THE KINEMATIC AND THERMODYNAMIC STABILITY OF VORTICES IN SUPERFLUIDS

BY

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ABSTRACT

The equations of motion of vortices in an ideal two-dimensional fluid are derived, and the laws of conservation associated with the motion are discussed. Two regions of flow are considered, an infinite region and a region bounded on the outside by a circle. The kinetic energy and angular momentum of a vortex fluid in these two regions are calculated. A Lagrangian formalism is introduced in order to discuss the symmetry transformations of vortex systems and their associated conservation laws. The kinematic stability of rigidly rotating polygonal configurations of quantized vortices is determined, and the states of thermodynamic equilibrium of a rotating superfluid are found for low angular velocities, resulting in the calculation of the spectrum of critical angular velocities for the creation of one, two, and three vortices.

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1. INTRODUCTION

1.1 A Brief Review of the Relevant Literature

Interest in the motion of vortices in a two-dimensional ideal fluid was initiated in the late 19th century by mathematicians and physicists such as Helmholtz, Kirchoff, Lord Kelvin, Stokes, and Routh. Much of this early work is covered in the classic texts on hydrodynamics by Lamb(1932) and Milne-Thomson(1968). While considerable effort was invested in finding solutions of the equations of motion of vortices in various configurations with various boundaries, it became apparent that certain configurations of vortices moved as a rigid body, preserving their relative displacements, generally by rotating with a constant angular velocity. The stability of these rigidly rotating configurations against small deformations became of interest. Thomson(1883) studied the stability of vortices of equal strength placed at the vertices of regular polygons. He found that such configurations are stable up to and including N=6 (N being the number of vortices in the configuration) independent of the size of the polygon, whereas all configurations with N>7 are unstable, N=7 being of indeterminant stability to first order in perturbation theory. Further work on the stability of various rigidly rotating configurations has been done by Morton(1934,1935) and Havelock(1931). A treatise on the motion of vortices in two dimensions with arbitrary boundaries has been produced by Lin(1943) who also reviews some of the early literature.

Interest in the motion of vortices was rekindled by the conjecture by Onsager(1949) and Feynman(1955) that liquid helium in the superfluid state allowed persistent currents of circulatory flow whose circulation would be quantized in units of h/m, m being the mass of the helium atom. This curl-free

flow with net circulation could only be supported by singular vortices threading the fluid, such that the flow would be irrotational everywhere except at the vortex line itself. This conjecture was verified experimentally by Vinen(1961) who demonstrated that the circulation in He II could indeed only be in units of h/m. The presence of vortex lines cleared up many mysteries about He II in rotation, especially the "rotation paradox", discussed by Wilks(1970) in chapters 7 and 8 of his introductory book on liquid helium. The discovery precipitated much research on rotating superfluids and on the motion of vortices in cylindrical containers. The theoretical work of Hess (1967) and the experimental work of Reppy, Depatie, and Lane(1961) and Reppy and Lane(1965) is relevant to this work. Tkachenko(1966) determined the stability of infinite, doubly periodic vortex lattices. Putterman and Uhlenbeck (1969) produced the definitive paper on the thermodynamics of rotating superfluids upon which this work is based. The detection of the presence of discrete quantized vortices in He II was accomplished by Packard and Sanders(1969) and the spatial positions of vortices in rotating He II were photographed by Williams and Packard(1974), showing no discernable pattern. The work of Packard and Sanders(1969) also demonstrated that there exists a critical angular velocity of rotation, below which no vortices may be present in the fluid at equilibrium.

1.2 The Two-Fluid Model of Liquid Helium

It is not intended to cover the theory of liquid helium in this work, but some of the results and basic concepts of the two-fluid model due to Landau(1941) will be stated. The reader is again refered to Wilks(1970) for a more complete introduction.

In the two-fluid model of He II it is postulated that the actual superfluid may be described by two non-interacting, interpenetrating fluids, so

that at each point there is a "normal" fluid with density ρ_n and velocity $\underline{V}_n$ and also a "superfluid" with density ρ_s and velocity $\underline{V}_s$. The total density of the fluid is $\rho=\rho_n+\rho_s$, and the total momentum density is $\underline{j}=\rho_n\underline{V}_n+\rho_s\underline{V}_s$. The equilibrium ratio ρ_s/ρ is a function of temperature alone, being unity at T=0, and dropping to zero as $T\to T_\lambda$, the lambda transition temperature (above which helium is not a superfluid). The superfluid component may not experience viscosity, its entropy is postulated to be zero, whereas the normal component does experience viscosity and the entropy of the fluid resides in the normal component alone. The superfluid flow is also postulated to have no turbulence, this condition being satisfied by the statement $\mathrm{curl}\underline{V}_s=0$ (that is , the flow is "irrotational").

The results which are needed in this work concern the conditions on the flow of both normal and superfluid components for a rotating vessel of He II to be in thermal equilibrium. Putterman and Uhlenbeck use a variational principle to determine the equilibrium conditions, maximizing the entropy of the fluid while maintaining total mass, total energy, and total angular momentum constant. The conditions on the velocity fields are:

(a) The normal fluid rotates as a rigid body with the angular velocity of the container, i.e.

$$\underline{V}_n = \underline{\Omega}\times\underline{r} \tag{1-1}$$

(b) The superfluid velocity field is stationary in the frame of reference in which the normal fluid is at rest, i.e.

$$\frac{\partial\underline{V}_s}{\partial t} = 0 \qquad \text{(in the rotating frame)} \tag{1-2}$$

(c) The superfluid velocity field is irrotational, i.e.

$$\nabla\times\underline{V}_s = 0 \tag{1-3}$$

(d) With the approximation that ρ_s and ρ_n are everywhere constant, the condition for thermodynamic equilibrium of the fluid is that the "free

energy" of the superfluid component be at an absolute minimum, i.e.

$$F = E - \underline{\Omega}.\underline{L} = \text{minimum} \qquad (1\text{-}4)$$

in which E is the kinetic energy of the superfluid and $\underline{L}$ is the angular momentum of the superfluid. This statement is equivalent to the statement that, in the rotating frame, the energy of the superfluid is at an absolute minimum.

Conditions (b) and (c) are satisfied by singular vortices of the classical type being present in the superfluid and rotating in rigid configurations with angular velocity Ω. If the vortex strengths are not quantized, but take any value in a continuous range, condition (d) would be satisfied by an infinite number of vortices of infinitesimal strength, approaching a state of solid body rotation. The problem of satisfying condition (d) becomes interesting when it is recognized that the vortex strengths are in reality quantized, and that there is a minimum non-zero strength.

1.3 Some Mathematical Preliminaries

The experiments on rotating He II generally contain the liquid in cylindrical vessels whose diameter is much smaller than their length, so that the effects of the vapour-liquid interface at the top and the solid boundary at the bottom are ignorable. Furthermore, a theoretical assumption is made that the vortex lines are rectilinear and parallel in the fluid, not a spaghetti-like tangle, so that the fluid has translational symmetry along the axis of rotation of the container. A cross-section of the fluid perpendicular to the axis may be regarded as typical of the fluid as a whole, and the hydrodynamics of the fluid may be reduced to a two-dimensional problem, in which vortex lines are singular points in the two-dimensional velocity field.

The condition that the superfluid velocity field is irrotational allows the introduction of a velocity potential ϕ, the derivatives of which give the components of $\underline{V}_s$:

$$v_1 = \frac{\partial \phi}{\partial x_1} \qquad v_2 = \frac{\partial \phi}{\partial x_2} \tag{1-5}$$

In addition, the approximation ρ_s=constant is tantamount to

$$\nabla . \underline{V} = 0 \tag{1-6}$$

which is satisfied in two dimensions by the introduction of a stream function ψ and writing

$$v_1 = \frac{\partial \psi}{\partial x_2} \qquad v_2 = -\frac{\partial \psi}{\partial x_1} \tag{1-7}$$

Equating (1-6) and (1-7) gives equations which may be viewed as the Cauchy-Riemann equations governing the conjugate functions ϕ and ψ which are the real and imaginary parts of a complex velocity potential Φ whose complex derivative is the complex velocity field w:

$$\frac{d\Phi}{dz} = v_1 - iv_2 = w(z) \tag{1-8}$$

Consequently, the differential equation governing the potential Φ is Laplace's equation

$$\nabla^2 \Phi = 0 \tag{1-9}$$

with the boundary condition that any solid boundary must be a streamline of the flow, i.e. ψ=constant on the boundary. A fluid whose velocity field may be expressed in such a manner is called an ideal fluid.

The hydrodynamics of a two-dimensional, ideal fluid may now be approached using the techniques of complex variable theory and analytic functions. This is the starting point of section 2 on vortex motion.

1.4 The Object of this Work

The questions which must be answered are "what configurations of vortex lines inside a rotating cylinder rotate rigidly with the cylinder and are stable against small perturbations, and which of these represent the state of thermodynamic equilibrium at a given angular velocity of rotation?" The first question is answered for configurations of equal strength vortices forming regular polygons: configurations with $N \geq 7$ are unstable, and configurations with $N<7$ are stable if the radius of the circle formed by the vertices of the polygon is smaller than a certain fraction of the cylinder radius, that fraction depending on N. This translates into a minimum angular velocity above which the configuration is stable. The state of thermodynamic equilibrium as a function of Ω is determined for low values of Ω, and the calculation of the critical angular velocities for the creation of one, two, and three vortices is performed.

In section 2 the equations of motion of vortices in an infinite fluid and in a fluid contained within a circular boundary are derived and discussed. The kinetic energy and angular momentum of a vortex fluid are calculated in section 3, showing their relation to the constants of the motion of vortex systems. In section 4 a Lagrangian formalism is introduced, allowing the constants of the motion to be derived from symmetries of the Lagrangian. The equations of motion of vortices perturbed slightly from rigidly rotating configurations are derived in section 5 and solved for the regular polygonal configurations. In section 6 the state of thermodynamic equilibrium of He II at low values of Ω is determined. Section 7 contains the conclusion of this work.

Various portions of the material contained in this work can be found in the literature cited. The original contributions of the author include the method of calculating the kinetic energy and the angular momentum in sections 3.1 and 3.2, and the Lagrangian technique in section 4.2. The method of calculation of the spectrum of critical angular velocities in section 6.2 is due to Putterman(1974), but the actual calculations are not available in the literature.

2. MOTION OF VORTICES IN A TWO-DIMENSIONAL IDEAL FLUID

2.1 Equations of Motion and Conservation Laws in an Infinite Fluid

The velocity field of a two-dimensional ideal fluid (inviscid and incompressible) can be expressed as the derivative of a complex, scalar potential function

$$\Phi(z) = \phi(z)+i\psi(z) \tag{2-1}$$

ϕ and ψ being real-valued functions. The velocity field is obtained from

$$w(z) = v_1(z)-iv_2(z) = \frac{d\Phi(z)}{dz} . \tag{2-2}$$

Of interest is a velocity potential of the form

$$\Phi(z) = \frac{\gamma}{i} \ln(z) \tag{2-3}$$

which gives rise to the velocity field

$$w(z) = \frac{\gamma}{iz} \tag{2-4}$$

which, in polar coordinates, is

$$v_r = 0, \qquad v_\theta = \frac{\gamma}{r} . \tag{2-5}$$

This is the velocity field of a vortex situated at z=0 in an infinite fluid. The flow is irrotational, since curl $\underline{v}$ = 0 everywhere except at the pole at the origin. The circulation of the fluid about any closed contour enclosing the pole is

$$\Gamma = \oint \underline{v}.\underline{d\ell} = 2\pi\gamma \tag{2-6}$$

hence γ is characteristic of the vortex and is called the strength of the vortex.

Of more interest is a configuration of N vortices situated at time t at positions $z_k(t)$ (k=1,2,...N) with strengths γ_k, whose velocity potential appears as the sum

$$\Phi(z) = \frac{1}{i} \sum_k \gamma_k \ln(z-z_k) \tag{2-7}$$

and whose corresponding instantaneous velocity field is

$$w(z) = \frac{1}{i} \sum_k \frac{\gamma_k}{z-z_k} \tag{2-8}$$

The vortices themselves move in the fluid, due the presence of the other vortices. The usual prescription for obtaining the equations of motion of the vortices, available in any of the classic texts on hydrodynamics, is to assign to the vortex at position z_n the velocity obtained by a superposition of the velocity contributions of all the other vortices, evaluated at $z=z_n$. That is,

$$i \frac{d\bar{z}_n(t)}{dt} = \sum_k{}' \frac{\gamma_k}{z_n(t)-z_k(t)} \tag{2-9}$$

where Σ_k' is the sum over all k=1,2,...N excluding k=n. Equation (2-9) represents N ordinary differential equations, of first order in t, whose solutions are the vortex trajectories. N initial conditions are required, the initial positions of the vortices. Unlike the mechanics of point particles, the initial velocities are not needed, since the equations are of first order in t. The initial velocities of the vortices are uniquely determined by their initial positions by the equations of motion (2-9).

The equations of motion can be put into canonical form through the introduction of the vortex stream function

$$\Psi_o = -\tfrac{1}{2} \sum_n \sum_k{}' \gamma_n \gamma_k \ln(z_n-z_k)(\bar{z}_n-\bar{z}_k) \tag{2-10}$$

along with the canonical coordinates and momenta

$$q_k = z_k, \qquad p_k = \gamma_k \bar{z}_k \tag{2-11}$$

It is easily verified that equations (2-9) and their complex conjugates may be written

$$i\,\frac{dq_k}{dt} = \frac{\partial \Psi_o}{\partial p_k} \tag{2-12a}$$

$$i\,\frac{dp_k}{dt} = -\,\frac{\partial \Psi_o}{\partial q_k} \tag{2-12b}$$

suggesting that Ψ_o be identified as the Hamiltonian of the vortex system. This claim will be substantiated in section 4, which deals with symmetries and conservation laws of the vortex system from a Lagrangian viewpoint.

By a simple application of the chain rule of differentiation and the canonical equations of motion, the time rate of change of any function $A(q_k,p_k,t)$ may be written

$$i\,\frac{dA}{dt} = \{A,\Psi_o\} + i\frac{\partial A}{\partial t} \tag{2-13}$$

in which

$$\{A,\Psi_o\} = \sum_k \left(\frac{\partial A}{\partial q_k}\frac{\partial \Psi_o}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial \Psi_o}{\partial q_k}\right) \tag{2-14}$$

is the Poisson Bracket (P.B.) of A and Ψ_o . Functions A which have no explicit time dependence are conserved quantities of the motion if

$$\{A,\Psi_o\} = 0 \tag{2-15}$$

The following conserved quantities of vortex motion can be found:

Centre of Circulation $\quad Z_o = \frac{1}{\gamma_o}\sum_k \gamma_k z_k \qquad (\gamma_o = \sum_k \gamma_k)$ (2-16a)

Moment of Circulation $\quad \Theta_o = \sum_k \gamma_k |z_k|^2$ (2-16b)

Vortex Stream Function $\quad \Psi_o = -\sum_n \sum_k{}' \gamma_n \gamma_k \ln|z_n - z_k|$ (2-16c)

Angular Moment of Circulation

$$W_o = \frac{1}{2i}\sum_k \gamma_k(\bar{z}_k \dot{z}_k - \dot{\bar{z}}_k z_k) \tag{2-16d}$$

As an independent check, these can be shown to be constant by direct substitution of the equations of motion.

2.2 The Velocity Potential of a Vortex Fluid Inside the Unit Circle

The previous section treated vortices in an infinite fluid. A more realistic situation is to have the vortices contained in a bounded fluid, and of particular interest in this work is the case of a circular boundary, since many experiments on liquid He use cylindrical vessels. For an ideal fluid, the velocity potential must be the solution of the differential equation

$$\nabla^2\Phi(z) = 0 \tag{2-17}$$

with the condition at the boundary that there be no normal component of the velocity field w(z). This is equivalent to requiring that the boundary be a streamline of the flow, i.e. $\psi(z) = 0$ along the boundary. Since the addition of a constant to the potential does not affect the velocity field, the constant which parametrizes the boundary streamline may be chosen to be zero. It has also been assumed that the boundary is continuous and closed. The boundary condition may then be expressed as

$$\Phi(z) = \bar{\Phi}(z) \qquad \text{along the boundary.} \tag{2-18}$$

The problem of finding the appropriate velocity potential for a vortex fluid with arbitrary boundaries has been approached by Lin(1943), but for simple boundaries such as the circular boundary the technique of conformal mapping is more convenient. The boundary value problem (2-17) with (2-18) can be mapped into a region for which the solution is already known or is easier to solve; the inverse mapping gives the correct solution in the original region. In this case, the region inside the unit circle can be mapped onto the upper half plane using the transformation

$$g(z) = i\left(\frac{1 - z}{1 + z}\right) \tag{2-19}$$

The same transformation maps poles in the z-plane into poles in the g-plane, and the circular boundary is mapped into the real axis. The solution to (2-17) in the upper half plane is well-known, cf. Milne-Thomson(1968), and is

$$\Phi(g) = \frac{1}{i}\sum_k \gamma_k \ln(g-g_k) - \frac{1}{i}\sum_k \gamma_k \ln(g-\bar{g}_k) \tag{2-20}$$

Except at the poles, this is a solution to the potential equation (2-17), and it is straightforward to show that, on the real axis ($g=\bar{g}$), the solution is totally real, satisfying the condition (2-18). Using the transformation (2-19) gives the required solution inside the unit circle:

$$\Phi(z) = \frac{1}{i}\sum_k \gamma_k \ln(z-z_k) - \frac{1}{i}\sum_k \gamma_k \ln(1-z\,\bar{z}_k) \tag{2-21}$$

where a real constant has been dropped. This solution also satisfies the boundary condition, which can be checked using the fact that $z=1/\bar{z}$ on the boundary.

It is interesting to note that the conformal mapping method is not useful in determining the potential outside the unit circle, that is, in deriving the circle theorem of Milne-Thomson (page 157 of reference). The transformation (2-19) maps the outside of the unit circle into the lower half g-plane, but the derivative (dg/dz) vanishes at $z\to\infty$ ($g=-i$), so the jacobian of the transformation vanishes there and the inverse mapping is not defined at that point.

2.3 Equations of Motion and Conservation Laws Inside the Unit Circle

Once the velocity potential for a vortex fluid has been determined for any region, the equation of motion of the vortex at z_n, say, is obtained by subtracting from the potential the contribution of the nth vortex in an infinite fluid, and assigning the velocity of the resulting field at $z=z_n$ to the vortex at z_n. This is expressed mathematically as

$$\frac{d}{dt}\bar{z}_n = \frac{d}{dz}\left(\Phi(z) - \frac{1}{i}\gamma_n \ln(z-z_n)\right)_{z=z_n} \tag{2-22}$$

Using the velocity potential from (2-21), the equations of motion of vortices inside the unit circle are

$$i \frac{d}{dt} \bar{z}_n = \sum_k{}' \frac{\gamma_k}{z_n - z_k} - \sum_k \frac{\gamma_k}{z_n - z_k'} , \qquad z_k' = 1/\bar{z}_k . \qquad (2\text{-}23)$$

This may be interpreted as each vortex having induced an image vortex of opposite sign at the reciprocal point. Note that the second sum contains the term k=n, representing the interaction of the nth vortex with its image.

As in section 2.1, the equations of motion can be put into canonical form by the introduction of the vortex stream function

$$\Psi_o = -\sum_n \sum_k{}' \gamma_n \gamma_k \ln \left| z_n - z_k \right| + \sum_n \sum_k \gamma_n \gamma_k \ln \left| 1 - z_n \bar{z}_k \right| \qquad (2\text{-}24)$$

The equations of motion (2-23) and their complex conjugates follow from the canonical equations (2-12a) and (2-12b). Using relation, (2-15), the conserved quantities of the motion can be determined. All of the quantities defined in (2-16) remain constants of the motion excluding the centre of circulation. (This is because the equations of motion are no longer invariant under translation in space; see section 4.)

3. DYNAMICAL QUANTITIES OF A VORTEX FLUID

The kinetic energy and angular momentum of a two-dimensional vortex fluid will be calculated for the case of the fluid contained inside the unit circle.

The density is assumed constant throughout the fluid, which is a reasonable assumption for the superfluid component of liquid He, this density dropping rapidly to zero only within microscopic distances from vortex cores and boundaries (see Putterman,1974). It will be shown that the kinetic energy is related to the vortex stream function, Ψ_o, and that the angular momentum is related to the moment of circulation, Θ_o, and so these dynamical quantities of the fluid are functions only of the configuration of vortices in the fluid.

3.1 Kinetic Energy

The kinetic energy of an ideal fluid of constant density ρ is

$$E = \tfrac{1}{2}\rho\iint_R (\nabla\psi)^2 dxdy$$

$$= \tfrac{1}{2}\rho\iint_R \nabla.(\psi\nabla\psi)dxdy$$

$$= \tfrac{1}{2}\rho \oint_C \psi(\nabla\psi.\underline{\hat{n}})d\ell \qquad (3\text{-}1)$$

in which ψ is the stream function of the flow ($\mathrm{Im}\Phi(z)$), R is the region of flow, and C is the contour enclosing R, with outward normal $\underline{\hat{n}}$. The vector identity $\nabla.(A\nabla B) = \nabla A.\nabla B + A\nabla^2 B$ along with the fact that $\nabla^2\psi=0$ gives the second line, while application of Green's Theorem gives the final line. The contour C is chosen to be the unit circle, and small circles of radius $\varepsilon<<1$ surrounding the poles at $z=z_k$; the straight segments joining the outer and inner contours are ignorable, since their contribution to the integral

cancels. Small disks of radius ε have been excluded from the region of flow in calculating the integral, but this is reasonable on physical grounds since real vortices have finite cores containing no liquid. Even without this fact, one would be unjustified in shrinking ε to below an inter-atomic distance in the fluid, the distance at which the use of classical hydrodynamics becomes questionable. The approximation used here is acceptable as long as the vortices do not approach each other or the boundary at distances of the order ε, since then the energy of deformation of the vortex cores must be taken into account. The parameter ε becomes important in the calculation of the critical angular velocity for the appearance of a single quantized vortex line in liquid He II, in section 6.

The judicious choice of $\Phi(z)$ in (2-21), in which $\psi(z)=0$ on the unit circle, reduces the integral (3-1) to

$$E = -\tfrac{1}{2}\rho \sum_m \oint_{C_m} \psi(\nabla\psi.\underline{\hat{n}})\,d\ell \tag{3-2}$$

which consists of the sum of all the contour integrals around the vortex positions (i.e. C_m surrounds z_m). In order to evaluate one of these integrals about the vortex at z_m, say, it is convenient to define the quantities

$$\begin{aligned} \varepsilon &= |z-z_m| & r_k &= |z_k| \\ r_{mk} &= |z_m-z_k| & r'_{mk} &= |z_m-1/\bar{z}_k| \end{aligned} \tag{3-3}$$

so that $\psi(z)$ may be expanded in powers of ε about z_m. The details of this calculation are found in Appendix A. In anticipation of allowing ε to become very small, only $\ln\varepsilon$ and zero order terms in ε are retained in this expansion, obtaining

$$\psi = -\gamma_m \ln\varepsilon - \sum_k{}' \gamma_k \ln r_{mk} + \sum_k \gamma_k \ln r'_{mk} + \sum_k \gamma_k \ln r_k + O(\varepsilon) \tag{3-4}$$

Similarly, in the vicinity of z_m,

$$\nabla\psi . \underline{\hat{n}} d\ell = \frac{\partial\psi}{\partial\varepsilon} \varepsilon d\theta = -(\gamma_m + O(\varepsilon)) d\theta \tag{3-5}$$

Combining these in (3-1), and again retaining only zero order and $\ln\varepsilon$ terms,

$$E = \tfrac{1}{2}\rho \sum_m \gamma_m \int_0^{2\pi} \psi d\theta \tag{3-6}$$

$$= \rho\pi\{- \sum_m \sum_k{}' \gamma_m \gamma_k \ln r_{mk} + \sum_m \sum_k \gamma_m \gamma_k \ln r'_{mk} + \sum_m \sum_k \gamma_m \gamma_k \ln r_k - \sum_m \gamma_m^2 \ln\varepsilon\}$$

which acquires a familiar form when complex notation is adopted:

$$E = \rho\pi\{- \sum_m \sum_k{}' \gamma_m \gamma_k \ln\left|z_m - z_k\right| + \sum_m \sum_k \gamma_m \gamma_k \ln\left|1 - z_m \bar{z}_k\right| - \sum_m \gamma_m^2 \ln\varepsilon\}$$

$$= \rho\pi(\Psi_o - \Sigma\gamma^2 \ln\varepsilon) \tag{3-7}$$

Thus the energy of a vortex fluid appears as the sum of what has been identified as the Hamiltonian of the vortex system and a term which, for a given set of vortices (γ's constant), does not depend on the configuration. The first sum in (3-7) represents the energy of interaction between the vortices, the second sum is the energy of interaction between the vortices and all the images, while the third sum may be considered as the self-energy of the vortices. This term diverges logarithmically as $\varepsilon \to 0$, but an argument has already been given that this parameter must remain finite, although small.

3.2 Angular Momentum

The angular momentum of a two-dimensional ideal fluid is

$$L = \rho \iint_R (\underline{r} \times \underline{V}) r . dr d\theta = \rho \iint_R \mathrm{Re}\{izw\} r . dr d\theta \tag{3-8}$$

Using the velocity field for vortices inside the unit circle, the integrand is

$$\mathrm{Re}\{izw\} = \sum_k \gamma_k \left(\mathrm{Re}\{z/(z - z_k)\} + \mathrm{Re}\{z\bar{z}_k/(1 - z\bar{z}_k)\}\right) \tag{3-9}$$

The first set of terms contain singularities in the region of integration at the points $z=z_k$, so they must be expanded in different convergent series in the two regions $|z|<|z_k|$ and $|z|>|z_k|$:

$$\mathrm{Re}\{z/(z-z_k)\} = -\sum_{n=1}^{\infty}\left(\frac{r}{r_k}\right)^n \cos n(\theta-\theta_k) \qquad |z|<|z_k|$$

$$= \sum_{n=0}^{\infty}\left(\frac{r_k}{r}\right)^n \cos n(\theta-\theta_k) \qquad |z|>|z_k| \qquad (3\text{-}10)$$

The remaining terms do not contain poles in the region of integration $|z|<1$, so one series suffices, namely

$$\mathrm{Re}\{z\bar{z}_k/(1-z\bar{z}_k)\} = \sum_{n=1}^{\infty}(rr_k)^n \cos n(\theta-\theta_k) \qquad |z|<1 \qquad (3\text{-}11)$$

The only terms in these series that contribute to the angular momentum are those for which n=0, due to the fact that

$$\int_0^{2\pi} \cos n(\theta-\theta_k)d\theta = \begin{cases} 2\pi & n=0 \\ 0 & n\neq 0 \end{cases} \qquad (3\text{-}12)$$

so the angular momentum integral becomes

$$L = \rho\sum_k \gamma_k \; 2\pi\int_{r_k}^{1} r\,dr = \rho\pi\sum_k \gamma_k(1-r_k^2)$$

$$= \rho\pi(\gamma_o-\Theta_o) \qquad (3\text{-}13)$$

The angular momentum is the sum of a configuration-free term (total circulation) and the negative of the moment of circulation. Note that the poles of the velocity field do not present any divergences in the final result.

3.3 Renormalizing the Energy

Expression (3-7) for the kinetic energy differs from the vortex stream function (2-24) only by what has been identified as the self-energy contributions $-\Sigma\gamma_m^2 \ln\varepsilon$ from the vortices. In cases for which the vortex stream function serves as the hamiltonian of the system, the γ_k's are

regarded as being fixed, hence the energy may be "renormalized" by subtracting the self-energies, the energy and vortex stream function becoming identical. Since $\Sigma\gamma_m^2$ is a positive definite quantity, it may not vanish by a suitable choice of γ's. For use in applications to liquid helium physics, the contribution of the self-energies may not be ignored, since Putterman's criterion for the thermodynamic equilibrium of superfluids (Putterman and Uhlenbeck(1969)) requires the comparison of free energies of vortex line configurations with unequal total circulations. Even for quantized systems with each γ_k an integral multiple of some unit strength, the $\Sigma\gamma_m^2$ may change due to a change in total circulation, or a re-distribution of vortex strengths giving the same total circulation. Thus the renormalized energy is useful only in determining the relative stability of configurations having the same total circulation and distribution of vortex strengths, that is, for those which differ only in their geometrical configuration. In this work (section 6) only quantized vortices of unit strength will be considered, so that $\gamma_o \propto N$ and $\Sigma\gamma_m^2 \propto N$, where N is the total number of vortices.

3.4 In the Limit of an Infinite Fluid

The kinetic energy and angular momentum of an infinite vortex fluid may be obtained by repeating the previous calculations, replacing the boundary at $|z|=1$ by a boundary at $|z|=a$, and taking the limit as $a\to\infty$. It is not necessary to repeat the calculation, since the appropriate expressions result from replacing z_k by z_k/a, and ε by ε/a in expressions (3-7) and (3-13) revealing the behaviour at large values of a to be

$$E/\rho\pi = -\sum_m \sum_k{}' \gamma_m\gamma_k \ln|z_m - z_k| - \sum_m \gamma_m^2 \ln\varepsilon + \gamma_o^2 \ln a + O(a^{-2}) \qquad (3\text{-}14a)$$

$$L/\rho\pi = -\sum_m \gamma_m |z_m|^2 + \gamma_o a^2 \qquad (3\text{-}14b)$$

Unless the total circulation is zero, the kinetic energy and angular momentum of an infinite vortex fluid contain infinite terms which behave as $\ln a$ and a^2, respectively. These diverging terms appear in the form of the energy and angular momentum of a single vortex of strength γ_0, due to the net circulation of the flow at large distances . If the condition $\gamma_0=0$ is imposed, the infinite terms vanish, corresponding to the cancellation of the velocity fields of the vortices at large distances. The presence of these terms in the case $\gamma_0 \neq 0$ is tantamount to the statement that no net circulation may be introduced into an infinite vortex fluid due to the infinite energy and angular momentum that would be required to do so.

Apart from these infinite terms, the only difference remaining between the energy and the vortex stream function (2-10) for an infinite fluid is the contribution from the self-energies of the vortices. Again, the energy may be "renormalized" , although the procedure seems uncertain in this case, since the question of renormalizing the energy to exclude the infinite terms is raised. This question, and whether it makes sense to discuss such infinite quantities seriously, will be left open.

4. THE LAGRANGIAN MECHANICS OF VORTEX SYSTEMS

4.1 The Lagrangian Formalism

It is possible to view the equations of motion of point objects as arising from a principle of least action, in cases for which a Lagrangian function may be defined for the system. Once a Lagrangian has been defined, the symmetry properties of the system and the conservation laws of the motion may be extracted with a minimum of effort. For a review of the relation between Hamilton's principle of least action and the conservation theorems of mathematical physics, see Hill(1951). The basic concepts of the formalism are introduced below.

The Lagrangian for a system of point objects is a function $L(q,\dot{q},t)$ of the coordinates $q(t)$ and their time derivatives $\dot{q}(t)$, and possibly of t itself. (In the following, all the coordinates have been denoted generically by q, and the sums over coordinate label are omitted, for clarity.) The action is the functional

$$S = \int_{t_2}^{t_1} L(q,\dot{q},t)dt \tag{4-1}$$

computed between the two points $(q_1(t_1),\dot{q}_1(t_1))$ and $(q_2(t_2),\dot{q}_2(t_2))$ of configuration space along all curves joining the points. The trajectory for which the value of S is a minimum, compared with all other trajectories, is the actual motion of the system between the points. The techniques of variational calculus, along with this principle, show that the equations of motion may be obtained from the Lagrangian by the Euler-Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{4-2}$$

Under a transformation $t \to t'$, $q \to q'$, the functional form of the Lagrangian must alter to preserve its numerical invariance, however, some transformations leave the Lagrangian form-invariant, or at most add the total derivative of some function of the coordinates (which leaves δS invariant). That is,

$$L'(q') = L(q') + \frac{d}{dt}\Lambda(q') \tag{4-3}$$

Such transformations are symmetry transformations, since they result in the transformation of a solution of the equations of motion into another possible solution of the equations of motion. As shown by Hill, the test for a symmetry transformation is that

$$\left\{\delta t\frac{\partial}{\partial t} + \frac{\partial}{\partial t}(\delta t) + \delta q\frac{\partial}{\partial q} + \delta\dot{q}\frac{\partial}{\partial \dot{q}}\right\}L(q,\dot{q},t) = -\frac{d}{dt}\,\delta\Lambda(q,t) \tag{4-4}$$

where δt, δq, and $\delta\dot{q}$ are infinitesimal quantities. The interpretation of (4-4) is that the RHS must be expressible as the total derivative of some infinitesimal function $\delta\Lambda(q,t)$. A given infinitesimal symmetry transformation of L, with the equations of motion derived from an action principle, gives rise to an associated conservation law, written

$$\frac{d}{dt}\left\{(L - \frac{\partial L}{\partial \dot{q}}\dot{q})\delta t + \delta q\frac{\partial L}{\partial \dot{q}} + \delta\Lambda\right\} = 0. \tag{4-5}$$

In this way, each symmetry transformation of L leads to a conservation law, although the reverse is not necessarily true. There exist conservation laws which do not correspond to symmetries of the Lagrangian, such as the Runge-Lenz vector of the Kepler problem, which is discussed by Greenberg(1966).

4.2 The Symmetry Transformations and Conservation Laws of Vortex Motion

That the equations of motion (2-8) of vortices in an infinite fluid result from an action principle using the Lagrangian

$$L(z_k,\bar{z}_k,\dot{z}_k,\dot{\bar{z}}_k) = \frac{1}{2i}\sum_k \gamma_k(\bar{z}_k\dot{z}_k - z_k\dot{\bar{z}}_k) - \sum_m \sum_k{}' \gamma_m\gamma_k \ln|z_m - z_k| \qquad (4\text{-}6)$$

via the Euler-Lagrange equations (regarding z_k and $\bar{z}_k$ as independent variables) is straightforward, in fact, this Lagrangian has been constructed so that this would be so. The Lagrangian (4-6) displays some unusual features: The velocities do not appear in quadrature, but in bilinear combination with the coordinates, ensuring that the equations of motion are first order in time . The Lagrangian is the sum of two constants of the motion (previously shown), W_o and Ψ_o, so the Lagrangian itself is a constant of the motion. In Newtonian point mechanics, the Lagrangian is the difference between the kinetic and potential energies, which is not conserved; this Lagrangian allows no such interpretation, the total energy of the system having been shown to be Ψ_o alone. With these comments in mind, the Lagrangian may be investigated for its symmetries, which are already known from section 2, with the intent of associating a conservation law with each symmetry transformation.

The test for a symmetry transformation of this Lagrangian is written, from (4-4),

$$L\frac{\partial}{\partial t}(\delta t) + \frac{1}{2i}\sum_k \gamma_k(\dot{z}_k\delta\bar{z}_k - \dot{\bar{z}}_k\delta z_k + \bar{z}_k\delta\dot{z}_k - z_k\delta\dot{\bar{z}}_k)$$
$$- \tfrac{1}{2}\sum_m \sum_k{}' \gamma_m\gamma_k\left(\frac{\delta z_m - \delta z_k}{z_m - z_k} \quad \frac{\delta\bar{z}_n - \delta\bar{z}_k}{z_m - z_k}\right) = -\frac{d}{dt}(\delta\Lambda) \qquad (4\text{-}7)$$

and the associated conservation law is

$$\frac{d}{dt}\left\{\Psi_o\delta t + \frac{1}{2i}\sum_k \gamma_k(\bar{z}_k\delta z_k - z_k\delta\bar{z}_k) + \delta\Lambda\right\} = 0 \qquad (4\text{-}8)$$

These expressions will now be used to test proposed infinitesimal symmetry transformations and to find their associated conservation laws.

(a) Space Translation Symmetry and Conservation of Centre of Circulation

The transformation

$$\delta t = 0, \quad \delta z_k = a, \quad \delta \dot{z}_k = 0 \qquad \text{(a infinitesimal)} \qquad (4\text{-}9)$$

is a symmetry transformation on L, but does not leave L invariant in form, requiring the introduction of the infinitesimal function

$$\delta \Lambda = \frac{1}{2i}(a\Sigma_k \gamma_k \bar{z}_k - \bar{a}\Sigma_k \gamma_k z_k) \qquad (4\text{-}10)$$

The associated conservation law follows from allowing a to be arbitrary. Letting a be completely real or completely imaginary results in the conservation of the imaginary or real parts, respectively, of the centre of circulation, Z_o.

(b) Rotational Symmetry and the Conservation of Moment of Circulation

The transformation

$$\delta t = 0, \quad \delta z_k = i\alpha z_k, \quad \delta \dot{z}_k = i\alpha \dot{z}_k \quad (\alpha \text{ infinitesimal}) \qquad (4\text{-}11)$$

leaves L form-invariant, resulting in the conservation of the moment of circulation Θ_o.

(c) Time translation Symmetry and Conservation of Vortex Stream Function

The transformation

$$\delta t = \tau, \quad \delta z_k = 0, \quad \delta \dot{z}_k = 0 \qquad (\tau \text{ infinitesimal}) \qquad (4\text{-}12)$$

leaves L form invariant, resulting in the conservation of vortex stream function, Ψ_o.

These three symmetries of L are the ones which, in Newtonian mechanics, lead to the conservation of linear momentum, angular momentum, and energy, respectively. It is easy to see that in a vortex fluid, the linear momentum is zero, due to the symmetry of the velocity field contributed by each vortex.

It has been shown that the moment of circulation, Θ_o, is a measure of the angular momentum in an infinite fluid, and that the vortex stream function, Ψ_o, is the energy of the fluid (minus the self energy of the vortices). Also, in Newtonian mechanics, it is the Hamiltonian which generates translations in time, so this justifies the identification of Ψ_o with the Hamiltonian of the vortex system.

(d) The Absence of a Symmetry Transformation Leading to Conservation of W_o

At the time of writing, it has not been possible to find an extra symmetry transformation associated with the conservation of W_o. There is one remaining symmetry of the equations of motion, namely, the transformation

$$t' = e^{2\beta}t, \quad z_k' = e^{\beta}z_k, \quad \dot{z}_k' = e^{-\beta}\dot{z}_k \quad (\beta \text{ real}) \tag{4-13}$$

which is a scale transformation, and does not lead to a conservation law. Since W_o is a constant of the motion which apparently does not reflect a symmetry of L, it may belong to that class of conservation laws exemplified by the Runge-Lenz vector, mentioned earlier. This question remains open.

As a final note, the Lagrangian for the vortex system bounded by the unit circle is obtained by adding to L the terms necessary to alter Ψ_o, i.e.

$$\sum_m \sum_k \gamma_m \gamma_k \ln|1-z_m\bar{z}_k| \tag{4-14}$$

The correct equations of motion are obtained from the Euler-Lagrange equations. The addition of these terms reduces the symmetry of the system, since the Lagrangian is no longer invariant under translations in space, as is easily verified. This agrees with the result stated in section 2.3, that the centre of circulation, Z_o, is not conserved in a fluid bounded by the unit circle. This result may be generalized to all bounded fluids.

5. THE STABILITY OF RIGIDLY ROTATING VORTEX CONFIGURATIONS

In this section, it will be shown that the state of kinematic equilibrium of the vortex fluid is one of rigid rotation of the vortex configuration. These configurations must then be examined for stability against small perturbations from kinematic equilibrium, which is effected by expanding the equations of motion in small quantities about a rigidly rotating configuration, obtaining linear equations of motion of the perturbations (as a first approximation). This method is used in the case of regular polygonal configurations of quantized vortices lying equally spaced on a circle of radius $r<1$ inside the unit circle. The symmetry of these configurations makes the normal modes of vibration apparent, and the criterion of stability is reduced to finding the roots of two polynomials in r^2 of degree 2N, where N is the number of vortices.

5.1 The Equilibrium State of a Rotating Vortex Fluid

The thermodynamic equilibrium of rotating superfluids based on the Landau(1941,1947) macroscopic two-fluid model has been treated by Putterman and Uhlenbeck(1969). Their results show that, for a superfluid contained in a vessel rotating with angular velocity Ω, the equilibrium state of the normal fluid component is one of rigid rotation ($\underline{V}_n = \underline{r}\times\underline{\omega}$), while the superfluid must be irrotational ($\mathrm{curl}\,\underline{V}_s = 0$) and stationary in the frame of reference rotating at Ω, and that the "free energy" of the superfluid, $F=E-\Omega L$, must be at a minimum. If the condition of irrotational flow is satisfied by the presence of vortex lines in the fluid, the condition of stationary $\underline{V}_s$ in the rotating frame requires that the configuration of vortex lines rotate rigidly with angular velocity Ω.

This can be shown easily for a two-dimensional vortex fluid, for

which the equations of motion and the constants of the motion are already known. The conserved quantities for the vortex fluid inside the unit circle are Ψ_o, the energy, Θ_o, and W_o. The state of equilibrium of the system must minimize Ψ_o, keeping Θ_o and W_o constant as auxiliary conditions. It is sufficient to relax these conditions , retaining only Θ_o constant in the variational problem as the solution obtained thereby automatically guarantees the conservation of W_o. The state of stable equilibrium is given by

$$F = \Psi_o + \Omega\Theta_o = \text{minimum} \tag{5-1}$$

in which Ω is , for the moment, an undetermined multiplier. The first derivatives of F with respect to the independent variables z_k and $\bar{z}_k$ must vanish, leading immediately to the differential equations

$$\dot{z}_k = i\Omega z_k \tag{5-2}$$

and their complex conjugates, whose solutions are

$$z_k = z_k^0 e^{i\Omega t} \tag{5-3}$$

in which the z_k^0 are constants, and Ω may now be identified as the angular velocity of a rigidly rotating configuration of vortices. The value of Ω is calculated by substituting (5-3) into the equations of motion, obtaining a constraint on the z_k^0 in the bargain, namely

$$\bar{z}_n^0\Omega = \Sigma_k'\gamma_k(z_n^0-z_k^0)^{-1} + \Sigma_k\gamma_k\bar{z}_k^0(1-z_n^0\bar{z}_k^0)^{-1}. \tag{5-4}$$

This is a kind of eigenvalue problem, the general solution to which would enumerate all possible configurations which rotate rigidly and their angular velocities. If the problem is restricted to the infinite fluid case with unit strength vortices, it can be shown that the rigidly rotating configurations of N vortices can be put into one-to-one correspondence with the N×N symmetric matrices with zero determinant.

Returning again to W_o, it is evident that for rigidly rotating systems, $W_o=\Omega\Theta_o$, and is therefore automatically conserved, as expected.

5.2 The Linearized Equations of Motion of the Perturbations

The stable rigidly rotating configurations are now to be found by looking for oscillatory motions of the vortices about their equilibrium positions. This is effected by expanding the equations of motion in terms of small perturbations about the rigidly rotating configuration, and terminating the expansion at the first order terms, so that linear equations of motion of the perturbations result. The substitution

$$z_k = (z_k^0 + b_k(t))e^{i\Omega t} \tag{5-5}$$

is made in (2-23), and the equations of motion are

$$i\frac{d}{dt}\bar{b}_n = -\sum_k{}'\gamma_k(b_n-b_k)/(z_n^0-z_k^0)^2 + \sum_k\gamma_k\bar{b}_k/(1-z_n^0\bar{z}_k^0)^2 + b_n\sum_k\gamma_k(\bar{z}_k^0)^2/(1-z_n^0\bar{z}_k^0)^2 - \Omega\bar{b}_n \tag{5-6}$$

Due to the substitution (5-5), the vortices are now being viewed from the frame of reference rotating at Ω, so the equilibrium positions z_k^0 are stationary.

It is necessary that the perturbed system leave the constants of the motion Ψ_o, Θ_o, and W_o unchanged to first order in the b_k's, so the solutions to (5-6) need some constraints. The constants of the motion may be expanded about a rigidly rotating configuration in terms of the b_k's, and the first order terms must vanish for the conservation laws to remain valid. The resulting equations of constraint are

$$\begin{aligned} \Psi_o, W_o: &\quad \textstyle\sum_k \gamma_k(\bar{z}_k b_k - z_k\bar{b}_k) = 0 \\ \Theta_o: &\quad \textstyle\sum_k \gamma_k(\bar{z}_k b_k + z_k\bar{b}_k) = 0 \end{aligned} \tag{5-7}$$

which combine to give the simpler, single constraint

$$\Sigma_k \gamma_k \bar{z}_k b_k = 0 \tag{5-8}$$

The general solution to (5-6) is not readily apparent, especially if the γ_k's are of arbitrary value, It has already been pointed out, however, that if all the vortices are of some unit strength, a certain symmetry is imposed on the system. For example, one class of rigidly rotating configurations is that of N vortices of equal strength lying equally spaced on a circle of radius r<1 (since the boundary is at r=1), forming an N-sided polygon.The high degree of symmetry of these configurations suggests that the normal modes of vibration will possess a similar symmetry. The solutions to (5-6) for the polygonal configurations will be investigated.

The positions of the unperturbed vortices forming an N-gon are

$$z_k^0 = re^{i(2\pi/N)k} \tag{5-9}$$

The N coupled equations (5-6) may be uncoupled by proposing the solutions

$$b_k(L,t) = e^{i(2\pi/N)k}\{a_L e^{i\omega t} e^{i(2\pi/N)Lk} + c_L e^{-i\omega t} e^{-i(2\pi/N)Lk}\}$$

$$(L=1,2,..,N) \tag{5-9}$$

in which ω is real, L labels the mode of vibration, and a_L and c_L are real constants to be determined by the initial conditions. Since the functions $e^{i\omega t}$ and $e^{-i\omega t}$ are linearly independent, substitution of (5-9) into (5-6) yields the two homogeneous equations

$$\begin{aligned} (pT_2 - p^{-1}S_{L+1})a_L - (T_{N-L+1} - \Omega - \omega)c_L &= 0 \\ -(T_{L+1} - \Omega - \omega)a_L + (pT_2 - p^{-1}S_{N-L+1})c_L &= 0 \end{aligned} \qquad (L=1,2,..,N-1) \tag{5-10}$$

in which $p=r^2$ and the trigonometric sums T_L, S_L, and Ω are defined and given in closed form in Table I, the calculations for which appear in Appendix B. For non-trivial solutions to exist for a_L and c_L, ω must be the solution of the quadratic equation

$$\omega^2 - \omega(T_{N-L+1} - T_{L+1}) - \Omega^2 + (pT_2 - p^{-1}S_{L+1})^2 - T_{N-L+1} \cdot T_{L+1}$$

$$+ \Omega(T_{N-L+1} + T_{L+1}) = 0 \tag{5-11}$$

The roots of this equation are

$$\omega = C \pm \sqrt{AB} \tag{5-12}$$

where

$$2pA = N^2 p^{N-L} \frac{(1+p^L)^2}{(1-p^N)^2} + NL \frac{p^L - p^{N-L}}{1-p^N} - 4N \frac{p^N}{1-p^N} - 2(N-1) + L(N-L)$$

$$2pB = N^2 p^{N-L} \frac{(1-p^L)^2}{(1-p^N)^2} + NL \frac{p^L - p^{N-L}}{1-p^N} - L(N-L) \tag{5-13}$$

$$2pC = NL \frac{p^L + p^{N-L}}{1-p^N} - N^2 p^{N-L} \frac{(1-p^{2L})}{(1-p^N)^2}$$

Applying the constraint (5-8) to the solutions (5-9) gives

$$(a_L e^{i\omega t} + c_L e^{-i\omega t}) \sum_k e^{i(2\pi/N)Lk} = 0 \tag{5-14}$$

The constraint is identically satisfied for L=1,2,...N-1 and places no conditions on a_L and c_L for these modes. However, for L=N, it is necessary that $a_L = c_L = 0$ for the constraint to be satisfied. In other words, the L=N mode must be removed from consideration if the perturbations are not to alter the constants of the motion.

Havelock(1931) has used a similar approach to the problem and his expressions giving the frequencies of the normal modes are the same, apart from some misprints in the publication. These misprints are listed in Appendix D.

5.3 The Stability of the Polygonal Configurations

For the polygonal configurations to have stable oscillatory motion about equilibrium, it is necessary that ω be real for all allowed modes, so the product AB must be positive for these modes. To demonstrate instability, it suffices to find one mode which is unstable. From the quadratic equation (5-11) governing ω, A and B may be written in the form

$$\left\{\begin{matrix}A\\B\end{matrix}\right\} = \tfrac{1}{2}(T_{N-L+1} + T_{L+1}) - \Omega \pm (pT_2 - p^{-1}S_{L+1}) \tag{5-15}$$

This expression is more convenient than (5-13) in demonstrating the fact that A and B are symmetric under the interchange of L and N-L (recognizing that S_{L+1} has this symmetry itself). The consequence of this is that the least stable mode or modes are the "central" ones, that is, L=N/2 for N even and L=(N±1)/2 for N odd. The stability of the polygonal configurations is then determined by the stability of these modes, which are to be examined separately for N even and N odd.

(a) N even

$$\text{Let} \qquad N=2n, \quad L=n, \quad N-L=n \qquad (n=1,2,\ldots) \tag{5-16}$$

Substituting these into (5-13) gives, for the least stable mode

$$\begin{aligned} a &= 2p(1-p^N)^2 A = (n^2-4n+2) + 4n^2p^n + 2(3n^2-2)p^{2n} + 4n^2p^{3n} \\ &\qquad + (n^2+4n+2)p^{4n} \qquad\qquad (5\text{-}17) \\ b &= 2p(1-p^N)^2 B = -n^2(1-p^n)^4 \end{aligned}$$

Since b<0 always, the criterion for instability is a>0, which is satisfied for all p for $n \geq 4$. Since a(p=0)<0 for n=1,2,3 and $\lim_{p\to 1} a(p)>0$, there exists a root $p_c(n)$ of a. For $p<p_c(n)$, the configurations n=1,2,3 are stable; for $p>p_c(n)$, they are unstable.

(b) N odd

$$\text{Let} \qquad N=2n+1, \quad L=n, \quad N-L=n+1 \qquad (n=1,2,\ldots) \tag{5-18}$$

Then, for the least stable modes,

$$a = n(n-3) + n(2n+1)p^{n} + (n+1)(2n+1)p^{n+1} + 2(3n^2+3n-1)p^{2n+1} + (n+1)(2n+1)p^{3n+1} + n(2n+1)p^{3n+2} - (n+1)(n+4)p^{4n+2}$$

$$b = -n(n+1) + n(2n+1)p^{n} + (n+1)(2n+1)p^{n+1} - 2(3n^2+3n+1)p^{2n+1} + (n+1)(2n+1)p^{3n+1} + n(2n+1)p^{3n+2} - n(n+1)p^{4n+2} \tag{5-18}$$

Again, $b<0$ always in the region $0\leq p<1$, reducing the criterion for instability to $a>0$, which is satisfied for all p for $n\geq 3$. Again, a sign change in a occurs for the special cases n=1,2, so there exists a root $p_c(n)$ such that these configurations are stable if $p<p_c(n)$, and are unstable if $p>p_c(n)$.

In summary, the regular polygonal configurations inside the unit circle are unstable for all values of r, except for N=1,2,3,4,5, and 6. For these values of N there exists a critical value of r, called r_c, such that in the region $r<r_c$, the configuration is stable, and for $r>r_c$, the configuration is unstable. The value of r_c for the different polygons is available from a numerical calculation of the roots of the polynomial a in each case. The results of such a computation appear in Table II. In addition, r_c gives a minimum value of Ω for which the polygon is stable, via the relation in Table I. These values of Ω_{min} for each N also appear in Table II.

S_L	$\sum_{k=1}^{N-1} \frac{1-e^{i(2\pi/N)Lk}}{(1-e^{i(2\pi/N)k})^2}$	$-\frac{1}{2}(L-2)(N-L)$
T_L	$\sum_{k=1}^{N} \frac{e^{i(2\pi/N)Lk}}{(1-pe^{i(2\pi/N)k})^2}$	$N^2\frac{p^{N-L}}{(1-p^N)^2} - N(L-1)\frac{p^{N-L}}{1-p^N}$
Ω	$p^{-1}S_1 + T_1 - pT_2$	$\frac{1}{2}p^{-1}(N-1) + N\frac{p^{N-1}}{1-p^N}$

TABLE I SOME RELEVANT TRIGONOMETRIC SUMS

(defined and in closed form)

N	r_c	Ω_{min}
2	.462	2.79
3	.567	3.43
4	.574	4.70
5	.588	5.86
6	.547	8.37

TABLE II THE VALUES OF r_c AND Ω_{min} FOR THE STABLE POLYGONS

6. THERMODYNAMIC EQUILIBRIUM OF A ROTATING SUPERFLUID

The states of stable kinematic equilibrium of quantized vortices forming regular polygons were found in the last section, however, this is insufficient to determine the state of thermodynamic equilibrium of the rotating vortex fluid. The criterion for thermodynamic equilibrium according to Putterman and Uhlenbeck(1969) is that the vortex fluid must have an absolute minimum of the free energy $F=E-\Omega L$. Experimentally, vessels of HeII are rotated at some constant angular velocity Ω, so the minima of F for the various configurations must be found for given Ω, and these minima compared in order to find that configuration which represents the absolute minimum of F for that Ω. As Ω is increased from zero, there is a range of Ω for which the absence of vortices represents the state of thermodynamic equilibrium. Above the critical angular velocity Ω_1, the free energy of one vortex at the centre is less than the free energy of zero vortices, so this becomes the state of thermodynamic equilibrium. As Ω is increased further, there appears a spectrum of critical angular velocities as it becomes energetically favourable to introduce new vortices into the fluid. That such is the case in reality has been shown by the work of Packard and Sanders(1969). The purpose of this section is to show that such a spectrum can be reproduced from theory, at least for small numbers of vortices.

Since the energy of the vortex fluid involves products of the vortex strengths γ_k, it is reasonable to assume that the equilibrium state of the fluid would require that all the vortices have the minimum non-zero quantum strength (i.e. $\hbar/m$ for helium). For small numbers of vortices, attention is restricted to the regular polygons, whose stability has been determined, or regular polygons with an extra vortex at the centre. The stability of the latter has not been investigated. The calculations of the minimum of

F as a function of Ω is a task in numerical analysis for every configuration except in the case of one vortex, which may be treated analytically. This important calculation relates the critical angular velocity for the creation of one vortex, Ω_1, to the vortex core radius parameter, ε.

The unit of length has already been chosen to be a, the cylinder radius. The unit of time will be ma^2/γ, where γ is the unit quantum strength. A dimensionless F is introduced by choosing the unit of energy to be $\rho\pi\gamma^2\ell$, ℓ being the depth of the liquid, and ρ being the density of the liquid (in this case the density used is the density of the superfluid component of HeII). The ratio of the vortex core radius to the cylinder radius is ε, so the vortex core radius is $R=\varepsilon a$. The ratio of the polygon radius to the cylinder radius will continue to be r.

6.1 One Vortex in a Rotating Cylinder

The free energy of a single vortex in the cylinder is, from (3-7) and (3-13),

$$F = E-\Omega L = \ln(1-r^2) - \ln\varepsilon - \Omega(1-r^2) \tag{6-1}$$

For a given Ω, equilibrium is given by

$$\frac{dF}{dr} = 0 \quad \rightarrow \quad \begin{cases} r=0 \\ r^2 = 1-1/\Omega \end{cases} \tag{6-2}$$

The second extremum does not exist for $\Omega<1$. The first extremum is a minimum for $\Omega>1$, whereas the second extremum is always a maximum. The stable position of a single vortex is at the centre, for $\Omega>1$. The equilibrium free energy of a single vortex as a function of Ω becomes

$$F_1 = -\Omega - \ln\varepsilon \tag{6-3}$$

The free energy of zero vortices in the fluid is zero, so the condition that

one vortex at the centre is thermodynamically stable over zero vortices present is

$$F_1 = -\Omega - \ln\varepsilon < 0 \qquad (6\text{-}4)$$

leading to the definition of the critical angular velocity for the creation of one vortex

$$\Omega_1 = -\ln\varepsilon \qquad (6\text{-}5)$$

(In conventional units, this is $\Omega_1 = (\gamma/a^2)\ln(a/R)$.)

In the region $1<\Omega<\Omega_1$, one vortex at the centre is said to be metastable, since that configuration is stable against small perturbations from equilibrium, but it does not represent the state of thermodynamic equilibrium, which must minimize F absolutely. In this region of angular velocity, zero vortices in the fluid represents the state of thermodynamic equilibrium: the superfluid component is at rest.

The quantity Ω_1 has been measured by Packard and Sanders(1969) to be 1.6 sec^{-1}(25.2 in dimensionless units) with a=0.05 cm, giving $R=5\times10^{-5}$Å. A more realistic value of the vortex core radius is R=0.3 Å, from Wilks(1970), giving Ω_1=1.0 sec^{-1}(16.6 in dimensionless units) with the same cylinder radius. The measurement of Ω_1 cannot be considered a good determination of R since R is very sensitive to small variations in Ω_1. Also, there is evidence in Packard and Sanders' work that their experimental system was subject to strong effects of metastability and the measurement of Ω_1 is in some doubt. The measurement of subsequent Ω_2,Ω_3, etc was certainly not reproducible.

6.2 The Spectrum of Critical Angular Velocities in He II

The free energy of N vortices in a regular polygonal configuration in a vessel rotating at Ω may be calculated from (3-3) and (3-13) using (5-9) and (6-4). The details appear in Appendix C, the result being

$$F_N(\Omega,p) = N\ln(1-p^N) - \tfrac{1}{2}N(N-1)\ln p - N\Omega(1-p) - N\ln N + N\Omega_1 \qquad (6\text{-}6)$$

For N=1,2,3,4,5,6 the polygons are stable, so this expression represents a minimum of F with respect to deformations of the polygon. The variables $p=r^2$ and Ω are independent in this expression, but it was shown that in equilibrium the polygon must rotate with the same angular velocity as the vessel, giving the relation

$$\Omega = \frac{N-1}{2p} + \frac{N}{p}\frac{p^N}{1-p^N} \tag{6-7}$$

For a given N and Ω, (6-6) is the free energy of the configuration, p being the solution to (6-7), taking care that Ω is greater than Ω_{min} given in Table II, section 5. Operationally, (6-7) may be solved for a specified value of Ω by an iterative technique; the value of p obtained is substituted into (6-6) with Ω to obtain F. Thus a curve of the equilibrium value of F vs Ω may be computed for each configuration.

The addition of a vortex at the centre alters the form of the free energy and angular velocity. The free energy of an N-gon with a central vortex is

$$\tilde{F}_N(\Omega, p) = N\ln(1-p^N) - \tfrac{1}{2}N(N+1)\ln p - N\ln N + (N+1)\Omega_1 - \Omega(1+N(1-p)) \tag{6-8}$$

with the new angular velocity, due to the central vortex

$$\Omega = \frac{N+1}{2p} + \frac{N}{p}\frac{p^N}{1-p^N} \tag{6-9}$$

The curves of $\tilde{F}$ vs Ω may be obtained in the same way as above.

Once these curves have been plotted, the states of thermal equilibrium can be determined by visual inspection, and the critical angular velocities obtained by the intersection of the appropriate curves. The free energies of one vortex (F_1), two vortices (F_2), three vortices in an equilateral triangle (F_3), and three colinear vortices ($\tilde{F}_2$) are plotted as a function of Ω in Figure 1, using R=0.3 Å. The intersection of these curves give the theoretical values of Ω_1, Ω_2, and Ω_3 to be 16.6, 20.1, and 22.1

respectively (1.05 sec^{-1}, 1.27 sec^{-1}, and 1.40 sec^{-1} in conventional units). Figure 1 also shows that, in this region of angular velocity, the triangular configuration of three vortices is thermodynamically stable over three co-linear vortices , although both configurations are stable with respect to small perturbations from equilibrium.

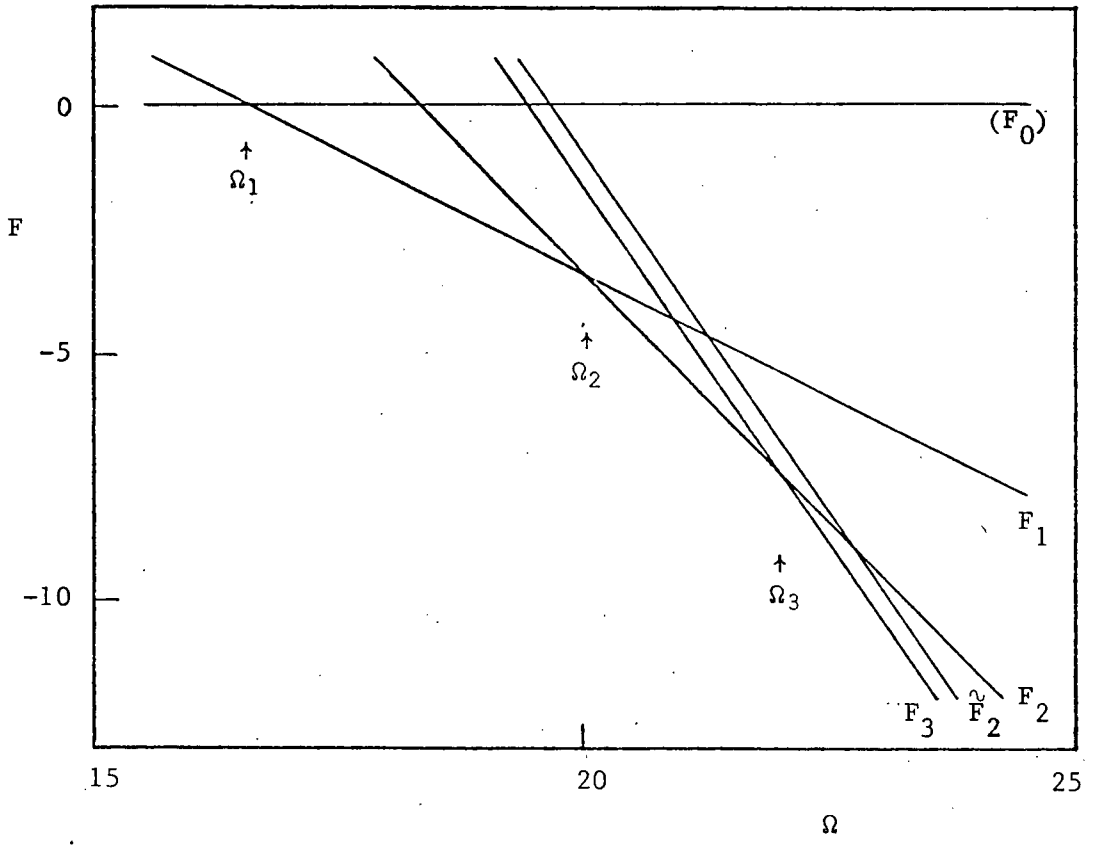

Figure 1: The equilibrium free energies of one, two, and three vortices. F and Ω are plotted in dimensionless units (see text). The curves F_1, F_2, $\tilde{F}_2$, and F_3 are the equilibrium free energies of one vortex, two vortices, three colinear vortices, and three vortices in an equilateral triangle, respectively. The quantities Ω_1, Ω_2, and Ω_3 are the critical velocities for the creation of one, two, and three vortices, respectively.

7. CONCLUSION

The results of section 5 concerning the stability of the regular polygonal configurations in an ideal fluid bounded by a circle are a departure point for the solution of the general problem of the stability of all rigidly rotating configurations. Before this is attempted, the sub-problem of classifying all possible rigidly rotating configurations must be solved. The acceptance of quantized vortex strengths simplifies this task somewhat, since it is expected that the resulting configurations will exhibit some degree of symmetry; these symmetries would indicate the normal modes of vibration and the solution of the equations of motion of the perturbations would proceed in a straightforward manner.

One question which has been nagging the author throughout this research is the nature of the law of conservation of angular moment of circulation. It has been demonstrated that the other conservation laws arise naturally from the symmetries of the system of vortices, but that no symmetry associated to W_o has been found. It may be the case that this is not an independent conservation law, but that it is a consequence of the others. If this is a fact, it has not emerged in the 100 years of literature on the subject. It would be satisfying to have this cleared up once and for all.

The approach used here may prove cumbersome when large numbers of vortices are being considered. Recent work by Su(1973) and Pointin and Lundgren(1976) involving methods of statistical mechanics may bear more fruit for workers in liquid helium physics.

The relevance of this work to the actual behaviour of vortex lines in He II depends on the validity of the assumptions made concerning the properties of He II: that the two-fluid model is adequate in describing the macroscopic properties of He II, that the problem is essentially two- dimen-

sional, that the normal fluid and superfluid may be treated as incompressible, and that the details of what occurs at the vortex core is ignorable. The ability to reproduce quantitatively the spectrum of critical angular velocities for vortex creation at low angular velocity seems to support the assumptions made - more accurate experimental results are needed before a stronger statement can be made.

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APPENDIX A: STREAM FUNCTION APPROXIMATION FOR ENERGY CALCULATION

The stream function for a vortex fluid inside the unit cicle is

$$\psi(z) = -\Sigma_k \gamma_k \ln|z-z_k| + \Sigma_k \gamma_k \ln|1-z\bar{z}_k|$$

The following notation is introduced:

$$z-z_m = \varepsilon e^{i\beta_m} \qquad z_k = r_k e^{i\theta_k}$$

$$z_m - z_k = r_{mk} e^{i\theta_{mk}} \qquad z_m - 1/\bar{z}_k = r'_{mk} e^{i\theta'_{mk}}$$

In the vicinity of z_m, the two terms in the stream function may be written

$$\ln|z-z_k| = \ln|(z-z_m)+(z_m-z_k)| = \ln|\varepsilon e^{i\beta_m} + r_{mk} e^{i\theta_{mk}}|$$

$$= \tfrac{1}{2}\ln(\varepsilon^2 + r_{mk}^2 + 2\varepsilon r_{mk}\cos(\beta_m - \theta_{mk}))$$

$$= \begin{cases} \ln\varepsilon & (k=m) \\ \ln r_{mk} + O(\varepsilon/r_{mk}) & (k\neq m) \end{cases}$$

$$\ln|1-z\bar{z}_k| = \ln|\bar{z}_k| + \ln|z-1/\bar{z}_k| = \ln|\bar{z}_k| + \ln|(z-z_m)+(z_m-1/\bar{z}_k)|$$

$$= \ln r_k + \ln|\varepsilon e^{i\beta_m} + r'_{mk} e^{i\theta'_{mk}}|$$

$$= \ln r_k + \tfrac{1}{2}\ln(\varepsilon^2 + r'^2_{mk} + 2\varepsilon r'_{mk}\cos(\beta_m - \theta'_{mk}))$$

$$= \ln r_k + \ln r'_{mk} + O(\varepsilon/r'_{mk})$$

If $\varepsilon \ll r_{mk}$ and $\varepsilon \ll r'_{mk}$ then the stream function in the vicinity of z_m may be written

$$\psi = -\gamma_m \ln\varepsilon - \Sigma'_k \gamma_k \ln r_{mk} + \Sigma_k \gamma_k \ln r'_{mk} + \Sigma_k \gamma_k \ln r_k + O(\varepsilon)$$

APPENDIX B: SOME RELEVANT TRIGONOMETRIC SUMS

It is necessary to make use of the relation

$$\sum_{k=1}^{N} e^{i(2\pi/N)mk} = N\delta_{m,\ell N} \qquad \ell = 1,2,\ldots \infty$$

and the series

$$(1-x)^{-2} = \sum_{n=0}^{\infty}(n+1)x^n \qquad\qquad x(1-x)^{-1} = \sum_{n=1}^{\infty} x^n$$

$$x(1-x)^{-2} = \sum_{n=1}^{\infty} nx^n$$

Evaluation of T_L:

$$T_L = \sum_{k=1}^{N} \frac{e^{i(2\pi/N)Lk}}{(1-pe^{i(2\pi/N)k})^2} \qquad p<1 \qquad L = 1,2,\ldots N$$

$$= \sum_{k=1}^{N} \sum_{n=0}^{\infty}(n+1)p^n e^{i(2\pi/N)(L+n)k}$$

$$= N \sum_{n=0}^{\infty}(n+1)p^n \delta_{L+n,\ell N} \qquad \ell = 1,2,\ldots \infty$$

$$= N^2 p^{-L} \sum_{\ell=1}^{\infty} \ell p^{N\ell} - N(L-1)p^{-L} \sum_{\ell=1}^{\infty} p^{N\ell}$$

$$= N^2 \frac{p^{N-L}}{(1-p^N)^2} - N(L-1) \frac{p^{N-L}}{1-p^N}$$

Evaluation of S_L:

$$S_L = \sum_{k=1}^{N-1} \frac{1-e^{i(2\pi/N)Lk}}{(1-e^{i(2\pi/N)k})^2} = \lim_{p\to 1} \{T_N - T_L\}$$

$$= \lim_{p\to 1} \left\{ \frac{N + N(N-1)p^N - N^2 p^{N-L} + N(L-1)\, p^{N-L}(1-p^N)}{(1-p^N)^2} \right\}$$

$$= \frac{1}{2N} \lim_{p\to 1} \{NL(N-L)p^{-N-L} - 2N(n-L)(L-1)p^{-L}\} \quad \text{(using l'Hopital's rule)}$$

$$= -\tfrac{1}{2}(N-L)(L-2)$$

APPENDIX C: THE FREE ENERGY OF REGULAR POLYGONAL CONFIGURATIONS

It is necessary to use the same relation used in Appendix B and the series

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

to find the following sums:

$$\sum_{m=1}^{N-1} \ln\left|1-e^{i(2\pi/N)m}\right| = \lim_{p\to 1} \left\{ \sum_{m=1}^{N-1} \ln\left|1-pe^{i(2\pi/N)m}\right| \right\}$$

$$= -\tfrac{1}{2} \lim_{p\to 1} \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{N-1} \frac{p^n}{n} e^{i(2\pi/N)mn} + \text{c.c.} \right\}$$

$$= -\lim_{p\to 1} \left\{ \sum_{n=1}^{\infty} \frac{p^n}{n} (N\delta_{n,\ell N} - 1) \right\} \qquad \ell = 1,2,\ldots \infty$$

$$= -\lim_{p\to 1} \left\{ \sum_{\ell=1}^{\infty} \frac{p^{\ell N}}{\ell} - \frac{p^{\ell}}{\ell} \right\}$$

$$= \lim_{p\to 1} \ln\left(\frac{1-p^N}{1-p}\right) = \ln N \qquad \text{(using l'Hopital's rule)}$$

and

$$\sum_{m=1}^{N} \sum_{k=1}^{N} \ln\left|1-pe^{i(2\pi/N)(m-k)}\right|$$

$$= -\tfrac{1}{2} \sum_{m=1}^{N} \sum_{k=1}^{N} \sum_{n=1}^{\infty} \frac{p^n}{n} e^{i(2\pi/N)(m-k)n} + \text{c.c.}$$

$$= -N^2 \sum_{n=1}^{\infty} \frac{p^n}{n} \delta_{n,\ell N} \qquad \ell = 1,2,\ldots \infty$$

$$= N\ln(1-p^N)$$

Using these it is straightforward to show that (with $z_k = e^{i(2\pi/N)k}$)

$$E = -\Sigma_m \Sigma_k' \ln|z_m - z_k| + \Sigma_m \Sigma_k \ln|1-z_m \bar{z}_k| - \Sigma_m \ln\varepsilon$$

$$= -\tfrac{1}{2}N(N-1)\ln p - N\ln N + N\ln(1-p^N) - N\ln\varepsilon$$

and that

$$L = \Sigma_m (1-z_m \bar{z}_m) = N(1-p)$$

APPENDIX D: SOME MISPRINTS IN HAVELOCK'S PAPER

The published version of the work by Havelock(1931) on the stability of various configurations of vortices contains some misleading misprints. The important ones are listed here(the equation numbers are Havelock's).

(i) In (23), the second term in the last line should read ...

$$\tfrac{1}{2}n^2 p^{n-1} \frac{p^k+p^{-k}}{(1-p^n)^2}$$

(ii) In (25), the second term in Q should read " $-2(n-1)$ ", and there is a missing term:

$$-\frac{4n}{1-p^n}$$

(iii) In (28), the second term should read " $-2(n-1)$ ", but the term missing from (25) has been replaced. The first term should read " $(\tfrac{1}{2}n)^2$ ".