THE EFFECT OF GROUND CONDUCTIVITY AND PERMITTIVITY ON THE MODE
PROPAGATION CONSTANTS OF AN OVERHEAD TRANSMISSION LINE

by

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ABSTRACT

A general analytical method to derive the distributed circuit parameters and mode propagation constants for an n-conductor transmission line is developed. The analysis uses electromagnetic field concepts and the results are interpreted in terms of distributed circuit parameters. The procedure involves transforming the problem of the n-conductor line above a ground with finite conductivity into that of an n-conductor above a ground with infinite conductivity. Correction factors are added to account for the finite conductivity of the ground. The distributed circuit parameters thus calculated are used to calculate the mode propagation constants over a frequency range from 10 Hz to 1 MHz for values of ground conductivity varying between 1 mho/m and $10^{-5}$ mho/m and relative permittivity varying between 10 and 50.

Numerical results for the distributed circuit parameters and mode propagation constants for a typical 500 kV single circuit transmission line and various ground conditions are given. The results show that one mode has a higher attenuation and a lower velocity than either of the other two modes, suggesting the zero sequence mode for a completely balanced system.
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<tr>
<td>(a^2)</td>
<td>(-\varepsilon_0 \omega (n-1)/\sigma + j)</td>
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<tr>
<td>(a_i)</td>
<td>conductor radius</td>
<td></td>
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<tr>
<td>(b)</td>
<td>(1/r^2 = n - j\sigma/\varepsilon_0)</td>
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<td>(b_{ij})</td>
<td>horizontal conductor spacing</td>
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<td>(I_j)</td>
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<td>(m)</td>
<td>propagation constant</td>
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<td>(m_o, m_l)</td>
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<td>(n)</td>
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<td>(P_{ij}, Q_{ij})</td>
<td>real and imaginary parts of the correction terms for the elements of the impedance matrix</td>
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<td>(R)</td>
<td>radial distance in spherical coordinates</td>
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**R_1** = radial distance from the current element

**R_2** = radial distance from the image of the current element

**R_{ij}, R_{1j}** = resistance matrix, elements

**r** = radial distance in cylindrical coordinates

**S_{ij}** = distance from conductor i to the image of conductor j

**S_{ij}** = distance from conductor i to conductor j

**s** = elemental length

**u** = quantity defined in Appendix B

**V_j** = phasor voltage

**V_c, V_p** = phasor voltage vectors (mode, line)

**v** = voltage vector

**v_i** = modal velocity

**Y, Y_{ij}** = admittance matrix, elements

**Z, Z_{ij}** = impedance matrix, elements

**z_i** = internal conductor impedance

**α** = attenuation constant

**β** = phase constant

**γ** = propagation constant = \( u^2 + m^2 \)

**γ_0, γ_1** = propagation constant (in air, in the ground)

**ε_c** = complex permittivity

**ε_0, ε_1** = permittivity (of free space, of the ground)

**μ_0** = magnetic permeability of free space

**π, π_x** = Hertz vector, components

**ρ** = radial distance in cylindrical coordinates

**ρ_1** = radial distance from conductor i

**ρ_2** = radial distance from the image of conductor

**σ** = conductivity
\[ \gamma^2 = \frac{m_1^2}{m_0^2} \]

\[ \phi = \text{scalar function of position} \]

\[ \omega = \text{angular frequency} \]

\[ \gamma = e^c = 1.7811 \]
ACKNOWLEDGEMENT

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Chapter 1. Introduction.

The advent of extra-high-voltage transmission has renewed the interest in the study of energy propagation along transmission lines. These studies are associated with fault current calculations, system stability, switching and restriking overvoltages, and the propagation characteristics of carrier waves along power lines. A necessary prerequisite for investigations of the above problems is a thorough knowledge of the multiconductor transmission line distributed circuit parameters. Although these parameters may be calculated from the conductor conductivity, the frequency and the geometry of the system, they are also dependent on the ground conductivity, permittivity, and permeability. While an exact calculation of the parameters is impossible, due to irregularities in the shape of the ground surface and the lack of uniform conductivity, a fairly accurate calculation is possible by replacing the ground with a plain or multi-layer homogeneous media. After the distributed circuit parameters have been found the propagation problem is solved from the transmission line equations by meeting the boundary conditions.

The first important engineering study of the effect of ground with finite conductivity on the electromagnetic propagation produced by current-carrying conductors above the ground is due to J.R. Carson\(^1\)(2). Applying electromagnetic
field theory, he calculated the field due to an alternating current in a straight infinitely long wire above and parallel to a plain and electrically homogeneous ground. The derivation contains four basic assumptions;

1. The ground relative permeability is unity.
2. The ground is electrically homogeneous.
3. The polarization currents may be neglected.
4. The current is propagated without attenuation at the speed of light.

The first three assumptions greatly simplify the results. The fourth assumption implies reasonably efficient energy transmission, as indicated in reference (2). In the same publication Carson derived resistance and inductance correction factors. They are valid only for frequencies below 10 kHz because of the omission of the ground relative permittivity in the derivation.

Rudenberg\(^{(6)}\) analysed the same problem. He used a model of the transmission system which has a semi-circular ground surface, with the axis coincidental with that of a conductor. In addition to Carson's assumptions he assumed that the current distribution in the ground is a function of the radial distance from the conductor. The resistance and inductance correction terms are expressed in terms of the zero and first order Bessel functions of the third kind. At low frequencies these functions have small arguments. Using only the first term of the power series expansions of the Bessel functions similar results to those given by Carson are obtained.
Rudenberg did not derive the mutual inductance correction terms and he neglected the ground permittivity effects.

Carson's results were extended by Wise\(^{(9)}\)(10)(11). His first paper removed the restriction of assuming a ground relative permeability of unity. His subsequent works\(^{(10)}\)(11) included the effect of polarization currents, gave correction terms for the line admittance, and extended Carson's impedance correction terms to a broader frequency range. Carson's impedance correction terms were also extended by Sunde\(^{(7)}\) to include the effect of a multi-layer ground.

The investigation of multi-conductor transmission, using matrices, was started by Bewley\(^{(4)}\). His method was first to consider an ideal lossless line and then to expand the analysis to include losses. Rice\(^{(3)}\) and Pipes\(^{(20)}\) followed Bewley's approach but added the use of Laplace transform methods. A major contribution has been the work of Hayashi\(^{(5)}\) who extended analysis to include transient phenomena. He also developed new techniques in matrix calculus to facilitate the solutions, for example his extension of Sylvester's expansion theorem. Although his analysis included conductor skin effects at higher frequencies, he used ground impedance correction from Rudenberg's model which neglected ground permittivity effects. A systematic mathematical procedure for handling the multi-conductor transmission line problem was recently published by Wedepohl\(^{(8)}\). In two subsequent publications\(^{(21)}\)(22) he gave numerical procedures including the correction of the impedance parameters following Wise's methods. A matrix analysis of
multi-conductor transmission systems with various boundary conditions was done by Dowdeswell in his thesis, however his work was restricted to steady state analysis at power frequency. Arismunandar also derived admittance correction terms with the assumption that the lines are close to the ground surface, from which he calculated characteristic impedances for switching surge studies.

This thesis derives correction factors for the distributed circuit parameters including the effects of conductivity and relative permittivity of the ground, and gives a systematic method of calculating the mode propagation constants of a multi-conductor transmission line. A general description of the system and the assumptions required in the derivation are given in Chapter 2. The electromagnetic field surrounding a current-carrying conductor above a plain ground is derived in Chapter 3 in terms of an E-type Hertz vector, valid for the frequency range from 10 Hz to 1 MHz. The distributed circuit impedance parameters for the transmission line are derived in Chapter 4 and the distributed circuit admittance parameters in Chapter 5, from the electromagnetic field solution given in Chapter 3. These distributed circuit parameters include the effects of ground relative permittivity and conductivity. The derivation follows closely that of Carson, Wise and Sunde. The detailed solution of the transmission line equation, following Wedepohl's solution, is given in Chapter 6. The distributed circuit parameters for a typical 500 kV transmission line for various frequencies and ground
conditions are tabulated in Chapter 7. Chapter 7 also includes graphs giving the magnitude of the distributed circuit parameter correction factors over the stated frequency range for various ground conditions. A detailed description of the solution, with numerical results, of the wave equation for the typical transmission line are included in Chapter 8. The rationalized MKS system of units is used throughout the thesis, unless specified.
Chapter 2. Assumptions and Approaches.

2.1 General Description of the System.

The system under study consists of a number of separate conductors and ground return lines situated above and parallel to a plane ground. No geometric symmetry is assumed in the transverse plane.

The transmission line equations for a line with n conductors may be written in terms of the distributed circuit parameters in the general form as

\[- \frac{\partial V}{\partial z} = R_{i} + L_{t} \frac{\partial i}{\partial t} = (R + Lp)i = Z_{i} \]

\[- \frac{\partial i}{\partial z} = G_{v} + C_{t} \frac{\partial V}{\partial t} = (G + Cp)v = Y_{v} \]

where Z and Y are \( n \times n \) symmetric matrices, and \( v \) and \( i \) column matrices with \( n \) elements. Each element of the Z and Y matrices includes the effect of ground return(4) and is illustrated in the system shown in fig. 2-1.

![Distributed Circuit Parameters of an n-conductor Line]

Fig. 2-1 Distributed Circuit Parameters of an n-conductor Line.
2.2 Basic Assumptions.

The effect of finite ground conductivity and relative permittivity on the transmission of energy along a transmission line can be derived in terms of field concepts. In engineering, however, circuit concepts are more practical, since current and voltage can be easily measured. Hence the problem is to analyse the field and express the results in terms of the distributed circuit parameters.

In development of the analysis in this thesis the ground is considered homogeneous with a relative permeability of unity. The extension of the theory to multi-layer grounds with different permeabilities, does not require any new concepts although it becomes much more involved algebraically.

The electromagnetic field surrounding the transmission line consists primarily of the TM mode with some departure from this due to the losses in the ground. An Hertz vector of the \( \Pi_e \) type is used to describe the field. It has two components, one oriented in the direction of propagation and the other vertically into the ground. In the construction of the field solution the following assumptions are made:

1. At the frequency of interest \( \omega^2 \varepsilon \mu \) in air is much less than unity. This assumption is valid for frequencies in the range from 10 Hz to 1 MHz.
2. In the solution of the wave equation a first order approximation is considered sufficiently accurate for practical purposes in view of the physical irregularities of the transmission line.
3. The principle of superposition is used to find the complete solution for the system.

4. Air conductivity is considered negligible.

2.3 **Circuit Parameters**.

The ideal circuit parameter is derived first by assuming that the ground has infinite conductivity. For a ground of finite conductivity the circuit parameter is expressed as the sum of the ideal parameter plus a correction term for the ground condition. This analysis transforms an n-conductor line over a ground of finite conductivity to an n-conductor line over a ground of infinite conductivity.
Chapter 3. Derivation of the Hertz Vector due to Current in a Straight Wire.

The Hertz vector for a current along an infinite straight wire parallel to a flat ground is obtained by integrating the elemental Hertz vectors for current elements along the wire from minus infinity to plus infinity.

Fig. 3-1 Current Element Orientation.

Fig. 3-2 Cylindrical Coordinate System for the External Field.
3-1 Hertz Vector due to a Current Element in a Medium of Infinite Extent.

The Hertz vector due to a current element Idz in an isotropic homogeneous medium of infinite extent is derived in Appendix A, and is given by

\[
\begin{align*}
\Pi_x &= 0 \\
\Pi_y &= 0 \\
\Pi_z &= \frac{Ie^{\frac{-m_o R}{R}}}{4\pi m_o^2 R} \\
\end{align*}
\]

3-2 Hertz Vector due to a Current Element above a Ground with Infinite Conductivity.

The total field above ground due to a current element and its image, as shown in fig. 3-1, consists of a primary field, plus a secondary field due to the finite conductivity of the ground. Let the total field be

\[
\Pi_z = \Pi^1_{oz} + \Pi^2_{oz}
\]

where \( \Pi^1_{oz} \) is the primary field, and \( \Pi^2_{oz} \) is the secondary field vector. The suffix "o" designates the field vector above the ground surface.

From equation 3-4 we have

\[
\Pi^1_{oz} = \frac{j\omega Idz}{4\pi m_o^2} \left( \frac{e^{-m_o R_1}}{R_1} - \frac{e^{-m_o R_2}}{R_2} \right)
\]

where \( R_1 \) and \( R_2 \) are the distances from the point of interest to the current element and its image respectively.
3-3 Hertz Vector due to a Current Element above a Ground with Finite Conductivity.

When the ground has finite conductivity some energy will be dissipated, and the field may be described by means of two Hertz vector components, in the y and z directions. Equation B-15 gives the general solution in the coordinate system of fig. 3-2. However, the particular solution must meet the boundary conditions which may be stated as follows:

a. The field vanishes at an infinite distance from the source.

b. At the source only \( \tau_z \) exists and is finite, since the current element is orientated in the z direction.

c. The tangential E and H fields are continuous across the ground surface.

A further restriction is imposed due to the geometry of the system;

d. The fields are symmetrical about the y-z plane.

The form of the field functions may be constructed from equation B-15, Appendix B, by applying the above four restrictions. Due to (a) and (b) only Bessel functions of the first kind can appear in the solution, and due to (b) \( \tau_z \) can contain the zero order function but \( \tau_y \) cannot. Due to (d) the solutions can contain terms in \( \cos P \) only. Assume as a first approximation that only the leading term of the summation over the possible values of \( P \) is of significance. Finally in accordance with equation 3-6 let the vector above the ground be the sum of two functions. The Hertz vector for the region \( y \leq h \) may now be given by
where the suffix "1" designates the field below the ground surface. There now remains the evaluation of the functions $f_0, g_0, g_1, p_0$ and $p_1$ using condition (c). Condition (c) may be stated in terms of the Hertz vector components by means of equations A-10 and A-3 as follows:

\[
\frac{\partial}{\partial x} \left[ \frac{\partial \Pi_{ox}}{\partial y} + \frac{\partial \Pi_{oz}}{\partial z} \right] = \frac{\partial}{\partial x} \left[ \frac{\partial \Pi_{1y}}{\partial y} + \frac{\partial \Pi_{1z}}{\partial z} \right] \quad 3-11
\]

\[
- m_0^2 \frac{\partial \Pi_{oz}}{\partial z} + \frac{\partial}{\partial z} \left[ \frac{\partial \Pi_{ox}}{\partial y} + \frac{\partial \Pi_{oz}}{\partial z} \right] = - m_0^2 \Pi_{1z} + \frac{\partial}{\partial z} \left[ \frac{\partial \Pi_{1y}}{\partial y} + \frac{\partial \Pi_{1z}}{\partial z} \right] \quad 3-12
\]

\[
m_0^2 \left[ \frac{\partial \Pi_{oz}}{\partial y} - \frac{\partial \Pi_{ox}}{\partial z} \right] = m_1^2 \left[ \frac{\partial \Pi_{1z}}{\partial y} - \frac{\partial \Pi_{1x}}{\partial z} \right] \quad 3-13
\]

\[
m_0^2 \frac{\partial \Pi_{ox}}{\partial x} = m_1^2 \frac{\partial \Pi_{1y}}{\partial x} \quad 3.14
\]

Equation 3-11 may be written as

\[
\frac{\partial}{\partial x} \nabla \cdot \Pi_0 = \frac{\partial}{\partial x} \nabla \cdot \Pi_1
\]

integrating both sides yields
\[ \nabla \cdot \pi_o = \nabla \cdot \pi_l + f(z,y,u) \]

However, \( \pi_o \) and \( \pi_l \) are zero when \( x \) becomes infinite and hence \( f(z,y,u) \) is zero. Integrating both sides of equation 3-14 yields

\[ m_o^2 \pi_{oy} = m_1^2 \pi_{ly} + g(z,y,u) \]

Again both \( \pi_{oy} \) and \( \pi_{ly} \) are zero when \( x \) becomes infinite, hence \( g(z,y,u) \) is zero. With these results equations 3-11 through 3-14 reduce to the following equations

\[ \frac{\partial \pi_{oy}}{\partial y} + \frac{\partial \pi_{oz}}{\partial z} = \frac{\partial \pi_{ly}}{\partial y} + \frac{\partial \pi_{lz}}{\partial z} \quad 3-15 \]

\[ m_o^2 \pi_{oz} = m_1^2 \pi_{lz} \quad 3-16 \]

\[ m_o^2 \frac{\partial \pi_{oz}}{\partial y} = m_1^2 \frac{\partial \pi_{lz}}{\partial y} \quad 3-17 \]

\[ m_o^2 \pi_{oy} = m_1^2 \pi_{ly} \quad 3-18 \]

The various field functions may now be evaluated in terms of the function \( f_o(u) \) they are

\[ g_o(u) = \frac{\gamma_o - \gamma_1}{\gamma_o + \gamma_1} \quad f_o(u) \quad e^{-h\gamma_o} \quad 3-19 \]

\[ g_1(u) = \frac{2u\gamma_o\gamma^2}{\gamma_o + \gamma_1} \quad f_o(u) \quad e^{-h\gamma_o} \quad 3-20 \]

\[ p_o(u) = \frac{2u\gamma_o(1 - \gamma^2)}{(\gamma_1 + \gamma_o)(\gamma^2\gamma_1 + \gamma_o)} \quad f_o(u) \quad e^{-h\gamma_o} \quad 3-21 \]
\[ p_1(u) = r^2 p_0(u) \quad 3-22 \]

where

\[ r^2 = \frac{m_1^2}{m_0^2} \quad 3-23 \]

The function \( f_0(u) \) may be determined by letting

\[ \gamma_1 = \gamma_0 \quad \text{and} \quad m_1 = m_0, \]

the solution must then be the same as for a uniform medium of infinite extent. Then

\[ \int_{0}^{8} f_0(u) \ e^{(y-h)\gamma_0} J_0(r u) \ du = \frac{i\omega I \ e^{-m_0 R_1} dz}{4\pi m_0^2 R_1} \quad 3-24 \]

Then by transforming the left-hand-side of equation 3-24 to the same origin as the right-hand-side and letting

\[ f_0(u) = B \ \frac{u}{\gamma_0} \quad 3-25 \]

where \( B \) is a constant, and by making the substitution

\[ \gamma_0^2 = u^2 + m_0^2 \quad 3-26 \]

equation 3-29 becomes

\[ \int_{0}^{8} B e^{-y'y_0} J_0(r \sqrt{y_0^2 - m_0^2}) \ dy_0 \ = \frac{i\omega I \ e^{-m_0 R_1} dz}{4\pi m_0^2 R_1} \quad 3-27 \]

Assuming that \( m_0 \) is a small quantity in air compared with \( \gamma_0 \) and restricting the solution to frequencies below 1 MHz equation 3-27 becomes

\[ \int_{0}^{8} B e^{-y'y_0} J_0(r \gamma_0) \ dy_0 = \frac{i\omega I dz}{4\pi m_0^2 R_1} \quad 3-28 \]

which is Lipschitz's integral\(^{(19)}\). Hence
Using the same reasoning
\[ \int \frac{j \omega u I e^{-m_0 R_2} dz}{4 \pi m_0^2 R_2} = \int_0^\infty f_0(u) e^{-(y+h) \gamma_0} J_0(\rho u) \, du \] 3-30

Then combining equation 3-30 with 3-19 to obtain the second part of equation 3-7
\[ \int_0^\infty \xi_0(u) e^{-y \gamma_0} J_0(\rho u) \, du = -\frac{j \omega u I e^{-m_0 R_2} dz}{4 \pi m_0^2 R_2} \]
\[ + \int_0^\infty \frac{2 \gamma_0}{\gamma_1 + \gamma_0} f_0(u)e^{-y \gamma_0} J_0(\rho u) \, du \] 3-31

The last term of 3-31 corresponds to the secondary field due to finite ground conductivity
\[ \Pi_{o z}^2 = \int_0^\infty \frac{2 \gamma_0}{\gamma_1 + \gamma_0} f_0(u)e^{-y \gamma_0} J_0(\rho u) \, du \] 3-32

Then the complete Hertz vector in the region 0 \leq y \leq h is
\[ \Pi_{ox} = 0 \] 3-33
\[ \Pi_{oy} = \int_0^\infty \cos \gamma \frac{2 j \omega u (1 - \gamma^2) u^2 e^{-(y+h) \gamma_0}}{4 \pi m_0^2 (\gamma_1 + \gamma_0)(\gamma^2 \gamma_1 + \gamma_0)} J_1(\rho u) \, du \] 3-34
\[ \Pi_{oz} = \frac{j \omega u I dz}{4 \pi m_0^2} \left[ \frac{e^{-m_0 R_1}}{R_1} - \frac{e^{-m_0 R_2}}{R_2} + \int_0^\infty \frac{2 u e^{-y \gamma_0}}{(\gamma_1 + \gamma_0)} J_0(\rho u) \, du \right] \] 3-35

The current at any point z on the line is given by
\[ I = I_o e^{-yz} \] 3-36
where \( \gamma \) is the propagation constant in the \( z \) direction.

Noting that

\[
- \cos \gamma \ uJ_1(\gamma z) = \frac{\beta}{\delta z} \ J_0(\beta z)
\]

equations 3-34 and 3-35 become

\[
\Pi_{oy} = - \frac{2j\omega \mu I_o e^{-\gamma z}}{4\pi \mu_0} \frac{\beta}{\delta z} \int_0^\infty \frac{u(1 - \gamma^2)e^{-(y+h)\gamma_0}}{(\gamma_0 + \gamma_1)(\gamma^2\gamma_1 + \gamma_0)} J_0(\beta z)du \\
\Pi_{oz} = \frac{j\omega \mu I_o e^{-\gamma z}}{4\pi \mu_0} \left[ -\frac{m R_1}{R_1} e^{-(y+h)\gamma_0} - \frac{m R_2}{R_2} + \int_0^\infty \frac{2ue^{-(y+h)\gamma_0}}{(\gamma_1 + \gamma_0)} J_0(\beta z)du \right]
\]

3-4 Complete Hertz Vector for a Current in a Straight Conductor above Ground

The complete Hertz vector for the current-carrying conductor is now obtained from integration of the Hertz vectors due to the current elements along the line from minus infinity to plus infinity. The components of the Hertz vector are

\[
\Pi_x = 0  \quad 3-40
\]

\[
\Pi_y = - \frac{2j\omega \mu I_o e^{-\gamma z}}{4\pi \mu_0} \int_0^\infty e^{-\gamma z} \delta \int_0^\infty \frac{u(1 - \gamma^2)e^{-(y+h)\gamma_0}}{(\gamma_0 + \gamma_1)(\gamma^2\gamma_1 + \gamma_0)} J_0(\beta z)du \\
\]

\[
\Pi_z = \frac{j\omega \mu I_o e^{-\gamma z}}{4\pi \mu_0} \int_0^\infty e^{-\gamma z} \left[ -\frac{m R_1}{R_1} e^{-(y+h)\gamma_0} - \frac{m R_2}{R_2} \right] + \int_0^\infty \frac{2ue^{-(y+h)\gamma_0}}{(\gamma_1 + \gamma_0)} J_0(\beta z)du \right] dz  \quad 3-42
\]

4.1 Derivation of the Series Impedance Formulae.

Fig. 4-1 Two-Conductor Configuration.

Fig. 4-1 shows two typical conductors of an n-conductor transmission system. The total electric field on the surface of conductor $i$ is the sum of the electric fields due to all $n$ conductors. From equation A-10 the electric field on the $i$-th conductor due to the $j$-th conductor in the $z$ direction is given by

$$E_{iz} = -m_o \pi_{jz} + \frac{\partial}{\partial z} \left[ \frac{\partial \pi_{iv}}{\partial y} + \frac{\partial \pi_{iz}}{\partial z} \right]$$  \hspace{1cm} 4-1

Then the complete electric field on the surface of conductor $i$ due to all conductors is

$$E_{iz} = \sum_{j=1}^{n} -m_o \pi_{jz} + \frac{\partial}{\partial z} \left[ \frac{\partial \pi_{iv}}{\partial y} + \frac{\partial \pi_{iz}}{\partial z} \right]$$  \hspace{1cm} 4-2
The fields inside and outside conductor \( i \) must be equal at the boundary. If the internal impedance of conductor \( i \) is \( z_i \), then

\[
 z_i I_{10} e^{-\gamma z} = \sum_{j=1}^{n} m^2 \Pi_{jz} + \frac{\partial}{\partial z} \left[ \frac{\partial \Pi_{jy}}{\partial y} + \frac{\partial \Pi_{jz}}{\partial z} \right] \tag{4-3}
\]

where \( z_i \) is derived in Appendix C. The second term on the right-hand-side of equation 4-1 may be expressed as a gradient of the scalar potential, \( V_i \), where

\[
 \sum_{j=1}^{n} \frac{\partial}{\partial z} \left[ \frac{\partial \Pi_{jy}}{\partial y} + \frac{\partial \Pi_{jz}}{\partial z} \right] = -\frac{\partial V_i}{\partial z} \tag{4-4}
\]

From the general transmission line equation E-8 the voltage equation in phasor form can be written

\[
 -\frac{\partial V_i}{\partial z} = 2 \sum_{j=1}^{n} Z_{ij} I_{10} e^{-\gamma z} \tag{4-5}
\]

Hence by combining equations 4-3 and 4-5 the following formulae can be established for the self and mutual impedances, respectively

\[
 Z_{ii} = z_i + m^2 \Pi_{1z} \frac{1}{I_{10} e^{-\gamma z}} \tag{4-6}
\]

\[
 Z_{ij} = m^2 \Pi_{jz} \frac{1}{I_{10} e^{-\gamma z}} \tag{4-7}
\]

where \( j = 1, 2, \ldots, n \) except \( i \), and all functions are evaluated on the surface of conductor \( i \).

4.2 Evaluation of \( Z_{ii} \) and \( Z_{ij} \)

The evaluation of these impedances requires the evaluation of the infinite integrals in the \( \Pi_z \) function given
by equation 3-42. To evaluate $\Pi_z$ the assumption must be made that attenuation along the line is negligible, then

$$\mathcal{Y} = m_o$$

where

$$m_o = j \sqrt{\varepsilon \mu \omega^2} = jk$$

This is an ideal value for $\mathcal{Y}$ but is necessary due to the following two considerations:

1. To assume that the attenuation is not zero on an infinitely long line amounts to assuming a source of infinite energy and makes the integral infinite.
2. To assume a propagation velocity less than that of light makes the integral extremely difficult to evaluate.

Substituting $jk$ for $m_o$ and $\mathcal{Y}$ in equation 3-42

$$\Pi_z = I_o e^{-jkz} \int_{-\infty}^{\infty} e^{-jkz} \left[ \frac{j\omega}{4\pi m_o^2} \left( \frac{e^{-jkR_1}}{R_1} - \frac{e^{-jkR_2}}{R_2} \right) \right. \\
+ \left. \int_{0}^{\infty} \frac{j\omega}{4\pi m_o^2} \frac{-\mathcal{Y}_0^{(h+y)} - \mathcal{Y}_0^{(ru)}}{\mathcal{Y}_0 + \mathcal{Y}_1} du \right] dz$$

From equation 4-6 we have

$$z_{ii} = z_i + I_1 + I_2$$

where

$$I_1 = \frac{j\omega}{4\pi} \int_{-\infty}^{\infty} e^{-jkz} \left[ \frac{e^{-jkR_1}}{R_1} - \frac{e^{-jkR_2}}{R_2} \right] dz$$

$$I_2 = \frac{j\omega}{4\pi} \int_{-\infty}^{\infty} e^{-jkz} \int_{0}^{\infty} \frac{-\mathcal{Y}_0^{(h+y)} J_0^{(ru)} du}{\mathcal{Y}_0 + \mathcal{Y}_1} dz$$
To evaluate $I_1$, let

$$P_1 = R_1 + z \quad \text{4-13}$$

$$P_2 = R_2 + z$$

where

$$R_1 = \sqrt{\rho_1^2 + z^2} \quad \text{and} \quad R_2 = \sqrt{\rho_2^2 + z^2}$$

$$\rho_1^2 = b_{ij}^2 + (h_i - h_j)^2, \quad \rho_2^2 = b_{ij}^2 + (h_i + h_j)^2 \quad \text{4-14}$$

where $b_{ij}$ is the horizontal distance between the $i$th and $j$th conductors, assuming this distance to be much greater than the radius of conductor $i$, $h_i$ is the height of the $i$th conductor above ground and $h_j$ that of $j$th conductor. The general case is considered because it is required in the evaluation of $Z_{ij}$. When $j$ equals $i$, $b_{ij}$ becomes the radius of conductor $i$, and

$$\rho_1 = a_i, \quad \rho_2 = 2h_i,$$

where $a_i$ is the radius of conductor $i$, and the height $h_i$ is assumed to be much greater than the radius $a_i$.

With the change of variable from $z$ to $P_1$ and $P_2$ we have

$$dP_1 = \frac{P_1}{R_1} \, dz, \quad dP_2 = \frac{P_2}{R_2} \, dz \quad \text{4-16}$$

and $P_1 \to \infty$, $P_2 \to \infty$ when $z \to \infty$

$$P_1 \to \alpha_1, \quad P_2 \to \alpha_2 \quad \text{when } z \to - \infty$$

where $\alpha_1 = \frac{\rho_1^2}{2z}$, $\alpha_2 = \frac{\rho_2^2}{2z} \quad \text{4-17}$
for a very large $z$. Then

$$I_1 = \frac{\omega}{4\pi} \left[ \int_{\alpha_1}^{\infty} \frac{-jk\rho_1}{\rho_1} d\rho_1 - \int_{\alpha_2}^{\infty} \frac{-jk\rho_2}{\rho_2} d\rho_2 \right]$$

where the integrals are defined as the exponential integrals. Hence equation 4-18 can be written as

$$I_1 = \frac{\omega}{4\pi} \left[ \text{li}(e^{-jk\alpha_1}) - \text{li}(e^{-jk\alpha_2}) \right]$$

where

$$\text{li}(e^{-t}) = c + \ln(t) + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!}$$

and $c$ is Euler's constant. For a small value of $t$ the first two terms in the above expression predominate and all other terms can be neglected, then

$$I_1 = \frac{\omega}{4\pi} \left[ \ln(jk\alpha_2) - \ln(jk\alpha_1) \right]$$

$$= \frac{\omega}{2\pi} \ln \frac{\rho_2}{\rho_1}$$

Equation 4-20 gives the self reactance per unit length of line conductor when the ground has infinite conductivity.

To evaluate $I_2$ let the order of integration be changed, and $I_2$ becomes

$$I_2 = \frac{\omega}{4\pi} \int_0^{\infty} 2ue^{\frac{-\gamma_0}{\gamma_0 + \gamma_1}} \int e^{-jkz} J_0(\rho u) \, dz \, d\rho$$

Since $J_0(\rho u)$ and $\cos(kz)$ are even functions, and $\sin(kz)$ is an odd function of $kz$
\[
\int_{-\infty}^{\infty} e^{-j\alpha z} J_0(\alpha z) dz = 2 \int_{0}^{\infty} \cos(\alpha z) J_0(\alpha z) dz
\]
\[
= 0, \quad k > u
\]
\[
= \frac{2\cos(x \sqrt{u^2 - k^2})}{\sqrt{u^2 - k^2}}
\]
\[
= \frac{2\cos \gamma_o x}{\gamma_o}, \quad k \leq u
\]

Hence for \( k \leq u \)
\[
I_2 = \frac{j\omega u}{\pi} \int_{k}^{\infty} \frac{ue \cos \gamma_o x}{\gamma_o (\gamma_1 + \gamma_o)} du
\]

To change the lower limit of the above integration, substitute
\[
\gamma_o^2 = u^2 - k^2, \quad du = \frac{\gamma_o d \gamma_o}{u}
\]
then
\[
I_2 = \frac{j\omega u}{\pi} \int_{0}^{\infty} \frac{e^{-\gamma_o (h+y)}}{\gamma_1 + \gamma_o} \cos \gamma_o x d\gamma_o
\]
\[
= \frac{j\omega u}{\pi} \frac{\gamma_1 - \gamma_o}{\gamma_1^2 - \gamma_o^2} e^{-\gamma_o (h+y)} \cos \gamma_o x d\gamma_o
\]

To find results similar to Carson's but including the effect of relative ground permittivity, in accordance with reference (9) let
\[
s^2 = 1 - \frac{j \varepsilon_0 \omega (n-1)}{\sigma}
\]

since
\[
\gamma_1^2 = u^2 + \sigma^2 = u^2 + j \omega \sigma - \varepsilon_1 \omega^2
\]
and
\[
\gamma_o^2 = u^2 - \varepsilon_0 \omega^2
\]
hence
\[ \gamma _1^2 - \gamma _0^2 = j \omega \sigma s^2 \]

Further let
\[ y' = \sqrt{\omega \sigma} y, \quad h' = \sqrt{\omega \sigma} h, \quad x' = \sqrt{\omega \sigma} x, \quad v = \frac{\sigma_0}{\sqrt{\omega \sigma}} \]

Then
\[ I_2 = \frac{\omega u}{\pi s^2} \int_0^\infty (\sqrt{v^2 - j u^2} - v) e^{-v(h' + y')} \cos x' v dv \]

Finally let
\[ y'' = sy', \quad h'' = sh', \quad x'' = sx', \quad v = su \]

then
\[ I_2 = \frac{\omega u}{\pi} \int_0^\infty (\sqrt{u^2 + j u} - u) e^{-u(h'' + y'')} \cos x'' u du \]

Equation 4-30 is the same equation as given by Carson\(^{(2)}\) except for \( s \) (the ground permittivity correction factor). In his case \( s = 1 \) because he assumed that the relative permittivity of the ground equals one.

Then
\[ I_2 = \frac{\omega u}{\pi} (P + jQ) \]

and
\[ Z_{ii} = z_i + \frac{j \omega u}{2 \pi} \ln \frac{2h_i}{a_i} + \frac{\omega u}{\pi} (P_{ii} + jQ_{ii}) \]

The last term of equation 4-32 is the correction factor for the finite ground conductivity and relative permittivity. The exact solutions for \( P_{ii} \) and \( Q_{ii} \) are given in Appendix D, and numerical results are included in chapter 7.

Using the same procedure as above, we have for \( Z_{ij} \)
\[ Z_{ij} = \frac{\lambda i u}{2\pi} \ln \frac{S'_{ij}}{S_{ij}} + \frac{\omega u}{\pi} (P_{ij} + Q_{ij}) \] 4-33

where \( S'_{ij} \) is the distance from conductor \( i \) to the image of conductor \( j \), and \( S_{ij} \) is the distance from conductor \( i \) to conductor \( j \).
Chapter 5. Evaluation of the Shunt Admittance.

5.1 Derivation of Shunt Admittance Formulae.

From equation 2-1 the transmission line equation of the n-conductor system can be written

$$- \frac{\partial I_i}{\partial z} = j\omega \sum_{j=1}^{n} C_{ij} V_j$$  \hspace{1cm} 5-1

where \( i = 1, 2, 3, 4, \ldots, n. \)

Making the same assumptions as in equation 4-8, and noting that \( \frac{\partial}{\partial z} = -jk \), differentiation yields

$$\sum_{j=1}^{n} C_{ij} V_j = \frac{k}{\omega} I_i \hspace{1cm} i = 1, 2, (3, 4, \ldots), n.  \hspace{1cm} 5-2$$

Then solving this system of equations for the voltages, we have

$$V_i = \frac{k}{\omega} \sum_{j=1}^{n} K_{ij} I_j \hspace{1cm} i = 1, 2, \ldots, n.  \hspace{1cm} 5-3$$

where \( K_{ij} \) is Maxwell's potential coefficient. Substituting these results into equation 4-4 yields

$$\frac{k}{\omega} K_{ij} I_j = - \frac{\partial \Pi_{iv}}{\partial y} - \frac{\partial \Pi_{iz}}{\partial z}$$

hence

$$K_{ij} = - \frac{\omega}{k I_j} \left[ \frac{\partial \Pi_{iv}}{\partial y} + \frac{\partial \Pi_{iz}}{\partial z} \right]  \hspace{1cm} 5-4$$

Thus the potential coefficients can be derived from the field components.
5.2 Evaluation of the Maxwell Potential Coefficients $K_{ii}$ and $K_{1j}$

The same assumptions are made here regarding the evaluation of the infinite integrals as stated in the preceding chapter for equation 4-8.

From equations 5-4, 3-41 and 3-42

$$K_{ii} = I_1 + I_2 + I_3$$  \hspace{1cm} 5-5

where

$$I_1 = \frac{1}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} e^{-jkz} \left[ \frac{e^{-jkR_1}}{R_1} - \frac{e^{-jkR_2}}{R_2} \right] dz$$  \hspace{1cm} 5-6

$$I_2 = \frac{1}{4\pi \varepsilon_0} \int_{-\infty}^{\infty} e^{-jkz} \left( \frac{2ue}{\gamma_0 + \gamma_1} \right) J_0(ru) \ dudz$$  \hspace{1cm} 5-7

$$I_3 = \frac{1}{2\pi \varepsilon_0} \frac{d}{dy} \int_{-\infty}^{\infty} e^{-jkz} \frac{d}{dz} \left( \frac{u(1-\gamma^2)e^{-(y+h)\gamma}}{\gamma_1^2 + \gamma_0^2} \right) J_0(ru) \ dudz$$  \hspace{1cm} 5-8

$I_1$ and $I_2$ have been evaluated in chapter 4.

$$I_1 + I_2 = \frac{1}{2\pi \varepsilon_0} \ln \frac{\rho_2}{\rho_1} + \frac{1}{\pi \varepsilon_0} (P + jQ) = \frac{1}{2\pi \varepsilon_0} \ln \frac{\rho_2}{\rho_1} + \frac{1}{\pi \varepsilon_0} (Q - jP)$$  \hspace{1cm} 5-9

$I_3$ is evaluated as follows

$$I_3 = \frac{-1}{2\pi \varepsilon_0} \int_{0}^{\infty} \gamma'(1-\gamma^2)e^{-(y+h)\gamma} \left( \frac{u}{\gamma_1 + \gamma_0} \right) \left( \frac{1}{\gamma_1^2 + \gamma_0^2} \right) e^{-jkz} \frac{d}{dz} J_0(ru) \ dudz$$  \hspace{1cm} 5-10

$$= \frac{-1}{2\pi \varepsilon_0} \int_{0}^{\infty} \frac{u\gamma'(1-\gamma^2)e^{-(y+h)\gamma}}{(\gamma_1 + \gamma_0)(\gamma_1^2 + \gamma_0^2)} \left[ e^{-jkz} J_0(\rho_2) + \int_{0}^{\infty} jke^{-jkz} J_0(\rho_2) \ dz \right] du$$  \hspace{1cm} 5-11
Since the first term in the bracket is zero and the second term has been evaluated in equation 4-22, equation 5-11 becomes

\[
I_3 = -\frac{1}{\pi \varepsilon_0} \int_k^\infty \frac{u(1-\gamma^2) e^{-(y+h)\gamma_o \cos \gamma_o x}}{(\gamma_1 + \gamma_o)(\gamma^2 \gamma_1 + \gamma_o)} \, du
\]

Noting that \( \gamma_o^2 = u^2 - k^2 \), \( u \, du = \gamma_o \, d\gamma_o \)

and \( \gamma_o - \gamma_o \gamma^2 = \gamma_o + \gamma_1 \gamma^2 - \gamma^2(\gamma_1 + \gamma_o) \)

the integral breaks into two parts, and

\[
I_3 = -\frac{1}{\pi \varepsilon_0} \int_0^\infty \frac{e^{(y+h)\gamma_o \cos \gamma_o x}}{\gamma_1 + \gamma_o} \, d\gamma_o + \frac{1}{\pi \varepsilon_0} \int_0^\infty \frac{2e^{-(y+h)\gamma_o \cos \gamma_o x}}{\gamma^2 \gamma_1 + \gamma_o} \, d\gamma_o
\]

\[
= -\frac{1}{\pi \varepsilon_0} (Q-JP) + \frac{1}{\pi \varepsilon_0} (M+JN) = -I_2 + \frac{1}{\pi \varepsilon_0} (M+JN)
\]

where

\[
M+JN = \int_0^\infty \frac{e^{-(y+h)\gamma_o \cos \gamma_o x}}{\gamma_1 + \frac{1}{\gamma^2} \gamma_o} \, d\gamma_o
\]

Then

\[
K_{ii} = \frac{1}{2\pi \varepsilon_0} \ln \frac{2h_i}{a_i} + \frac{1}{\pi \varepsilon_0} (M_{ii}+JN_{ii})
\]

and

\[
K_{ij} = \frac{1}{2\pi \varepsilon_0} \ln \frac{S_{ij}}{S_{ij}} + \frac{1}{\pi \varepsilon_0} (M_{ij}+JN_{ij})
\]

The last terms in equations 5-15 and 5-16 give the correction factors for finite ground conductivity and relative permittivity to Maxwell's potential coefficients. An approximation of \( M \) and \( N \) is given in section 5.3 of this chapter and the numerical results are included in chapter 7.
5.3 Evaluation of $M_{ij}$ and $N_{ij}$

Making the same substitutions as in 4-25, 4-26 and 4-27, equation 5-14 becomes

$$M + jN = \int_0^\infty \frac{e^{-v(h'+y') \cos x' v}}{\sqrt{v^2 + j s^2 + \frac{v}{\gamma^2}}} \, dv$$  \hspace{1cm} 5-17

Next let $js^2 = a^2, \quad \frac{1}{\gamma^2} = b, \quad \text{then} \quad (12)$

$$M + jN = \int_0^\infty \frac{e^{-v(h'+y') \cos x' v}}{\sqrt{v^2 + a^2 + bv}} \, dv$$  \hspace{1cm} 5-18

The numerical computation of this integral is involved due to the large number of variables. An approximate solution will be developed below.

For large values of $v$ the integrand vanishes, hence 5-18 may be written as

$$M + jN \approx \int_0^v \frac{e^{-v(h'+y') \cos x' v}}{\sqrt{v^2 + a^2 + bv}} \, dv$$  \hspace{1cm} 5-19

If within the range of values from $0$ to $v_0$,

$$\left| 2av \right| < \left| v + a \right|^2$$  \hspace{1cm} 5-20

then

$$\sqrt{v^2 + a^2} \approx v + a - \frac{av}{v + a}$$  \hspace{1cm} 5-21

The limitations of this assumption will be discussed in chapter 7. Substituting 5-21 into 5-19 (11)
\[ M + jN = \int_0^\infty \frac{(v + a) e^{-v(h' + y')}}{(v + r_1)(v + r_2)(b + 1)} dv \]

\[ = \frac{1}{(b+1)(r_2-r_1)} \int_0^\infty \left[ \frac{r_2}{v + r_1} - \frac{r_1}{v + r_2} \right] e^{v(h' + y')} \cos x' \, dv \]

where

\[ r_1 = \frac{a}{2} \left( 1 - \sqrt{1 - \frac{4}{l + b}} \right), \quad r_2 = \frac{a}{2} \left( 1 + \sqrt{1 - \frac{4}{l + b}} \right) \]

Next let

\[ g_1 = h' + y' - jx' = \sqrt{\omega i \theta} \quad \text{Re}^j \theta = R'e^{-j\theta} \]

\[ g_2 = h' + y' + jx' = \sqrt{\omega i \theta} \quad \text{Re}^j \theta = R'e^{j\theta} \]

\[ M + jN = \frac{1}{2(b+1)(r_2-r_1)} \int_0^\infty \left[ \frac{r_2}{v + r_1} - \frac{r_1}{v + r_2} \right] \left[ e^{-g_1 v} + e^{-g_2 v} \right] dv \]

Equation 5-25 includes four separate integrals of similar form. Let

\[ I_1 = \frac{1}{2(b+1)(r_2-r_1)} \int_0^\infty \frac{e^{-g_1 v}}{v + r_1} dv \]

and

\[ v + r_1 = w \]

then

\[ I_1 = \frac{-r_2 e^{g_1 r_1}}{2(b+1)(r_2-r_1)} \int_{r_1}^\infty \frac{e^{-g_1 w}}{w} dw = \frac{-r_2 e^{g_1 r_1}}{2(b+1)(r_2-r_1)} \ln(e^{-g_1 r_1}) \]

The other three integrals in 5-25 have similar solutions.

Therefore
\[ M + jN = \frac{1}{2(b+1)(r_2-r_1)} \left[ -r_2 e^{g_1 r_1 i(i)} e^{-g_1 r_1} + r_1 e^{g_1 r_2 i(i)} e^{-g_1 r_2} ight. \\
\left. -r_2 e^{g_2 r_1 i(i)} e^{-g_2 r_1} + r_1 e^{g_2 r_2 i(i)} e^{-g_2 r_2} \right] \] 5-29

The numerical results of the evaluation of \( M \) and \( N \) as given by equation 5-29 are included in chapter 7.

6.1 Calculation of the Propagation Constants.

The transmission line equations for an n-conductor system may be written in matrix and phasor form as follows

\[
\frac{\partial^2 V_p}{\partial z^2} = Z(\omega) Y(\omega) V_p \tag{6-1}
\]

\[
\frac{\partial^2 I_p}{\partial z^2} = Y(\omega) Z(\omega) I_p \tag{6-2}
\]

where \( V_p \) and \( I_p \) are column vectors of n terms and \( Z \) and \( Y \) are \( n \times n \) symmetric matrices. The suffix \( p \) denotes the line coordinate system.

The systems of second order differential equations represented by equations 6-1 and 6-2 can be solved by transforming them into a new coordinate system wherein the transformed \( ZY \) or \( YZ \) matrix becomes a diagonal matrix.

Then equations 6-1 and 6-2 become

\[
\frac{\partial^2 V_c}{\partial z^2} = \gamma^2 V_c \tag{6-3}
\]

\[
\frac{\partial^2 I_c}{\partial z^2} = \gamma^2 I_c \tag{6-4}
\]

where the suffix \( c \) denotes the mode coordinate system.

The eigenvalues of the system may be denoted by \( \gamma_i^2, \ i = 1, 2, 3, \ldots, n \), and are obtained from

\[
| ZY - I \gamma^2 | = 0 \quad \text{or} \quad | YZ - I \gamma^2 | = 0 \tag{6-5}
\]
since \((ZY)^t = Y^t Z^t = YZ\)

Here \(I\) is a unit diagonal matrix. The solutions of equations 6-3 and 6-4 are

\[
V_c = e^{\gamma_1 z_1} + e^{-\gamma_1 z_2} \quad 6-6
\]

\[
I_c = e^{\gamma_1 d_1} + e^{-\gamma_1 d_2} \quad 6-7
\]

where \(\gamma_i\) represents the \(n\) propagation constants in the \(n\)-modes of the system. The solution in the line coordinate system, designated by the suffix \(p\), is now determined as follows

\[
V_p = R V_c \quad \text{and} \quad I_p = S I_c \quad 6-8
\]

where \(R\) and \(S\) represent \(n \times n\) transformation matrices, each consisting of \(n\) columns of eigenvectors of the system. Each eigenvector is determined from the corresponding eigenvalue by means of the following equations

\[
(ZY - \gamma_i^2 I) R(i) = 0 \quad 6-9
\]

\[
(YZ - \gamma_i^2 I) S(i) = 0 \quad 6-10
\]

where \(R(i)\) is the \(i\)th column of the \(R\) matrix, and \(S(i)\) the \(i\)th column of the \(S\) matrix. In general

\[
R^t S = S^t R = D \quad 6-11
\]

where \(D\) is a diagonal matrix, where the elements may be complex.

6.2 **Formation of the Impedance Matrix \(Z\).**

The impedance matrix \(Z\) has two types of elements, the diagonal and the off-diagonal elements. The diagonal elements are given by equation 4-32 and the off-diagonal elements
by equation 4-33. Let the diagonal elements be

\[ Z_{ii} = R_{ii} + j\omega L_{ii} \]  

where

\[ R_{ii} = R_{int} + \frac{j\omega}{\pi} P_{ii} \]

where \( R_{int} \) is the internal AC resistance of the conductor or the real part of \( z_i \) in equation 4-32, and \( \frac{j\omega}{\pi} P_{ii} \) is the real part of the ground correction factor. Let

\[ z_i = R_{int} + j\omega L_{int} \]

then

\[ L_{ii} = \frac{\mu}{2\pi} \ln \frac{2h_i}{a_i} + \frac{\mu}{\pi} Q_{ii} + L_{int} \]

where \( \frac{\mu}{\pi} Q_{ii} \) is the imaginary part of the ground correction factor.

The off-diagonal elements are given by equation 4-33

\[ Z_{ij} = j \frac{\omega \mu}{2\pi} \ln \frac{S_{ij}^i}{S_{ij}} + \frac{\omega \mu}{\pi} (P_{ij} + jQ_{ij}) \]

Now if we write

\[ Z_{ij} = R_{ij} + j\omega L_{ij} \]

then

\[ R_{ij} = \frac{j\omega}{\pi} P_{ij} \]

and

\[ L_{ij} = \frac{\mu}{2\pi} \ln \frac{S_{ij}^i}{S_{ij}} + \frac{\mu}{\pi} Q_{ij} \]

For all off-diagonal terms

\[ Z_{ij} = Z_{ji} \]
6.3 Formation of the Admittance Matrix $Y$.

The general form of the potential coefficients was derived in chapter 5. There is no distinction in form between diagonal and off-diagonal elements.

$$K_{ij} = \frac{1}{2\pi \varepsilon} \ln \frac{S_i^1}{S_j^1} + \frac{1}{\pi \varepsilon} (M_{ij} + jN_{ij}) \quad 6-21$$

If $K_{ij}$ is written as

$$K_{ij} = K_{ij}^l + \Delta K_{ij} \quad 6-22$$

where $\Delta K_{ij}$ is the correction term for the ground effect. Then the capacitance matrix of the transmission system may be written as

$$C' - \Delta C = (K' + \Delta K)^{-1} = K'^{-1} - K'^{-1} \Delta KK'^{-1} \quad 6-23$$

Defining

$$C' = K'^{-1}$$

gives

$$\Delta C = C'\Delta KC' = \frac{1}{\pi \varepsilon} C'(M + jN)C' \quad 6-24$$

Hence the corrected capacitance matrix is

$$C = C' - \frac{C'}{\pi \varepsilon} (M + jN)C' = C'(I - \frac{1}{\pi \varepsilon} (M + jN)C') \quad 6-25$$

The admittance matrix $Y$ can be written

$$Y = j\omega C = j\omega C'(I - \frac{1}{\pi \varepsilon} (M + jN)C') = G + j\omega C'' \quad 6-26$$

where

$$G = \frac{1}{\pi \varepsilon} \omega C'NC' \quad 6-27$$

and

$$C'' = C'(I - \frac{1}{\pi \varepsilon} MC') \quad 6-28$$
6.4 The A, B, C, D Parameters of the System.

By analogy to the single conductor case, the behaviour of an n-conductor transmission line can be described by the following equation:

\[
\begin{bmatrix}
V_s \\
I_s \\
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix} \begin{bmatrix}
V \_l \\
I \_l \\
\end{bmatrix}
\]

where the parameters A, B, C and D are \( n \times n \) matrices. The column vectors \( V \_l \) and \( I \_l \) are voltages and currents at a distance \( l \) from the sending end, and \( V \_s \) and \( I \_s \) those at the sending end. The A, B, C, D and the characteristic impedance matrices are derived in reference (8) and are

\[
A = R(\cosh \gamma \_1 l) R^{-1}
\]

\[
B = R \left[ \frac{\sinh \gamma \_1 l}{\gamma \_1} \right] R^{-1} Z
\]

\[
C = S \left[ \frac{\sinh \gamma \_1 l}{\gamma \_1} \right] S^{-1} Y
\]

\[
D = S(\cosh \gamma \_1 l) S^{-1}
\]

\[
Z_0 = R \gamma \_1^{-1} R^{-1} Z
\]
Chapter 7. Numerical Results for a 500 kV Transmission Line.

7.1 Description of the Transmission Line.

The ground effect on the distributed circuit parameters for an overhead transmission line is dependent on line geometry, ground conductivity, ground relative permittivity, and frequency. The line geometry can have a large number of variations. In order to illustrate the significance of the formulae developed in Chapters 4, 5 and 6 a typical 500 kV line is chosen. For the given geometry the ground conductivity, relative permittivity, and frequency are varied. The 500 kV line is chosen because recent developments in long distance energy transmission have called for more detailed study of the properties of lines at this voltage level than has previously been available. The single circuit line without overhead ground wires consists of:

A bundle of four conductors at the corners of an 18 inch square, per phase.
Conductor size ................ 583.2 MCM ACSR
Conductor DC resistance ........ 0.1764 ohm/mile at 50°C
Conductor diameter (including stranding factor) ............ 0.948 in
Average phase spacing .......... 40 ft.
Average conductor height ....... 54 ft.

The line conductors are at equal heights above the ground in a flat array. The ground conditions were considered to vary from dry rock to wet marsh land. This represents a ground relative permittivity range from 10 to 50 and a ground
conductivity range from $10^{-5}$ to 1 mho/m.

7.2 **Ideal Parameters.**

From equation 4-33

$$L_{1j} = \frac{\mu l}{2\pi} \ln \frac{S_{ij}}{S_{1j}} \quad \text{h/m} \quad 7-1$$

Inserting the lengths in the above equation and converting the units to milli-henries per mile gives the following inductance matrix, not including internal conductor inductance

$$L = \begin{bmatrix} 1.63 & .339 & .167 \\ .339 & 1.63 & .339 \\ .167 & .339 & 1.63 \end{bmatrix} \quad \text{mh/mile} \quad 7-2$$

Inverting this matrix and multiplying by a known constant gives the capacitance matrix in units of micro-farads per mile

$$C = \begin{bmatrix} .0183 & -.00302 & -.00148 \\ -.00302 & .0189 & -.00302 \\ -.00148 & -.00302 & .0183 \end{bmatrix} \quad \text{uf/mile} \quad 7-3$$

with the inductance in henries per meter and the potential coefficients in darafs per meter the matrices are,

$$L = 2 \times 10^{-7} \begin{bmatrix} 5.10 & 1.06 & 0.520 \\ 1.06 & 5.10 & 1.06 \\ 0.520 & 1.06 & 5.10 \end{bmatrix} \quad \text{h/m} \quad 7-4$$

$$K = 18 \times 10^9 \begin{bmatrix} 5.10 & 1.06 & 0.520 \\ 1.06 & 5.10 & 1.066 \\ 0.520 & 1.06 & 5.10 \end{bmatrix} \quad \text{darafs/m} \quad 7-5$$
7.3 **Skin Effect.**

For the conductor with a radius of .479 in and with a bundle spacing of 18 in, the proximity effect is considered negligible. The internal conductor impedance per bundle is taken as one quarter of the internal impedance per conductor and is calculated from C-16, C-17, or C-21, C-22 or C-26 and C-27 depending on the magnitude of $\sqrt{\omega \mu} \delta a$.

7.4 **Impedance Correction.**

The impedance correction terms depend on the frequency, ground conductivity, and relative permittivity. Figs. 7.1 and 7.2 show the correction factors for the diagonal elements of the impedance matrix, according to the formulae,

$$L'_{ii} = L_{ii} + 2Q_{ii}$$  \hspace{1cm} 7-6

$$R'_{ii} = R_{ii} + \frac{\omega \mu}{\pi} P_{ii}$$  \hspace{1cm} 7-7

assuming no permittivity correction. The correction factors for the relative permittivity of 10 are shown in figs. 7.3 and 7.4. Since the difference in values between the diagonal elements and the off-diagonal elements is not great the curves of the off-diagonal elements have been omitted from the remaining figures. Figs. 7.3 through 7.6 show the correction factors for relative permittivities of 10 and 40. By comparing fig. 7.3 through 7.6 with figs. 7.1 and 7.2 it may be seen that for conductivities above $10^{-2}$ mho/m the relative permittivity has negligible effect, but the effect becomes quite pronounced for
conductivities below $10^{-3}$ mho/m. For example, the correction matrices without ground permittivity correction at a frequency of 10 kHz are

$R = \begin{bmatrix}
123 & 123 & 121 \\
123 & 123 & 123 \\
121 & 123 & 123
\end{bmatrix}$ ohm/mile 7-8

$L = \begin{bmatrix}
.623 & .605 & .558 \\
.605 & .623 & .605 \\
.558 & .605 & .623
\end{bmatrix}$ mh/mile 7-9

when $\sigma = 10^{-5}$ mho/meter. For the same frequency but $\sigma = 10^{-2}$ mho/meter

$R = \begin{bmatrix}
60.8 & 58.0 & 50.6 \\
58.0 & 60.8 & 58.0 \\
50.6 & 58.0 & 60.8
\end{bmatrix}$ ohm/mile 7-10

$L = \begin{bmatrix}
.137 & .126 & .099 \\
.126 & .137 & .126 \\
.099 & .126 & .137
\end{bmatrix}$ mh/mile 7-11

For ground relative permittivity of 10 and at the frequency of 10kHz.

$R = \begin{bmatrix}
279 & 279 & 278 \\
279 & 279 & 279 \\
278 & 279 & 279
\end{bmatrix}$ ohm/mile 7-12
when \( \sigma = 10^{-5} \text{ mho/meter} \), for the same frequency but \( \sigma = 10^{-2} \text{ mho/meter} \)

\[
L = \begin{bmatrix}
0.698 & 0.677 & 0.629 \\
0.677 & 0.698 & 0.677 \\
0.629 & 0.677 & 0.698 \\
\end{bmatrix} \text{ mh/mile} \quad 7-13
\]

\[
R = \begin{bmatrix}
61.2 & 58.4 & 51.0 \\
58.4 & 61.2 & 58.4 \\
51.0 & 58.4 & 61.2 \\
\end{bmatrix} \text{ ohm/mile} \quad 7-14
\]

\[
L = \begin{bmatrix}
0.137 & 0.126 & 0.099 \\
0.126 & 0.137 & 0.126 \\
0.099 & 0.126 & 0.137 \\
\end{bmatrix} \text{ mh/mile} \quad 7-15
\]

From figs. 7.1 through 7.6 it may be seen that for conductivity below \( 10^{-2} \text{ mho/m} \) and a frequency above 1 kHz the ground relative permittivity should be included in the calculation.

7.5 Admittance Correction.

The admittance correction terms were evaluated by the two methods described in Chapter 5. First the integral in equation 5-14 was evaluated using numerical methods and then the correction terms were obtained from equation 5-29. Figs. 7.7 through 7.12 show a comparison of the correction terms obtained by the two methods. The discrepancy between the two sets of values increases with decrease in ground conductivity. For the range of conductivities greater than \( 10^{-5} \text{ mho/m} \) and the range of relative permittivities between 10 and 50 the maximum
discrepancy is of the order of 10%. This is considered tolerable for propagation calculations in view of the relatively small magnitude of the correction terms as compared with the potential coefficients. The difference in the magnitude between the diagonal elements and the off-diagonal elements is of the order of 2% at the lowest conductivities and thus the off-diagonal elements are not included in figs. 7.7 through 7.12. For conductivities of $10^{-3}$ mho/m or greater all the correction factors become negligibly small. In calculating the propagation constants a correction of 3% or greater in the diagonal elements is considered significant. A comparison of exact and approximate values of the correction factors and their magnitude are shown in figs. 7.7 through 7.12. This requires correction factors to be calculated for conductivities less than $10^{-3}$ mho/m and frequencies greater than 100 Hz. The potential coefficient terms are corrected according to the formula

$$K_{ij}' = K_{ij} + 18 \times 10^9 (2M_{ij} + j2N_{ij})$$

7-16

To illustrate the magnitude, the potential coefficient correction terms for the diagonal elements at frequencies of 1 kHz, 10 kHz and 100 kHz and for conductivities of $10^{-3}$ mho/m and $10^{-5}$ mho/m for a relative permittivity of 10 are shown in table 7.1.
Table 7.1 Potential coefficient correction terms.

<table>
<thead>
<tr>
<th></th>
<th>2M</th>
<th>2N</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f = 1 \text{ kHz} )</td>
<td>( \sigma = 10^{-3} \text{ mho/m} )</td>
<td>( 0.03 )</td>
</tr>
<tr>
<td></td>
<td>( \sigma = 10^{-5} \text{ mho/m} )</td>
<td>( 0.03 )</td>
</tr>
<tr>
<td>( f = 10 \text{ kHz} )</td>
<td>( \sigma = 10^{-3} \text{ mho/m} )</td>
<td>( 0.003 )</td>
</tr>
<tr>
<td></td>
<td>( \sigma = 10^{-5} \text{ mho/m} )</td>
<td>( 0.43 )</td>
</tr>
<tr>
<td>( f = 100 \text{ kHz} )</td>
<td>( \sigma = 10^{-3} \text{ mho/m} )</td>
<td>( 0.028 )</td>
</tr>
<tr>
<td></td>
<td>( \sigma = 10^{-5} \text{ mho/m} )</td>
<td>( 0.66 )</td>
</tr>
</tbody>
</table>

7.6 Conclusion.

It has been found that the impedance should always be corrected for ground conductivity. In addition when \( \sigma/\omega\varepsilon \) is less than 180, the impedance should be corrected for the ground relative permittivity and the capacitance should be corrected for both ground conductivity and relative permittivity. When \( \sigma/\omega\varepsilon \) is greater than 180, these correction terms are small, less than 3% approximately, and can be neglected.
**FIG 7-1**

IMPEDANCE CORRECTION FACTOR \( Q \) WITH \( \theta = 0 \), \( d = 0 \)

**FIG 7-2**

IMPEDANCE CORRECTION FACTOR \( P \) WITH \( \theta = 0 \), \( d = 0 \)
FIG 7-3
IMPEDANCE CORRECTION FACTOR Q WITH \( n = 10 \)

FIG 7-4
IMPEDANCE CORRECTION FACTOR P WITH \( n = 10 \)
FIG 7-5
IMPEDANCE CORRECTION FACTOR $Q$ WITH $n=50, \theta=0$

FIG 7-6
IMPEDANCE CORRECTION FACTOR $P$ WITH $n=50, \theta=0$
FIG 7-7
POTENTIAL COEFFICIENT CORRECTION FACTOR N WITH \( \theta = 0 \), \( d = 10^5 \) mho/m

![Graph of N vs Frequency for different values of n]

FIG 7-8
POTENTIAL COEFFICIENT CORRECTION FACTOR M WITH \( \theta = 0 \), \( d = 10^5 \) mho/m

![Graph of M vs Frequency for different values of n]
FIG 7-9
POTENTIAL COEFFICIENT CORRECTION FACTOR \( N \) WITH \( \theta = 0, \sigma = 10^{-4} \text{ mho/m} \)

\[ N \]

\[ 10^{-10} \]

\[ 10^{-20} \]

FREQUENCY HZ

\[ n=50 \]
\[ n=40 \]
\[ n=30 \]
\[ n=20 \]
\[ n=10 \]

FIG 7-10
POTENTIAL COEFFICIENT CORRECTION FACTOR \( M \) WITH \( \theta = 0, \sigma = 10^{-4} \text{ mho/m} \)

\[ M \]

\[ 10^{-10} \]

\[ 10^{-20} \]

FREQUENCY HZ

\[ n=50 \]
\[ n=40 \]
\[ n=30 \]
\[ n=20 \]
\[ n=10 \]
FIG 7-11
POTENTIAL COEFFICIENT CORRECTION FACTOR N WITH θ = 0, \( d = 10^{-3} \text{ mho/m} \)

FIG 7-12
POTENTIAL COEFFICIENT CORRECTION FACTOR M WITH θ = 0, \( d = 10^{-3} \text{ mho/m} \)
Chapter 8. **Calculation of the Mode Propagation Constants.**

The correction factors given in Chapter 7 are inserted into the transmission line equations, 6-1, and the mode propagation constants are calculated from equation 6-5. Over the frequency range from 10 Hz to 1 MHz two of the mode propagation constants are nearly equal and over the range from 100 kHz to 1 MHz all three propagation constants are nearly equal. To increase the accuracy of the calculation both the impedance and the admittance matrices are treated as follows:

\[ Z = \frac{j\omega l}{2\pi} Z' \quad 8-1 \]

\[ Y = j2\pi\varepsilon_0 Y' \quad 8-2 \]

Then let

\[ A = ZY = -\omega^2\varepsilon_0 Z'Y' = \omega^2\varepsilon_0 A' \quad 8-3 \]

The real part of the diagonal elements of the matrix \( A' \) are nearly equal to \(-1\) while the imaginary part of the diagonal elements and the off-diagonal elements are relatively small in magnitude. Let

\[ A' = A'' - I \quad 8-4 \]

then from

\[ |A' - \gamma I| = 0 \quad 8-5 \]

one has

\[ |A'' - (\gamma' + 1)I| = 0 \]

or

\[ |A'' - \gamma'' I| = 0 \quad 8-6 \]
Then the eigenvalues of $A''$ will not be as close together as those of $A'$. The propagation constants can now be obtained from,

$$\alpha_j + j\beta_j = \gamma_j = \sqrt{\omega^2 \varepsilon_\mu (\sigma_{\mu}^2 - 1)} \quad 8-7$$

where $\alpha_j$ is attenuation in nepers per meter and $\beta_j$ is the phase constant in radians per-meter. The mode velocity is obtained as follows

$$v_j = \frac{\omega}{\beta_j} \quad \text{m/sec} \quad 8-8$$

In terms of the velocity of light in free space, the normalized mode velocity is

$$v'_j = \frac{2\pi f}{\beta_j} \sqrt{\varepsilon_0 \mu_0} \quad 8-9$$

The velocities at modes 1 and 2 are very close to unity under all conditions, and all three mode velocities are close to unity for a perfectly conducting ground. However, for finite ground conductivity and relative permittivity, the velocity of mode 3 varies from 0.48 to nearly 1 per unit.

Fig. 8-1 shows the velocity of mode 3 without permittivity correction for conductivities from $10^{-5}$ to 1 mho/m. Figs. 8-2 through 8-4 show the velocity of mode 3 with impedance correction but without capacitance correction, for relative permittivities from 10 to 50 over the same range of conductivities as in fig. 8-1. Figs. 8-5 through 8-7 show the velocity of mode 3 with capacitance correction over the same range of conductivities and permittivities. From the figures it can be seen that relative permittivity has little effect on the
velocity of mode 3 for a conductivity greater than $10^{-3}$ mho/m, and the capacitance correction only affects the calculation where the conductivity is less than $10^{-3}$ mho/m. In figs. 8-1 through 8-7 it can be seen that the permittivity correction of both impedance and capacitance tends to increase the velocity of mode 3 at high frequencies.

Fig. 8-8 shows the mode attenuation constants, in nepers per mile, for a perfectly conducting ground, in which case the attenuation is due entirely to internal conductor a.c. resistance. Fig. 8-9 shows the mode attenuation constants for a range of ground conductivity from $10^{-5}$ to 1 mho/m without permittivity correction of the impedance and without capacitance correction. Figs. 8-10 through 8-12 show mode attenuation constants over the same conductivity range and over a range of relative permittivities from 10 to 50, with the impedance corrected for conductivity and relative permittivity, but without capacitance correction. Figs. 8-13 through 8-15 show the mode attenuation constants with conductivity and relative permittivity correction for the impedance and the capacitance over the same ranges of relative permittivity and conductivity. The attenuation of mode 3 is most affected by changes in ground conductivity, the attenuation of mode 2 is affected to a lesser degree and the attenuation of mode 1 is least affected. From figs. 8-9 through 8-15 it can be seen that for conductivities greater than $10^{-3}$ mho/m the relative permittivity has little effect on the mode attenuation constants. For lower conductivities both the impedance correction and the
capacitance correction have an increasing effect on mode attenuation for frequencies greater than 100 Hz. Tables 8-1 through 8-3 show a comparison of variations in the attenuation for a conductivity of $10^{-5}$ mho/m and at frequencies of 100 kHz, 10 kHz and 1 kHz.

Table 8.1 Attenuation in nepers/mile for $\sigma = 10^{-5}$ mho/m and $f = 100$ kHz

<table>
<thead>
<tr>
<th></th>
<th>Mode 3</th>
<th>Mode 2</th>
<th>Mode 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permittivity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td>3.6 x $10^{-1}$</td>
<td>1.5 x $10^{-3}$</td>
<td>1.3 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>7.0 x $10^{-1}$</td>
<td>2.7 x $10^{-3}$</td>
<td>1.4 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>7.2 x $10^{-1}$</td>
<td>4.0 x $10^{-3}$</td>
<td>1.4 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 40$</td>
<td>7.3 x $10^{-1}$</td>
<td>5.2 x $10^{-3}$</td>
<td>1.4 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>7.4 x $10^{-1}$</td>
<td>6.5 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
<tr>
<td>Impedance</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td>7.0 x $10^{-1}$</td>
<td>2.7 x $10^{-3}$</td>
<td>1.4 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>7.2 x $10^{-1}$</td>
<td>4.0 x $10^{-3}$</td>
<td>1.4 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>7.3 x $10^{-1}$</td>
<td>5.2 x $10^{-3}$</td>
<td>1.4 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 40$</td>
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<td>6.5 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>7.5 x $10^{-1}$</td>
<td>7.6 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
<tr>
<td>Capacitance</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td>4.6 x $10^{-1}$</td>
<td>2.7 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>5.8 x $10^{-1}$</td>
<td>4.0 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>6.2 x $10^{-1}$</td>
<td>5.2 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 40$</td>
<td>6.6 x $10^{-1}$</td>
<td>6.5 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>6.6 x $10^{-1}$</td>
<td>7.6 x $10^{-3}$</td>
<td>1.5 x $10^{-3}$</td>
</tr>
</tbody>
</table>
Table 8.2 Attenuation in nepers/mile for $\sigma = 10^{-5}$ mho/m and $f = 10$ kHz

<table>
<thead>
<tr>
<th></th>
<th>Mode 3</th>
<th>Mode 2</th>
<th>Mode 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>No permittivity correction</td>
<td>$3.7 \times 10^{-2}$</td>
<td>$3.3 \times 10^{-4}$</td>
<td>$4.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>Impedance correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td>$4.4 \times 10^{-2}$</td>
<td>$3.4 \times 10^{-4}$</td>
<td>$4.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>$5.2 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>$5.7 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>$n = 40$</td>
<td>$6.0 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>$6.3 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>Capacitance correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td>$3.3 \times 10^{-2}$</td>
<td>$3.4 \times 10^{-4}$</td>
<td>$4.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>$4.0 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>$4.6 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>$n = 40$</td>
<td>$5.1 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>$5.4 \times 10^{-2}$</td>
<td>&quot;</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

From table 8-3 it can be seen that capacitance correction is significant for low conductivities even at a frequency 1 kHz for the attenuation of mode 3. For the attenuation calculation, conductivity and relative permittivity correction of impedance and capacitance matrices is required for conductivities below $10^{-3}$ mho/m and frequencies above 100 Hz. For conductivities above $10^{-3}$ mho/m no relative permittivity correction of the impedance and no capacitance correction are required and Carson's formulae $^{(2)}$ are sufficient to obtain
the ground correction terms for the circuit parameters.

Table 8.3 Attenuation in nepers/mile for $\varepsilon = 10^{-5}$ mho/m and $f = 1$ kHz

<table>
<thead>
<tr>
<th></th>
<th>Mode 3</th>
<th>Mode 2</th>
<th>Mode 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>No permittivity</td>
<td>$3.6 \times 10^{-3}$</td>
<td>$1.15 \times 10^{-4}$</td>
<td>$1.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Impedance correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td>$3.3 \times 10^{-3}$</td>
<td>$1.15 \times 10^{-4}$</td>
<td>$1.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>$3.4 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 30$</td>
<td>$3.4 \times 10^{-3}$</td>
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<tr>
<td>$n = 40$</td>
<td>$3.5 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 50$</td>
<td>$3.6 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capacitance</td>
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<td></td>
</tr>
<tr>
<td>correction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td>$7.8 \times 10^{-3}$</td>
<td>$1.15 \times 10^{-4}$</td>
<td>$1.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>$n = 20$</td>
<td>$4.5 \times 10^{-3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 30$</td>
<td>$3.5 \times 10^{-3}$</td>
<td></td>
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</tr>
<tr>
<td>$n = 40$</td>
<td>$3.4 \times 10^{-3}$</td>
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<td></td>
</tr>
<tr>
<td>$n = 50$</td>
<td>$3.4 \times 10^{-3}$</td>
<td></td>
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</tr>
</tbody>
</table>

An insight into the behaviour of the mode propagation constants may be obtained by comparing the transmission line with a line that is transposed at short intervals. Then all the off-diagonal terms in both the impedance and admittance matrices would be equal and the diagonal elements in each matrix would also be equal. The transformation matrices, S and R, would be equal and would be the matrix used for transformation
to symmetrical components. Thus the mode 3 propagation constants can be compared with the zero sequence propagation constants in a balanced system. To illustrate this similarity the R and S matrices for a frequency of 100 kHz, a ground conductivity of $10^{-4}$ mho/m and a relative permittivity of 10 are

\[ R = \begin{bmatrix}
0.484 - j.161 & -0.763 + j.106 & 0.592 - j.003 \\
0.787 + j.177 & 0.115 - j.102 & 0.566 + j.014 \\
0.299 + j.00 & 0.619 + j.00 & 0.570 + j.00
\end{bmatrix} \]

\[ S_t = \begin{bmatrix}
0.223 + j.246 & -0.750 - j.246 & 0.517 + j.00 \\
-0.603 - j.115 & -0.143 + j.126 & 0.766 + j.00 \\
0.571 + j.002 & 0.570 + j.017 & 0.591 + j.00
\end{bmatrix} \]

\[ R^{-1} = \begin{bmatrix}
0.275 + j.246 & -0.833 - j.147 & 0.543 + j.020 \\
-0.656 - j.064 & -0.125 + j.067 & 0.808 + j.00 \\
0.568 - j.005 & 0.572 + j.005 & 0.593 - j.011
\end{bmatrix} \]
FIG 8-1
MODE 3 PROPAGATION VELOCITY WITHOUT PERMITTIVITY CORRECTION

FIG 8-2
MODE 3 PROPAGATION VELOCITY, NO CAPACITANCE CORRECTION WITH $n=10$.
FIG 8-3
MODE 3 PROPAGATION VELOCITY, NO CAPACITANCE CORRECTION WITH n=30

FIG 8-4
MODE 3 PROPAGATION VELOCITY, NO CAPACITANCE CORRECTION WITH n=50
FIG 8.5
MODE 3 PROPAGATION VELOCITY. APPROXIMATE CAPACITANCE CORRECTION WITH
FREQUENCY HZ

P.U. VELOCITY

FIG 8.6
MODE 3 PROPAGATION VELOCITY. APPROXIMATE CAPACITANCE CORRECTION WITH
FREQUENCY HZ

P.U. VELOCITY
FIG 8-7
MODE 3 PROPAGATION VELOCITY, APPROXIMATE CAPACITANCE CORRECTION

WITH
n=50

P. U. VELOCITY

\[ \sigma = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5} \]

FREQUENCY HZ

\[ 10, 10^2, 10^3, 10^4, 10^5, 10^6 \]
FIG 8-8
ATTENUATION WITH INFINITELY CONDUCTING GROUND

ATTENUATION NP/MIILE

FREQUENCY HZ
FIG 8-9
MODE ATTENUATION WITH NO PERMITTIVITY CORRECTION

ATTENUATION NP/MILE

FREQUENCY HZ
FIG 8-11
MODE ATTENUATION, NO CAPACITANCE CORRECTION WITH $n=30$

ATTENUATION NP/MILE

FREQUENCY HZ

$\sigma=10^{-5}$

$\sigma=10^{-3}$

$\sigma=10^{-1}$

MODE 2

$\sigma=10^{-3}$

$\sigma=10^{-1}$

MODE 1

$\sigma=10^{-5}$

MODE 3
FIG 8-12
MODE ATTENUATION. NO CAPACITANCE CORRECTION WITH n=50

ATTENUATION NP/MILE

FREQUENCY HZ

\( d = 10^{-5} \)
\( d = 10^{-3} \)
\( d = 10^{-1} \)
\( d = 10^{-2} \)
\( d = 10^{-4} \)
\( d = 10^{-5} \)
FIG 8-13

MODE ATTENUATION, APPROXIMATE CAPACITANCE CORRECTION WITH n = 10

ATTENUATION NP/MILE

FREQUENCY HZ

MODE 1

σ = 10

σ = 10^{-5}

MODE 2

σ = 10^{-3}

σ = 10^{-1}

MODE 3

σ = 10
FIG 8-14

MODE ATTENUATION. APPROXIMATE CAPACITANCE CORRECTION WITH n = 30

ATTENUATION NP/MILE

FREQUENCY HZ
FIG 8-15
MODE ATTENUATION. APPROXIMATE CAPACITANCE CORRECTION WITH n = 50

ATTENUATION NP/MILE

FREQUENCY HZ
Chapter 9. **Conclusions.**

The methods of Carson\(^{(2)}\) and Wise\(^{(11)}\) have been extended to calculate the distributed circuit parameters for a multi-conductor transmission line. The correction terms for the distributed circuit parameters have been derived for variations in line geometry, ground conductivity, ground relative permittivity, and for a frequency range up to 1 MHz. Their corrected parameters have been used in the transmission line equation to calculate the mode propagation velocities and attenuation constants.

A practical example has been investigated in detail to show the variations in the distributed circuit parameters and the mode propagation constants due to various ground conditions. For a typical 500 kV transmission line the impedance should always be corrected for ground conductivity. When \(\phi/\omega\epsilon\) is less than 180, the impedance should, in addition, be corrected for the relative permittivity of the ground and the capacitance should be corrected for both the ground conductivity and the relative permittivity of the ground. When \(\phi/\omega\epsilon\) equals 180, the maximum error is approximately 3% in Carson's correction of the impedance parameters and the uncorrected potential coefficients.

The initial rate of rise of recovery voltage in circuit breakers clearing faults in power systems can be evaluated solely in terms of the characteristic impedance of the transmission line and the energizing conditions. The former depends in turn on the corrected impedance and admittance
matrices and the natural frequency of the energizing system. Thus it may be possible to investigate the effect of ground conditions on the initial rate of recovery voltages for circuit breakers opening under fault conditions.

The program developed for calculating the mode propagation constants in this thesis may be extended to include terminal conditions. This allows the investigation of the various permutations to find the optimum condition for energizing and terminating a line for carrier communication.

In the calculation of the mode propagation constants in this thesis the conventional formulae for the internal impedance of the line conductors have been used with the assumption that the equivalent radius is the maximum radius over the strands. For bundled conductors further investigations would be required to determine the internal impedance more accurately. Since the internal impedance affects the diagonal elements of the impedance matrix it has a direct effect on the mode propagation constants.

Further developments in this field should be directed to find the effects of corona and tower footing resistance on the transmission system parameters. Corona disturbs the electric field adjacent to the conductors and may have a noticeable effect on the parameters. The effect is non-linear since it is voltage dependent. The tower footing resistance may become important for lines with overhead ground conductors where the tower spacing approaches a quarter of the wave length of the impressed signal.
The procedure developed in this thesis is solely for overhead transmission lines. In the present form it is unsuitable for the calculation of propagation constants of underground systems, which presents quite a different problem due to their geometric configuration with respect to the ground.
Appendix A. The Field due to an Isolated Oscillating Current Element.

In an isotropic homogeneous medium Maxwell's field equations are stated as

\[ \nabla \times \mathbf{E} = -j \omega \mu \mathbf{H} \]  
\[ \nabla \times \mathbf{H} = \mathbf{J} + j \omega \varepsilon_c \mathbf{E} \]

where, the time dependence is sinusoidal and the conductivity of the medium is accounted for by the complex permittivity \( \varepsilon_c \), where \( \varepsilon_c = \varepsilon - j \frac{\sigma}{\omega} \).

Define a vector \( \mathbf{\Pi} \) such that

\[ \mathbf{H} = j \omega \varepsilon_c \nabla \times \mathbf{\Pi} \]

Then from equation A-1 and A-2

\[ \nabla \times \mathbf{E} = \omega^2 \varepsilon_c \mu \nabla \times \mathbf{\Pi} = -m^2 \nabla \times \mathbf{\Pi} \]

hence

\[ \mathbf{E} = -m^2 \mathbf{\Pi} + \nabla (\phi) \]

where \( \phi \) is a scalar function of position. Then from equation A-2

\[ j \omega \varepsilon_c \nabla \times (\nabla \times \mathbf{\Pi}) = \mathbf{J} + j \omega \varepsilon_c (-m^2 \mathbf{\Pi} + \nabla (\phi)) \]

Dividing by \( j \omega \varepsilon_c \) and expanding the first term of equation A-6 gives

\[ -\nabla^2 \mathbf{\Pi} + \nabla \cdot \nabla \times \mathbf{\Pi} = \frac{\mathbf{J}}{j \omega \varepsilon_c} - m^2 \mathbf{\Pi} + \nabla (\phi) \]

Let

\[ \nabla (\phi) = \nabla \cdot \mathbf{\Pi} \]

a Lorentz type condition, \((19)\) and equation A-7 becomes
\[ \nabla^2 \Pi = m^2 \Pi - \frac{J}{j\omega \varepsilon_c} \quad \text{(A-9)} \]

and

\[ E = -m^2 \Pi + \nabla \nabla \cdot \Pi \quad \text{(A-10)} \]

Let the solution of equation A-9 be

\[ \Pi = \int_{\mathcal{V}_o} \frac{\text{Ae}^{-m \cdot R}}{R} \, dv \quad \text{(A-11)} \]

where \( \mathcal{V}_o \) is a small closed surface at the point \((x,y,z)\) in Cartesian coordinates. Then

\[
\begin{align*}
\nabla^2 \Pi &= \nabla^2 \left( \int_{\mathcal{V}_o} \frac{\text{Ae}^{-m \cdot R}}{R} \, dv \right) \\
&= \int_{\mathcal{V}_o} \left[ \frac{m^2 \text{Ae}^{-m \cdot R}}{R} + \text{Ae}^{-m \cdot R} \nabla^2 \left( \frac{1}{R} \right) \right] \, dv \\
&= m^2 \Pi + \int_{\mathcal{V}_o} \text{Ae}^{-m \cdot R} \nabla^2 \left( \frac{1}{R} \right) \, dv \quad \text{(A-12)}
\end{align*}
\]

Substituting A-12 into A-9 gives

\[
J = -j\omega \varepsilon_c \int_{\mathcal{V}_o} \text{Ae}^{-m \cdot R} \nabla^2 \left( \frac{1}{R} \right) \, dv \quad \text{(A-13)}
\]

The solution of A-13 is

\[
J = \begin{cases} 
0 & \text{when } R \neq 0 \\
-j\omega \varepsilon_c \frac{A4\pi}{a} & \text{when } R = 0 
\end{cases} \quad \text{(A-14)}
\]

The current flows only in the z direction, hence \( J_x \) and \( J_y \) are both zero at \( r \) equals zero. When the element has a current \( I \) and a cross section \( a \),

\[
J_z = \frac{I}{a} \quad \text{at } R = 0 \quad \text{(A-15)}
\]

Hence

\[
A = \frac{1}{4\pi j\omega \varepsilon_c a} \quad \text{(A-16)}
\]
and
\[ \Pi_z = \int_v^\infty Ie^{-mR} \frac{1}{4\pi j\omega e_R} dv \]
Assuming \( v_o \) is small and independent of \( r \), then
\[ \Pi_z = \frac{Ie^{-mR}}{4\pi j\omega e_R} v_o \]
where
\[ v_o = sa \]
and \( s \) is the elemental length which is replaced by \( dz \). Hence
\[ \Pi_z = \frac{Ie^{-mR} dz}{4\pi j\omega e_R} \] \hspace{1cm} A-17
\[ \Pi_x = 0 \] \hspace{1cm} A-18
\[ \Pi_y = 0 \] \hspace{1cm} A-19
Appendix B. The General $\Pi$ Vector in Cylindrical Coordinates.

Fig. B-1 Cylindrical coordinate system for the internal field.

The wave equation for the $\Pi_z$ vector in the coordinate system shown in fig. B-1 is

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial \Pi_z}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2 \Pi_z}{\partial \phi^2} + \frac{\partial^2 \Pi_z}{\partial z^2} = m^2 \Pi_z \quad B-1$$

The solution is obtained by separating the variables. Let

$$\Pi_z = R(\rho) \Phi(\phi)Z(z) \quad B-2$$

Then

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial R}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = m^2 \quad B-3$$

Since the first two terms of B-3 are independent of $z$, the equation becomes

$$\frac{\partial^2 Z}{\partial z^2} = \gamma^2 Z \quad B-4$$

where

$$\gamma^2 = u^2 + m^2 \text{ and } u \text{ is a constant.} \quad B-5$$
The solution of B-4 is

\[ Z = A_1 e^{-\gamma z} + A_2 e^{\gamma z} \]  \hspace{1cm} B-6

Then B-3 reduces to

\[ \frac{\rho}{R} \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial R}{\partial \rho} \right] + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} + u^2 \rho^2 = 0 \]  \hspace{1cm} B-7

Since the second term of the equation is independent of \( r \), B-7 can be written as

\[ \frac{\partial^2 \Phi}{\partial \phi^2} + p^2 \Phi = 0 \]  \hspace{1cm} B-8

where \( p^2 \) is a constant. The solution of B-8 is

\[ \Phi = B_1 \cos P\phi + B_2 \sin P\phi \]  \hspace{1cm} B-9

Equation B-7 now becomes

\[ \rho \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial R}{\partial \rho} \right] + \left[ (\rho u)^2 - p^2 \right] R = 0 \]  \hspace{1cm} B-10

which can be transformed to the form

\[ (\rho u) \frac{\partial}{\partial (\rho u)} \left[ (\rho u) \frac{\partial R}{\partial (\rho u)} \right] + \left[ (\rho u)^2 - p^2 \right] R = 0 \]  \hspace{1cm} B-11

This is Bessel's equation of order \( P \). The solution has the following form

\[ R = C Z_p(\rho u) \]  \hspace{1cm} B-12

where \( Z_p(\rho u) \) indicates the generalized form the the Bessel function. The exact types of Bessel functions required for the solution depend on the boundary conditions of the particular problem. Then the complete solution for \( \Pi_z \) is the sum of all
possible solutions and may be written as

$$\Pi_z = \sum_{p=0}^{\infty} (B_1 p \cos \theta + B_2 p \sin \theta) x \int_0^\infty (A_1(u)e^{-Yz} + A_2(u)e^{Yz})\, xC z\, <\rho u> du$$  

The wave equation in the Cartesian coordinate system can be separated into three components. Each component satisfies

$$\frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial y^2} + \frac{\partial^2 \Pi}{\partial z^2} = m^2 \Pi$$  

where $\Pi$ may be either $\Pi_x$, $\Pi_y$ or $\Pi_z$. Thus the solutions will be similar functions. For the coordinate system shown in fig. B-2. The solution for $\Pi_z$ and $\Pi_y$ will be similar and of the general form shown in equation B-15. This is the form of the solution used in the ground effect calculations in Chapter 3.

Fig. B-2 Cylindrical Coordinate System for the External Field
\[ \Pi = \sum_{p=0}^{\infty} \left( B_1 p \cos \psi + B_2 p \sin \psi \right) \int_0^\infty \left( A_1(u) e^{-\gamma y} + A_2(u) e^{\gamma y} \right) x Z_p(r u) \, du \]
Appendix C. Derivation of the Internal Impedance of Cylindrical Conductors.

In this Appendix we shall be concerned only with the fields inside the conductors. Consider a solid cylindrical conductor with the axis along the z-axis of the coordinate system shown in fig. B-1. The direction of propagation is along the positive z-axis.

The electromagnetic field inside the conductor consists of the TM mode only. A Hertz vector of the $\text{TE}_e$ type in the z direction is used to describe the field. Due to the circular symmetry of the system the field components are independent of $\phi$, and hence the Hertz vector is also independent of $\phi$. The solution of the wave equation for $\Pi_z$ is given in equation B-13. However as $\Pi_z$ is independent of $\phi$ and must be finite at the origin only the first term of the solution exists and the required Bessel function is of the first kind of order zero. Therefore

$$\Pi_z = Ae^{-\gamma z} J_0(\rho u) \quad C-1$$

Using equations A-3 and A-10 the field components $E_z$ and $H_\phi$ are

$$E_z = (-m^2 + \gamma^2) \Pi_z \quad C-2$$

$$H_\phi = j\omega \epsilon_0 c A e^{-\gamma z} J_1(\rho u) \quad C-3$$

Let the external radius of the cylinder be a, then at the boundary

$$H_\phi = \frac{I_0 e^{-\gamma z}}{2\pi a} \quad C-4$$
Consequently
\[ A = \frac{I_0}{j\omega \epsilon_0 2\pi au} \]  

When \( \sigma \) is very large, \( \gamma^2 \) is negligible as compared to \( m^2 \). Then for practical purposes
\[ u^2 = -m^2 \]

and
\[ E_z = \frac{-m^2 I_0 J_0(\rho u)e^{-\gamma z}}{j\omega \epsilon_0 2\pi au J_1(au)} \]

Let \( E_z \) at radius \( a \) be given by
\[ E_z = z_i I_0 e^{-\gamma z} \]

where \( z_i \) is the equivalent internal impedance of the cylindrical conductor. Then
\[ z_i = \frac{-m^2 J_0(au)}{j\omega \epsilon_0 2\pi au J_1(au)} \]

For a conductor
\[ \epsilon_c = -j \frac{\sigma}{\omega} \]

and
\[ m = j^{0.5} a' \]

where
\[ a' = \sqrt{\omega \mu \sigma} \]

Then
\[ z_i = \frac{j^{1.5} a' J_0(j^{1.5} a')}{2\pi a \sigma J_1(j^{1.5} a')} \]

The numerical evaluation of \( z_i \) may be divided into three regions, dependent on the value of \( a'a \).
1) When \( \alpha' a < 0.1 \)

For small values of \( \alpha' a \) the following expansion holds,

\[
z_1 = \left[ 1 + j \frac{(\alpha' a)^2}{8} + \frac{(\alpha' a)^4}{192} - j \frac{(\alpha' a)^6}{3072} \right] \frac{1}{\pi a^2} \cos \theta
\]

Let

\[
Z_i = R_{int} + jX_{int}
\]

Then for small values of \( \alpha' a \)

\[
R_{int} = \frac{1}{\pi a^2} = R_{dc}
\]

and

\[
L_{int} = \frac{\omega M}{8\pi}
\]

which are the formulae obtained when a uniform current distribution is assumed.

2) When \( \alpha' a > 10 \)

For large values of \( \alpha' a \) \( z_1 \) may be written as

\[
z_1 = \frac{\alpha'(1 + 1)}{2\sqrt{2} \pi a \delta}
\]

Let the skin depth \( \delta \) be defined as

\[
\delta = \frac{\sqrt{2}}{\alpha}
\]

Then

\[
z_1 = \frac{(1 + 1)}{2\pi a \delta}
\]

which is Rayleigh's formula. Then

\[
R_{int} = \frac{1}{2\pi a \delta}
\]

\[
L_{int} = \frac{1}{2\pi a \delta \omega}
\]
3) When \( 0.1 \leq \alpha'a \leq 10 \)

Equation C-13 may be changed into a more convenient form for numerical computation by letting

\[
J_0(j^{1.5}\alpha'a) = \text{Ber}(\alpha'a) + j\text{Bei}(\alpha'a) \quad \text{C-23}
\]

The first order Bessel function is obtained by differentiating equation C-23

\[
J_1(j^{1.5}\alpha'a) = j^{0.5} \left[ \text{Ber}'(\alpha'a) + j\text{Bei}'(\alpha'a) \right] \quad \text{C-24}
\]

Then the internal impedance \( z_i \) becomes

\[
z_i = \frac{\alpha'}{2\pi\alpha} \left[ \frac{-\text{Bei}(\alpha'a) + j\text{Ber}(\alpha'a)}{\text{Ber}'(\alpha'a) + j\text{Bei}'(\alpha'a)} \right] \quad \text{C-25}
\]

and the resistance and inductance are

\[
R_{\text{int}} = \frac{\alpha'}{2\pi\alpha} \left[ \frac{\text{Ber}(\alpha'a)\text{Bei}'(\alpha'a) - \text{Bei}(\alpha'a)\text{Ber}'(\alpha'a)}{\text{Ber}'^2(\alpha'a) + \text{Bei}'^2(\alpha'a)} \right] \quad \text{C-26}
\]

\[
L_{\text{int}} = \frac{\mu}{2\pi\alpha} \left[ \frac{\text{Ber}(\alpha'a)\text{Bei}'(\alpha'a) + \text{Bei}(\alpha'a)\text{Bei}'(\alpha'a)}{\text{Ber}'^2(\alpha'a) + \text{Bei}'^2(\alpha'a)} \right] \quad \text{C-27}
\]

The numerical values of the functions \( \text{Ber}, \text{Bei}, \text{Ber}', \) and \( \text{Bei}' \) are tabulated(17), or they may be calculated from power series(15).
Appendix D. Solution of the Integral in Equation 4-30.

The integral is

\[ I = \frac{\omega u}{\pi} \int_0^\infty (\sqrt{u^2 + j - u}) e^{-u} (h'' + y'') \cos x''u \, du \]

where

\[ x'' = x's = s\sqrt{\omega \mu} x \]
\[ y'' = y's = s\sqrt{\omega \mu} y \]
\[ h'' = h's = s\sqrt{\omega \mu} h \]

and

\[ s^2 = 1 + \frac{j \varepsilon \omega (n-1)}{\delta} \]

Equation D-1 has at least two possible solutions, one suggested by J.R. Carson\(^2\) for small values of \(R''\) and the other obtained by repeated partial integration for large values of \(R''\). The first solution is

\[ I = P + jQ \]

\[ = \frac{j}{R + e^{j(\theta + \delta)}} \left[ K_1(jR''e^{j(\theta + \delta)}) + G(jR''e^{j(\theta + \delta)}) \right] \]

\[ + \frac{1}{R''e^{j(-\theta + \delta)}} \left[ K_1(jR''e^{-j(\theta + \delta)}) + G(jR''e^{-j(\theta + \delta)}) \right] \]

\[ - \frac{1}{R''^2} \left[ e^{2j(\theta - \delta)} + e^{-2j(\theta + \delta)} \right] \]

Where \(K_1(x)\) is a Bessel function of the second kind and the function \(G(x)\) is defined as\(^2\)

\[ G(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n} 2^{2n-1} n!(n-1)!}{(2n+1)! (2n-1)} \]
From D-4 and D-5

\[ P = 0.125\pi(1-s_{3r}+s_{11}) + 0.5\ln\left(\frac{R}{R''}\right)(s_{1r}+s_{31}) - 0.5\theta(1-s_{11}-s_{3r}) - 0.5\theta(s_{2r}+s_{4i}) + \frac{1}{\sqrt{2}}(r_{3r}-r_{1r}-r_{3i}-r_{1i}) + 0.5(r_{2r}+r_{4i}) \]  

\[ Q = 0.25-0.125\pi(s_{1r}-s_{31}) - 0.5\theta(1+s_{31}+s_{1r}) - 0.5\theta(s_{4r}-s_{2i}) + 0.5\ln\left(\frac{2}{R''}\right)(1-s_{3r}+s_{11}) + \frac{1}{\sqrt{2}}(r_{1r}+r_{3r}+r_{3i}-r_{1i}) + 0.5(r_{2i}-r_{4r}) \]

where

\[ s_{1r} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n-1)!} \left(\frac{R''}{2}\right)^{4n-2} \cos(4n-2)\theta \cos(4n-2)\theta \]  

\[ s_{11} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n-1)!} \left(\frac{R''}{2}\right)^{4n-2} \cos(4n-2)\theta \sin(4n-2)\theta \]  

\[ s_{2r} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n-1)!} \left(\frac{R''}{2}\right)^{4n-2} \sin(4n-2)\theta \cos(4n-2)\theta \]  

\[ s_{2i} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n-1)!} \left(\frac{R''}{2}\right)^{4n-2} \sin(4n-2)\theta \sin(4n-2)\theta \]  

\[ s_{3r} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n+1)!} \left(\frac{R''}{2}\right)^{4n} \cos 4n\theta \cos 4n\theta \]  

\[ s_{3i} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n+1)!} \left(\frac{R''}{2}\right)^{4n} \cos 4n\theta \sin 4n\theta \]  

\[ s_{4r} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n+1)!} \left(\frac{R''}{2}\right)^{4n} \sin 4n\theta \cos 4n\theta \]  

\[ s_{4i} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n+1)!} \left(\frac{R''}{2}\right)^{4n} \sin 4n\theta \sin 4n\theta \]
\[ s_{4i} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n! (2n+1)!} \left( \frac{R^n}{2} \right) 4^n \sin 4n \theta \sin 4n \phi \]  

\[ r_{1r} = \sum_{n=1}^{\infty} \frac{(2n-2)!^2 2^4 (n-1)! R^n 4n-3 (-1)^{n+1}}{(4n-3)!^2 (4n-1)!^2} \cos (4n-3) \theta \cos (4n-3) \phi \]  

\[ r_{1i} = \sum_{n=1}^{\infty} \frac{(2n-2)!^2 2^4 (n-1)! R^n 4n-3 (-1)^{n+1}}{(4n-3)!^2 (4n-1)!^2} \cos (4n-3) \theta \sin (4n-3) \phi \]  

\[ r_{2r} = \sum_{n=1}^{\infty} \left[ \sum_{j=1}^{n+1} \frac{1}{j} - \frac{1}{4n} \right] \frac{(-1)^{n+1}}{2n! (2n-1)!} \left( \frac{R^n}{2} \right) 4^n \cos (4n-2) \theta \cos (4n-2) \phi \]  

\[ r_{2i} = \sum_{n=1}^{\infty} \left[ \sum_{j=1}^{n+1} \frac{1}{j} - \frac{1}{4n} \right] \frac{(-1)^{n+1}}{2n! (2n-1)!} \left( \frac{R^n}{2} \right) 4^n \cos (4n-2) \theta \sin (4n-2) \phi \]  

\[ r_{3r} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2n-1)!^2 2^4 (4n-2)! R^n 4n-1}{(4n-1)!^2 (4n+1)!} \cos (4n-1) \theta \cos (4n-1) \phi \]  

\[ r_{3i} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2n-1)!^2 2^4 (4n-2)! R^n 4n-1}{(4n-1)!^2 (4n+1)!} \cos (4n-1) \theta \sin (4n-1) \phi \]  

\[ r_{4r} = \sum_{n=1}^{\infty} \left[ \sum_{j=1}^{2n+1} \frac{1}{j} - \frac{1}{4n+2} \right] \frac{(-1)^{n+1}}{2n! (2n+1)!} \left( \frac{R^n}{2} \right) 4^n \cos 4n \theta \cos 4n \phi \]  

\[ r_{4i} = \sum_{n=1}^{\infty} \left[ \sum_{j=1}^{2n+1} \frac{1}{j} - \frac{1}{4n+2} \right] \frac{(-1)^{n+1}}{2n! (2n+1)!} \cos 4n \theta \sin 4n \phi \]
The second solution is

\[ P = - \frac{\cos 2\theta \cdot \cos 2\phi}{R''^2} \sqrt{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\phi}{R''^{2n-1}} \left[ \cos(2n-1)\phi \right] 
+ (-1)^{n+1} \sin(2n-1)\phi \]  

\[ Q = \frac{\cos 2\theta \cdot \sin 2\phi}{R''^2} + \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\phi}{R''^{2n-1}} \left[ \cos(2n-1)\phi \right] 
+ (-1)^n \sin(2n-1)\phi \]  

For numerical computation the solutions are divided into two ranges

1) When \( R'' \leq 5.0 \)

In this range use equations D-6 through D-23

2) When \( R'' > 5.0 \)

In this range use equations D-24 and D-25.

These equations reduce to J.R. Carson's solutions when \( s_0 \) is equal to one and \( \phi \) is equal to zero.
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