TOPICS IN STOCHASTIC DOMINANCE: THEORY 
AND APPLICATION 

by 

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ABSTRACT

The dissertation investigates some important aspects of managerial decision making under conditions of uncertainty.

In the last three decades two prominent general approaches have evolved to deal explicitly with risk in managerial decisions. They are:

(1) the central tendency-dispersion trade off approach, and
(2) expected utility analysis.

The first task undertaken in this investigation is to integrate these two approaches. This is accomplished by identifying those situations in which decision rules obtained by either approach are equivalent.

Once equivalence between the two basic approaches to decision making under uncertainty is established, the focus shifts to the extension of these decision theories into situations involving multi-attribute outcome spaces. In particular, stochastic dominance rules for multivariate outcome distributions are developed.

Two applications of stochastic dominance criteria are then presented, illustrating the relevance of the approach to theory development and management of resource systems.

The first illustration demonstrates the application of stochastic dominance to portfolio diversification problems. Several results are obtained describing the sensitivity of optimal mixes with respect to changes in opportunities for investment.
The second illustration demonstrates the role stochastic dominance criteria can play in ecosystem policy analysis. A methodology of stochastic dominance policy screening for forest management systems is developed and applied.
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Chapter 1

INTRODUCTION

The complexities characterizing modern managerial decision environments have intensified the need to develop appropriate decision structures dealing explicitly with uncertainties. A typical characterization of a decision problem under uncertainty is as follows: a decision maker must choose an action from a set of possible actions associated with a probability distribution over possible consequences. This problem has a long history dating back to the early work in the field of "games of chance" [18,85] where the expected value was thought to be an appropriate criterion in inducing the preference order among distributions.

The first attempt to deal formally with the above problem was made by Bernoulli [6] who introduced some of the fundamental ideas of expected utility theory. Bernoulli's solution to his now famous paradox of gambling was to choose a form with diminishing marginal utility for the utility function (e.g. choosing the logarithmic utility function indicates that marginal utility is inversely proportional to wealth). Risk taking attitudes were not explicitly incorporated into his definition of the function.

Two major schools of thought have subsequently evolved attempting to deal with risk attitudes explicitly:
(1) Central tendency-dispersion ("mean-risk analysis")

(2) Expected utility analysis.

Central tendency-dispersion analysis presupposes a risk measure and indifference curves between this risk measure and expected payoffs. The decision maker will prefer an action (or a consequence) which is associated with a higher indifference curve ("less risk and more payoff").

There have been many proposals for alternative combinations of central tendency and dispersion measurements. Markowitz [62] proposed the variance as a dispersion measure in trade off with expected payoff (mean). Baumol [4] extended Markowitz's mean-variance criterion to the mean-lower confidence rule, where the difference between the mean and a constant multiple of standard deviation is used as a risk measure. The possible use of the semivariance, first suggested by Markowitz, was explored and developed by Mao [61]. Other risk measures were suggested by Roy [89], and Philippatos and Wilson [74]. Assuming that investors are principally concerned with avoiding a possible disaster, Roy suggested that risk should be defined as the probability of occurrence of a disaster. The mean-entropy rule was recently proposed by Philippatos and Wilson where higher entropy implies higher risk. No axiomatic system underlies these proposals for mean-risk analysis. They are primarily based upon observations of decision heuristics.

In contrast the development of expected utility theory has emphasized the development of a satisfactory normative axiomatic system to describe "rational" choice. Expected utility theory is based on the premise that decision making is primarily a subjective process and the appropriate criterion is a "moral expectation." Here, the term "moral
"expectation" means that value judgements on consequences and/or probabilities have been incorporated in calculating the expectation. Under this theory, the value of consequences is measured by a utility function and the information on the distribution of consequences is represented by the subjective probability. The development of the weakest axiomatic system is not a trivial problem since a completely preordered mixture set does not guarantee the existence of a utility function [19, 25, 39, 110]. The most important theoretical development in this respect is the work of Von Neumann and Morgenstern who laid the foundation for a comprehensive theory of "rational decision making" facing uncertainty [110]. Unfortunately, the applied value of their work is somewhat constrained by the fact that the focus of the theory is upon individual decision-maker's preference profiles, hence the theory does not apply to organizational decisions and measurement procedures derived on the basis of the theory are difficult to apply and are often unreliable [105].

Recently, utilizing the consensus on the properties of utility functions [3, 81], various authors have proposed and developed decision rules which require limited knowledge of utility functions and are consistent with expected utility theory [30, 35, 36, 63, 84, 87, 108, 111]. These authors have proposed that certain characteristics of utility functions are common to a large number of decision makers and can be verified easily. These characteristics define four classes of utility functions.

1. \( U_1 = \{ u | u \text{ is an increasing utility function} \} \), [30, 35, 36, 63, 84, 87].
2. \( U_2 = \{ u | u \text{ is a concave utility function} \} \cap U_1 \), [30, 35, 36, 63, 84, 87].
3. \( U_3 = \{ u | u \text{ has a non-negative third derivative} \} \cap U_2 \), [100, 111].
4. \( U_D = \{ u | r' = \left( \frac{-u''}{u'} \right)' \leq 0 \} \cap U_2 \), [3, 81, 106, 107, 108].
For each class, rules which permit some comparison between probability distributions of consequences have been devised. The rules for comparing cumulative distributions of consequences for each of the above classes of utility functions are known as

1. **First degree Stochastic Dominance (FSD)**
2. **Second degree Stochastic Dominance (SSD)**
3. **Third degree Stochastic Dominance (TSD)**
4. **DARA dominance.**

The stochastic dominance ordering rules are consistent with the ordering based upon expected utility. The random variable \( X \) is said to dominate the random variable \( Y \) for the utility function \( u \) if and only if

\[
E_F u(X) = \int u(x) dF(x) \geq \int u(x) dG(x) = E_G u(Y).
\]

Stochastic dominance rules for classes \( U_1, U_2, U_3 \) and \( U_D \) are listed below.

Let \( F = \{F|F \) is a cumulative distribution of a random variable \( x \} \) and \( U = \{u|u: \mathbb{R} \rightarrow \mathbb{R}, \int udF < \infty, F \in F\} \), then the stochastic dominance ordering corresponding to \( U_1, U_2, U_3 \) and \( U_D \) contained in \( U \) are:

1. Let \( F, G \in F \), then \( F \) dominates \( G \) in first degree (\( >_1 \)) if and only if \( F(x) \leq G(x) \) for all \( x \in \mathbb{R} \) and a strict inequality holds for some \( x \in \mathbb{R} \).

2. Let \( F, G \in F \), then \( F \) dominates \( G \) in second degree (\( >_2 \)) if and only if

\[
\int_{-\infty}^{x} F(t) dt \leq \int_{-\infty}^{x} G(t) dt
\]

for all \( x \in \mathbb{R} \) and a strict inequality holds for some \( x \in \mathbb{R} \).
(3) Let \( F, G \in \mathcal{F} \), then \( F \) dominates \( G \) in third degree \( (\succ_3) \)
if and only if

\[
\begin{align*}
(\text{i}) & \quad \int_{-\infty}^{x} \int_{-\infty}^{y} F(z) \,dz \,dy \leq \int_{-\infty}^{x} \int_{-\infty}^{y} G(z) \,dz \,dy \quad \text{for all } x \in \mathbb{R} \\
& \text{and a strict inequality holds for some } x \in \mathbb{R}, \text{ and}

(\text{ii}) & \quad \int_{-\infty}^{\infty} x \,dF(x) > \int_{-\infty}^{\infty} x \,dG(x).
\end{align*}
\]

Let \( H = F - G \in S_{n} \) if there exists \((n+1)\) intervals \([0,a_1), [a_1,a_2), \ldots, [a_n,\infty)\) with \(0 < a_1 < a_2, \ldots, < a_n\) such that:

i) In each interval, \( H \not\equiv 0 \) and have constant signs and

ii) The sign alternates between successive intervals.

Then the following rule holds.

(4) i) If \( H \in S_{n}(a_1, \ldots, a_n) \) and \( H < 0 \) on \([0,a_1)\), then

\( F > D G \).

ii) If \( H \in S_{2}(a_1, a_2) \), then \( F > D G \) if and only if

\( E_F u > E_G u \) for all concave exponential \( u \).

iii) If \( H \in S_{3}(a_1, a_2, a_3) \), let \( b = \text{Sup}\{z|z < a_3; \int_{z}^{\infty} H(x) \,dx = 0\} \) then \( F > D G \) if and only if

\( E_F u > E_G u \) on \([0,b]\) for all concave exponential \( u \).

iv) If \( H \in S_{2n}(a_1, a_2, \ldots, a_{2n}) \) with \( n \geq 2 \), then \( F > D G \)
if and only if \( \int_{0}^{\infty} H(x) \,u'(x) \,dx > 0 \) for all

\[
u'(x) = u'(a) \exp\left[-\int_{a}^{x} r(y) \,dy\right]
\]

where

\[
r(y) = k_i, \ x \in [0,y_1), \ i=1, \ldots, n-1
\]

\[
= k_i, \ x \in [y_{i-1}, y_i), \ i=2, \ldots, n-1
\]

\[
= k_n, \ x \in [y_{n-1}, \infty),
\]

with \( k_1 > \cdots > k_n > 0 \) and \( y_i \in (a_{2i}, a_{2i+1}) \), \( i=1,2,\ldots,n-1 \).
If \( H \in S_{2n+1}(a_1, a_2, \ldots, a_{2n+1}) \) with \( n \geq 2 \), let \( b = \sup\{z | z \geq a_{2n+1}, \int_{z}^{\infty} H(x)dx = 0\} \), then \( F \succ_D G \) if and only if \( F \succ_D G \) on \([0, b] \in S_{2m}\) with \( m \leq n \).

Each of the classes of utility functions \( U_1, U_2, U_3 \) and \( U_D \) represent preference structures considered interesting from the economic point of view [3], and many of the studies in stochastic dominance have been concerned with \( U_1, U_2, \) and \( U_D \).

The stochastic dominance orderings related to \( U_1 \) and \( U_2 \) have a long history dating back at least to Hardy-Littlewood-Polya [38]. They induced a second degree stochastic dominance on a set of simple measures defined on a one-dimensional space. Other important mathematical developments concerning first and second degree stochastic dominance are included in Sherman [96], Blackwell [9], Lehmann [48], Cartier-Fell-Meyer [16] and Strassen [99]. The progress made by these authors was the generalization of consequence space from one-dimensional to a multi-dimensional. The results in the mathematical literature cited above were later reintroduced to the economic literature by Massé and Morlat [63], Quirk and Saposnik [84], Hadar and Russell [30], Hanoch and Levy [36], Fishburn [26] and Hammond [35]. In all these references the consequence space was taken to be a subset of \( \mathbb{R}^n \), the \( n \)-dimensional real space except for Fishburn's work which extended the results to a set of finite consequences.

If one accepts the argument in Arrow-Pratt [3,81] that the risk-averse decision maker should have non-increasing absolute risk-aversion \((r'(x) \leq 0)\), then the stochastic dominance concept should be extended to the class \( U_D \). Whitmore [111] initiated this line of investigation.
by considering the class $U_3 \supset U_D$. The analysis was later completed by Vickson [108].

Several authors have attempted to provide a theoretical basis to central tendency-dispersion approaches to risk decision analysis. These attempts considered conditions under which criteria obtained by employing central tendency-dispersion analysis are equivalent to employment of maximization of expected utility. Philippatos and Gressis [73] showed that the mean-variance, second degree dominance and mean-entropy criteria are equivalent when the random variables have uniform or normal distributions. Hanoch-Levy [36] showed that when random variables are restricted to distributions having two parameters that are independent monotone functions of mean and variance, SSD is equivalent to the mean-variance criterion. The equivalence between the mean-semivariance and the stochastic dominance criteria has been also investigated by Porter [76] and Jean [44]. Porter showed that second degree dominance implies semivariance preference under reasonably general conditions. Jean showed that under certain extremely restrictive conditions, third degree dominance implies mean-semivariance preference.

The first task which is undertaken in this dissertation (Chapter 2) is to bridge the remaining gaps between criteria derived on the basis of central tendency-dispersion approaches and those derived on the basis of expected utility theory. A summary theorem presents equivalence relationships among the various criteria of choice for a variety of important probability distributions of consequences. The appendix presents some additional results on equivalences between stochastic dominance criteria and choice criteria based upon particular parameters of a distribution.
The applicability of choice rules developed by either approach has been limited by the focus upon single attribute consequences. Yet many important decision situations are described by a multi-attribute outcome space. Levy [51] and Levy and Paroush [56,57] pioneered in the economic literature in developing stochastic dominance rules for comparing multivariate distributions. Huang et al. [43] extended these results to other classes of utility function (\(U_3\) and \(U_D\)). Further extensions, focusing primarily on \(U_2\), were made by Lehmann [48], Sherman [96], Strassen [99], Meyer [65], Brummelle-Vickson [15] and Levhari et al. [50]. Unfortunately these authors provided necessary and sufficient conditions which are difficult to apply. The prime contribution of Chapter 3 is to further generalize the results in Levy [51], Levy and Paroush [56] and Huang et al. [43] hence extending the applicability of stochastic dominance.

The last two chapters present applications of stochastic dominance. Chapter 4 presents a theoretical application of stochastic dominance to two period portfolio selection problems. In particular the chapter investigates the effects upon portfolio composition of:

1. introduction of a stochastically dominating security to the choice set, and
2. a change in location and scale parameters of a security in that set.

Chapter 5 presents an application of stochastic dominance rules to a system's management problem. This chapter adds to applications of stochastic dominance an important area characterized by uncertainties, the area of ecosystem management.
In this chapter we attempt to integrate choice criteria for situations involving uncertainty: those which are derived from the dichotomy of central tendency-dispersion and those which are based on maximization of expected utility. We first provide the formal definitions of the various criteria and then the summary theorem representing their interrelationships.

Let $X$ and $Y$ be random variables with right continuous cumulative distribution functions $F$ and $G$ having finite means $\mu_F$, $\mu_G$ and finite non-zero variances $\sigma^2_F$ and $\sigma^2_G$, respectively. Then the following definitions can be stated.

**Mean-Variance ($E-V$).** $X$ is preferred to $Y$ in ($E-V$) iff

(i) $\mu_F \geq \mu_G$, and  
(ii) $\sigma^2_F \leq \sigma^2_G$

with strict inequality holding for at least one inequality. As a selection rule, this criterion has serious drawbacks [3,4,8,11,24,30,36,84,87]. In practice, however, it is a very popular and useful rule because of its simplicity. For theoretical support of $E-V$ rule, see [8,12,53,94,103,104].
Mean-Lower Confidence (E - L). Baumol [4] introduced this rule in order to eliminate some shortcomings in the E - \sigma^2 rule. Let \( L_F = \mu_F - \alpha \sigma_F \) and \( L_G = \mu_G - \alpha \sigma_G \), where \( \alpha > 0 \) is a given constant. Then \( X \) is said to be preferred to \( Y \) in E - L iff

(i) \( \mu_F \geq \mu_G \)

(ii) \( L_F \geq L_G \)

with strict inequality holding for at least one inequality.

Moeske [66] independently proposed this criterion and called it the truncated minimax criterion. For a recent statement of theory of this method and an algorithm for calculating the efficient frontier, see [67]. A more efficient algorithm appears in Hohenbalken [41]. Hanoch and Levy [37] criticize this rule stating the fact that available information for the choice of an optimal \( \alpha \) is not utilized. In addition, E - L is not entirely an efficient criterion since it depends on individual tastes through the choice of \( \alpha \).

Mean-Semivariance (E - SV(h)). Let \( SV_F(h) = \int_{-\infty}^{h} (x-h)^2 \, dF(x) \) and \( SV_G(h) = \int_{-\infty}^{h} (x-h)^2 \, dG(x) \) be the semivariance of \( F \) and \( G \). Then \( X \) is preferred to \( Y \) in E - SV(h) iff

(i) \( \mu_F \geq \mu_G \)

(ii) \( SV_F(h) \leq SV_G(h) \),

where one inequality is strict.

Markowitz [62] considered the semivariance as a risk measure, but because of computational difficulties he rejected it.

Mao [61] applied this rule to a capital budgeting problem. An efficient
algorithm to compute the efficient frontier in mean-semivariance space appears in Hogan-Warren [40].

**Mean-Aspiration (E - F(s)).** Let s be the individual's aspiration or target level. Then X is said to be preferred to Y in E - F(s) iff

(i) $\mu_X \geq \mu_Y$

(ii) $F(s) \leq G(s),$

where one inequality is strict.

This criterion is similar in spirit to the chance constrained rules [72]. E - SV(h) and E - F(s) rules suffer from a similar deficiency to the E - L rule (Hanoch and Levy [37]).

**Mean-Entropy (E - H).** Let $H_F \equiv \int_{-\infty}^{\infty} \ln(F'(x))dF(x)$ and $H_G \equiv \int_{-\infty}^{\infty} \ln(G'(x))dG(x)$ be the entropies of X and Y, respectively. Then X is preferred to Y in E - H iff

(i) $\mu_F \geq \mu_G$

(ii) $H_F \leq H_G$

with strict inequality holding for at least one inequality.

In general, H is large whenever the distributions are completely random, i.e. distributions are uniform.

**Stochastic Dominance**

Let

$$F_1(x) = \int_{-\infty}^{x} F(t)dt, G_1(x) = \int_{-\infty}^{x} G(t)dt$$

$$F_2(x) = \int_{-\infty}^{x} F_1(y)dy \text{ and } G_2(x) = \int_{-\infty}^{x} G_1(y)dy,$$
then $X$ is said to dominate $Y$ in 

- **(FSD)** iff $F(x) \leq G(x)$ for all $x \in \mathbb{R}$ and $F(x_0) < G(x_0)$ for some $x_0 \in \mathbb{R}$,
- **(SSD)** iff $F_1(x) \leq G_1(x)$ for all $x \in \mathbb{R}$ and $F_1(x_0) < G_1(x_0)$ for some $x_0 \in \mathbb{R}$,
- **(TSD)** iff $F_2(x) < G_2(x)$ for all $x \in \mathbb{R}$, with strict inequality $\mu_F < \mu_G$ holding for at least one inequality.

Theorems which relate the above rules to expected utility theory can be found in [30,36,84,87,111].

We can now state the following theorem which demonstrates the equivalence of various efficient sets.

**Theorem 2.1.** Let the random variables $X$ and $Y$ have distributions $F$ and $G$, respectively. Suppose

1. $F$ intersects $G$ only at $y_0$.
2. $F$ and $G$ are members of a two parameter family whose parameters are independent increasing functions of mean and variance.
3. The aspiration and semivariance levels $s$ and $h$ are less than $y_0$.

Then holds,
where the solid line denotes general implications and the dotted line denotes the uniform and normal cases.

**Proof:**

1. See [36].

2. Let \( F >_2 G \), then \( \int_{-\infty}^{x} \left[ G(t) - F(t) \right] dt \geq 0 \) for all \( x \in \mathbb{R} \).

   Hence by integration by parts \( \int_{-\infty}^{h} \int_{-\infty}^{x} [G(t) - F(t)] dt \, dx \leq 0 \). Thus \( X \) dominates \( Y \) in \( E - \text{SV}(h) \). Now suppose \( X \) dominates \( Y \) in \( E - \text{SV}(h) \), then \( \int_{-\infty}^{h} \int_{-\infty}^{X} [G(t) - F(t)] dt \, dx > 0 \) and \( \int_{-\infty}^{\infty} [G(t) - F(t)] dt > 0 \).

   Let \( H(t) = G(t) - F(t) \). There are two possibilities: either \( H(t) > 0 \) and changes to \( H(t) < 0 \) or \( H(t) < 0 \) and changes to \( H(t) > 0 \). It is impossible for \( H(t)'s \) initial sign to be negative and \( \int_{-\infty}^{h} \int_{-\infty}^{x} H(t) dt \, dx > 0 \) since \( h < y_0 \).

   If \( H(t)'s \) initial sign is positive then \( \int_{-\infty}^{x} H(t) dt > 0 \) for all \( x \in \mathbb{R} \) since \( \int_{-\infty}^{\infty} H(t) dt > 0 \) and only one sign change is possible for \( \int_{-\infty}^{x} H(t) dt \).

3. Follows from 1 and 2.

4. Suppose \( X(>_2)Y \), then \( \mu_F \geq \mu_G \) and \( F(x) \leq G(x) \) for all \( x \leq y_0 \). Thus \( F(s) < G(s) \) since \( s < y_0 \). If \( \mu_F \geq \mu_G \) and \( F(s) \leq G(s) \) then \( F(>_2)G \). This follows from [36; Thm. 3].

5. Follows from 2 and 4.

6. Follows from 1 and 4.
(7) Let $F(x) = \frac{x}{b_F-a_F}$ and $G(x) = \frac{x}{b_G-a_G}$, then $H_F = \ln(b_F-a_F)$ and $H_G = \ln(b_G-a_G)$. Therefore $F(E-V)G \Leftrightarrow F(E-H)G$. Similar implication follows for $F(x)$ and $G(x)$.

Then the equivalence follows from 1.

(8) Follows from 4 and 7.

(9) Since $L_F-L_G = (\mu_F - \alpha\sigma_F) - (\mu_F - \alpha\sigma_G)$

$= (\mu_F - \mu_G) + \alpha(\sigma_G - \sigma_F)$.

(10) This follows immediately from the assumption (1) and the definitions of SSD and TSD.

Let the sets of efficient portfolios under $E - V$, $E - L$, SSD, TSD, $E - F(s)$, $E - SV(h)$ and $E - H$ be $e(E - V)$, $e(E - L)$, $e(\text{SSD})$, $e(\text{TSD})$, $e(E - F(s))$, $e(E - SV(h))$ and $e(E - H)$ respectively, then

$$e(\text{TSD}) = e(\text{SSD}) = e(E - V) = e(E - F(s)) = e(E - SV(h)) \subset e(E - L)$$

$$e(\text{SSD}) = e(\text{TSD}) = e(E - H) = e(E - F(s)).$$
Chapter 3

MULTIVARIATE STOCHASTIC DOMINANCE

3.1 Introduction

In Chapter 2 we have attempted to integrate the previous results in the area of choice under uncertainty with a univariate outcome space. In this chapter we generalize results in stochastic dominance theory to choices involving multivariate outcome spaces. The multivariate decision problem investigated is as follows.

Let \( F^n \) = \{ \( F^n \mid F^n \) is the joint distribution of random vector \( X = (X_1, \ldots, X_n) \) \} and \( U^n = \{ u^n \mid u^n \) is a multi-attribute utility function} \}. Then, what is a partial ordering \( >_p \) on \( F^n \), \( G^n \in F^n \) such that \( F^n >_p G^n \) if and only if \( E_u^n(X) \geq E_u^n(Y) \) for all \( u^n \in U^n_p \subset U^n \) and \( > \) for some \( u^n \), where \( p \) determines the property of utility function \( u^n \)?

Several attempts have been made recently to extend the general framework of stochastic dominance from the single variable to the multivariate case. Levy [51] developed sufficient rules for first and second degree dominance when utility functions are defined on terminal wealth and there is independence among outcomes in different periods. Levy and Paroush [56] extended these results for rules of first degree dominance omitting the independence requirement. They also developed necessary and
sufficient rules for first degree dominance for additive utility functions. Huang et al. [43] extended all the results in Levy [51] and the results obtained for additive utility functions in Levy and Paroush [56] to classes $U_1, U_2, U_3$ and $U_0$.

One can also easily obtain the additive results utilizing Fishburn's work [25]. He showed that $u^n$ separates into an additive form if the desirability of any lottery $X$ depends only on the marginal probability distributions of the $X_i$'s and not on their joint distributions.

Of course, the restriction imposed by the additivity or the independence assumptions on the members of $U^n$, $F^n$ respectively, severely limits the usefulness of such a result.

Lehmann [48], Sherman [96], Strassen [99], Meyer [65] and Brumelle-Vickson [15] analyzed stochastic orderings in general settings. In their studies the utility functions were assumed to be either nondecreasing or nondecreasing and concave without addition of any other assumptions. Brumelle-Vickson base their analysis on a different characterization of stochastic dominance conditions due to Rothschild and Stiglitz [87]. Levhari et al. [50] provide sufficient and necessary conditions for the general case of monotone increasing (first degree dominance) and quasi concave utility functions. Unfortunately, the conditions developed in the above studies are difficult to apply.

In our analysis we investigate stochastic dominance rules for multi-attribute utility functions. After establishing (by means of theorem 3.3.1) the equivalence of rules for multivariate utility functions $u = u^n(x_1, \ldots, x_n)$ and univariate utility functions defined on multivariate outcome space $u = u_i(\phi(x_1, \ldots, x_n))$, we focus upon the latter.
3.2 Equivalence of Utility Functions

Define the following classes of multi-attribute utility functions.

\[ U_1^n = \{ u^n | \frac{\partial u^n}{\partial x_i} \geq 0 \text{ for all } i \} \]

\[ U_2^n = \{ u^n | u^n \in U_1^n \text{ and } \frac{\partial^2 u^n}{\partial x_i^2} \leq 0 \text{ for all } i \} \]

\[ U_3^n = \{ u^n | u^n \in U_2^n \text{ and } \frac{\partial^2 u^n}{\partial x_i^2} \geq 0 \text{ for all } i \} \]

\[ U_D^n = \{ u^n | u^n \in U_3^n \text{ and } \frac{\partial}{\partial x_i} \left[ - \frac{\partial^2 u^n}{\partial x_i^2} \frac{\partial u^n}{\partial x_i} \right] \leq 0 \text{ for all } i \} \]

These classes correspond to the important classes of utility functions investigated extensively in the literature for the single variable case. In this section we show that they are identical to the following classes:

\[ V_i^n = \{ u^n(x_1, \cdots, x_n) = u_i(\phi(x_1, \cdots, x_n)) | u_i \in U_i \text{ and } \phi \in U_i^n \} \]

for each \( i=1,2,3,D \).

**Theorem 3.2.1** \( V_i^n = U_i^n \) for \( i=1,2,3,D \).

**Proof:** First we show \( V_i^n \subset U_i^n \) for \( i=1,2,3,D \).

---

\(^1\) It is assumed that the chain rule of differentiation holds. For sufficient conditions, see e.g. [112]. We also assume that the functions \( u^n, u_i \) and \( \phi \) are thrice differentiable unless stated otherwise.
\( \frac{\partial u}{\partial x_i} = \frac{\partial u_i}{\partial \phi} \frac{\partial \phi}{\partial x_i} \)

\( \geq 0 \text{ since } u_i \in U_1 \text{ and } \phi \in U_1^n. \)

\( \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u_i}{\partial \phi^2} \frac{\partial \phi}{\partial x_i} + \frac{\partial u_i}{\partial \phi} \frac{\partial^2 \phi}{\partial x_i^2} \)

\( \leq 0 \text{ since } u_i \in U_2 \text{ and } \phi \in U_2^n. \)

\( \frac{\partial^3 u}{\partial x_i^3} = \frac{\partial^3 u_i}{\partial \phi^3} \left( \frac{\partial \phi}{\partial x_i} \right)^3 + 3 \frac{\partial^2 u_i}{\partial \phi^2} \frac{\partial \phi}{\partial x_i} \frac{\partial^2 \phi}{\partial x_i^2} + \frac{\partial u_i}{\partial \phi} \frac{\partial^3 \phi}{\partial x_i^3} \)

\( \geq 0 \text{ since } u_i \in U_3 \text{ and } \phi \in U_3^n. \)

\( \frac{\partial}{\partial x_i} \left[ - \frac{\partial^2 u}{\partial x_i^2} \frac{\partial u}{\partial x_i} \right] = \left[ \frac{\partial^3 u}{\partial x_i^3} \frac{\partial u}{\partial x_i} \left( \frac{\partial \phi}{\partial x_i} \right)^3 + 3 \frac{\partial^2 u_i}{\partial \phi^2} \frac{\partial u_i}{\partial \phi} \frac{\partial^2 \phi}{\partial x_i^2} \left( \frac{\partial \phi}{\partial x_i} \right)^2 \right] / \left( \frac{\partial u}{\partial x_i} \right)^2. \)

By substituting (1) and (3)

\[ \frac{\partial^3 u}{\partial x_i^3} \frac{\partial u}{\partial x_i} = \frac{\partial^3 u_i}{\partial \phi^3} \frac{\partial u_i}{\partial \phi} \left( \frac{\partial \phi}{\partial x_i} \right)^3 + 3 \frac{\partial^2 u_i}{\partial \phi^2} \frac{\partial u_i}{\partial \phi} \frac{\partial^2 \phi}{\partial x_i^2} \left( \frac{\partial \phi}{\partial x_i} \right)^2 \]

\[ + \left( \frac{\partial u_i}{\partial \phi} \right)^2 \frac{\partial^3 \phi}{\partial x_i^3} \frac{\partial \phi}{\partial x_i} \]

\[ \geq \left( \frac{\partial^2 u_i}{\partial \phi^2} \right)^2 \left( \frac{\partial \phi}{\partial x_i} \right)^4 + 3 \frac{\partial^2 u_i}{\partial \phi^2} \frac{\partial u_i}{\partial \phi} \frac{\partial^2 \phi}{\partial x_i^2} \left( \frac{\partial \phi}{\partial x_i} \right)^2 \]

\[ + \left( \frac{\partial u_i}{\partial \phi} \right)^2 \frac{\partial^2 \phi}{\partial x_i^2} \] \quad \text{since } u_i \in U_D \text{ and } \phi \in U_D^n.

\[ = \left( \frac{\partial^2 u_i}{\partial x_i^2} \right)^2 + \frac{\partial^2 u_i}{\partial \phi^2} \frac{\partial u_i}{\partial \phi} \frac{\partial^2 \phi}{\partial x_i^2} \left( \frac{\partial \phi}{\partial x_i} \right)^2 \]

\[ \geq \left( \frac{\partial^2 u_i}{\partial x_i^2} \right)^2. \]
Therefore \( \frac{3}{3x_i} \lt 0. \)

Hence \( V_i^n \subseteq U_i^n \) for \( i=1,2,3,D. \)

Letting \( u_i(\phi) = \phi \) and \( u_i \in U_i \) yields \( V_i^n = U_i^n \), for \( i=1,2,3,D. \)

In what follows the stochastic dominance rules are obtained for the classes \( V_i^n \), \( i=1,2,3,D. \) This is done mainly for consistency with respect to the current literature.

### 3.3 Independent Attributes

Consider two policy options, with impacts upon attribute levels \( (X_1,\ldots,X_n) \). The impacts of these policies are described by the cumulative distributions \( F^n = F(x_1,\ldots,x_n) \) for the first option and \( G^n = G(x_1,\ldots,x_n) \) for the second.

In this section we consider only the case of independence of \( x_i \)'s.

**Lemma 3.3.1** Let \( u_i \in U_i \) and \( \phi \in U_i^n \), then \( H(x_i) \equiv \int u_i(\phi) \, dF^{n-1} \in U_i \) for \( i=1,2,3,D. \)

**PROOF:** By differentiation,

\[
\begin{align*}
1) \quad \frac{\partial H}{\partial x_i} & = \int \frac{\partial u_i(\phi)}{\partial x_i} \, dF^{n-1} \geq 0 & \text{since } u_i(\phi) \in U_i^n \\
2) \quad \frac{\partial^2 H}{\partial x_i^2} & = \int \frac{\partial^2 u_i(\phi)}{\partial x_i^2} \, dF^{n-1} \leq 0 & \text{since } u_i(\phi) \in U_i^n
\end{align*}
\]

\(^2\)We assume that the interchange of integration and differentiation is legitimate, for details see e.g. Loeve [60].
\[
\frac{\partial^3 H}{\partial x^3_n} = \int \frac{\partial^3 u_i(\phi)}{\partial x^3_n} \, dF^{n-1} \geq 0 \quad \text{since } u_i(\phi) \in U^n_i
\]

\[
\frac{\partial H}{\partial x_n} \frac{\partial^3 H}{\partial x^3_n} = \left( \int \frac{\partial u_i(\phi)}{\partial x_n} \, dF^{n-1} \right) \left( \int \frac{\partial^3 u_i(\phi)}{\partial x^3_n} \, dF^{n-1} \right)
\]

\[
\geq \left[ \int \frac{\partial^2 u_i(\phi)}{\partial x^2_n} \, dF^{n-1} \right] \quad \text{(by the Cauchy Schwartz inequality)}
\]

\[
= \left( \frac{\partial^2 H}{\partial x^2_n} \right)^2 \quad \text{(since } u_i(\phi) \in U^n_i \text{)}
\]

Therefore, \( H(x_n) \in U_i \) for \( i=1,2,3,D \).

The following theorem gives necessary and sufficient conditions for \( F^n \) to dominate \( G^n \). First we need the following definition.

**Definition 3.3.1** \( F^n \) is said to be preferred to \( G^n \) in first, second, third and DARA degree if and only if \( E_{\text{F}^n} u_i[\hat{\phi}(x_1, \ldots, x_n)] \geq E_{\text{G}^n} u_i[\hat{\phi}(x_1, \ldots, x_n)] \) for all \( u_i \in U_j, j=1,2,3 \) and D, respectively (\( > \) holds for at least one \( u_i \in U_j \)).

We denote such relationship of preference by \( F^n >_i G^n \) for \( u \in U_j, i=1,2,3 \) and D.

**Theorem 3.3.1** If \( u \in U_j \) and \( \phi \in U^n_j \), then \( F^n >_j G^n \) if and only if \( F_i >_j G_i \) for all \( i, j=1,2,3,D \).
**PROOF:** The proof of sufficiency is by induction on $n$. Let $n=2$, then

$$E_{F^2}u_i(\phi) = \int_0^\infty \int_0^\infty u_i(\phi) \ dF_1(x_1) \ dF_2(x_2)$$

$$\geq \int_0^\infty \int_0^\infty u_i(\phi) \ dG_1(x_1) \ dF_2(x_2) \quad \text{(by theorem 3.2.1 and } F^1 >_j G_1)$$

$$\geq \int_0^\infty \int_0^\infty u_i(\phi) \ dG_1(x_1) \ dG_2(x_2) \quad \text{(by lemma 3.3.1 and } F^2 >_j G_2)$$

$$= E_{G^2}u_i(\phi).$$

Assume that the theorem holds for $n=k$. Let $n = k+1$, then

$$E_{F^{k+1}}u_i(\phi) = \int_0^\infty E_{F^k}u_i(\phi) \ dF_{k+1}(x_{k+1})$$

$$\geq \int_0^\infty E_{G^k}u_i(\phi) \ dF_{k+1}(x_{k+1}) \quad \text{(by induction assumption)}$$

$$\geq \int_0^\infty E_{G^k}u_i(\phi) \ dG_{k+1}(x_{k+1}) \quad \text{(by lemma 3.3.1 and } F^k+1 >_j G_{k+1})$$

$$= E_{G^{k+1}}u_i(\phi)$$

For the converse, let $u_i(\phi(x_1, \ldots, x_n)) = \phi(x_1, \ldots, x_n) = \sum_{i=1}^n \phi_i(x_i)$ with $\phi_i \in U_j$. Then

$$E_{F^n}u_i(\phi) = \sum_{i=1}^n E_{F^n} \phi_i(x_i)$$

$$E_{G^n}u_i(\phi) = \sum_{i=1}^n E_{G^n} \phi_i(x_i).$$
If \( F_i \prec_j G_i \) for some \( i \), then \( E_{F_i} \phi_i < E_{G_i} \phi_i \) for \( i = i^* \). Then by replacing \( \phi_i^* \) by \( \alpha \phi_i^* \),
\[
E_{F^n} u_i(\tilde{x}) - E_{G^n} u_i(\tilde{x}) = \sum_{i=1}^{n} \left[ E_{F_i} \phi_i(x_i) - E_{G_i} \phi_i(x_i) \right] + (E_{F_i} \phi_i^* - E_{G_i} \phi_i^*) < 0 \text{ for some } \alpha > 0
\]

But \( \phi \) and \( \tilde{\phi} \) have the same preference ordering. Therefore \( F_i \prec_j G_i \) for some \( i \) implies \( F^n \prec_j G^n \) for \( j = 1, 2, 3, D \). Thus \( F_i \succ_j G_i \) for all \( i \) for \( j = 1, 2, 3, D \) is a necessary condition for \( F^n \succ_j G^n \) for \( j = 1, 2, 3, D \).

The above theorem is a natural result for \( U_j \) since the calculation of \( \int \cdots \int u^n(x_1, \ldots, x_n) \, dF_1(x_1) \cdots dF_n(x_n) \) for any index \( i \) depends essentially on the general properties of \( u^n \) with respect to \( x_i \). Note that any separable utility function \( u^n \in U^n \) belongs to \( U^n_j \) if each component utility function \( u_i \in U_j \) for \( j = 1, 2, 3, D \).

### 3.4 Dependent Attributes

Levy and Paroush [56] have developed sufficient conditions for the special multivariate function \( \phi(x_1, x_2) = x_1 \cdot x_2 \). In this section we consider the more general problem of stochastic dominance where \( \phi \) satisfies

\[
\frac{\partial \phi}{\partial x_i} > 0 \quad \forall \ i.
\]

If the independence assumption of \( F^n \) and \( G^n \) is weakened, then the following first degree dominance relation holds.
Theorem 3.4.1\textsuperscript{3} If $\phi \in U^2_1$ and the following three conditions hold

(i) $F_1 >_1 G_1$

(ii) $F_2|1 >_1 G_2|1$

(iii) $\frac{\partial G_2|1}{\partial x_1} \leq 0$ then

$F^2 >_1 G^2$.

PROOF: Writing $x_2$ implicitly, $x_2 = x_2(\phi, x_1)$.

\[
F^2(\phi) = \int_0^\infty F_2|1(x_2(\phi, x_1)|x_1) \, dF_1(x_1) \quad (\phi \in U^2_1)
\]

\[
\leq \int_0^\infty G_2|1(x_2(\phi, x_1)|x_1) \, dF_1(x_1) \quad \text{(from ii)}
\]

\[
\leq \int_0^\infty G_2|1(x_2(\phi, x_1)|x_1) \, dG_1(x_1) = G^2(\phi)
\]

The last inequality is due to the following fact.

\[
\frac{dG_2|1(x_2(\phi, x_1)|x_1)}{dx_1} = \frac{\partial G_2|1}{\partial x_2} \frac{\partial x_2}{\partial x_1} + \frac{\partial G_2|1}{\partial x_1} \leq 0 \quad \text{(from iii and } \phi \in U^2_1)
\]

The result extends to the multivariate case. In the following $H_{i|1}$ denotes the conditional cumulative distribution of random variable $z_i$ given the realization $z_1, \ldots, z_{i-1}$.

\textsuperscript{3}It is assumed that the inverse function of $\phi$ exists (a sufficient condition for this is to have strictly increasing $\phi$). It is also assumed that the expectation with respect to conditional distribution is well defined.
Theorem 3.4.2 \[ F_{i,1} > G_{i,1} \] \( \forall i=1,2,\ldots,n \Rightarrow F_{i} > G_{i} \) if \( \phi(x_{1},\cdots,x_{n}) \in U_{i}^{n} \) and \( \frac{\partial G_{i}}{\partial x_{j}} < 0 \) \( \forall i=1,2,\cdots,n \) and \( j < i \).

**Proof:** Let \( \phi = \phi(x_{1},\cdots,x_{n}) \), we have

\[
F_{n}(\phi) = \int \cdots \int F_{n,1}(x_{n}(x_{n-1},\phi)|x_{n-1})dF_{n-1,1}(x_{n-1}|x_{n-2}) \cdots dF_{1,1}(x_{1})
\]

since \( \frac{\partial \phi(x_{1},\cdots,x_{n})}{\partial x_{n}} \geq 0 \) \( (\phi \in U_{1}^{n}) \)

\[
\leq \int \cdots \int G_{n,1}(x_{n}(x_{n-1},\phi)|x_{n-1})dF_{n-1,1}(x_{n-1}|x_{n-2}) \cdots dF_{1,1}(x_{1})
\]

since \( F_{n,1} \leq G_{n,1} \) by assumption

\[
\equiv \int \cdots \int H_{\phi}(x_{n},x_{n-1}) \ dF_{n-1,1}(x_{n-1}|x_{n-2}) \cdots dF_{1,1}(x_{1})
\]

where \( H_{\phi}(x_{n},x_{n-1}) \equiv G_{n,1}(x_{n}(x_{n-1},\phi)|x_{n-1}) \). \( H_{\phi}(x_{n},x_{n-1}) \) has the following property.

\[
\frac{dH_{\phi}(x_{n},x_{n-1})}{dx_{j}} \leq 0, \ j \leq n-1.
\]

Proof of this statement is parallel to the proof that \( \frac{dG_{2,1}(x_{2}(x_{1},\phi)|x_{1})}{dx_{1}} \leq 0 \) in theorem 3.4.1. Thus
\[ F^n(\phi) \leq \int \cdots \int H_\phi(x_n, x_{n-1}) \, dG_{n-1}|.(x_{n-1}|x^{n-2}) \, dF_{n-2}|.(x_{n-2}|x^{n-3}) \cdots dF_1(x_1) \]

since \( \frac{dH_\phi(x_n, x_{n-1})}{dx_{n-1}} \leq 0 \) and \( F_{n-1}|.| > 1 \, G_{n-1}|.| \)

\[ = \int \cdots \int H_\phi(x_n, x_{n-2}) \, dF_{n-2}|.(x_{n-2}|x^{n-3}) \cdots dF_1(x_1) \]

where

\[ H_\phi(x_n, x_{n-2}) = \int H_\phi(x_n, x_{n-1}) \, dG_{n-1}|.(x_{n-1}|x^{n-2}) \]

\[ \frac{dH_\phi(x_n, x_{n-2})}{dx_j} = \frac{d}{dx_j} \left[ \int H_\phi(x_n, x_{n-1}) \, dG_{n-1}|.(x_{n-1}|x^{n-2}) \right] \]

Integration by parts yields

\[ = \left[ \frac{d}{dx_j} \left[ H_\phi(x_n, x_{n-1}) \right] G_{n-1}|.(x_{n-1}|x^{n-2}) \right]_0^\infty \]

\[ - \int G_{n-1}|.|.(x_{n-1}|x^{n-2}) \frac{dH_\phi(x_n, x_{n-1})}{dx_{n-1}} \, dx_{n-1} \]

\[ = \frac{d}{dx_j} H_\phi(x_n, \infty, x_{n-2}) - \int \frac{dG_{n-1}|.|.(x_{n-1}|x^{n-2})}{dx_j} \frac{dH_\phi(x_n, x_{n-1})}{dx_{n-1}} \, dx_{n-1} \]

\[ - \int G_{n-1}|.|.(x_{n-1}|x^{n-2}) \frac{d^2H_\phi(x_n, x_{n-1})}{dx_j \, dx_{n-1}} \, dx_{n-1} \]

Integration by parts of the third term yields
\[
\begin{align*}
&= -\int \frac{dG_{n-1}}{dx_j} \frac{\partial H_\phi(x_n,x_{n-1})}{\partial x_{n-1}} d\xi_{n-1} \\
&+ \int \frac{dG_{n-1}}{dx_j} \frac{\partial H_\phi(x_n,x_{n-1})}{\partial x_{n-1}} d\xi_{n-1} \\
&\leq 0 \quad \forall j \leq n-2 \\ &\text{since } \frac{\partial H_\phi(x_n,x_{n-1})}{\partial x_j} \leq 0 \quad \forall j \leq n-1, \\
&\frac{dG_{n-1}}{dx_j} \leq 0 \quad \forall j \leq n-2 \text{ by assumption,} \\
&\text{and } \frac{dG_{n-1}}{dx_{n-1}} \geq 0 \quad \text{by definition.} \\
\end{align*}
\]

i.e. \( \frac{\partial H_\phi(x_n,x_{n-1})}{\partial x_j} \leq 0 \quad \forall j \leq n-1 \) and \( \frac{dG_{n-1}}{dx_j} \leq 0 \quad \forall j \leq n-2 \)

imply \( \frac{\partial H_\phi(x_n,x_{n-2})}{\partial x_j} \leq 0 \quad \forall j \leq n-2. \)

Hence, by repeating the above procedure we obtain

\[
F^n(\phi) \leq \int \cdots \int H_\phi(x_n,x_n) dG_{n-2} | (x_{n-2} | x_{n-3}) dF_{n-3} | (x_{n-3} | x_{n-4}) \cdots dF_1(x_1)
\]

\[
= \int \cdots \int H_\phi(x_n,x_3) dF_{n-3} | (x_{n-3} | x_{n-4}) \cdots dF_1(x_1)
\]

\[
\vdots
\]

\[
= \int H_\phi(x_n,x_1) dF_1(x_1)
\]

\[
\leq \int H_\phi(x_n,x_1) dG_1(x_1)
\]

\[
= G^n(\phi).
\]
Remarks

(1) The results in Levy [51], Levy and Paroush [56] and Huang et al. [43]'s are the special case of theorems 3.3.1, 3.4.1 and 3.4.2 with
\[ \phi(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i. \]
It seems that a sufficient condition under which \( F^n \) dominates \( G^n \) in \( >_2, >_3 \) and \( >_D \) degree is difficult to obtain.

(2) These theorems also apply directly to utility functions of the following specific forms

(a) logadditive [75]

(b) quasi separable [22,46]

(3) For independent \( x_i \)'s \( (F_i\| = F_i, G_i\| = G_i \) and \( \frac{\partial G_i}{\partial x_i} = \frac{\partial G_i}{\partial x_i} = 0 \) theorem 3.4.2 yields the same sufficient conditions as stated in theorem 3.3.1.
Chapter 4

QUALITATIVE ASPECTS OF OPTIMAL PORTFOLIO BALANCING

4.1 Introduction

Since a central theme in portfolio theory is the investigation of the role of diversification we now focus upon the implication of stochastic dominance rules to the problem of optimal portfolio balancing. In particular, the effect on the optimal mix under the following two conditions is considered:

(1) Portfolios formed by one safe asset and one risky asset, and

(2) Portfolios formed by two risky assets.

We examine the effect on the optimal mix due to changes in the probability distribution of the return of a risky asset in cases (1) and (2). For case (1) Fishburn and Porter [28] have analyzed the effect on the optimal mix of the two assets due to changes in the rate of return of the riskless asset and shifts in the probability distribution of return of the risky asset. They obtained a general criterion, in terms of the index of absolute risk aversion which indicates that any improvement (in FSD sense) in the probability distribution for the risky asset must lead to an increase, or at least no decrease, in the optimal proportion invested in the risky
asset. However, their result is restricted to the case where the cumulative distribution of return has compact support \([0,h]\), \(h\) finite. In this analysis we extend their result in the following manner. Improvements in probability distributions in SSD and TSD as well as FSD sense and the distributions which we consider are general nonnegative random variables. The effect of location and scale parameter changes in the probability distribution of return for a risky asset is examined. For a special class of distributions this type of change is equivalent to FSD and SSD improvements in cumulative distributions. We also generalize Fishburn and Porter's work (FSD sense) by considering the effect on the optimal mix when changes in the probability distribution occur under condition (2). In addition to these analyses we examine the time effect on the optimal mix. Formally the problem upon which we focus is

\[
\begin{align*}
\max_{\lambda_1 \in \Lambda_1} E_{H_2} \max_{\lambda_2 \in \Lambda_2} E_{H_2|1} u(W_1 + \sum_{j=1}^{n} \lambda_2 j X_{2j}) \\
W_1 = W_0 + \sum_{j=1}^{n} \lambda_1 j X_{1j}
\end{align*}
\]

(P)

\[
u \in U_2
\]

where \(X_t \equiv (X_{t1}, \ldots, X_{tn})\) and the nonnegative returns in period \(t\), \(t=1,2\); \(H_t\) represent cumulative joint distributions of \(X_t\), \(t=1,2\); \(H_{2|1}\) represents the conditional distribution of \(X_2\) given \(X_1\); \(\Lambda_t \equiv \{\lambda_t = (\lambda_{t1}, \ldots, \lambda_{tn})| \sum_{j=1}^{n} \lambda_{tj} = W_{t-1}, \lambda_{tj} \geq 0\}, t=1,2\); \(E_k\) represents the expectation operator for a cumulative distribution \(K\), and \(W_{t+s}\) are the initial wealth at the beginning of period \(t+1\), \(t=0,1,2\).
It is convenient to assume that the random variables are net rather than the gross returns (i.e. 1 + return). We first consider case 1.

### 4.2 One Safe and One Risk Assets

We consider the simple problem where at the beginning of each period the choice of portfolio mix is limited to one safe and one risky asset.

Let

- $\rho_t$ = the return on the riskless asset at the beginning of period $t$, $t=1,2$.
- $X_t$ = the random variable representing the return on the risky asset at the beginning of period $t$, $t=1,2$.
- $F_t$ = the cumulative distribution of $X_t$, $t=1,2$.
- $F_{2/1}$ = the conditional distribution of $X_2$ given $X_1 = x_1$.

Then (P) becomes

$$\max_{\lambda_1 \in \Lambda_1} \max_{\lambda_2 \in \Lambda_2} \mathbb{E}_{F_1} \mathbb{E}_{F_{2/1}} u(W_1(1 + \rho_2) + \lambda_2(X_2 - \rho_2))$$

s.t. $W_i = W_0(1 + \rho_0) + \lambda_1(X_1 - \rho_1)$, where $\lambda_2 = \lambda_{22}$ and $\lambda_{21} = 1 - \lambda_2$.

Let $\lambda_t^*$ denote an optimal $\lambda_t$ at the beginning of period $t=1,2$. The inner maximization problem gives a "derived" utility function $\phi(W_1)$ which represents the preference of period one distribution incorporating the uncertain nature of period two prospects [69]. Note that $\lambda_2^* = 0$ if and only if the slope of $\mathbb{E}_{F_{2/1}} u(W_2)$ is nonpositive at $\lambda_2^* = 0$ and $\lambda_2^* = W_i$ if
and only if this slope is nonnegative at $\lambda_2^* = W_1$. Thus nontrivial allocation of $W_1$ occurs if and only if

$$\int \frac{x_2 u'(W_1(l + x_2)) dF_{2/1}(x_2|x_1)}{u'(W_1(l + x_2)) dF_{2/1}(x_2|x_1)} < p_2 < E(X_2|X_1 = x_1).$$

Clearly, $p_4 > E(X_2|X_1 = x_1)$ implies $\lambda_1^* = 0$. In fact, Hadar and Russell prove

**Theorem 4.2.1** If $\rho > E(X)$, then $W + \lambda X + (W - \lambda)\rho > W + \lambda' X + (W - \lambda')$; where $0 \leq \lambda < \lambda' \leq W$.

**Proof:** See [31, Thm. 10].

If the distribution assessment of $F_{2/1}$ changes to $G_{2/1}$ and $G_{2/1}$ is preferred to $F_{2/1}$ in stochastic dominance sense, then do we necessarily have $\lambda_{G_{2/1}}^* \geq \lambda_{F_{2/1}}^*$? A solution to this query was obtained by Fishburn and Porter [28] when the preference relation is FSD and the distributions are restricted to those defined on $[0, h]$. They showed that the appropriate sufficient condition is $r(w^*)w_0^*(x-\rho) \leq 1$, $w^* = W_0[\lambda^*(x-\rho) + \rho]$. We show that their result can be generalized to FSD as well as the SSD and TSD cases with arbitrary nonnegative random variables. The setting for our analysis is over a two period horizon. The results for period 2 are comparable to those of Fishburn and Porter [28] who utilized a static model.

Let $f(x_2) = u'(w_1(l+\rho_2) + \lambda^*(x_2-\rho_2))(x_2-\rho_2)$

$w_2^* = W_1(l+\rho_2) + \lambda^*(x_2-\rho_2)$.

Consider the following conditions:
(C1) \( r(w_2^*) \lambda^*(x_2 - \rho_2) \leq 1 \quad x_2 \in (\rho_2, \infty) \)

(C2) \(-\frac{u''''(w_2^*)}{u''(w_2^*)} \lambda^*(x_2 - \rho_2) \leq 2 \quad x_2 \in (\rho_2, \infty) \)

(C3) \(-\frac{u''''(w_2^*)}{u''''''(w_2^*)} \lambda^*(x_2 - \rho_2) \leq 3 \quad x_2 \in (\rho_2, \infty) \)

**Lemma 4.2.1** If C1, C1 and C2, and C1, C2 and C3 hold then \( f \in U_j \), \( j=1,2,3 \), respectively.

**PROOF:**

\[
\begin{align*}
    f'(x_2) &= u''(w_2^*) \lambda^*(x_2 - \rho_2) + u'(w_2^*) \\
    f''(x_2) &= u''''(w_2^*) (\lambda^*)^2 (x_2 - \rho_2) + 2 \lambda^* u''(w_2^*) \\
    f'''(x_2) &= u''''''(w_2^*) (\lambda^*)^3 (x_2 - \rho_2) + 3 (\lambda^*)^2 u'''(w_2^*) .
\end{align*}
\]

Therefore if C1 holds then \( f'(x_2) > 0 \), i.e. \( f \in U_1 \);

if C1 and C2 hold then \( f''(x_2) < 0 \) and \( f'(x_2) > 0 \),

i.e. \( f \in U_2 \);

if C1, C2, and C3 hold then \( f''' > 0 \), \( f'' < 0 \) and \( f' > 0 \),

i.e. \( f \in U_3 \).

We can now state the following theorem.

**Theorem 4.2.2** If \( u \in U_3 \) and \( G_{2/1} > F_{2/1} \) then sufficient conditions for

\[
\lambda^*_{G_{2/1}} > \lambda^*_{F_{2/1}}
\]

are
(1)  \( r(w^*) \lambda^* F_{2/1} (x_2-\rho_2) \leq 1 \)  \( x_2 \in (\rho_2, \infty) \)

(2)  \( -\frac{u''(w^*)}{u''(w^*)} \lambda^* F_{2/1} (x_2-\rho_2) \leq 2 \)  \( x_2 \in (\rho_2, \infty) \)

\[ \frac{dE_G u(W_2)}{d\lambda_2} \bigg|_{\lambda_2=\lambda^*_F} = \frac{dE_G u(W_2)}{d\lambda_2} - \frac{dE_F u(W_2)}{d\lambda_2} \bigg|_{\lambda_2=\lambda^*_F} \]

\[ = \int_0^\infty u'(w^*)(x_2-\rho_2) dG_{2/1}(x_2|x_1) - F_{2/1}(x_2|x_1) \]

\[ = \int_0^\infty f(x_2) dG_{2/1}(x_2|x_1) - F_{2/1}(x_2|x_1) \]

\[ \geq 0 \text{ since } f \in U_2 \text{ and } G_{2/1} > F_{2/1} \]

**Theorem 4.2.3** If \( u''' \leq 0,4 u \in U_3 \) and \( G_{2/1} > F_{2/1} \) then sufficient conditions for \( \lambda_{G_{2/1}}^* \geq \lambda_{F_{2/1}}^* \) are

(1)  \( r(w^*) \lambda^* F_{2/1} (x_2-\rho_2) \leq 1 \)  \( x_2 \in (\rho_2, \infty) \)

(2)  \( -\frac{u''(w^*)}{u''(w^*)} \lambda^* F_{2/1} (x_2-\rho_2) \leq 2 \)  \( x_2 \in (\rho_2, \infty) \)

(3)  \( -\frac{u''(w^*)}{u''(w^*)} \lambda^* F_{2/1} (x_2-\rho_2) \leq 3 \)  \( x_2 \in (\rho_2, \infty) \)

**PROOF:** The proof parallels the proof of theorem 4.2.2.

---

4 \( u''' \leq 0 \) assumption is related to the peakness measurement.
Remark: Note that if \( u \in U_D \) then \( -\frac{u^{''''}}{u''} \geq -\frac{u''}{u'} \), i.e. conditions C1 and C2 are noncontradictory. Fishburn and Porter's FSD result follows immediately from \( f \in U_1 \).

If we rewrite the conditions C1, C2 and C3 as

\[
\text{(C1')} \quad \frac{1}{\lambda_{F_2/1}} \frac{1}{(x_2 - \rho_2)} > r(w^*),
\]

\[
\text{(C2')} \quad \frac{2}{\lambda_{F_2/1}} \frac{2}{(x_2 - \rho_2)} > -\frac{u^{''''}(w^*)}{u''(w^*)},
\]

\[
\text{(C3')} \quad \frac{3}{\lambda_{F_2/1}} \frac{3}{(x_2 - \rho_2)} > -\frac{u^{''''}(w^*)}{u''''(w^*)},
\]

then each of these restrictions gives us a band for acceptable utility functions. Figure 4.1 on the following page illustrates these conditions.

Consider the following utility function.

\[
u_1(x) = -\exp(-\alpha x), \ \alpha > 0,\]

\[
u_2(x) = (x + \beta)\gamma, \ \beta > 0, \ 0 < \gamma < 1,\]

\[
u_3(x) = \log(x + \delta); \ \delta > 0, \text{ and}\]

\[
u_4(x) = -(x + \phi)^{-\psi}; \ \phi > 0, \ \psi > 0.
\]

Note that \( u_1 \) and \( u_4 \) do not satisfy conditions C1-C3. \( u_2 \) satisfies conditions C1-C3 while \( u_3 \) satisfies C1 and C2. A basic reason why the conditions are satisfied by \( u_2 \) and \( u_3 \) and not by \( u_1 \) and \( u_4 \) is that the rate of...
Figure 4.1. Illustration of assumptions C1', C2' and C3'.
convergence as $x_2$ approaches infinity of $r(w)$, $\frac{-u'''(w)}{u''(w)}$ and $\frac{-u'''(w)}{u''(w)}$ functions is faster than the rate of convergence of $\frac{1}{\lambda_F(x_2-\rho_2)}$.

In the above analysis we require additional properties for $u$ as we expand our analysis from FSD to SSD and TSD.

The restriction C1, C2, and C3 on $u$ prevent "unexpected movement" of the optimal mix when the improvement in the probability distribution of return for the risky asset is introduced with the rate of return of the safe asset held fixed. Figure 4.2 depicts the situation.

![Figure 4.2. The optimal mix for changing portfolio composition.](image)
We now consider the effect of changes in location and scale parameters $\delta$ and $\beta$ on $\lambda^*$, respectively. Tobin [101] and Rothschild and Stiglitz [88] analyzed changes in the optimal portfolio mix due to changes in mean and variance of return for the risky asset. Tobin claimed (incorrectly) that if $u(x)$ is quadratic then an increase in the expected rate of return, $E(X)$, will lead to an increase in $\lambda$. Tobin [101] and Rothschild and Stiglitz [88] showed that an increase in the variance, $V(X)$, of the rate of return of the risky asset with $E(X)$ held fixed reduces $\lambda$, if $u(x)$ is quadratic. Rothschild and Stiglitz [88] also showed that for any $u \in U_D$, an increase in $V(X)$ with $E(X)$ held fixed will reduce $\lambda$ if the relative risk aversion, $R(x)$, is less than or equal to one and $R'(x) > 0$.

Mossin [68], Tobin [101], Richter [86], Stiglitz [98] and Näslund [70] deal with the effect of taxation on social risk-taking. Results obtained by these authors are of the following form: If an increase in income tax is introduced by the government, then the public will take a greater risk ($\lambda$ increases) provided the public utility function is of the type which belongs to $U_D$ and has increasing relative risk aversion. Feldstein [23] showed that the tax has no effect on $\lambda$ when $R(x)$ is constant (e.g. $u(x) = \ln(x)$). Leland [79] deals with the allocation problem between safe and risky foreign exchange. He also obtained a similar risk-taking behaviour by the public. Since the stochastic dominance relations are determined by the above parameters for the special class of distributions (see Chapter 2) we shall consider the location and scale parameter changes.

The first and second order optimality conditions are:

$$\int_0^\infty u'(w_2)(x_2-p_2) \ dF_{2/1}(x_2|x_1) = 0,$$  and

$$\int_0^\infty u'(w_2)(x_2-p_2) \ dF_{2/1}(x_2|x_1) = 0.$$  (4.2.1)
\[ \int_0^\infty u''(w_2)(x_2-\rho_2)^2 \, dF_{2/1}(x_2 | x_1) \leq 0. \]  

(4.2.2)

If \( u \) is strictly concave then \( \lambda_2^* \) is unique. Let the risky asset \( Z_2(\delta, \beta) = \beta x_2 + \delta \), where \( \delta, \beta > 0 \) and \( \rho_2 > \delta \). The effect of changes in the location parameter \( \delta \) on \( \lambda_2^* \) is determined by differentiating (4.2.1) with respect to \( \delta \).

\[
\frac{d\lambda_2^*}{d\delta} = \int_0^\infty u''(w_2^*) \lambda_2^* \beta x_2 + \delta - \rho_2 \, dF_{2/1}(x_2 | x_1) + \int_0^\infty u'(w_2^*) \, dF_{2/1}(x_2 | x_1) \\
\phantom{\frac{d\lambda_2^*}{d\delta}} - \int_0^\infty u''(w_2)(\beta x_2 + \delta - \rho_2)^2 \, dF_{2/1}(x_2 | x_1)
\]

where \( w_2^* = w_1(1+\rho_2) + \lambda_2^* (\beta x_2 + \delta - \rho_2) \). The sign of \( \frac{d\lambda_2^*}{d\delta} \) depends on the value of the numerator, since \( \int_0^\infty u''(w_2^*)(\beta x_2 + \delta - \rho_2)^2 \, dF_{2/1}(x_2 | x_1) < 0. \)

**Theorem 4.2.4** If \( u \in U_3 \) and \( E(x_2 | x_1) \leq \frac{\rho_2 - \delta}{\beta} \) then \( \frac{d\lambda_2^*}{d\delta} \geq 0. \)

**Proof:**

\[
\int_0^\infty u''(w_2^*) \lambda_2^* (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2 | x_1)
\]

\[
\frac{\rho_2 - \delta}{\beta} \int_0^\infty u''(w_2^*) \lambda_2^* (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2 | x_1)
\]

\[
+ \int_0^\infty u''(w_2) \lambda_2^* (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2 | x_1)
\]

\[
\frac{\rho_2 - \delta}{\beta} \int_0^\infty u''(w_2^*) \lambda_2^* (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2 | x_1)
\]

\[
\frac{\rho_2 + \delta}{\beta} \int_0^\infty u''(w_1(1+\rho_2)) \lambda_2^* (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2 | x_1)
\]
\[ + \int_{\beta}^{\infty} u''(w_1(1+\rho_2)) \lambda^*_2(\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1) \text{ (since } u \in U_3) \]

\[ = u''(w_1(1+\rho_2)) \lambda^*_2 \int_0^{\infty} (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1). \]

Therefore \( \frac{d\lambda^*_2}{d\delta} > 0 \) if \( \int_0^{\infty} (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1) < 0. \) Since \( E(X_2|X_1) \leq \frac{\rho_2 - \delta}{\beta} \), the result follows.

Note that if we consider the change in location parameter from \( \delta = 0 \) and \( \beta = 1 \), then the condition \( E(X_2|X_1) < \frac{\rho_2 - \delta}{\beta} \) restricts \( \lambda^*_2 \) from becoming \( W_2 \).

The following theorem gives an effect of the scale parameter change on \( \lambda^*_2 \).

**Theorem 4.2.5** If \( u \in U_3 \) and the ratio of conditional second moment and the first moment,

\[ R = \frac{x_2^2 \, dF_{2/1}(x_2|x_1)}{x_2 \, dF_{2/1}(x_2|x_1)} < \frac{\rho_2 - \delta}{\beta} \text{ then } \frac{d\lambda^*_2}{d\beta} > 0. \]

**PROOF:** since

\[ \frac{d\lambda^*_2}{d\beta} = \frac{u''(w_2) x_2 \lambda^*_2(\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1) + u'(w_2) x_2 \lambda^*_2 \, dF_{2/1}(x_2|x_1)}{-u''(w_2)(\beta x_2 + \delta - \rho_2)^2 \, dF_{2/1}(x_2|x_1)} \]

\[ \frac{d\lambda^*_2}{d\beta} > 0 \text{ if } \int u''(w_2) x_2 (\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1) > 0. \]

But
\[ \int_0^\infty u''(w_2^*)x_2l_2^*(\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1) \]

\[ = - \frac{\rho_2 - \delta}{\beta} \int_0^\infty u''(w_2^*)l_2 x_2(\rho_2 - \beta x_2 - \delta) \, dF_{2/1}(x_2|x_1) \]

\[ + \int_0^\infty u''(w_2^*) l_2 x_2(\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1) \]

\[ > - \frac{\rho_2 - \delta}{\beta} \int_0^\infty u''(w_1(1+\rho_2)) l_2 x_2(\rho_2 - \beta x_2 - \delta) \, dF_{2/1}(x_2|x_1) \] (since \( u \in U_3 \))

\[ + \int_0^\infty u''(w_1(1+\rho_2)) l_2 x_2(\beta x_2 + \delta - \rho_2) \, dF_{2/1}(x_2|x_1) \]

\[ = u''(w_1(1+\rho_2)) l_2 \int_0^\infty [x_2^2\beta + x_2(\delta - \rho_2)] \, dF_{2/1}(x_2|x_1) \]

\[ = u''(w_1(1+\rho_2)) l_2 [\beta \int_0^\infty x_2^2 \, dF_{2/1}(x_2|x_1) + (\delta - \rho_2) \int_0^\infty x_2 \, dF_{2/1}(x_2|x_1)] \]

\[ > 0 \] (from \( R < \frac{\rho_2 - \delta}{\beta} \))

We now consider the effects of changes in \( \lambda_1, F_1 \) and \( \rho_1 \) upon \( \lambda_2^* \) (this will introduce the time element to the portfolio selection problem). We first analyze the effect of \( W_1 \) on \( \lambda_2^* \) since \( W_1 \) summarizes all the relevant information included in parameters of the first period.

The effect of changes in \( W_1 \) on \( \lambda_2^* \) is obtained by differentiating (4.2.1) with respect to \( W_1 \).
\[ \frac{d\lambda^*_2}{dW_1} = -\int_{0}^{\infty} \frac{u''(w_2^*)(1+\rho_2)(x_2-\rho_2)}{u''(w_2^*) (x_2-\rho_2)^2} \ dF_{2/1}(x_2|x_1) \]

where \( w_2^* \equiv w_1(1+\rho_2) + \lambda^*_2(x_2-\rho_2) \).

The sign of \( \frac{d\lambda^*_2}{dW_1} \) depends on the value of the numerator in the above equation since \( \int u''(w_2^*)(x_2-\rho_2) \ dF_{2/1}(x_2|x_1) < 0 \). The sign of \(-\int u''(w_2^*)(1+\rho_2)(x_2-\rho_2) \ dF_{2/1}(x_2|x_1)\) cannot be determined from \( u'' \) alone and information on the risk aversion index is needed. The following result is due to Arrow [3] and Cass and Stiglitz [17]. For completeness we state and prove the result for the model under consideration.

**Theorem 4.2.6** If \( r' \geq 0 \) then \( \frac{d\lambda^*_2}{dW_1} \leq 0 \).

**Proof:**

\[ -\int_{0}^{\infty} u''(w_2^*)(1+\rho_2)(x_2-\rho_2) \ dF_{2/1}(x_2|x_1) = \int_{0}^{\rho_2} u''(w_2^*)(1+\rho_2)(\rho_2-x_2) \ dF_{2/1}(x_2|x_1) - \int_{\rho_2}^{\infty} u''(w_2^*)(1+\rho_2)(x_2-\rho_2) \ dF_{2/1}(x_2|x_1) \]

\[ = \int_{0}^{\rho_2} -u'(w_2^*) \cdot r(w_2^*)(\rho_2-x_2)(1+\rho_2) \ dF_{2/1}(x_2|x_1) \]

\[ + \int_{\rho_2}^{\infty} u'(w_2^*) \cdot r(w_2^*)(1+\rho_2)(x_2-\rho_2) \ dF_{2/1}(x_2|x_1) \]

\[ \forall \lambda^* \int_{0}^{\rho_2} -u'(w_2^*) \cdot r(w_1(1+\rho_2))(\rho_2-x_2)(1+\rho_2) \ dF_{2/1}(x_2|x_1) \]

\[ + \int_{\rho_2}^{\infty} u'(w_2^*) \cdot r(w_1(1+\rho_2))(x_2-\rho_2)(1+\rho_2) \ dF_{2/1}(x_2|x_1) \] (since \( r' \geq 0 \))

Since the last integral is equal to zero \( r' \geq 0 \) implies \( \frac{d\lambda^*_2}{dW_1} \leq 0 \).
Let $E$ be the expected value operator with respect to the joint distribution of $X_1$ and $X_2$, then we have

**Corollary 4.2.1** If $r' \leq 0$ then $\frac{dE(W_2)}{dW_1} \leq 0$.

**Proof:** Since $\frac{dE(W_2)}{dW_1} = \frac{d\lambda_1}{dW_1} (E(X_2) - \rho_1)$ and the nontrivial solution requires $E(X_2 | X_1 = x_1) > \rho_1$ for each $x_1$. Then from the above theorem

$$\frac{d\lambda_1}{dW_1} (E(X_2) - \rho_2) \leq 0 \quad \text{if} \quad r' \leq 0$$

A rational individual is typically assumed to have $r' \leq 0$ [3,81]. Examples where $r' > 0$, $r' = 0$ and $r' < 0$ can be found in [34,81,99,101]. Note that $r' < 0$ implies $u'' > 0$ but not conversely.

Now consider the effect of changes in $W_0$ on $\lambda_2^*$. 

$$\frac{dW_1^*}{dW_0} = (1+\rho_1) + \frac{d\lambda_1^*}{dW_0} (X_1 - \rho_1) \quad \text{and}$$

$$\frac{dE(W_1^*)}{dW_0} = (1+\rho_1) + \frac{d\lambda_1^*}{dW_0} (E(X_1) - \rho_1)$$

where $\lambda_1^*$ is the optimal parameter of allocation for period one.

Therefore, we need to determine the sign of $\frac{d\lambda_1^*}{dW_0}$ in order to identify the signs of $\frac{dW_1^*}{dW_0}$ and $\frac{dE(W_1^*)}{dW_0}$.

First, consider the following lemma.
Lemma 4.2.1 \( \phi(w_1) \) is strictly increasing and concave.

**Proof:** 
\[ \phi'(w_1) = E_{F_{2/1}} u'(w_1(l+p_2) + \lambda^*_2(x_1-p_2)) \]
\[ = E_{F_{2/1}} u'(w_1(l+p_2) + \lambda^*_2(x_2-p_2))[(l+p_2) + \frac{d\lambda^*_2}{dw_1} (x_2-p_2)] \]
\[ = E_{F_{2/1}} u'(w_2)(l+p_2) > 0 \]

\[ \phi''(w_1) = E_{F_{2/1}} u''(w_1(l+p_2) + \lambda^*_2(x_2-p_2))[(l+p_2) + \frac{d\lambda^*_2}{dw_1} (x_2-p_2)]^2 \]
\[ + E_{F_{2/1}} u'(w_1(l+p_1) + \lambda^*_2(x_2-p_2))\left[\frac{d^2\lambda^*_2}{dw_1^2} (x_2-p_2)\right] \leq 0 \]

Note that if \( u \) is quadratic, then \( \phi \) is quadratic.

Then the first period solution \( \lambda^*_1 \) satisfies

\[ E_{F_1} \phi'(W_1)(x_1-p_1) = 0 \] (4.2.3)
\[ E_{F_1} \phi''(W_1)(x_1-p_1)^2 < 0 \] (4.2.4)

Hence,

\[ \frac{d\lambda^*_1}{dw_0} = \frac{-E_{F_1} \phi''(W^*_1)(x_1-p_1)(l+p_1)}{E_{F_1} \phi''(W^*_1)(x_1-p_1)^2} \]

**Theorem 4.2.7** If

i) \( r' > 0 \),
ii) $-\frac{\phi''}{\phi'} > 0$ for all $x_1$, and

iii) $x_1 < \rho_1$.

Then

$$\frac{d\lambda_2^*}{dW_0} > 0,$$

**PROOF:**

$$\frac{d\lambda_2^*}{dW_0} = \frac{d\lambda_2^*}{dW_1^*} \frac{dW_1^*}{dW_0}
= \frac{d\lambda_2^*}{dW_1^*} \left[ (1+\rho_1) + \frac{d\lambda_1^*}{dW_0} (x_1-\rho_1) \right],$$

$$> 0.$$

The last inequalities follow since $\frac{d\lambda_2^*}{dW_1^*} > 0$ from (i) and

$$\frac{d\lambda_1^*}{dW_0} (x_1-\rho_1) > 0$$ from (ii) and (iii).

We note that $r' = 0$ then $\frac{d\lambda_2^*}{dW_0} = 0$ since $\frac{d\lambda_2^*}{dW_1^*} = 0$, if $-\frac{\phi''}{\phi'} = 0$ then $\frac{d\lambda_2^*}{dW_0} = 0$, if $x_1 = \rho_1$, then the sign of $\frac{d\lambda_2^*}{dW_0}$ depends on $r'$.

The condition (iii) requires the actual realization of a risky venture in period one, however, since the change in $W_0$ must take place before the realization, the proper quantity to be considered is the expected change in $\lambda_2^*$. From the corollary 4.2.1, the expected wealth change, $\frac{dE(W_1)}{dW_0}$, is nonnegative if $-\frac{\phi''}{\phi'} > 0$. Hence the expected change of $\lambda_2^*$ must be positive, given the assumptions $r' > 0$ and $-\frac{\phi''}{\phi'} > 0$.

4.3. The Two Risky Asset Case

We now consider the effect on the optimal mix when both assets are random. The model is
Max \( \lambda_1 \in \Lambda_1 \) \( E_{H_0} \left[ \max_{\lambda_2 \in \Lambda_2} E_{H_2/1} u(W_1(1 + X_{22}) + \lambda_2(X_{21} - X_{22})) \right] \)

s.t. \( W_1 = W_0(1 + X_{12}) + \lambda_1(X_{11} - X_{12}) \).

The optimal allocation \( \lambda^*_2 \) in period two satisfies

\[
E_{H_2/1} u'(W_2)(X_{21} - X_{22}) = 0 \tag{4.3.1}
\]

\[
E_{H_2/1} u''(W_2)(X_{21} - X_{11})^2 \leq 0 \tag{4.3.2}
\]

and

\[
\frac{d\lambda^*_2}{dW_1} = \frac{-E_{H_2/1} u''(W_2^*)(1 + X_{22})(X_{21} - X_{22})}{H_{2/1} u''(W_2^*)(X_{21} - X_{22})^2}.
\]

Clearly, the sign of \( \frac{d\lambda^*_2}{dW_1} \) is determined by the numerator of the above fraction. However, the sign of \( E_{H_2/1} u''(W_2^*)(1 + X_{22})(X_{21} - X_{22}) \) is difficult to determine since the calculation of the above integral involves a joint distribution. It is convenient to assume that investment returns are serially independent. A similar model was considered by Brunelle [14] and Pyle [83].

Let \( \psi(x_{22}) = \int_0^\infty -u''(w^*)(1 + x_{22})(x_{21} - x_{22}) \, dH(x_{21}|x_{22}) \), where \( H(x_{21}|x_{22}) \) denotes the conditional distribution of \( X_{21} \) given \( X_{22} = x_{22} \).

Similar results to those in Section 4.2 can be derived but are not included here. We state and prove the following two results to illustrate the method of proof used.
Theorem 4.3.1 If \( r' \geq 0 \) and

\[
\int_0^\infty u'(w_2)(1 + x_{22})(x_{21} - x_{22}) \, dH(x_{21} | x_{22}) \geq 0 \quad \forall \ x_{22}
\]

then

\[
\frac{d\lambda_2^*}{d\lambda_1} \leq 0.
\]

**Proof:**

\[
-\int_0^\infty \int_0^\infty u''(w_2^*)(1 + x_{22})(x_{21} - x_{22}) \, dH_1(x_{21}, x_{22} | x_{11}, x_{12})
\]

\[
= -\int_0^\infty \int_0^\infty u''(w_2^*)(1 + x_{22})(x_{21} - x_{22}) \, dP(x_{21} \leq x_{21}, x_{22} \leq x_{22})
\]

\[
= \int_0^\infty \psi(x_{22}) \, dP(x_{22} \leq x_{22})
\]

Now \( \psi(x_{22}) = -\int_0^{x_{22}} r(w_2^*)(u(w_2^*)(1 + x_{22})(x_{22} - x_{21}) \, dH(x_{21} | x_{22})
\]

\[
+ \int_{x_{22}}^\infty r(w_2^*)u'(w_2^*)(1 + x_{22})(x_{22} - x_{21}) \, dH(x_{21} | x_{22})
\]

\[
\geq r(w_1(l + x_{22})) \int_0^\infty u'(w_2^*)(1 + x_{22})(x_{21} - x_{22}) \, dH(x_{21} | x_{22}) \quad (\text{from } r' \geq 0)
\]

Therefore \( \int_0^\infty u'(w_2)(1 + x_{22})(x_{21} - x_{22}) \, dH(x_{21} | x_{22}) \geq 0 \) implies \( \frac{d\lambda_2^*}{d\lambda_1} \leq 0 \).

The following theorem generalizes Fishburn and Porter's result.

Theorem 4.3.2 Let \( G(x, y) = H(y | x)G_1(x) \) and \( F(x, y) = H(y | x)F_1(x) \) be the joint distributions of random variables \( X \) and \( Y \), then \( \lambda_G^* \geq \lambda_F^* \) whenever the following conditions hold.
(i) \( G_1 >_1 F_1 \)

(ii) \( \frac{d^2 u(w_2)}{dx d\lambda} \frac{dH(y|x)}{dy} + \frac{du(w_2)}{d\lambda} \frac{d^2 H(y|x)}{dy dx} > 0 \) for each \( x \)

(iii) \( \frac{d}{dx} \left[ \lim_{a \to \infty} H(a|x) \frac{du(w_2)}{d\lambda} \right] = \lim_{a \to \infty} \left[ \frac{dH(a|x)}{dx} \frac{du(w_2)}{d\lambda} \right] \), and

\[ \lim_{a \to \infty} H(a|x) \frac{d^2 u(w_2)}{dx d\lambda} < \infty \text{ and } \lim_{a \to \infty} \frac{dH(a|x)}{dx} \frac{du(w_2)}{d\lambda} < \infty, \]

where \( w_2 = w_1(l+y) + \lambda(x-y) \) and \( \lambda^*_G, \lambda^*_F \) denote the optimal mix under \( G \) and \( F \), respectively.

**PROOF:**

\[ \frac{dE}{d\lambda} \left. u(w_1(l+y) + \lambda(x-y)) \right|_{\lambda=\lambda^*_F} \]

\[ = \frac{dE}{d\lambda} \left[ u(w_1(l+y) + \lambda(x-y)) \right] - \frac{dE}{d\lambda} \left[ u(w_1(l+y) + \lambda(x-y)) \right] \bigg|_{\lambda=\lambda^*_F} \]

\[ = \frac{d}{d\lambda} \left[ \int_0^\infty \int_0^\infty u(w_1(l+y) + \lambda(x-y)) \frac{dH(y|x)}{d\lambda} \right. \left. \frac{dG_1}{dx} \right|_{\lambda=\lambda^*_F} \]

\[ = \int_0^\infty \int_0^\infty \frac{d}{d\lambda} u(w_1(l+y) + \lambda(x-y)) \bigg| \lambda=\lambda^*_F \frac{dH(y|x)}{d\lambda} \frac{dG_1}{dx} \] > 0 since

\[ \frac{d}{dx} \left[ \int_0^\infty \frac{du(w_2)}{d\lambda} dH(y|x) \right] \]

\[ = \frac{d}{dx} \left[ H(y|x) \frac{du(w_2)}{d\lambda} \bigg|_{y=0} \right] - \frac{d}{dx} \int_0^\infty H(y|x) \frac{d^2 u(w_2)}{dy d\lambda} dy \]
\[
\begin{aligned}
&= \frac{d}{dx} \left[ H(y|x) \left. \frac{du(w_2)}{d\lambda} \right|_{y=0} \right] - \left. \frac{d^2 u(w_2)}{dx d\lambda} \right. H(y|x) \right|_{y=0} \\
+ & \int_0^\infty \frac{du(w_2)}{d\lambda} \frac{dH(y|x)}{dy dx} \ dy \\
+ & \int_0^\infty \frac{d^2 u(w_2)}{dx d\lambda} \frac{dH(y|x)}{dy} \ dy + \int_0^\infty \frac{du(w_2)}{d\lambda} \frac{d^2 H(y|x)}{dy dx} \ dy \\
& \text{(from assumption iii)} \\
& > 0 \quad \text{(from assumption ii)}. \end{aligned}
\]

Therefore \( \lambda^*_G > \lambda^*_F \) (from \( u \in U_2 \)).
5.1 Introduction

While the last chapter focused upon a theoretical application of stochastic dominance, in this chapter we develop a methodology to widen the scope of managerial applications of the theory. Most previous managerial applications focused upon financial problems (see [13, 52, 58, 77, 78, 79, 80, 95, and 109]). In this chapter we focus upon ecosystem management.

Two major approaches to incorporate risk attitudes into policy analysis have been commonly utilized in ecosystem management.

(1) **Fail-safe and chance constraint formulations**: Risk aversion is indicated by posting either absolute constraints on problems or probability constraints. Absolute level constraints are incorporated to ensure margins of safety through built-in inefficiencies (fail-safe system design) while probabilistic constraints ensure satisfaction of a particular objective (or ensure against a constraint violation) with a given minimal probability.

(2) **Substitution of individual for social preferences and risk neutrality assumptions**: Risk attitudes of particular individual
decision makers are used to represent social preferences, or in other cases expectations of payoff are taken as the objective function, assuming implicitly a utility linear in payoff.

Stochastic dominance theory offers an improvement in the area of policy analysis. This approach to the problem alleviates the above deficiencies by taking explicit account of common features of risk attitudes in policy comparisons.

In this chapter we outline a methodology for evaluation of forest management policies and provide an example of its application in the analysis of alternative management policies for the New Brunswick forests of Canada. The methodology consists of (1) construction of a validated simulation model, from which (2) reward probability distributions for alternative policies are inferred, which are (3) compared to identify possible preference hierarchies among policies.

5.2 The Forest Management System

The boreal forests of North America have, for centuries, experienced periodic outbreaks of a defoliating insect, the spruce budworm. In any one outbreak cycle a major proportion of the mature softwood forest in affected areas can die, with major consequences to the economy and employment of regions like New Brunswick, which are highly dependent on the forest industry. An extensive insecticide spraying programme initiated in New Brunswick in 1951 has succeeded in minimizing tree mortality, but at the price of maintaining incipient outbreak conditions over an area considerably more extensive than in the past. The present management approach
is, therefore, particularly sensitive to unexpected shifts in economics, social and regulatory constraints, and to unanticipated behaviour of the forest ecosystem.

A simulation model of the budworm/forest ecosystem was developed by IIASA and IARE scientists [42] to be used as a laboratory world to aid in the design and evaluation of the alternative policies. The key requirement of that laboratory world was that it capture the essential qualitative behaviour of the budworm/forest ecosystem in both space and time. Extensive data concerning forest-pest and economic interrelations has been collected over the past 30 years by Environment Canada as one of the earliest interdisciplinary efforts in the field of renewable resource management. The essential qualitative behaviour in time has been identified through an analysis of tree ring studies. Four outbreaks have been detected since 1770 each lasting 7 to 16 years, with a 34 to 72 year period between the outbreaks. During the inter-outbreak periods the budworm is present in barely detectable densities which, when appropriate conditions occur, can increase explosively over four orders of magnitude during a 3 to 4 year period.

The distinctive pattern in time is paralleled by one in space. The historical outbreaks typically initiated in one to three or four local areas of Eastern Canada and from those centres spread to contaminate progressively larger areas. Collapse of the outbreaks occurred in the original centres of infestation in conjunction with mortality of the trees and similarly spread to the areas infested at later times. The resulting high degree of spatial heterogeneity in the forest age and species composition is closely coupled to the "contamination" feature caused by the high dispersal properties of this insect.
Figure 5.1 depicts the basic model structure for the budworm simulation model. In our investigation we have used the model for a single site (site-model), that is Budworm Survival Model, Forest Response Model, Budworm Control Policy and Forest Management Policy. Dispersal between sites is modelled as stochastic without direct reference to other sites.

In the unmanaged situation budworm numbers remain low until better weather induces an outbreak. Once beyond the control of natural enemies, budworm numbers grow rapidly regardless of weather, and mortality to older trees is high. However, budworm larvae have poor survival on trees less than thirty years old. Hence, budworm numbers decline dramatically after the few years taken to destroy the older age classes of fir. Following the population crash, budworm remains at a low endemic level until the forest recovers and an ample supply of older trees becomes available to support another outbreak. The time between outbreaks is typically thirty to seventy years. In a managed forest the above pattern can be altered principally in two ways. First, the budworm's food supply can be reduced by logging. Second, the survival of large larvae can be reduced by spraying. The principal policy problem is to determine the appropriate levels of logging and spraying.

5.3 The Forest System Problem

The most apparent objective in forest management is maximization of profit from logging. Additional considerations are maintenance of high recreational value, low level of pesticide spraying

\[^{5}\text{From [42].}\]
Figure 5.1. The basic structure of the budworm/forest model.
and stable levels of employment in the forest industries. In this case high recreational value proved to be commensurate with profit maximization [42]. Thus, we will confine ourselves to comparing policies in terms of discounted profit, spraying frequency and employment stability. A time horizon of one hundred years was chosen in order to allow time for two outbreaks. The three performance measures are defined as follows.

1. The total discounted profit is

\[ P = \sum_{i=1}^{100} (1-\rho)^{i-1} R_i, \]

where \( \rho \) is the discount rate and

\[ R_i = \sum_{j=1}^{75} H_{ij} Q_j (V-L_j) \]

is the net profit in year \( i \),

\( H_{ij} \) is harvest in year \( i \) of trees of age \( j \),

\( Q_j \) is the average value of lumber from one tree of age \( j \),

\( V \) is the sale price of one unit minus the stumpage charge minus the average cost of transporting a log to the mill, and

\( L_j \) is the cost of logging one unit of wood from trees of age \( j \).

2. The spraying index is

\[ S = 1-N/100 \]

where \( N \) is the number of years with spraying.
(3) The employment stability index is

\[ E = \frac{100}{\sum_{i=2}^{75} T_i} \]

where \( T_i = \sum_{j=1}^{\max(H_{i-1,j} - H_{ij}, 0)} Q_j \)

Two approaches have been taken to derive management policies for the budworm system. In the first a simplified version of the budworm site model was used to formulate a dynamic programming problem [42]. This model required a high degree of aggregation of variables and the elimination of dispersal. The following policies were derived. In the absence of budworm and for a given discount rate \( \rho \) and price \( P \) ($45 per cunit hereafter), there is an age of trees \( i^*(\rho, P) \) such that in each year all trees of age \( i \geq i^*(\rho, P) \) would be harvested. Trees of age \( i < i^*(\rho, P) \) should be left. In the presence of budworm, the age of cutting \( i^*(\rho, P) \) still holds. That is, trees of age \( i \geq i^*(\rho, P) \) should be harvested. But for trees of age \( i < i^*(\rho, P) \) the correct policy depends upon not only \( \rho \) and \( P \), but also on the condition of the foliage, the budworm density level, and the age of the tree. Examples of these policies are given by Holling et al. [40]. In general they take on the following form:

(i) At some critical age (20-25 years) begin spraying when budworm density is high and foliage is in moderate to good condition.

(ii) With advancing age of trees, spray also when budworm density is high and foliage is in fair to moderate condition.

(iii) At another critical age (50-60 years) log when budworm density is extremely high.
(iv) At another slightly higher age, also log whenever foliage is in poor to fair condition.

(v) At another slightly higher age, also log whenever foliage is in fair to moderate condition.

(vi) Finally age \(i^*(\rho, P)\) is reached at which point trees are harvested regardless of budworm density and forest conditions.

Under this policy there are no budworm outbreaks over a 100 year period for price of lumber greater than $45 per unit and discount rate \(\rho \leq .05\). Under these conditions it is economically sound to prevent budworm outbreaks through intensive spraying. For higher discount rates or lower lumber prices outbreaks do occur. Two of the policies will be examined here; W1 - 0% discount rate - and W2 - 10% discount rate.

The second approach has been to apply heuristic "optimization" techniques to the simulation model for one site. The resulting policies are structured somewhat differently. Each specifies some age \(i^*(\rho, P)\) at which to log and that logging shall not be done on younger trees. Furthermore, each computes a hazard index \(H\) such that all stands of trees for which \(H \geq H^*\) are sprayed, and stands with the \(H < H^*\) are left as is. These policies differ only in their specification for the functional forms of \(i^*(\rho, P)\) and \(H\), and the critical level \(H^*\).

The first one has the following specification:

a) \(H^* = 1000\),

b) \(H = a + b \times \bar{T}\),

c) \(i^*(\rho, P) = p + q\bar{T} + rH\),

where \(\bar{T}\) is the mean age of trees and \(x\) is the budworm density.
The parameters $p$, $q$, $r$, and $b$ were chosen to maximize logging profit. Using an heuristic search technique, parameter values $a = 206$, $b = .17$, $p = 55$, $q = .08$, and $r = -.004$ were found. One should note that this policy is independent of both the price of lumber and the prevailing discount rate. Fourteen examples of this policy will be examined. The first five cases — $H_1, \ldots, H_5$ — will have $H^* = 200, 400, 600, 1000$ and 2000 respectively. The second five — $H_6, \ldots, H_{10}$ — will again have $H^* = 200, 400, 600, 1000$ and 2000 respectively and $p = 70$ (rather than 55). All ten of the preceding start with current forest conditions. The four remaining cases start with an 'ideal' forest, i.e. a forest with equal numbers of trees in each age class. Policies I1 and I2 have 55 age classes, $p = 55$ and $H^* = 200$, $H^* \geq 400$ respectively. Policies I3 and I4 have 70 age classes, $p = 70$ and $H^* = 200$, $H^* \geq 400$ respectively.

The second set of policies of the above type specify:

a) $H^* = 60$

b) $H = \text{budworm egg density}$

c) $i^*(\rho, P) = p + qT + r(a + bxT)$

with parameters $p$, $q$, $r$, $a$ and $b$ as above.

Again we have a policy independent of lumber price and discount rate.

We examine three cases - J1, J2 and J3 - where $H^* = 30$, 60 and 120 respectively.

Finally, four policies are examined which convert a forest with the current age structure into a forest with an ideal age structure. Policies T1 and T2 specify cutting the oldest 1/55th (or 1/70th) of the trees per year for the first 55 years. Policies T3 and T4 are similar using 1/70th and 70 years. Thereafter, T1 through T4 are identical to I1.
through I4 respectively. Spraying during the transition period is determined as in I1 through I4 respectively. All together, twenty-three policies will be examined.

In order to compare these policies using stochastic dominance rules, the distributions associated with these policies must be determined. Moreover, a satisfactory comparison of these random variables requires development of efficient algorithms. Topics of the next two sections deal with these subjects.

5.4 Derivation of Outcome Distributions

The complexity of the model makes it impossible to directly determine the probability distribution of outcomes for a given policy. Hence, we must use Monte Carlo Techniques to estimate the distribution. By sampling from the distribution of dispersal possibilities and the distribution of weather sequences, we can use the site model to obtain a sample point in the outcome distribution. As the number of independent samples increases, the sample outcome distribution converges to the probability outcome distribution.

The sampling schemes for dispersal and weather were as follows. Since we are considering only one site (6 x 9 miles), the dispersal over the whole area can only be treated stochastically. A satisfactory fit was obtained by setting I = E·e^X, where I is immigrants to the site, E is emigrants from the site and X is a normally distributed random variable with mean 0 and standard deviation .04. Weather sequences for New Brunswick have been analysed [42] and little year to year correlation was found. An adequate model is that poor weather occurs with probability .25, average
weather with probability .50, and good weather with probability .25. Sample weather sequences were randomly generated using the above probabilities for the three weather types.

5.5 The Screening Algorithm

One can screen those policies which are superior to others (in the expected utility sense) by comparing: 1) cumulative simulated performance distributions at all points, or 2) areas under the cumulative distributions at all points, or 3) areas under the integrals of the cumulative distributions at all points.

For the purpose of empirical studies, a discrete version of stochastic dominance rules must be adopted [77]. Let the points \( x_1, x_2, \ldots, x_n \) be the realizations of the random variate \( X \) with distribution \( F \), and let \( x_1 \leq x_2 \leq \cdots \leq x_n \). Then the cumulative empirical distribution \( F_n(x) \) is a step function with steps at \( x = x_i \), \( i = 1, 2, \ldots, n \).

\[
F_n(x) = P(X \leq x) = \sum_{x_i \leq x} p_i, \quad p_i = P(X = x_i)/q \quad \text{and} \quad q = \sum_{i=1}^{n} P(X = x_i).
\]

If the observations are from completely random experiments, then the probability \( p_i = \frac{1}{n} \) for all \( i \) can be used as an approximation to the true distribution [29].

We are now prepared to restate FSD, SSD and TSD conditions in discrete form. Let \( S_X = \{x_1, x_2, \ldots, x_n\} \) be the set of sample points determining \( F_n(x) \) and \( S_Y = \{y_1, y_2, \ldots, y_n\} \) be the set of sample points determine \( G_n(y) \). Let \( S_Z = S_X \cup S_Y = \{z_1, z_2, \ldots, z_{2n}\} \) with \( z_1 \leq z_2 \leq \cdots, \leq z_{2n} \), that is, the \( z \)'s are the \( x \)'s and \( y \)'s combined into one ordered set.
Rule 1 \[ F_n \geq_1 G_n \] if and only if \[ F_n^{(1)}(z_i) \leq G_n^{(1)}(z_i), \quad i = 1, 2, \ldots, 2n. \]

Rule 2 \[ F_n \geq_2 G_n \] if and only if \[ F_n^{(2)}(z_i) \leq G_n^{(2)}(z_i), \quad i = 1, 2, \ldots, 2n. \]

where \[ F_n^{(1)} = F_n \]

\[ F_n^{(2)}(z_i) = \frac{1}{2} \sum_{j=2}^{i} [F_n(z_j) + F_n(z_{j-1})](z_j - z_{j-1}), \quad i = 2, \ldots, 2n \]

\[ F_n^{(2)}(z_1) = 0 \] and \[ G_n^{(1)}(z_i), G_n^{(2)}(z_i) \] are defined similarly.

For both first and second degree dominance the distributions are compared only at the observation points. Such comparisons are sufficient since \[ G_n(x) - F_n(x) \] is a step function and \[ G_n^{(2)}(x) - F_n^{(2)}(x) \] is piecewise linear. Let

\[ F_n^{(3)}(z_i) = \frac{1}{2} \sum_{j=2}^{i} [F_n^{(2)}(z_j) + F_n^{(2)}(z_{j-1})](z_j - z_{j-1}), \quad i = 2, \ldots, 2n \]

\[ F_n^{(3)}(z_1) = 0, \] and \[ G_n^{(3)}(z_i) \] is defined similarly. Then the comparison at the observations points only is not sufficient to conclude the dominance of \[ F_n^{(3)} \] over \[ G_n^{(3)} \] (TSD). This follows from the nonlinearity of \[ G_n^{(3)} - F_n^{(3)} \]. It has a local minima at all points \( z^* \) where \[ G_n^{(2)}(z^*) - F_n^{(2)}(z^*) = 0 \] and \[ G_n(z^*) - F_n(z^*) > 0. \] Thus the algorithm of Porter et al. must be modified to include such points.

Rule 3 \[ F_n \geq_3 G_n \] if and only if \[ F_n^{(3)}(z_i) \leq G_n^{(3)}(z_i), \quad i = 1, 2, \ldots, 2n, \]

and \[ F_n^{(2)}(z_{2n}) \leq G_n^{(2)}(z_{2n}), \] and
\[ F_n^3(z*) \leq G_n^3(z*), \text{ where} \]
\[ F_n^3(z*) = F_n^3(z_k) + \frac{1}{\lambda} \left[ F_n^2(z_k) + F_n^2(z*) \right] (z* - z_k). \]

Points \( z* \) must satisfy four conditions:

i) \( z* \in (z_k, z_{k+1}) \),

ii) \( F_n^2(z*) - G_n^2(z*) = 0 \),

iii) \( F_n^2(z_k) - G_n^2(z_k) > 0 \) and

iv) \( F_n^2(z_{k+1}) - G_n^2(z_{k+1}) < 0 \).

\( G_n^3(z*) \) is defined similarly.

5.6 Stochastic Dominance Test Results

Policy rankings under all three stochastic dominance criteria for four performance measures are given in Figures 5.2-5.5. Tabulated values of mean, standard deviation and range are in Tables I and II and a summary of dominance relations is given in Table III. Examination of these results yields the following general observation: policies which promote a high spraying frequency do poorly under both economic criteria (policies J2, J3, \( I_1, I_3, T_1 \) and \( T_3 \)).

It is also noted that those policies specifically designed to yield nearly equal harvests each year are successful in minimizing socio-economic dislocation, but at the expense of low profit levels.

If we first restrict our attention to profit, we find the optimal policy is to cut trees at age 70 and never spray with an 'ideal' forest. However, such a policy cannot be applied until the ideal forest
Figure 5.2. Dominance relations derived for budworm/forest policies - profit/yr at 0% discount.
Figure 5.3. Dominance relations derived for budworm/forest policies - profit/yr at 10% discount.
Figure 5.4. Dominance relations derived for budworm/forest policies - frequency with no spraying.
Figure 5.5. Dominance relations derived for budworm/forest policies - employment stability index.
Table I
Policy Returns. Mean, Standard Deviation and Range of Return are Given for each Policy. Results were obtained from Twenty Simulation Runs of One Hundred Years Time Horizon using Twenty Different Stochastic Weather Sequences:

<table>
<thead>
<tr>
<th>Policy</th>
<th>Mean ± 1 S.D.</th>
<th>Range</th>
<th>Mean ± 1 S.D.</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$ Profit/yr. @ 0% Discount</td>
<td></td>
<td>$ Profit/yr. @ 10% Discount</td>
<td></td>
</tr>
<tr>
<td>H 1</td>
<td>- 77 818 ± 0</td>
<td>890, 6382</td>
<td>- 8349 ± 0</td>
<td>102, 71</td>
</tr>
<tr>
<td>2</td>
<td>3 134 ± 2413</td>
<td>3972, 6382</td>
<td>1 ± 58</td>
<td>- 174, 71</td>
</tr>
<tr>
<td>3</td>
<td>2 155 ± 3144</td>
<td>3322, 6382</td>
<td>13 ± 71</td>
<td>- 237, 71</td>
</tr>
<tr>
<td>4</td>
<td>1 738 ± 3402</td>
<td>3898, 6848</td>
<td>27 ± 86</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6 158 ± 689</td>
<td>3898, 6848</td>
<td>71 ± 13</td>
<td>27, 91</td>
</tr>
<tr>
<td>6</td>
<td>- 51 670 ± 0</td>
<td>8312 ± 0</td>
<td>- 9 137</td>
<td>155, 70</td>
</tr>
<tr>
<td>7</td>
<td>18 929 ± 2063</td>
<td>15 093, 21 853</td>
<td>50 ± 51</td>
<td>- 160, 22</td>
</tr>
<tr>
<td>8</td>
<td>16 787 ± 1923</td>
<td>13 099, 19 977</td>
<td>40 ± 51</td>
<td>- 147, 27</td>
</tr>
<tr>
<td>9</td>
<td>14 531 ± 3582</td>
<td>10 588, 18 883</td>
<td>10 ± 40</td>
<td>- 52, 84</td>
</tr>
<tr>
<td>10</td>
<td>14 853 ± 3478</td>
<td>10 714, 20 768</td>
<td>49 ± 39</td>
<td>- 42, 91</td>
</tr>
<tr>
<td>J 1</td>
<td>2 931 ± 3634</td>
<td>9 137, 6382</td>
<td>23 ± 120</td>
<td>- 455, 70</td>
</tr>
<tr>
<td>2</td>
<td>- 46 664 ± 0</td>
<td>2761 ± 0</td>
<td>- 4831 ± 0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>- 58 452 ± 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>W 1</td>
<td>7 118 ± 2684</td>
<td>2 469, 12 115</td>
<td>48 ± 13</td>
<td>24, 72</td>
</tr>
<tr>
<td>2</td>
<td>9 212 ± 2912</td>
<td>4 792, 15 883</td>
<td>74 ± 20</td>
<td>36, 112</td>
</tr>
<tr>
<td>I 1</td>
<td>- 79 927 ± 0</td>
<td>4 457 ± 0</td>
<td>- 7963 ± 0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4 973 ± 0</td>
<td>4 457 ± 0</td>
<td>457 ± 0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>- 38 763 ± 0</td>
<td>2 414 ± 0</td>
<td>- 2414 ± 0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>45 990 ± 161</td>
<td>45 302, 46 075</td>
<td>6016 ± 2</td>
<td>6011, 6020</td>
</tr>
<tr>
<td>T 1</td>
<td>-105 115 ± 0</td>
<td></td>
<td>-13 225 ± 0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>- 20 915 ± 0</td>
<td>- 4 806 ± 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>- 80 593 ± 0</td>
<td>-12 386 ± 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3 607 ± 0</td>
<td>- 3 966 ± 0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table II
Policy Returns 2. Mean, Standard Deviation and Range of Return are Given for each Policy. Results were obtained from Twenty Simulation Runs of One Hundred Years Time Horizon using Twenty Different Stochastic Weather Sequences

<table>
<thead>
<tr>
<th>Non-spray Frequency</th>
<th>Employment Stability Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Policy</td>
<td>Mean ± 1 S.D.</td>
</tr>
<tr>
<td>--------</td>
<td>----------------</td>
</tr>
<tr>
<td><strong>H 1</strong></td>
<td>0 ± 0</td>
</tr>
<tr>
<td>2</td>
<td>.972 ± .0218</td>
</tr>
<tr>
<td>3</td>
<td>.975 ± .0199</td>
</tr>
<tr>
<td>4</td>
<td>.986 ± .0136</td>
</tr>
<tr>
<td>5</td>
<td>.993 ± .0047</td>
</tr>
<tr>
<td>6</td>
<td>0 ± 0</td>
</tr>
<tr>
<td>7</td>
<td>.848 ± .0237</td>
</tr>
<tr>
<td>8</td>
<td>.842 ± .0217</td>
</tr>
<tr>
<td>9</td>
<td>.896 ± .0246</td>
</tr>
<tr>
<td><strong>.10</strong></td>
<td>.979 ± .0095</td>
</tr>
<tr>
<td><strong>J 1</strong></td>
<td>1 ± 0</td>
</tr>
<tr>
<td>2</td>
<td>.370 ± 0</td>
</tr>
<tr>
<td>3</td>
<td>.230 ± 0</td>
</tr>
<tr>
<td><strong>W 1</strong></td>
<td>1 ± 0</td>
</tr>
<tr>
<td>2</td>
<td>1 ± 0</td>
</tr>
<tr>
<td><strong>I 1</strong></td>
<td>0 ± 0</td>
</tr>
<tr>
<td>2</td>
<td>1 ± 0</td>
</tr>
<tr>
<td>3</td>
<td>0 ± 0</td>
</tr>
<tr>
<td>4</td>
<td>1 ± 0</td>
</tr>
<tr>
<td><strong>T 1</strong></td>
<td>0 ± 0</td>
</tr>
<tr>
<td>2</td>
<td>1 ± 0</td>
</tr>
<tr>
<td>3</td>
<td>0 ± 0</td>
</tr>
<tr>
<td>4</td>
<td>1 ± 0</td>
</tr>
</tbody>
</table>
Table III
Dominance Rankings of Policies under Four Performance Criteria

The criteria are:
1. profit at 0% discount rate - 0%
2. profit at 10% discount rate - 10%
3. frequency of years with no spraying - F.N.S.
4. employment stability index - E.S.I.

Policy rankings are indicated by asterisks:
* means the policy was dominated by only one other policy;
** means the policy was not dominated.

All other policies were dominated by two or more others. No separate table is given for third degree dominance since its table was identical to that for second degree dominance, in this case.

<table>
<thead>
<tr>
<th>Policy</th>
<th>1st Degree</th>
<th>2nd &amp; 3rd Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0%</td>
<td>10%</td>
</tr>
<tr>
<td>H 1</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>**</td>
</tr>
<tr>
<td>3</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>**</td>
</tr>
<tr>
<td>5</td>
<td>**</td>
<td>*</td>
</tr>
<tr>
<td>6</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>**</td>
</tr>
<tr>
<td>8</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>J 1</td>
<td></td>
<td>**</td>
</tr>
<tr>
<td>2</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>W 1</td>
<td>*</td>
<td>**</td>
</tr>
<tr>
<td>2</td>
<td>**</td>
<td>**</td>
</tr>
</tbody>
</table>
is created. A plausible policy then is one which combines a transition strategy to an 'ideal' forest and then a switch to a long term steady state strategy. With a time horizon of one hundred years, none of the tested transition strategies worked well. Invariably the losses accrued in the early years of transition were not repaid by the later years of high profit. Thus, for the remainder of the discussion we shall restrict our attention to policies for the management of the forest with its current age distribution.

Low discount rate favours cutting older trees and spraying with moderate frequency, while higher discount rate favours earlier cutting and lower frequency of spraying. If we are constrained to cut at age 55, the same policies are favoured regardless of discount rate. But, if we are constrained to cut at age 70, then a shift from low to high discount rate is accompanied by a shift from low to high spray threshold.

Additional points of interest include: the best policies derived via formal techniques (W1 and W2, dynamic programming with simplified model) did no better than the best heuristically derived policies; the dynamic programming policy derived under 10% discount rate was stochastically superior to the one derived under 0% discount rate regardless of the discount rate used in the simulation model; policies which differed only in H* tended to produce bimodal distributions of profit, suggesting that spraying as soon as budworm numbers pass a small threshold or only when budworm numbers are high are both superior to spraying when budworm numbers are moderate and threatening to outbreak. This observation arises because budworm numbers are controlled naturally in the many years of poor to average weather, provided that the population is not 'too large'. By keeping
numbers down in good years through spraying, this natural control is
effectively utilized. Less frequent spraying allows budworm epidemics
anyway. Hence, one might as well drop spraying entirely. Examining those
policies which rarely spray (say less than 5% of the years), we find the
most profitable are W2, regardless of discount rate, and H5, if discount
rate is high.

Finally, if we examine dislocation, we find that policies which
specify equal or nearly equal harvests per year dominate. However, such
policies are not reasonable economic choices under current forest condi-
tions. Setting them aside, we find a large number of policies clustered
together. Employing the second degree dominance criterion reduces this
set to four non-dominated policies: H10, H1, J2 and J3. The latter three
policies are poor choices since they require spraying in over 50% of the
years.

5.7 Sensitivity Analysis and Robustness of Results

Letting F and G be the given theoretical distributions, the
question of dominance relationship between F and G is merely an exercise
in calculation. In empirical studies these distributions are estimated
by sample distributions $F_n$ and $G_n$. If $F_n$ dominates $G_n$, then we proceed
to conclude that F dominates G. However, it is clear that F may or may
not dominate G. In this section the "confidence" level attached to the
statement $F_n$ dominates $G_n$ implies F dominates G is discussed.

The probability of finding $F_n \geq_1 G_n$ or $G_n \geq_1 F_n$ when $F_n$ and
$G_n$ have the same underlying distribution can be computed directly. Let
the sample size be $2n$, $n$ points each for F and G. Since $F_n$ and $G_n$ are
from the same distributions, each of the $2n$ sample points are observed for $F$ or $G$ with equal likelihood. Thus, there are $\binom{2n}{n}$ sequences of observations with equal probability. Then $F_n \succeq G_n$ if and only if for each subsequence $(z_i)_{i=1}^r$, $r=1,2,\ldots,2n$, the number of $x$'s belonging to $F_n$ is no larger than the number of $y$'s belonging to $F_n$.

Let

$$W_i = \begin{cases} 1 & \text{if } z_i = x_i \\ -1 & \text{if } z_i = y_i \end{cases}$$

Then $F_n \succeq G_n$ if and only if $\sum_{i=1}^r W_i < 0$ for all $r = 1,2,\ldots,2n$.

Similarly $G_n \succeq F_n$ if and only if $\sum_{i=1}^r W_i > 0$ for all $r = 1,2,\ldots,2n$.

There are $\binom{2n}{n-1}$ sequences for which $\sum_{i=1}^r W_i = -1$. Thus the probability of observing $F \succeq G$ or $G \succeq F$ is $\frac{2}{n+1}$.

This result may be used as one criterion for selecting sample size. However, no such nonparametric result is available for second and third degree dominance. This is due to the fact that the calculation of areas, $F_2(x)$, $G_2(x)$, requires the measurement of $z_i - z_{i-1}$, $i=1,2,\ldots,2n$.

Lacking any proven statistical tests, we chose to compare truncated sample distributions. Since the tails of a distribution are generally the least accurately estimated part of the distribution, we recompared the dominance relations after removing the smallest 5% and largest 5% of the observations for each empirical distribution.

Results of this analysis appear in Table IV. Two types of changes were obtained in the dominance relations. In only one case was a dominance relation reversed; for profit at 0% discount $H_{10} \succeq H_9$ was reversed (reduction in risk of H9, see Table IV). Second and more common
Table IV
Changes Greater than 10% in Mean and/or Standard Deviation obtained by Truncating Policies. The Maximum and Minimum Value for Policy was Deleted, leaving 18 Outcomes Per Policy of the Original 20

<table>
<thead>
<tr>
<th>Index</th>
<th>Policy</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit/yr.</td>
<td>J1</td>
<td>$2.93 \times 10^3 \rightarrow 3.41 \times 10^3$</td>
<td>$3.63 \times 10^3 \rightarrow 2.39 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>H9</td>
<td>$3.63 \times 10^3 \rightarrow 3.19 \times 10^3$</td>
<td>$3.58 \times 10^3 \rightarrow 1.69 \times 10^3$</td>
</tr>
<tr>
<td>Profit/yr.</td>
<td>H5</td>
<td>$-23 \rightarrow -4$</td>
<td>13 $\rightarrow$ 7</td>
</tr>
<tr>
<td>@ 10%</td>
<td>J1</td>
<td>120 $\rightarrow$ 69</td>
<td></td>
</tr>
<tr>
<td>Employment</td>
<td>H5</td>
<td>435 $\rightarrow$ 299</td>
<td></td>
</tr>
<tr>
<td>Stability</td>
<td>H10</td>
<td>1817 $\rightarrow$ 1530</td>
<td>1695 $\rightarrow$ 791</td>
</tr>
</tbody>
</table>
were changes of the form $F \mid_{1} G$ to $F \geq_{1} G$. Such changes occur primarily from truncation when cumulative distribution functions cross at the far left.

5.8 Conclusions

Of primary interest is the question of whether application of stochastic dominance rules enhance the ability to rationalize resource management. As more and more of our natural resources become an integral part of the public domain (either directly through ownership and management or indirectly through regulation), the need to consider social preferences in their management intensifies. Rather than making arbitrary assumptions about the wishes of the amorphous public or spending resources excessively to obtain information about prevailing preferences among possible system options, it is useful to utilize some of the more universal and stabler attributes of these preferences in forming policy. As demonstrated in this chapter, stochastic dominance offers a theoretically attractive and practically feasible tool for policy option evaluation. Reasonable, rather weak and easily verifiable assumptions about the commonality among individuals' preferences are used to produce an effective algorithm with good screening power.
We have attempted in this dissertation to integrate the two traditional approaches to managerial decision making under uncertainty: the central tendency-dispersion approaches and expected utility maximization.

We then proceeded to advance the frontiers of these theories by extending stochastic dominance rules obtained for uni-dimensional outcome spaces to multi-attribute decision situations. The dissertation concluded with demonstration of the use of stochastic dominance criteria in development of theories (portfolio selection problems) and managerial decision making (forest management).
BIBLIOGRAPHY


[47] La Cava, C.J., "Improving the Mean-Variance Criterion Using Stochastic Dominance," Georgia State University, mimeo.


APPENDIX

ADDITIONAL RESULTS

From the development of necessary and sufficient stochastic dominance orderings for $U_1, U_2, U_3$ and $U_D$, it is clear that the $F_1 \succ F_2 \succ F_3 \succ F_D$ since $U_1 \succ U_2 \succ U_3 \succ U_D$. Furthermore, one can obtain higher degrees of dominance relations by considering the set $U_n$ where $U_n$ is a convex cone contained in $U_D$. The existence of a stochastic ordering follows from the existence of a dual cone in $M_1$, the set of probability measures [65]. However, the economic significance of higher stochastic orderings is not clear. The preceding discussion indicates that the stronger ordering can be obtained if the set of utility functions becomes small. In fact, in the literature dealing with decision making under uncertainty, utility functions are often assumed to be of a specific form. For example,

$$U_{\log} = \{u|u(x) = \log(x + d), d > 0\}$$

$$U_{\exp} = \{u|u(x) = -e^{-d x}, d > 0\}$$

$$U_{\pow} = \{u|u(x) = (x+d)^{1-c}, d > 0, 0 < c < 1\}$$

$$U_{\quad} = \{u|u(x) = x - cx^2, c > 0, x \leq \frac{1}{2} c\}$$
The members of $U_{\log}$, $U_{\exp}$ and $U_{\text{pow}}$ exhibit all the essential characteristics necessary for rational decision making. The members of the last set do not belong to $U_D$. The economic analysis which utilizes the special properties of the above utility functions in decision making under uncertainty can be found in [7,30,34,35,37,71,82].

Similarly, a stronger stochastic ordering on $F$ can be achieved by placing some additional conditions on the distribution functions rather than on $U$, as was done in theorem 2.1.

Let

\[ F_E = \{ F | F \text{ has an equal mean} \} \]

\[ F_6^6 = \{ F | F(x) = \phi((x - e_F)/s_F), s_F > 0 \} \]

\[ F_1 = \{ F | F(x) = \phi((t(x) - e_F)/s_F), t'(x) > 0, s_F > 0 \} \]

\[ F_{\chi^2} = \{ F | F \text{ is a } \chi^2\text{-distribution} \} \]

\[ F_\Gamma = \{ F | F \text{ is a } \Gamma\text{-distribution} \} \]

\[ F_N = \{ F | F \text{ is a normal distribution} \} \]

\[ F_{\log N} = \{ F | F \text{ is a lognormal distribution} \} \]

\[ F_{\text{stable}} = \{ F | F \text{ is a stable distribution} \} \]

\[ F_{\text{uniform}} = \{ F | F \text{ is a uniform distribution} \} \]

The stochastic dominance analysis within these subclasses was initiated by various authors [1,5,30,36]. In particular, Ali [1] and Bawa [5] give simple stochastic dominance rules in terms of specific parameters of the distributions belonging to $F_0$, $F_1$, $F_{\chi^2}$ and $F_\Gamma$. Levy and Hanoch [36] and La Cava [47] have also obtained similar results on $F_0$, $F_1$.

$e_F$ and $s_F$ denote a location and scale parameter of $F$, respectively.
In studies of diversification analysis, rates of returns on common stocks are often assumed to be normally distributed. Recent empirical studies, however, have shown that the returns are often better described by lognormal (see Lintner [59] and references therein), or stable distributions [20,21].

In view of these empirical results, the examination of stochastic orderings on the restricted class of $F_{\log N}$ or $F_{\text{stable}}$ is useful. Levy provided the first and second degree dominance criteria for lognormal distributions [55]. Since lognormal distributions intersect at most at one point, the extension of SSD to TSD is immediate.

If $X$ and $Y$ have symmetric stable distributions with $|\xi_k| < \infty$, $0 < s_k < \infty$, common characteristic exponent $\alpha$, $1 < \alpha \leq 2$, $\mu_F \geq \mu_G$ and $F(x) > G(x)$ for some $x \in R$, then Ziemba [113] shows that $E_{F} u > E_{G} u$ for all convergent concave non-decreasing $u$ if and only if $s_1^{1/\alpha} < s_2^{1/\alpha}$. This result is used to derive the efficient frontier in mean-\(\alpha\) dispersion space, a generalization of Tobin's separation theorem, and an algorithm for computing approximately optimal portfolio allocations.

Let $F(\mu, s, \beta, \alpha)$ be a cumulative distribution of a stable random variable, where parameters $\mu$, $s$, $\beta$, and $\alpha$ are location, scale, skewness, and characteristic exponent respectively. Then the following additional results can be obtained.

**Theorem.** Suppose $F(x) \sim F(0,1,0,\alpha_F)$ and $G(x) \sim G(0,1,0,\alpha_G)$. Then $E_{F} u > E_{G} u$ for all concave non-decreasing differentiable utility functions if and only if $\alpha_F > \alpha_G$.

**Proof:**

\[
\Delta u = E_{F} u - E_{G} u = \int_{-\infty}^{\infty} u(x) d(F-G) = -\int_{-\infty}^{\infty} (F-G)u'(x)dx = \int_{-\infty}^{\infty} (G-F)u'(x)dx
\]

\[
= \int_{-\infty}^{0} (G-F)u'(x)dx + \int_{0}^{\infty} (G-F)u'(x)dx.
\]
Suppose \( \Delta u \geq 0 \) and \( \alpha_F < \alpha_G \), then \( F(x) > G(x) \) on \((-\infty, 0)\). Then from symmetry (\( \beta = 0 \)), \( G(x) \geq F(x) \) on \((0, \infty)\). Thus \( \int_0^\infty (G-F)u'(x)dx < 0 \) and \( \int_{-\infty}^0 (G-F)u'(x)dx > 0 \). But then \( \Delta u < 0 \) since \( u'(x) \) decreasing implies \( \left| \int_{-\infty}^0 (G-F)u'(x)dx \right| > \int_0^\infty (G-F)u'(x)dx \), a contradiction. Therefore \( \alpha_F \geq \alpha_G \) is a necessary condition. If \( \alpha_F > \alpha_G \), then we have the opposite situation, thus \( \Delta u \geq 0 \).

This result is similar to that of La Cava [47]. La Cava gives a necessary and sufficient condition for FSD and SSD in terms of the location and the scale parameters. He also shows that in order to relate the SSD rule to the scale parameter, it is necessary to assume that the distributions are symmetric.

Jean [44] attempted to show the following TSD and E-SV(h) relation. If \( F_1(b) < G_1(b) \), \( R = [a, b] \), \( F_2(x) = G_2(x) \) for all \( x \in R \), then \( \mu_F > \mu_G \) and \( SV(\mu_F) \leq SV(\mu_G) \).

However, the above result cannot hold, since \( 2F_2(\mu_F) = SV(\mu_F) \), \( 2G_2(\mu_G) = SV(\mu_G) \) and, for some \( F \) and \( G \), \( F_2(\mu_F) > F_2(\mu_G) = G_2(\mu_G) \). Thus \( SV(\mu_F) > SV(\mu_G) \). The above problem stems from the fact that the distributions \( X \) and \( Y \) do not have a common \( h \) point. If \( X \) and \( Y \) have a single intersection property, then the SSD efficient set is equal to the TSD efficient set. Hence \( e(\text{TSD}) = e(\text{E-SV}(h)) \) from Theorem 2.1.

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7 See page 11.