# MATRIX ANALYSIS OF STEADY STATE, MULTI-CONDUCTOR, DISTRIBUTED PARAMETER TRANSMISSION SYSTEMS

by

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B.A.Sc., University of British Columbia, 1961

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in the Department of Electrical Engineering

We accept this thesis as conforming to the required standard

Members of the Department of Electrical Engineering

THE UNIVERSITY OF BRITISH COLUMBIA

November , 1965

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Date 3 : xii : 65

## ABSTRACT

Problems concerning transmission lines have been solved in the past by treating the line in terms of lumped parameters.

Pioneering work was done by L. V. Bewley and S. Hayashi in the application of matrix theory to solve polyphase multi-conductor distributed parameter transmission system problems. The availability of digital computers and the increasing complexity of power systems has renewed the interest in this field.

With this in mind, a systematic procedure for handling complex transmission systems was evolved. Underlying the procedure is the significant concept of a complete system which defines how the parametric inductance, capacitance, leakance and resistance matrices must be formed and used. Also of significance is the use of connection matrices for handling transpositions and bonding, together with development of the manipulation of these matrices and the complex (Z) and (Y) matrices. In the numerical procedure, methods were found to transform complex matrices into real matrices of twice the order and to determine the coefficients in the general solution systematically. The procedure was used to deal with phase asymmetry and mixed end boundary conditions.

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# ACKNOWLEDGEMENT

The author would like to express his sincere gratitude to his supervising professors Dr. F. Noakes and Dr. Y. N. Yu for guidance, perseverance and continued inspiration throughout the course of the research. The author also wishes to thank the members of the Department of Electrical Engineering.

The author is indebted to the National Research Council of Canada for financial support of the research.

# 1. INTRODUCTION

The purpose of this thesis is to develop a procedure for solving the problem of polyphase, distributed parameter transmission systems, during steady state operation. Historically, this problem has been attacked by treating the line configuration in terms of lumped circuit parameters, obtained through transformations from the distributed parameters, leading to various closed form solutions.

Early work in the development of matrix methods for analysis was done by L.V. Bewley (1). The approach taken was to analyze the lossless polyphase line and to expand the analysis to include lines with losses. From this, travelling wave solutions were developed which led to a study of surges by matrix methods. L.A. Pipes (2) followed Bewley's approach but used Laplace transform methods. Parallel developments were made by S. Hayashi (3) who extended the analysis to transient phenomena, including travelling wave properties of surges.

The increasing complexity and interconnection of modern power systems, together with the flexibility and availability of digital computers, makes the use of matrix methods both imperative and practical.

With this in mind, a systematic mathematical and numerical procedure for handling the complex system is evolved in this thesis. The rationalized M.K.S. system of units is used throughout.

## 2. GENERAL DIFFERENTIAL EQUATIONS FOR MULTI-CONDUCTOR SYSTEMS

Consider a system of (n + 1) parallel conductors mutually coupled electrostatically and electromagnetically. By definition, this is a complete system if and only if the sum of the currents over the whole system is zero,

$$\sum_{i=1}^{n+1} i_i = 0 \qquad 2\text{-}1$$

This definition precludes radiation effects, but this is an acceptable approximation at low frequencies.

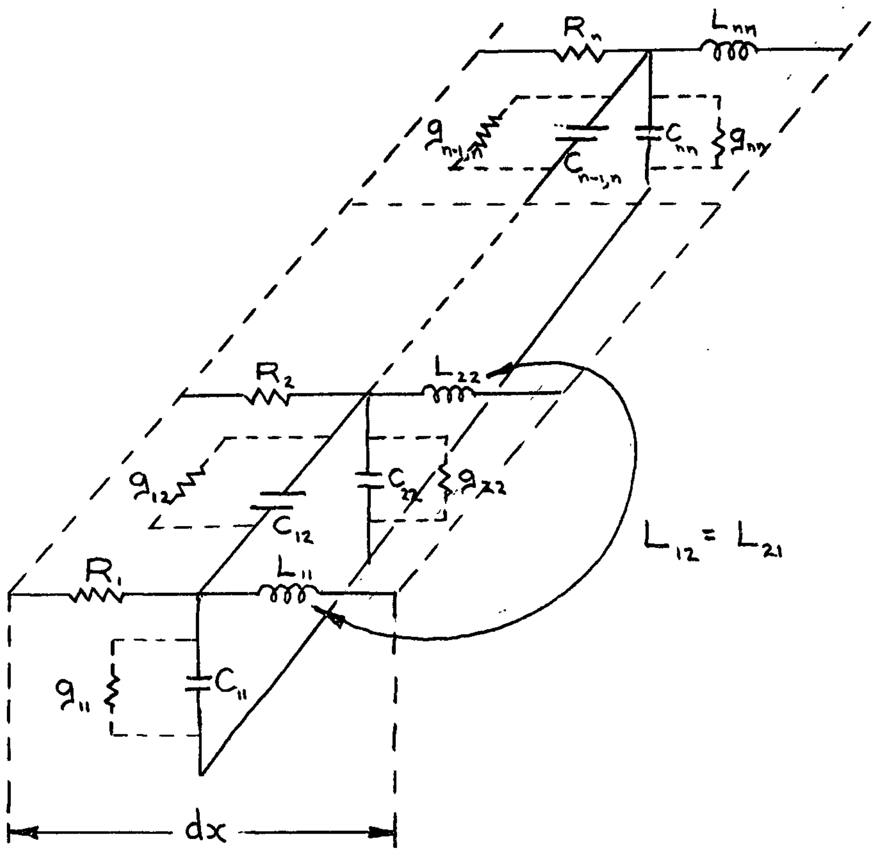



Fig 2.1 Part of Mutually Coupled Circuit of (n + 1) conductors.

Of the (n + 1) conductors, n will be defined as being independent; the (n + 1)th conductor becomes the reference conductor for voltages and the "return" path for unbalanced currents. Depending on the physical arrangement of the transmission system, this reference conductor would normally be taken as a ground conductor or an equivalent earth conductor (4).

The voltage and current equations for the ith conductor may be written

$$-\frac{\partial v_i}{\partial x} = \frac{\partial \psi_i}{\partial t} + R_i i_i \qquad 2\text{-}2$$

$$-\frac{\partial i_i}{\partial x} = \frac{\partial q_i}{\partial t} + i_i^1 \qquad 2\text{-}3$$

where $v_i$ = potential of conductor i with respect to some arbitrary reference

$\psi_i$ = total flux linkages per unit length of conductor i due to currents in all conductors

$R_i$ = series resistance per unit length of conductor i

$i_i$ = current in conductor i

$q_i$ = charge per unit length on conductor i

and $i_i^1$ = leakage current per unit length from conductor i

The system is assumed to be linear in the following analysis. Let p be the differential operator $p = \partial/\partial t$ Associated with each unit length of conductors i and j are

$$Z'_{ii} = R_i + pL_{ii}$$

$$Z'_{ij} = pL_{ij}$$

$$Y'_{ii} = G_{ii} + pC_{ii}$$

$$Y'_{ij} = G_{ij} + pC_{ij}$$

where $R_i$ = series resistance of conductor i

$L_{ii}$ = self inductance coefficient of conductor i

$L_{ij}$ = mutual inductance coefficient of conductors i & j

$C_{ii}$ = self capacitance coefficient of conductor i

$C_{ij}$ = mutual capacitance coefficient between conductors i & j

$G_{ii}$ = leakance from the ith conductor to the arbitrary reference

$G_{ij}$ = leakance between conductors i & j

The differential equations of the ith conductor become

$$-\frac{\partial v_i}{\partial x} = Z'_{i1}\, i_1 + Z'_{i2}\, i_2 + \dots + Z'_{ii}\, i_i + \dots + Z'_{i,n+1}\, i_{n+1} \qquad 2\text{–}4$$

$$-\frac{\partial i_i}{\partial x} = Y'_{i1}\, v_1 + Y'_{i2}\, v_2 + \dots + Y'_{ii}\, v_i + \dots + Y'_{i,n+1}\, v_{n+1} \qquad 2\text{–}5$$

In matrix form, the 2(n + 1) equations for the (n + 1) conductors may be written

$$-\frac{\partial (v)}{\partial x} = (Z'(p))\ (i) \qquad 2\text{–}6$$

$$-\frac{\partial (i)}{\partial x} = (Y'(p))\ (v) \qquad 2\text{–}7$$

where (v) and (i) are column vectors, (Z') and (Y') are square matrices which are functions of time (the differential operator p).

By differentiation with respect to x and substitution, equations 2-6 and 2-7 may be combined to give

$$\frac{\partial^2}{\partial x^2}(v) = (Z'(p))\ (Y'(p))\ (v) \qquad 2\text{-}8$$

$$\frac{\partial^2}{\partial x^2}(i) = (Y'(p))\ (Z'(p))\ (i) \qquad 2\text{-}9$$

where $(Z')$ and $(Y')$ are both independent of x.

## 3. THE DIFFERENTIAL EQUATIONS FOR STEADY STATE ANALYSIS

Consider the system of conductors operating under a-c steady state conditions such that the voltages of the (n+1) conductors at a position x, with respect to some arbitrary reference, are given by 3-1

$$\begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} V_1 \; e^{j(\omega t+\theta_1)} \\ V_2 \; e^{j(\omega t+\theta_2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_{n+1} \; e^{j(\omega t+\theta_{n+1})} \end{pmatrix} = \begin{pmatrix} \bar{V}'_1 \\ \bar{V}'_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{V}'_{n+1} \end{pmatrix} e^{j\omega t} = (\bar{V}')\, e^{j\omega t}$$

Since the system is linear, the current response will have the same form with different phase angles $\phi_1, \phi_2, \ldots, \phi_{n+1}$

$$\begin{pmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ i_{n+1} \end{pmatrix} = \begin{pmatrix} I_1 \; e^{j(\omega t+\phi_1)} \\ I_2 \; e^{j(\omega t+\phi_2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_{n+1} \; e^{j(\omega t+\phi_{n+1})} \end{pmatrix} = (\bar{I}')\, e^{j\omega t} \qquad 3\text{-}2$$

where $(\bar{V}')$ and $(\bar{I}')$ are phasor vectors.

Substitution of these phasor vectors into equations 2-6 and 2-7 respectively, with the operator p replaced by $j\omega$ yields

$$-\frac{d}{dx}(\overline{V}') = (Z'(\omega)) \quad (\overline{I}') \qquad 3\text{-}3$$

$$-\frac{d}{dx}(\overline{I}') = (Y'(\omega)) \quad (\overline{V}') \qquad 3\text{-}4$$

Equations 3-3 and 3-4 are written in terms of voltages with respect to some arbitrary reference. For a complete system, some reference within the system, such as a "ground" conductor may be used. If the (n+1)th conductor is chosen as the reference conductor, then the voltage phasor vector becomes

$$(\overline{V}) = \begin{pmatrix} \overline{V}_1 \\ \overline{V}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overline{V}_n \end{pmatrix} = \begin{pmatrix} \overline{V}_1' - \overline{V}'_{n+1} \\ \overline{V}_2' - \overline{V}'_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overline{V}_n' - \overline{V}'_{n+1} \end{pmatrix} \qquad 3\text{-}5$$

and

$$\overline{I}'_{n+1} = -\sum_{i=1}^{n} \overline{I}_i' \qquad 3\text{-}6$$

Applying these constraints to equation 3-3 yields the reduced system of equations for n independent conductors.

$$-\frac{d}{dx}(\overline{V}) = (Z(\omega)) \quad (I) \qquad 3\text{-}7$$

where $Z_{ij}(\omega) = Z_{ij}'(\omega) + Z'_{n+1,n+1}(\omega) - Z'_{i,n+1}(\omega) - Z'_{n+1,j}(\omega)$

$i, j = 1, 2, \ldots, n$

Since $(Y'(\omega))$ in equation 3-4 is not a function of x,

$$-(\overline{V}') = \frac{d}{dx}\left(Y'(\omega)\right)^{-1}(\overline{I}')$$

Let $(Y(\omega))^{-1}$ be the reduced form of $(Y'(\omega))^{-1}$, then,

$$-(\overline{V}) = (Y(\omega))^{-1}\frac{d}{dx}(\overline{I})$$

and the current equation for n independent conductors may be written

$$-\frac{d}{dx}(\overline{I}) = (Y(\omega))\ (\overline{V}) \qquad 3\text{-}8$$

This analysis indicates that the reduction must be achieved with the $(Y'(\omega))$ matrix in its inverse form.

The leakance matrix (G) may be separated into two parts,

(a) the leakance empirically derived from the losses due to the supporting mechanism (towers, conduits etc.) of the transmission system,

(b) the leakance due to the geometrical configuration of the conductors and to the conductivity of the surrounding media.

If part (b) alone is considered, then since the field distribution and leakage current distribution are the same for any given linear system of conductors (5),

$$(Y'(\omega)) = \frac{1}{(\sigma + j\omega\varepsilon)}(P')$$

where (P') is the potential coefficient matrix for the (n+1) conductors

$\sigma$ is the conductivity of the medium surrounding the conductors

and $\varepsilon$ is the permittivity of the medium surrounding the conductors.

Therefore, the matrix (P) for the reduced system of n independent conductors will have elements of the form

$$P_{ij} = P'_{ij} + P'_{n+1,n+1} - P'_{i,n+1} - P'_{n+1,j} \ , \qquad i,j = 1,\ 2,\ \ldots,\ n$$

and

$$(Y(\omega)) = (\sigma + j\omega\varepsilon)\ (P)^{-1}$$

Note that for most transmission systems, $\sigma/\omega\varepsilon \ll 1$

Finally, the reduced equations 3-7 and 3-8 may be combined as before to give

$$\frac{d^2}{dx^2}(\bar{V}) = (Z(\omega))\ (Y(\omega))\ (\bar{V}) \triangleq (A(\omega))\ (\bar{V}) \qquad 3\text{-}9$$

$$\frac{d^2}{dx^2}(\bar{I}) = (Y(\omega))\ (Z(\omega))\ (\bar{I}) \triangleq (B(\omega))\ (\bar{I}) \qquad 3\text{-}10$$

# 4. SOLUTION OF THE DIFFERENTIAL EQUATIONS

## 4.1 Characteristic Root and Characteristic Vector Analysis (6)

Equation 3-9, the voltage equation, may be written

$$\begin{pmatrix} A_{11} - d^2/dx^2 & A_{12} \cdot\cdot\cdot\cdot\cdot\cdot & A_{in} \\ A_{21} & A_{22} - d^2/dx^2 \cdot\cdot\cdot\cdot & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ A_{n1} \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & & A_{nn} - d^2/dx^2 \end{pmatrix} \begin{pmatrix} \bar{V}_1 \\ \bar{V}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{V}_n \end{pmatrix} = 0 \qquad 4\text{-}1$$

This has a non-trivial solution if and only if the determinant

$$\det \left| (A) - d^2/dx^2 (U) \right| = 0 \qquad 4\text{-}2$$

where (U) is the identity matrix. This determinantal equation is the characteristic equation whose solution yields the characteristic roots.

Consider again equation 3-9. There are n ordinary second order linear differential equations with constant coefficients which are homogeneous in $\frac{d^2}{dx^2}$.

Hence the form of the solution is

$$\bar{V}_i = \sum_{r=1}^{n} \left( C_{ir} e^{\gamma_r x} + C'_{ir} e^{-\gamma_r x} \right) , \quad i = 1, 2, \ldots, n \qquad 4\text{-}3$$

where the C's and C' 's are the complex constants of integration and $\gamma_i = \sqrt{\lambda_i}$, where the $\lambda$'s are the characteristic roots of the determinantal equation 4-2.

There are $2n^2$ constants of integration in the above form of the solution but it will be shown that only 2n of these constants are independent. Substitution of the general solution, equation 4-3 into the equation 3-9 yields n equations of the form

$$\sum_{r=1}^{n} \gamma_r^2 (C_{ir}e^{\gamma_r x} + C'_{ir}e^{-\gamma_r x}) = \sum_{r=1}^{n} A_{i1}(C_{ir}e^{\gamma_r x} + C'_{ir}e^{-\gamma_r x})$$
$$+ \cdots\cdots\cdots$$
$$+ \sum_{r=1}^{n} A_{in}(C_{nr}e^{\gamma_r x} + C'_{nr}e^{-\gamma_r x})$$

Collecting terms in $e^{\gamma_r x}$ and $e^{-\gamma_r x}$ we have

$$\sum_{r=1}^{n}\Big\{((\gamma_r^2 - A_{ii})\, C_{ir} - A_{i1}C_{1r} - A_{i2}C_{2r} - \cdots - A_{in}C_{nr})e^{\gamma_r x}$$
$$+ ((\gamma_r^2 - A_{ii})C'_{ir} - A_{i1}C'_{1r} - A_{i2}C'_{2r} - \cdots - A_{in}C'_{nr})e^{-\gamma_r x}\Big\} = 0 \tag{4-4}$$

Hence each of the coefficients in equation 4-4 is individually equal to zero for a non-trivial solution, $(i = 1, 2, \ldots, n)$. This provides $n^2$ equations for the unprimed constants, C, and $n^2$ equations in the primed constants, $C'$, as both i and r vary.

For the unprimed constants, C

$$\begin{pmatrix} A_{11} - \lambda_r & A_{12} & A_{1n} \\ A_{21} & A_{22} - \lambda_r \cdots & A_{2n} \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn} - \lambda_r \end{pmatrix} \cdot \begin{pmatrix} C_{1r} \\ C_{2r} \\ \vdots \\ C_{nr} \end{pmatrix} = 0 \ , \ r = 1, 2, \ldots, n \tag{4-5}$$

Similar equations may be written for the constants, $C^1$, and since these systems of equations are homogeneous, there are (n-1) independent relations between the constants C and also between the constants C' for each choice of r. This leaves 2n independent constants to be found from the boundary conditions.

It is apparent from the above discussion that the determination of the relationship between the constants, C yields the characteristic vectors, with one vector corresponding to each choice of λ. This may be shown explicitly by rewrit~~t~~ing equations 4-5 as follows,

$$\begin{pmatrix} A_{11}-\lambda_r & A_{12} \cdots & A_{1n} \\ A_{21} & A_{22}-\lambda_r & \cdots \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn}-\lambda_r \end{pmatrix} \begin{pmatrix} C_{1r}/C_{nr} \\ C_{2r}/C_{nr} \\ \vdots \\ 1 \end{pmatrix} C_{nr} = (0)$$

$$\begin{pmatrix} A_{11}-\lambda_r & A_{12} \cdots & A_{1,n-1} \\ A_{21} & A_{22}-\lambda_r \cdots & \cdots \\ \cdots & \cdots & \cdots \\ A_{n-1,1} & \cdots & A_{n-1,n-1}-\lambda_r \end{pmatrix} \begin{pmatrix} D_{1r} \\ D_{2r} \\ \vdots \\ D_{n-1,r} \end{pmatrix} = - \begin{pmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{n-1,n} \end{pmatrix}$$

4-6

where $D_{ir} = C_{ir}/C_{nr}$ and $C_{nr} \neq 0$.

This analysis indicates a method of determining the characteristic vectors numerically. The same vectors hold for both the constants C and $C^1$.

4.2 The General Solution

The voltage solution may be written as

$$\bar{V}_j = C_{nj} \sum_{r=1}^{n} D_{jr}\, e^{\gamma_r x} + C'_{nj} \sum_{r=1}^{n} D_{jr}\, e^{-\gamma_r x} \quad , \quad r = 1, 2, \ldots n \qquad 4\text{-}7$$

where $D_{nr} = 1$, and $C_{nj}$, $C'_{nj}$ are unknown constants to be determined from the boundary conditions.

By a similar analysis, the current solutions may be written as

$$\bar{I}_j = F_{nj} \sum_{r=1}^{n} G_{jr}\, e^{\gamma_r x} + F'_{nj} \sum_{r=1}^{n} G_{jr}\, e^{-\gamma_r x} \quad , \quad r = 1, 2, \ldots n \qquad 4\text{-}8$$

where $G_{nr} = 1$, and $F_{nj}$, $F'_{nj}$ are unknown constants. It will be shown that the $\gamma_r$'s are the same for both voltage and current solutions. The constants $C_{nj}$, $C'_{nj}$ and $F_{nj}$, $F'_{nj}$ are not independent but are related through equations 3-7 and 3-8.

This solution may also be written in the alternative hyperbolic form using the hyperbolic sine and cosine.

The general solution for voltage and current may be written in matrix form

$$(\bar{V}) = (D)\left\{(C_{nr} e^{\gamma_r x}) + (C'_{nr} e^{-\gamma_r x})\right\} \qquad 4\text{-}9$$

$$(\bar{I}) = (G)\left\{(F_{nr} e^{\gamma_r x}) + (F'_{nr} e^{-\gamma_r x})\right\} \qquad 4\text{-}10$$

where (D) and (G) are square matrices containing characteristic vectors as columns. For example, the first column of (D) is the characteristic vector which satisfies

$$((A) - \lambda_1 \ (U)) \ (D_1) = 0$$

and hence is associated with the characteristic value $\lambda_1$. Note that the entries $D_{nr}$ (and $G_{nr}$) for $r = 1, 2, \ldots, n$ will be unity.

The matrices $(C_{nr}\, e^{\pm\gamma_r x})$ represent column vectors

$$\left(C_{nr}\, e^{\pm\gamma_r x}\right) \triangleq \begin{pmatrix} C_{n1}\, e^{\pm\gamma_1 x} \\ C_{n2}\, e^{\pm\gamma_2 x} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ C_{nn}\, e^{\pm\gamma_n x} \end{pmatrix} \qquad 4\text{-}11$$

As stated previously, the voltage and current solutions are related through equations 3-7 and 3-8; these equations imply that the current solution may be obtained from the voltage solution and vice versa.

Let the current solution be known, then by rearrangement of equation 3-8

$$\begin{aligned} \overline{V} &= -\left(Y(\omega)\right)^{-1} \frac{d}{dx}\left(\overline{I}\right) = -\left(Y(\omega)\right)^{-1} (G) \left\{ \left(F_{nr} e^{\gamma_r x}\, \gamma_r\right) - \left(F'_{nr}\, \gamma_r e^{-\gamma_r x}\right) \right\} \\ &= (D) \left\{ \left(C_{nr} e^{\gamma_r x}\right) + \left(C'_{nr} e^{-\gamma_r x}\right) \right\} \end{aligned}$$

By equating coefficients of $e^{\gamma_r x}$ and $e^{-\gamma_r x}$ respectively, we obtain

$$\begin{aligned} (D)(C_{nr}) &= -\left(Y(\omega)\right)^{-1} (G)(F_{nr}\gamma_r) \qquad 4\text{-}12 \\ &= -\left(P(\omega)\right)(G)(F_{nr}\gamma_r) \end{aligned}$$

and $$\left( D \right)\left( C'_{nr} \right) = \left( Y(\omega) \right)^{-1} \left( G \right)\left( F^{1}_{nr}\gamma_r \right) \qquad 4\text{-}13$$

$$= \left( P(\omega) \right) \left( G \right)\left( F'_{nr}\gamma_r \right)$$

where leakance has been ignored and $\left( Y(\omega) \right)\left( P(\omega) \right) = \left( U \right)$.

Thus the general form for the voltage and current solutions may be written

$$\left( \bar{V} \right) = -\left( P(\omega) \right) \left( G \right)\left\{\left( F_{nr}\gamma_r e^{\gamma_r x} \right) - \left( F'_{nr}\gamma_r e^{-\gamma_r x} \right)\right\}, \quad r=1,2,..n \qquad 4\text{-}14$$

$$\left( \bar{I} \right) = \left( G \right)\left\{\left( F_{nr} e^{\gamma_r x} \right) + \left( F'_{nr} e^{-\gamma_r x} \right)\right\}, \quad r=1,2,..,n \qquad 4\text{-}15$$

In a similar manner the current constants may be determined in terms of the voltage constants by use of equation 3-7 using the properties of duality,

$$\left( \bar{V} \right) = \left( D \right)\left\{\left( C_{nr} e^{\gamma_r x} \right) + \left( C'_{nr} d^{-\gamma_r x} \right)\right\}, \quad r=1,2,..,n \qquad 4\text{-}16$$

$$\left( \bar{I} \right) = -\left( Z(\omega) \right)^{-1} \left( D \right)\left\{\left( C_{nr}\gamma_r e^{\gamma_r x} \right) - \left( C'_{nr}\gamma_r e^{-\gamma_r x} \right)\right\}$$

$$r=1,2,..,n \qquad 4\text{-}17$$

For equations 4-16 and 4-17, the voltage solutions will be defined as the "primary" solution; the current solution is a "derived" solution. Conversely, for equations 4-14 and 4-15, the current solution is the primary solution from which the voltage is derived. This latter form will be chosen to illustrate the following analysis of the boundary conditions.

## 5. BOUNDARY CONDITIONS

Consider the boundary conditions for the complete conductor system. There must be 2n such conditions which may be specified as constraints on the voltage, current or both at the boundaries.

For such an n-conductor system, there are in general 4n conditions at the boundaries, 2n at each end of the line. These are

$$\left.\begin{array}{l} (\bar{V})_{x=-\ell} \triangleq (V_s) \\ (\bar{I})_{x=-\ell} \triangleq (I_s) \end{array}\right\} \text{, sending end conditions,}$$

and

$$\left.\begin{array}{l} (\bar{V})_{x=0} \triangleq (V_r) \\ (\bar{I})_{x=0} \triangleq (I_r) \end{array}\right\} \text{, receiving end conditions.}$$

Alternatively, the origin of x may be defined at the sending end, in which case the receiving end is designated by $x = \ell$ ; in both cases, x increases from the sending end to the receiving end.

Of the 4n boundary conditions, 2n must be known in order to obtain a unique system solution. Several special cases may be considered,

(i) $(V_s)$ and $(I_s)$ or $(V_r)$ and $(I_r)$

(ii) $(V_s)$ or $(I_s)$ and $(V_r)$ or $(I_r)$

(iii) $(V_s)$ or $(I_s)$ and $(Z_r)$

(iv) any 2n conditions of $(V_s)$, $(I_s)$, $(V_r)$ and $(I_r)$

where $(Z_r)$ is the receiving end impedance matrix defined by $(V_r) \triangleq (Z_r)(I_r)$.

The primary current form of solution is used to illustrate these special cases.

(i) $(V_r)$ and $(I_r)$ known.

"n" equations may be written for $(I_r)$.

$$(G)\left\{(F_{nr}) + (F'_{nr})\right\} = (I_r)$$

and n equations for $(V_r)$,

$$-(P(\omega))(G)\left\{(F_{nr}\gamma_r) - (F'_{nr}\gamma_r)\right\} = (V_r)$$

or $$(G)\left\{(F_{nr}\gamma_r) - (F'_{nr}\gamma_r)\right\} = -(Y(\omega))(V_r)$$

The 2n equations in 2n unknowns may be rewritten in the form

$$\begin{pmatrix} (U) & (U) \\ (U\gamma) & -(U\gamma) \end{pmatrix} \begin{pmatrix} F_{nr} \\ F'_{nr} \end{pmatrix} = \begin{pmatrix} (G)^{-1}(I_r) \\ -(G)^{-1}(Y(\omega))(V_r) \end{pmatrix} \tag{5-1}$$

where $$(U\gamma) \triangleq \begin{pmatrix} \gamma_1 & & 0 \\ & \gamma_2 & \\ 0 & & \ddots \; \gamma_n \end{pmatrix} \tag{5-2}$$

An explicit solution may be obtained for the column vectors $(F_{nr})$ and $(F'_{nr})$,

$$(F_{nr}) = \tfrac{1}{2}\left\{(G)^{-1}(I_r) - (U\gamma)(G)^{-1}(Y(\omega))(V_r)\right\}$$

$$(F'_{nr}) = \tfrac{1}{2}\left\{(G)^{-1}(I_r) + (U\gamma)(G)^{-1}(Y(\omega))(V_r)\right\}$$

Let $(X) \triangleq ((G)(U\gamma))^{-1}$

$$\therefore \quad (F_{nr}) = \tfrac{1}{2}\left\{(U\gamma)(X)(I_r) - (X)(Y(\omega))(V_r)\right\}$$

and $$(F'_{nr}) = \tfrac{1}{2}\left\{(U\gamma)(X)(I_r) + (X)(Y(\omega))(V_r)\right\}$$

which is the simplest form for numerical solution using a digital computer.

(iia) $(I_r)$ and $(I_s)$ known.

The 2n equations are

$$(G)\left\{(F_{nr}) + (F'_{nr})\right\} = (I_r)$$

and $$(G)\left\{(F_{nr}\, e^{-\gamma_r \ell}) + (F'_{nr}\, e^{\gamma_r \ell})\right\} = (I_s)$$

or

$$\begin{pmatrix} (U) & (U) \\ (Ue^{-\gamma\ell}) & (Ue^{\gamma\ell}) \end{pmatrix} \begin{pmatrix} (F_{nr}) \\ (F'_{nr}) \end{pmatrix} = \begin{pmatrix} (G)^{-1}(I_r) \\ (G)^{-1}(I_s) \end{pmatrix} \qquad 5\text{-}3$$

which may be solved as in case (i).

(iib) $(I_r)$ and $(V_s)$ known.

The matrix equation is

$$\begin{pmatrix} (U) & (U) \\ (U\gamma e^{-\gamma\ell}) & -(U\gamma e^{\gamma\ell}) \end{pmatrix} \begin{pmatrix} (F_{nr}) \\ (F'_{nr}) \end{pmatrix} = \begin{pmatrix} (G)^{-1}(I_r) \\ -(G)^{-1}(Y(\omega))(V_s) \end{pmatrix} \qquad 5\text{-}4$$

which may be solved as in case (i).

(iii) $(I_s)$ and $(Z_r)$ known.

From $(I_s)$,

$$(G)\left\{(F_{nr}\, e^{-\gamma_r \ell}) + (F'_{nr}\, e^{\gamma_r \ell})\right\} = (I_s)$$

$$\text{From } (V_r) = (Z_r)(I_r)\ ,$$

$$-(P(\omega))(G)\{(F_{nr}\gamma_r) - (F'_{nr}\gamma_r)\} = (Z_r)(G)\{(F_{nr}) + (F'_{nr})\}\ ,$$

or in matrix form

$$\left(\begin{array}{c|c} (G)(Ue^{-\gamma\ell}) & (G)(Ue^{\gamma\ell}) \\ (Z_r)(G) + P(\omega)\ (G)(U\gamma) & (Z_r)(G) - P(\omega)\ (G)(U\gamma) \end{array}\right)\begin{pmatrix} (F_{nr}) \\ (F'_{nr}) \end{pmatrix} = \begin{pmatrix} (I_s) \\ (0) \end{pmatrix} \qquad 5\text{-}5$$

(iv) Bonding of cables.

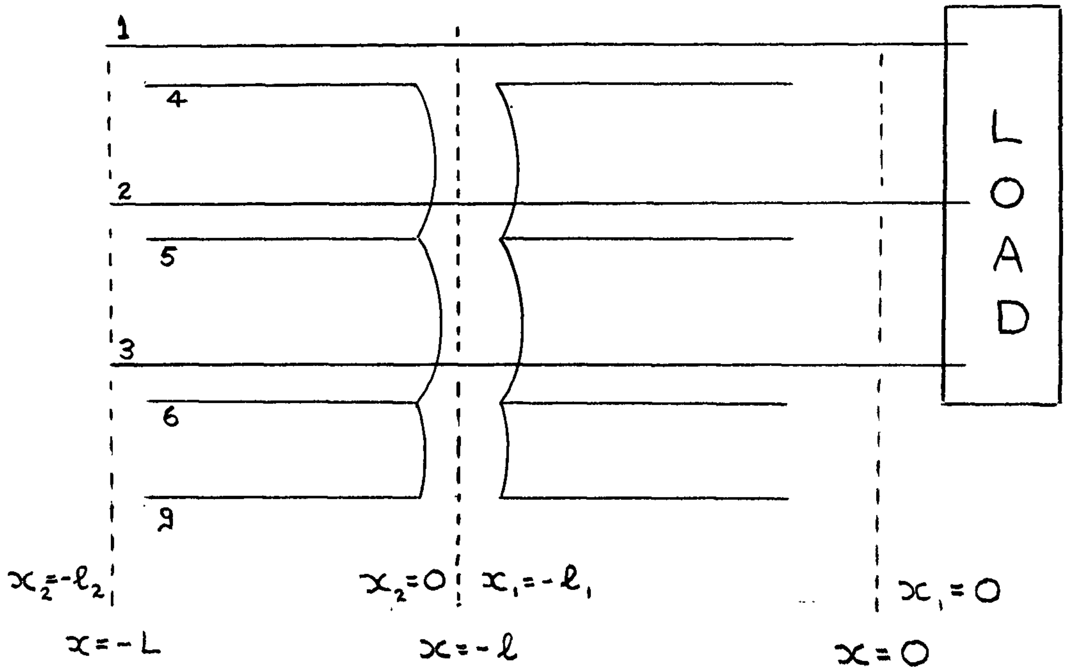



Fig. 5-1 Single Section of a Doubly Bonded Cable Transmission System with Six Independent Conductors.

Consider a section of a doubly bonded cable transmission system as shown, where conductors 1, 2, and 3 are the cable cores, conductors 4, 5, and 6 are the respective sheaths which are open circuited at x = 0 and x = −L. The sheaths are doubly bonded to the equivalent ground conductor, g, at $x = -\ell$, some point between the terminals. This ground conductor will be used as the voltage reference conductor. If the earth is to be considered as interacting with this system, then a further equivalent earth conductor would be necessary—this would be used as the reference conductor, giving a system of seven independent conductors.

Consider this system of conductors where the core currents and the core voltages at the load are specified. In this case the double-bonding junction must be treated as a "new" boundary and the given section of the transmission system must be treated as two sub-sections. If the load end sub-section is designated as sub-section ①, $-\ell_1 \leq x_1 \leq 0$, and the remaining sub-section as ②, $-\ell_2 \leq x_2 \leq 0$, then for section ① the system of equations to be solved for the boundary conditions

$$\left(\bar{V}_{1,2,3,}\right)_{x_1 = 0} \triangleq \left(V_r^o\right)_1 \quad \text{, receiving end conditions}$$

$$\left(\bar{V}_{4,5,6}\right)_{x_1 = -\ell_1} \triangleq \left(V^s_{\ell_1}\right)_1 = (0), \quad \text{sending end conditions}$$

and

$$\left(\bar{I}_j\right)_{x_1 = 0} \triangleq \left(I_r\right),\ j = 1, 2, \ldots, 6 \quad \text{, receiving end conditions}$$

is

$$\begin{pmatrix} -(P(\omega))(G)(U\gamma) & (P(\omega))(G)(U\gamma) \\ \text{(rows 1,2,\&3)} & \text{(rows 1,2,\&3)} \\ \hline -(P(\omega))(G)(U\gamma e^{-\gamma\ell_1}) & (P(\omega))(G)(U\gamma e^{\gamma\ell_1}) \\ \text{(rows 4,5,\&6)} & \text{(rows 4,5,\&6)} \\ \hline (G) & (G) \\ \text{(all rows)} & \text{(all rows)} \end{pmatrix} \begin{pmatrix} (F_{nr})_1 \\ \hline (F'_{nr})_1 \end{pmatrix} = \begin{pmatrix} (V_r^o)_1 \\ (V^s)_1 \\ (I_r) \end{pmatrix}$$

5-6

which may be expanded as

$$
\left(\begin{array}{cccc|ccc}
-\ell_{11}\gamma_1 & -\ell_{12}\gamma_2 & \cdots & -\ell_{16}\gamma_6 & \ell_{11}\gamma_1 & \cdots & \ell_{16}\gamma_6 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\ell_{31}\gamma_1 & \cdots & \cdots & -\ell_{36}\gamma_6 & \ell_{31}\gamma_1 & \cdots & \ell_{36}\gamma_6 \\
\hdashline
-\ell_{41}\gamma_1 e^{-\gamma_1\ell_1} & -\ell_{42}\gamma_2 e^{-\gamma_2\ell_1} & \cdots & -\ell_{46}\gamma_6 e^{-\gamma_6\ell_1} & \ell_{41}\gamma_1 e^{\gamma_1\ell_1} & \cdots & \ell_{46}\gamma_6 e^{\gamma_6\ell_1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\ell_{61}\gamma_1 e^{-\gamma_1\ell_1} & \cdots & \cdots & -\ell_{66}\gamma_6 e^{-\gamma_6\ell_1} & \ell_{61}\gamma_1 e^{\gamma_1\ell_1} & \cdots & \ell_{66}\gamma_6 e^{\gamma_6\ell_1} \\
\hdashline
g_{11} & g_{12} & \cdots & g_{16} & g_{11} & \cdots & g_{16} \\
g_{21} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
g_{61} & \cdots & \cdots & g_{66} & g_{61} & \cdots & g_{66}
\end{array}\right)
\begin{pmatrix} (F_{nr}) \\ \\ (F'_{nr}) \end{pmatrix}
=
\begin{pmatrix} (V_r^c)_1 \\ (V_l^s)_1 \\ (I_r) \end{pmatrix}
$$

where $\ell_{ij}$ is the $(i,j)^{th}$ element of the product $(P(\omega))(G)$ and $g_{ij}$ is the $(i,j)^{th}$ element of the characteristic vector matrix $(G)$.

The system of equations 5-6 can be solved for the constant vectors $(F_{nr})_1$ and $(F'_{nr})_1$ for sub-section 1 . Hence the remaining unknown currents and voltages at the left-hand boundary of sub-section 1 , $x_1 = -\ell_1$, can be found. The core voltages and currents are continuous and the boundary conditions for sub-section 2 are

$$(\overline{V}_j)_{x_2=0} \triangleq (V_r)_2 = (\overline{V}_j)_{x_1=-\ell_1}, \quad j = 1, 2, \ldots, 6$$

$$(\overline{I}_{1,2,3})_{x_2=0} \triangleq (I_r^c)_2$$

and $$(\overline{I}_{4,5,6})_{x_2=-\ell_2} \triangleq (I_s^s) = (0).$$

The system of equations

$$\begin{pmatrix} -\ \begin{matrix}P(\omega)(G)(U\gamma)\\ \text{(all rows)}\end{matrix} & \begin{matrix}P(\omega)\ (G)(U\gamma)\\ \text{(all rows)}\end{matrix} \\ \begin{matrix}(G)\\ \text{(rows 1,2,\&3)}\end{matrix} & \begin{matrix}(G)\\ \text{(rows 1,2,\&3)}\end{matrix} \\ \begin{matrix}(G)(Ue^{-\gamma \ell_2})\\ \text{(rows 4,5,\&6)}\end{matrix} & \begin{matrix}(G)(Ue^{+\gamma \ell_2})\\ \text{(rows 4,5,\&6)}\end{matrix} \end{pmatrix} \begin{pmatrix} (F_{nr})_2 \\ \\ (F'_{nr})_2 \end{pmatrix} = \begin{pmatrix} (V_r)_2 \\ (I^c_r)_2 \\ (I^s_s)_2 \end{pmatrix} \qquad 5\text{-}7$$

can be solved for the constant vectors $(F_{nr})_2$ and $(F'_{nr})_2$

Knowing the constant vectors for each sub-section of the transmission line, the complete solution for the complete section can be determined using equations 4-14 and 4-15.

If the sheath bondings for each sub-section are connected, additional constraints are imposed on the system. In this case, the constraint equation is

$$\sum_{j=4}^{6} (I_j)_1 = -\sum_{j=4}^{6} (I_j)_2 \qquad 5\text{-}8$$

Since the load boundary conditions for the case are specified, then the connection of the bonding causes a constraint to be imposed of sub-section 1 by sub-section 2.

Such a constraint can be handled by using a transposition matrix as specified in the next chapter. However, since the entries in a transposition matrix are unity, this represents a "lossless" internal boundary or transposition point. This is satisfactory as a first approximation.

For a "lossy" transposition, where voltage and current magnitude and/or phase angle for a given conductor does change, the corresponding entry in the transposition matrix will be in general a complex number with absolute value different from unity. Such an entry, $t_{ij}$, may be represented by

$$t_{ij} = e^{\pm(\rho + j\eta)}$$

where $\rho$ is the attenuation factor, and $\eta$ gives the phase angle shift.

# 6. TRANSPOSITION MATRICES AND THE COMPLEX CHARACTERISTIC MATRIX

## 6.1 The Transposition Matrix.

Consider a transmission line with multiple sections where at each or any junction, two or more of the conductors have their physical locations in space interchanged. Such an interchange is indicated in the resistance, ( R ), inductance, ( L ) and capacitance, ( C ) coefficient matrices by a corresponding interchange of the appropriate rows and columns.

For example, the interchange of two conductors (i&j) at one junction may be made in the ( R ), ( L ) and ( C ) matrices by using the transformation matrix $( E_r )$ which is formed from the identity matrix ( U ) by interchanging the $i^{th}$ and $j^{th}$ rows or columns

$$E_r = \begin{array}{cc} & \begin{array}{cccccc} \phantom{1} & \phantom{0} & i & j & \phantom{0} & \phantom{1} \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & \vdots & \vdots & & \\ & \cdots & 0 & 1 & \cdots & \\ & \cdots & 1 & 0 & \cdots & \\ & & \vdots & \vdots & & 1 \end{pmatrix} \end{array} \tag{6-1}$$

Premultiplication of any matrix ( A ), (say), by the matrix $( E_r )$, (which is compatible with matrix ( A ),) causes the $i^{th}$ and $j^{th}$ rows of ( A ) to be interchanged, and postmultiplication causes the corresponding columns of ( A ) to be interchanged.

i.e. $(E_r)(A)$ yields $(A)$ with $i^{th}$ and $j^{th}$ rows interchanged,
$(A)(E_r)$ yields $(A)$ with $i^{th}$ and $j^{th}$ columns interchanged,
and $(E_r)(A)(E_r)$ yields $(A)$ with $i^{th}$ and $j^{th}$ rows and columns interchanged.

Consider some properties of the set of matrices $(E_r)$. By inspection, these matrices are symmetric,

$$(E_r) = (E_r)_t \qquad 6\text{-}2$$

If the matrix $(E_r)$ of rank r, is partitioned as follows,

$$(E_r) = \begin{pmatrix} (U_i) & & 0 \\ & (J) & \\ 0 & & (U_{r-j}) \end{pmatrix}$$

where $(U_i)$ is an identity matrix of rank i, and $(J)$ is a matrix of rank $(j - i - 1)$,

$$(J) = \begin{pmatrix} \emptyset & \cdots & & & & 1 \\ \vdots & 1 & & & & \vdots \\ \vdots & & 1 & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & 1 & \vdots \\ 1 & \cdots & & & & \emptyset \end{pmatrix} = \begin{pmatrix} \emptyset & (0) & 1 \\ (0) & (U_{j-i-2}) & (0) \\ 1 & (0) & \emptyset \end{pmatrix}$$

where the symbol $\emptyset$ represents the scalar zero,

$$\therefore \quad (J)^2 = \begin{pmatrix} 1 & & 0 \\ & (U_{j-i-2}) & \\ 0 & & 1 \end{pmatrix} = (U_{j-i-1})$$

and $$(E_r)^2 = \begin{pmatrix} (U_i) & & 0 \\ & (J)^2 & \\ 0 & & (U_{r-j}) \end{pmatrix} = (U_r)$$

$$\therefore \quad (E_r)(E_r)_t = (U_r) \qquad \text{6-3}$$

and $(E_r)$ is orthogonal.

For this matrix $(E_r)$ the distributive law holds, i.e.

$$(E_r)(R)(E_r) + j\omega(E_r)(L)(E_r) = (E_r)(Z)(E_r)$$

and hence,

$$(E_r)(Z)(E_r)(E_r)(Y)(E_r) = (E_r)(A)(E_r)$$

where $(Z)(Y) = (A)$.

Thus the transformation is valid on the product, $(A)$, without reverting to the separate parts of the matrix $(A)$.

From these properties of the transposition matrix, it may be seen that the operation $(E_r)(A)(E_r)$ is a similarity transformation which leaves the characteristic roots of $(A)$ and $(E_r)(A)(E_r)$ unchanged, but not the characteristic vectors. The fact that the characteristic roots of the different sections of the line are the same is to be expected from physical considerations of the transmission line as a whole, since the geometrical configuration of the conductors is unchanged.

The relationship between characteristic vectors for two sections of line may be found from the defining equations of the characteristic vectors for each section,

$$(A)(x) = \lambda(x),$$

where $(x)$ is the characteristic vector corresponding to the root $\lambda$ for matrix $(A)$, and $(E_r)(A)(E_r)(y) = \lambda(y)$

$$\therefore \quad (A)(E_r)(y) = \lambda(E_r)^{-1}(y) = \lambda(E_r)(y)$$

Hence the required relationship which depends on the orthogonality of $(E_r)$ is

$$(x) = (E_r)(y).$$

Consider the physical situation where at a junction, more than one pair of conductors is interchanged. Implicit here is the assumption that the necessary elementary transformation matrices of the type $(E_r)$ may interchange at different times the same rows and columns more than once.

For example, in order to interchange the $i^{th}$, $j^{th}$, and $k^{th}$ conductors cyclically, the $i^{th}$ and $j^{th}$ rows and columns must be interchanged, followed by an interchange of the $k^{th}$ and "old" $i^{th}$ rows and columns.

Consider n such interchanges at one junction and the associated elementary matrices $(E_1)$, $(E_2)$, ..., $(E_n)$. The relationship between the original section of transmission line with associated characteristic matrix $(A')$ is given by

$$\begin{aligned}(A') &= (E_n)(E_{n-1}) \dots (E_2)(E_1)(A)(E_1)(E_2) \dots (E_n) \\ &\triangleq (\omega)(A)(X)\end{aligned}$$

The relationships between the characteristic roots and vectors of the matrices $(A)$ and $(A')$ are valid if it can be proved that $(\omega)^{-1} = (X)$, i.e. that such a transformation is a similarity transformation. It will also be shown that $(\omega)$ is an orthogonal matrix, $(\omega)(\omega)_t = (U)$.

$$\begin{aligned}(\omega)^{-1} &= ((E_n)(E_{n-1}) \dots (E_2)(E_1))^{-1} \\ &= (E_1)^{-1}(E_2)^{-1} \dots (E_{n-1})^{-1}(E_n)^{-1}\end{aligned}$$

$$= (E_1)(E_2) \cdot \cdot \cdot \cdot (E_{n-1})(E_n) = (X)$$

Also, $(\omega)_t = (E_1)_t (E_2)_t \cdot \cdot \cdot \cdot (E_{n-1})_t (E_n)_t$

$$= (E_1)(E_2) \cdot \cdot \cdot \cdot (E_{n-1})(E_n) = (X)$$

$\therefore$ $(\omega)(\omega)_t = (U)$ as required. Note that $(\omega)$ is not symmetric.

Thus the characteristic roots of $(A)$ and $(A')$ are the same and the characteristic vectors of these two matrices are related by

$$(A)(x) = \lambda (x)$$

$$(A')(y) = \lambda (y)$$

where $(A') = (X)^{-1}(A)(X)$

$\therefore$ $(x) = (X)(y)$

or $(y) = (\omega)(x)$.

Although the $(A)$ matrices are complex symmetric matrices, only real elementary transformations have been used.

## 6.2 Expansion of Complex Matrices to Real Matrices of Twice the Order. (8, 9, 10, 11)

In the numerical determination of the characteristic roots of the complex matrix $(A)$, it is found convenient to expand this matrix into a real matrix of twice the order, i.e. if $(A)$ is of rank n, then $(A)_{expanded}$ will be of rank 2n.

Hence for the expanded matrix there will be twice as many characteristic roots as for the original matrix. It will be shown that the 2n roots of $(A)_{exp}$ comprise the n roots of $(A)$ and n conjugates of these roots.

The usual way of expanding a complex matrix $(T)$ into the real matrix $(S_o)$ which doubles the number of rows and columns, is to expand each element, a + jb, of $(T)$ as follows,

$$a + jb \sim \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

A more convenient form of the matrix $(S_o)$ is

$$(S) = \begin{pmatrix} (A) & (B) \\ -(B) & (A) \end{pmatrix}$$

where $(A)$ is a matrix comprising the real parts of each complex element of $(T)$ in the same order, and $(B)$ is the corresponding matrix for the imaginary part.

Now, it may be shown that the matrices $(S_o)$ and $(S)$ are similar by use of transformation matrices of the type $(E)$ of the last section.

i.e. $\quad (S) = (P)^{-1}(S_o)(P) = (P)(S)(P)$

where $\quad (P) = (E_1)(E_2) \cdot \cdot \cdot \cdot (E_{n-1})(E_n)$

and $\quad (P)(P)_t = (U)$

Hence since similar matrices have the same characteristic polynomials and characteristic roots, either of the above forms may be used.

The relationship between the characteristic values, $\lambda$, of the complex matrix $(A) = (B) + j(C)$, and the characteristic roots of the expanded real matrix $(A)_{exp}$ may be shown as follows. The characteristic equations of the complex matrix $(A)$ and its conjugate matrix $(A)^*$ are

$$\det \{(A) - \lambda (U)\} = 0$$

and $\det \left\{ (A)^* - \lambda^*(U) \right\} = 0$

The characteristic equation of the following expanded complex matrix has twice the rank of the original complex matrix $(A)$,

$$\det \begin{pmatrix} (A) - \lambda(U) & 0 \\ 0 & (A)^* - \lambda^*(U) \end{pmatrix} = 0$$

or

$$\det \left\{ (A) - \lambda(U) \right\} \cdot \det \left\{ (A)^* - \lambda^*(U) \right\} = 0$$

Therefore the characteristic roots of this expanded complex matrix are the roots of $(A)$ and their conjugates.

Consider the similarity transformation $(T)^{-1}(K)(T)$ where

$$(T) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -j \\ 1 & j \end{pmatrix}$$

$$\text{For } (K) = \begin{pmatrix} (B) + j(C) & 0 \\ 0 & (B) - j(C) \end{pmatrix}$$

this transformation becomes

$$(T)^{-1}(K)(T) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ j & -j \end{pmatrix} \begin{pmatrix} (B) + j(C) & -j(B) + (C) \\ (B) - j(C) & j(B) - (C) \end{pmatrix} = \begin{pmatrix} (B) & (C) \\ (-C) & (B) \end{pmatrix} = (A)_{exp}$$

which is real.

The expanded matrix $(A)_{exp}$ can also be reduced to the form

$$\begin{pmatrix} (B) & -(C) \\ (C) & (B) \end{pmatrix}$$

by a further similarity transformation using the matrix $T_o$ where

$$(T_o) = (T_o)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

These expanded matrices being similar, they have the same characteristic roots and these roots will be the roots of $(A)$ and their conjugates, as required for the roots of a real matrix. Since this process yields 2n characteristic roots and only the n roots associated with the complex matrix $(A)$ are required, then it becomes necessary to separate these n required roots from the 2n roots obtained.

One approach to the selection of the required roots is to determine the complex characteristic polynomial by evaluating det $|(A) - \lambda(U)| = 0$. Only n of the roots would satisfy this equation; the remaining roots must be discarded. Such a numerical process was devised using the numerical evaluation of the characteristic polynomial attributable to A.M. Danilevsky.[12]

As an alternative approach, it will be shown that if the rank of the original complex matrix, $(A)$ is small ($n \leq 10$ say) then it is possible to determine the required roots by inspection.

Since the roots must appear as conjugate pairs, then the real parts of the roots, $Re(\lambda_i)$ will be repeated. For the imaginary parts, $Im(\lambda_i)$ there will be a change in sign. Hence the problem becomes one of separating from the 2n known imaginary parts, the n required imaginary parts. This may be achieved by comparing the aggregate of the n imaginary parts of the roots to the imaginary part of the trace of the complex matrix $(A)$ since

$$Im\ (tr(A)) = \sum_{i=1}^{n} \pm Im\ (\lambda_i)$$

## 6.3 Transposition and Connection Matrices for Multiple Section Lines.

Consider the partially transposed transmission line shown in Fig. 6.1 where the geometrical configuration of the line is the same for both sections and losses due to the transposition itself are negligible.

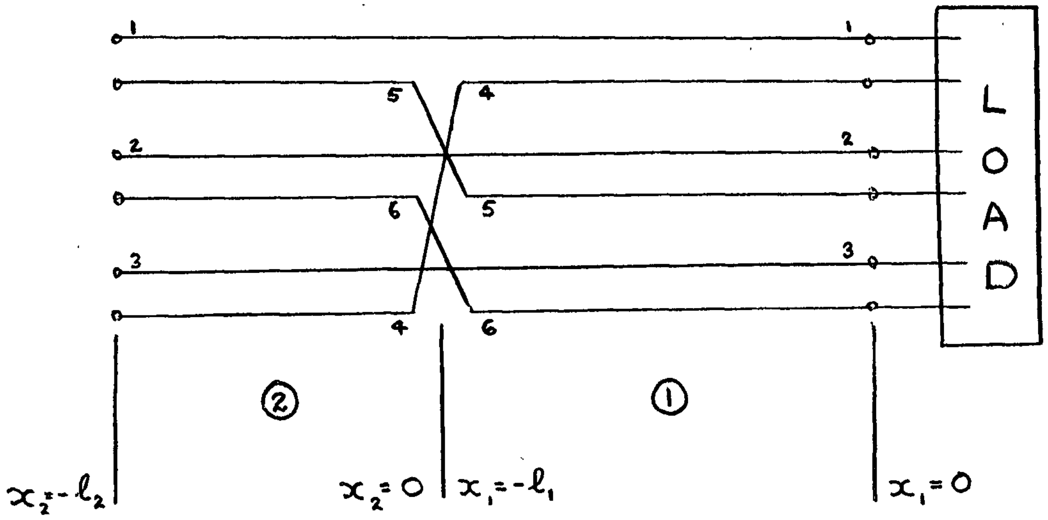



Fig. 6.1. Partially Transposed Transmission Line.

The relationships between the two sections are:

(i) $$(X)^{-1}(Z_1)(X) = (Z_2)$$

and $$(X)^{-1}(Y_1)(X) = (Y_2)$$

(ii) $$(X)^{-1}(Y_1)(Z_1)(X) = (Y_2)(Z_2)$$

(iii) the characteristic roots of the two sections are the same

$$\lambda_{1,r} = \lambda_{2,r}$$

(iv) the characteristic vectors of the two sections are related by

$$(X)^{-1}(d_{1i}) = (d_{2i})$$

or $$(X)^{-1}(D_1) = (D_2)$$

and $$(X)^{-1}(g_{1i}) = (g_{2i})$$

or $$(X)^{-1}(G_1) = (G_2)$$

where $(d)$, $(g)$ represent one characteristic vector of the voltage and current solutions respectively, and $(D)$, $(G)$ represent the square arrays of the n characteristic column vectors of the voltage and current solutions respectively.

For the two sections of line, the voltage and current solution equations 4-14 and 4-15 become

$$(\bar{V}_1) = -(P_1(\omega))(G_1)(U\gamma)\left\{(F_{1,nr}e^{\gamma_r x_1}) - (F'_{1,nr}e^{-\gamma_r x_1})\right\}$$

$$(\bar{I}_1) = (G_1)\left\{(F_{1,nr}e^{\gamma_r x_1}) + (F'_{1,nr}e^{-\gamma_r x_1})\right\}$$

for section 1 and

$$(\bar{V}_2) = -(P_2(\omega))(G_2)(U\gamma)\left\{(F_{2,nr}e^{\gamma_r x_2}) - (F'_{2,nr}e^{-\gamma_r x_2})\right\}$$

$$(\bar{I}_2) = (G_2)\left\{(F_{2,nr}e^{\gamma_r x_2}) + (F'_{2,nr}e^{-\gamma_r x_2})\right\}$$

for section 2. But,

$$(P_2(\omega)) = (X)^{-1}(P_1(\omega))(X) = (X)_t(P_1(\omega))(X)$$

and $$(G_2) = (X)^{-1}(G_1) = (X)_t(G_1)$$

$$\therefore (P_2(\omega))(G_2) = (X)_t(P_1(\omega))(G_1)$$

Hence the current and voltage solutions for section 2 become

$$(\bar{V}_2) = -(X)_t(P_1(\omega))(G_1)(U\gamma)\left\{(F_{2,nr}e^{\gamma_r x_2}) - (F'_{2,nr}e^{-\gamma_r x_2})\right\}$$

and $$(\bar{I}_2) = (X)_t(G_1)\left\{(F_{2,nr}e^{\gamma_r x_2}) + (F'_{2,nr}e^{-\gamma_r x_2})\right\}$$

Since the sequence in which the propagation constants, $\gamma_r$ are taken is unchanged by the transformation, the matrix $(U\gamma)$ is the same for both sections.

At the transposition boundary,

$$(\bar{I}_2)_{x_2 = 0} = (\bar{I}_1)_{x_1 = -\ell_1}$$

and $$(\bar{V}_2)_{x_2 = 0} = (\bar{V}_1)_{x_1 = -\ell_1}$$

$$\therefore \begin{pmatrix} (G_1)(Ue^{-\gamma\ell_1}) & (G_1)(Ue^{\gamma\ell_1}) \\ -(P_1(\omega))(G_1)(U\gamma e^{-\gamma\ell_1}) & (P_1(\omega))(G_1)(U\gamma e^{\gamma\ell_1}) \end{pmatrix} \begin{pmatrix} F_{1,nr} \\ F'_{1,nr} \end{pmatrix} =$$

$$\begin{pmatrix} (X)_t(G_1) & (X)_t(G_1) \\ -(X)_t(P_1(\omega))(G_1)(U\gamma) & (X)_t(P_1(\omega))(G_1)(U\gamma) \end{pmatrix} \begin{pmatrix} (F_{2,nr}) \\ (F'_{2,nr}) \end{pmatrix} =$$

$$(X)_t \begin{pmatrix} (G_1) & (G_1) \\ -(P_1(\omega))(G_1)(U\gamma) & (P_1(\omega))(G_1)(U\gamma) \end{pmatrix} \begin{pmatrix} (F_{2,nr}) \\ (F'_{2,nr}) \end{pmatrix} \qquad 6.4$$

This system of equations may be solved for $(F_{2,nr})$ and $(F'_{2,nr})$ in terms of $(F_{1,nr})$ and $(F'_{1,nr})$. The first n equations represent current continuity and the last n equations represent continuity of the space derivative of current.

Note that since the matrices $(U\gamma)$ and $(Ue^{\pm\gamma\ell})$ are diagonal, they commute and may be combined to form

$$(U\gamma)(Ue^{\pm\gamma\ell}) = (Ue^{\pm\gamma\ell})(U\gamma) = (U\gamma e^{\pm\gamma\ell}) .$$

Since equation 6-4 provides a relationship between the constants of integration for the two sections of the transmission line, then the analysis can be extended to give similar relationships between all sections of a multiple section line.

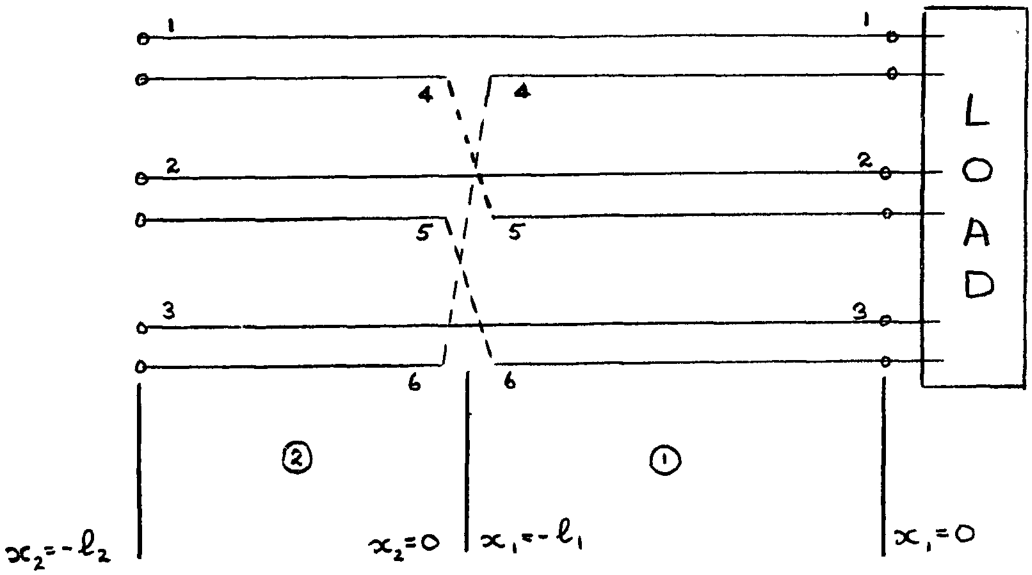



Fig. 6.2. Two Sections of Transmission Line.

The form of the system of equations 6-4 indicates an alternative and simpler method of considering the transmission line sections. For the transmission line shown in Fig. 6.2, the sections are identical and hence have the same characteristic equation, roots and vectors; only the constants of integration are different. At the transposition boundary, the respective currents and voltages are related through a connection matrix[13].

This connection matrix is identical to the matrix $(X)_t$ as observed in equation 6-3

i.e. $$(X)_t\,(\bar{I}_1) = (\bar{I}_2)$$

and $$(X)_t\,(\bar{V}_1) = (\bar{V}_2)$$

This is an invariant power transformation as required since

$$(\bar{I}_2)_t(\bar{V}_2) = (\bar{I}_1)_t(X)(X)_t(\bar{V}_1) = (\bar{I}_1)_t(\bar{V}_1)$$

Use of the connection matrix facilitates solution for multiple section transmission lines by greatly reducing the complexity of the numerical analysis.

# 7 THE ( Z ) AND ( Y ) MATRICES FOR A COMPLETE SYSTEM

## 7.1 The (Z) & (Y) Matrices

Under the constraints that the voltage reference is within the system, and that the sum of the currents over all conductors is zero, it has been shown that the matrices ( Z' ) and ( Y' ) of rank (n + 1) reduce to the matrices ( Z ) and ( Y ) of rank n. For the ( Z ) matrix,

$$Z_{rs} = Z'_{rs} + Z'_{n+1,n+1} - Z'_{r,n+1} - Z'_{n+1,s}$$

Hence,

$$Z_{rr} = R_r + R_n + j\omega(L_{rr} + L_{n+1,n+1} - L_{r,n+1} - L_{n+1,r})$$

$$= (R_r + R_n) + \frac{j\omega\mu}{2\pi}\left(\ln\frac{D'_{rx}}{D'_{rr}} + \ln\frac{D'_{n+1,x}}{D'_{n+1,n+1}} - \ln\frac{D'_{n+1,x}}{D'_{r,n+1}} - \ln\frac{D'_{rx}}{D'_{n+1,r}}\right)$$

and since $D'_{r,n+1} = D'_{n+1,r}$

$$Z_{rr} = (R_r + R_n) + \frac{j\omega\mu}{2\pi}\ln\frac{(D'_{r,n+1})^2}{D'_{rr}.D'_{n+1,n+1}} \qquad 7\text{-}1$$

where $D_{ij}$ is the geometric mean distance between conductors i and j and $D_{ii}$ is the geometric mean radius of conductor i.

Similarly,

$$Z_{rs} = R_n + \frac{j\omega\mu}{2\pi}\ln\frac{D'_{r,n+1}.D'_{n+1,s}}{D'_{rs}.D'_{n+1,n+1}} \qquad 7\text{-}2$$

and since $D'_{ij} = D'_{ji}$, the matrix ( Z ) is symmetric.

The reduced matrix ( Y ) may be found from the reduced form of the potential coefficient matrix ( P ) where

$$P_{rr} = P'_{rr} + P'_{n+1,n+1} - P'_{r,n+1} - P'_{n+1,r}$$

$$= \frac{1}{2\pi\varepsilon}\ln\frac{(D_{r,n+1})^2}{D_{rr}.D_{n+1,n+1}} \qquad 7\text{-}3$$

$$P_{rs} = \frac{1}{2\pi\varepsilon} \ln \frac{D_{r,n+1} \cdot D_{n+1,s}}{D_{rr} \cdot D_{n+1,n+1}} \qquad 7\text{-}4$$

and since $D_{ij} = D_{ji}$, the matrix $(P)$ is symmetric. The reduced matrix, $(Y)$ is given by

$$(Y) = j\omega (P)^{-1} = j\omega (C) \qquad 7\text{-}5$$

Since the matrix $(P)$ is symmetric, $(Y)$ is symmetric.

7.2 Properties of the $(Z)$ and $(Y)$ Matrices.

Associated with the voltage equations we have

$$(A) = (Z)(Y) ,$$

and for the current equations

$$(B) = (Y)(Z)$$

Since $$(Z) = (Z)_t$$

and $$(Y) = (Y)_t,$$

$$(Z)(Y) = (Z)_t (Y)_t = ((Y)(Z))_t$$

$\therefore$ $$(A) = (B)_t \qquad 7\text{-}6$$

Thus $(A)$ and $(B)$ are similar and hence have the same characteristic values.

Since the matrices $(D)$ and $(G)$ are the characteristic vector matrices corresponding to the matrices $(A)$ and $(B)$ respectively, we may write

$$(D)^{-1} (A)(D) = (U\lambda) \qquad 7\text{-}7$$

and $$(G)^{-1} (A)(G) = (U\lambda) \qquad 7\text{-}8$$

The transpose of equation 7-8 gives

$$(G)_t (B)_t (G)_t^{-1} = (U\lambda)_t = (U\lambda)$$

and from equation 7-6,

$$(G)_t(A)(G)_t^{-1} = (U\lambda) \qquad 7\text{-}9$$

Hence, from equations 7-7 and 7-9, we obtain

$$(G)_t(D) = (U) \qquad 7\text{-}10$$

which is a sufficient but not necessary condition. It may also be seen that if this condition is satisfied, the matrices $(G)$ and $(D)$ commute.

Since the voltage and current forms of solution are related to the matrices $(A)$ and $(A)_t$ respectively, there exists a matrix $(\Upsilon)$ such that

$$(\Upsilon)^{-1}(A)(\Upsilon) = (A)_t \qquad 7\text{-}11$$

Hence equation 7-8 becomes

$$(G)^{-1}(\Upsilon)^{-1}(A)(\Upsilon)(G) = (U\lambda) \qquad 7\text{-}12$$

and from equations 7-7 and 7-12

$$(D) = (\Upsilon)(G). \qquad 7\text{-}13$$

7-3 Restrictions on the Use of the Distributed Parameters.

Fundamental to any derivation or use of the distributed parameters is the assumption that there is a relationship with Maxwell's electromagnetic equations[14].

The application of circuit concepts to electromagnetic field phenomena, is restricted to those frequencies where the wavelength is far greater than the physical dimensions of the circuit. This condition is satisfied for power systems operating at low frequencies.

For a transmission line in a medium with homogeneous dielectric, the distribution of leakage current in the space surrounding the conductors follows the same pattern as the electric flux distribution and thus the conductance matrix $(G)$ has the same form as the capacitance matrix $(C)$ with conductivity in place of dielectric constant (5);

$$(G) + j\omega(C) \quad \alpha(\sigma_d + j\omega\varepsilon_d)$$

where the subscript "d" denotes dielectric. Hence ignoring the conductance $(G)$ implies that the displacement current is far greater than the conduction current in the dielectric,

$$\text{i.e.} \qquad \frac{\sigma_d}{\omega\varepsilon_d} \ll 1$$

There is a further contribution to the matrix $(G)$ due to the supporting mechanism of the conductor system. This can only be expressed empirically. In the physical model used, the effect of resistance $(R)$ but not of conductance $(G)$ was included. This implies that within the conductor the displacement currents are negligible compared to the conduction currents

$$\text{i.e.} \qquad \frac{\sigma_c}{\omega\varepsilon_c} \gg 1$$

where the subscript "c" denotes conductor and $\sigma_c$ is finite.

Since there is a component of electric field in the direction of propagation to force the current through the conductors, then the electric and magnetic field distributions must be disturbed which in turn affects the original inductance and capacitance parameters. However, this may be neglected if the axial electric field components within the homogeneous dielectric are small compared to the transverse components

$$\frac{\omega\varepsilon_d}{\sigma_c} \ll 1$$

Proximity effect involves all the parameters but has the most appreciable effect on the inductance and the capacitance. If the separation between conductors, D, is much greater than the conductor radius, $r_0$

i.e. $$\frac{r_0}{D} \ll 1$$

the effect is negligible.

These restrictions are applicable to the mathematical model developed in preceeding chapters. Increased sophistication of the model would require more stringent restrictions.

# 8 EXAMPLES OF APPLICATION AND RESULTS

Two examples of application were considered to test the validity of the theory. The first was an aerial double-line three phase transmission system with an overhead ground wire. The second was a three phase sheathed cable underground transmission system with a separate ground wire. It was assumed that homogeneous media surrounded the transmission system in both cases. Effects of the earth on distribution parameters were ignored. The rational M.K.S. system of units was used in the calculations. Leakance was ignored in both examples.

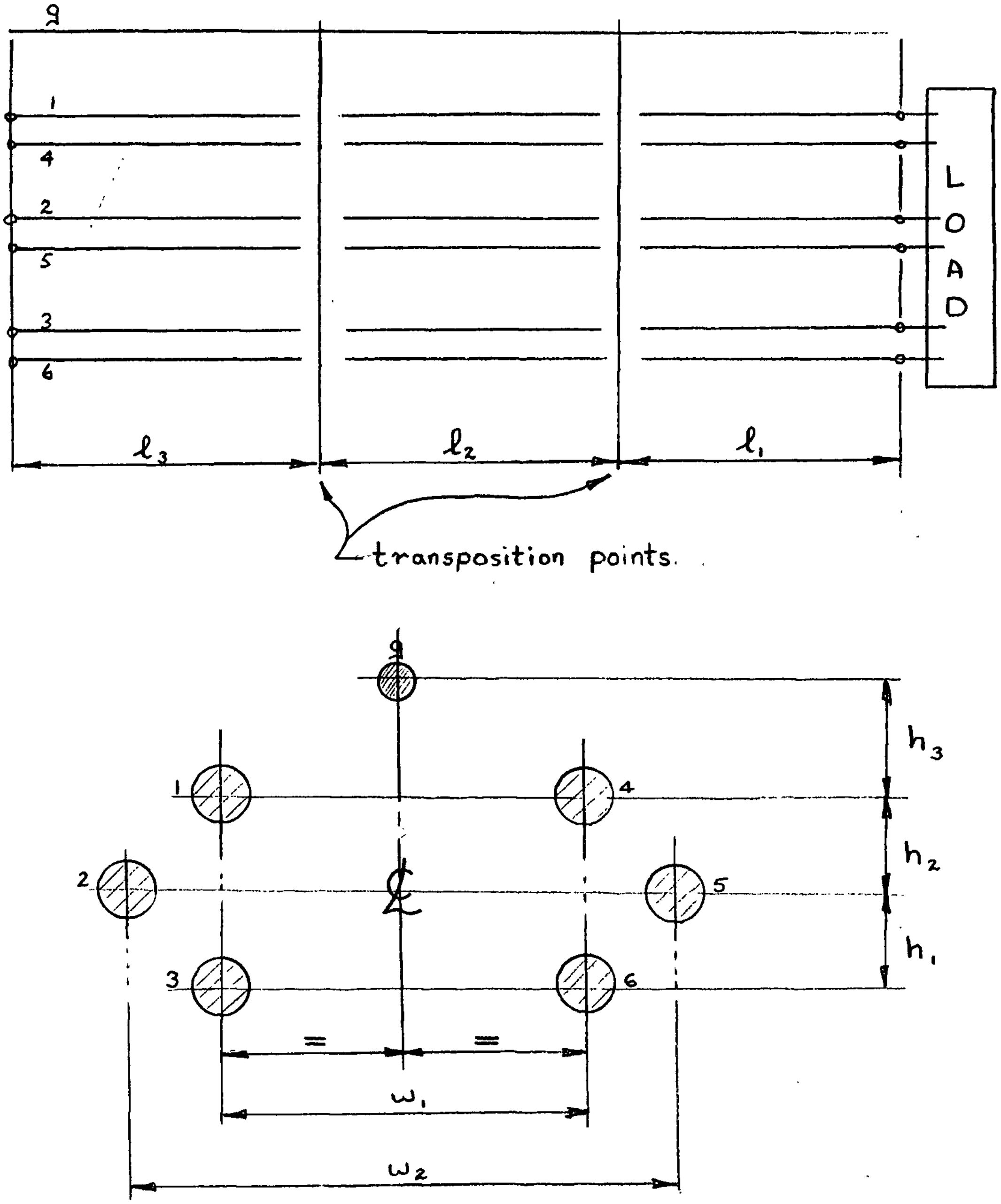
## 8.1 The Overhead Transmission System

The overhead transmission line in Figure 8.1 consists of six hollow aluminum conductors, 1-6, with inside radius 0.00622 metres and outside radius 0.0145 metres, and a copper ground conductor, g, with radius 0.00636 metres. The geometrical configuration of this system is shown in Figure 8.1.

Both transposed and untransposed systems were considered. Transposition points are shown in Figure 8.1. The system was assumed to operate at a constant temperature.

The transmission line has a capacity of 200 M.V.A. at 230 KV phase to phase and operates at 60 cycles per second. The load is assumed to have a 0.8 lagging power factor with three phase Y-connected balanced impedances.

Solutions for full load and no load conditions were found for current, voltage and power.




$\ell_1 = \ell_2 = \ell_3 = 133{,}200$ metres

$w_1 = 9{\cdot}45$ metres

$w_2 = 11{\cdot}59$ metres

$h_1 = 6{\cdot}19$ metres

$h_2 = 6{\cdot}19$ meters

$h_3 = 6{\cdot}83$ meters

Fig. 8.1 A three section, six conductor with ground, overhead system.

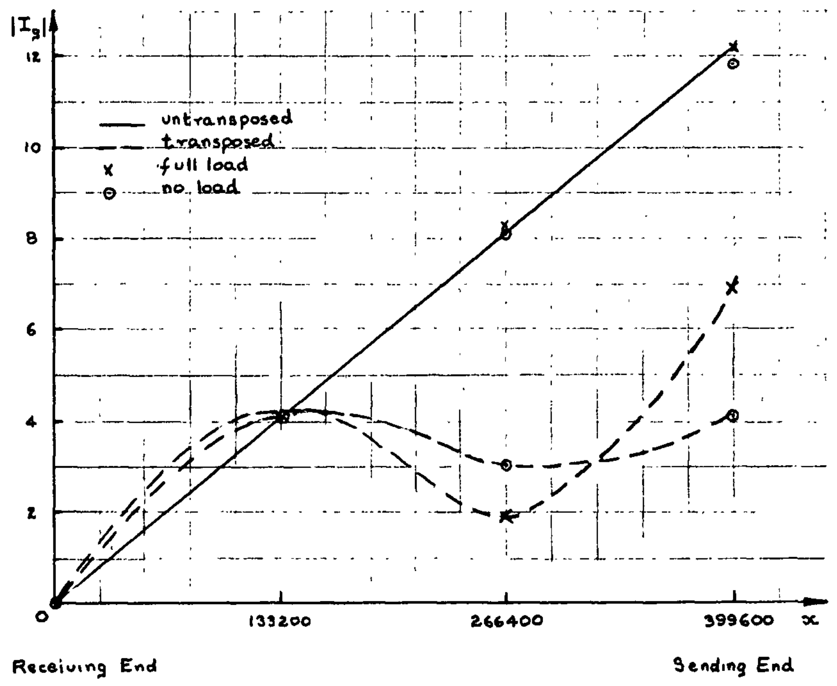



Fig. 8.2 Ground Wire Current, $|I_g|$, amps vs Distance Along the Line, $x$, metres

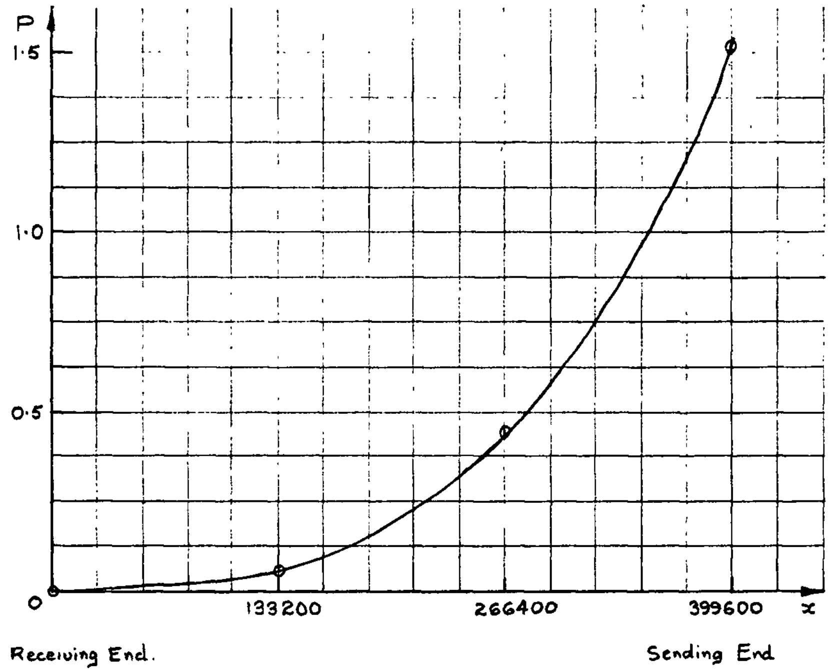



Fig. 8.3 Power Consumption, P, m.w. at No Load vs. Distance Along the Line, $x$, metres.

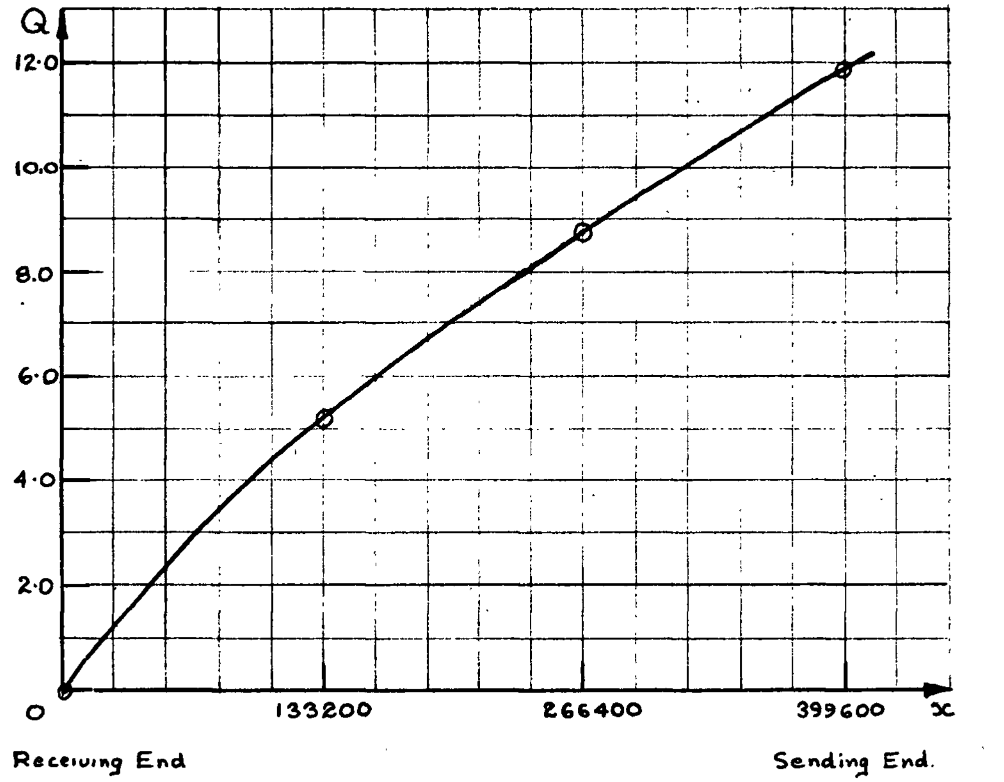



Fig. 8.4 Reactive Capacitative Power Consumption, Q, mva. at No Load vs Distance Along the Line, x, metres.

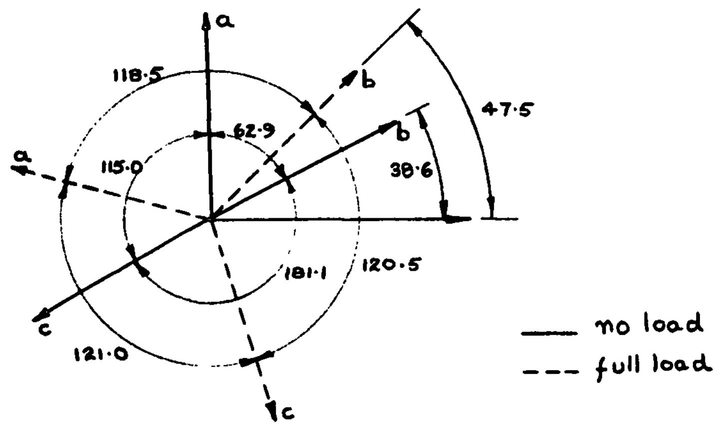



Fig. 8.5 Current Phase Angle Differences at Sending End for both Transposed and Untransposed Systems.

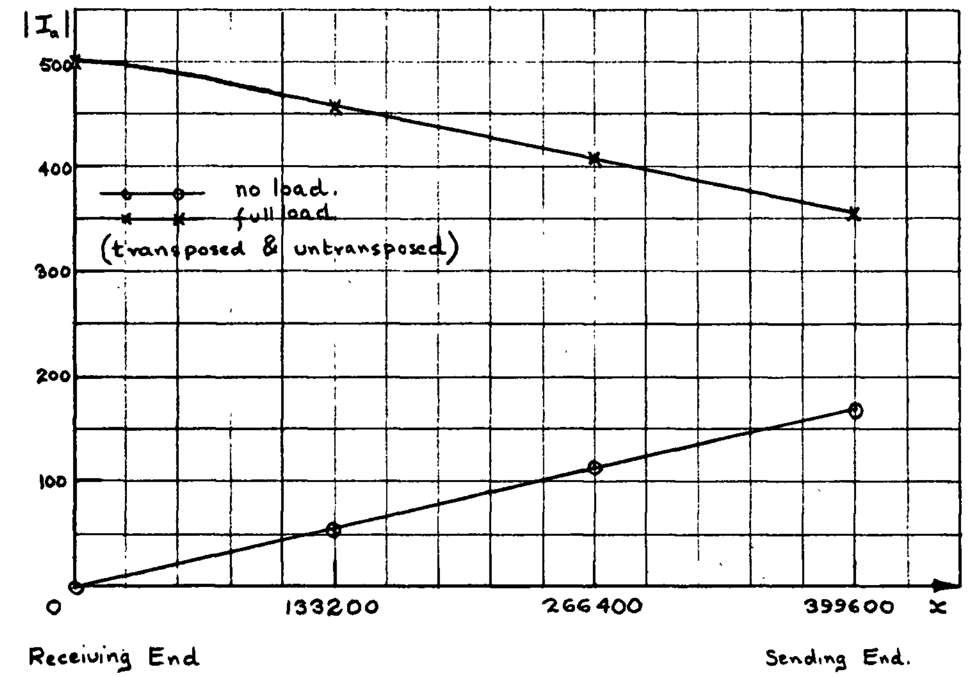



Fig. 8.6 a-phase Current, $|I_a|$, amps. vs Distance Along the Line, $x$, metres.

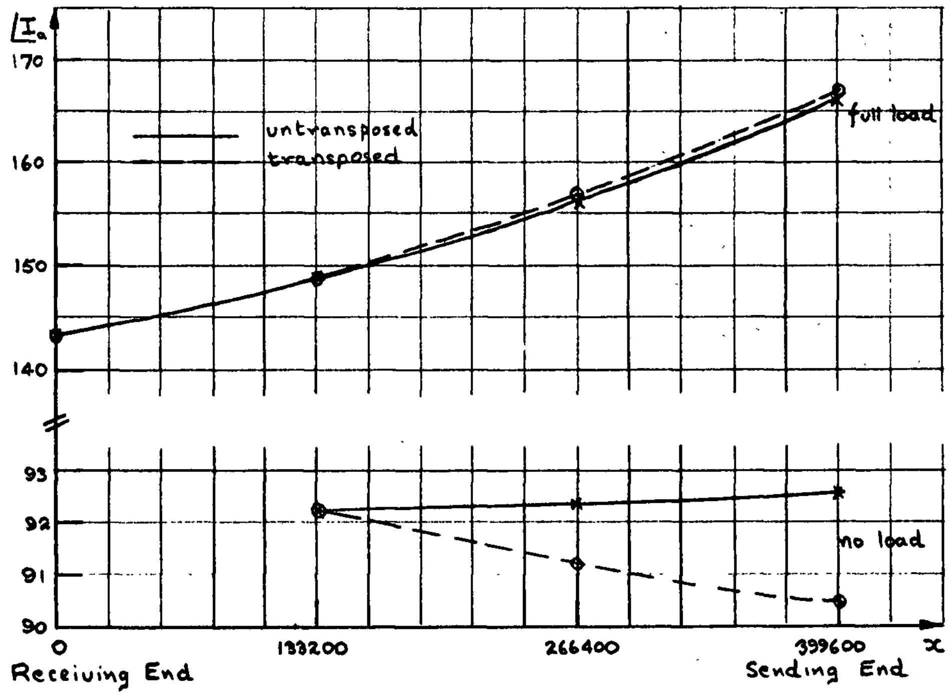



Fig. 8.7 a-phase Current Argument, $\angle I_a$, degrees vs Distance Along the Line, $x$, metres.

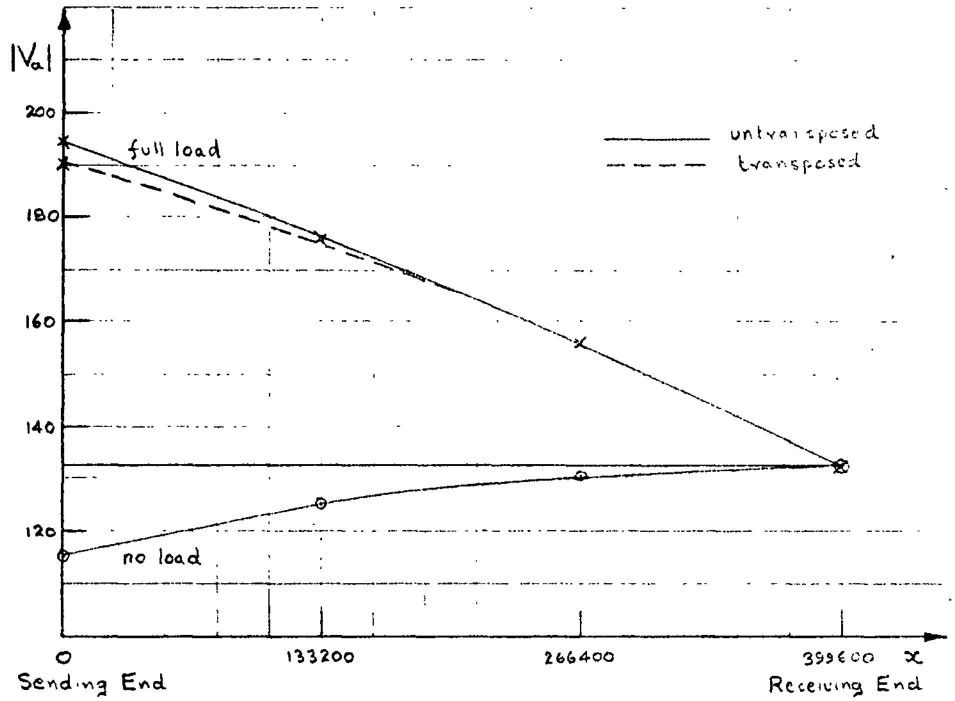



Fig. 8.8 a-phase Voltage, $|V_a|$, kv vs Distance Along the Line, x, metres

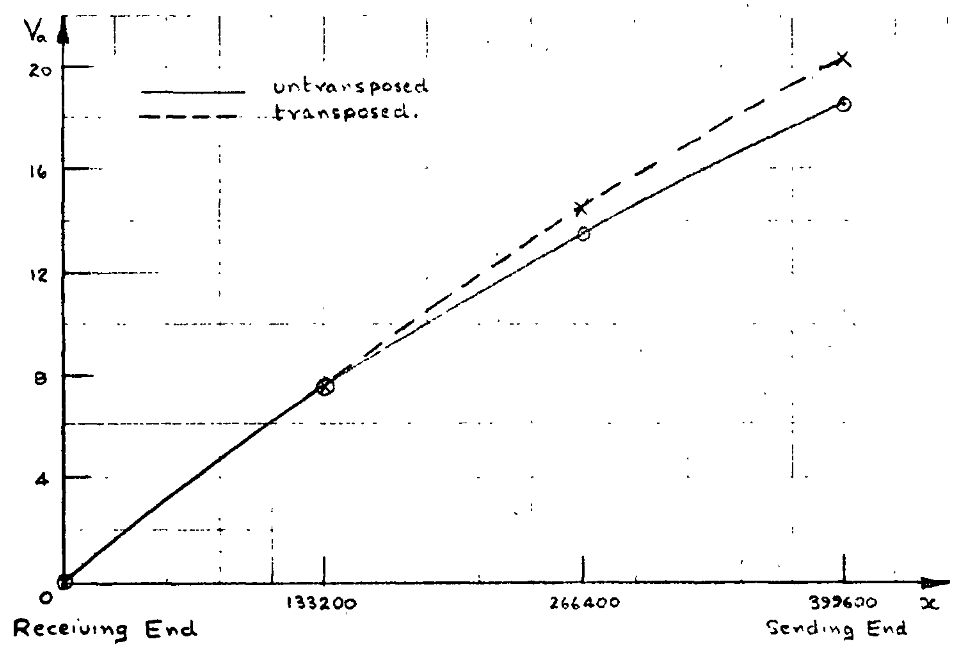



Fig. 8.9 a-phase Voltage Argument, $\angle V_a$, degrees vs Distance Along the Line, x, metres.

## 8.2 Results

In Figure 8.2, the ground wire current variation along the transmission line from the receiving end is plotted. For the untransposed system, the current varies linearly with distance as might be expected, and little variation is evident between no load and full load conditions. For the transposed system, the current variation is cyclic with maxima and minima occurring at the transposition points for both no load and full load, although the general trend is an increase in current from the receiving end towards the sending end. The trend can be explained on the basis of the different voltages occurring at the transposition points. The reversal of current magnitude at the transposition points is attributable to the effect of the mechanically abrupt transposition on current continuity.

Figure 8.3 and Figure 8.4 show the variation of power and reactive power respectively of the transmission system at no load. The real power increases rapidly at the sending end; the capacitative reactive power increases less rapidly.

Figure 8.5 shows the current phase angle differences at no load and full load. Although the three phase current phasors are balanced at full load, they are quite unbalanced at no load due to the effect on the charging currents of the asymmetrical geometry of the transmission line.

Figure 8.6 and Figure 8.7 show the a-phase current variation in magnitude and phase respectively along the transmission line. As expected, the no load current distribution increases linearly from the receiving end. At full load the current decreases towards the sending end, which is an indication of the compensation effect of the charging current on the load current. Little difference was observed at full load between the phase angle variation along the line of the transposed and untransposed systems. A difference can be seen however, under no load conditions.

The a-phase voltage magnitude variation and phase angle shift are indicated in Figure 8.8 and Figure 8.9 respectively. Under full load, the voltage difference between the transposed and untransposed systems was slight, and for no load, no difference was detectable. Similar statements can be made for phase angle shifts along the transmission line.

The data for the above graphs is included in Appendix C.1. In addition, it may be seen that at full load, while the power consumption increases towards the sending end, the inductive reactive power decreases.

## 8.3 The Underground Transmission System

The underground transmission system in Figure 8.10 consists of three sheathed conductors and a ground wire. The solid copper conductors, 1-3, have a radius of 0.0132 metres and the aluminum sheaths, 4-6, have an inside radius of 0.0239 metres and an outside radius of 0.0247 metres. The solid copper ground wire has a radius of 0.00318 metres. The geometrical configuration of the system is shown in Figure 8.10.

The system was assumed to operate at a constant temperature in a medium with relative dielectric constant, $\varepsilon_r = 4.0$.

The transmission line has a capacity of 10 M.V.A. at 13.2 KV phase to phase and operates at 60 cycles per second. The load is assumed to have a 0.9 lagging power factor with three-phase Y-connected balanced impedances.

Solutions for full load and no load conditions were found for current, voltage and power.

## 8.4 Results

In contrast to the overhead transmission system, the ground wire current of the underground system is independent of load and decreases in magnitude slightly from the sending end to the receiving end. A similar small change was observed in the phase angle. These results may be seen in Appendix C.2.

Figure 8.11 and Figure 8.12 show that the variation of power and reactive power respectively along the transmission line at no load is linear.

The conductor current phase angles are symmetric for full load conditions but not for no load conditions, as shown in Figure 8.13. The phase angle drift along the line is negligible for no load and is slight for full load, as indicated in the data of Appendix C.2.

The conductor current variation along the line from the receiving end, as shown in Figure 8.14, increases for both no load and full load.

In Figure 8.15, the sheath current phase angle differences are shown. These are the same for both no load and full and there is no phase angle drift along the line for any phase.

The variation of the sheath current along the line is shown in Figure 8.16.

From Appendix C.2 it may be seen that conductor voltage variation along the line in both magnitude and phase is small at no load, but increases slightly at full load. The three phase voltages are always balanced.

Figure 8.17 shows the phase angle differences of the sheath voltages. Considerable imbalance is apparent at no load but is less severe at full load. The phase angle drift along the line is small.

In Figure 8.18 and Figure 8.19 the sheath voltage variations along the line for no load and full load respectively are shown. Linear increase from the receiving end is observed at full load, but the increase is not linear at no load.

The data for the above graphs is included in Appendix C.2. In addition it may be seen that at full load, while the power consumption increases towards the sending end, the inductive reactive power decreases.

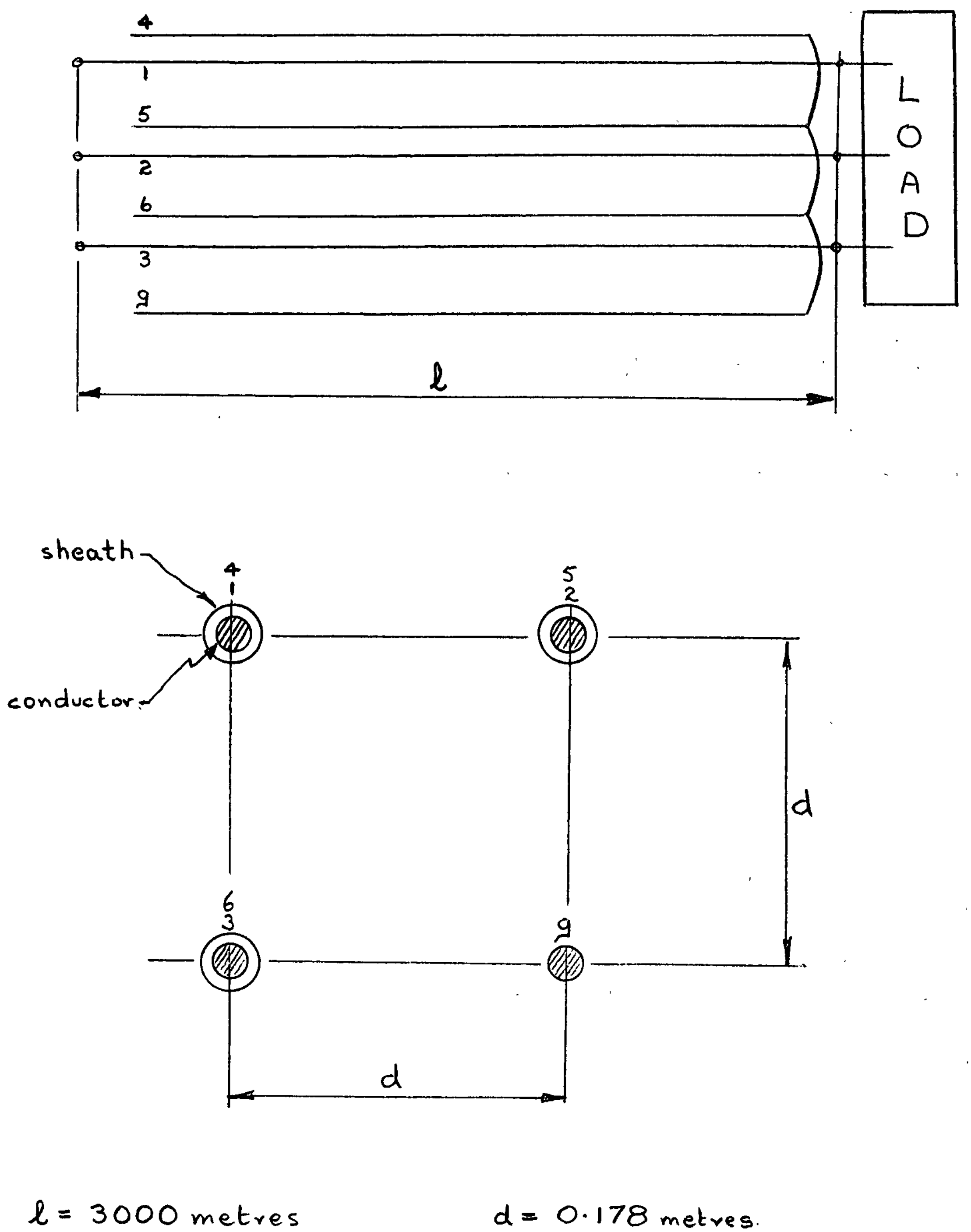



Fig. 8.10 An underground, three phase cable system with separate ground wire and sheaths around each conductor.

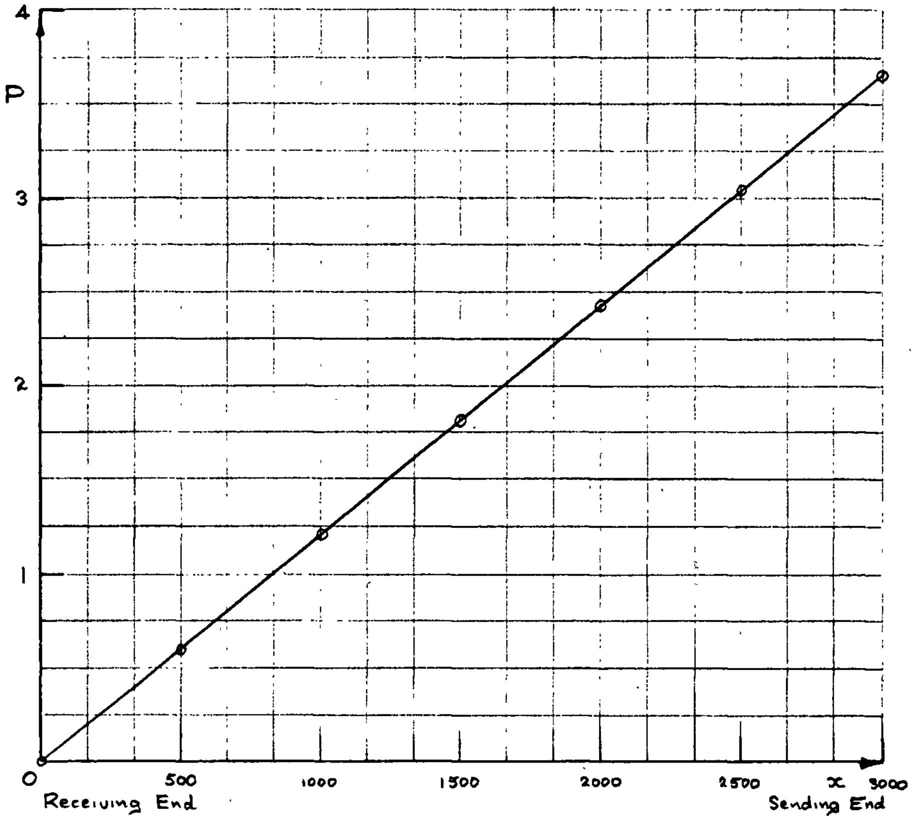



Fig. 8.11 No Load Power, P, megawatts vs. Distance Along the Line, x, metres.

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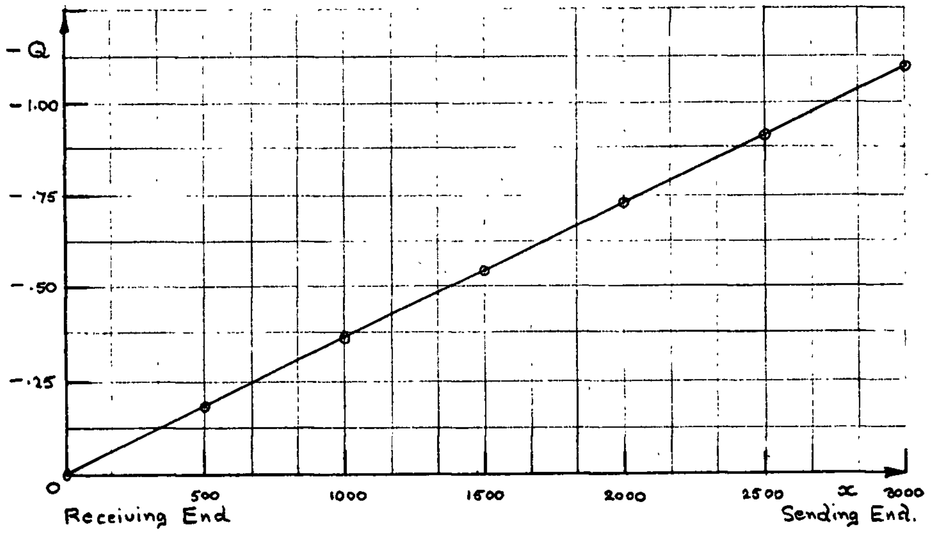


Fig. 8.12 No Load Reactive Power, Q, m.v.a. vs Distance Along the Line, x, metres.

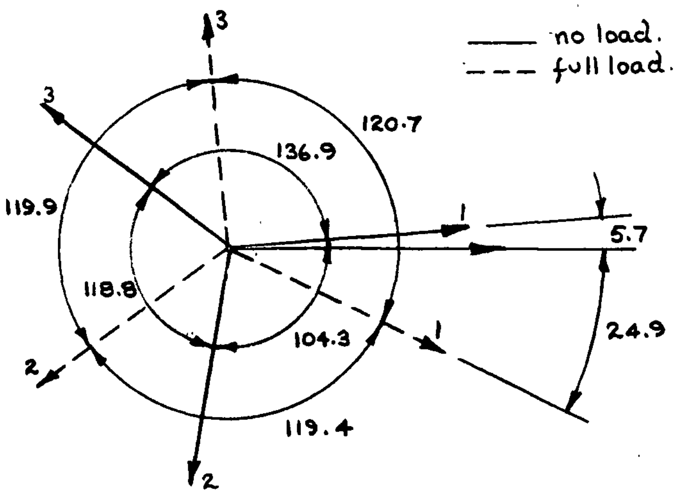


Fig. 8.13 Conductor Current Phase Angle Differences at the Sending End.

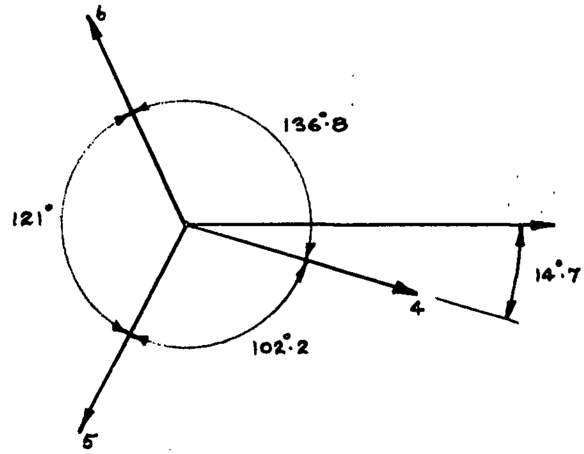


Fig 8.15 No Load and Full Load Sheath Current Phase Angle Differences at the Receiving End.

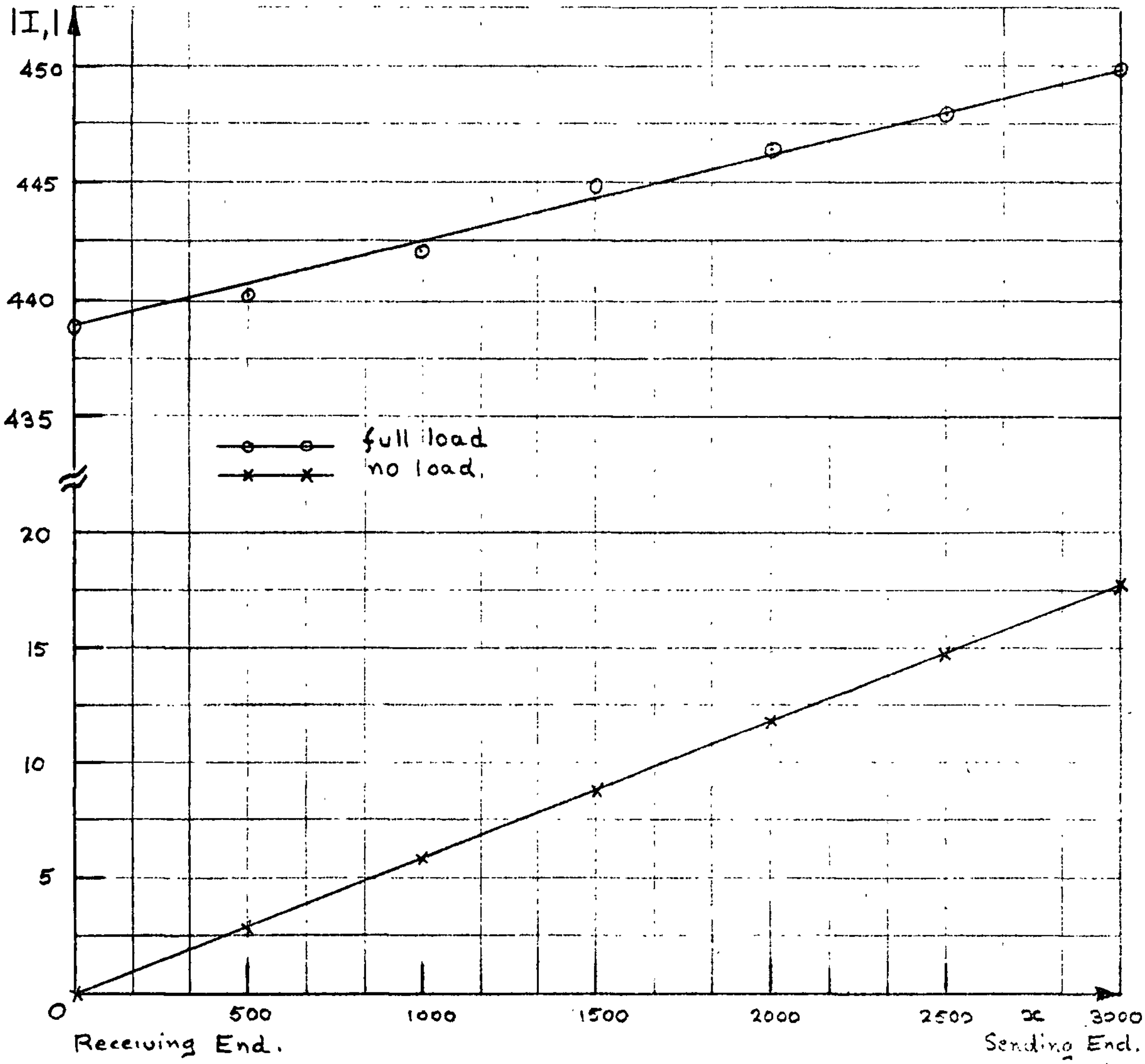


Fig. 8.14 Conductor Current, |I,|, amps vs. Distance Along the Line, x, metres.

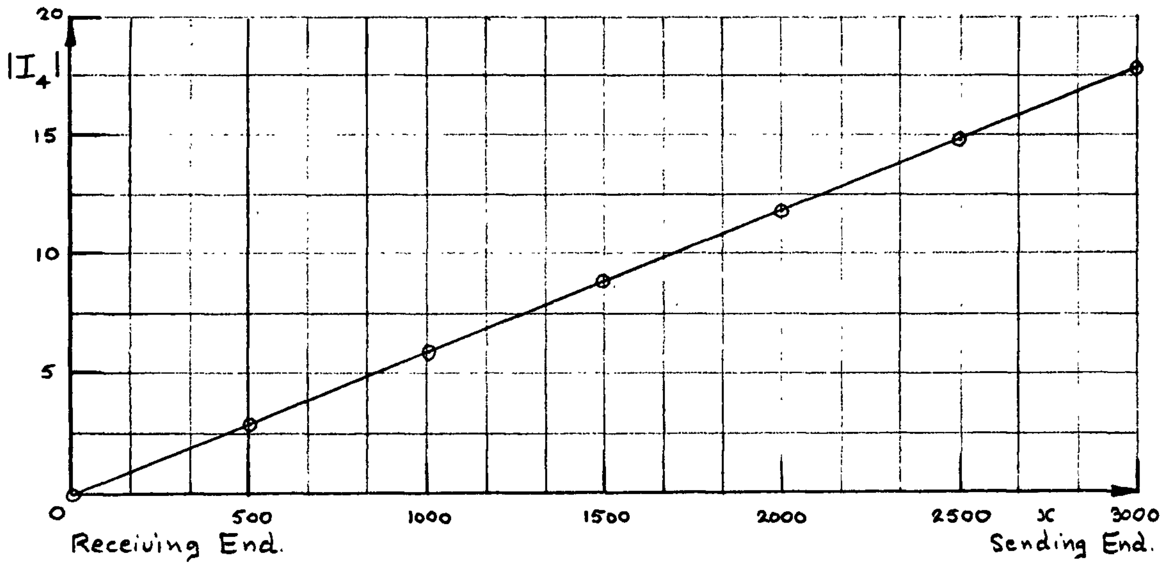


Fig. 8.16 No Load and Full Load Sheath Current, $|I_4|$, amps vs. Distance Along the Line, $x$, metres.

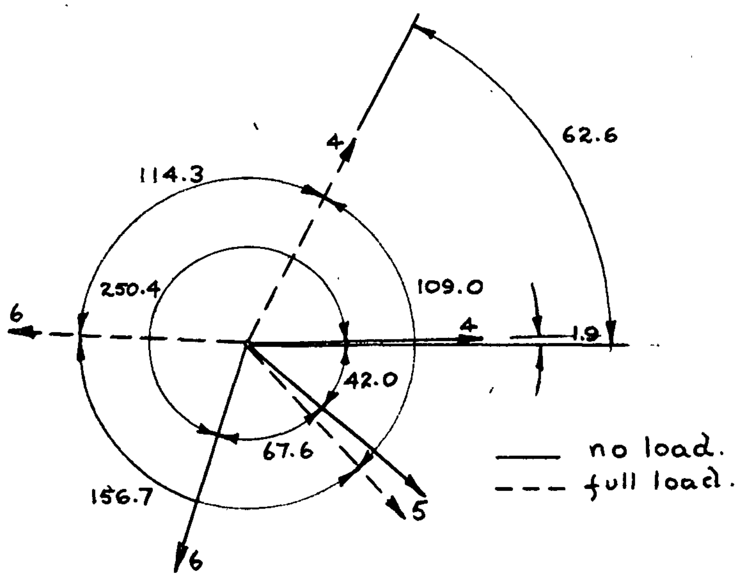


Fig. 8.17 Sheath Voltage Phase Angle Differences at the Sending End.

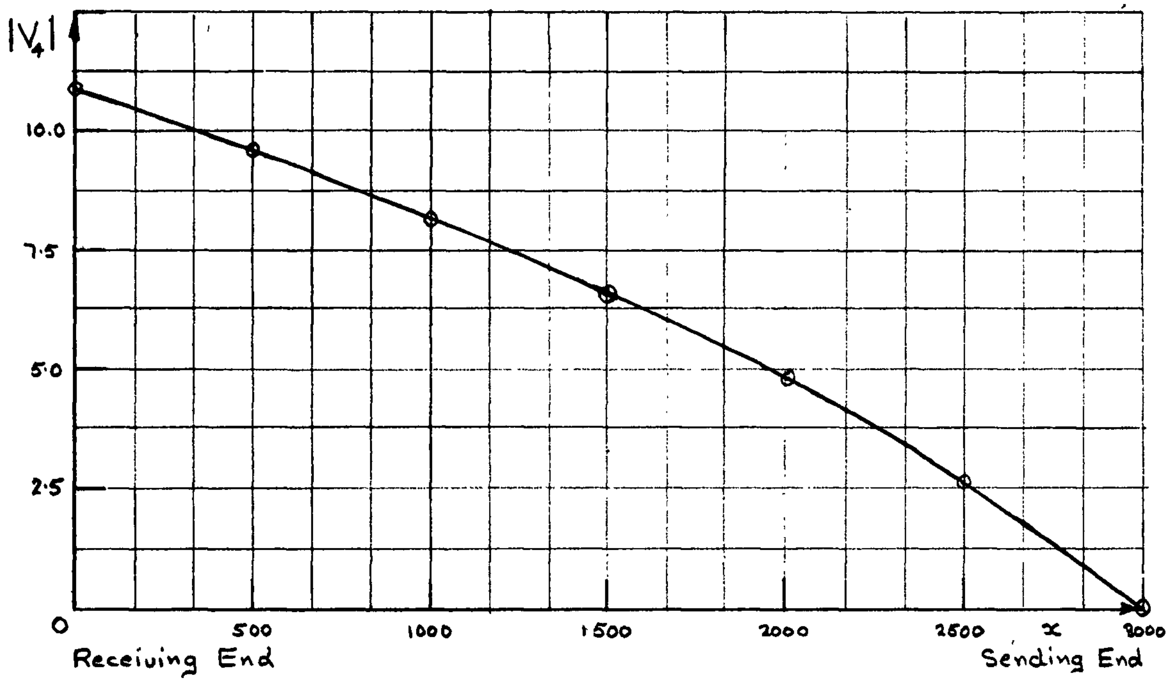


Fig. 8.18 No Load Sheath Voltage, $|V_4|$, volts vs Distance Along the Line, $x$, metres.

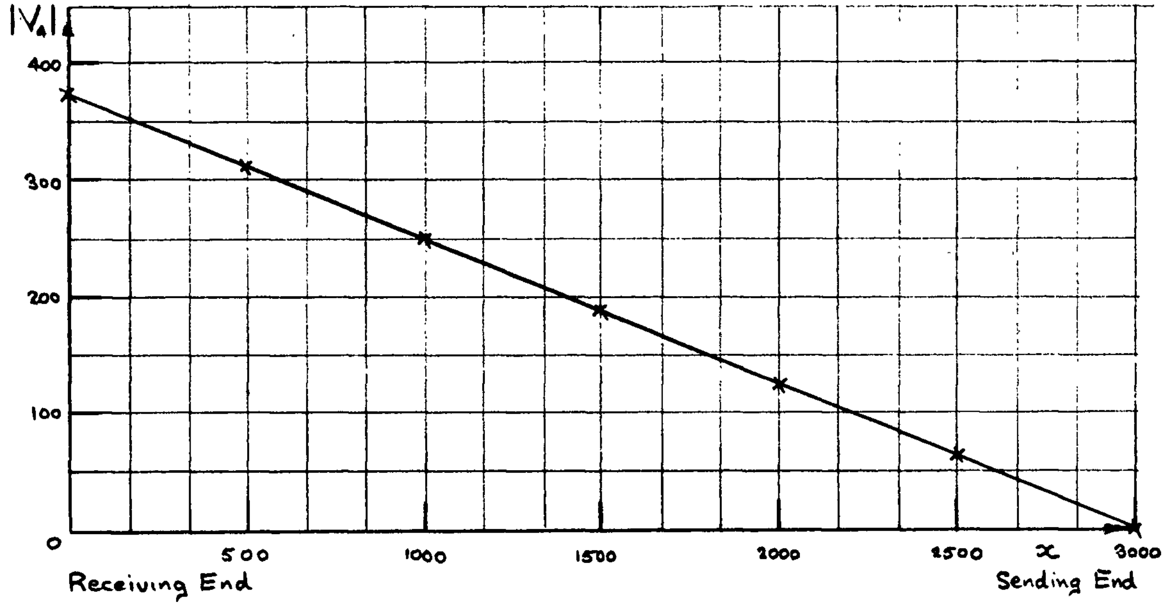


Fig 8.19. Full Load Sheath Voltage, $|V_4|$, volts vs. Distance Along the Line, $x$, metres

# 9 CONCLUSIONS

An accurate mathematical procedure was developed to be used in the analysis and design of multiconductor transmission systems under various loading or boundary conditions. The validity of the theory was substantiated using two numerical examples. The results of these two analysis are given in the report.

The sequence in which the parts of the numerical analysis must be performed is shown in Appendix B. It is apparent that given the conductor and geometrical specifications of a particular transmission system, a variety of terminal or boundary conditions can be analysed for that system without repeating the steps which lead to the general solution. In developing this procedure, an important theoretical concept was evolved; the concept of a complete system.

Consider the example of the overhead conductor system, which comprises seven conductors, including the ground wire but excluding any earth effects. Only six of these are independent. By choice, the ground conductor was used as a voltage reference, but the magnetic, electric and loss effects due to this conductor, which may not be ignored, appear in the system parametric matrices. The reduced system resistance matrix for example is not a diagonal matrix since the resistance of the ground wire appears as a component of all matrix elements.

Had the earth effect been included in the model as an equivalent earth conductor, then there would have been eight conductors, seven of which would have been independent. The ground wire or the equivalent earth conductor would be chosen as a reference conductor. Hence definition of the complete system requires the specification of a closed system of conductors, one of which will be used as a voltage reference conductor.

This approach to transmission line analysis suggests that it is ideally suited to time shared machine aided design. The optimum boundary terminations or the best locations for the transpositions, for example, could be arrived at by using a computer to verify an analyst's heuristic reasoning.

Future research into this field should include analysis of the full significance of the location of the characteristic values in the complex plane with respect to propagation and attenuation. Further development will lead to the superposition of analysis of the same system at various frequencies for transient studies or for carrier wave transmission studies. A more precise formulation and method for finding the complex transposition matrices which occur at lossy transposition boundaries will also be required, particularly where optimum solutions are to be found.

# APPENDIX A

## THE PARAMETERS R, L AND C.

### A.1 Assumptions

In the derivation(14, 16) of the matrices $(R)$, $(L)$ and $(C)$, it will be assumed that skin effects, proximity effects and saturation may be ignored. It will be assumed also that derivation using static fields will not invalidate the application to a slowly changing or quasi-static system. The rationalized M.K.S. system of units will be used.

Initially, all the parameters for a system of $(n + 1)$ conductors will be related to some arbitrary external reference, but in the final form, the restriction that the sum of the currents within the system is zero will be applied, and all voltages will be referred to some conductor within the system (e.g. a ground conductor, or an equivalent earth conductor).

### A.2 The Resistance, $R^l$.

The matrix of resistance per unit length is

$$(R^l) = \begin{pmatrix} R_1^l & & & & 0 \\ & R_2^l & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & \cdot & R_{n+1}^l \end{pmatrix}$$

where $R_i^l = \rho_i^t / A_i$,

$A_i$ = cross sectional area of the $i^{th}$ conductor and $\rho_i^t = \rho_o(1 + \alpha t)$ = resistivity of conductor i at a temperature $t^oC$

where $\rho_o$ = resistivity of the conductor material at a temperature $t_o{}^oC$

and $\alpha$ = thermal coefficient of resistivity.

A-3 The Inductance, $L^1$

Consider the group of (n + 1) conductors shown in Fig. A.1 The axes are set up through conductor " $a$ ", about which the flux linkages are to be computed. The point X is some remote point where magnetic effects may be considered to be negligible. The total number of linkages produced by flux which crosses the x - axis between the origin and the point X is given by

$$\Psi_a = \frac{\mu}{2\pi} \left( \left(\frac{1}{4} + \ln \frac{D_x}{v}\right) I + I_b \ln \frac{D_{bx}}{D_b} + \ldots\ldots + I_j \ln \frac{D_{jx}}{D_j} + \ldots\ldots + I_{n+1} \ln \frac{D_{n+1,\,x}}{D_{a,\,n+1}} \right) \qquad \text{A-2}$$

where $\mu$ = permeability of the surrounding medium

$r_a$ = radius of conductor j

$D_{jx}$ = distance between conductor j and the point X

$D_{ij}$ = distance between conductors i and j

$I_j$ = current in $i^{th}$ conductor

A similar expression can be written for the flux linkages surrounding the remaining conductors.

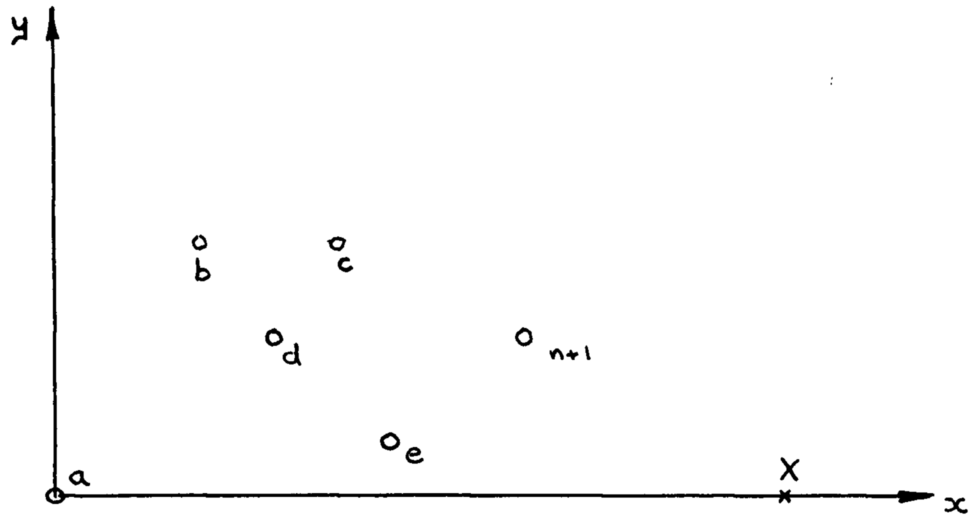


Fig. A.1 Group of n + 1 Current Carrying Conductors.

For two parallel cylindrical conductors of arbitrary cross-section, the total flux linkage about one of the conductors is given by (15),

$$\Psi_T = \frac{\mu I}{2\pi} \ln \frac{D^1_{ij}}{D^1_{ii}} \qquad \text{A-3}$$

where I is the current in the conductor

$D^1_{ij}$ is the G.M.D., the geometric mean distance between the conductors

$D^1_{ii}$ is the G.M.R., the geometric mean radius of the conductor.

Replacing the distances D and r of equation A-2 by the geometric mean distances as defined in equation A-3 gives

$$\begin{pmatrix} \Psi_a \\ \Psi_b \\ \\ \Psi_{n+1} \end{pmatrix} = \frac{\mu}{2\pi} \begin{pmatrix} \ln \frac{D^1_{ax}}{D^1_{aa}} & \ln \frac{D^1_{bx}}{D^1_{ab}} \; \cdots\cdots & \ln \frac{D^1_{n+1,\,x}}{D_{a,\,n+1}} \\ \ln \frac{D^1_{ax}}{D^1_{ba}} & \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ \ln \frac{D^1_{ax}}{D_{n+1,\,a}} & \cdots\cdots\cdots & \ln \frac{D^1_{n+1,\,x}}{D^1_{n+1,\,n+1}} \end{pmatrix} \begin{pmatrix} I_a \\ I_b \\ \cdot\cdot \\ I_{n+1} \end{pmatrix} \qquad \text{A-4}$$

For a linear system, the inductance coefficient, may be defined as

$$L_{jj} \triangleq \left.\frac{\Psi_j}{I_j}\right|_{\substack{I_i = 0 \\ i \neq j}} = \frac{\mu}{2\pi} \ln \frac{D^1_{jx}}{D^1_{jj}} \qquad \text{A-5}$$

$$L_{ij} \triangleq \left.\frac{\Psi_i}{I_j}\right|_{\substack{I_i = 0 \\ i \neq j}} = \frac{\mu}{2\pi} \ln \frac{D^1_{jx}}{D^1_{ij}} \qquad \text{A-6}$$

and hence the inductance coefficient matrix $(L')$ becomes

$$(L') = \begin{pmatrix} L_{11} & L_{12} \cdots\cdots\cdots & L_{1,\,n+1} \\ L_{21} & \cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots \\ L_{n+1,\,1} & \cdots\cdots & L_{n+1,\,n+1} \end{pmatrix} \qquad \text{A-7}$$

A-4 The Capacitance, C'

For a system of (n + 1) parallel conductors, Maxwell's potential coefficients are defined by the equation

$$\begin{pmatrix} V_a \\ V_b \\ \cdot\cdot \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} P_{aa} & P_{ab} & \cdots\cdots & P_{a,\,n+1} \\ P_{ba} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P_{n+1,\,a} & \cdots & \cdots & P_{n+1,\,n+1} \end{pmatrix} \begin{pmatrix} Q_a \\ Q_b \\ \cdot\cdot \\ Q_{n+1} \end{pmatrix} \qquad \text{A-8}$$

where the $P'$, are the coefficients of potential.

Solving for the charges $Q_j$ we have

$$\begin{pmatrix} Q_a \\ Q_b \\ \cdot\cdot \\ Q_{n+1} \end{pmatrix} = \begin{pmatrix} C_{aa} & C_{ab} & \cdots\cdots & C_{a,\,n+1} \\ C_{ba} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ C_{n+1,\,a} & \cdots & \cdots & C_{n+1,\,n+1} \end{pmatrix} \begin{pmatrix} V_a \\ V_b \\ \cdot\cdot \\ V_{n+1} \end{pmatrix} \qquad \text{A-9}$$

Where the C', are the capacitance coefficient

For the system of (n + 1) conductors,

$$P_{jj} \triangleq \left.\frac{V_j}{Q_j}\right|_{\substack{Q_i = 0 \\ i \neq j}} = \frac{1}{2\pi\varepsilon} \ln \frac{D_{j,\,x}}{D_{j,\,j}} \qquad \text{A-10}$$

$$P_{ij} \triangleq \left.\frac{V_i}{Q_j}\right|_{\substack{Q_i = 0 \\ i \neq j}} = \frac{1}{2\pi\varepsilon} \ln \frac{D_{jx}}{D_{ij}} \qquad \text{A-11}$$

wherethe D's represent the distances between the points denoted by the subscripts, and $\varepsilon$ is the permittivity of the medium surrounding the conductors. Clearly, the matrices of potential, capacitance and inductance coefficients are symmetric.

Since, for the potential coefficient matrix

$$P_{rr} \geqslant P_{rs} \geqslant 0$$

then for the capacitance coefficient matrix

$$C_{rr} \geqslant 0$$

and $$C_{rs} \leqslant 0$$

For a system of conductors containing coaxial cables where one conductor is completely enclosed by another as shown in Fig A.3, then because conductor j is shielded by conductor i,

$$C_{jk} = C_{j\ell} = 0$$

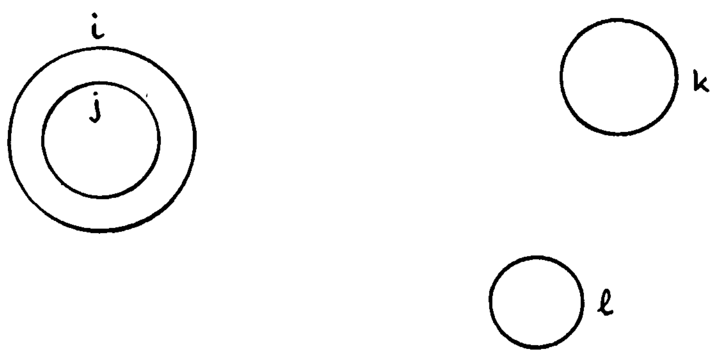


Fig. A.2 Cross-section of part of a system of conductors where one conductor completely encloses another.

For the system,

$$(Q) = (C)(V),$$

then
$$C_{jj} = \frac{Q_j}{V_j}\Bigg|_{\substack{V_m = 0 \\ m \neq j}}$$

In particular, for $V_i = 0$, $C_{jj}$ becomes the capacitance between two concentric cylinders. Also,

$$C_{ij} = \frac{Q_i}{V_j}\Bigg|_{\substack{V_m = 0 \\ m \neq j}}$$

$$= -\frac{Q_j}{V_j}\Bigg|_{\substack{V_m = 0 \\ m \neq j}}$$

and since the matrix $(C)$ is symmetric, then

$$C_{ij} = C_{ji} = -C_{jj}$$

In terms of the potential coefficients, $P_{ik} = P_{jk}$ and $P_{i\ell} = P_{j\ell}$ imply that $C_{jk} = C_{j\ell} = 0$, and $P_{ij} = P_{ji} = P_{ii}$ implies $C_{ij} = C_{ji} = -C_{jj}$. In this discussion, it has been assumed that the permittivity of the medium surrounding the conductors is constant.

# APPENDIX B

A flowsheet of the solution procedure for (n + 1) conductor system.

| | Procedure | Checks |
|---|---|---|
| Part I | Compute R, L, C, G coefficient matrices from transmission line configuration. | Manual calculation of selected elements. |
| | Form reduced Z and Y matrices. | " |
| | Find characteristic values of characteristic equation $(A) - \lambda\ (U) = 0$ where $(A) = (Z)(Y)$ | Trace $(A) = \sum_{i=1}^{r} \lambda_i$ |
| | Find characteristic vectors D - voltage vectors G - current vectors | $(D)^{-1}(A)(D) = (\lambda\ U)$ <br> $(G)^{-1}(A)_t(G) = (\lambda\ U)$ |
| Part II | Solve for 2n unknown constants using known boundary conditions and the connection matrices at the transpositions. | The voltage form of the solution must give the same results as the current form of solution |
| | Generate required output from the particular solution. | |

Part I of the procedure gives the general solution for the given transmission line; Part II provides the particular solutions for the specified sets of boundary conditions and connection matrices at the transpositions.

Appendix C1.1 Overhead Conductor System, No Load Voltage and Power

| Distance L (km) | $\lvert\bar{V}_a\rvert$ (kv) | $\lvert\bar{V}_b\rvert$ (kv) | $\lvert\bar{V}_c\rvert$ (kv) | $\angle\bar{V}_a$ (deg) | $\angle\bar{V}_b$ (deg) | $\angle\bar{V}_c$ (deg) | P/Q (m.v.a.) |
|---|---|---|---|---|---|---|---|
| UNTRANSPOSED | | | | | | | |
| 0 | 132.8 | 132.8 | 132.8 | 0 | -117.2 | 117.2 | 0 |
| 133.2 | 130.8 | 127.6 | 127.7 | 0.58 | -117.1 | 117.3 | 0.06/ - 46.2 |
| 266.4 | 125.0 | 121.9 | 122.2 | 0.50 | -116.8 | 117.5 | 0.46/ - 87.1 |
| 399.6 | 115.6 | 112.8 | 113.5 | 0.30 | -116.3 | 119.1 | 1.52/ -118.1 |
| TRANSPOSED | | | | | | | |
| 266.4 | 124.8 | 120.0 | 122.0 | 0.30 | -116.7 | 118.0 | 0.46/ - 87.0 |
| 399.6 | 115.7 | 113.4 | 112.7 | 0.04 | -115.1 | 118.6 | 1.51/ -117.9 |

Appendix C1.2 Overhead Conductor System, No Load Current.

| Distance L (km) | $\lvert\bar{I}_a\rvert$ (amp) | $\lvert\bar{I}_b\rvert$ (amp) | $\lvert\bar{I}_c\rvert$ (amp) | $\angle\bar{I}_a$ (deg) | $\angle\bar{I}_b$ (deg) | $\angle\bar{I}_c$ (deg) | $\angle\bar{I}_g/\lvert\bar{I}_g\rvert$ (deg)/ampx10$^4$ |
|---|---|---|---|---|---|---|---|
| UNTRANSPOSED | | | | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 133.2 | 58.5 | 62.0 | 59.6 | 92.2 | 28.7 | -152.8 | -47.7/ 41.0 |
| 266.4 | 115.0 | 121.0 | 117.0 | 92.3 | 28.8 | -152.7 | -47.4/ 81.0 |
| 399.6 | 169.0 | 178.0 | 172.0 | 92.5 | 28.6 | -152.5 | -47.0/ 118.0 |
| TRANSPOSED | | | | | | | |
| 266.4 | 177.0 | 120.0 | 118.0 | 91.2 | 29.7 | -150.8 | -27.7/ 31.0 |
| 399.6 | 170.0 | 174.0 | 174.0 | 90.5 | 28.6 | -150.6 | -89.2/ 42.0 |

Appendix C1.3 Overhead Conductor System, Full Load Voltage and Power.

| Distance L (km) | $\lvert\bar{V}_a\rvert$ (kv) | $\lvert\bar{V}_b\rvert$ (kv) | $\lvert\bar{V}_c\rvert$ (kv) | $\angle\bar{V}_a$ (deg) | $\angle\bar{V}_b$ (deg) | $\angle\bar{V}_c$ (deg) | P/Q (m.v.a.) |
|---|---|---|---|---|---|---|---|
| UNTRANSPOSED | | | | | | | |
| 0 | 132.8 | 132.8 | 132.8 | 0 | -117.2 | 117.2 | 312.9/ 234.7 |
| 133.2 | 156.1 | 153.0 | 148.0 | 7.6 | -108.7 | 126.8 | 324.2/ 266.2 |
| 266.4 | 176.8 | 175.0 | 165.5 | 13.5 | -102.2 | 134.7 | 333.5/ 264.8 |
| 399.6 | 193.9 | 193.4 | 181.0 | 18.5 | - 96.9 | 141.2 | 341.0/ 230.6 |
| TRANSPOSED | | | | | | | |
| 266.4 | 176.1 | 172.0 | 169.1 | 14.5 | -102.0 | 133.5 | 333.5/ 264.9 |
| 399.6 | 190.9 | 191.0 | 186.5 | 20.4 | - 97.3 | 139.6 | 341.0/ 230.5 |

Appendix C1.4 Overhead Conductor System, Full Load Current.

| Distance L (km) | $\lvert\bar{I}_a\rvert$ (amp) | $\lvert\bar{I}_b\rvert$ (amp) | $\lvert\bar{I}_c\rvert$ (amp) | $\angle\bar{I}_a$ (deg) | $\angle\bar{I}_b$ (deg) | $\angle\bar{I}_c$ (deg) | $\angle\bar{I}_g/\lvert\bar{I}_g\rvert$ (deg)/amp$\times 10^4$ |
|---|---|---|---|---|---|---|---|
| UNTRANSPOSED | | | | | | | |
| 0 | 500.0 | 500.0 | 500.0 | 143.1 | 23.1 | - 86.9 | 0 |
| 133.2 | 458.0 | 458.0 | 462.0 | 148.9 | 29.4 | - 90.6 | -47.7/ 41.6 |
| 266.4 | 409.0 | 408.0 | 415.0 | 156.3 | 37.2 | - 97.1 | -47.6/ 82.8 |
| 399.6 | 357.0 | 358.0 | 372.0 | 166.0 | 47.5 | -107.0 | -47.2/ 122.5 |
| TRANSPOSED | | | | | | | |
| 266.4 | 410.0 | 412.0 | 414.0 | 156.7 | 37.1 | - 96.8 | -36.0/ 19.7 |
| 399.6 | 364.0 | 359.0 | 365.0 | 166.7 | 46.6 | -107.0 | -77.0/ 69.4 |

Appendix C2.1 Underground Conductor System, No Load Voltage and Power

Distance Voltage: Magnitude and Argument for conductors 1-6, $\underline{/V}/|V|$ — System Real and Reactive Power

| L (m) | (deg)/(kv) | (deg)/(kv) | (deg)/(kv) | (deg)/(kv) | (deg)/(kv) | (deg)/(kv) | P/Q (m.v.a.) |
|---|---|---|---|---|---|---|---|
| 0 | 0.0/ 76.20 | -120.0/ 76.20 | 120.0/ 76.20 | 0.0/ 0.0 | 0.0/ 0.0 | 0.0/ 0.0 | 0.0/ 0.0 |
| 500 | 0.0/ 76.20 | -120.0/ 76.20 | 120.0/ 76.20 | -7.6/ 2.52 | -54.6/ 3.41 | -125.6/ 1.41 | 0.61/ -0.18 |
| 1000 | 0.0/ 76.20 | -120.0/ 76.20 | 120.0/ 76.20 | -6.1/ 4.75 | -51.6/ 6.85 | -120.5/ 3.13 | 1.22/ -0.37 |
| 1500 | 0.0/ 76.205 | -120.0/ 76.205 | 120.0/ 76.205 | -4.5/ 6.69 | -48.7/ 10.35 | -116.4/ 5.20 | 1.83/ -0.55 |
| 2000 | 0.0/ 76.21 | -120.0/ 76.21 | 120.0/ 76.21 | -2.65/ 8.35 | -45.7/ 14.00 | -112.9/ 7.61 | 2.44/ -0.73 |
| 2500 | 0.0/ 76.21 | -120.0/ 76.21 | 120.0/ 76.21 | -0.55/ 9.73 | -42.7/ 17.70 | -110.1/ 10.41 | 3.05/ -0.92 |
| 3000 | 0.04/ 76.21 | -120.4/ 76.21 | 119.6/ 76.21 | 1.9/ 10.84 | -40.1/ 21.60 | -107.7/ 13.60 | 3.66/ -1.10 |

Appendix C2.2 Underground Conductor System, No Load Current.

Distance Current: Magnitude and Argument for conductors 1-7, $\underline{/I}/|I|$

| L (m) | (deg)/(A) | (deg)/(A) | (deg)/(A) | (deg)/(A) | (deg)/(A) | (deg)/(A) | (deg)/(A) |
|---|---|---|---|---|---|---|---|
| 0 | 0.0/ 0.0 | 0.0/ 0.0 | 0.0/ 0.0 | -14.7/ 17.8 | -116.9/ 16.9 | 122.1/ 17.0 | 84.5/ 5.15 |
| 500 | 5.7/ 2.75 | -98.6/ 2.94 | 142.6/ 2.76 | -14.6/ 14.8 | -116.9 14.1 | 122.0/ 14.2 | 84.4/ 5.25 |
| 1000 | 5.7/ 5.50 | -98.6/ 5.87 | 142.6/ 5.50 | -14.6/ 11.85 | -116.9 11.27 | 122.1/ 11.35 | 84.25 5.35 |
| 1500 | 5.7/ 8.23 | -98.6/ 8.80 | 142.6/ 8.28 | -14.7/ 8.88 | -116.9/ 8.44 | 122.1/ 8.51 | 84.1 5.45 |
| 2000 | 5.7/ 10.99 | -98.6/ 11.75 | 142.6/ 11.05 | -14.7/ 5.92 | -116.9/ 5.63 | 122.1/ 5.68 | 84.0/ 5.55 |
| 2500 | 5.7/ 13.71 | -98.6/ 14.67 | 142.6/ 13.80 | -14.7/ 2.96 | -116.9/ 2.82 | 122.1/ 2.84 | 83.9/ 5.66 |
| 3000 | 5.7/ 16.45 | -98.6/ 17.60 | 142.6/ 16.55 | 0.0/ 0.0 | 0.0/ 0.0 | 0.0/ 0.0 | 83.75/ 5.76 |

Appendix C2.3 Underground Conductor System, Full Load Voltage and Power

Distance — Voltage: Magnitude and Argument for conductors 1-6, $\underline{/\bar{V}}/|\bar{V}|$ — System Real and Reactive Power

| L (m) | (deg) / kv | (deg) / kv | (deg) / kv | (deg) / v | (deg) / v | (deg) / v | P/Q (m.v.a.) |
|---|---|---|---|---|---|---|---|
| 0 | 0.0/ 76.20 | -120.0/ 76.20 | 120.0/ 76.20 | 0.0/ 0.0 | 0.0/ 0.0 | 0.0/ 0.0 | 90.05/ 43.82 |
| 500 | 0.0/ 76.23 | -120.0/ 76.23 | 120.0/ 76.23 | 61.8/ 62.4 | -47.2/ 65.5 | 176.15/ 63.0 | 90.67/ 43.70 |
| 1000 | 0.0/ 76.26 | -120.0/ 76.26 | 120.0/ 76.26 | 62.1/ 124.8 | -47.0/ 131.1 | 176.20/ 124.9 | 91.29/ 43.59 |
| 1500 | 0.0/ 76.29 | -120.0/ 76.29 | 120.0/ 76.29 | 62.2/ 187.0 | -46.9/ 197.1 | 176.40/ 188.9 | 91.91/ 43.48 |
| 2000 | 0.0/ 76.33 | -120.0/ 76.33 | 120.0/ 76.33 | 62.3/ 250.0 | -46.8/ 262.5 | 176.60/ 251.3 | 92.53/ 43.36 |
| 2500 | 0.0/ 76.36 | -120.0/ 76.36 | 120.0/ 76.36 | 62.4/ 312.0 | -46.6/ 328.5 | 176.75/ 314.5 | 93.15/ 43.25 |
| 3000 | 0.2/ 76.39 | -119.75 76.4 | 120.15/ 76.4 | 62.6/ 374.0 | -46.4/ 395.0 | 176.95/ 376.6 | 93.77/ 43.13 |

Appendix C2.4 Underground Conductor System, Full Load Current

Distance — Current: Magnitude and Argument for Conductors 1-7, $\underline{/\bar{I}}/|\bar{I}|$

| L (m) | (deg) / A | (deg) / A | (deg) / A | (deg) / A | (deg) / A | (deg) / A | (deg) / A |
|---|---|---|---|---|---|---|---|
| 0 | -26.00/ 439.0 | -146.00/ 439.0 | 94.00/ 439.0 | -14.7/ 17.75 | -117.0/ 16.90 | 122.0/ 17.00 | 84.5/ 5.15 |
| 500 | -25.80/ 440.0 | -145.65/ 440.0 | 94.30/ 440.0 | -14.7/ 14.80 | -117.0/ 14.10 | 122.0/ 14.20 | 84.3/ 5.25 |
| 1000 | -25.60/ 443.0 | -145.40/ 442.0 | 94.60/ 441.0 | -14.7/ 11.90 | -117.0/ 11.30 | 122.0/ 11.30 | 84.251 5.35 |
| 1500 | -25.45/ 446.0 | -145.10/ 445.0 | 94.90/ 443.0 | -14.7/ 8.90 | -117.0/ 8.45 | 122.0/ 8.40 | 84.10/ 5.45 |
| 2000 | -25.25/ 448.0 | -144.90/ 446.5 | 95.20/ 445.0 | -14.7/ 5.68 | -117.0/ 5.64 | 122.0/ 5.67 | 84.00/ 5.55 |
| 2500 | -25.10/ 450.0 | -144.60/ 448.0 | 95.40/ 447.0 | -14.7/ 2.84 | -117.0/ 2.82 | 122.01 2.84 | 83.91 5.66 |
| 3000 | -24.9/ 452.0 | -144.25/ 450.0 | 95.80/ 449.0 | 0.0/ 0.0 | 0.0/ 0.0 | 0.0/ 0.0 | 83.75/ 5.76 |

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