ON THE ABSOLUTE STABILITY OF NONLINEAR CONTROL SYSTEMS

by

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ABSTRACT

Sufficient conditions for the absolute stability of a class of non-linear sampled-data systems are derived using the techniques of system transformations due to Aizerman and Gantmacher. These criteria are based on different forms used to approximate the area under the nonlinear characteristic. It is also shown that the stability criterion can be improved by relaxing a restriction on the slope of nonlinear element.

In multiple nonlinearily continuous systems, by the application of numerical techniques, it was shown that some improvement over previous stability bounds can be made.

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1. INTRODUCTION

In recent years much attention has been given to the consideration of the absolute stability of nonlinear control systems. The most fruitful approaches to the derivation of stability criteria have been Liapunov's Second Method and Popov's frequency domain method.

Before the publication of V.M. Popov's paper [1] the investigation of nonlinear system was usually based on the Liapunov function of Lurie type [2] - consisting of a quadratic form plus an integral of the nonlinearity. This second method was extended to discrete systems by Kalman and Bertram [3]. Although the method yields sufficient conditions for system stability and is general in application, nevertheless it is sensitive to the choice of suitable functions.

In 1961, Popov [1] developed an entirely new approach to the classical problem of absolute stability and obtained a number of very powerful results. One of the main advantages of Popov's method is that a stability criterion is obtained in terms of the frequency response of the linear part of the system, similar in many respects to the Nyquist stability criterion for linear systems. Kalman [4] obtained an equivalent result and showed that the frequency stability criterion derived through Popov's approach is a necessary and sufficient condition for the existance of a Liapunov function of the Lurie type.

Recently, Tsypkin [5] and Jury and Lee [6], [7] used the Z-transform method to extend Popov's method to a class of sampled-data systems with a single nonlinearity. Szegö and Pearson [8] obtained essentially the same result using the Liapunov method. Jury and Lee [9] then extended their results [6], [7] further to multivariable systems, and Anderson [10] has used the Liapunov method for multivariable continuous systems.

In chapter II, two sufficient conditions for the absolute stability of nonlinear sampled-data systems are derived. Although these criteria are indentical to the results of Jury and Lee [6], [7], the derivation, using the technique of system transformations of Aizerman and Gantmacher [11], is more straightforward. Further, an example of the application of the theorem, when compared with previous work, [6], [8], is more general.

In chapter III, the constraints on the slope of the nonlinearity of Theorem I are relaxed. The theorem for absolute stability of nonlinear sampled-data systems is proved using the same method applied in chapter II.

Several examples illustrating the best choice of the free parameters, q_i, and maximum gains, k_i, of the multi-variable nonlinear continuous systems are considered in chapter IV. In the general case, since the evaluation of the principal minors of criteria matrices is very complex, some simplification is necessary in order to reduce the number of variable parameters.

2. THE STABILITY OF NONLINEAR SAMPLED-DATA SYSTEMS

2.1 Description of System

Consider the single input-single output sampled-data feedback system S shown in Figure 2.1. It consists of a memoryless nonlinearity N and a linear, time-invariant plant G subject to the following conditions:

(N.1) The nonlinear function $\phi(\sigma)$ is assumed to be piecewise continuous and to satisfy the conditions:

$$\phi(0) = 0,$$

$$0 < e \leq \phi(\sigma)/\sigma \leq k-e, \quad \forall \sigma \neq 0 \tag{2.1}$$

and

$$-k' \leq d\phi/d\sigma \leq k''. \tag{2.2}$$

The output of N is

$$u(n) = \phi[\sigma(n)]. \tag{2.3}$$

Eq. (2.1) restricts the nonlinear function to lie in a sector [e, k−e] with e arbitrarily small as shown in Figure 2.2. Eq. (2.2) bounds the slope of the nonlinear function.

(G) The system at the nth sampling instant is described by

$$\sigma(n) = r(n) - \eta(n) - \sum_{i=0}^{n} g(n-i)u(i). \tag{2.4}$$

Here $g(n)$ is the sampled impulse response and $\eta(n)$ is the zero-input response of the linear plant G. The input $r(n)$ is assumed to tend to zero as $n\to\infty$ and to be bounded. In addition, the following conditions are imposed:

(G.1) For any initial state, $\eta(n)$ and its first backward difference $\nabla\eta(n)$ are bounded, are elements of $L_2(0,\infty)$ and so go to zero with time.

(G.2) $g(n)$ and its first backward difference $\nabla g(n)$ are bounded, are elements of $L_1(0,\infty)$ and so go to zero with time. It is assumed further that for some $\alpha > 1$

$$\sum_{n=0}^{\infty} \alpha^n g(n) < \infty . \tag{2.5}$$

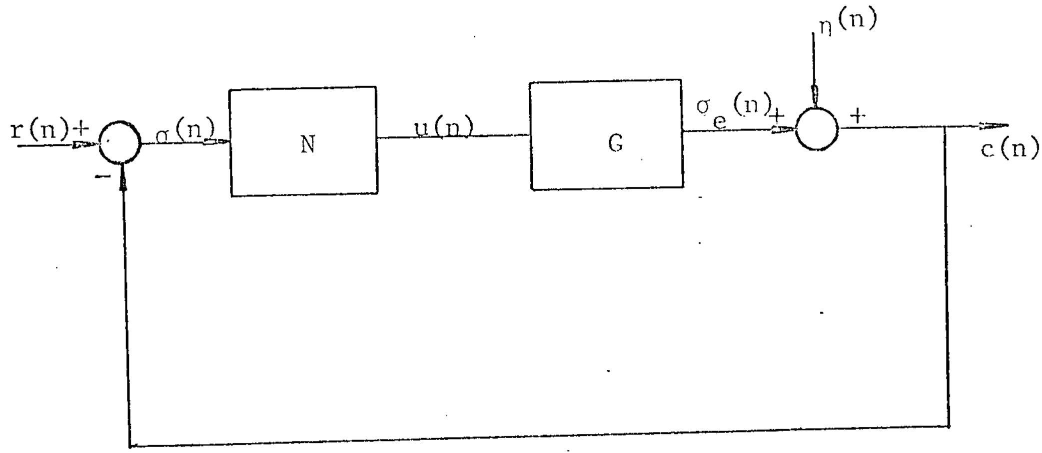


Figure 2.1 System S.

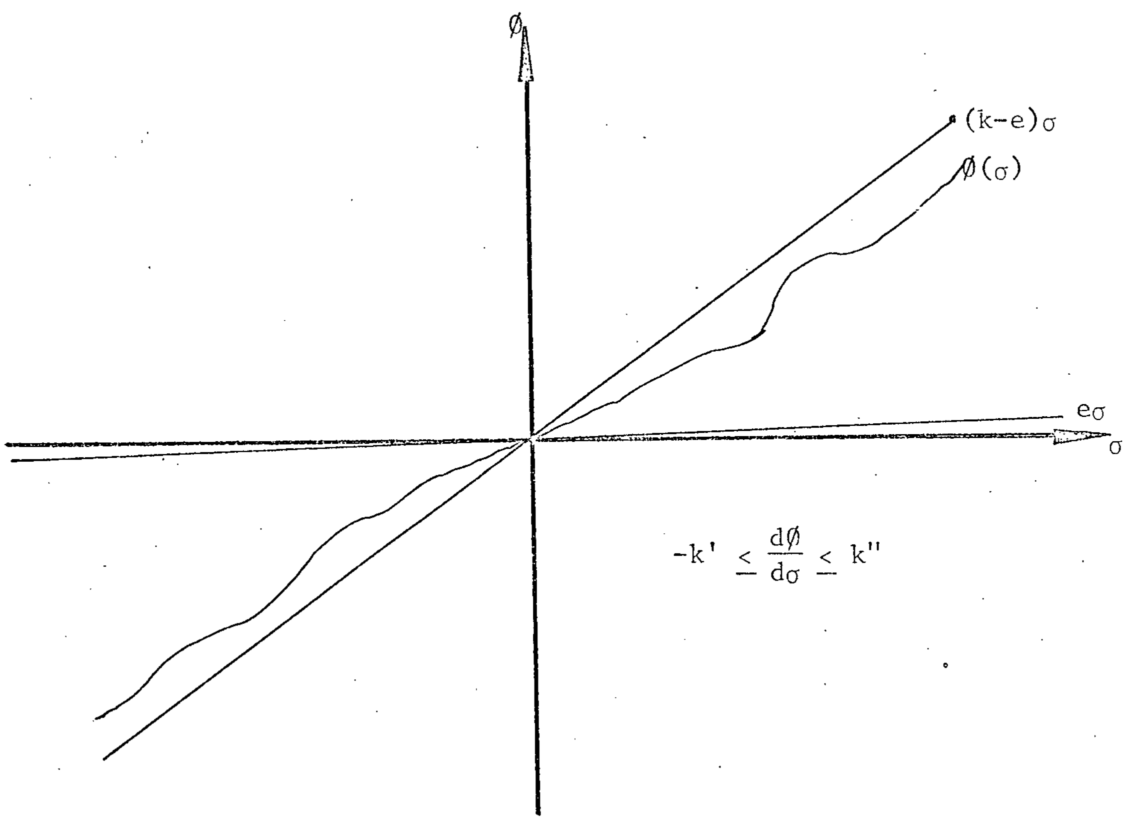


Figure 2.2 Nonlinear Function.

Thus, $G(z)$, the Z-transform of $g(n)$, is analytic in $|z|>\alpha$, and has poles within circle $|z| < \alpha$, i.e. the linear plant is strictly stable.

According to the notation of Jury and Lee [7], a nonlinear sampled-data (NSD) system S satisfying these assumptions for specific nonnegative k, k', k'' is referred to as an NSD system S of $(0,k;\ -k',k'')$.

In the following section, sufficient conditions are derived for the absolute stability of the equilibrium point $c(n) = \sigma(n) = u(n) = 0$. The zero equilibrium point of the system described above is said to be absolutely stable if, for any $\phi[\sigma(n)]$ satisfying Eqs. (2.1) and (2.2), the zero solution is globally asymptotically stable. That is

$$\lim_{n\to\infty} c(n) = \lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} u(n) = 0. \tag{2.6}$$

2.2 Sufficient Condition for Absolute Stability

Theorem I

The NSD system S of $(0,k;\ -\infty,k'')$ is absolutely stable if there exists a nonnegative number q such that

$$\mathrm{Re}H_1(z) = \mathrm{Re}\{[1+(z-1)q]G(z)\} - \frac{k''q}{2}\,|(z-1)G(z)|^2 + \frac{1}{k} \geq \delta > 0 \tag{Q}$$

for all $|z| = 1$.

Corollary I

The NSD system of $(0,k;\ -k',\infty)$ is absolutely stable if there exists a nonnegative q such that

$$\mathrm{Re}H'(z) = \mathrm{Re}\{[1+\frac{(z-1)}{z}q]G(z)\} - \frac{k'q}{2}|(z-1)G(z)|^2 + \frac{1}{k} \geq \delta > 0 \tag{Q'}$$

for all $|z| = 1$.

Note: If q=0, (Q) and (Q') reduce to the Tsypkin criterion [5]

$$\mathrm{Re}G(z) + \frac{1}{k} \geq \delta > 0 \tag{P}$$

for all $|z| = 1$, where δ is arbitrarily small.

Proof of Theorem I

Rewriting the system equation (2.4) equivalently as

$$\sigma(n) = x(n) - \sum_{i=0}^{n} g(n-i)u(i) \tag{2.7}$$

where

$$x(n) = r(n) - \eta(n). \tag{2.8}$$

From Eq. (2.7), the first backward difference of $\sigma(n)$ is

$$\nabla\sigma(n) = \nabla x(n) - \sum_{i=0}^{n} \nabla g(n-i)u(i). \tag{2.9}$$

Truncate the variable $u(n)$ at any positive integer N, then define the following auxiliary functions such that

$$u_N(n) = \begin{cases} u(n), & 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases} \tag{2.10}$$

$$\sigma_N(n) = x(n) - \sum_{i=0}^{n} g(n-i)u_N(i), \tag{2.11}$$

$$\nabla\sigma_N(n) = \nabla x(n) - \sum_{i=0}^{n} \nabla g(n-i)u_N(i). \tag{2.12}$$

Therefore, because of the truncation, $u_N(n)$ are elements of $L_2(0,\infty)$. In addition, by assumption (G1), $x(n)$ and $\nabla x(n)$ clearly belong to $L_2(0,\infty)$, and by (G2), $g(n)$ and $\nabla g(n)$ belong to $L_1(0,\infty)$. Then, making use of the fact that the convolution of L_1 and L_2 sequences is an L_2 sequence, $\sigma_N(n)$ and $\nabla\sigma_N(n)$ belong to $L_2(N,\infty)$.

Let us introduce the notation

$$f_1(n-1) = \sigma_N(n-1) + q\nabla\sigma_N(n) - k^{-1}u_N(n-1), \tag{2.13}$$

$$f_2(n-1) = x(n-1) + q\nabla x(n). \tag{2.14}$$

It is obvious that f_1 and f_2 belong to $L_2(0,\infty)$. This guarantees the existance of Z-transform denoted by $F_1(z)$ and $F_2(z)$.

By substituting Eqs. (2.11) and (2.12) into Eq. (2.13) and using Eq. (2.14), we obtain

$$f_1(n-1) = f_2(n-1) - \sum_{i=0}^{n} [g(n-i-1) + q\nabla g(n-i)]u_N(i) - k^{-1}u_N(n-1). \quad (2.15)$$

Consider the summation

$$\sum_{n=1}^{\infty} f_1(n-1)u_N(n-1)\alpha^{2(n-1)} \quad (2.16)$$

where $\alpha > 1$.

Then, from Eq. (2.16),

$$\sum_{n=1}^{\infty} f_1(n-1)u_N(n-1)\alpha^{2(n-1)} = \sum_{n=1}^{\infty} f_2(n-1)u_N(n-1)\alpha^{2(n-1)}$$

$$- \sum_{n=1}^{\infty} u_N(n-1)\alpha^{(n-1)}\{\sum_{i=0}^{n} [g(n-i-1) + q\nabla g(n-i)]u_N(i)\}\alpha^{(n-1)}$$

$$- k^{-1}\sum_{n=1}^{\infty} u_N^2(n-1)\alpha^{2(n-1)}. \quad (2.17)$$

It is now to be noted that

$$\sum_{n=1}^{N} [\nabla\sigma(n)]^2\alpha^{2(n-1)} = \sum_{n=1}^{N} [\nabla\sigma_N(n)]^2\alpha^{2(n-1)} \le \sum_{n=1}^{\infty} [\nabla\sigma_N(n)]^2\alpha^{2(n-1)} \quad (2.18)$$

and, by using Eq. (2.12) we obtain

$$\frac{k''q}{2}\sum_{n=1}^{N} [\nabla\sigma(n)]^2\alpha^{2(n-1)} \le \frac{k''q}{2}\sum_{n=1}^{\infty} [\nabla x(n) - \sum_{i=0}^{n} \nabla g(n-i)u_N(i)]^2\alpha^{2(n-1)} \quad (2.19)$$

Adding the quantities on both sides of the inequality (2.19) to those of Eq. (2.17), the following key inequality is established:

$$\sum_{n=1}^{\infty} f_1(n-1)u_N(n-1)\alpha^{2(n-1)} + \frac{k''q}{2}\sum_{n=1}^{N} [\nabla\sigma(n)]^2\alpha^{2(n-1)}$$

$$\ge \sum_{n=1}^{\infty} f_2(n-1)u_N(n-1)\alpha^{2(n-1)}$$

$$- \sum_{n=1}^{\infty} u_N(n-1)\alpha^{(n-1)}\{\sum_{i=0}^{n} [g(n-i-1) + q\nabla g(n-i)]u_N(i)\}\alpha^{(n-1)}$$

$$-k^{-1}\sum_{n=1}^{\infty} u_N^2(n-1)\alpha^{2(n-1)}$$

$$+ \frac{k''q}{2} \sum_{n=1}^{\infty} [\nabla x(n) - \sum_{i=0}^{n} \nabla g(n-i)u_N(i)]^2 \alpha^{2(n-1)} \quad . \tag{2.20}$$

Applying a modified form of Parseval's theorem (Appendix A) to the right-hand side (r.h.s.) of inequality (2.20) and collecting like terms, the right-hand side of Eq. (2.20) can be expressed as:

$$\text{r.h.s.} = \frac{-1}{2\pi j} \oint_c \{[1+(z_1-1)q]G(z_1) - \frac{k''q}{2} |(z_1-1)G(z_1)|^2 + k^{-1}\}$$

$$\cdot |U_N(z_1)|^2 z^{-1} dz$$

$$+ \frac{1}{2\pi j} \oint_c [F_2^*(z_1) - k''q|z_1-1|^2 X^*(z_1)G(z_1)]U_N(z_1)z^{-1}dz$$

$$+ \frac{k''q}{4\pi j} \oint_c |(z_1-1)X(z_1)|^2 z^{-1}dz, \tag{2.21}$$

where $z = \exp(j\omega)$, $z_1 = \frac{\exp(j\omega)}{\alpha}$ and in general [12] for any $x(n)$ with Z-transform $X(z_1)$,

$$Z[x(n)\alpha^n] = X(z_1), \tag{2.22}$$

and the asterisk (*) is used to indicate the conjugate of a complex function.

It is convenient to define the following functions

$$H_1(z_1) = [1 + (z_1-1)q]G(z_1) - \frac{k''q}{2} |(z_1-1)G(z_1)|^2 + k^{-1}, \tag{2.23}$$

$$F_4^*(z_1) = F_2^*(z_1) - k''q|z_1-1|^2 X^*(z_1)G(z_1). \tag{2.24}$$

Nothing that the right-hand side must be a real quantity, after substituting $z_1 = e^{j\omega}/\alpha$ and $z^{-1}dz = jd\omega$ in Eq. (2.21), it can be shown that

$$\text{r.h.s.} = \frac{-1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Re}H_1(e^{j\omega}/\alpha)| U_N(e^{j\omega}/\alpha)|^2 d\omega$$

$$+\frac{1}{4\pi} \int_{-\pi}^{\pi} F_4^* (e^{j\omega}/\alpha)U_N(e^{j\omega}/\alpha)d\omega + \frac{1}{4\pi} \int_{-\pi}^{\pi} F_4(e^{j\omega}/\alpha)U_N^*(e^{j\omega}/\alpha)d\omega$$

$$+\frac{k''q}{4\pi} \int_{-\pi}^{\pi} \left|(\frac{e^{j\omega}}{\alpha} - 1)X(e^{j\omega}/\alpha)\right|^2 d\omega \tag{2.25}$$

where

$$\frac{1}{2}[F_4^*(e^{j\omega}/\alpha)U_N(e^{j\omega}/\alpha) + F_4(e^{j\omega}/\alpha)U_N^*(e^{j\omega}/\alpha)] = \mathrm{Re}[F_4^*(e^{j\omega}/\alpha)U_N(e^{j\omega}/\alpha)]. \tag{2.26}$$

If

$$\mathrm{ReH}_1(e^{j\omega}/\alpha) \geq \delta_\alpha > 0, \quad \text{for} \quad -\pi \leq \omega \leq \pi, \tag{2.27}$$

then

$$\mathrm{Re}\{[1 + (z_1-1)q]G(z_1)\} - \frac{k''q}{2} |(z_1-1)G(z_1)|^2 + k^{-1} \geq \delta_\alpha. \tag{Q}$$

Since satisfaction of (Q) is assumed, hence (Q) implies (Q_α) for sufficiently small $\alpha > 1$ [13].

Now completing the square in the right-hand side of Eq. (2.25), we obtain

$$\text{r.h.s.} = -(2\pi)^{-1} \int_{-\pi}^{\pi} \left| [\mathrm{ReH}_1(e^{j\omega}/\alpha)]^{1/2} U_N(e^{j\omega}/\alpha) + \frac{F_4(e^{j\omega}/\alpha)}{2[\mathrm{ReH}_1(e^{j\omega}/\alpha)]^{1/2}} \right|^2 d\omega$$

$$+ \frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{|F_4(e^{j\omega}/\alpha)|^2}{\mathrm{ReH}_1(e^{j\omega}/\alpha)} d\omega + \frac{k''q}{4\pi} \int_{-\pi}^{\pi} \left|(\frac{e^{j\omega}}{\alpha} - 1)X(e^{j\omega}/\alpha)\right|^2 d\omega. \tag{2.28}$$

Removing the first integration, which is always negative, from the right-hand side and making use of the condition (2.27) yields

$$\text{r.h.s.} \leq \frac{1}{8\pi\delta_\alpha} \int_{-\pi}^{\pi} |F_4(e^{j\omega}/\alpha)|^2 d\omega + \frac{k''q}{4\pi} \int_{-\pi}^{\pi} \left|(\frac{e^{j\omega}}{\alpha} - 1)X(e^{j\omega}/\alpha)\right|^2 d\omega. \tag{2.29}$$

From Appendix B, Eq. (2.29) can be expressed as:

$$\text{r.h.s.} \leq \sum_{n=1}^{\infty} \frac{E}{4\delta_\alpha} \{[x(n-1) + q\nabla x(n)]^2 + \frac{k''q}{2} [\nabla x(n)]^2\}\alpha^{2(n-1)}. \tag{2.30}$$

It is obvious that the right-hand side of Eq. (2.30) does not depend on N, but depends quadratically on the initial conditions and tends to zero with the initial condition tending to zero.

Now returning to the left-hand side (l.h.s.) of Eq. (2.20) and substituting the expression for $f_1(n-1)$, we obtain

$$\text{l.h.s.} = \sum_{n=1}^{\infty} [\sigma_N(n-1) - k^{-1}u_N(n-1)]u_N(n-1)\alpha^{2(n-1)}$$

$$+ q \sum_{n=1}^{\infty} \nabla\sigma_N(n)u_N(n-1)\alpha^{2(n-1)} + \frac{k''q}{2} \sum_{n=1}^{N} [\nabla\sigma(n)]^2\alpha^{2(n-1)}, \tag{2.31}$$

or

$$\text{l.h.s.} = \sum_{n=1}^{N} [1 - \frac{\emptyset[\sigma(n-1)]}{k\sigma(n-1)}]\sigma(n-1)\emptyset[\sigma(n-1)]\alpha^{2(n-1)}$$

$$+ q\,\{\sum_{n=1}^{N} \emptyset[\sigma(n-1)]\nabla\sigma(n) + \frac{k''q}{2}\sum_{n=1}^{N} [\nabla\sigma(n)]^2\}\alpha^{2(n-1)} \quad . \tag{2.32}$$

The first sum in Eq. (2.32) is always positive, that is

$$\sum_{n=1}^{N} (1 - \frac{\emptyset[\sigma(n-1)]}{k\sigma(n-1)})\sigma(n-1)\emptyset[\sigma(n-1)] \geq 0, \qquad \text{for all } N > 0. \tag{2.33}$$

For the second and third sums, because of the constraint $-\infty \leq \frac{d\emptyset}{d\sigma} \leq k''$, the following area inequality, due to Jury and Lee [6], applies.

$$\emptyset[\sigma(n-1)]\nabla\sigma(n) + \frac{k''}{2}[\nabla\sigma(n)]^2 \geq \int_{\sigma(n-1)}^{\sigma(n)} \emptyset(\sigma)d\sigma, \text{ for all } n > 0, \tag{2.34}$$

and, since $q \geq 0$, it follows that

$$q\sum_{n=1}^{N} \emptyset[\sigma(n-1)]\nabla\sigma(n) + \frac{k''q}{2}\sum_{n=1}^{N} [\nabla\sigma(n)]^2 \geq q[\Phi(\sigma[N]) - \Phi(\sigma[0])] \geq -q\Phi(\sigma[0]) \tag{2.35}$$

where

$$\Phi(\sigma) = \int_0^{\sigma} \emptyset(\sigma)d\sigma.$$

Therefore, by using the inequality (2.35) in (2.32), it follows that

$$\text{l.h.s.} \geq \sum_{n=1}^{N} [1 - \frac{\emptyset[\sigma(n-1)]}{k\sigma(n-1)}]\sigma(n-1)\emptyset[\sigma(n-1)]\alpha^{2(n-1)} - q\Phi(\sigma[0])\alpha^{2(n-1)} \quad . \tag{2.36}$$

Finally, returning again to Eq. (2.20) and using the Eqs. (2.30) and (2.36) the inequality is rearranged to yield

$$\sum_{n=1}^{N} [1 - \frac{\emptyset[\sigma(n-1)]}{k\sigma(n-1)}]\sigma(n-1)\emptyset[\sigma(n-1)]$$

$$\leq \sum_{n=1}^{\infty} \{\frac{E}{4\delta\alpha}[x(n-1) + q\nabla x(n)]^2 + \frac{k''q}{2}[\nabla x(n)]^2\} + q\Phi(\sigma[0]) \quad . \tag{2.37}$$

or

$$\sum_{n=1}^{N} [1 - \frac{\emptyset[\sigma(n-1)]}{k\sigma(n-1)}]\ \sigma(n-1)\emptyset[\sigma(n-1)] \leq C. \tag{2.38}$$

Here the constant C, which denotes the right-hand side of the inequality (2.37), depends only on the initial values and tends to zero together with them. Note this constant is obviously independent of N. Consequently, the inequality (2.38) is valid for all N, and implies that this sum is uniformly bounded for all N [6]. Therefore

$$\lim_{n \to \infty} \sigma(n-1)\emptyset(n-1)[1 - \frac{\emptyset(n-1)}{k\sigma(n-1)}] = 0. \qquad (2.39)$$

From Eq. (2.1),

$$[1 - \frac{\emptyset(n-1)}{k\sigma(n-1)}] > 0, \qquad \text{for all} \quad \sigma \neq 0. \qquad (2.40)$$

Hence

$$\lim_{n \to \infty} \sigma(n) = 0, \text{ and } \lim_{n \to \infty} \emptyset[\sigma(n)] = 0, \qquad (2.41)$$

and the theorem is proved.

Proof of the Corollary

For the NSD system S of $(0,k,-k',\infty)$, the constraints of nonlinear element are satisfied by the conditions (2.1) and

$$-k' \leq \frac{d\emptyset(\sigma)}{d\sigma} \leq \infty \qquad (2.42)$$

Since $q > 0$, these conditions ensure the existance of the following inequality area [7], that is,

$$q\emptyset[\sigma(n)]\nabla\sigma(n) + \frac{k'q}{2}[\nabla\sigma(n)]^2 \geq q\int_{\sigma(n-1)}^{\sigma(n)} \emptyset(\sigma)d\sigma, \qquad \forall\, n \geq 0. \qquad (2.43)$$

The auxiliary functions f_1 and f_2, first defined in Eqs. (2.13) and (2.14), are replaced by

$$f_1(n) = \sigma_N(n) + q\Delta\sigma_N(n) - k^{-1}u_N(n) \qquad (2.44)$$

$$f_2(n) = x(n) + q\nabla x(n). \qquad (2.45)$$

Now, following the steps of the derivation given in Theorem I, it can be shown that if the condition,

$$\mathrm{Re}H'(e^{j\omega}/\alpha) \geq \delta_\alpha > 0, \quad \text{for } -\pi \leq \omega \leq \pi \qquad (2.46)$$

is satisfied the following inequality, first shown in Eq. (2.37), follows.

$$\sum_{n=0}^{N} [1 - \frac{\emptyset[\sigma(n)]}{k\sigma(n)}]\sigma(n)\emptyset[\sigma(n)]$$

$$\leq \sum_{n=0}^{\infty} \{\frac{E}{4\delta_\alpha} [x(n) + q\nabla x(n)]^2 + \frac{k'q}{2}[\nabla x(n)]^2\} + \Phi(\sigma(0)) \tag{2.47}$$

or

$$\sum_{n=0}^{N} [1 - \frac{\emptyset[\sigma(n)]}{k\sigma(n)}]\sigma(n)\emptyset[\sigma(n)] < C'. \tag{2.48}$$

Here the constant C', just like the constant C, depends only on the initial values and tends to zero together with them.

This completes the proof of the corollary.

2.3 Application of Theorem I

The examples given in [6] and [8] all satisfy the constraint

$$\mathrm{Re}(z-1)G(z) > 0, \qquad \text{for } |z| = 1 \tag{2.49}$$

and the maximum k", denoted by k''_m, is just equal to the minimum positive value of -x" given by:

$$x'' = \frac{-\mathrm{Re}(z-1)G(z)}{|(z-1)G(z)|^2}.$$

Furthermore, the inequality (Q) does depend on the choice of q and no simple graphical procedure has yet been developed for $q \neq 0$.

In this thesis, the case of $q \neq 0$ is considered and the following step by step method for determining maximum k of k" is suggested. First rewrite the inequality (Q) as

$$\frac{2[\mathrm{Re}G(z) + k^{-1}]}{|(z-1)G(z)|^2} - q[\frac{-\mathrm{Re}[(z-1)G(z)]}{|(z-1)G(z)|^2} + k''] \geq 0, \tag{2.50}$$

and let

$$y'' = \frac{2(\mathrm{Re}G(z) + k^{-1})}{|(z-1)G(z)|^2} , \tag{2.51}$$

$$x'' = \frac{-\mathrm{Re}(z-1)G(z)}{|(z-1)G(z)|^2} . \tag{2.52}$$

Then Eq. (2.50) can be expressed as

$$y'' - q(x''+k'') \geq 0. \tag{2.53}$$

But the equation

$$y'' - q(x''+k'') = 0 \tag{2.54}$$

is the equation of a straight line with slope q, passing through the point $-k''$ on the real axis of x''-y'' plane shown in Figure 2.3.

This straight line divides the plane into two half planes. The inequality (2.53) is satisfied if a real positive q may be found such that the plot of $x''+jy''$ as a function of $\omega(0$ to $\pi)$ in the x''-y'' plane, lies entirely to the left of this straight line.

From Figure 2.3, for $y'' = 0$ at $\omega = \omega_m$, Eq. (2.54) becomes

$$k'' = -x'' \tag{2.55}$$

Hence

$$k''_m = \left.\frac{2[\mathrm{Re}(z-1)G(z)]}{|(z-1)G(z)|^2}\right|_{z=e^{j\omega_m}} \tag{2.56}$$

It is clear that k''_m may be found from the x''-y'' plane for different values of k starting with the value at q=0. Then, if k'' is given, it is easy to find the largest permissible k from k-k'' curve.

Example 1.

The open loop transfer function of the linear part of system S is

$$G(s) = \frac{1}{(s+0.1)(s+0.3)}. \tag{2.57}$$

Consider the sampling period T = 1 second and take the Z-transform of G(s)

$$G(z) = 5\left[\frac{z}{z-e^{-.1}} - \frac{z}{z-e^{-.3}}\right]. \tag{2.58}$$

Substituting $z = e^{j\omega}$ into Eq. (2.54). Then setting q = 0 in inequality (0), one can find

$$\mathrm{Min.Re}G(e^{j\omega}) = -3.44. \tag{2.59}$$

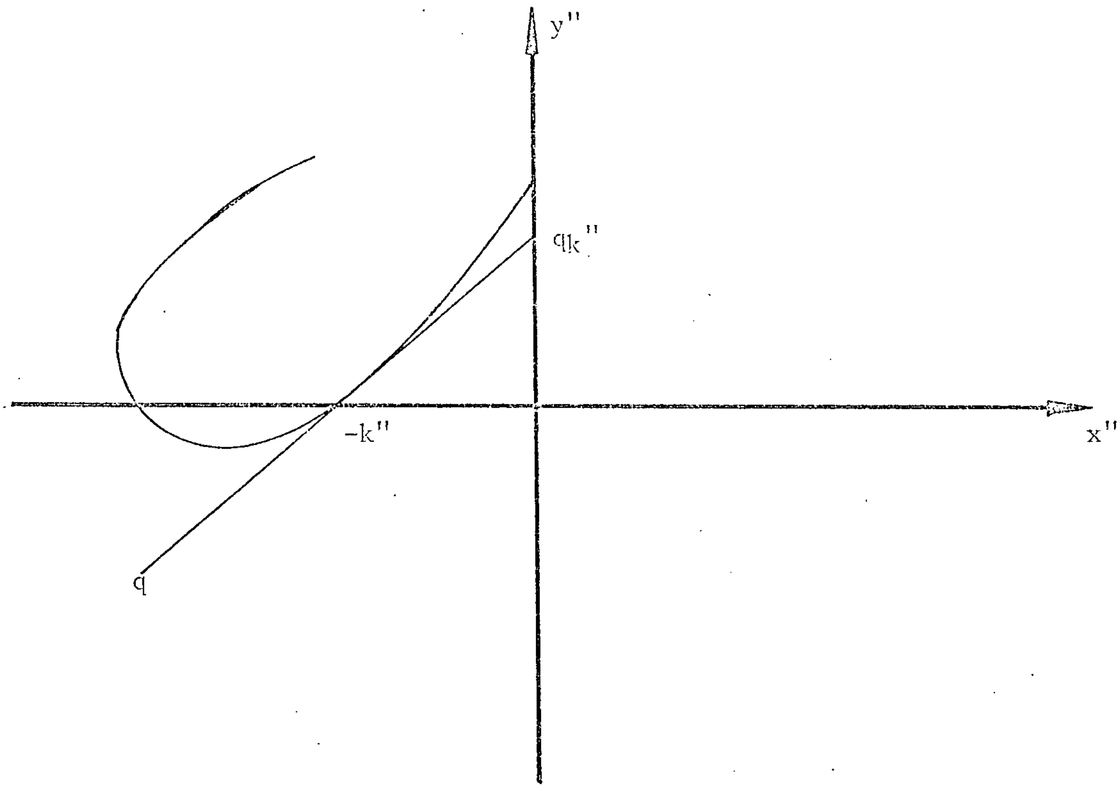


Figure 2.3 x" - y" Plane.

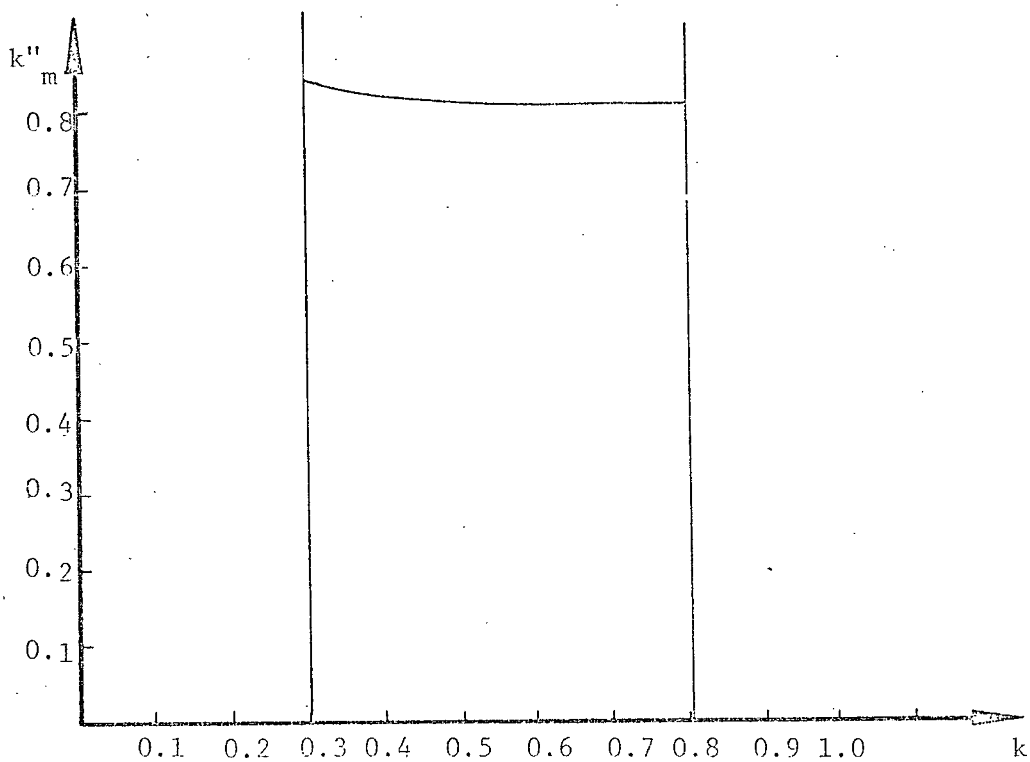


Figure 2.4 k - k''_m Plane.

Therefore

$$k = 0.29. \tag{2.60}$$

Now, according to the procedure mentioned above, the relation between k and k''_m, is shown in Table 1, and is plotted in Figure 2.4.

Table 1

k	k''_m
0.3	0.837
0.5	0.809
0.7	0.805
0.8	0.804
0.81	0.804

From Figure 2.4, the minimum positive value of k''_m that satisfies the constraint $k_r < k''_m$ is equal to 0.804. Hence, the system S is absolutely stable if

$$k \leq k'' \leq 0.804. \tag{2.61}$$

3. THE STABILITY OF NONLINEAR SAMPLED-DATA SYSTEMS WITH A MONOTONE NONLINEARITY

3.1 Description of System

In this chapter the stability of NSD systems with a monotone increasing nonlinear function will be considered. The system is similar to that treated in Chapter II.

(N) The nonlinear function $\emptyset[\sigma(n)]$, is assumed to be a piecewise continuous and a montonically increasing relation, and is described by the equations (2.1), (2.3) and

$$0 \le \frac{\emptyset[\sigma_1] - \emptyset[\sigma_2]}{\sigma_1 - \sigma_2} \le k, \qquad \text{for } \sigma_1 \neq \sigma_2 \tag{3.1}$$

(G) By setting the input $r = 0$, the system is described by

$$\sigma(n) = -\eta(n) - \sum_{i=0}^{n} g(n-i)u(i), \tag{3.2}$$

or using the symbol $*$ to denote the convolution, we obtain

$$\sigma(n) = -\eta(n) - (g*u)(n). \tag{3.3}$$

Here η and g satisfy the conditions (G1) and (G2) of Chapter II.

Denoting

$$\eta_M = \sup_{n \ge 0} |\eta(n)|, \qquad g_M = \sup_{n \ge 0} |g(n)|, \tag{3.4}$$

and using $\|\cdot\|$ to denote L_1 norms and $\|\cdot\|_2$ to denote L_2 norms individually,

$$\|y\| = \sum_{n=0}^{\infty} |y|, \qquad \|\eta\|_2 = \left(\sum_{n=0}^{\infty} \eta^2(n)\right)^{1/2}. \tag{3.5}$$

3.2 Sufficient Condition for Absolute Stability

Theorem II.

Consider a system satisfying the above conditions. Let $y(n)$ be any real-valued function such that

$$\begin{aligned} &\text{(i)} && y(n) = 0 && \text{for } n < 0, \\ &\text{(ii)} && y(n) \le 0 && \text{for } n \ge 0, \\ &\text{(iii)} && \|y\| < 1. \end{aligned} \tag{3.6}$$

If, for some $q > 0$,

$$\mathrm{Re}H_2(z) = \mathrm{Re}\{[1+(z-1)q+Y(z)]G(z)\} - \frac{kq}{2} |(z-1)G(z)|^2$$

$$+ \mathrm{Re}[1+Y(z)]k^{-1} > \delta \geq 0, \quad \text{for all } |z| = 1 \tag{3.7}$$

then

$$\text{(i)} \sup_{n \geq 0} |\sigma(n)| < \infty, \tag{3.8}$$

$$\text{(ii)} \lim_{n \to \infty} \sigma(n) = 0, \tag{3.9}$$

(iii) as $\| \eta \|_2 + \| \nabla\eta \|_2 \to 0$, the corresponding $\sigma(n)$ has the property that $\sup_{n \geq 0} |\sigma(n)| \to 0$.

Lemma.

Let $x(n)$ and $f[x(n)]$ be functions in $L_2(-\infty,\infty)$. If $f[x(n)]$ is monotonically increasing, then for all i,

$$\sum_{n=-\infty}^{\infty} x(n-i)f[x(n)] \leq \sum_{n=-\infty}^{\infty} x(n)f[x(n)]. \tag{3.10}$$

If, in addition, $f[x(n)]$ is odd $(f(x) = -f(-x))$, then the inequality above holds with absolute value taken on the left sum.

The proof of this lemma is obtained with slight modification from that contained in [14]. This important result is used in the proof of Theorem II.

Proof of Theorem II

For any positive integer $N \geq 0$, the following auxiliary functions are defined:

$$u_N(n) = \begin{cases} u(n) & 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases} \tag{3.11}$$

$$u_N(n) = \emptyset[\sigma_N(n)], \tag{3.12}$$

$$\sigma_{eN}(n) = \sum_{i=0}^{n} g(n-i)u_N(i), \tag{3.13}$$

$$\Delta\sigma_{eN}(n) = \sum_{i=0}^{n} \nabla g(n-i)u_N(i). \tag{3.14}$$

Thus, $\sigma_{eN}(n)$ and $\nabla\sigma_{eN}(n)$ belong to $L_2(N,\infty)$.

Let

$$\sigma_m = \sigma + \sigma * y, \quad u_m = u + u*y \tag{3.15}$$

and, in general, given any function x, we define [15]

$$x_m = x + x*y.$$

Then, let us introduce the notation

$$f_1(n-1) = \sigma_{mN}(n-1) + q\nabla\sigma_N(n) - k^{-1}u_{mN}(n-1), \tag{3.16}$$

$$f_2(n-1) = -\eta_m(n-1) - q\nabla\eta(n). \tag{3.17}$$

Because

$$\sigma_N(n) = -\eta(n) - \sigma_{eN}(n), \tag{3.18}$$

therefore,

$$f_1(n-1) = f_2(n-1) - [\sigma_{eNm}(n-1) + q\nabla\sigma_{eN}(n) + k^{-1}u_{mN}(n-1)], \tag{3.19}$$

or

$$\sum_{n=0}^{\infty} f_1(n-1)u_N(n-1) = \sum_{n=0}^{\infty} f_2(n-1)u_N(n-1)$$

$$- \sum_{n=0}^{\infty} [\sigma_{eNm}(n-1)+q\nabla\sigma_{eN}(n)+k^{-1}u_{mN}(n-1)]u_N(n-1). \tag{3.20}$$

It is now to be noted that

$$\frac{k\,q}{2}\sum_{n=0}^{\infty}[\nabla\sigma_N(n)]^2 = \frac{k\,q}{2}\sum_{n=0}^{\infty}[\nabla\sigma_{eN}(n) + \nabla\eta(n)]^2. \tag{3.21}$$

Making use of Eqs. (3.20) and (3.21), we obtain

$$\sum_{n=0}^{\infty} f_1(n-1)u_N(n-1) + \frac{k\,q}{2}\sum_{n=0}^{\infty}[\nabla\sigma_N(n)]^2$$

$$= \sum_{n=0}^{\infty} f_2(n-1)u_N(n-1) - \sum_{n=0}^{\infty}[\sigma_{eNm}(n-1)+q\nabla\sigma_{eN}(n)+k^{-1}u_{mN}(n-1)]u_N(n-1)$$

$$+ \frac{k\,q}{2}\sum_{n=0}^{\infty}[\nabla\sigma_{eN(n)} + \nabla\eta(n)]^2. \tag{3.22}$$

Then, applying Parseval's theorem [18] to the right-hand side (r.h.s.) of Eq. (3.22), and collecting like terms, the r.h.s. of Eq. (3.22) can be expressed as:

$$\text{r.h.s.} = \frac{-1}{2\pi j} \oint_c \{[1+(z-1)q + Y(z)]G(z) - \frac{kq}{2} |(z-1)G(z)|^2$$

$$+ [1+Y(z)]k^{-1}\} |u_N(z)|^2 z^{-1} dz$$

$$+ \frac{1}{2\pi j} \oint_c [F_2(z^{-1}) - kq|z-1|^2 X(z^{-1})G(z)]U_N(z)z^{-1} dz$$

$$+ \frac{qk}{4\pi j} \oint_c |(z-1)X(z)|^2 z^{-1} dz \tag{3.23}$$

where

$$X(z) = -Z[\eta(n)].$$

Now, define

$$H_2(z) = [1+(z-1)q+Y(z)]G(z) - \frac{kq}{2}|(z-1)G(z)|^2 + [1+Y(z)]k^{-1}, \tag{3.24}$$

$$F_4^*(z) = F_2(z^{-1}) - kq|z-1|^2 X(z^{-1})G(z). \tag{3.25}$$

Since the imaginary part of the right-hand side must be equal to zero, and by the condition of the theorem,

$$\text{Re}H_2(z) \geq \delta.$$

Then, following the steps of the derivation given in Theorem I (from Eq. (2.25) to (2.30)), it can be shown that

$$\text{r.h.s.} \leq \sum_{n=0}^{\infty} \{\frac{E}{4\delta}[\eta_m(n-1) + q\nabla\eta(n)]^2 + \frac{kq}{2}[\nabla\eta(n)]^2\}. \tag{3.26}$$

Now, returning to the left-hand side (l.h.s.) of Eq. (3.22) and substituting the expression for $f_1(n-1)$, we have

$$\text{l.h.s.} = \sum_{n=0}^{\infty} [\sigma_{mN}(n-1) + q\nabla\sigma_N(n) - k^{-1}u_{mN}(n-1)]U_N(n-1)$$

$$+ \frac{kq}{2} \sum_{n=0}^{N} [\nabla\sigma(n)]^2. \tag{3.27}$$

Consider

$$\sum_{n=0}^{\infty} [\sigma_{mN}(n) - \frac{u_{mN}(n)}{k}]u_N(n)$$

$$= \sum_{n=0}^{\infty} [\sigma_N(n) - \frac{u_N(n)}{k}]u_N(n) + \sum_{n=0}^{\infty} [y^*(\sigma_N - \frac{u_N}{k})](n)u_N(n). \tag{3.28}$$

Define

$$R(i) = \sum_{n=0}^{\infty} [\sigma_N(n-i) - \frac{u_N(n-i)}{k}]u_N(n). \tag{3.29}$$

Since $\emptyset(\sigma)$ is a monotonically increasing function, by lemma,

$$R(i) \leq R(0). \tag{3.30}$$

Considering the last sum in Eq. (3.28),

$$\sum_{n=0}^{\infty} (y*(\sigma_N - \frac{u_N}{k}))(n)u_N(n) = \sum_{n=0}^{\infty} \{ \sum_{i=0}^{\infty} y(i)[\sigma_N(n-i) - \frac{u_N(n-i)}{k}]\}u_N(n)$$

$$= \sum_{i=0}^{\infty} y(i)\{ \sum_{n=0}^{\infty} [\sigma_N(n-i) - \frac{u_N(n-i)}{k}]u_N(n)\}$$

$$= \sum_{i=0}^{\infty} y(i)R(i). \tag{3.31}$$

Since $y \leq 0$, substituting Eq. (3.30) into Eq. (3.31) yields

$$\sum_{i=0}^{\infty} y(i)R(i) \geq -R(0) \| y \|. \tag{3.32}$$

Let i=0. Then substituting Eqs. (3.29) and (3.32) into eq. (3.28) gives

$$\sum_{n=0}^{\infty} [\sigma_{mN}(n) - \frac{u_{mN}(n)}{k}]u_N(n) \geq [1.-\|y\|]R(0). \tag{3.33}$$

For $\|y\| \leq 1$, there is a finite $b(N) \geq (1. - \|y\|) > 0$, such that

$$\sum_{n=0}^{N} [\sigma_m(n) - \frac{u_m(n)}{k}]u(n) = b(N)R(0) \geq 0. \tag{3.34}$$

Remember that $u_N(n) = \emptyset[\sigma_N(n)]$. By using Eq. (3.34) in Eq. (3.27), it follows that

$$\text{l.h.s.} \geq q \sum_{n=0}^{\infty} \emptyset[\sigma_N(n-1)]\nabla\sigma_N(n) + \frac{kq}{2} \sum_{n=0}^{\infty} [\nabla\sigma_N(n)]^2. \tag{3.35}$$

Also, because of the constraint $0 \leq \frac{d\emptyset}{d\sigma} \leq k$, the inequality (2.35) can be used in Eq. (3.35). Hence

$$\text{l.h.s.} \geq q\Phi[\sigma(N)] - q\Phi[\sigma(0)]. \tag{3.36}$$

Finally, returning again to Eq. (3.22) and using the Eqs. (3.26) and (3.36) and rearranging the inequality on both sides yields

$$q\Phi[\sigma(N)] - q\Phi[\sigma(0)] \leq \sum_{n=0}^{\infty} \{\frac{E}{4\delta}[\eta_m(n-1) + q\nabla\eta(n)]^2 + \frac{kq}{2}[\nabla\eta(n)]^2\}. \tag{3.37}$$

Letting

$$M(\|\eta\|_2, \|\nabla\eta\|_2) = \sum_{n=0}^{\infty}\{\frac{E}{4\delta}[\eta_m(n-1) + q\nabla\eta(n)]^2 + \frac{kq}{2}[\nabla\eta(n)]^2\} \tag{3.38}$$

and adding to both sides of the inequality (3.37) the positive quantity $q\Phi[\sigma(0)]$, also, for convenience, writing M for $M(\|\eta\|_2, \|\nabla\eta\|_2)$, Eq. (3.37) becomes

$$q\Phi[\sigma(N)] \leq M + q\Phi[\sigma(0)]. \tag{3.39}$$

Since $\emptyset[\sigma(n)]$ is monotonic

$$\Phi[\sigma(n)] \geq [\emptyset(\sigma(n))]^2 / 2k, \tag{3.40}$$

hence

$$\frac{1}{2k}|u(N)|^2 \leq Mq^{-1}+\Phi[\sigma(0)]. \tag{3.41}$$

It is clear that the right-hand side of inequality (3.41) is independent of N, thus $|u(N)|^2$ is uniformly bounded for all N. Therefore

$$\sup_{n \geq 0} |u(n)| \leq \{2K[Mq^{-1} + \Phi(\sigma(0))]\}^{1/2} \tag{3.42}$$

and $\sup_{n \geq 0}|u(n)|$ tends to zero as $\|\eta\|_2 + \|\nabla\eta\|_2 \to 0$.

Now, let us show that $\sigma(n) \to 0$ as $n \to \infty$. From Eqs. (3.27), (3.34) and (3.37), we have

$$b(N) \sum_{n=0}^{N} (\sigma(n-1) - \frac{u(n-1)}{k})u(n-1) + q\Phi(\sigma(N)) - q\Phi(\sigma(0)) \leq M. \tag{3.43}$$

Thus

$$\sum_{n=0}^{N} [\sigma(n-1) - \frac{u(n-1)}{k}]u(n-1) \leq [M + q\Phi(\sigma(0))](1 - \|y\|)^{-1}. \tag{3.44}$$

Since the right-hand side of Eq. (3.44) is obviously independent of N, and tends to zero with together $\|\eta\|_2 + \|\nabla\eta\|_2 \to 0$. This property implies that the sum of Eq. (3.44) is uniformly bounded for all N. Therefore, using the same arguments as in Theorem I one can show that

$$\lim_{n \to \infty} \sigma(n) = 0. \tag{3.45}$$

In addition, since $u(n) \to 0$ only if $\sigma(n) \to 0$, inequality (3.42) implies that

$\text{SUP}_{n \geq 0} |\sigma(n)| < \infty$ and tends to zero as $\| \eta \|_2 + \| \nabla\eta \|_2 \to 0$.

This completes the proof of Theorem II.

3.3 Example

Example 2.

Consider the system and transfer function of example 1. By taking $q = 0$, the inequality (3.7) becomes

$$\text{Re}[1 + Y(z)][G(z) + k^{-1}] \geq 0. \tag{3.46}$$

or

$$[1 + \text{Re}Y(z)]\text{Re}G(z) - \text{Im}Y(z)\text{Im}G(z) + [1 + \text{Re}Y(z)]k^{-1} \geq 0. \tag{3.47}$$

Assuming

$$1 + \text{Re}Y(z) > 0, \tag{3.48}$$

then, dividing $1 + \text{Re}Y(z)$ on both sides of inequality (3.47) yields

$$\text{Re}G(z) = \text{Im}Y(z)\text{Im}G(z)/[1 + \text{Re}Y(z)] + k^{-1} \geq 0. \tag{3.49}$$

Let

$$\text{Re}G'(z) = \text{Re}G(z) - \text{Im}Y(z)\text{Im}G(z)/[1 + \text{Re}Y(z)]. \tag{3.50}$$

Hence

$$\text{Re}G'(z) + k^{-1} \geq 0. \tag{3.51}$$

Suppose $y(n) = -\frac{1}{d}e^{-cn}$ with a choice of c and d to satisfy

$$\|y(n)\| = \sum_{n=0}^{\infty} \left|-\frac{1}{d}e^{-cn}\right| < 1. \tag{3.52}$$

The Z-transform of $y(n)$ is

$$Y(z) = -\frac{z}{d(z - e^{-c})}. \tag{3.53}$$

By choosing $c = 1$, $d = 1.7$ and substituting $z = e^{j\omega}$ into Eq. (3.53), it can be shown that

$$\text{Min}\ [1 + \text{Re}Y(e^{j\omega})] > 0. \tag{3.54}$$

Then, making use of Eqs. (2.58) and (3.53) in Eq. (3.47), we find

$$\text{Min } \mathrm{Re}G'(e^{j\omega}) = -0.8075 \tag{3.55}$$

at $\omega = 0.8$ rad..

Therefore

$$k \leq 1.237. \tag{3.56}$$

Due to the compensation of $\mathrm{Im}Y(z)\mathrm{Im}G(z)/[1+\mathrm{Re}Y(z)]$ in Eq. (3.49), the value of k is larger than $k = 0.29$, the value found from Theorem I.

In this example, q was taken zero in order to reduce the complexity of calculation.

4. THE STABILITY OF NONLINEAR CONTINUOUS SYSTEMS WITH SEVERAL NONLINEARITIES

4.1 Formulation of the Problem

The system under consideration is shown in Figure 4.1, where $\underline{r}$, $\underline{\sigma}$, $\underline{u}$ and $\underline{c}$ are n-vectors.

N is a time-invariant memoryless nonlinearity. The ith components of its input, $\sigma_i(t)$, and output, $\phi_i(\sigma_i(t))$ are assumed to be characterized as follows.

$$0 \leq \phi_i\sigma_i \leq k_i\sigma_i^2, \qquad \text{for} \quad \sigma_i \neq 0, \tag{4.1}$$

$$\phi_i(0) = 0, \qquad (i = 1,2,\ldots\ldots\ldots,n), \tag{4.2}$$

where k_i is the ith element of diagonal matrix k.

G is a linear-time invariant sub-system, the relation of its input, $\underline{\phi}(t)$ and output, $\underline{c}(t)$, is described by the equation:

$$\underline{c}(t) = \underline{\eta}(t) - \int_0^t g(t-\tau)\underline{\phi}(\tau)d\tau, \tag{4.3}$$

where g(t) is the nxn impulse response matrix of G, and $\eta(t)$ is n-vector and is the zero-input response of G. It is assumed that the following conditions are satisfied:

(G1) For all initial states, the zero input response (a) $\eta_i(t)$ is bounded on $(0,\infty)$, (b) $\eta_i(t),\dot{\eta}_i(t) \in L_2(0,\infty)$ and (c) $\eta_i(t) \to 0$ as $t \to \infty$.

(G2) Impulse response of G (a) $g_{ij}(t) \in L_1(0,\infty)$ and (b) $g_{ij}(t) \to 0$ as $t \to \infty$.

In the finite-dimensional case, (G1) and (G2) are implied by the requirement that G(s), the Laplace transform of g(t), has no singularity in the right-hand plane or on the real frequency axis except for a single pole at the origin.

4.2 Statement of Theorem

Theorem III

Let k be a real diagonal matrix with positive elements. If the system under consideration satisfies the above assumptions and there exists a real diagonal matrix q, such that

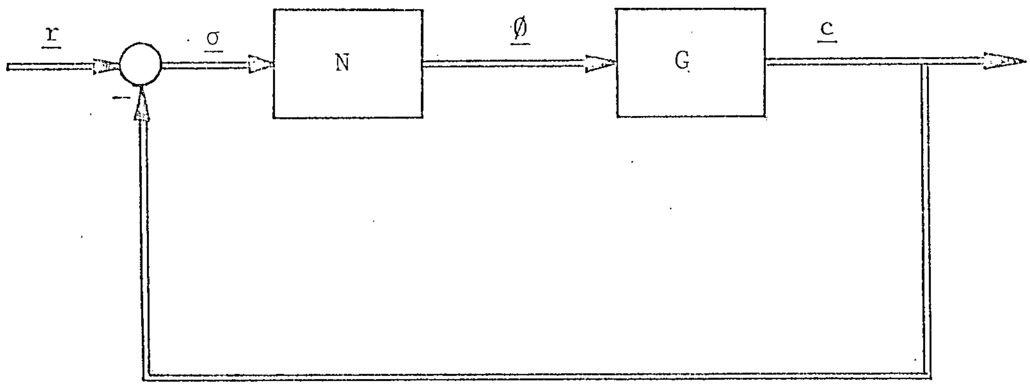


Figure 4.1 Nonlinear Multi-variable Feedback System.

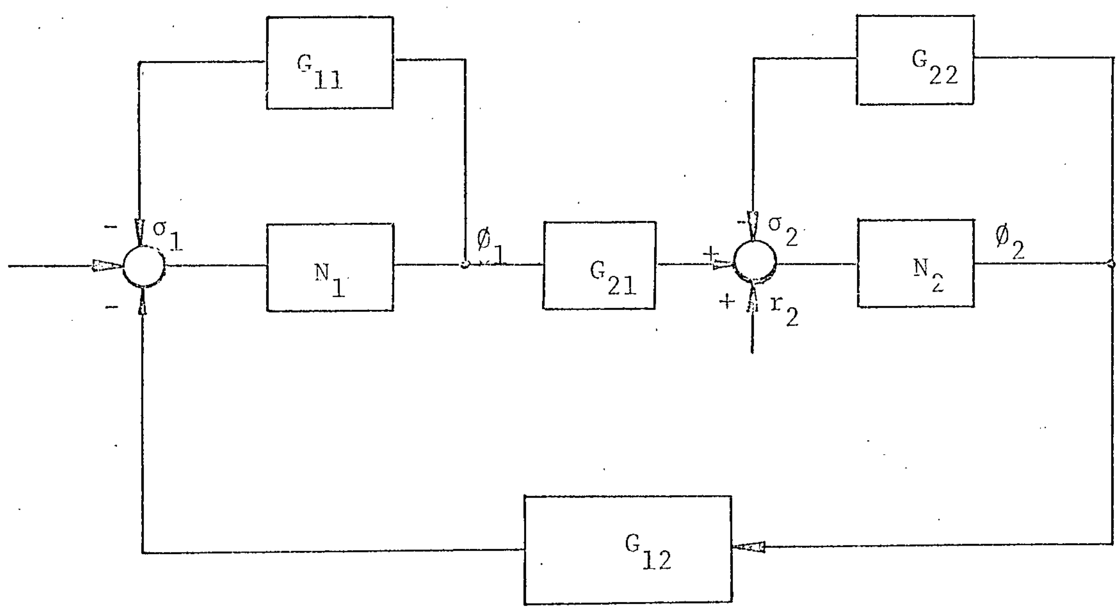


Figure 4.2 Two-Variable Cascade System.

$$P(\omega) \triangleq W(j\omega) + W^*(j\omega) > 0. \qquad \forall\ \omega, \tag{4.4}$$

then the system is absolutely stable.

W^* is the complex conjugate transpose of W and

$$W(j\omega) = [I + j\omega q]G(j\omega) + k^{-1}. \tag{4.5}$$

The proof of the theorem was given by Jury and Lee [9], using the Popov approach, and by Anderson [10], using the Liapunov approach.

4.3 Specific Structures

Let us consider the general case of transfer matrix $G(s)$ as

$$G(s) = [g_{ij}(s)], \qquad (i,j = 1,n), \tag{4.6}$$

for which the corresponding system (with $n = 2$) shown in Figure 4.2 is a cascade connection.

However, when

$$G_{ij}(s) = [G_{1n}(s)], \qquad (i = 2,n;\ j = 1,n), \tag{4.7}$$

the system also implies a parallel connection as shown in Figure 4.3.

In addition, one series case is considered for which

$$G(s) = \begin{bmatrix} 0 & 0 & 0 & \cdots & G_{1n}(s) \\ -G_{21}(s) & 0 & \cdots & \cdots & 0 \\ 0 & -G_{32}(s) & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & -G_{n\,n-1}(s) & 0 \end{bmatrix}. \tag{4.8}$$

The corresponding system is shown in Figure 4.4.

4.4 Application of Theorem III

In this section, the system shown in Figures 4.2 and 4.4 will be considered as example. For the convenience of future use, the $G(j\omega)$ matrix appearing in the stability criterion takes the general form as shown in Eq. (4.6).

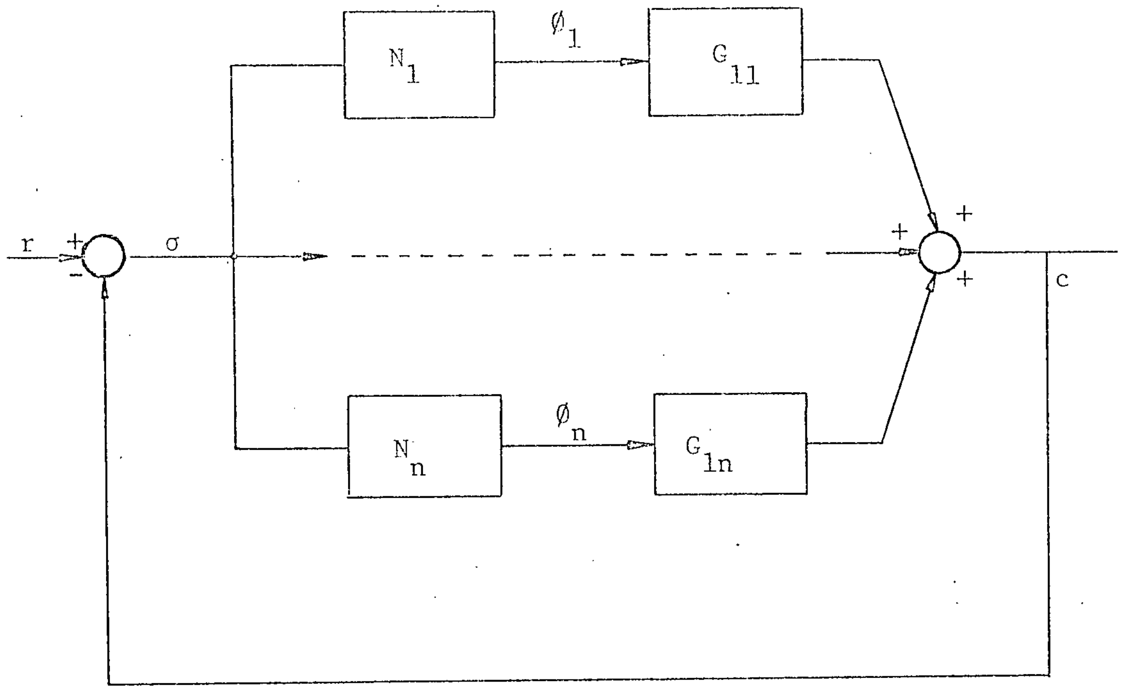


Figure 4.3 n - Variable Parallel System.

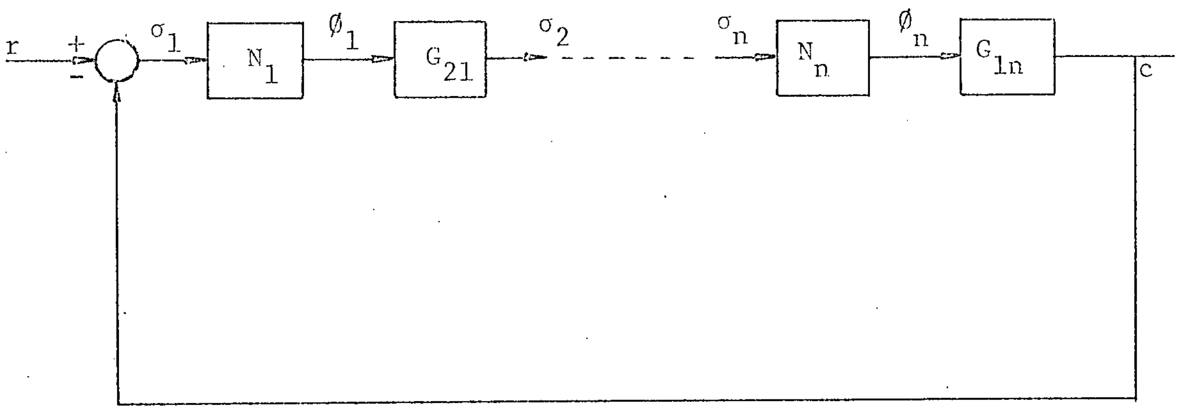


Figure 4.4 n - Variable Series System.

$$W(j\omega) = \begin{bmatrix} (1.+j\omega q_1)G_{11}(j\omega)+K_1^{-1} & (1.+j\omega q_1)G_{12}(j\omega) & \cdots & (1.+j\omega q_1)G_{1n}(j\omega) \\ (1.+j\omega q_2)G_{21}(j\omega) & (1.+j\omega q_2)G_{22}(j\omega)+k_2^{-1} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ (1.+j\omega q_n)G_{n1}(j\omega) & (1.+j\omega q_n)G_{n2}(j\omega) & \cdots & (1.+j\omega q_n)G_{nn}(j\omega)+k_n^{-1} \end{bmatrix} \quad (4.9)$$

then by the theorem, a sufficient condition for absolute stability of the system is that the following criteria matrix

$$\begin{bmatrix} W_{11}(j\omega)+W_{11}(-j\omega) & W_{12}(j\omega)+W_{21}(-j\omega) & \cdots & W_{1n}(j\omega)+W_{n1}(-j\omega) \\ W_{21}(j\omega)+W_{12}(-j\omega) & W_{22}(j\omega)+W_{22}(-j\omega) & \cdots & W_{2n}(j\omega)+W_{n2}(-j\omega) \\ \vdots & \vdots & & \vdots \\ W_{n1}(j\omega)+W_{1n}(-j\omega) & \cdots & & W_{nn}(j\omega)+W_{nn}(-j\omega) \end{bmatrix} \quad (4.10)$$

be positive definite for all ω.

It is seen that even for the very simple system, the polynomials produced by the principal minors of Eq. (4.10) are far too complex for manual evaluation. Even with q_i $(i = 1,2,\ldots,n)$ given there is no known general method for exact finding the largest range on the nonlinearity gains k_i $(i = 1,2,\ldots,n)$.

Our main purpose is to find the best q_i with the largest gains k_i by applying two geometric techniques, namely

(i) Gradient Method

(ii) Projected Gradient Method [16].

The calculations are made by a digital computer, on the basis of the flow-chart shown in Figure 4.5.

The outline of flow-chart is as follows

(1) The coefficients of transfer matrix G(s) and the initial values of q_i and k_i are read.

(2) Frequency is changed from zero until the minimum positive values of the principal minors of the criterion matrix $P(\omega)$ with the corresponding frequency ω_m, are found.

(3) Frequency is fixed at ω_m, and k_i is increased along the normal gradient direction until one of the minors falls into the positive constraint range e_i.

(4) Procedure (2) is repeated again.

(5) If one of the minors is negative at some frequency, then k_i is decreased and the procedures of (2) to (4) are repeated until one of the minimum minors converging in the range e_i, all other minimum minors are positive.

(6) In the general case, q_i will be increased, and the procedures of (2) - (5) are repeated until the maximum values of k_i are found.

(7) In the series case, k_i and q_i will be changed in the projected gradient direction on the basis of ω_m.

(8) Procedures of (2) and (7) are repeated until k_i cannot be increased.

The symbols used in the flow-chart are explained as follows:

Nl: the number of nonlinearity.

M: the order of numerator of G(s) ; N: the order of denominator of G(s).

$a_{i,j,k}$: the coefficients of numerator of G(s).

$b_{i,j,\ell}$: the coefficients of denominator of G(s).

dk: the increment of k_i in the normal gradient direction.

de: the converge factor

xk_{ni}: the coefficients of function $f = \sum_{ni=NO}^{NJ} xk_{ni} q_{ni}$

e_i: the constraint range of Δ_i.

α_{jj}: the increment of projected gradient direction.

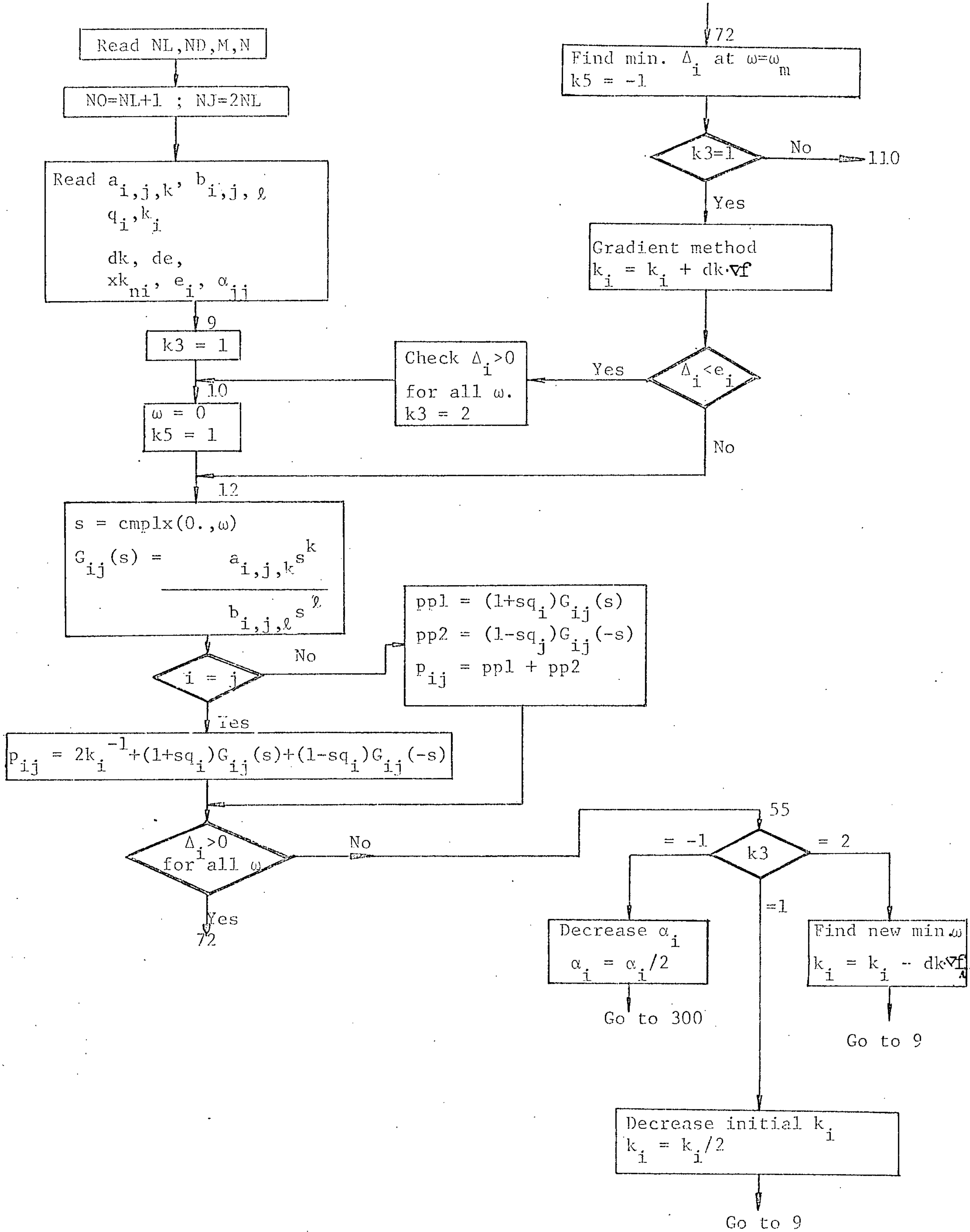
P_{ij}: the elements of criterion matrix P(s).

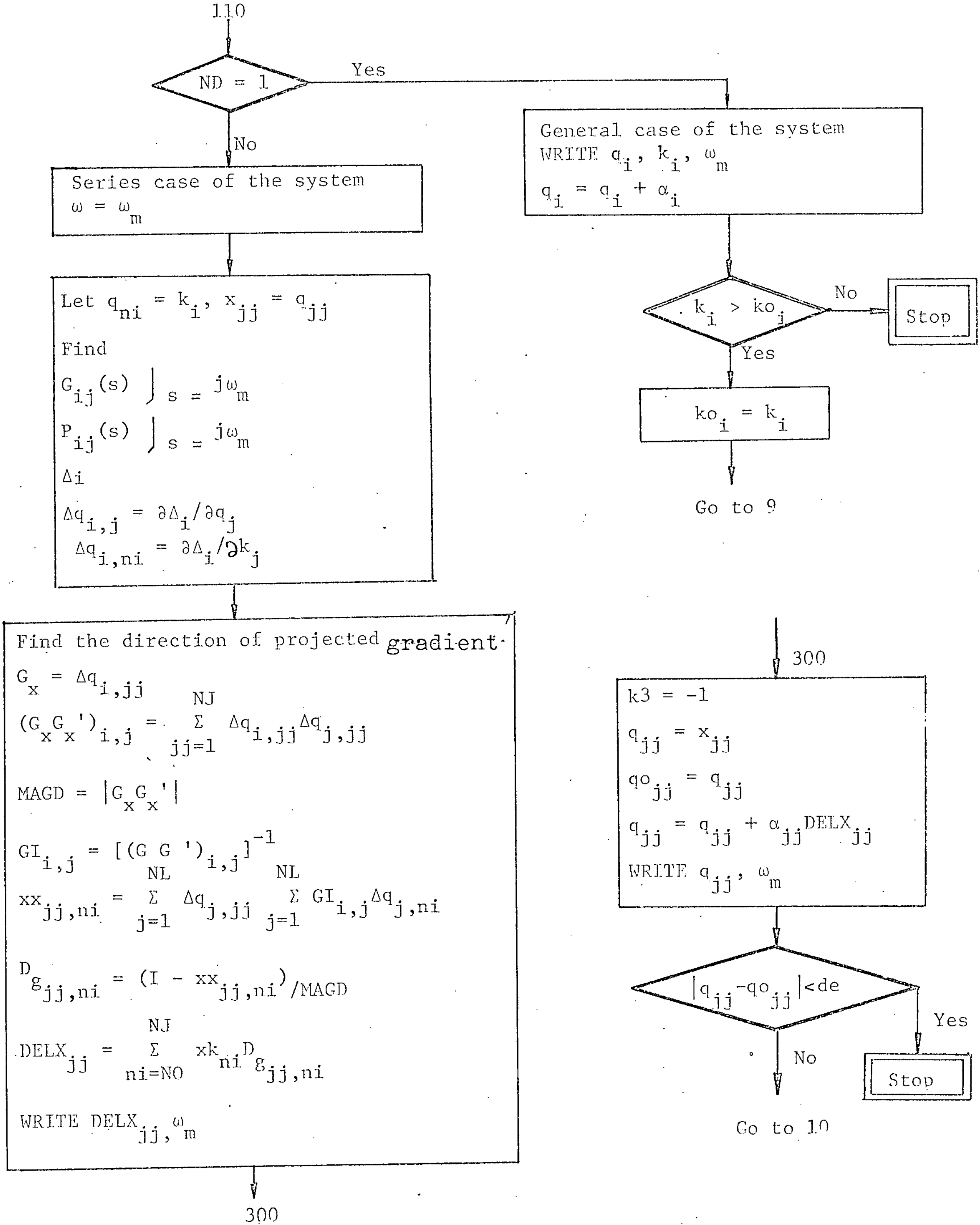
Δ_i: the real principal minors of P(s).

$DELX_{jj}$: projected gradient direction.

NOTE: i,j=(1,NL), jj=(1,NJ), ni=(NO,NJ), k=(0,M), ℓ=(0,N).

Figure 4.5 Flow-Chart of the Multiple Nonlinear System.




110
ND = 1
Yes
No
Series case of the system
$\omega = \omega_m$
Let $q_{ni} = k_i$, $x_{jj} = q_{jj}$
Find
$G_{ij}(s) \Big|_{s = j\omega_m}$
$P_{ij}(s) \Big|_{s = j\omega_m}$
Δi
$\Delta q_{i,j} = \partial\Delta_i/\partial q_j$
$\Delta q_{i,ni} = \partial\Delta_i/\partial k_j$
Find the direction of projected gradient
$G_x = \Delta q_{i,jj}$
$(G_x G_x')_{i,j} = \sum_{jj=1}^{NJ} \Delta q_{i,jj} \Delta q_{j,jj}$
$MAGD = |G_x G_x'|$
$GI_{i,j} = [(G\ G')_{i,j}]^{-1}$
$xx_{jj,ni} = \sum_{j=1}^{NL} \Delta q_{j,jj} \sum_{j=1}^{NL} GI_{i,j} \Delta q_{j,ni}$
$D_{g_{jj,ni}} = (I - xx_{jj,ni})/MAGD$
$DELX_{jj} = \sum_{ni=NO}^{NJ} xk_{ni} D_{g_{jj,ni}}$
WRITE $DELX_{jj}$, ω_m
300
General case of the system
WRITE q_i, k_i, ω_m
$q_i = q_i + \alpha_i$
$k_i > ko_j$
No
Stop
Yes
$ko_i = k_i$
Go to 9
300
$k3 = -1$
$q_{jj} = x_{jj}$
$qo_{jj} = q_{jj}$
$q_{jj} = q_{jj} + \alpha_{jj} DELX_{jj}$
WRITE q_{jj}, ω_m
$|q_{jj} - qo_{jj}| < de$
Yes
No
Stop
Go to 10

Example 1

Consider a series system as shown in Fig. 4.4 with

$$n = 2, \qquad G_{21}(s) = \frac{1}{s+5}, \qquad G_{12}(s) = \frac{s+1}{(s+2)(s+3)} \tag{4.11}$$

The linearized system, that is, the system with the nonlinear elements replaced by linear elements with gains k_i $(i = 1,2)$, is stable for $k_1k_2 > -30$. A.G. Dewey [17] applied the theorem in section 4.2, and found that $q_1 = q_2 = 0.19$, the nonlinear system is stable with $k_1k_2 < 2390$.

Substituting Eq. (4.11) into Eq. (4.10), it follows that the nonlinear system is stable if

(i) $k_1 > 0$,

(ii) $4(k_1k_2)^{-1} - \left| \frac{(1+j\omega q_1)(1+j\omega)}{(2+j\omega)(3+j\omega)} - \frac{(1-j\omega q_2)}{(5-j\omega)} \right|^2 > 0,$

for all ω. (4.12)

It is clear that at $\omega = 0$, the inequality becomes

$$4(k_1k_2)^{-1} - \frac{1}{900} > 0, \tag{4.13}$$

or

$$k_1k_2 < 3600. \tag{4.14}$$

Thus the best that we can expect from inequality (4.12) as the values of q_1 and q_2 are varied is Eq. (4.14). By using of both the normal and projected gradient method, values of $q_1 = 0.20$, $q_2 = 0.166$ are found, for which the system is absolutely stable with $k_1k_2 < 3599$.

Example 2

Consider a three-variable general system as shown in Fig. 4.2, where the matrix transfer function of the linear part is

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+4} & \frac{1}{s+6} \\ \frac{1}{s+9} & \frac{1}{s+.2} & \frac{1}{s+8} \\ \frac{1}{s+10} & \frac{1}{s+12} & \frac{1}{s+3} \end{bmatrix}. \tag{4.15}$$

It can be shown that the linearized system is stable for all positive k_i $(i = 1,2,3)$. A.G. Dewey [17] has used the theorem shown in section 4.2, for a choice of $[q] = 0$, he establishes stability for the nonlinear system with $k_1 = k_2 = k_3 = 10$.

By using of the normal gradient method in this example, it was found that the nonlinear system is stable with

$$k_1 = k_2 = k_3 = \begin{cases} 18, & \text{for } [q] = 0 \\ 323. & \text{for } q_1 = q_2 = q_3 = 0.2 \end{cases} \tag{4.16}$$

Although, the results are much better than the previous work, since the minimum values of principal minors of criteria matrix (4.10) do not simultaneously converge to zero, it is assumed that $q_1 = q_2 = q_3$, so simplifying the computation. Indeed, only the highest principal minor converges to zero.

5. CONCLUSIONS

Sufficient conditions for the absolute stability of nonlinear sampled-data systems have been derived. The method used is the technique of system transformation of Aizerman and Gantmacher, adapted to sampled-data system.

The criterion in Theorem I was expressed in terms of the frequency transfer function of the linear elements and the bounds on the gain and on the slopes of the nonlinear elements. Corollary I was expresses in similar form except that the slope of nonlinear element was bounded from below. These criteria were based on different forms used to approximate the area under the nonlinear characteristic. A simple graphical method for testing stability of the case $q \neq 0$ was suggested.

Theorem II relaxed the restriction on the slope of nonlinear element and introduced a auxiliary function $y(n)$ which may be used to increase the maximum gain k. However, some difficulties occurred in choosing the function $y(n)$.

In the multiple nonlinear continuous systems, by the application of numerical techniques, it was shown that some improvement over previous stability bounds can be made. But, in the general case, the computational difficulties are significant. For this reason, simplified assumptions, such as taking $q_1 = q_2 = q_3$, are often necessary to simplify the conputation.

The extension of the nonlinear sampled-data system criteria given in theorems I and II to systems involving time-varying gain is desirable. Such an extension has been made for the continuous case.

APPENDIX A

The Z-transform of $f_2(n)$ is

$$F_2(z) = \sum_{n=0}^{\infty} f_2(n) z^{-n}. \tag{A.1}$$

By inverse integral,

$$f_2(n) = \frac{1}{2\pi j} \oint_c F_2(z) z^{(n-1)} dz. \tag{A.2}$$

The Z-transform of $f_2(n)\alpha^n$ gives

$$Z[f_2(n)\alpha^n] = F(z/\alpha) = F(z_1). \tag{A.3}$$

By inverse integral,

$$[f_2(n)\alpha^n] = \frac{1}{2\pi j} \oint_c F(z/\alpha) z^{(n-1)} dz. \tag{A.4}$$

Therefore,

$$\sum_{n=0}^{\infty} [f_2(n)\alpha^n][u_N(n)\alpha^n] = \sum_{n=0}^{\infty} u_N(n)\alpha^n \frac{1}{2\pi j} \oint_c F(z/\alpha) z^{(n-1)} dz$$

$$= \frac{1}{2\pi j} \oint_c F(z/\alpha) \left[\sum_{n=0}^{\infty} u_N(n)(\alpha z)^n \right] z^{-1} dz$$

$$= \frac{1}{2\pi j} \oint_c F(z/\alpha) U_N(1/\alpha z) z^{-1} dz. \tag{A.5}$$

Let

$$z_1 = z/\alpha. \tag{A.6}$$

Thus

$$\sum_{n=0}^{\infty} [f_2(n)\alpha^n][u_N(n)\alpha^n] = \frac{1}{2\pi j} \oint_c F(z_1) U_N^*(z_1) z^{-1} dz. \tag{A.7}$$

This is the modified Parseval's theorem.

APPENDIX B

Inequality (2.29) is repeated for convenience

$$\text{r.h.s.} \leq \frac{1}{8\pi\delta_\alpha} \int_{-\pi}^{\pi} |F_4(e^{j\omega}/\alpha)|^2 d\omega + \frac{qk''}{4\pi} \int_{-\pi}^{\pi} |(\frac{e^{j\omega}}{\alpha} - 1)X(e^{j\omega}/\alpha)|^2 \, d\omega. \tag{B.1}$$

By modified Parseval's theorem

$$(2\pi^{-1}) \int_{-\pi}^{\pi} |X(e^{j\omega}/\alpha)|^2 d\omega = \sum_{n=0}^{\infty} x^2(n-1)\alpha^{2(n-1)},$$

$$(2\pi^{-1}) \int_{-\pi}^{\pi} |(\frac{e^{j\omega}}{\alpha} - 1)X(e^{j\omega}/\alpha)|^2 d\omega = \sum_{n=0}^{\infty} [\nabla x(n)]^2 \alpha^{2(n-1)}. \tag{B.2}$$

Replacing F_4 in the first integral in (B.1) by the expression in Eq. (2.24), therefore,

$$\int_{-\pi}^{\pi} |F_4(e^{j\omega}/\alpha)|^2 = \int_{-\pi}^{\pi} |F_2(e^{-j\omega}/\alpha) + qk''|(e^{j\omega}/\alpha) - 1|^2 X(e^{-j\omega}/\alpha)G(e^{j\omega}/\alpha)|^2 dw. \tag{B.3}$$

Since

$$f_2(n)\alpha^{(n-1)} = [x(n-1) + \hat{q}\nabla x(n)]\alpha^{(n-1)}, \tag{B.4}$$

then, by modified Parseval's theorem

$$F_2(e^{j\omega}/\alpha) = [1 + q(1 - \frac{e^{-j\omega}}{\alpha})]X(e^{j\omega}/\alpha), \tag{B.5}$$

and

$$F_2(e^{-j\omega}/\alpha) = [1 + q(1 - \frac{e^{j\omega}}{\alpha})] \; X(e^{-j\omega}/\alpha). \tag{B.6}$$

Therefore (B.3) may be rewritten as

$$\int_{-\pi}^{\pi} |F_4(e^{j\omega}/\alpha)|^2 \, d\omega$$

$$= \int_{-\pi}^{\pi} |[1+q(1 - \frac{e^{j\omega}}{\alpha})]X(e^{-j\omega}/\alpha)|^2 \; \left|1 - \frac{k''q|(\frac{e^{j\omega}}{\alpha} - 1)|^2 G(e^{j\omega}/\alpha)}{1 + q(1 - \frac{e^{j\omega}}{\alpha})}\right|^2 d\omega. \tag{B.7}$$

Since $G(z_1)$ is analytic in the domain $\alpha > 1$ for α sufficiently small, therefore

$$\sup \left|1 - \frac{k''q|(e^{j\omega}/\alpha) - 1|^2 G(e^{j\omega}/\alpha)}{1 + q(1 - \frac{e^{j\omega}}{\alpha})}\right|^2 \leq E < \infty. \tag{B.8}$$

Because

$$|F_4|^2 = |F_4^*|^2,$$

hence

$$\int_{-\pi}^{\pi} |F_4(e^{j\omega}/\alpha)|^2 d\omega \leq E \int_{-\pi}^{\pi} |[1 + q(1 - \frac{e^{-j\omega}}{\alpha})]X(e^{j\omega}/\alpha)|^2 \, d\omega$$

$$\leq 2\pi E \sum_{n=0}^{\infty} [x(n-1) + q\nabla x(n)]^2 \alpha^{2(n-1)}. \qquad \text{(B.9)}$$

Making use of (B.2) and (B.9), inequality (B.1) becomes

$$\text{r.h.s.} \leq \sum_{n=0}^{\infty} \{\frac{E}{4\delta_\alpha} [x(n-1) + q\nabla x(n)]^2 + \frac{k''q}{2}[\nabla x(n)]^2\}\alpha^{2(n-1)}, \qquad \text{(B.10)}$$

which is inequality used in Eq. (2.30).

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