ESSAYS IN CAPITAL MARKET EQUILIBRIUM

by

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Abstract.

The thesis consists of three essays dealing with related problems of capital market equilibrium under conditions of uncertainty. The method of analysis is the construction and use of two period general equilibrium models. The first essay develops a set of necessary and sufficient conditions, for the determination of equilibrium prices in the well-known Capital Asset Pricing model. A number of structural properties of the equilibrium are determined and applied in investigation of the comparative statics of the model. In particular, the effect on equilibrium prices of changes in the stochastic parameters and number of investors is established. The second essay deals with problems of financing and investment which occur when production and firms are added to the model. It is demonstrated that a "financial inconsistency" may arise as firms pursuing a particular objective, may attempt to invest more than the households in the economy are willing to finance. This possibility has implications for the existence of general equilibrium, admissible decision rules for firms, and the firm's problem of choosing a feasible financial policy. In order to deal with these issues a number of properties of the 'stock market economy' are established. The final essay relaxes the assumption, maintained in the first two essays, that all debt in the economy is default free. A number of simple capital market models are developed which determine an equilibrium interest rate on (risky) debt and an equilibrium probability of default. The comparative statics of these variables are investigated with respect to changes in the riskless rate of interest, the productivity of investment, the expectations of investors, and the transactions costs of default. It is shown that when equity is introduced, then provided
bankruptcy is not costless, the market also determines an equilibrium debt/equity ratio.
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Chapter I
INTRODUCTION

The following three essays are concerned with capital market equilibrium under conditions of uncertainty. The emphasis in each case is the effect of uncertainty on the individual market participants, and its implications for the economy as a whole. Capital markets, in addition to their risk bearing function, have as a primary task the intertemporal allocation of commodities — in particular they facilitate the savings investment process. In order to deal with the intertemporal aspect, the models considered in the following essays are two-period models. This is primarily to put the problems in as simple form as possible, but also to preserve comparability between the models constructed herein and those in the fast growing literature on the subject.

The models all have the common feature of having less than complete markets in the Arrow-Debreu sense. In general, the purchase of some type of security serves as a substitute for trading in both futures markets and Arrow's [1964] complete insurance markets. This change in market organization, first introduced by Diamond [1967], has some rather strong implications for capital theory. The essays are concerned with these implications.

Much of the recent interest, both theoretical and empirical, in the effect of uncertainty on capital-market equilibrium is due to the seminal paper of Sharpe [1964] on the general equilibrium pricing of assets under conditions of risk. This paper, which used the mean-variance portfolio model of Tobin-Markowitz, has generated a vast and still expanding literature on extensions of the model, and of its implications and applications.
It is probably fair to say that most current capital market theory is taught in the context of this particular model, now known as the Capital Asset Pricing model.

It is with some apologies then that my first essay is concerned precisely with the Capital Asset Pricing model. My reasons are as follows. All the literature has been concerned with the implications of the market equilibrium assumption for the structure of the asset prices. This type of analysis produced or used, in some form or another, what became known as the Capital Asset Pricing equation: an equation relating equilibrium asset prices to various parameters of the model. The first essay, 'The Structure of a Capital Market Equilibrium', demonstrates, while it is true that the equilibrium prices must satisfy this equation, it is not true that this equation determines equilibrium prices, i.e., it is a necessary but not a sufficient condition. The essay is concerned with developing a complete (i.e., sufficient) determination of equilibrium prices and exploring some of the implications of this derivation, both for the structure of equilibrium prices and portfolios and for the comparative statics of this general equilibrium model.

The Capital Asset Pricing Model is a model of a pure exchange economy in which individuals trade bundles of asset claims in order to attain individual portfolio equilibrium. Diamond [1967] introduced production in this type of model, although he did not employ the mean-variance assumption. The introduction of production and firms raises the issue as to what criteria firms should or would use in making their production decisions, as the profits of a firm under uncertainty are a random variable and, consequently, cannot be maximized. Diamond argued that stock market value maximization was the appropriate criteria and showed that in
his model the equilibrium was Pareto efficient in a certain restricted sense. This paper prompted a number of responses questioning the generality of Diamond's result and attempting to elucidate the conditions under which it may or may not be true. In this debate all the models have a capital or stock market in which individuals trade the shares of different firms and have the feature that, while production occurs, investment does not. Consequently on the real (production) side of the economy there is no intertemporal allocation problem. The sole function of stock markets in these models is to facilitate risk bearing. They do not serve the traditional role of a capital market as a mechanism which allocates real investment funds.

The second essay is concerned with the implications of integrating the risk-bearing function of a stock market with the savings-investment function. This is done by requiring that firms raise the investment funds necessary for production in the same capital market as households trade equity shares. The consequences of this are rather strong. Perhaps the most striking result is that when a firm pursues an objective independent of the preferences for risk held by the household sector an equilibrium may not exist. Therefore, when a capital market serves both a risk bearing function and investment allocation function, the investment policies firms may actually carry out are constrained by what the capital market will allow them to do. In order to examine these and related issues some new properties of the stock market model are developed in the second essay.

The securities, which are traded by households in the first two essays and are issued by firms in the second essay, could be best described as common stocks. That is, purchase of the security provides
a claim on the income stream of the asset against which the security is issued. The risk or uncertainty of holding such a security is, thus, the uncertainty as to the income produced by the primary asset. Many of the contracts written in capital markets, however, are of the debt type; that is, the repayment is of fixed amount in value terms and there is no uncertainty as to this amount. The sole source of uncertainty on this type of contract is the possibility of default in which case the loan will not be repaid at the terms originally agreed to in the contract. This type of uncertainty, or default risk, is significantly different from the variability of the income stream of the equity type of securities, as the probability of default on a debt contract is directly related to the price of the contract. For a given investment project financed by a debt instrument, the probability of default increases as the price of the debt falls, or equivalently, increases as the gross interest rate on the debt rises. The increase in the interest rate has another effect of course in that when the debt does pay off, it pays off at a higher rate. This particular feature of contracts on which default is a possibility is dealt with in the third essay.

Some simple equilibrium models of a capital market are constructed in which trade occurs in a single type of debt contract in order to finance investment in a risky project. Any loans secured by the project have some positive probability of defaulting. The models deal with the determination of the equilibrium loan price and the equilibrium probability of default, and with their relationship to the risk free interest rate, the productivity of investment and the expectations of the capital market participants. In the latter part of the essay an equity type of security is introduced in addition to the debt type security. The introduction of
equity to the model has the consequence that as the capital structure of an investment project becomes more highly levered, the probability of the debt portion of the total capital structure paying off at its original price is diminished. Furthermore, when default or bankruptcy occurs there is a real cost involved, in that the returns from the investment project are less than they would have been had default not occurred. Due to the possibility of costly default, the model has the feature that not only is an equilibrium price of debt determined, but in addition an equilibrium debt/equity ratio. This is in strong contrast of course to the financial irrelevance propositions of the Miller-Modigliani [1958] sort, which state that when debt is risk free the economy is indifferent between varying debt/equity ratios. The relationship between equilibrium debt/equity ratios and various parameters is investigated.
Footnote

1. Throughout the thesis the terms uncertainty and risk will be used interchangeably. Both denote the situation in which the consequence of a certain act or 'experiment' may be described by a probability distribution, either objective or subjective.
References


Chapter II
THE STRUCTURE OF A CAPITAL MARKET EQUILIBRIUM

1. Introduction

The mean-variance portfolio model of Markowitz and Tobin has been the most substantive contribution to the theory of individual asset demand under uncertainty, in terms of comparative static results and testable implications.\(^1\) Although subject to a number of criticisms at the axiomatic level, it still stands as the classic portfolio model. The general equilibrium extension of the Tobin-Markowitz model due to Sharpe [1964], Lintner [1965] and Mossin [1966] has led to important propositions about the nature of risk in general equilibrium and its effect on the pricing of assets, and the model has subsequently been subjected to extensive empirical testing.\(^2\) It would appear then that the Capital Asset Pricing model, as it is referred to in the literature, is to remain with us, as it is the only general equilibrium model of asset pricing under uncertainty which can be econometrically implemented. It has been used for a variety of purposes in areas ranging from corporate-finance theory to the debate on the social discount rate.\(^3\)

What is somewhat remarkable, given the extensive use of this model, is that for a long time neither the existence question nor the comparative static properties of the model were investigated. Hart [1974] has recently treated the existence question; it is the purpose of this paper to investigate the comparative static properties of the model. In doing so we shall also establish a number of propositions on the structure of the equilibrium which have not been discussed in the existing literature.
In order to carry out the comparative statics, it proved to be necessary to provide a complete description of the determination of the equilibrium prices in the mean-variance general equilibrium model. The mode of analysis initiated by Sharpe [1964], Lintner [1965] and Mossin [1966] was to use the first-order conditions of individual portfolio equilibrium and a market clearing assumption to derive a set of linear equations which the equilibrium asset-prices satisfied. To carry the analysis this far was sufficient to determine a general-equilibrium version of the well-known Tobin [1958] separation theorem and to derive some implications about the pricing of 'risk'. These linear equations, or the capital-asset pricing equation, do not, however, determine equilibrium prices, and it is necessary to consider explicitly the excess demand functions of the economy in order to give a complete determination of equilibrium prices. It is somewhat surprising that, in such a general model of arbitrary dimensions allowing n assets and H investors, we are able to establish some rather strong qualitative propositions, since, as is generally well known, these are usually quite difficult to come by.\(^4\)

The rest of the paper proceeds as follows. In section 2 we introduce some notation and the assumptions of the model. Section 3 contains the derivation of a set of equations which the equilibrium prices must satisfy and some of the implications of these equations. In section 4 we establish the determination of equilibrium prices and portfolios, and some characteristics of the equilibrium. Finally in section 5 we investigate the comparative static properties of the model with section 6 presenting a summary and discussion of the results obtained.
2. **The Model**

The model consists of \( H \) investors, \( H \) finite and integer valued, who hold portfolios of \( n + 1 \) assets, where the 0th asset is riskless. By holding a portfolio at the beginning of the period, an individual realizes a random return on his portfolio at the end of the period. The following assumptions are basic to the Capital Asset Pricing Model.

A.1. All investors are single-period, expected-utility-of-terminal-wealth maximizers who choose among alternative portfolios on the basis of the mean and variance of the portfolio.

A.2. All investors are price takers, and no production takes place, i.e., a classical competitive exchange model.

A.3. There are no short sales restrictions on any of the \( n+1 \) assets.

A.4. All investors have identical subjective probability distributions on asset returns.

A.5. The distribution of the asset endowments across investors is given.\(^5\)

Our notation and some further assumptions are as follows:\(^6\)

R.1. \( V \) denotes the variance-covariance matrix of the probability distribution of risky asset returns; \( V \) is an \( n \times n \), real symmetric positive definite matrix.

R.2. \( \mu \) denotes the mean vector of returns on the risky assets, \( \mu \in \mathbb{R}^n_+ \), and we assume \( \mu >> 0^n \). By definition the return on the riskless asset is 1.

R.3. \( U^h(r_h, v_h) \) denotes the \( h^{th} \) individual's utility function, defined over mean portfolio return \( r_h \), and standard deviation of the portfolio return, \( v_h \), for all \( h = 1, \ldots, H \). \( U^h \) is defined
over the interior $\mathbb{R}^2_+ \text{ with } U^h_r > 0 \text{ and } U^h_v < 0 \text{ everywhere with } U^h_r \text{ denoting the partial derivative of } U^h \text{ with respect to } r^h \text{ and } U^h_v \text{ denoting the partial derivative of } U^h \text{ with respect to } v^h; \ U^h \text{ strictly quasi-concave}; \ \lim \limits_{v^h \to 0} [U^h_v(r^h,v^h)/U^h_r(r^h,v^h)] = 0 \text{ for any } r^h > 0.7 \n
\text{R.4. } \hat{q}'=(q_0,q') \text{ denotes the price vector on the } n+1 \text{ assets; } \hat{q} \in \mathbb{R}^{n+1}. \n
\text{R.5. } \hat{s}'_h=(s'_h,s'_h) \text{ denotes the portfolio of the } h^{th} \text{ investor; } \hat{s}_h \in \mathbb{R}^{n+1} \text{ by A.3.} \n
\text{R.6. } w'_h=(s'_h,s'_h) \text{ denotes the endowment of the } h^{th} \text{ investor; } w_h >> 0_{n+1}. \n
\text{R.7. } \sum_{h=1}^{H} w_h = \hat{i}, \text{ where } \hat{i} = (1, ..., 1), \hat{i} \in \mathbb{R}^{n+1} \text{ is the unit vector.}^8 \n
\text{These assumptions are quite usual and need no explanation here. All vectors are taken to be column vectors, with row vectors indicated by a transpose superscript, i.e., } x' \text{ is the transpose of } x \text{ where } x \text{ is an } n \times 1 \text{ vector.} \n
\text{The individual investor maximizes his utility with respect to his portfolio vector subject to the restriction that his portfolio be feasible at current prices } \hat{q}. \text{ Thus we define:} \n
\text{D.1. The } h^{th} \text{ investor's budget set at prices } q \text{ is} \n
B^h(\hat{q}) = \{\hat{s}_h | \hat{s}_h \in \mathbb{R}^{n+1}; \hat{q}'\hat{s}_h \leq \hat{q}'w_h\}. \n
\text{An equilibrium is defined as a set of prices and portfolio allocations such that each individual maximizes his utility and all asset markets clear. Therefore:} \n
\text{D.2. An equilibrium is an } H+1 - \text{ tuple } (q^*,\hat{s}^*_1,\hat{s}^*_2, ..., \hat{s}^*_H) \text{ such that}
a) \( \hat{s}_h^* \in B^h(q^*) \) for all \( h \), and \( \hat{s}_h^* \) maximizes \( U^h \) over all \( \hat{s}_h^* \in B^h(q^*) \),

\[
\sum_{h=1}^{H} \hat{s}_h^* = i \quad \text{and} \quad \sum_{h=1}^{H} \hat{s}_{ho}^* = 1,
\]

b) \( \hat{q}^* \in \mathbb{R}^{n+1} \).

c) We may, of course, normalize \( \hat{q}^* \), but note, since we admit negative returns on portfolios, we may not restrict \( \hat{q}^* \) to \( \mathbb{R}_+^{n+1} \). Often we shall normalize \( q_0 = 1 \) for convenience, and this is clearly admissible, since the asset demand functions have the property of being homogeneous of degree zero in \( \hat{q} \). We shall further define:

D.3. \( E \) denotes the set of equilibrium prices \( \hat{q}^* \) for the previously described economy. \( E = \mathbb{R}^{n+1} \).

Using the results of Hart [1974], Appendix 2 we shall assume

A.6. \( E \) is non-empty.

That is we assume that an equilibrium exists. Given the assumption A.4., that all probability beliefs are identical, this can be proved. \( E \) of course depends on \((\mu, V)\) and consequently we shall sometimes write \( E(\mu, V) \).

The rest of the paper concerns the nature of \( E \) and its relation to \((\mu, V)\).

3. A General Equilibrium Equation

We shall now proceed on the assumption that we are dealing with equilibrium prices, that is that \( \hat{q} \in E \), and derive a set of linear equations which the equilibrium prices must satisfy. Consider then the individuals, choice problem:

\[
\max_{\hat{s}_h} U^h(r_h, v_h) \text{ subject to } \hat{s}_h \in B^h(\hat{q}) \quad (3.1)
\]

which can be formulated as a Lagrangean maximization problem as we have
no non-negativity constraints by A.3. Hence we have the Lagrangean

\[ L_h = u^h(r_h, v_h) - \lambda_h(q's_h + q_o s_{oh} - q's_h - q_o s_{oh}) \]  

(3.2)
giving us the first-order conditions:

\[ u^h_r + v^h \frac{V_{sh}}{(s_{h}V_{sh})^{\frac{1}{2}}} - \lambda_h q = \frac{0}{n}, \quad (3.3) \]

\[ u^h_v - \lambda_h q_o = 0, \quad (3.4) \]

\[ q's_h + q_o s_{oh} - q's_h - q_o s_{oh} = 0. \quad (3.5) \]

In order to make use of the first-order conditions (3.3) - (3.5), we must assume that a solution exists to problem (3.1). However since we allow short sales, the feasible portfolio set \( B^h \) is unbounded below and there is some possibility that a maximum may not exist.

Bertsekas [1974] has recently shown that under quite reasonable conditions on the underlying expected utility function and probability distribution a finite solution always exists. Hence in assuming A.6 we have implicitly assumed that for all \( q \in E \), a finite solution to (3.1) exists. Therefore we may use the first-order conditions (3.3) - (3.5) to derive some implications about \( q \in E \).

Substituting (3.4) into (3.3) we have

\[ q_o u^h_r + q_o \frac{u^h_v}{u^h_r} \frac{V_{sh}}{(s_{h}V_{sh})^{\frac{1}{2}}} - q = \frac{0}{n}, \quad (3.6) \]

which upon re-arranging gives

\[ q_o u^h_r / u^h_v (s_{h}V_{sh})^{\frac{1}{2}} + q_o V_{sh} - q u^h_r / u^h_v (s_{h}V_{sh})^{\frac{1}{2}} = \frac{0}{n}. \quad (3.7) \]

Premultiplying both sides of (3.7) by \( i' \) and summing over \( h=1, \ldots, H \),
yields,
\[
q_o i' \mu \sum_{h=1}^{H} \frac{U_h^h}{V(s_h^*)^{1/2}} + q_o i'Vi - i'q \sum_{h=1}^{H} \frac{U_h^h}{V(s_h^*)^{1/2}} = 0 \quad (3.8)
\]
where we use the fact that at an equilibrium price \( \sum_{h=1}^{H} s_h = i \) (we have implicitly assumed \( s_h = s_h^* \), an equilibrium portfolio). Following the procedure used in describing the pricing equation of the capital asset pricing model we define:

\[
V_m = q'i \quad ( = i'q)
\]

\[
E(V_m) = \mu'i \quad (3.9)
\]

\[
\sigma^2_m = i'Vi,
\]

where the subscript \( m \) refers to a market aggregate and \( E(V_m) \) is the expected value of the return on the aggregate market portfolio. Solving equation (3.8) we get

\[
\sum_{h=1}^{H} \frac{U_h^h}{V(s_h^*)^{1/2}} = q_o (i'Vi) / [i'q - q_o i'\mu] = q_o \sigma^2_m / [V_m - q_o E(V_m)] = -q_o \sigma^2_m / [q_o E(V_m) - V_m]. \quad (3.10)
\]

We now solve equation (3.6) for the individual's risky portfolio and sum over all \( h=1, \ldots, H; \)

\[
\sum_{h=1}^{H} s_h = i = -\{\sigma^2_m / [q_o E(V_m) - V_m]\}(V^{-1}q - q_o V^{-1} \mu). \quad (3.11)
\]

Premultiplying both sides of (3.11) by \( V \) gives

\[
Vi = \frac{-\sigma^2_m}{[q_o E(V_m) - V_m]} (q - q_o \mu). \quad (3.12)
\]
Let us define
\[ \theta = \frac{E(\tilde{V}_m) - 1/q_o V_m}{\sigma_m^2}. \]

Solving for \( q \) we get the well-known capital asset pricing equation:
\[ q = q_o [\mu - \theta Vi]. \] (3.13)

Usually, the "market risk" of the \( j \)th asset is defined as \( \sigma_j \equiv \sigma_{jm} \) where \( V_j \) is the \( j \)th row of \( V \), and consequently the \( j \)th equation of (3.13) is written as,
\[ q_j = \frac{1}{1+r} [\mu_j - \theta \sigma_{jm}], \quad q_o = \frac{1}{1+r} \] (3.14)

where \( r \) is called the riskless rate of interest. The term \( \sigma_{jm} \) is often called the risk characteristic of the \( j \)th asset, as it represents the marginal contribution to aggregate market risk, by increasing the \( j \)th asset. Hence, if we let
\[ \sigma_m(x) = \sum_{ij} \sigma_{ji} x_i x_j, \quad x \in \mathbb{R}^n, \] (3.15)

then
\[ \sigma_{jm}(i) = \frac{\partial \sigma_m}{\partial x_j} = \Sigma_{ji} \] (3.16)
evaluated at \( x = i \).

We now re-write equation (3.13) in a more basic form as.
\[ q = q_o [\mu - \{(i'\mu - 1/q_o i'q)/(i'Vi)V_i}], \] (3.17)

which upon re-arranging gives us
\[ (i'Vi)q - i'q Vi = i'Viq_o \mu - i'\mu Viq_o. \]

Taking the transpose and using the fact that \( V \) is symmetric,
\[ (i'Vi)q' - q' ii'V = q_o (i'Vi)\mu' - \mu' ii'V q_o, \] (3.18)
or
\[ q'[I - \frac{ii'\gamma}{i'i\gamma}] = q_o u' [I - \frac{ii'\gamma}{i'i\gamma}] \]  
(3.19)

Now define
\[ A' = \frac{ii'\gamma}{i'i\gamma} \]  
(3.20)

Hence, we may write the final equation system as
\[ [I - A] q = [I - A] q_o u. \]  
(3.21)

Let us denote the solution set of (3.21) as
\[ Q = \{(q, q_o) \mid [I - A] q = [I - A] q_o u; q \in \mathbb{R}^n, q_o \in \mathbb{R}\}. \]

Then we have by construction, the following lemma:

**Lemma 3.1:** \( E \subseteq Q \), i.e., if \( \hat{q} \in E \), then \( \hat{q} \in Q \).

Note that the converse is not true. That is if \( \hat{q} \) is a solution to (3.21) it is not necessarily the case that \( \hat{q} \) is an equilibrium price.

4. **The Structure of Equilibrium Prices**

In this section we investigate the nature of the equilibrium set of prices and the associated set of equilibrium portfolios. It will be convenient in this and subsequent sections to make the normalization \( q_o = 1 \), and throughout this section we shall take the statement \( q \in E \) to imply \( (q,1) \in E \) and similarly for \( Q \).

Crucial to the subsequent theorem is the following assumption made on the variance-covariance matrix.

**A.7.** \( \gamma_i \gg 0 \) .

The implication of the above assumption is that in a well defined sense all assets are positively risky. Intuitively, if one examines the capital asset pricing equation (3.14), assuming risk is negatively valued, i.e.,
is greater than zero, then A.7. implies $\sigma_{jm} > 0$ for all $j = 1, \ldots, n$, and therefore all assets contribute positively at the margin to total market variance. All subsequent lemmas and theorems stated are implicitly understood to have assumptions A.1. - A.6. and R.1 - R.7 holding.

The following lemma will be quite useful.

**Lemma 4.1**: The matrix $A$ defined by (3.20) has the following properties:

1. $A$ is idempotent.
2. $A$ is of rank 1.

And if A.7. holds,

3. $A \gg O_{nxn}$.

(proof): Property (1) follows since $AA = A$. $A$ is of rank 1, since $ii'$ is of rank 1, and the fact that the rank of a product of two matrices is not greater than the minimum of the two ranks; recall rank $V = n$. (3) follows straightforwardly from A.7. Q.E.D.

We shall now establish a theorem which puts some bounds on the set $E$.

**Theorem 4.1**: (a) If $q \in E$, then

i) $q - \mu \neq O_n$.

(b) If $q \in E$, and A.7 holds then

ii) $q \ll \mu$.

(proof): Let $x = q - \mu$ and $y = \mu - q$. The equation system (3.21) may be written as

$$Ax = x \quad (4.1)$$

or

$$Ay = y. \quad (4.2)$$

That is, $x$ and $y$ are eigenvectors of the matrix $A$ corresponding to an eigenvalue of 1. Since $A$ is idempotent it can have eigenvalues of only zero or one, and furthermore it has at least one eigenvalue equal to unity. (Lang [1966] pg. 193 Ex. 3).
(i) Now suppose that \( q = \mu \), and we consider an arbitrary individual's equilibrium equation (3.6). Re-writing this, assuming \( q = \mu \), gives us

\[
\frac{U_r^h}{U_v^h} \cdot \frac{V_s^h}{(s_h^\prime V_s^h)^{\frac{1}{2}}} = q - \mu = 0_n. \tag{4.3}
\]

Now suppose \( s_h \neq 0_n \). As \( U_r^h / U_v^h < 0 \), the system (4.3) may be written as

\[
\frac{V_s^h}{(s_h^\prime V_s^h)^{\frac{1}{2}}} = 0_n.
\]

Since \( s_h \neq 0_n \), \( (s_h^\prime V_s^h)^{\frac{1}{2}} > 0 \), as \( V \) is positive definite, and therefore \( V_s^h = 0_n \). But \( V \) is non-singular. Hence \( s_h = 0_n \) giving a contradiction to the supposition that \( s_h \neq 0_n \). Therefore \( s_h = 0_n \) is the equilibrium portfolio if \( q = \mu \). Since \( h \) was arbitrary, \( s_h = 0_n \) for all \( h = 1, \ldots, H \).

But this contradicts the assumption that \( q \in E \), as \( \sum_{h=1}^H s_h = 0_n \neq i \).

Therefore \( q \neq \mu \).

(ii) From (i) we have that \( x \neq 0_n \), \( y \neq 0_n \). Since \( A \) is a positive matrix, we know these exists a vector \( z \gg 0_n \) and a unique positive eigenvalue \( \lambda > 0 \), such that \( Az = \lambda z \). (Karlin [1959], Theorem 8.2.1.) Furthermore since \( A \) is of rank 1, and idempotent, \( \lambda = 1 \) is the only non-zero eigenvalue, and its associated eigenspace is the image space of \( A \). Using the fact that \( A \) is idempotent, \( q \neq \mu \), and \( A \) is a positive matrix, we have that the Frobenius eigenvalue = 1, and that either \( x \gg 0_n \) or \( y \gg 0_n \), or both. Note that \( A \) has an eigenvalue \( 0 \) of multiplicity \( n-1 \). Suppose \( y \gg 0_n \), i.e., \( q \gg \mu \), and \( q - \mu \) is the eigenvector corresponding to the Frobenius eigenvalue 1. Consider an arbitrary individual's equilibrium equation (3.6), assuming \( q \gg \mu \).

\[
\frac{U_r^h}{U_v^h} \cdot \frac{V_s^h}{(s_h^\prime V_s^h)^{\frac{1}{2}}} = q - \mu = A(q - \mu) \tag{4.5}
\]
Re-arranging this we have

$$V \left[ \frac{U^h_v}{U^h_r} \frac{s_h}{(s^t_h V s_h)^{\frac{1}{2}}} - \frac{i_i^t}{i^t V_1} (q-\mu) \right] = 0_n.$$  (4.6)

Since $V$ is non-singular the equation in brackets must be identically zero, and hence

$$s_h = \frac{U^h_r}{U^h_v} (s^t_h V s_h)^{\frac{1}{2}} \frac{i_i^t}{i^t V_1} (q-\mu) << 0_n.$$  (4.7)

This clearly contradicts the assumption that $q \in E$. Therefore $q << \mu$. Q.E.D.

Basically, theorem 4.1 says that risk "matters" in a non-trivial sense. All asset prices must be strictly less than their expected values. As a by-product of the proof of theorem 4.1 we establish the well-known separation theorem (Tobin [1958], Sharpe [1964], Lintner [1965], Mossin [1966]), which states that in equilibrium all individuals hold positive fractions of an identical risky portfolio consisting of the vector of fixed aggregate supplies of risky assets.

**Theorem 4.2**: If $q \in E$, then $s_h(q) = \lambda_h i$, $\lambda_h > 0$ for all $h=1, \ldots, H$.

(proof): Let $\lambda_h = \frac{U^h_r}{U^h_v} (s^t_h V s_h)^{\frac{1}{2}}$, then from (4.5) we have

$$s_h(q) = \gamma_h V^{-1}(q-\mu).$$ Note that $\gamma_h < 0$. Since $q \in E$, we have

$$\sum_{h=1}^{H} s_h(q) = i = V^{-1}(q-\mu) \left[ \sum_{h=1}^{H} \gamma_h \right].$$ Therefore $V^{-1}(q-\mu) = i \left[ \sum \gamma_h \right]^{-1}$

and $s_h(q) = \frac{\gamma_h}{\sum \gamma_h} i$. Let $\lambda_h = \frac{\gamma_h}{\sum \gamma_h} > 0$, then $s_h(q) = \lambda_h i$. Q.E.D.

Since short sales were allowed in constructing the individual's portfolio,
it is of some interest to consider whether in equilibrium asset prices are
such that some individuals choose to go short in some assets. The separa-
tion theorem (Theorem 4.2) tells us that in a general equilibrium indivi-
duals will not go short in risky assets. The following theorem gives
sufficient conditions to ensure individuals do not go short in the risk-
less asset.

**Theorem 4.3**: If \( q \in E, A.7 \) holds, and either

\[
\left(1\right) \quad \frac{-U^h_v}{U^h_r} > \frac{i'u - i'q}{(i'Vi)^{1/2}} \quad \text{along the ray}
\]

\[r_h = Mv_h, \text{ where } M = \frac{i'u}{(i'Vi)^{1/2}} \text{ for all } h=1, \ldots, H,
\]

or \( \left(2\right) U^h(r_h, v_h) \) is identical for all \( h \in \{1, \ldots, H\} \) and
homothetic in \((r,v),\n\]
then \( s_{ho}(q) > 0 \) for all \( h=1, \ldots, H.\)

(proof): (1) Consider the first-order conditions, for the individual's
portfolio problem:

\[
\text{Max } U^h(r_h, v_h) \text{ subject to } s_h \in B^h(q)
\]

which are from (4.5), and Theorem 4.2.ii.

\[
- \frac{U^h_v}{U^h_r} = \frac{i'u - i'q}{(i'Vi)^{1/2}} \equiv \pi > 0. \quad (4.8)
\]

Consider figure 1. From the hypothesis (1) of the theorem, and the equilib-
rium condition (4.8), the equilibrium point of the individual, \( e_h \) in
\((r_h,v_h)\) space, occurs to the left of the line \( r_h = Mv_h \) and tangent to a
line \( r_h = \pi v_h + y_h \) where \( y_h = q's_h + s_{ho} \). From theorem 4.2 we know
\( s_h(q) = \lambda_h i, \) and this corresponds to the point \((\lambda_h(i'Vi)^{1/2}, \lambda_h(i'u)) = d_h\)
Figure 1.

Individual Portfolio Equilibrium
in \((r_h,v_h)\) space, where \(d_h\) lies on the line \(r_h = M v_h\) directly below the equilibrium point \(e_h\). As \(r_h(q) = \lambda_h(i'\mu) + s_{oh}\), it therefore must be the case that \(s_{ho}(q) > 0\). As \(y_h \to 0\), \(s_{ho} \to 0\). Hence if endowment income is identically zero, \(s_{ho}(q) = 0\). We have excluded this possibility, however, by R.6.

(2) Let \(W(s_h) = U_h(u's_h + s_{ho} + (s'Vs_h)^{1/2})\). Since \(U_h\) is homothetic it is straightforward to verify that \(W(s_h)\) is homothetic, and quasi-concave. The (asset) demand functions may therefore be written as

\[
\hat{s}_h^* = \hat{s}(q) y_h
\]

where

\[
y_h = q's_h + s_{ho} > 0.
\]

by using the well-known properties of homothetic functions. Since \(q \in E\),

\[
\sum_{h=1}^H \hat{s}_{ho}(q) = \sum_{h=1}^H \hat{s}_o(q) y_h = \hat{s}_o(q) \sum_{h=1}^H y_h = 1.\]

Clearly then \(s_o(q) > 0\), and hence \(s_{ho}(q) > 0\) if \(y_h > 0\). Q.E.D.

The conditions of the theorem are quite strong, and they suggest that the likelihood of short sales in the riskless asset is quite great. An open question which remains is the possibility of finding a weaker set of sufficient conditions which ensure the absence of short sales in the riskless asset.

Let us denote by \(\hat{Q}(V)\) the following set:

\[
\hat{Q}(V) = \{w \mid Aw = w, w \neq 0\}.
\]

This is the set of non-zero eigenvectors of the matrix \(A\). From the previous discussion we have determined that if \(q \in E\), then \(q = u - w^*\) for some \(w^* \in \hat{Q}(V)\). Since \(\hat{Q}(V)\) denotes the solution set to the capital asset pricing equation, we cannot claim in general that the capital asset pricing
equation "determines" equilibrium prices. An equilibrium corresponds to
a particular \( w^* \in \hat{Q}(V) \). We shall now consider the manner in which \( w^* \) is
determined.

Once \( \hat{Q}(V) \) has been determined, we may choose \( w_0 \in \hat{Q}(V) \), such that
\[ ||w_0|| = 1, \] where \( || \cdot || \) denotes the euclidean norm. Consider then the
set of non-negative real numbers \( \Pi \equiv \{ \pi | \pi > 0, \pi = \frac{\lambda i'w_0}{(i'Vi)^{1/2}}, \lambda > 0 \} \).

Dropping the \( h \) subscript, the individual's choice problem can be reformu-
lated, given \( w_0 \), as

\[
\text{Max } U(r,v) \text{ subject to } r - \pi v \leq y. \tag{4.12}
\]

Recall that \( y_h = q'i' \hat{s}_h + \hat{s}_{ho} = \mu'i' \hat{s}_h - \lambda w_0'i' \hat{s}_h + \hat{s}_{ho}, \) when \( q = \mu - \lambda w_0 \).
Therefore, \( y_h = \tilde{r}_h - \lambda w_0'i' \hat{s}_h \) where \( \tilde{r}_h = \mu'i' \hat{s}_h + \hat{s}_{ho} \). Let \( \tilde{v}_h = \frac{w_0'i' \hat{s}_h}{w_0'i' (i'Vi)^{1/2}}, \)
then \( \tilde{\pi} \tilde{v}_h = \lambda w_0'i' \hat{s}_h \). Hence we may write problem (4.12) as

\[
\text{Max } U(r,v) \text{ subject to } r - \pi v \leq \tilde{r} - \tilde{\pi} \tilde{v} \tag{4.13}
\]

This gives us the individual demand functions for \((r,v)\) as

\[
r = f_r(\pi), \quad v = f_v(\pi). \tag{4.14}
\]

The individual demand functions for 'risk' \( v \), and 'return' \( r \) depend upon
the 'price of risk' \( \pi \) and the endowment income \( y = \tilde{r} - \tilde{\pi} \tilde{v} \), which in turn
depends upon the price of risk. These demand functions satisfy the usual
properties implied by Hicks-Allen demand theory. In addition, they have
the property that 'return' \( r \) is always a normal good, that is \( \frac{\partial r(\pi,y)}{\partial \pi} > 0 \)
and \( \frac{\partial r(\pi,y)}{\partial y} > 0 \). Note that it is not true in general that \( f_r'(\pi) > 0 \).

Of course, \( f_r(\pi) \) and \( f_v(\pi) \) are related by the budget constraint, \( f_r(\pi) - \pi f_v(\pi) = y \) and hence the equation system (4.14) has only one independent
equation. Note that these demand functions are not defined at \( \pi = 0 \). If \( \pi = 0 \) then by the argument of theorem 4.2, \( s_h(q) = 0 \), for all \( h \), and by the positive slope of indifference curves \( f_r(0) = y_h = \mu' \bar{s}_h \). Now, however, \( f(V)(0) = 0 \). But, in fact, this demand is infeasible in the actual commodity space, as the economy's demand for risky assets is identically zero. Therefore if \( q = \mu \), then \( y_h = \bar{s}_h \). Suppose that \( q = \mu \), or, equivalently, \( \pi = 0 \). The demand functions at this point are given as the solution to:

\[
\max U(r,v) \text{ subject to } v = (\bar{s}'V\bar{s})^{\frac{1}{2}} \\
r, v \geq 0 \\
r \leq \bar{s}_0. 
\]

At \( q = \mu \), then all individuals will be in equilibrium holding their endowments at this 'constrained' equilibrium, or equilibrium with rationing which is not an equilibrium in the Walrasian sense of desired supply being greater than or equal to desired demand.

We define the excess demand functions in the usual fashion. The individual excess demand function for mean return is defined as

\[
Z_r(\pi) = f_r(\pi) - \mu' \bar{s}_h + \bar{s}_\mu 
\]

and the aggregate excess demand function is given by

\[
Z(\pi) = \sum_h f_r(\pi) - \mu' \mu + 1. 
\]

In proving the main theorem of this section we shall have to make use of a partial converse to Lemma 3.1.

**Lemma 4.2:** If \( w \in \hat{Q}(V) \), then \( s_h(q) = \delta_h i \), for some scalar \( \delta_h \) and all \( h \in \{1, \ldots, H\} \), when \( \mu - q = w \).

**(proof):** Consider an arbitrary individual's portfolio equilibrium equation (4.5), which is reproduced here for convenience.
Let \( \beta_h = -\frac{u^h_r}{u^h_r} \left( s^h Vs_h \right)^{-\frac{1}{2}} \), then (4.5) may be written as

\[
Vs_h = \frac{1}{\beta_h} \tilde{w}
\]

where \( \tilde{w} \in \hat{Q}(V) \). As \( \beta_h > 0 \), if \( w_h = 1/\beta_h \tilde{w} \), then \( w_h \in \hat{Q}(V) \) from the basic properties of eigenvectors. Therefore,

\[
Vs_h = w_h = Aw_h = \frac{Vii'w_h}{i'Vq}.
\]

Upon re-arranging this last equation we have

\[
V[s_h - \frac{ij'w_h}{i'Vq}] = 0_n.
\]

As \( V \) is a positive definite matrix and consequently non-singular, the vector in brackets must be the zero vector. Letting \( \delta_h = \frac{ij'w_h}{i'Vq} \), \( s_h = \delta_h i \). Q.E.D.

The lemma has the following interpretation. If an asset price vector satisfies the basic eigenvector equation, then the portfolio of risky assets held by every household must be proportional to the aggregate endowment of risky assets. The lemma is stronger in one sense than theorem 4.2 as the price vector \( q \) is not required to be an equilibrium price vector, but rather it must satisfy only the weaker eigenvector property.

We now prove the central theorem which provides a complete determination of equilibrium prices.

**Theorem 4.4:** If \( Z_r(\pi^*) = 0 \) for \( \pi^* \in \Pi \) then \( q \in E \)

where \( q = \mu - \lambda w_0 \), \( \pi^* = \frac{\lambda i'w_0}{(i'Vq)^{\frac{1}{2}}} \) and \( w^* = \lambda w_0 \).
(proof): Since $Z_{r}(\pi^{*}) = 0$, and $\pi^{*} > 0$, then by Walras' law $Z_{v}(\pi^{*}) = 0$. Therefore $\sum_{h} f^{h}(\pi^{*}) = (i'V i)^{1/2}$, and $f^{h}(\pi^{*}) = v_{h} = [s_{h}(q)'V s_{h}(q)]^{1/2}$.

As $\mu - q = \lambda^{*}w_{o}$, and $\lambda^{*}w_{o} \in \hat{Q}(V)$, then by lemma 4.2 $s_{h}(q) = \delta_{h}i$, $\delta_{h} \in R$, for all $h=1, \ldots, H$. Substituting, we have $v_{h} = \delta_{h}(i'V i)^{1/2}$ and thus market clearing in the 'risk' market implies $\sum_{h} \delta_{h} = 1$. Therefore $\sum_{h} s_{h}(q) = i$ and consequently $q \in E$. Q.E.D.

What this theorem says is that, in addition to the matrix equation $Aw_{o} = w_{o}$, additional information on the form of utility functions and the distribution of endowments across individuals, or equivalently the aggregate demand functions for risk or return is needed in order to determine equilibrium prices. This additional information is particularly critical when considering questions of comparative statics.

5. Comparative Statics

A fundamental criteria of the usefulness of a model in economics has been whether or not it yields unambiguous comparative static results, and it is the purpose of this section to see how far one can go in this direction with the Capital Asset Pricing Model. In order to do so we shall use a different mode of analysis than is generally followed.

At this point it is necessary to make some further assumptions on investors' preferences in order to proceed with the comparative statics. This is necessary for two reasons. The first is that without a more stringent set of conditions on the excess demand functions $Z_{v}(\pi)$, $Z_{r}(\pi)$ other than Walras' law and some differentiability assumptions, we are not assured in general that the model has a unique equilibrium price $q^{*}$. This requirement is generally needed in order to make our comparative static
results meaningful. Secondly, in order to get qualitative results it is necessary to sign the derivatives of $Z_v$ and $Z_r$ in some manner.

The class of preferences we choose to work with is a convenient generalization of the class of expected utility functions which exhibit constant absolute risk aversion. Basically, the requirement is that the marginal rate of substitution between risk and return is independent of return. Therefore in $(r,v)$ space all indifference curves have the same slope along vertical lines, as in Figure 2. We express this requirement by the equation $-U_v/U_r = k(v)$. Clearly by the convexity assumption $k'(v) > 0$, and the assumption of risk aversion implies $k(v) > 0$. Therefore we assume

A.8. Investors have utility functions, $U^h(r,v)$, where $U^h(r_h,v_h)$ satisfies the condition that $-U^h_v/U^h_r = k^h(v_h)$ with $k^h(v_h)$ continuous and $\lim_{v \to 0} k^h(v_h) = 0$, for all $h=1, \ldots, H$.

Thus all investors have preferences over risk and return, which are of the same class given by A.8, although different investors may have different preferences within this class.

Consider the problem of the individual investor, given by

$$\max_{r_h,v_h \geq 0} U^h(r_h,v_h) \text{ subject to } r_h - \pi v_h \leq y_h.$$ (5.1)

It is straightforward to verify that the demand functions generated by this problem take the form

$$r_h = f^h_r(\pi) + y_h$$

$$v_h = f^h_v(\pi).$$ (5.2)

Using the budget constraint, we have that

$$f^h_r(\pi) - f^h_v(\pi) = 0.$$ (5.3)
Figure 2.

Individual Preferences with the Property that the Marginal Rate of Substitution Between Risk and Return is Independent of Return
The excess demand function for return is given by

\[ Z_r(\pi) = \sum_h f_r^h(\pi) + Y - i'\mu + 1 \quad (5.3) \]

where \( Y = \sum_h y_h \), and the excess demand for risk is given by

\[ Z_v(\pi) = \sum_h f_v^h(\pi) - (i'V_i)^{\frac{1}{2}}. \quad (5.4) \]

The functions \( f_r^h(\pi) \) and \( f_v^h(\pi) \) have the property that \( f_r^{h'} > 0 \) and \( f_v^{h'} > 0 \). Consider then the equation \( Z_v(\pi^*) = 0 \), where \( \pi^* \) is defined as in theorem 4.4. If \( \pi^* \) is unique, in the sense that for all \( \pi \in \Pi, \pi \neq \pi^*, Z_v(\pi) \neq 0 \), then the equilibrium price is unique; \( q^* = \mu - \lambda^* w_0 \). Therefore we have

Lemma 5.1: Given A.8, the set \( E(\mu, v) = \{ q^* \} \). The equilibrium price vector is unique.

(proof): By A.8 \( Z_v(\pi) = \sum_h f_v^h > 0 \) for all \( \pi \in \Pi \). Therefore \( Z_v \) is globally univalent, which implies \( \pi^*, Z_v(\pi^*) = 0 \), is unique. Q.E.D.

Having assured ourselves that the equilibrium is well defined, we now consider questions of comparative statics.

The equilibrium price \( q^* \) is determined by two equations:

\[ A w_o = w_o \quad ||w_o|| = 1 \quad (5.5) \]

and

\[ Z_r(\pi^*) = 0 \quad \text{or} \quad Z_v(\pi^*) = 0, \quad (5.6) \]

where

\[ \pi^* = \lambda^* \frac{i'w_0}{(i'V_i)^{\frac{1}{2}}}. \]

We will consider first changes in the equilibrium price with respect to changes in the mean vector \( \mu \). In particular we consider a change in the mean of the \( i^{th} \) asset, \( \mu_i \). We note first

Lemma 5.2: The positive eigenvectors of \( A \) are invariant with respect to changes in \( \mu \), and hence \( \frac{dw_o}{d\mu} = [0]_{n \times n} \).
This is a trivial consequence of the fact that $A$ does not depend on $y$. Therefore $\frac{dw_i}{d\mu_i} = 0$, for all $i=1, \ldots, n$. A consequence of this is the following theorem.

**Theorem 5.1:** Given $A.8$, if $q^* \in E(\mu, \Sigma)$, then

$$\frac{dq_i}{d\mu_i} = 1 \quad \text{for } i=1, \ldots, n, \quad (5.7)$$

$$\frac{dq_i}{d\mu_j} = 0 \quad \text{for } i \neq j \quad i,j=1, \ldots, n. \quad (5.8)$$

**Remark:** Since we have normalized $q_0 = 1$, $q_i^*$ is the price of the $i^{th}$ asset relative to the riskless asset.

**(proof):** By definition $q^* = \mu - \lambda^* w_0$. Therefore, using Lemma 5.2

$$\frac{dq_i}{d\mu_k} = \delta_{ik} - \frac{d\lambda^*}{d\mu_k} w_0, \quad \text{where } \delta_{ik} = 1 \quad \text{if } i=k, \quad \delta_{ik} = 0 \quad \text{if } i \neq k.$$

Now $\lambda^* = \pi^* \left( \frac{(i'\Sigma_i)^{1/2}}{i'w_0} \right)$. Differentiating equation (5.6) and using (5.4) we have that $\Sigma f_{\nu}(\pi^*) \frac{d\pi^*}{d\mu_k} = 0$. Since $f_{\nu} > 0$, $\frac{d\pi^*}{d\mu_k} = 0$, and hence $\frac{d\lambda^*}{d\mu_k} = 0$. Therefore $\frac{dq_i}{d\mu_k} = \delta_{ik}$. Q.E.D.

The result of theorem 5.1 is quite strong. It says that changes in asset means are asset specific, and moreover an absolute change in the mean return of an asset results in an equal absolute change in the price of the asset, of the same sign. Recall that $\theta = \frac{i'\mu - i'q^*}{i'\Sigma_i}$ was termed the "price" of risk in section 3. A simple consequence of the theorem is that $d\theta/d\mu_i = 0$; that is the "price" of risk is unaffected by changes in the means of asset returns. If one defines the price of risk as $\frac{i'\mu - i'q^*}{(i'\Sigma_i)^{1/2}}$, our $\pi^*$, then the same result holds.

We wish now to consider the consequence of changes in the variance -
covariance structure of the returns on assets. In particular we focus on a change in \( \sigma_{ij} \), the \((i,j)\)th and \((j,i)\)th elements of \( V \). Henceforth we will take \( dx \) to mean \( \frac{dx}{d\sigma_{ij}} \) where \( x \) may be a scalar, vector or matrix with the correct interpretation clear from the context. In order to carry out the analysis, we break it down into two steps. We first examine the changes in the eigenvector \( w_0 \) of equation (5.5), and then the change in \( \lambda^* \) of equation (5.6). Combining these two effects we get the total effect on \( q^* \).

When changing the elements of the variance - covariance matrix, it is necessary to restrict the possible parameter changes. Therefore we will assume henceforth

A.9: \( d\sigma_{ij} \) is restricted in such a way that \( V \) remains a positive definite matrix.

Consider then the equation system (5.5). Differentiating with respect to \( \sigma_{ij} \), and noting that \( w_0 \) must lie on the unit sphere gives us two equations:

\[
A dw_0 + Dw_0 = dw_0 \quad (5.9)
\]
\[
dw_0' w_0 = 0. \quad (5.10)
\]

\( D \) is a matrix with elements \( \frac{d\sigma_{kl}}{d\sigma_{ij}} \). Recalling that \( A = \frac{V_{ii}'}{(i'iV_i)} \)

straightforward calculations yield

\[
D = \frac{Cii'}{i'iV_i} - \frac{V_{ii}'}{(i'iV_i)^2} i'Ci \quad (5.11)
\]

where

\[
C = \begin{bmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1
\end{bmatrix}_{nxn},
\]

i.e., the zero matrix with exception of 1's in the \((i,j)\)th and \((j,i)\)th places. Note that \( i'Ci = 1 \) if \( i=j \) and 2 otherwise.
Thus we may re-write (5.9) as

$$\begin{bmatrix} C_{ii'} - \frac{V_{ii'}}{V_{ii'}} & i'C_i \\ \frac{i'I}{i'I} & \end{bmatrix} w_0 = \begin{bmatrix} I - \frac{V_{ii'}}{V_{ii'}} \end{bmatrix} dw_0$$  \hspace{1cm} (5.13)

Now equation (5.10) implies that $dw_0$ is orthogonal to $w_0$, and hence orthogonal to the linear subspace containing non-zero eigenvectors of $A$. Since $A$ is of rank 1, $dw_0$ is in the kernel of $A$, i.e., it is orthogonal to the image space of $A$. Thus we have

**Lemma 5.3:** If $dw_0$ is a solution to equations (5.9) and (5.10) then

$$A dw_0 = 0_n.$$  

Using this lemma we now derive some properties of $dw_0$. Premultiplying both sides of (5.13) by $dw_0^T$ we get

$$\begin{bmatrix} dw_{oi} + \frac{dw_{oi}}{d\sigma_{ij}} & dw_{oj} + \frac{dw_{oj}}{d\sigma_{ij}} \end{bmatrix} i'w_0 = dw_0^T dw_0 \geq 0.$$  \hspace{1cm} (5.14)

This inequality in turn implies

$$\frac{dw_{oi}}{d\sigma_{ij}} + \frac{dw_{oi}}{d\sigma_{ij}} \geq 0$$  \hspace{1cm} (5.15) \quad \text{for } i\neq j, \quad i,j=1, \ldots, n,$n,$ and

$$\frac{dw_{oi}}{d\sigma_{ii}} \geq 0; \quad i=1, \ldots, n.$$  \hspace{1cm} (5.16)

From equation (5.13) we have also

$$\frac{d\omega_{ok}}{d\sigma_{ij}} = \frac{-2}{V_{ii'}} w_{ok} < 0 \quad \text{for } k\neq i,j.$$  \hspace{1cm} (5.17)

These results give us the qualitative implications for changes in $\sigma_{ij}$ on the non-zero eigenvectors of $A$.

We consider now equation (5.6) and in particular $Z_r(\pi^*) = 0$. From (5.3) we may write this as
Differentiating with respect to \( \sigma_{ij} \) we have

\[
\sum_{h} f_{r}^{h}(\pi^{*}) \text{ } d\pi^{*} + dY = 0. \tag{5.19}
\]

Now \( \pi^{*} = \frac{i'w^{*}}{(i'V_i)^{1/2}} \), and \( Y = \mu'i - i'w^{*} + 1 \) where \( q^{*} = \mu - w^{*} \), thus

\[
\frac{d\pi^{*}}{d\sigma_{ij}} = \frac{i'dw^{*}}{(i'V_i)^{1/2}} - \frac{i'w^{*}}{(i'V_i)^{3/2}}, \tag{5.20}
\]

and

\[
dY = -i'dw^{*}. \tag{5.21}
\]

Substituting (5.20) and (5.21) into (5.19) and re-arranging gives

\[
[\sum_{h} f_{r}^{h}(i'V_i) - (i'V_i)^{3/2}] \text{ } i'dw^{*} = \sum_{h} f_{r}^{h} \text{ } i'w^{*}. \tag{5.22}
\]

Now using Walras' law and differentiating (5.3) at \( \pi^{*} \) implies

\[
\sum_{h} f_{r}^{h}(\pi^{*}) = \sum_{h} f_{r}^{h} + \pi^{*} \sum_{h} f_{r}^{h}(\pi^{*})(5.23)
\]

Substituting this into (5.22) implies

\[
i'dw^{*} = \frac{\sum_{h} f_{r}^{h} \text{ } (i'V_i)^{1/2}}{\sum_{h} f_{r}^{h}} > 0. \tag{5.24}
\]

Equation (5.24) is the qualitative restriction placed on the equilibrium system by equation (5.6). We now wish to develop the total effect.

Since \( q^{*} \) is an equilibrium, \( q^{*} = \mu - \lambda^{0}w_{0}, \lambda^{0} > 0 \), and since \( q^{*} + dq^{*} \) is an equilibrium \( q^{*} + dq^{*} = \mu - \lambda^{1}(w_{0} + dw_{0}) \). Let \( q^{*} + dq^{*} = \mu - w_{1}^{*} \), and by definition \( w_{0}^{*} = \lambda^{0}w_{0} \). Then \( w_{1}^{*} = \lambda^{1}(w_{0} + dw_{0}), \lambda^{1} > 0 \). Now \( dw_{*}^{*} \) by definition is \( w_{1}^{*} - w_{0}^{*} \).
Substituting and re-arranging we have then that
\[ dw^* = \lambda^1 [w_o + dw_o] - \lambda^0 w_o \]
\[ = [\lambda^1 - \lambda^0] w_o + \lambda^1 dw_o. \]  
(5.25)

Pre-multiplying (5.25) by \( \mathbf{i}' \) and using (5.24) gives us
\[ [\lambda^1 - \lambda^0] \mathbf{i}'w_o + \lambda^1 \mathbf{i}'dw_o > 0. \]  
(5.26)

Consider now the term \( \mathbf{i}'dw_o \). Using Lemma 5.3 we may re-write (5.13) as
\[ \left[ C_{ii'} - \frac{\mathbf{v} \mathbf{i}' \mathbf{v}}{(\mathbf{i}' \mathbf{v})} \right] \mathbf{w}_o = \mathbf{dw}_o. \]  
(5.27)

Pre-multiplying both sides of (5.27) by \( \mathbf{i}' \) gives, using the fact that
\[ A\mathbf{w}_o = \mathbf{w}_o, \]
\[ \mathbf{i}'\mathbf{C}_{ii'} \left[ \frac{\mathbf{i}'w_o}{(\mathbf{i}' \mathbf{v})} - \frac{\mathbf{i}'w_o}{(\mathbf{i}' \mathbf{v})} \right] = 0 = \mathbf{i}'\mathbf{dw}_o. \]  
(5.28)

Thus substituting \( \mathbf{i}'\mathbf{dw}_o = 0 \) into (5.26) implies
\[ \lambda_1 - \lambda_0 > 0.18 \]  
(5.29)

Using (5.25), (5.29) and (5.14) through (5.16) we have the following results

I. \[ \frac{dw_i}{d\sigma_{ii}} = [\lambda_1 - \lambda_0] \mathbf{w}_o + \mathbf{dw}_{oi} \geq 0 \]  
(5.30)

II. \[ \frac{dw^*_i}{d\sigma_{ij}} + \frac{dw^*_j}{d\sigma_{ij}} = [\lambda_1 - \lambda_0] \mathbf{v}_{oi} + \mathbf{v}_{oj} + \lambda_1 \left[ \frac{dw_{oi}}{d\sigma_{ij}} + \frac{dw_{oj}}{d\sigma_{ij}} \right] \geq 0 \]  
(5.31)

III. \[ \frac{dw^*_k}{d\sigma_{ij}} = [\lambda_1 - \lambda_1 \frac{2}{(\mathbf{i}' \mathbf{v})} - \lambda_0] \mathbf{w}_{ok} \]
\[ = [\lambda_1 (1 - \frac{2}{(\mathbf{i}' \mathbf{v})}) - \lambda_0] \mathbf{w}_{ok} \quad k \neq i, j \]  
(5.32)

Note that the last result (III) is not signed due to the offsetting effect of the term \( -\frac{2}{(\mathbf{i}' \mathbf{v})} \). However if aggregate market variance is large, such
that $2/iV_i$ is sufficiently small then $\frac{dw_k^{*}}{dq_{ij}} > 0$ for all $k \neq i, j$. Using the above results and noting that $dq^{*} = -dw^{*}$, we have the following theorem.

**Theorem 5.2**: a) An increase in the covariance between assets $i$ and $j$ has the effect of

1. decreasing (weakly) the sum of the $i^{th}$ and $j^{th}$ asset prices
2. decreasing (weakly) at least one of the $i^{th}$ or $j^{th}$ prices

b) If aggregate market variance is sufficiently large such that $2/iV_i$ is "close" to zero in the sense that $\lambda_1(1-2/iV_i) - \lambda_0 \geq 0$, then an increase in the covariance between assets $i$ and $j$ has the effect of decreasing (weakly) the price of all others assets $k \neq i$ or $j$.

c) The effect of an increase in the covariance between assets $i$ and $j$ has the same qualitative effect on all other assets $k \neq i$ or $j$.

An analogous result holds for an increase in the variance of an asset. We state only the part analogous to a) of Theorem 5.2 but note that similar statements to b) and c) hold.

**Theorem 5.3**: An increase in the variance of the $i^{th}$ asset has the effect of decreasing (weakly) the price of the $i^{th}$ asset.

These results seem to be intuitively confortable. Note that for an increase in variance of the $i^{th}$ asset $\frac{dw_{ok}}{dq_{ij}} = -\frac{1}{iV_i} w_{0k}$, and therefore the requirement that $\lambda_1(1 - \frac{1}{iV_i}) - \lambda_0 \geq 0$ is much weaker than the requirement that $\lambda_1(1 - \frac{2}{iV_i}) - \lambda_0 \geq 0$. Thus an increase in the variance of an asset, for large market variance, is more likely to decrease the price of other
assets than an increase in the covariance between two assets.

The "price" of risk in our model has been termed $\theta$, or $\pi^*$, depending on one's definition. However it turns out that in only one case an increase in either variance or covariance has an unambiguous effect on the "price" of risk. From (5.19) and (5.24)

$$d\pi^* = \frac{dY}{\Sigma f^h_r} = \frac{i'dw^*}{\Sigma f^h_r} > 0. \quad (5.33)$$

Clearly since $\theta = \frac{\pi^*}{(i'Vi)^{1/2}} = \frac{i'w^*}{i'Vi}$, $d\theta = \frac{i'dw^*}{i'Vi} - \frac{i'w^*}{(i'Vi)^2}$ which is ambiguous. Thus one might question whether in fact $\theta$ was the correct measure of the price of risk, since one feels that with risk averse investors, an increase in risk should imply that the risk margin should increase. In any case we have shown

**Lemma 5.4:** The "price" of risk interpreted as $\pi^*$ unambiguously increases with increases in either the variance or covariance of assets.

This completes our program of comparative statics with respect to the variance-covariance matrix. At this point we should mention two further comparative static exercises. One would be to consider increases in risk aversion on the part of the investors, and the other would be to consider supply effects; i.e., changing the aggregate endowment of assets. Although in the general model this would be quite difficult, in the case of all consumers having identical constant absolute risk aversion utility functions, with risk aversion parameter $\lambda$, it is well known that the equilibrium prices $q$, given $q_0 = 1$ can be written as

$$q = \mu - \lambda Vx/(x'Vx)^{1/2} \quad (5.34)$$

where $x$ is the aggregate endowment vector of the economy. It is straightforward to verify that increases in risk aversion, i.e., increase in $\lambda$,
uniformly increases the risk margin on all assets, and hence decreases all asset prices. Secondly, an increase in the supply of, say the first asset, \( dx_1 > 0 \), will increase or decrease the price of the \( j^{th} \) asset, \( \frac{dq_j}{dx_1} > 0 \), as \( \sigma_{j1} > 0 \), \( j=1, \ldots, n \). This result indicates that even in a strongly specified model supply effects are dependent on the particular parameters of the model.

Another interesting case is to examine the effects of increasing the number of investors \( H \). In particular we pose the question, as \( H \) tends to infinity, what happens to equilibrium prices, \( q^* \). The result one obtains clearly depends on the assumptions one makes about the behaviour of aggregate market variance. Let \( t \) indicate an economy with an aggregate endowment vector \( x_t >> 0 \), and we assume that \( t \) also indicates the number of households in each economy. Furthermore, we assume that all households or investors have identical preferences within the class of preferences described in A.8. Thus as \( t \) grows large we have more individuals all possessing the same preferences but possibly different endowments. Thus for any \( t \), \( t \in \mathbb{N} \), we know that \( k(v_t) \rightarrow 0 \) as \( v_t \rightarrow 0 \), \( \pi_t^* = k(v_t) \), where \( v_t = 1/t \left[ x_t'Vx_t \right]^{1/2} \). Suppose that \( \lim \{ \sup x_s \} \leq M \), where \( M \) is a large but finite vector in \( \mathbb{R}^n \). Hence we consider an increasing sequence of economies, where ultimately resources are bounded. Thus \( \lim v_t = \lim \frac{1}{t} \left[ x_t'Vx_t \right]^{1/2} = 0 \), given the assumption of bounded resources. Since, by A.8, \( k \) is continuous function of \( v \), \( \lim_{t \rightarrow \infty} k(v_t) = k(\lim v_t) = 0 \), and hence \( \lim_{t \rightarrow \infty} \pi_t^* = 0 \). Recall that \( q_t^* = \mu - \frac{\pi_t^*Vx_t}{\left( x_t'Vx_t \right)^{1/2}} \), and thus \( q_t^* \rightarrow \mu \), or asset prices tend to the mean.
values of the asset returns as the number of identical investors tends to infinity.

Lemma 5.5: In a sequence of economies, such that all households have identical preferences, the number of households tends to infinity, and the sequence of aggregate endowment vectors is bounded, the equilibrium price vector tends to the expected value of asset returns.

This argument is quite similar to one made by Arrow and Lind [1970] in a different context. It could be interpreted as saying that for large but finite economies expected values become good approximations to equilibrium prices. An important question is whether such a result obtains in more general models, because once the assumption of fixed asset supplies is dropped and a production side is introduced to the model, a fundamental problem is the decision rule for a firm under uncertainty. A result such as Lemma 5.5 is important because it says that the firm may maximize expected profits if the economy is "large" and this decision criterion will closely approximate the neoclassical criterion of maximizing market value. The firm may thus behave in a risk neutral manner without affecting its market value to a significant degree.

6. Conclusions

A number of conclusions were obtained about the nature of the equilibrium of the well known capital asset pricing model which is based on mean-variance portfolio analysis. In order to carry out the analysis we assumed the existence of an equilibrium and considered explicitly the determination of equilibrium asset prices. This was accomplished by deriving the aggregate excess demand functions for risk and expected
return which, together with a matrix equation derived from the capital-
asset pricing equation, gave a complete determination of equilibrium
prices.

Some conclusions were derived about the structure of the set of
equilibrium prices. In particular, assuming that the price of the risk-
less asset was normalized to unity, we have found that the equilibrium
price vector was not equal to the vector of expected asset returns. If
a further assumption is made to the effect that all assets contribute
positively at the margin to aggregate risk, then the equilibrium price
set is bounded (strictly) from above by the vector of expected asset
returns.

To proceed with the comparative statics, we have assumed all investors
have preferences from a particular class, where the preferences are such
that the marginal rate of substitution between return and risk is indepen-
dent of the level of return. This is a convenient generalization of the
constant absolute risk aversion utility function. The comparative static
results for the model were quite satisfactory in the sense that they were
intuitively plausible. Changes in the mean return vector produces equal,
absolute percentage changes in the corresponding asset prices, and a
change in, for example, the jth mean affects only the jth price leaving
all other prices unaffected.

A positive increase in the variance of an asset induces a reduction
in the price of the same asset. All other asset prices are affected in
the same qualitative manner, however the sign of the qualitative effect
is ambiguous. An increase in the covariance between two assets decreases
the sum of the two asset prices, and again induces the same qualitative
but ambiguous effect on all other asset prices.
Finally we considered the effects of increasing the number of investors in the model, by examining a sequence of economies with the number of investors increasing in such a way that the sequence converges to an economy having a finite aggregate endowment of assets. It was shown that as the number of investors tends to infinity, equilibrium asset prices tend to the expected returns of assets. Thus, in large but finite economies risk has a negligible effect on equilibrium prices.

The results indicate the power of the mean-variance assumption, even in a general equilibrium context. An important question as to the robustness of these results remains to be answered. Whether similar qualitative results hold in a more general expected utility framework without the strong stochastic assumption of normality is an open question. To the extent that the mean-variance model is, however, a good approximation to an arbitrary capital market model, the comparative static results derived can be viewed in a similar approximate fashion.
Footnotes

1. The comparative statics of the portfolio holdings of individuals have been investigated by Bierwag and Grove [1968], Jones - Lee [1971], and Royama and Hamada [1967].

2. For a review of the empirical literature and theoretical contributions to this model see Jensen [1972]. This paper also contains a bibliography of the numerous applications of the capital asset pricing model.

3. Ibid.

4. Arrow and Hahn [1971], Chapter 10 discuss the comparative statics of general equilibrium models, and the results available to date.

5. The model differs somewhat from the conventional capital asset pricing model in that we do not assume an infinitely elastic supply of the safe asset at a fixed price. In the model which follows the supply of the safe asset is fixed and its price is determined in the general equilibrium of the economy.

6. Our vector notation is as follows. For $x, y \in \mathbb{R}^n$, $x \gg y$ if and only if $x_i > y_i$ for all $i = 1, \ldots, n$; $x > y$ if and only if $x_i > y_i$ for all $i = 1, \ldots, n$ and for at least one $j \in \{1, 2, \ldots, n\}$, $x_j > y_j$; $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, \ldots, n$.

7. This is a regularity assumption on the utility function which ensures an interior solution to the individual's utility maximization problem.

8. Assumption R.6., which requires that each individual endowment vector be strictly positive, is stronger than necessary. Actually, for the purpose of the analysis undertaken here, all that is required is that the endowment income of each investor be strictly positive at equilibrium prices. Assumption R.7 which implies that aggregate asset
endowments are equal to unity for each asset is merely a convenience and it makes no substantial difference to the analysis.

9. The necessity of admitting negative prices when the returns from an asset may be negative, as in the case of normally distributed returns, is noted by Hart [1974], Appendix 2.

10. See Hart [1974], Theorem 2.2 and Appendix 2.

11. There is an additional problem in that a finite solution may exist which is not interior to (r,v) space. However, it can be shown that provided \( q \ll \mu \), and given the regularity conditions R.3 on the utility function, the optimal solution is always interior to (r,v) space. Theorem 4.1 below assures us that, under certain conditions, for \( q \in E \), \( q \ll \mu \).

12. This fact is well known and can be easily demonstrated in the mean-variance diagram. See Tobin [1958].


14. The basic difficulty in relaxing this assumption (A.8) and attempting to do comparative statics is that the excess demand functions for risk and return do not decompose as in (5.3) and (5.4) below. Generally we should have \( Z_r(\pi, \mu, V; w_0) \). Comparative statics in this general case are exceedingly complicated and do not give unambiguous results.

15. \( f^h_r \) denotes \( \frac{\partial f^h_r}{\partial \pi} \) and \( f^h_v \) denotes \( \frac{\partial f^h_v}{\partial \pi} \). The strict positivity of these derivatives follows from A.8 and R.3.

16. We shall assume throughout this section that the equilibrium price \( q^* \) is a continuously differentiable function of the parameters (\( \mu, V \)).

17. Actually, we need in addition that \( i'w_0 > 0 \). As \( \pi^*(i'V)^{1/2} = i'w^* > 0 \).
and \( w_0 \) is a positive multiple of \( w^* \), this condition holds.

18. In showing \( \lambda' - \lambda^0 > 0 \) we have assumed \( i'w_0 > 0 \). As
\[
\pi^*(i'V_i)^\frac{1}{2} = i'w^* > 0
\]
and \( w_0 \) is a positive multiple of \( w^* \), this condition in fact holds.

19. This problem is discussed in a number of papers in a recent Symposium
on the Optimality of Competitive Capital Markets [1974].
References


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Chapter III
RISK BEARING, INVESTMENT AND FINANCING
IN STOCK MARKETS

1. Introduction

Recently a number of contributions have been made to the analysis of production decisions under uncertainty. They have all been made within the context of a two-period general equilibrium model under uncertainty in which the household sector is assumed to make portfolio decisions on the holding of claims against different firms' future output. This type of economy, which we shall distinguish by the term 'stock market' economy, is characterized by the feature that markets are assumed to be incomplete in the Arrow-Debreu sense; that is, the number of contracts traded in the economy is fewer than the number of states of nature. This characteristic has the effect of linking the production or real investment decisions of firms to the portfolio or financial decisions of households. Since the 'equity shares' of the different firms are traded on the stock market by the households attempting to achieve an optimal portfolio balance, the market value of the firm, or the total value of its equity, becomes a price determined in the general equilibrium of the economy.

These contributions to the stock market economy have concentrated solely upon the risk-bearing function of the stock market and on the allocative significance of asset prices in making efficient production decisions. The complications we consider in this paper arise from attempting to integrate the risk-bearing functions of the capital market with its more traditional role as an intermediary in the savings-investment process.
In the existing literature, two questions have been asked of this model. The first is supposing that firms make their production-investment decisions such as to maximize the market value of the firm, would the resulting equilibrium allocation of investment be Pareto efficient? The reason market value maximization is postulated as the firm's decision rule is that it corresponds closely to the neo-classical criterion of maximizing profits under certainty, and has been presumed in the corporate finance literature to be the only goal of the firm generally consistent with the owners' interests. Interestingly enough, the answer to this question has generally been in the negative; that is, firms acting on this criterion make inefficient production decisions. The natural question which comes out of this negative conclusion is whether or not, in the context of the stock market economy, there exist production decisions which would be unanimously approved by all shareholders of the firm. The conditions under which such 'unanimity theorems' are true turn out to be rather stringent, and one would not expect these conditions to hold in general. King [1974] has, in any case, severely criticized the usefulness or realistic nature of the concept of unanimity, independent of the conditions under which it may or may not hold.

The general feeling one has after examining this literature is that the criterion which the firm should use in making its production-investment decisions such as to be consistent with its owners' interests is very much unsettled. It would seem that the decision rules which firms actually use may in fact be quite independent of any economy-wide efficiency considerations or may bear no close relation to the preferences of their stockholders. As is well known, the number of decision rules proposed as explaining firms' behaviour is quite large. Under uncertainty the problem
is compounded in that the firm must take an attitude towards risk and form expectations regarding the probability of future events. The diversion of interests between management and ownership is an even greater possibility under uncertainty. Different firms may, of course, have different attitudes towards risk and different probability beliefs, all leading to the likelihood that different firms will choose their production-investment decisions based on different objectives. The resulting diversity of decision rules may be the norm rather than the exception in an economy with significant uncertainties.

It appears we are left with the impression that firms have substantial independence in their decision rules whence the connection between the portfolio decisions of the household sector and the 'real' investment decisions of the firms in the economy has been substantially weakened, if not totally eliminated, at the firm decision level. What we propose to argue is that such independence does not in fact exist and that the capital market of the economy forces some very real constraints upon the firm. The reason our analysis differs from previous ones is that in their consideration of the stock market economy all previous authors have ignored a basic constraint in analyzing the investment decision. The constraint is a budget inequality in the first-period which requires that the cost of the investment projects undertaken by the firm cannot exceed the value of investment funds which must be raised in the capital market. This essential institutional or market constraint binds the firm's behaviour, just as the technology of the firm is a constraint to be taken into account. Thus the capital market of the economy forces a basic budget constraint upon the firm, and the 'real' decisions of firms become closely related to the 'financial' decisions of households, regardless of whether the firm
desires to act independently of its shareholders or in their interest. We shall show that consideration of this basic constraint has some interesting implications for the existence of equilibrium, the class of admissible decision rules for the firm, and the problem of making the financing decision.

The subsequent sections are as follows. In section 2 we develop the basic model and the concept of the financial-equilibrium correspondence. In section 3 we introduce the basic constraint and examine the property of financial consistency in the investment-financing decision of the firm. The next section considers two examples which demonstrate the non-vacuous nature of the capital market constraint on investment choices. In section 5 we consider the implications of the capital-market constraint for financial decisions and the problem of the capital structure of the firm when there exist both bonds and equities in the model. The last section gives an explanation of how the constraint was circumvented in previous models of the stock market economy and finally a summary of this essay's findings.

2. **The Basic Model**

In this section we shall construct the basic stock market equilibrium model and, in particular, the concept of the financial-equilibrium correspondence. The model is a two-period model with technological uncertainty occurring in the second period. In the first period firms purchase a single homogeneous investment good from the households. The households in turn both trade their existing shares of the firms' equities and receive new equity shares in exchange for selling the firms their endowment of the homogeneous investment good. The different firms, which we assume to be fixed in number, all produce a single homogeneous consumption good in the
second period, the amount of the good produced depending upon both the investment made in the first period and the state of nature in the second period. The consumption good is distributed to shareholders in proportion to the percentage share of the firm they purchased in the first period. The equilibrium in this model refers to the equilibrium of the portfolio decisions of households and investment decisions of firms which occurs in the first period. In the second period no trade takes place, production and distribution of the consumption good being the only activities undertaken.

It is convenient in this model to analyze the financial equilibrium, separately from the total equilibrium of the economy. The basic idea is that the market is organized in a sequential fashion. Firms announce their investment decisions and their offerings of new equity. Once the investment decision of a firm is known, its pattern of next period's output across states is known, and, given its announced offering of new equity, each investor knows his percentage endowment of the total outstanding stock of shares. Each individual, then, knows his budget constraint and can make the necessary adjustments in his portfolio. A financial equilibrium is characterized by a vector of market values, one on each firm, and a set of equilibrium portfolio holdings, one for each individual, such that the supply of equities equals the demand. Note that a financial equilibrium is conditional upon the announced investment and financing decisions of firms. It is important to realize that a financial equilibrium does not imply total equilibrium for the economy. Furthermore, the notion that the market is organized in a sequential fashion such that financial markets clear first is only a convenient analytical device and is not to be taken as an actual description of the way a stock market
economy functions. In actual fact total equilibrium must be reached simulta­
neously on all markets. This point will be of particular importance in
Section 4 below.

We introduce now the notation and basic assumptions of the model. There are J+1 firms in the economy, indexed with subscripts $j=0, 1, \ldots, J$. Each firm uses the input of a single homogeneous investment good in the first period to produce a homogeneous output, after the realization of the state of nature, in the second period. Inputs are denoted by $x_j$, outputs by $y_j$, and the state of nature variable by $\theta$, where $\theta$ is an element of a measurable space $(\Omega, \Sigma)$. Here $\Sigma$ denotes the Borel field of events in $\Omega$. Note that since no trade occurs in the second period the only uncertainty which is present is technological uncertainty; that is, the realization of the variable $\theta$ which cannot be observed until the second period. To each firm $j$, there is a technology $f_j(x_j, \theta)$ which gives the output $y_j$ in the second period, given that $x_j$ was chosen in the first period, and that event $\theta$ obtains in the second period.

We assume that $f_j$ satisfies the following:

A.1. $f_j : R_+ \times \Omega \to R_+$.

A.2. $f_j$ is twice differentiable, strictly monotone increasing and strictly concave in $x_j$.

A.3. $\lim_{x_j \to 0^+} f'_j(x_j, \theta) = +\infty$, $f_j(0, \theta) = 0$, for all $\theta \in \Omega$.

A.4. $f_j$ is a measurable function in $\Omega$.\(^{10}\)

Note that assumption A.3. implies that it is necessary and sufficient that a firm make some positive investment in order to produce some positive output in all states of nature. The concavity of $f_j$ implies there is some
old capital stock on which the firm is earning rents, but the old capital stock is taken to be imperative without some new investment.

The vector function \( f(x, \theta) \), maps \( \mathbb{R}^{J+1} \times \Omega \) into \( \mathbb{R}^{J+1} \) taking on values \([f_0(x, \theta), \ldots, f_J(x, \theta)]\) in \( \mathbb{R}^{J+1} \); i.e., the \( J+1 \) vector of the firms' outputs. The firm, of course, is characterized by more than just a technology and we shall return to this point below.

The investors or households of the economy are assumed to sell their first period endowments of the investment good plus their existing ownership claims to firms in order to purchase a desired portfolio of equities, which is determined by maximizing the expected utility of second period consumption. Investors are indexed by \( i \), where \( i \) runs from 1 to \( I \). We make the following assumptions on investors preferences.

B.1. Each investor has a strictly concave, non-decreasing, twice differentiable utility function of consumption, \( u_i(c_i) \).

B.2. \( \lim_{c_i \to 0^+} u'_i(c_i) = +\infty \).

B.3. \( u_i \) is bounded.

B.4. Each investor has a subjective probability measure \( \mu_i \) on the state space \((\Omega, \Sigma)\). Thus each investor's expectations are characterized by a probability space \((\Omega, \Sigma, \mu_i)\).

Each investor has an endowment \( e_i \) of the investment good in the first period. We assume

B.5. \( e_i > 0 \) for all \( i = 1, \ldots, I \).

Each investor \( i \), has an initial holding of \( S_{ij} \) of shares of firm \( j \), where \( \sum_{i=1}^{I} S_{ij} = \bar{S}_j \geq 0 \). Each firm \( j \), in addition to making an investment decision as to \( x_j \), must make a financing decision in terms of the quantity of new equities it plans to offer on the market. We shall denote this
quantity of new equities by $\Delta S_j$, and we shall require without loss of generality that $\Delta S_j$ be non-negative. Thus, the total supply of equities of the $j$th firm is given by $S_j = \bar{S}_j + \Delta S_j > 0$. We can then define $\bar{s}_{ij} = \bar{S}_{ij}/S_j$ as the percentage share of $j$th firm, initially owned by the $i$th investor. Similarly we define $d_j = \Delta S_j/S_j > 0$, as the percentage share of the $j$th firm, which the $j$th firm itself offers in new issues of stock. From the above definitions we have that

$$\sum_{i=1}^{I} \bar{s}_{ij} + d_j = 1. \quad (2.1)$$

The total market value of the $j$th firm is denoted by $V_j$; i.e., $V_j$ is the total value of the outstanding equities $S_j$ in the $j$th firm. We can now define the budget set of feasible portfolios of the $i$th investor. Let $V = (V_0, \ldots, V_J)' \in \mathbb{R}^{J+1}$ denote the vector of the market values of the firms in the economy, and $\bar{s}_i = (\bar{s}_{i0}, \ldots, \bar{s}_{iJ})'$ denote the vector of initial percentage, ownership shares held by the $i$th investor.

**D.1.** The budget set of the $i$th investor is

$$B_i(V, w, d) \equiv \{s_i | s_i \in \mathbb{R}^{J+1}; V's_i \leq V's_i + we_i\}.$$ 

Note that since $\bar{s}_i$ depends upon the firms' announcement of $\Delta S_j's$, the budget set depends upon this set of parameters and consequently upon $d$, the $J+1$ vector $(d_0, \ldots, d_J)'$. The price of the first-period investment good is denoted by $w$. Often we shall normalize $w$ to unity, as we are allowed one price normalization in the model by virtue of homogeneity. Note also that the budget set is unbounded as short sales are allowed, and this allows certain difficulties as to the existence of optimal portfolios.

We wish now to characterize the investors asset demand functions.
Before proceeding however we shall make two further assumptions on the nature of the economy.

A.5. There exists a 'riskless' firm in the economy, in the sense that \( f(x; \theta) \) is independent of the realization of \( \theta \in \Omega \).

This firm shall be denoted with the subscript 0.\(^{15}\)

B.6. For each investor \( i = 1, \ldots, I \), \( \mu_i \{ \theta \in \Omega | f_j(x_j; \theta) > 0, \text{ for any } x_j > 0 \} > 0 \).

B.6. implies that if any firm undertakes positive investment, each individual assigns some positive probability to the set of states in which the firm produces some positive output. We could replace B.6 by a stronger assumption, that each probability measure \( \mu_i \) was 'atomless' which would require in conjunction with A.3 that the above inequality be replaced with \( \mu_i \{ \theta \in \Omega | f_j(x_j; \theta) > 0 \text{ for any } x_j > 0 \} = 1 \), but this turns out not to be necessary. Note that given the vector function \( f(x, \theta) \), and \( s_i \), the individual's portfolio holdings, his consumption if the state \( \theta \) obtains is given by \( c_i(s_i; x; \theta) \equiv s_i f(x; \theta) \). Thus the ith investor has the following problem

\[
\max_{s_i} \mathbb{E}_{\mu_i} [u_i(c_i)] = \int_{\Omega} u_i[c_i(s_i; x, \theta)]d\mu_i
\]

subject to \( s_i \in B^i(V,w;d) \). \( (2.2) \)

Here \( \mathbb{E}_{\mu_i} \) denotes the expectation operator with respect to the probability measure \( \mu_i \), and the integral is taken to be the usual Lesbegue integral.

Now since \( B^i \) was unbounded, a finite solution to \( (2.2) \) may not always exist. However, Berstekas [1974] has shown that, given some reasonable conditions on the utility function, a finite solution always exists. Thus we assume
C.1. Given that $(V,w) > 0_{i+2}$, a finite solution to (2.2) exists for all $i=1, \ldots, I$.

Let $g^i(s_i;x) \equiv \mathbb{E}^{i} [u_i(c_i)]$ be the 'induced utility function' of portfolio holdings conditional on the vector of production decisions $x$. We then have the following lemma.

**Lemma 2.1**: Given A.1-5, B.1-7 the induced utility function $g^i(s_i;x)$ has the following properties:

1. **Concavity**: $g^i(s_i;x)$ is concave in $s_i$ for fixed $x$;
2. **Concavity in $x$**: $g^i(s_i;x)$ is concave in $x$ for fixed $s_i$;
3. **Monotonicity in $s$**: $g^i(s_i;x)$ is monotonically increasing in each $s_{ij}$, if $x_j > 0$;
4. **Monotonicity in $x$**: $g^i(s_i;x)$ is monotonically increasing in each $x_j$ if $s_{ij} > 0$;
5. **Differentiability**: $g^i(s_i;x)$ is continuously differentiable in $(s_i,x)$, on $\text{int } \mathbb{R}_+^{J+1}$.

**Proof**: Concavity follows from the facts that $u_i(c_i)$ is a concave function, and $s_i^j f(x,\theta)$ is a linear function in $s_i$ and a concave function in $x$, since each $f_j(x_j;\theta)$ is concave. 2.1c. is proved by noting that if $x_j > 0$, then $s_i^j f_j(x_j;\theta)$ is increasing in $s_{ij}$ for all $\theta \in \Omega$ such that $f_j(x_j;\theta) > 0$. Since $\{x_j \in \Omega : f_j(x_j;\theta) > 0\}$, for any $x_j > 0 > 0$ by B.6., in evaluating $\int_{\Omega} u_i[s_i^j f(x,\theta)]d\mu_i$ there is a set of positive measure in $\Omega$ on which $f_j(x_j,\theta)$ takes on positive values. Thus the monotonicity of $u_i(c_i)$ ensures monotonicity in $s_{ij}$. Similarly, monotonicity in $x_j$ follows using the monotone increasing property of $f_j$ with respect to $x_j$. Differentiability is ensured by the differentiability of $u_i(c_i)$ and $s_i^j f(x,\theta)$ in $(s_i,x)$. Q.E.D.

The investor's problem (2.2) can be expressed as

$$\max_{s_i} g^i(s_i;x) \text{ subject to } s_i \in B^i(V,v;\delta).$$

(2.3)
From this problem we derive the consumer's demand functions for assets, which we denote by:

\[ h_j^i(V, y_i; x, d) \quad j = 0, \ldots, J. \tag{2.4} \]

where \( y_i \) denotes the income of the \( i \)th individual given by \( V s_i + w e_i \).

These demand functions, given Lemma 2.1, can be shown to have the usual properties.

**Lemma 2.2:** For all \( i = 1, \ldots, I \), given A.1-5, B.1-7 and C.1, then the demand functions \( h_j^i(V, y_i; x, d) \) satisfy the following properties:

1. Homogeneity of degree zero in \((V, y_i)\).
2. Budget identity, \( \sum_j V_j h_j^i(V, y_i; d, x) = y_i \).
3. Symmetry and negative semi-definiteness of the Slutsky matrix \( S_i = [b^i_{k\ell}] \)
   where \( b^i_{k\ell} = \frac{\partial h^i}{\partial V_k} + h^i \frac{\partial h^i}{\partial y_i}, \quad k, \ell = 0, \ldots, J. \)
4. Jointly continuous in \((d, x)\) on \( \text{int} \ R^2_{+}(J+1) \).

(proof): The theorem follows from the usual theorems in neoclassical consumer theory.\(^1\) We can now define the aggregate excess-demand function for asset shares as

\[ Z(V, w; x, d) = \left[ \sum_{i=1}^{I} h_i^i(V, y_i; d, x) - s_i \right] - d, \tag{2.5} \]

recalling that \( d \) is the \( J+1 \) vector of percentages of total equity offered by firms in new issues. It will be convenient to normalize \( w = 1 \), and this is clearly admissible given the homogeneity property of the demand functions. We can now define a financial equilibrium.

D.2. A financial equilibrium, conditional upon a set of production
decisions $x \in \mathbb{R}^{J+1}_+$, and a set of financing decisions $d \in \mathbb{R}^{J+1}_+$, is an $I+1$-tuple \( \{s_1^*, \ldots, s_I^*; V^*\} \) such that

a) $s_i^*$ is maximal with respect to $E_u^i \left[u_i(c_i)\right]$ over all set $s_i \in B_i(V^*, 1; d, x)$, for all $i=1, \ldots, I$.

b) $V^* \in \mathbb{R}^{J+1}_+$

c) $z(V^*; x, d) \leq 0_{J+1}$.

Thus a financial equilibrium, corresponds to a set of market values on all firms, and portfolio decisions by investors, such that each investor's portfolio choice maximizes his expected utility and is feasible in his budget set, and that the market for financial assets clears. We re-emphasize at this point that a financial equilibrium does not imply total equilibrium in the economy, as the vectors $(x, d)$ are so far rather arbitrary. We shall generally assume

C.2. Given any vector $(x, d)$ such that $x \gg 0_{J+1}$ and $d \succeq 0_{J+1}$ a financial equilibrium conditional on these values exists.

Note that the assumption that a financial equilibrium exists is much weaker than the assumption that a global equilibrium exists. 10

In the financial sector of this economy a version of Walras' law is satisfied due to the requirement that households satisfy their budget constraint. Walras' law can be written as

$$
\sum_{i=1}^{I} \sum_{j=0}^{J} V_j z_{ij}(V;d,x) = \sum_{j=0}^{J} V_j d_j. \quad (2.6)
$$

This says simply that the value of total equities demanded by households, in excess of their own equity holdings, is equal to the value of the total equities supplied by the firms. This, of course, holds regardless of
whether the economy is financial equilibrium or not, i.e., some securities may be in excess supply and others in excess demand. If the economy is in financial equilibrium a much stronger result obtains. Recall the individual's budget equation,

$$\sum_{j=0}^{J} V_j s_{ij} = \sum_{j=0}^{J} V_j s_{ij} + e_i. \quad (2.6)$$

Adding these up across individuals at market clearing prices and portfolios we have that

$$\sum_{j=0}^{J} V^*_j = \sum_{j=0}^{J} V^*_j (1-d_j) + \sum_{i=1}^{I} e_i, \quad (2.7)$$

or upon re-arranging

$$\sum_{j=0}^{J} d_j V^*_j = \sum_{i=1}^{I} e_i. \quad (2.8)$$

Now $d_j V^*_j$ is the funds obtained by the $j$th firm in the stock market at financial equilibrium market values. $e_i$ is the supply of first period investment goods by the $i$th individual. Therefore (2.8) implies that in a financial equilibrium the total demand for funds by the real sector or firms will equal the total fixed supply of investment goods to the firm sector. Stated another way, the consequence of a financial equilibrium is that the total flow of funds to the firm sector must equal the total supply of investment goods to that sector. Hence if firms chose to act collectively they could use the funds supplied to them by households to purchase the aggregate endowment of investment goods. This relationship, while certainly not implying equilibrium in the firm sector, is a real constraint upon the equilibrium $V^*_j$ in that, given $J$ market values, the $(J+1)$th $V^*_j$ is automatically determined by (2.8). Thus, for example, in
searching for financial-equilibrium market values in $\mathbb{R}^{J+1}$, we may restrict our search to the linear manifold

$$\Delta^J(d,e) = \{V \mid V \in \mathbb{R}^{J+1}; \sum_{j=0}^{J} d_j V_j = \sum_{i=1}^{I} e_i = e\}.$$  \hspace{1cm} (2.9)

Thus in effect, the equilibrium assumption allows us to make a particular normalization given by (2.9) above, on equilibrium market values. Note that this normalization is additional to setting $w=1$, which we did by virtue of the usual homogeneity property of the excess demand functions.

A useful concept is that of the financial-equilibrium correspondence. This is simply a mapping from the set of investment-financing decisions by firms, $(x,d)$, to the set of equilibrium market values. Hence we define

D.3. The financial-equilibrium correspondence is a mapping from $(x,d) \in \mathbb{R}^{2(J+1)}$ to the set

$$E(x,d) = \{V^* \mid V^* \in \Delta^J(d,e); Z(V^*;x,d) \leq 0_{J+1}\}.$$  \hspace{1cm} (2.10)

Assumption C.2 assures us that $E(x,d)$ is non-empty, and furthermore it can be shown that $E(x,d)$ is an upper hemi continuous correspondence. This implies that "small" changes in $(x,d)$ produce "small" changes in the equilibrium price set. In general $E(x,d)$ will not be single-valued. Just as in the case of the Walras correspondence, there is the general presumption that the number of equilibria is, at least, finite.

The fact that the market value of a firm is a price in a general equilibrium model has some rather strong implications for the theory of the firm. As noted in the introduction, a common suggestion has been that the appropriate decision criterion for a firm, even under uncertainty, is the maximization of the stock market value of the firm. Recently a number of objections has been raised about this criterion on the basis that it
leads to inefficient production decisions from the viewpoint of shareholders. The observation, however, that the stock market value of a firm is one element of a vector which lies in the range of the financial-equilibrium correspondence raises two further points about the proposed criterion. Suppose first that \( E(x,d) \) were single-valued so that we could write the vector function \( V = E(x,d) \). Then the net market value of the \( j \)th firm would be \( N_j \equiv (1-d_j)V_j = (1-d_j)E_j(x,d) \), i.e., the value of the firm to the original shareholders. Now, viewed as an oligopoly or non-cooperative game problem, the maximization of \( N_j \) would be feasible provided the firm knows the vector function \( E_j(x,d) \) and the vectors \((x,d)\) or the decisions of all other firms. This, of course, is equivalent to assuming the firm can compute the general equilibrium of the economy. From both an information and computational viewpoint, maximization of stock market value appears to be a non-operational criterion, unless we endow the firm with rather exceptional abilities that we normally do not regard economic agents as possessing. Secondly, the fact that \( E(x,d) \) is generally not single-valued implies that the term maximization here has no meaning. A correspondence cannot be maximized. For a given vector of production and financing decisions the equilibrium market value of the firm may take on a number of possible values depending upon the adjustment processes which are operating in the stock market. At best then maximization of market value can be viewed as a criterion which might be used in a monopolistically competitive framework where the "market value functions" are perceived as opposed to being actual market value functions. Postulating this type of theory has its own difficulties of course, in that some mechanism must be introduced which relates the perceived demand functions to the true demand functions. It would seem that the market
value criterion poses more problems than solutions, not only in relation to its non-optimality properties, but also in its intractable nature.

We conclude this section with some final comments on the nature of a financial equilibrium, a concept which is conditional upon both the financing and investment decisions made by firms in the economy. The concept of a financial equilibrium seems to have first been introduced into economic analysis by Tobin [1969] in his general equilibrium treatment of a monetary economy. Tobin argued it was a useful concept in that it was not unreasonable that in the short-run the production decisions of the economy could be taken as given. He seems to implicitly assume that financial markets clear much faster than markets which deal in "real" commodities. In the context of our model this interpretation may be used but it is certainly not necessary. Rather it may be treated as a useful analytical device to separate the production/firm side of the economy from the portfolio/household side. In actual fact, of course, both sides of the economy operate simultaneously, and if a stock market economy has a full equilibrium then all markets would have to clear simultaneously, both real and financial. Whether or not a stock market economy has a full equilibrium is an issue treated in Section 4.

3. Financial Consistency

Most authors dealing with models of the stock market economy have avoided the financing decision of the firm, which in the context of our model is the choice of $d_j$, concentrating solely upon the real investment decision, or choice of $x_j$, and its relation to the risk-bearing function of the market. Presumably, however, what makes the stock market economy model an interesting one is the feature of the capital market which
requires that firms raise the necessary investment funds in the capital market. The possibility then exists that the financing decision and investment decision of the firm are incompatible, given a financial equilibrium. The amount of funds allocated to the firm in a financial equilibrium simply may not be sufficient to cover the costs of the proposed investment project.

In the Arrow-Debreu economy with complete markets, the possibility of inconsistency between financing and production decisions does not arise. Given the relevant prices on all date and event pairs, firms choose their production plans in order to maximize market value. Since the firm is always assumed to have the possibility of inaction, its market value must be non-negative. Furthermore, the completeness of markets guarantees that the firm can meet any obligations it incurs in the first period by selling forward contracts on its future output. Since market value is non-negative, the total receipts from selling its future output must exceed its total expenses in purchasing current inputs. In the stock market economy, however, due to the absence of futures markets the firm must pay for its current inputs by selling claims to the future output in the form of common shares. The firm has no knowledge as to the marginal change in its market value by purchasing an additional unit of investment goods, or to the marginal change in its market value by issuing another share of new equity. Consequently, for a given investment decision the firm has no means of evaluating how much stock must be issued in order to cover investment costs, or whether in fact any amount of stock issued will cover the cost of investment.

The choice of the investment plan, which is in our model the quantity of investment goods $x_j$, depends upon the decision criterion of the firm.
There have been a number proposed in the literature. To mention a few, choose the production plan such as to a) maximize the net market value of the firm, b) maximize expected profits, c) maximize an expected utility function of profits, d) maximize stockholder welfare or e) minimize the probability of a take over. It may be that some or all of these decision rules suffer from the above mentioned problem. Thus we shall define a decision rule to have the property of \textit{financial consistency} if it leads to simultaneous investment and financing decisions which result in sufficient funds being available to the firm in order to carry out its proposed investment plan.

There is a certain type of myopia implicit in firms' behaviour in this world. It may not seem unreasonable to require that they recognize the effect their production decision and financing decision has on their market value, and thus require that they maximize their objective function, subject not only to their technology, but also to the capital market constraint. There are, however, some good reasons firms may not do so. First, there are the information and computational costs involved in evaluating the general equilibrium effects. Secondly, if firms undertake such a policy, they have to consider not only the impact of their own decisions, but also the impact of other firms' decisions on the viability of their plans. This leads immediately to oligopolistic type implications in the competition for funds in the capital market. An interesting question, which arises then, is whether we can tell a competitive-like story in this economy. We shall say that firms are \textit{financially myopic} if they choose an investment plan without explicitly recognizing the capital market constraint. All the contributions made thus far to
the stock-market economy have implicitly assumed financial myopia on the part of firms. Financial myopia is certainly one way to maintain a competitive structure within the stock market economy. Firms perceive no interdependence between their financing decisions and other firms' actions. Of course financial myopia also requires that firms take no cognizance of the fact that their financing decision and investment decision are related in some fashion. If we admit that firms do recognize this interdependence, then we must also logically admit that they recognize the general equilibrium nature of the stock market and thus their interdependence with other firms. Therefore, if we do not allow that firms are financially myopic then we must drop the competitive assumption. For much of the analysis which follows, however, we shall maintain the assumption of financial myopia.

In terms of the notation of the previous section, the jth firm chooses \((x_j, \Delta S_j)\), i.e., a quantity of investment goods, \(x_j\), and a quantity of new equities, \(\Delta S_j\), to supply on the capital market. The possibility of bond financing will be considered in Section 5 below. The total outstanding supply of equities is given by \(S_j = \bar{S}_j + \Delta S_j\), and therefore \(d_j = \Delta S_j / S_j\) is the proportion of total equity supplied which consists of new issues by the firm. We assume throughout that shares are infinitely divisible. If \(V_j^*\) is the equilibrium market value for the jth firm conditional on the decisions \((x, d)\) of all firms, and \(p_j^*\) is the equilibrium price per share, then we have the identity \(S_jp_j^* = V_j^*\).

D.4. The investment and financing decisions of firms, \((x_j, d_j)\), for \(j=0, \ldots, J\) are financially consistent if
a) \(V^* \in E(x, d)\)
and

\[ d_j V_j^* \geq x_j \quad \text{for } j = 0, \ldots, J. \]

The constraint in b) says simply that the total funds going to the firm in a financial equilibrium must exceed the costs of the proposed investment. Clearly as all investment is assumed to be non-negative, \( x_j \geq 0 \), and the non-negativity of asset prices requires \( d_j \geq 0 \). Since \( \Delta S_j \geq 0 \), i.e., we have excluded the possibility of share repurchase, and \( \bar{S}_j \geq 0 \), then \( d_j \leq 1 \). Therefore we also have

\[
0 \leq d_j \leq 1 \tag{3.1}
\]

\[
x_j \geq 0 \quad \text{for all } j = 0, \ldots, J. \tag{3.2}
\]

A natural question to ask is whether a set of financially consistent decisions exists. An affirmative answer to this question is necessary to ensure that the proposed criterion of financial consistency is logically compatible with the assumptions of the model. Note that since we have not specified firm behaviour the existence of financially consistent decisions is a weaker requirement than the existence of a complete general equilibrium.

**Theorem 3.1:** Given assumptions A, B and C, a set of financially consistent decisions exists.

(proof): In order to prove this theorem it is convenient to use a different normalization of prices, than the one used thus far, \( w = 1 \). Therefore, by virtue of the homogeneity of the excess demand functions, which follows from Lemma 2.2 (1), we may choose \((V, w) \in \Delta^{J+1}\), the unit simplex in \( \mathbb{R}_+^{J+2} \). \( \Delta^{J+1} \) is a compact, convex subset of \( \mathbb{R}_+^{J+2} \). It is necessary to bound the feasible set of \( x_j \) and we may assume without loss of generality that

\[
\sum_{j=0}^{J} x_j \leq e \equiv \sum_{i=1}^{I} e_i. \tag{3.3}
\]
Let $S \equiv \{ (x,d) | \text{such that (3.1), (3.2) and (3.3) hold}\}$. $S$ is a compact convex subset of $\mathbb{R}^2 \Delta^{J+1}$. Thus $T \equiv SX \Delta^{J+1}$ is a compact convex subset of $\mathbb{R}^{3J+4}$. We shall now construct two correspondences. Let

$$\psi_1(V,w) \equiv \{ (x,d) | (x,d) \in S; d_j V_j \geq wx_j, j=0, \ldots, J \},$$

be a correspondence defined on $\Delta^{J+1}$. The requirement that $d_j V_j \geq wx_j$, is simply the requirement of financial consistency. Clearly $\psi_1(V,w)$ is non-empty. We shall demonstrate that $\psi_1(V,w)$ is convex valued. Let $x_\lambda^1 = \lambda x_1^1 + (1-\lambda)x_2^1$ and $d_\lambda^1 = \lambda d_1^1 + (1-\lambda)d_2^1$, and suppose $(x_1^1,d_1^1)$ and $(x_2^2,d_2^2)$ are in $\psi_1(V,w)$: that is, $d_j^1 V_j \geq wx_j^1$ and $d_j^2 V_j \geq wx_j^2$ for $j=0, \ldots, J$. Now $[\lambda d_j^1 + (1-\lambda)d_j^2] V_j \geq w[\lambda x_j^1 + (1-\lambda)x_j^2]$ and therefore $(x_\lambda^1,d_\lambda^1) \in \psi_1(V,w)$. Thus $\psi_1(V,w)$ is convex valued. $\psi_1(V,w)$ is an upper hemi-continuous correspondence as the graph of $\psi_1$ is closed. A correspondence $\psi_2(x,d)$ is constructed as follows. Recall that $E(x,d)$ is an upper hemi-continuous correspondence, which by C.2 is non-empty, defined on $S$, and takes values in $\Delta^{J+1}$. By a theorem of Michel [1956] there exists a continuous function $f(x,d)$, defined on $S$, such that $f(x,d) \in E(x,d)$. Let $\psi_2(x,d) \equiv f(x,d)$. Clearly $\psi_2(x,d)$ is non-empty, upper hemi-continuous since it is continuous, and convex valued since it is a function. Now define

$$\psi(x,d,V,w) \equiv \psi_1(V,w)X \psi_2(x,d).$$

$\psi$ is non-empty, upper hemi-continuous, convex valued and maps $T$ into itself. Thus we may apply Kakutani's fixed point theorem which says there exists a fixed point $(x^*,d^*,V^*,w^*) \in T$, such that

$$(x^*,d^*) \in \psi_1(V^*,w^*) \text{ and } (V^*,w^*) \in \psi_2(x^*,d^*).$$

By construction $(x^*,d^*)$ are financially consistent at prices $(V^*,w^*)$. Q.E.D.

Thus the definition proposed is non-vacuous in that the set of financially consistent decisions was shown to be non-empty. However, we have
said nothing yet about the decision rules by which the firm arrives at its choice of \((x_j, d_j)\). It is important to note that in order to be viable the decision rule which the firm uses must result in financially consistent decisions. It seems not unreasonable to suspect that some financially myopic decision rules may result in inconsistent decisions. The above existence theorem, can be interpreted as saying there exists some decision rules which result in financially consistent decisions. The next section considers two important examples of consistent and inconsistent behaviour on the part of firms.

4. Some Examples of Firm Behaviour

We wish to demonstrate first that some reasonable financially myopic decision-rules lead to inconsistent choices on the part of firms and consequently the failure of a full general equilibrium to exist. The second example illustrates a situation in which a set of financially myopic decision rules lead to a set of consistent investment and financing decisions. The examples will be constructed on the basis of the explicit mean-variance asset-pricing model. Furthermore, in order to specify explicitly the financing-investment decision it is necessary to choose a particular set of \(f_j(x_j; \theta)\) and a choice rule for \((x_j, d_j)\).

First we shall assume that all firms resort to an extreme sort of new equity financing, such that they offer such a large offering of new equities, that to all intent purposes \(d_j = 1\). This can also be interpreted as a case in which all firms are just starting up, and firms sell their equity in order to purchase investment goods. In this case the financial constraint becomes

\[
V_j \geq x_j \quad j = 0, \ldots, J. \tag{4.1}
\]
In this case the total market value of the firm must exceed its input costs. It is somewhat analogous to the Marshallian rule that a firm under perfect competition must in the short run have total revenues which exceed total variable costs.

The production side is specified by assuming that the stochastic technology of the firm sector is given by

\[ E[f_j(x_j; \theta)] = a_j x_j^{1/2}, a_j > 0; \quad (4.2) \]

\[ \text{Cov} [f_j(x_j; \theta) f_i(x_i; \theta)] = a_{ij} x_i^{1/2} x_j^{1/2} \]

for all \( i, j = 1, \ldots, J \). \quad (4.3)

All economic agents are assumed to have identical expectations, both managers and investors. The 0th firm is assumed to be riskless as it produces a certain second period output given by \( y_0 = a_0 x_0^{1/2} \). The "riskless" interest rate (gross) in the economy is defined by \( 1+r = y_0/V_0 \) where \( V_0 \) is the market value of the riskless firm. All individuals are assumed to have identical constant absolute risk-aversion utility functions, with risk aversion parameter \( \lambda > 0 \). Assuming the distribution of firms' second period output is normally distributed, the financial-equilibrium market values are given by \(^{26}\)

\[
\text{a) } V_j = \frac{1}{1+r} \left[ E(y_j) - \lambda \sum_{i=1}^{J} \frac{\text{cov}(y_i, y_j)}{\left( \sum \sum \text{cov}(y_i, y_j) \right)^{1/2}} \right] \\
= \frac{1}{1+r} \left[ a_j x_j^{1/2} - \lambda \sum_{i=1}^{J} a_{ij} x_i^{1/2} x_j^{1/2} \right] \\
\frac{\left( \sum \sum a_{ij} x_i^{1/2} x_j^{1/2} \right)^{1/2}}{\left( \sum \sum a_{ij} x_i^{1/2} x_j^{1/2} \right)^{1/2}} \\
\text{for } j = 1, \ldots, J 
\] \quad (4.4.1)
b) \( 1+r = a_0 x_0^{\frac{1}{2}} / V_0 \) \hspace{1cm} (4.4.2)

c) \( \sum_{j=0}^{J} V_j = \sum_{i=1}^{I} e_i = e. \) \hspace{1cm} (4.4.3)

Given the \( J+1 \)-vector of production decisions \( x = (x_0, \ldots, x_J)' \), financial equilibrium market values are specified completely by equations a), b) and c) of 4.4. The larger is the parameter \( \lambda \), the more risk-averse individuals are said to be.

The financing constraint, relating investment decisions \( x_j \) and market values \( V_j \) can thus be written as

\[
\frac{1}{1+r} [a_j x_j - \lambda \sum_{i=1}^{J} a_{ij} x_i^{\frac{1}{2}} / (\sum_{i} a_{ij} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}})] > x_j
\]

for all \( j=1, \ldots, J \). \hspace{1cm} (4.5)

The financial constraint of the riskless firm is

\[
V_0 = \frac{a_0 x_0^{\frac{1}{2}}}{1+r} \geq x_0.
\] \hspace{1cm} (4.6)

An additional constraint upon the model is that the covariance matrix of the \( J \) risky firms' output patterns be positive definite. Hence

\[
A(x) = [a_{ij} x_i^{\frac{1}{2}} x_j^{\frac{1}{2}}]_{JxJ}
\]

is a positive-definite matrix for all \( x >> 0 \). A sufficient condition for this to hold is that \( A(x) \) be a positive diagonal matrix, i.e. \( a_{ij} = 0 \) for \( i \neq j \), \( a_{ii} > 0 \) \( i=1, \ldots, J \).

We shall assume that all firms are expected-profit maximizers. This assumption is probably the most common one employed in the partial equilibrium literature on the theory of the firm under uncertainty and has been
used extensively in the management science literature. Expected profits of the jth firm are given by

$$\Pi_j = \frac{1}{1+r} a_j x_j^{1/2} - x_j . \quad (4.8)$$

Maximization of $\Pi_j$ with respect to $x_j$ yields

$$x_j^* = \frac{a_j^2}{4} (1+r)^{-2} > 0 . \quad (4.9)$$

Substituting (4.9) into the valuation formulae (4.4.1) yields

$$V_j = \frac{1}{1+r} \left[ a_j \left( \frac{1+r}{2} \right)^{-1} - \frac{1}{\lambda} \sum_{i=1}^{J} a_i a_i a_j^{1/2} (1+r)^{-2} / (\sum_{i,j} a_{ij} a_i a_j^{1/2} (1+r)^{-1/2}) \right]$$

$$= \frac{1}{(1+r)^2} \left[ a_j^2/2 - 2\lambda \sum_{i,j} a_{ij} a_j / (\sum_{i,j} a_{ij} a_i a_j^{1/2} (1+r)^{-2}) \right]. \quad (4.10)$$

Now if $V_j \geq x_j^* = \frac{a_j^2}{4} (1+r)^{-2}$, then

$$a_j^2/(1+r)^2 \geq 2\lambda \sum_{i,j} a_{ij} a_i a_j / (\sum_{i,j} a_{ij} a_i a_j^{1/2} (1+r)^{-2}) \geq a_j^2/(1+r)^2 4,$$

or, upon re-arranging,

$$a_j^2/4 - 2\lambda \sum_{i,j} a_{ij} a_i a_j / (\sum_{i,j} a_{ij} a_i a_j^{1/2}) \geq 0. \quad (4.11)$$

Consider now the "diagonal" capital market model, which assumes $a_{ij} = 0$ for $i \neq j$. Then (4.11) becomes

$$a_j^2/4 - 2\lambda \sum_{i,j} a_{jj} a_j^2 / (\sum_{i,j} a_{ij} a_i a_j^2) \geq 0$$

or

$$\lambda \geq 2\lambda a_{jj} / (\sum_{i,j} a_{ij} a_i a_j^2)^{1/2} > 0. \quad (4.12)$$

Since $a_{jj} > 0$, the second term on the left-hand side of inequality (4.12) is positive. Clearly for $\lambda$ large enough this inequality cannot hold, and
thus financial consistency does not obtain. The more risky a firm is relative to the others, that is the larger \( a_{jj} \), the greater the likelihood of the firm choosing financially inconsistent investment decisions.

We present here a simple numerical example to illustrate the problem of non-existence. The example has two risky firms with technological parameters

\[
\begin{align*}
    a_1 &= 2 & a_{11} &= 1 \\
    a_2 &= 2 & a_{22} &= 2 \\
    a_{12} &= a_{21} &= 0.
\end{align*}
\]

(4.13)

If we let \( R = (1+r) = 1 \), then the optimal investment by firms one and two is given by \( x_1^* = 1 \) and \( x_2^* = 1 \). Using the valuation formulae with a risk aversion coefficient \( \lambda = 3^{\frac{1}{2}}/6 \), we get the financial equilibrium market values of \( V_1 = 4/3 \) and \( V_2 = 2/3 \). As \( V_1 = 4/3 > x_1^* \), firm 1 is financially consistent. However \( V_2 = 2/3 < x_1^* \) and is therefore financially inconsistent.

This example corresponds to the case in which the interest rate is arbitrarily set, for example by a government willing to sell riskless bonds in any amount at a fixed interest rate. This is the case usually considered in the capital asset pricing model. What happens if the government changes the interest rate?

The following example considers the case in which the interest rate is endogenously determined, and shows financial inconsistency can occur. This example uses the same parameters as used above but in addition we must specify \( a_0 \) and \( e \), then using (4.4.1), (4.4.2) and (4.4.3) the complete financial equilibrium can be determined. Let \( a_0 = 2 \) and \( e = 2 \). Straightforward calculations yield
\[ V_0^* = 1 \quad \text{and} \quad x_0^* = \frac{1}{2} \]
\[ V_1^* = \frac{2}{3} \quad \text{and} \quad x_1^* = \frac{1}{2} \]
\[ V_2^* = \frac{1}{3} \quad \text{and} \quad x_2^* = \frac{1}{2} \]

and consequently \((1+r^*) = \sqrt{2}\). The riskless firm and firm 1 are financially consistent, while firm 2 is financially inconsistent. Since \(x'_j(r) < 0\) from (4.9), there is a possibility equilibrium might be established by sufficient change in the interest rate. Note however that (4.12) is independent of \(r\), and consequently changes in \(r\) will not affect the inconsistency result. In the example constructed the demand for investment funds by firms from equation (4.9) depends only upon the riskless interest rate, and has the property that changing the interest rate changes all firms' demand for investment proportionately. The risk factor of each firm \(j\) is given by
\[
\sum_{i} a_{ij} x_i^* x_j^* / \left( \sum_{i} \sum_{j} a_{ij} x_i^* x_j^* \right)^{\frac{1}{2}}
\]  
(4.14)

and its value is equal to (4.14) multiplies by \(\lambda/(1+r)\). The risk discount, for a change in \(r\), therefore changes by the same factor of proportionality for all firms \(j=1, \ldots, J\). Similarly the value of the expected return factor of each firm is
\[
\frac{1}{1+r} a_j x_j^*^{\frac{1}{2}},
\]  
(4.15)

which changes by the same factor of proportionality for a change in \(r\). Therefore, with expected-profit maximizing firms, changes in \(r\) affect only the absolute and not the relative market values of the risky firms. If some risky firm is financially inconsistent at some \(r\), then by lowering \(r\) you increase its demand for investment funds, but simultaneously
you increase its market value by the same factor from equation (4.10). Similarly for increases in \( r \). Thus once financially inconsistent, always financially inconsistent.

The failure of a set of financially consistent choices on the part of firms to exist for any set of financial equilibrium prices \((r,V)\) implies that with respect to at least one firm its demand for investment funds will always exceed the supply of investment funds to that firm by the capital market, in any financial equilibrium. Therefore although a financial equilibrium exists a full general equilibrium does not. Alternatively stated, there exists no set of prices on securities and real commodities which clear all markets simultaneously. If a tatonnement process were to run in the economy there is no possibility that it would ever converge as markets would not clear for any set of prices.

The source of the non-existence of equilibrium in the above example is clear. It can obviously be attributed to the financial myopia of firms. If firms were to maximize expected profits subject to (4.1), then firms could never demand more investment goods than the stock market wished to finance. As noted, it is, however, difficult to develop a theory of firm behaviour, which may be described as "competitive", when this constraint is taken into account. Alternatively, the source of non-existence is the decision rule of the firm. In the above example, firms in attempting to maximize expected profits wish to take on more risk than the household sector is willing to bear. The fact that firms are financially myopic is equivalent to saying that firms do not take into account the economy's willingness to bear risk, and this is the cause of the basic market failure.28

If there were an additional fictitious "price of risk" which firms
responded to in calculating their investment demand function, then by appropriately manipulating this price the market for investment funds by all firms might be cleared. "Risk", however, is not a market commodity and there does not exist a market price for it. Alternatively firms might base their decisions on their own market value $V_j$, but this type of behaviour would violate financial myopia on the part of firms and the competitive assumptions of the model. The example we construct now is similar to the one above, with the exception that firms use a different decision rule. This example has the characteristic that financial myopia persists, but the resulting investment decisions satisfy the consistency property.

Suppose the decision criteria of all firms is to maximize

$$g_j(x_j) = \frac{1}{1+r} \left[ \alpha_j x_j^2 - \beta a_{jj} x_j \right] - x_j$$

where $\beta > 0$. The function $g_j$ might be interpreted as a utility function of profits (net) with a risk aversion parameter $\beta$, on the variance of the firm's own gross returns. In this example we shall maintain the 'diagonal' model assumption. The maximization of (4.13) yields two possible solutions. Let

$$\frac{a_j}{2(\beta a_{jj} + \beta j + r)^2} = x_j(r, \beta).$$

Then

$$x_j^* = x_j(r, \beta) \quad \text{if} \quad g_j(x_j[r, \beta]) \geq 0,$$

$$x_j^* = 0 \quad \text{if} \quad g_j(x_j[r, \beta]) < 0.$$

Now if the risk aversion parameter of the firm, $\beta$, is greater than $\lambda / (\sum_{i,j} a_{ij} x_j^* x_j^*)^{1/2}$, the community risk aversion index, then from (4.17), it
follows immediately that the resulting production decisions are financially consistent. If all firms are sufficiently risk averse, then financial consistency is ensured. Over the long run firms may learn from experience that it is necessary to adopt a certain risk-averse attitude in order to be competitive in the capital market for funds. Flexibility in decision rules is desirable, as the firm could adjust its degree of risk aversion to be consistent with that reflected in the capital market.

The above examples demonstrate that we are in somewhat of a dilemma as regards the firm in this type of economy. As the first example demonstrate ignoring the capital market constraint leads to certain obvious difficulties. If the firm is willing to adopt flexible attitudes towards risk, the second example suggests that these difficulties might be circumvented. A major problem with the latter formulation is the problem of the firm learning the "right" $\beta$. As $\beta$ changes moreover the firm's behaviour would change. It is not obvious why the firm's preferences should change, rather one would suspect that the firm would learn from its experience about the capital market constraint.

5. The Capital Structure Problem

Until now we have assumed a single source of financing, by new equities only. We now examine what happens upon the introduction of bonds which are issued by each firm. The firm faces a problem in choosing a capital structure, in that it may finance a proposed investment project by issuing new equities, bonds, or combinations of both. For the moment let us require that firms issue bonds only up to the point at which the probability of default is zero. Thus we may assume that all potential bondholders regard the probability of default on bond payments as zero.
If a firm sells a bond in period one for one unit of account which in our model is a dollar's worth of investment goods, then the bond pays \((1+r)\) in units of consumption goods the next period, where \(r\) is the riskless rate of interest.

The \(j\)th firm, given that it has undertaken the investment plan \(x_j\), yields a pattern of returns given by \(f_j(x_j; \theta)\) over all \(\theta \in \Omega\). The no bankruptcy requirement implies that

\[
f_j(x_j; \theta) \geq (1+r)B_j, \quad \text{for all } \theta \in \Omega \tag{5.1}
\]

when the riskless interest rate is \(r\) and the \(j\)th firm issues \(B_j\) dollars worth of bonds (measured in terms of investment goods). Now for some interest rates, the set of \(x_j \in \mathbb{R}_+\) which satisfy (5.1) may be the empty set. But in order to avoid the bankruptcy issue in the capital-structure problem, it is necessary that there are some investment plans which yield strictly positive output in all states of nature, and therefore some possibility exists for the issuing of default free bonds. Assumptions A.3 and B.6 ensure this is possible.

Assume each firm has no outstanding debt obligations, and consequently any bonds issued in the first period constitute the total debt in the capital structure of the firm. For each firm \(j\), these bonds will be denoted by \(B_j\). Recall that \(\Delta S_j\) denotes the quantity of new equities issued by the \(j\)th firm and \(p_j\) denotes the price of one share in the \(j\)th firm. The requirement of financial consistency may be written as

\[
\Delta S_j p_j + B_j \geq x_j \tag{5.2}
\]

where (5.1) is assumed to hold under the no bankruptcy assumption.

It might be useful at this stage to recall the basic Modigliani-Miller
The total market value of the firm is now \( V_j = E_j + B_j \), where \( E_j \) is the value of the firm's equity, in our model equal to \( S_j B_j \). Assuming no bankruptcy and that individuals can borrow and lend at the same rate of interest as firms and ignoring transactions costs, then, for a given production plan the total market value of the firm, \( V_j \), is unchanged with respect to changes in the debt/equity ratio \( B_j/E_j \). Note that the theorem holds for a given production plan. The theorem does not assure that a given financial/investment plan satisfies the property of financial consistency. It is often claimed that the M+M theorem 'proves' the irrelevancy of corporate financial policy. Under what conditions might this be true? Certainly if an economy has a complete set of Arrow-Debreu markets, financial policy is irrelevant. In this type of situation the maximization of the market value of the firm is a desirable objective from the stockholders' viewpoint, and the M+M theorem tells us the capital structure of the firm makes no difference as to the market value of the firm. Furthermore, in an economy with complete markets the problem of financial consistency is irrelevant. In the stock-market economy the firm, however, has no obvious reason for maximizing its market value, and the financial consistency problem is very real. The question then arises as to what financial irrelevance (or more accurately the irrelevance of the firm's capital structure) means in the stock-market economy.

We might make the following proposition: if a financially consistent set of decisions has been made with respect to \( x_j, \Delta S_j \) and \( B_j \), then any other values of \( (\Delta S_j^1, B_j^1) \), such that the no bankruptcy condition was met, will continue to be financially consistent. A positive answer to the above proposition would go a long way to justifying the irrelevance of
the firm's capital structure. For if it were true, the problem of a firm choosing a capital structure could be divorced from the problem of achieving financial consistency. Furthermore, if a firm were fortunate enough to have a decision criterion which allowed it to obtain the necessary financing for its investment projects, then the means by which it obtained this financing would be a matter of indifference; any combination of new equity issues and debt would suffice. The main theorem of this section demonstrates that the proposition is, in fact, true.

Before proceeding with the theorem, let us point out again the arguments made in the last section. As we did there we could certainly construct examples of decision rules which are financially inconsistent. The introduction of bonds in no way changes that possibility. Suppose, for example, a firm used bonds as its sole source of financing. The no-bankruptcy condition requires that \((5.1)\) hold and financial consistency requires

\[
B_j \geq x_j. \tag{5.3}
\]

There are certainly decision rules which would violate one of these constraints. What is perhaps more interesting is that there may exist no \(x_j > 0\) which is compatible with both \((5.3)\) and \((5.1)\). Suppose \(y_j(\theta) \equiv x_j h(\theta)\), where \(\frac{1}{2} \leq h(\theta) \leq \frac{3}{2}\) for all \(\theta \in \Omega\). Then \((5.1)\) requires that \(\frac{1}{2} x_j \geq (1+r) B_j\), and \((5.3)\) requires \(B_j \geq x_j\) or \((1+r) B_j \geq (1+r)x_j\). Combining both yields

\[
\frac{1}{2} x_j \geq (1+r)x_j. \tag{5.4}
\]

If the interest rate (gross), \(1+r\), is greater than \(\frac{1}{2}\), the only investment choice which is feasible is \(x_j = 0\).

What is 'peculiar' about this example? First, of course, the require-
ment of no-bankruptcy with probability one is no doubt unrealistic. Almost all debt has attached to it some probability of default, although in practice it may be so close to zero that it may be ignored. What the example does show is that the requirement of financial consistency provides an incentive for firms to diversify their capital structure. In the above example, the firm by issuing some equity can clearly realize investment choices other than \( x_j = 0 \), avoiding the no-bankruptcy constraint, and still yield a positive market value. The example demonstrates that provided all bonds issued must have a zero probability of default, then a role for financial policy exists in choosing a capital structure such that the desired investment policy is also feasible with the no-bankruptcy requirement.\(^3\)

Suppose that a financially consistent set of decisions has been made by the \( j \)th firm given by the triple \( \{ x_j, d_j, B_j \} \) which satisfies the basic inequality

\[
d_j E_j + B_j \geq x_j
\]  

We wish to prove the following theorem.

**Theorem 5.1**: Given the conditions of 1) no bankruptcy, 2) all individuals can borrow and lend at the riskless interest rate, 3) conditions of financial equilibrium hold and 4) for some firm \( k \in \{ 0, 1, ..., J \} \) the initial set of decisions \( (x_k, d_k, B_k) \) were financially consistent, then there exists another financial equilibrium with \( B_k/E_k \) taking any value and such that financial consistency continues to hold for the \( k \)th firm.

**Remark**: The proof of the theorem relies on a leverage type argument used by Modigliani and Miller in their original paper. The only new notation is \( D_i \), which is the debt (positive or negative) held by the \( i \)th investor. The output patterns produced by firms, given investment decisions \( x_j \) are
given by \( y_j(\theta) \).

(proof): Recall that the budget constraint of the \( i \)th investor is given by

\[
\sum_{j=0}^{J} S_{ij} p_j + D_i = \sum_{j=0}^{J} \bar{S}_{ij} p_j + e_i. \tag{5.5}
\]

Consumption of the \( i \)th investor in state \( \theta \in \Omega \), given his share-holdings \( \{S_{i0}, \ldots, S_{iJ}\} \), debt \( D_i \) and interest rate (gross) \( R = 1+r \), is given by

\[
c_i(\theta) = \sum_{j=0}^{J} \left( \frac{S_{ij}}{S_j} \right) [y_j(\theta) - R \bar{B}_j] + RD_i
\] 

\[
= \sum_{j=0}^{J} \left( \frac{S_{ij}}{S_j} \right) [y_j(\theta) - R \bar{B}_j] + R \left[ e_i - \left( \sum_{j=0}^{J} (S_{ij} - \bar{S}_{ij}) p_j \right) \right]. \tag{5.6}
\]

In financial equilibrium the demand for shares equals the supply of shares, so \( \sum_{i=1}^{I} S_{ij} = S_j = \bar{S}_j + \Delta S_j \). Now adding individual budget constraints \( i=1, \ldots, I \) at financial equilibrium prices \( p = (p_0, \ldots, p_J) \) and \( R \), we get

\[
\sum_{i,j} S_{ij} p_j + \sum_i D_i = \sum_{i,j} \bar{S}_{ij} p_j + \sum_i e_i \\
= \sum_{j} S_j p_j + \sum_i D_i = \sum_{j} \bar{S}_j p_j + \sum_i e_i. \tag{5.7}
\]

Re-arranging (5.7) yields

\[
\sum_j \Delta S_j p_j + \sum_i D_i = \sum_i e_i \tag{5.8}
\]

The net excess demand for bonds is given by

\[
Z_B = \sum_i [e_i - \left( \sum_{j=0}^{J} (S_{ij} - \bar{S}_{ij}) p_j \right)] - \sum_i B_j. \tag{5.9}
\]

where the first-term in square brackets on the right represents \( \sum_i D_i \) and
the second term represents the supply of bonds by firms. In equilibrium
\[ \sum_i S_{ij}p_j = E_j = V_j - B_j. \]
Hence using (5.8)
\[
Z_B = \sum_i e_i - \sum_j (V_j - B_j) + \sum_j \tilde{S}_j p_j - \sum_j B_j
\]
\[= \sum_i e_i + \sum_j \tilde{S}_j p_j - \sum_j V_j = 0 \quad (5.10)\]
Thus, if the market for equities clears, this implies equilibrium in the
bond market.

Now we take the kth firm to be k=0, without loss of generality.\(^{33}\)
Suppose that this firm issues no debt, but issues new equities (in addition
to those offered in the first equilibrium), \(\Delta S_0\) such that \(\Delta S_0 p_0 = B_0\). That
is the value of new equities at the equilibrium prices of the first situ­
tion is equal to the value of the bonds it issued in the first situation.
All variables in the second situation are denoted with carets. Thus
\(\hat{B}_0 = 0\). Now suppose
a) \(\hat{R} = R\)
b) \(\hat{p}_j = p_j \quad j=0, 1, \ldots, J\)
c) \(\hat{S}_j = S_j \quad j \geq 1\)
d) \(\hat{B}_j = B_j \quad j \geq 1. \quad (5.11)\)
That is we hold the prices of all securities constant and hold the supplies
of all securities by firms other than \(j=0\), at the same level. Now
\[\hat{V}_0 = S_0 p_0 + B_0\]
and
\[\hat{V}_0 = \hat{S}_0 \hat{p}_0\]
\[= S_0 p_0 + B_0 \quad (5.12)\]
\[= V_0\]
using the above assumptions. Hence we change the debt/equity ratio of
the oth firm, but by assumption we do not change the total market value
of the firm. More importantly we do not change the price per share of
the oth firm's equity. This implies that since
\[ \Delta S_o p_o + B_o \geq x_o, \]
\[ \Delta S_o p_o + \Delta S_o p_o \geq x_o. \]  
(5.13)

The construction is such that the new financial decisions of the firm
remain financially consistent. The proof is completed by showing that
the constructed variables are in fact equilibrium ones.

The consumption possibilities of the ith investor in the second
situation are given by
\[ \hat{c}_i(\theta) = \sum_{j=0}^{J} (\hat{S}_{ij}/\hat{S}_j)[y_j(\theta) - \hat{R} \hat{B}_j] + \hat{R} e_i \left[ \sum_{j=0}^{J} (\hat{S}_{ij} - \hat{S}_j)\hat{p}_j \right]. \]  
(5.14)

Assume now that the individual undertakes some home-made leverage.
Specifically, assume that for every dollar of equity the investor owned
in the oth firm in the initial situation, the individual borrows \( B_o/p_o S_o \)
in addition to \( D_i \). Therefore
\[ \hat{D}_i = D_i + \left( \frac{B_o}{p_o S_o} \right) S_{io} p_o, \]  
(5.15)
and with the proceeds of this loan he increases his holdings of equity in
the first firm. Thus
\[ \hat{S}_{io} p_o = S_{io} p_o + \left( \frac{B_o}{p_o S_o} \right) S_{io} p_o \]
\[ = S_{io} p_o + S_{io} \left( \frac{B_o}{S_o} \right). \]  
(5.16)
Now using assumptions (5.11) and the above leverage we re-calculate the individual's consumption opportunity set given by (5.14).

\[
\hat{c}_i(\theta) = \frac{\hat{S}_{i0}}{S_0} y_0(\theta) + \sum_{j=1}^{J} \frac{S_{ij}}{S_j} [y_j(\theta) - R B_j] + R[e_i - \sum_{j=1}^{J} (S_{ij} - \hat{S}_{ij})p_j
\]

\[-\{S_{i0}p_o + S_{i0}(\frac{B_o}{S_0}) - \hat{S}_{i0}p_o\}]

\[
= \frac{\hat{S}_{i0}}{S_0} y_0(\theta) - R \frac{S_{i0}}{S_0} B_o + \sum_{j=1}^{J} \frac{S_{ij}}{S_j} [y_j(\theta) - R B_j]
\]

\[+ R[e_i \sum_{j=1}^{J} (S_{ij} - \hat{S}_{ij})p_j].
\]

(5.17)

From (5.16) we have that \(\hat{S}_{i0}p_o = S_{i0}(p_o + \frac{B_o}{S_0}) = S_{i0}(\frac{S_o p_o + B_o}{S_0})\). But \(S_0 p_o + B_o = V_1 = \hat{V}_1 = \hat{S}_{i0} p_o\) from (5.12). Therefore \(S_{i0}/S_0 = \hat{S}_{i0}/\hat{S}_0\).

Substituting this in (5.17) we have that

\[
\hat{c}_i(\theta) = \sum_{j=0}^{J} \frac{S_{ij}}{S_j} [y_j(\theta) - R B_j] + R[e_i - \sum_{j=0}^{J} (S_{ij} - \hat{S}_{ij})p_j],
\]

(5.18)

and therefore from (5.6) \(\hat{c}_i(\theta) = c_i(\theta)\). Thus the individual's opportunity set has not changed as a result of the firm changing its capital structure. Since the initial situation was optimal for all investors as they were all in portfolio equilibrium, the final situation must be optimal. To establish that the second situation is an equilibrium, it only remains to show that markets are clearing.

The demand for the equity shares of the oth firm by the ith individual is given by \(\hat{S}_{i0} = S_{i0} + \frac{B_o}{p_o S_0} S_{i0}\). Summing over all individuals \(i=1, \ldots, I\) we get that
Thus the demand for shares has increased by a factor of \( (B_o/E_o) \). However \( \Delta S_o/S_o = B_o/p_o S_o \) since \( \Delta S_o p_o = B_o \), and thus the supply of shares for the first firm increases by the same factor and hence all share markets clear. Similarly in the bond market, the decrease in the demand for bonds by the \( o \)th firm is \( B_o \), but the increase in the demand for bonds by individuals is \( \Sigma S_{i0} \left( \frac{B_o}{S_o} \right) = B_o \) from (5.15) and thus the bond market clears. Q.E.D.

Thus it is possible that the firm, having initially made a financially consistent set of decisions, can change its debt/equity ratio and still remain financially consistent. The crux of the proof here is that not only total market value remains constant as in the Modigliani-Miller theorem, but the actual share price remains constant, in spite of changes in the capital structure of the firm. An interesting question then arises as to the robustness of the above irrelevance theorem. Is it possible that, when the firm substitutes new equity for debt in its financing mix, the economy will settle at an equilibrium with the same total market value, but a different share price? The answer is no, provided the substitution of equity for debt takes place at a particular ratio. Consider the set of possible equilibria with constant market value. Suppose the firm takes \( -\Delta B_k \) dollars of bonds off the market. If it issues \( \Delta S_k = \Delta B_k/p_k \) new equities, where \( p_k \) is the original equilibrium price of equity, then the new equilibrium price of equity shares must be \( \bar{p}_k = p_k \). The over variables denotes the equilibrium values after the substitution of equity for debt by the firm. The proof is straightforward. Using the identity \( V_k = S_k p_k + B_k \), and the fact that we restrict ourselves to equilibrium with
constant market value, we have

\[(S_k + \Delta S_k)(p_k + \Delta p_k) = -\Delta B_k + \tilde{V}_k - B_k\]

or

\[S_k p_k + \Delta S_k p_k + S_k \Delta p_k + \Delta S_k \Delta p_k = -\Delta B_k + \tilde{V}_k - B_k.\]

Substituting \(\Delta S_k = -B_k/p_k\), \(\tilde{V}_k = V_k = S_k p_k + B_k\), we have

\[S_k \Delta p_k - B_k \Delta p_k / p_k = 0\]

or

\[(S_k + \Delta S_k) \Delta p_k = 0\]

which implies \(\Delta p_k = 0\), since \(S_k + \Delta S_k > 0\). Therefore \(\tilde{p}_k = p_k\). The possibility of an equilibrium with \(\Delta p_k = 0\) is ensured by the main theorem of this section. Thus holding the investment decisions of firms and their total market values constant, we can substitute new equity for debt in the ratio \(\Delta S_k / \Delta B_k = \frac{1}{p_k}\), and maintain share price constant and ensure financial consistency is maintained, once established. If the firm does not follow this particular financial strategy then, even comparing equilibria with constant market values, there is no assurance the share price of the firm will remain constant. Consider an extreme case in which the jth firm reduces its demand for debt, with no offsetting increase in new equity. The M+M theorem tells us that there exists a new equilibrium with the same market value on the jth firm as in the original equilibrium; consequently we may have \(dV_j = dE_j + dB_j = 0\). The funds available for investment are \(F_j = d_j E_j + B_j\). In this case \(d_j\) is constant so \(dF_j = d_j (-dB_j) + dB_j = (1-d_j)dB_j < 0\). Thus financial consistency is violated, the total increase in the market value of the equity going to existing shareholders. There is a possibility of intermediate cases in which the
firm issues some new equity, but some of the increase in equity value goes to the firm, and some to existing shareholders. In this case a set of decisions which were previously financially consistent could now be inconsistent. Thus the force of the irrelevance theorem is diminished somewhat as the firm has no way of establishing a prior what its share price will be in equilibrium. Of course we have restricted ourselves so far to comparing equilibria with constant market values on all firms. As the economy may well have multiple financial equilibria, there is no assurance that when a firm changes its financing mix, the same equilibrium will be re-established. Therefore the idea of comparing equilibria with constant market values is not completely justified without specifying the market adjustment processes. For "small" parameter changes we might assume, however, that financial equilibria are locally unique and stable, and consequently our comparative static exercises would be valid.

The actual process by which the firm selects \((d_j, B_j)\) has been left unspecified in this section. This is the most unsatisfactory aspect of the stock market economy, and further work in this area is clearly desirable.

6. Summary and Conclusion

We compare now the type of market arrangement of this model, with previous models of the stock market.

(i) The original stock market model was that of Diamond [1967]. Diamond's model had incomplete markets, but the technological/stochastic specification was such that the firm made efficient decisions by maximizing market value. In this model, participation in the stock market amounted essentially to taking part in a gambling game. Individuals trade
equity shares and prices are established but no real productive activity takes place until the state of nature is realized. After the state of nature obtains in the second period, firms engage in production using inputs which were contracted in the first period. Each firm is assumed to avoid becoming insolvent in that it never contracts in the first period to hire more inputs than it could pay for in the second period with output in each state of nature. Thus the firm makes no real investment decision, and consequently the stock market puts no real constraint on the firm.

(ii) The stock market models of Stiglitz [1972] and Radner [1972] differ from the Diamond-type market in that real investment is undertaken by the firm in the first period in order to produce output (uncertain) in the subsequent period. The method of financing in these models is, however, rather odd. The initial shareholders are assumed to bear the full costs of the investment decision of the firm, each shareholder bearing a percentage of the cost in proportion to his initial shareholdings. In this manner the budget constraint on the firm in the first period is shifted to the shareholders, it being assumed implicitly that the firm never undertakes so much investment as to drive a shareholder's net wealth to zero. In these models then, no financing by the firm takes place, and consequently the firm faces no constraints in the capital market. Consider an economy which operates in the following manner. Firms pay for investment by issuing riskless bonds only, and no firm issues so much debt as to violate the bankruptcy-condition. This type of financing arrangement is equivalent to that in the Stiglitz and Radner economies, in the sense that in equilibrium the portfolio holdings of individuals will be identical in both situations.
(iii) A third type of model is Leland's [1974]. His stochastic specification is such that a certain cost is subtracted from a random output. Although he does not state the type of market organization he has in mind there are two possible interpretations. One is that the market works in the same way as in Diamond's model, all contracts being paid for after the uncertain outcome. An alternative interpretation is that all financing is bond financing with the implicit assumption that no debt is issued beyond the point of no bankruptcy.

Thus the existing models of the stock market economy have completely avoided treating the capital market as an institution which serves to facilitate the saving/investment process. Rather the treatment of the stock market in these models focuses only upon the risk-bearing function of equity markets, and as such only half of the story has been told.

We have attempted in this paper to suggest a number of interesting problems which arise when the functions of the stock market are expanded to include both risk bearing and the financing of real investment. The examples presented suggest that these two functions are closely related. The household sector's desire to bear risk clearly affects the type of investment projects it is willing to fund, and the quantities of funding available to different investment projects. The firms in the economy can only avoid the influence of the household sector to the extent that they can finance their production activities by issuing riskless debt. The possibilities for this may be extremely limited.

The major problem in constructing a complete model of the stock market economy in which investment is financed in the stock market is in providing a theory of firm behaviour which avoids financial inconsistency. That is, financial myopia is a property which must clearly be eliminated,
given the possibilities for non-existence of equilibrium. It would seem
that there are two possibilities here. The first is a type of monopolistic
competition theory in which the firm maximizes its objective function sub-
ject to a perceived financial constraint. We have discussed this type of
a theory already. The second possibility is more in the competitive frame-
work, in which it is assumed that the firm faces a budget constraint which
requires that the firm cannot invest more than the market currently allo-
cates to it in investment funds. The firm then acts as if this amount
were fixed. Equilibrium in this sort of capital-rationing model would be
characterized by an equilibrium rationing of investment funds, together
with the usual equilibrium conditions and financial consistency would be
automatically ensured. There may of course be other possible descriptions
of firm behaviour.
Footnotes

1. The original contribution was that of Diamond [1967]. Subsequent contributions have been made by Ekern and Wilson [1974], Jensen and Long [1972], Leland [1973][1974], Radner [1972][1974] and Stiglitz [1972].

2. The standard reference to the Arrow-Debreu economy is Debreu [1959]. Chapter 7 in particular introduces uncertainty and deals with the case of complete contingent markets.

3. In this paper the terms 'capital market' and 'stock market' are used synomously, as this market serves both a risk-bearing and investment-allocative function, a feature which distinguishes 'real world' stock markets.

4. This issue was raised initially by Diamond [1967] and given an answer in the affirmative. Stiglitz [1972] and Jensen and Long [1972] both came to the contrary conclusion by relaxing some of Diamond's stringent assumptions.


7. See section 3.6 below for an explanation of how the constraint was avoided.

8. For the distinction between technological and price uncertainty see Diamond [1967], page 760.

9. Notice the strong informational requirements of this model. It must be assumed either that all investors know both the technologies and
investment decisions of all firms, or that firms announce their patterns of output across states of nature, and that they are truthful in their announcements.

10. \( R^+ \times \Omega \) denotes the cross product of \( R^+ \) and \( \Omega \). \( \lim_{x \to y}^+ \) denotes the left-hand limit as \( x \) approaches \( y \) from the right. A.4 is a technical assumption which ensures that in an individual's maximization problem, (2.2) below, his utility function is measurable with respect to his subjective probability measure.

11. Note we are assuming the utility function is state-independent. By dealing with a finite-state model the expected utility hypothesis could be relaxed, as in Debreu [1959], chapter 7.

12. This simply excludes the possibility of share repurchase. This is not a serious restriction given the two-period nature of the model and the absence of alternative means of finance.

13. \( x' \) denotes the transpose of the vector \( x \).

14. See Berstekas [1974].

15. The riskless firm assumption is really quite inessential. We use it only when dealing with the examples of section 4.

16. Our vector notation is as follows. For \( x,y \in R^n \), \( x \gg y \) if and only if \( x_i > y_i \) for all \( i = 1, \ldots, n \); \( x > y \) if and only if \( x_i \geq y_i \) for all \( i = 1, \ldots, n \) and for at least one \( j \in \{1,2, \ldots, n\} \), \( x_j > y_j \); \( x \geq y \) if and only if \( x_i \geq y_i \) for all \( i = 1, \ldots, n \). \( 0_n \) denotes the zero vector in \( R^n \).


18. The existence of a financial equilibrium could be proved in a straightforward fashion, from a theorem of Hart [1974]. Note that
from A.3 all security returns are strictly positive, and thus equilibrium market values can be taken to be non-negative.

19. A correspondence $\psi$ from a metric space $T$ into a metric space $S$ is said to be upper hemi-continuous at $x \in S$, if $\psi(x) \neq \phi$ and if for every neighbourhood $V$ of $\psi(x)$ there exists a neighbourhood $V$ of $x$ such that $\psi(z) \subset U$ for every $z \in V$. The correspondence $\psi$ is said to be upper hemi-continuous if it is upper hemi-continuous at every $x \in S$. The upper hemi-continuity of $E(x,d)$ on int $R^2_{(j+1)}$ follows immediately from the continuity of $Z(V,x,d)$ on int $R^3_{(j+1)}$.

20. See Debreu [1970].

21. For example Stiglitz [1972] and Jensen and Long [1972].

22. Radner [1972] appears to be one of the first to have noted this problem.

23. For a) see Miller and Modigliani 1958; b) and c) see the literature on the firm under uncertainty, a representative sample being Baron [1970], Leland [1972] and Sandmo [1971]; d) see the literature on unanimity; note 6 above; e) see King [1974].

24. Actually, the definition D.3. required that $E(x,d)$ take values in $R^k_{(j+1)}$. With the alternative normalization on prices used here, however, $E(x,d)$ is understood to take values in $A^J_{(j+1)}$.

25. The basic developments of this model were made by Sharpe [1974], Lintner [1965] and Mossin [1966].

26. This is the basic valuation equation of the capital asset pricing model. See note 25 above.

27. If the technology of the riskless firm was constant returns to scale, for example $y_0 = a_0x_0$, then the only interest rate (riskless) compatible with full equilibrium would be $1+r = a_0$. As the technology used
in this example is strictly concave, there is no reason this should be the case.

28. When the assumption of identical expectations for firms and investors is relaxed another possibility of market failure arises. If both firm managers and investors have identical degrees of risk aversion, but differ in their probability beliefs, firms will attempt to invest in projects which yield returns in states of nature which have a high probability of occurrence relative to the firms' beliefs. Investors will, however, value those firms lower than otherwise, if they do not assign the same states the same high degree of probability. This type of possibility could easily be a source of financial inconsistency.


30. In previous sections $V_j = E_j$ as the firm issued equity only.

31. The no bankruptcy assumption is severely restrictive. The consequences of dropping this assumption are quite strong. See chapter four in this thesis.

32. This theorem is similar in its method of proof to the general equilibrium version of the M+M theorem proved by Stiglitz [1969].

33. Note that we do not mean to imply that A.5. holds. This can be an arbitrary firm with uncertainty as to its distribution of output across states.
References


Chapter IV

THE COSTS AND CONSEQUENCES OF DEFAULT

1. Introduction

The analysis of default on contracts in capital markets is a subject of increasing interest to economists. The reasons for this interest are many. Before enumerating them, a brief discussion of contract types may, however, be in order.

Capital-market relations between economic agents typically deal with facilitating the savings-investment process; as a consequence, an essential feature of a capital market contract is the specification of inter-temporal terms of exchange. Because of the absence of complete markets in the Arrow-Debreu sense and since markets operate sequentially at each date, two important features of a capital market contract are uncertainty and imperfect information. The uncertainty and imperfect information may relate to the date of re-payment, the identity of the other party, the probability of re-payment, the amount of payment, and so forth.

Most capital-market contracts can be classified into two types. The type of contract most familiar in economic theory is the insurance contract. This states that, conditional on a certain event occurring which is identifiable to both parties, a certain amount of money or goods shall be paid or delivered. A second type of contract is the unconditional contract which merely provides a claim against some income stream and usually signifies ownership of some identifiable asset. The most common type of contract of this sort is the common stock or equity of a corporation. There is neither an event or date conditioning clause to this type
of contract which promises payment of any set amount. Some contracts are of course a mixture of the two.²

An important feature of conditional contracts is the possibility of default; that is, the terms of the contract will not be met by one party or the other. Much of the uncertainty involved in these type of contracts relates to the possibility and the circumstances under which default may occur. As a result of the default feature many so called capital-market imperfections can be readily explained. For example, the collateral in a borrower-lender relation becomes important as a means of providing insurance against default, and as a means of providing information to the lender about the probability of the borrower re-paying.³

The purpose of this chapter is to consider the nature of the equilibrium in a capital market when default is a possibility. In particular, we shall be concerned with the determination of the equilibrium price of loans on which default may occur and the determination of the probability of default, which is a market-determined variable in the models considered. Furthermore, we shall consider how these variables relate to certain parameters, such as the riskless interest rate, the productivity of investment, the collateral value of the loan, and the expectations of borrowers and lenders in the market. The analysis is partial equilibrium to the extent that the capital market is treated in isolation, but it could be given a simple general equilibrium interpretation.⁴

The rest of the essay proceeds as follows. In section 2 there is a general discussion of default and its economic significance. Section 3 develops a simple model of a capital market with one type of security available for investment purposes. The focus here is on the demand side of the loan market. Section 4 considers the role of risk aversion and
the nature of the demand function for investment loans when default is a possibility. The next section considers the same market, only this time it has the feature that the equilibrium price and probability of default are determined by supply factors. Section 6 then considers the complications when an additional security, equity, is introduced. The last section presents some conclusions.

2. Default: Some General Considerations

The significance of default for matters which have traditionally concerned economists, namely resource allocation, welfare, and distribution is not generally appreciated. Some might argue that the orders of magnitude involved are at the second or third decimal place; consequently, it is not a matter we should be much concerned about. There are, however, reasons to believe such a view is not correct. The reasons are as follows.

1. Empirically default is closely correlated to the business cycle and economic growth. Corporate bankruptcies are a common phenomena. For example, in the United States approximately 44 per 10,000 businesses failed in 1970, with an average liability per failure of $175,000. Default on consumer credit and mortgages is a phenomenon we are all aware of, and specialized institutions have arisen to deal with it. There appears to be a difficult identification problem in determining the order of magnitude involved in default. Some takeovers and mergers, for example, may in fact be partial substitutes for default. Financial institutions often extend credit to their borrowers in situations they would normally feel it was not prudent to do so; the reason being, that were credit not extended, the borrower would default on his loan and the financial company involved would have to bear the supervision and administration costs
associated with the default. Therefore many contracts which do default may not be identified as so doing because institutions have developed which effectively disguise default by writing new contracts or modifying the terms of the old contracts.

2. The literature on corporate finance has considered the effect of bankruptcy on the firm's optimal financial mix. As soon as the possibility of bonds becoming risky assets is admitted, the Modigliani-Miller theorem is not generally true and by varying the debt/equity ratio the firm can affect its market value. Thus default on corporate bonds should be an important factor in explaining their valuation and in the relative supplies of debt and equity. Secondly Stiglitz [1972] has demonstrated that with bankruptcy the real production decisions of the firm are not independent of its financing decisions. Thus in explaining the investment behaviour of the corporate sector, one should pay attention to the capital structure of corporations and the default risk on their debt.

3. From the point of view of general equilibrium theory, default on contracts has important implications in economies which have markets operating sequentially over time. At any date there are pre-existing contracts which were written at an earlier date and whose terms must be met in the current period. Depending upon the values of current (spot) equilibrium prices some of these pre-existing contracts may default. An economy consists of not only of a set of markets, but also a system of rules which determines the distribution of the loss on defaulting contracts amongst agents in the economy. In addition the set of rules must be consistent; once agents take these rules into account in making contracts the resulting demand and supply correspondences must have the appropriate properties
in order that existence of an equilibrium can be proved.

4. A great deal of the resource costs of transactions involved in running capital markets are due to the default feature of contracts. Presumably, if default were not a possibility, then anyone entering a contract at the existing price would be 100% certain of meeting the terms of the contract. Under what conditions might this be true? Clearly it would require an individual having very detailed information as to his future position and the relative prices at future dates. Secondly, it would require a code of ethics adhered to by all agents participating in the market under which no one would choose to default on a contract. The fact that both features are hardly characteristic of any 'real world' capital markets indicates that default is a significant possibility.

Because of the possibility of default a firm or individual making a loan desires certain information which he would not otherwise require. First he needs information about the circumstances under which default will occur. These relate to both external circumstances, i.e., those beyond the control of the borrower, and those circumstances which the behaviour of the borrower might modify. For example, if the borrower is a firm then some information about its choice of technique is relevant to determining the possibility of default. A problem of the 'moral hazard' variety may face the lender in this case since a borrower may choose to default in circumstances he could avoid, were he more directly responsible for the outcome of his actions. The loan company, consequently, may incur costs monitoring the borrower's actions in an attempt to reduce the probability of default. Second, information is required about the extent to which the loan company can recover its losses if a borrower defaults. This involves
some detailed knowledge of the collateral securing the loan, its future value, and the ease with which it can be marketed.\(^9\)

5. Investment in real capital, including consumer durables, real estate, resource extraction, and plant and equipment, is significantly affected by the default risk associated with the financial contracts underlying the real capital formation. The probability of default is, of course, a subjective matter, and thus differences in expectations between borrowers and lenders, and differences in risk aversion undoubtedly affect the outcome of the investment process. For example, consider the situation Keynes\(^10\) considered of a 'collapse' in investors expectations; this might be thought of as a deepening pessimism, either on the part of savers or on the part of firms or individuals undertaking the real investment. Surely the quantity and allocation of investment should depend upon who thinks what. The significance of default in affecting real capital accumulation is enhanced by the observation that there are very few second-hand capital good markets or rental markets for capital goods as traditional neoclassical investment theory assumes. If a purchase of a capital good is irreversible, or capital is of a putty-clay nature, and this purchase must be financed with a loan, then, should conditions turn out to be unfavourable to the investment, there would be little the loan company could do to cover its losses; the capital good is neither saleable, nor would it be possible to rent it out to firms experiencing more favourable conditions.

Summarizing this section, it is fair to say default on contracts is not an insignificant matter and a complete examination of the causes and consequences of default is necessary to a better understanding of the manner in which capital markets function.
3. **Equilibrium in a Loan Market: Demand Determined Loan Price**

In this section a simple model of a capital market with a single type of contract is developed in a partial equilibrium context, although it could be extended to a simple general equilibrium context. The model of this section has the characteristic that the equilibrium loan price and probability of default are determined by the demand side of the capital market, that is, the borrowers of investment funds. Borrowers have access to real investment opportunities, but they need to acquire the necessary funds in order to undertake the investment. In most of the analysis we shall maintain the assumption of risk neutrality in order to focus on the default issue and the role of expectations. The type of loan available is a simple one-period debt contract which is repaid in the second period, provided the borrower does not default.

Borrowers have access to a constant-returns-to-scale technology or investment project. For a real investment or input in the first period, denoted by \( x \), the investment project gives a return in the second period of \( y(x, \theta) \), where \( \theta \) denotes a state of nature variable, an element of the state space \( \Omega \). More specifically it is assumed

A.1. \[ y(x, \theta) = r(\theta)x, \quad x \geq 0 \]

\[ r(\theta) > 0 \text{ for all } \theta \in \Omega. \]

A.2. \[ r(\theta) = rs, \quad r > 0 \]

\[ s \in [0, M]. \]

The positive scalar \( s \) parameterizes the state space and takes values in the interval \([0, M]\).

It is assumed that the economy consists of many borrowers or firms with access to the identical technology given by A.1 and A.2. Entry and
exit of firms is costless and is presumed to take place if the expected rate of profit on the investment projects differs from an exogenous rate of return, $\rho$, which is taken to be greater than zero. We shall show that a $\rho$ equal to zero precludes the existence of an equilibrium in the capital market when default on loan contracts is admissible. A positive opportunity cost can be rationalized on two grounds. If one views these borrowers as risk neutral firms, then $\rho$ represents the profit rate to the industry on fixed factors. For example, if a unit of investment requires a unit of land, then $\rho$ represents the competitive rental to land. On the other hand if borrowers are individuals, the $\rho$ can be viewed simple as their opportunity cost to investing, perhaps given by the rate of return on a riskless security. We shall assume that the competitive process, through the demand for investment funds, forces the loan price, denoted by $q$, to such a level that the expected profit rate equals $\rho$.

Borrowers may choose to default on their loans if they wish, the only penalty being that they lose all claims to returns from the investment project, it being assumed that the lenders in the market have acquired these claims. This transfer of ownership will be treated more explicitly in section 5. With regard to expectations we assume:

B.1. All borrowers have identical expectations as to the state of the world. These are represented by the distribution function $F(s)$, with support in the interval $[0,M]$. The distribution may also be represented by a continuous probability density function, $p(s)ds = dF(s)$.

In much of the subsequent analysis we shall be concerned with changes in expectations, in which case the distribution is parameterized with a scalar
t, F(s,t) also being presumed differentiable in t.

The gross returns to an investment project, given an outcome s and input x, are by A.1 and A.2 rsx. The borrowing costs or the amount promised to repay on a loan of x are given by qx, where 1/q is the price of a loan or q = 1+r is the gross interest rate on these loans. The price of investment goods will always be set equal to unity. As borrowers are assumed to maximize expected end-of-period profits, they will choose to default on their loans in any state in which net receipts are negative or rsx < qx. Thus the net returns to a borrower from undertaking an investment project are given by

$$\Pi(s,x,q) = \begin{cases} 
rsx - qx & \text{if } s \geq q/r \\
0 & \text{if } s < q/r .
\end{cases}$$

Therefore, expected profits to a borrower are given by

$$E\Pi = \int_{q/r}^{M} [rsx - qx]dF(s)$$

$$= rx \int_{q/r}^{M} s dF(s) - qx \int_{q/r}^{M} dF(s) , \quad (3.1)$$

where the capital E denotes the expectation operator.\(^{12}\)

With the free entry/exit assumption, in order for the demand for loans to be a positive finite amount, which is a necessary condition for loan market equilibrium, the expected rate of profit must be equal to the exogenous rate of return \(\rho\).

$$r \int_{q/r}^{M} sdF(s) - q \int_{q/r}^{M} dF(s) = \rho . \quad (3.2)$$

The price q is now presumed to be the equilibrium price. Let q/r \(\equiv z\) and \(\rho/r \equiv v\). From basic probability theory we can re-write (3.2) as

...
\[ rE(s|s \geq z) = q + \frac{p}{R(z)} \] (3.3)

where \( E(s|s \geq z) \) is the conditional mean of \( s \), conditioned on the fact that the loan is not defaulted, and \( R(z) = \int_z^M dF(s) \), is the probability of not defaulting. (3.2) can also be written as

\[ f(z) = \int_z^M s dF(s) - z \int_z^M dF(s) = v. \] (3.4)

Thus, the market has an equilibrium if there exists a \( z^* \in [0,M] \) such that \( f(z^*) = v \). Note that once \( z^* \) is determined, both an equilibrium loan price \( q = z^*r \), and an equilibrium probability of default \( F(z^*) \) are determined. Henceforth \( z^* \) will be referred to as the default point. Notice also that changes in the loan rate \( q \) have two effects. First, \( q \) affects the direct productivity of the investment by affecting net receipts; second, it affects the probability with which default occurs. This second feature makes the default problem, and the supply and demand of securities on which default can occur, different from the treatment of securities on which default cannot occur.

Now \( f(z) \) by assumption B.1 is continuous and differentiable on the interval \([0,M]\). Furthermore, \( f(0) = E(s) = \bar{s} \) and \( f(M) = 0 \). Thus, by the mean-value theorem \( f(z) \) takes on all values in \([0,\bar{s}]\). Taking the derivative of \( f(z) \) we have

\[ f'(z) = -zdF(z) - \int_z^M dF(s) + zdF(z) \]
\[ = -\int_z^M dF(s) < 0, \text{ for all } z \in [0,M]. \] (3.5)

This implies that \( f(z) \) is uniquely invertible on \([0,M]\) and the only solution of \( f(z) = 0 \), is \( z = M \). Therefore if \( p = 0 \), the only loan rate compatible with equilibrium is \( q = Mr \), with the probability of default being
equal to one. Thus all loans would default with certainty, and this would not appear to be a reasonable sort of capital-market equilibrium. Now if \( v \in (0, \tilde{s}) \), then \( f(z) = v \) possesses a unique solution \( z^* \in (0, M) \) and \( q = z^* r \). Hence we have demonstrated

**Proposition 3.1:** If \( r \in (0, \tilde{s}) \), then the loan market has a unique equilibrium loan rate and a unique equilibrium probability of default, \( F(z^*) \), such that \( 0 < F(z^*) < 1 \). Thus default occurs with positive probability but not with certainty.

This analysis gives us a model of a loan market which has its price determined solely by demand conditions. Notice that because of the constant returns assumption the individual levels of \( x \) are indeterminate. Introduction of a supply side would determine the aggregate amount of investment.

We wish now to consider the effects of changes in the parameters of the market on the equilibrium variables \( q^* \) and \( F(z^*) \). Given the assumption of risk neutrality the parameters are \( r \), the productivity of investment, \( p \), the exogenous rate of profit or opportunity cost of investing, and \( F(s, t) \), the expectations of the borrowers.

In order to do the comparative statics we differentiate the equilibrium equation \( f(z^*) = v \) and the identities \( q^* = z^* r \) and \( p = vr \).

\[
\begin{align*}
f'(z^*) dz^* &= dv \\
dq &= z^* dr + r dz^* \\
dp &= cdr + rdv.
\end{align*}
\]

Consider now changes in \( r \), i.e., \( dr > 0 \), \( dp = 0 \). Recalling that \( f'(z^*) < 0 \) we have \( dr = \frac{-vdv}{r} < 0 \) and thus \( dz^* > 0 \). From (3.7) then \( dq^* > 0 \).
Proposition 3.2: An increase in the productivity of investment causes an increase in the equilibrium loan rate and an increase in the equilibrium probability of default.

This proposition is rather interesting in that the more productive the investment, the higher the probability of default. It is sometimes argued on the basis of casual empiricism that projects with high expected returns tend to be "riskier". If the risk of a project to the lender is in some sense positively related to the size of the probability of default, then the above proposition shows that the correlation between "risk" and productivity may be a result of the nature of equilibrium in the capital market; that is, it may be an endogenously generated phenomenon rather than a technological fact.

Considering changes in $\rho(dp > 0, dr = 0)$ we have using (3.6) through (3.7) that $dz^* < 0$ and $dq^* < 0$. Hence

Proposition 3.3: An increase in the required profit rate of the industry (or the opportunity cost of investing) decreases the equilibrium loan rate and decreases the equilibrium probability of default.

If we view $\rho$ as "the" riskless interest rate (gross) of the economy then proposition 3.3 says that higher interest rates on riskless ventures is associated with lower interest rates on risky investments, an inverse relationship. Notice that the higher the riskless interest rate the less risky are the projects as measured by the probability of default. In this case, the rise in the expected return to investing by decreasing the probability of default is greater than the fall in the expected return due to the lowering of the loan rate. The default effect of lowering the loan
rate swamps the revenue effect.

We have been associating the term "risk" with the probability of default. We wish now to consider changes in the expectations of borrowers such that from the economy's viewpoint the investment project appears riskier. To do this we shall use the Rothschild-Stiglitz notion of a mean-preserving increase in risk. Let $F(s,t)$ be a family of distribution functions with support in $[0,M]$ and indexed by $t$. Then as Diamond and Stiglitz [1974] have shown, increases in $t$ correspond to increases in risk in the sense of Rothschild and Stiglitz [1970] if

$$\int_0^m dF(s,t) = 1,$$

$$\int_0^x F_t(s,t)ds \geq 0, \quad 0 \leq x \leq M,$$

and

$$\int_0^m F_t(s,t)ds = 0.$$

Now, using integration by parts we re-write the equilibrium equation (3.4) as

$$M - z^*F(z^*,t) - \int z^* F(s,t)ds - z^*[1-F(z^*,t)] = \nu. \quad (3.9)$$

Cancelling terms and re-writing

$$M - \int z^* F(s,t)ds - z^* = \nu. \quad (3.10)$$

Now differentiating (3.10) with respect to $z^*$ and $t$ we have

$$F(z^*,t)dz^* - dz^* - \{\int z^* F_t(s,t)ds\}dt = 0$$

or

$$[F(z^*,t) - 1]dz^* = \int z^* F_t(s,t)ds$$
\[ z^* = \int_0^t F_t(s, t) ds < 0. \]  \hspace{1cm} (3.11)

where the last inequality follows from the definition of an increase in risk. Now as \( F(z^*, t) < 1 \), \( dz^*/dt > 0 \). Thus we have shown

Proposition 3.4: A change in the expectations of borrowers in the sense of an increase in risk does not decrease the equilibrium loan rate.

Thus, from the point of view of a capital market whose loan rate is determined on the demand side, a change in expectations of borrowers, in that they perceive the investment projects to be riskier, will not decrease the interest rate on risky loans. This is, perhaps, somewhat paradoxical in that one might suspect a collapse in the expectations of borrowers might cause a decrease in the demand for investment funds and hence a drop in the interest rate on risky loans. The reason this is not true, is that the change in expectations in the sense of an increase in risk actually increases expected profits. Because of the asymmetry of the returns of the distribution but holding the mean constant, we increase the probability of states which pay off well in the upper ends of the distribution. But the increased probability of states with a low pay-off does not affect expected profits as the borrower can choose to default in those states. Thus, as a consequence of default we have a situation where from the economy's viewpoint the investment project is riskier, yet from the individual borrower's viewpoint the project is more desirable in terms of expected return.

4. Risk Aversion and the Demand for Loans

This section is a brief digression on the properties of the demand function for loans when borrowers are risk averse. In the last section
using the assumptions of risk neutrality, stochastic constant returns to scale, and no barriers to entry or exit we in effect generated a demand curve for loans which was horizontal at the equilibrium loan rate. The equilibrium quantity of investment was left indeterminate. Here we assume essentially the same situation with the following changes. All borrowers are alike in the sense that they have identical utility functions and expectations. The number of borrowers is fixed and they maximize the expected utility of net receipts from the investment project. There is in this model no exogenous rate of profit, and instead of generating an equilibrium loan price we generate a demand curve for loans.

The following assumptions are made on the utility functions of borrowers, $U(\Pi)$.

C.1 $U(\Pi)$ is a bounded, strictly concave, twice differentiable non-decreasing function defined on $R_+$ with finite first and second derivatives everywhere.

Note that this is not a portfolio choice problem. The borrowers, or firms as you may choose to call them, have only one decision to make - how much to invest: Their net receipts, $\Pi$, are never negative as they may always choose to default on the loan. Recall that net profits are given by

$$\Pi(s,q,x) = \begin{cases} rsx - qx & \text{if } s \geq q/r \\ 0 & \text{if } s < q/r. \end{cases}$$

Thus the expected utility of profits is given by

$$EU(\Pi) = \int_{q/r}^{M} U[rsx - qx]dF(s), \quad (4.1)$$

which can be re-written as
Maximizing with respect to $x$, we have the first-order condition, assuming an interior solution\(^{13}\) ($x^* > 0$),

\[
\frac{d\text{EU}(\Pi)}{dx} = \int_{Z} U'[rx(s-z)]r(s-z)dF(s) = 0, \tag{4.3}
\]

where $x^*$ is the optimal quantity of investment or size of loan.

Of some interest is whether or not the demand curve for loans, $x^*(q,r)$, is downward sloping with respect to the loan price $q$ and increasing with respect to the productivity of investment. From (4.3) we see that $x^*$ is a function of $z$ and $r$. First we differentiate (4.3) with respect to $x^*$ and $z$.

\[
\frac{M}{Z} \left[ U''r^2(s-z)^2dx^* - U''r(s-z)rx^*dz - U'rdz \right]dF(s)
\]
\[
- U'[rx^*(z-z)]r(z-z)dF(z) = 0. \tag{4.4}
\]

The last term in (4.4) will vanish provided $U'(0)$ is finite which is assumed by C.1. Thus (4.4) can be re-written as

\[
\frac{M}{Z} \left\{ \int_{Z} U''r^2(s-z)^2dF(s) \right\}dx^* = \frac{M}{Z} \left\{ \int_{Z} [U''r(s-z)x^* + U'r]dF(s) \right\}dz. \tag{4.5}
\]

As $U'' < 0$ by concavity the term in brackets on the left-hand side of (4.5) is negative. Consider the term

\[
U''r^2(s-z)x^* + U'r. \tag{4.6}
\]

The term in brackets on the right-hand side of (4.5) is positive or negative as (4.6) is greater or less than zero. Suppose $U''\Pi + U' > 0$. Then we have $- \frac{UU''}{U'} < 1$. But $R(\Pi) = - \frac{UU''(\Pi)}{U'(\Pi)}$ is nothing but the Arrow-Pratt
measure of relative risk aversion. We have thus shown

Proposition 4.1: The demand for loans is an increasing or decreasing function of the loan rate as the coefficient of relative risk aversion is greater or less than unity everywhere.

Therefore, we have the surprising proposition that the demand function for loans may in fact be upward sloping if the utility function has a coefficient of relative risk aversion greater than unity. There is no general way one can preclude such a situation. A corollary of proposition 4.1 is that if the utility function exhibits constant relative risk aversion, with a relative risk aversion coefficient equal to unity then the demand for loans is price inelastic.

In general it is not possible to sign the effect of $x^*$ with respect to changes in $r$. In subsequent sections when dealing with a supply determined loan price, the effect of parameter changes on the equilibrium quantity of investment will depend upon whether the demand curve for loans has positive or negative slope.

5. Loan Market Equilibrium: Supply Determined Loan Price

In this section we consider a capital market, again with a single type of loan contract available, only in this situation the equilibrium loan price and equilibrium probability of default is determined by the supply side of the market.

The suppliers of investment funds are assumed to be risk neutral and to be sufficiently large in number that the market is considered to be competitive. These lenders may put their funds in either a riskless security paying a gross rate of interest $\rho$, or in loans to individuals or firms
having access to technologies such as those described by A.1 and A.2 in section 3. These loans are risky in that the borrowers may choose to default, and, as we assume the lenders know the technological coefficient \( r \), they realize that if \( s \), the 'state-of-the-world' parameter should take a value less than \( q/r \), the loan will not pay the promised amount. Should default occur, however, the rights to investment project and the capital associated with it are transferred to the lender. We assume though that the investment project to the lender is worth less than to the original borrower. This could be for two reasons. If a lender has to take over an investment project and actually supervise its operation he may incur substantial costs doing so, or he may lack the expertise necessary for a fully efficient operation. Alternatively, if the lender chooses not to complete the investment project but rather to sell off the capital equipment associated with it, he may incur marketing costs in the process and in any case will not recover the full amount due on the loan.

We shall term the value of the investment project to the lender, should he have to take it over, the \textit{collateral value}. The collateral return function is \( c(x,s) = cx s \), where \( c > 0 \). The assumption that the collateral value of the project is always less than its investment value implies that \( c \) is less than \( r \). All lenders are assumed to have the same collateral function.

The expectations of lenders are represented by a distribution function \( F(s) \) with support in the interval \([0,M]\). All lenders have the same expectations. Again, changes in expectations will be represented by changes in \( t \), where \( F(s,t) \) is a family of distributions. The definition of an increase in risk used in this section is the same as in the previous section with the additional requirement that \( F_t(z^*,t) \geq 0 \). That is, we assume
that the increase in risk is always such that, from the viewpoint of lenders, at the equilibrium default point, $z^*$, the probability of default does not decrease. Sometimes we shall refer to this type of increase in risk as an increase in default risk.

Competition forces all suppliers of investment funds to charge the same price $q$ on risky loans, and capital-market equilibrium requires that the lowest $q$ compatible with the expected return on risky loans being equal to $\rho$ be charged. Should this not be true, for example if the expected return to these loans exceeded $\rho$, then there would be an excess supply of funds for risky investment projects driving down the loan rate so as to restore capital market equilibrium.

The return function to lenders is denoted by $\beta(q,s,x)$ where

$$
\beta(q,s,x) = \begin{cases} 
q x & \text{if } s \geq q/r \\
cxs & \text{if } s < q/r.
\end{cases}
$$

Therefore the expected return per dollar loaned out is given by

$$
\frac{q}{r} c \int_0^s s dF(s) + \frac{q}{q/r} dF(s). \quad (5.1)
$$

Again we let $z = q/r$, and capital market equilibrium requires that (5.1) be equal to $\rho$. Thus we have

$$
g(z) = c \int_0^z s dF(s) + rz \int_z^M dF(s) = \rho \quad (5.2)
$$

as the basic equilibrium equation. We are looking for solutions $z^* \in [0,M]$, to the equation $g(z^*) = \rho$. Note that $g(z)$ is continuous and differentiable, $g(0) = 0$ and $g(M) = cS$, and thus by the mean-value theorem $g(z)$ takes on all values in the interval $[0,cS]$. Therefore, if $\rho \in (0,cS)$ a solution
$z^* \in (0, M)$ exists for equation (5.2). What of uniqueness? Evaluating the derivative of $g(z)$ we have

$$g'(z) = czdF(z) + r\int_z^M dF(s) - rzdF(z)$$

$$= (c-r)zdF(z) + r\int_z^M dF(s). \quad (5.3)$$

As $r > c$, since investment value exceeds collateral value, (5.3) is of indeterminate sign. Changing the default point has two effects of opposite sign. The term $r\int_z^M dF(s)$ is the direct profitability effect and is positive. This term accounts for the added return by raising the loan price. The term $(c-r)zdF(z)$ is the default effect, and is always negative. As the default point is raised collateral value is substituted for loan value and as $c$ is less than $r$ the terms of this substitution effect are negative. As $g'(z)$ cannot be signed it appears we cannot claim that a unique equilibrium price exists on technical arguments alone.

In Figure 1 $g(z)$ is graphed. Note that its slope may have both positive and negative values. For an exogenous rate of return $\rho$ the equation $g(z) = \rho$ has three solutions $z_1$, $z_2$, and $z_3$. Now, we shall argue $z_1$ is the only possible equilibrium solution. Why? Suppose for example all lenders are charging $q_2 = rz_2$. Then any lender could charge $\hat{q}$, where $q_1 < \hat{q} < q_2$, attract a large number of loan customers and earn an expected rate of return above $\rho$. Thus $z_1$ is the only possible equilibrium solution, and all lenders must be charging this price. Notice at $z_1$ that $g'(z_1) > 0$. Is it possible that at some equilibrium $z^*$, $g'(z^*) < 0$? The answer is no. Notice first that from (5.3) $g'(0) = r\int_0^M dF(s) = r > 0$. Thus the function $g(z)$ always starts at 0 with a rising segment. Now suppose there existed an equilibrium $z^*$ such that $g'(z^*) < 0$. But then by
Figure 1.

Expected Return Function
the mean value theorem there would exist a \( \hat{z} < z^* \), such that \( g(\hat{z}) = \rho \), and \( g'(\hat{z}) > 0 \). Thus \( z^* \) could not have been an equilibrium, proving our proposition. Therefore we have shown 15

**Proposition 5.1**: There exists a unique equilibrium default point \( z^* \), and furthermore in the neighbourhood of this equilibrium the expected return on risky loans, \( g(z) \) is increasing with \( z \).

Now differentiating (5.2) at the equilibrium point \( z^* \) we have

\[
g'(z^*) dz^* = d\rho. \tag{5.4}
\]

Since \( g'(z^*) > 0 \), the following proposition holds.

**Proposition 5.2**: An increase in the exogenous rate of return causes an increase in the equilibrium loan rate and an increase in the equilibrium probability of default.

Notice that if we have a downward sloping demand curve for risky loans, say via the analysis of section 4, then an increase in the riskless interest rate would cause a decline in the equilibrium quantity of investment. Contrary to the demand-determined model, an increase in the interest rate (riskless) causes an increase in the probability of default. The supply determined model thus posits a positive correlation between high interest rates and high levels of default.

We turn now to consideration of a change in the productivity coefficient \( r \). Differentiating (5.2) with respect to \( r \) and \( z^* \) we get the equation

\[
g'(z^*) dz^* + \left\{ z^* \int_{z^*}^{M} dF(s) \right\} dr = 0. \tag{5.5}
\]

As \( g'(z^*) > 0 \), we have \( \frac{dz^*}{dr} < 0 \). Differentiating the identity \( q = z^* r \) we
have \( \frac{dz^*}{dr} = r \frac{dz^*}{dr} + z^* \). From (5.5)

\[
\frac{dz^*}{dr} = \frac{-z^* \int_{z^*}^{s} dF(s)}{g'(z^*)}
\]

and from (5.3)

\[
g'(z^*) = (c - r)z^* dF(z^*) + r \int_{z^*}^{s} dF(s).
\]

Therefore

\[
\frac{dg^*}{dr} = \frac{-rz^* \int_{z^*}^{s} dF(s)}{g'(z^*)} + \frac{g'(z^*)z^*}{g'(z^*)} + \frac{M}{g'(z^*)} - \frac{rz^* \int_{z^*}^{s} dF(s)}{g'(z^*)} + \frac{(c - r)z^* dF(z^*) + rz^* dF(z^*)}{g'(z^*)}
\]

\[
= \frac{(c - r)z^* dF(z^*)}{g'(z^*)} < 0.
\]

Thus we have shown

**Proposition 5.3:** An increase in the productivity of investment causes a) a decrease in the equilibrium default rate and b) a decrease in the equilibrium loan rate.

Notice again that the results of the supply model are completely opposite to those of the demand model (proposition 3.2). In this case more productive investments are associated with lower default probabilities. It would seem that in a supply-determined case higher productivity projects have a lower cost of capital; that is, a lower cost of funds to borrowers.

Changes in the collateral value of the project are treated similarly. Differentiating (5.2) with respect to c and z* we get
\[ g'(z^*)dz^* + \int_0^{z^*} s dF(s) dc = 0 \]

which implies \[ \frac{dz^*}{dc} < 0. \] Hence

**Proposition 5.4:** An increase in the collateral value of projects causes a decrease in the equilibrium probability of default and a decrease in the equilibrium loan rate.

This result makes a great deal of intuitive sense. It seems not unreasonable that the better the collateral value of the project the lower the loan rate should be. In fact, this is a rather common phenomena observed in capital markets.

Finally consider changes in the expectations of lenders. The type of change we wish to consider is an increase in risk in the Rothschild-Stiglitz [1970] sense, such that probability of default does not decrease at the equilibrium default point. We characterize this by the conditions

(i) \[ \int_0^x F_t(s,t) ds > 0 \quad \text{for} \quad 0 \leq x \leq M \]

and

(ii) \[ F_t(z^*,t) > 0. \]

Using integration by parts (5.2) may be re-written as

\[ g(z^*,t) = c[z^*F_t(z^*,t) - \int_0^{z^*} F(s,t)ds] + rz^*[1-F(z^*,t)] = \rho. \quad (5.6) \]

Now \[ g_z(z^*,t) > 0 \] from proposition 5.1. Differentiating (5.6) with respect to \( t \) we have

\[ g_t(z^*,t) = (c-r)z^*F_t(z^*,t) + c \int_0^{z^*} F_t(s,t)ds \leq 0, \]

where the inequality follows from the definition of an increase in default.
risk. Thus differentiation of (5.6) with respect to \( z^* \) and \( t \) implies
\[
\frac{dz^*}{dt} > 0.
\]
Now \( F(z^*,t) \) is the probability of default as viewed by lenders, and hence
\[
\frac{dF(z^*,t)}{dt} = F_s(z^*,t) \frac{dz^*}{dt} + F_t(z^*,t) > 0.
\]
Therefore we have shown

Proposition 5.5: A change in the expectations of lenders, represented by an increase in default risk, does not decrease the equilibrium loan rate and does not decrease the equilibrium probability of default.

Thus a collapse in the expectations of lenders may cause an increase in the equilibrium loan rate. If the demand curve for loans were downward sloping this would cause a reduction in the equilibrium quantity of risky investment. Recall, however, there is no \textit{a priori} reason for the demand curve for risky loans to have a negative slope.

This completes our program of comparative statics. We wish to ask one other question of this model. Suppose there are two types of firms or technologies in existence, one unambiguously more productive than the other, for example with productivity coefficients \( r_1 \) and \( r_2 \), where \( r_1 > r_2 \). Does there exist a competitive capital market equilibrium with default, in which the market allocates some investment funds to both firms, or technologies? Let \( g_1(z) \) and \( g_2(z) \) be the expected return functions for firms 1 and 2 respectively. It is straightforward to verify
\[
g_1(0) = g_2(0),
\]
\[
g_1(M) = g_2(M),
\]
and
\[
g_1(z) > g_2(z) \quad \text{for} \quad 0 < z < M.
\]
These functions are graphed in Figure 2.
Figure 2.

Expected Return Functions for Firms 1 and 2
If only type two firms existed $q_2 = z_2 r_2$ would be the equilibrium loan price. Since type one firms exist, however, and lenders can distinguish type one from type two firms, an equilibrium with two loan rates $q_1 = z_1 r_1$ and $q_2$ can exist, with both types of firms yielding expected rates of return on their loans equal to $\rho$. This possibility indicates that competitive equilibrium with default may in a very strong sense be inefficient. Any investment taken out of type two firms and put in type one firms would raise the total expected return to a fixed quantity of investment goods. From an efficiency viewpoint the best solution is to allocate all investment to type one firms. The introduction of default on loan contracts is one step towards inefficiency, in that if the loan is defaulted, then the collateral return is less than the investment return. The problem is compounded further, however, as firms of different basic technological productivity, may yield the same expected return on their loans. This occurs as lenders may charge firms of different types different loan rates. In a competitive equilibrium any attempt to charge a higher productivity firm a lower loan rate would result, as a consequence of proposition 5.1, in the lender earning less than the opportunity cost, $\rho$, on his loan. The solution mechanism of the capital market sustains a competitive inefficiency. Note that a government with the same information as the market participants could, by simply excluding the loan requests of type two firms and guaranteeing the loans of type one firms, raise aggregate expected return.

This brief digression on the efficiency aspects of default indicates that there may be some serious questions as to the optimality of the capital market allocation of investment, quite independent as to any issues of risk bearing. Clearly a more complete equilibrium treatment is
6. The Introduction of Equity

Our analysis so far has concentrated on a capital market with a single type of contract which facilitates the savings-investment process. In this section we consider some possible complications of introducing an alternative means of investing in risky investment projects—the purchase of equity or limited liability stocks in risky firms.

The model in this section will be a simple general model of the standard two-period variety, investment taking place in the first period and output (random) occurring in the second period. The economy consists of two types of firms with all firms of each type having identical technologies. The riskless firms have a constant-returns-to-scale technology which fixes the riskless interest rate (gross) provided some positive investment occurs in the riskless technology. Risky firms have technologies identical to those investment projects outlined in section 3. We shall require, however, that in states in which the firm goes bankrupt, i.e., defaults on its payments to bondholders, a change in technique occurs. That is, in states of bankruptcy the firm does not operate with the same technology as in those states in which the firm is solvent. The technology in bankrupt states will be identical to the collateral return function of section 5, and has the property that it pays off at a rate below that of the technology in solvent states.

The reason for making the above assumption is that it introduces a deadweight cost to the economy of bankruptcy. In previous analyses of bankruptcy, the bankruptcy always occurs at zero resource cost to the economy. Yet, it would seem that the transactions costs involved in
administering bankruptcies, and perhaps more important the loss of output due to the change of management during a bankruptcy, would not be insignificant. In many cases a firm which goes bankrupt will actually shut down, and if the investment made was of the putty-clay variety the cost to the economy of bankruptcy is the entire output lost, as the capital invested in the firm has no consumption or production value. Of course since bankruptcies incur real costs on the economy, this implies that from the social-efficiency viewpoint the organization of production should be such as to eliminate bankruptcies. How this might be done is a difficult question and we shall not be concerned with it here. Bankruptcies are an institutional feature of market economies and our primary concern is the positive implications of this fact.

As a result of bankruptcy inducing real costs, we shall show in this section that it is possible for the economy to have equilibria with a determinate debt-equity ratio, while maintaining the assumptions of identical expectations and risk neutrality on the part of investors. This is in contrast to the well-known Modigliani-Miller proposition, which states that provided the debt issued has no default risk the economy as a whole is indifferent between alternative debt/equity ratios of the firms in the economy. We shall show that, even with the strong assumptions made, the debt/equity ratio of the economy may be a market determined variable. In addition to the resource costs of bankruptcy, the result hinges on the fact that any bond issued by a risky firm has a positive probability of not paying off at the nominal interest rate, as in some states arbitrarily small amounts of output are obtained. Furthermore, changes in the debt/equity ratio of a risky firm affect the default point and this becomes an additional mechanism by which expected rates of return are equalized.
Recall that the gross return on an investment project or to a risky firm's technology is given by \( rsx \), where \( x \) is total investment, provided the firm does not default. If the firm does default on its bond payments, then in those states \( s \) in which it is bankrupt output is given by \( csx \). We have assumed that bankruptcy has real resource costs, or equivalently induces deadweight losses which is expressed by the assumption that \( c \) is less than \( r \). Let \( B \) stand for dollars worth of bonds (measured in units of investment goods) and \( E \) the dollars amount of equity for a particular firm. Thus \( x = B + E \), or the amount invested by a single firm is equal to the value of its debt plus equity. This assumes all firms start up in the first period and are dissolved in the second period. Bondholders are to receive \( qB \) in the second period where \( q \) is the nominal interest rate (gross) on bonds. Therefore \( \Pi(q, B, s, x) \), the return to total equity \( E \), is given by

\[
\Pi(q, B, s, x) = \begin{cases} 
rsx - qB & \text{if } s > \frac{qB}{rx} \\
0 & \text{if } s < \frac{qB}{rx}.
\end{cases}
\]  

(6.1)

Recall that \( q/r = z \). Let \( B/x = w \), where \( w \) is the proportion of investment which is debt financed, and \( (1 - w) \) the proportion which is equity financed; an increase in \( w \) will correspond to an increase in the debt/equity ratio. Using this notation the return on a dollar's worth of investment in equity, \( R_E(z, w) \), is given by

\[
R_E(z, w) = \begin{cases} 
\frac{rs}{1 - w} - \frac{qw}{1 - w} & \text{if } s \ge zw \\
0 & \text{if } s < zw.
\end{cases}
\]  

(6.2)
Note again that the constant-returns assumption makes the return to equity independent of the scale of investment. The default point is given by $zw$ in this model. Therefore, using (6.2), the expected rate of return on investing in the equity of risky firms is given by

$$\text{ER}_E(z, w) = \frac{r}{1 - w} \int_{zw}^{M} s \, dF(s) - \frac{qw}{1 - w} \int_{zw}^{M} dF(s) \quad (6.3)$$

where $F(s)$ is distribution function representing the expectations of investors.

The return to a dollar's investment in the bonds of a risky firm is

$$R_B(z, w) = \begin{cases} 
q & \text{if } s \geq zw \\
\frac{cs}{w} & \text{if } s < zw.
\end{cases} \quad (6.4)$$

Hence the expected return to investment in risky bonds is given by

$$\text{ER}_B(z, w) = \frac{c}{w} \int_{0}^{zw} s \, dF(s) + q \int_{zw}^{M} dF(s). \quad (6.5)$$

The rate of return to investing in either the bonds or equity of the riskless firm is given by $\rho$, where $\rho x$ is the production function for riskless firms. Capital market equilibrium, given the assumption of risk neutrality, requires that the expected rate of return on all securities in which positive investment takes place be equal. Supposing that investment occurs in all securities the market equilibrium conditions can be written as

$$\text{ER}_E(z, w) = \rho \quad (6.6)$$

$$\text{ER}_B(z, w) = \rho.$$
Dividing both of these equations through by \( r \), and letting \( \rho/r \equiv v \), we get

\[
\begin{align*}
H_1(z, w) &\equiv \frac{1}{1 - w} \int_{zw}^{M} \text{sd}F(s) - \frac{zw}{1 - w} \int_{zw}^{M} \text{d}F(s) = v \quad (6.7) \\
H_2(z, w) &\equiv \frac{c}{r} \int_{zw}^{1} \text{sd}F(s) + z \int_{zw}^{M} \text{d}F(s) = v. \quad (6.8)
\end{align*}
\]

We wish to investigate the possibility of an equilibrium in which investment occurs in risky bonds, equity and the riskless firm and thus the equations (6.7) and (6.8) have a solution \((z^*, w^*)\). If this is the case, then capital market equilibrium determines both a nominal interest rate on debt and an equilibrium debt/equity ratio. It is quite conceivable of course that equilibrium may entail investment in all of one type of security.

Note that if an investor holds a portfolio with the fraction \( w \) held in risky bonds and the remainder \((1 - w)\) in equities of the risky firm his expected return on the portfolio is given by

\[
\text{ER}_p(z, w) \equiv c \int_{0}^{zw} \text{sd}F(s) + r \int_{zw}^{M} \text{sd}F(s). \quad (6.9)
\]

Again, provided investment occurs in all securities, capital-market equilibrium requires \( \text{ER}_p(zw) = \rho \). Let \( zw = y \), then the function \( h(y) \equiv \frac{c}{r} \int_{0}^{y} \text{sd}F(s) + r \int_{y}^{M} \text{sd}F(s) \), can easily be shown to take values in the interval \([c\bar{s}, r\bar{s}]\) and to be uniquely invertible on \([0, M]\). Therefore, a necessary condition for such an equilibrium to exist is that \( \rho \in (c\bar{s}, r\bar{s}) \), and we assume this condition holds henceforth. Clearly one possible equilibrium is for no investment to occur in either the riskless firm or in the bonds of risky firms, in which case the expected return on all
portfolios is given by \( r_s \). In terms of efficiency this equilibrium is the most desirable, provided \( \rho < r_s \).

The case \( w^* = 1 \) is excluded in this section as it corresponds to a completely debt financed equilibrium, which we treated in section 5. Such an equilibrium is possible as demonstrated in that section. In order to prove that an intermediate case is possible, we must show there exists \( w^*, 0 < w^* < 1 \) and \( z^* > 0, 0 < z^* w^* < M \), such that (6.7) and (6.8) hold. In fact this is the case.

**Proposition 6.1:** If \( \rho \in (c_s, r_s) \) there exists a capital market equilibrium with \( 0 < w^* < 1 \), i.e., a strictly positive debt equity ratio.

**(Proof):** The proof is based on the fact that

\[
wrH_2(z, w) + (1 - w)rH_1(z, w) = ER_p(zw). \tag{6.10}
\]

Therefore of the three equations (6.7), (6.8) and (6.9) only two are independent, the third being determined by (6.10). As \( \rho \in (c_s, r_s) \), we know there exists a unique \( y^*, 0 < y^* < M \), such that

\[
ER_p(y^*) = c \int_0^{y^*} sdF(s) + r \int_{y^*}^M sdF(s) = \rho. \tag{6.11}
\]

As \( y^* = zw \), we may write \( z \) as an implicit function of \( w \), \( z(w) \), with the property that \( z(w)w = y^* \). Taking the solution \( y^* \) from (6.11) and substituting into \( H(z, w) \) we have

\[
H_1[z(w), w] = \frac{1}{1 - w} \left[ \int_{y^*}^M sdF(s) - y^* \int_{y^*}^M dF(s) \right]. \tag{6.12}
\]

We shall show now there exists a \( w^*, 0 < w^* < 1 \) such that \( H_1[z(w^*), w^*] = v \).
Consider the expression in brackets on the right-hand side of (6.12),

\[ M \int_{y^*}^{y} sdF(s) - y^* \int_{y^*}^{M} dF(s). \]  

(6.13)

Dividing through by \([1 - F(y^*)] > 0\) we get

\[ E(s|s \geq y^*) - y^* > y^* - y^* = 0. \]

Hence the expression in brackets is strictly positive in (6.12). Now from (6.11)

\[ \frac{C}{r} \int_{0}^{y^*} sdF(s) + \int_{y^*}^{M} sdF(s) = \frac{\rho}{r} = v. \]  

(6.14)

Subtracting (6.13) from (6.14), we have

\[ \frac{C}{r} \int_{0}^{y^*} sdF(s) + y^* \int_{y^*}^{M} dF(s) > 0, \]  

(6.15)

which implies that (6.13) is strictly less than \(v\). Letting (6.13) equal \(\alpha\) we have shown \(0 < \alpha < v\). (6.12) can be written as

\[ H_1[z(w), w] = \frac{\alpha}{1 - w}. \]  

(6.16)

Now

\[ \lim_{w \to 0^+} H_1[z(w), w] = \alpha \]

and

\[ \lim_{w \to 0^-} H_1[z(w), w] = +\infty \]

where the plus and minus superscripts denote right and left hand limits.
respectively. Since $H_1(z(w), w)$ is a continuous function of $w$ and $\alpha < \nu$, there exists a $w^*, 0 < w^* < 1$ such that $H_1(z(w^*), w^*) = \nu$. Using (6.10) and substituting for $w^*$ and $z^* = z(w^*)$ we have that $H_2(z^*, w^*) = \nu$, and the proposition is proved. Q.E.D.

Thus we have established that provided the riskless firms' technologies have a productivity of an appropriate value, there exists a capital market equilibrium with a determinate debt/equity ratio. Notice that the debt/equity ratio of this economy is an equilibrium one, and any attempt by firms to change their debt/equity ratios will result in arbitrage taking place in the capital market such as to force the debt/equity ratio back to its equilibrium value. In equilibrium the market value of any firm investing an amount $x$ is $px$.

In order to do the comparative statics of the general equilibrium, it is necessary to work with two of the three equilibrium equations. We shall use (6.8) multiplied by $r$ and (6.11) to give us

$$G_1(z^*, w^*) = \frac{c}{w^*} \int_0^{z^*W^*} z^* dF(s) + rz^* \int_{z^*W^*}^{M} dF(s) = \rho, \quad (6.17)$$

$$G_2(z^*, w^*) = c \int_0^{z^*W^*} z^* dF(s) + r \int_{z^*W^*}^{M} dF(s) = \rho. \quad (6.18)$$

Evaluating the partial derivatives of these equations we have

$$G_{11}(z,w) = (c - r)zw dF(zw) + r \int_{zw}^{M} dF(s),$$

$$G_{12}(z,w) = \frac{c}{w} \int_0^{zw} dF(s) + (c - r)z^2 dF(zw) < 0,$$

$$G_{21}(z,w) = (c - r)zw^2 dF(zw) < 0,$$

and

$$G_{22}(z,w) = (c - r)z^2 w dF(zw) < 0,$$
where \( G_{ij} \) denotes the partial derivative of \( G_i \), \( i = 1, 2 \) with respect to \( j = z, w \). The last three inequalities follow from the fact that \( c < r \). \( G_{11} \) is of indeterminate sign as the first term is negative and the second positive. Recall, however, that \( G_1(z, w) \) is the expected return to investing in the bonds of risky firms at the nominal interest rate of \( q = rz \). Now, just as was argued in section 5 with regard to proposition 5.1, competition amongst lenders will force \( q = rz^* \), the interest rate on bonds, to the lowest one compatible with (6.17) for any value of \( w \), which to the individual investor is taken as given. Note that \( G_1(0, w) = 0 \), \( G_1(M, w) > 0 \) and \( G_{11}(0, w) = 1 > 0 \). Thus by the same reasoning as used to prove proposition 5.1, it must be the case that at an equilibrium \((z^*, w^*)\), \( G_{11}(z^*, w^*) > 0 \).

Therefore the Jacobian matrix

\[
\begin{bmatrix}
  G_{11} & G_{12} \\
  G_{21} & G_{22}
\end{bmatrix}
\]

has the sign pattern

\[
\begin{bmatrix}
  + & - \\
  - & -
\end{bmatrix}
\]

at equilibrium values of \((z^*, w^*)\). Letting \(|G|\) denote the determinant of this matrix we see that \(|G| < 0\).

Proceeding with the comparative statics we differentiate (6.17) and (6.18) with respect to \( z, w \) and \( \rho \) at the equilibrium point \((z^*, w^*)\) to get

\[
\begin{bmatrix}
  G_{11} & G_{12} \\
  G_{21} & G_{22}
\end{bmatrix}
\begin{bmatrix}
  dz^* \\
  dw^*
\end{bmatrix}
= \begin{bmatrix}
  d\rho \\
  d\rho
\end{bmatrix}
\]

(6.19)
Using Cramer's rule we have

$$\frac{dz^*}{d\rho} = \frac{G_{22} - G_{12}}{|G|}$$

$$\frac{dw^*}{d\rho} = \frac{G_{11} - G_{21}}{|G|} < 0.$$

Now

$$G_{22} - G_{12} = (c - r)z^2w dF(zw) + \frac{c}{w^2} \int_0^{zw} sdF(s) - (c - r)z^2 dF(zw)$$

$$= (c - r)z^2 dF(zw) [w - 1] + \frac{c}{w^2} \int_0^{zw} sdF(s)$$

$$> 0$$

at \((z^*, w^*)\) as \(0 < w^* < 1\). Therefore \(\frac{dz^*}{d\rho} < 0\) and as \(q^* = z^* r\), \(\frac{dq^*}{d\rho} < 0\).

Hence

**Proposition 6.2:** An increase in the riskless interest rate \(\rho\) (the productivity of the riskless firm) causes 
a) a decrease in the nominal rate of interest on risky bonds and 
b) a decrease in the equilibrium debt/equity ratio.

This proposition states that as a result of market equilibrium the nominal interest rate on risky bonds is negatively correlated with the riskless interest rate. Furthermore, risky firms will tend to have relatively highly levered capital structures when interest rates are low, even though the nominal interest rates on risky bonds will tend to be high.

We consider now increases in the productivity of risky firms. Letting \(G_{1r}\) and \(G_{2r}\) stand for the partial derivatives of (6.17) and (6.18) with respect to \(r\) we get
\[ G_{1r} = z \int_{zw}^{M} dF(s) > 0 \]

and

\[ G_{2r} = \int_{zw} \sigma dF(s) > 0. \]

Again using Cramer's rule we have that

\[ \frac{dw^*}{dr} = \frac{\begin{vmatrix} G_{11} - G_{1r} \\ G_{21} - G_{2r} \end{vmatrix}}{|G|} > 0. \]

Therefore

**Proposition 6.3:** An increase in the productivity of risky firms causes an increase in the equilibrium debt/equity ratio.

Thus economies with more productive risky technologies will have more highly levered financial structures. The effect of changes in productivity on the price of bonds is ambiguous in this model.

We turn now to changes in the cost of bankruptcy. Recall that an increase in \( c \) corresponds to lowering the cost of bankruptcy. Now

\[ G_{1c} = \frac{1}{w} \int_{0}^{zw} \sigma dF(s) > 0 \]

and

\[ G_{2c} = \int_{0}^{zw} \sigma dF(s) > 0. \]

So

\[ \frac{dz^*}{dc} = \frac{\begin{vmatrix} -G_{1c} & G_{12} \\ -G_{2c} & G_{22} \end{vmatrix}}{|G|} \]
Proposition 6.3: A decrease in the cost of bankruptcy causes a) an increase in the nominal interest rate on risky bonds and b) an increase in the equilibrium debt/equity ratio.

This proposition makes a good deal of intuitive sense. As the costs of default to bondholders diminishes, both the rate of interest paid on bonds goes up, and the economy's relative security mix shifts so as to raise the proportion of risky investment financed by issuing debt. Note that in the limiting case of zero costs to bankruptcy, \( c = r \), the Modigliani-Miller theorem is true and there is no equilibrium determined debt/equity ratio for the economy, in the sense that any debt/equity ratio is compatible with an investor earning an expected rate of return of \( r^5 \) on his portfolio.

We propose now to examine the changes in the equilibrium probability of default, with respect to the parameters of the model. In order to do so it suffices to examine the equilibrium equation (6.14) letting \( z^w = y^* \), the equilibrium default point.

\[
h(y^*) \equiv c \int_0^{y^*} s dF(s) + r \int_{y^*}^{M} s dF(s) = \rho.
\]

Note that \( \frac{\partial h(y^*)}{\partial y} < 0 \), and it is straightforward to show

\[
\int_0^{z^w} s dF(s) = \frac{1}{|G|} \left[ G_{12} - \frac{1}{w} G_{22} \right]
\]

\[
\int_0^{z^w} s dF(s) = \frac{c}{w} \int_0^{z^w} s dF(s)
\]

Similarly \( \frac{d w^*}{d c} > 0 \). We have shown.
Proposition 6.4: The equilibrium probability of default or bankruptcy of risky firms increases with a) decreases in the riskless interest rate, b) increases in the productivity of risky investment and c) decreases in the cost of bankruptcy.

Therefore low interest rates on riskless securities are associated with high bankruptcy rates on risky firms, the reason being that, from proposition 6.2, with low interest rates firms tend to have highly levered capital structures and therefore bankruptcy becomes more likely. Note also that economies with more highly productive technologies tend to have higher bankruptcy rates.

This completes the analysis of the two security market. There is one problem we have left untreated. This is the question of the production decision on the part of the firm. If the firm has a production decision to make, for example the scale of investment or choice of technique, which it doesn't in the simple economy considered here, then this decision may be closely related to the debt/equity ratio and the price of debt. This is an important question to tackle, as real decisions become closely related to financial variables, and the presumed independence of these variables in neoclassical theory is no longer valid.

7. Conclusion

We have considered three models of a capital market which has a type of debt contract through which investment is made on a risky investment project. The significant feature of this contract is that, given some positive amount of investment was undertaken, there is always some positive probability that the borrower will default on the loan. Thus, lending via this type of debt contract becomes a risky venture as well. In
each case we considered the determination of the equilibrium interest rate on the loan and the equilibrium probability of default. A number of comparative static results were developed and these are summarized in Table 1 below.

Some observations on these results follow.

1. Note that with respect to changes in the interest rate and the productivity of investment the results of the supply and demand models are completely opposite. This might suggest that in some intermediate cases, for example where both demand and supply curves were not perfectly elastic the results would probably be ambiguous. Unless one can make a good case for one side of the market dominating it would seem difficult to argue on theoretical grounds any qualitative comparative static relationship between riskless interest rates and the price of loans in the money-market.

2. Notice that in the supply and demand models the effect of an increase in risk is to raise the equilibrium interest rate on loans. Therefore, even though we have assumed risk neutrality on the part of firms and investors, increases in risk on investment projects correspond to higher loan rates. Hence the cost of capital to investment projects unambiguously contains a risk discount when the possibility of default is admitted. In other words, when all real investment must be financed by issuing debt or taking out a loan, then comparing two firms, one riskier than the other in the Rothschild-Stiglitz sense, the riskier firm will have to pay a higher interest rate on its loans or the debt it issues than the less risky firm. Recall that the increases in risk considered hold the expected value of the investment project constant, and thus differences in the loan rate cannot be attributed to differences in expected values.
Table 1
SUMMARY OF COMPARATIVE STATIC RESULTS

A. Demand Determined Model

Equilibrium Variables Response

\[
\begin{array}{c|cc}
\text{Parameter Increases} & q & p \\
\hline
r & + & + \\
p & - & - \\
risk & + & ? \\
\end{array}
\]

B. Supply Determined Model

Equilibrium Variable Response

\[
\begin{array}{c|cc}
\text{Parameter Increases} & q & p \\
\hline
r & - & - \\
p & + & + \\
c & - & - \\
risk & + & + \\
\end{array}
\]

C. Debt-Equity Model/Supply Determined

Equilibrium Variable Response

\[
\begin{array}{c|ccc}
\text{Parameter Increases} & q & w & p \\
\hline
r & ? & + & + \\
p & - & - & - \\
c & + & + & + \\
\end{array}
\]

NOTE: The p in each column denotes the equilibrium probability of default. A "?" signifies an ambiguous comparative static result.
3. The introduction of equity into the model gives the market an additional equilibrating mechanism, in that movement between bonds, equity and the riskless security affect not only the relative quantities of these securities but also the expected return to holding them by altering the default point on debt.

4. In the supply model and two-security model the costs associated with default on loans or the bankruptcy of limited liability firms have rather different implications. In the one-security loan model lowering the costs of default to lenders lowers both the nominal interest rate on loans and the equilibrium probability of default. Conversely in the two security model, lowering the resource cost of bankruptcy raises nominal interest rates on bonds and raises the probability of bankruptcy. This suggests that the introduction of a limited liability security as a means of financing investment may be a mixed blessing from a welfare point of view. Should lowering the costs of bankruptcy become a policy issue, for example by a reform of the bankruptcy laws, then this will have the unfortunate effect of raising the probability of default or bankruptcy. The expected gain from such an action may be either positive or negative.

5. The introduction of transactions costs to bankruptcy has the implication that a competitive capital market may have equilibria with a determinate debt/equity ratio. The inability to develop models which provide an explanation of debt/equity ratio determination has long been a problem in economic theory and corporation finance. Stiglitz [1972] has recently given an explanation in terms of divergent expectations. The explanation offered in section 6 is based on a cost to bankruptcy argument, in addition to the necessary assumption that all corporate debt in risky firms
has some positive probability of default. It was established that the higher the transactions costs of bankruptcy are, the lower is the equilibrium debt/equity ratio. Further effort in developing testable hypotheses which provide an explanation as to how this important economic variable is determined seems called for.
Footnotes

1. See notes 5-10 below. The increasing attention paid to risk by economi­
   mists studying capital markets is no doubt responsible for this con­
   cern, as the risk of default is often the most significant risk facing
   the lender. The literature which treats this topic most extensively
   is the credit rationing literature, although it is usually at the
   individual firm or investor level. See for example Hodgman [1960],
   Jaffee and Modigliani [1969], Jaffee [1971] and the references con­
   tained therein.

2. For example preferred stock, convertible debentures, warrants or stock
   options.

3. This particular aspect of the loan market has been treated by Barro
   [1974].

4. The general equilibrium extension could be given by introducing a risk­
   less production technology with constant returns to scale in capital
   investment. This would tie down the riskless interest rate. This
   type of general equilibrium treatment is used in section 6 below.

5. Altman [1971] documents the empirical relationship between corporate
   bankruptcy and general economic variables. These figures are from
   his Table 1-3, page 15.

6. The literature here is very large. For example see Miller and

7. General equilibrium models which operate sequentially over time have
   been developed by Radner [1972], Hahn [1971], Diewert [1974], Grandmont
   [1974], and Green [1973]. Green [1972] and Arrow and Hahn [1971],
   pp. 119-122 have dealt particularly with the issue of pre-existing
   contracts.
8. The introduction of information into economic analysis is still in its relative infancy. Some progress has been made in developing the concept of 'screening'. See Spence [1974] and Arrow [1973] for example.


10. See Keynes [1936], Chapter 12.

11. The case of risk averse borrowers is treated in Section 4 below.

12. All integrals can be taken to be Riemann integrals as the distribution F(s) has a continuous density function and the region of integration is a compact interval.

13. In fact an interior solution can always be shown to exist, provided

\[ \frac{dE}{dx} \bigg|_{x=0} = U'(0) \left[ \int_{-\infty}^{0} r s dF(s) - \int_{-\infty}^{0} r z dF(s) \right] > 0. \]

From C.1. \( U'(0) > 0. \) Therefore \( \frac{dE}{dx} \bigg|_{x=0} \) is greater than zero if

\[ \int_{-\infty}^{0} s dF(s) - \int_{-\infty}^{0} z dF(s) > 0. \]

Dividing both sides by \( \int_{-\infty}^{0} dF(s) \) we have

\[ E(s|s \geq z) - z > 0 \]

provided \( z > 0. \) Therefore the demand function for loans is defined implicitly by (4.3) provided \( q > 0. \)

14. We maintain the assumption that \( F(s) \) has a continuous density function for reasons mentioned above. See note 12.

15. This proposition excludes the degenerate case \( g'(z^*) = 0 \) at an equilibrium default point \( g(z^*) \neq 0. \) This case is degenerate in the sense that the set of points in \([0, M]\) on which it may occur is of measure zero by Sard's theorem.
16. As we have maintained the assumption of risk neutrality in this section, expected value is the only relevant moment of the distribution which needs to be considered. The optimality of the capital market allocation of investment when investors are risk averse has been called into question by a number of people. See the recent Symposium on the Optimality of Competitive Capital Markets [1974]. They are concerned solely with the optimal allocation of risk bearing, however, and deal with capital markets in which common stocks are the only type of securities available; consequently default is not a possibility.

17. As for example outlined in Chapter III, section 2 in this thesis.

18. See for example Smith [1972] and Stiglitz [1972]. Stigler [1967] has an interesting discussion of transaction costs in capital markets and their relation to the much used term 'capital market imperfections'.


20. This follows as \( h(y) \) is a continuously differentiable function on \((0, M)\) and \( h'(y) = (c - r)y dF(y) < 0, \) for \( y \in (0, M) \).

21. If \( \rho > r \bar{s} \), then equilibrium requires that all investment occur in the riskless firms. If \( \rho = r \bar{s} \), then provided no debt is issued the allocation of investment between risky and riskless firms is a matter of indifference to risk neutral investors. If \( \rho < c \bar{s} \), then no investment will be made in the riskless industry. In this case it might be possible to have an equilibrium with a positive debt/equity ratio, but in this case equating the expected returns of bonds and equity would determine only one of the two variables, \( q \) and \( w \). Therefore the nominal interest rate on bonds would be a function of the debt/equity
ratio, and there would exist a continuum of equilibria with varying
debt/equity ratios.

22. Stiglitz [1972] and Smith [1969] have made some progress on this
issue.

23. Strictly speaking this conclusion does not follow automatically from
the comparative static propositions 3.4 and 5.5. It can be shown,
however, that if there are two firms which differ in their risk
characteristics in the Rothschild-Stiglitz sense, then capital market
equilibrium is characterized by two loan rates which bear the same
qualitative relationship to one another as given in propositions 3.4
and 5.5. This argument can be extended to any number of firms.

24. As Stiglitz notes, one of the problems with the divergent expectations
assumption, is that it can explain almost anything, and empirically is
very difficult to test for. Lintner [1969] also deals with diverse
expectations in a different context.
References


Jaffee, D.M. [1971], Credit Rationing and the Commercial Loan Market, New York: Wiley.


