A STEADY STATE SELF-CONSISTENT MODEL
FOR PULSAR MAGNETOSPHERES
by
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Date Oct. 6, 1976
A steady state self-consistent model for a pulsar magnetosphere is developed. It is shown that the central neutron star of a pulsar should possess a magnetosphere. In the first approximation, the inertia of the magnetospheric particles is neglected. Steady state corotating models are developed to calculate the structure of the magnetosphere for the axisymmetric case and the case of the arbitrarily oriented dipole. Two results are that charge density is proportional to the $z$ component of the magnetic field and that the $z$ component of the magnetic field vanishes at the light cylinder. The light cylinder is where the corotation velocity reaches the speed of light. The pulsar spin axis is aligned with the $z$ axis. Illustrations of the fields are presented for the cases of magnetic dipole axis parallel and perpendicular to the spin axis.

Next these models are altered to take into account the non-zero mass of the particles in the magnetosphere. An extra electric field is required to hold the particles in corotation. Charge separation is assumed. The following results are found: 1) Field lines which previously were horizontal inside the light cylinder, now have a cusp in them where they were horizontal. This cusp increases in size as its location approaches the light cylinder. 2) Field lines no longer are horizontal at the light cylinder but vertical, and divide into two groups -- those with positive $B_z$
(carrying negative charge) and those with negative \(B_z\) (carrying positive charge).

We next cease to require that the particles be fixed in the corotating frame. First the single particle motion is calculated for arbitrary fields, assuming small velocities in the rotating frame. We find that the motion can be separated into a slow drift along streamlines (which very nearly follow magnetic field lines) and a spiraling about these streamlines. The energy is conserved, and can be separated into a longitudinal and a transverse energy associated with the two types of motion. The transverse energy divided by the frequency of spiraling is an adiabatic invariant. For the axisymmetric case, a model is developed from which the fields, charge density, and velocities can be computed. With a restriction on the boundary conditions, an analytical model is outlined for the case of arbitrary magnetic multipoles.
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INTRODUCTION

Pulsars are astronomical objects which regularly give out radio pulses with a period, which varies from pulsar to pulsar, of about a second. The pulses have variable amplitude but a precise period. Over 100 pulsars are now catalogued, with periods ranging from 33 milliseconds to 4 seconds. D. ter Haar summarizes the observational results.

The first pulsar was discovered late in 1967 by Hewish et al at Cambridge. They used a new radio telescope built to study scintillations, caused by plasma clouds in the solar wind, of radio sources of small angular size. Since the time scale of the scintillations is a fraction of a second, the instrument was ideal for recording pulsar signals.

Some ideas about pulsars are firmly established. Because of the short period and its slow increase with the passage of time due to loss of energy, a rotating neutron star is most certainly the central object. Its rotational energy is coupled to its surroundings by a huge magnetic field, having a value at the surface of \( B_s = 10^{12} \) Gauss. The reason for the pulsation is the asymmetry introduced by non-alignment of the magnetic dipole and spin axes. A coherent emission process is necessary for the intense radio emission. However, the emission mechanism and the structure of the magnetic fields and of matter surrounding the neutron star is poorly understood. Many different theoretical models have been proposed but no conclusive model has yet emerged.
In the following work, a particular, self-consistent, steady state model for the pulsar magnetosphere is presented. The emission mechanism is not considered here.
The neutron star is a rotating, conducting body and thus will have an interior electric field:

\[ E = -\nabla \psi \times \hat{B} \]

where \( \nabla = \Omega \hat{\phi} \) in cylindrical polar coordinates (\( \hat{\rho}, \hat{\phi}, \hat{z} \) unit vectors) centred on the neutron star, with the \( z \)-axis along the rotation axis. \( \Omega \) is the angular frequency of rotation. For an exterior dipole magnetic field, an interior field is \( \mathbf{B} = B_0 \hat{z} \) with \( B_0 \) a constant. Thus from (1) one obtains:

\[ E = -\left( \Omega B_0 / c \right) \hat{\phi} \]

This can be obtained from the interior electrostatic potential

\[ \Psi = \Omega^2 B_o / 2c + \Phi_0 \]

\( \Phi_0 \) a constant, by \( E = -\nabla \psi \). For a typical pulsar \( B_0 \) is of the order of \( 10^{12} \) gauss, the radius \( R \) is approximately 7 to 10 kilometers and \( \Omega \) is of the order of 20 sec \(^{-1} \). This gives a potential difference between poles and equator of \( \Delta \Psi = \Omega R^2 B_0 / 2c = 10^{17} \) volts. More precisely \( \Delta \Psi = 3.1 \times 10^{16} B_{12} R_6^2 / P \) volts, where \( B_{12} = B_0 / 10^{12} \) gauss, \( R_6 = R / 10^6 \) cm and \( P = 2\pi / \Omega \) is the period of rotation.

Early pulsar models assumed a vacuum surrounding the magnetic neutron star. The exterior field in this case is given by solving Laplace's equation: \( \nabla^2 \Psi = 0 \), and fitting this solution to the boundary conditions at the neutron star surface. The tangential component of \( E \) is continuous across the surface. In spherical polar coordinates it is, from (2):

\[ E_{\text{tan}} = -\left( \Omega R B_0 / c \right) \sin \theta \cos \theta \hat{\phi} \]

One obtains for the exterior electrostatic potential

\[ \Psi = \left( B_0 \Omega R^3 / 3c r^3 \right) P_2(\cos \theta) + \left( C_R + \Omega R^3 B_0 / 3c \right) 1/r \]
where $P_2$ is the second Legendre polynomial and $C_Q$ is a constant. This results in a surface charge on the neutron star due to the discontinuity in the normal component of the electric field, of:

\[ \sigma = -(B_0 R/4\pi c)(5\cos^2\theta - 3)/2 + (C_Q R + \Omega R^3 B_0/3c)/4\pi R^2 \]

The second term vanishes if the net charge on the neutron star is zero ($Q = C_Q R + \Omega R^3 B_0/3c$). The resulting electric field ($Q = 0$) is:

\[ \vec{E} = -\nabla \phi = -(B_0 \Omega R^5/c^4) P_2(\cos\theta) \hat{r} - (B_0 \Omega^2 R^5/2c^4) \sin 2\theta \hat{\theta} \]

The magnitude of the surface component parallel to the magnetic field $B$ is

\[ E = \Omega^3 R/c \approx 6 \times 10^{10} B_{12} R_6/P \text{ volts/cm} \]

To understand what effect this enormous electric field would have on the surface material of the neutron star, one must examine other factors. The gravitational field on a neutron star is large. The attractive force at the surface is:

\[ F_g = 1.6 \times 10^{-13} M/R^2 \text{ dynes} \]

for an electron mass ($M$ is the neutron star mass in solar mass units) and correspondingly greater for ions (about $10^5$ times, assuming nuclei near Fe$^{56}$). The ratio of electromagnetic to gravitational force is:

\[ eE/F_g = 7.5 \times 10^{11} B_{12} R_6^3/PM \]

Thus gravitational binding is insignificant. It has been shown that Fe$^{56}$ nuclei in a magnetic field of $2 \times 10^{12}$ Gauss will form an anisotropic, very tightly bound lattice with a binding energy of $\varepsilon_f \approx 14 \text{ kev per ion}$ and $\varepsilon_e \approx 750 \text{ eV per electron}$. Lattice spacing is of the order of $10^{-9} \text{ cm}$. To remove ions requires a field of

\[ E_0 = \varepsilon_f / Ze \approx 5 \times 10^{11} \text{ volts/cm} \]
The nuclear charge, $Z$, of Fe$^{56}$ is 26, and $l$ is the lattice spacing. Compare this to the maximum electric field available, calculated for vacuum in equation (8). Probably only the fastest pulsar (the Crab pulsar, with $P = 0.033$ sec) will be able to pull ions from its surface. Electrons are removed more easily than ions but the details have not yet been calculated. In any case, it is assumed that a source of charged particles for the magnetosphere exists and serves to reduce the huge value of $E \cdot B$ in (8). A possible source is the region outside the magnetosphere. More likely, particles are trapped in the magnetosphere during the formation of the pulsar in a supernova explosion, so that a huge $E \cdot B$ never develops.
Another 'force' enters the problem. Due to the enormous rotating magnetic field and the propensity of charged particles to follow field lines, an inertial force due to the acceleration of the particles arises. For a particle in corotation with the central neutron star at distance from the axis \( r \), the energy is \( \gamma \phi m c^2 \), with:

\[
\gamma \phi = (1 - \frac{\Omega^2 r^2}{c^2})^{-\frac{1}{2}}
\]

This becomes singular at the 'light cylinder' where the corotation velocity \( \Omega r \) is the speed of light \( c \). The magnetic energy density is much greater than \( \gamma \phi m c^2 : \quad B^2/8\pi \gg \gamma \phi m c^2 \), except for very close to \( r = c/\Omega \). For an electron these are equal for \( \gamma \phi = B^2/8\pi mc^2 = 2 \times 10^5 B_6^2 \), where \( B_6 \) is \( B \) in units of \( 10^6 \) gauss. Thus \( \Omega r \) is very nearly \( c \) at this point. Estimates for \( B \) at the light cylinder (from energy loss by a radiation dipole) vary from \( 10^6 \) gauss for the Crab pulsar downwards.

The force free assumption is:

\[
\vec{E} + \vec{v}/c \times \vec{B} = 0
\]

i.e. that the electromagnetic forces on a particle vanish. We neglect non-electromagnetic forces and assume zero mass particles. This is a good approximation everywhere in the magnetosphere (which is here considered as being limited to the region between the neutron star and the light cylinder) except very near the light cylinder.
The basic equations are the force free assumption (13), and Maxwell's equations:

(14) \[ \mathbf{v} \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} = 0 \quad \nabla \times \mathbf{B} = 4\pi/c \mathbf{j} \quad \mathbf{v} \cdot \mathbf{E} = 4\pi \rho \]

Note that because only steady state \((\partial / \partial t = -\Omega \partial / \partial \phi)\) in the stationary reference frame is being considered, in combination with axial symmetry \((\partial / \partial \phi = 0)\), Maxwell's equations take the form (14), i.e. \(\partial / \partial t = 0\). In addition we assume only toroidal particle velocities and that the velocities of all particles at one point are the same

(15) \[ \mathbf{v} = \omega (r, z) \mathbf{r} \quad \phi \]

The current is given by

(16) \[ \mathbf{j} = \rho \mathbf{v} \]

Cylindrical coordinates are used. Since \(\nabla \times \mathbf{E} = 0\), one writes \(\mathbf{E} = -\mathbf{V} V\) or:

(17) \[ E_r = -\partial \mathbf{V} / \partial r \quad E_z = -\partial \mathbf{V} / \partial z \]

\(\mathbf{v} \cdot \mathbf{B} = 0\) gives \(\partial (r B_z) / \partial z + \partial (r B_r) / \partial r = 0\). Thus one can define \(\psi\) by:

(18) \[ B_r = 1/r \partial \psi / \partial z \quad B_z = -1/r \partial \psi / \partial r \]

The flux through a circle perpendicular to the z-axis is

\[ \int_0^R B_z 2\pi r dr = -2\pi \int_0^R \partial \psi / \partial r dr = -2\pi \psi (R, z) - \psi (0, z) \]

The flux through the sides of a cylinder about the z-axis from \(z = 0\) to \(z = Z\) is

\[ \int_0^Z B_r 2\pi r dz = 2\pi \psi (r, Z) - \psi (r, 0) \]

Thus the stream function \(\psi\) is directly related to the magnetic flux.

Using (17) & (18), equation (13) has components \(-\partial \mathbf{V} / \partial r \cdot (\omega r / c) \partial \psi / \partial r = 0\) and \(-\partial \mathbf{V} / \partial z + (\omega r / c)(-1/r) (\partial \psi / \partial z) = 0\). Therefore one has:

(19) \[ \partial \mathbf{V} / \partial r = (-\omega / c) \partial \psi / \partial r \quad \partial \mathbf{V} / \partial z = (-\omega / c) \partial \psi / \partial z \]
From (19) one gets
\[ \frac{\partial^2 V}{\partial r \partial z} = \frac{\partial}{\partial z} \left( \frac{\omega}{c} \frac{\partial \psi}{\partial r} \right) = \frac{\partial}{\partial r} \left( \frac{\omega}{c} \frac{\partial \psi}{\partial z} \right) \]
or
\[ \frac{\omega}{\partial z} \frac{\partial \psi}{\partial r} - \frac{\omega}{\partial r} \frac{\partial \psi}{\partial z} = 0. \]
But this is just the Jacobian
\[ J(\omega, \psi, r, z) = 0, \]
so that \( \omega \) is a function of \( \psi \). Now, one has:
\[ (20) \quad \nabla \psi \cdot \mathbf{B} = \frac{\partial \psi}{\partial r} \left( \frac{1}{r} \right) \frac{\partial \psi}{\partial z} + \frac{\partial \psi}{\partial z} \left( -\frac{1}{r} \right) \frac{\partial \psi}{\partial r} = 0 \]
so that \( \psi \) and thus \( \omega \) are constants along field lines. Since the field lines originate in the neutron star which is rigidly rotating at angular frequency \( \Omega \), we must have \( \omega = \Omega \) everywhere. I.e. the magnetosphere rigidly corotates with the neutron star.

From (19) we have:
\[ (21) \quad V = \left( -\frac{\Omega}{c} \right) \psi \]

From the inhomogeneous Maxwell's equations (14) one gets:
\[ (22) \quad \nabla \times \mathbf{B} = \left( \frac{\partial B_z}{\partial r} - \frac{\partial B_r}{\partial z} \right) = \left( -4\pi / \mu \right) \left( \frac{1}{r} \right) \left( \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} \right) = \left( 4\pi / c \right) \frac{\partial \Omega r}{\partial r} \frac{\partial \phi}{\partial z} \]
\[ (23) \quad \nabla \cdot \mathbf{E} = -\nabla^2 V = \left( 4\pi / c \right) \frac{\partial \Omega r}{\partial r} \frac{\partial \phi}{\partial z} = 4\pi \rho \]
Eliminating \( \rho \) gives a differential equation for \( V \):
\[ (24) \quad \left( 1 - \frac{\Omega^2 r^2}{c^2} \right) \left( \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} \right) = \left( 4\pi / c \right) \left( 1 + \frac{\Omega^2 r^2}{c^2} \right) \frac{\partial V}{\partial r} = 0 \]
The solution of this equation is obtained in appendix 1. The result is:
\[ (25) \quad V(r,z) = \left( \frac{\Omega r}{c} \right) \left( \frac{1}{\mu} \right) \int_{-\infty}^{\infty} X(1 - \frac{\Omega^2 r^2}{c^2}) / X(1) \ e^{iaz/c} \ da \]
where \( \mu \) is the magnetic dipole moment of the neutron star and \( X(1 - \frac{\Omega^2 r^2}{c^2}) \) is given in appendix 1. The boundary conditions assumed were a dipole magnetic field at the origin, and finite fields at the light cylinder.

For each value of \( (r,z) \), the series solution for \( V \) was evaluated using a hand calculator and pencil, to 3 figures of accuracy, for 5 values of \( a \). The integration in (25) was then carried out using Simpson's Rule for numerical integration. The calculation results
are presented in table 1. Magnetic field lines are parallel to lines of constant V and electric field lines perpendicular to lines of constant V. Figures 1, 2 and 3 illustrate the magnetic and electric fields and the charge density. As x→1 i.e. as one approaches the light cylinder, ψ = 1 - \frac{\Omega^2 r^2}{c^2} approaches zero so that X(ψ) approaches a constant plus terms of second degree and higher in ψ (see appendix 1). Thus:

\[ B_z = -\frac{1}{r} \frac{\partial \psi}{\partial r} = -(1/r)(dv/dr)\frac{\partial \psi}{\partial \psi} = \times \text{ approaches zero. I.e. field lines approach the light cylinder horizontally. Also from the fact that } X(\psi) \text{ involves only even powers of } \alpha, \text{ one sees that } B_r = 1/r \frac{\partial \psi}{\partial \psi} \text{ is zero at } z=0. \text{ This results in a null point in the magnetic field at } z=0,r=c/\Omega \text{ (see figure 1).}

The charge density can be computed from (23), using (24) to eliminate the second derivative of V. Using the definition of \( \psi \) (18), the result is:

\[ 4 \pi \rho = (-2 \frac{\partial \psi}{\partial \psi}) \frac{\partial^2 \psi}{\partial^2 \phi} B_z \]

Thus \( \rho \), the charge density, is directly proportional to the z component of the magnetic field.
### TABLE I  NUMERICAL EVALUATION OF $V^*(x,y)$, WHERE

$$V(x,y) = \frac{\mu}{\pi} \left( \frac{\mu}{c} \right)^2 V^*(x,y)$$

<table>
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<th>$x$</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
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<tr>
<td>0</td>
<td>singular</td>
<td>6.0</td>
<td>4.3</td>
<td>3.2</td>
<td>2.9</td>
</tr>
<tr>
<td>0.25</td>
<td>0</td>
<td>6.0</td>
<td>2.9</td>
<td>2.5</td>
<td>1.8</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>-0.2</td>
<td>0.8</td>
<td>1.2</td>
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</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>0.5</td>
<td>0.1</td>
<td>-0.1</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

### FIGURE 1  MAGNETIC FIELD LINES; AXISYMMETRIC

FORCE FREE CASE
FIGURE 2: ELECTRIC FIELD LINES, AXISYMMETRIC FORCE FREE CASE

FIGURE 3: CHARGE DENSITY, AXISYMMETRIC FORCE FREE CASE
COVARIANT FORMULATION

So far we have considered only the axisymmetric situation. In that case all field quantities are time independent. For the more general case of non-aligned spin and magnetic dipole axes, or the existence of higher order multipoles, time independence is invalidated, except in the reference frame corotating with the neutron star. However this frame is non-inertial and even becomes singular at the light cylinder. In order to express the equations in their correct form we use the covariant formalism of general relativity.

We work in the coordinate system rotating at the angular velocity of the pulsar using cylindrical coordinates \( \vec{x} = (t, z, r, \phi) \). The non-rotating coordinates are: cartesian \( \vec{x} = (t, x, y, z) \); and cylindrical \( \vec{x}_c = (t, z, r, \phi_c) \). Thus \( \phi = \phi_c + \Omega t \). The metric is then

\[
\begin{align*}
\text{ds}^2 &= g_{\alpha\beta} \text{d}x^\alpha \text{d}x^\beta = g_{\alpha\beta} \text{d}x^\alpha \text{d}x^\beta \\
&= c^2 (\text{dt})^2 - (\text{dx})^2 - (\text{dy})^2 - (\text{dz})^2 \\
&= c^2 (\text{dt})^2 - (\text{dz})^2 - (\text{dr})^2 - (r \text{d}\phi)^2 \\
&= (c^2 - \Omega^2 r^2) (\text{dt})^2 - (\text{dz})^2 - (\text{dr})^2 - (r \text{d}\phi)^2 - \Omega^2 r^2 \text{d}t
\end{align*}
\]

Thus one has:

\[
(27) \quad g_{\alpha\beta} = \begin{bmatrix}
c^2 - \Omega^2 r^2 & 0 & 0 & -\Omega^2 r^2 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-\Omega^2 r^2 & 0 & 0 & -r^2
\end{bmatrix}
\]
and its inverse: $$g_{\alpha\beta} = \begin{bmatrix} 1/c^2 & 0 & 0 & -\Omega/c^2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\Omega/c^2 & 0 & 0 & \Omega^2/c^2 - 1/r^2 - 1/(\phi^2 r^2) \end{bmatrix}$$

The Einstein summation convention has been used.

Greek indices range from 0 to 3, Latin indices from 1 to 3.

The field variables $F^{\mu\nu}$ are expressed in terms of the fields $E$ and $B$ in the stationary frame, in appendix 2. The covariant current density has the components, in an inertial frame, $J^\alpha = (\rho, \nu)$ and satisfies the equation of continuity (conservation of charge):

$$J^\alpha_{\beta;\alpha} = 0$$

The comma (semicolon) denotes ordinary (covariant) differentiation.

Maxwell's equations are then:

$$(30) \quad F^{\mu\nu}_{\;\;\nu} = (\pi/c) J^\mu$$

$$(31) \quad \epsilon^{\alpha\mu\nu\lambda} F_{\mu\nu\;\lambda} = 0$$

(30) can also be written:

$$(32) \quad (4\pi/c)(-g)^{\frac{1}{2}} J^\mu = ((-g)^{\frac{1}{2}} f_{\mu\nu})_{\nu}$$

where $g$ = determinant $g_{\alpha\beta} = -c^2 r^2$ for the metric (27).
THE FORCE FREE MAGNETIC POTENTIAL

We now assume corotation so that the current in the rotating frame is zero: \( J^k = 0 \). This assumption has some justification: 1) It holds in the axisymmetric case, of which this is a generalization. 2) A highly conducting magnetosphere will have the magnetic flux frozen into it, thus will be dragged by the field. The field is so large that it will be locked into corotation.

Since we have time independence in the rotating frame (the steady state assumption), (32) reads:

\[ (rF^{kl})_{,l} = 0 \]  

Since \( F \) is an antisymmetric tensor, write \( rF^{kl} = \varepsilon^{klm} A_m \). Then (33) becomes

\[ A_m,1 - A_1,m = 0 \]  

so that \( A \) is derivable from a potential: \( A_m = \chi_m \). Thus we have:

\[ F^{kl} = \varepsilon^{klm} (1/r) \chi_m \]  

\( \chi \) is called the magnetic potential.

Now we use the force free assumption, i.e. the Lorentz force \( f \) on each particle vanishes: \( f = \mu / \varepsilon F \) \( u \nu = 0 \) where \( u \) is the particle's velocity. As \( \rho u_\nu \) is \( J^\nu \) this gives:

\[ F_{\alpha \nu} J^\nu = 0 \]  

We use the metric (27) to relate \( J^\nu \) to \( J^\nu = \varepsilon_{\nu \lambda} J^\lambda \), so that we have:

\[ J^\nu = (c^2 - \Omega^2 r^2) J^\nu \quad J_1 = J_2 = 0 \quad J_3 = -\Omega r^2 J^\nu \]  

One gets from (35): \( (F_{\alpha \nu} (c^2 - \Omega^2 r^2) - F_{\alpha \nu}) J^\nu = 0 \) or since \( J^\nu = \rho \) (in the rest frame) is not identically zero, we have:

\[ F_{\alpha \nu} \varepsilon_{\nu \lambda} J^\lambda = 0 \]  

The covariant form of (35): \( \varepsilon_{\nu \lambda} J^\lambda = 0 \) gives:

\[ F_{\alpha 0} = 0 \]
Thus the homogeneous Maxwell's equations are just:

\[(39) \quad F_{12}' 3 + F_{23}' 1 + F_{31}' 2 = 0\]

We know \(F_{\alpha \beta}\) in terms of \(\chi\) from (34) and (37). To get the desired equation for \(\chi\) from (39) use:

\[(40) \quad F_{ij} = g_{i \alpha} g_{j \beta} F_{\alpha \beta}\]

The results are:

\[(41) \quad F_{12}' = F_{12}^1 = 2F_{10}^1 + F_{12}^1 = (\gamma_0^2 r^2/c^2) \gamma_2^2 + 1) r^2 F_{13}' = r^2 \gamma_0^2 F_{13}\]

\[(42) \quad F_{23}' = F_{23}^2 = 2F_{20}^2 + F_{23}^2 = (\gamma_0^2 r^2/c^2) \gamma_2^2 + 1) r^2 F_{23}' = r^2 \gamma_0^2 F_{23}\]

Thus (39) becomes \((/(r x, 3), (r y^2 x, _1), (r y^2 x, 2), 2 = 0\) or:

\[(42) \quad \gamma_0^2 \chi / \partial \phi^2 + r^2 \gamma_0^2 (\partial^2 \chi / \partial z^2 + \partial^2 \chi / \partial r^2) = \gamma_0^2 (2\gamma_0^2 - 1) \partial \chi / \partial r = 0\]

The solution to this equation is obtained in appendix 3. For a dipole magnetic field, with axis oriented at angle \(\theta_0\) with respect to the rotation axis (z-axis), at the neutron star. The second boundary condition was the requirement of finite fields at the light cylinder. No attempt was made in matching the solution within the magnetosphere to an outgoing wave solution in the region outside the magnetosphere. This is a possibility for further study. The magnetic potential obtained for the stated boundary conditions is:

\[(43) \quad \chi (z, r, \phi) = \sum_{-\infty}^{\infty} R_0 (v) e^{i\alpha y} \varphi + (c^{i\phi} + e^{-i\phi}) \sum_{-\infty}^{\infty} R_1 (v) e^{i\alpha y} \varphi \]

\[= (\cos \phi + e^{-i\phi}) \sum_{-\infty}^{\infty} R_0 (v) e^{i\alpha y} \varphi \]

where \(R_0 (v)\) and \(R_1 (v)\) are given explicitly in appendix 3: by equations \((23)\) and \((31)\) respectively, where \(v = 1 - \Omega r^2 / c^2\), \(\gamma = \Omega z / c\). \(\mu\) is the magnetic dipole moment of the neutron star.
From \( F^\mu_\nu \), in terms of \( E \) and \( B \) in the stationary frame from appendix 2, we have:

\[
E_z = (\frac{r}{c}) \gamma_\phi \gamma^2 \frac{\partial \gamma}{\partial r} \quad E_r = (\frac{r}{c}) \gamma_\phi \gamma^2 \frac{\partial \gamma}{\partial z} \quad E_\phi = 0
\]

\[
B_z = \gamma_\phi \gamma^2 \frac{\partial \gamma}{\partial z} \quad B_r = \gamma^2 \frac{\partial \gamma}{\partial r} \quad B_\phi = \frac{1}{r} \gamma^2 \frac{\partial \gamma}{\partial \phi}
\]

(44) gives \( E \cdot \nabla \gamma = 0 \): lines of constant \( \gamma \) are electric field lines. Also in the \( r, z \) plane, \( B \) is proportional to \( \nabla \gamma \). The charge density \( \rho \) can be calculated using: \( \frac{d}{dr}(r^2 \gamma^2) = 2r \gamma^4 \) to get:

\[
4 \pi \rho = v \cdot E = (-1/r^2 \gamma^2) \nabla \left( \frac{\gamma^2}{r} \frac{\partial \gamma}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\gamma^2}{r} \frac{\partial \gamma}{\partial r} \right)
\]

\[
= (-2 \Omega/c) \gamma^4 \frac{\partial \gamma}{\partial z} = -\left( \frac{2 \Omega/c}{r} \right) \gamma^2 B_z
\]

Thus the relation between \( \rho \) and \( B_z \) found for the axisymmetric case (26) is still valid in the more general case.

In figure 4 are shown lines of constant \( \gamma \) for the axisymmetric case. Note that these are orthogonal to lines of constant \( \psi \), the stream function, as given in figure 2.

For the \( m=1 \) case, a dipole oriented at 90° with respect to the rotation axis, see figure 5. The electric and magnetic fields and charge density in the plane containing both rotation and dipole axes, are shown in figures 6, 7 and 8. The electric field is basically a quadrupole field. As one approaches the light cylinder, \( v \) approaches 0.

Thus from (43) we have:

\[
\chi = \left( -\mu/\eta \right) (\Omega/c)^2 v^2 \cos \phi \int_\alpha \left( \frac{\gamma^2}{c} \right) \left( 1 + \frac{\alpha}{c} \right) \frac{i \gamma}{(c \beta)} d \alpha
\]

Using (40) and \( \alpha/\gamma = 2(v-1)(\gamma-1) \): one obtains:

\[
E_z = -\frac{\mu}{\eta} (\Omega/c)^3 \left( v - 1 \right) \gamma \cos \phi \int_\alpha \left( \frac{\gamma^2}{c} \right) \left( 1 + \frac{\alpha}{c} \right) \frac{i \gamma}{(c \beta)} d \alpha
\]

\[
E_z = -\frac{\mu}{\eta} (\Omega/c)^3 (1-v)^{\frac{1}{2}} \gamma \cos \phi \int_\alpha \left( \frac{\gamma^2}{c} \right) \left( 1 + \frac{\alpha}{c} \right) \frac{i \gamma}{(c \beta)} d \alpha
\]

(48) Also from (44) & (45), as \( v \to 0 \), one has \( B_z \to B_z \) and \( B_z \to E_z \). \( E_z \) and \( B_z \)
are zero at the light cylinder. Thus at the light cylinder where $E_z$ changes sign (see figure 6), there is a null in the electric and magnetic fields. The null follows two circles parallel to the equatorial plane, since the $\phi$-dependence is contained entirely in the $\cos\phi$ term of (47).

Some estimates of the order of magnitude of the various quantities can now be given. The magnetic dipole moment of the neutron star is roughly $B_s R^3 \approx 10^{-30}$ esu for $B\approx 10^{12}$ gauss, $R=10^6$ cm. From (44) & (45) we have, for regions intermediate between the neutron star and the light cylinder: $E \approx (\Omega/c)^3 \approx 10^3$ esu $= 10^5$ volts/cm for $\Omega=30$ sec$^{-1}$. Then we have $B \approx E \approx 10^3$ Gauss, and $\rho \approx (\Omega/c)B \approx 10^{-6}$ esu $= 10^4$ electronic charges per cm$^3$. 
TABLE II: NUMERICAL EVALUATION OF $X(x, y)$, WHERE

$$X(x, y) = \left(\frac{4}{n}\right) (a/c)^2 X(x, y) \cos \phi$$

<table>
<thead>
<tr>
<th>$y$</th>
<th>$x$</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>singular</td>
<td>-50</td>
<td>-11</td>
<td>-1.8</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\pi/8$</td>
<td>0</td>
<td>-11</td>
<td>-5.2</td>
<td>-1.2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>0</td>
<td>6.0</td>
<td>-0.1</td>
<td>-0.2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>0</td>
<td>25</td>
<td>4.0</td>
<td>0.6</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
FIGURE 5: LINES OF CONSTANT X, FORCE FREE m=1 CASE
FIGURE 6: ELECTRIC FIELD LINES, FORCE FREE $m=1$ CASE
FIGURE 7: MAGNETIC FIELD LINES, FORCE FREE m=1 CASE
FIGURE 8: CHARGE DENSITY, FORCE FREE m=1 CASE
RELAXATION OF THE FORCE FREE ASSUMPTION

Up until now all effects of a finite particle mass have been ignored because of the presence of huge electromagnetic fields. To maintain corotation net electromagnetic force on the particles is necessary. This means the charge to mass ratio of all particles in any arbitrarily small volume must be the same. Thus we assume charge separation and the identity of all particles at a given point in the magnetosphere.

In rotating cylindrical coordinates the Lagrangian for single particle motion is:

\[
L = -mc^2/\gamma - (q/c) (A_1 \dot{x} + A_2 \dot{r} + A_3 \dot{\phi} + A_0)
\]

where \( \dot{} = \frac{d}{dt} \)\, m and q are the mass and charge of the particle. We define

\[
\gamma = c \left( c^2 - \Omega^2 r^2 - z^2 - \Omega r^2 \phi - r^2 \phi^2 \right)^{-1/2}
\]

Since \( L \) does not explicitly depend on time (assuming that the covariant four potential \( A_\mu \) is time independent in the rotating frame), the energy \( H \) is conserved:

\[
H = (\partial L/\partial \dot{x}^i) \dot{x}^i - L = m((c^2 - \Omega^2 r^2) - \Omega r^2 \phi + \Omega^2 r^2)\frac{q}{c} A_0
\]

The Lagrangian equations of motion are:

\[
\begin{align*}
\frac{d}{dt} (\gamma \dot{z}) &= (q/c) F_{1\alpha} \dot{x}^\alpha \\
\frac{d}{dt} (\gamma \dot{r}) - \gamma \dot{r} (\Omega + \phi) &= (q/c) F_{2\alpha} \dot{x}^\alpha \\
\frac{d}{dt} (\gamma \dot{\phi} (\phi + \Omega)) &= (q/c) F_{3\alpha} \dot{x}^\alpha
\end{align*}
\]

Corotation has: \( \dot{z} = \dot{r} = \dot{\phi} = 0 \). Using \( F_{\alpha \beta} = \partial A_\alpha / \partial x^\beta - \partial A_\beta / \partial x^\alpha \), (52) reads

\[
0 = \partial A_\alpha / \partial z \\
-\gamma \dot{r} \Omega^2 = (q/c) \partial A_\alpha / \partial r \\
0 = \partial A_\alpha / \partial \phi
\]

Thus apart from a trivial constant we have:

\[
A_\alpha = (mc^2/q) (c^2 - \Omega^2 r^2)^{1/2} = (-c/q) mc^2/\gamma
\]

This provides the necessary electromagnetic field to hold the particles in corotation.
The previous force free models involving the stream function \( \psi \) (axisymmetric case) and the magnetic potential \( \chi \) can now be modified.

For the former case write:

\[
(54) \quad A^\mu = \left( (-c/q) mc^2/\gamma_\phi, 0, 0, \psi \right)
\]

so that \( B \) is given by (18) and \( E \) by (17) when \( m=0 \) (see appendix 2 for the relation between \( F^{\alpha \beta} \) and \( E \& B \)). Using \( g^{\alpha \beta} \) from (28) one gets:

\[
(55) \quad F_{01} = (-1/c^2) F_{01} + (\Omega/c^2) F_{31} = \left( \Omega / c^2 \right) \partial \psi / \partial z \\
F_{02} = (-1/c^2) F_{02} + (\Omega/c^2) F_{32} = (-c/q) (\Omega^2 r/c^2) \gamma_\phi (\Omega/c^2) \partial \psi / \partial r \\
F_{03} = 0 \\
F_{12} = F_{12} = 0 \\
F_{23} = (\Omega/c^2) F_{20} + (1/r^2 \gamma_\phi^2) F_{23} = (-m c \Omega / q) (\Omega^2 r/c^2) \gamma_\phi (\gamma_\phi^2 r^2) \partial \psi / \partial r \\
F_{31} = (\Omega/c^2) F_{01} + (1/r^2 \gamma_\phi^2) F_{31} = (1/r^2 \gamma_\phi^2) \partial \psi / \partial z
\]

Thus Maxwell's equations:

\[
(56) \quad (4 \pi/c) F_{\mu \nu} = \nabla \times \mathbf{J} = \mathbf{E}
\]

with \( J^0 = \rho \) and \( J^1 = 0 \) (corotation) read:

\[
(57) \quad \mu = 0: \quad (4 \pi/c) \rho = \nabla \times (\nabla \times (\nabla \times \gamma_\phi^2) \partial \psi / \partial z) + \nabla \times (\nabla \times (\nabla \times \gamma_\phi^2) \partial \psi / \partial r) \\
\mu = 3: \quad 0 = \nabla \times (1/\gamma_\phi^2 r) \partial \psi / \partial z + \nabla \times (m c \Omega / q) (\Omega^2 r/c^2) \gamma_\phi (1/\gamma_\phi^2 r) \partial \psi / \partial r
\]

These give, respectively, the equations for \( \psi \) and \( \rho \) when inertia is included. When the mass \( m \) is zero, these reduce to (23) and (24), using (21). To find the solution to (57) write:

\[
(58) \quad \psi = \psi_h + \psi_p
\]

where \( \psi_h \) is the solution to the homogeneous equation (\( m=0 \)) as given by (21) and (25). Since the inhomogeneous term is a function of \( r \) alone, we have: \( d/dr(1/\gamma_\phi^2 r) d\psi_p/dr + (mc \Omega / q)(\Omega^2 r/c^2) \gamma_\phi = 0 \). Setting the constant of integration to 0, we get:

\[
\frac{d\psi_p}{dr} = -mc \Omega / q)(\Omega^2 r/c^2) \gamma_\phi
\]
\[ \psi_p = (-mc^2/\Omega q) (\gamma^2 + 1/\gamma) \]

For the magnetic potential, \( F_{ij} \) are known from (34) in terms of \( \chi \) but \( F_{0j} \) are not. We calculate \( F_{0j} \) as follows:

\[ F_{01} = \left(-c^2/r^2\right) F_{01} (\Omega/c^2) F_{31} = 0 \phi r (\Omega^2 F_{01} + r F_{31}) \] gives \( F_{01} = \gamma^2 \left(\Omega r/c^2\right) \partial \chi/\partial r \)

Similarly \( F_{02} = -\gamma^2 \left(c/r^2\right) (m^2 r/c^2) \gamma^2 \left(\Omega r/c^2\right) \partial \chi/\partial z \) and \( F_{03} = 0 \) are found.

The only Maxwell's equation that changes is for \( \mu = 0 \), in (56):

\[ (4\pi r/c) \rho = -mc^2/(r/c^2) \gamma^2 \left(\gamma^2 - 1\right) - \left(\Omega/c^2\right)^2 r \gamma^4 \partial \chi/\partial z \]

The equation for \( \chi \) given by the \( \mu = 3 \) equation of (56) is unchanged so that \( F_{ij} \) are unchanged. Thus only \( \rho \) and \( F_{01} \) are modified.

However, referring to appendix 2, one sees that both \( E \) and \( B \) are altered.

In both cases, for \( \psi \) and \( \chi \), one has the result:

\[ 4\pi \rho (q/c/\Omega) \gamma^2 = -mc^2 \gamma^2 (\gamma^2 + 1) - 2qB_z \]

The left hand side of (60) is never negative (\( \rho q > 0 \)). Thus we have

\[ 2qB_z < -mc^2 \gamma^2 (\gamma^2 + 1) < 0 \] so that \( q \) and \( B_z \) are always of opposite sign. Thus one has \( |B_z| > mc^2 (\gamma^2 + 1) \). In this model \( q \) may have different values in different regions. Field lines along which \( q \) does not change sign do not have a change in the sign of \( B_z \). If \( q \) changes sign, \( B_z \) must change discontinuously, resulting in a cusp in the field line. Since \( \rho = 0 \) at the point of change of \( q \), (60) yields

\[ \Delta B_z = c\Omega (m_2/2q_2 - m_1/2q_1) \gamma^2 (\gamma^2 + 1) \]

For a change from electrons to protons in the intermediate magnetosphere, one has:

\[ \Delta B_z = c\Omega (m_p + m_e)/e^2 2 \times 10^{-3} \text{ gauss (for } \Omega = 20 \text{ sec}^{-1}) \].
Thus the cusp in the field lines is negligible until one gets close to the light cylinder. From (61) and the estimate of \( B \) on page 17: \( \frac{B_z}{B} \approx 10^{-3} \frac{\gamma_\phi^3}{10^3} \). This is of the order of unity when \( \gamma_\phi = 10^6 \). At this point the particle energy, \( \gamma_\phi mc^2 \), exceeds the magnetic energy density. Corotation is no longer valid, so this model is no longer valid. Neglecting the fact that the model is no longer physical for such large values of \( \gamma_\phi \), the change in field lines is illustrated in figure 9 for the axisymmetric case. Compare this to figure 1. Now the field lines are not horizontal at the light cylinder but vertical and thus do not penetrate the light cylinder.
FIGURE 9: MAGNETIC FIELD LINES FOR AXISYMMETRIC COROTATION CASE WITH INERTIA
If one no longer assumes corotation, what particle motions can occur and how will the fields be altered? To answer this question one proceeds as follows: Calculate the single particle motion in an arbitrarily given field assuming small departures from corotation. I.e. particle velocities in the rotating frame are assumed to be of order $\epsilon$ where $\epsilon << 1$. Next make assumptions which give the collective motion in terms of the single particle motion and use this to calculate altered fields. The altered fields can be used to recalculate the single particle motion. One repeats this procedure until the desired accuracy of results in orders of $\epsilon$ is attained. The model is then self-consistent to that order.
SINGLE PARTICLE MOTION

For single particle motion we assume the fields are such that corotation is possible. I.e. we imagine that all the particles except the test particle in question are in corotation. Then by (53) we have \((q/c)A_o = -mc^2/\gamma_\phi\) since all particles at any particular place in the magnetosphere are identical, with mass \(m\) and charge \(q\).

We now write the single particle Lagrangian (49):

\[
L = -mc^2/\gamma + mc^2/\gamma_\phi \cdot \frac{q}{c} (A_1 \dot{z} + A_2 \dot{r} + A_3 \dot{\phi}) = (m/2X) (z^2 + r^2) + \gamma_\phi r^2 \dot{\phi}^2 - (q/c) A_1 \dot{z} - (q/c) A_2 \dot{r} - (q/c) A_3 \dot{\phi} + \text{terms cubic in the velocities}
\]

Since we are assuming the velocities are of order \(\varepsilon, \varepsilon \ll 1\), we keep terms to second order and write \(L\) in the form:

\[
L = \frac{1}{2} (A \dot{x}^2 + B \dot{y}^2 + C \dot{z}^2) - \Phi_2 \dot{z} - \Phi_3 \dot{\phi} - \Phi_1 \dot{x} - \Phi_2 \dot{y} - \Phi_3 \dot{\phi}
\]

where \((x, y, z) = (z, r, \phi), A = B = m\gamma_\phi, C = m\gamma_\phi^2 r^2, \Phi_1 = (q/c) A_1, \Phi_2 = (q/c) A_2, \Phi_3 = (q/c) A_3 - m\gamma_\phi r^2 \Omega\). Lagrangians of this form are treated in appendix 4. The main results are:

1) The motion consists of a slow drift tangent to streamlines, specified by \(dx/X = dy/Y = dz/Z\) where:

\[
X = (q/c) r B_z + m\Omega r \gamma_\phi (\gamma_\phi^2 + 1) \\
Y = (q/c) r B_r \\
Z = (q/c) B_\phi
\]

and a spiraling motion around these streamlines.

2) The energy \(E\) is conserved.

3) The energy can be separated into a longitudinal energy associated with the drift \(-1/2 p \dot{z}^2\), and a transverse energy assoc-
4) An adiabatic invariant exists for the motion, namely \( M = \frac{N_\sigma^*}{\omega} \), the transverse energy divided by the angular frequency of the spiraling.

\[ P, \omega N \text{ and } \sigma \text{ are defined by equations (8), (9), (10), and (16) of appendix 4. Essentially, } \tfrac{1}{2} P \ell^2 \text{ is the longitudinal kinetic energy} \]

\[ \tfrac{1}{2} m \gamma_\phi \left( \dot{z}^2 + \ddot{r}^2 + r^2 \dot{\gamma}_\phi^2 + \dot{\phi}^2 \right) \text{ and } N_\sigma^* \text{ is the transverse kinetic energy} \]

\[ \tfrac{1}{2} m \gamma_\phi \left( \dot{z}^2 + \ddot{r}^2 + r^2 \dot{\gamma}_\phi^2 + \dot{\phi}^2 \right). \text{ The subscripts refer to motion parallel to, and perpendicular to streamlines.} \]
If, instead of allowing just one particle to move, one allows all to move, then the fields will be altered. In the formalism of the single particle treatment this can be taken into account since the four potential, except for $A_0$, was not assumed to be of a particular form. Thus one only needs to allow for a different $A_0$.

We write:

\begin{equation}
A_0 = (-c/q)mc^2/\gamma_\phi + \tilde{A}_0
\end{equation}

This modifies the Lagrangian $L$ and thus the energy, which becomes:

\begin{equation}
E = \frac{1}{2}p^2 + (q/c)\tilde{A}_0 \quad \text{(for M=0)}
\end{equation}

Thus one sees that $\tilde{A}_0$ is second order in $\epsilon$ since both $E$ and $L^2$ are of order $\epsilon^2$ (see (67) below).

To determine the fields we must make assumptions about how the currents derive from the single particle motion. Since we have steady state, all particles which move along a given streamline must be of the same type. The Coulomb interaction between streaming particles results in no spiraling motion (M=0) and ensures that all particles along the same streamline have the same energy $E$.

If M were not zero, collisions would occur between particles on neighboring streamlines which damp out the transverse motion.

Thus we have convection current only: $j = \rho v$ or:

\begin{equation}
J = \rho(1, \hat{x}, \hat{y}, \hat{z}) = \rho(1, \ell X, \ell Y, \ell Z)
\end{equation}

using (16) of appendix 4 (with $\sigma=0$). $X, Y, Z$ are given by (64). The equation of continuity (y$J^\mu$)$_\mu = 0$ reads:

\begin{equation}
y(\rho \ell X)_x + (y\rho \ell Y)_y + y(\rho \ell Z)_z = 0
\end{equation}
From the definition (2) of appendix 4, one has the identity (corresponding to $v \cdot \mathbf{B} = 0$):

$$X, x + Y, y + Z, z = 0,$$

thus (68) becomes:

$$x(p^\pi y), x + y(p^\pi y), y + z(p^\pi y), z = 0$$

Therefore $p^\pi y$ is constant along the streamlines specified by $dx/X = dy/Y = dz/Z$. We write:

$$p^\pi y = (c^2/4\pi q) K$$

where $K$ is constant along a streamline, the value of which is determined by the boundary conditions. Write $p = p_0 + p_1$ where $p_0$ is zero order in $\epsilon$ and $p_1$ includes all higher orders. (70) gives $\lambda$ to first order in $\epsilon$ as a function of position along a streamline since $p_0$ is known from the corotation case as given by (60). Maxwell's equations (56) written out are:

$$\left( rF^\mu_j \right)_j = (4\pi r/c) J^\mu = (4\pi r_\rho/c)(1, \xi X, \xi Y, \xi Z)$$

$$= (4\pi r_\rho/c, (c/q) KX, (c/q) KY, (c/q) KZ)$$

Equation (70) requires that $K$ and $\lambda$ be zero along a streamline where $p$ changes sign. In steady state, and with charge separation, no streaming can occur along any streamline along which the charge changes sign. From (60) $p_0$ changes sign, to a good approximation except very near the light cylinder, when $B_z$ changes sign. Thus the pulsar must have a corotating 'dead' zone, where no streaming occurs, which very nearly coincides with the region containing field lines which go horizontal inside the light cylinder. This situation is illustrated for the axisymmetric case in figure 10.
FIGURE 10. Streaming and Corotating 'Dead' Zones

Axisymmetric Case

LEGEND -

- Corotation Region

Streaming Region

Note: the sign of charge is indicated by + and - above.
AXISYMMETRIC CASE

Write (72): $\Lambda_\mu = (c^2/q\gamma c^2/\gamma_\phi, 0, 0, \psi) + (\vec{\Lambda}_0, \vec{\Lambda}_1, \vec{\Lambda}_2, \vec{\Lambda}_3)$ so that the zero order part is identical to (54). The terms higher order in $\varepsilon$ are contained in the second set of brackets.

For axial symmetry the partial derivative of any quantity, with respect to $\phi$, is zero. Thus $\vec{\Lambda}_1$ and $\vec{\Lambda}_2$ always appear in the combination $\partial \vec{\Lambda}_1/\partial r - \partial \vec{\Lambda}_2/\partial z = F_{12} = B_\phi$. Now from (64) we have:

(73) $X = (q/c)(\partial \vec{\Lambda}_3/\partial r + \partial \psi/\partial r) + m\omega \gamma_\phi (\gamma_\phi^2 + 1)$

Thus $\vec{\Lambda}_1$ and $\vec{\Lambda}_2$ always appear in the combination $\partial \vec{\Lambda}_1/\partial r - \partial \vec{\Lambda}_2/\partial z = F_{12} = B_\phi$. Now from (64) we have:

(74) $F_{\alpha\beta}^0 = \partial \vec{\Lambda}_\beta/\partial x^\alpha$ is calculated from (72) and $F^{\alpha\beta}$ can be found using $g^{\alpha\beta}$ from (28). The result is:

(75) $\mu = 0$: $(4\pi r/c)(\rho_0 + \rho_1) = \partial/\partial z (c^{-2/r} \partial \vec{\Lambda}_0/\partial z + \partial \vec{\Lambda}_1/\partial z)$

(76) $\mu = 1$: $(4\pi r/c)(\rho_0 + \rho_1) = \partial/\partial r (m/r \partial \vec{\Lambda}_0/\partial r + \partial \vec{\Lambda}_1/\partial r)

(77) $\mu = 2$: $(4\pi r/c)(\rho_0 + \rho_1) = \partial/\partial z (\partial \vec{\Lambda}_0/\partial z + \partial \vec{\Lambda}_1/\partial z)$

(78) $\mu = 3$: $(4\pi r/c)(\rho_0 + \rho_1) = \partial/\partial r (\partial \vec{\Lambda}_0/\partial r + \partial \vec{\Lambda}_1/\partial r)$

The zero order parts of (74) are identical to (55), as necessary for consistency.
The zero order parts of (75) and (78) are the same as (57) and give \( \psi \) and \( \rho_0 \).

Using (70) and (73), (76) and (77) have first order parts:

\[
\frac{c}{q} K \frac{\partial}{\partial r} \left( (-q/c) \psi + m \Omega r^2 \gamma \phi \right) = \frac{\partial}{\partial r} (r B_\phi)
\]

\[
\frac{c}{q} K \frac{\partial}{\partial r} ((q/c) \psi) = -\frac{\partial}{\partial z} (r B_\phi)
\]

We define:

\[
\psi' = \psi - \left( \frac{cm \Omega}{q} \right) r^2 \gamma \phi
\]

so that these equations become:

\[
K \frac{\partial}{\partial x^1} \psi' = -\frac{\partial}{\partial x^1} (r B_\phi) \quad \text{with} \quad (x^1, x^2) = (r, z).
\]

Thus we have the following relation among \( K, \psi', \) and \( B_\phi \):

\[
K = -\frac{d(r B_\phi)}{d\psi'}
\]

Note that \( K, \psi', \) and \( r B_\phi \) are all constant along streamlines (actually the surfaces \( dx/X = dy/Y \) for this case of axial symmetry). From (81) it is seen that in regions where streaming does not occur (\( K = 0 \)), \( r B_\phi \) is constant everywhere.

\( \tilde{A}_0 \) is known from the energy equation (66) to second order since \( \tilde{A}_0 \) is known to first order in \( \varepsilon \) from (70). From (78) we can calculate \( \tilde{A}_3 \) to second order:

\[
(1/\gamma^2_\phi) \frac{\partial^2}{\partial z^2} \tilde{A}_3 + \frac{\partial}{\partial r} ((1/\gamma^2_\phi) \frac{\partial \tilde{A}_3}{\partial r}) = K B_\phi \frac{\psi}{c^2} \nabla^2 \tilde{A}_0
\]

where \( \nabla^2 \) is the Laplacian operator in cylindrical coordinates.

(75) gives \( \rho_1 \) to second order:
In summary: The zero order quantities $\psi$ and $\rho_0$ are given by (57). The first order quantities $K,  \xi$, and $B_\phi$ are given by boundary conditions, (70), and (81) respectively. The second order quantities $\tilde{A}_0, \tilde{A}_3$, and $\rho_1$ are given by (66), (82), and (83).
THE GENERAL CASE

The magnetic potential $\chi$ cannot be used in the case where streaming currents occur since the definition (34), when inserted in Maxwell's equations (56), implies zero current $J^i$.

However this problem can be handled in the following manner. We assume a $z,\phi$ dependence of the form $e^{iu}$, where $u=\bar{m}z+k\phi$ ($\bar{m}$ an integer, $k$ a real number). From Maxwell's equations (71) we have:

$$K_{ij}(rF^{ij})_j = (c/q)K(K_x X + K_y Y + K_z Z)$$

This is identically zero, from (69). We have, from the assumption concerning the $z,\phi$ dependence: $\partial/\partial z = k\partial/\partial u$ and $\partial/\partial \phi = \bar{m}\partial/\partial u$ so that (84) becomes:

$$0 = K_{ij}(rF^{ij})_j = \partial K/\partial u \partial/\partial r (krF^{12} - mrF^{23})$$

$$= J(K, krF^{12} - mrF^{23} ; r, u)$$

I.e. the Jacobian of $K$ and $krF^{12} - mrF^{23}$ with respect to the variables $r$ and $u$ vanishes. Thus the Jacobian of the general stream function $\Lambda$ also vanishes, where $\Lambda$ is defined by:

$$Kk\Lambda = mrF^{23} - krF^{12}$$

Since $K$ is constant along streamlines $dx/X = dy/Y = dz/Z$, $\Lambda$ is also constant along streamlines.

With a $z,\phi$ dependence of $e^{iu}$ the solutions will be helical waves. To get a physically acceptable solution, solutions of different $k$ and $\bar{m}$ must be superposable through Fourier analysis. In this way the total solution can fit the boundary conditions. Thus Maxwell's equations (71) must be linear. This is turn requires $K$ to be constant or at
least piecewise constant in different regions of the pulsar magnetosphere, e.g. $K=0$ for the corotating dead zone and $K=K_0$ (a constant) outside the corotating dead zone. We restrict $K$ to be piecewise constant for the calculation of $A$. In the following the lower case letters refer to Fourier components of the fields:

\[(87) \quad F^\alpha_\beta (z,r,\phi) = \sum_{m=0}^\infty \cos(m\phi) \int_{-\infty}^{\infty} C_m(k) \alpha^\alpha_\beta (r) \ e^{ikz} \ dk \]

Note that (86) is no longer valid but instead we have:

\[(88) \quad Kk\lambda = -mr^2 f^{23} - kr^1 \]

Now $f_\alpha^\beta$ and $g_\alpha^\beta$, using $A_0$ from (65) and $g_\alpha^\beta$ from (28), are:

\[(89) \quad f_{01} = \imath k a_0, \quad f_{02} = da_0/dr \left( \frac{mc}{r^2} \right), \quad f_{03} = \imath m a_0 \]

\[(90) \quad f_{12} = b, \quad f_{23} = rb_z, \quad f_{31} = rb_r \]

\[(91) \quad f_{01} = (\imath k/c^2)^2 a_0 + (\imath /c^2) f_31, \quad f_{03} = -\imath m a_0 /c^2 r^2 \]

\[(92) \quad f_{12} = f_{12}, \quad f_{23} = (\imath k/c^2) a_0 (1/r^2 \gamma_\phi^2), \quad f_{31} = (\imath k/c^2) a_0 (1/r^2 \gamma_\phi^2) \]

where $\delta(k)$ is the Dirac delta function. Writing out the components of Maxwell's equations (71) using (64), we get:

\[(93) \quad \mu = 1: \quad d/dr(\tilde{f}_{12}) + \imath m r f_{13} = K(f_{23} c m /q) d/dr(r^2 \gamma_\phi^2) \delta(k) \]

\[\mu = 2: \quad \imath k r f^{21} + \imath m r f_{31} = K f_{31} \]

\[\mu = 3: \quad \imath k r f^{31} + d/dr(r f_{23}) = K f_{12} \]

From the $\mu = 2$ equation one has:

\[(94) \quad f_{31} = \imath k \lambda \]

Combining the $\mu = 1$ and $\mu = 3$ equations, one obtains:

\[K(\tilde{f}_{12} + K f_{23}) + K(c m /q) d/dr(r^2 \gamma_\phi^2) \delta(k) = d/dr(k f^{23} + \imath m r^2 /c^2) = -K k \lambda /d \]

From (88) and (91) one finds:

\[K k \lambda = (\tilde{m} /r^2 c^2) f_{23} (\tilde{m} m r^2 /c^2) d a_0 /d r - (\tilde{m} m r^2 /c^2) \gamma_\phi^2 \delta(k) - kr^1 f_{23} \]
Using the previous two relations, one obtains:

\[
(95) \quad f_{12} = \frac{(-Kk^2 - \frac{m\Omega}{\gamma_0 r} / c^2) \frac{d\lambda}{dr} - (\frac{m\Omega}{\gamma_0 r} / c^2) \lambda_0}{(k^2 r^2 + \frac{m^2}{\gamma_0 r^2})} \frac{d\tilde{a}_0}{dr} - (cm\Omega m^2 k / q) \gamma_0 \delta (k)
\]

\[
(96) \quad f_{23} = \left( \frac{K m \lambda - k^2 r \lambda_0}{d\lambda / dr + (m^2 \Omega / r / c^2) \lambda_a \gamma_0 / d\gamma_0 / dr} - \frac{(cm\Omega / q)}{d\gamma_0 / dr} \right) (m^2 (\gamma_0 + 1 / \gamma_0) \gamma_0 \phi (\gamma_0^2 + 1)) \delta (k) \left( k^2 r^2 + \frac{m^2}{\gamma_0 r^2} / (\gamma_0^2 r) \right)
\]

Thus (94), (95), and (96) give \( f_{ij} \) in terms of \( \lambda \) and \( \tilde{a}_0 \).

To obtain the differential equation for \( \lambda \), take the \( \mu=3 \) component in (93) and substitute for \( f^3 \), \( f^2 \), and \( f^1 \). We define:

\[
(97) \quad \zeta = k^2 r^2 (\gamma_0^2 + 1 + m^2)
\]

The differential equation for \( \lambda \) is:

\[
(98) \quad \frac{d^2}{d\phi^2} \left[ (k^2 r^2 / c^2) \lambda_0 \right] + k^2 (\lambda_0 / \gamma_0^2 r) \gamma_0^2 (2m \lambda / c^2) \gamma_0 \phi \left( \gamma_0^2 + K - 1 / \gamma_0^2 r \right)
\]

\[
= - \frac{d}{d\phi} \left[ \left( \frac{2m \lambda / c^2}{\gamma_0^2 r} \right) \frac{d\lambda_0}{d\phi} - (cm\lambda r / c^2) \lambda_0 \right] - \frac{k^2 r^2 \gamma_0 \phi (\gamma_0^2 + 1)}{cm\lambda r / c^2} \delta (k) \left[ (d / d\phi) \left( m^2 / c^2 \right) (\gamma_0 - 1 / \gamma_0^2) - \frac{k^2 r^2 \gamma_0 \phi (\gamma_0^2 + 1)}{cm\lambda r / c^2} \delta (k) \left( 2m \lambda / c^2 \right) \gamma_0 \phi \phi \gamma_0 (1 - \gamma_0^2 r) \right]
\]

For the force free case (\( \tilde{a}_0 = 0, m = 0 \)) the right hand side of (98) vanishes to give the equation for the force free stream function.

Write \( \lambda = \lambda_0 + \lambda_1 \) where \( \lambda_0 \) is zero order in the streaming velocities and \( \lambda_1 \) includes all other orders. The zero order part of (98) is:

\[
(99) \quad \frac{d}{d\phi} \left( k^2 r^2 / c^2 \right) \lambda_0 = \frac{cm\lambda_0^4 \delta (k) (k^2 r^2 + q^2 \gamma_0^2 + 2 \gamma_0^2 \gamma_2 + 2k^2 r^2 \gamma_0 \phi (1 - \gamma_0^2 r))}{c^2 m^2 r^2}
\]

For the axisymmetric case (\( m = 0 \)) (99) becomes:

\[
(100) \quad \frac{d}{d\phi} \left( 1 / r \gamma_0^2 \right) \lambda_0 = - k^2 \gamma_0 / \gamma_0^2 r = \frac{cm\lambda_0 r / q r}{d\gamma_0 / dr} 2 \gamma_0 \phi (1 - \gamma_0^2 r) \delta (k)
\]

The Fourier transform of this is not the same as the equation for \( \psi \) (57), but is the same as the equation for \( \psi' = (-cm\lambda / q) r^2 \gamma_0 \phi + \psi \)

(from (79)). We have:

\[
(101) \quad \frac{d}{d\phi} \left( 1 / r \gamma_0^2 \right) \frac{d}{d\phi} \left( -cm\lambda / q \right) r^2 \gamma_0 \phi = \frac{cm\lambda^3 / q \gamma_0 (1 - \gamma_0^2 r)}{d\gamma_0 / dr} 2 \gamma_0 \phi (1 - \gamma_0^2 r)
\]

When this is added to the right hand side of (57) one gets the equation for \( \psi' \):

\[
(102) \quad \frac{d}{d\phi} \left( 1 / r \gamma_0^2 \right) \frac{d}{d\phi} \left( -cm\lambda / q \right) r^2 \gamma_0 \phi = \frac{cm\lambda^3 / q \gamma_0 (1 - \gamma_0^2 r)}{d\gamma_0 / dr} 2 \gamma_0 \phi (1 - \gamma_0^2 r)
\]
This is just the Fourier transform of (100). This result is consistent since it is \(\psi\) and not \(\psi\) which is constant along streamlines.

The first order part of (98) is:

\[
\frac{d}{dr}(k^2 r/\xi) \lambda_1 \phi - k^2 /\xi \phi^2 =
\]
\[
(-2Kk/r) \phi^2 /\xi \phi \phi (cm\Omega/q) \delta(k) \phi^2 + k^2 \phi^2 u_0 /\xi
\]

This gives \(\lambda=\lambda_0 + \lambda_1\) up to second order in the streaming velocities.

The \(\mu=0\) equation of (71) can be written, using (91), as:

\[
(4\pi r/c) (\rho_0 + \rho_1) = (\xi/c^2) a_0 \phi (\phi^2/c^2) d/dr(r \rho_0/dr) - (m^2 r/cq) \phi^3 (\phi^2+1) \delta(k)
\]

where \(\rho_0\) and \(\rho_1\) are the Fourier components of \(\rho_0\) and \(\rho_1\). The zero order part of this equation is:

\[
4\pi r\rho_0/c=(\xi/c) \phi^2 (m\Omega/q) \phi^2 (\phi^2+1) \delta(k) + 2b_2
\]

The Fourier transform of this is identical to (60), as necessary for consistency. The remainder is:

\[
4\pi r\rho_1/c=(\phi/c) \phi^2 (m\Omega/q) \phi^2 (\phi^2+1) \delta(k) + 2b_2
\]

This has first order and second order order parts, in contrast to the axisymmetric case. This is because \(b_\phi\) now has a zero order part. \(a_0\) is still second order, as given by the Fourier transform of (66).

The field components given by (94), (95), (96) have zero order parts:

\[
b_{\phi} = f_{120} = (\tilde{m} k/\xi) (d\lambda_0/dr + (cm\Omega/q) 2\gamma_3^3 r \delta(k))
\]
\[
r_{\phi} = f_{210} = i k \lambda_0
\]
\[
r_{\phi}^0 = f_{230} = -k^2 (\phi^2/\xi) \phi^2 (d\lambda_0/dr + (cm\Omega/q c r) \phi^2 r \delta(k) \phi^2 (\phi^2+1))
\]
\[
r_{\phi}^0 = f_{230} = -k^2 (\phi^2/\xi) \phi^2 (d\lambda_0/dr + (cm\Omega/q c r) \phi^2 r \delta(k) \phi^2 (\phi^2+1))
\]
The first order parts of the Fourier components of the fields are given by:

\[(106)\]

\[b_{\phi 1} = -\kappa k^2 r Y_0^2 / \zeta \cdot \{ \tilde{m} k / \zeta \} d\lambda_1 / dr\]

\[r b_{r 1} = i k \lambda_1\]

\[r b_{z 1} = k \tilde{m} r Y_0^2 / \zeta - \{ k^2 r^2 / \zeta \} d\lambda_1 / dr\]

One can calculate the magnetic flux through a vertical slice, width \(r \Delta \phi\), of a cylinder with axis coinciding with the spin axis (z-axis), extending from \(z=0\) to \(z=z_0\). This is:

\[(107)\]

\[r \Delta \phi \int_0^{z_0} B_r dz = \Delta \phi \int_0^{z_0} F_{31} dz = \Delta \phi (\Lambda(z_0,r,\phi) - \Lambda(0,r,\phi))\]

-a simple result. If one integrates over \(\phi\), this will vanish except for the \(m=0\) case, for which one obtains the same result as for the stream function \(\psi\):

\[2 \pi \tilde{m}(z_0,r) - \Lambda(0,r)\]

From the Fourier transform (87) and the formulae for \(f_{\alpha \beta}\) in terms of \(\lambda_0\) and \(\tilde{a}_0\) it is possible to calculate all the fields. To calculate \(\lambda_0\) for a specific set of boundary conditions there are two approaches: i) Calculate the series solution to (100), and compute series for \(b_{z 0}, b_{r 0}, b_{\phi 0}\). Match these to the Fourier transforms of the fields at the neutron star to get \(C_m(k)\). ii) Use the result (107) and calculate the cylindrical slice flux near the neutron star (e.g. for an oblique dipole field). Then differentiate with respect to \(\phi\) to get \(\Lambda(z,r,\phi)\) at the boundary, Inverse Fourier transform to get \(\lambda\) at the boundary. Match this to the series solution to (100) to determine \(C_m(k)\). The first procedure is similar to that used in solving explicitly the axisymmetric force free corotation case for a dipole field at the neutron star, in appendix 1.
DISCUSSION

After arguing for the existence of a magnetosphere we started with a simple but restrictive model utilizing the stream function $\psi$. This was generalized to fit any boundary field at the neutron star via the magnetic potential $\chi$. Next the force free assumption was relaxed, which had the effect of altering the fields to hold the particles in corotation. Finally the particles were allowed to move. The type of motion that occurs was calculated as was its resulting effect on the fields.

The two major assumptions are: 1) the model is steady state (time independence in the rotating frame); 2) particle motions are small departures from corotation. The first assumption seems quite reasonable. However some pulsar models, most notably those involving some type of relaxation oscillation to explain the emissions, such as the spark gap model of Ruderman and Sutherland, abandon assumption 1) entirely. In any case, keeping 1) is reasonable in order to calculate the conditions under which non-steady processes occur.

The second assumption is on less solid footing. It is quite reasonable well within the light cylinder, but as one approaches regions where $\gamma_\phi$ is very large, the physical situation is unclear. Many other processes become important. Others have proposed a shock front near the light cylinder. Associated with 2) is the assumption of charge separation, which follows naturally from the steady state corotation case. Again some authors assume countering views.
The last models presented here, allow slow particle motions. The continuity equation gives rise to the parameter $K$ which is constant along a streamline. Except for the axisymmetric case, $K$ is further restricted to be piecewise constant. $K$ must be known before one can proceed with the calculation of a model. It is determined by the conditions at the neutron star surface. Since one knows the position of the streamlines from the zero order calculation, $K$ is then known everywhere inside the light cylinder. From (64), (67) and (70) we have:

$$K(B_z^4c m\Omega/q)\gamma_\phi (\gamma_\phi^2 + 1), B_r, B_\phi = (4\pi \rho/c)(\hat{z}, \hat{r}, \hat{\phi})$$

The zero order charge density and fields are known. The physics of the emission process enters the problem in determining the initial velocities and thus $K$. If the neutron star has a uniform surface and a pure dipole field, the surface properties and thus $K$ should be a function only of the angle from the dipole axis. Precisely what this function is depends on a knowledge of the physics of the surface, which is outside the scope of this investigation. The restriction to the use of a piecewise constant $K$ for the case of the general stream function $A$, may or may not be an unreasonable approximation to the actual physical situation.
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APPENDIX 1

Series Solution of:

\[(1-\Omega^2 r^2/c^2)(\partial^2 V/\partial x^2 + \partial^2 V/\partial z^2) - (1/r)(1+\Omega^2 r^2/c^2)\partial V/\partial r = 0\]

Since this equation is separable in \( r \) and \( z \), we write:

\[V(r,z) = X(x) \ Y(y)\]

where \( x = \Omega r/c \), \( y = \Omega z/c \). Separation of variables gives:

\[\frac{1}{Y} \frac{d^2 Y}{dy^2} = -\alpha^2 = \frac{1}{X} \left( \frac{1}{x} \frac{1+x^2}{1-x^2} \frac{dX}{dx} - \frac{d^2 X}{dx^2} \right)\]

Writing \( \frac{d}{dx} = ' \), one has

\[x'' - \frac{(1/x)(1+x^2)}{(1-x^2)} \cdot x' - \alpha^2 \ X = 0\]

(4) has regular singular points at \( x = 0, 1, \) and \( \infty \).

The boundary conditions we use are: 1) The magnetic field at the neutron star is a dipole field aligned with the spin axis. 2) The fields at the light cylinder are finite.

From (21) of the main text, we have \( V(x,z) = -\left( \Omega/c \right) \psi(x,z) \)

\( \psi \) has the same behavior as \( x, z \to 0 \), as the stream function for a dipole field. In cylindrical coordinates the dipole field is:

\[B_r = 3 \mu r z/(r^2 + z^2)^{3/2} \quad B_z = \mu (2z^2 - r^2)/(r^2 + z^2)^{5/2}\]

This can be obtained from the stream function:

\[\psi_{\text{dipole}} = -\mu r^2/(r^2 + z^2)^{3/2} = -\left( \Omega/c \right) x^2/(x^2 + y^2)^{3/2}\]

Thus we have:

\[\psi_{\text{dipole}} = \left( \Omega/c \right) \mu x^2/(x^2 + y^2)^{3/2}\]

At the origin we have: \( V(x,y) \to \int_{-\infty}^{\infty} X(x)e^{i \alpha y} \ d\alpha V_{\text{dipole}} \) as \( x, y \to 0 \). We perform the inverse Fourier transform to get

\[\int_{-\infty}^{\infty} X(x) e^{-i \alpha y}/(x^2 + y^2)^{3/2} \ dy\]
We need the Fourier transform:

\( F\left( \frac{\pi}{2} \right) = \frac{1}{i} \frac{x}{(x^2+y^2)^{3/2}} = -\frac{d}{dx} K_0(\alpha x) = a \ K_1(\alpha x) \)

Here \( K_0, K_1 \) are the first and second Hankel functions. Thus we have:

\[ 2\pi \to (\gamma/c)^2 \mu (2\pi)^{1/2} F(\pi/(x^2+y^2)^{3/2}) = (\gamma/c)^2 2\mu_1 \alpha K_1(\alpha) \]

We have: \( K_1 \to (1/\alpha) \log(\alpha/2) \) as \( x \to 0 \). Thus one obtains:

\[ X \to (\gamma/c)^2 (\mu_1/\pi) (1 + (\gamma x^2/2) \log(\alpha/2) + \cdots) = (\gamma/c)^2 \mu_1/\pi \] as \( x \to 0 \).

The series for the two independent solutions to (4) were computed:

(7) \( X(x) = A(\alpha)X_0(x) + B(\alpha)X_1(x) \)

At the origin these series have the values:

\( X_0(0) = 0, \ X_1(0) = 1 \)

From (7) we then have:

(8) \( B = (\gamma/c)^2 \mu_1/\pi \)

\( A \) is left undetermined. Both solutions diverge as one approaches the light cylinder, so the requirement of finite fields at \( x = 1 \) cannot be applied in any simple manner.

We consider the behaviour of \( X(x) \) near the light cylinder using the variable \( v \):

(9) \( v = 1 - x^2 \)

We have the relations:

(10) \( \frac{dv}{dx} = 2x \frac{x}{1+x^2} = 2-v \quad \frac{d}{dx} = -2x \frac{d}{dv} \)

\( \frac{d^2}{dx^2} = -2\frac{d}{dv} + 4(1-v) \frac{d^2}{dv^2} \)

With these (4) becomes (with \( \frac{d}{dv} = \)):

(11) \( (v-v_0^2)X + (1-v)\frac{dX}{dv} - (v/\alpha)^2 vX = 0 \)

We write \( X(v) = \sum_{n=1}^{\infty} a_n v^{n+\beta} \) then becomes:

(12) \( \sum_{n=1}^{\infty} v^{n+\beta-1} (v/\alpha)^2 \frac{d}{dv} (v^{n+\beta-1}) - (\alpha/2)^2 a_{n-2} = 0 \)
The indicial equation \((n=0)\) is \(\beta^2=0\). Since it has a double root the second solution is \(\frac{\partial X}{\partial \beta}\big|_{\beta=0} = X(\nu)\log(\nu) + \) regular series , where \(X(\nu)\) is the solution for \(\beta=0\). This second solution is divergent at the light cylinder \((\nu \to 0)\). Thus we only need to consider the one solution, \(X(\nu)\). The recursion relation for the coefficients \(a_n\), from (16), is:

\[
(13) \quad a_n = \frac{(n-1)^2 a_{n-1} - (\alpha/2)^2 a_{n-2}}{n^2}
\]

Taking \(a_0 = 1\) we have:

\[
V(r, z) = \int_{-\infty}^{\infty} C(\alpha) X(\nu) e^{i\alpha y} d\alpha = \int_{-\infty}^{\infty} (A(\alpha) X_0(x) + (\Omega/c)^2 (\mu/\pi) X_1(x)) e^{i\alpha y} d\alpha
\]

By Fourier transforming this relation, we obtain:

\[
(14) \quad C(\alpha) X(\nu) = A(\alpha) X_0(x) + (\Omega/c)^2 (\mu/\pi) X_1(x)
\]

for all \(\alpha\) and \(x\). For \(x=0\) \((\nu=1)\), (18) gives:

\[
(15) \quad C(\alpha) = (\Omega/c)^2 (\mu/\pi) 1/X(1)
\]

Thus the explicit solution for \(V(r, z)\) is:

\[
(16) \quad V(r, z) = (\Omega/c)^2 (\mu/\pi) \int_{-\infty}^{\infty} (X(1 - \Omega^2 r^2/c^2)/X(1)) e^{i\alpha z/c} d\alpha
\]

Note that we could have set \(x\) to zero in the equation following (13), then performed the Fourier transform to get (15) directly, without any reference to the solutions \(X_0(x)\) and \(X_1(x)\).
APPENDIX 2 The Physical Meaning of $F^{\mu \nu}$

In non-rotating cartesian coordinates $x^\alpha(t, x, y, z)$, the field quantities, for Gaussian units, are:

\[ F^{\mu \nu} = \begin{pmatrix}
0 & E_x/c & E_y/c & E_z/c \\
-E_x/c & 0 & B_z & -B_y \\
-E_y/c & -B_z & 0 & B_x \\
-E_z/c & B_y & -B_x & 0
\end{pmatrix} \tag{1} \]

In transforming to cylindrical (stationary) coordinates: $x^\alpha(t, z, r, \phi_0)$, $F^{\alpha \beta} = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} F^{\mu \nu}$ is used. We have:

\[
\begin{align*}
\vec{x}^0 &= \begin{pmatrix} t \\ 0 \end{pmatrix}, & z = \vec{x}^3, & x = \vec{x}^1, & \vec{x}^2 = \vec{x}^1 \cos(\vec{x}^3), & \vec{x}^3 = \tan^{-1}(\vec{x}^2 / \vec{x}^1) \\
\end{align*}
\]

and

\[
\begin{align*}
\partial t / \partial \vec{x}^x &= x = \cos \phi_0 & \partial t / \partial \vec{x}^y &= y = \sin \phi_0 \\
\partial \phi_0 / \partial \vec{x}^x &= \partial / \partial x (\tan^{-1}(y / x)) = -y / (x^2 + y^2) = -\sin \phi_0 / r \\
\partial \phi_0 / \partial \vec{x}^y &= \cos \phi_0 / r
\end{align*}
\]

Thus one has:

\[ \frac{\partial x^\alpha}{\partial \vec{x}^\mu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & \cos \phi_0 & \sin \phi_0 & 0 \\
0 & -\sin \phi_0 / r & -\cos \phi_0 / r & 0
\end{pmatrix} \tag{2} \]

Using (2) one can now calculate $F^{\mu \nu}$ in cylindrical coordinates from (1). The result is:

\[ F^{\alpha \beta} = \begin{pmatrix}
0 & E_z/c & E_r/c & E_\phi/c \\
-E_z/c & 0 & B_\phi & -B_r/r \\
-E_r/c & -B_\phi & 0 & B_z/r \\
-E_\phi/c & B_r/r & -B_z/r & 0
\end{pmatrix} \tag{3} \]
where:

\[ E_x = E_x \cos \phi + E_y \sin \phi \]
\[ E_\phi = -E_x \sin \phi + E_y \cos \phi \]
\[ B_x = B_x \cos \phi + B_y \sin \phi \]
\[ B_\phi = -B_x \sin \phi + B_y \cos \phi \]

The change to rotating cylindrical coordinates is simply accomplished by using \( \phi = \phi_0 - \Omega t \) to obtain the derivatives:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\Omega & 0 & 0 & 1
\end{bmatrix}
\]

Using (3) and (4) we calculate \( F^{\mu \nu} \):

\[
F^{\mu \nu}(\partial x^\mu / \partial x^\alpha)(\partial x^\nu / \partial x^\beta) \hat{a}^{\alpha \beta}:
\]

\[
\begin{bmatrix}
0 & E_z/c & E_r/c & E_\phi/cr \\
-E_z/c & 0 & B_\phi & \Omega E_z/c - B_r/r \\
-E_r/c & -B_\phi & 0 & \Omega E_r/c + B_z/r \\
-E_\phi/cr & \frac{B_r}{r} - \frac{\Omega E_r}{c} & -\frac{B_z}{r} - \frac{\Omega E_r}{c} & 0
\end{bmatrix}
\]

Note that \( E \) and \( B \) are quantities as measured in the stationary frame. Using \( g_{\alpha \beta} \) from (27) of the main text, one gets:

\[
\begin{bmatrix}
0 & -cE_z + \Omega rB_r & -cE_r - \Omega rB_z & -crE_\phi \\
cE_z - \Omega rB_r & 0 & B_\phi & -rB_r \\
cE_r - \Omega rB_z & -B_\phi & 0 & rB_z \\
-crE_\phi & rB_r & -rB_z & 0
\end{bmatrix}
\]
APPENDIX 3 Series Solution of:

\[ \delta^2 \chi / \delta \phi^2 + r^2 \gamma^2 (\delta^2 \chi / \delta r^2 + \delta^2 \chi / \delta z^2) + \gamma^2 (2 \gamma^2 - 1) \delta \chi / \delta r = 0 \]

The \( \phi \) and \( z \) dependences can be separated by writing \( \chi \) as:

\[ (1) \quad \chi(z, r, \phi) = e^{i\Omega} e^{im\phi} R(x) \]

with \( x = \Omega r / c \), \( y = \Omega z / c \). \( m \) is an integer, and \( \alpha \) is a real number.

For \( R \), with \( d/dx' = ' \), one obtains the equation:

\[ (2) \quad x^2 (1-x^2) R'' + x (1+x^2) R' - (m^2 (1-x^2) + \alpha^2 x^2) (1-x^2) R = 0 \]

where \( 2\gamma^2 - 1 = (1-x^2) / (1-x^2) \) has been used.

Before solving (2) for \( R(x) \), we consider the problem of matching to an oblique magnetic dipole field at the origin. As \( z, r \to 0 \) \( \chi \) approaches \( \chi_{\text{dipole}} \). To calculate \( \chi_{\text{dipole}} \), we must know what \( F^\alpha_\beta \) is in the rotating coordinates. From appendix 2, we have:

\[ (3) \quad \chi_{1} = B_{z} \]

\[ (4) \quad \chi_{2} = B_{r} \]

\[ (5) \quad \chi_{3} = B_{\phi} \]

The field of a dipole, in coordinates \( z', r', \phi' \) with \( z' \) aligned with the magnetic axis, is given by the equations:

\[ B_{z'} = \mu (2z'^2 - r'^2) / (z'^2 + r'^2)^{3/2} = \partial \chi_{\text{dipole}} / \partial z' \]

\[ B_{r'} = 3\mu r' z' / (z'^2 + r'^2)^{5/2} = \partial \chi_{\text{dipole}} / \partial r' \]

These are satisfied by the magnetic potential:

\[ \chi_{\text{dipole}} = -uz' / (r'^2 + z'^2)^{3/2} \]

The diagram next page illustrates the relation between
cartesian coordinates \((x,y,z)\) and \((x',y',z')\), and cylindrical coordinates \((z,r,\phi)\) and \((z',r',\phi')\). The primed coordinates are related to the unprimed coordinates by a rotation of \(\theta_0\) about the y axis. One has the relations:

\[
z' = z \cos \theta_0 + r \cos \phi \sin \theta_0 \]
\[
r'^2 + z'^2 = r^2 + z^2 \]

(5) \(X_{\text{dipole}} = -\mu (z \cos \theta_0 + rcos \phi \sin \theta_0) / (r^2 + z^2)^{3/2}\)

From the separation of variables (1), we have

(6) \(\chi(z,r,\phi) = e^{im\phi} \int_{-\infty}^{\infty} e^{i\alpha z/c} d\alpha\)

We invert this to get:

(7) \(R(x) = (1/4\pi^2) \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} e^{-i(m\phi + \alpha y)} \chi(cy/\Omega, cx/\Omega, \phi)\)

As \(x\) approaches zero, one has \(X \to X_{\text{dipole}}\). Thus, from (5), the integral in (7) is non-zero only for \(m=0, 1, -1\).

\[\text{m=0: } R_0(x) \to (\mu/2\pi) (\Omega/c)^2 x \cos \theta_0 \int_{-\infty}^{\infty} e^{-iy} (-y/(x^2 + y^2)^{3/2})\]

Here we use integration by parts. The integral is then:

\[\int_{-\infty}^{\infty} e^{-iy} d(1/(x^2 + y^2)^{1/2}) = i\alpha \int_{-\infty}^{\infty} (x^2 + y^2)^{-1/2} e^{-iy} dy\]

This is just \(i\alpha(2\pi)^{1/2}\) times the Fourier transform of \((x^2 + y^2)^{-1/2}\) which is \((2/\pi)^{1/2} K_0(ax)\). Thus we have:

(8) \(R_0(x) \to (\mu/\pi) (\Omega/c)^2 x \cos \theta_0 \left\{ \begin{array}{l} i\alpha K_0(ax) \text{ as } x \to 0 \\ - (\mu/\pi) (\Omega/c)^2 i\alpha \log(ax) \end{array} \right\}\)

where \(K_0(ax) \to -\log(ax)\) as \(x \to 0\) has been used.
Now, the integral in $\phi$ is just $\pi$. The integral in $y$ is found directly from appendix 1 to be $2aK_1(ax)$. As $x \to 0$, we have

$$K_1(ax) \to (1/ax) + (ax/2) \log(ax/2)$$

Thus one demands that

$$R_{\pm1}(x) \to (\mu/4\pi^2)(\Omega/c)^2 \sin \theta_0 \int_0^{2\pi} e^{i \phi} \phi \int_0^{\infty} x(x^2+y^2)^{-3/2} e^{-i \alpha y} dy$$

The solutions $R(x)$ were found as series about $x=0$ (two for $m^2=0$, two for $m^2=1$). However, as in appendix 1, the boundary condition at the neutron star determines the coefficient of only one solution of each pair.

We shall now consider the behaviour near the light cylinder using the variable $v=1-x^2$ (see appendix 1, equations (9) & (10)).

Writing $d/dv=v$, (2) becomes:

$$v(1-v)\frac{dR}{dv} + vR - (m^2 + (a/2)^2 v)R = 0$$

Writing $R(v) = \sum a_n v^n$, (18) becomes:

$$v \frac{d}{dv} \sum a_n v^{n+\beta} = \sum (a_n \frac{d}{dv} v^{n+\beta})$$

The indicial equation is $\beta(\beta-2)=0$, so that $\beta=0,2$. For $\beta=0$$a_1=0; a_2, a_3$ etc. diverge. Thus we set $a_0=\beta b$ so that $(\partial R/\partial \beta)_{\beta=0}$ is also a solution. But we have:

$$R_m(v) = \sum a_{nm} v^{n+2}$$

with $a_{nm} = ((n+1)(2n-1)a_{n-1} + ((\alpha/2)^2 + (m/2)^2 - n(n-1))a_{n-2} - (\alpha/2)^2 a_{n-3}) \gamma/(n(n+2))$.

Here $R_m(v)$ is the solution for $\beta=2$, as follows:

$$\frac{\partial R_m(v)}{\partial \beta} = R_m(v) \log(v) + \frac{\partial}{\partial m} v^n$$
Both series are finite as \( v \) goes to zero. From appendix 2, we have:

\[ F^0 = \frac{E_r}{c} \]

Using this and (37) from the main text, we have:

\[ E_r = -(\Omega r^2/c^2) \gamma^2 F^2 = -(\Omega r/c) \gamma^2 \partial_x \partial_z = -(\Omega r/c) \gamma^2 i \alpha (\Omega/c) \chi \]

Since we assume that \( E_r \) is well behaved as \( \Omega r/c \) approaches unity, \( \chi \) must approach zero as one approaches the light cylinder \((v=0)\).

Thus the solution (12) is rejected. We are left with, for \( m=0 \):

\[ (14) \quad R_0(v) = a_{00}(v^2 + (2/3)v^3 + (1/8)((\alpha/2)^2 + 4)v^4 + \cdots) \]

and, for \( m=\pm 1 \):

\[ (15) \quad R_1(v) = a_{01}(v^2 + (2/3)v^3 + (1/8)((\alpha/2)^2 + (17/4))v^4 + \cdots) \]

The magnetic potential \( \chi \) is given by:

\[ (16) \quad \chi(v, y) = \int_{-\infty}^{\infty} R_0(v) e^{i\gamma \alpha} d\alpha + (e^{i\phi} + e^{-i\phi}) \int_{-\infty}^{\infty} R_1(v) e^{i\gamma \alpha} d\alpha \]

with \( R_0 \) and \( R_1 \) from (13) as given in (14) and (15). Also we have

\[ x \rightarrow x_{\text{dipole}} \quad \text{as} \quad x, y \rightarrow 0 \]

By multiplying by \( e^{-i\alpha u} \) and \( e^{-i\alpha u} e^{-i\phi} \) and integrating over \( y \) and \( \phi \) one gets, for \( x \rightarrow 0 (v \rightarrow 1) \):

\[ (17) \quad R_0(v \rightarrow 1) \rightarrow -\frac{\alpha}{(\Omega/c)^2} \cos \theta_0 \frac{\alpha}{(\Omega/c)^2} \log(x) \]

\[ (18) \quad R_1(v \rightarrow 1) \rightarrow -\frac{\alpha}{(\Omega/c)^2} \sin \theta_0 \frac{1}{(1/x)} \]

Recall \( x \frac{d}{dx} = 2(v-1)\frac{d}{dv} \) ((10) in appendix 1) so that (17) is:

\[ (19) \quad 2(v-1)\frac{d}{dv} R_0(v \rightarrow 1) \rightarrow -\frac{\alpha}{(\Omega/c)^2} \cos \theta_0 \frac{\alpha}{(\Omega/c)^2} \]

Write the quantity on the left hand side as the series:

\[ a_{00} \sum b_n \cdot n^2 \quad \text{with} \quad b_n = 2(n+1) a_{n-1}/a_{00} - 2(n+2) a_n/a_{00} \quad a_n \] are the coefficients of \( R_0(v) \) as given by (13). Then as \( v \rightarrow 1 \), we have:

\[ (20) \quad a_{00} \sum b_n \rightarrow -\frac{\alpha}{(\Omega/c)^2} \cos \theta_0 \frac{\alpha}{(\Omega/c)^2} \]

This determines the coefficient of \( R_0(v) \), since \( b_n \) are known.
To determine the coefficient of $R_1(v)$ from (18), we solve the differential equation for $W(v)$. We define:

(21) \[ W(v) = xR_1(v) = (1-v)^{-3/2} R(v) \]

$W(v)$ will be finite as $x \to 0$. With $R_1(v) = (1-v)^{-3/2} W(v)$, we have:

\[ R_1 = (1-v)^{-3/2} W + (3/4) (1-v)^{-5/2} W \]

Thus (10) (with $m=1$) becomes, upon multiplying by $(1-v)^{3/2}$:

(22) \[ v(1-v)\ddot{W} - (1-v)\dot{W} - \left( \frac{\alpha^2}{4} v \right) W = 0 \]

Writing $W = a_0 c_n v^{n+\beta}$, $c_0 = 1$, (22) becomes:

(23) \[ v^{n+\beta-1} \left( -\frac{\alpha^2}{4} c_{n-2} - (\frac{3}{2} + n + \beta - \frac{1}{4}(n + \beta - 3))c_{n-1} + (n + \beta)(n + \beta - 2)c_n \right) = 0 \]

The indicial equation is $\beta(\beta-2) = 0$. The $\beta = 0$ solution is rejected since as $v \to 0$, $R$ (and thus $W$) must approach zero. We are left with:

(24) \[ W(v) = a_0 c_n v^{n+\beta} \text{ with } c_n = \left( (n+1)(n-1) + \frac{3}{2} + (n+\beta)(n+\beta-2) \right) c_{n-1} + (n+\beta)(n+\beta-2)c_n \]

As $v \to 0$, from (18) and (21) we get:

(25) \[ a_0 c_n = -\left( \frac{\mu}{\pi} \right)(\Omega/c)^2 \sin \theta \]

Here $a_0$ is the coefficient of $R_1(v)$.

Now $\chi$, as given in (16), is completely determined:

\[
\chi(v, \gamma, \phi) = -\left( \frac{\mu}{\pi} \right)(\Omega/c)^2 \left( \cos \theta \int_{-\infty}^{\infty} (i \alpha \beta c_n)^{\frac{3}{2}} \sum (a_{n0} / a_{00}) v^{n+2} e^{i \gamma y} d\alpha 
+ \sin \theta \cos \phi \int_{-\infty}^{\infty} (\Omega/c_n)^{\frac{3}{2}} \sum (a_{n1} / a_{01}) v^{n+2} e^{i \gamma y} d\alpha \right)
\]
We consider the Lagrangian:

$$L = \frac{1}{2}(A\dot{x}^2 + B\dot{y}^2 + C\dot{z}^2) + \phi_1 \dot{x} + \phi_2 \dot{y} + \phi_3 \dot{z}$$

Here $A, B, C, \phi_1, \phi_2, \phi_3$ are functions of $x, y, z$ but not explicitly of $t$ or the velocities. We define:

$$X = \frac{\partial \phi_3}{\partial y} - \frac{\partial \phi_2}{\partial z}$$
and cyclically for $Y$ and $Z$.

Then the Lagrange equations of motion are:

$$\frac{d}{dt}(A\dot{x}) = Z\dot{y} - Y\dot{z} + \frac{1}{2}(A, x \dot{x}^2 + B, y \dot{y}^2 + C, z \dot{z}^2)$$

$$\frac{d}{dt}(B\dot{y}) = X\dot{z} - Z\dot{x} + \frac{1}{2}(A, y \dot{y}^2 + B, y \dot{y}^2 + C, y \dot{z}^2)$$

$$\frac{d}{dt}(C\dot{z}) = Y\dot{x} - X\dot{y} + \frac{1}{2}(A, z \dot{z}^2 + B, z \dot{z}^2 + C, z \dot{z}^2)$$

with $A, x = \frac{\partial A}{\partial x}$ etc. The Hamiltonian energy function is constant:

$$E = \frac{1}{2}(A\dot{x}^2 + B\dot{y}^2 + C\dot{z}^2)$$
since $L$ is not explicitly a function of time.

The equations (3) have $\dot{x} = \dot{y} = \dot{z} = 0$ as a possible solution, for arbitrary $x, y, z$. We are looking for small deviations from these particular motions. If the quantities $A, B, C, X, Y$ and $Z$ were constant, then the motion would be a superposition of solutions of normal mode type, each of which has a time dependence of the form $e^{-i\omega t}$. The characteristic frequencies are given by

$$\text{determinant } \begin{vmatrix} i\omega A & Z & -Y \\ -Z & i\omega B & X \\ Y & -X & i\omega C \end{vmatrix} = 0$$

or (6) $\omega(ABC\omega^2 - AX^2 - BY^2 - CZ^2) = 0$

The root $\omega = 0$ corresponds to uniform motion with the velocities $(\dot{x}, \dot{y}, \dot{z})$ proportional to $(x, y, z)$. The other roots correspond to transverse elliptical motion.
One sees that $A, B, C, X, Y$ and $Z$, which are functions of position, will vary to first order in the velocities. They vary slowly, due to the uniform motion, and rapidly, with frequency $\omega$, due to the oscillating motion. We define:

$$\omega = \left( \frac{AX^2 + BY^2 + CZ^2}{AEC} \right)^{1/2}$$

Thus the roots of (6) are $0, \pm \omega$. With this value for $\omega$, we write the matrix equation for the amplitudes of the transverse motion:

$$i\omega A \quad Z \quad -Y$$
$$-Z \quad i\omega B \quad X$$
$$Y \quad -X \quad i\omega C$$

This has non-trivial solutions $(\xi, \eta, \zeta)$ which are determined up to an arbitrary complex multiplying factor.

We shall use the vectors: $(X, Y, Z), (\xi, \eta, \zeta)$ and $(\xi^*, \eta^*, \zeta^*)$, as a basis for describing the general motion. We introduce the definitions:

$$AX^2 + BY^2 + CZ^2 = P$$

$$A\xi^* + B\eta^* + C\zeta^* = N$$

Multiplying (8) on the left by $(X, Y, Z)$ and $(\xi, \eta, \zeta)$, we have:

$$AX\xi + BY\eta + CZ\zeta = 0$$

$$A\xi^2 + B\eta^2 + C\zeta^2 = 0$$

From (8), we can express any one of $\xi, \eta, \zeta$ in terms of another:

$$\xi = A\left( (XZ - i\omega BY) / (BY^2 + CZ^2) \right) \xi \quad \text{and cyclic relations}$$

Thus we can express $N$, from (10), as a function of only one of $\xi, \eta, \zeta$ and one complex conjugate quantity $\xi^*, \eta^*, \zeta^*$. Inverting these relations gives:
(14) \[ \xi^* = \frac{1}{\sqrt{\Lambda}} \frac{\chi^2}{P} \] and cyclic relations

(15) \[ \xi^* = (-i\omega CZ - XY)N/2P \] and cyclic relations

We now express the velocity vectors as a superposition of the instantaneous normal modes. These have a real amplitude \( \xi \) describing the longitudinal component and a complex amplitude \( \sigma \) representing the elliptical component, as follows:

(16) \[
\begin{align*}
\dot{x} &= \xi x + \sigma \xi^* e^{-i\chi} \\
\dot{y} &= \xi y + \sigma \eta e^{-i\chi} \\
\dot{z} &= \xi z + \sigma \xi^* e^{-i\chi}
\end{align*}
\]

The factor \( e^{i\chi} \) incorporates the rapid time dependence of the transverse elliptical motion.

These equations can be inverted to give \( \xi \) and \( \sigma \) in terms of the velocity components. The results are:

(17) \[ P \xi = AX\dot{x} + BY\dot{y} + CZ\dot{z} \]

(18) \[ N \sigma e^{-i\chi} = \xi\eta x + \sigma \eta y + \xi \sigma^* z \]

Using (16), the energy (4) is given by:

(19) \[ E = \frac{1}{2} P \xi^2 + N \sigma^* \]

This shows that the particle energy is unambiguously expressible as the sum of a longitudinal part \( \frac{1}{2} P \xi^2 \) and a transverse part \( N \sigma^* \).

We can obtain the equation of motion for \( \xi \) and \( \sigma \) by differentiating (17) and (18), using the equations of motion (3), and then using (16) to express the results in terms of \( \xi \) and \( \sigma \). The results are expressions for \( \dot{\xi} \) and \( \dot{\sigma} \) which contain slowly varying terms, plus oscillating terms proportional to \( e^{i\chi}, e^{2i\chi} \). In considering
the "adiabatic" aspects of the motion the oscillating terms can
be neglected, to second order in the velocities. When one integrates
over many cycles the first order parts of the oscillating terms
vanish, whereas the first order parts of the slowly varying terms
do not. The slowly varying terms only contain $\xi, \eta, \text{ or } \zeta$ as a product
with one of $\xi^*, \eta^*, \text{ or } \zeta^*$. (14) and (15) are used to eliminate these terms.
The result of the calculation for $\dot{\lambda}$ is written:

$$P \dot{\lambda} = -\frac{1}{2} \delta^2 (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}) P$$
$$-N \sigma^* (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}) \log \omega + \text{oscillating terms}$$

The rate of change of any field quantity, along the path of a
particle, is given by applying the operator:

$$\frac{d}{dt} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

The particle's velocities, $\dot{x}, \dot{y}, \dot{z}$, are given explicitly by (16).
When we average over several cycles, this operator becomes, to
first order:

$$\lambda (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})$$

Thus from (20), we can write, for a particle moving along a stream-
line:

$$P \dot{\lambda} + \frac{1}{2} \lambda^2 dP + N \sigma^* d(\log \omega) = 0$$

or

$$d (\frac{1}{2} P \lambda^2 ) + N \sigma^* d(\log \omega) = 0$$

(21)
From (19) and the constancy of $E$, this yields:
\[ d(N^* \sigma) = N^* \sigma d \log \omega \quad \text{or} \quad d \log (N^* \sigma / \omega) = 0, \ i.e. \]
(22) \quad N^* \sigma / \omega = M \quad \text{M is a constant.}

This result is a generalization of a result familiar in many aspects of particle motions in a magnetic field. Namely, the magnetic moment is an adiabatic invariant. Combining (19) and (20) gives:
(23) \quad p^2 = 2(E - M\omega)

in the adiabatic limit.