JACOBI POLYNOMIAL TRUNCATIONS AND APPROXIMATE SOLUTIONS TO CLASSES OF NONLINEAR DIFFERENTIAL EQUATIONS

by

RONALD EDWARD DODD

B.A.Sc., University of British Columbia, 1964.

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

in the Department of
Electrical Engineering

We accept this thesis as conforming to the
required standard

Research Supervisor .....

Members of the Committee .....

.....

Head of the Department .....

Members of the Department
of Electrical Engineering

THE UNIVERSITY OF BRITISH COLUMBIA

AUGUST, 1966

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of *Electrical Engineering*

The University of British Columbia
Vancouver 8, Canada

Date *September 9, 1966*

ABSTRACT

Solutions to classes of second-order, nonlinear differential equations of the form

$$\ddot{x} + 2\delta\dot{x} + f(x) = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0$$

are approximated in this work. The techniques which are developed involve the replacement of the characteristic, $f(x)$, in the nonlinear model by piecewise-linear or piecewise-cubic approximations. From these, closed-form time solutions in terms of the circular trigonometric functions or the Jacobian elliptic functions may be obtained. Particular examples in which $f(x)$ is grossly nonlinear and asymmetric are considered. The orthogonal Jacobi and shifted Jacobi polynomials are introduced for the approximation in order to satisfy criteria which are imposed on the error and on the use of symmetry.

Error bounds are then developed which demonstrate that the maximum error in the normalized time solution is bounded, no matter how large the coefficients of the nonlinear terms in the model become. Because of these error-bound results, an heuristic measure of the departure from linearity is defined for classes of symmetric oscillations, and the weighting of convergence of the Jacobi and shifted Jacobi polynomial expansions is set according to this measure.

For asymmetric conservative models, shifted Chebychev polynomials are used to obtain near-uniform approximations to the characteristic in the nonlinear differential equation.

Based on the equivalence of the classical approximation techniques which is given for the symmetric, conservative models, extension of the polynomial approximation to classes of non-conservative models is considered.

Throughout the work, by comparison with classical approximation methods, the polynomial approximation techniques are shown to provide an improved, direct and more general attack on the approximation problem with a decrease in tedious labor.

TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

Page

LIST OF TABLES

LIST OF PRINCIPAL SYMBOLS

a	=	a constant
a_k	=	coefficients of a polynomial characteristic
$a(t)$	=	amplitude parameter in K-B approximation
A_{mn}	=	coefficient of a Jacobi polynomial
b	=	a constant
b_k	=	coefficients of an approximate polynomial characteristic
B_{n-m}	=	a sum of coefficients of Jacobi polynomials
c	=	$\sqrt{g/h}$
c_k	=	constant coefficients
C	=	capacitance
C_o	=	a constant
Cn	=	the Jacobian elliptic cosine function
Dn	=	a Jacobian elliptic function
EMAX	=	the maximum value of $E(z + \varepsilon)$
$E(x)$	=	the error in the approximation to $f(x)$
E_t	=	a constant proportional to the total energy
$f(x)$	=	a nonlinear characteristic in the differential equation
$\tilde{f}(x)$	=	an approximation to $f(x)$
F	=	the nonlinear factor
g	=	a parameter in Legendre's transformation
$G_k^{(\alpha,\beta)}(x)$	=	the k^{th} shifted Jacobi polynomial
h	=	a parameter in Legendre's transformation
H_k	=	a function of the coefficients of the Jacobi polynomials
i_L	=	current through an inductor

J = a mass or moment of inertia

k = the modulus of a Jacobian elliptic function

k_o = a constant

K_k = a function of the coefficients of the Jacobi polynomials

L = inductance

m = $\sqrt{(p - \underline{\theta})(p - \underline{\lambda})}$

$M(x)$ = a mathematical model

n = $\sqrt{(q - \underline{\theta})(q - \underline{\lambda})}$

p = a parameter in the Legendre transformation

$P_k^{(\mu)}(x)$ = an ultraspherical Jacobi polynomial

PM_k = the absolute value of the k^{th} Jacobi or shifted Jacobi polynomial at the point where $E(z + \varepsilon)$ is a maximum

q = a parameter in the Legendre transformation

q_C = the charge on a capacitor

r = $\sqrt{(p - \underline{\mu})(p - \underline{\pi})}$

R = $\int_0^1 f(x)x^{\alpha-1}(1 - x)^{\beta-1}dx$

s = $\sqrt{(q - \underline{\mu})(q - \underline{\pi})}$

S = $\int_0^1 f(x)x^{\alpha}(1 - x)^{\beta-1}dx$

Sn = the Jacobian elliptic sine function

t = time

$\bar{t}$ = shifted time

T = the period of an oscillation

$\tilde{T}$ = an approximation to T

Tn = the Jacobian elliptic tangent function

$T(\theta)$ = a mechanical torque or restoring force

$U(t)$ = the unit step function

$v(t), w(t), x(t)$	=	dependent variables
$\tilde{x}(t)$	=	an approximation to $x(t)$
x_{min}	=	a bound on an asymmetric oscillation
X_o	=	a constant
$y(t), z(t)$	=	dependent variables
α, β	=	scalar parameters which determine the weighting of the shifted Jacobi polynomials
$\Gamma(x)$	=	the Gamma function
δ	=	the coefficient of viscous damping
Δt	=	an increment in time
$\varepsilon(t)$	=	$x(t) - \tilde{x}(t)$
ε_p	=	the particular integral in a differential equation for $\varepsilon(t)$
$\underline{\theta}, \underline{\lambda}$	=	roots of the denominator polynomial to which Legendre's transformation is applied
μ	=	a scalar parameter which determines the weighting of the ultraspherical Jacobi polynomials
$\underline{\mu}$	=	a root of the denominator polynomial to which Legendre's transformation is applied
π	=	3.14159... radians
$\underline{\pi}$	=	a root of the denominator polynomial
ϕ	=	flux density in an inductor
ω	=	angular frequency in radians per second

ACKNOWLEDGEMENT

Grateful acknowledgement is given to the National Research Council of Canada for the assistance received under Block Term Grant A68 from 1964 to 1966 and to the British Columbia Telephone Company for the award of a scholarship in 1964-1965.

The author would like to express appreciation to his supervising professor, Dr. A. C. Soudack, for his inspiration and guidance throughout the course of the work.

Thanks is also given to Dr. J. S. MacDonald for reading the manuscript and for making many helpful suggestions.

The author is also indebted to Mr. A. G. Longmuir and Mr. E. Lewis for discussion of the manuscript, to Mr. B. Wilbee for proof-reading, to Mr. R. Proudlove for photographing curves and to Miss L. Blaine for typing the thesis.

JACOBI POLYNOMIAL TRUNCATIONS AND APPROXIMATE SOLUTIONS TO CLASSES OF NONLINEAR DIFFERENTIAL EQUATIONS

1. INTRODUCTION

1.1 Description of the Mathematical Model

In order to obtain mathematical representations which approach the natural behaviour of some real systems, nonlinear relations among variables are often required. Many of these systems are described by mathematical models for which closed-form solutions cannot be found in terms of known functions, and a numerical scheme must be employed to obtain an accurate solution. There is a considerable advantage in having a closed-form solution to a problem because one achieves insight into changes in the solution with variation of particular parameters in the model. In engineering, the concern with approximation techniques is important because many models derived from the physical world are themselves only approximations. Therefore, it is often useful to approximate a nonlinear system model by one from which a closed-form solution may be obtained.

Initial consideration in this study is given to the second-order differential equation

$$M(x) = \ddot{x} + f(x) = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0, \tag{1.1}$$

which describes undamped or conservative oscillations with a single degree of freedom. It is not required that $f(x)$ have zero-point symmetry in this model. Thus, the zero on the right-hand side of equation (1.1) does not necessarily imply that the

oscillations are free - a constant or step function driving term is allowed. The principle of superposition does not apply to general nonlinear systems. In particular, nonlinear oscillations described by equation (1.1) show a dependence of the frequency on the amplitude of the oscillation. To provide a constant frame of reference from which the examples to be considered may be compared, the initial conditions are normalized as shown in equation (1.1). Conversion of models with arbitrary initial conditions to this form is outlined in paragraph (2.1.1).

Common examples of the model in equation (1.1) occur in the description of electrical and mechanical oscillating systems. Suppose, for example, that $i_L(\phi) = g(\phi)$ where ϕ is the flux density for the nonlinear inductor in the LC circuit shown in Figure (1.1).

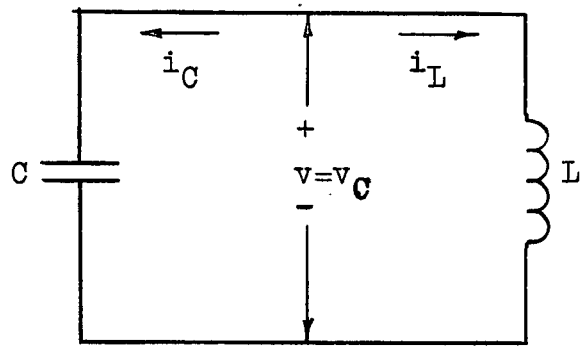


Figure 1.1 A Nonlinear LC Circuit

If the inductor alone is nonlinear, by Kirchhoff's current law one obtains $C\dot{v} + g(\phi) = 0$. Now $v = N\dot{\phi}$ for an N-turn coil from Faraday's Law. Hence,

$$\ddot{\phi} + \frac{1}{NC} g(\phi) = 0. \qquad (1.2)$$

Carrying out a dual analysis for the case in which the capacitor alone is nonlinear, one obtains

$$\ddot{q} + \frac{1}{L} h(q) = 0, \tag{1.3}$$

where q is the charge on the capacitor and the voltage across the capacitor is defined by the relation $v_C(q) = h(q)$. Hayashi[1] gives practical examples of inductors for which the current-flux relationship has the form $i_L(\phi) = c_1\phi + c_3\phi^3 + c_5\phi^5 + c_7\phi^7$, and for which the coefficients c_5 and c_7 dominate over c_1 and c_3 for the larger values of ϕ on the interval of interest.

In mechanical systems the nonlinear characteristic represents a restoring force or torque. Suppose that such a nonlinear force is given by $T(\theta) = g(\theta)$ where θ is a displacement. By d'Alembert's principle for a restrained element of mass or moment of inertia, J,

$$\ddot{\theta} + \frac{1}{J} T(\theta) = 0$$

is obtained as the system equation. For example, if the motion of a body in a central force field depends only on the distance, r, from some fixed point, then the equation of motion is

$$\frac{d^2r}{d\lambda^2} + f(r) = 0, \qquad r(0) = 1, \qquad \dot{r}(0) = 0, \tag{1.4}$$

where λ is the azimuthal angle in spherical coordinates. Goldstein[2] shows that determination of the motion of a system consisting of two interacting particles may be reduced to the problem of determining the motion of a single particle in an exter-

nal field, such that the motion is governed by equation (1.4). Pipes[3] gives a model of a mechanical system executing free asymmetric oscillations. The model is $\ddot{x} + n^2x + hx^2 = 0$, and it is of importance in the theory of seismic vibrations. As a final example, the application of models of the form $\ddot{x} + cx|x|^{k-1} = 0$ to the description of motion in principal modes of certain classes of nonlinear systems having many degrees of freedom has been shown by Rosenberg[4].

The electrical and mechanical examples quoted above all have the form of the generic model in equation (1.1). In Chapter 3 an extension is made to the case in which the model in equation (1.1) assumes light, viscous damping. Primary consideration is given to the conservative system, however, so that the approximation of the nonlinear amplitude-frequency relationship may be studied with variation of the characteristic, f(x), alone.

1.2 Some Existing Approximation Techniques

Classical first-order approximation techniques, such as the perturbation method[5] and the averaging method of Krylov and Bogoliubov[6] (K-B method), require an explicit linear term which must dominate over the nonlinear terms in the model. These techniques are unsuitable for the models in this work because the "quasi-linear" nature of the model is undefined. The Ritz-Galerkin[7] averaging method or the Principle of Harmonic Balance[8], makes no such restrictions on the deviation of the model from linearity, but it will be shown that this method fails to yield practical approximate solutions when the deviation

from linearity becomes appreciable. Moreover, in Appendix C the one-term Ritz method is shown to give the same first-order approximate solution as the K-B averaging technique.

As an example, consider the nonlinear model $M(x) = \ddot{x} + f(x) = 0$, $x(0) = 1$, $\dot{x}(0) = 0$, in which $f(x)$ may be asymmetric, or biassed. A first-order Ritz approximate solution $\tilde{x}(t) = X_o + A \cos \omega t$ is assumed for this model. The Ritz conditions plus a constraint from the initial conditions determine the parameters X_o, A and ω in this assumed approximate solution. These conditions are

$$X_o + A = 1,$$

$$\int_0^{2\pi} M(\tilde{x}) d(\omega t) = 0$$

and

$$\int_0^{2\pi} M(\tilde{x}) \cos(\omega t) d(\omega t) = 0.$$

Satisfaction of these full-period averaging integrals on the residual, $M(\tilde{x})$, bears no direct relation to the error in the approximate time solution. In this work the absolute error $\varepsilon(t) = x(t) - \tilde{x}(t)$ is considered. Another indirect property of the Ritz method is that the form of the approximate solution must be assumed. Hence any features not assumed in the approximate solution will not be found. Also, the Ritz method yields nonlinear algebraic equations for which the

solution is difficult. If the above first-order solution is refined to the form $\tilde{x}(t) = X_o + A\cos\omega t + B\cos 3\omega t$, then four nonlinear equations in X_o, A, B and ω are obtained from the Ritz conditions.

In a recent paper[9], Denman and Lui considered the approximate solution of the equation $\ddot{x} + ax + bx^3 = 0$. The nonlinear characteristic was expanded to a linear polynomial in terms of ultraspherical polynomials. The techniques given in Appendix A allow a closed-form solution to be written for this cubic equation by inspection.

Soudack[10] has given techniques for the approximation of a nonlinear model in the form of equation (1.1), in which the nonlinear characteristic is a polynomial with odd symmetry. The techniques replace the nonlinear polynomial by a cubic one and the closed-form solution of the resulting differential equation is obtained using the Jacobian elliptic functions. Approximate models obtained from a Chebychev expansion of the nonlinear characteristic were found to give much better approximate time solutions than models with cubic characteristics obtained by a least-square error or Legendre polynomial fit to the nonlinear characteristics.

This work investigates piecewise-linear and piecewise-cubic approximations to nonlinear characteristics. A direct approach toward making the error in the approximate closed-form time solutions "small" is undertaken. In Chapter 2, criteria for the closeness of the approximate time solutions are given, and some existing approximation techniques are

investigated under these criteria. The derivation of bounds on the approximation error is then made. In Chapter 3 new approximation techniques which employ the Jacobi and shifted Jacobi polynomials are introduced. The restriction that the nonlinear oscillation have a small amplitude or that the nonlinear characteristic be quasi-linear is not imposed.

2. DEVELOPMENT OF THE APPROXIMATION TECHNIQUES

2.1 Considerations of Approximation Range and Criteria for Closeness of Fit in the Approximate Time Solutions.

2.1.1 Determination of Bounds on the Oscillation and Normalization of the System Equation.

The first integral of equation (1.1) is a statement of the law of conservation of energy for a conservative system with a single degree of freedom. Writing equation (1.1) in the form

$$\dot{x}\frac{d\dot{x}}{dx} + f(x) = 0 \qquad x(0) = 1, \qquad \dot{x}(0) = 0, \tag{2.1}$$

and integrating, one obtains

$$\frac{\dot{x}^2}{2} + \int f(x)dx = C_o,$$

or

$$\frac{\dot{x}^2}{2} + V(x) = E_t. \tag{2.2}$$

The integration constant associated with this first integral may be evaluated from the initial conditions given to equation (2.1). In equation (2.2) the constant, E_t, is equal to the initial value of the potential function, V(1). The solution of

$$V(x) = E_t \tag{2.3}$$

for two real roots, ($x = 1$, $x = x_{min}$), determines the turning

points, $(1, x_{min})$, of a bounded oscillation for a normalized differential equation with an integrable nonlinear characteristic, $f(x)$.

The above discussion assumes that there are at least two real roots of equation (2.3) over the range of interest in the dependent variable, and that the solution to equation (2.1) is bounded. Cunningham[11] has given a technique which determines the position and nature of singular points in the phase plane from extrema of the potential function, $V(x)$. A singularity at a relative minimum of $V(x)$ is a centre point and the motion is locally bounded. This "potential well" which exhibits a local minimum is the most common one encountered in this study. A singularity at a local maximum of $V(x)$ is a saddle point and the motion is locally unstable. At a point of inflection of $V(x)$, the local behaviour of the singularity is like that of both a centre point and a saddle point and is thus unstable. In attempting to solve a nonlinear problem, some knowledge of the kind of solution to be expected is almost essential. Using the potential function thus provides useful information about the solution in various regions of x even before a solution of the system equation is attempted.

No generality has been lost by fixing the initial conditions as in equation (2.1). Consideration is given to bounded, periodic oscillations in this work. These oscillations may be started, arbitrarily, at a point where $\dot{x}(t)$ is zero by making a shift in the independent variable. One special case of interest serves to illustrate the normalization which is

carried out. Consider a system which is initially at rest to be acted upon by a step input at t = 0. That is,

$$\frac{d^2y}{dt^2} + h(y) = kU(t), \qquad y(0) = \dot{y}(0) = 0. \tag{2.4}$$

The bounds on the oscillation are obtained by finding the real roots of potential function

$$V(y) = \int (h(y) - k)dy + C_o.$$

The constant, C_o, is obtained from the initial conditions and the first integral. In this case C_o must be chosen so that $V(0) = 0$, and it is evident that $y = 0$ is one bound on the oscillation. If the real root of $V(y)$ closest to zero is $y_m = \lambda$ and a minimum of $V(y)$ is enclosed on the interval $(0,\lambda)$, then a bounded oscillation limited by zero and λ is obtained.

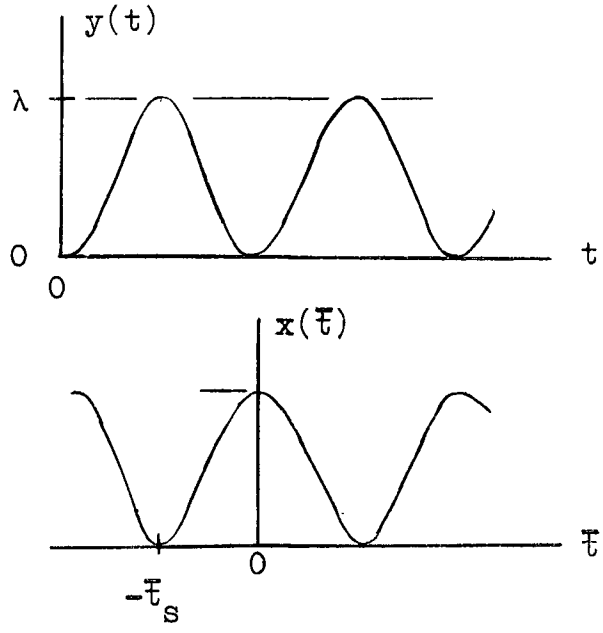


Figure 2.1 Normalization of an Asymmetric, Nonlinear Oscillation

The transformation $y(t) = \lambda x(\bar{t})$ along with a shift in the independent variable, $\bar{t} = t - \bar{t}_s$, may be used to transform equation (2.4) to one of the form

$$\frac{d^2x}{d\bar{t}^2} + \frac{1}{\lambda}h(\lambda x) - \frac{k}{\lambda} = 0 \qquad x(\bar{t}=0) = 1, \qquad \dot{x}(\bar{t}=0) = 0. \tag{2.5}$$

Figure (2.1) illustrates how this normalization is carried out. Equation (2.5) may now be written as

$$\frac{d^2x}{d\bar{t}^2} + f(x) = 0 \qquad x(0) = 1, \qquad \dot{x}(0) = 0, \tag{2.6}$$

which is the form of equation (1.1). Normalization to (0,1) is useful both for comparison of mathematical models on (0,1) and for comparison of errors in the approximate time solutions.

If the normalized $f(x)$ in equation (2.6) has odd symmetry on (-1, 1) and is monotonically increasing on (0, 1), then the oscillatory time solution is symmetric with respect to the first quarter period about the $x(t) = 0$ axis. It will be shown in section (2.3) that the maximum error in the approximate time solution over a fixed time interval depends on the maximum error, $E(x)_{max}$, in the approximation to $f(x)$ in equation (2.6). Because of the above symmetry in the normalized time solution, it is only necessary to make approximations to $f(x)$ on ($x = 0$, $x = 1$) for the first quarter period. Taking advantage of this symmetry in $f(x)$ thus seems an obvious choice because it is intuitive that $E(x)_{max}$ is smaller for a smaller approximation interval in the x-$f(x)$ plane. The Ritz and

ultraspherical polynomial truncation techniques make approximations on (-1, 1) for this symmetric case.

The smallest interval of symmetry for an oscillation derived from a model with an asymmetric, nonlinear characteristic is one half period. A piecewise approximation to f(x) may therefore be made on the range (1, x_{min}), and the remainder of the approximate time solution may be obtained by symmetry with this half period approximate solution. Figure (2.4) in section (2.2) shows linear approximations to symmetric and asymmetric f(x).

2.1.2 Criteria for Closeness of Fit in the Time Solution Approximations

For a model normalized to (0,1) the error in the time solution approximation is defined by $\varepsilon(t) = x(t) - \tilde{x}(t)$. This error function is not, in general, obtainable because x(t) cannot always be found in closed form. An IBM 7040 digital computer was employed to obtain numerical approximate solutions to the nonlinear differential equations. The numerical solutions were obtained using a fourth-order Runge-Kutta-Gill integration subroutine. Direct comparison of the numerical solutions with closed-form Jacobian elliptic function solutions for cubic models shows that the numerical solutions are accurate to six decimal digits. A technique suggested by Fröberg[12] was also employed to check the accuracy. Using this technique, a change in the numerical integration step size from 0.01 sec. to 0.001 sec. produced no change in the first five significant digits of the solutions for the time

intervals which were considered.

A goal of this work is to make the approximation error, $\varepsilon(t)$, as small as possible with a minimum amount of mathematical labor. The choice of the smallest intervals of time symmetry for the approximations in section (2.1.1) is directed toward this goal. The principal error criterion chosen is that the error at the ends of an interval of approximation be small compared to the maximum error over the approximation interval. Since the numerical solution and the approximate closed-form solution are matched at $t = 0$, the object is to have the error small at the end of the first approximation interval. This is important because the solution is constructed by matching each successive, partial approximate solution with the final value of the previous approximation.

In section (2.3) it is shown that the maximum error in the approximate time solution for a given model has an upper bound. To justify the choice of the above error criteria, suppose that the error is exactly zero at the end of a symmetric approximation interval, such as the quarter period. Then, by symmetry, the bound on the error does not increase on subsequent approximation intervals as the solution is extended in time. Also, the frequency of the nonlinear oscillation is determined exactly.

2.1.3 Examples of Some Existing Approximation Techniques Under the Given Error Criteria

Approximate solutions to the nonlinear differential

equation,

$$\ddot{x} + x + x^3 + 10x^5 = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0, \tag{2.7}$$

are carried out throughout the remainder of this chapter. The characteristic in this equation is grossly nonlinear for the larger values of $|x|$ on $(-1, 1)$. The following analysis indicates the importance of the error criteria in section (2.1.2):

(a) First-order Ritz and Linear Approximations

The first-order Ritz approximate solution to equation (2.7) is $\tilde{x}(t) = \cos\sqrt{8}\,t$. A linear, least-square error fit to the characteristic in this model over the interval (0,1) yields the approximate equation

$$\ddot{\tilde{x}} + 9.043\tilde{x} - 2.105 = 0, \qquad \tilde{x}(0) = 1, \qquad \dot{\tilde{x}}(0) = 0.$$

Solution of this equation gives $\tilde{x}(t) = 0.767 \cos(3.007t) + 0.233$ which is valid for the first quarter cycle of the oscillation. In Figure (2.2) these first-order approximations are plotted with the numerical solution for the first quarter cycle.

(b) Cubic Polynomial Truncations

The characteristic in equation (2.7) is now replaced by odd-cubic characteristics obtained by a Chebychev polynomial approximation and a least-square error fit. Using the procedure in Appendix A, solution of the Chebychev model gives $\tilde{x}(t) = Cn(0.7703, 3.373t)$, and the least-square error model gives $\tilde{x}(t) = Cn(0.7512, 3.276t)$, where Cn is the Jacobian

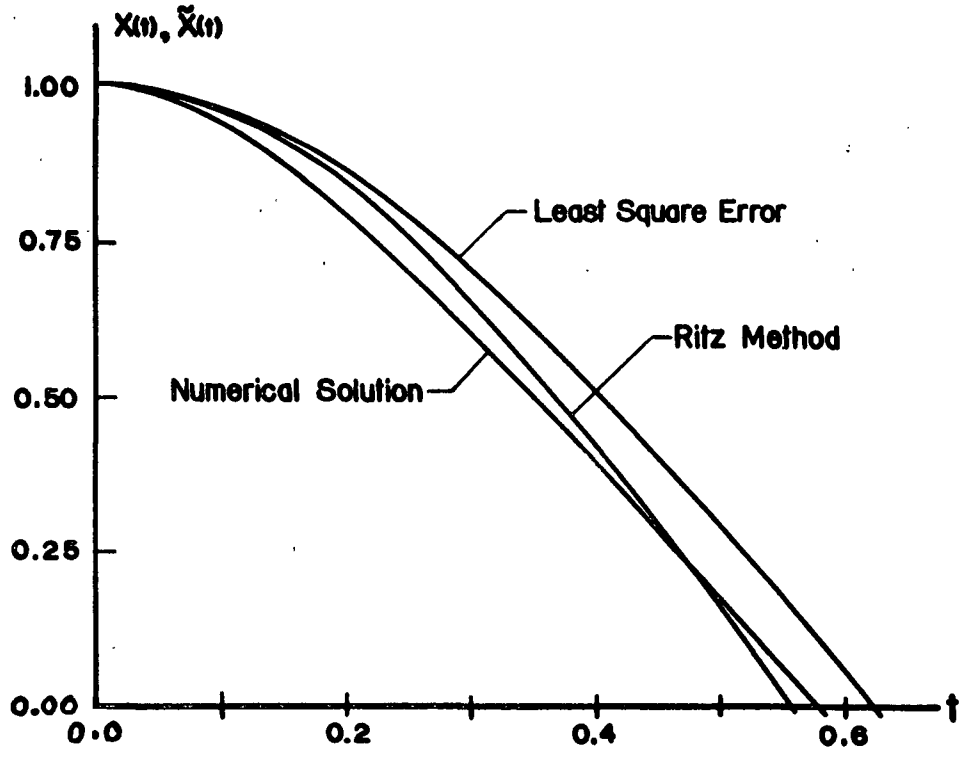


Figure 2.2 Comparison of the First-Order Ritz Approximate Solution and a Linear, Least-Square Error Approximation

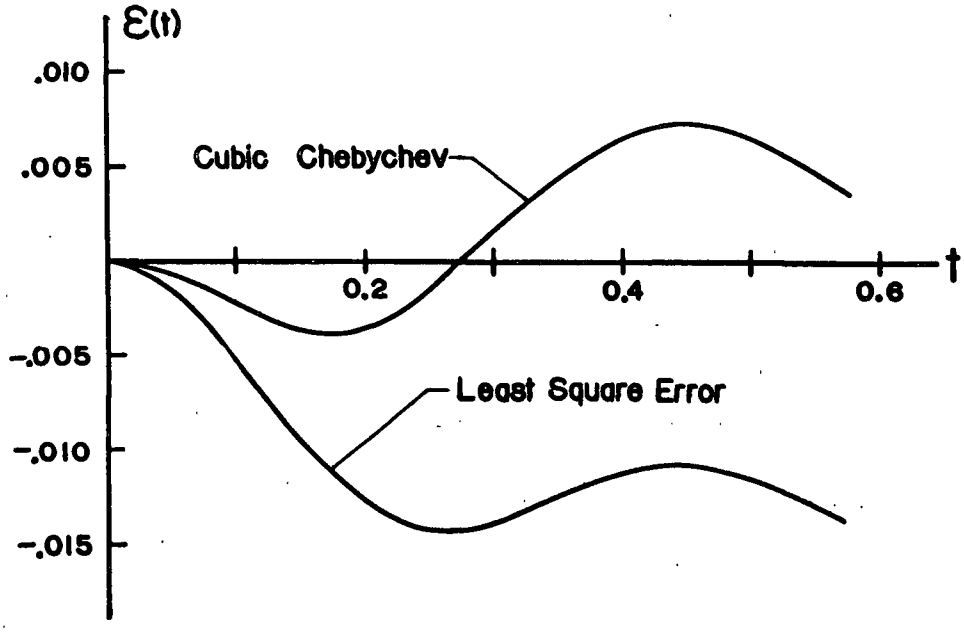


Figure 2.3 Comparison of Chebychev and Least-Square Error Cubic Approximations to $\ddot{x} + x + x^3 + 10x^5 = 0$

elliptic cosine function. Curves of the error, $\varepsilon(t)$, over the quarter period for these closed-form approximate solutions are shown in Figure (2.3).

(c) A Two-term Ritz Approximation

If a two-term Ritz approximation in the form $x(t) = A\cos\omega t + B\cos 3\omega t$ is assumed, then the Ritz conditions give three algebraic equations in A, B and ω. These are:

$$A + B = 1,$$

$$-\omega^2 A + A + \frac{3}{4}A^3 + \frac{50}{8}A^5 = 0 \qquad (2.8)$$

and

$$-9\omega^2 B + B + \frac{3}{4}B^3 + \frac{50}{8}B^5 + \frac{A^3}{4} + \frac{25}{8}A^5 = 0.$$

Solution of these equations is difficult without some knowledge of values of A and B to be expected. The error curves in Figure (2.3) show the Jacobian elliptic cosine to be a close approximation to the nonlinear oscillation. Soudack[13] suggested a device which facilitates the solution of equations (2.8) for A, B and ω. The Jacobian elliptic cosine solution has approximately 5% third harmonic, so the choice $A = 0.95$ and $B = 0.05$ is made as a first guess in equations (2.8). After a trial and error procedure with various values of A and B, the two-term Ritz approximate solution is found to be

$$\tilde{x}(t) = 0.955 \cos (2.624t) + 0.045 \cos (7.872t).$$

(d) Summary

Approximation	$\varepsilon(0 - \frac{\tilde{T}}{4})_{max}$	$\varepsilon(\frac{\tilde{T}}{4})$	$\varepsilon(0-\tilde{T})_{max}$	Frequency Error
Ritz First-Order	-0.06	+0.049	-0.25	3.9%
Linear Least-Squares	-0.12	-0.10	+0.33	-8.2%
Cubic Chebychev	0.0073	0.0036	0.014	0.3%
Cubic Least-Squares	-0.014	-0.014	0.05	-1.1%
Two-term Ritz	-0.045	-0.045	+0.14	-3.65%

Table 2. Summary of Approximation Errors

The results are summarized in Table 2. Column three of this table shows the maximum error over the first period. For all the approximations considered, the error at the approximate quarter period, $\varepsilon(\tilde{T}/4)$, is not small compared to the maximum relative error over the first quarter period. Thus, the numerical and closed-form solutions show a large phase difference after only one cycle of the oscillation. The cubic Chebychev approximation gives an approximate time solution which comes closest to the error criteria discussed above. Still, the maximum error over the first quarter period is 0.0073 and the maximum error over the first period grows to 0.014. Column four of the table shows that the magnitude of the relative error in the frequency of the approximation depends on the amplitude of the error at the approximate quarter period. Under the above error criteria a poorer approximate time solution is obtained from the two-term Ritz method than from either of the cubic polynomial approximations. Also, the

labor for the two-term Ritz method is considerably greater.

The approximation techniques outlined in Table 2 do not satisfy the error criteria which have been imposed because the error at the quarter period is not small compared to the maximum error over the quarter period. In Appendix C, replacement of an odd-symmetric, monotonically increasing characteristic in equation (2.1) by an ultraspherical, Chebychev linear approximation is shown to give the same solution as the first-order Ritz method. This equivalence of the Ritz average over time with an orthogonal polynomial approximation in the x - f(x) plane provides motivation for an investigation of the approximating properties of orthogonal polynomials more general than the ultraspherical Chebychev polynomials.

2.2 Use of the Jacobi and Shifted Jacobi Polynomials for Approximation

The object of the remainder of this chapter is to give insight into the relation between the approximation of the nonlinear characteristic in the x versus f(x) plane and the resulting error in the approximate time solution. The goal is then to obtain approximate time solutions which will satisfy the error criteria imposed in paragraph (2.1.2). To achieve these ends, the Jacobi and shifted Jacobi polynomials are chosen for the piecewise approximation of the characteristic in the nonlinear model. Approximations more general than those obtained from Chebychev and Ritz approximation techniques may be obtained using these polynomials.

The expansion of a function which is absolutely

integrable[14] in terms of a set of polynomials, $P_k(x)$, orthogonal with weight function, $W(x)$, on an interval (a,b) to a polynomial of degree, n, has the form

$$\tilde{f}(x) = \sum_{k=0}^{n} c_k P_k(x), \tag{2.9}$$

where

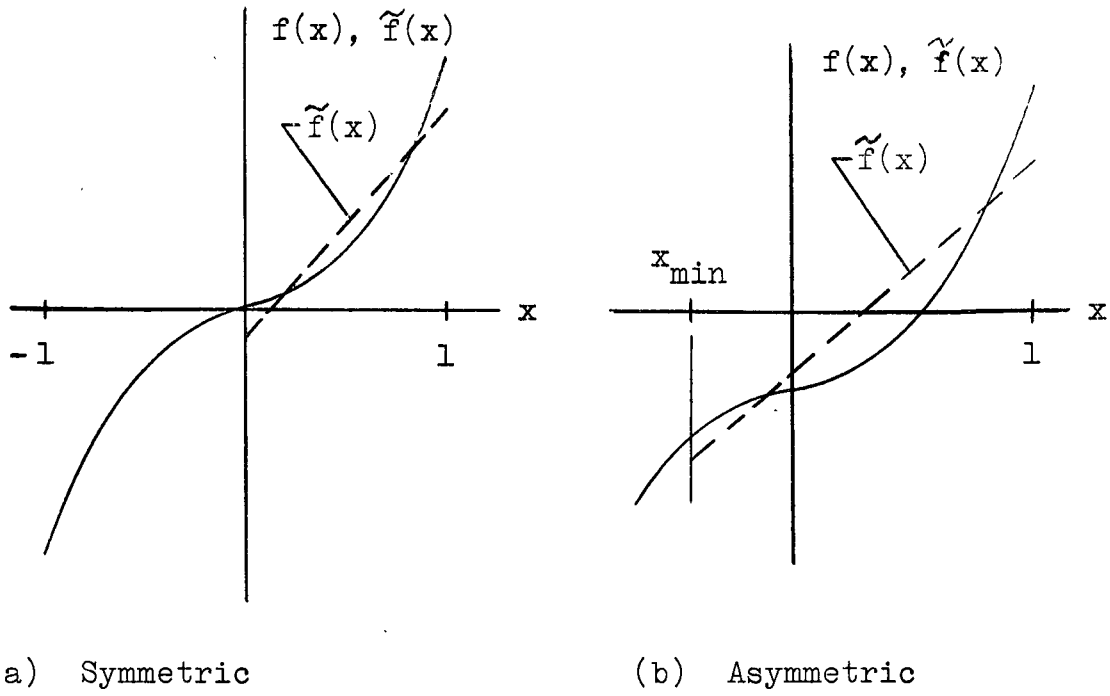
$$c_k = \frac{\int_a^b f(x)P_k(x)W(x)dx}{\int_a^b \left[P_k(x)\right]^2 W(x)dx}. \tag{2.10}$$

The shifted Jacobi polynomials are orthogonal on (0,1) with respect to the weighting function $W(x) = (1-x)^{\beta-1} x^{\alpha-1}$. For the ultraspherical Jacobi polynomials the weighting function is $(1-x^2)^{\mu-1}$ and the interval of orthogonality is (-1,1). Thus normalization of the oscillations to a maximum amplitude of unity is necessary for the expansion of f(x) in terms of these polynomials. Derivations and closed-form expressions for these polynomials are given in Appendix B.

Only the shifted Jacobi polynomials, $G_k^{(\alpha,\beta)}(x)$, are considered for the linear or first-order approximation of the models in this work. Figure (2.4) shows how these approximations are carried out. For the symmetric characteristic in Figure (2.4a), the linearization is on the interval (0,1). In contrast, Ritz and ultraspherical polynomial approximations are on the interval (-1,1) for this symmetric case. For both asymmetric and symmetric nonlinear characteristics an asymmetric linear

differential equation of the form $\ddot{\tilde{x}} + b_1\tilde{x} + b_o = 0$, $\tilde{x}(0) = 1$, $\dot{\tilde{x}}(0) = 0$, is obtained on (0,1). The solution to this equation is

$$\tilde{x}(t) = (1 + b_o/b_1) \cos\sqrt{b_1}\, t - b_o/b_1.$$



(a) Symmetric (b) Asymmetric

Figure 2.4 The Approximation of Two Nonlinear Characteristics

For the approximation of the nonlinear characteristic by a cubic characteristic, both the ultraspherical Jacobi and shifted Jacobi polynomials are employed. The ultraspherical Jacobi polynomials, $P_k^{(\mu)}(x)$, may be used to approximate odd-symmetric characteristics. This approximation produces the equation $\ddot{x} + b_1 x + b_3 x^3 = 0$. From this cubic model with zero-point symmetry, the closed-form solutions may be written by inspection using the techniques given in Appendix A.

Approximation in terms of the shifted Jacobi polynomials yields differential equations with asymmetric, cubic characteristics of the form $\ddot{x} + b_3x^3 + b_2x^2 + b_1x + b_0 = 0$. These models are obtained from the refined approximation on (0,1) of a characteristic with odd symmetry on (-1,1), or from the approximation of an asymmetric characteristic normalized to (0,1). Distinct from the linear case, the asymmetric cubic differential equation requires somewhat more labor to obtain the closed form approximate solution than does the odd-symmetric, cubic equation. Nevertheless, the techniques in Appendix A may be applied directly, and the labor is less than that required for the two-term Ritz method shown in paragraph (2.1.3) for an odd-symmetric characteristic. Examples of the closed-form solution of differential equations with cubic characteristics are given in Chapter 3.

From the expressions for an orthogonal expansion given in equations (2.9) and (2.10), a linear approximation to a given f(x) in terms of the shifted Jacobi polynomials may be written

$$\tilde{f}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left[R + \left\{\frac{\beta(\alpha+\beta+1)}{\alpha}\left(\frac{[\alpha+\beta]}{\alpha}\cdot S - \frac{\alpha}{\beta}\cdot R\right)\right\}\left(\frac{[\alpha+\beta]}{\alpha}x - \frac{\alpha}{\beta}\right)\right]. \qquad (2.11)$$

$\Gamma(\beta)$ is the gamma function. Also in this expansion

$$R = \int_0^1 f(x)x^{\alpha-1}(1-x)^{\beta-1}dx$$

and

$$S = \int_0^1 f(x)\, x^{\alpha}(1 - x)^{\beta-1}\, dx.$$

For arbitrary α, β and $f(x)$ the integrals R and S do not, in general, have closed-form solutions. One special case has been found, however, in which the expansion technique in equation (2.9) is general. If

$$f(x) = x^q, \qquad q + \alpha > 0,$$

then R and S reduce to the form

$$\int_0^1 x^{(m-1)}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)},$$

which is the integral for the Beta function. This result allows one to make approximations to classes of hardening and saturating characteristics which have $q > 1$ and $q < 1$ respectively. The method is quite general because q need not be an integer.

When $f(x)$ is itself a polynomial, a truncation or an expansion in terms of the shifted Jacobi polynomials may be made using Lanczos' Economization[15]. Lanczos[16] shows that this truncation technique gives identical coefficients to the expansion determined by equations (2.8) and (2.9) when orthogonal polynomials are used for the truncation.

To illustrate the procedure, consider the truncation of

$$f(x) = x + 3x^3 \tag{2.12}$$

to a linear polynomial on (0,1) using the shifted Chebychev polynomials, $G_k^{(0.5,\ 0.5)}(x)$. For simplicity, the superscripts are dropped and the complete expansion of (2.12) may be written

$$f(x) = c_0G_0(x) + c_1G_1(x) + c_2G_2(x) + c_3G_3(x).$$

A linear polynomial is desired so $G_3(x)$ and $G_2(x)$ are used successively to obtain expressions for the cubic and the quadratic in terms of a linear polynomial. From $G_3(x) = -1 + 18x - 48x^2 + 32x^3$, one obtains

$$x^3 = \frac{1}{32}(1 - 18x + 48x^2) + \frac{1}{32}G_3(x);$$

therefore,

$$x + 3x^3 = \frac{3}{32} - \frac{11}{16}x + \frac{9}{2}x^2 + \frac{3}{32}G_3(x).$$

Similarly,

$$x^2 = \frac{1}{8}(-1 + 8x) + \frac{1}{8}G_2(x);$$

hence,

$$x + 3x^3 = -\frac{15}{32} + \frac{61}{16}x + \frac{9}{16}G_2(x) + \frac{3}{32}G_3(x). \tag{2.13}$$

For a linear truncation $G_2(x)$ and $G_3(x)$ are set equal to zero. Hence,

$$\tilde{f}(x) = -\frac{15}{32} + \frac{61}{16}x.$$

Carrying through the orthogonal polynomials to equation (2.13) in the expansion makes the distribution of the error in the function approximation explicit. Since the maximum value of $G_2(x)$ or $G_3(x)$ is standardized to be unity, the maximum error produced by this truncation on (0,1) is $E_{max}(x) = 9/16 + 3/32 = 21/32$, and it occurs at $x = 1$ where the polynomials have a maximum oscillation.

The distribution of the error in a convergent expansion can be predicted roughly from the first term neglected. The first term neglected in (2.13), $c_2G_2(x)$, oscillates with six times the amplitude of $c_3G_3(x)$. Therefore, the error on (0,1) behaves like $G_2(x)$ with three, near-equal error maxima. The two shifted Jacobi polynomials in Figure (2.5) oscillate with a larger value near $x = 1$ than near $x = 0$ on their range of orthogonality. Thus, the distribution of the error in the $x - f(x)$ plane for a linear approximation behaves like $G_2^{(0.5,\ 0.6)}(x)$, and the distribution of the error in a cubic polynomial approximation behaves like $G_4^{(0.5,\ 0.6)}(x)$ with four, unequal error maxima. Since the exact and approximate time solutions are matched initially where $x = 1$ and $x = 0$, heuristic arguments justify variation of the error in the approximation of the characteristic near $x = 1$ in order to obtain an improvement in the time solution at some point such as the approximate quarter period. Change in the weighting function of the shifted Jacobi polynomials allows this variation in distribution of the error over the orthogonal range (0,1) in the $x - f(x)$ plane.

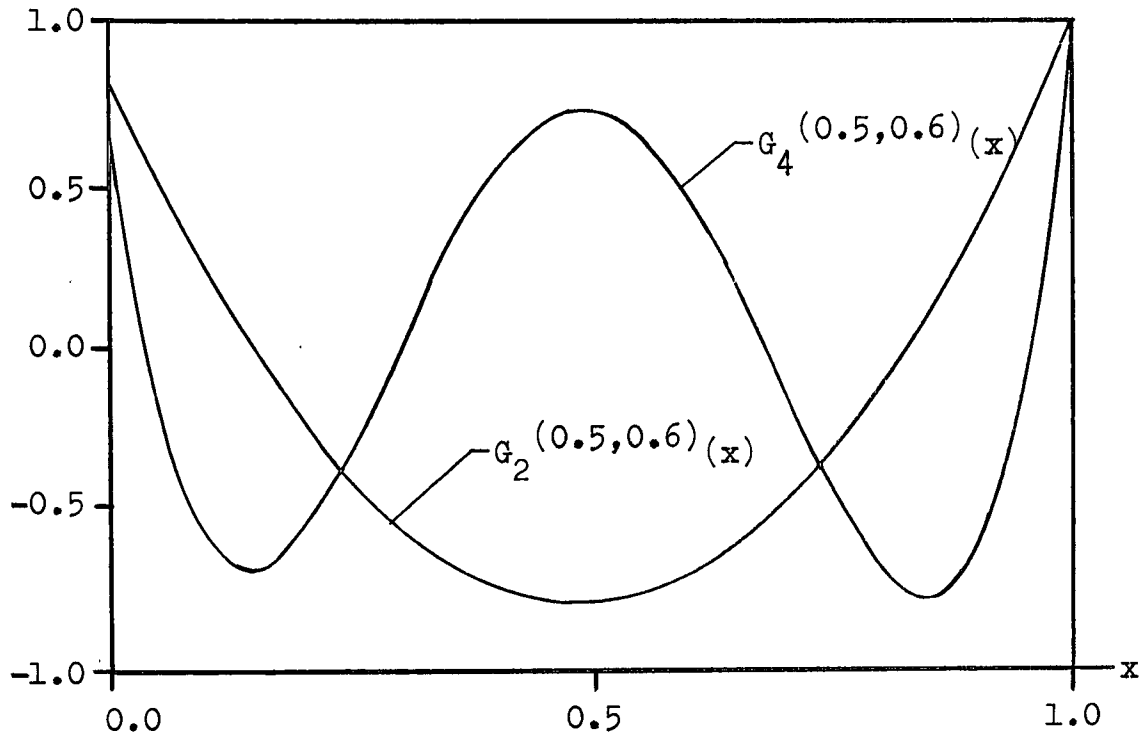


Figure 2.5 A Plot of Two Shifted Jacobi Polynomials

2.3 Determination of Upper Bounds on the Error in the Approximate Time Solution

A method was developed by Soudack[17] which gives a rough upper bound on the error in the time solution approximation from the error in a cubic polynomial function approximation to a polynomial of higher degree in the model. A similar approach is used here to determine the upper bound when a linear approximation is made to a higher degree polynomial. For simplicity replace $\tilde{x}$ by z, then the approximate equation may be written as

$$\ddot{z} + \tilde{f}(z) = 0 \qquad z(0) = 1, \qquad \dot{z}(0) = 0,$$

and the original differential equation may be written in the form

$$\ddot{x} + \tilde{f}(x) + E(x) = 0 \qquad x(0) = 1, \qquad \dot{x}(0) = 0,$$

where

$$x(t) = z(t) + \varepsilon(t)$$

and

$$E(x) = f(x) - \tilde{f}(x).$$

Note that $\varepsilon(t)$ is the error in the normalized time solution (which is not in general obtainable in closed form) and $E(x)$ is the error in the function approximation. Now,

$$\begin{aligned} \ddot{x} = \ddot{z} + \ddot{\varepsilon} &= -\tilde{f}(z) + \ddot{\varepsilon} \\ &= -\tilde{f}(z + \varepsilon) - E(z + \varepsilon); \end{aligned}$$

hence,

$$\ddot{\varepsilon} = \tilde{f}(z) - \tilde{f}(z + \varepsilon) - E(z + \varepsilon).$$

For the linear approximation

$$\tilde{f}(z) = b_o + b_1 z,$$

we have

$$\begin{aligned} \ddot{\varepsilon} &= b_o + b_1 z - b_o - b_1(z + \varepsilon) - E(z + \varepsilon) \\ &= -b_1 \varepsilon - E(z + \varepsilon). \end{aligned}$$

The differential equation for the error is thus

$$\ddot{\varepsilon} + b_1 \varepsilon = -E(z + \varepsilon) \qquad \varepsilon(0) = \dot{\varepsilon}(0) = 0. \tag{2.14}$$

The initial conditions are obtained as shown because the numerical solution and approximate solution are matched initially. The solution to equation (2.14) has the form

$$\varepsilon(t) = \varepsilon_p - \varepsilon_p \cos(\sqrt{b_1}\, t). \tag{2.15}$$

The particular integral, ε_p, is determined from equation (2.14) using convolution[18], that is

$$\varepsilon_p = -\frac{1}{b_1} \int_0^t \left[E^*(\beta) \right] \sin(t - \beta)\, d\beta,$$

where $E^*(t)$ is the function obtained by replacing z and ε by their respective time functions. Since a bound on the error is of interest here, consider

$$\varepsilon_p \leq \frac{1}{|b_1|} \int_0^t \left| E^*(\beta) \right| \cdot \left| \sin(t - \beta) \right| d\beta.$$

Now $\left|E^*(\beta)\right|$ is bounded because f(x) and $\tilde{f}(z)$ are bounded on (0,1). Let $\left|E^*(t)\right|_{max} = \left|E(z + \varepsilon)\right|_{max} \triangleq$ EMAX. EMAX can be found from f(x) and $\tilde{f}(x)$. Thus, the bound on the particular integral now becomes

$$\left|\varepsilon_p\right| \leq \frac{EMAX}{b_1} \int_0^t \left| \sin\,(t - \beta) \right| d\beta.$$

Since the sine function is less than or equal to one, for a time

interval Δt, we have

$$|\varepsilon_p| \leq \frac{EMAX}{|b_1|}\,\Delta t. \tag{2.16}$$

Specifically, the time interval, Δt, will be an approximate quarter period or an approximate half period depending on the interval over which the truncation is made.

From equation (2.15) one obtains

$$|\varepsilon(t)| \leq |\varepsilon_p| \cdot |1 - \cos\sqrt{b_1}\, t\,|$$

$$\leq 2\,\frac{EMAX}{|b_1|}\,\Delta t.$$

This relation determines an upper bound on the error in a first-order approximate time solution from the error in the $x - f(x)$ plane matching of the characteristic. Using a similar method for a cubic polynomial truncation

$$\tilde{f}(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3.$$

Soudack[19] has shown that a rough upper bound for the error in the time solution approximation is

$$|\varepsilon(t)|_{max} \leq 2\,\frac{EMAX}{|b_1|}\,\Delta t. \tag{2.17}$$

These results are for extreme upper bounds because the above approach is a pessimistic one in which maximum or worst possible errors are considered. In practice, the actual error is much smaller than that predicted by equations (2.16)

and (2.17). As an example, consider the linear, least-square error approximation to equation (2.6) in paragraph (2.1.3). The upper bound predicted for the first quarter-period by equation (2.16) is

$$|\varepsilon(t)|_{max} \leq 2 \frac{EMAX}{|b_1|} \Delta t$$

$$= 2 \frac{5.06}{9.04} \quad 0.62$$

$$= 0.70.$$

From Table 2 the actual maximum error obtained over the first quarter period is $\varepsilon(t)_{max} = -0.12$.

The value of the above bounds on the error for approximations to polynomial characteristics arises from a proof by Soudack[20]. It was shown that $|\varepsilon(t)|_{max}$ is bounded when the polynomial characteristic is truncated down to a cubic polynomial using the shifted Chebychev polynomials. Below, this result is extended to the expansion of a polynomial in terms of the orthogonal Jacobi or shifted Jacobi polynomials to either a linear or a cubic polynomial. Let

$$P_n(x) = -A_{on} - A_{1n}x + \ldots - A_{mn}x^m + \ldots - A_{(n-1)n}x^{n-1} + A_{nn}x^n \quad (2.18)$$

represent an orthogonal Jacobi or shifted Jacobi polynomial as given in Appendix B. The coefficient A_{mn} is positive if n-m is even; the signs have been chosen as shown for convenience. Also, for convenience

$$B_{n-2} = A_{(n-2)n} + \frac{A_{(n-1)n}A_{(n-2)(n-1)}}{A_{(n-1)(n-1)}}$$

and

$$B_{n-m} = A_{(n-m)n} + \frac{A_{(n-2)n}A_{(n-m)(n-1)}}{A_{(n-1)(n-1)}} + \frac{B_{(n-2)}A_{(n-3)(n-2)}}{A_{(n-2)(n-2)}} + \cdots$$

$$+ \frac{B_{(n-m+1)}A_{(n-m)(n-m+1)}}{A_{(n-m+1)(n-m+1)}}$$

are defined.

Application of Lanczos' Economization to the nonlinear characteristic $f(x) = \sum a_n x^n$ yields

$$\tilde{f}(x) = \sum_{j=0}^{p-1} b_j x^j,$$

where

$$b_j = a_j + \sum_{k=p}^{n} \frac{a_k}{A_{kk}} \left[A_{jk} + \sum_{m=1}^{n-p} \frac{B_{(k-m)}A_{j(k-m)}}{A_{(k-m)(k-m)}} \right]. \qquad (2.19)$$

The error in this approximation is $E(x) = f(x) - \tilde{f}(x)$. In terms of the nonlinear characteristic and the orthogonal polynomials

$$E(x) = \sum_{p}^{n} \left[\frac{a_k}{A_{kk}} \sum_{p}^{n} \frac{A_{kn}P_k(x)}{A_{kk}} \right] . \qquad (2.20)$$

In both (2.19) and (2.20), $p = 2$ for a linear approximation and $p = 4$ for a cubic polynomial approximation.

Let PM_k be the absolute value of the k^{th} Jacobi or

shifted Jacobi polynomial at the point x_m where $E(x)$ assumes a maximum oscillation, EMAX. From equations (2.16), (2.17), (2.19) and (2.20)

$$|\varepsilon(t)|_{max} \leq \frac{2\Delta t\left[\sum_{p}^{n} \frac{a_k}{A_{kk}} \sum_{p}^{n} \frac{A_{kn}}{A_{kk}} PM_k\right]}{\left|a_1 + \sum_{p}^{n} \frac{a_k}{A_{kk}}\left[A_{1k} + \sum_{m=1}^{n-2} \frac{B_{(k-m)}A_{1(k-m)}}{A_{(k-m)(k-m)}}\right]\right|}.$$

Expanding the summations one obtains

$$|\varepsilon(t)|_{max} \leq 2\Delta t\left|\frac{a_n H_1(Aqr)+a_{n-1}H_2(Aqr)+\ldots+a_p H_{n-p+1}(Aqr)}{a_1+a_n K_1(Aqr)+a_{n-1}K_2(Aqr)+\ldots+a_p K_{n-p+1}(Aqr)}\right|. \tag{2.21}$$

This error bound depends only on the coefficients, a_k, in the polynomial which is being approximated, the coefficients, Aqr, of the orthogonal Jacobi polynomials and the time interval over which the approximation is made. If a_k gets very large in the polynomial being approximated, then from equation (2.21)

$$|\varepsilon(t)|_{max} \longrightarrow 2\Delta t\left|\frac{H_{n-k+1}(Aqr)}{K_{n-k+1}(Aqr)}\right|,$$

which is constant. If more than one of the a_k such as a_k and a_{k-1} are increased, then

$$|\varepsilon(t)|_{max} \longrightarrow 2\Delta t\left|\frac{\ldots+a_k H_{n-k+1}(Aqr)+a_{k-1}H_{n-k+2}(Aqr)+\ldots}{\ldots+a_k K_{n-k+1}(Aqr)+a_{k-1}K_{n-k+2}(Aqr)+\ldots}\right|$$

$$\longrightarrow 2\Delta t \left| \frac{H_{n-k+1}(Aqr)+\frac{a_{k-1}}{a_k} H_{n-k+2}(Aqr)}{K_{n-k+1}(Aqr)+\frac{a_{k-1}}{a_k} K_{n-k+2}(Aqr)} \right|,$$

and this error bound is also a constant as a_k and a_{k-1} get very large. These results apply to both linear and cubic approximations to $f(x)$, and they may be extended inductively to the case where more than two of the a_k grow large. The form of these error bounds depends on the fact that both the truncation of $f(x)$ by Lanczos' Economization and the maximum error in the truncation have the same dependence on the coefficients, a_k. Thus, when the quotient is taken in equation (2.21), the dependence on the a_k is removed, in the limit, as the a_k grow large.

From these results it follows that if the maximum error in the approximate time solution over a given time interval is bounded, then the error at any point on the approximate time solution, such as at the approximate quarter period, must also be bounded. The approximate time solutions which have been obtained in this work agree with these error bounds. As an example, consider the nonlinear model which is given in equation (2.7), that is

$$\ddot{x} + x + x^3 + 10x^5 = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0, \tag{2.7}$$

If the amplitude of the oscillation is now doubled, then the transformation $x = 2y$ normalizes this equation to unity

amplitude in the form

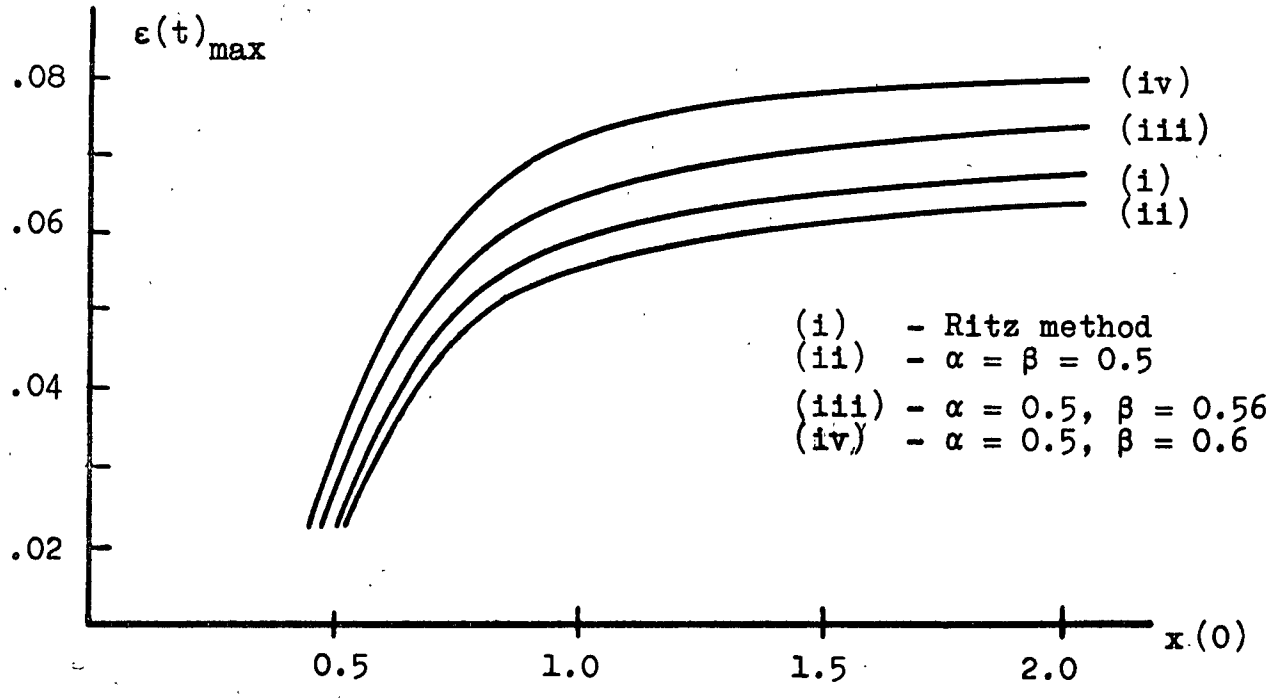
$$\ddot{y} + y + 4y^3 + 160y^5 = 0, \qquad y(0) = 1, \qquad \dot{y}(0) = 0.$$

Similarly, if the amplitude of the oscillation in (2.7) is halved, the normalization yields

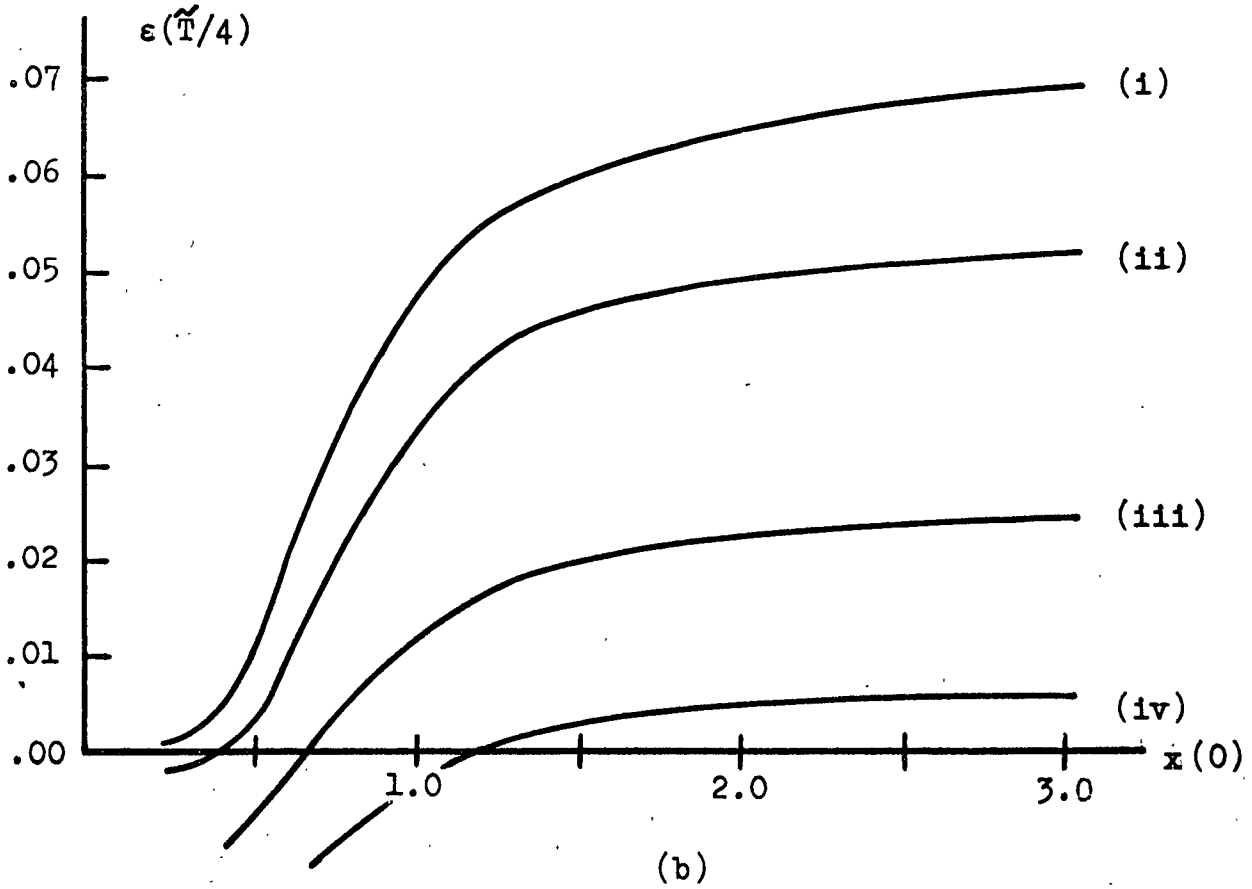
$$\ddot{w} + w + 0.25w^3 + 0.156w^5 = 0, \qquad w(0) = 1, \qquad \dot{w}(0) = 0,$$

where $x = 0.5w$. Clearly, these normalizations show considerable differences in the characteristics as $x(0)$ is varied. Other normalizations similar to these have been made on equation (2.7), and then first-order approximations to the models obtained have been carried out using the one-term Ritz method and the shifted Jacobi polynomials $G_k^{(0.5,0.5)}(x)$, $G_k^{(0.5,0.56)}(x)$ and $G_k^{(0.5,0.6)}(x)$.

Figure (2.6) shows that both the maximum errors, $\varepsilon(t)_{max}$, over the first quarter period and the errors, $\varepsilon(\tilde{T}/4)$, at the approximate quarter period are bounded as the amplitude of the nonlinear terms is increased by increasing $x(0)$. The saturation of the error at the approximate quarter period occurs at a lower value for the nonsymmetrically-weighted, shifted Jacobi polynomial approximations than for the ultraspherical, shifted Chebychev or one-term Ritz method approximations. When $x(0) = 1$, as in equation (2.7), for example, the errors at the approximate quarter period are 0.049 and -0.004 for the one-term Ritz and $G_k^{(0.5,0.6)}(x)$ approximations respectively. Extension of these solutions to a full period by symmetry yields the maximum error of 0.25 for the one-term Ritz solution,



(a)



(b)

Figure 2.6 Saturation of the Maximum and Quarter Period Errors for Linear Approximations as the Initial Amplitude is Increased.

and 0.08 for the $G_k^{(0.5,0.6)}(x)$ approximation. A numerical solution of equation (2.7) gives a derivative at the quarter period of -2.1981. The relative errors in the derivative at the approximate quarter periods are -29% for the first-order Ritz approximation and -13% for the $G_k^{(0.5,0.6)}(x)$ approximate solution. Also, from a numerical solution a quarter period of 0.5777 seconds is obtained. The approximate quarter periods are: 0.553 sec., a relative error of 3.9% and 0.5728 sec., a relative error of 0.8% from the Ritz and $G_k^{(0.5,0.6)}(x)$ polynomial approximations, respectively. The improvement over the Ritz method in Figure (2.6) is mainly in the phase or frequency. This is because the errors at the approximate quarter periods saturate at considerably smaller values for the weighted approximations. The saturation of the maximum error over the first approximate quarter-period does not show such large differences.

In Figure (2.6), the near-uniform, shifted Chebychev approximation in curve (ii) shows the effect of an approximation on the interval (0,1). The near-uniform Ritz or Chebychev approximation on (-1,1) does not account for the symmetry in the quarter-period. Thus, $E(x)_{max}$ obtained from the match to f(x) by the Ritz or Chebychev approximation on (-1,1) is larger than $E(x)_{max}$ from the shifted Chebychev approximation on (0,1). As a result, both $\varepsilon(t)_{max}$ and $\varepsilon(\tilde{T}/4)$ in curve (i) are larger for the Chebychev approximation than for the shifted Chebychev approximation shown in curve (ii). The improvement obtained using the nonsymmetrically-weighted shifted Jacobi polynomials shows, however, that near-uniform or Chebychev matching in the

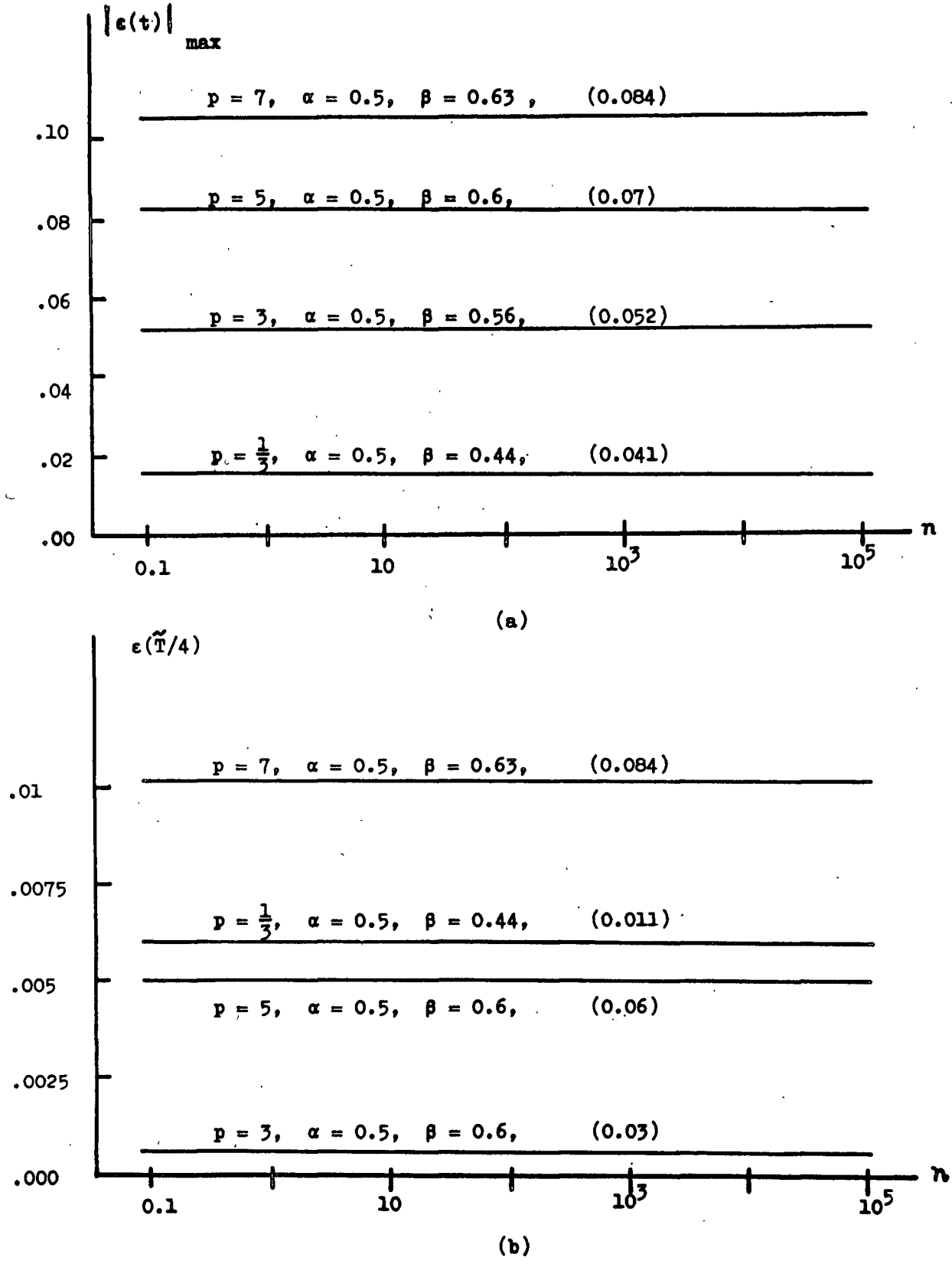


Figure 2.7 Saturation of the Maximum and Quarter Period Error for Linear Approximations to $\ddot{x} + nxP = 0$, $x(0) = 1$, $\dot{x}(0) = 0$.

$x - f(x)$ plane does not provide the best approximation to the nonlinear amplitude-frequency relationship under the error criteria which have been imposed.

Now consider maximum and quarter-period values of $\varepsilon(t)$ for linear approximations over the first quarter period to equations of the form

$$\ddot{x} + nx^p = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0.$$

In this case, $x(0)$ is fixed at unity and different values of n and p are used.

In Figure (2.7), curves showing the errors for various values of p and for different linear shifted Jacobi polynomial approximations are plotted versus an increase in n in the characteristic. For clarity, the errors in the Ritz approximations, which are much larger than those plotted in Figure (2.7), have been placed in brackets on the curves. The quarter-period amplitude errors are 0.010, 0.03, 0.06 and 0.09 for the Ritz approximation to models with p equal to one-third, three, five and seven, respectively. These error curves do not show the same increase toward saturation as those in Figure (2.6). This is because the characteristic in equation (2.7) has a linear term present which becomes dominant when the effect of the nonlinear terms is made small by decreasing the initial amplitude. As predicted by equation (2.21), however, the curves in Figure (2.7) demonstrate that the error is bounded as the coefficients of the nonlinear terms increase.

3. APPLICATION OF THE APPROXIMATION TECHNIQUES TO SPECIFIC NONLINEAR MODELS

3.1 Techniques for a Class of Odd-Symmetric Nonlinear Characteristics

3.1.1 Definition of the Nonlinear Factor and Choice of Weighting Functions

Characteristics with odd symmetry on (-1,1) and with a monotonic increasing property on (0, 1) are often called hardening when $f''(x) > 0$ and softening when $f''(x) < 0$. This class of characteristics is approximated in this section. For particular cases, the variable weighting of the error in the x-f(x) plane afforded by shifted Jacobi polynomial expansions has been used in the previous chapter. These weighted polynomial approximations have yielded approximate time solutions with a small error in the quarter period. Extension of this weighting property to more general characteristics in this class requires a means for specifying the deviation from linearity of different, normalized characteristics on the interval (0,1). For this purpose, the nonlinear factor

$$F \triangleq 1 - \frac{2\int_0^1 f(x)dx}{f(1)} \qquad (3.1)$$

is defined. Figure (3.1) shows graphically that F is the difference between the area under the curve and the area under the chord on (0,1), relative to the area under the chord. In value, the nonlinear factor lies between -1 and +1 on (0,1). For hardening characteristics $F > 0$ and for softening charac-

teristics $F<0$.

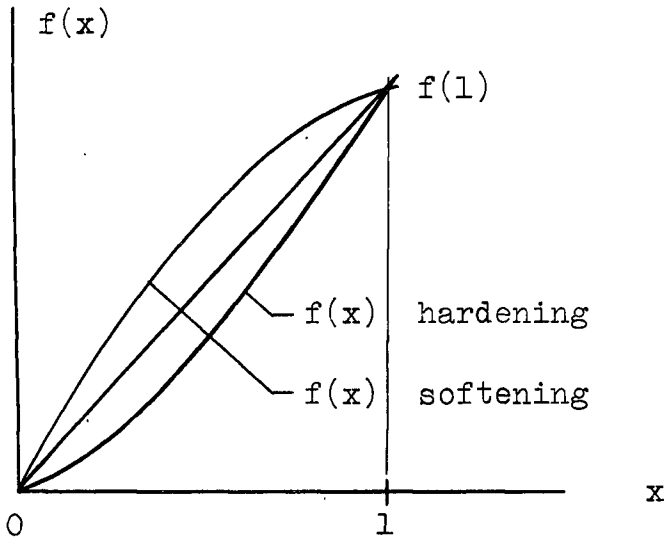


Figure 3.1 A Graphical Representation of the Measure of Departure from Linearity

Now consider variation of the parameters α, β and μ in the weighting functions

$$W_1(x) = x^{(\alpha-1)}(1-x)^{(\beta-1)} \tag{3.2}$$

and

$$W_2(x) = (1-x^2)^{\mu-1}, \tag{3.3}$$

according to the value of F. A derivation by Denman and Lui[21] shows that Chebychev approximation to "quasi-linear" f(x) yields approximate time solutions with a small relative error in the period. Chebychev approximations are obtained when $\alpha = \beta = \mu = 0.5$. Thus, the weighting has been varied starting from this case with "quasi-linear" characteristics. Heuristically, the initial matching of the exact and approximate time solutions at $x(0) = 1$ and $\dot{x}(0) = 0$ allows variation of the

error in the approximation to f(x) near x = 1. What is more important, as the amplitude of the nonlinear terms increases, the deviation from linearity is most severe for the larger values of x on the interval (0,1). From equation (3.2), variation of β is seen to have the greatest effect on the weighting near x = 1. Setting $\alpha = 0.5$ makes the weighting close to that obtained from a shifted Chebychev truncation for the small values of x on the interval (0,1). This choice is reasonable because Chebychev approximations in the x-f(x) plane give the closest approximate time solutions for small values of F or for "quasi-linear" characteristics.

Particular values of β and μ have been chosen empirically from error saturation curves similar to those in Figures (2.6) and (2.7) but with the nonlinear factor, F, as the abscissa. Approximation with different weighting using the shifted Jacobi polynomials has been shown to produce large differences in the error

Nonlinear Measure	β	μ
$F \leq -0.40$	0.37	0.35
$-.40 < F \leq -0.25$	0.44	0.42
$-0.25 < F \leq +0.25$	0.50	0.50
$0.25 < F \leq 0.60$	0.56	0.58
$F \geq 0.60$	0.63	0.65

Table 3.1 The Choice of Weighting for the Jacobi and Shifted Jacobi Polynomial Approximations

at the approximate quarter period. Hence, considerable improvement in the amplitude-frequency relationship has been found possible by choosing the weighting according to the value of F given in Table (3.1). From a model with a saturating or softening characteristic, the frequency decreases as the initial amplitude increases. In contrast, from a model with a hardening characteristic, the frequency increases with the amplitude of the oscillation. Variation of the weighting in a linear approximation varies the slope and thus the approximate frequency changes. Hence, variations of β or μ are opposite for models with softening and hardening characteristics.

3.1.2 Some Particular Cases

The nonlinear factor is now applied to weighting the error in Jacobi and shifted Jacobi polynomial expansions. In Tables (3.2) and (3.3) the maximum error over the first quarter period and the error at the quarter period are given for approximate solutions to the equations $\ddot{x} + f(x) = 0$, $x(0) = 1$, $\dot{x}(0) = 0$. The nonlinear functions in Table (3.2) have all been truncated down to an asymmetric linear polynomial on (0,1) using Lanczos' Economization and the shifted Jacobi polynomials. Initially, either a five-term Taylor series expansion to a ninth degree polynomial or a four-term Legendre polynomial expansion to a seventh degree polynomial is used to obtain polynomial approximations to the functions in Table (3.2). The Legendre approximation is carried out for the grossly nonlinear functions marked by asterisks because the Taylor expansion converges slowly for these functions. The maximum error in these pre-

liminary Taylor and Legendre polynomial expansions was kept below 0.5% of the value of the function at x = 1. Under this criterion, the linear shifted Jacobi polynomial truncations carried out in the x - f(x) plane agree with corresponding linear expansions obtained by numerical integration of equations (2.9) to three significant figures.

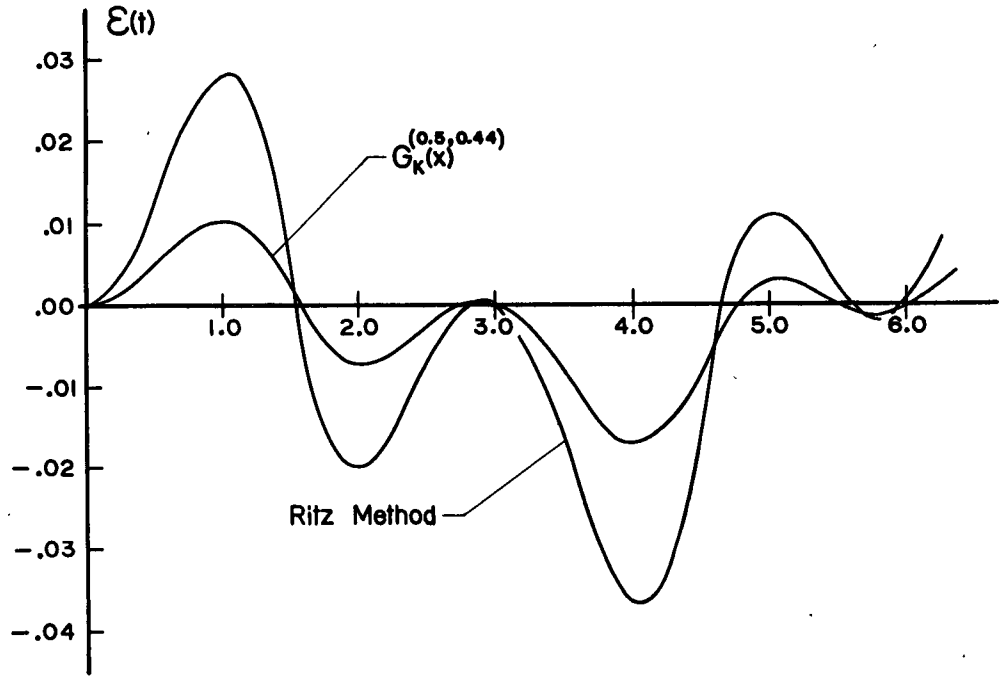


Figure 3.2 A Comparison of the Error in Two Linear Approximations to $\ddot{x} + \sqrt{|x|}\ \operatorname{sgn} x = 0$

The models with characteristics of the form $x^{1/q}$, and with q equal to 2,3,5, and 7 are given to show the behaviour of approximations to models with softening characteristics. Rigorously, these models are not Lipschitz at x = 0, but this non-Lipschitz character at x = 0 will not be present in any

Table 3.2 Comparison of the First-Order Ritz Method and Linear Jacobi Polynomial Approximations

Nonlinear Characteristic	F	Ritz Method $\varepsilon(t)_{max}$	Ritz Method $\varepsilon(\tilde{T}/4)$	β	Shifted Jacobi $\varepsilon(t)_{max}$	Shifted Jacobi $\varepsilon(\tilde{T}/4)$
$x+0.16x^3+0.256x^5$	0.17	-0.0170	+0.0027	0.50	-0.015	0.0005
$x+x^3+x^5$	0.39	-0.0390	+0.0160	0.56	-0.041	-0.0063
$x+x^3+5x^5$	0.55	-0.0560	+0.0380	0.56	-0.060	+0.0040
$x+x^3+10x^5$	0.59	-0.0610	+0.0500	0.56	-0.065	+0.0100
$x+4x^3+160x^5$	0.66	-0.0680	+0.0660	0.63	-0.090	-0.0080
$x+9x^3+810x^5$	0.66	-0.0700	+0.0670	0.63	-0.090	-0.0080
$x+0.25x^3+0.31x^5+0.16x^7$	0.26	-0.0250	+0.0064	0.50	-0.022	+0.0015
$x+x^3+x^5+x^7$	0.48	-0.0480	+0.0270	0.56	-0.051	-0.0016
$x+x^3+10x^5+10x^7$	0.67	-0.0680	+0.0690	0.63	-0.090	-0.0040
$x+x^3+10x^5+100x^7$	0.73	+0.0970	+0.0970	0.63	-0.100	+0.0060
$x+0.3x^3$	0.11	-0.0120	+0.0011	0.50	-0.009	-0.0010
$x+0.75x^3$	0.21	-0.0220	+0.0040	0.50	-0.018	-0.0003
$x+3x^3$	0.38	-0.0380	+0.0140	0.56	-0.039	-0.0070
$x+12x^3$	0.46	-0.0480	+0.0260	0.56	-0.049	-0.0040
$x+1000x^3$	0.50	-0.0520	+0.0300	0.56	-0.053	-0.0010
x^2 sgn x	0.34	-0.0340	+0.0100	0.56	-0.032	-0.0070
$x^{2.5}$ sgn x	0.43	-0.0440	0.0190	0.56	-0.044	-0.0050
x^3	0.50	-0.0520	+0.0300	0.56	-0.053	-0.0010
x^5	0.67	-0.0700	+0.0600	0.63	-0.090	-0.0080
x^7	0.75	+0.0840	+0.0840	0.63	-0.100	+0.0090
sinh (x)	0.10	0.0073	+0.0005	0.50	-0.006	-0.0009
sinh (2x)	0.24	-0.0240	+0.0060	0.50	-0.020	+0.0004

Table 3.2 (Continued)

Nonlinear Characteristic	F	Ritz Method $\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$	β	Shifted Jacobi $\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$
sinh (4x)**	0.48	-0.0520	+0.0320	0.56	-0.0540	+0.00050
sinh (4x)**	0.67	-0.0660	+0.0640	0.63	+0.0900	-0.00900
tan (x)	0.21	-0.0180	+0.0033	0.50	-0.0174	-0.00006
tan (1.3x)**	0.45	-0.0410	+0.0210	0.56	-0.0440	-0.00500
tan (1.5x)**	0.62	+0.0840	+0.0840	0.63	-0.0900	+0.00100
$\|x\|^{0.8}\operatorname{sgn} x$	-0.11	+0.0100	+0.0050	0.50	+0.0060	+0.00100
$\sqrt{\|x\|}\operatorname{sgn} x$	-0.33	+0.0280	+0.0060	0.44	+0.0120	+0.00300
$2\sqrt{\|x\|}\operatorname{sgn} x$	-0.33	+0.0290	+0.0060	0.44	+0.0120	+0.00300
$x^{1/3}$	-0.50	+0.0410	+0.0120	0.44	+0.0150	+0.00500
$x^{1/5}$	-0.67	+0.0520	+0.0170	0.37	+0.0100	+0.00015
tanh (x)	-0.10	+0.0140	+0.0016	0.50	+0.0110	+0.00300
tanh (2x)**	-0.38	+0.0380	+0.0080	0.44	+0.0230	+0.00400
$2x-0.5x^3$	-0.17	+0.0150	+0.0020	0.50	+0.0130	+0.00300
$2x-0.9x^2\cdot\operatorname{sgn} x$	-0.27	+0.0250	+0.0040	0.44	+0.0170	+0.00140
sin(1.5x)**	-0.26	+0.0230	+0.0040	0.44	+0.0150	+0.00030

physical problems or in any digital simulation. As an example from Table (3.2), the first-order approximation of $\ddot{x} + \sqrt{|x|}\,\operatorname{sgn} x = 0$ is shown in Figure (3.2). The error over the first period is shown for the one-term Ritz approximation and the shifted Jacobi polynomial expansion, $G_k^{(0.5,0.44)}(x)$.

Approximation to the classical pendulum model, $\ddot{x} + \sin(1.5x) = 0$, is the final example in Table (3.2). An analytical solution to this model may be obtained in terms of the Jacobian elliptic functions but the linearization in Table (3.2) is given to show an example of the improvement in the shifted Jacobi truncation technique over the Ritz averaging method.

Figure (3.3) compares results for the relative error in the frequency or period. One-term Ritz and linear shifted Jacobi approximations to the equation $\ddot{x} + x + x^3 + 10x^5 = 0$, $x(0) = 1$, $\dot{x}(0) = 0$, are shown in this figure. The normalization with changes in $x(0)$ has been carried out as described in paragraph (2.3).

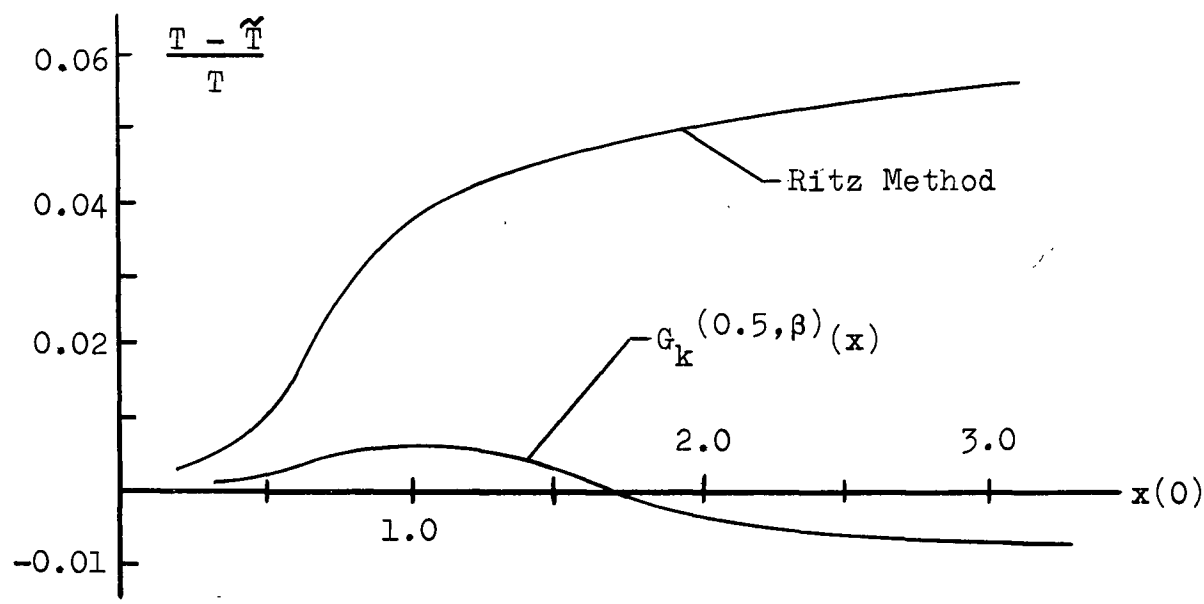


Figure 3.3 The Relative Error in the Frequency for Linear Approximations to $\ddot{x} + x + x^3 + 10x^5 = 0$, $x(0) = 1$, $\dot{x}(0) = 0$

In Figure (3.3) the $G_k^{(\alpha,\beta)}(x)$ approximations have a smaller error than the Ritz method approximations. This means that the shifted Jacobi polynomial approximate solutions do not go out of phase as rapidly as do the Ritz approximations.

In Table (3.3), the functions to which Lanczos' Economization does not apply have been expanded to cubic polynomials by numerical integration of equation (2.9). This allows the behaviour of the amplitude-frequency approximations to non-linear models with characteristics more general than polynomials to be studied under cubic, shifted Jacobi polynomial expansions. Figure (3.4) gives the relative errors over the first period for three cubic polynomial truncations to the model $\ddot{x} + x + x^3 + 10x^5 = 0$, plotted for the first period. Under the error criteria of this work, the time solutions obtained from the ultraspherical, Jacobi approximations $P_k^{(0.5)}(x)$ and $P_k^{(0.58)}(x)$ on (-1,1) are poorer than the refined, shifted Jacobi, $G_k^{(0.5,0.56)}(x)$ closed-form time solution. Also, Table (2.1) shows that over the first period for this same model, the maximum error in a two-term Ritz approximation is 0.14, and the error in a cubic, least-square error approximation is 0.05. For the cubic $G_k^{(0.5,0.56)}(x)$ approximation in Figure (3.4), the maximum error is -0.0028 over the first period. This latter approximation also shows that the error in the extended solution does not grow large with respect to the error over the first quarter period because the error at the approximate quarter period is +0.0005.

As predicted by the relation obtained between $\varepsilon(t)_{max}$

Table 3.3 Cubic Polynomial Approximation of the Nonlinear Characteristics

Nonlinear Characteristics	F	Chebychev		Ultraspherical Jacobi			Shifted Jacobi		
		$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$	μ	$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$	β	$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$
$x + 0.25x^3 + 0.3125x^5$	0.21	+0.00080	+0.00030	0.50	0.00080	+0.00030	0.50	-0.00020	0.00000
$x + x^3 + x^5$	0.39	+0.00190	+0.00090	0.58	-0.00200	-0.00009	0.56	-0.00050	+0.00000
$x + 0.562x^3 + 3.1641x^5$	0.51	+0.00500	+0.00240	0.58	-0.00400	+0.00060	0.56	-0.00100	+0.00020
$x + x^3 + 10x^5$	0.58	-0.00750	+0.00360	0.58	-0.00500	+0.00100	0.56	-0.00120	+0.00050
$x + 4x^3 + 160x^5$	0.66	+0.01000	0.00470	0.65	-0.00700	+0.00200	0.63	-0.00180	+0.00006
$x + 9x^3 + 810x^5$	0.66	+0.01000	0.00480	0.65	-0.00720	-0.00010	0.63	-0.00180	+0.00013
$x + 0.25x^3 + 0.31x^5 + 0.15x^7$	0.25	+0.00140	+0.00600	0.50	+0.00140	+0.00600	0.50	-0.00050	+0.00018
$x + x^3 + x^5 + x^7$	0.48	+0.00480	+0.00250	0.58	-0.00400	+0.00040	0.56	-0.00130	+0.00020
$x + x^3 + 10x^5 + 10x^7$	0.67	+0.01300	+0.00700	0.65	-0.00900	+0.00030	0.63	-0.00300	+0.00040

Table 3.3 (Continued)

Nonlinear Characteristics	F	Chebychev		Ultraspherical Jacobi			Shifted Jacobi		
		$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$	μ	$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$	β	$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$
$x + x^3 + 10x^5 + 100x^7$	0.73	+0.02100	+0.01000	0.65	-0.0130	+0.0040	0.63	-0.0043	+0.00100
x^5	0.67	+0.01000	+0.00450	0.65	-0.0074	-0.0002	0.63	-0.0019	+0.00015
$x^5 + x^7$	0.71	+0.01600	+0.00740	0.65	-0.0100	+0.0007	0.63	-0.0033	+0.00080
x^7	0.75	+0.02400	0.01100	0.65	-0.0137	+0.0024	0.63	-0.0047	+0.00180
sinh(x)	0.10	-0.00003	0.00000	0.50	-0.00003	0.00000	0.50	0.0000	0.00000
sinh(2x)	0.24	-0.00400	+0.00013	0.50	-0.00400	+0.00013	0.50	-0.0001	+0.00003
sinh(4x)	0.48	+0.00440	+0.00240	0.58	-0.00360	+0.00050	0.56	-0.0010	+0.00028
tan(x)	0.21	-0.00070	+0.00020	0.50	-0.00700	+0.00020	0.50	-0.0002	+0.00004
tan(1.3x)	0.45	-0.00430	+0.00350	0.58	-0.00570	+0.00250	0.56	-0.0023	+0.00040
tan(1.5x)	0.62	+0.03800	+0.02900	0.65	-0.02100	+0.00700	0.63	-0.0100	+0.00700
$x^{1/3}$	-0.50	-0.00800	-0.00310	0.42	-0.00540	+0.00200	0.44	+0.0009	-0.00040

Table 3.3 (Continued)

Nonlinear Characteristics	F	Chebychev		Ultraspherical Jacobi			Shifted Jacobi		
		$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$	μ	$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$	β	$\varepsilon(t)_{max}$	$\varepsilon(\tilde{T}/4)$
tanh(x)	-0.10	-0.00020	+0.00004	0.50	-0.00022	+0.00004	0.50	-0.00009	-0.00002
tanh(2x)	-0.38	-0.00340	-0.00130	0.42	-0.00240	+0.00090	0.44	-0.00030	-0.00012
$x^{1/5}$	-0.67	-0.01100	-0.00500	0.42	-0.00760	+0.00180	0.44	-0.00100	-0.00050

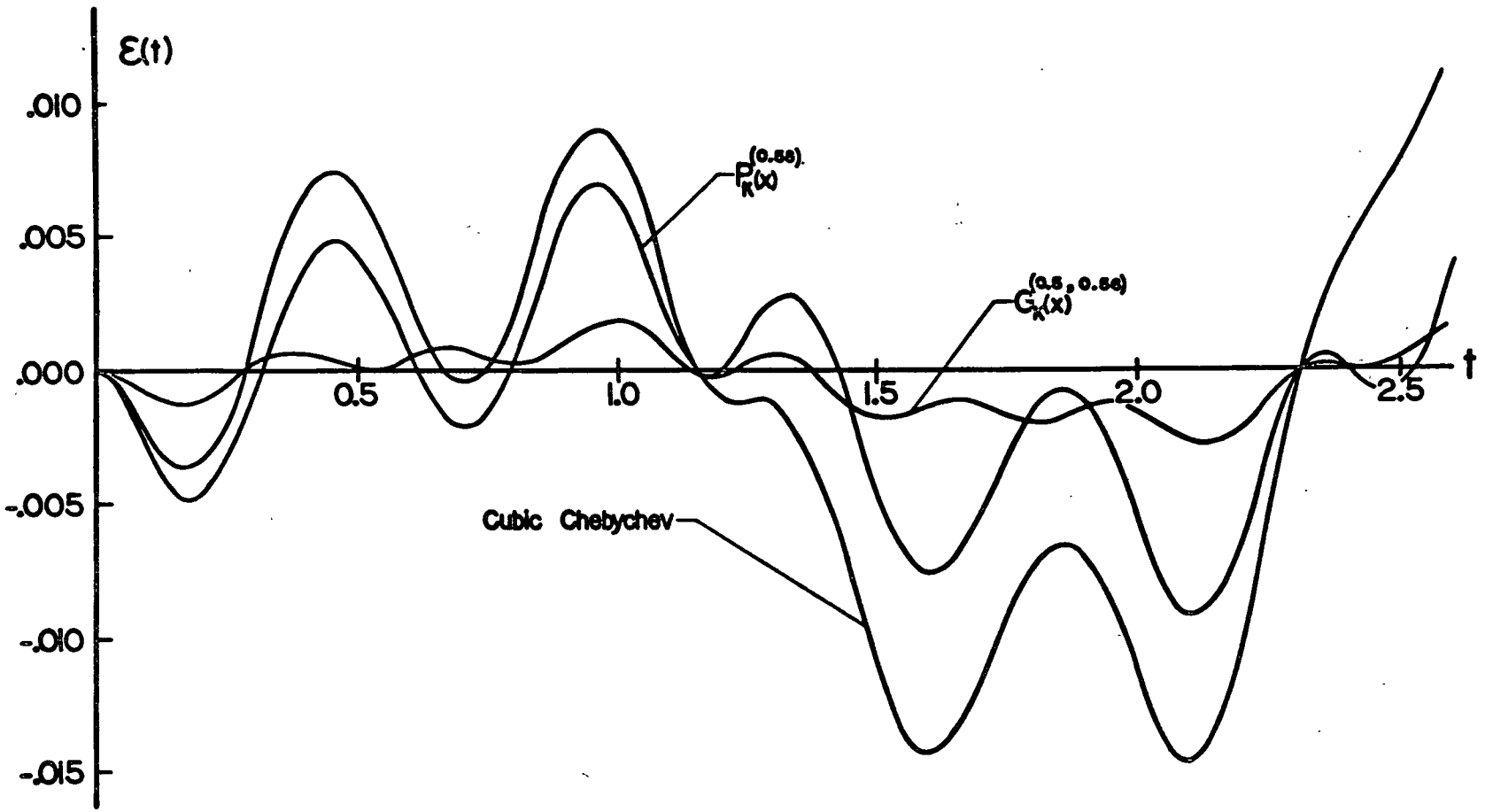


Figure 3.4 Comparison of Solutions to Models Obtained from Chebychev, Ultraspherical Jacobi and Shifted Jacobi Approximations to $\ddot{x} + x + x^3 + 10x^5 = 0$

and $E(x)_{max}$ in equations (2.16) and (2.17), the time solution errors for the cubic function approximations in Table (3.3) are much smaller than errors from the linear approximations in Table (3.2). This is because the cubic is always a closer fit to f(x) than is the straight line in the x - f(x) plane. Weighting the convergence in the x - f(x) plane according to the value of F in Table (3.1) has resulted in improvement over the classical Ritz-averaging and Chebychev polynomial approximations. Particular examination of the refined, shifted Jacobi approximations in Tables (3.2) and (3.3) shows that the error at the approximate quarter period is less than 10% of the maximum error over the first quarter period for most of the examples considered.

3.2 Techniques for Models with Asymmetric Nonlinear Characteristics

When the nonlinear characteristic does not have the symmetry possessed by those considered in paragraph (3.1), the oscillation often has a large dc component. The procedure given in paragraph (2.1.1) permits determination of the range of oscillation of the dependent variable and normalization of the initial conditions to $x(0) = 1$ and $\dot{x}(0) = 0$. The transformation $x = (1-x_{min})w + x_{min}$ is used to transform the range $(1, x_{min})$ to the range $(w = 1, w = 0)$. Otherwise, the weighting of the error in a shifted Jacobi polynomial approximation in the x - f(x) plane will not be controlled. The nonlinear characteristic now has the form $g(w) = f(\left[1 - x_{min}\right]w + x_{min})$. An approximate solution to the model $\ddot{w} + g(w) = 0$ is valid for the

first half period.

Approximations to the characteristics in this asymmetric class have been attempted using the shifted Jacobi polynomials. Some models yield approximate solutions which show improvement over the near-uniform, shifted Chebychev approximations. Notably, the characteristics in these models have terms of even symmetry which are small compared to the amplitude of the odd-symmetric terms. For other characteristics which have significant asymmetries, a common property which would allow nonsymmetrically-weighted approximations to be used has not been found. Thus, near-uniform or near equal-ripple shifted Chebychev linear and cubic expansions are considered for models with asymmetric, nonlinear characteristics.

3.2.1 First-Order or Linear Approximations

The Ritz-Chebychev equivalence has been shown for models with symmetric characteristics. Linear, shifted Chebychev polynomial approximations to equations with asymmetric characteristics normalized to (0,1) give similar results to the two-term Ritz method discussed in paragraph (1.2). The distribution of the error over the full period shown in Figure (3.5) compares these two techniques. Our asymmetric example in this case is $\ddot{x} + x + x^2 + 3x^3 = 0$, and its range of oscillation is ($x = 1$, $x = -1.176$). The two-term Ritz approximation is

$$\ddot{\tilde{x}} + 3.542\tilde{x} + 0.335 = 0,$$

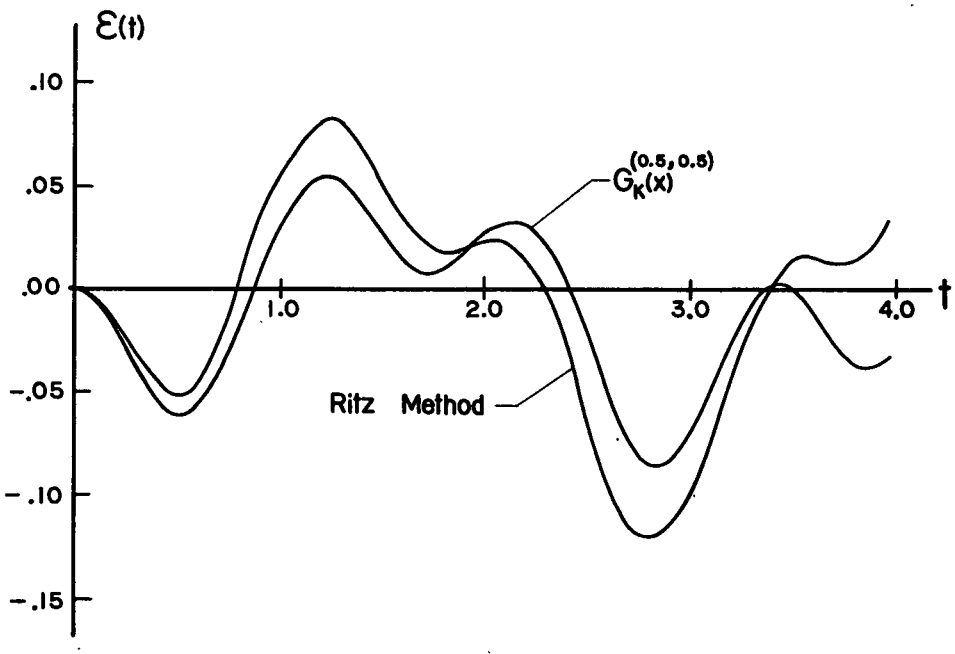


Figure 3.5 Error in the Ritz and Shifted Chebychev Approximations to $\ddot{x} + x + \dot{x}^2 + 3x^3 = 0$

while direct truncation of the nonlinear characteristic in terms of the shifted Chebychev polynomials gives

$$\ddot{\tilde{x}} + 3.512\tilde{x} + 0.328 = 0.$$

In general, the shifted Chebychev polynomial truncation technique is preferred over the two-term Ritz approximation because the predetermination of x_{min} makes the Chebychev approximate solution closer to the numerical half-period than does the Ritz approximation technique. This property may be observed in Figure (3.5). Also, the Ritz method requires the solution of nonlinear, algebraic equations to determine the parameters in the assumed solution.

3.2.2 Cubic Approximations to Asymmetric Models

When the normalized characteristic, g(w), is approximated by a cubic polynomial on (0,1), the techniques given in Appendix A may be used to obtain a closed-form time solution. As an example, consider the approximate step response of a system which is initially at rest and is described by the model

$$\ddot{x} + 10x^5 + x^3 + x = U(t), \qquad x(0) = \dot{x}(0) = 0. \quad (3.4)$$

U(t) is the unit step function. From the initial conditions and the first integral

$$\frac{\dot{x}^2}{2} + 1.6667x^5 + 0.25x^3 + 0.5x^2 - x = 0,$$

one obtains (x = 0, x = 0.7825) as the range of the oscillation. The transformation x = 0.7825 w gives the normalized model

$$\ddot{w} + 3.7492w^5 + 0.6123w^3 + w - 1.278 = 0, \qquad w(0) = \dot{w}(0) = 0,$$

with (w = 0, w = 1) as the range of the dependent variable. Lanczos' Economization of the characteristic in this equation using the shifted Chebychev polynomials on (0,1) plus a shift in the independent variable, defined by $t = \bar{t} + t_s$, produces the approximate equation

$$\frac{d^2\tilde{w}}{d\bar{t}^2} + 11.1569\tilde{w}^3 - 8.7872\tilde{w}^2 + 2.9771\tilde{w} - 1.3490 = 0, \qquad \tilde{w}(0) = 1,$$

$$\dot{\tilde{w}}(0) = 0. \quad (3.5)$$

The first integral of (3.5) is

$$\frac{1}{2}\left(\frac{d\tilde{w}}{d\tilde{t}}\right)^2 + 2.7892\tilde{w}^4 - 2.9290\tilde{w}^3 + 1.4885\tilde{w}^2 - 1.3490\tilde{w} = 0.$$

Integration of this first integral gives

$$\int_0^{\tilde{t}} dt = -\int_1^{\tilde{w}} \frac{d\tilde{w}}{\sqrt{-\tilde{w}(5.5784\tilde{w}^3 - 5.8582\tilde{w}^2 + 2.9771\tilde{w} - 2.6980)}};$$

hence,

$$t' \triangleq \tilde{t} + \tilde{T}/2 = -\int_0^{\tilde{w}} \frac{d\tilde{w}}{\sqrt{-\tilde{w}(5.5784\tilde{w}^3 - 5.8582\tilde{w}^2 + 2.9771\tilde{w} - 2.6980)}} \qquad (3.6)$$

In general $t' \neq t$, but the shifted time, t', is equal to the real time, t, in this special case because the oscillation starts one half period before the first maximum and thus, $t_s = \tilde{T}/2$. The procedure described in Appendix A may be used to transform the integrand in equation (3.6) to the form of the Elliptic Integral of the First Kind. Then a closed-form expression for $\tilde{w}(t)$ may be obtained from Table A according to the parameters in this elliptic integral. In this example, both w = 0 and w = 1 are turning points of the motion, so both zero and one are roots of the polynomial in equation (3.6). After the roots are found, p = 0.3673 and q = -1.3843 are obtained from Appendix A. The transformation w = (p + qy)/(1 + y) applied to equation (3.6) gives

$$1.3484\ dt = \frac{dy}{0.3312\sqrt{(1 - 14.2029y^2)(1 + 4.1273y^2)}}.$$

Choosing $h^2 = 14.2049$ and $g^2 = 4.1273$ so that $c^2 = g^2/h^2 < 1$, and making the transformation $v = hy$, gives the form

$$1.6828\ dt = \frac{dv}{\sqrt{(1 - v^2)(1 + 0.2906v^2)}}, \tag{3.7}$$

which corresponds to entry II in Table A. From equation (3.7) and Table A, the modulus of the elliptic integral is $k = \sqrt{c^2/(1 + c^2)} = 0.4745$, and the complementary modulus is $k' = \sqrt{1 - k^2} = 0.8802$. In the form of the elliptic integral, equation (3.7) becomes

$$1.6828\ t = \int_0^{\phi} 0.8802\ \frac{d\phi}{\Delta\phi}.$$

In this case, to obtain a closed-form time solution, the important function is $\phi(t)$ rather than $t(\phi)$. Under the transformation $v = \cos\phi$ for inversion of the elliptic integral given in Table A, the Jacobian elliptic cosine function, Cn, is defined. That is,

$$\frac{1.6828t}{0.8802} = 1.9117\ t = Cn^{-1}(v).$$

Thus, $v(t) = Cn(k,\omega t) = Cn(0.4745,\ 1.9117t)$. Retracing through the transformations which have been made, the closed-form solution for $\tilde{w}$ is $\tilde{w}(t) = \frac{p + (q/h)v}{1 + v/h}$

$$= \frac{0.3673\left[1 - Cn(0.4745,\ 1.9117t)\right]}{1 + 0.2653\ Cn(0.4745,\ 1.9117t)} . \qquad (3.8)$$

Figure (3.6) compares the closed-form, shifted Chebychev polynomial approximate solution given in equation (3.8) with an

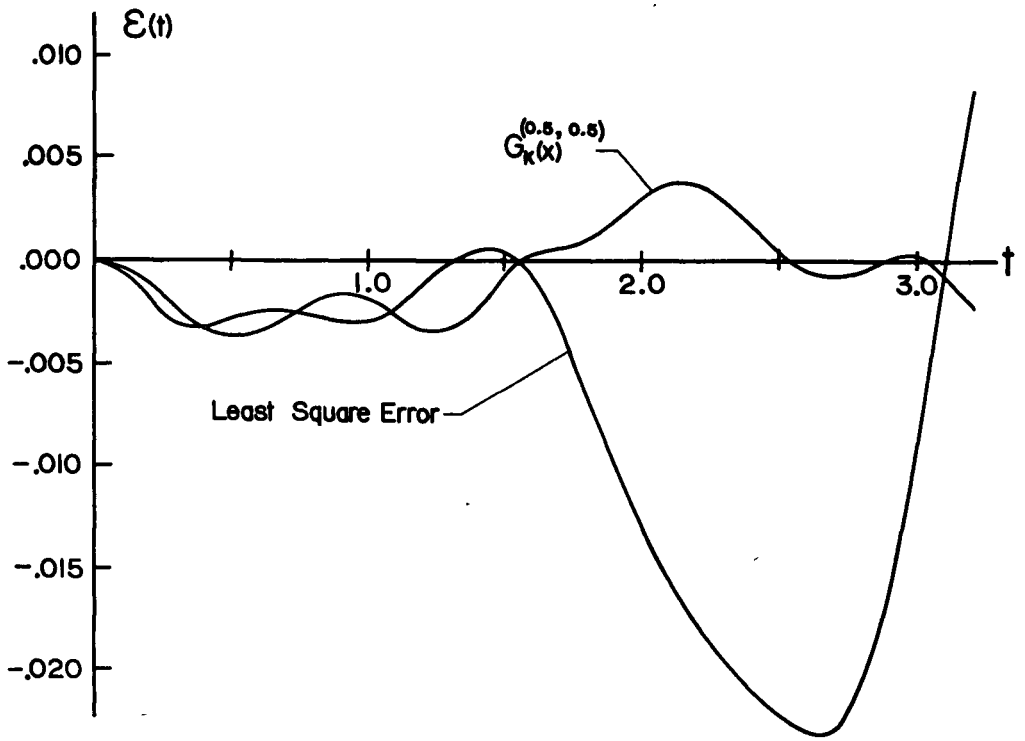


Figure 3.6 The Error Distribution in Shifted Chebychev and Least-Square Error Cubic Approximation to $\ddot{x} + x + x^3 + 10x^5 = U(t)$

approximate solution obtained from a cubic least-square error approximation to the characteristic in this same model. The distribution of the error over the first period of the oscillation is shown in this figure. In comparison, a linear shifted Chebychev approximation to this model gives a maximum error of 0.16 over the first half-period and this grows to -0.26 over the

first period.

The original normalization, $x = 0.7825w$, was performed on equation (3.1) to change the range of oscillation to (0,1). Therefore, from equation (3.8),

$$\tilde{x}(t) = \frac{0.2874\,[1 - Cn(0.4745,\ 1.9117t)]}{1 + 0.2653Cn(0.4745,\ 1.9117t)}$$

is the shifted Chebychev, cubic approximate solution to $\ddot{x} + x + x^3 + 10x^5 = U(t)$, $x(0) = \dot{x}(0) = 0$. This approximate oscillation is on the range (0, 0.7824).

3.3 An Extension of Lanczos' Economization to the Transient Response of Lightly-Damped Models

It has been shown that the amplitude-frequency relation in a conservative, nonlinear oscillation need only be determined over an interval of symmetry. Nonlinear, non-conservative oscillations show a more interesting, continuous change in frequency as the amplitude of the oscillation is damped. Based on the Ritz - K-B equivalence for symmetric, conservative models and the improvement in phase obtained from shifted Jacobi polynomial truncations, an extension of these nonsymmetrically-weighted approximations to models of the form

$$\ddot{x} + 2\delta\dot{x} + f(x) = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0$$

is now undertaken.

The K-B averaging method requires $f(x)$ to have an explicit linear term and imposes criteria for the "lightness" of damping and for the "quasi-linear" nature of the characteristic.

The parameters a(t) and $\theta(t)$, given in Appendix C, for the change in amplitude and phase, respectively, are assumed constant over the first cycle of the oscillation. In this work, the exponential decay predicted by the first-order approximate solution in equation (C.1) of Appendix C is assumed, and a direct piecewise-linearization of the characteristic in terms of the shifted Jacobi polynomials is carried out. No restriction is placed on the change in frequency of the nonlinear oscillation over the first cycle or on the presence of a linear term in f(x). Only f(x) with an odd-symmetric, monotonic-increasing property are considered for the approximation because the nonsymmetrically-weighted approximations have been applied only to this class. Also, the K-B approximation of the form $\tilde{x}(t) = a(t)\cos(\omega_o t + \theta(t))$, applicable to $f(x) = \omega_o^2 x + \mu g(x)$, would not be expected to fit an asymmetric oscillation.

To show the application of the linearization procedure, consider approximate solutions to the equation

$$\ddot{x} + 0.4\dot{x} + x + x^3 + 5x^5 = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0. \tag{3.9}$$

The K-B approximate solution to this equation is

$$\tilde{x}(t) = e^{-0.2t} \cos\left(\sqrt{4.875}\, t\right). \tag{3.10}$$

This solution is valid for the first cycle of the oscillation. As shown in Table (3.2), the characteristic in equation (3.9) has a nonlinear factor F = 0.55. A Lanczos' Economization of this characteristic in terms of the shifted Jacobi polynomials,

$G_k^{(0.5,0.56)}(x)$, on (0,1) yields the linear approximation

$$\ddot{\tilde{x}}_1 + 0.4\dot{\tilde{x}}_1 + 5.782x_1 - 0.900 = 0, \quad \tilde{x}_1(0) = 1, \quad \dot{\tilde{x}}_1(0) = 0.$$

Hence the approximate solution, valid for the first quarter-cycle, is

$$\tilde{x}_1(t) = 0.847e^{-0.2t} \cos (2.40t - 0.083) + 0.153. \tag{3.11}$$

From equation (3.10) an approximation to the first minimum is $x_{min} = -0.76$. A truncation of the characteristic in equation (3.9) on (0, -0.76) in terms of the $G_k^{(0.5,0.56)}(x)$ polynomials yields the approximation

$$\ddot{\tilde{x}}_2 + 0.4\dot{\tilde{x}}_2 + 2.64\tilde{x}_2 + 0.340 = 0, \quad \tilde{x}_2(0.78) = 0,$$

$$\dot{\tilde{x}}_2(0.78) = -1.66.$$

The initial conditions for this equation are obtained from equation (3.11) when $\tilde{x}_1(t) = 0$. The approximate solution for the next half-period on (0, -0.76) is thus

$$\tilde{x}_2(t) = 1.19e^{-0.2t} \cos (1.61t - 2.70) - 0.13. \tag{3.12}$$

The approximate solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are valid for the range $0 \leq t \leq 2.93$ sec. and this is three quarters of the first cycle. Using the amplitude decay predicted by the K-B averaging method, the approximate amplitude after one cycle is $x_{max} = 0.56$. The normalization of equation (3.9) from the range (0, 0.56) to the range (0,1), under the transformation $x = 0.56y$, gives

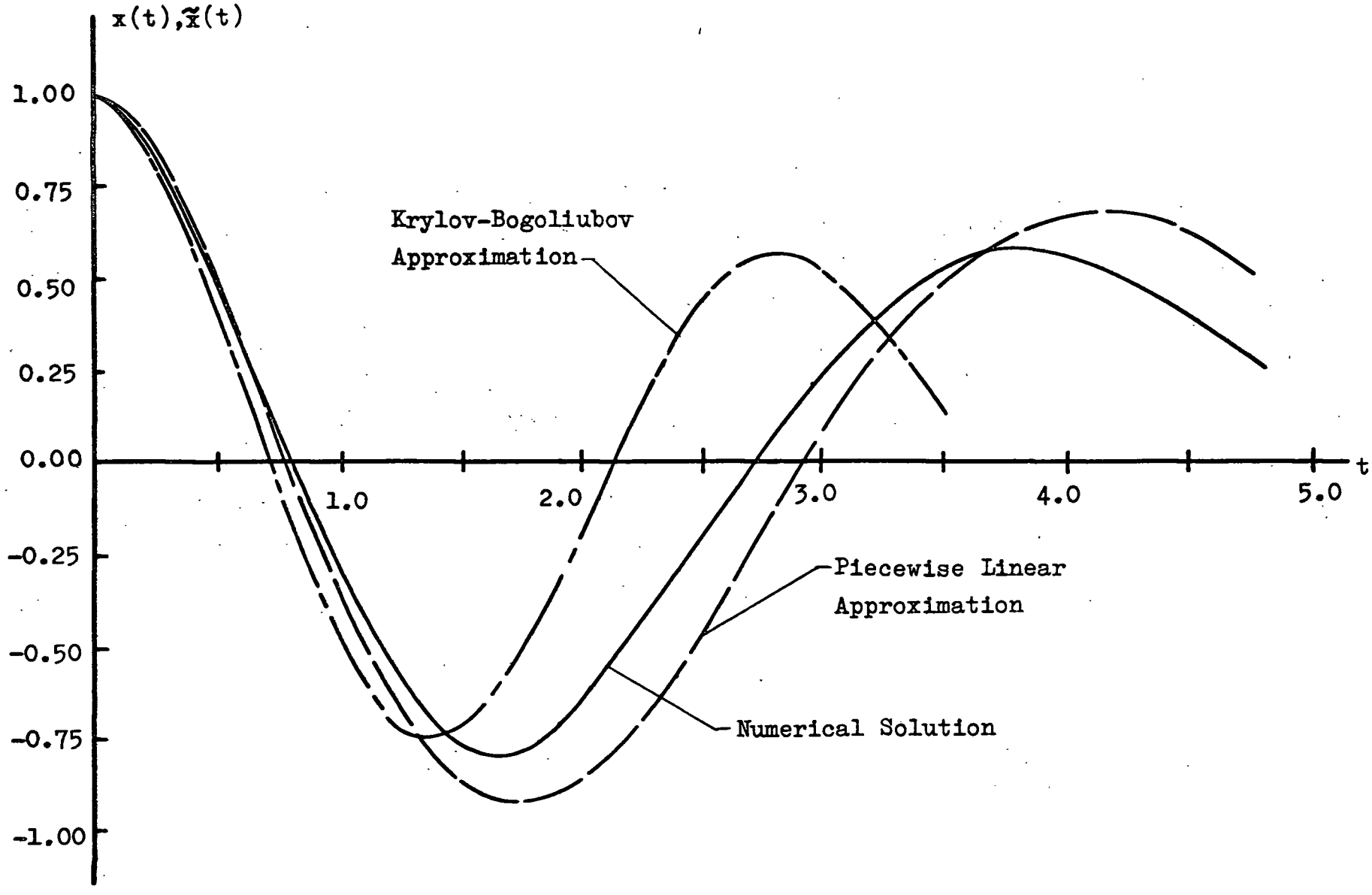


Figure 3.7 Linear Approximations to $\ddot{x} + 0.4\dot{x} + x + x^3 + 5x^5 = 0$

$$\ddot{y} + 0.4\dot{y} + y + 0.314y^3 + 0.492y^5 = 0. \qquad (3.13)$$

As the oscillation is damped, the amplitude of the nonlinear terms decreases and the value of F, which has been defined on (0,1), also decreases. The normalized characteristic in equation (3.13) has F = 0.28. After truncating this characteristic on (0,1) in terms of the $G_k^{(0,5,0.56)}(y)$ polynomials, and then changing back to the range (x = 0, x = 0.56), we obtain

$$\ddot{\tilde{x}}_3 + 0.4\dot{\tilde{x}}_3 + 1.08\tilde{x}_3 - 0.091 = 0, \qquad \tilde{x}_3(2.93) = 0,$$
$$\dot{\tilde{x}}_3(2.93) = 1.00$$

as the third piecewise approximation, valid for the next half cycle. The K-B solution and the above piecewise-linear solution for equation (3.9) are compared in Figure (3.7).

In Figure (3.8), piecewise-linear approximations to

$$\ddot{x} + 0.2\dot{x} + \tanh(2x) = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0,$$

have been carried out in terms of the shifted Jacobi polynomials, $G_k^{(0.5,0.44)}(x)$. The characteristic tanh(x) arises physically as a model for nonlinear barium titinate capacitor characteristics [22]. For this equation the K-B method does not apply because an explicit linear term is not present. Instead, Lanczos' Economization of a least-square error, seventh-degree polynomial approximation to tanh(2x) is carried out for each linearization. To get an approximation to the amplitude at each half cycle of this damped oscillation, a linear Chebychev polynomial truncation is made on (-1,1), and then the solution of the resulting

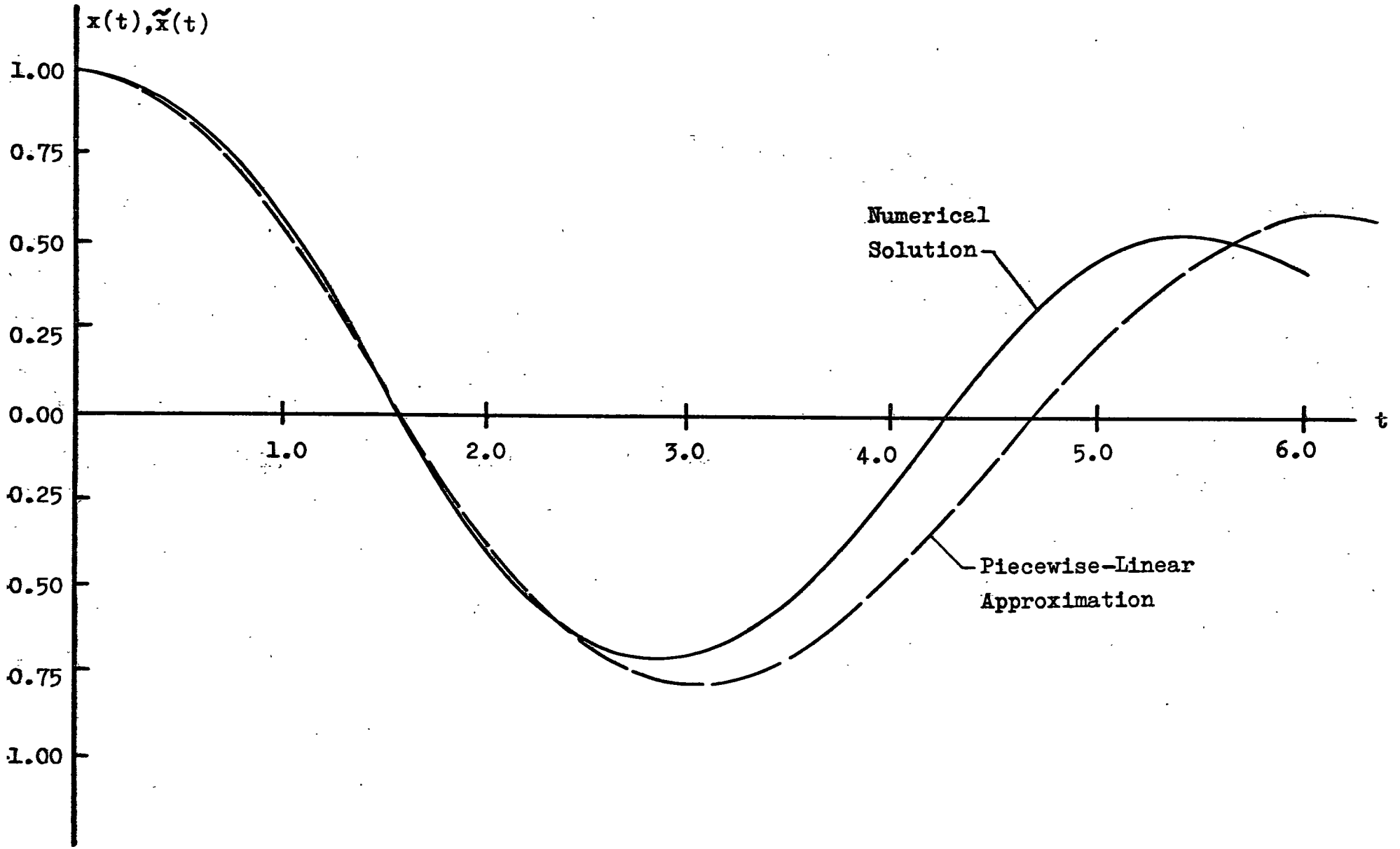


Figure 3.8 The Piecewise-Linear Approximation of
$\ddot{x} + 0.2\dot{x} + \tanh(2x) = 0$

approximation is used to determine x_{min}. This technique gives an approximate amplitude of $x_{min} = -0.717$ after one half-cycle and an amplitude of $x_{max} = 0.538$ after a full cycle. Linear truncations in terms of the shifted Jacobi polynomials, $G_k^{(0.5,\ 0.44)}(x)$, are then made for the first quarter-cycle and for each successive half cycle. The procedure is the same as for the example in equation (3.9). Figure (3.8) compares the numerical and approximate solution.

In the above two examples, the first piecewise-linearization is over only the first quarter cycle of the oscillation. This is useful because the nonlinear terms in the characteristic have a large magnitude where the amplitude of the oscillation is greatest, and thus a small approximation interval is important. The piecewise-linearization also allows different, weighted approximations to be made as the oscillation damps out.

The possibility of using cubic truncations to improve the approximation has not been explored. Only a first-order approximation to the decay in amplitude of the nonlinear oscillation has been found. Therefore the Jacobian elliptic functions are not considered in the approximation of non-conservative oscillations.

3.4 Discussion and Possible Extension of the Results

An advantage of the orthogonal polynomial truncation techniques given in this chapter is that they are generally applicable to a wide range of both conservative and non-conservative models for common physical systems. Also, the

refinement of the approximate model from a linear one to a cubic one is direct, and the same techniques are employed for symmetric, asymmetric and non-conservative models. In contrast, the classical averaging techniques refer indirectly to the residual error obtained when the assumed solution is substituted into the model. Refined solutions are also difficult to obtain from the classical techniques.

Application of polynomial truncation to first-order differential equations was considered. In some cases, weighted, Jacobi approximate solutions, improved over those obtained by near-uniform expansion of the characteristic, have been found but no relation to the weighting of the truncated approximation to a second-order model has been noted. This is because the same arguments for a small error at the end of an interval of symmetry do not apply to first-order systems. For this reason, approximation of first-order systems has not been studied in this work.

Extension of this work to second-order models driven by harmonic time-functions is also possible. Results obtained by Klotter[23] show that the backbone of the nonlinear amplitude-frequency response curves is obtained by setting the driving term to zero. Hence, approximations to the backbone curve, improved over the one-term Ritz method, may be obtained directly from the results of this work. The arguments for the use of symmetry and the insight gained into weighting the approximations in this work could be investigated for the symmetric, steady-state response. Also, the error criterion on the error at the

end of an interval of symmetry is important for the steady-state approximation. Near resonance, where the amplitude of the oscillation is large, the effect of the nonlinear terms is important. For lightly-damped models, the amplitude of the steady-state response approaches the backbone curve near resonance. Thus, an improvement over the classical one-term Ritz method could, perhaps, be significant using different weighting in Jacobi polynomial approximations.

4. CONCLUSIONS

Piecewise-linear and piecewise-cubic approximations, from which analytical solutions to classes of second-order nonlinear differential equations may be obtained, have been developed in this work. Initially, criteria for the error in the approximate time solutions were imposed. The introduction of Jacobi and shifted Jacobi polynomials has given a flexibility to the approximation techniques which is not possessed by the classical approximation methods.

Error bounds were then given which prove that the maximum relative error in the solution to an approximate Jacobi or shifted Jacobi model is bounded, no matter how large the coefficients in the original nonlinear model become. An empirical measure of the departure from linearity, based on these error bound results, permitted approximations to be made using the shifted Jacobi polynomials, $G_k^{(0.5,\ \beta)}(x)$. Approximate time solutions for which the relative error at the approximate quarter period is of the order of 10% of the maximum relative error over the quarter period have been obtained. Thus, it has been possible to obtain quantitative approximations to the nonlinear amplitude-frequency relationship. Unrefined approximations using the ultraspherical Jacobi polynomials allowed closed-form solutions to be written for symmetric, cubic models by inspection.

To provide a general approach for the approximation of asymmetric, conservative models the shifted Chebychev polynomials, $G_k^{(0.5,0.5)}(x)$, were employed. This approximation

technique was found to provide improved solutions over the one-term Ritz method in the first-order approximation case, and over the two-term Ritz method and least-square error approximation in the cubic case.

Improvement in the amplitude-frequency approximation obtained from the K-B method has been shown possible for second-order nonlinear models with light, viscous damping. Again the $G_k^{(0.5,\beta)}(x)$ polynomials have been used for a direct, piecewise-linearization of the characteristic in the nonlinear model.

The polynomial truncation techniques have provided a direct attack on the approximation problem. The results which have been obtained using these techniques show improvement over the classical averaging methods with a decrease in tedious labor.

APPENDIX A

The Closed Form Solution of Second-Order, Conservative System Models with Cubic, Nonlinear Characteristics

For the special case in which the cubic characteristic has zero-point symmetry, Soudack[24] has derived closed-form solutions in terms of the Jacobian elliptic functions. The cases of interest are:

(i) $\ddot{x} + ax - bx^3 = 0, \quad x(0) = X_o, \quad \dot{x}(0) = 0, \quad a,b>0.$

The solution is $x(t) = X_o Sn(k, \omega t + K(k))$ for $X_o \leq \sqrt{\frac{a}{b}}$, where $\omega^2 = a - 0.5\, bX_o^2$ and $k^2 = bX_o^2/(2a - bX_o^2)$.

(ii) $\ddot{x} + ax + bx^3 = 0, \quad x(0) = X_o, \quad \dot{x}(0) = 0, \quad a,b<0.$

The solution is $x(t) = X_o Cn(k, \omega t)$, where $\omega^2 = a + bX_o^2$ and $k^2 = bX_o^2/2(a + bX_o^2)$.

(iii) $\ddot{x} - ax + bx^3 = 0, \quad x(0) = X_o, \quad \dot{x}(0) = 0, \quad a,b> 0.$

An oscillatory solution, symmetric in the origin, is obtained in the form of case (ii) when a is small. For $a<bX_o^2/2$, the solution is $x(t) = X_o Cn(k, \omega t)$, where $\omega^2 = -a + bX_o^2$ and $k^2 = bX_o^2/2(-a + bX_o^2)$. An oscillatory solution for larger a has the form $x(t) = X_o Dn(k, \omega t)$, where $k^2 = 2(1 - a/bX_o^2)$ and $\omega^2 = bX_o^2/2$. The conditions on the initial amplitude are $0<X_o<\sqrt{2a/b}$ and $X_o \neq \sqrt{a/b}$.

When the nonlinear characteristic does not have the above symmetry the closed-form solution may not be written by

inspection. The system model now has the form

$$\frac{d^2w}{d\bar{t}^2} + b_o + b_1 w + b_2 w^2 + b_3 w^3 = 0, \quad w(0) = 1, \quad \dot{w}(0) = 0. \tag{A.1}$$

The range of oscillation of w in this model will always be (0,1) after the normalizations described in paragraphs (2.1.1) and (3.2) are carried out. From equation (2.1), the first integral of the motion is

$$\frac{1}{2}\left(\frac{dw}{d\bar{t}}\right)^2 = E_t - \left(b_o w + b_1 \frac{w^2}{2} + b_2 \frac{w^3}{3} + b_3 \frac{w^4}{4}\right) \tag{A.2}$$

$$\triangleq E_t - V(w).$$

The normalized initial conditions of (A.1) give $E_t = V(1)$, and hence

$$\frac{dw}{d\bar{t}} = -\sqrt{2\left[V(1) - V(w)\right]} \tag{A.3}$$

from (A.2). The minus sign is chosen from physical considerations because w decreases as t increases from zero for this bounded oscillation. Integration of (A.3) gives

$$\int_0^{\bar{t}} dt = -\int_1^w \frac{dw}{\sqrt{2\left[V(1) - V(w)\right]}}, \quad 0 \leq w \leq 1 \tag{A.4}$$

or

$$t' = \bar{t} + \Delta\bar{t} = -\int_0^w \frac{dw}{\sqrt{2\left[V(1) - V(w)\right]}}, \quad 0 \leq w \leq 1, \tag{A.5}$$

where

$$\Delta\bar{t} = -\int_0^1 \frac{dw}{\sqrt{2\left[V(1) - V(w)\right]}} \cdot \qquad (A.6)$$

The quantity, $\Delta\bar{t}$, obtained by dividing the range of integration in equation (A.4), is either one quarter period or one half period depending on the symmetry of the problem being considered. Equation (A.6) is a complete elliptic integral of the first kind. A technique given by Hancock[25] allows the integrand in (A.5) to be transformed to the Jacobian elliptic Integral of the first kind using Legendre's transformation, and then a closed-form expression for $w(t')$ may be obtained. Evaluation of (A.6) is not necessary because the quarter period or half period may be obtained from $w(t')$.

The integrand in (A.5) may be written in factored form as

$$\frac{dw}{\sqrt{W}} = \frac{dw}{\sqrt{\frac{b_3}{4}\left[(x-\underline{\theta})(x-\underline{\lambda})(x-\underline{\mu})(x-\underline{\pi})\right]}} \cdot \qquad (A.7)$$

It is clear from (A.2) and (A.3) that $x = 1$ is a root of the denominator polynomial in (A.7). Hence, this polynomial can always be reduced to a cubic and the roots may be found. The roots in (A.7) are ordered so that $\underline{\theta} > \underline{\lambda} > \underline{\mu} > \underline{\pi}$ with the real roots ordered first. Legendre's transformation is

$$w = (p + qy)/(1 + y),$$

where

$$\frac{p + q}{2} = \frac{\underline{\theta}\,\underline{\lambda} - \underline{\mu}}{\underline{\theta} + \underline{\lambda} - \underline{\mu} - \underline{\pi}} \quad \text{and}$$

$$pq = \frac{\underline{\theta}\underline{\lambda}(\underline{\mu} + \underline{\pi}) - \underline{\mu}\underline{\pi}(\underline{\theta} + \underline{\lambda})}{\underline{\theta} + \underline{\lambda} - \underline{\mu} - \underline{\pi}}.$$

Under this transformation

$$\frac{dw}{\sqrt{W}} = \frac{(q - p)}{\sqrt{Y}} = \frac{dy}{\sqrt{(\pm m^2y^2 \pm n^2y^2)(\pm r^2 \pm s^2y^2)}} \qquad (A.8)$$

is obtained, where

$$m^2 = (p - \underline{\theta})(p - \underline{\lambda}), \qquad n^2 = (q - \underline{\theta})(q - \underline{\lambda}),$$
$$r^2 = (p - \underline{\mu})(p - \underline{\pi}), \qquad s^2 = (q - \underline{\mu})(q - \underline{\pi}).$$

For the special case in which $\underline{\theta} + \underline{\lambda} = \underline{\mu} + \underline{\pi}$ the transformation $w = y + (\underline{\theta} + \underline{\lambda})/2 = y + (\underline{\mu} + \underline{\pi})/2$ is used. It is shown by Hancock that $p + q$ and pq are always real. From equation (A.7), equation (A.5) may now be written in the form

$$dt' = \frac{(q - p)dy}{mr\sqrt{\pm(1 \pm g^2y^2)(1 \pm h^2y^2)}}.$$

On the right hand side of this expression, h and g are defined so that $h > g$ and then the integrand may be further reduced to

$$dt' = \frac{(q - p)dv}{mrh\sqrt{\pm(1 \pm \frac{g^2}{h^2}v^2)(1 \pm v^2)}}$$

$$= \frac{1}{N}\frac{dv}{\sqrt{\pm(1 \pm c^2v^2)(1 \pm v^2)}}, \qquad (A.9)$$

where $v = hy$ and $c^2 = g^2/h^2 < 1$. Hancock shows that N is always real. It is also shown that of the eight possible combinations of sign under the radical in equation (A.9), $\sqrt{-(1 + v^2)(1 + c^2v^2)}$

may be neglected, since W, which is positive for some of the original w, cannot be transformed to a function which is always negative by a real substitution.

Using Table A below, closed-form expressions may be obtained for $v(t')$ according to each of the seven possible sign combinations in equation (A.9). In this table $\Delta\phi = \sqrt{1 - k^2\sin^2\phi}$, where k and ϕ are defined as the modulus and the amplitude of t' respectively. The trigonometric substitutions in Table A define the Jacobian elliptic functions. The quantity k is defined as the modulus of the elliptic function, and ϕ is defined as the amplitude of the elliptic function. For example, the substitution $v = \tan\phi \triangleq Tn(Nt')$ $= Tn(k,Nt')$ for entry I in Table A defines the Jacobian elliptic tangent function from

$$t' = \frac{1}{N}\int^{v} \frac{dv}{\sqrt{(1 + v^2)(1 + c^2v^2)}}.$$

The complementary modulus is defined to be $k' = \sqrt{1 - k^2}$. Entries VI and VIa in Table A have the same form. In VI, $v \leq 1$, while in VIa, $v \geq \frac{1}{c}$. The elliptic integrals of the first kind and the elliptic functions are obtainable from works such as the "Smithsonian Elliptic Function Tables".[26]

TABLE A

Standard Forms for Jacobian Elliptic Integrals of the First Kind

I	$\frac{dv}{\sqrt{(1+v^2)(1+c^2v^2)}} = \frac{d\phi}{\Delta\phi}$	$v = \tan\phi$	$k^2 = 1-c^2$
II	$\frac{dv}{\sqrt{(1-v^2)(1+c^2v^2)}} = \frac{-k'd\phi}{\Delta\phi}$	$v = \cos\phi$	$k^2 = \frac{c^2}{1+c^2}$
III	$\frac{dv}{\sqrt{(v^2-1)(1+c^2v^2)}} = \frac{kd\phi}{\Delta\phi}$	$v = \sec\phi$	$k^2 = \frac{1}{1+c^2}$
IV	$\frac{dv}{\sqrt{(1+v^2)(1-c^2v^2)}} = \frac{-kd\phi}{\Delta\phi}$	$v = \frac{\cos\phi}{c}$	$k^2 = \frac{1}{1+c^2}$
V	$\frac{dv}{\sqrt{(1+v^2)(c^2v^2-1)}} = \frac{k'd\phi}{\Delta\phi}$	$v = \frac{\sec\phi}{c}$	$k^2 = \frac{c^2}{1+c^2}$
VI	$\frac{dv}{\sqrt{(1-v^2)(1-c^2v^2)}} = \frac{d\phi}{\Delta\phi}$	$v = \sin\phi$	$k^2 = c^2$
VIa	$\frac{dv}{\sqrt{(v^2-1)(c^2v^2-1)}} = \frac{-d\phi}{\Delta\phi}$	$v = \frac{1}{c\sin\phi}$	$k^2 = c^2$
VII	$\frac{dv}{\sqrt{(v^2-1)(1-c^2v^2)}} = \frac{-d\phi}{\Delta\phi}$	$v^2 = \sin^2\phi + \frac{\cos^2\phi}{c^2}$ $k^2 = 1-c^2$	$k^2 = 1-c^2$

APPENDIX B

Derivation of the Jacobi and Shifted Jacobi Polynomials

Lanczos[27] gives a development of the Jacobi polynomials from the hypergeometric series

$$F(\lambda,\theta,a;x) = 1 + \frac{\lambda\theta}{\alpha}x + \frac{\lambda(\lambda+1)\theta(\theta+1)}{\alpha(\alpha+1)\cdot 1\cdot 2}x^2$$

$$+ \frac{\lambda(\lambda+1)(\lambda+2)\theta(\theta+1)(\theta+2)}{\alpha(\alpha+1)(\alpha+2)\cdot 1\cdot 2\cdot 3}x^3 + \ldots.$$

This series terminates with the power x^n if $\lambda = -n$. The choice of $\theta = n + \alpha + \beta - 1$ yields the set of orthogonal polynomials

$$G_n'^{(\alpha,\beta)}(x) = F(-n,\ n+\alpha+\beta-1,\ \alpha\ ;\ x),$$

which are orthogonal on (0,1) with respect to the weight factor $W(x) = x^{\alpha-1}(1-x)^{\beta-1}$. In this study the shifted Jacobi polynomials have been formed by standardizing the above polynomials so that $G_n^{(\alpha,\beta)}(1) = 1$. Defining the quantities $\theta_n = n + \alpha + \beta - 1$, $\beta_n = \beta + n$ and $\alpha_n = \alpha + n$, the first five shifted Jacobi polynomials may be written

$$G_0(x) = 1$$

$$G_1(x) = \frac{\alpha}{\beta}\left[-1 + \frac{\theta_1}{\alpha}x\right]$$

$$G_2(x) = \frac{\alpha\alpha_1}{\beta\beta_1}\left[1 - \frac{2\theta_2}{\alpha}x + \frac{\theta_2\theta_3}{\alpha\alpha_1}x^2\right]$$

$$G_3(x) = \frac{\alpha\alpha_1\alpha_2}{\beta\beta_1\beta_2}\left[-1 + \frac{3\theta_3}{\alpha}x - \frac{3\theta_3\theta_4}{\alpha\alpha_1}x^2 + \frac{\theta_3\theta_4\theta_5}{\alpha\alpha_1\alpha_2}x^3\right]$$

$$G_4(x) = \frac{\alpha\alpha_1\alpha_2\alpha_3}{\beta\beta_1\beta_2\beta_3}\left[1 - \frac{4\theta_4}{\alpha}x + \frac{6\theta_4\theta_5}{\alpha\alpha_1}x^2 - \frac{4\theta_4\theta_5\theta_6}{\alpha\alpha_1\alpha_2}x^3 + \frac{\theta_4\theta_5\theta_6\theta_7}{\alpha\alpha_1\alpha_2\alpha_3}x^4\right].$$

Some special cases of the above shifted Jacobi polynomials are: $\alpha = \beta = 0.5$, shifted Chebychev polynomials; $\alpha = \beta = 1.0$, shifted Legendre polynomials.

The ultraspherical Jacobi polynomials, orthogonal on (1,-1) with respect to the weight factor $(1 - x^2)^{\mu-1}$, are obtained from the hypergeometric series

$$P_n'^{(\mu)}(x) = F(-n, n + 2\mu - 1, \mu; \frac{1 - x}{2}).$$

The first eight ultraspherical Jacobi polynomials, standardized so that $P_n^{(\mu)}(1) = 1$, are:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2\mu}\left[x^2(2\mu + 1) - 1\right]$$

$$P_3(x) = \frac{1}{2\mu}\left[(2\mu + 3)x^3 - 3x\right]$$

$$P_4(x) = \frac{1}{4\mu(\mu + 1)}\left[(2\mu + 5)(2\mu + 3)x^4 - 6(2\mu + 3)x^2 + 3\right]$$

$$P_5(x) = \frac{1}{4\mu(\mu + 1)}\left[(2\mu + 7)(2\mu + 5)x^5 - 10(2\mu + 5)x^3 + 15x\right]$$

$$P_6(x) = \frac{1}{8\mu(\mu + 1)(\mu + 2)}\left[(2\mu + 9)(2\mu + 7)(2\mu + 5)x^6 - 15(2\mu + 7)(2\mu + 5)x^4 + 45(2\mu + 5)x^2 - 15\right]$$

$$P_7(x) = \frac{1}{8\mu(\mu + 1)(\mu + 2)}\left[(2\mu + 11)(2\mu + 9)(2\mu + 7)x^7 - 21(2\mu + 9)(2\mu + 7)x^5 + 105(2\mu + 7)x^3 - 105\,x\right].$$

Some special cases of these ultraspherical Jacobi polynomials are: $\mu = 0.5$, Chebychev polynomials; $\mu = 1.0$, Legendre polynomials; and $\mu = 1.5$, Chebychev polynomials of the second kind.

APPENDIX C

The Krylov-Bogoliubov Approximation; its Equivalence with the One-Term Ritz Method and with a Linear Expansion in Ultraspherical Chebychev Polynomials.

The method of Krylov and Bogoliubov or the K-B approximation applies to the second order model $\ddot{x} + \omega_o^2 x + \mu f(x, \dot{x}) = 0$, where μ is a "small" parameter. For the model $\ddot{x} + \omega_o^2 x + \mu \left[2\delta\dot{x} + g(x)\right] = 0$, the K-B approximation may be assumed in the form $\tilde{x}(t) = a(t) \cos(\omega_o t + \theta(t)) \triangleq a(t) \cos \alpha(t)$. Cunningham[28] shows the evaluation of $a(t)$ and $\theta(t)$ by averaging over one cycle of the oscillation. The result is

$$a(t) = A_o e^{-\mu\delta t} \tag{C.1}$$

and

$$\dot{\theta}(t) = \frac{\mu}{2\pi\omega_o a} \int_0^{2\pi} g(a \cos \alpha) \cos\alpha \, d\alpha. \tag{C.2}$$

For the case in which the oscillations are damped, the K-B method applies only to models with an explicit linear term. Consider the special case in which the damping term is zero and the model takes the form $\ddot{x} + \omega_o^2 x + \mu g(x) = 0$. Then

$$\frac{d\alpha}{dt} = \omega_o + \frac{\mu}{2\pi\omega_o a} \int_0^{2\pi} g(a \cos \alpha) \cos\alpha d\alpha \triangleq \omega(a). \tag{C.3}$$

Integrating this expression for the phase, one obtains $\alpha(t) = \omega(a)t + C$. If $\dot{x}(0) = 0$ in the model, then $C = 0$.

From (C.3) we have

$$\omega^2(a) = \omega_o^2 + \frac{\mu}{\pi a} \int_0^{2\pi} g(a\cos\alpha) \cos\alpha d\alpha,$$

where second-order terms in μ are neglected. Thus,

$$\omega^2(a) = \frac{1}{\pi a} \left[\omega_o^2 a \int_0^{2\pi} \cos^2\alpha d\alpha + \mu \int_0^{2\pi} g(a\cos\alpha)\cos\alpha d\alpha \right]$$

$$= \frac{1}{\pi a} \int_0^{2\pi} \left[\omega_o^2 a\cos\alpha + \mu g(a\cos\alpha) \right] \cos\alpha d\alpha$$

$$= \frac{1}{\pi a} \int_0^{2\pi} F(a\cos\alpha)\cos\alpha d\alpha,$$

where $F(x) = \omega_o^2 x + \mu g(x)$.

This expression thus determines the approximate frequency, $\omega(a)$, for a nonlinear model $\ddot{x} + F(x) = 0$.

For the case in which $f(x)$ has odd-symmetry in $\ddot{x} + f(x) = 0$, $x(0) = 1$, $\dot{x}(0) = 0$, Soudack[29] has shown that a one-term Ritz approximation is equivalent to a linear expansion in terms of the Chebychev polynomials. The Ritz conditions give

$$\omega^2 = \frac{1}{\pi} \int_0^{2\pi} f(\cos\theta)\cos\theta d\theta. \qquad (C.5)$$

Similarly a Chebychev polynomial expansion gives

$$\omega^2 = \frac{2}{\pi} \int_{-1}^{1} x\, f(x)(1 - x^2)^{(-0.5)} dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} f(\cos\theta)\cos\theta\, d\theta \qquad (C.6)$$

for an odd $f(x)$ under the substitution $x = \cos\theta$. The one-term Ritz and Chebychev approximation techniques are applicable to models more general than those restricted to have an explicit linear term by the K-B approximation. Nevertheless, the expansions in (C.4), (C.5) and (C.6) show that, for conservative models to which the K-B method applies, the same approximate solution is obtained from the K-B method and the Ritz method.

REFERENCES

1. Hayashi, C.H., Nonlinear Oscillations in Physical Systems , McGraw-Hill Book Co., New York, 1964, pp. 158-160.

2. Goldstein, H., Classical Mechanics , Addison-Wesley Publishing Co., Reading, Mass., 1950, pp. 58-59.

3. Pipes, L.A., Operational Methods in Nonlinear Mechanics , Dover Publications, New York, 1965, p. 7.

4. Rosenberg, R.M., "The Ateb (H) - Functions and Their Properties", Quarterly of Applied Mathematics, Vol. XXI, No. 1, April, 1963.

5. Cunningham, W.J., Introduction to Nonlinear Analysis , McGraw-Hill Book Co., New York, 1958, pp. 123-124.

6. Minorsky, N., Introduction to Nonlinear Mechanics , J.W. Edwards, Ann Arbor, Michigan, 1947, pp. 186-190.

7. Cunningham, W.J., op. cit., p. 157.

8. Cunningham, W.J., op. cit., p. 164.

9. Denman, H.H., and King Lui, Y., "Applications of Ultraspherical Polynomials to Nonlinear Oscillations II. Free Oscillations", Quarterly of Applied Mathematics, Vol. XXII, No. 4, January, 1965.

10. Soudack, A.C., "Jacobian Elliptic and Other Functions as Approximate Solutions to a Class of Grossly Nonlinear Differential Equations", Tech. Report No. 2054-1, Stanford Electronics Laboratories, Stanford University, Stanford, California, April 24, 1961.

11. Cunningham, W.J., op. cit., pp. 114-117.

12. Fröberg, C.E., Introduction to Numerical Analysis , Addison-Wesley Publishing Co. Inc., 1965, p. 245.

13. Soudack, A.C., op. cit., p. 41.

14. Lanczos, C., Applied Analysis , Prentice Hall Inc., Englewood Cliffs, N.J., 1961, p. 451.

15. Davis, P.J., Interpolation and Approximation , Blaisdell Publishing Co., New York, 1963, p. 174.

16. Lanczos, C., op. cit., pp. 457-463.

17. Soudack, A.C., op. cit., pp. 50-52.

18. Rainville, E., Elementary Differential Equations , The Macmillan Co., New York, 1958, p. 169.

19. Soudack, A.C., op. cit., p. 52.

20. Soudack, A.C., op. cit., p. 53-55.

21. Denman, H.H., and King Lui, Y., op. cit.

22. Weber, E., "Introduction, Nonlinear Physical Phenomena," Proceedings of the Symposium on Nonlinear Circuit Analysis, Vol. II, Polytechnic Press of the Polytechnic Institute of Brooklyn, New York, April 23, 1953.

23. Klotter, K., "Steady State Vibrations in Systems Having Arbitrary Restoring and Arbitrary Damping Forces", Proceedings of the Symposium on Nonlinear Circuit Analysis, Vol. II, Polytechnic Press of the Polytechnic Institute of Brooklyn, New York, April 23, 1953.

24. Soudack, A.C., "Some Nonlinear Differential Equations Satisfied by the Jacobian Elliptic Functions", Mathematics Magazine, Vol. 37, No. 3, May, 1964.

25. Hancock, H., Elliptic Integrals , Dover Publications Inc., New York, 1958, pp. 10-13.

26. Spenceley G.W., and Spenceley, R.M., Smithsonian Elliptic Function Tables , Smithsonian Institution, Washington, D.C., 1947.

27. Lanczos, C., op. cit., pp. 367-370.

28. Cunningham, W.J., op. cit., pp. 135-137.

29. Soudack, A.C., "Equivalence of the One-term Ritz Approximation and Linearization in a Chebychev Sense", American Mathematical Monthly, Vol. 72, No. 2, February, 1965.