hbar^2}[P^{(1)}, H] + \cdots$$

$$\begin{aligned}(x^{(1)},x^{(2)},x^{(3)},t) &\longrightarrow (x^{(1)}+ a,x^{(2)},x^{(3)},t)\\ &\longrightarrow (x^{(1)}+ a,x^{(2)},x^{(3)},t-\sigma)\\ &\longrightarrow (x^{(1)}+ a-a,x^{(2)},x^{(3)},t-\sigma)\\ &\longrightarrow (x^{(1)},x^{(2)},x^{(3)},t-\sigma+\sigma)\\ &= (x^{(1)},x^{(2)},x^{(3)},t)\end{aligned}$$

Again, the initial and final placements of the preparation apparatus are the same, so we can take

$$[P^{(1)}, H] = i\hbar c_1 I$$

Similarly,

$$[P^{(j)}, H] = i\hbar c_j I \qquad j=1,2,3 \qquad (2.2.6)$$

Case 3

Consider next the successive transformations consisting of a translation through a along the 1-axis, a rotation through θ about the 2-axis, and their inverses.

$$e^{i\theta J^{(2)}/\hbar}\, e^{iP^{(1)}a/\hbar}\, e^{-iJ^{(2)}\theta/\hbar}\, e^{-iP^{(1)}a/\hbar} = 1 + \frac{a\theta}{\hbar^2}[P^{(1)}, J^{(2)}] + \dots$$

$$(x^{(1)},x^{(2)},x^{(3)},t) \longrightarrow (x^{(1)}+ a,x^{(2)},x^{(3)},t)$$

$$\longrightarrow ((x^{(1)}+ a)\cos\theta + x^{(3)}\sin\theta ,x^{(2)},-(x^{(1)}+ a)\sin\theta + x^{(3)}\cos\theta ,t)$$

$$\longrightarrow ((x^{(1)}+ a)\cos\theta + x^{(3)}\sin\theta -a,x^{(2)},-(x^{(1)}+ a)\sin\theta + x^{(3)}\cos\theta ,t)$$

$$\longrightarrow ((x^{(1)}+ a)\cos^2\theta +x^{(3)}\sin\theta\cos\theta -a\cos\theta + (x^{(1)}+a)\sin^2\theta$$

$$-x^{(3)}\cos\theta\sin\theta ,x^{(2)}, (x^{(1)}+ a)\cos\theta\sin\theta + x^{(3)}\sin^2\theta -a\sin\theta$$

$$-(x^{(1)}+ a)\sin\theta\cos\theta + x^{(3)}\cos^2\theta ,t)$$

$$= (x^{(1)}+ a-a\cos\theta ,x^{(2)},x^{(3)}-a\sin\theta ,t)$$

In the limit of a,θ infinitesmal,

$$(x^{(1)},x^{(2)},x^{(3)},t) \longrightarrow (x^{(1)},x^{(2)},x^{(3)}-a\theta ,t)$$

This corresponds to a space translation through $-a\theta$ along the 3-axis. Thus,

$$[P^{(1)}, J^{(2)}] = i\hbar\,(P^{(3)} + e_{12}I)$$

Similarly,

$$[J^{(j)}, P^{(k)}] = i\hbar\,(\epsilon_{jk\ell}P^{(\ell)} + e_{jk}I) \qquad j,k,\ell = 1,2,3 \qquad (2.2.7)$$

Case 4

Consider a rotation through θ about the $x^{(1)}$-axis followed by a time translation of σ. Their inverses succeed them.

$$e^{iH\sigma/\hbar}\, e^{iJ^{(1)}\theta/\hbar}\, e^{-iH\sigma/\hbar}\, e^{-iJ^{(1)}\theta/\hbar} = 1 + \frac{\theta\sigma}{\hbar^2}[J^{(1)}, H] + \dots$$

$$(x^{(1)},x^{(2)},x^{(3)},t) \longrightarrow (x^{(1)},x^{(2)}\cos\theta -x^{(3)}\sin\theta ,x^{(2)}\sin\theta +x^{(3)}\cos\theta ,t)$$

$$\longrightarrow (x^{(1)}, x^{(2)}\cos\theta - x^{(3)}\sin\theta, x^{(2)}\sin\theta + x^{(3)}\cos\theta, t-\sigma)$$

$$\longrightarrow (x^{(1)}, x^{(2)}\cos^2\theta - x^{(3)}\sin\theta\cos\theta + x^{(2)}\sin^2\theta + x^{(3)}\cos\theta\sin\theta,$$

$$-x^{(2)}\cos\theta\sin\theta + x^{(3)}\sin^2\theta + x^{(2)}\sin\theta\cos\theta + x^{(3)}\cos^2\theta, t-\sigma)$$

Taking the infinitesmal limit,

$$\longrightarrow (x^{(1)}, x^{(2)}, x^{(3)}, t-\sigma+\sigma)$$

$$= (x^{(1)}, x^{(2)}, x^{(3)}, t)$$

The initial and final placements of the preparation apparatus are unchanged so we can take

$$[J^{(1)}, H] = i\hbar f_1 I$$

Similarly,

$$[J^{(j)}, H] = i\hbar f_j I \qquad j = 1, 2, 3 \qquad (2.2.8)$$

Case 5

Under rotations, the space coordinates of a point on the preparation apparatus transform as

$$x \longrightarrow x' = \mathcal{M} x$$

where x is the column matrix $\begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{pmatrix}$ and $\mathcal{M}$ is a three-by-three matrix which may be written in terms of products of the unitary matrices:

$$\mathcal{M}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathcal{M}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathcal{M}_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{M}_1(\theta)$, $\mathcal{M}_2(\theta)$, and $\mathcal{M}_3(\theta)$ describe rotations about the 1-,2-, and 3-axis, respectively. Since

$$\mathcal{M}_j(\theta_1 + \theta_2) = \mathcal{M}_j(\theta_1)\,\mathcal{M}_j(\theta_2) \qquad \text{for real } \theta_1, \theta_2$$

and

$$\mathcal{M}_j(0) = 1 \qquad j = 1, 2, 3$$

Stone's Theorem can be used to write

$$\mathcal{M}_j(\theta) = e^{-iM^{(j)}\theta}$$

where the $M^{(j)}$ are Hermitian operators.[3] Now

$$i\,\frac{d\,\mathcal{M}_j(\theta)}{d\theta}\bigg|_{\theta=0} = M^{(j)}$$

so it follows from equation (2.2.9) that

$$M^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$M^{(2)} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$M^{(3)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These matrices satisfy the following commutation relation.

$$[M^{(j)}, M^{(k)}] = i\,\epsilon_{jk\ell}\, M^{(\ell)} \qquad j, k, \ell = 1, 2, 3$$

Consider a rotation of the preparation apparatus of a system about the 1-axis through φ, followed by a rotation through θ about the 2-axis. These are followed by the inverse rotations.

$$\mathcal{M}_2(-\theta)\,\mathcal{M}_1(-\varphi)\,\mathcal{M}_2(\theta)\,\mathcal{M}_1(\varphi) = 1 + \varphi\theta\,[M^{(1)}, M^{(2)}] + \cdots$$

$$= 1 + i\varphi\theta\, M^{(3)} + \cdots$$

This corresponds to a rotation about the 3-axis through an angle of $-\varphi\theta$. Therefore, the operator

$$e^{i\theta J^{(2)}/\hbar}\, e^{i\varphi J^{(1)}/\hbar}\, e^{-i\theta J^{(2)}/\hbar}\, e^{-i\varphi J^{(1)}/\hbar} = 1 + \frac{\varphi\theta}{\hbar^2}\,[J^{(1)}, J^{(2)}] + \cdots$$

must equal

$$e^{i\theta\varphi J^{(3)}/\hbar} = 1 + i\frac{\varphi\theta}{\hbar}\, J^{(3)} + \cdots$$

Consequently, we can write

$$[J^{(1)}, J^{(2)}] = i\hbar\,(J^{(3)} + d_{12} I)$$

and similarly,

$$[J^{(j)}, J^{(k)}] = i\hbar\,(\epsilon_{jk\ell}\, J^{(\ell)} + d_{jk} I) \qquad (2.2.10)$$

Case 6

A space displacement through a along the 1-axis is followed by a Lorentz boost through v, also along the 1-axis. These transformations are followed by their inverses.

$$e^{icK^{(1)}u/\hbar}\, e^{iP^{(1)}a/\hbar}\, e^{-icK^{(1)}u/\hbar}\, e^{-iP^{(1)}a/\hbar} = 1 + \frac{auc}{\hbar^2}\left[P^{(1)}, K^{(1)}\right] + \cdots$$

$$\begin{aligned}
(x^{(1)},x^{(2)},x^{(3)},t) &\longrightarrow (x^{(1)}+ a,x^{(2)},x^{(3)},t) \\
&= (x^{(1)\prime},x^{(2)\prime},x^{(3)\prime},t^{\prime}) \\
&\longrightarrow (\gamma\,(x^{(1)\prime}+ vt^{\prime}),x^{(2)\prime},x^{(3)\prime}, \gamma\,(t^{\prime}+ v/c^2\, x^{(1)\prime})) \\
&= (x^{(1)\prime\prime},x^{(2)\prime\prime},x^{(3)\prime\prime},t^{\prime\prime}) \\
&\longrightarrow (x^{(1)\prime\prime} -a,x^{(2)\prime\prime},x^{(3)\prime\prime},t^{\prime\prime}) \\
&= (x^{(1)\prime\prime\prime},x^{(2)\prime\prime\prime},x^{(3)\prime\prime\prime},t^{\prime\prime\prime}) \\
&\longrightarrow (\gamma\,(x^{(1)\prime\prime\prime} -vt^{\prime\prime\prime}),x^{(2)\prime\prime\prime},x^{(3)\prime\prime\prime}, \gamma\,(t^{\prime\prime\prime} -v/c^2\, x^{(1)\prime\prime\prime})) \\
&= (x^{(1)\mathrm{IV}},x^{(2)\mathrm{IV}},x^{(3)\mathrm{IV}},t^{\mathrm{IV}})
\end{aligned}$$

$$x^{(1)\mathrm{IV}} = (x^{(1)}+ a + vt)\gamma^2 -a\gamma -(vt + v^2/c^2\,(x^{(1)}+ a))\gamma^2$$

$$x^{(2)\mathrm{IV}} = x^{(2)}$$

$$x^{(3)\mathrm{IV}} = x^{(3)}$$

$$t^{\mathrm{IV}} = (t + v/c^2\,(x^{(1)}+ a))\gamma^2 -v/c^2\,(x^{(1)}+ a)\gamma -v^2/c^2\, t\gamma + va/c^2$$

In the infinitesmal limit,

$$(x^{(1)},x^{(2)},x^{(3)},t) \longrightarrow (x^{(1)},x^{(2)},x^{(3)},t + va/c^2)$$

This corresponds to a time translation through $-va/c^2$ so we write

$$\left[P^{(1)}, K^{(1)}\right] = i\hbar\,(H/c^2 + h_1 I)$$

Similarly,

$$[K^{(j)}, P^{(k)}] = -i\hbar(\delta_{jk} H/c^2 + h_j I) \tag{2.2.11}$$

Case 7

A Lorentz boost of velocity v along the 1-axis is followed by a time translation of σ and then by their inverses.

$$e^{iH\sigma/\hbar} e^{icK^{(1)}u/\hbar} e^{-iH\sigma/\hbar} e^{-icK^{(1)}u/\hbar} = 1 + \frac{uc\sigma}{\hbar^2}[K^{(1)}, H] + \cdots$$

$$(x^{(1)},x^{(2)},x^{(3)},t) \longrightarrow (\gamma(x^{(1)}+vt),x^{(2)},x^{(3)},\gamma(t+v/c^2 x^{(1)}))$$

$$= (x^{(1)\prime},x^{(2)\prime},x^{(3)\prime},t')$$

$$\longrightarrow (x^{(1)\prime},x^{(2)\prime},x^{(3)\prime},t'-\sigma)$$

$$= (x^{(1)\prime\prime},x^{(2)\prime\prime},x^{(3)\prime\prime},t'')$$

$$\longrightarrow (\gamma(x^{(1)\prime\prime}-vt''),x^{(2)\prime\prime},x^{(3)\prime\prime},\gamma(t''-v/c^2 x^{(1)\prime\prime}))$$

$$= (x^{(1)\prime\prime\prime},x^{(2)\prime\prime\prime},x^{(3)\prime\prime\prime},t''')$$

$$\longrightarrow (x^{(1)\prime\prime\prime},x^{(2)\prime\prime\prime},x^{(3)\prime\prime\prime},t'''+\sigma)$$

$$= (\gamma^2(x^{(1)}+vt)-v(t+v/c^2 x^{(1)})\gamma^2+v\sigma\gamma,x^{(2)},x^{(3)},$$

$$(t+v/c^2 x^{(1)})\gamma^2-\sigma\gamma-v/c^2(x^{(1)}+vt)\gamma^2+\sigma^2)$$

In the infinitesmal limit,

$$(x^{(1)},x^{(2)},x^{(3)},t) \longrightarrow (x^{(1)}+v\sigma,x^{(2)},x^{(3)},t)$$

This corresponds to a translation through $v\sigma$ along the 1-axis. Thus,

$$[K^{(1)}, H] = -i\hbar(P^{(1)} + j_1 I)$$

and similarly,

$$[K^{(j)}, H] = -i\hbar\,(P^{(j)} + j_j I) \qquad (2.2.12)$$

Case 8

A rotation through θ about the 1-axis is followed by a Lorentz boost through v along the 2-axis and then by the inverse transformations.

$$e^{icK^{(2)}u/\hbar}\, e^{iJ^{(1)}\theta/\hbar}\, e^{-icK^{(2)}_u/\hbar}\, e^{-iJ^{(1)}\theta/\hbar} = 1 + \frac{\theta u c}{\hbar^2}\,[J^{(1)}, K^{(2)}] + \cdots$$

$$(x^{(1)}, x^{(2)}, x^{(3)}, t) \longrightarrow (x^{(1)}, x^{(2)}\cos\theta - x^{(3)}\sin\theta\ , x^{(2)}\sin\theta + x^{(3)}\cos\theta\ , t)$$

$$= (x^{(1)\prime}, x^{(2)\prime}, x^{(3)\prime}, t^{\prime})$$

$$\longrightarrow (x^{(1)\prime}, \gamma\,(x^{(2)\prime} + vt^{\prime}), x^{(3)\prime}, \gamma\,(t^{\prime} + v/c^2\, x^{(2)\prime}))$$

$$= (x^{(1)\prime\prime}, x^{(2)\prime\prime}, x^{(3)\prime\prime}, t^{\prime\prime})$$

$$\longrightarrow (x^{(1)\prime\prime}\ , x^{(2)\prime\prime}\cos\theta + x^{(3)\prime\prime}\sin\theta\ , -x^{(2)\prime\prime}\sin\theta + x^{(3)\prime\prime}\cos\theta\ , t^{\prime\prime})$$

$$= (x^{(1)\prime\prime\prime}, x^{(2)\prime\prime\prime}, x^{(3)\prime\prime\prime}, t^{\prime\prime\prime})$$

$$\longrightarrow (x^{(1)\prime\prime\prime}, \gamma\,(x^{(2)\prime\prime\prime} - vt^{\prime\prime\prime}), x^{(3)\prime\prime\prime}, \gamma\,(t^{\prime\prime\prime} - v/c^2\, x^{(2)\prime\prime\prime}))$$

$$= (x^{(1)IV}, x^{(2)IV}, x^{(3)IV}, t^{IV})$$

In the limit of v,θ infinitesmal,

$$(x^{(1)}, x^{(2)}, x^{(3)}, t) \longrightarrow (x^{(1)}, x^{(2)}, x^{(3)} - \theta vt, t - v\,\theta\,/c^2\, x^{(3)})$$

This corresponds to an infinitesmal Lorentz boost through $-v\theta$ along the 3-axis so we write

$$[J^{(1)}, K^{(2)}] = i\hbar\,(K^{(3)} + k_{12} I)$$

Similarly,

$$[J^{(j)}, K^{(k)}] = i\hbar(\epsilon_{jk\ell}\, K^{(\ell)} + k_{jk} I) \qquad (2.2.13)$$

Case 9

A Lorentz boost along the 1-axis of velocity v is followed by a second Lorentz boost, also of velocity v (for simplicity), along the 3-axis. These are succeeded by their inverses.

$$e^{icK^{(3)}u/\hbar}\, e^{icK^{(1)}u/\hbar}\, e^{-icK^{(3)}u/\hbar}\, e^{-icK^{(1)}u/\hbar} = 1 + \frac{u^2c^2}{\hbar^2}\left[K^{(1)}, K^{(3)}\right] + \ldots$$

$$(x^{(1)}, x^{(2)}, x^{(3)}, t) \longrightarrow (\gamma\,(x^{(1)} + vt), x^{(2)}, x^{(3)}, t\gamma + \gamma\, v/c^2\, x^{(1)})$$

$$= (x^{(1)\prime}, x^{(2)\prime}, x^{(3)\prime}, t^{\prime})$$

$$\longrightarrow (x^{(1)\prime}, x^{(2)\prime}, \gamma\,(x^{(3)\prime} + vt^{\prime}),\ \gamma\,(t^{\prime} + v/c^2\, x^{(3)\prime}))$$

$$= (x^{(1)\prime\prime}, x^{(2)\prime\prime}, x^{(3)\prime\prime}, t^{\prime\prime})$$

$$\longrightarrow (\gamma\,(x^{(1)\prime\prime} - vt^{\prime\prime}), x^{(2)\prime\prime}, x^{(3)\prime\prime},\ \gamma\,(t^{\prime\prime} - v/c^2\, x^{(1)\prime\prime}))$$

$$= (x^{(1)\prime\prime\prime}, x^{(2)\prime\prime\prime}, x^{(3)\prime\prime\prime}, t^{\prime\prime\prime})$$

$$\longrightarrow (x^{(1)\prime\prime\prime}, x^{(2)\prime\prime\prime}, \gamma\,(x^{(3)\prime\prime\prime} - vt^{\prime\prime\prime}),\ \gamma\,(t^{\prime\prime\prime} - v/c^2\, x^{(3)\prime\prime\prime}))$$

$$= (x^{(1)IV}, x^{(2)IV}, x^{(3)IV}, t^{IV})$$

$$x^{(1)IV} = \gamma^2(x^{(1)} + vt) - \gamma^3(vt + v^2/c^2\, x^{(1)}) - \gamma^2(v^2/c^2\, x^{(3)})$$

$$x^{(2)IV} = x^{(2)}$$

$$x^{(3)IV} = \gamma^2 x^{(3)} + \gamma^3 v(t + v/c^2\, x^{(1)}) - \gamma^4 v(t + v/c^2\, x^{(1)}) - \gamma^3 v^2/c^2\, x^{(3)} + \gamma^3 v^2/c^2\,(x^{(1)} + vt)$$

$$t^{IV} = \gamma^4(t + v/c^2\, x^{(1)}) + \gamma^3 v/c^2\, x^{(3)} - \gamma^3 v/c^2\,(x^{(1)} + vt) - \gamma^2 v/c^2\, x^{(3)} - \gamma^3 v/c^2\,(vt + (v/c)^2 x^{(1)})$$

In the limit of v infinitesmal,

$$(x^{(1)}, x^{(2)}, x^{(3)}, t) \longrightarrow (x^{(1)} - x^{(3)}v^2/c^2, x^{(2)}, x^{(3)} + x^{(1)}v^2/c^2, t)$$

This corresponds to a rotation through an infinitesmal angle $-v^2/c^2$ about the 2-axis. Therefore, we can write

$$[K^{(1)}, K^{(3)}] = i\hbar\,(J^{(2)}/c^2 + g_{13}I)$$

Similarly,

$$[K^{(j)}, K^{(k)}] = -i\hbar\,(\epsilon_{jk\ell}\, J^{(\ell)}/c^2 + g_{jk}I) \tag{2.2.14}$$

We now show how the multiplés of the identity are eliminated from the commutation relations. Using the Jacobi identity and equation (2.2.5), we have

$$i\hbar\,[P^{(3)}, P^{(1)}] = [[J^{(1)}, P^{(2)}], P^{(1)}]$$

$$= [[P^{(1)}, P^{(2)}], J^{(1)}] + [[J^{(1)}, P^{(1)}], P^{(2)}]$$

$$= [a_{12}I, J^{(1)}] + [e_{11}I, P^{(2)}]$$

$$= 0$$

so $a_{31} = 0$ and, similarly, $a_{jk} = 0$, $j, k = 1, 2, 3$. Using the Jacobi identity with equation (2.2.11), we find

$$h_{jk} = \delta_{jk} h$$

so

$$[K^{(j)}, P^{(k)}] = -i\hbar\,\delta_{jk}\,(H/c^2 + hI)$$

Here, we eliminate the multiple of the identity by adding a term to the Hamiltonian H. That is,

$$H/c^2 \longrightarrow H/c^2 + hI$$

Then,

$$[K^{(j)}, P^{(k)}] = -i\hbar\, \delta_{jk}\, H/c^2$$

All the multiples of the identity are eliminated from the commutation relations either by using the Jacobi identity or by adding real numbers to the generators. In summary, the generators of the Poincaré transformations satisfy the following commutation relations:

$$[P^{(j)}, P^{(k)}] = 0 \qquad [P^{(j)}, H] = 0$$

$$[J^{(j)}, P^{(k)}] = i\hbar\, \epsilon_{jk\ell} P^{(\ell)} \qquad [J^{(j)}, H] = 0 \qquad [J^{(j)}, J^{(k)}] = i\hbar\, \epsilon_{jk\ell} J^{(\ell)}$$

$$[K^{(j)}, P^{(k)}] = -i\hbar\, \delta_{jk} H/c^2 \qquad [K^{(j)}, H] = -i\hbar\, P^{(j)} \qquad [K^{(j)}, J^{(k)}] = i\hbar\, \epsilon_{jk\ell} K^{(\ell)} \qquad [K^{(j)}, K^{(k)}] = i\hbar\, \epsilon_{jk\ell} J^{(\ell)}$$

$$j, k, \ell = 1, 2, 3 \qquad (2.2.15)$$
a - i

Thus, to describe a system which is Lorentz invariant, one must construct from the fundamental dynamical variables of the system ten Hermitian operators satisfying the above commutation relations.

2.3 Various Transformed Operators

Recall that an observable A' measured by a Poincaré transformed measuring apparatus is related to A measured by the original apparatus in this manner.

$$A' = \mathcal{U} A \mathcal{U}^{+} \tag{2.3.1}$$

where $\mathcal{U}$ is the unitary operator which corresponds to the Poincaré transformation. Further, $\mathcal{U}$ may be written in terms of products of one or more of the ten one-parameter operators $\mathcal{U}_\alpha(\tau)$.

$$\mathcal{U}_\alpha(\tau) = e^{-i C_\alpha \tau/\hbar} \tag{2.3.2}$$

We define $A(\tau, j)$ as

$$A(\tau, j) \equiv \mathcal{U}_\alpha(\tau) A \mathcal{U}_\alpha^{\dagger}(\tau) \tag{2.3.3}$$

The parameter τ and axis, j , along or about which the transformation takes place specify the label α . That is, they specify the transformation. Thus, in this section we use the notation $A(\tau, j)$ to identify the observable measured by the transformed apparatus. Then

$$i\hbar \frac{d}{d\tau} A(\tau, j) = \mathcal{U}_\alpha(\tau) \left[C_\alpha, A \right] \mathcal{U}_\alpha^{+}(\tau) \tag{2.3.4}$$

Equation (2.3.4), along with the initial condition

$$A(0, j) = A$$

can be solved for $A(\tau, j)$ when the commutator $[C_\alpha, A]$ is known. Some results follow.

$$\mathcal{U}(t)\,H\,\mathcal{U}^{\dagger}(t) = H \tag{2.3.5}$$

$$\mathcal{U}(t)\,\underline{P}\,\mathcal{U}^{\dagger}(t) = \underline{P} \tag{2.3.6}$$

$$D(\underline{a})\,H\,D^{\dagger}(\underline{a}) = H \tag{2.3.7}$$

$$D(\underline{a})\,\underline{P}\,D^{\dagger}(\underline{a}) = \underline{P} \tag{2.3.8}$$

$$\mathcal{R}(\underline{\theta})\,H\,\mathcal{R}^{\dagger}(\underline{\theta}) = H \tag{2.3.9}$$

$$\mathcal{R}^{(1)}(\alpha)\,P^{(1)}\,\mathcal{R}^{(1)\dagger}(\alpha) = P^{(1)} \tag{2.3.10a}$$

$$\mathcal{R}^{(1)}(\alpha)\,P^{(2)}\,\mathcal{R}^{(1)\dagger}(\alpha) = P^{(2)}\cos\alpha + P^{(3)}\sin\alpha \tag{2.3.10b}$$

$$\mathcal{R}^{(1)}(\alpha)\,P^{(3)}\,\mathcal{R}^{(1)\dagger}(\alpha) = P^{(3)}\cos\alpha - P^{(2)}\sin\alpha \tag{2.3.10c}$$

$$L^{(1)}(u)\,H\,L^{(1)\dagger}(u) = H\cosh u - cP^{(1)}\sinh u \tag{2.3.11}$$

$$L^{(1)}(u)\,P^{(1)}\,L^{(1)\dagger}(u) = P^{(1)}\cosh u - \tfrac{1}{c}H\sinh u \tag{2.3.12a}$$

$$L^{(1)}(u)\,P^{(2)}\,L^{(1)\dagger}(u) = P^{(2)} \tag{2.3.12b}$$

$$L^{(1)}(u)\,P^{(3)}\,L^{(1)\dagger}(u) = P^{(3)} \tag{2.3.12c}$$

$$\mathcal{U}(t)\,\underline{J}\,\mathcal{U}^{\dagger}(t) = \underline{J} \tag{2.3.13}$$

$$\mathcal{U}(t)\,\underline{K}\,\mathcal{U}^{\dagger}(t) = \underline{K} + \underline{P}t \tag{2.3.14}$$

$$D(\underline{a})\,\underline{J}\,D^{\dagger}(\underline{a}) = \underline{J} - \underline{a}\times\underline{P} \tag{2.3.15}$$

$$D(\underline{a})\,\underline{K}\,D^{\dagger}(\underline{a}) = \underline{K} + \underline{a}\,H/c^2 \tag{2.3.16}$$

$$\mathcal{R}^{(1)}(\alpha)\, J^{(1)}\, \mathcal{R}^{(1)\dagger}(\alpha) = J^{(1)} \tag{2.3.17a}$$

$$\mathcal{R}^{(1)}(\alpha)\, J^{(2)}\, \mathcal{R}^{(1)\dagger}(\alpha) = J^{(2)}\cos\alpha + J^{(3)}\sin\alpha \tag{2.3.17b}$$

$$\mathcal{R}^{(1)}(\alpha)\, J^{(3)}\, \mathcal{R}^{(1)\dagger}(\alpha) = J^{(3)}\cos\alpha - J^{(2)}\sin\alpha \tag{2.3.17c}$$

$$\mathcal{R}^{(1)}(\alpha)\, K^{(1)}\, \mathcal{R}^{(1)\dagger}(\alpha) = K^{(1)} \tag{2.3.18a}$$

$$\mathcal{R}^{(1)}(\alpha)\, K^{(2)}\, \mathcal{R}^{(1)\dagger}(\alpha) = K^{(2)}\cos\alpha + K^{(3)}\sin\alpha \tag{2.3.18b}$$

$$\mathcal{R}^{(1)}(\alpha)\, K^{(3)}\, \mathcal{R}^{(1)\dagger}(\alpha) = K^{(3)}\cos\alpha - K^{(2)}\sin\alpha \tag{2.3.18c}$$

$$L^{(1)}(u)\, J^{(1)}\, L^{(1)\dagger}(u) = J^{(1)} \tag{2.3.19a}$$

$$L^{(1)}(u)\, J^{(2)}\, L^{(1)\dagger}(u) = J^{(2)}\cosh u + c K^{(3)}\sinh u \tag{2.3.19b}$$

$$L^{(1)}(u)\, J^{(3)}\, L^{(1)\dagger}(u) = J^{(3)}\cosh u - c K^{(2)}\sinh u \tag{2.3.19c}$$

$$L^{(1)}(u)\, K^{(1)}\, L^{(1)\dagger}(u) = K^{(1)} \tag{2.3.20a}$$

$$L^{(1)}(u)\, K^{(2)}\, L^{(1)\dagger}(u) = K^{(2)}\cosh u - \tfrac{1}{c} J^{(3)}\sinh u \tag{2.3.20b}$$

$$L^{(1)}(u)\, K^{(3)}\, L^{(1)\dagger}(u) = K^{(3)}\cosh u + \tfrac{1}{c} J^{(2)}\sinh u \tag{2.3.20c}$$

where $D(\underset{\sim}{a}) = D^{(1)}(a^{(1)})\, D^{(2)}(a^{(2)})\, D^{(3)}(a^{(3)}) = e^{-i\underset{\sim}{P}\cdot\underset{\sim}{a}/\hbar}$ (2.3.21)

and $\mathcal{R}(\underset{\sim}{\theta}) = \mathcal{R}^{(3)}(\alpha)\, \mathcal{R}^{(2)}(\beta)\, \mathcal{R}^{(3)}(\gamma) = e^{-i\underset{\sim}{J}\cdot\underset{\sim}{\theta}/\hbar}$ (2.3.22)

where $\underset{\sim}{\theta} = \theta\, \underset{\sim}{n}$ and $\underset{\sim}{n}$ is a unit vector along the axis of rotation.

Before deriving any of the above equations, we introduce the space-time displacement operator, $\mathcal{D}(x)$, where

$$\mathcal{D}(x) \equiv D(\underline{x})\, U(t) \tag{2.3.23}$$

Taking $L^{(1)\dagger}(u) \cdots L^{(1)}(u)$ on the displacement operator and using the transformation properties of $\underline{P}$ and H from above,

$$L^{(1)\dagger}(u)\,\mathcal{D}(x)\,L^{(1)}(u) = e^{-i L^{(1)\dagger}(u)\,\underline{P}\,L^{(1)}(u)\cdot \underline{x}/\hbar}\; e^{-i L^{(1)\dagger}(u)\,H\,L^{(1)}(u)\,t/\hbar}$$

$$= e^{-i\left(\left(P^{(1)}\cosh u + \frac{1}{c} H \sinh u\right)x^{(1)} + P^{(2)}x^{(2)} + P^{(3)}x^{(3)}\right)/\hbar}\; e^{-i\left(H\cosh u + c P^{(1)}\sinh u\right)t/\hbar}$$

$$= e^{-i\underline{P}\cdot\underline{x}'/\hbar}\; e^{-iHt'/\hbar}$$

$$= D(\underline{x}')\,U(t')$$

$$L^{(1)\dagger}(u)\,\mathcal{D}(x)\,L^{(1)}(u) = \mathcal{D}(x') \tag{2.3.24}$$

where x' is the point on a Lorentz boosted preparation apparatus given by equation (2.1.5). Thus the measurement of an observable by a measuring apparatus which has undergone a space-time displacement through x' yields the same result as the measurement of that observable by an apparatus which has undergone a Lorentz boost through u, a space-time displacement through x, and, finally, a Lorentz boost through $-u$. That is,

$$\mathcal{D}(x')\,A\,\mathcal{D}^{\dagger}(x') = L^{(1)\dagger}(u)\,\mathcal{D}(x)\,L^{(1)}(u)\,A\,L^{(1)\dagger}(u)\,\mathcal{D}^{\dagger}(x)\,L^{(1)}(u) \tag{2.3.24}$$

We now derive some of the equations (2.3.5) through (2.3.20). Equations (2.3.5) through (2.3.9) and equation (2.3.13) follow immediately since the corresponding commutators in equation (2.3.4) vanish. Equations (2.3.10) are now derived.

For a rotation through α about the 1-axis, define

$$P^{(j)}(\alpha,1) \equiv \mathcal{R}^{(1)}(\alpha)\,P^{(j)}\,\mathcal{R}^{(1)\dagger}(\alpha) \qquad j=1,2,3$$

Then

$$i\hbar \frac{d}{d\alpha} P^{(j)}(\alpha,1) = \mathcal{R}^{(1)}(\alpha)\,[J^{(1)}, P^{(j)}]\,\mathcal{R}^{(1)\dagger}(\alpha)$$

and $P^{(j)}(0,1) = P^{(j)}$

Since $[J^{(1)}, P^{(1)}] = 0$, we have

$$\mathcal{R}^{(1)}(\alpha)\, P^{(1)}\, \mathcal{R}^{(1)\dagger}(\alpha) = P^{(1)}$$

For $j=2$ and $j=3$, we obtain the following coupled equations.

$$i\hbar \frac{d}{d\alpha} P^{(2)}(\alpha,1) = \mathcal{R}^{(1)}(\alpha)\,(i\hbar P^{(3)})\,\mathcal{R}^{(1)\dagger}(\alpha) = i\hbar P^{(3)}(\alpha,1)$$

$$i\hbar \frac{d}{d\alpha} P^{(3)}(\alpha,1) = -\mathcal{R}^{(1)}(\alpha)\,(i\hbar P^{(2)})\,\mathcal{R}^{(1)\dagger}(\alpha) = -i\hbar P^{(2)}(\alpha,1)$$

Differentiating with respect to α,

$$\frac{d^2}{d\alpha^2} P^{(2)}(\alpha,1) = -P^{(2)}(\alpha,1) \qquad \frac{d^2}{d\alpha^2} P^{(3)}(\alpha,1) = -P^{(3)}(\alpha,1)$$

Also,

$$P^{(2)}(0,1) = P^{(2)} \qquad P^{(3)}(0,1) = P^{(3)}$$

These equations have equations (2.3.10b) and (2.3.10c) as their respective solutions. Thus $(P^{(1)}, P^{(2)}, P^{(3)})$ transform under rotations like the components of a three-vector so we write $\underline{P} = (P^{(1)}, P^{(2)}, P^{(3)})$.

Equations (2.3.17) and (2.3.18) are derived in exactly the same way. Thus, we write $\underline{J} = (J^{(1)}, J^{(2)}, J^{(3)})$ and $\underline{K} = (K^{(1)}, K^{(2)}, K^{(3)})$.

We now derive equations (2.3.11) and (2.3.12). Under a Lorentz boost through u along the 1-axis, define

$$P^{(j)}(u,1) \equiv L^{(1)}(u)\, P^{(j)} L^{(1)\dagger}(u) \qquad\qquad j=1,2,3$$

$$i\hbar \frac{d}{du} P^{(j)}(u,1) = L^{(1)}(u)\left[cK^{(1)}, P^{(j)}\right] L^{(1)\dagger}(u) = -L^{(1)}(u)\left(i\hbar\, \delta_{j1}\, H/c\right) L^{(1)\dagger}(u)$$

$$P^{(j)}(0,1) = P^{(j)}$$

Obviously, for $j=2$ and $j=3$ we have the results

$$L^{(1)}(u)\, P^{(2)} L^{(1)\dagger}(u) = P^{(2)} \qquad\qquad L^{(1)}(u)\, P^{(3)} L^{(1)\dagger}(u) = P^{(3)}$$

since the commutators vanish. We also define

$$H(u,1) \equiv L^{(1)}(u)\, H\, L^{(1)\dagger}(u)$$

Then $\quad i\hbar \frac{d}{du} H(u,1) = L^{(1)}(u)\left[cK^{(1)}, H\right] L^{(1)\dagger}(u)$

$$= -L^{(1)}(u)\left(i\hbar c P^{(1)}\right) L^{(1)\dagger}(u)$$

$$= -i\hbar c P^{(1)}(u,1)$$

and $H(0,1) = H$

The coupled equations

$$\frac{d}{du} P^{(1)}(u,1) = -\frac{1}{c} H(u,1) \qquad\qquad \frac{d}{du} H(u,1) = -c P^{(1)}(u,1)$$

have as their solutions equations (2.3.11) and (2.3.12a), respectively.

We now derive equation (2.3.14) above. For a time translation through t, define

$$\underline{K}(t) \equiv \mathcal{U}(t)\, \underline{K}\, \mathcal{U}^{\dagger}(t)$$

Then $i\hbar \frac{d}{dt}\underline{K}(t) = U(t)[H,\underline{K}]U^{+}(t)$

$$= U(t)\,(i\hbar\underline{P})\,U^{+}(t)$$

$$= i\hbar\underline{P}$$

and $\underline{K}(0) = \underline{K}$

Therefore, we have equation (2.3.14) as the solution.

To derive equation (2.3.15), consider one component of $\underline{J}$, $J^{(j)}$, displaced by $a^{(K)}$ along the K-axis. Define

$$J^{(j)}(a^{(K)}, K) \equiv D^{(K)}(a^{(K)})\, J^{(j)}\, D^{(K)+}(a^{(K)})$$

Then $i\hbar \frac{d J^{(j)}(a^{(K)}, K)}{d a^{(K)}} = D^{(K)}(a^{(K)})\,[P^{(K)}, J^{(j)}]\,D^{(K)+}(a^{(K)})$

$$= -D^{(K)}(a^{(K)})\,(i\hbar\,\epsilon_{jK\ell}\,P^{(\ell)})\,D^{(K)+}(a^{(K)})$$

$$= -i\hbar\,\epsilon_{jK\ell}\,P^{(\ell)}$$

Also, $J^{(j)}(0, K) = J^{(j)}$

Therefore, we have

$$D^{(K)}(a^{(K)})\, J^{(j)}\, D^{(K)+}(a^{(K)}) = J^{(j)} - \epsilon_{jK\ell}\,P^{(\ell)}\,a^{(K)}$$

so, for a general displacement through $\underline{a} = a^{(1)}\underline{e}_1 + a^{(2)}\underline{e}_2 + a^{(3)}\underline{e}_3$, we have

$$D(\underline{a})\,J^{(j)}\,D^{+}(\underline{a}) = J^{(j)} - (a^{(K)}P^{(\ell)} - a^{(\ell)}P^{(K)})\epsilon_{jK\ell} \qquad K, \ell \neq j$$

Thus equation (2.3.15) follows.

Equation (2.3.16) is derived simply as (2.3.14) was. Equations (2.3.19) and (2.3.20) are derived below. Under Lorentz boosts through u along the 1-axis, define

$$J^{(j)}(u,1) \equiv L^{(1)}(u)\, J^{(j)}\, L^{(1)\dagger}(u)$$

Then $$i\hbar \frac{dJ^{(j)}(u,1)}{du} = L^{(1)}(u)\,[cK^{(1)}, J^{(j)}]\, L^{(1)\dagger}(u)$$

$$= -i\hbar c\, \epsilon_{j1\ell}\, L^{(1)}(u)\, K^{(\ell)}\, L^{(1)\dagger}(u)$$

$$\equiv -i\hbar c\, \epsilon_{j1\ell}\, K^{(\ell)}(u,1)$$

and $J^{(j)}(0,1) = J^{(j)}$

In the simplest case, $j=1$ and

$$\frac{dJ^{(1)}(u,1)}{du} = 0$$

so that equation (2.3.15a) follows. For $j=2$, we have

$$\frac{dJ^{(2)}(u,1)}{du} = c\,K^{(3)}(u,1)$$

The corresponding equation for $K^{(3)}(u,1)$ is

$$i\hbar \frac{dK^{(3)}(u,1)}{du} = L^{(1)}(u)\,[cK^{(1)}, K^{(3)}]\, L^{(1)\dagger}(u)$$

$$= L^{(1)}(u)\,(i\hbar J^{(2)}/c)\, L^{(1)\dagger}(u)$$

$$= \frac{i\hbar}{c}\, J^{(2)}(u,1)$$

Also, $K^{(3)}(0,1) = K^{(3)}$

Now we have a system of two coupled equations.

$$\frac{d}{du} J^{(2)}(u,1) = c\, K^{(3)}(u,1) \qquad \frac{d}{du} K^{(3)}(u,1) = \frac{1}{c} J^{(2)}(u,1)$$

$$J^{(2)}(0,1) = J^{(2)} \qquad K^{(3)}(0,1) = K^{(3)}$$

This system has as its solution equations (2.3.19b) and (2.3.20c). Similarly, we have

$$\frac{d}{du} J^{(3)}(u,1) = -c\, K^{(2)}(u,1) \qquad \frac{d}{du} K^{(2)}(u,1) = -\frac{1}{c} J^{(3)}(u,1)$$

$$J^{(3)}(0,1) = J^{(3)} \qquad K^{(2)}(0,1) = K^{(2)}$$

with equations (2.3.19c) and (2.3.20b) as solutions. Also,

$$L^{(1)}(u)\, K^{(1)} L^{(1)\dagger}(u) = K^{(1)}$$

2.4 Poincaré Invariants M^2 and $\mathcal{W}^2$

There are certain combinations of operators which are invariant under all Poincaré transformations. Equivalently, such invariants commute with all the generators of the Poincaré group.

Let the operator M^2 be defined by

$$M^2c^4 = H^2 - c^2 P^2 \tag{2.4.1}$$

From the commutation relations of H and $\underline{P}$ it is clear that this object commutes with all the generators of the Poincaré group. M^2 is therefore an invariant under all Poincaré transformations. The nonnegative quantity $(M^2)^{1/2} = M$ is called the total mass of the system.

We also define the Pauli-Lubanski four-vector $(\mathcal{H}, \underline{W})$ where

$$\mathcal{H} = \underline{J} \cdot \underline{P} \tag{2.4.2}$$

$$\underline{W} = c\,(\underline{K} \times \underline{P}) + \tfrac{1}{c} H \underline{J} \tag{2.4.3}$$

is called the total helicity of the system. Also, we note that

$$\tfrac{1}{c}\mathcal{H} H - \underline{W} \cdot \underline{P} = 0 \tag{2.4.4}$$

$\mathcal{H}$ and $\underline{W}$ transform under Poincaré transformations as follows:

$$U(t)\,\mathcal{H}\,U^\dagger(t) = \mathcal{H} \quad (2.4.5) \qquad\qquad U(t)\,\underline{W}\,U^\dagger(t) = \underline{W} \quad (2.4.6)$$

$$D(\underline{a})\,\mathcal{H}\,D^\dagger(\underline{a}) = \mathcal{H} \quad (2.4.7) \qquad\qquad D(\underline{a})\,\underline{W}\,D^\dagger(\underline{a}) = \underline{W} \quad (2.4.8)$$

$$\mathcal{R}^{(1)}(\alpha)\,\mathcal{H}\,\mathcal{R}^{(1)\dagger}(\alpha) = \mathcal{H} \quad (2.4.9) \qquad\qquad \mathcal{R}^{(1)}(\alpha)\,W^{(1)}\,\mathcal{R}^{(1)\dagger}(\alpha) = W^{(1)} \quad (2.4.10a)$$

$$\mathcal{R}^{(1)}(\alpha)\,W^{(2)}\,\mathcal{R}^{(1)\dagger}(\alpha) = W^{(2)}\cos\alpha + W^{(3)}\sin\alpha \tag{2.4.10b}$$

$$R^{(1)}(\alpha)W^{(3)}R^{(1)\dagger}(\alpha) = W^{(3)}\cos\alpha - W^{(2)}\sin\alpha \tag{2.4.10c}$$

$$L^{(1)}(u)\mathcal{H}L^{(1)\dagger}(u) = \mathcal{H}\cosh u - W^{(1)}\sinh u \tag{2.4.11}$$

$$L^{(1)}(u)W^{(1)}L^{(1)\dagger}(u) = W^{(1)}\cosh u - \mathcal{H}\sinh u \tag{2.4.12a}$$

$$L^{(1)}(u)W^{(2)}L^{(1)\dagger}(u) = W^{(2)} \tag{2.4.12b}$$

$$L^{(1)}(u)W^{(3)}L^{(1)\dagger}(u) = W^{(3)} \tag{2.4.12c}$$

The above transformation equations are all obtained from equations (2.4.2) and (2.4.3) and from the transformation properties of the generators H, $\underline{P}$, $\underline{J}$, and $\underline{K}$ of the previous section. For example,

$$L^{(1)}(u)\mathcal{H}L^{(1)\dagger}(u) = (L^{(1)}(u)\underline{J}L^{(1)\dagger}(u))\cdot(L^{(1)}(u)\underline{P}L^{(1)\dagger}(u))$$

$$= J^{(1)}(P^{(1)}\cosh u - \tfrac{1}{c}H\sinh u) + (J^{(2)}\cosh u + cK^{(3)}\sinh u)P^{(2)}$$

$$+ (J^{(3)}\cosh u - cK^{(2)}\sinh u)P^{(3)}$$

$$= \underline{J}\cdot\underline{P}\cosh u - (c(\underline{K}\times\underline{P})^{(1)} + \tfrac{1}{c}HJ^{(1)})\sinh u$$

$$= \mathcal{H}\cosh u - W^{(1)}\sinh u$$

We now define an operator $\mathcal{W}^2$ in terms of the operators $\mathcal{H}$ and $\underline{W}$.

$$\mathcal{W}^2 \equiv \mathcal{H}^2 - \underline{W}\cdot\underline{W} \tag{2.4.13}$$

$\mathcal{W}^2$ is invariant under the Poincaré transformations. That is,

$$U(t)\mathcal{W}^2U^\dagger(t) = \mathcal{W}^2 \tag{2.4.14}$$

$$D(\underline{a})\,\mathcal{W}^2\,D^{\dagger}(\underline{a}) = \mathcal{W}^2 \qquad (2.4.15)$$

$$\mathcal{R}(\underline{\theta})\,\mathcal{W}^2\,\mathcal{R}^{\dagger}(\underline{\theta}) = \mathcal{W}^2 \qquad (2.4.16)$$

$$L^{(j)}(u)\,\mathcal{W}^2\,L^{(j)\dagger}(u) = \mathcal{W}^2 \qquad (2.4.17)$$

These invariance relations are arrived at easily. For example, under a Lorentz boost along the 1-axis, $\mathcal{W}^2$ transforms as

$$\begin{aligned} L^{(1)}(u)\,\mathcal{W}^2\,L^{(1)\dagger}(u) &= L^{(1)}(u)\,(\mathcal{H}^2 - \underline{W}\cdot\underline{W})\,L^{(1)\dagger}(u) \\ &= (\mathcal{H}\cosh u - W^{(1)}\sinh u)^2 - (W^{(1)}\cosh u - \mathcal{H}\sinh u)^2 - (W^{(2)})^2 - (W^{(3)})^2 \\ &= \mathcal{H}^2 - \underline{W}\cdot\underline{W} \\ &= \mathcal{W}^2 \end{aligned}$$

2.5 Four-vector Notation

Equations (2.3.6), (2.3.13), and (2.3.14) show that $\underline{P}$, $\underline{J}$, and $\underline{K}$ transform like three-vectors under rotations. In this section we show that combinations of the generators transform under Lorentz boosts like four-vectors and like a rank two tensor.

Equations (2.3.7) and (2.3.8) may be written as

$$L^{(1)}(u)P^{\mu}L^{(1)\dagger}(u) = \ell^{\mu}{}_{\nu}\, P^{\nu} \qquad \mu,\nu = 0,1,2,3 \tag{2.5.1}$$

where

$$(\ell^{\mu}{}_{\nu}) = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.5.2}$$

and $P^{\mu} = (\frac{1}{c}H, \underline{P})$ (2.5.3)

Thus P^{μ} transforms under Lorentz boosts like a contravariant vector. We take the metric tensor $g^{\mu\nu}$ to be such that $g^{00} = -g^{ii} = 1$; $g^{\mu\nu} = 0$ if $\mu \neq \nu$; $g^{\mu\nu} = g_{\mu\nu}$. Thus, $P_{\mu} = g_{\mu\nu}P^{\nu} = (\frac{1}{c}H, -\underline{P})$. One can thus write equation (2.4.1) as

$$P^{\mu}P_{\mu} = M^2 c^4 \tag{2.5.4}$$

Equations (2.4.4), (2.4.11), (2.4.12), and (2.4.13) may be written as

$$P^{\mu}W_{\mu} = 0 \tag{2.5.5}$$

$$L^{(1)}(u)W^{\mu}L^{(1)\dagger}(u) = \ell^{\mu}{}_{\nu}\, W^{\nu} \tag{2.5.6}$$

$$W^{\mu}W_{\mu} = w^{2} \tag{2.5.7}$$

where $W^{\mu} = (\mathcal{H}, \underline{W})$

Furthermore, the equations (2.3.15) and (2.3.16) may be written as

$$L^{(1)}(u) M^{\mu\nu} L^{(1)\dagger}(u) = \ell^{\mu}{}_{\mu'} \, \ell^{\nu}{}_{\nu'} \, M^{\mu'\nu'} \tag{2.5.8}$$

where $(M^{23}, M^{31}, M^{12}) = \underline{J} = (J^{(1)}, J^{(2)}, J^{(3)})$ (2.5.9)

and $(M^{01}, M^{02}, M^{03}) = c\underline{K} = (cK^{(1)}, cK^{(2)}, cK^{(3)})$ (2.5.10)

Also, $M^{\mu\nu} = -M^{\nu\mu}$

Thus, the $M^{\mu\nu}$ transform under Lorentz boosts as a rank two contravariant tensor. The commutation relations for the generators may be written more compactly in terms of P^{μ} and $M^{\mu\nu}$ as follows.[4]

$$[P^{\mu}, P^{\nu}] = 0 \tag{2.5.11}$$

$$[M^{\mu\nu}, P^{\sigma}] = i\hbar \left(g^{\nu\sigma} P^{\mu} - g^{\mu\sigma} P^{\nu} \right) \tag{2.5.12}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\hbar \left(g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma} + g^{\mu\sigma} M^{\rho\nu} - g^{\nu\sigma} M^{\rho\mu} \right); \quad \mu, \nu, \rho, \sigma = 0,1,2,3 \tag{2.5.13}$$

Equation (2.5.11) corresponds to

$$[P^{(j)}, P^{k}] = 0 \quad \text{and} \quad [P^{(j)}, H] = 0 \quad ,$$

equation (2.5.12) to

$$[J^{(j)}, H] = 0 \quad , \quad [K^{(j)}, H] = -i\hbar P^{(j)} \quad ,$$

$$[J^{(j)}, P^{(k)}] = i\hbar \epsilon_{jk\ell} P^{(\ell)} \quad , \quad \text{and} \quad [P^{(j)}, K^{(k)}] = i\hbar \delta_{jk} H/c^2 \quad ,$$

and equation (2.5.13) to

$$[J^{(j)}, J^{(k)}] = i\hbar\, \epsilon_{jkl} J^{(l)} \quad , \qquad [K^{(j)}, K^{(k)}] = -i\hbar\, \epsilon_{jkl} J^{(l)}/c^2$$

and $$[J^{(j)}, K^{(k)}] = i\hbar\, \epsilon_{jkl} K^{(l)} \qquad\qquad j, k, l = 1, 2, 3$$

Equations (2.5.11), (2.5.12), and (2.5.13) can be rewritten in terms of the covariant quantities P_μ and $M_{\mu\nu}$ as

$$[P_\mu, P_\nu] = 0 \tag{2.5.14}$$

$$[M_{\mu\nu}, P_\sigma] = i\hbar\,(g_{\nu\sigma} P_\mu - g_{\mu\sigma} P_\nu) \tag{2.5.15}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\hbar\,(g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\rho\nu} - g_{\nu\sigma} M_{\rho\mu}) \tag{2.5.16}$$

2.6 The Space Inversion Operator $\mathcal{P}$

We now extend the above considerations to include also space inversion. That is, we consider the transformation

$$(\underline{x},t) \longrightarrow (\underline{x}',t') = (-\underline{x},t) \tag{2.6.1}$$

This is not a Poincaré transformation (2.1.1) and it is not characterized by a continuous parameter as are the Poincaré transformations. The transformation (2.6.1) corresponds to viewing the preparation or measuring process in a mirror.

We assume that equations (2.1.8) and (2.1.9) hold for all states and observables prepared and measured by apparatuses related by equation (2.6.1). Thus, by Wigner's Theorem,[2] one has

$$|\psi'\rangle = \mathcal{P}|\psi\rangle \tag{2.6.2}$$

$$A' = \mathcal{P} A \mathcal{P}^{\dagger} \tag{2.6.3}$$

where $\mathcal{P}$ is either a linear unitary operator or an antilinear antiunitary operator which corresponds to the transformation (2.6.1). Moreover, since the product of two inversions leaves the apparatus unchanged one can write

$$\mathcal{P}^2 = 1 \tag{2.6.4}$$

so

$$\mathcal{P}^{\dagger} = \mathcal{P} \tag{2.6.5}$$

We now investigate various combinations of Poincaré transformations and space inversions.

Obviously, a space inversion followed by a time translation and then by a second space inversion is equivalent to a time translation alone.

$$(\underline{x},t) \longrightarrow (-\underline{x},t) \longrightarrow (-\underline{x},t-\epsilon) \qquad (\underline{x},t-\epsilon)$$

Therefore,

$$\mathcal{P}\, \mathcal{U}(t)\, \mathcal{P}^{\dagger} = \mathcal{U}(t) \tag{2.6.6}$$

A space inversion followed by a space displacement and then by another space inversion yields the transformation

$$(\underline{x},t) \longrightarrow (-\underline{x},t) \longrightarrow (-\underline{x} + \underline{a},t) \longrightarrow (\underline{x}-\underline{a},t)$$

which corresponds to the inverse space displacement so that

$$\mathcal{P}\, D(\underline{a})\, \mathcal{P}^{\dagger} = D(-\underline{a}) \tag{2.6.7}$$

For the case of an inversion followed by a rotation, then another inversion, we have

$$(\underline{x},t) \longrightarrow (-\underline{x},t) \longrightarrow (-\underline{x}_{R},t) \longrightarrow (\underline{x}_{R},t)$$

where $\underline{x}_{R}$ is the point on the rotated apparatus. Therefore,

$$\mathcal{P}\, \mathcal{R}(\underline{\theta})\, \mathcal{P}^{\dagger} = \mathcal{R}(\underline{\theta}) \tag{2.6.8}$$

For an inversion followed by a Lorentz boost through a velocity v along the 1-axis, then another inversion,

$$(\underline{x},t) \longrightarrow (-\underline{x},t) \longrightarrow (\gamma\,(-x^{(1)}+ vt), -x^{(2)}, -x^{(3)}, (t-v/c^{2}\, x^{(1)})\,\gamma\,)$$

$$\longrightarrow (\gamma\,(x^{(1)}-vt), x^{(2)}, x^{(3)}, (t-v/c^{2}\, x^{(1)})\,\gamma\,)$$

This corresponds to a Lorentz boost through -v along the 1-axis so

$$\mathcal{P} L^{(1)}(u) \mathcal{P}^{\dagger} = L^{(1)}(-u) \tag{2.6.9}$$

In summary, we have

$$\mathcal{P} \mathcal{U}(t) \mathcal{P}^{\dagger} = \mathcal{U}(t) \tag{2.6.6}$$

$$\mathcal{P} D(\underline{a}) \mathcal{P}^{\dagger} = D(-\underline{a}) \tag{2.6.7}$$

$$\mathcal{P} \mathcal{R}(\underline{\theta}) \mathcal{P}^{\dagger} = \mathcal{R}(\underline{\theta}) \tag{2.6.8}$$

$$\mathcal{P} L^{(j)}(u) \mathcal{P}^{\dagger} = L^{(j)}(-u) \tag{2.6.9}$$

Since $\mathcal{U}(t) = e^{-iHt/\hbar}$, we have from equation (2.6.6) that

$\mathcal{P} H \mathcal{P}^{\dagger} = H$ if $\mathcal{P}$ is linear

or

$\mathcal{P} H \mathcal{P}^{\dagger} = -H$ if $\mathcal{P}$ is antilinear

However, H and $\mathcal{P} H \mathcal{P}^{\dagger}$ have the same spectrum so, if H and $-H$ have different spectra (for example, if H is bounded below but not above), then $\mathcal{P}$ must be linear. It follows that the action of $\mathcal{P}$ on the generators of the Poincaré group is

$$\mathcal{P} H \mathcal{P}^{\dagger} = H \tag{2.6.10}$$

$$\mathcal{P} \underline{P} \mathcal{P}^{\dagger} = -\underline{P} \tag{2.6.11}$$

$$\mathcal{P} \underline{J} \mathcal{P}^{\dagger} = \underline{J} \tag{2.6.12}$$

$$\mathcal{P}\underline{K}\mathcal{P}^{+} = -\underline{K} \qquad (2.6.13)$$

These are consistent with the commutation relations for the generators. For example,

$$\mathcal{P}[P^{(j)}, K^{(k)}]\mathcal{P}^{+} = [\mathcal{P}P^{(j)}\mathcal{P}^{+}, \mathcal{P}K^{(k)}\mathcal{P}^{+}] = [P^{(j)}, K^{(k)}] = \mathcal{P}(i\hbar\delta_{jk}H/c^2)\mathcal{P}^{+} = i\hbar\delta_{jk}H/c^2$$

as required.

2.7 The Time Reversal Operator $\mathcal{T}$

We now extend the above considerations to include time reversal as well. That is, we consider the transformation

$$(\underline{x},t) \rightarrow (\underline{x}',t') = (\underline{x},-t) \tag{2.7.1}$$

This is not a Poincaré transformation (2.1.1) and it is not characterized by a continuous parameter as are the Poincaré transformations. The transformation (2.7.1) corresponds to viewing in reverse a motion picture of the preparation-measurement process.

We assume that equations (2.1.8) and (2.1.9) hold for all states and observables prepared and measured by apparatuses related by equation (2.7.1). Then, one has

$$|\psi'\rangle = \mathcal{T}|\psi\rangle \tag{2.7.2}$$

$$A' = \mathcal{T} A \mathcal{T}^{\dagger} \tag{2.7.3}$$

where $\mathcal{T}$ is either a linear unitary operator or an antilinear antiunitary operator which corresponds to the transformation (2.7.1). Moreover, since the product of two time reversals leaves the apparatus unchanged one can write

$$\mathcal{T}^2 = 1 \tag{2.7.4}$$

so

$$\mathcal{T}^{\dagger} = \mathcal{T} \tag{2.7.5}$$

We now investigate various combinations of Poincaré transformations and time reversals.

A time reversal followed by a time translation and then by a second time reversal yields the transformation

$$(\underline{x},t) \longrightarrow (\underline{x},-t) \longrightarrow (\underline{x},-t-\epsilon) \longrightarrow (\underline{x},t+\epsilon)$$

This is equivalent to time translation through $-\epsilon$. In terms of the unitary operators,

$$\mathcal{T}\,\mathcal{U}(t)\mathcal{T}^{\dagger} = \mathcal{U}(-t) \tag{2.7.6}$$

Obviously, if the time translation is replaced by a space displacement in the sequence of transformations above the net transformation is just the space displacement. That is,

$$\mathcal{T}\,D(\underline{a})\mathcal{T}^{\dagger} = D(\underline{a}) \tag{2.7.7}$$

The same is true for a time reversal followed by a rotation, then by a time reversal since the rotation only transforms the space coordinates and the time reversals act on only the time coordinate. Therefore,

$$\mathcal{T}\,\mathcal{R}(\underline{\theta})\mathcal{T}^{\dagger} = \mathcal{R}(\underline{\theta}) \tag{2.7.8}$$

For a time reversal followed by a Lorentz boost through v along the 1-axis, followed by another time reversal,

$$(\underline{x},t) \longrightarrow (\underline{x},-t) \longrightarrow (\gamma\,(x^{(1)}-vt),x^{(2)},x^{(3)},\gamma\,(-t+v/c^2\,x^{(1)}))$$

$$\longrightarrow (\gamma\,(x^{(1)}-vt),x^{(2)},x^{(3)},\gamma\,(t-v/c^2\,x^{(1)}))$$

This corresponds to a Lorentz boost through $-v$ along the 1-axis so

$$\mathcal{T}\,L^{(1)}(v)\mathcal{T}^{\dagger} = L^{(1)}(-v) \tag{2.7.9}$$

In summary, we have

$$\mathcal{T}\, U(t)\, \mathcal{T}^{\dagger} = U(-t) \tag{2.7.6}$$

$$\mathcal{T}\, D(\underline{a})\, \mathcal{T}^{\dagger} = D(\underline{a}) \tag{2.7.7}$$

$$\mathcal{T}\, \mathcal{R}(\underline{\theta})\, \mathcal{T}^{\dagger} = \mathcal{R}(\underline{\theta}) \tag{2.7.8}$$

$$\mathcal{T}\, L^{(j)}(u)\, \mathcal{T}^{\dagger} = L^{(j)}(-u) \tag{2.7.9}$$

The action of $\mathcal{T}$ on the generators of the Poincaré group will be investigated. From equation (2.7.6) above, we see that

$$\mathcal{T} H \mathcal{T}^{\dagger} = -H \qquad \text{if } \mathcal{T} \text{ is linear}$$

$$\mathcal{T} H \mathcal{T}^{\dagger} = H \qquad \text{if } \mathcal{T} \text{ is antilinear}$$

The spectra of $\mathcal{T} H \mathcal{T}^{\dagger}$ and H are identical. If H and $-H$ have different spectra, then $\mathcal{T}$ must be antilinear. The action of $\mathcal{T}$ on the generators of the Poincaré group is such that

$$\mathcal{T} H \mathcal{T}^{\dagger} = H \tag{2.7.10}$$

$$\mathcal{T}\, \underline{P}\, \mathcal{T}^{\dagger} = -\underline{P} \tag{2.7.11}$$

$$\mathcal{T}\, \underline{J}\, \mathcal{T}^{\dagger} = -\underline{J} \tag{2.7.12}$$

$$\mathcal{T}\, \underline{K}\, \mathcal{T}^{\dagger} = \underline{K} \tag{2.7.13}$$

These are consistent with the commutation relations for the generators. For example,

$$\mathcal{T}[P^{(j)}, K^{(k)}]\mathcal{T}^{\dagger} = -[P^{(j)}, K^{(k)}] = \mathcal{T} i \mathcal{T}^{\dagger} \hbar \delta_{jk} \mathcal{T}(H/c^2)\mathcal{T}^{\dagger} = -i\hbar \delta_{jk} H/c^2$$

so that

$$[P^{(j)}, K^{(k)}] = i\hbar \delta_{jk} H/c^2$$

as required.

2.8 The Centre of Mass Position $\underline{\mathcal{X}}$, Internal Angular Momentum $\underline{\$}$, and Centre of Mass Velocity $\underline{V}$ Operators

We now introduce new operators $\underline{\mathcal{X}}$ and $\underline{\$}$ which we call the position of the centre of mass of the system and the total internal angular momentum of the system, respectively. (It is shown, in the next section that $\underline{\mathcal{X}}$ and $\underline{\$}$ transform like three-vectors under rotations.) In terms of these operators, we express the total angular momentum of the system as

$$\underline{J} = \underline{\mathcal{X}} \times \underline{P} + \underline{\$} \qquad (2.8.1)$$

where $\underline{P}$ is the total momentum of the system. To retain the commutation relations (2.2.15) of the generators conditions must be imposed on $\underline{\mathcal{X}}$ and $\underline{\$}$. In particular, the commutation relations (2.2.15c), (2.2.15d), and (2.2.15e) follow when

$$[\mathcal{X}^{(j)}, \mathcal{X}^{(k)}] = 0 \qquad [\mathcal{X}^{(j)}, P^{(k)}] = i\hbar\,\delta_{jk} \qquad [\mathcal{X}^{(j)}, H] = i\hbar c^2 P^{(j)} H^{-1}$$

$$[\$^{(j)}, \$^{(k)}] = i\hbar\,\epsilon_{jk\ell}\,\$^{(\ell)} \qquad [\$^{(j)}, P^{(k)}] = [\$^{(j)}, H] = [\$^{(j)}, \mathcal{X}^{(k)}] = 0 \qquad j,k,\ell = 1,2,3 \qquad (2.8.2)\ \text{a-g}$$

Moreover, as we show in Appendix B, the remaining commutation relations involving $\underline{J}$ and $\underline{K}$ follow without further conditions on $\underline{\mathcal{X}}$ and $\underline{\$}$ when

$$\underline{K} = -\frac{1}{2c^2}(\underline{\mathcal{X}} H + H \underline{\mathcal{X}}) + (H + \not{\$} M c^2)^{-1}(\underline{\$} \times \underline{P}) \qquad (2.8.3)$$

wherein $$M c^2 = (H^2 - c^2 P^2)^{1/2} \qquad (2.8.4)$$

and $$\not{\$} = E H^{-1}$$

where $$E = +\sqrt{H^2}$$

Also shown in Appendix B are the inverse relations

$$\not{\beta} M \underline{\mathcal{X}} = -\frac{\not{\beta} M c^2}{2}\left(H^{-1}\underline{K} + \underline{K}H^{-1}\right) - \frac{\underline{P}\times\underline{J}}{H+\not{\beta}Mc^2} + \frac{c^2}{H(H+\not{\beta}Mc^2)}\,\underline{P}\times(\underline{P}\times\underline{K}) \qquad (2.8.5)$$

$$\not{\beta} M \underline{\$} = \underline{K}\times\underline{P} + \frac{H}{c^2}\underline{J} - \frac{(\underline{J}\cdot\underline{P})\underline{P}}{(H+\not{\beta}Mc^2)} \qquad (2.8.6)$$

Equation (2.8.5) is the same as the total position operator given by Osborn for a system of two noninteracting relativistic particles and equation (2.8.6) agrees with Schwinger's expression for the total angular momentum.[5]

Thus if one has determined the ten generators $H, \underline{P}, \underline{J}$, and $\underline{K}$ in terms of the fundamental dynamical variables of the system then $\underline{\mathcal{X}}$ and $\underline{\$}$ are given by equations (2.8.5) and (2.8.6). Alternatively, if one has determined ten operators $H, \underline{P}, \underline{\mathcal{X}}, \underline{\$}$ which satisfy

$$[P^{(j)}, P^{(k)}] = 0 \qquad\qquad [P^{(j)}, H] = 0 \qquad (2.8.7)$$

in addition to the commutation relations (2.8.2), then $\underline{J}$ and $\underline{K}$ are given by equations (2.8.1) and (2.8.3). That is, equations (2.8.2) and (2.8.7) are sufficient conditions for a physical system to be Lorentz invariant.

It is clear from equations (2.8.1) and (2.8.3) that the operator $\$^2 = \underline{\$}\cdot\underline{\$}$ commutes with $\underline{J}$ and $\underline{K}$, as well as with $\underline{P}$ and H. It is therefore a Lorentz invariant. In fact,

$$\mathcal{W}^2 = -c^2 M^2 \$^2 \qquad (2.8.8)$$

This relation is derived as follows. Substituting equations (2.8.1) and (2.8.3) for $\underline{J}$ and $\underline{K}$ into equations (2.4.2) and (2.4.3) for $\mathcal{H}$ and $\underline{\mathcal{W}}$,

$$\mathcal{H} = \underline{P}\cdot\underline{\$} \qquad (2.8.9)$$

$$\underline{W} = \tfrac{1}{c} H(\underline{X} \times \underline{P}) + \tfrac{1}{c} H \underline{\$} + \tfrac{1}{2c}\, \underline{P} \times (\underline{X} H + H \underline{X}) - \frac{c}{(H+\beta M c^2)}\, \underline{P} \times (\underline{\$} \times \underline{P}) \tag{2.8.10}$$

Using the identity

$$\underline{P} \times (\underline{\$} \times \underline{P}) = P^2 \underline{\$} - (\underline{P} \cdot \underline{\$}) \underline{P} \tag{2.8.11}$$

and the commutator

$$[\underline{X}, H] = i\hbar c^2 \underline{P} H^{-1} \tag{2.8.12}$$

we rewrite $\underline{W}$ as

$$\underline{W} = \tfrac{1}{c} H \underline{\$} - \frac{c}{(H+\beta M c^2)} (P^2 \underline{\$} - (\underline{P} \cdot \underline{\$}) \underline{P})$$

$$= c \beta M \underline{\$} + \frac{c}{(H+\beta M c^2)} (\underline{P} \cdot \underline{\$}) \underline{P}$$

$$= c \beta M \underline{\$} + c\,(H+\beta M c^2)^{-1} \mathcal{H} \underline{P} \tag{2.8.13}$$

Substituting equations (2.8.9) and (2.8.13) above into equation (2.4.13) for $\mathcal{W}^2$,

$$\mathcal{W}^2 = \mathcal{H}^2 - c^2 M^2 \$^2 - \frac{c^2 \mathcal{H}^2 P^2}{(H+\beta M c^2)^2} - \frac{2 c^2 \beta M \mathcal{H}}{(H+\beta M c^2)} (\underline{\$} \cdot \underline{P})$$

$$= -c^2 M^2 \$^2$$

It is convenient to define a Hermitian operator $\underline{V}$ by

$$[\underline{X}, H] = i\hbar \underline{V} \tag{2.8.14}$$

Then the corresponding Heisenberg operators

$$\underline{V}(t) = U^{\dagger}(t)\,\underline{V}\,U(t) \tag{2.8.15}$$

and

$$\underline{\mathcal{X}}(t) = U^{\dagger}(t)\,\underline{\mathcal{X}}\,U(t) \tag{2.8.16}$$

are related via equation (2.8.15) by

$$\underline{V}(t) = \frac{d\underline{\mathcal{X}}(t)}{dt} \tag{2.8.17}$$

Thus $\underline{V}$ is the velocity of the centre of mass of the system. Identifying equation (2.8.12) and equation (2.8.14), we have

$$\underline{V} = c^2\,\underline{P}\,H^{-1} \tag{2.8.18}$$

Finally, in addition to the commutation relations (2.8.2) we can write

$$[J^{(j)}, \mathcal{X}^{(k)}] = i\hbar\,\epsilon_{jk\ell}\,\mathcal{X}^{(\ell)} \tag{2.8.19}$$

$$[K^{(j)}, \mathcal{X}^{(k)}] = i\hbar\left\{-\frac{1}{2c^2}(\mathcal{X}^{(k)}V^{(j)} + V^{(j)}\mathcal{X}^{(k)}) + V^{(k)}(H+\beta Mc^2)^{-1} + (H+\beta Mc^2)^{-1}\epsilon_{jk\ell}\$^{(\ell)}\right\} \tag{2.8.20}$$

$$[J^{(j)}, \$^{(k)}] = i\hbar\,\epsilon_{jk\ell}\,\$^{(\ell)} \tag{2.8.21}$$

$$[J^{(j)}, V^{(k)}] = i\hbar\,\epsilon_{jk\ell}\,V^{(\ell)} \tag{2.8.22}$$

$$[K^{(j)}, V^{(k)}] = -i\hbar\left(\delta_{jk} + \frac{1}{c^2}V^{(j)}V^{(k)}\right) \tag{2.8.23}$$

$$[V^{(j)}, V^{(k)}] = [V^{(j)}, P^{(k)}] = [V^{(j)}, \$^{(k)}] = [V^{(j)}, M] = [V^{(j)}, H] = 0 \tag{2.8.24}$$

2.9 Transformation Properties of the Observables $\underline{X}$, $\underline{\$}$, and $\underline{V}$

An observable A' measured by a transformed apparatus is related to A measured by the original apparatus by

$$A' = \mathcal{U} A \mathcal{U}^{\dagger} \tag{2.9.1}$$

where $\mathcal{U}$ is the unitary or antiunitary operator which corresponds to the transformation. The observables $\underline{X}$, $\underline{\$}$, $\underline{V}$ transform as follows under the Poincaré transformations, space inversion, and time reversal.

$$U(t)\,\underline{X}\,U^{\dagger}(t) = \underline{X} - \underline{V}t \tag{2.9.2}$$

$$D(\underline{a})\,\underline{X}\,D^{\dagger}(\underline{a}) = \underline{X} - \underline{a} \tag{2.9.3}$$

$$R^{(1)}(\alpha)\,X^{(1)}\,R^{(1)\dagger}(\alpha) = X^{(1)} \tag{2.9.4a}$$

$$R^{(1)}(\alpha)\,X^{(2)}\,R^{(1)\dagger}(\alpha) = X^{(2)}\cos\alpha + X^{(3)}\sin\alpha \tag{2.9.4b}$$

$$R^{(1)}(\alpha)\,X^{(3)}\,R^{(1)\dagger}(\alpha) = X^{(3)}\cos\alpha - X^{(2)}\sin\alpha \tag{2.9.4c}$$

$$L^{(1)}(u)\,X^{(1)}\,L^{(1)\dagger}(u) = \frac{X^{(1)}}{\gamma} + \frac{v}{2c^2\gamma}\left\{X^{(1)}\frac{V^{(1)}}{1-V^{(1)}v/c^2} + \frac{V^{(1)}}{1-V^{(1)}v/c^2}X^{(1)}\right\} \tag{2.9.5a}$$

$$L^{(1)}(u)\,X^{(K)}\,L^{(1)\dagger}(u) = X^{(K)} + \frac{v}{2c^2}\left\{X^{(1)}\frac{V^{(K)}}{1-V^{(1)}v/c^2} + \frac{V^{(K)}}{1-V^{(1)}v/c^2}X^{(1)}\right\} \quad K=2,3 \tag{2.9.5b}$$

$$\mathcal{P}\,\underline{X}\,\mathcal{P}^{\dagger} = -\underline{X} \tag{2.9.6}$$

$$\mathcal{T}\,\underline{X}\,\mathcal{T}^{\dagger} = \underline{X} \tag{2.9.7}$$

$$U(t)\,\underline{\$}\,U^{\dagger}(t) = \underline{\$} \tag{2.9.8}$$

$$D(\underline{a})\underline{\$}\,D^{\dagger}(\underline{a}) = \underline{\$} \tag{2.9.9}$$

$$\mathcal{R}^{(1)}(\alpha)\,\$^{(1)}\mathcal{R}^{(1)\dagger}(\alpha) = \$^{(1)} \tag{2.9.10a}$$

$$\mathcal{R}^{(1)}(\alpha)\,\$^{(2)}\mathcal{R}^{(1)\dagger}(\alpha) = \$^{(2)}\cos\alpha + \$^{(3)}\sin\alpha \tag{2.9.10b}$$

$$\mathcal{R}^{(1)}(\alpha)\,\$^{(3)}\mathcal{R}^{(1)\dagger}(\alpha) = \$^{(3)}\cos\alpha - \$^{(2)}\sin\alpha \tag{2.9.10c}$$

$$L^{(1)}(u)\,\underline{\$}\,L^{(1)\dagger}(u) = \underline{\$}\cos\eta + (1-\cos\eta)(\underline{\tilde{N}}\cdot\underline{\$})\,\underline{\tilde{N}} - \sin\eta\,(\underline{\tilde{N}}\times\underline{\$}) \tag{2.9.11}$$

where $\underline{\tilde{N}} = -\dfrac{\underline{P}\times\underline{e}_1}{|\underline{P}\times\underline{e}_1|}$ (2.9.12)

and $\frac{1}{2}\eta = \tan^{-1}\left(\dfrac{e^{u}(H - c\underline{e}_1\cdot\underline{P}) + \beta Mc^2}{c|\underline{P}\times\underline{e}_1|}\right) - \tan^{-1}\left(\dfrac{H - c\underline{e}_1\cdot\underline{P} + \beta Mc^2}{c|\underline{P}\times\underline{e}_1|}\right)$ (2.9.13)

and $\underline{e}_1$ is a unit vector along the 1-axis.

$$\mathcal{P}\,\underline{\$}\,\mathcal{P}^{\dagger} = \underline{\$} \tag{2.9.14}$$

$$\mathcal{T}\,\underline{\$}\,\mathcal{T}^{\dagger} = -\underline{\$} \tag{2.9.15}$$

$$\mathcal{U}(t)\,\underline{V}\,\mathcal{U}^{\dagger}(t) = \underline{V} \tag{2.9.16}$$

$$\mathcal{R}^{(1)}(\alpha)\,V^{(1)}\,\mathcal{R}^{(1)\dagger}(\alpha) = V^{(1)} \tag{2.9.17a}$$

$$\mathcal{R}^{(1)}(\alpha)\,V^{(2)}\,\mathcal{R}^{(1)\dagger}(\alpha) = V^{(2)}\cos\alpha + V^{(3)}\sin\alpha \tag{2.9.17b}$$

$$\mathcal{R}^{(1)}(\alpha)\,V^{(3)}\,\mathcal{R}^{(1)\dagger}(\alpha) = V^{(3)}\cos\alpha - V^{(2)}\sin\alpha \tag{2.9.17c}$$

$$L^{(1)}(u)\,V^{(1)}L^{(1)\dagger}(u) = \frac{V^{(1)} - v}{1 - V^{(1)}v/c^2} \tag{2.9.18a}$$

$$L^{(1)}(u)V^{(K)}L^{(1)\dagger}(u) = \frac{V^{(K)}}{\gamma(1-V^{(1)}v/c^2)} \qquad K=2,3 \tag{2.9.18b}$$

$$\mathcal{P}\underline{V}\mathcal{P}^\dagger = -\underline{V} \tag{2.9.19}$$

$$\mathcal{T}\underline{V}\mathcal{T}^\dagger = -\underline{V} \tag{2.9.20}$$

Also, since $\mathcal{B}$ commutes with all the generators,

$$\mathcal{U}(t)\mathcal{B}\,\mathcal{U}^\dagger(t) = \mathcal{B} \tag{2.9.21}$$

$$D(\underline{a})\mathcal{B}\,D^\dagger(\underline{a}) = \mathcal{B} \tag{2.9.22}$$

$$\mathcal{R}^{(j)}(\alpha)\mathcal{B}\,\mathcal{R}^{(j)\dagger}(\alpha) = \mathcal{B} \qquad j=1,2,3 \tag{2.9.23}$$

$$L^{(j)}(u)\mathcal{B}\,L^{(j)\dagger}(u) = \mathcal{B} \qquad j=1,2,3 \tag{2.9.24}$$

Thus, $\mathcal{B}$ is a Lorentz invariant.

Most of the above equations are derived as in Section 2.3. We derive the exceptions below. Equations (2,9.5) are found as follows.[6] Let us consider the commutator

$$[\mathcal{X}^{(1)}, L^{(1)\dagger}(u)P^{(1)}L^{(1)}(u)] = [\mathcal{X}^{(1)}, P^{(1)}\cosh u + \tfrac{1}{c}H\sinh u]$$

$$= i\hbar\left(\cosh u + \tfrac{1}{c}V^{(1)}\sinh u\right)$$

where we have used equations (2.3.12a) and (2.8.2). Taking $L^{(1)}(u)\ldots L^{(1)\dagger}(u)$ on both sides of the equation above yields

$$[L^{(1)}(u)\mathcal{X}^{(1)}L^{(1)\dagger}(u), P^{(1)}] = i\hbar\left(\cosh u + \tfrac{1}{c}L^{(1)}(u)V^{(1)}L^{(1)\dagger}(u)\sinh u\right)$$

$$= i\hbar\gamma\left(1 + \frac{(V^{(1)}-v)v/c^2}{1-V^{(1)}v/c^2}\right)$$

or $$[L^{(1)}(u)\,X^{(1)}L^{(1)\dagger}(u),\,P^{(1)}] = \frac{i\hbar}{\gamma(1-V^{(1)}v/c^2)} \tag{2.9.25}$$

Similarly,

$$[L^{(1)}(u)\,X^{(1)}L^{(1)\dagger}(u),\,H] = \frac{i\hbar V^{(1)}}{\gamma(1-V^{(1)}v/c^2)} \tag{2.9.26}$$

Equations (2.9.25) and (2.9.26) are satisfied when $L^{(1)}(u)X^{(1)}L^{(1)\dagger}(u)$ is given by equation (2.9.5a). Similarly,

$$[L^{(1)}(u)\,X^{(K)}L^{(1)\dagger}(u),\,P^{(1)}] = \frac{i\hbar V^{(K)}v}{c^2(1-V^{(1)}v/c^2)} \qquad K=2,3 \tag{2.9.27}$$

and $$[L^{(1)}(u)X^{(K)}L^{(1)\dagger}(u),\,H] = \frac{i\hbar V^{(K)}}{1-V^{(1)}v/c^2} \qquad K=2,3 \tag{2.9.28}$$

Equations (2.9.27) and (2.9.28) are satisfied when $L^{(1)}(u)\,X^{(K)}L^{(1)\dagger}(u)$ is given by equation (2.9.5b).

Equations (2.9.6) and (2.9.7) follow from

$$[X^{(j)},\,P^{(K)}] = i\hbar\,\delta_{jK}$$

That is,

$$\mathcal{P}[X^{(j)},\,P^{(K)}]\mathcal{P}^\dagger = -[\mathcal{P}X^{(j)}\mathcal{P}^\dagger,\,P^{(K)}] = \mathcal{P}\,i\,\mathcal{P}^\dagger\hbar\,\delta_{jK} = i\hbar\,\delta_{jK}$$

so $\mathcal{P}\,\underline{X}\,\mathcal{P}^\dagger = -\underline{X}$

Similarly,

$$\mathcal{T}[X^{(j)},\,P^{(K)}]\mathcal{T}^\dagger = -[\mathcal{T}X^{(j)}\mathcal{T}^\dagger,\,P^{(K)}] = \mathcal{T}\,i\,\mathcal{T}^\dagger\hbar\,\delta_{jK} = -i\hbar\,\delta_{jK}$$

so $\mathcal{T}\,\underline{X}\,\mathcal{T}^\dagger = \underline{X}$

Equations (2.9.14) and (2.9.15) follow similarly from

$$[\$^{(j)}, \$^{(k)}] = i\hbar \epsilon_{jk\ell} \$^{(\ell)}$$

and equations (2.9.19) and (2.9.20) follow from

$$[\underline{X}, H] = i\hbar \underline{V}$$

We now derive equation (2.9.11).[7] For a Lorentz boost of velocity v along the 1-axis, we define

$$\underline{\$}(u,1) = L^{(1)}(u)\, \underline{\$}\, L^{(1)\dagger}(u)$$

Then,

$$i\hbar \frac{d}{du} \underline{\$}(u,1) = L^{(1)}(u) [cK^{(1)}, \underline{\$}] L^{(1)\dagger}(u)$$

The commutator can be written as

$$[cK^{(1)}, \underline{\$}] = i\hbar c \left(H + \beta Mc^2\right)^{-1} \left((\underline{e}_1 \times \underline{P}) \times \underline{\$} \right)$$

where $\underline{e}_1$ is a unit vector along the 1-axis. Thus,

$$\frac{d}{du} \underline{\$}(u,1) = c \left(H(u,1) + \beta Mc^2\right)^{-1} \left((\underline{e}_1 \times \underline{P}(u,1)) \times \underline{\$}(u,1) \right)$$

where $H(u,1)$ is given by equation (2.3.11). Also,

$$\underline{e}_1 \times \underline{P}(u,1) = \underline{e}_1 \times \underline{P}$$

Now, we let

$$\underline{\tilde{N}} = \frac{\underline{e}_1 \times \underline{P}}{|\underline{e}_1 \times \underline{P}|}$$

and $\frac{d\eta}{du} = \frac{c\,|\underline{e}_1 \times \underline{P}|}{H(u,1)+\beta Mc^2}$ $\qquad \eta(0) = 0 \qquad$ (2.9.29)

Then,

$$\frac{d}{du}\,\underline{\$}(u,1) = \frac{d\eta}{du}\,\tilde{\underline{N}} \times \underline{\$}\,(u,1)$$

The solution of the above equation is given by equation (2.9.11). Equation (2.9.29) integrates to give equation (2.9.13).

We now derive equations (2.9.18). They follow from equation (2.8.18). That is,

$$L^{(1)}(u)\,\underline{V}\,L^{(1)+}(u) = c^2 L^{(1)}(u)\,\underline{P}\,L^{(1)+}(u)\,L^{(1)}(u)\,H^{-1}\,L^{(1)+}(u)$$

Since

$$H\,H^{-1} = H^{-1}\,H = 1$$

it follows that

$$1 = L^{(1)}(u)\,H\,L^{(1)+}(u)\,L^{(1)}(u)\,H^{-1}\,L^{(1)+}(u)$$

so $\quad L^{(1)}(u)\,H^{-1}L^{(1)+}(u) = \left(L^{(1)}(u)\,H\,L^{(1)+}(u)\right)^{-1}$

Therefore,

$$L^{(1)}(u)\,\underline{V}\,L^{(1)+}(u) = c^2 L^{(1)}(u)\,\underline{P}\,L^{(1)+}(u)\,\left(H\cosh u - cP^{(1)}\sinh u\right)^{-1}$$

Thus,

$$L^{(1)}(u)\,V^{(1)}L^{(1)+}(u) = c^2\,\frac{\left(P^{(1)}\cosh u - \frac{1}{c}H\sinh u\right)}{\left(H\cosh u - cP^{(1)}\sinh u\right)}$$

$$= \frac{c^2 P^{(1)} H^{-1} - c \tanh u}{1 - c P^{(1)} H^{-1} \tanh u}$$

$$= \frac{V^{(1)} - v}{1 - V^{(1)} v / c^2}$$

Equation (2.9.18b) follows similarly.

2.10 The Operators T and $\underline{X}(T)$

We have seen that the centre of mass velocity $\underline{V}$ transforms in the familiar manner under Lorentz boosts. That is, equations (2.9.18) represent the Einstein addition of velocities. However, as is seen from equations (2.9.5), the centre of mass position operator does not transform in a familiar or even simple way under Lorentz boosts. We have not yet given an operator equation analogous to the familiar space-time transformation (2.1.6). We now rectify this matter. In so doing we construct operators T and $\underline{X}(T)$ whose physical interpretation requires some comment.[6] This is taken up at the end of this section.

Let the Hermitian operator T be defined by

$$L^{(1)}(u)\,T\,L^{(1)\dagger}(u) = t\cosh u - \tfrac{1}{c}\,X^{(1)}(t)\sinh u \tag{2.10.1}$$

where $X^{(1)}(t)$ is the component of the Heisenberg position operator along the 1-axis. That is,

$$X^{(1)}(t) = U^{\dagger}(t)\,X^{(1)}\,U(t) = X^{(1)} + V^{(1)}t \tag{2.10.2}$$

Further, we define the Hermitian operator $\underline{X}(T)$ by

$$\underline{X}(T) = \underline{X} + \tfrac{1}{2}\left(\underline{V}\,T + T\,\underline{V}\right) \tag{2.10.3}$$

Upon taking $L^{(1)}(u)\cdots L^{(1)\dagger}(u)$ on both sides of equation (2.10.3) and using equations (2.9.5), (2.9.18), and (2.10.1), we find

$$L^{(1)}(u)\,X^{(1)}(T)\,L^{(1)\dagger}(u) = X^{(1)}(t)\cosh u - ct\,\sinh u \tag{2.10.4}$$

$$L^{(1)}(u)\,X^{(K)}(T)\,L^{(1)\dagger}(u) = X^{(K)}(t) \qquad K = 2,3 \tag{2.10.5}$$

Along with equation (2.10.1), these equations are the desired operator analogs to the space-time transformation (2.1.6). One also has the rearranged equations which follow.

$$\mathcal{X}^{(1)}(t) = L^{(1)}(u)\,\mathcal{X}^{(1)}(T)\,L^{(1)\dagger}(u)\cosh u + c\,L^{(1)}(u)\,T\,L^{(1)\dagger}(u)\sinh u \qquad (2.10.6)$$

$$\mathcal{X}^{(k)}(t) = L^{(1)}(u)\,\mathcal{X}^{(k)}(T)\,L^{(1)\dagger}(u) \qquad k = 2,3 \qquad (2.10.7)$$

$$t = L^{(1)}(u)\,T\,L^{(1)\dagger}(u)\cosh u + \frac{1}{c}\,L^{(1)}(u)\,\mathcal{X}^{(1)}(T)\,L^{(1)\dagger}(u)\sinh u \qquad (2.10.8)$$

We now derive equation (2.10.4). From equation (2.10.3), we have

$$\mathcal{X}^{(1)}(T) = \mathcal{X}^{(1)} + \frac{1}{2}\left(V^{(1)}T + T V^{(1)}\right) \qquad (2.10.9)$$

Taking $L^{(1)}(u) \ldots L^{(1)\dagger}(u)$ on both sides of this equation yields

$$L^{(1)}(u)\mathcal{X}^{(1)}(T)L^{(1)\dagger}(u) = L^{(1)}(u)\mathcal{X}^{(1)}L^{(1)\dagger}(u) + \frac{1}{2}\left(L^{(1)}(u)V^{(1)}T L^{(1)\dagger}(u) + L^{(1)}(u)T V^{(1)}L^{(1)\dagger}(u)\right)$$

which, on using equations (2.9.5), (2.9.18), and (2.10.1), may be rewritten as

$$= (\cosh u)^{-1}\left[\mathcal{X}^{(1)} + \frac{v}{2c^2}\left(\mathcal{X}^{(1)}\frac{V^{(1)}}{1 - V^{(1)}v/c^2} + \frac{V^{(1)}}{1 - V^{(1)}v/c^2}\mathcal{X}^{(1)}\right)\right] + \frac{1}{2}\left[\frac{V^{(1)} - v}{1 - V^{(1)}v/c^2}\left(t\cosh u - \frac{1}{c}(\mathcal{X}^{(1)} + V^{(1)}t)\sinh u\right)\right.$$

$$\left. + \left(t\cosh u - \frac{1}{c}(\mathcal{X}^{(1)} + V^{(1)}t)\sinh u\right)\frac{V^{(1)} - v}{1 - V^{(1)}v/c^2}\right]$$

After some algebra, this reduces to equation (2.10.4) as required. Equation (2.10.5) follows similarly.

It must be stressed that equations (2.10.1), (2.10.4), and (2.10.5) are only analogous to the transformation (2.1.6). They do incorporate the correct quantum mechanical asymmetry between position and time; that is, on the right hand side of each equation, position appears as a Hermitian operator while time is a parameter of the theory. The equations look appropriate in form but contain the objects T and $\underline{X}(T)$. These operators do not have an immediate physical interpretation but we will now obtain from the above results equations which depend only on objects which are physically meaningful. Assume that $|\psi\rangle$ is practically an eigenvector of $L^{(1)}(u)\underline{V}L^{(1)\dagger}(u)$. (although this operator has no real eigenvectors) Taking $L^{(1)}(u)\ldots L^{(1)\dagger}(u)$ on both sides of equation (2.10.3),

$$L^{(1)}(u)\underline{X}(T)L^{(1)\dagger}(u) = L^{(1)}(u)\underline{X}L^{(1)\dagger}(u) + \tfrac{1}{2}L^{(1)}(u)(\underline{V}T+T\underline{V})L^{(1)\dagger}(u)$$

Also,

$$\begin{aligned}\langle\psi|L^{(1)}(u)\underline{V}L^{(1)\dagger}(u)L^{(1)}(u)TL^{(1)\dagger}(u)|\psi\rangle &\simeq \langle\psi|L^{(1)}(u)TL^{(1)\dagger}(u)L^{(1)}(u)\underline{V}L^{(1)\dagger}(u)|\psi\rangle \\ &\simeq \langle\psi|L^{(1)}(u)\underline{V}L^{(1)\dagger}(u)|\psi\rangle\langle\psi|L^{(1)}(u)TL^{(1)\dagger}(u)|\psi\rangle\end{aligned}$$

Therefore,

$$\langle\psi|L^{(1)}(u)\underline{X}(T)L^{(1)\dagger}(u)|\psi\rangle \simeq \langle\psi|L^{(1)}(u)\underline{X}L^{(1)\dagger}(u) + L^{(1)}(u)\underline{V}L^{(1)\dagger}(u)\,t'|\psi\rangle \quad (2.10.11)$$

where $t' = \langle\psi|L^{(1)}(u)TL^{(1)\dagger}(u)|\psi\rangle$

Since the Heisenberg position operator is

$$\underline{X}(t) = \underline{X} + \underline{V}t$$

it follows that

$$L^{(1)}(u)\underline{X}L^{(1)\dagger}(u) = L^{(1)}(u)\underline{X}(t')L^{(1)\dagger}(u) - L^{(1)}(u)\underline{V}L^{(1)\dagger}(u)\,t'$$

Substituting the above equation into equation (2.10.11), we find

$$\langle\psi|L^{(1)}(u)\underline{X}(T)L^{(1)\dagger}(u)|\psi\rangle \simeq \langle\psi|L^{(1)}(u)\underline{X}(t')L^{(1)\dagger}(u)\rangle = \overline{L^{(1)}(u)\underline{X}(t')L^{(1)\dagger}(u)}$$

Thus, when averaged over the "eigenvectors" $|\psi\rangle$ of $L^{(1)}(u)\underline{V}L^{(1)\dagger}(u)$, equations (2.10.1), (2.10.4), (2.10.5), (2.10.6), (2.10.7), and (2.10.8) become, respectively,

$$t' \simeq t\cosh u - \frac{1}{c}\overline{X^{(1)}(t)}\sinh u \tag{2.10.12}$$

$$\overline{L^{(1)}(u)X^{(1)}(t')L^{(1)\dagger}(u)} \simeq \overline{X^{(1)}(t)}\cosh u - ct\sinh u \tag{2.10.13}$$

$$\overline{L^{(1)}(u)X^{(K)}(t')L^{(1)\dagger}(u)} \simeq \overline{X^{(K)}(t)} \qquad K=2,3 \tag{2.10.14}$$

$$\overline{X^{(1)}(t)} \simeq \overline{L^{(1)}(u)X^{(1)}(t')L^{(1)\dagger}(u)}\cosh u + ct'\sinh u \tag{2.10.15}$$

$$\overline{X^{(K)}(t)} \simeq \overline{L^{(1)}(u)X^{(K)}(t')L^{(1)\dagger}(u)} \qquad K=2,3 \tag{2.10.16}$$

$$t \simeq t'\cosh u + \frac{1}{c}\overline{L^{(1)}(u)X^{(1)}(t')L^{(1)\dagger}(u)}\sinh u \tag{2.10.17}$$

The equations above do not contain the operators T and $\underline{X}(T)$. Rather, they depend on the parameter t and the average value of the Heisenberg position operator.

Chapter 3 Descriptions of Some Lorentz Invariant Systems

In this chapter, a set of ten generators satisfying the commutation relations (2.2.15) is constructed for each of several systems. First, in Section 3.1, we construct from the fundamental dynamical variables the generators for a system consisting of a single free spinless particle. We then consider a system consisting of a single free particle with spin in Section 3.2. In each of these sections we find that the centre of mass position $\underset{\sim}{\mathcal{X}}$ is equal to the Cartesian coordinates of the particle and that the total internal angular momentum $\underset{\sim}{\mathcal{S}}$ is equal to the spin $\underset{\sim}{S}$ of the particle. Also, the unitary operators corresponding to the Poincaré transformations and to space inversion along with the antiunitary operator corresponding to time reversal are expressed in integral form in these sections.

The generators of the Poincaré group are constructed in Section 3.3 for a system consisting of a single free Dirac particle in terms of the fundamental dynamical variables of the system. Expressions for $\underset{\sim}{\mathcal{X}}$ and $\underset{\sim}{\mathcal{S}}$ in terms of the fundamental dynamical variables are obtained. Finally, the operators $\underset{\sim}{\mathcal{X}}$ and $\underset{\sim}{\mathcal{S}}$ are seen to be related to the fundamental dynamical variables $\underset{\sim}{X}$ and $\underset{\sim}{S}$ via the well known Pryce-Foldy-Wouthuysen transformation.[8]

In Section 3.4, a system of n noninteracting particles is considered wherein the generators of the Poincaré group are seen to be sums of single particle generators. Interaction is introduced to the system in Section 3.5. In specifying the interaction, it becomes necessary to define internal operators and expressions are given for some of these along with various commutation relations which they satisfy. The results are compared with those for a system of n interacting nonrelativistic particles in Appendix I.

3.1 Single Spinless Free Particle

Let us consider a single spinless free particle of mass m. We take the fundamental dynamical variables of this system to be the Cartesian coordinates $X^{(1)}, X^{(2)}, X^{(3)}$ and the momenta $P^{(1)}, P^{(2)}, P^{(3)}$ of the particle which satisfy the fundamental quantum conditions.

$$[X^{(j)}, X^{(k)}] = 0$$

$$[P^{(j)}, P^{(k)}] = 0$$

$$[X^{(j)}, P^{(k)}] = i\hbar\,\delta_{jk} \qquad j,k = 1,2,3 \tag{3.1.1}$$

We assume that the operators $X^{(1)}, X^{(2)}, X^{(3)}$ and $P^{(1)}, P^{(2)}, P^{(3)}$ each form a complete set of compatible observables. We denote their eigenkets by $|\underline{x}\rangle = |x^{(1)}, x^{(2)}, x^{(3)}\rangle$ and $|\underline{p}\rangle = |p^{(1)}, p^{(2)}, p^{(3)}\rangle$. The eigenkets may be used as bases for the Hilbert space.

$$X^{(j)} = \int d^3x \; |\underline{x}\rangle\, x^{(j)} \langle \underline{x}| \tag{3.1.2}$$

$$P^{(j)} = \int d^3p \; |\underline{p}\rangle\, p^{(j)} \langle \underline{p}| \tag{3.1.3}$$

$$1 = \int d^3x \; |\underline{x}\rangle\langle\underline{x}| = \int d^3p \; |\underline{p}\rangle\langle\underline{p}| \tag{3.1.4}$$

$$\langle \underline{x} | \underline{x}' \rangle = \delta(\underline{x} - \underline{x}') \tag{3.1.5}$$

$$\langle \underline{p} | \underline{p}' \rangle = \delta(\underline{p} - \underline{p}') \tag{3.1.6}$$

Also it follows from the commutation relations (3.1.1) that

$$\langle \underline{x} | \underline{p} \rangle = \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{i\underline{p}\cdot\underline{x}/\hbar} \tag{3.1.7}$$

We now seek a construction of the generators of the Poincaré group in terms of the fundamental dynamical variables $X^{(k)}$ and $P^{(k)}$ (k=1,2,3). All of the commutation relations (2.2.15) hold when

$$\underline{J} = \underline{X} \times \underline{P} \tag{3.1.8}$$

$$\underline{K} = \frac{-1}{2c^2} (\underline{X} H + H \underline{X}) \tag{3.1.9}$$

$$H = (P^2c^2 + m^2c^4)^{1/2} = \int d^3p \ |\underline{p}\rangle \, \epsilon(p) \langle \underline{p}| \tag{3.1.10}$$

where $\epsilon(p) = (p^2c^2 + m^2c^4)^{1/2}$ (3.1.11)

Thus the particle is said to be a Lorentz invariant system when $\underline{J}$, $\underline{K}$, and H are given by the above equations.

Furthermore, it follows from equation (3.1.10) that the velocity $\underline{V}$ of the particle is

$$\underline{V} = c^2 \underline{P} H^{-1} = [\underline{X}, H]/i\hbar \tag{3.1.12}$$

Thus $\underline{X}$ satisfies the commutation relations (2.8.2) and indeed substitution of equations (3.1.8), (3.1.9), and (3.1.10) into equations (2.8.4), (2.8.5) and (2.8.6) shows that

$$\mathcal{B} = 1 \qquad\qquad M = m$$

$$\underline{\mathcal{X}} = \underline{X} \qquad\qquad \underline{\mathcal{S}} = 0 \tag{3.1.13}$$

Finally, the corresponding unitary operators for the Poincaré transformations, the space inversion operator, and the antilinear time reversal operator may be written simply in terms of space and momentum integrals.[6] We show in Appendix D that

$$D(\underline{a}) = e^{-i\underline{P}\cdot\underline{a}/\hbar} = \int d^3x \; |\underline{x}+\underline{a}\rangle\langle\underline{x}| \tag{3.1.14}$$

$$\mathcal{R}^{(1)}(\alpha) = e^{-iJ^{(1)}\alpha/\hbar} = \int d^3x \; |\underline{x}_R\rangle\langle\underline{x}| = \int d^3p \; |\underline{p}_R\rangle\langle\underline{p}| \tag{3.1.15}$$

where $|\underline{x}_R\rangle = |x^{(1)}, x^{(2)}\cos\alpha - x^{(3)}\sin\alpha, x^{(3)}\cos\alpha + x^{(2)}\sin\alpha\rangle$ (3.1.16)

and $|\underline{p}_R\rangle = |p^{(1)}, p^{(2)}\cos\alpha - p^{(3)}\sin\alpha, p^{(3)}\cos\alpha + p^{(2)}\sin\alpha\rangle$ (3.1.17)

$$L^{(1)}(u) = e^{-icK^{(1)}u/\hbar} = \int d^3p \left(\frac{\epsilon_L}{\epsilon}\right)^{1/2} |\underline{p}_L\rangle\langle\underline{p}| \tag{3.1.18}$$

where $\epsilon = (p^2c^2 + m^2c^4)^{1/2}$ (3.1.19)

$$\epsilon_L = \epsilon\cosh u + c\,p^{(1)}\sinh u \tag{3.1.20}$$

$$|\underline{p}_L\rangle = |p^{(1)}\cosh u + \tfrac{1}{c}\epsilon\sinh u, p^{(2)}, p^{(3)}\rangle \tag{3.1.21}$$

$$\mathcal{P} = \int d^3x \; |-\underline{x}\rangle\langle\underline{x}| \tag{3.1.22}$$

$$\mathcal{T}|\psi\rangle = \int d^3x \; |\underline{x}\rangle\langle\psi|\underline{x}\rangle = \int d^3p \; |-\underline{p}\rangle\langle\psi|\underline{p}\rangle \qquad \forall\,|\psi\rangle \tag{3.1.23}$$

Thus

$$D(\underline{a})|\underline{x}\rangle = |\underline{x}+\underline{a}\rangle \tag{3.1.24}$$

$$D(\underline{a})|\underline{p}\rangle = e^{-i\underline{p}\cdot\underline{a}/\hbar}|\underline{p}\rangle \tag{3.1.25}$$

$$\mathcal{R}^{(1)}(\alpha)|\underline{x}\rangle = |\underline{x}_R\rangle \tag{3.1.26}$$

$$\mathcal{R}^{(1)}(\alpha)|\underline{p}\rangle = |\underline{p}_R\rangle \tag{3.1.27}$$

$$L^{(1)}(u)|\underline{p}\rangle = \left(\frac{\epsilon_L}{\epsilon}\right)^{1/2}|\underline{p}_L\rangle \tag{3.1.28}$$

$$\mathcal{P}|\underline{x}\rangle = |-\underline{x}\rangle \tag{3.1.29}$$

$$\mathcal{P}|\underline{p}\rangle = |-\underline{p}\rangle \tag{3.1.30}$$

$$\mathcal{T}|\underline{x}\rangle = |\underline{x}\rangle \tag{3.1.31}$$

$$\mathcal{T}|\underline{p}\rangle = |-\underline{p}\rangle \tag{3.1.32}$$

The various integral forms of the space-time operators have a certain intuitive appeal in that they make the correspondence with equations (2.1.3), (2.1.4), (2.1.5), and (2.6.1) more manifest. In addition they may be used directly to give the various transformation equations derived in Section 2.3. For example, we derive $L^{(1)}(u)\, H L^{(1)\dagger}(u)$.

$$L^{(1)}(u) H L^{(1)\dagger}(u) = \int d^3p \left(\frac{\epsilon'(p)}{\epsilon(p)}\right)^{1/2} |\underline{p}'\rangle\langle\underline{p}| \int d^3\hat{p}\, |\hat{\underline{p}}\rangle\, \epsilon(\hat{p})\, \langle\hat{\underline{p}}| \int d^3q \left(\frac{\epsilon'(q)}{\epsilon(q)}\right)^{1/2} |\underline{q}\rangle\langle\underline{q}'|$$

where

$$|\underline{p}\rangle = |p^{(1)}, p^{(2)}, p^{(3)}\rangle$$

$$|\underline{p}'\rangle = |p^{(1)}\cosh u + \tfrac{1}{c}\,\epsilon(p)\sinh u,\ p^{(2)}, p^{(3)}\rangle$$

$$|\hat{\underline{p}}\rangle = |\hat{p}^{(1)}, \hat{p}^{(2)}, \hat{p}^{(3)}\rangle$$

$$|\underline{q}\rangle = |q^{(1)}, q^{(2)}, q^{(3)}\rangle$$

$$\langle\underline{q}'| = \langle q^{(1)}\cosh u + \tfrac{1}{c}\,\epsilon(q)\sinh u,\ q^{(2)}, q^{(3)}|$$

$$\epsilon(p) = (p^2c^2 + m^2c^4)^{1/2}$$

$$\epsilon'(p) = \epsilon(p)\cosh u + c\,p^{(1)}\sinh u$$

$$\epsilon'(q) = \epsilon(q)\cosh u + c\,q^{(1)}\sinh u$$

Thus,

$$L^{(1)}(u)HL^{(1)\dagger}(u) = \int d^3p\, d^3\hat{p}\, d^3q \left(\frac{\epsilon'(p)}{\epsilon(p)}\right)^{1/2} |p'\rangle \delta(p-\hat{p})\, \epsilon(\hat{p})\, \delta(\hat{p}-q) \left(\frac{\epsilon'(q)}{\epsilon(q)}\right)^{1/2} \langle q'|$$

Integrating over q,

$$L^{(1)}(u)HL^{(1)\dagger}(u) = \int d^3p\, d^3\hat{p} \left(\frac{\epsilon'(p)}{\epsilon(p)}\right)^{1/2} |p'\rangle\, \delta(p-\hat{p})\, \epsilon(\hat{p}) \left(\frac{\epsilon'(\hat{p})}{\epsilon(\hat{p})}\right)^{1/2} \langle \hat{p}'|$$

Integrating over $\hat{p}$,

$$L^{(1)}(u)HL^{(1)\dagger}(u) = \int d^3p \left(\frac{\epsilon'(p)}{\epsilon(p)}\right)^{1/2} |p'\rangle\, \epsilon(p) \left(\frac{\epsilon'(p)}{\epsilon(p)}\right)^{1/2} \langle p'|$$

$$= \int d^3p\ |p'\rangle\, \epsilon'(p)\, \langle p'|$$

Now a change of variables is made. Put

$$\tilde{p}^{(1)} = p^{(1)}\cosh u + \frac{1}{c}\,\epsilon(p)\sinh u$$

$$\tilde{p}^{(2)} = p^{(2)}$$

$$\tilde{p}^{(3)} = p^{(3)}$$

Since $\dfrac{d^3\tilde{p}}{\epsilon(\tilde{p})} = \dfrac{d^3p}{\epsilon(p)}$,

and $\epsilon(\tilde{p}) = \epsilon'(p)$,

$$L^{(1)}(u)HL^{(1)\dagger}(u) = \int d^3\tilde{p}\ |\tilde{p}\rangle\, \epsilon(p) \langle \tilde{p}|$$

But $\epsilon(p) = \epsilon(\tilde{p})\cosh u - c\,\tilde{p}^{(1)}\sinh u$

so $L^{(1)}(u)HL^{(1)\dagger}(u) = H\cosh u - c\,P^{(1)}\sinh u$

This is the result obtained in Section 2.3. The other transformation equations of that section are obtained similarly.

3.2 Single Free Particle with Spin

For a single free particle of mass m and spin s the fundamental dynamical variables include the spin operators $S^{(1)}, S^{(2)}, S^{(3)}$ as well as the Cartesian coordinates $X^{(k)}$ and the momenta $P^{(k)}$ ($k=1,2,3$) of the particle. In addition to the commutation relations (3.1.1), these variables satisfy the following relations.

$$[X^{(j)}, S^{(k)}] = [P^{(j)}, S^{(k)}] = 0$$

$$[S^{(j)}, S^{(k)}] = i\hbar\epsilon_{jk\ell}S^{(\ell)} \qquad j,k,\ell=1,2,3 \tag{3.2.1}$$

We assume that the operators $X^{(1)}, X^{(2)}, X^{(3)}, S^2=\underline{S}\cdot\underline{S}, S^{(3)}$ and $P^{(1)}, P^{(2)}, P^{(3)}, S^2, S^{(3)}$ each form a complete set of compatible observables. We denote their eigenkets by $|\underline{x},s,m_s\rangle$ and $|\underline{p},s,m_s\rangle$. The eigenkets may be used as bases for the Hilbert space.

$$\underline{X} = \sum_{m_s=-s}^{s}\int d^3x\ |\underline{x},s,m_s\rangle\ \underline{x}\ \langle\underline{x},s,m_s| \tag{3.2.2}$$

$$\underline{P} = \sum_{m_s=-s}^{s}\int d^3p\ |\underline{p},s,m_s\rangle\ \underline{p}\ \langle\underline{p},s,m_s| \tag{3.2.3}$$

$$S^2 = \sum_{m_s=-s}^{s}\int d^3x\ |\underline{x},s,m_s\rangle\ s(s+1)\hbar^2\langle\underline{x},s,m_s| \tag{3.2.4a}$$

$$= \sum_{m_s=-s}^{s}\int d^3p\ |\underline{p},s,m_s\rangle\ s(s+1)\hbar^2\ \langle\underline{p},s,m_s| \tag{3.2.4b}$$

$$S^{(3)} = \sum_{m_s=-s}^{s}\int d^3x\ |\underline{x},s,m_s\rangle\ m_s\hbar\ \langle\underline{x},s,m_s| \tag{3.2.5a}$$

$$= \sum_{m_s=-s}^{s}\int d^3p\ |\underline{p},s,m_s\rangle\ m_s\hbar\ \langle\underline{p},s,m_s| \tag{3.2.5b}$$

$$1 = \sum_{m_s=-s}^{s}\int d^3x\ |\underline{x},s,m_s\rangle\langle\underline{x},s,m_s| = \sum_{m_s=-s}^{s}\int d^3p\ |\underline{p},s,m_s\rangle\langle\underline{p},s,m_s| \tag{3.2.6}$$

$$\langle \underline{x}, s, m_s \mid \underline{x}', s, m_s' \rangle = \delta(\underline{x} - \underline{x}')\, \delta_{m_s m_s'} \tag{3.2.7}$$

$$\langle \underline{p}, s, m_s \mid \underline{p}', s, m_s' \rangle = \delta(\underline{p} - \underline{p}')\, \delta_{m_s m_s'} \tag{3.2.8}$$

Also,

$$\langle \underline{x}, s, m_s \mid \underline{p}, s, m_s' \rangle = \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{i\underline{p}\cdot\underline{x}/\hbar}\, \delta_{m_s m_s'} \tag{3.2.9}$$

The commutation relations (2.2.15) are all satisfied when the generators are expressed in terms of the fundamental dynamical variables as

$$\underline{J} = \underline{X} \times \underline{P} + \underline{S} \tag{3.2.10}$$

$$\underline{K} = -\frac{1}{2c^2}\left(\underline{X}H + H\underline{X}\right) + \left(H + mc^2\right)^{-1}\left(\underline{S} \times \underline{P}\right) \tag{3.2.11}$$

$$H = \left(P^2c^2 + m^2c^4\right)^{1/2} \tag{3.2.12a}$$

$$= \sum_{m_s=-s}^{s} \int d^3p \mid \underline{p}, s, m_s \rangle\, \epsilon(p) \langle \underline{p}, s, m_s \mid \tag{3.2.12b}$$

where $\epsilon(p) = \left(p^2c^2 + m^2c^4\right)^{1/2}$ (3.2.13)

Also, it follows from the commutation relations (3.1.1) and equation (3.2.12) for H that

$$[\underline{X}, H] = i\hbar c^2 \underline{P} H^{-1} \tag{3.2.14}$$

so the velocity of the particle, $\underline{V}$, is

$$\underline{V} = \frac{1}{i\hbar}[\underline{X}, H] = c^2 \underline{P} H^{-1} \tag{3.2.15}$$

Thus $\underline{X}$ satisfies the commutation relations (2.8.2) and substitution of equations (3.2.10), (3.2.11), and (3.2.12) into equations (2.8.4), (2.8.5), and (2.8.6) yields

$$\not{B} = 1 \tag{3.2.16}$$

$$M = m \tag{3.2.17}$$

$$\underline{\not{X}} = \underline{X} \tag{3.2.18}$$

$$\underline{\$} = \underline{S} \tag{3.2.19}$$

The unitary operators corresponding to the space-time transformations under consideration and the antiunitary operator $\mathcal{T}$ can be written in integral form as follows.

$$D(\underline{a}) = \int d^3x \; |\underline{x}+\underline{a}, s, m_s\rangle\langle \underline{x}, s, m_s| \tag{3.2.20}$$

$$\mathcal{R}(\underline{\theta}) = \sum_{m_s, m_s'} D^s_{m_s', m_s}(\alpha,\beta,\gamma) \int d^3p \; |\underline{p}_R, s, m_s'\rangle\langle \underline{p}, s, m_s| \tag{3.2.21}$$

where (α,β,γ) are the Euler angles corresponding to $\underline{\theta}$. The matrix $D^s(\alpha,\beta,\gamma)$ is the (2s + 1)-dimensional irreducible representation of the rotation group. It ensures the proper transformation of the spin under rotations. To verify that the above equation is the correct form for $\mathcal{R}(\underline{\theta})$ we separate the spin and momentum parts of the expression.

$$\mathcal{R}(\underline{\theta}) = \int d^3p \; |\underline{p}_R\rangle\langle \underline{p}| \otimes \sum_{m_s, m_s'} |s, m_s'\rangle \, D^s_{m_s', m_s}(\alpha,\beta,\gamma) \, \langle s, m_s|$$

$$= \int d^3p \, |\underline{p}_R\rangle\langle \underline{p}| \otimes \sum_{m_s, m_s'} |s, m_s'\rangle\langle s, m_s'| \, D^s(\alpha,\beta,\gamma) \, |s, m_s\rangle\langle s, m_s|$$

From Section 3.1, we have

$$\int d^3p \; |\underline{p}_R\rangle\langle \underline{p}| = e^{-i(\underline{X}\times \underline{P})\cdot\underline{\theta}/\hbar}$$

and it is well known that

$$D^s(\alpha,\beta,\gamma) = e^{-i\underline{S}\cdot\underline{\theta}/\hbar}$$

so $\mathcal{R}(\underline{\theta}) = e^{-i(\underline{X}\times\underline{P}+\underline{S})\cdot\underline{\theta}/\hbar} = e^{-i\underline{J}\cdot\underline{\theta}/\hbar}$

as required.

$$L^{(1)}(u) = \sum_{m_s,m_s'} D^s_{m_s',m_s}(\alpha_\omega,\beta_\omega,\gamma_\omega)\int d^3p\left(\frac{\epsilon_L}{\epsilon}\right)^{1/2}|\underline{p}_L,s,m_s'\rangle\langle\underline{p},s,m_s| \qquad (3.2.22)$$

where $(\alpha_\omega,\beta_\omega,\gamma_\omega)$ are the Euler angles corresponding to the Wigner rotation of the spin, $\underline{S}$.[9] Recall the transformation of $\underline{\$}$ under Lorentz boosts given by equation (2.9.11). Here, the operator $\underline{\$}$ is equal to the spin, $\underline{S}$. The spin undergoes a rotation under Lorentz boosts. That is, taking matrix elements of the operator equation (2.9.11) would yield the Wigner rotation. Thus, the Lorentz boost operator $L^{(1)}(u)$ must take the form above to ensure the proper transformation of the spin.

$$\mathcal{P} = \sum_{m_s=-s}^{s}\int d^3x\ |-\underline{x},s,m_s\rangle\langle\underline{x},s,m_s| \qquad (3.2.23)$$

$$\mathcal{T}|\psi\rangle = \sum_{m_s=-s}^{s}\int d^3x\ |\underline{x},s,-m_s\rangle\langle\psi|\underline{x},s,m_s\rangle \qquad (3.2.24a)$$

$$= \sum_{m_s=-s}^{s}\int d^3p\ |-\underline{p},s,-m_s\rangle\langle\psi|\underline{p},s,m_s\rangle \qquad \forall|\psi\rangle \qquad (3.2.24b)$$

Thus,

$$D(\underline{a})|\underline{x},s,m_s\rangle = |\underline{x}+\underline{a},s,m_s\rangle$$

$$\mathcal{R}(\underline{\theta})|\underline{x},s,m_s\rangle = \sum_{m_s'=-s}^{s} D^s_{m_s',m_s}(\alpha,\beta,\gamma)|\underline{x}_R,s,m_s'\rangle$$

$$L^{(1)}(u)|\underline{p},s,m_s\rangle = \sum_{m_s'=-s}^{s} D^s_{m_s',m_s}(\alpha_\omega,\beta_\omega,\gamma_\omega)\left(\frac{\epsilon_L}{\epsilon}\right)^{1/2}|\underline{p}_L,s,m_s'\rangle$$

$$\mathcal{P}|\underline{x},s,m_s\rangle = |-\underline{x},s,m_s\rangle$$

$$\mathcal{P}|\underline{p},s,m_s\rangle = |-\underline{p},s,m_s\rangle$$

$$\mathcal{T}|\underline{x},s,m_s\rangle = |\underline{x},s,-m_s\rangle$$

$$\mathcal{T}|\underline{p},s,m_s\rangle = |-\underline{p},s,-m_s\rangle$$

3.3 Single Free Dirac Particle

For a system consisting of a single free Dirac particle of mass m, we take the Cartesian coordinates $\underline{X}$, the momenta $\underline{P}$, and four Dirac operators $\underline{\alpha}$ and β as the fundamental dynamical variables. These operators satisfy the following conditions.

$$[X^{(j)}, X^{(k)}] = 0 \qquad [P^{(j)}, P^{(k)}] = 0 \qquad [X^{(j)}, P^{(k)}] = i\hbar\,\delta_{jk}$$

$$(\alpha^{(k)})^2 = 1 \qquad \beta^2 = 1$$

$$\alpha^{(j)}\alpha^{(k)} + \alpha^{(k)}\alpha^{(j)} = 0 \quad ; \quad j \neq k \qquad \underline{\alpha}\beta + \beta\underline{\alpha} = 0$$

$$[X^{(j)}, \alpha^{(k)}] = [X^{(j)}, \beta] = 0 \qquad [P^{(j)}, \alpha^{(k)}] = [P^{(j)}, \beta] = 0 \qquad j, k = 1, 2, 3 \tag{3.3.1}$$

We now seek a construction of the generators of the Poincaré group in terms of the fundamental dynamical variables. Following Dirac,[10] we write the Hamiltonian as

$$H = c\,\underline{\alpha}\cdot\underline{P} + \beta m c^2 \tag{3.3.2}$$

The conditions (3.3.1) on $\underline{\alpha}$ and β ensure that

$$H^2 = P^2 c^2 + m^2 c^4 \tag{3.3.3}$$

Thus M defined in equation (2.8.4) is equal to m, the mass of the particle. Finally, it is straightforward to check that the commutation relations (2.2.15) are satisfied when H is given by equation (3.3.2) above and

$$\underline{P} = \underline{P} \tag{3.3.4}$$

$$\underline{K} = -\frac{1}{2c^2}\left(\underline{X}H + H\underline{X}\right) \tag{3.3.5}$$

$$\underline{J} = \underline{X} \times \underline{P} + \underline{S} \tag{3.3.6}$$

where $\underline{S} = \frac{1}{2}\hbar\hat{\underline{\sigma}}$ (3.3.7)

and $\hat{\underline{\sigma}} = \frac{-i}{2}(\underline{\alpha} \times \underline{\alpha})$ (3.3.8)

We note that

$$H \neq E$$

where E is given by equation (B.16). Therefore,

$$\beta \neq 1$$

This is, of course, expected since the Dirac Hamiltonian (3.3.2) has both positive and negative spectrum.

Also, since

$$[\underline{X}, H] = i\hbar c\underline{\alpha} \tag{3.3.9}$$

and $[\underline{S}, H] = i\hbar c\underline{\alpha} \times \underline{P}$ (3.3.10)

the operators $\underline{X}$ and $\underline{S}$ are not equal to the centre of mass position operator $\underline{\mathcal{X}}$ and the internal angular momentum operator $\underline{\mathcal{S}}$. Indeed, as is shown in Appendix E, substitution of the above expressions for the generators into the definitions (2.8.5) and (2.8.6) of $\underline{\mathcal{X}}$ and $\underline{\mathcal{S}}$ yields

$$\underline{\mathcal{X}} = \underline{X} - \frac{c^2\,\underline{S} \times \underline{P}}{E(E+mc^2)} + \frac{c\hbar}{2E}\left\{ i\beta\underline{\alpha} - \frac{ic^2\beta\,\underline{P}(\underline{P}\cdot\underline{\alpha})}{E(E+mc^2)} \right\} \tag{3.3.11}$$

$$\underline{\mathcal{S}} = \underline{S} - \frac{i\hbar c\beta}{2E}\,\underline{\alpha} \times \underline{P} - \frac{c^2\,\underline{P} \times (\underline{S} \times \underline{P})}{E(E+mc^2)} \tag{3.3.12}$$

We recognize that these operators are those given by Pryce, Foldy and Wouthuysen, Jordan and Mukunda, and de Groot and Suttorp.[11] We show in Appendix F that

$$F \underline{X} F^{\dagger} = \underline{X} \tag{3.3.13}$$

$$F \underline{S} F^{\dagger} = \underline{S} \tag{3.3.14}$$

$$F \underline{P} F^{\dagger} = \underline{P} \tag{3.3.15}$$

$$F \underline{J} F^{\dagger} = \underline{J} \tag{3.3.16}$$

$$F H F^{\dagger} = \beta E \tag{3.3.17}$$

$$F \beta F^{\dagger} = \beta \tag{3.3.18}$$

where

$$F = \frac{E + \beta H}{(2E(E+mc^2))^{1/2}} \tag{3.3.19}$$

The unitary operator F is the Pryce-Foldy-Wouthuysen transformation operator.[12]

3.4 n Noninteracting Particles with Spin

Let us consider a system of a fixed number, n, of free particles of masses $m_1, m_2, \ldots, m_n$ and spins $s_1, s_2, \ldots, s_n$. The fundamental dynamical variables of this system are the components of the Cartesian coordinates, of the momentum, and of the spin of the individual particles, $\underline{X}_1, \underline{P}_1, \underline{S}_1, \cdots, \underline{X}_n, \underline{P}_n, \underline{S}_n$. These operators satisfy the following commutation relations.

$$[X_\alpha^{(j)}, X_\beta^{(k)}] = 0 \qquad [P_\alpha^{(j)}, P_\beta^{(k)}] = 0 \qquad [X_\alpha^{(j)}, P_\beta^{(k)}] = i\hbar\, \delta_{\alpha\beta}\delta_{jk}$$

$$[X_\alpha^{(j)}, S_\beta^{(k)}] = 0 \qquad [P_\alpha^{(j)}, S_\beta^{(k)}] = 0 \qquad [S_\alpha^{(j)}, S_\beta^{(k)}] = i\hbar\, \delta_{\alpha\beta}\epsilon_{jk\ell} S_\alpha^{(\ell)} \qquad \alpha,\beta = 1,\ldots,n \quad j,k,\ell = 1,2,3 \tag{3.4.1}$$

Here the Greek subscripts are the particle labels while the Roman superscripts label the component of the variable.

Since the particles are noninteracting each generator must be a sum of single free particle generators. That is,

$$H_0 = \sum_{\alpha=1}^{n} H_\alpha \tag{3.4.2}$$

$$\underline{P}_0 = \sum_{\alpha=1}^{n} \underline{P}_\alpha \tag{3.4.3}$$

$$\underline{J}_0 = \sum_{\alpha=1}^{n} \underline{J}_\alpha \tag{3.4.4}$$

$$\underline{K}_0 = \sum_{\alpha=1}^{n} \underline{K}_\alpha \tag{3.4.5}$$

where

$$H_\alpha = (P_\alpha^2 c^2 + m_\alpha^2 c^4)^{1/2} \tag{3.4.6}$$

$$\underline{J}_\alpha = \underline{X}_\alpha \times \underline{P}_\alpha + \underline{S}_\alpha \tag{3.4.7}$$

$$\underline{K}_\alpha = -\frac{1}{2c^2}(\underline{X}_\alpha H_\alpha + H_\alpha \underline{X}_\alpha) + (H_\alpha + m_\alpha c^2)^{-1}(\underline{S}_\alpha \times \underline{P}_\alpha) \tag{3.4.8}$$

We have appended a subscript zero to these generators to distinguish them from the generators for an interacting system, which we consider in the next section. These generators satisfy the commutation relations (2.2.15). It is convenient to henceforth denote the total internal angular momentum by the symbol $\tilde{\underline{J}}$. That is,

$$\underline{\$} = \tilde{\underline{J}} \tag{3.4.9}$$

The total mass M_0, the centre of mass position $\underline{X}_0$, and the total internal angular momentum $\tilde{\underline{J}}_0$ are given by equations (2.8.4), (2.8.5), and (2.8.6). We write these equations here explicitly for later convenience.

$$M_0 c^2 = (H_0^2 - P_0^2 c^2)^{1/2} \tag{3.4.10}$$

$$M_0 \underline{X}_0 = -\frac{M_0 c^2}{2}(H_0^{-1}\underline{K}_0 + \underline{K}_0 H_0^{-1}) - \frac{\underline{P}_0 \times \underline{J}_0}{H_0 + M_0 c^2} + \frac{c^2}{H_0(H_0 + M_0 c^2)}\underline{P}_0 \times (\underline{P}_0 \times \underline{K}_0) \tag{3.4.11}$$

$$M_0 \tilde{\underline{J}}_0 = \underline{K}_0 \times \underline{P}_0 + \frac{H_0}{c^2}\underline{J}_0 - \frac{(\underline{J}_0 \cdot \underline{P}_0)\underline{P}_0}{(H_0 + M_0 c^2)} \tag{3.4.12}$$

$(\cancel{\beta} = 1)$

3.5 n Particles in Interaction

Let us consider a system of n interacting particles of masses $m_1, \ldots, m_n$ and spins $s_1, \ldots, s_n$. We assume that no particle creation or annihilation takes place. That is, the number of particles, n, is fixed. Therefore, we may take the fundamental dynamical variables to be the individual particle position, momentum, and spin operators $\underline{X}_1, \ldots, \underline{X}_n, \underline{P}_1, \ldots, \underline{P}_n, \underline{S}_1, \ldots, \underline{S}_n$. These operators satisfy the commutation relations (3.4.1).

We seek a construction of the generators of the Poincaré group for this interacting system. Following Bakamjian and Thomas,[1] we see that the commutation relations (2.2.15) are satisfied when

$$H = (P^2c^2 + M^2c^4)^{1/2} \tag{3.5.1}$$

$$\underline{P} = \underline{P}_0 \tag{3.5.2}$$

$$\underline{J} = \underline{J}_0 \tag{3.5.3}$$

$$\underline{K} = -\frac{1}{2c^2}(\underline{X}_0 H + H\underline{X}_0) + (H + Mc^2)^{-1}(\tilde{\underline{J}}_0 \times \underline{P}) \tag{3.5.4}$$

where $\underline{P}_0$, $\underline{J}_0$, $\underline{X}_0$, and $\tilde{\underline{J}}_0$ are given by equations (3.4.3), (3.4.4), (3.4.10), and (3.4.11), respectively, and where the mass operator M satisfies

$$[M, \underline{P}] = [M, \underline{X}_0] = [M, \tilde{\underline{J}}_0] = 0 \tag{3.5.5}$$

The interaction in the system is, thus, specified completely by the mass operator M. If $M = M_0$, given by equation (3.4.10), the particles are free whereas, if $M \neq M_0$, the particles interact amongst themselves.

Finally, we may define the potential V of the system by

$$V \equiv H - H_0 \tag{3.5.6}$$

One should notice that the potential for a Poincaré invariant n particle system will in general be a function of the total momentum of the system.

To complete the description of the system, we must show how to construct an operator M satisfying equation (3.5.5). This is a nontrivial problem. One can easily find many operators which commute with $\underline{P}$ (eg. $\underline{P}_\alpha$, $\underline{X}_\alpha - \underline{X}_\beta, \underline{X}_0 - \underline{X}_\alpha, \underline{S}_\alpha$) but it is more difficult to find operators which also commute with the complicated operators $\underline{X}_0$ and $\tilde{\underline{J}}_0$. Following Osborn, we solve this problem using a transformation analogous to that introduced by Gartenhaus and Schwartz.[13]

We first define a unitary operator $\mathcal{B}(\lambda)$ by

$$\mathcal{B}(\lambda) = e^{-i\lambda B/\hbar} \tag{3.5.7}$$

where λ is a real number and

$$B = -\tfrac{1}{2}(\underline{X}_0 \cdot \underline{P} + \underline{P} \cdot \underline{X}_0) \tag{3.5.8}$$

Using equation (3.4.10) for $\underline{X}_0$, one can also write B in the form

$$B = \tfrac{1}{2}(\underline{K}_0 \cdot \underline{V}_0 + \underline{V}_0 \cdot \underline{K}_0) \tag{3.5.9}$$

where

$$\underline{V}_0 = c^2 \underline{P} H_0^{-1} \tag{3.5.10}$$

Thus, $\mathcal{B}(\lambda)$ reminds one of a Lorentz boost operator. It follows on using the methods of Section 2.3 that

$$\mathcal{B}(\lambda)\underline{P}\,\mathcal{B}^{\dagger}(\lambda) = e^{-\lambda}\underline{P} \tag{3.5.11}$$

$$\mathcal{B}(\lambda)\underline{X}_0\,\mathcal{B}^{\dagger}(\lambda) = e^{\lambda}\underline{X}_0 \tag{3.5.12}$$

$$\mathcal{G}(\lambda)\,\tilde{\underline{J}}_0\,\mathcal{G}^\dagger(\lambda) = \tilde{\underline{J}}_0 \tag{3.5.13}$$

so that for any operator A ,

$$[\underline{P}, \mathcal{G}(\lambda)A\mathcal{G}^\dagger(\lambda)] = \mathcal{G}(\lambda)[\mathcal{G}^\dagger(\lambda)\underline{P}\mathcal{G}(\lambda), A]\mathcal{G}^\dagger(\lambda) = e^{\lambda}\mathcal{G}(\lambda)[\underline{P}, A]\mathcal{G}^\dagger(\lambda) \tag{3.5.14}$$

and

$$[\underline{K}_0, \mathcal{G}(\lambda)A\mathcal{G}^\dagger(\lambda)] = e^{-\lambda}\mathcal{G}(\lambda)[\underline{K}_0, A]\mathcal{G}^\dagger(\lambda) \tag{3.5.15}$$

Thus, if

$$[\underline{P}, A] = 0 \tag{3.5.16}$$

and if

$$\lim_{\lambda\to\infty}(\mathcal{G}(\lambda)[\underline{K}_0, A]\mathcal{G}^\dagger(\lambda)) \quad \text{exists} \tag{3.5.17}$$

then

$$[\underline{P}, \hat{A}] = [\underline{K}_0, \hat{A}] = 0 \tag{3.5.18}$$

where

$$\hat{A} \equiv \lim_{\lambda\to\infty}(\mathcal{G}(\lambda)A\mathcal{G}^\dagger(\lambda)) \tag{3.5.19a}$$

$$\equiv \mathcal{G}A\mathcal{G}^\dagger \tag{3.5.19b}$$

The above gives the conditions for an operator to commute with both $\underline{P}$ and $\underline{K}_0$. As we shall see, some well chosen operators which satisfy these conditions will allow us to find operators commuting with $\tilde{\underline{J}}_0$ as well.

Osborn states that the condition for the existence of $\hat{A}$ is that A commute with $\underline{P}$.[14] He gives as an example the operator $\underline{K}_0$ which does not commute with $\underline{P}$ and which does indeed diverge under $\mathcal{G}(\lambda)$ in the limit $\lambda\to\infty$ since

$$\mathcal{G}(\lambda)\,\underline{X}_0\,\mathcal{G}^{\dagger}(\lambda) = e^{\lambda}\,\underline{X}_0 \tag{3.5.20}$$

However, consider the operator P^{-2} which commutes with $\underline{P}$. By the methods of Section 2.3, it follows that

$$\mathcal{G}(\lambda)\,P^{-2}\,\mathcal{G}^{\dagger}(\lambda) = e^{\lambda}\,P^{-2} \tag{3.5.21}$$

Thus $\hat{P}^{-2}$ diverges. Also, consider the operator X_0^{-2} which does not commute with $\underline{P}$. We find that

$$\mathcal{G}(\lambda)\,X_0^{-2}\,\mathcal{G}^{\dagger}(\lambda) = e^{-\lambda}\,X_0^{-2} \tag{3.5.22}$$

so that $\hat{X}_0^{-2}$ exists. Clearly, Osborn's condition does not ensure the existence of $\hat{A}$. That is,

$$[[\underline{X}_0, A], \underline{P}] = 0$$

does not guarantee that condition (3.5.17) hold.

It follows from equation (3.5.11) that

$$\hat{\underline{P}} = \mathcal{G}\,\underline{P}\,\mathcal{G}^{\dagger} = 0 \tag{3.5.23}$$

In view of this equation, we call $\hat{A}$ for Hermitian A "the observable A in the zero-momentum picture". If equation (3.5.16) holds, we call $\hat{A}$ a "displacement invariant zero-momentum picture operator". If, in addition, condition (3.5.17) holds, we call $\hat{A}$ an "internal operator".

Of particular interest are the following displacement invariant zero-momentum picture operators.

$$\hat{\underline{P}}_\alpha = \mathcal{U}\,\underline{P}_\alpha\,\mathcal{U}^\dagger \qquad \hat{H}_\alpha = \mathcal{U}\,H_\alpha\,\mathcal{U}^\dagger$$

$$\hat{\underline{X}}_{\alpha\beta} = \mathcal{U}(\underline{X}_\alpha - \underline{X}_\beta)\mathcal{U}^\dagger \qquad \hat{H}_0 = \mathcal{U}\,H_0\,\mathcal{U}^\dagger$$

$$\hat{\underline{S}}_\alpha = \mathcal{U}\,\underline{S}_\alpha\,\mathcal{U}^\dagger$$

$$\hat{\underline{S}} = \mathcal{U}\,\underline{S}\,\mathcal{U}^\dagger \qquad \text{where} \qquad \underline{S} = \sum_{\alpha=1}^{n} \underline{S}_\alpha \tag{3.5.24}$$

(One might suppose that the operator $\mathcal{U}(\underline{X}_0 - \underline{X}_\alpha)\mathcal{U}^\dagger$ should be included in this list. However, this operator is not an internal operator.) Thus, for example, $\hat{\underline{P}}_\alpha$ is the momentum of particle α in the zero-momentum picture. It follows from the commutation relations (3.4.1) that the following commutation relations are satisfied.

$$[\hat{P}_\alpha^{(j)}, \hat{P}_\beta^{(k)}] = [\hat{X}_{\alpha\beta}^{(j)}, \hat{X}_{\nu\mu}^{(k)}] = 0 \qquad [\hat{X}_{\alpha\beta}^{(j)}, \hat{P}_\nu^{(k)}] = i\hbar(\delta_{\alpha\nu} - \delta_{\beta\nu})\delta_{jk}$$

$$[\hat{P}_\alpha^{(j)}, \hat{S}_\beta^{(k)}] = [\hat{X}_{\alpha\beta}^{(j)}, \hat{S}_\nu^{(k)}] = 0 \qquad [\hat{S}_\alpha^{(j)}, \hat{S}_\beta^{(k)}] = i\hbar\,\delta_{\alpha\beta}\,\epsilon_{jkl}\,\hat{S}_\alpha^{(l)} \tag{3.5.25}$$

We show in Appendix G that

$$\hat{H}_\alpha = (H_0 H_\alpha - \underline{P}\cdot\underline{P}_\alpha)/M_0 c^2 \tag{3.5.26}$$

$$\hat{\underline{P}}_\alpha = \underline{P}_\alpha - \left\{\frac{H_\alpha + \hat{H}_\alpha}{H_0 + M_0 c^2}\right\}\underline{P} \tag{3.5.27}$$

from which

$$\sum_{\alpha=1}^{n} \hat{H}_\alpha = \hat{H}_0 = M_0 c^2 \tag{3.5.28}$$

and

$$\sum_{\alpha=1}^{n} \hat{\underline{P}}_\alpha = 0 \tag{3.5.29}$$

as required. Bakamjian and Thomas give operators of this form.[15]

We show in Appendix H that $\hat{\underline{P}}_\alpha$, $\hat{\underline{X}}_{\alpha\beta}$, and $\hat{\underline{S}}_\alpha$ are indeed internal operators. (This is true providing that $\hat{\underline{X}}_{\alpha\beta}$ and $\hat{\underline{S}}_\alpha$ exist. Osborn has given explicit expressions for $\hat{\underline{X}}_{12}$ and $\hat{\underline{S}}_\alpha$ for the two-body problem.)[16] Thus,

$$[P^{(j)}, \hat{P}_\alpha^{(k)}] = [P^{(j)}, \hat{X}_{\alpha\beta}^{(k)}] = [P^{(j)}, \hat{S}_\alpha^{(k)}] = 0$$

$$[\mathcal{X}_0^{(j)}, \hat{P}_\alpha^{(k)}] = [\mathcal{X}_0^{(j)}, \hat{X}_{\alpha\beta}^{(k)}] = [\mathcal{X}_0^{(j)}, \hat{S}_\alpha^{(k)}] = 0 \qquad (3.5.30)$$

Moreover we show that

$$[\tilde{J}_0^{(j)}, \hat{P}_\alpha^{(k)}] = i\hbar\,\epsilon_{jk\ell}\,\hat{P}_\alpha^{(\ell)} \qquad [\tilde{J}_0^{(j)}, \hat{X}_{\alpha\beta}^{(k)}] = i\hbar\,\epsilon_{jk\ell}\,\hat{X}_{\alpha\beta}^{(\ell)}$$

$$[\tilde{J}_0^{(j)}, \hat{S}_\alpha^{(k)}] = i\hbar\,\epsilon_{jk\ell}\,\hat{S}_\alpha^{(\ell)} \qquad (3.5.31)$$

Thus, for example, the following internal operators commute with $\tilde{\underline{J}}_0$.

$$\hat{\underline{P}}_\alpha \cdot \hat{\underline{P}}_\alpha\,,\ \hat{\underline{X}}_{\alpha\beta} \cdot \hat{\underline{X}}_{\alpha\beta}\,,\ \hat{\underline{X}}_{\alpha\beta} \cdot \hat{\underline{P}}_\alpha\,,\ \hat{\underline{P}}_\alpha \cdot \hat{\underline{S}}_\alpha\,,\ \hat{\underline{X}}_{\alpha\beta} \cdot \hat{\underline{S}}_\alpha \qquad (3.5.32)$$

This completes the problem of constructing the mass operator M. That is, M may be a function of the operators (3.5.32) above.

In Appendix I, we consider a system of n interacting particles invariant under the Galilean transformations.

Chapter 4 Two Particle Scattering

4.1 Lorentz Invariance for Two Particle Scattering Systems

We now consider a system of two spinless particles of masses m_1 and m_2 scattering off each other. We assume that no particle creation or annihilation takes place. The fundamental dynamical variables of this system can be taken to be the Cartesian coordinates of the particles, $\underline{X}_1$, $\underline{X}_2$, and the individual particle momenta, $\underline{P}_1$, $\underline{P}_2$, which obey the commutation relations (3.4.1). Since the fundamental dynamical variables of different particles commute, the Hilbert space $\mathcal{H}$ of the system may be expressed as the direct product of the individual particle Hilbert spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$.[17] That is,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{4.1.1}$$

We also define centre of mass position and momentum, $\underline{X}_0$ and $\underline{P}$, and internal (relative) variables $\hat{\underline{X}}_{\alpha\beta}$ and $\hat{\underline{P}}_\alpha$, as in Section 3.5:

$$\underline{X}_0 \qquad\qquad \underline{P}$$

$$\hat{\underline{X}}_{12} \equiv \hat{\underline{X}} \qquad\qquad \hat{\underline{P}}_1 = -\hat{\underline{P}}_2 \equiv \hat{\underline{P}} \tag{4.1.2}$$

where

$$\underline{P} = \underline{P}_1 + \underline{P}_2 \tag{4.1.3}$$

and $\underline{X}_0$ and $\hat{\underline{P}}_\alpha$ are given by equations (3.4.11) and (3.5.27), respectively. These operators obey the commutation relations (2.8.2), (3.5.25), and (3.5.20). That is,

$$[X_0^{(j)}, X_0^{(k)}] = 0 \qquad [X_0^{(j)}, P^{(k)}] = i\hbar\,\delta_{jk} \qquad [P^{(j)}, P^{(k)}] = 0$$

$$[\hat{X}^{(j)}, \hat{X}^{(k)}] = 0 \qquad [\hat{X}^{(j)}, \hat{P}^{(k)}] = i\hbar\,\delta_{jk} \qquad [\hat{P}^{(j)}, \hat{P}^{(k)}] = 0$$

$$[X_0^{(j)}, \hat{X}^{(k)}] = [X_0^{(j)}, \hat{P}^{(k)}] = [P^{(j)}, \hat{X}^{(k)}] = [P^{(j)}, \hat{P}^{(k)}] = 0 \qquad j,k = 1,2,3 \tag{4.1.4}$$

The centre of mass variables commute with the internal (relative) variables. Thus, the Hilbert space $\mathcal{H}$ may be expressed as the direct product of the centre of mass and relative Hilbert spaces, $\mathcal{H}_{cm}$ and $\mathcal{H}_{rel}$.[17] That is,

$$\mathcal{H} = \mathcal{H}_{cm} \otimes \mathcal{H}_{rel} = \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{4.1.5}$$

We now define the important operators of scattering theory. The Moller operators $\Omega_\pm$ on the two body Hilbert space $\mathcal{H}$ are

$$\Omega_\pm \equiv \lim_{t \to \mp\infty} U^\dagger(t)\, U_0(t) \tag{4.1.6}$$

where

$$U(t) = e^{-iHt/\hbar} \tag{4.1.7}$$

$$U_0(t) = e^{-iH_0 t/\hbar} \tag{4.1.8}$$

wherein H is the Hamiltonian for the system and H_0 is the Hamiltonian for two free particles. That is,

$$H_0 = \sum_{\alpha=1}^{2} (P_\alpha^2 c^2 + m_\alpha^2 c^4)^{1/2} \tag{4.1.9}$$

The Moller operators are isometric.[18] That is,

$$\Omega_\pm^\dagger \Omega_\pm = 1 \tag{4.1.10}$$

$$\Omega_\pm \Omega_\pm^\dagger = 1 - \mathcal{B} \tag{4.1.11}$$

where $\mathcal{B}$ is the projection operator onto the bound states of H. Of course, if H has no bound states, the Moller operators are unitary. Furthermore, the Moller operators satisfy the "intertwining relation":[19]

$$\mathcal{U}(t)\Omega_\pm = \Omega_\pm \mathcal{U}_0(t) \tag{4.1.12}$$

In the above, we have, of course, assumed that the Moller operators exist. This requirement ("the asymptotic condition") places restrictions on the Hamiltonian H which we do not investigate in detail here.[20] At the very least, the potential V given by

$$H = H_0 + V \tag{4.1.13}$$

must vanish sufficiently rapidly as the interparticle distance becomes large. We have also assumed "asymptotic completeness"; that is, the ranges of Ω_+ and Ω_- are identical.[21]

The scattering operator S is defined as

$$S \equiv \Omega_-^\dagger \Omega_+ \tag{4.1.14}$$

and it follows from equations (4.1.10) and (4.1.11) that S is unitary. That is,

$$SS^\dagger = S^\dagger S = 1 \tag{4.1.15}$$

Moreover, it follows from equations (4.1.12) and (4.1.14) that

$$\mathcal{U}_0(t)\, S\, \mathcal{U}_0^\dagger(t) = S \tag{4.1.16}$$

and thus,

$$[H_0, S] = 0 \tag{4.1.17}$$

As stressed by Taylor, this equation corresponds to the statement that the scattering process conserves energy.[22] That equation (4.1.17) involves the free Hamiltonian H_0 corresponds to the fact that the scattering operator is a mapping of the asymptotic free orbits which label the particles' state only when they are far apart and do not feel the potential, when their energy is given by H_0.

The invariance of the system under the Poincaré transformations, in the sense of Section 2.1, imposes conditions on $\Omega_\pm$ and S. We also assume the system to be invariant under space inversion and time reversal. We give the resulting conditions on $\Omega_\pm$ and S below.

$$\begin{aligned} &D(\underline{a})\Omega_\pm D^\dagger(\underline{a}) = \Omega_\pm && D(\underline{a}) S D^\dagger(\underline{a}) = S \\ &\mathcal{R}(\underline{\theta})\Omega_\pm \mathcal{R}^\dagger(\underline{\theta}) = \Omega_\pm && \mathcal{R}(\underline{\theta}) S \mathcal{R}^\dagger(\underline{\theta}) = S \\ &\mathcal{P}\Omega_\pm \mathcal{P}^\dagger = \Omega_\pm && \mathcal{P} S \mathcal{P}^\dagger = S \\ &\mathcal{T}\Omega_\pm \mathcal{T}^\dagger = \Omega_\mp && \mathcal{T} S \mathcal{T}^\dagger = S^\dagger \end{aligned} \tag{4.1.18}$$

The conditions (4.1.18) above follow from equations (2.3.7), (2.3.9), (2.6.6), and (2.7.6). We note in particular that it follows from the conditions (4.1.18) that the scattering operator is invariant under space displacements and rotations and, thus, that

$$[\underline{P}, S] = [\underline{J}, S] = 0 \tag{4.1.19}$$

Therefore, S conserves total momentum and total angular momentum.

It will be noticed that equations (4.1.18) do not involve Lorentz boosts. Following Fong and Sucher, we now assume that

$$L_\circ S L_\circ^\dagger = S \tag{4.1.20}$$

where $L_\circ$ is the Lorentz boost operator for the free particles.[23] That is,

$$L_\circ = e^{-ic\underline{K}_\circ\cdot\underline{u}/\hbar} \tag{4.1.21}$$

wherein $\underline{K}_\circ$ is given by equation (3.4.5). The scattering operator is said to be "asymptotically covariant" when equation (4.1.20) holds. In analogy with equation (4.1.16), equation (4.1.20) involves the free Lorentz boost operators $L_\circ$ since the scattering operator is a mapping of asymptotic free orbits. (We note that it follows from the obvious two free particle equation which generalizes the single free particle equation (3.1.30) that, in the momentum representation, equation (4.1.20) is

$$(\epsilon_L(p_{1L})\epsilon_L(p_{2L}))^{1/2}\langle \underline{p}_{1L}, \underline{p}_{2L}|S|\underline{p}'_{1L}, \underline{p}'_{2L}\rangle(\epsilon_L(p'_{1L})\epsilon_L(p'_{2L}))^{1/2} = (\epsilon(p_1)\epsilon(p_2))^{1/2}\langle \underline{p}_1, \underline{p}_2|S|\underline{p}'_1, \underline{p}'_2\rangle(\epsilon(p'_1)\epsilon(p'_2))^{1/2} \tag{4.1.22}$$

which is the familiar equation specifying "the Lorentz invariance of the scattering matrix".)[24] Equation (4.1.20) guarantees that one measures the same values for observables on the average in a scattering experiment when one Lorentz boosts both the preparation and measuring apparatuses. As pointed out by Fong and Sucher,[25] one needs to introduce equation (4.1.20) as an extra assumption at this stage because a scattering experiment involves a concept not incorporated in the discussion of Chapter 2 which led to the commutation relations of the Poincaré group; namely, a scattering experiment involves a comparison between interacting and free systems.

Condition (4.1.20) can be expressed in another way as follows. Rearranging equation (4.1.20),

$$L_\circ S = S L_\circ$$

$$L_0\Omega_-^\dagger\Omega_+ = \Omega_-^\dagger\Omega_+L_0$$

Multiplying by Ω_- from the left and $\Omega_+^\dagger$ from the right,

$$\Omega_-L_0\Omega_-^\dagger(1-\mathcal{B}) = (1-\mathcal{B})\Omega_+L_0\Omega_+^\dagger$$

But, since

$$\Omega_\pm^\dagger\mathcal{B} = \mathcal{B}\Omega_\pm = 0 \tag{4.1.23}$$

we find

$$\Omega_-L_0\Omega_-^\dagger = \Omega_+L_0\Omega_+^\dagger \tag{4.1.24}$$

whence

$$\Omega_-\underline{K}_0\Omega_-^\dagger = \Omega_+\underline{K}_0\Omega_+^\dagger \tag{4.1.25}$$

This condition is equivalent to condition (4.1.20).

Fong and Sucher have shown the Lorentz boost operator

$$L = e^{-ic\underline{K}\cdot\underline{u}/\hbar} \tag{4.1.26}$$

for an interacting system with an asymptotically covariant scattering operator to be uniquely given,[26] in the absence of bound states (in which case $\Omega_\pm$ are unitary), by

$$L = \Omega_-L_0\Omega_-^\dagger = \Omega_+L_0\Omega_+^\dagger \tag{4.1.27}$$

so that

$$\underline{K} = \Omega_-\underline{K}_0\Omega_-^\dagger = \Omega_+\underline{K}_0\Omega_+^\dagger \tag{4.1.28}$$

Thus, from equations (4.1.16), (4.1.17), and (4.1.19), it follows that the generators of the Poincaré group for an interacting system with an asymptotically covariant scattering operator and without bound states are

$$H = \Omega_\pm H_0 \Omega_\pm^\dagger$$

$$\underline{P}_0 = \Omega_\pm \underline{P}_0 \Omega_\pm^\dagger$$

$$\underline{J}_0 = \Omega_\pm \underline{J}_0 \Omega_\pm^\dagger$$

$$\underline{K} = \Omega_\pm \underline{K}_0 \Omega_\pm^\dagger \tag{4.1.29}$$

The set of generators (4.1.29) obeys the commutation relations (2.2.15).

We now consider a two particle system with the Poincaré group generators of the Bakamjian-Thomas form and we show that equations (4.1.29) hold when the system has no bound states. That is, we show that the scattering operator of the system is asymptotically covariant. It follows from equations (3.5.23) and (4.1.9) that we can write

$$H_0 = (P^2c^2 + \hat{H}_0^2)^{1/2} \tag{4.1.30}$$

where

$$\hat{H}_0 = \sum_{\alpha=1}^{2} (\hat{P}_\alpha^2 c^2 + m_\alpha^2 c^4)^{1/2} \tag{4.1.31}$$

Moreover, the Bakamjian-Thomas form for the Hamiltonian is given by equation (3.5.1). Thus,

$$H = (P^2c^2 + \hat{H}^2)^{1/2} \tag{4.1.32}$$

where

$$\hat{H} = \hat{H}_o + \hat{V} \tag{4.1.33}$$

where

$$\hat{V} = \hat{V}(\underset{\sim}{X}, \underline{\hat{P}}) \tag{4.1.34}$$

and

$$[\hat{V}, \underset{\sim}{\tilde{J}}_o] = 0 \tag{4.1.35}$$

We now define operators $\hat{\Omega}_\pm$ by

$$\hat{\Omega}_\pm \equiv \lim_{t\to\mp\infty} e^{i\hat{H}t/\hbar} e^{-i\hat{H}_o t/\hbar} \tag{4.1.36}$$

These operators will exist if $\hat{V}$ is suitably restricted. Moreover, they are unitary if $\hat{H}$ has no bound states and they satisfy the intertwining relation:

$$\hat{H}\hat{\Omega}_\pm = \hat{\Omega}_\pm \hat{H}_o \tag{4.1.37}$$

Furthermore, since $\hat{\Omega}_\pm$ are rotationally invariant functions of internal variables, it follows that

$$[\hat{\Omega}_\pm, \underset{\sim}{X}_o] = [\hat{\Omega}_\pm, \underline{P}] = [\hat{\Omega}_\pm, \underset{\sim}{\tilde{J}}_o] = 0 \tag{4.1.38}$$

Kato has proven the remarkable result ("invariance of the wave operators") that

$$\lim_{t\to\mp\infty} e^{i\phi(\hat{H})t/\hbar} e^{-i\phi(\hat{H}_o)t/\hbar} = \lim_{t\to\mp\infty} e^{i\hat{H}t/\hbar} e^{-i\hat{H}_o t/\hbar} \tag{4.1.39}$$

for a wide class of functions ϕ.[27] In particular, for H_o and H given by equations (4.1.30) and (4.1.32), respectively, it follows that

$$\Omega_{\pm} = \hat{\Omega}_{\pm} \tag{4.1.40}$$

Thus when $\hat{H}$ has no bound states, it follows from equations (3.4.5), (3.5.4), (4.1.12), (4.1.30), (4.1.32), (4.1.37), and (4.1.38) that

$$H = \Omega_{\pm} H_0 \Omega_{\pm}^{\dagger} \qquad \underline{P}_0 = \Omega_{\pm} \underline{P}_0 \Omega_{\pm}^{\dagger}$$

$$\underline{J}_0 = \Omega_{\pm} \underline{J}_0 \Omega_{\pm}^{\dagger} \qquad \underline{K} = \Omega_{\pm} \underline{K}_0 \Omega_{\pm}^{\dagger} \tag{4.1.41}$$

where H, $\underline{K}$, H_0, and $\underline{K}_0$ are given by equations (3.5.1), (3.5.4), (3.4.2), and (3.4.5), respectively. Thus the scattering operator for a two particle system with no bound states and with Poincaré group generators of the Bakamjian-Thomas form is asymptotically covariant.

We wish to emphasize the importance of the role of Kato's theorem and of the result (4.1.40). This result allows one to determine the scattering operator solely by solving the scattering problem for $\hat{H}_0 + \hat{V}$ on $\mathcal{H}_{rel}$. This is exactly analogous to the procedure one follows in the nonrelativistic case. There, however, the $P^2/2m$ term in H_G and H_{0G} cancels out in the expression for the Moller operators and one does not need to appeal to Kato's theorem.

Finally, we note that Fong and Sucher and also Heller, Bohannon, and Tabakin assume the result (4.1.40) without discussion or proof.[28]

Fong and Sucher have generalized the above to include a discussion of bound states.[29]

Footnotes

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Appendix A Proof that Equation (2.1.8) Implies Equation (2.1.9)

Consider a set of apparatuses which prepares a set of orthonormal basis vectors $\{|a_n\rangle\}$. Let $\{|a'_n\rangle\}$ be a set of orthonormal basis vectors prepared by the apparatuses after they have undergone a change in placement. Then,

$$\sum_n |a_n\rangle\langle a_n| = \sum_n |a'_n\rangle\langle a'_n| = 1 \quad .$$

and arbitrary states prepared by the apparatuses in their two placements may be written as

$$|\psi\rangle = \sum_n c_n |a_n\rangle$$

$$|\psi'\rangle = \sum_n c'_n |a'_n\rangle$$

From equation (2.1.8) it follows that

$$\langle\psi|A|\psi\rangle = \sum_{n,m} \langle a_n| c_n^* A c_m |a_m\rangle = \langle\psi'|A'|\psi'\rangle = \sum_{n,m} \langle a'_n| c'^*_n A' c'_m |a'_m\rangle$$

Rewriting,

$$\sum_n |c_n|^2 \langle a_n|A|a_n\rangle + \sum_{\substack{n,m \\ n\neq m}} c_n^* c_m \langle a_n|A|a_m\rangle = \sum_n |c'_n|^2 \langle a'_n|A'|a'_n\rangle + \sum_{\substack{n,m \\ n\neq m}} c'^*_n c'_m \langle a'_n|A'|a'_m\rangle \quad \text{(A.1)}$$

Now equation (2.1.8) must hold for all observables and states so we may choose some particular observable, $\tilde{A}$, say, in order to obtain restrictions on the coefficients c'_n . For instance, let

$$\tilde{A} = |a_\ell\rangle\langle a_\ell|$$

Then it follows on substituting $\tilde{A}$ into equation (A.1) that

$$|c_\ell| = |c'_\ell|$$

We obtain a second condition on the coefficients by choosing a particular state $|\varphi\rangle$ while the observable A is left arbitrary. For instance, let

$$|\varphi\rangle = c_1|a_1\rangle + c_n|a_n\rangle$$

$$|\varphi'\rangle = c_1'|a_1'\rangle + c_n'|a_n'\rangle$$

Since the overall phase of $|\varphi'\rangle$ is arbitrary, we choose $c_1' = c_1$. Then

$$\langle\varphi|A|\varphi\rangle = |c_1|^2\langle a_1|A|a_1\rangle + |c_n|^2\langle a_n|A|a_n\rangle + c_1^* c_n\langle a_1|A|a_n\rangle + c_n^* c_1\langle a_n|A|a_1\rangle$$

$$= \langle\varphi'|A'|\varphi'\rangle = |c_1|^2\langle a_1'|A'|a_1'\rangle + |c_n'|^2\langle a_n'|A'|a_n'\rangle + c_1^* c_n'\langle a_1'|A'|a_n'\rangle + c_n'^* c_1\langle a_n'|A'|a_1'\rangle$$

Thus

$$c_1^* c_n\langle a_1|A|a_n\rangle + c_n^* c_1\langle a_n|A|a_1\rangle = c_1^* c_n'\langle a_1'|A'|a_n'\rangle + c_n'^* c_1\langle a_n'|A'|a_1'\rangle$$

Since A represents an observable, it is Hermitian so

$$c_1^* c_n + c_n^* c_1 = c_1^* c_n' + c_n'^* c_1$$

Multiplying by c_n',

$$c_1^*(c_n')^2 + c_1|c_n'|^2 - c_n'(c_1^* c_n + c_n^* c_1) = 0$$

and solving for c_n',

$$c_n' = c_n \quad \underline{\underline{\text{or}}} \quad (c_1/c_1^*)\, c_n^*$$

Choosing the phase of $|\varphi\rangle$ so that c_1 is real, we have

$$c_n' = c_n \quad \underline{\underline{\text{or}}} \quad c_n^*$$

Now, let $|\varphi_1\rangle$ and $|\varphi_2\rangle$ be two arbitrary states.

$$|\varphi_1\rangle = \sum_n b_n |a_n\rangle \qquad\qquad |\varphi_2\rangle = \sum_m d_m |a_m\rangle$$

so $$|\varphi_1'\rangle = \sum_n b_n' |a_n'\rangle \qquad\qquad |\varphi_2'\rangle = \sum_m d_m' |a_m'\rangle$$

Then

$$\begin{aligned}
|\langle\varphi_1|\varphi_2\rangle|^2 &= \sum_n |b_n|^2 |d_n|^2 + \sum_{\substack{n,m \\ n\neq m}} b_n^* d_n d_m^* b_m \\
&= \sum_n |b_n'|^2 |d_n'|^2 + \sum_{\substack{n,m \\ n\neq m}} b_n'^* d_n' d_m'^* b_m \\
&= |\langle\varphi_1'|\varphi_2'\rangle|^2
\end{aligned}$$

Appendix B Derivation of Equation (2.8.3) for the Generator $\underline{K}$

It is clear that the generator $\underline{K}$ will satisfy the commutation relations (2.2.15f) and (2.2.15g) if it takes the form

$$\underline{K} = -\tfrac{1}{2c^2}(\underline{X}H + H\underline{X}) + a(\underline{\$} \times \underline{P}) \qquad (B.1)$$

where $a = a(H)$ is an operator which may depend on H. (as well as upon the invariants M^2 and $\$^2$.) The operator $a(H)$ is determined by requiring that the commutation relation (2.2.15i) holds. That is, by requiring that

$$\underline{K} \times \underline{K} = -i\hbar\, \underline{J}/c^2 \qquad (B.2)$$

where $\underline{J}$ is given by equation (2.8.1). The quantity $\underline{K} \times \underline{K}$ involves three terms which may be simplified as follows. One can use equations (2.8.2a), (2.8.2b), and (2.8.2c) to write

$$(\underline{X}H + H\underline{X}) \times (\underline{X}H + H\underline{X}) = -4i\hbar c^2 (\underline{X} \times \underline{P}) \qquad (B.3)$$

Also one can write

$$(\underline{X}H + H\underline{X}) \times a(\underline{\$} \times \underline{P}) + a(\underline{\$} \times \underline{P}) \times (\underline{X}H + H\underline{X}) =$$

$$= H\{a[\underline{X} \times (\underline{\$} \times \underline{P}) + (\underline{\$} \times \underline{P}) \times \underline{X}] + [\underline{X}, a] \times (\underline{\$} \times \underline{P})\}$$

$$+ \{a[\underline{X} \times (\underline{\$} \times \underline{P}) + (\underline{\$} \times \underline{P}) \times \underline{X}] + [\underline{X}, a] \times (\underline{\$} \times \underline{P})\} H \qquad (B.4)$$

the right hand side of which may be simplified using

$$\underline{X} \times (\underline{\$} \times \underline{P}) + (\underline{\$} \times \underline{P}) \times \underline{X} = 2i\hbar\, \underline{\$} \qquad (B.5)$$

$$[\underline{X}, a] = [\underline{X}, H]\frac{da}{dH} = i\hbar c^2\, \underline{P} H^{-1}\left(\frac{da}{dH}\right) \qquad (B.6)$$

$$\underline{P} \times (\underline{\$} \times \underline{P}) = -\underline{P}(\underline{P}\cdot\underline{\$}) + P^2\underline{\$} \tag{B.7}$$

to

$$2i\hbar\left\{\left(2aH + P^2c^2\left(\frac{da}{dH}\right)\right)\underline{\$} - c^2(\underline{P}\cdot\underline{\$})\left(\frac{da}{dH}\right)\underline{P}\right\} \tag{B.8}$$

Furthermore, one has

$$(\underline{\$} \times \underline{P}) \times (\underline{\$} \times \underline{P}) = i\hbar(\underline{P}\cdot\underline{\$})\underline{P} \tag{B.9}$$

Thus

$$\underline{K} \times \underline{K} = -\frac{i\hbar}{c^2}\left\{\underline{\mathbb{X}} \times \underline{P} + \left(2aH + P^2c^2\left(\frac{da}{dH}\right)\right)\underline{\$} - c^2(\underline{P}\cdot\underline{\$})\left(\frac{da}{dH} + a^2\right)\underline{P}\right\} \tag{B.10}$$

Hence equation (B.2) is satisfied when

$$P^2c^2\left(\frac{da}{dH}\right) + 2aH = 1 \tag{B.11}$$

and

$$\frac{da}{dH} + a^2 = 0 \tag{B.12}$$

That is, if

$$P^2c^2a^2 - 2aH + 1 = 0 \tag{B.13}$$

Solving this quadratic operator equation for $a(H)$ yields

$$a(H) = (H \pm \beta Mc^2)^{-1} \tag{B.14}$$

where $\beta = EH^{-1}$

and $E \equiv +\sqrt{H^2}$

Choosing the nonsingular solution gives equation (2.8.3).

Finally, it is straightforward to show that the commutation relation (2.2.15h) is satisfied when $\underset{\sim}{J}$ and $\underset{\sim}{K}$ are given by equations (2.8.1) and (2.8.3), respectively.

Appendix C Derivation of Equations (2.8.5) and (2.8.6) for $\underline{X}$ and $\underline{\$}$

We determine our expression for $\underline{X}$ by manipulating equations (2.8.1) and (2.8.3) for $\underline{J}$ and $\underline{K}$. This latter equation may be rewritten using equation (2.8.2c) as

$$\underline{K} = -\frac{1}{c^2} H \underline{X} - \frac{i\hbar}{2} H^{-1} \underline{P} - (H + \not{\beta} M c^2)^{-1} (\underline{P} \times \underline{\$}) \tag{C.1}$$

The quantity ($\underline{P} \times \underline{\$}$) may be eliminated from this equation using

$$\underline{P} \times \underline{J} = \underline{P} \times (\underline{X} \times \underline{P}) + \underline{P} \times \underline{\$} = P^2 \underline{X} - \underline{P}(\underline{P} \cdot \underline{X}) + \underline{P} \times \underline{\$} \tag{C.2}$$

and the quantity $\underline{P} \cdot \underline{X}$ may be eliminated from this equation using

$$\underline{P} \cdot \underline{K} = -\frac{1}{c^2} H \underline{P} \cdot \underline{X} - \frac{i\hbar}{2} H^{-1} P^2 \tag{C.3}$$

Thus solving equation (C.1) for $\underline{X}$ yields

$$\not{\beta} M \underline{X} = -\underline{K} + \frac{c^2}{H(H + \not{\beta} M c^2)} \underline{P}(\underline{P} \cdot \underline{K}) - \frac{1}{H + \not{\beta} M c^2} \underline{P} \times \underline{J} - \frac{i\hbar \not{\beta} M c^2}{2 H^2} \underline{P} \tag{C.4}$$

This latter equation may be written as equation (2.8.5) using

$$\underline{P}(\underline{P} \cdot \underline{K}) = \underline{P} \times (\underline{P} \times \underline{K}) + P^2 \underline{K} \tag{C.5}$$

and $$i\hbar \underline{P} H^{-2} = [\underline{K}, H^{-1}] \tag{C.6}$$

Finally, $\underline{\$}$ may be determined on writing

$$\underline{\$} = \underline{J} - \underline{X} \times \underline{P} \tag{C.7}$$

and using equation (C.4) for $\underline{X}$. Our expression (2.8.6) follows when one uses

$$\underline{P}(\underline{P}\cdot\underline{K})\times\underline{P} = 0 \tag{C.8}$$

$$(\underline{P}\times\underline{J})\times\underline{P} = -(\underline{J}\cdot\underline{P})\underline{P} + \underline{J}\,P^2 \tag{C.9}$$

Appendix D Integral Forms of the Space-time Transformation Operators

The unitary operators corresponding to the Poincaré transformations are determined by the equation

$$\frac{d}{d\tau}\langle \underline{p}|\mathcal{U}_\alpha(\tau)|\underline{p}'\rangle\Big|_{\tau=0} = -\frac{i}{\hbar}\langle \underline{p}|\, C_\alpha\, |\underline{p}'\rangle \tag{D.1}$$

and the condition

$$\mathcal{U}_\alpha(0) = 1 \tag{D.2}$$

We now show that equations (3.1.14), (3.1.15), and (3.1.18) for $D(\underline{a})$, $\mathcal{R}^{(1)}(\alpha)$, and $L^{(1)}(u)$, respectively, satisfy these conditions. In the case of Lorentz boosts we must show that

$$\frac{d}{du}\langle \underline{p}|L^{(1)}(u)|\underline{p}'\rangle\Big|_{u=0} = -\frac{i}{\hbar}c\langle \underline{p}|K^{(1)}|\underline{p}'\rangle \tag{D.3}$$

and $$L^{(1)}(0) = 1 \tag{D.4}$$

When $u=0$, we have

$$\epsilon_L = \epsilon$$

$$|\underline{p}_L\rangle = |\underline{p}\rangle$$

so that equation (3.1.18) becomes

$$L^{(1)}(0) = \int d^3p\, |\underline{p}\rangle\langle\underline{p}| = 1$$

as required. Thus, the second condition is fulfilled. Also,

$$\langle \underline{p}|L^{(1)}(u)|\underline{p}'\rangle = \left(\frac{\epsilon_L}{\epsilon}\right)^{1/2}\delta(\underline{p}-\underline{p}_L')$$

Differentiating,

$$\frac{d}{du}\langle \underset{\sim}{p}|L^{(1)}(u)|\underset{\sim}{p}'\rangle\Big|_{u=0} = \frac{c\,p^{(1)'}}{2\epsilon}\,\delta(\underset{\sim}{p}-\underset{\sim}{p}') - \delta_1^{(1)}(\underset{\sim}{p}-\underset{\sim}{p}')\,\frac{\epsilon}{c}$$

where

$$\delta_1^{(1)}(\underset{\sim}{p}-\underset{\sim}{p}') = \delta(p^{(1)}-p^{(1)'})\,\delta(p^{(2)}-p^{(2)'})\,\delta(p^{(3)}-p^{(3)'})$$

Using

$$[\underset{\sim}{X}, H] = i\hbar c^2 \underset{\sim}{P} H^{-1}$$

and the equations (3.1.2) and (3.1.10), we have

$$\langle \underset{\sim}{p}|X^{(1)}H - HX^{(1)}|\underset{\sim}{p}'\rangle = i\hbar\,\delta(\underset{\sim}{p}-\underset{\sim}{p}')\,\frac{c^2}{\epsilon}\,p^{(1)'}$$

and $\langle \underset{\sim}{p}|X^{(1)}H|\underset{\sim}{p}'\rangle = i\hbar\,\delta_1^{(1)}(\underset{\sim}{p}-\underset{\sim}{p}')\,\epsilon$

so that

$$\frac{d}{du}\langle \underset{\sim}{p}|L^{(1)}(u)|\underset{\sim}{p}'\rangle\Big|_{u=0} = \frac{i}{2\hbar c}\langle \underset{\sim}{p}|X^{(1)}H + HX^{(1)}|\underset{\sim}{p}'\rangle = -\frac{i}{\hbar}\,c\,\langle \underset{\sim}{p}|K^{(1)}|\underset{\sim}{p}'\rangle$$

Therefore, equation (3.1.18) does indeed satisfy the conditions (D.3) and (D.4). Similarly, we have, for the cases of space displacements and rotations,

$$\frac{d}{da}\langle \underset{\sim}{p}|D^{(1)}(a)|\underset{\sim}{p}'\rangle\Big|_{a=0} = -\frac{i}{\hbar}\langle \underset{\sim}{p}|P^{(1)}|\underset{\sim}{p}'\rangle \qquad D^{(1)}(0) = 1$$

$$\frac{d}{d\alpha}\langle \underset{\sim}{p}|\mathcal{R}^{(1)}(\alpha)|\underset{\sim}{p}'\rangle\Big|_{\alpha=0} = -\frac{i}{\hbar}\langle \underset{\sim}{p}|J^{(1)}|\underset{\sim}{p}'\rangle \qquad \mathcal{R}^{(1)}(0) = 1$$

for $D^{(1)}(\alpha)$ and $R^{(1)}(\alpha)$ given by equations (3.1.14) and (3.1.15), respectively.

Equation (3.1.14) can be obtained more directly from the exponential form of the space displacement operator. That is,

$$D(\underline{a}) = e^{-i\underline{P}\cdot\underline{a}/\hbar} = \int d^3p \ |\underline{p}\rangle e^{-i\underline{p}\cdot\underline{a}/\hbar} \langle\underline{p}|$$

$$= \int d^3p \, d^3x \ |\underline{p}\rangle e^{-i\underline{p}\cdot\underline{a}/\hbar} \langle\underline{p}|\underline{x}\rangle\langle\underline{x}|$$

$$= \left(\frac{1}{2\pi\hbar}\right)^{3/2} \int d^3p \, d^3x \ |\underline{p}\rangle e^{-i\underline{p}\cdot(\underline{x}+\underline{a})} \langle\underline{x}|$$

$$= \int d^3x \ |\underline{x}+\underline{a}\rangle\langle\underline{x}|$$

To obtain the integral form of the space inversion operator, we seek a linear operator $\mathcal{P}$ such that

$$\mathcal{P}\underline{X}\mathcal{P}^{\dagger} = -\underline{X}$$

That is,

$$\int d^3x \ |\mathcal{P}\underline{x}\rangle \ \underline{x} \ \langle\mathcal{P}\underline{x}| = -\int d^3x \ |\underline{x}\rangle \ \underline{x} \ \langle\underline{x}|$$

The above equation is satisfied if

$$\mathcal{P}|\underline{x}\rangle = |-\underline{x}\rangle$$

Multiplying each side by $\langle\underline{x}|$ and integrating,

$$\mathcal{P}\int d^3x \ |\underline{x}\rangle\langle\underline{x}| = \mathcal{P} = \int d^3x \ |-\underline{x}\rangle\langle\underline{x}|$$

Similarly, for the time reversal operator, we seek an antilinear operator $\mathcal{T}$ such that

$$\mathcal{T}\underline{X}\mathcal{T}^{\dagger} = \underline{X}$$

That is,

$$\int d^3x\ |\mathcal{T}\underline{x}\rangle\ \underline{x}\ \langle\mathcal{T}\underline{x}| = \int d^3x\ |\underline{x}\rangle\ \underline{x}\ \langle\underline{x}|$$

This holds if

$$\mathcal{T}|\underline{x}\rangle = |\underline{x}\rangle \tag{D.9}$$

We also require

$$\mathcal{T}\,\underline{P}\,\mathcal{T}^{\dagger} = -\underline{P}$$

That is,

$$\int d^3p\ |\mathcal{T}\underline{p}\rangle\ \underline{p}\ \langle\mathcal{T}\underline{p}| = -\int d^3p\ |\underline{p}\rangle\ \underline{p}\ \langle\underline{p}|$$

This holds if

$$\mathcal{T}|\underline{p}\rangle = |-\underline{p}\rangle \tag{D.10}$$

An antilinear form for $\mathcal{T}$ which satisfies equations (D.9) and (D.10) above is

$$\mathcal{T}|\psi\rangle = \int d^3x\ \ |\underline{x}\rangle\langle\psi|\underline{x}\rangle \tag{D.11}$$

In terms of momentum kets we rewrite equation (D.11) as

$$\mathcal{T}|\psi\rangle = \int d^3p\ |-\underline{p}\rangle\langle\psi|\underline{p}\rangle$$

Appendix E Centre of Mass Position $\underset{\sim}{\mathbb{X}}$ and Internal Angular Momentum $\underset{\sim}{\$}$ Operators for a Free Dirac Particle

$\underset{\sim}{\mathbb{X}}$ is given in terms of the generators by equation (2.8.5). Into this expression, we substitute

$$\underset{\sim}{K} = -\frac{1}{2c^2}\left(2H\underset{\sim}{X} + i\hbar c\underset{\sim}{\alpha}\right) \tag{E.1}$$

and equation (3.3.6) for $\underset{\sim}{J}$ to obtain

$$\underset{\sim}{\mathbb{X}} = \underset{\sim}{X} + \frac{i\hbar}{2c^2\beta m}\underset{\sim}{\alpha} - \frac{i\hbar c}{2\beta m H(H+\beta mc^2)}\underset{\sim}{P}(\underset{\sim}{P}\cdot\underset{\sim}{\alpha}) - \frac{1}{\beta m(H+\beta mc^2)}\underset{\sim}{P}\times\underset{\sim}{S} - \frac{i\hbar c^2}{2H^2}\underset{\sim}{P} \tag{E.2}$$

where we have made use of the identity

$$\underset{\sim}{P}\times(\underset{\sim}{X}\times\underset{\sim}{P}) = P^2\underset{\sim}{X} - \underset{\sim}{P}(\underset{\sim}{P}\cdot\underset{\sim}{X}) \tag{E.3}$$

Now, since $\beta = E/H = H/E$

and $(\underset{\sim}{\alpha}\cdot\underset{\sim}{P})(\underset{\sim}{P}\cdot\underset{\sim}{\alpha}) = P^2$ (E.4)

then $\underset{\sim}{\mathbb{X}}$ can be rewritten

$$\underset{\sim}{\mathbb{X}} = \underset{\sim}{X} + \frac{i\hbar c\beta\underset{\sim}{\alpha}}{2E} - \frac{i\hbar c^3\beta}{2E^2(E+mc^2)}\underset{\sim}{P}(\underset{\sim}{P}\cdot\underset{\sim}{\alpha}) + \frac{i\hbar}{2Em}(\underset{\sim}{\alpha}\cdot\underset{\sim}{P})\underset{\sim}{\alpha} - \frac{1}{m(E+mc^2)}\underset{\sim}{P}\times\underset{\sim}{S} - \frac{i\hbar}{2Em}\underset{\sim}{P} \tag{E.5}$$

We now use equations (3.3.7) and (3.3.8) to write $\underset{\sim}{S}$ in terms of $\underset{\sim}{\alpha}$ and use the identities

$$\underset{\sim}{P}\times(\underset{\sim}{\alpha}\times\underset{\sim}{\alpha}) = \underset{\sim}{\alpha}(\underset{\sim}{P}\cdot\underset{\sim}{\alpha}) - (\underset{\sim}{P}\cdot\underset{\sim}{\alpha})\underset{\sim}{\alpha} \tag{E.6}$$

$$\underset{\sim}{P} = \tfrac{1}{2}\left((\underset{\sim}{\alpha}\cdot\underset{\sim}{P})\underset{\sim}{\alpha} + \underset{\sim}{\alpha}(\underset{\sim}{\alpha}\cdot\underset{\sim}{P})\right) \tag{E.7}$$

to write the last three terms in equation (E.5) as

$$\frac{i\hbar(\underline{\alpha}\cdot\underline{P})\underline{\alpha}}{2Em} - \frac{1}{m(E+mc^2)}\underline{P}\times\underline{S} - \frac{i\hbar\underline{P}}{2Em} = \frac{i\hbar c^2(\underline{\alpha}\times\underline{\alpha})\times\underline{P}}{4E(E+mc^2)} = \frac{-c^2\,\underline{S}\times\underline{P}}{E(E+mc^2)} \tag{E.8}$$

We thus obtain equation (3.3.11) for $\underline{X}$.

$\underline{\$}$ is given in terms of the generators by equation (2.8.6). Substituting equations (E.1) and (3.3.6) into equation (2.8.6), we obtain

$$\underline{\$} = \frac{H}{Em}\left(\frac{-1}{2c^2}\right)(2H\underline{X}\times\underline{P} + i\hbar c\,\underline{\alpha}\times\underline{P}) + \frac{E}{mc^2}(\underline{X}\times\underline{P}+\underline{S}) - \frac{1}{m(E+mc^2)}(\underline{S}\cdot\underline{P})\underline{P} \tag{E.9}$$

where we have used the definition of $\underline{\$}$. From equation (3.3.2) for H and the identity

$$(\underline{S}\cdot\underline{P})\underline{P} = P^2\underline{S} - \underline{P}\times(\underline{S}\times\underline{P}) \tag{E.10}$$

we have

$$\underline{\$} = \frac{-i\hbar c\beta}{2E}(\underline{\alpha}\times\underline{P}) + \underline{S} - \frac{i\hbar}{2Em}(\underline{\alpha}\cdot\underline{P})(\underline{\alpha}\times\underline{P}) + \frac{1}{m(E+mc^2)}(\underline{P}\times(\underline{S}\times\underline{P})) \tag{E.11}$$

We now use equations (3.3.7) and (3.3.8) to write $\underline{S}$ in terms of $\underline{\alpha}$ and use equations (E.4), (E.6), and (E.7) to obtain

$$\underline{P}\times(\underline{S}\times\underline{P}) = -\frac{i\hbar}{4}\left((\underline{P}\cdot\underline{\alpha})(\underline{P}\times\underline{\alpha}) - (\underline{P}\times\underline{\alpha})(\underline{P}\cdot\underline{\alpha})\right) \tag{E.12}$$

From equation (E.7), we have

$$(\underline{P}\cdot\underline{\alpha})(\underline{\alpha}\times\underline{P}) = -(\underline{\alpha}\times\underline{P})(\underline{P}\cdot\underline{\alpha}) \tag{E.13}$$

so that

$$\underline{\$} = \frac{-i\hbar c\beta}{2E}(\underline{\alpha}\times\underline{P}) + \underline{S} + \frac{i\hbar}{2Em}(\underline{\alpha}\cdot\underline{P})(\underline{P}\times\underline{\alpha}) - \frac{i\hbar}{4m(E+mc^2)}(2(\underline{\alpha}\cdot\underline{P})(\underline{P}\times\underline{\alpha}))$$

$$= -\frac{i\hbar c\beta}{2E}(\underline{\alpha} \times \underline{P}) + \underline{S} + \frac{i\hbar}{4}\;\frac{2(E+mc^2)(\underline{\alpha}\cdot\underline{P})(\underline{P}\times\underline{\alpha}) - 2E(\underline{\alpha}\cdot\underline{P})(\underline{P}\times\underline{\alpha})}{Em(E+mc^2)}$$

$$= \underline{S} - \frac{i\hbar}{2E}c\beta(\underline{\alpha}\times\underline{P}) - \frac{c^2(\underline{P}\times(\underline{S}\times\underline{P}))}{E(E+mc^2)} \tag{E.14}$$

We thus obtain equation (3.3.12) for $\underline{\$}$.

Appendix F Derivation of Equations (3.3.13) through (3.3.18) for a Free Dirac Particle

The operator $\underset{\sim}{X}$ is given by equation (3.3.11). We now show that it can be written as equation (3.3.13). One has

$$F^{\dagger}\underline{X}F = F^{\dagger}(F\underline{X} - [F,\underline{X}]) = \underline{X} - F^{\dagger}[F,\underline{X}] \tag{F.1}$$

where F is given by equation (3.3.19). Using the commutators

$$[E^{1/2},\underline{X}] = -\frac{i\hbar}{2}c^2\underline{P}E^{-3/2} \tag{F.2}$$

$$[E^{-1/2},\underline{X}] = \frac{i\hbar}{2}c^2\underline{P}E^{-5/2} \tag{F.3a}$$

$$[(E+mc^2)^{-1/2},\underline{X}] = \frac{i\hbar}{2}c^2\underline{P}E^{-1}(E+mc^2)^{-3/2} \tag{F.3b}$$

we obtain

$$[F,\underline{X}] = -i\hbar\left\{\frac{c^2\underline{P}E^{-1}+c\beta\underline{\alpha}}{2E(E+mc^2)^{1/2}} - \frac{Fc^2\underline{P}(E^{-1}(E+mc^2)+1)}{2E(E+mc^2)}\right\} \tag{F.4}$$

Since

$$F^{\dagger} = \frac{E+mc^2-c\beta\underline{\alpha}\cdot\underline{P}}{(2E(E+mc^2))^{1/2}} \tag{F.5}$$

we can write

$$F^{\dagger}[F,\underline{X}] = -i\hbar\left\{\frac{(E+mc^2-c\beta\underline{\alpha}\cdot\underline{P})(c^2\underline{P}E^{-1}+c\beta\underline{\alpha})-c^2\underline{P}(E^{-1}(E+mc^2)+1)}{2E(E+mc^2)}\right\}$$

$$= -i\hbar\left\{\frac{c\beta\underline{\alpha}}{2E} - \frac{c^3\beta(\underline{\alpha}\cdot\underline{P})\underline{P}}{2E^2(E+mc^2)} + \frac{c^2((\underline{\alpha}\cdot\underline{P})\underline{\alpha}-\underline{P})}{2E(E+mc^2)}\right\} \tag{F.6}$$

Using the identity

$$(\underline{\alpha}\cdot\underline{P})\underline{\alpha}-\underline{P}=2i\,\underline{S}\times\underline{P} \tag{F.7}$$

in equation (F.6) and substituting that equation and equation (3.3.11) into equation (F.1), we obtain equation (3.3.13) for $\underline{X}$.

We now derive equation (3.3.14) for $\underline{\$}$ from equation (3.3.12). One has

$$F^{\dagger}\underline{S}F = F^{\dagger}(F\underline{S}-[F,\underline{S}]) = \underline{S}-F^{\dagger}[F,\underline{S}] \tag{F.8}$$

and

$$[F,\underline{S}]=\frac{\beta}{(2E(E+mc^2))^{1/2}}\,(i\hbar c\,\underline{\alpha}\times\underline{P}) \tag{F.9}$$

substitution of which into equation (F.8) yields

$$F^{\dagger}\underline{S}F = \underline{S}-\left(\frac{E+mc^2-c\beta\underline{\alpha}\cdot\underline{P}}{(2E(E+mc^2))^{1/2}}\right)\frac{\beta}{(2E(E+mc^2))^{1/2}}\,(i\hbar c(\underline{\alpha}\times\underline{P}))$$

$$=\underline{S}-\frac{i\hbar c\beta\,\underline{\alpha}\times\underline{P}}{2E}+\frac{i\hbar c^2\,(\underline{\alpha}\cdot\underline{P})(\underline{P}\times\underline{\alpha})}{2E(E+mc^2)} \tag{F.10}$$

On comparison with equation (E.14), we see that equation (3.3.14) follows.

Similarly, since

$$[F,\underline{P}]=[F,\underline{J}]=0 \tag{F.11}$$

we obtain equations (3.3.15) and (3.3.16).

We now derive equation (3.3.17). One has

$$FHF^{\dagger}= F(F^{\dagger}H-[F^{\dagger},H]) = H-F[F^{\dagger},H] \tag{F.12}$$

and

$$[F^{\dagger}, H] = \frac{1}{(2E(E+mc^2))^{1/2}}(2mc^2\underline{\alpha}\cdot\underline{P} - 2\beta c^2 P^2) = \frac{2}{(2E(E+mc^2))^{1/2}}(mc^2H - \beta E^2) \tag{F.13}$$

substitution of which into equation (F.12) yields

$$FHF^{\dagger} = H - \frac{(E+\beta H)}{E(E+mc^2)}(mc^2H - \beta E^2)$$

$$= H + \frac{(-mc^2EH + \beta E^2(E+mc^2) - HE^2)}{E\ (E+mc^2)}$$

$$= \beta E$$

where we have used

$$\beta H = -H\beta + 2mc^2 \tag{F.14}$$

Finally,

$$F^{\dagger}\beta F = H/(F^{\dagger}EF) = H/E = \beta \tag{F.15}$$

Appendix G Derivation of $\hat{\underline{P}}_\alpha$, $\hat{H}_\alpha$, and $\hat{\underline{S}}_\alpha$

We use the method of Section 2.3 to determine $\hat{\underline{P}}_\alpha$. First, we define $A(\lambda)$ by

$$A(\lambda) = \mathcal{G}(\lambda)\, A\, \mathcal{G}^{\dagger}(\lambda) \tag{G.1}$$

where A is some arbitrary operator. It follows that

$$i\hbar \frac{d}{d\lambda} \underline{P}_\alpha(\lambda) = \mathcal{G}(\lambda)\,[B, \underline{P}_\alpha]\, \mathcal{G}^{\dagger}(\lambda)$$

$$= -i\hbar\, \underline{P}(\lambda) \frac{H_\alpha(\lambda)}{H_o(\lambda)}$$

Using equation (3.5.11) for $\underline{P}(\lambda)$, we find

$$\frac{d}{d\lambda} \underline{P}_\alpha(\lambda) = -e^{-\lambda} \frac{H_\alpha(\lambda)\underline{P}}{H_o(\lambda)}$$

Similarly,

$$i\hbar \frac{d}{d\lambda} H_\alpha(\lambda) = \mathcal{G}(\lambda)\,[B, H_\alpha]\, \mathcal{G}^{\dagger}(\lambda)$$

$$= -i\hbar c^2 \frac{\underline{P}(\lambda)\cdot\underline{P}_\alpha(\lambda)}{H_o(\lambda)}$$

which we rewrite as

$$\frac{d}{d\lambda} H_\alpha(\lambda) = -c^2 e^{-\lambda} \frac{\underline{P}_\alpha(\lambda)\cdot\underline{P}}{H_o(\lambda)}$$

Now, let

$$\frac{d\mathcal{E}}{d\lambda} = e^{-\lambda} Pc / H_o(\lambda) \qquad \underline{N} = \underline{P}/P \tag{G.2}$$

Then

$$\frac{d}{d\xi} \underline{P}_\alpha(\lambda) = -\frac{1}{c} H_\alpha(\lambda) \underline{N} \tag{G.3}$$

and

$$\frac{d}{d\xi} H_\alpha(\lambda) = -c\, \underline{P}_\alpha(\lambda) \cdot \underline{N} \tag{G.4}$$

These equations are like those for Lorentz boosts. Thus a solution is

$$\underline{P}_\alpha(\lambda) = \underline{P}_\alpha + (\cosh\xi - 1)(\underline{P}_\alpha \cdot \underline{N})\underline{N} - \frac{1}{c}(\sinh\xi) H_\alpha \underline{N} \tag{G.5}$$

$$H_\alpha(\lambda) = H_\alpha \cosh\xi - c\,(\underline{P}_\alpha \cdot \underline{N}) \sinh\xi \tag{G.6}$$

Since

$$H_0(\lambda) = (P^2c^2 e^{-2\lambda} + M_0^2 c^4)^{1/2}$$

we have

$$d\xi = \frac{-dx}{(M_0^2 c^4 + x^2)^{1/2}} \tag{G.7}$$

where $x = Pc\, e^{-\lambda}$ (G.8)

and also $\xi(0) = 0$

Therefore,

$$\xi(\lambda) = -\sinh^{-1}\left(\frac{Pc\,e^{-\lambda}}{M_0c^2}\right) + \sinh^{-1}\left(\frac{Pc}{M_0c^2}\right) \tag{G.9}$$

Thus

$$\xi(\infty) = \sinh^{-1}(Pc/M_0c^2)$$

so $\sinh \xi(\infty) = Pc/M_o c^2$

and $\cosh \xi(\infty) = H_o/M_o c^2$

Therefore,

$$\hat{H}_\alpha = \lim_{\lambda \to \infty} H_\alpha(\lambda) = (H_\alpha H_o - c^2 \underline{P}_\alpha \cdot \underline{P})/M_o c^2 \tag{G.10}$$

$$\hat{\underline{P}}_\alpha = \lim_{\lambda \to \infty} \underline{P}_\alpha(\lambda) = \underline{P}_\alpha + \frac{1}{P^2}\left[\frac{H_o}{M_o c^2} - 1\right](\underline{P}_\alpha \cdot \underline{P})\underline{P} - \frac{1}{M_o c^2} H_\alpha \underline{P}$$

$$= \underline{P}_\alpha + \left\{\frac{c^2 \underline{P}_\alpha \cdot \underline{P}}{H_o + M_o c^2} - H_\alpha\right\}\frac{\underline{P}}{M_o c^2} \tag{G.11}$$

From equation (G.10),

$$c^2 \underline{P}_\alpha \cdot \underline{P} = H_\alpha H_o - M_o c^2 \hat{H}_\alpha$$

so

$$\hat{\underline{P}}_\alpha = \underline{P}_\alpha - \left\{\frac{H_\alpha + \hat{H}_\alpha}{H_o + M_o c^2}\right\}\underline{P} \tag{G.12}$$

By the method of Section 2.3, we have

$$i\hbar \frac{d}{d\lambda} \underline{S}_\alpha(\lambda) = \mathcal{G}(\lambda)[B, \underline{S}_\alpha]\mathcal{G}^\dagger(\lambda)$$

$$= \frac{1}{(H_\alpha(\lambda) + m_\alpha c^2)}\left[(\underline{V}_o(\lambda) \times \underline{P}_\alpha(\lambda)) \times \underline{S}_\alpha(\lambda)\right]$$

where

$$\underline{V}_o(\lambda) = c^2 \underline{P}(\lambda) H_o^{-1}(\lambda) = c^2 e^{-\lambda} \underline{P} H_o^{-1}(\lambda)$$

Let $\underline{N} = \underline{P}/P$

and recall that

$$\frac{d\xi}{d\lambda} = \frac{e^{-\lambda} P_c}{H_0(\lambda)}$$

Then

$$\frac{d}{d\xi}\underline{S}_\alpha(\lambda) = \frac{c}{(H_\alpha(\lambda)+m_\alpha c^2)}\left[(\underline{N}\times\underline{P}_\alpha(\lambda))\times\underline{S}_\alpha(\lambda)\right]$$

From equation (G.5), we see that

$$\underline{N}\times\underline{P}_\alpha(\lambda) = \underline{N}\times\underline{P}_\alpha$$

Now let

$$\frac{d\eta}{d\xi} = \frac{c\,|\underline{N}\times\underline{P}_\alpha|}{(H_\alpha(\lambda)+m_\alpha c^2)} \qquad \underline{\tilde{N}} = \frac{\underline{N}\times\underline{P}_\alpha}{|\underline{N}\times\underline{P}_\alpha|} \tag{G.13}$$

Then

$$\frac{d}{d\xi}\underline{S}_\alpha(\lambda) = \frac{d\eta}{d\xi}\,\underline{\tilde{N}}\times\underline{S}_\alpha(\lambda)$$

A solution of this equation is

$$\underline{S}_\alpha(\lambda) = (\cos\eta)\underline{S}_\alpha + (1-\cos\eta)(\underline{\tilde{N}}\cdot\underline{S}_\alpha)\underline{\tilde{N}} - (\sin\eta)(\underline{\tilde{N}}\times\underline{S}_\alpha) \tag{G.14}$$

We have

$$d\eta = \frac{d\xi\; c\,|\underline{N}\times\underline{P}_\alpha|}{H_\alpha\cosh\xi - c\,\underline{P}_\alpha\cdot\underline{N}\,\sinh\xi + m_\alpha c^2}$$

This integrates to

$$\frac{\eta}{2} = \tan^{-1}\left(\frac{e^{\xi}\left(H_\alpha - c\left(\underline{P}_\alpha \cdot \underline{N}\right)\right) + m_\alpha c^2}{|\underline{N} \times \underline{P}_\alpha|}\right) - \tan^{-1}\left(\frac{H_\alpha - c\left(\underline{P}_\alpha \cdot \underline{N}\right) + m_\alpha c^2}{|\underline{N} \times \underline{P}_\alpha|}\right) \qquad \text{(G.15)}$$

We leave the calculation at this point as it becomes more complicated than useful.

Appendix H Verification that $\hat{\underline{P}}_\alpha$, $\hat{\underline{X}}_{\alpha\beta}$, and $\hat{\underline{S}}_\alpha$ are Internal Operators and Derivation of the Commutation Relations (3.5.31)

We have shown that if

$$\mathcal{G}[\underline{X}_0, A]\mathcal{G}^\dagger \quad \text{exists} \tag{H.1}$$

then

$$[\underline{X}_0, \hat{A}] = 0 \tag{H.2}$$

To check that equation (H.1) is satisfied for $A = \underline{P}_\alpha, \underline{X}_\alpha - \underline{X}_\beta, \underline{S}_\alpha$ we compute the commutators of $\underline{X}_0$ with these operators.

$$[X_0^{(j)}, P_\alpha^{(k)}] = \frac{i\hbar H_\alpha}{M_0 c^2}\delta_{jk} - \frac{i\hbar}{M_0(H_0+M_0c^2)} P^{(j)}P^{(k)}\frac{H_\alpha}{H_0} - \frac{i\hbar}{M_0(H_0+M_0c^2)}(\underline{P}\cdot\underline{P}_\alpha - P^{(j)}P_\alpha^{(j)})\delta_{jk}$$

$$-\frac{i\hbar}{M_0(H_0+M_0c^2)}\epsilon_{jk\ell}P^{(k)}P_\alpha^{(j)} \tag{H.3}$$

Thus, we have

$$\lim_{\lambda\to\infty}\mathcal{G}(\lambda)[X_0^{(j)}, P_\alpha^{(k)}]\mathcal{G}^\dagger(\lambda) = \mathcal{G}[X_0^{(j)}, P_\alpha^{(k)}]\mathcal{G}^\dagger = i\hbar\frac{\hat{H}_\alpha}{M_0c^2}\delta_{jk} \tag{H.4}$$

We have given $\hat{H}_\alpha$ explicitly (i.e. $\hat{H}_\alpha$ exists.) so $A = \underline{P}_\alpha$ satisfies condition (H.2). Similarly,

$$[X_0^{(j)}, S_\alpha^{(k)}] = \frac{i\hbar}{M_0(H_\alpha+m_\alpha c^2)}(\underline{S}_\alpha\cdot\underline{P}_\alpha - S_\alpha^{(j)}P_\alpha^{(j)})\delta_{jk} + \frac{i\hbar}{M_0(H_0+M_0c^2)}(\underline{S}_\alpha\cdot\underline{P} - S_\alpha^{(j)}P^{(j)})\delta_{jk}$$

$$-\frac{i\hbar}{M_0(H_\alpha+m_\alpha c^2)}\epsilon_{jk\ell}S_\alpha^{(j)}P_\alpha^{(k)} + \frac{i\hbar}{M_0(H_0+M_0c^2)}\epsilon_{jk\ell}S_\alpha^{(j)}P^{(k)} \tag{H.5}$$

Thus, we have

$$\lim_{\lambda\to\infty} \mathcal{U}(\lambda)[\mathcal{X}_0^{(j)}, S_\alpha^{(k)}]\mathcal{U}^\dagger(\lambda) = \mathcal{U}[\mathcal{X}_0^{(j)}, S_\alpha^{(k)}]\mathcal{U}^\dagger = \frac{i\hbar}{M_0(\hat{H}_\alpha + m_\alpha c^2)}\left\{(\hat{\underline{S}}_\alpha\cdot\hat{\underline{P}}_\alpha - \hat{S}_\alpha^{(j)}\hat{P}_\alpha^{(j)})\delta_{jk} - \epsilon_{jk\ell}\hat{S}_\alpha^{(j)}\hat{P}_\alpha^{(k)}\right\} \quad (H.6)$$

We have given $\hat{H}_\alpha$, $\hat{\underline{P}}_\alpha$, and $\hat{\underline{S}}_\alpha$ explicitly ($\hat{\underline{S}}_\alpha$ is given in Appendix G.) so $A = \underline{S}_\alpha$ satisfies condition (H.2).

Finally, we consider $A = \underline{X}_\alpha - \underline{X}_\beta$.

$$[\mathcal{X}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}] = -\frac{c^2}{2}[(H_0^{-1}\underline{K}_0 + \underline{K}_0 H_0^{-1})^{(j)}, (X_\alpha - X_\beta)^{(k)}]$$

+ terms which contain a factor of $P^{(\ell)}$ (and hence vanish under $\mathcal{U}$)

$$= -c^2 H_0^{-1}[K_0^{(j)}, (X_\alpha - X_\beta)^{(k)}] - c^2[H_0^{-1}, (X_\alpha - X_\beta)^{(k)}]K_0^{(j)}$$

+ terms which contain a factor of $P^{(\ell)}$

From equations (2.8.20), (2.8.12), and (3.4.5),

$$[\mathcal{X}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}] = -c^2 H_0^{-1}\left\{\frac{i\hbar}{2c^2}(X_\alpha^{(j)}V_\alpha^{(k)} + V_\alpha^{(k)}X_\alpha^{(j)}) - \frac{i\hbar}{2c^2}(X_\beta^{(j)}V_\beta^{(k)} + V_\beta^{(k)}X_\beta^{(j)})\right\}$$

$$+ i\hbar(c^2 P_\alpha^{(k)}H_\alpha^{-1} - c^2 P_\beta^{(k)}H_\beta^{-1})H_0^{-2}(H_\alpha X_\alpha^{(j)} + H_\beta X_\beta^{(j)})$$

+ terms which contain factors of $P^{(\ell)}$

Since

$$V_\alpha^{(j)} = c^2 P_\alpha^{(j)} H_\alpha^{-1}$$

we find

$$[\mathcal{X}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}] = -i\hbar\left\{c^2\frac{P_\alpha^{(k)}H_\beta}{H_0^2 H_\alpha}(X_\alpha - X_\beta)^{(j)}\right\} - i\hbar\left\{c^2\frac{P_\beta^{(k)}H_\alpha}{H_0^2 H_\beta}(X_\alpha - X_\beta)^{(j)}\right\}$$

+ terms which contain a factor of $P^{(\ell)}$ (H.7)

Thus,

$$\lim_{\lambda\to\infty} \mathcal{U}(\lambda)[\mathbb{X}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}]\,\mathcal{U}^\dagger(\lambda) = \mathcal{U}[\mathbb{X}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}]\,\mathcal{U}^\dagger = \frac{-i\hbar c^2}{M_0^2 c^4}\left\{\hat{P}_\alpha^{(k)}\frac{\hat{H}_\beta}{\hat{H}_\alpha} + \hat{P}_\beta^{(k)}\frac{\hat{H}_\alpha}{\hat{H}_\beta}\right\}\hat{X}_{\alpha\beta}^{(j)} \tag{H.8}$$

We assume that $\hat{\underline{X}}_{\alpha\beta}$ exists so $A = \underline{X}_\alpha - \underline{X}_\beta$ satisfies equation (H.2).

Therefore,

$$[\mathbb{X}_0^{(j)}, \hat{P}_\alpha^{(k)}] = [\mathbb{X}_0^{(j)}, \hat{S}_\alpha^{(k)}] = [\mathbb{X}_0^{(j)}, \hat{X}_{\alpha\beta}^{(k)}] = 0 \tag{H.9}$$

We now compute the commutators of $\tilde{\underline{J}}_0$ with the operators $\underline{P}_\alpha$, $\underline{S}_\alpha$, and $(\underline{X}_\alpha - \underline{X}_\beta)$.

$$[\tilde{J}_0^{(j)}, P_\alpha^{(k)}] = \frac{-i\hbar H_\alpha}{M_0 c^2}\epsilon_{jk\ell}P^{(\ell)} + \frac{i\hbar H_0}{M_0 c^2}\epsilon_{jk\ell}P_\alpha^{(\ell)} - \frac{i\hbar}{M_0(H_0 + M_0 c^2)}(\underline{P}_\alpha \times \underline{P})^{(k)}P^{(j)} \tag{H.10}$$

Therefore,

$$\lim_{\lambda\to\infty} \mathcal{U}(\lambda)[\tilde{J}_0^{(j)}, P_\alpha^{(k)}]\,\mathcal{U}^\dagger(\lambda) = [\tilde{J}_0^{(j)}, \hat{P}_\alpha^{(k)}] = i\hbar\,\epsilon_{jk\ell}\hat{P}_\alpha^{(\ell)} \tag{H.11}$$

Similarly,

$$[\tilde{J}_0^{(j)}, S_\alpha^{(k)}] = \frac{i\hbar H}{M_0 c^2}\epsilon_{jk\ell}S_\alpha^{(\ell)} - \frac{i\hbar}{M_0(H_0 + M_0 c^2)}(\underline{S}_\alpha \times \underline{P})^{(k)}P^{(j)} + \frac{i\hbar}{M_0(H_\alpha + m_\alpha c^2)}(\underline{S}_\alpha \times \underline{P})^{(j)}P_\alpha^{(k)}$$

$$-\frac{i\hbar}{M_0(H_\alpha + m_\alpha c^2)}\epsilon_{jk\ell}\,\underline{S}_\alpha\cdot\underline{P}_\alpha\,P^{(\ell)} \tag{H.12}$$

Therefore,

$$\lim_{\lambda\to\infty} \mathcal{U}(\lambda)[\tilde{J}_0^{(j)}, S_\alpha^{(k)}]\,\mathcal{U}^\dagger(\lambda) = [\tilde{J}_0^{(j)}, \hat{S}_\alpha^{(k)}] = i\hbar\,\epsilon_{jk\ell}\hat{S}_\alpha^{(\ell)} \tag{H.13}$$

Finally,

$$[\tilde{J}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}] = J^{(j)}\left[\frac{H_0}{M_0c^2}, (X_\alpha - X_\beta)^{(k)}\right] + [J^{(j)}, (X_\alpha - X_\beta)^{(k)}]\frac{H_0}{M_0c^2}$$

$$+ \text{ terms which contain a factor of } P^{(\ell)}$$

We simplify the first term. From

$$[M_0^2c^2, \underline{X}_\alpha - \underline{X}_\beta] = \frac{1}{c^2}[H_0^2 - P^2c^2, \underline{X}_\alpha - \underline{X}_\beta] = -\frac{2i\hbar}{c^2}(\underline{V}_\alpha - \underline{V}_\beta)H_0$$

we obtain

$$[(M_0c^2)^{-1}, \underline{X}_\alpha - \underline{X}_\beta] = \frac{i\hbar}{M_0^3c^6}(\underline{V}_\alpha - \underline{V}_\beta)H_0$$

so that

$$[H_0(M_0c^2)^{-1}, (X_\alpha - X_\beta)^{(k)}] = -\frac{i\hbar}{M_0c^2}(V_\alpha - V_\beta)^{(k)} + \frac{i\hbar}{M_0^3c^6}(V_\alpha - V_\beta)^{(k)}H_0^2$$

$$= \frac{i\hbar}{M_0c^2}\left\{(V_\beta - V_\alpha)^{(k)} + \frac{(V_\alpha - V_\beta)^{(k)}(P^2c^2 + M_0^2c^4)}{M_0^2c^4}\right\}$$

$$= \frac{i\hbar}{M_0}(V_\alpha - V_\beta)^{(k)}P^2$$

Since this term contains a factor of P^2, it vanishes under $\mathcal{G}$. Therefore,

$$[\tilde{J}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}] = i\hbar\epsilon_{jk\ell}(X_\alpha - X_\beta)^{(\ell)}H_0(M_0c^2)^{-1}$$

$$+ \text{ terms containing factors of } P^{(\ell)} \qquad \text{(H.14)}$$

so

$$\lim_{\lambda\to\infty}\mathcal{G}(\lambda)[\tilde{J}_0^{(j)}, (X_\alpha - X_\beta)^{(k)}]\mathcal{G}^\dagger(\lambda) = [\tilde{J}_0^{(j)}, \hat{X}_{\alpha\beta}^{(k)}] = i\hbar\epsilon_{jk\ell}\hat{X}_{\alpha\beta}^{(\ell)} \qquad \text{(H.15)}$$

The commutation relations (3.5.31) follow at once from equations (H.11), (H.13), and (H.15).

Appendix I   The Galilean Group

The Galilean group consists of the transformations (2.1.2), (2.1.3), (2.1.4), and the Galilean boosts:

$$(\underline{x},t) \longrightarrow (\underline{x}',t') = (\underline{x} + \underline{v}t, t) \tag{I.1}$$

The unitary operator corresponding to a Galilean boost along the j-axis of speed v will be written as

$$G^{(j)}(v) = e^{-iK_G^{(j)}v/\hbar} \tag{I.2}$$

where $K_G^{(j)}$ is the generator of the boost. Using the methods of Section 2.2, one finds that the ten generators of the Galilean group obey the following set of commutation relations.

$$[P_G^{(j)}, P_G^{(k)}] = 0 \qquad [P_G^{(j)}, H_G] = 0$$

$$[J_G^{(j)}, P_G^{(k)}] = i\hbar\, \epsilon_{jk\ell} P_G^{(\ell)} \qquad [J_G^{(j)}, H_G] = 0 \qquad [J_G^{(j)}, J_G^{(k)}] = i\hbar\, \epsilon_{jk\ell} J_G^{(\ell)}$$

$$[K_G^{(j)}, P_G^{(k)}] = -i\hbar\, m_G\, \delta_{jk} \qquad [K_G^{(j)}, H_G] = -i\hbar\, P_G^{(j)} \qquad [K_G^{(j)}, J_G^{(k)}] = i\hbar\, \epsilon_{jk\ell} K_G^{(\ell)} \qquad [K_G^{(j)}, K_G^{(k)}] = 0 \qquad j,k,\ell = 1,2,3 \tag{I.3}$$

where m_G is a parameter characterizing the system (which cannot be eliminated using the Jacobi identity as in Section 2.2) and where we have appended a subscript G to the generators of the transformations (2.1.2), (2.1.3), and (2.1.4) as a reminder that we now consider the Galilean group. The only Galilean group commutation relations which differ in form from those for the Poincaré group are, not surprisingly, the first and last ones in the last line of equations (I.3). (In the Poincaré case, these commutators contain factors of $1/c^2$.)

We now introduce the centre of mass position operator $\underline{X}_G$ and the total internal angular momentum $\tilde{\underline{J}}_G$ by requiring that

$$\underline{J}_G = \underline{X}_G \times \underline{P}_G + \tilde{\underline{J}}_G \tag{I.4}$$

Then the commutation relations (I.3) follow when

$$\underline{K}_G = -m_G \underline{X}_G \tag{I.5}$$

providing that

$$[X_G^{(j)}, X_G^{(k)}] = 0 \qquad [X_G^{(j)}, P_G^{(k)}] = i\hbar\delta_{jk} \qquad [X_G^{(j)}, H_G] = i\hbar P_G^{(j)}/m_G$$

$$[\tilde{J}_G^{(j)}, \tilde{J}_G^{(k)}] = i\hbar\epsilon_{jk\ell}\tilde{J}_G^{(\ell)} \qquad [\tilde{J}_G^{(j)}, P_G^{(k)}] = [\tilde{J}_G^{(j)}, H_G] = [\tilde{J}_G^{(j)}, X_G^{(k)}] = 0 \qquad j,k,\ell = 1,2,3 \tag{I.6}$$

Equations (I.4) and (I.5) may be solved immediately for $\underline{X}_G$ and $\tilde{\underline{J}}_G$ to yield

$$\underline{X}_G = -\frac{1}{m_G}\underline{K}_G \tag{I.7}$$

$$\tilde{\underline{J}}_G = \underline{J}_G + \frac{1}{m_G}\underline{K}_G \times \underline{P}_G \tag{I.8}$$

which equations are the definitions of $\underline{X}_G$ and $\tilde{\underline{J}}_G$ in terms of the generators of the Galilean group. It also follows that

$$[J_G^{(j)}, X_G^{(k)}] = i\hbar\epsilon_{jk\ell}X_G^{(\ell)}$$

$$[K_G^{(j)}, X_G^{(k)}] = 0$$

$$[J_G^{(j)}, \tilde{J}_G^{(k)}] = i\hbar\epsilon_{jk\ell}\tilde{J}_G^{(\ell)}$$

$$[K_G^{(j)}, \tilde{J}_G^{(k)}] = 0 \tag{I.9}$$

A system is said to be Galilean invariant if one can construct operators satisfying equations (I.3) in terms of the fundamental dynamical variables of the system. For a system of n particles of masses $m_1, \ldots, m_n$ and spins $s_1, \ldots, s_n$ with fundamental dynamical variables the Cartesian coordinates, $\underline{X}_1, \ldots, \underline{X}_n$, momenta, $\underline{P}_1, \ldots, \underline{P}_n$, and spins, $\underline{S}_1, \ldots, \underline{S}_n$, satisfying fundamental commutation relations (3.4.1), the commutation relations (I.3) are satisfied when

$$\underline{P}_G = \sum_{\alpha=1}^{n} \underline{P}_\alpha \tag{I.10}$$

$$\underline{J}_G = \sum_{\alpha=1}^{n} (\underline{X}_\alpha \times \underline{P}_\alpha + \underline{S}_\alpha) \tag{I.11}$$

$$\underline{K}_G = -\sum_{\alpha=1}^{n} m_\alpha \underline{X}_\alpha \tag{I.12}$$

$$H_G = \sum_{\alpha=1}^{n} P_\alpha^2/2m_\alpha + V_G \tag{I.13}$$

provided the operator V_G satisfies

$$[\underline{P}_G, V_G] = [\underline{X}_G, V_G] = [\underline{\tilde{J}}_G, V_G] = 0 \tag{I.14}$$

The operator V_G is called the potential of the system and specifies the interactions in the system. Also, from

$$[K_G^{(j)}, P_G^{(k)}] = -i\hbar m_G \delta_{jk}$$

we have that

$$m_G = \sum_{\alpha=1}^{n} m_\alpha \tag{I.15}$$

The centre of mass position $\underline{X}_G$ and the total internal angular momentum $\underline{\tilde{J}}_G$ can be written explicitly in terms of the fundamental dynamical variables of the system. In particular, it follows from equations (I.7) and (I.15) that

$$\underset{\sim}{X}_G = \frac{\sum_{\alpha=1}^{n} m_\alpha \underset{\sim}{X}_\alpha}{\sum_\alpha m_\alpha} \tag{I.16}$$

To complete the problem, one must find rotationally invariant "internal variables" for the system, operators commuting with $\underset{\sim}{P}_G$, $\underset{\sim}{X}_G$, and $\tilde{\underset{\sim}{J}}_G$. This can be done at length using the method of Section 3.5. Thus, one defines the unitary operator

$$\mathcal{G}_G(\lambda) \equiv e^{-i B_G \lambda/\hbar} \tag{I.17}$$

where

$$B_G = -\tfrac{1}{2}\left(\underset{\sim}{X}_G \cdot \underset{\sim}{P}_G + \underset{\sim}{P}_G \cdot \underset{\sim}{X}_G\right) \tag{I.18}$$

(This is precisely the Gartenhaus-Schwartz transformation.)[30] It follows that

$$\mathcal{G}_G(\lambda)\, \underset{\sim}{P}_G\, \mathcal{G}_G^{+}(\lambda) = e^{-\lambda}\, \underset{\sim}{P}_G \tag{I.19}$$

$$\mathcal{G}_G(\lambda)\, \underset{\sim}{X}_G\, \mathcal{G}_G^{+}(\lambda) = e^{\lambda}\, \underset{\sim}{X}_G \tag{I.20}$$

so that if $[\underset{\sim}{P}_G, A] = 0$ (I.21)

and if $\lim_{\lambda\to\infty} \mathcal{G}_G(\lambda)[\underset{\sim}{X}_G, A]\,\mathcal{G}_G^{+}(\lambda)$ exists (I.22)

then $[\underset{\sim}{P}_G, \hat{\hat{A}}] = [\underset{\sim}{X}_G, \hat{\hat{A}}]$ (I.23)

where

$$\hat{\hat{A}} \equiv \lim_{\lambda\to\infty} \mathcal{G}_G(\lambda)\, A\, \mathcal{G}_G^{+}(\lambda) \tag{I.24}$$

Conditions (I.21) and (I.22) are satisfied by the operators $\underline{P}_\alpha$, $\underline{X}'_\alpha \equiv \underline{X}_\alpha - \underline{X}_G$, $\underline{X}_{\alpha\beta} \equiv \underline{X}_\alpha - \underline{X}_\beta$, $\underline{S}_\alpha$. Proceeding as in Section 3.5, one finds that

$$\hat{\hat{\underline{P}}}_\alpha = \underline{P}_\alpha - \frac{m_\alpha}{m_G} \underline{P}_G \tag{I.25}$$

$$\hat{\hat{\underline{X}}}'_\alpha = \underline{X}_\alpha - \underline{X}_G = \underline{X}'_\alpha \tag{I.26}$$

$$\hat{\underline{X}}_{\alpha\beta} = \underline{X}_\alpha - \underline{X}_\beta = \underline{X}_{\alpha\beta} \tag{I.27}$$

$$\hat{\hat{\underline{S}}}_\alpha = \underline{S}_\alpha \tag{I.28}$$

The above operators do indeed commute with $\underline{P}_G$ and $\underline{X}_G$. Moreover,

$$[\tilde{J}_G^{(j)}, \hat{\hat{P}}_\alpha^{(k)}] = i\hbar\, \epsilon_{jk\ell}\, \hat{\hat{P}}_\alpha^{(\ell)}$$

$$[\tilde{J}_G^{(j)}, \hat{\hat{X}}_\alpha^{\prime(k)}] = i\hbar\, \epsilon_{jk\ell}\, \hat{\hat{X}}_\alpha^{\prime(\ell)}$$

$$[\tilde{J}_G^{(j)}, \hat{\hat{X}}_{\alpha\beta}^{(k)}] = i\hbar\, \epsilon_{jk\ell}\, \hat{\hat{X}}_{\alpha\beta}^{(\ell)}$$

$$[\tilde{J}_G^{(j)}, \hat{\hat{S}}_\alpha^{(k)}] = i\hbar\, \epsilon_{jk\ell}\, \hat{\hat{S}}_\alpha^{(\ell)} \tag{I.29}$$

Rotationally invariant internal variables (operators commuting also with $\tilde{J}_G$) may thus be found by taking scalar products of $\hat{\hat{P}}_\alpha$, $\hat{\hat{X}}'_\alpha$, $\hat{\hat{X}}_{\alpha\beta}$, and $\hat{\hat{\underline{S}}}_\alpha$. Finally, the internal angular momentum $\tilde{J}_G$ may be written as

$$\tilde{\underline{J}}_G = \sum_{\alpha=1}^{n} (\hat{\hat{\underline{X}}}'_\alpha \times \hat{\hat{\underline{P}}}_\alpha + \hat{\hat{\underline{S}}}_\alpha) \tag{I.30}$$

The Galilean case is clearly simpler than the Poincaré case. This is due to the fact that the Galilean boost (I.1) does not involve a change in the time coordinate whereas the Lorentz boost (2.1.5) does. This leads to less coupling in the commutation relations (I.3) for the generators of the

Galilean group than in the commutation relations (2.2.15) for the Poincaré generators. The Galilean generators (I.10)-(I.13) for the particle system are correspondingly less complicated than the Poincaré generators (3.5.1)-(3.5.4), as are the centre of mass position, internal angular momentum, and the internal variables. In both cases, the interactions in the system are specified by a single operator. In the Galilean case only the Hamiltonian is changed by the interaction, whereas, in the Poincaré case, both the Hamiltonian and the generator of Lorentz boosts are altered by an interaction.

One can also arrive at the above particle results for the Galilean case by taking the limit $c\to\infty$ of the appropriate equations for the Poincaré case. Thus, one has

$$\lim_{c\to\infty} M_0 = m_G \tag{I.31}$$

$$\lim_{c\to\infty} \underline{K}_0 = \underline{K}_G \tag{I.32}$$

$$\lim_{c\to\infty} \underline{X}_0 = \underline{X}_G \tag{I.33}$$

$$\lim_{c\to\infty} \hat{\underline{P}}_\alpha = \hat{\hat{\underline{P}}}_\alpha \tag{I.34}$$

$$\lim_{c\to\infty} \underline{K} = \underline{K}_G \tag{I.35}$$

if $\lim\limits_{c\to\infty} M$ exists

Moreover, if

$$\lim_{c\to\infty} V = V_G \tag{I.36}$$

then

$$\lim_{c \to \infty} H = H_G \qquad (I.37)$$

where we have omitted the (infinite) term $m_G c^2$ on the right hand side of the above equation.