A DIFFUSION MODEL FOR A TWO PRODUCT INVENTORY SYSTEM

by

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ABSTRACT

This thesis presents the results of an investigation of a continuous-time two product inventory model in which the stock level of two divisible commodities is represented by a two dimensional diffusion process. Two classes of replenishment policies are considered. One is a two dimensional analog of the stationary one dimensional (s,S) policy; i.e., when either the inventory of product one declines to s_1 or when the inventory of product two declines to s_2, both stocks are instantaneously replenished, product one up to S_1, and product two up to S_2. This is referred to as the (s_1,s_2,S_1,S_2) policy. The inventory is then allowed to decline again and is replenished. These cycles continue indefinitely. There are costs associated with the replenishment of stock and maintaining a given inventory. The objective is to choose values for (s_1,s_2,S_1,S_2) to minimize the long-run average cost of operating such a system. The appropriate theory of diffusion processes is heuristically developed and then applied to evaluate this cost. In general, analytic solutions cannot be obtained. Classical numerical analysis methods are used to obtain the average costs for given (s_1,s_2,S_1,S_2) values and to select the best such values. One dimensional diffusion models are a special case of the present model and Puterman's [21] results are used to verify the results obtained. The other policy examined differs from the two dimensional (s_1,s_2,S_1,S_2) policy in that the lower levels, s_1 and s_2, of the stock levels are coupled in the form of an elliptic arc. Numerical solution of this policy can be obtained and comparisons of the two policies are made.

TABLE OF CONTENTS

1. Introduction And Summary 1

2. Literature Survey 3

3. Diffusion Processes 5

4. Model Description And Renewal Structure 9

5. Evaluation Of The Long-run Average Cost 14

6. Methods Of Solution 21

7. The (s_1, s_2, S_1, S_2) Policy 25

8. The Modified Policy 39

9. Recommendations For Future Research 44

References 46

Appendix 48

LIST OF TABLES

Table 1. Search Of Global Optimal (s_1, s_2, S_1, S_2) Policy ... 33

LIST OF FIGURES

Fig. 1. A Control Region Of The (s_1, s_2, S_1, S_2) Policy 13

Fig. 2. A Control Region Of The Modified Policy 13

Fig. 3. Sample Output Of (s_1, s_2, S_1, S_2) Policy Code 34

Fig. 4. Convexity Of $\hat{\theta}(\Delta_1, \Delta_2)$ With Respect To (s_1, s_2) 35

Fig. 5. Effects Of Correlation On $\hat{\theta}(\Delta_1, \Delta_2)$ 36

Fig. 6. The Phenomenon Of Splitting 37

Fig. 7. Effects Of Unequal Variance 38

Fig. 8. Sample Output Of Modified Policy Code 41

Fig. 9. The Convexity Of θ With Respect To (x_γ, y_γ) 42

Fig. 10. Effect Of Correlation On The Modified Policy 43

Fig. 11. The Nine-point Star Approximation 48

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1. Introduction And Summary

In this thesis a continuous-time stochastic model of a storage system for two divisible commodities is studied. The stock level of the two commodities is represented by a two dimensional diffusion process (Markov process with continuous sample path), with a negative drift vector. The continuous sample paths naturally suggests facilities storing liquids such as gas, oil, petroleum products, but the model may also be used to approximate non-divisible quantities such as automobiles, televisions, books, or pens, especially if the numbers are large.

The stock level of the system is observed continuously through time and two stationary operating policies are considered. One is the two dimensional analog, (s_1, s_2, S_1, S_2), of the one dimensional (s,S) policy; i.e., when the stock level of commodity i falls to s_i, both commodities are instantaneously replenished to the level (S_1, S_2). The process is then allowed to drift again with the switching repeated indefinitely. There are costs associated with re-ordering as well as maintaining a given inventory. The other policy examined differs from the (s_1, s_2, S_1, S_2) policy in that the lower levels s_1 and s_2 of the re-order curve (In two dimensions it is refered to as the re-order curve rather than the re-order point) are coupled in the form of an elliptic arc. This is refered to as the modified policy. Our objective is to characterize the long-run average cost, determine the values of (s_1, s_2, S_1, S_2) that minimize this cost, examine the properties of the solution and note the difference between the two classes of operating policies.

This thesis is organized as follows. In Section 2 the related literature is surveyed. In Section 3 relevant results from the theory of multi-dimensional diffusion processes are reviewed. In Section 4 the model is formulated and the underlying renewal structure described. The appropriate theory of diffusion processes is heuristically developed in Section 5 to obtain partial differential equations which characterize the expected values of the random functions used in the evaluation of the long-run average cost. Sufficient conditions for these expected values to be finite are also given. Section 6 discusses the methods available to solve these partial differential equations and shows that in general solutions cannot be obtained in closed form. In Section 7 the properties of the (s_1, s_2, S_1, S_2) policy is studied. Classical numerical analysis methods are used to obtain the average cost for given (s_1, s_2, S_1, S_2) values and to select the best values. Puterman's [21] results for the one-dimensional problem are used to verify the results obtained. Qualitative features of the numerical solution are also presented. In Section 8 the modified policy is examined and comparisons between the two policies are made. Some possible future research areas are discussed in Section 9.

2. Literature Survey

In this thesis we study generalizations of the one-dimensional (s,S) inventory model. Much of the relavent literature on (s,S) policies are surveyed extensively in [21].

The diffusion process model for inventory systems was introduced by Bather [3] and further studied by Gimon [14] and Puterman [21], similiar results for a related model have been obtained by Constantinides and Richard [6].

The literature of multi-product inventory systems is more scarce and less extensive than the one product literature. Some of the recent works are that of Goswick and Sivazlian [15], and Ignall [17].

Ignall [17] investigated a two product continuously reviewed inventory system where demand of the two goods come from two independent Poisson processes. The cost structure consists of shortage and holding costs and a variable cost of reordering. A (s,c,S) policy was considered; if the stock of any product drops to its re-order point, s, all the products with stock less than its can-order point, c, are re-ordered back to its order-to point, S. Ignall showed that the (s,c,S) policy is not always optimal, in certain cases the optimal policy has the characteristic that the quantity ordered of the product that triggered the re-order depends on the inventory of the other product.

Goswick and Sivazlian [15] also examined a model with variable re-order cost. The model studied is a two-product periodic review model with uniform demand distribution. The steady state behaviour of the system was analyzed and comparisons between mixed re-ordering, individual re-ordering and joint re-ordering are made. It was shown that each of the three policy was optimal in some situations and not in others.

The two policies examined in this thesis belong to the class of joint replenishment policies, i.e., when an order is placed, both products are replenished. This thesis can be considered as an extension of Puterman's [21] results to higher dimensions.

3. Diffusion Processes

Diffusion processes are special cases of strong Markov processes with almost certainly continuous sample paths. The simplest example is the motion of very small particles suspended in a fluid, the so-called Brownian motion. The study of systems with white noise and continuous models for random-walk problems also lead to diffusion processes. Much of the material presented below are treated in great detail in Arnold[2] and Dynkin[9].

Let I denote an nonempty index set and $(\Omega, \mathfrak{U}, P)$ a probability space. A family $\{ D_t ; t \in I \}$ of $\mathbb{R}^n$ valued random variables is a stochastic process with index set I and state space $\mathbb{R}^n$. Let I be an interval of the extended real line and $\{ D_t ; t \in I \}$ a stochastic process, then $D_t(\bullet)$ is, for every fixed $t \in I$, an $\mathbb{R}^n$ valued random variable whereas, for every fixed ω, $D_{\bullet}(\omega)$ is an $\mathbb{R}^n$ valued function defined on I. It is called a sample path of the stochastic process.

Let B be an open connected differential manifold in $\mathbb{R}^n$ with boundary ∂B and $\underline{x} = (x_1, \ldots, x_n)$ a n-vector in B. To uniquely define a diffusion process we must specify its behaviour on B and ∂B. First we discuss B. A n-dimensional Markov process with probability transition function $p(s,\underline{x},t,A)$ ($p(s,\underline{x},t,A)$ is the conditional probability of $D_t \in A$ given that $D_s = \underline{x}$, i.e., $p(s,\underline{x},t,A) = p(D_t \in A \mid D_s = \underline{x}))$ on B is a diffusion if it satisfies the following conditions;

1) For any $\epsilon > 0$, $t \geq 0$, $\underline{x}$ B,

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{|\underline{y} - \underline{x}| > \epsilon} P(t, \underline{x}, t+\delta, d\underline{y}) = 0$$

2) There exists $\mu(\underline{x},t)$ and $\Sigma(\underline{x},t)$ such that for any $\epsilon>0$, $t\geq 0$ $\underline{x}\in B$,

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \int_{|\underline{y}-\underline{x}|<\epsilon} (y_j - x_j) P(t,\underline{x},t+\delta,d\underline{y}) = \mu_j(\underline{x},t) \qquad j=1,\cdots,n.$$

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} \int_{|\underline{y}-\underline{x}|<\epsilon} (y_i - x_i)(y_j - x_j) P(t,\underline{x},t+\delta,d\underline{y}) = \sigma_{ij}(\underline{x},t) \qquad i,j=1,\cdots,n.$$

the n-vector $\mu(\underline{x},t)$ with components $\mu_i(\underline{x},t)$ the <u>drift coefficients</u>, is the <u>drift vector</u>. The nxn matrix $\Sigma(\underline{x},t)$ of elements $\sigma_{ij}(\underline{x},t)$ the <u>diffusion coefficients</u>, is the <u>diffusion matrix</u>. The <u>correlation</u> between the i-th and the j-th component is defined by

$$\rho_{ij}(\underline{x},t) = \frac{\sigma_{ij}(\underline{x},t)}{\sqrt{\sigma_{ii}(\underline{x},t) \quad \sigma_{jj}(\underline{x},t)}}$$

Corresponding to each diffusion, there exists an <u>infinitesimal operator</u> on the class of twice continuously differentiable functions of the form

$$A(\underline{x},t) = \sum_{i,j=1}^{n} \frac{\sigma_{ij}(\underline{x},t)}{2} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i(\underline{x},t) \frac{\partial}{\partial x_i} \tag{3}$$

The infinitesimal operator has the following interpretation. Let f(•) be a function defined on the sample paths of the diffusion. Then Af(•) can be thought of as the expected time rate of change of f(•). The drift coefficients and the diffusion coefficients have the following interpretation. If $D_{\Delta} = \underline{x}$, then for small $\delta>0$, $D_{\Delta+\delta} - \underline{x}$ is distributed according to the n-dimensional normal distribution $N(\mu(\underline{x},t)\delta, \Sigma(\underline{x},t)\delta)$. Conditions (4)-(5) are imposed on the drift and diffusion coefficients to ensure the existence of a diffusion corresponding to (3)[9,p.162].([13,p.32] give analogous existence conditions which are more restrictive).

(4) $\mu_i(\underline{x},t)$, $(\underline{x},t)$ $(i,j=1,\ldots,n)$ are bounded and satisfy a Hölder condition on B, $t\geq 0$.

(5) there exists a positive constant γ such that, for all $\underline{x} \in B$, $t\geq 0$ and all n-tuples $(\lambda_1,\ldots,\lambda_n)$ of real numbers,

$$\sum_{i,j=1}^{n} \sigma_{ij}(\underline{x},t)\lambda_i\lambda_j \geq \gamma \sum_{i=1}^{n} \lambda_i^2 .$$

If an infinitesimal operator, A, satisfies conditions (5) at $\underline{x}$,t, then it is a <u>non-degenerate</u> operator at $\underline{x}$,t. Otherwise it is a <u>degenerate</u> operator. For instance condition (5) is violated if $\rho_{ij}(\underline{x},t) = \pm 1$. If A is degenerate for all $\underline{x}$ in B and $t\in I$, then some component(s) of the diffusion process are deterministic functions of the other components. Thus the diffusion process can be represented by another diffusion process of lower dimension. For example, if D_t is a degenerate diffusion process in $\mathbb{R}^2$, then knowledge of one component of D_t completely specifies the other component. Thus D_t, a diffusion process in $\mathbb{R}^2$, can be represented by another diffusion process in $\mathbb{R}^1$.

We now discuss the behaviour of diffusion processes on ∂B. To treat this topic properly ∂B should be a sufficiently smooth surface so that the notions introduced below make sense. The behaviour of diffusion process may be modified by imposing <u>boundary conditions</u> on ∂B. Wentzell [23] has shown that the possible behaviours near the boundaries are termination, reflection, adhesion, jump and any suitable combination of these behaviour. With each type of boundary behaviour, there is a corresponding condition on the diffusion process. This correspondes to restricting the domain of the infinitesimal operator (3). For instance, if in addition to requiring that the domain include all twice continuously differentiable functions f on B, we require that $f(\underline{x}) = 0$ on ∂B, then the process will terminate upon reaching the boundary. If we require that the normal derivative of $f(\underline{x}) = 0$ on ∂B, the process will be reflected at the point of contact on ∂B. These two conditions are of considerable importance in our study.

4. Model Description And Renewal Structure

We now restrict our attention to $\mathbb{R}^2$. Let $\{ D_t^i, t \geq 0, i \geq 1 \}$ be a sequence of i.i.d. (independent and identically distributed) two-dimensional diffusion processes with a negative drift vector. The process D_t^i is allowed to take values in $B \subseteq \mathbb{R}^2$. Assume that the process is time homogeneous, i.e., the transition probabilities are stationary in time. Mathematically this means $p(s,\underline{x},t,A) = p(t-s,\underline{x},A)$, $\mu(\underline{x},t) = \mu(\underline{x})$ and $\Sigma(\underline{x},t) = \Sigma(\underline{x})$. Let D_0^i, the initial position of the i-th process, be $\underline{S} = (S_1,S_2)$ for all $i \geq 1$. The process D_t^i represent the stock level during the i-th cycle. A stopping criterion is imposed on D_t^i that when satisfied, terminates D_t^i. In the inventory context, this correspondes to placing an order to replenish the stock on hand. The set of points that satisfies the stopping criterion is the <u>stopping curve</u>. In the inventory context it is also refered to as the <u>re-order curve</u>. Let a sequence of random variables $\{ T_i, i \geq 1 \}$ be defined by

$$T_i = \inf\{ t \geq T_{i-1} : D_t^i \in \text{stopping curve} \},\ i \geq 1$$

where T_0 equals zero. T_i is a stopping time for D_t^i. Additional assumptions will be made in Section 5 to ensure that $ET_i < \infty$.

Define the process Y_t by

$$Y_t = D_t^i \quad, T_{i-1} \leq t < T_i .$$

Y_t represents the stock level at time t. The parameters of the two dimensional diffusion process can be interpreted as follows. The drift vector represents the net demand rate of the two products, the diagonal elements of the diffusion matrix represents the demand variation per unit time of each product, and $\rho_{ij}(x) = \sigma_{ij}(x) / \sqrt{\sigma_{ii}(x)\, \sigma_{jj}(x)}$ is the correlation between the mean demand rate of the two products. Diffusion processes allow positive increments even though the drift is negative, i.e., the net demand in small time intervals may be positive even though the average demand is negative. However, for large intervals such reversals are very improbable.

The economic parameters of the system are as follows. There is a fixed cost $K > 0$ of re-ordering and a variable cost $g(\bullet)$ that depends on the quantities re-ordered. There also is a cost $c(\underline{x})$ per unit time the stock level is $\underline{x}$. In Section 5 conditions on $c(\underline{x})$ and $g(\underline{x})$ are presented to ensure that the expected shortage and holding costs per cycle and the expected variable cost per cycle will be finite.

Note that the sequence $\{ T_i,\ i \geq 1 \}$ constitute regeneration points for the process $\{ Y_t,\ t \geq 0 \}$. Thus the process Y_t can be viewed as a sequence of i.i.d. cycles, where each cycle consists of a sojourn from $\underline{S}$, the initial position, to the stopping curve. Thinking of the problem this way allows results from the theory of regenerative processes to be applied; c.f. Ross[22]. As a consequence, the long-run average cost can be evaluated by dividing the expected cost per regeneration cycle by the expected length of the regeneration cycle.

Let the random function $C(t)$, the cost incurred in $[0,t]$, be defined by

$$C(t) = \int_0^t c(Y_u)\,du + \sum_{i=1}^{N(t)} g(\underline{x}_{T_i}) + KN(t)$$

where N(t) is the number of replenishments in [0,t], T_i is the stopping time of the i-th sojourn, and $\underline{x}_{T_i} \in$ stopping curve is the terminating position of Y_t on the i-th sojourn.

The following theorem follows from proposition 5.9 of Ross[22,p.98].

Theorem 1.

If $E_{\underline{S}}\ T_1 < \infty$, $E_{\underline{S}} \int_0^{T_1} c(Y_u)\ du < \infty$,

and $E_{\underline{S}}\ g(\underline{x}_{T_1}) < \infty$,

then

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{E_{\underline{S}} \int_0^{T_1} c(Y_u)\,du + E_{\underline{S}}\, g(x_{T_1}) + K}{E_{\underline{S}}\ T_1} \tag{7}$$

with probability one.

The result is also valid if $\frac{EC(t)}{t}$ replaces $\frac{C(t)}{t}$.

Here $E_{\underline{S}}$ denotes the expectation with respect to a process starting at $\underline{S}$, i.e., $Y_0 = (S_1, S_2)$. The limiting probability of being in a set Q not on the boundary can also be obtained by using the indicator of Q as the stock level cost function and setting K and the variable cost equal to zero.

Let the limit in (7) be denoted by θ . θ depends on the stopping curve implicitely through the the random variable T_1. We will study two classes of operating policies. One is the two-dimensional analog of the (s,S) policy - the (s_1, s_2, S_1, S_2)

policy. A typical control region of the (s_1, s_2, S_1, S_2) policy is shown in Fig.1. The jagged line indicates the evolution of the stock level over time until it is necessary to re-order. Its starting position is (S_1, S_2) and it terminated when the supply of the goods reached (x_{T_1}, y_{T_1}). One of the objectives will be to find the values of s_1, s_2, S_1 and S_2 that minimize θ. The modified policy differs from the (s_1, s_2, S_1, S_2) policy in that the lower levels, s_1 and s_2, are coupled via an elliptic arc. A typical control region is shown in Fig.2.

There exists an important difference between one dimensional and two dimensional processes. In one dimension the variable cost of a (s,S) policy is not a random function because the re-order point is a singleton set, i.e., $Y_{T_1} = s$. However, in two dimensions Y_{T_1} is stochastic thus the variable cost is a random function. This difference arises because the point at which the process reaches the boundary is not determined in advance.

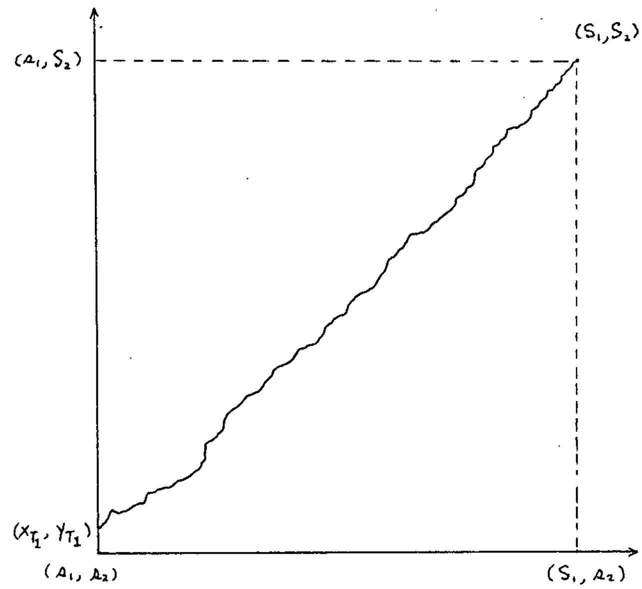

Fig. 1. A Control Region Of The (s_1, s_2, S_1, S_2) Policy

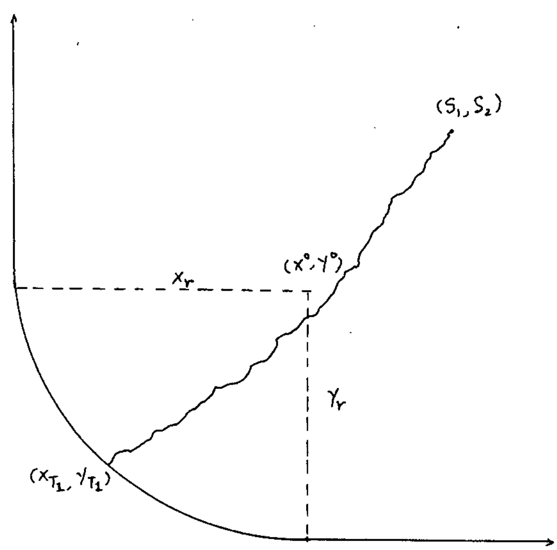

Fig. 2. A Control Region Of The Modified Policy

5. Evaluation Of The Long-run Average Cost

Let the function c(x,y) mapping $B \subseteq \mathbb{R}^2$ to $\mathbb{R}^1$ be the shortage and holding cost rate function. Define V(x,y,s) for $(x,y) \in B$ and $s \in [0,\infty)$ by

$$V(x,y,s) = E\left\{ \int_s^{T_1} c(D_u)\,du \;\middle|\; D_s = (x,y) \right\}.$$

This is the expected shortage and holding cost from time s onward given that $D_s = \underline{x} = (x,y)$. We formally show that V(x,y,s) satisfies a nonhomogeneous Kolmogorov backward equation on the interior of B. A rigorous proof of this result is an easy consequence of Theorem 5.1 of Dynkin [9,p.132]. The proof here is in the spirit of Chernoff [5] and gives insight into the meaning of the model parameters. Let the process drift for an infinitesimal time δ, a cost $c(x,y)\delta$ is incurred, and the process drifts to $D_{s+\delta}$. Thus

$$V(x,y,s) = c(x,y)\delta + E\left\{ V(D_{s+\delta}, s+\delta) \;\middle|\; D_s = (x,y) \right\}.$$

Assuming V(x,y,s) is sufficiently differentiable to apply Taylor's theorem,

$$V(x,y,s) = c(x,y)\delta + E\Big\{ V(D_s, s+\delta) + \nabla V(D_s, s+\delta)^T [D_{s+\delta} - D_s]$$

$$+ \frac{1}{2}[D_{s+\delta} - D_s]^T HV(D_s, s+\delta)[D_{s+\delta} - D_s] + o(\delta)$$

$$\Big|\; D_s = (x,y) \Big\}$$

where ∇, H denotes the gradient and the Hessian operators respectively and superscript T denote transpose. Hence

$$V(x,y,s) = c(x,y)\,\delta + E\left\{ V(D_s, s+\delta) \mid D_s = (x,y) \right\}$$

$$+ E\left\{ \nabla V(D_s, s+\delta)^T \left[D_{s+\delta} - D_s \right] \mid D_s = (x,y) \right\}$$

$$+ E\left\{ \tfrac{1}{2} \left[D_{s+\delta} - D_s \right]^T HV(D_s, s+\delta) \left[D_{s+\delta} - D_s \right] \mid D_s = (x,y) \right\}$$

$$+ o(\delta)$$

Rearranging terms dividing both sides by δ and applying (2) yields

$$\frac{V(x,y,s) - V(x,y,s+\delta)}{\delta} = c(x,y) + \mu_1(x,y,s)\frac{\partial V}{\partial x}(x,y,s+\delta)$$

$$+ \mu_2(x,y,s)\frac{\partial V}{\partial y}(x,y,s+\delta) + \frac{1}{2}\sigma_{11}(x,y,s)\frac{\partial^2 V}{\partial x^2}(x,y,s+\delta)$$

$$+ \sigma_{12}(x,y,s)\frac{\partial^2 V}{\partial x \partial y}(x,y,s+\delta) + \sigma_{22}(x,y,s)\frac{\partial^2 V}{\partial y^2}(x,y,s+\delta)$$

$$+ \frac{o(\delta)}{\delta}$$

Taking limit as δ approaches 0,

$$
\begin{aligned}
-\frac{\partial V}{\partial s}(x,y,s) &= c(x,y) + \mu_1(x,y,s)\frac{\partial V}{\partial x}(x,y,s) + \mu_2(x,y,s)\frac{\partial V}{\partial y}(x,y,s) \\
&\quad + \frac{1}{2}\sigma_{11}(x,y,s)\frac{\partial^2 V}{\partial x^2}(x,y,s) + \sigma_{12}(x,y,s)\frac{\partial^2 V}{\partial x \partial y}(x,y,s) \qquad (8) \\
&\quad + \frac{1}{2}\sigma_{22}(x,y,s)\frac{\partial^2 V}{\partial y^2}(x,y,s)
\end{aligned}
$$

provided the derivatives are right-continuous with respect to s. This is the backward equation because the time derivative is with respect to s, the initial time. The analogous forward equation is derived similarly, see [12]. Since the process is time homogeneous, (the drift vector and the diffusion matrix are independent of time), $V(x,y,s)$ is constant in s and the left hand side of (8) is zero. Letting $V(x,y)$ be $V(x,y,0)$, this yields the following equation

$$(9) \qquad AV(x,y) = -c(x,y)$$

this is shorthand for the partial differential equation

$$
\begin{aligned}
&\frac{1}{2}\sigma_{11}(x,y)\frac{\partial^2 V}{\partial x^2}(x,y) + \sigma_{12}(x,y)\frac{\partial^2 V}{\partial x \partial y}(x,y) + \frac{1}{2}\sigma_{22}(x,y)\frac{\partial^2 V}{\partial y^2}(x,y) \\
&+ \mu_1(x,y)\frac{\partial V}{\partial x}(x,y) + \mu_2(x,y)\frac{\partial V}{\partial y}(x,y) = -c(x,y)
\end{aligned}
$$

which together with the following boundary conditions determines the expected cost. The diffusion process is required to terminate on reaching the stopping curve, therefore (10) is imposed on the diffusion

(10) $V(\underline{x}) = 0$ for $\underline{x} \in$ stopping curve.

this condition is intuitively satisfying since if the stock level starts out on the stopping curve $T_1 = 0$ and no cost will be incurred during the cycle. Additional boundary condition are needed at infinity to uniquely determine solutions to (9). Below are two dimensional analogs of the result of Puterman [21].

(11)
$$\lim_{x \to \infty} e^{\left\{ \int_0^x \frac{\mu_1(x,y)}{\sigma_{11}(x,y)} dx \right\} x} \frac{\partial V}{\partial x}(x,y) = 0 \qquad \forall y$$

$$\lim_{y \to \infty} e^{\left\{ \int_0^y \frac{\mu_2(x,y)}{\sigma_{22}(x,y)} dy \right\} y} \frac{\partial V}{\partial y}(x,y) = 0 \qquad \forall x$$

These conditions say that the growth of $V(x,y)$ is exponentially bounded. (11) is also the condition on $V(x,y)$ if reflecting boundaries if placed near infinity, c.f.[4]. If the cost rate, $c(\underline{x})$, is set equal to one, then

$$V(x,y) = E_{x,y} \int_0^{T_1} 1 \, dt = E_{x,y} T_1$$

We denote this expected sojourn length by $T(x,y)$. The expected variable cost incurred in one cycle given that $D_0 = (x,y)$, denoted by $Z(x,y)$, is the solution of the following partial differential equation

$$AZ(x,y) = 0$$

(12) SUBJECT TO $Z(x,y) = \delta(x,y) \quad x,y \in \partial B$

AND (11) APPLIED TO $Z(x,y)$.

Thus all the expectations that are needed to calculate the long-run average cost rate can be computed by solving the following partial differential equations.

$$\frac{1}{2}\sigma_{11}(x,y)\frac{\partial^2 V}{\partial x^2}(x,y) + \sigma_{12}(x,y)\frac{\partial^2 V}{\partial x \partial y}(x,y) + \frac{1}{2}\sigma_{22}(x,y)\frac{\partial^2 V}{\partial y^2}(x,y)$$

$$+ \mu_1(x,y)\frac{\partial V}{\partial x}(x,y) + \mu_2(x,y)\frac{\partial V}{\partial y}(x,y) = -c(x,y)$$

SUBJECT TO (10), (11).

$$V(x,y) \quad \text{IS} \quad E_{x,y}\left\{ \int_0^{T_1} c(D_u)\,du \right\}$$

(13) $$\frac{1}{2}\sigma_{11}(x,y)\frac{\partial^2 T}{\partial x^2}(x,y) + \sigma_{12}(x,y)\frac{\partial^2 T}{\partial x \partial y}(x,y) + \frac{1}{2}\sigma_{22}(x,y)\frac{\partial^2 T}{\partial y^2}(x,y)$$

$$+ \mu_1(x,y)\frac{\partial T}{\partial x}(x,y) + \mu_2(x,y)\frac{\partial T}{\partial y}(x,y) = -1.$$

SUBJECT TO (10), (11) APPLIED TO $T(x,y)$.

$$T(x,y) \quad \text{IS} \quad E_{x,y}\, T_1$$

$$\frac{1}{2}\sigma_{11}(x,y)\frac{\partial^2 Z}{\partial x^2}(x,y) + \sigma_{12}(x,y)\frac{\partial^2 Z}{\partial x \partial y}(x,y) + \frac{1}{2}\sigma_{22}(x,y)\frac{\partial^2 Z}{\partial y^2}(x,y)$$

$$+ \mu_1(x,y)\frac{\partial Z}{\partial x}(x,y) + \mu_2(x,y)\frac{\partial Z}{\partial y}(x,y) = 0$$

SUBJECT TO (11) APPLIED TO $Z(x,y)$

AND

$$z(x,y) = g(x,y) \qquad (x,y) \in \partial B.$$

$$z(x,y) \text{ is } E_{x,y}\, g(\underline{x}_{T_1}).$$

To apply Theorem 1 we need sufficient conditions to ensure that the solutions of the above partial differential equations are finite. If the solutions of the partial differential equations are indeed finite (by obtaining and examining the solutions), then the expected values of shortage and holding cost per sojourn, sojourn length, and variable cost incurred per sojourn are finite. The following conditions give analytic conditions to ensure that the solutions of (13) are finite. Let G^1, G^2 denote

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2},$$

respectively. Assume $G^1\mu$, $G^2\mu$, G^1R, G^2R exist and are continuous where R is the square root of Σ. Let $c(\bullet)$ have two continuous derivatives and

$$(14) \qquad |G^iR| + |G^i\mu| \le a^i(1+|x|^{b^i}) \qquad i=1,2$$

$$(15) \qquad |G^ic| \le \alpha^i(1+|x|^{\beta^i}) \qquad i=1,2$$

where a^1, a^2, b^1, b^2, α^1, α^2, β^1, and β^2 are constants. Then by

Theorem 5.5 of Friedman [11, p. 122], $V(x,y)$ has two continuous derivatives and therefore it is also continuous. Thus for any finite (x,y), $V(x,y)$ is finite. From partial differential equation theory [8], the maximum attained actually occurs on the boundary. The function $c(x,y) = 1$ satisfies (15) trivially. Therefore, $T(x,y)$ is finite for finite (x,y) if the diffusion process satisfies (14). Since the variable cost $g(x,y)$ is incurred only from reordering, if $g(\bullet)$ is finite for finite (x,y), then $Z(x,y)$ is finite.

6. Methods Of Solution

The problem we study is that of solving the three partial differential equations in (13). Each partial differential equation is a boundary value problem and is elliptic if the correlation of the two diffusions is not ±1 (non-degenerate) and is parabolic if the correlation is equal to ±1 (degenerate).

There are four main techniques available for solving partial differential equations. They are transform methods, eigenfunction methods, Green's function methods, and numerical methods. Each method will be considered in terms of its applicability to the problem.

The classical Fourier and Laplace transform methods are generally used in solving problems involving infinite or semi-infinite regions, c.f.[8], [20]. Typically one assumes that the function values and the derivatives at infinity are equal to zero. However, due the presence of the first order derivative (the drift term), the transforms of the partial differential equations in (13) all involve the function value at infinity. Since $V(x,y)$, $T(x,y)$, and $Z(x,y)$ are the expected shortage and holding cost during one sojourn, sojourn length, and variable reordering cost per sojourn, we cannot set the function value at infinity to zero. In fact we expect it to be infinite. Thus transform methods does not seem to be appropriate.

The method of eigenfunctions usually arises when the partial differential equation is separable in a suitable coordinate system in a finite domain, i.e., by seeking solutions of the form $F^1(\bullet)F^2(\bullet)$ where $F^1(\bullet)$, $F^2(\bullet)$ are functions of the two coordinates only, the partial differential equation can be reduced to separate ordinary differential equations, c.f.[8]. For the (s_1, s_2, S_1, S_2) policy, the region is rectangular thus the Cartesian coordinate system is the natural coordinate system. However, the partial differential equation is not separable in the Cartesian coordinate system unless the covariance of the diffusion process is zero (<u>independent diffusions</u>). Given that the covariance is zero, the control region may be made finite by replacing (11) with (16)

$$\frac{\partial V}{\partial x}(S_1, y) = 0 \qquad (S_1, y) \in \partial B$$

(16)

$$\frac{\partial V}{\partial y}(x, S_2) = 0 \qquad (x, S_2) \in B$$

this correspondes to placing barriers at $x=S_1$ and $y=S_2$. However, (10),(16) are not suitable because there does not exist any rectangular harmonics (eigenfunctions) relative to them. Rectangular harmonics of (13) involve trigonometric functions in one coordinate and hypertrigonometric functions in the other. The trigonometric functions can be made to fit (10),(16), but the hypertrigonometric functions cannot. Therefore, eigenfunction methods do not yield solutions.

A Green's function is an apparatus to solve general partial differential equations, c.f.[8], [20]. Replacing the non-homogeneous right hand side of the partial differential equation with a delta or source function, the solution is the Green's function of the partial differential equation. The solution to the original equation is obtained by integrating the nonhomogeneous term with the Green's function over the region. Green's functions are known for standard partial differential equations with nice boundary conditions. However, to obtain the Green's function for (13) with boundary conditions (10),(11) or (10),(16) may be just as difficult as the original problem if not more, and one obtains only an integral representation of the solution which may not be integrable in closed form. Thus although Green's function method will yield a 'solution', it is not easily obtainable and does not help us in solving the problem.

Where classical analysis methods fail, numerical analysis come in. There are two main types of techniques used in solving partial differential equations. They are finite difference and finite element (<u>collocation</u>). They both require finite regions. Thus boundary conditions (10),(16) are used. The method of finite element works as follows. First, the region is partitioned into a number of cells. Then a solution basis usually consisting of Hermite polynomials is chosen in each cell to represent the solution. The partial differential operator is then applied to each cell which yields a system of algebraic equations that determines the coefficients of the solution basis in each cell. One also has to impose conditions on the boundaries of the cells to ensure the continuity of the solution, and/or the derivatives. Once the coefficients in each cell is know, the solution in the whole region is known. Finite difference, unlike finite element, only considers the function value at discrete points in the region. It approximates the derivatives by suitable 'finite' differences, c.f.[1], [16].

This results in an algebraic system which when solved, yields the approximate function values at the discrete points considered.

One of the advantages of numerical methods is that approximate solutions can be obtained when the region is irregularly shaped and/or when the problem is analytically untractable. The method of finite difference is used to solve the partial differential equations in (13) subject to (10),(16). As mentioned previously, diffusion processes allow positive increments even though the drift is negative, however, for large time intervals such reversals are very improbable. Thus replacing (11) by (16) and changing from a semi-infinite region to a finite region may not be a bad approximation. Also if the ratio of drift to variance is large, i.e., the system does not have great variations, the probability of such reversals for any given time interval is exceedingly small.

7. The (s_1, s_2, S_1, S_2) Policy

In this section, the (s_1, s_2, S_1, S_2) policy is examined. We will restrict ourselves to the class of spatially-homogeneous diffusion processes in our computations, i.e., the drift and diffusion coefficients are scalar constants and the infinitesimal operator is given by

$$A = \frac{1}{2}\sigma_{11}\frac{\partial^2}{\partial x^2} + \sigma_{12}\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\sigma_{22}\frac{\partial^2}{\partial y^2} + \mu_1\frac{\partial}{\partial x} + \mu_2\frac{\partial}{\partial y}$$

Recall from Section 3 that the drift vector, (μ_1, μ_2), is negative. We will also assume that $g(x,y) = 0$ for ease of computation. Our objective is to obtain θ for a set of given system parameters : μ, Σ, s_1, s_2, S_1, S_2, $c(x,y)$, K, and study the effects of different parameter values on θ.

A code was developed which, utilizing the method of finite difference, calculates the optimal long-run average cost rate of a specified diffusion and cost structure for a specified stopping curve, i.e., it determines an optimal reorder point S_1, S_2 for a fixed boundary $x = s_1$, $y = s_2$. Boundary conditions (10), (16) are used in the computation, (16) has an additional side benefit of being one of the necessary conditions of optimality. The code works as follows.

The program has as its input μ, Σ, s_1, s_2, $c(x,y)$, K, and trial values of S_1, S_2.

(1) $V(x,y)$, the expected shortage and holding cost per sojourn given that $D_0 = (x,y)$, and $T(x,y)$, the expected sojourn length given that $D_0 = (x,y)$ are calculated by solving the following differential equations.

$$\frac{1}{2}\sigma_{11}\frac{\partial^2 V}{\partial x^2}(x,y) + \sigma_{12}\frac{\partial^2 V}{\partial x \partial y}(x,y) + \frac{1}{2}\sigma_{22}\frac{\partial^2 V}{\partial y^2}(x,y) + \mu_1\frac{\partial V}{\partial x}(x,y) + \mu_2\frac{\partial V}{\partial y}(x,y)$$

$$= - c(x,y)$$

SUBJECT TO (10), (16).

(17)

$$\frac{1}{2}\sigma_{11}\frac{\partial^2 T}{\partial x^2}(x,y) + \sigma_{12}\frac{\partial^2 T}{\partial x \partial y}(x,y) + \frac{1}{2}\sigma_{22}\frac{\partial^2 T}{\partial y^2}(x,y) + \mu_1\frac{\partial T}{\partial x}(x,y)$$

$$+ \mu_2\frac{\partial T}{\partial y}(x,y) = -1$$

SUBJECT TO (10), (16)

APPLIED TO $T(x,y)$.

These partial differential equations are solved by the method of finite difference. A grid is chosen for the new control region bounded by the stopping curve, $x = s_1$, $y = s_2$, and the reflecting boundaries $x=S_1$, $y=S_2$ on which approximate values of $V(x,y)$ and $T(x,y)$ are calculated at the grid points. See the appendix for details of the calculation.

(2) The long run average cost rate given that $D_0 = (x,y)$, θ, is calculated by

$$\theta(x,y) = \frac{V(x,y) + K}{T(x,y)}$$

(3) The minimum θ on the grid is found by searching the grid sequentially.

(4) If this optimal θ is positioned on the boundary, the binding boundary(s) is relaxed by increasing S_1 and/or S_2 and steps (1), (2) and (3) are repeated.

Step (4) is repeated until the optimum is found to be strictly in the interior. Step (5) is then performed.

(5) The grid is shrunk by reducing S_1 and/or S_2 to place the reflecting boundaries at this optimum position and steps (1), (2), (3) are repeated.

This shrinking process continues until the position of the optimum remained stationary, i.e., the position of the optimal did not change from one iteration of step (1), (2), (3) to the next. This value is taken to be the optimal long-run average cost rate for a fixed stopping curve. It is denoted by

$\hat{\theta}(s_1,s_2)$ and the optimal S_1, S_2 are denoted by $\hat{S}_1, \hat{S}_2$ (the dependence of $\hat{\theta}(s_1,s_2)$ on μ, Σ, $c(x,y)$ and K and the dependence of $\hat{S}_1$, $\hat{S}_2$ on s_1, s_2 are suppressed in the notation). The global optimal values of s_1, s_2, S_1, S_2 and θ are denoted by $\bar{s}_1, \bar{s}_2, \bar{S}_1, \bar{S}_2$ and $\bar{\theta}$ and are found by iteratively repeating the above process with different s_1, s_2 values. Table 1 demonstrates the search of a global optimal (s_1,s_2,S_1,S_2) the finite difference scheme used to approximate solutions of (17) is presented in the appendix along with a discussion of the errors involved. A sample output is shown in Fig.3. In order to verify the code, we consider the special case of <u>symmetric diffusion</u>, i.e., $\mu_1 = \mu_2$, $\sigma_{11} = \sigma_{22}$, and a correlation of +1. The infinitesimal generator, A, is degenerate and because the two dimensional diffusion is symmetric, it collapses to an one-dimensional diffusion. Puterman [21] has investigated the one-dimensional model in detail, and for the case of quadratic cost rate, analytic results were obtained in closed form. This was used to verify the code. It was observed that on the average, the numerical solution, θ, agrees with the theoretical result to 0.08%. The following summarizes our main findings.

The diffusion parameters used to obtain the results below were chosen for numerical efficiency and to represent moderate variation. They are as follows.

Drift coefficients : $\mu_1 = \mu_2 = -2$

diffusion coefficients : $\sigma_{11} = \sigma_{22} = 1$, $\rho \in [-1,1]$

the fixed re-ordering cost used is 125/12.

For (1)-(5) the cost structure used is $c(x,y) = ax^2+by^2$, where $a = b = 1/2$. This separable cost strusture allows comparisons with the one-dimensional results. Recall that ρ is the correlation between the two mean demand rates and is defined by

$$\rho = \frac{\sigma_{12}}{\sqrt{\sigma_{11} \cdot \sigma_{22}}}$$

The following qualitative results summarize the numerical computations for the above data.

1) $\hat{\theta}(s_1, s_2)$ is convex with respect to (s_1, s_2) for $\rho \in [-1,1]$. This is demonstrated in Fig.4. Puterman has shown that $\theta(s, S)$ is convex with respect to s,S for quadratic holding cost in one dimension. This property carries over to two dimensions. We expect the convexity property to carry over to higher dimensions as well.

2) $\hat{\theta}(s_1, s_2)$ increases as ρ decreases from 1 to -1 for a fixed stopping curve. This is demonstrated in Fig.5. This result is expected because as the correlation decreases from +1 to -1, the diffusiom sample paths become more erratic , the variation of the system generally increases thus increasing the cost of operating the system.

3) As K increases, $\bar{s}_1$, $\bar{s}_2$, the optimal values of s_1, s_2 decrease. This is because when computing the optimal policy, we try to balance the shortage and holding costs against the re-order cost. As the re-order cost increases it is better to re-order less frequently thus $S_i - s_i$, $i=1,2$ increases lowering s_1, s_2 and increasing S_1, S_2. Since the process can have positive increments, it is expected that s_1 and s_2 decrease more than S_1 and S_2 increase. Alternatively if $c(x,y)$ increases we expect the opposite to occur. The important quantity is the ratio of K to $c(x,y)$. This is analogous to the one-dimensional result.

4) As ρ decreases from 1 to -1, $\bar{s}_1$ and $\bar{s}_2$ decreases as well. This is also demonstrated in Fig.4. This is expected from the intuition that in general the system tries to counteract more variation by lowering $\bar{s}_1, \bar{s}_2$.

5) For $(s_1, s_2) > (\bar{s}_1, \bar{s}_2)$, $\hat{S}_1, \hat{S}_2$ is symmetric, i.e., $\hat{S}_1 = \hat{S}_2$, if the diffusion is symmetric. For $(s_1, s_2) < (\bar{s}_1, \bar{s}_2)$, it is possible that $\hat{S}_1 \neq \hat{S}_2$. This results in 'splitting', i.e., the optimal values are no longer on the main diagonal, but on the minor diagonals. This is depicted in Fig.6. Once splitting has ocurred, moving (s_1, s_2) further away from $(\bar{s}_1, \bar{s}_2)$ results in the optimal minor diagonals moving further away from the main diagonal.

6) As $(s_1, s_2) > (\bar{s}_1, \bar{s}_2)$ decrease toward $(\bar{s}_1, \bar{s}_2)$, $\hat{S}_i - s_i$ increases. This trend continues after (s_1, s_2) pass through $(\bar{s}_1, \bar{s}_2)$.

7) $\hat{S}$-s increases as ρ decreases from +1 to -1.

8) We were unable to predict the effect of unequal variances for a fixed stopping curve. This is shown in Fig.7. For some correlations $\hat{\theta}_{(\ell_1,\ell_2)}$ increased as one of the variances decreased and yet for other correlations the opposite is true. However, Fig.7 again demonstrates that as the correlation decreased, $\hat{\theta}_{(\ell_1,\ell_2)}$ increases.

9) For the case of unequal variances, 4) also holds and $\bar{s}_1,\bar{s}_2$ are relatively insensitive to changes in one of the variances.

Results 10)-11) are obtained using $c(x,y) = (ax+by)^2$, where $a = b = 1/2$. This cost structure is not separable and the system exhibits some quite different behaviour from the previous cost structure.

10) The splitting effect is sharp. For $(s_1,s_2) < (\bar{s}_1,\bar{s}_2)$, splitting always occur, this is not the case with the previous cost structure. The difference may stem from the fact that the iso-cost contours of the present cost structure is open while the iso-cost contours of the previous cost structure is closed.

11) As (s_1,s_2) move away from $(\bar{s}_1,\bar{s}_2)$ in either direction, $\hat{S}$-s decreases. This is opposite to 6).

Table 1 demonstrates a search for a global optimal (s_1, s_2, S_1, S_2) policy.

A search of a global optimal (s_1, s_2, S_1, S_2) policy

SYSTEM PARAMETERS

$\mu_1 = \mu_2 = -2$

$\sigma_{11} = \sigma_{22} = 1 \qquad \rho = 0$

$c(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 \qquad K = 125/12$

$s_1 = s_2 = s$

s	$\hat{\theta}(s_1, s_2)$
-2.45	7.24
-2.75	7.02
-3.05	6.88
-3.35	6.84
-3.65	6.86
-3.95	6.97
-4.25	7.16
-5.00	7.90

Fig. 3 Sample Output of (S_1, S_2, S_1, S_2) Policy Code

| P.D.E. COEFFICIENTS |

SIGMA X = 1.000 MUX = 2.000 COVARIANCE = 1.000

SIGMA Y = 1.000 MUY = 2.000

| COST STRUCTURE |

CX = 5.0000000E-01 CY = 5.0000000E-01 SWITCHING COST = 1.0416667E+01

| GRID EXPANSION PARAMETERS |

INCREMENTAL EXPANSION = 5 MULTIPLICATIVE EXPANSION = 1.5000000E+00

| RELAXATION ITERATION PARAMETERS |

OVER-RELAXATION PARAMETER = 1.31666666667 MAXIMUM ITERATIONS IS 100

COARSE RESIDUE TOLERANCE = 1.0000000E-04 FINE RESIDUE TOLERANCE = 1.0000000E-10

| PROGRAM OUTPUTS |

THE SMALLEST 10 AV COST RATES WILL BE PRINTED

****

RELAXATION GRID # 1 SPECIFICATIONS

X = (-1.25000 , 3.25000) X STEP = 0.25000

Y = (-1.25000 , 3.25000) Y STEP = 0.25000

COST CALCULATION

ITERATION	MAX RESIDUE IN VALUES
10	9.4844121069277E-03
20	3.1110692941999E-04
24	8.6044497832891E-05

RESIDUE TOLERANCE REACHED

TIME CALCULATION

ITERATION	MAX RESIDUE IN VALUES
10	6.95503459516 78E-03
20	3.7357491415377E-04
25	7.6513613215157E-05

RESIDUE TOLERANCE REACHED

```
***************
*   RESULTS   *
***************
```

RANK	X	Y	EXPECTED COST RATE
1	2.50000	2.50000	7.8730045108869E+00
2	2.75000	2.75000	7.8998892208495E+00
3	2.50000	2.75000	7.9234137481203E+00
4	2.75000	2.50000	7.9238585043857E+00
5	2.25000	2.25000	7.9355819042445E+00
6	2.25000	2.50000	7.9482904875564E+00
7	2.50000	2.25000	7.9486021501625E+00
8	2.75000	3.00000	7.9621402733680E+00
9	3.00000	2.75000	7.9625723503602E+00
1C	3.00000	3.00000	7.9767611645190E+00

```
**************************************************************************************************
```

RELAXATION GRID # 2 SPECIFICATIONS

X = (-1.25000 , 2.75000)   X STEP = 0.25000

Y = (-1.25000 , 2.75000)   Y STEP = 0.25000

COST CALCULATION

ITERATION	MAX RESIDUE IN VALUES
10	2.0432203265300E-03
20	6.4440346613322E-05
30	1.6393193287063E-06
40	9.4073375679028E-08
50	5.3210086263802E-09
60	3.4712021965796E-10
64	9.0133496437466E-11

RESIDUE TOLERANCE REACHED

TIME CALCULATION

ITERATION	MAX RESIDUE IN VALUES
10	1.7546274152017E-04
20	1.8517977703692E-06
30	5.5760024150214E-08
40	3.9103949474689E-09
50	2.2269430804449E-10
53	9.3586155024946E-11

RESIDUE TOLERANCE REACHED

```
***************
*   RESULTS   *
***************
```

RANK	X	Y	EXPECTED COST RATE
1	2.50000	2.50000	7.9200088715692E+00
2	2.25000	2.25000	7.9678791384453E+00
3	2.25000	2.50000	7.9709026492459E+00
4	2.50000	2.25000	7.9709026498949E+00
5	2.00000	2.25000	8.0893637971261E+00
6	2.25000	2.00000	8.0893637983244E+00
7	2.00000	2.00000	8.1262956914557E+00
8	2.00000	2.50000	8.1533888501528E+00
9	2.50000	2.00000	8.1533888518573E+00
10	1.75000	2.00000	8.3245074271854E+00

OPTIMAL FIXED LOWER BOUNDARY RECTANGLE HAS BEEN FOUND   *** OPFBRC5 ***

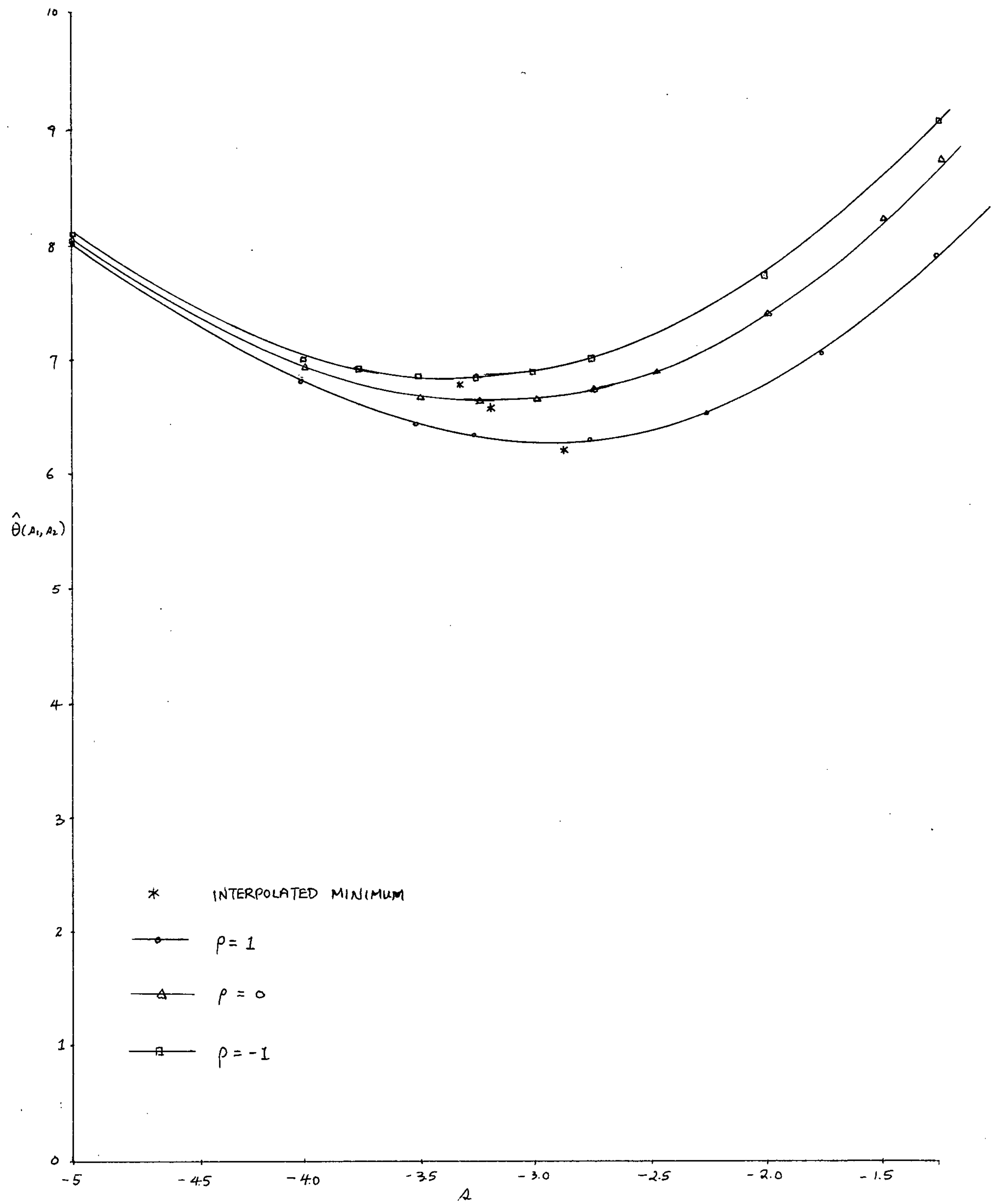

Fig. 4. Convexity Of $\hat{\theta}(s_1, s_2)$ With Respect To (s_1, s_2)

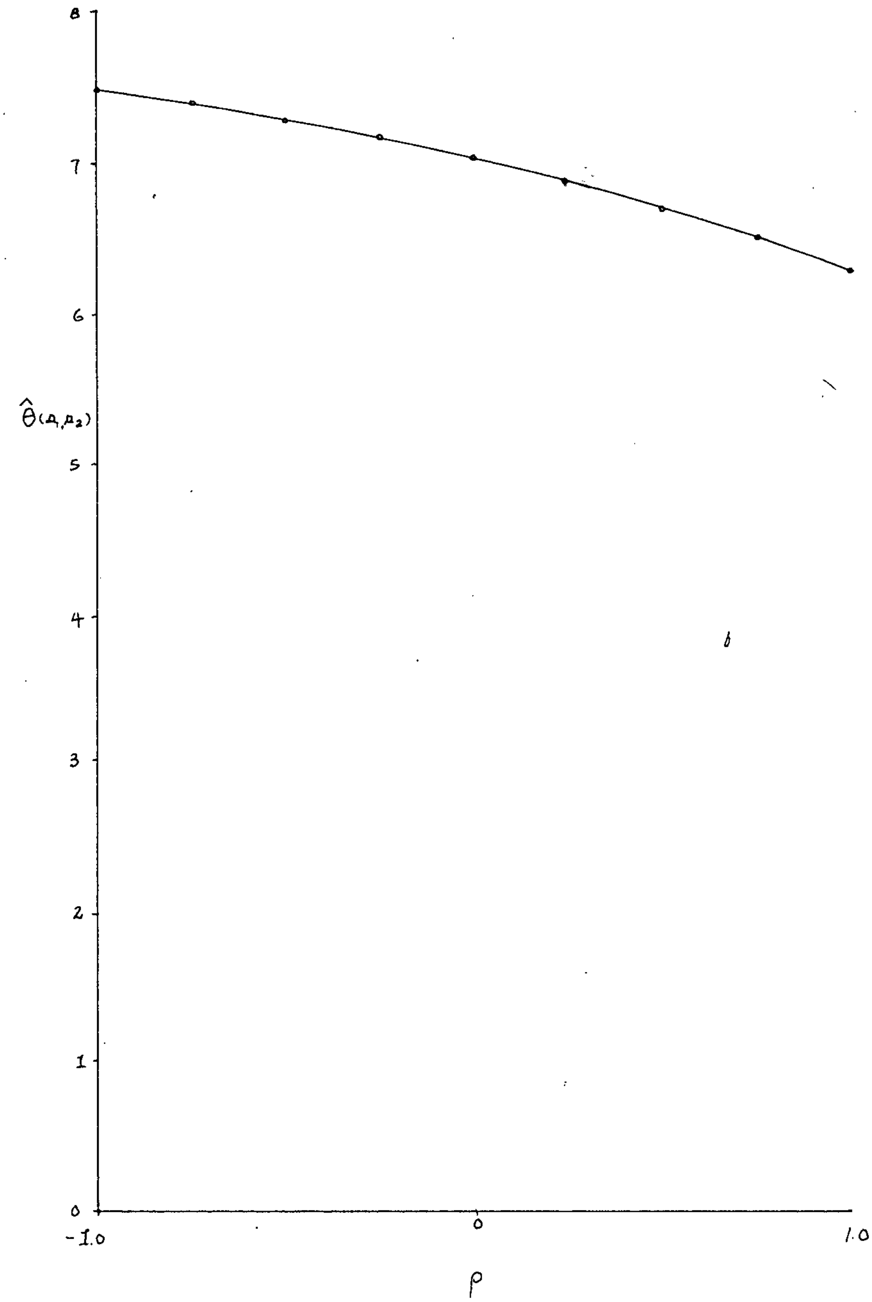

Fig.5. Effects Of Correlation On $\hat{\theta}(\rho_1,\rho_2)$ For The (s_1, s_2, S_1, S_2) policy.

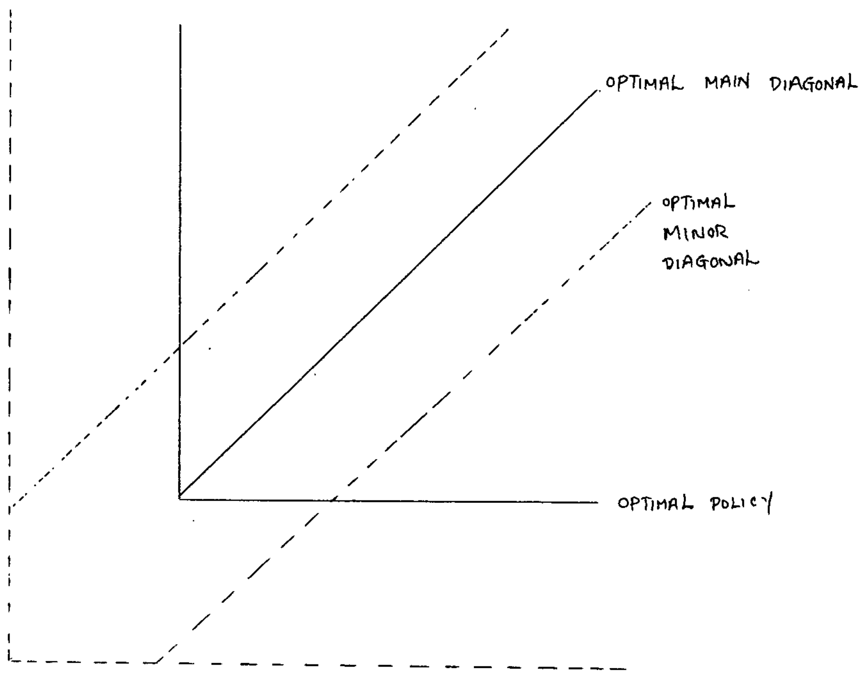

Fig.6. The Phenomenon Of Splitting For The (s_1, s_2, S_1, S_2) policy.

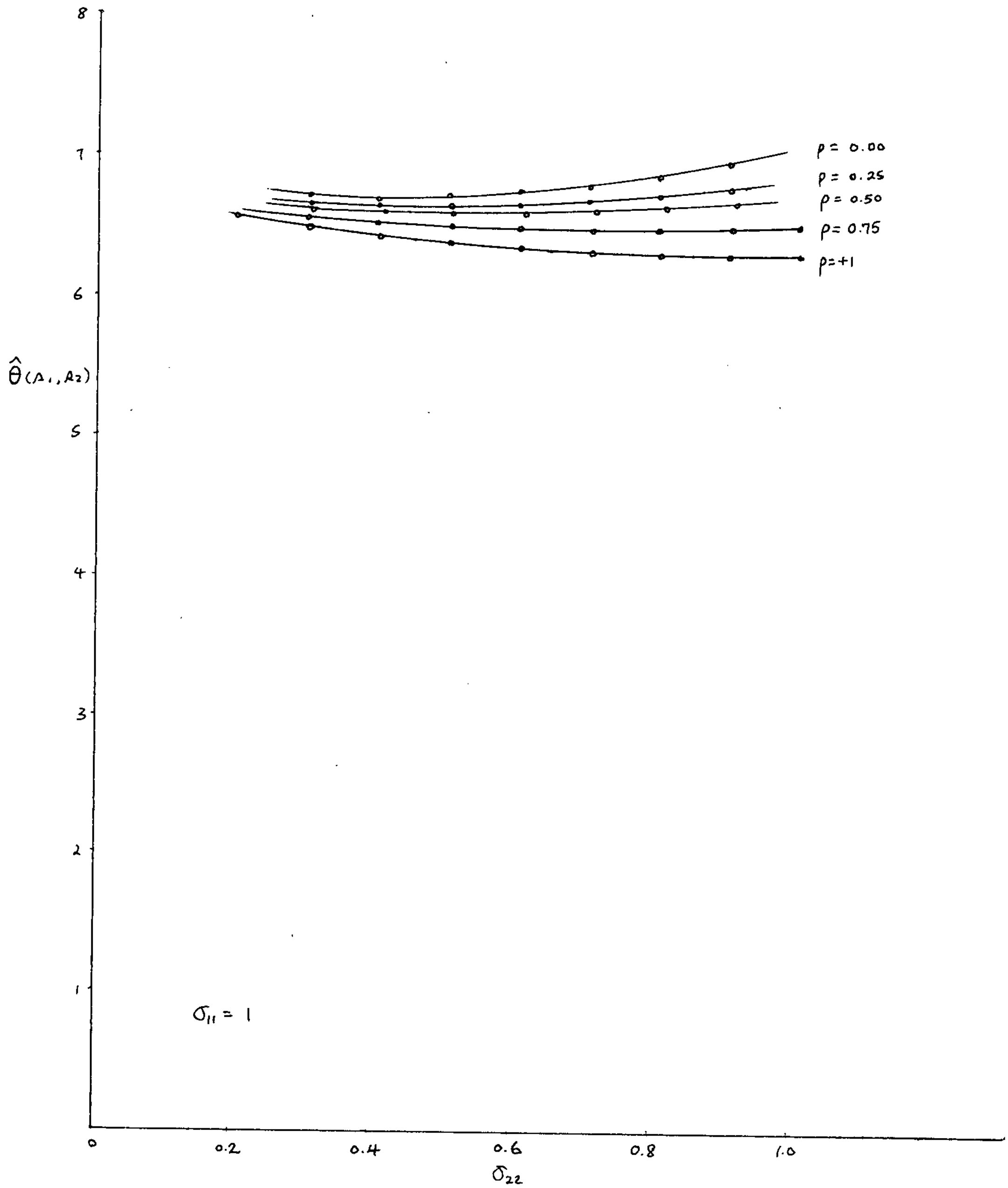

Fig.7. Effects Of Unequal Variance

8. The Modified Policy

This policy is described by six parameters: S_1, S_2, (x^0,y^0) the center of the ellipse, and (x_r,y_r) the length of the axis. Note that if x^0, y^0 = s_1, s_2 and $x_r = y_r = 0$ this reduces the modified policy to the (s_1,s_2,S_1,S_2) policy considered in the previous section. As with the (s_1,s_2,S_1,S_2) policy, we have a computer code which evaluate the long run average cost for a given policy. The program has as its input the following parameters: μ, Σ, $c(x,y)$, K, x^0, y^0, x_r, y_r, S_1, S_2. Reflecting boundaries again are imposed at $x = S_1$, $y = S_2$. This code has been observed to have accuracy of the order of 0.3%. A sample output is shown in Fig.8. The following summarizes our findings. Results 1)-5) are obtained using $c(x,y) = ax^2+by^2$ with $a = b = 1/2$.

1) Θ is convex with respect to (x_r,y_r) for $\rho\in[-1,1]$. This is demonstrated in Fig.9 for ths case of $\rho = 0$.

2) $\hat{\Theta}$ increases as ρ decreases from 1 to -1 for a fixed policy. This is demonstrated in Fig.10. This is consistent with results from the (s_1,s_2,S_1,S_2) policy.

3) As K increases, $\hat{x}_r,\hat{y}_r$ increases.

4) The modified policy is at least as cheap as the (s_1,s_2,S_1,S_2) policy if not cheaper.

5) The modified policy seems to be less sensitive to changes to

ρ.

Results 6)-7) are obtained using $c(x,y) = (ax+by)^2$ with $a = b = 1/2$.

6) As ρ decreased from +1 to -1, the modified policy does progressively better than the (s_1,s_2,S_1,S_2) policy. For the data used, at $\rho = 0$, it is at least 3.7% better, and at $\rho = -1$, it is at least 5.7% better.

7) $\bar{\theta}$ is quite insensitive to the correlation.

The results show that the modified policy is better than the (s_1,s_2,S_1,S_2) policy, especially if the products are substitutable. The implementation of this policy is not difficult. The user merely checks the inventory on hand and compares it to a chart which specifies the optimal stopping curve. If the inventory is lower than as specified on the chart, an order is placed to bring the stock up to (S_1,S_2).

The numerical results obtained depends on the cost structure used. For $c(x,y) = ax^2 + by^2$, the isocost contours are closed ellipses, while the isocost contours of $c(x,y) = (ax+by)^2$ are straight lines. We suspect this difference lead to observation (11) of the previous section, however the dependence of θ on this difference is still unresolved.

Fig. 8 Sample output of modified policy Code

| P.D.E. COEFFICIENTS |

SIGMA X = 1.000   MUX = 2.000   COVARIANCE = 0.750

SIGMA Y = 1.000   MUY = 2.000

| REGION SPECIFICATIONS |

UX = 2.50000   UY = 2.50000

XCNTR = 2.50000   YCNTR = 2.50000

XAXIS = 7.42462   YAXIS = 7.42462

BOUNDARY CLOSENESS TOLERANCE = 1.0000000000000E-10

| COST STRUCTURE |

CX = 5.0000000E-01   CY = 5.0000000E-01   SWITCHING COST = 1.0416667E+01

| RELAXATION ITERATION PARAMETERS |

OVER-RELAXATION PARAMETER = 1.31666666667   MAXIMUM ITERATIONS IS 100

RESIDUE TOLERANCE = 1.0000000E-10

| PROGRAM OUTPUTS |

THE SMALLEST 30 AV COST RATES WILL BE PRINTED

***** LIST PROCESSING *****

X	Y	REPRESENTATION
-4.75000	1.00000	00110110
-4.75000	1.25000	00100110
-4.75000	1.50000	00100110
-4.75000	1.75000	00100110
-4.75000	2.00000	00100110
-4.75000	2.25000	00100110
-4.50000	0.25000	00110110

-4.50000	0.50000	00100110
-4.50000	0.75000	00100010
-4.50000	1.00000	00000010
-4.25000	-0.50000	00110110
-4.25000	-0.25000	00100110
-4.25000	0.00000	00100010
-4.25000	0.25000	00000010
-4.00000	-1.00000	00110110
-4.00000	-0.75000	00100010
-4.00000	-0.50000	00000010
-3.75000	-1.50000	00110110
-3.75000	-1.25000	00100010
-3.75000	-1.00000	00000010
-3.50000	-1.75000	00110010
-3.50000	-1.50000	00000010
-3.25000	-2.00000	00110010
-3.25000	-1.75000	00000010
-3.00000	-2.25000	00110010
-3.00000	-2.00000	00000010
-2.75000	-2.50000	00110010
-2.75000	-2.25000	00000010
-2.50000	-2.75000	00110010
-2.50000	-2.50000	00000010
-2.25000	-3.00000	00110010
-2.25000	-2.75000	00000010
-2.00000	-3.25000	00110010
-2.00000	-3.00000	00000010
-1.75000	-3.50000	00110010
-1.75000	-3.25000	00000010
-1.50000	-3.75000	00110011
-1.50000	-3.50000	00000010
-1.25000	-3.75000	00010010
-1.00000	-4.00000	00110011
-1.00000	-3.75000	00000010
-0.75000	-4.00000	00010010
-0.50000	-4.25000	00110011
-0.50000	-4.00000	00000010
-0.25000	-4.25000	00010011
0.00000	-4.25000	00010010
0.25000	-4.50000	00110011
0.25000	-4.25000	00000010
0.50000	-4.50000	00010011
0.75000	-4.50000	00010010
1.00000	-4.75000	00110011
1.00000	-4.50000	00000010
1.25000	-4.75000	00010011
1.50000	-4.75000	00010011
1.75000	-4.75000	00010011
2.00000	-4.75000	00010011
2.25000	-4.75000	00010011
2.50000	-4.75000	00010011

COST CALCULATION

ITERATION   MAX RESIDUE IN VALUES

10	3.3709873202146E-03
20	1.2937098960727E-05
30	1.8686657003130E-07
40	3.0607594178849E-09

RESIDUE TOLERANCE REACHED

TIME___CALCULATION

ITERATION	MAX_RESIDUE_IN_VALUES
10	1.6458520708152E-03
20	5.5843052232559E-06
30	4.3038511746120E-08
40	5.8282092974198E-10

RESIDUE TOLERANCE REACHED

****
* RESULTS *
****

RANK	X	Y	EXPECTED___COST___RATE
1	2.25000	2.25000	6.3762236261967E+00
2	2.00000	2.25000	6.3999807554759E+00
3	2.25000	2.00000	6.4007983741412E+00
4	2.00000	2.00000	6.4156974333904E+00
5	1.75000	2.25000	6.4646047751437E+00
6	2.25000	1.75000	6.4665788742431E+00
7	1.75000	2.00000	6.4704844214667E+00
8	2.00000	1.75000	6.4715061315665E+00
9	1.75000	1.75000	6.5138816516727E+00
10	1.50000	2.00000	6.5719224401619E+00
11	2.00000	1.50000	6.5741694846511E+00
12	1.50000	2.25000	6.5755267937302E+00
13	2.25000	1.50000	6.5790064921227E+00
14	1.50000	1.75000	6.6028828347546E+00

15	1.75000	1.50000	6.6039099342572E+00
16	1.50000	1.50000	6.6798659360204E+00
17	1.25000	2.00000	6.7223388086662E+00
18	2.00000	1.25000	6.7261766610181E+00
19	1.25000	2.25000	6.7348224551381E+00
20	2.25000	1.25000	6.7403044197599E+00
21	1.25000	1.75000	6.7415214486472E+00
22	1.75000	1.25000	6.7437387537597E+00
23	1.25000	1.50000	6.8059680692621E+00
24	1.50000	1.25000	6.8069251968559E+00
25	1.25000	1.25000	6.9206084498766E+00
26	1.00000	2.00000	6.9229437070352E+00
27	2.00000	1.00000	6.9289353109676E+00
28	1.00000	1.75000	6.9312592347342E+00
29	1.75000	1.00000	6.9350479883355E+00
30	1.00000	2.25000	6.9436045891489E+00

***** ELLIPSE VERSION I *****

EXECUTION TERMINATED

$SIG

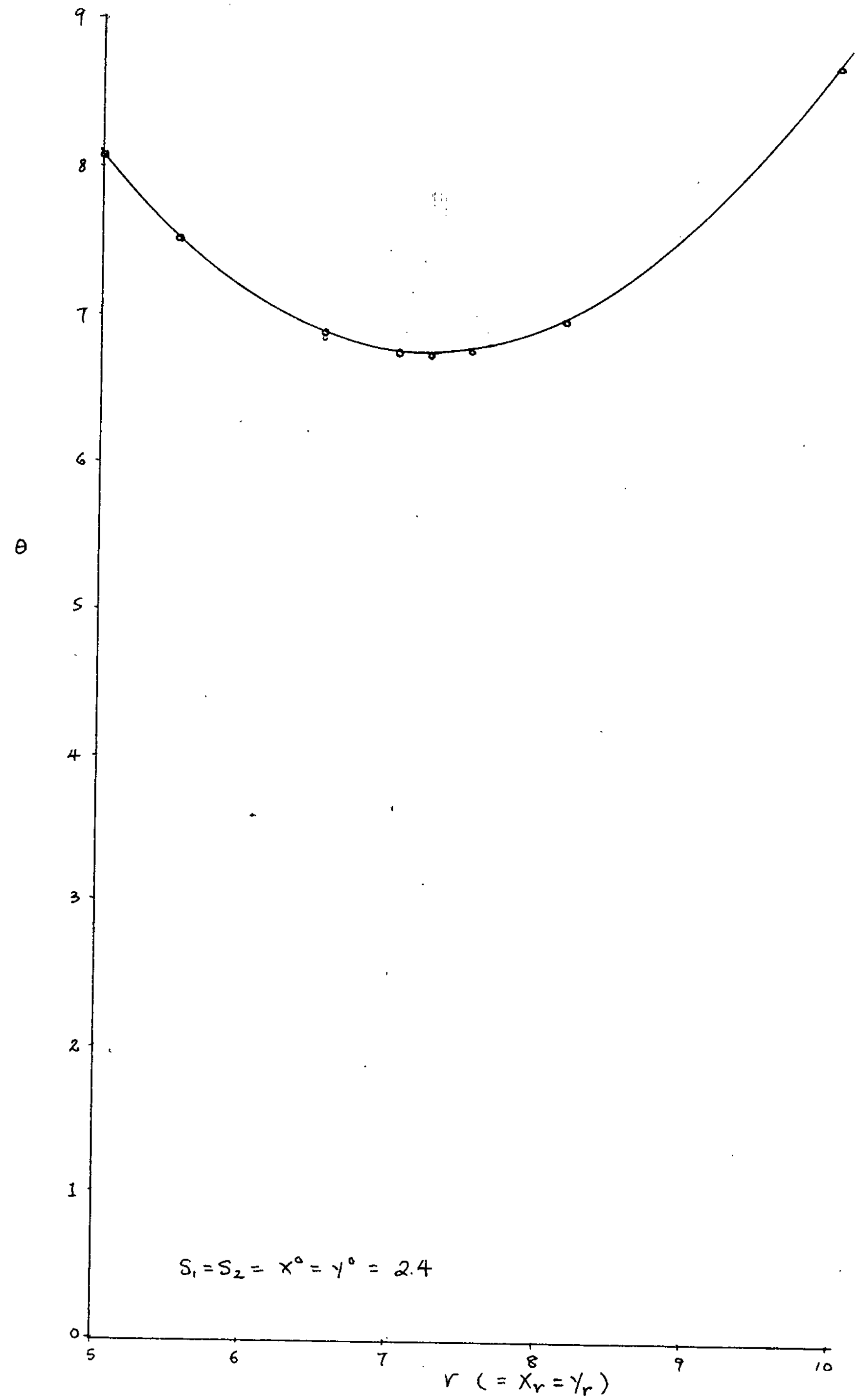

Fig. 9. The Convexity Of The Modified Policy.

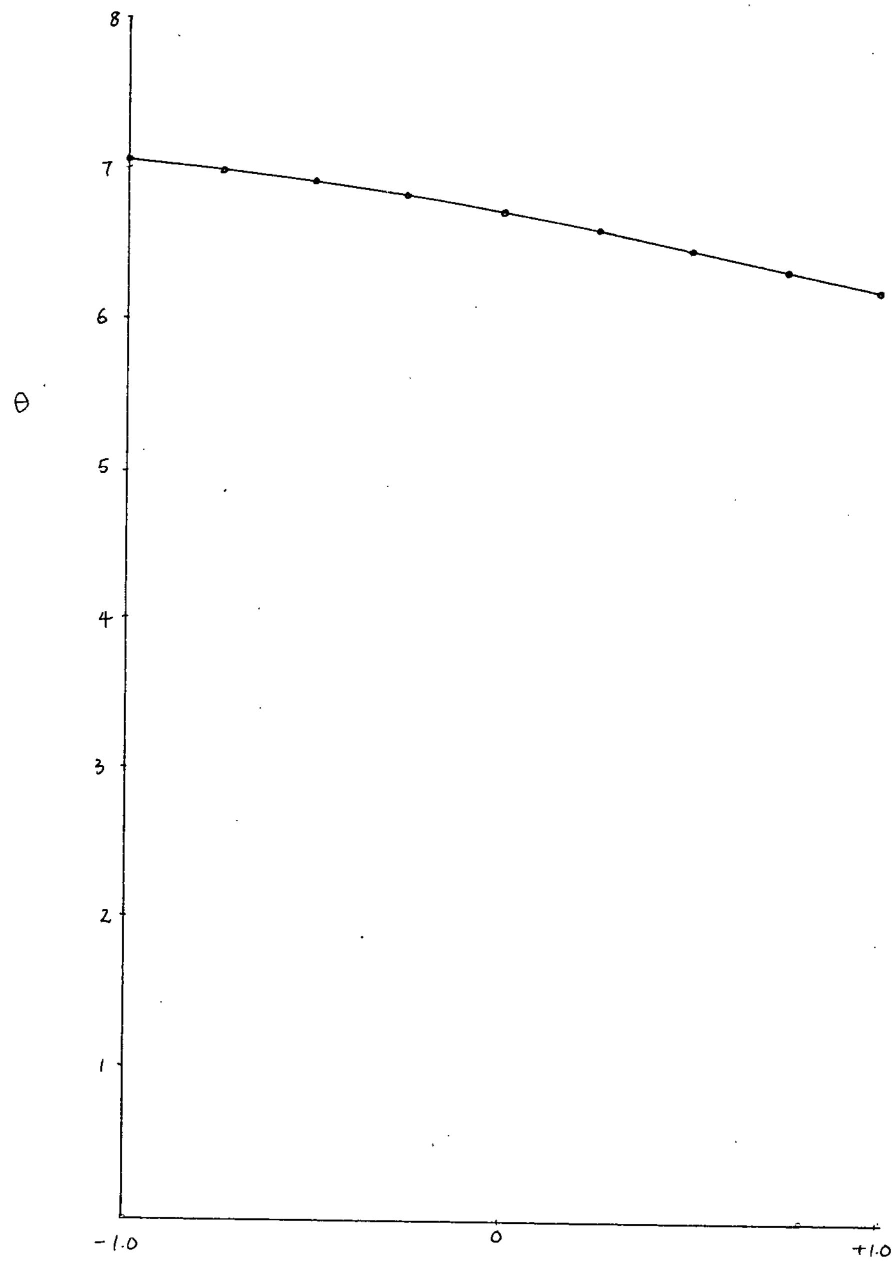

Fig. 10. Effects of correlation on the modified policy

9. Recommendations For Future Research

The case of a fixed length lagged delivery can be accomodated easily in the present model. Let $L > 0$ be the time lag between the time a re-order is decided and the time the actual replenishment occurs. In the inventory context, this may be thought of as the delivery time or the production setup time. The cumulative demand in the time interval $(t, t+L]$ is $Y_{t+L} - Y_t$, which by the stationarity of the diffusion process, has the same conditional distribution as $Y_L - Y_0$. Furthermore, if the demand is spatially homogeneous, then Y_0 can be set to zero. Defining $\bar{c}(\bullet)$ by $\bar{c}(\underline{y}) = c(\underline{y}+Y_L)$, this is the expected cost L time units in the future. To obtain results for this model, one simply replace $c(\bullet)$ by $\bar{c}(\bullet)$ in (2).

The terminating distribution of the diffusion path on the stopping curve can be obtained by solving the following partial differential equation:

$$AP(x,y) = 0$$

$$\text{SUBJECT TO} \qquad P(x,y) = 1 \qquad (x,y) \in I_Q$$

$$P(x,y) = 0 \qquad (x,y) \in \partial B \cap I_Q^c$$

where I_Q is the indicator of the set Q which is part of the boundary. $P(x,y)$ is the probability that $D_{T_1} \in I_Q$ given that $D_0 = (x,y)$. Knowledge of this terminating distribution should reveal great insight to the behaviour of the diffusion paths.

Through the backward equation we were able to characterize the long-run average cost rate of operating the inventory system, and calculate the expected variable re-ordering cost.

Programs have been developed which can calculate the expected variable cost, $Z(x,y)$, and the terminating distribution of the diffusion paths, $P(x,y)$. Other problems such as different cost structures can and should be investigated further to reveal the underlying dynamics of multi-dimensional diffusion processes. A very important and still unanswered question is what is the form of the optimal policy. In this research it was assumed that the order to point, $\underline{S}$, is independent of the terminating position of the sojourns on the stopping curve. In fact, the optimal policy may be to order to a point that depends on the terminating position of the sojourns. An important problem from the application point of view is that of the estimation of the diffusion parameters, this problem has not yet been investigated. This myriad of questions remain to be answered.

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APPENDIX

The purpose of this appendix is to present the computational scheme used to solve (9) subject to (10) ; (16). The method used is the method of finite difference. The main idea is to replace the derivatives by small finite differences which approximate the derivatives on a mesh obtained by discretizing the region of interest. Given a partial differential equation of the form

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = g(x,y)$$

consider the following <u>nine-point star approximation</u> of derivatives.

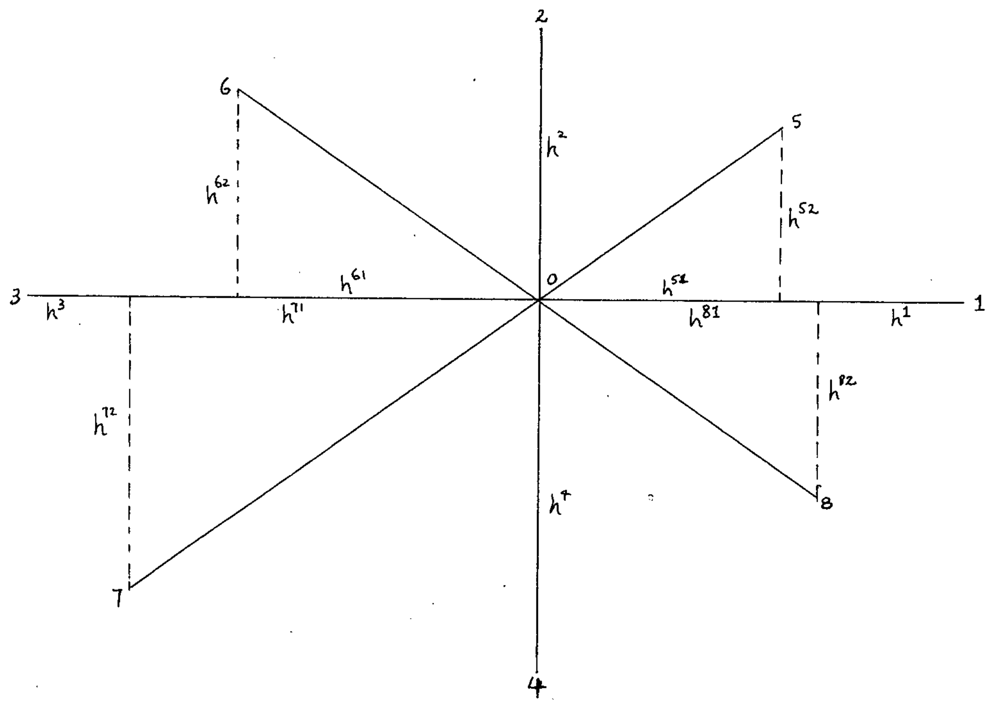

Fig. 11. The Nine-point Star Approximation Of Derivatives

The points are numbered 0 to 8 with the distances as shown in Fig. 11. These distances were chosen to allow for irregular grid points (points with unequal distances to its neighbour points) which occurs near the stopping curve of the modified policy. Let the function values at the nine points be $U^0, \ldots, U^8$. Constants $k^0, \ldots, k^8$ are to be determined such that

$$\sum_{i=0}^{8} k^i U^i = AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = g(x,y) \qquad (18)$$

Taylor expanding U^j, $j=1,\ldots,8$ about U^0, and collecting the terms yields the following system of 6 equations with 9 unknowns.

$$k^0+k^1+k^2+k^3+k^4+k^5+k^6+k^7+k^8 = F$$

$$h^1k^1-h^3k^3+h^{51}k^5-h^{61}k^6-h^{71}k^7+h^{81}k^8 = D$$

$$h^2k^2-h^4k^4+h^{52}k^5+h^{62}k^6-h^{72}k^7-h^{82}k^8 = E$$

$$(h^1)^2k^1+(h^3)^2k^3+(h^{51})^2k^5+(h^{61})k^6+(h^{71})^2k^7+(h^{81})^2k^8 = 2A$$

$$(h^2)^2k^2+(h^4)^2k^4+(h^{54})^2k^5+(h^{64})k^6+(h^{74})^2k^7+(h^{84})^2k^8 = 2C$$

$$h^{51}h^{52}k^5-h^{61}h^{62}k^6+h^{71}h^{72}k^7-h^{81}h^{82}k^8 = B$$

This system as it stands is indeterminate, therefore 3 more independent equations are added.

$$K^5+k^6+k^7+k^8 = 0$$

$$h^{51}k^5-h^{61}k^6-h^{71}k^7+h^{81}k^8 = 0$$

$$h^{52}k^5+h^{62}k^6-h^{72}k^7-h^{82}k^8 = 0$$

These equations serve to eliminate the contributations of points 5, 6, 7, 8 to the coefficients of the finite difference term which represent the first order derivatives. Let

$$h^{52}/h^{51} = h^{72}/h^{71} = r^1$$

$$h^{62}/h^{61} = h^{82}/h^{81} = r^2$$

The solution to this system is

$$k^0 = F-k^1-k^2-k^3-k^4$$

$$k^1 = [2A+Dh^3-K^1]/[h^1(h^1+h^3)]$$

$$k^2 = [2C+Eh^4-K^2]/[h^2(h^2+h^4)]$$

$$k^3 = [2A-Dh^1-K^1]/[h^3(h^1+h^3)]$$

$$k^4 = [2C-Eh^2-K^2]/[h^4(h^2+h^4)]$$

$$k^5 = B/\{[h^{51}h^{71}r^1+h^{61}h^{81}r^2](1+h^{51}/h^{71})\}$$

$$k^6 = -B/\{[h^{51}h^{71}r^1+h^{61}h^{81}r^2](1+h^{61}/h^{81})\}$$

$$k^7 = B/\{[h^{51}h^{71}r^1+h^{61}h^{81}r^2](1+h^{71}/h^{51})\}$$

$$k^8 = -B/\{[h^{51}h^{71}r^1+h^{61}h^{81}r^2](1+h^{81}/h^{61})\}$$

where

$$K^1 = (h^{51})^2k^5+(h^{61})^2k^6+(h^{71})^2k^7+(h^{81})^2k^8$$

$$K^2 = (h^{52})^2k^5+(h^{62})^2k^6+(h^{72})^2k^7+(h^{82})^2k^8$$

Substituting these coefficients into (18) reduces the problem to an algebraic system by requiring this formula to hold for all U^0 in the interior of B'. This algebraic system is most conveniently solved by the method of successive over-relaxation, c.f. [15]. The system can be written as

$$Ax = b$$

where A is a banded matrix and b is g(x,y) at the grid points. Using the coefficients derived above and successive over-relaxation, approximate solutions to the partial differential equations of (13) can be obtained.

This approximation has a truncation error of the order $o(h^3)$ from the finite Taylor expansion. This error can be reduced by taking h small. Greenspan provides conditions on the coefficients, $k^0, \ldots, k^8$, that guarantees the algebraic system will converge and satisfy the weak max-min property that the analytic solution posses, c.f.[15].