THE MAGNETIC MOMENT FORM FACTOR OF He^3

by

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B.S., Washington State University, 1973

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Physics

We accept this thesis as conforming to the

required standard

THE UNIVERSITY OF BRITISH COLUMBIA

October, 1975

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Abstract

The Derrick-Blatt expansion of the He^3 wave function is used to derive an expression for the magnetic moment form factor for He^3. The symmetric and mixed symmetric S states and all the D states of the expansion are retained in the calculation.

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Acknowledgements

I would like to thank Dr. J. M. McMillan for his suggestion of this problem and for his assistance in completing it.

1 Introduction

By studying light nuclei one hopes to gain information about the nuclear force. The natural starting point in this study is of course the deuteron, being the simplest nucleus. Indeed much has been learned from the study of the deuteron. For example, the nonzero quadrupole moment of the deuteron implies the existence of the nuclear tensor force.

The next step in complexity then is the study of the three-nucleon nuclei, that is, the triton or He^3. The three-nucleon system is in many ways better suited for the study of the nuclear force, being for example more tightly bound than the deuteron allowing one to probe the nuclear force more closely.

Derrick and Blatt(1958) have contributed significantly to the understanding of the three-nucleon system, constructing a complete set of states in terms of which the three-nucleon wave function may be expanded. The construction is analogous to the construction of the deuteron wave function found in Blatt and Weisskopf(1952). Further, Derrick(1960a,b) has derived a set of sixteen coupled partial differential equations in three variables for the expansion coefficients (the so-called internal wave functions). There has since been a great deal of work on the numerical calculations of these internal wave functions.

Investigation of the electromagnetic form factors provides one means of testing a given set of internal wave functions. Best(1966) has calculated an expression for the charge form factor for the triton using the Derrick-Blatt expansion of the wave function. Best retained in his calculation the symmetric and mixed-symmetric S states and all the D states. Derrick and Blatt have shown that the remaining states do not contribute significantly to the ground state wave function. Best also obtained some numerical results using a modified Feshbach-Rubinow(1955) approach to approximate the internal wave functions. In this work we calculate an expression for the magnetic moment form factor for He^3, again using the Derrick-Blatt expansion of the wave function. Like Best we retain the symmetric and mixed-symmetric S states and all the D states. Unlike the charge form factor, however, the magnetic moment form factor contains cross terms between S and D states. These arise because of the appearance of the spin operator in the magnetic moment density operator. Thus the magnetic moment form factor can be sensitive to the presence of the D states.

Schiff(1964) and Gibson(1965) have also carried out a calculation of the He^3 magnetic moment form factor. They, however, have used the Sachs(1953) expansion of the wave function, which is less convenient to work with since the angular momentum states are not orthogonal, wheras the angular momentum states in the Derrick-Blatt expansion are orthogonal. Also the Gibson and Schiff calculation lacks the generality of our calculation as their expression is based on a particular assumed form of the internal wave functions. No particular form is assumed in our calculation.

Numerical results may be obtained by evaluating the three-dimensional integrals included in our final expression using any one of the available sets of internal wave functions.

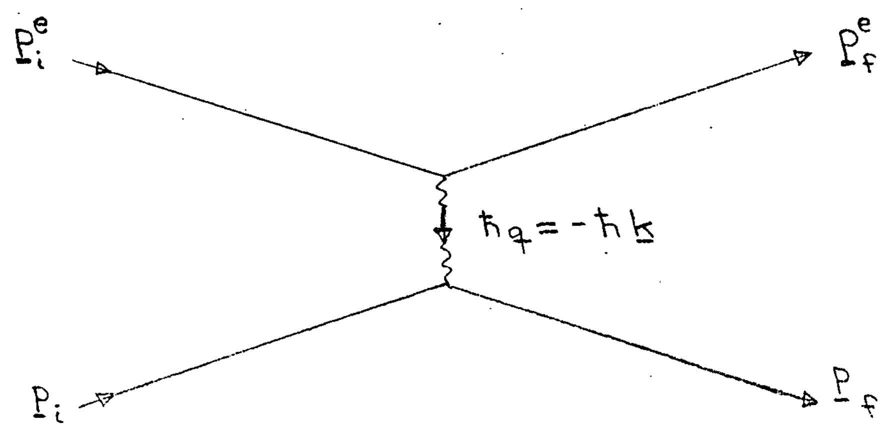
2 The Magnetic Moment Form Factor of He^3

Consider the elastic scattering of an electron with initial momentum $\underline{p}_i^e$ from He^3 having initial momentum $\underline{P}_i$. After the scattering takes place the electron emerges with final momentum $\underline{p}_f^e$ and the He^e recoils with final momentum $\underline{P}_f$. The momentum transfer q of the electron and the change $\underline{K}$ in momentum of the He^3 are defined as

$$q = (\underline{p}_i^e - \underline{p}_f^e)/\hbar \tag{2.1}$$

$$\underline{K} = (\underline{P}_i - \underline{P}_f)/\hbar \tag{2.2}$$

These terms are shown pictorially below:



The magnetic moment form factor for He^3 is taken by Schiff(1964) to be the Fourier transform of the expectation value of the magnetic moment density operator

$$\mu(He^3)\tilde{F}_{mag}(He^3) = \int e^{i\underline{q}\cdot\underline{x}} \langle \rho_{mag} \rangle \, d^3x \qquad (2.3)$$

where the magnetic moment density operator is

$$\rho_{mag} = \frac{1}{2}\sum_{k=1}^{3}\left[\sigma_{kz}(1+\tau_{kz})\mu_p f^p_{mag}(\underline{x}-\underline{r}_k) + \sigma_{kz}(1-\tau_{kz})\mu_n f^n_{mag}(\underline{x}-\underline{r}_k)\right]. \qquad (2.4)$$

The σ's and τ's are the unit amplitude Pauli matrices operating on the spin and isospin functions respectively. The f's are the spatial distribution functions for the moment densities about the centres of the nucleons, while the μ's are the static magnetic moments of the nucleons. The variables $\underline{x}$, $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_3$ are shown in figure (1).

The Derrick-Blatt wave function described in appendix A includes only the internal coordinates of He^3. For the calculation of the magnetic moment form factor, however, we need to include the centre of mass coordinates of the nucleus. Defining the centre of mass wave number to be $\underline{K}_G$ and the centre of mass position vector to be $\underline{R}_G$ the centre of mass wave function may be written as

$$\chi = \frac{1}{(2\pi)^{3/2}} e^{i\underline{K}_G\cdot\underline{R}_G} \qquad (2.5)$$

and the total wave function for He^3 including all coordinates is

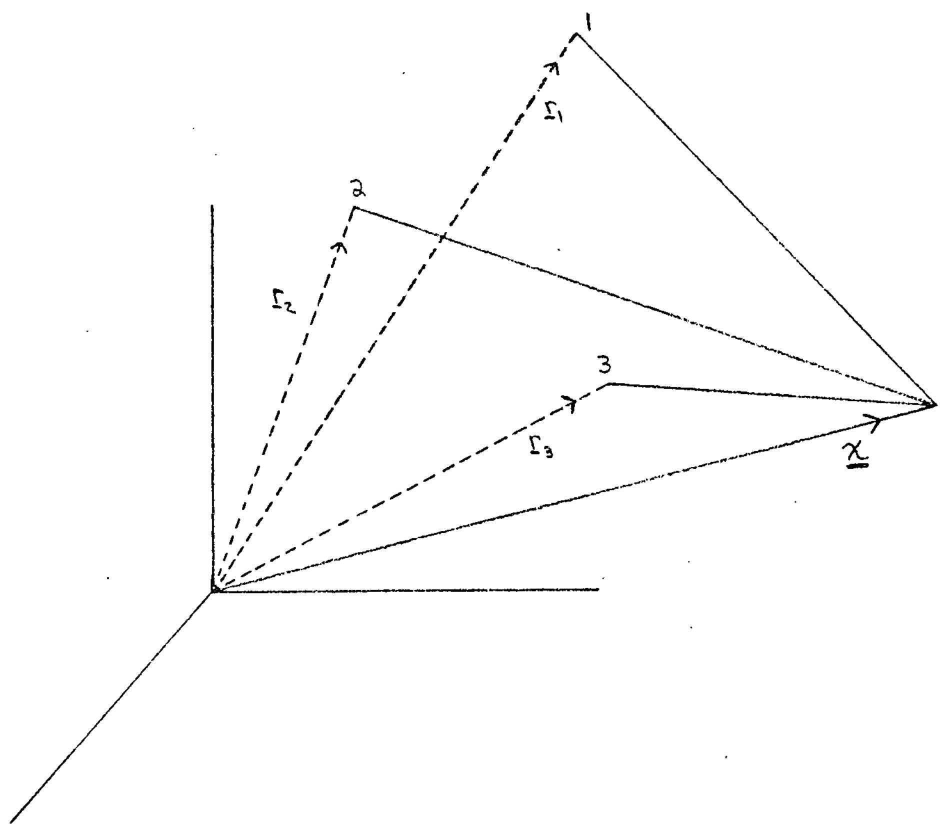


Figure 1 - The vectors $\underline{\chi}$, $\underline{r}_1$, $\underline{r}_2$, and $\underline{r}_3$.

then

$$\Psi = \chi \, \psi \, . \tag{2.6}$$

The form factor may then be written

$$\mu(He^3)\widetilde{F}_{mag}(He^3) = \tfrac{1}{2}\sum_{k=1}^{3} \iint e^{iq\cdot\rho}\, \Psi^* \Big[\sigma_{kz}(1+\tau_{kz})\mu_p f^p_{mag}(\underline{x}-\underline{r}_k)$$

$$+ \sigma_{kz}(1-\tau_{kz})\mu_n f^n_{mag}(\underline{x}-\underline{r}_k)\Big]\Psi \, dV\, d^3x \tag{2.7}$$

where

$$\int dV = \iint \cdots \int d^3\underline{r}_1\, d^3\underline{r}_2\, d^3\underline{r}_3 \, . \tag{2.8}$$

Defining the respective neutron and proton magnetic moment form factors as

$$F^n_{mag} = \int e^{iq\cdot\underline{u}} f^n_{mag}(\underline{u})\, d^3\underline{u} \tag{2.9}$$

and

$$F^p_{mag} = \int e^{iq\cdot\underline{u}} f^p_{mag}(\underline{u})\, d^3\underline{u} \tag{2.10}$$

and making the change of variables $\underline{u} = \underline{\chi} - \underline{r}_k$ enables one to write the form factor as

$$\mu(He^3)\tilde{F}_{mag}(He^3) = \tfrac{1}{2}\mu_p F^p_{mag} \sum_{k=1}^{3} \int e^{iq\cdot \underline{r}_k} \Psi^* [\sigma_{kz}(1+\tau_{kz})] \Psi \, dV$$

$$+ \tfrac{1}{2}\mu_n F^n_{mag} \sum_{k=1}^{3} \int e^{iq\cdot \underline{r}_k} \Psi^* [\sigma_{kz}(1-\tau_{kz})] \Psi \, dV. \qquad (2.11)$$

With the additional change of variables $\underline{r}_k = \underline{R}_G + \underline{r}'_k$, as shown in figure (2), we may write

$$\mu(He^3)\tilde{F}_{mag}(He^3) = \frac{1}{8\pi^3} \int e^{i(q+\underline{k})\cdot \underline{R}_G} \, d^3\underline{R}_G$$

$$\cdot \left\{ \tfrac{1}{2}\mu_p F^p_{mag} \sum_{k=1}^{3} \int e^{iq\cdot r'_k} \psi^* [\sigma_{kz}(1+\tau_{kz})] \psi \, dS \right.$$

$$\left. + \tfrac{1}{2}\mu_n F^n_{mag} \sum_{k=1}^{3} \int e^{iq\cdot r'_k} \psi^* [\sigma_{kz}(1-\tau_{kz})] \psi \, dS \right\}$$

$$= \delta(q+\underline{k})\, \mu(He^3) F_{mag}(He^3) \qquad (2.12)$$

where we have defined

$$\mu(He^3) F_{mag}(He^3) \qquad (2.13)$$

$$= \tfrac{1}{2}\mu_p F^p_{mag} \sum_{k=1}^{3} \int e^{iq\cdot r'_k} \psi^* [\sigma_{kz}(1+\tau_{kz})] \psi \, dS$$

$$+ \tfrac{1}{2}\mu_n F^n_{mag} \sum_{k=1}^{3} \int e^{iq\cdot r'_k} \psi^* [\sigma_{kz}(1-\tau_{kz})] \psi \, dS$$

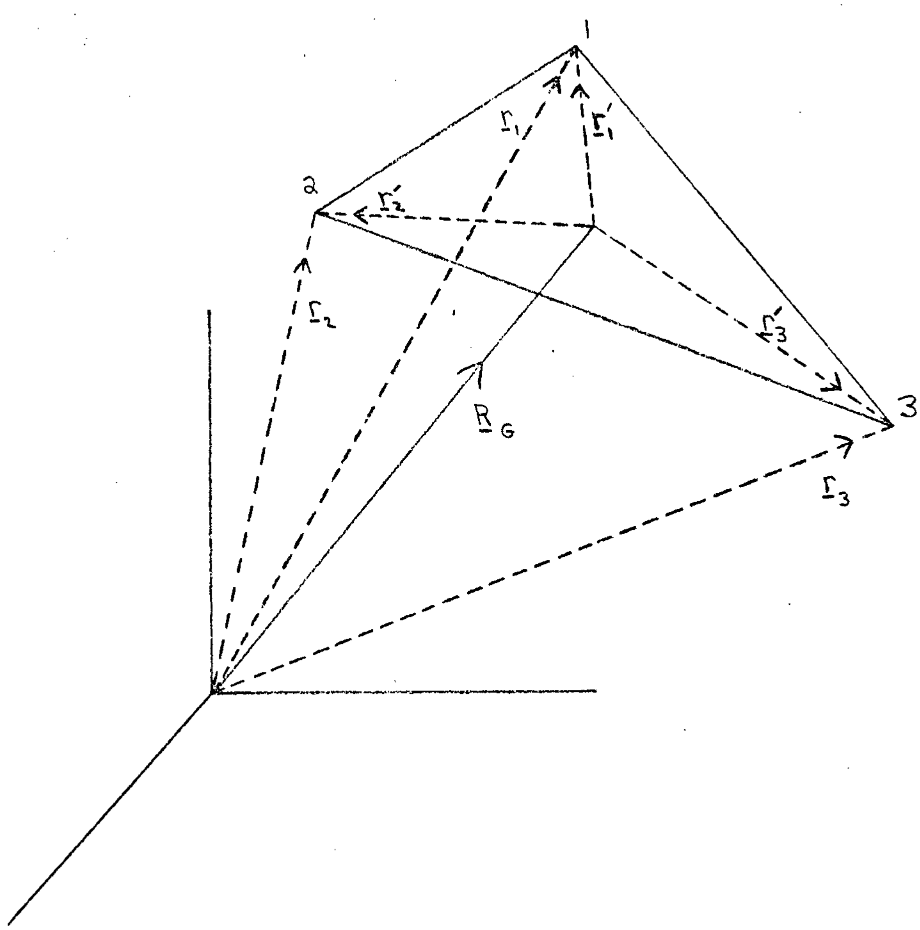


Figure 2 - The change of variables $\underline{r}_k = \underline{R}_G + \underline{r}_k'$.

and where

$$\int dS = \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\gamma \int_{0}^{\pi} \sin\beta \, d\beta \int_{0}^{\infty} dr_{13} \int_{0}^{\infty} dr_{23} \int_{|r_{13}-r_{23}|}^{r_{13}+r_{23}} dr_{12} \, r_{13} r_{23} r_{12} \, . \tag{2.14}$$

The angles α , β , and γ are the three Euler angles required to specify the spatial orientation of the triangle.[1]

The calculations of the matrix elements $\psi^*(\sigma_{kz} \pm \sigma_{kz} T_{kz})\psi$ and the integrations over the Euler angles may be done analyticaly. Writing the wave function as a sum of angular momentum states enables one to perform the calculations term by term, that is

$$\psi^*(\sigma_{kz} \pm \sigma_{kz} T_{kz})\psi = \sum_{i,j} \psi_i^*(\sigma_{kz} \pm \sigma_{kz} T_{kz})\psi_j \tag{2.15}$$

where the ψ_i are defined in equation (A.7.2). To illustrate how each of these terms is calculated we will determine here the terms $\psi_1^*(\sigma_{kz} \pm \sigma_{kz} T_{kz})\psi$. All other terms are found in a similar fashion. To simplify calculations each of the terms $\psi_1^* \sigma_{kz} \psi_8$ and $\psi_1^* \sigma_{kz} T_{kz} \psi_8$ are determined separately, then the sums and differences are found.

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[1] See appendix A.2 for more details.

Now

$$\sum_{k=1}^{3}\int e^{iq\cdot r_k'}\psi_1^*\sigma_{kz}\psi_8 =$$

$$\sum_{k=1}^{3}\int e^{iq\cdot r_k'} f_1 y_1^* \sigma_{kz}\frac{1}{\sqrt{2}}\left(f_{8,1}y_{8,2}-f_{8,2}y_{8,1}\right)dS. \tag{2.16}$$

We will calculate first the term $\sum_{k=1}^{3}\int e^{iq\cdot r_k'} f_1 y_1^*\sigma_{kz}\frac{1}{\sqrt{2}}f_{8,1}y_{8,2}$. Now $V^*_{M_TM_sK_t}(P_t,T,S)\,\sigma_{kz}V_{M_T'M_s'K_t}(P_t',T',S')$ is zero unless $M_s' = M_s$ so we may write

$$\sum_{k=1}^{3} 1/\sqrt{2}\int e^{iq\cdot r_k'} f_1 f_{8,1} y_1^*\sigma_{kz} y_{8,2}\,dS$$

$$=\sum_{k=1}^{3} 1/\sqrt{2}\int e^{iq\cdot r_k'} f_1 f_{8,1} Y_0^{0*}(S,0)\,1/\sqrt{5}\,Y_0^2(S,0)\,V^*_{\frac{1}{2}\frac{1}{2}1}(a\tfrac{1}{2}\tfrac{1}{2})\,\sigma_{kz}V_{\frac{1}{2}\frac{1}{2}2}(m\tfrac{1}{2}\tfrac{3}{2})dS$$

$$=\sum_{k=1}^{3} 1/\sqrt{10}\int e^{iq\cdot r_k'} f_1 f_{8,1} Y_0^{0*}(S,0)\,Y_0^2(S,0)\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}dS \tag{2.17}$$

where $\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}$ denotes

$$\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix} = V^*_{\frac{1}{2}\frac{1}{2}1}(a\tfrac{1}{2}\tfrac{1}{2})\,\sigma_{kz}V_{\frac{1}{2}\frac{1}{2}2}(m\tfrac{1}{2}\tfrac{3}{2}) = \begin{cases}1/3 & \text{for } k=1\\ 1/3 & \text{for } k=2\\ -2/3 & \text{for } k=3\end{cases} \tag{2.18}$$

Making use of the permutation properties of the functions we now eliminate the sum by changing the variables of integration to $\underline{r}_3'$. f_1, $Y_0^0(S,0)$, $Y_0^2(S,0)$, and dS are all symmetric under permutations while $f_{8,1}$ belongs to the first row of the mixed representation. Permuting the variables $\underline{r}_1'$ and $\underline{r}_3'$ results in[1]

$$(13)\, f_{8,1} = -\tfrac{1}{2} f_{8,1} - \tfrac{\sqrt{3}}{2} f_{8,2} \tag{2.19}$$

and permuting the variables r_2' and r_3' results in

$$(23)\, f_{8,1} = -\tfrac{1}{2} f_{8,1} + \tfrac{\sqrt{3}}{2} f_{8,2} \,. \tag{2.20}$$

Hence equation (2.17) becomes

$$\begin{aligned} &\frac{1}{\sqrt{10}} \int e^{iq\cdot r_3'} Y_0^{0*}(s,0)\, Y_0^2(s,0)\, f_1 \left\{ \tfrac{1}{3}\left(-\tfrac{1}{2} f_{8,1} - \tfrac{\sqrt{3}}{2} f_{8,2}\right) + \tfrac{1}{3}\left(-\tfrac{1}{2} f_{8,1} + \tfrac{\sqrt{3}}{2} f_{8,2}\right) - \tfrac{2}{3} f_{8,1} \right\} dS \\ &= -\frac{1}{\sqrt{10}} \int e^{iq\cdot r_3'} Y_0^{0*}(s,0)\, Y_0^2(s,0)\, f_1 f_{8,1}\, dS \\ &= -\frac{\sqrt{2}}{16\pi^2} \int e^{iq\cdot r_3'} f_1 f_{8,1} D_{00}^2(R)\, d\tau\, d\Omega \end{aligned} \tag{2.21}$$

where

$$\int d\tau = \int_0^\infty dr_{13} \int_0^\infty dr_{23} \int_{|r_{13}-r_{23}|}^{r_{13}+r_{23}} dr_{12}\, r_{13} r_{23} r_{12} \tag{2.22}$$

and

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[1] See appendix A.3.

$$\int d\Omega = \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\gamma \int_{0}^{\pi} \sin\beta \, d\beta \tag{2.23}$$

and where we have written

$$Y_0^0(s,0) = \sqrt{2}/4\pi \tag{2.24}$$

$$Y_0^2(s,0) = \sqrt{10}/4\pi \; D_{00}^2(R) \; . \tag{2.25}$$

We use here the argument R to denote the Euler angles (α,β,γ).

To perform the integration over the Euler angles we need to consider the Euler angle dependence of $e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}_3'}$.[1] expanding the exponential we have

$$e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}_3'} = 4\pi \sum_{l,m} i^l j_l(qr_3') Y_{lm}(\hat{q}) Y_{lm}^*(\hat{r}_3') \tag{2.26}$$

where the j_l's are the spherical Bessel functions and the Y_{lm}'s are the spherical harmonics. Furthermore $\hat{\underset{\sim}{q}}$ and $\hat{\underset{\sim}{r}}_3$ denote the spherical polar angles that $\underset{\sim}{q}$ and $\underset{\sim}{r}_3$, respectively, make with

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[1] The Euler angle dependence of the exponential seems to have been overlooked by Best(1966), but his final results in equations (5.23) and (5.35) are correct.

the space fixed coordinate system.

The Euler angles represent a rotation of the body fixed system into the space fixed system. From Wigner (1959) we find[1]

$$Y^*_{\ell m}(\hat{\underline{r}}'_3) = \sum_{m'} D^{\ell}_{mm'}(R^{-1})\, Y^*_{\ell m'}(\tilde{\underline{r}}_3) = \sum_{m'} D^{\ell}_{m'm}(R)\, Y^*_{\ell m'}(\tilde{\underline{r}}_3) \qquad (2.27)$$

so

$$e^{i\underline{q}\cdot\underline{r}_3} = 4\pi \sum_{\ell,m,m'} i^{\ell} j_{\ell}(qr'_3)\, Y_{\ell m}(\hat{q})\, Y^*_{\ell m'}(\tilde{\underline{r}}_3)\, D^{\ell *}_{m'm}(R) \qquad (2.28)$$

where now $Y^*_{\ell m'}(\tilde{\underline{r}}_3)$ is a function of the angles of $\underline{r}_3$ with respect to the body fixed system.[2]

Equation (2.21) now becomes

$$-\frac{\sqrt{2}}{4\pi} \int \sum_{\ell,m,m'} i^{\ell} j_{\ell}(qr'_3)\, Y_{\ell m}(\hat{q})\, Y^*_{\ell m'}(\tilde{\underline{r}}_3)\, D^{*\ell}_{m'm}(R)\, D^{2}_{00}(R)\, f_1 f_{8,1}\, dS$$

$$= \frac{2\sqrt{2}\,\pi}{5} Y^*_{20}(\tilde{\underline{r}}_3) \int j_2(qr'_3)\, Y_{20}(\hat{q})\, f_1 f_{8,1}\, d\tau \qquad (2.29)$$

where we have made use of the equation[3]

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1 We show in appendix C the relationship between the rotational representation coefficients of Wigner and of Derrick and Blatt.
2 See appendix B.
3 See Wigner (1959).

$$\int D^{j_3 *}_{\mu_3 m_3}(R)\; D^{j_2}_{\mu_2 m_2}(R)\; D^{j_1}_{\mu_1 m_1}(R)\; d\Omega$$

$$= \frac{8\pi^2}{2j_3+1}\; \delta_{\mu_1+\mu_2,\mu_3}\delta_{m_1+m_2,m_3}\langle j_1, j_2, \mu_1, \mu_2 | j_3, \mu_3\rangle\langle j_1, j_2, m_1, m_2 | j_3, m_3\rangle\,. \tag{2.30}$$

The z axis of our space fixed coordinate system lies perpendicular to the scattering plane, hence $\underline{q}$ lies in the x-y plane. We can thus write

$$Y_{20}(\hat{\underline{q}}) = -\frac{1}{4}\sqrt{\frac{5}{\pi}} \tag{2.31}$$

so that equation (2.29) becomes

$$-\sqrt{\frac{\pi}{10}}\; Y_{20}(\tilde{\underline{r}}_3)\int j_2(qr_3')\, f_1 f_{8,1}\, d\tau\,. \tag{2.32}$$

By a similar calculation we find that

$$-\sum_{k=1}^{3}\int e^{i\underline{q}\cdot\underline{r}_k'}\, f_1 f_{8,2}\, 1/\sqrt{2}\; y_1^*\, \sigma_{kz}\, y_{8,1}\, dS$$

$$= -\sqrt{\frac{\pi}{10}}\; Y_{20}(\tilde{\underline{r}}_3)\int j_2(qr_3')\, f_1 f_{8,1}\, d\tau \tag{2.33}$$

so that

$$\sum_{k=1}^{3}\int e^{i\mathbf{q}\cdot\mathbf{r}_k'}\,\psi_1^*\,\sigma_{kz}\,\psi_8\,dS =$$

$$= -2\sqrt{\frac{\pi}{10}}\,Y_{20}(\tilde{\mathbf{r}}_3)\int j_2(qr_3')\,f_1\,f_{8,1}\,d\tau\,. \tag{2.34}$$

Also one can show that

$$\sum_{k=1}^{3}\int e^{i\mathbf{q}\cdot\mathbf{r}_k'}\,\psi_1^*\,\sigma_{kz}\,\tau_{kz}\,\psi_8$$

$$= -2\sqrt{\frac{\pi}{10}}\,Y_{20}(\tilde{\mathbf{r}}_3)\int j_2(qr_3')\,f_1\,f_{8,1}\,d\tau \tag{2.35}$$

so

$$\sum_{k=1}^{3}\int e^{i\mathbf{q}\cdot\mathbf{r}_k'}\,\psi_1^*(\sigma_{kz}+\sigma_{kz}\tau_{kz})\,\psi_8\,dS$$

$$= -4\sqrt{\frac{\pi}{10}}\,Y_{20}(\tilde{\mathbf{r}}_3)\int j_2(qr_3')\,f_1\,f_{8,1}\,d\tau \tag{2.36}$$

and

$$\sum_{k=1}^{3}\int e^{i\mathbf{q}\cdot\mathbf{r}_k'}\,\psi_1^*(\sigma_{kz}-\sigma_{kz}\tau_{kz})\,\psi_8\,dS = 0. \tag{2.37}$$

Since σ_{kz} and $\tau_{kz}\sigma_{kz}$ are Hermitian and since the matrix elements

$V^{*}_{n_T n_S k_t}(P_t,T,S)\sigma_{k_z} V_{n_T' n_S' k_t'}(P_t',T',S')$ and $V_{n_T n_S k_t}(P_t,T,S)\sigma_{k_z}\tau_{k_z} V_{n_T' n_S' k_t'}(P_t',T',S')$ are real we see that

$$\sum_{k=1}^{3}\int e^{iq\cdot r_k'}\,\psi_1^{*}\,(\sigma_{k_z}\pm\sigma_{k_z}\tau_{k_z})\,\psi_8\,dS$$

$$=\sum_{k=1}^{3}\int e^{iq\cdot r_k'}\,\psi_8^{*}\,(\sigma_{k_z}\pm\sigma_{k_z}\tau_{k_z})\,\psi_1\,dS \tag{2.38}$$

Hence the contribution to the magnetic moment form factor from the cross terms of ψ_1 and ψ_8 is

$$\mu\,(He^3)\,F^{(1,8)}_{mag}\,(He^3) =$$

$$\tfrac{1}{2}\mu_p F^{p}_{mag}\left\{-8\sqrt{\tfrac{\pi}{10}}\;Y_{20}(\tilde{r}_3)\int j_2(qr_3')\,f_1 f_{8,1}\,d\tau\right\}. \tag{2.39}$$

From appendix B we have that

$$Y_{20}(\tilde{r}_3) = -\frac{1}{4}\sqrt{\frac{5}{\pi}} \tag{2.40}$$

so that we may write

$$\sum_{k=1}^{3} \int e^{i q \cdot r_k'} \psi_1^* (\sigma_{kz} + \sigma_{kz} \tau_{kz}) \psi_8 \, dS$$

$$= \sqrt{2}/4 \int j_2(q r_3') f_1 f_{8,1} \, d\tau$$

(2.41)

The Derrick-Blatt wave function has been approximated by using only the symmetric S, mixed symmetric S, and mixed symmetric D states of the total wave function. Table (1) lists all the terms calculated. The notation used in table (1) is

$$(i, j)_{A_k} = \sum_{k=1}^{3} \int e^{i q \cdot r_k'} \psi_i^* A_k \psi_j \, dS \tag{2.42}$$

where A_k denotes σ_{kz} or $\tau_{kz}\sigma_{kz}$. The functions Σ and Γ are defined in appendix B.

$$(1,1)_\sigma = \int d\tau\, j_0(qr_3')\, f_1^2$$

$$(1,3)_\sigma = \sqrt{2} \int d\tau\, j_0(qr_3')\, f_1 f_{3,1}$$

$$(1,8)_\sigma = \sqrt{2}/4 \int d\tau\, j_2(qr_3')\, f_1 f_{8,1}$$

$$(1,9)_\sigma = \frac{\sqrt{3}}{2\sqrt{2}}\, \Gamma \int d\tau\, j_2(qr_3')\, f_1 f_{9,1}$$

$$(1,10)_\sigma = 1/\sqrt{6} \sum \int d\tau\, j_2(qr_3')\, f_1 f_{10,2}$$

$$(3,3)_\sigma = \tfrac{1}{2} \int d\tau\, j_0(qr_3')\, (f_{3,1}^2 + f_{3,2}^2)$$

$$(3,8)_\sigma = -1/4 \int d\tau\, j_2(qr_3')\, (f_{3,1} f_{8,1} - f_{3,2} f_{8,2})$$

$$(3,9)_\sigma = -\sqrt{3}/4\, \Gamma \int d\tau\, j_2(qr_3')\, (f_{3,1} f_{9,1} - f_{3,2} f_{9,2})$$

$$(3,10)_\sigma = 1/4\sqrt{3} \sum \int d\tau\, j_2(qr_3')\, (f_{3,1} f_{10,2} + f_{3,2} f_{10,1})$$

Table 1 - The terms $\sum_{k=1}^{3} \int e^{i\mathbf{q}\cdot\mathbf{r}_k'}\, \psi_i^* A_k \psi_j\, ds$

$$(8,8)_\sigma = -\tfrac{1}{4}\int d\tau\, j_2(qr_3')\left(f_{8,1}^2 + f_{8,2}^2\right)$$

$$-\tfrac{1}{2}\int d\tau\, j_0(qr_3')\left(f_{8,1}^2 + f_{8,2}^2\right)$$

$$(8,9)_\sigma = \tfrac{\sqrt{3}}{2}\,\Gamma\int d\tau\, j_2(qr_3')\left(f_{8,1}f_{9,1} + f_{8,2}f_{9,2}\right)$$

$$(8,10)_\sigma = \tfrac{\sqrt{3}}{2}\,\Gamma\int d\tau\, j_2(qr_3')\left(f_{8,2}f_{10,1} - f_{8,1}f_{10,2}\right)$$

$$(9,9)_\sigma = -\tfrac{1}{4}\int d\tau\, j_2(qr_3')\left(f_{9,1}^2 + f_{9,2}^2\right)$$

$$-\tfrac{1}{2}\int d\tau\, j_0(qr_3')\left(f_{9,1}^2 + f_{9,2}^2\right)$$

$$(9,10)_\sigma = 0$$

$$(10,10)_\sigma = \tfrac{1}{4}\int d\tau\, j_2(qr_3')\left(f_{10,1}^2 + f_{10,2}^2\right)$$

$$+\tfrac{1}{2}\int d\tau\, j_2(qr_3')\left(f_{10,1}^2 + f_{10,2}^2\right)$$

Table 1 - continued

$$(1,1)_{\sigma\tau} = -(1,1)_{\sigma}$$

$$(1,3)_{\sigma\tau} = 0$$

$$(1,8)_{\sigma\tau} = (1,8)_{\sigma}$$

$$(1,9)_{\sigma\tau} = (1,9)_{\sigma}$$

$$(1,10)_{\sigma\tau} = (1,10)_{\sigma}$$

$$(3,3)_{\sigma\tau} = -1/2 \int d\tau\, j_0(qr_3')\left(f_{3,1}^2 - \tfrac{5}{3} f_{3,2}^2\right)$$

$$(3,8)_{\sigma\tau} = -1/4 \int d\tau\, j_2(qr_3')\left(f_{3,1} f_{8,1} + \tfrac{1}{2} f_{3,2} f_{8,2}\right)$$

$$(3,9)_{\sigma\tau} = -\sqrt{3}/4\,\Gamma \int d\tau\, j_2(qr_3')\left(f_{3,1} f_{9,1} + \tfrac{1}{2} f_{3,2} f_{9,2}\right)$$

$$(3,10)_{\sigma\tau} = 1/4\sqrt{3}\, \Sigma \int d\tau\, j_2(qr_3')\left(f_{3,1} f_{10,2} - \tfrac{1}{3} f_{3,2} f_{10,1}\right)$$

Table 1 - continued

$$(8,8)_{\sigma\tau} = -\tfrac{1}{4}\int d\tau\, j_2(qr_3')\left(f_{8,1}^2 - \tfrac{1}{3} f_{8,2}^2\right)$$

$$-\tfrac{1}{2}\int d\tau\, j_0(qr_3')\left(f_{8,1}^2 - \tfrac{1}{3} f_{8,2}^2\right)$$

$$(8,9)_{\sigma\tau} = \tfrac{\sqrt{3}}{4}\,\Gamma \int d\tau\, j_2(qr_3')\left(f_{8,1} f_{9,1} - \tfrac{1}{3} f_{8,2} f_{9,2}\right)$$

$$(8,10)_{\sigma\tau} = -\tfrac{\sqrt{3}}{4}\,\Sigma \int d\tau\, j_2(qr_3')\left(f_{8,1} f_{10,2} + \tfrac{1}{3} f_{8,2} f_{10,1}\right)$$

$$(9,9)_{\sigma\tau} = -\tfrac{1}{4}\int d\tau\, j_2(qr_3')\left(f_{9,1}^2 - \tfrac{1}{3} f_{9,2}^2\right)$$

$$-\tfrac{1}{2}\int d\tau\, j_0(qr_3')\left(f_{9,1}^2 - \tfrac{1}{3} f_{9,2}^2\right)$$

$$(9,10)_{\sigma\tau} = 0$$

$$(10,10)_{\sigma\tau} = \tfrac{1}{4}\int d\tau\, j_2(qr_3')\left(f_{10,2}^2 - \tfrac{1}{3} f_{10,1}^2\right)$$

$$+\tfrac{1}{2}\int d\tau\, j_0(qr_3')\left(f_{10,2}^2 - \tfrac{1}{3} f_{10,1}^2\right)$$

Table 1 - continued

With the aid of table (1) the magnetic moment form factor for He^3 may be written as:

$$\mu(He^3)\, F_{mag}(He^3) \tag{2.43}$$

$$= F^{n}_{mag} F_1 + F^{p}_{mag} F_2 + \left(F^{n}_{mag} + F^{p}_{mag}\right) F_3 + \left(F^{n}_{mag} - F^{p}_{mag}\right) F_4$$

where

$$F_1 = F_S + F_{S'}$$

$$F_2 = F_{SD} + F_{S'D} + F_D + F_{DD}$$

$$F_3 = F_{SS'} + \frac{1}{2} F_{DD}$$

$$F_4 = F_{S'1} + F_{S'D1} + F_{D1} + F_{DD1} \tag{2.44}$$

and

$$F_S = \int j_0 f_1^2$$

$$F_{S'} = \int j_0 \frac{f_{3,1}^2 + f_{3,2}^2}{2}$$

$$F_{SD} = \int j_2 \left(\tfrac{1}{\sqrt{2}} f_1 f_{8,1} + \sqrt{\tfrac{3}{2}}\, \Gamma f_1 f_{9,1} + \sqrt{\tfrac{2}{3}}\, \Sigma f_1 f_{10,2} \right)$$

$$F_{S'D} = \tfrac{1}{2} \int j_2 \Big\{ (f_{3,2} f_{8,2} - f_{3,1} f_{8,1}) + \sqrt{3}\, \Gamma (f_{3,2} f_{9,2} - f_{3,1} f_{9,1})$$

$$+ \tfrac{1}{\sqrt{3}}\, \Sigma \left(f_{3,1} f_{10,2} + f_{3,2} f_{10,1} \right) \Big\}$$

$$F_D = -\tfrac{1}{4} \int j_2 \left(f_{8,1}^2 + f_{8,2}^2 + f_{9,1}^2 + f_{9,2}^2 - f_{10,1}^2 - f_{10,2}^2 \right)$$

$$- \tfrac{1}{2} \int j_0 \left(f_{8,1}^2 + f_{8,2}^2 + f_{9,1}^2 + f_{9,2}^2 - f_{10,1}^2 - f_{10,2}^2 \right)$$

$$F_{DD} = \tfrac{\sqrt{3}}{2} \int j_2 \Big\{ \Gamma (f_{8,1} f_{9,1} + f_{8,2} f_{9,2}) + \Sigma (f_{8,2} f_{10,1} - f_{8,1} f_{10,2}) \Big\}$$

$$F_{SS'} = \sqrt{2} \int j_0 f_1 f_{3,1}$$

$$F_{S'1} = -\tfrac{2}{3} \int j_0 f_{3,2}^2$$

$$F_{S'D1} = \int j_2 \Big\{ \tfrac{3}{8} f_{3,2} f_{8,2} + \tfrac{3\sqrt{3}}{8} \Gamma f_{3,2} f_{9,2} + \tfrac{1}{3\sqrt{3}} \Sigma f_{3,2} f_{10,1} \Big\}$$

$$F_{D1} = -\tfrac{1}{6} \int j_2 \left(f_{8,2}^2 + f_{9,2}^2 - f_{10,1}^2 \right) - \tfrac{1}{3} \int j_0 \left(f_{8,2}^2 + f_{9,2}^2 - f_{10,1}^2 \right)$$

$$F_{DD1} = \tfrac{1}{\sqrt{3}} \int j_2 \left(\Gamma f_{8,2} f_{9,2} + \Sigma f_{8,2} f_{10,1} \right) \qquad (2.45)$$

and where

$$S_{j\ell} \equiv \int d\tau \, j_\ell(q r_3') \tag{2.46}$$

The form factor could be calculated numerically using, for example, the internal wave functions of McMillan(1970).[1]

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[1] The internal wave functions of McMillan actually correspond to somewhat different D states than ours. The relationship between the D states used by McMillan and the D states we use is given by Derrick(1960b).

3 Conclusions

Using the wave function expansion of Derrick and Blatt(1958) we have calculated an expression for the magnetic moment form factor of He^3. The symmetric and mixed-symmetric S states and all the D states were retained in the calculation. The exchange moment contribution to the magnetic moment form factor has not been included here.

A similar calculation of the charge form factor of the triton has been done by Best(1966) some time ago.

Our calculation is more general than the calculation of the magnetic moment form factor of He^3 by Schiff(1964) and Gibson(1965). The Gibson and Schiff calculation is based on a particular form of the internal wave functions, whereas no such choice is made in our calculation.

Unlike the charge form factor, the magnetic moment form factor contains cross terms between S and D states. This is a consequence of the spin operator appearing in the magnetic moment density operator. Through the SD cross terms the D state thus can make a more important contribution to the magnetic moment form factor than to the charge from factor.

As pointed out by McMillan and Landau(1974), an analogous situation holds for the scattering of low energy pions by He^3. That is, terms similar to the charge form factor terms arise from the non-spin flip part of the pion-nucleon interaction, and terms similar to the magnetic moment form factor terms arise from the spin-flip part of the pion-nucleon interaction. Indeed, many of the results derived here and by Best can be carried over quite directly to give the pion-He^3 elastic scattering cross section in the single-scattering form factor approximation. Work in this direction is, however, beyond the scope of this thesis.

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APPENDIX A The Derrick-Blatt Wave Function

A.1 Introduction

Derrick and Blatt (1958) have classified all angular momentum-isobaric spin functions which can be present in the ground state wave function of He^3. If the nuclear interaction was central only, each of the orbital angular momentum $\underline{L}$ and spin angular momentum $\underline{S}$ would be good quantum numbers and the ground state wave function of He^3 would be $\underline{L} = 0$ only. However with the inclusion of non-central forces in the nuclear interaction only the sum $\underline{J} = \underline{L} + \underline{S}$ remains a good quantum number, and the ground state wave function must be written as a mixture of several angular momentum states.

Derrick and Blatt write the He^3 ground state wave function as a sum of products:

$$\psi = \sum_{\chi} f_{\chi} \mathcal{Y}_{\chi} . \tag{A.1.1}$$

The functions $\mathcal{Y}_{\chi}$ are total angular momentum-isobaric spin functions, each with the experimentally observed $\underline{J} = 1/2$, $\underline{T} = 1/2$, even parity, and definite values of $\underline{L}$ and $\underline{S}$, while the functions f_{χ} are functions of the three interparticle distances. Each of the functions $\mathcal{Y}_{\chi}$ is in turn written as a sum of products, each product containing two factors:

1) a factor $\mathcal{Y}$ depending on the Euler angles which specify the orientation of the triangle in space,

2) a factor V depending on the spins and isobaric spins of the three particles.

The three Euler angles and the three interparticle distances are the six coordinates required to specify the spatial positions of the three particles after separating out the centre of mass coordinates.

To be consistent with charge independence of the nuclear force each of the functions y, V, $\mathcal{Y}$, and f are required to have a definite permutation symmetry, that is, each transforms according to one of the three irreducible representations of the symmetric group S(3). The functions are then combined so that each of the products $f_\chi \mathcal{Y}_\chi$ is overall antisymmetric with respect to permutation of any two particles.

A.2 The Body-Fixed and Space-Fixed Coordinate Systems

In order to specify the orientation in space of the triangle formed by the three particles, Derrick and Blatt first specify a body fixed coordinate system. The spatial orientation of the triangle is then determined by the set of three Euler angles required to rotate the body fixed frame into the space fixed frame.[1] the body fixed coordinate system is chosen as follows:

1) The triangle will lie in the x-y plane with the centre of gravity being the origin.

2) the x axis will be chosen as the principal axis associated

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[1] Derrick and Blatt refer to this as rotating the triangle from its "normal" position to its actual position.

with the largest moment of inertia. This does not uniquely specify a direction for the x axis so that some of the Euler angle wave functions could be double valued, however it turns out that wave functions of even parity are necissarily single valued so there is no ambiguity.

3) the z axis is chosen such that walking a path from particle 1 to 2, then to 3 and back to 1 would amount to a counterclockwise walk around the z axis.

4) once the direction of the x axis is chosen the y axis is chosen such that the coordinate system is right handed.

We will go into more detail of this coordinate system in appendix B.

We choose the space-fixed coordinate system to be a right-handed system with origin at the centre of gravity, with x-axis in the direction of the initial electron momentum, and z-axis perpendicular to the scattering plane.

A.3 The Symmetric Group S(3)

As the constituent wave functions of the total He^3 wave function are required to have definite permutation symmetries it is useful here to summarize some of the important properties of the group S(3) of permutations on three objects.

There are six elements in the group S(3). In the cyclic notation of Wigner(1959) these are (1), (132), (123), (12), (31), and (23). The first three of these are even permutations while the last three are odd. There are three irreducible representations of this group. One, the symmetric representation, represents each group element by 1. Another reresentation, the antisymmetric representation, represents the even group elements by 1 and the odd group elements by -1. The remaining irreducible representation is two dimensional and of mixed symmetry. The mixed symmetric irreducible representation represents the group elements in the above order as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

Now suppose a function $\Phi^{P_a}_{k_a}(1,2,3)$ transforms according to the k_a^{th} row of the P_a irreducible representation of S(3), and another function $\Phi^{P_b}_{k_b}(1',2',3')$ transforms according to the k_b^{th} row of the P_b irreducible representation of S(3). One may ask how the product functions transform under joint permutation of the indicies (1,2,3) and (1',2',3'). In analogy to the Clebsch-Gordan coefficients of the rotation group, there are addition coefficients for combining base functions of S(3). Sums of products can be formed like

$$\Phi^{P_a P_b P}_{k} = \sum_{k_1, k_2} \begin{pmatrix} P_a & P_b & P \\ k_a & k_b & k \end{pmatrix} \Phi^{P_a}_{k_a} \Phi^{P_b}_{k_b} \qquad (A.3.1)$$

which transform according to the k^{th} row of the P irreducible representation of S(3). The permutation addition coefficients $\begin{pmatrix} P_a & P_b & P \\ k_a & k_b & k \end{pmatrix}$ are unique apart from arbitrary phase factors which one chooses to make all the coefficients real. All non-zero permutation addition coefficients are listed in table (A.1).

$$\begin{pmatrix} s & s & s \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} s & a & a \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a & s & a \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a & a & s \\ 1 & 1 & 1 \end{pmatrix} = 1$$

$$\begin{pmatrix} s & m & m \\ 1 & k & \lambda \end{pmatrix} = \begin{pmatrix} m & s & m \\ k & 1 & \lambda \end{pmatrix} = \sqrt{2} \begin{pmatrix} m & m & s \\ k & \lambda & 1 \end{pmatrix} = \delta_{k\lambda}$$

$$\begin{pmatrix} a & m & m \\ 1 & k & \lambda \end{pmatrix} = \begin{pmatrix} m & a & m \\ k & 1 & \lambda \end{pmatrix} = -\sqrt{2} \begin{pmatrix} m & m & a \\ k & \lambda & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{k\lambda}$$

$$-\begin{pmatrix} m & m & m \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} m & m & m \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} m & m & m \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} m & m & m \\ 2 & 2 & 1 \end{pmatrix} = 1/\sqrt{2}$$

Table (A.1) - The non-zero permutation addition coefficients.

table (A.2) lists the sixteen possible direct product functions.

A.4 The Euler Angle Wave Functions

The Euler angle wave functions which describe the angular dependence of the He^3 wave function are simply the representation coefficients of the irreducible representations of the rotation group. These are[1]

$$D^L(\alpha\beta\gamma)_{\mu M_L} = \sum_k i^{2k+\mu-M_L} \frac{[(L+\mu)!(L-\mu)!(L+M_L)!(L-M_L)!]}{(L-\mu-k)!(L+M_L-k)!\,k!\,(k+\mu-M_L)!}$$

$$\times\, e^{i\mu\alpha} \cos^{2L+M_L-\mu-2k}\tfrac{1}{2}\beta \cdot \sin^{2k+\mu-M_L}\tfrac{1}{2}\beta \cdot e^{iM_L\gamma} \qquad \text{(A.4.1)}$$

where L is the orbital angular momentum with z component M_L and body z component μ. The experimentally observed total angular momentum for the ground state of He^3 is $\underset{\sim}{J} = 1/2$, so with a maximum possible spin of $\underset{\sim}{S} = 3/2$ the orbital angular momentum $\underset{\sim}{L}$ cannot exceed 2. Also it can be shown that the parity operator Π applied to the Euler angle function $D^L(\alpha\beta\gamma)_{\mu m_L}$ gives

$$\Pi\, D^L(\alpha\beta\gamma)_{\mu M_L} = (-1)^{\mu} D^L(\alpha\beta\gamma)_{\mu M_L} \qquad \text{(A.4.2)}$$

so that only functions with even μ have the required even parity. Requiring the Euler angle wave functions to be

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[1] See appendix C.

Symmetric	$\Phi_1^{ss's} = \Phi_1^{s}\,\Phi_{1'}^{s'}$ $\Phi_1^{aa's} = \Phi_1^{a}\,\Phi_{1'}^{a'}$ $\Phi_1^{mm's} = 1/\sqrt{2}\,\{\Phi_1^{m}\,\Phi_{1'}^{m'} + \Phi_2^{m}\,\Phi_{2'}^{m'}\}$
Antisymmetric	$\Phi_1^{as'a} = \Phi_1^{a}\,\Phi_{1'}^{s'}$ $\Phi_1^{sa'a} = \Phi_1^{s}\,\Phi_{1'}^{a'}$ $\Phi_1^{mm'a} = 1/\sqrt{2}\,\{\Phi_1^{m}\,\Phi_{2'}^{m'} - \Phi_2^{m}\,\Phi_{1'}^{m'}\}$
Mixed Symmetric	$\Phi_1^{sm'm} = \Phi_1^{s}\,\Phi_{1'}^{m'}$, $\Phi_2^{sm'm} = \Phi_1^{s}\,\Phi_{2'}^{m'}$ $\Phi_1^{ms'm} = \Phi_1^{m}\,\Phi_{1'}^{s'}$, $\Phi_2^{ms'm} = \Phi_2^{m}\,\Phi_{1'}^{s'}$ $\Phi_1^{am'm} = \Phi_1^{a}\,\Phi_{2'}^{m'}$, $\Phi_2^{am'm} = -\Phi_1^{a}\,\Phi_{1'}^{m'}$ $\Phi_1^{ma'm} = \Phi_2^{m}\,\Phi_{1'}^{a'}$, $\Phi_2^{ma'm} = -\Phi_1^{m}\,\Phi_{1'}^{a'}$ $\Phi_1^{mm'm} = 1/\sqrt{2}\,\{\Phi_2^{m}\,\Phi_{2'}^{m'} - \Phi_1^{m}\,\Phi_{1'}^{m'}\}$ $\Phi_2^{mm'm} = 1/\sqrt{2}\,\{\Phi_1^{m}\,\Phi_{2'}^{m'} + \Phi_2^{m}\,\Phi_{1'}^{m'}\}$

Table (A.2) - The direct product functions.

orthonormal and real under time reversal yields the following five functions:

$$Y_0^0(s,0) = \sqrt{2}/4\pi$$

$$Y_M^1(a,0) = i\sqrt{6}/4\pi \; D^1_{0M}(\alpha\beta\gamma)$$

$$Y_M^2(s,0) = \sqrt{10}/4\pi \; D^2_{0M}(\alpha\beta\gamma)$$

$$Y_M^2(s,2) = -\sqrt{5}/4\pi \left[D^2_{2,M}(\alpha\beta\gamma) + D^2_{-2,M}(\alpha\beta\gamma) \right]$$

$$Y_M^2(a,2) = -i\sqrt{5}/4\pi \left[D^2_{2,M}(\alpha\beta\gamma) - D^2_{-2,M}(\alpha\beta\gamma) \right]$$

We use here the notation $Y_M^L(P_e,|\mu|)$ for an Euler angle wave function with permutation symmetry P_e.

As a consequence of the highly symmetric choice of the body fixed coordinate system each of the Euler angle wave functions is either symmetric or antisymmetric; the mixed representation does not occur.

A.5 The Spin-Isospin Wave Functions

The spin functions for a three particle system may be calculated using the double Clebsch-Gordan series:

$$|SM_sP_sK_s\rangle =$$

$$\sum_{m_1}\sum_{m_2}\sum_{m_3} \langle s_1s_2m_1m_2|s'm'\rangle\langle s's_3m'm_3|SM_s\rangle |s_1m_1\rangle_1 |s_2m_2\rangle_2 |s_3m_3\rangle_3 \qquad \text{(A.5.1)}$$

where the $|s_i m_i\rangle_i$ are the eigenstates of s_i^2 and s_{i3} of the i^{th} particle, and the $|SM_sP_sK_s\rangle$ are the eigenstates of $\underline{S}\cdot\underline{S} = S^2$ and

$S_z = S_{1z} + S_{2z} + S_{3z}$ of the three particle system. k_s and P_s denote the row number and irreducible representation to which the eigenstate belongs. There are eight possible eigenstates $|SM_sP_sK_s\rangle$ for $\underline{S} = 1/2$ or $\underline{S} = 3/2$. These are listed in table (A.3), where we have set $\alpha(i) = |\frac{1}{2}\frac{1}{2}\rangle_i$ and $\beta(i) = |\frac{1}{2}-\frac{1}{2}\rangle_i$.

The isospin wave functions are constructed in a completely analogous way. With the assumption the the ground state of He^3 is $\underline{T} = 1/2$ only and setting $T_z = 1/2$, two isospin functions are obtained:

$$P_1 = 1/\sqrt{6}\left[\nu(1)\pi(2)\pi(3) + \pi(1)\nu(2)\pi(3) - 2\pi(1)\pi(2)\nu(3)\right] \qquad \text{(A.5.2)}$$

$$P_2 = 1/\sqrt{2}\left[\nu(1)\pi(2)\pi(3) - \pi(1)\nu(2)\pi(3)\right] \qquad \text{(A.5.3)}$$

where $\pi(i) = |\frac{1}{2}\frac{1}{2}\rangle_i$ represents a proton and $\nu(i) = |\frac{1}{2}-\frac{1}{2}\rangle_i$ represents a neutron.

From the eight spin functions and two isospin functions we can form sixteen linearly independent sums of products of spin and isospin functions. The sixteen spin-isospin functions, denoted by $V_{M_sK_t}(P_t,T,S)$, are listed in table (A.4). P_t and K_t denote the irreducible representation and corresponding row number to which the function belongs.

$$q_1 = \left|\tfrac{1}{2}\,\tfrac{1}{2}\,m\,1\right\rangle = 1/\sqrt{6}\left[\beta(1)\alpha(2)\alpha(3) + \alpha(1)\beta(2)\alpha(3) - 2\alpha(1)\alpha(2)\beta(3)\right]$$

$$q_2 = \left|\tfrac{1}{2}\,\tfrac{1}{2}\,m\,2\right\rangle = 1/\sqrt{2}\left[\beta(1)\alpha(2)\alpha(3) - \alpha(1)\beta(2)\alpha(3)\right]$$

$$q_3 = \left|\tfrac{3}{2}\,\tfrac{1}{2}\,s\,1\right\rangle = 1/\sqrt{3}\left[\alpha(1)\alpha(2)\beta(3) + \beta(1)\alpha(2)\alpha(3) + \alpha(1)\beta(2)\alpha(3)\right]$$

$$q_4 = \left|\tfrac{1}{2}\,-\tfrac{1}{2}\,m\,1\right\rangle = 1/\sqrt{6}\left[\alpha(1)\beta(2)\beta(3) + \beta(1)\alpha(2)\beta(3) - 2\beta(1)\beta(2)\alpha(3)\right]$$

$$q_5 = \left|\tfrac{1}{2}\,-\tfrac{1}{2}\,m\,2\right\rangle = 1/\sqrt{2}\left[\alpha(1)\beta(2)\beta(3) - \beta(1)\alpha(2)\beta(3)\right]$$

$$q_6 = \left|\tfrac{3}{2}\,\tfrac{3}{2}\,s\,1\right\rangle = \alpha(1)\alpha(2)\alpha(3)$$

$$q_7 = \left|\tfrac{3}{2}\,-\tfrac{1}{2}\,s\,1\right\rangle = 1/\sqrt{3}\left[\beta(1)\beta(2)\alpha(3) + \alpha(1)\beta(2)\beta(3) + \beta(1)\alpha(2)\beta(3)\right]$$

$$q_8 = \left|\tfrac{3}{2}\,-\tfrac{3}{2}\,s\,1\right\rangle = \beta(1)\beta(2)\beta(3)$$

Table (A.3) - The spin eigenstates.

$V_{\frac{1}{2}\frac{1}{2}1}(s,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_1p_1 + q_2p_2]$	$V_{\frac{1}{2}\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_6p_1$
$V_{\frac{1}{2}\frac{1}{2}1}(a,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_2p_1 - q_1p_2]$	$V_{\frac{1}{2}\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_6p_2$
$V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_2p_2 - q_1p_1]$	$V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_3p_1$
$V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_2p_1 + q_1p_2]$	$V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_3p_2$
$V_{\frac{1}{2}-\frac{1}{2}1}(s,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_4p_1 + q_5p_2]$	$V_{\frac{1}{2}-\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_7p_1$
$V_{\frac{1}{2}-\frac{1}{2}1}(a,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_5p_1 - q_4p_2]$	$V_{\frac{1}{2}-\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_7p_2$
$V_{\frac{1}{2}-\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_5p_2 - q_4p_1]$	$V_{\frac{1}{2}-\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_8p_1$
$V_{\frac{1}{2}-\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{1}{2}) = 1/\sqrt{2}\,[q_5p_1 + q_4p_2]$	$V_{\frac{1}{2}-\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) = q_8p_2$

Table (A.4) - The spin-isospin functions.

A.6 The Total Angular Momentum-Isospin Wave Functions

Making use of the permutation group addition coefficients $\begin{pmatrix} P_a & P_b & P \\ k_a & k_b & k \end{pmatrix}$ and Clebsch-Gordan coefficients $\langle S_1 S_2 m_1 m_2 | S\, m \rangle$, the spin-isospin functions $V_{M_S k_t}(P_t, T, S)$ are now combined with the Euler angle functions $Y^L_M(P_e, |\mu|)$ to obtain total angular momentum-isospin functions. These are

$$\mathcal{Y}_{M_J T_z k}(J, L, S, P, P_e, P_t, \mu)$$

$$= \sum_{M_L M_S k_t} \begin{pmatrix} P_e & P_t & P \\ 1 & k_t & k \end{pmatrix} (L S M_L M_S | L S J M_J)\, Y^L_{M_L}(P_e, \mu) \cdot V_{M_S k_t}(P_t, T, S). \tag{A.6.1}$$

With $J = 1/2$, $M_J = 1/2$, $T = 1/2$, and $T_z = 1/2$ we obtain ten distinct states. These are listed in table (A.5). The notation used in table (A.5) is that of Derrick (1960). Each pair of mixed symmetric functions is counted as one distinct state as both functions must be combined linearly to obtain one complete function.

From table (A.5) one sees that $\mathcal{Y}_1$, $\mathcal{Y}_2$, and $(\mathcal{Y}_{3,1}, \mathcal{Y}_{3,2})$ represent ${}^{1}_{2}S$ states; $\mathcal{Y}_4$, $\mathcal{Y}_5$, and $(\mathcal{Y}_{6,1}, \mathcal{Y}_{6,2})$ represent ${}^{1}_{2}P$ states; $(\mathcal{Y}_{7,1}, \mathcal{Y}_{7,2})$ represents a ${}^{3}_{2}P$ state; and $(\mathcal{Y}_{8,1}, \mathcal{Y}_{8,2})$, $(\mathcal{Y}_{9,1}, \mathcal{Y}_{9,2})$, and $(\mathcal{Y}_{10,1}, \mathcal{Y}_{10,2})$ represent ${}^{3}_{2}D$ states.

A.7 The Total Wave Function

The total angular momentum-isospin functions $\mathcal{Y}$ are now combined with the internal wave functions to yield an overall wave function for the ground state of He^3. We denote the

$$y_1 = Y_0^0(s,0)\,V_{\frac{1}{2}\frac{1}{2}1}(a,\tfrac{1}{2},\tfrac{1}{2})$$

$$y_2 = Y_0^0(s,0)\,V_{\frac{1}{2}\frac{1}{2}1}(s,\tfrac{1}{2},\tfrac{1}{2})$$

$$y_{3,1} = Y_0^0(s,0)\,V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{1}{2})$$

$$y_{3,2} = Y_0^0(s,0)\,V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{1}{2})$$

$$y_4 = {}^1\!/\!\sqrt{3}\,\{\sqrt{2}\,Y_1^1(a,0)V_{\frac{1}{2}-\frac{1}{2}1}(s,\tfrac{1}{2},\tfrac{1}{2}) - Y_0^1(a,0)V_{\frac{1}{2}\frac{1}{2}1}(s,\tfrac{1}{2},\tfrac{1}{2})\}$$

$$y_5 = {}^1\!/\!\sqrt{3}\,\{\sqrt{2}\,Y_1^1(a,0)V_{\frac{1}{2}-\frac{1}{2}1}(a,\tfrac{1}{2},\tfrac{1}{2}) - Y_0^1(a,0)V_{\frac{1}{2}\frac{1}{2}1}(a,\tfrac{1}{2},\tfrac{1}{2})\}$$

$$y_{6,1} = {}^1\!/\!\sqrt{3}\,\{\sqrt{2}\,Y_1^1(a,0)V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{1}{2}) - Y_0^1(a,0)V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{1}{2})\}$$

$$y_{6,2} = -{}^1\!/\!\sqrt{3}\,\{\sqrt{2}\,Y_1^1(a,0)V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{1}{2}) - Y_0^1(a,0)V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{1}{2})\}$$

$$y_{7,1} = {}^1\!/\!\sqrt{6}\,\{\sqrt{3}\,Y_{-1}^1(a,0)V_{\frac{1}{2}\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{2}\,Y_0^1(a,0)V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{1}{2}) - Y_1^1(a,0)V_{\frac{1}{2}-\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

$$y_{7,2} = -{}^1\!/\!\sqrt{6}\,\{\sqrt{3}\,Y_{-1}^1(a,0)V_{\frac{1}{2}\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{2}\,Y_0^1(a,0)V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{1}{2}) - Y_1^1(a,0)V_{\frac{1}{2}-\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

$$y_{8,1} = {}^1\!/\!\sqrt{10}\,\{2Y_2^2(s,0)V_{\frac{1}{2}-\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{3}\,Y_1^2(s,0)V_{\frac{1}{2}-\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) + \sqrt{2}\,Y_0^2(s,0)V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) - Y_{-1}^2(s,0)V_{\frac{1}{2}\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

$$y_{8,2} = {}^1\!/\!\sqrt{10}\,\{2Y_2^2(s,0)V_{\frac{1}{2}-\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{3}\,Y_1^2(s,0)V_{\frac{1}{2}-\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) + \sqrt{2}\,Y_0^2(s,0)V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) - Y_{-1}^2(s,0)V_{\frac{1}{2}\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

$$y_{9,1} = {}^1\!/\!\sqrt{10}\,\{2Y_2^2(s,2)V_{\frac{1}{2}-\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{3}\,Y_1^2(s,2)V_{\frac{1}{2}-\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) + \sqrt{2}\,Y_0^2(s,2)V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) - Y_{-1}^2(s,2)V_{\frac{1}{2}\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

$$y_{9,2} = {}^1\!/\!\sqrt{10}\,\{2Y_2^2(s,2)V_{\frac{1}{2}-\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{3}\,Y_1^2(s,2)V_{\frac{1}{2}-\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) + \sqrt{2}\,Y_0^2(s,2)V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) - Y_{-1}^2(s,2)V_{\frac{1}{2}\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

$$y_{10,1} = {}^1\!/\!\sqrt{10}\,\{2Y_2^2(a,2)V_{\frac{1}{2}-\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{3}\,Y_1^2(a,2)V_{\frac{1}{2}-\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) + \sqrt{2}\,Y_0^2(a,2)V_{\frac{1}{2}\frac{1}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2}) - Y_{-1}^2(a,2)V_{\frac{1}{2}\frac{3}{2}2}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

$$y_{10,2} = -{}^1\!/\!\sqrt{10}\,\{2Y_2^2(a,2)V_{\frac{1}{2}-\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) - \sqrt{3}\,Y_1^2(a,2)V_{\frac{1}{2}-\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) + \sqrt{2}\,Y_0^2(a,2)V_{\frac{1}{2}\frac{1}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2}) - Y_{-1}^2(a,2)V_{\frac{1}{2}\frac{3}{2}1}(m,\tfrac{1}{2},\tfrac{3}{2})\}$$

Table (A.5) - The total angular momentum-isospin functions.

internal wave functions by $f_{k'}^{P'}(\chi, r_{12}, r_{13}, r_{23})$, where k' and P' are the row number and irreducible representation to which the function f belongs, and χ denotes any of the ten possible values of $(J, L, S, P, P_e, P_t, \mu)$. Using the permutation group addition coefficients and requiring that the overall wave function be antisymmetric, the overall wave function is written

$$\Psi = \sum_{k'k} \begin{pmatrix} P' & P & a \\ k' & k & 1 \end{pmatrix} f_{k'}^{P'}(\chi, r_{12}, r_{13}, r_{23})\, y_{\frac{1}{2}\frac{1}{2}k}(\chi). \qquad \text{(A.7.1)}$$

Using the notation of Derrick (1959) the total wave function may be written

$$\begin{aligned} \Psi = {} & f_1 y_1 + f_2 y_2 + 1/\sqrt{2}\,(f_{3,1} y_{3,2} - f_{3,2} y_{3,1}) \\ & + f_4 y_4 + f_5 y_5 + 1/\sqrt{2}\,(f_{6,1} y_{6,2} - f_{6,2} y_{6,1}) \\ & + 1/\sqrt{2}\,(f_{7,1} y_{7,2} - f_{7,2} y_{7,1}) + 1/\sqrt{2}\,(f_{8,1} y_{8,2} - f_{8,2} y_{8,1}) \\ & + 1/\sqrt{2}\,(f_{9,1} y_{9,2} - f_{9,2} y_{9,1}) + 1/\sqrt{2}\,(f_{10,1} y_{10,2} - f_{10,2} y_{10,1}) \\ = {} & \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6 + \Psi_7 + \Psi_8 + \Psi_9 + \Psi_{10} \qquad \text{(A.7.2)} \end{aligned}$$

Some important properties of each of the ten states are summarized in table (A.6).

L	S	Permutation Symmetry			$\lvert\mu\rvert$
		Internal	Euler Angles	Spin-Isospin	
0	½	s	s	a	0
0	½	a	s	s	0
0	½	m	s	m	0
1	½	s	a	s	0
1	½	a	a	a	0
1	½	m	a	m	0
1	3/2	m	a	m	0
2	3/2	m	s	m	0
2	3/2	m	s	m	2
2	3/2	m	a	m	2

Table (A.6) - Permutation properties of the wave function.

A.8 Relative Importance of the States

For the purpose of further approximation it is of interest here to estimate which of the ten functions are likely to be dominant. In the absence of non-central forces the ground state would be S only. Furthermore if the interaction is spin independent the symmetry of the internal wave function is a good quantum number.

Now a symmetric internal wave function need not be zero for any shape of the triangle. A pair of mixed symmetric functions is necessarily zero whenever the triangle is equilateral, that is, whenever $r_{12} = r_{13} = r_{23}$. An antisymmetric function must be zero whenever the triangle is isosceles, that is, whenever $r_{12} = r_{13}$, $r_{12} = r_{23}$, or $r_{13} = r_{23}$. The more zeros a function possesses, the more it is forced to change, hence the higher its derivatives, and hence the higher its kinetic energy. Thus a purely symmetric function has the lowest kinetic energy associated with it. In the absence of non-central and spin-dependent forces the ground state would then be symmetric S, that is, the wave function would be of the type ψ_1.

With the inclusion of non-central and spin-dependent forces the symmetric S state is still expected to dominate. The next most important states are those which couple in the first order to ψ_1 under the interaction. Derrick (1959) has shown that the spin exchange operator couples the mixed symmetric S state ψ_3 directly to ψ_1, while the tensor operator couples the three mixed symmetric D states ψ_8, ψ_9, and ψ_{10} directly to ψ_1. Finally the $\underline{L} \cdot \underline{S}$ forces couple the P states ψ_5, ψ_6, and ψ_7

directly to ψ_1, but these are considered unimportant as the $\underline{L} \cdot \underline{S}$ force is believed to be of very short range.

With the inclusion of tensor and spin-dependent forces then the most important states present in the ground state will be ψ_1, ψ_3, ψ_8, ψ_9, and ψ_{10}. The least dominant states will be those with antisymmetric internal wave functions as those states are associated with very high kinetic energy.

APPENDIX B The Magnitude and Angles of r_3' in the Body-Fixed System

Consider the triangle formed by the three particles with sides $\underline{r}_{12} = \underline{r}_1 - \underline{r}_2$, $\underline{r}_{13} = \underline{r}_1 - \underline{r}_3$, and $\underline{r}_{23} = \underline{r}_2 - \underline{r}_3$. Define the vectors $\underline{r} = \underline{r}_{12}$ and $\underline{\rho} = \underline{r}_{13} + \underline{r}_{23}$ as shown in figure (B.1). It is easily shown that

$$\rho = \left(2r_{13}^2 + 2r_{23}^2 - r_{12}^2\right)^{1/2} \tag{B.1}$$

Now letting the centre of mass position vector be $R_G = \frac{1}{3}(r_{12} + r_{13} + r_{23})$ we have

$$\underline{r}_1' = \underline{r}_1 - \underline{R}_G = \frac{1}{3}\left(2\underline{r}_1 - \underline{r}_2 - \underline{r}_3\right) = \frac{1}{6}\left(\underline{\rho} + 3\underline{r}\right) \tag{B.2}$$

$$\underline{r}_2' = \underline{r}_2 - \underline{R}_G = \frac{1}{3}\left(2\underline{r}_2 - \underline{r}_1 - \underline{r}_3\right) = \frac{1}{6}\left(\underline{\rho} - 3\underline{r}\right) \tag{B.3}$$

$$\underline{r}_3' = \underline{r}_3 - \underline{R}_G = \frac{1}{3}\left(2\underline{r}_3 - \underline{r}_1 - \underline{r}_2\right) = -\frac{1}{3}\underline{\rho} \tag{B.4}$$

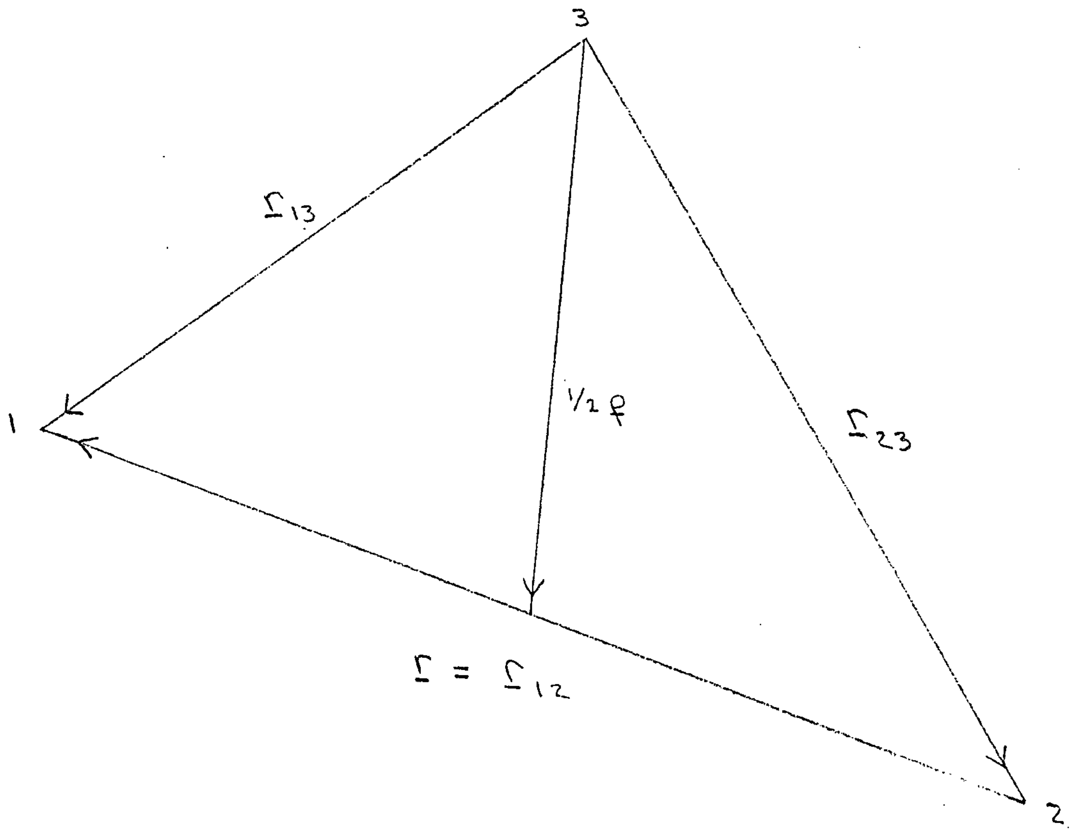

Figure (B.1) - The vectors $\underset{\sim}{r}$ and $\underset{\sim}{\rho}$.

Thus the magnitude of $\underline{r}_3'$ is

$$r_3' = \frac{1}{3}\left(2r_{13}^2 + 2r_{23}^2 - r_{12}^2\right)^{1/2} \tag{B.5}$$

Now consider a coordinate system with origin at the centre of mass, Z axis perpendicular to the plane of the triangle, and particle 3 on the X axis. The direction of the Z axis is chosen such that walking a path from particle 1 to 2, then to 3 and back to 1 would amount to a counterclockwise walk around the Z axis. This coordinate system is shown in figure (B.2). We have also defined η as the angle $(\underset{\sim}{\rho}, \underline{r})$, which written in terms of the triangle sides is

$$\cos\eta = \left(r_{13}^2 - r_{23}^2\right)/r\rho \tag{B.6}$$

It can also be shown that

$$\sin\eta = \left[(r_{12}+r_{13}+r_{23})(r_{12}+r_{13}-r_{23})(r_{12}-r_{13}+r_{23})(-r_{12}+r_{13}+r_{23})\right]^{1/2}/r\rho \tag{B.7}$$

The coordinates of $\underline{r}$ and $\underset{\sim}{\rho}$ in this coordinate system are

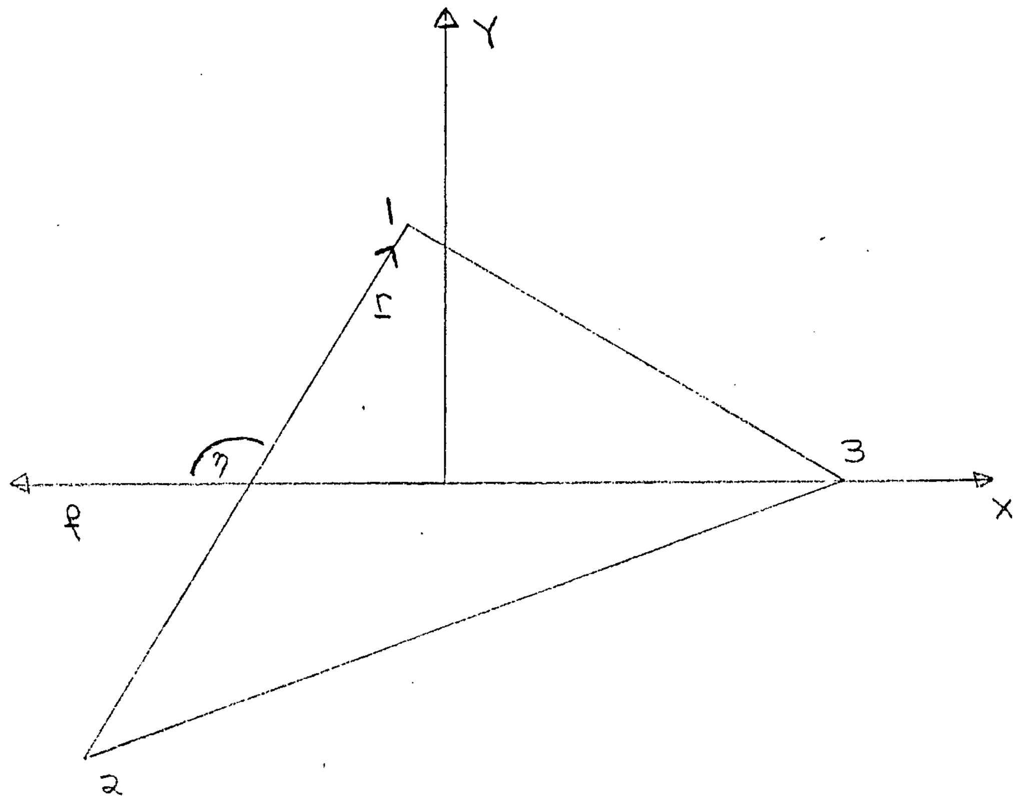

Figure (B.2) - The coordinate system with origin at the centre of mass, Z axis perpendicular to the plane of the triangle, and particle 3 on the X axis.

$$r_x = -r\cos\eta$$
$$r_y = r\sin\eta \qquad (B.8)$$
$$r_z = 0$$

$$\rho_x = -\rho$$
$$\rho_y = 0 \qquad (B.9)$$
$$\rho_z = 0$$

Suppose now the coordinate system is rotated about the z axis by an angle ζ, as in figure (B.3). The coordinates of $\underline{r}$ and $\underset{\sim}{\rho}$ in this new coordinate system are

$$r_x = -r\cos(\eta+\zeta)$$
$$r_y = r\sin(\eta+\zeta) \qquad (B.10)$$
$$r_z = 0$$

$$\rho_x = -\rho\cos\zeta$$
$$\rho_y = \rho\sin\zeta \qquad (B.11)$$
$$\rho_z = 0$$

The coordinates of $\underline{r}'_3$ in this coordinate system are hence

$$r'_{3x} = r'_3\cos\zeta$$
$$r'_{3y} = -r'_3\sin\zeta \qquad (B.12)$$
$$r'_{3z} = 0$$

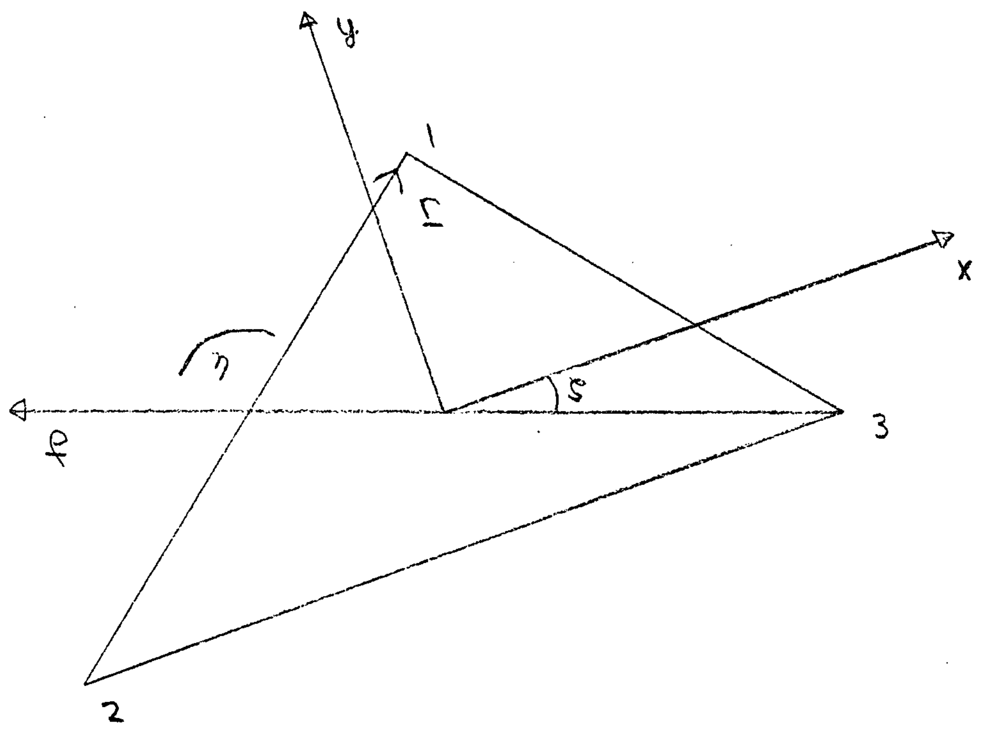

Figure (B.3) - The body fixed coordinate system.

In particular suppose that the axes of the rotated system are the principal axes, the x axis being associated with the larger principal moment of inertia. This new system then coincides with the body-fixed system described in appendix A.2. In this coordinate system the products of inertia must vanish:

$$I_{xy} = I_{xz} = I_{yz} = 0 \tag{B.13}$$

Now I_{xz} and I_{yz} are identicaly zero since $z_i = 0$, but $I_{xy} = 0$ is an equation for ζ:

$$
\begin{aligned}
I_{xy} &= -\sum_{i=1}^{3} m_i x_i y_i = -m\left[x_1 y_1 + x_2 y_2 + x_3 y_3\right] \\
&= -\frac{m}{36}\left[(\rho+3r)_x(\rho+3r)_y + (\rho-3r)_x(\rho-3r)_y + 4\rho_x\rho_y\right] \\
&= -\frac{m}{36}\left[6\rho_x\rho_y + 18 r_x r_y\right] \\
&= \frac{m}{36}\left[6\rho^2 \sin\zeta\cos\zeta + 18 r^2 \cos(\eta+\zeta)\sin(\eta+\zeta)\right] \\
&= \frac{m}{12}\left[\rho^2 \sin 2\zeta + 3r^2 \sin 2(\eta+\zeta)\right] = 0
\end{aligned} \tag{B.14}
$$

Solving the equation

$$\rho^2 \sin 2\zeta + 3r^2 \sin 2(\eta+\zeta) = 0 \tag{B.15}$$

yields

$$\sin 2\zeta = \pm \frac{3r^2 \sin 2\eta}{2\Lambda} \tag{B.16}$$

where we have defined

$$\Lambda = +\frac{1}{2}\left[\left(\rho^2 + 3r^2\cos 2\eta\right)^2 + \left(3r^2 \sin 2\eta\right)^2\right]^{1/2} \tag{B.17}$$

From equations (B.15) and (B.16) one can show that

$$\cos 2\zeta = \mp \frac{\left(\rho^2 + 3r^2 \cos 2\eta\right)}{2\Lambda} \tag{B.18}$$

If the sign of $\sin(2\zeta)$ is (+) then the sign of $\cos(2\zeta)$ must be (-), while if the sign of $\sin(2\zeta)$ is (-) then the sign of $\cos(2\zeta)$ must be (+).

To determine the signs of $\sin(2\zeta)$ and $\cos(2\zeta)$ we calculate the moments of inertia I_{xx} and I_{yy} and impose the condition $I_{xx} > I_{yy}$. Now

$$\begin{aligned} I_{xx} &= \sum_{i=1}^{3} m_i\left(r_i^2 - x_i^2\right) \\ &= \frac{m}{36}\left[(\rho+3r)^2 - (\rho+3r)_x^2 + (\rho-3r)^2 - (\rho-3r)_x^2 + 4\rho^2 - 4\rho_x^2\right] \\ &= \frac{m}{36}\left[6\rho^2 + 18r^2 - 6\rho_x^2 - 18r_x^2\right] \\ &= \frac{m}{6}\left[\frac{1}{2}\left(\rho^2 + 3r^2\right) - \frac{1}{2}\left\{\rho^2\cos 2\zeta + 3r^2 \cos 2(\eta+\zeta)\right\}\right] \end{aligned} \tag{B.19}$$

By a similar calculation we find

$$I_{yy} = \sum_{i=1}^{3} m_i (r_i^2 - y_i^2)$$

$$= \frac{m}{6}\left[\frac{1}{2}(\rho^2 + 3r^2) + \frac{1}{2}\left\{\rho^2 \cos 2\zeta + 3r^2 \cos 2(\eta + \zeta)\right\}\right] \tag{B.20}$$

From equations (B.19) and (B.20) we see that $I_{xx} > I_{yy}$ implies that

$$\rho^2 \cos 2\zeta + 3r^2 \cos 2(\eta + \zeta) < 0 \tag{B.21}$$

Now

$$\rho^2 \cos 2\zeta + 3r^2 \cos 2(\eta+\zeta)$$
$$= \frac{1}{2\Lambda}\left[\mp \rho^2 (\rho^2 + 3r^2 \cos 2\eta) \mp 3r^2 \cos 2\eta (\rho^2 + 3r^2 \cos 2\eta) \pm (3r^2 \sin 2\eta)^2\right]$$
$$= \mp \frac{1}{2\Lambda}\left[\rho^4 + 6\rho^2 r^2 \cos 2\eta + 9r^4\right] = \mp 2\Lambda \tag{B.22}$$

Hence in order that equation (B.21) be satisfied the above sign in equations (B.16) and (B.18) must hold, that is

$$\sin 2\zeta = \frac{3r^2 \sin 2\eta}{2\Lambda} \tag{B.23}$$

and

$$\cos 2\zeta = \frac{-(\rho^2 + 3r^2\cos 2\eta)}{2\Lambda} \tag{B.24}$$

Consider now the following four variables as defined by Derrick (1960b):

$$F = 1/\sqrt{3}\left(r_{13}^2 + r_{23}^2 - 2r_{12}^2\right) \tag{B.25}$$

$$G = r_{23}^2 - r_{13}^2 \tag{B.26}$$

$$R^2 = r_{12}^2 + r_{13}^2 + r_{23}^2 \tag{B.27}$$

Δ = area of triangle

$$= \frac{1}{4}\left[(r_{12}+r_{13}+r_{23})(r_{12}+r_{13}-r_{23})(r_{12}-r_{13}+r_{23})(-r_{12}+r_{13}+r_{23})\right]^{1/2} \tag{B.28}$$

With these variables we may write

$$\sin 2\zeta = \frac{-12\,\Delta G}{\Lambda \rho^2} \tag{B.29}$$

and

$$\cos 2\zeta = -\frac{(\Lambda^2 + \sqrt{3}\, F R^2)}{\Lambda \rho^2} \tag{B.30}$$

Note that Derrick's definition of ρ differs somewhat from ours.

In equation (2.27) we introduced $Y_{\ell m}(\tilde{\Gamma}_3)$. From the above it follows that

$$Y_{\ell m}(\tilde{\Gamma}_3) = Y_{\ell m}(\pi/2, -\zeta) \tag{B.31}$$

so that

$$Y_{20}(\tilde{\Gamma}_3) = -\frac{1}{4}\sqrt{\frac{5}{\pi}} \tag{B.32}$$

$$Y^*_{22}(\tilde{\Gamma}_3) + Y_{22}(\tilde{\Gamma}_3) = 2\left(\frac{15}{32\pi}\right)^{1/2} \cos 2\zeta \tag{B.33}$$

$$Y_{22}^{*}(\tilde{\Gamma}_3) - Y_{22}(\tilde{\Gamma}_3) = 2i\left(\frac{15}{32\pi}\right)^{1/2} \sin 2\zeta \tag{B.34}$$

It is also useful to define

$$\Sigma = \sin 2\zeta \tag{B.35}$$

$$\Gamma = \cos 2\zeta \tag{B.36}$$

APPENDIX C The Representation Coefficients $D^L(\psi,\theta,\phi)_{\mu m_L}$

Derrick's and Blatt's Euler angles (α, β, γ) actually are the angles (ψ, θ, ϕ) of Goldstein's(1950) Euler angle convention. We wish to show here that the representation coefficients $D^L(\psi,\theta,\phi)_{\mu m_L}$ given by Derrick and equation (A.4.1) of this work are equal to those given by Wigner(1959), who uses a different Euler angle convention, that is,

$$D^L(\psi,\theta,\phi)_{\mu m_L} = \mathcal{D}^L(\alpha,\beta,\gamma)_{\mu m_L} \tag{C.1}$$

where $\mathcal{D}^L(\alpha,\beta,\gamma)_{\mu m_L}$ is the representation coefficient given by Wigner.

Wigner writes $r' = Rr$, where

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & -\cos\alpha\sin\beta \\ -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ \sin\beta\cos\gamma & \sin\beta\cos\gamma & \cos\beta \end{pmatrix} \tag{C.2}$$

and where

$$0 \leqslant \alpha \leqslant 2\pi \, , \quad 0 \leqslant \beta \leqslant \pi \, , \quad 0 \leqslant \gamma \leqslant 2\pi \tag{C.3}$$

Goldstein writes $r' = Ar$, where

$$A = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{C.4}$$

and where

$$0 \leqslant \psi \leqslant 2\pi \, , \quad 0 \leqslant \theta \leqslant \pi \, , \quad 0 \leqslant \phi \leqslant 2\pi \tag{C.5}$$

Now clearly R and A are equal if their respective matrix elements are equal. This gives us equations to determine relations between (α, β, γ) and (ψ, θ, ϕ). In particular

$$\cos\beta = \cos\theta \quad \Rightarrow \quad \beta = \theta \tag{C.6}$$

Also

$$\sin\alpha \sin\beta = \cos\psi \sin\theta \Rightarrow \sin\alpha = \cos\psi \tag{C.7}$$

$$-\cos\alpha \sin\beta = \sin\psi \sin\theta \Rightarrow -\cos\alpha = \sin\psi \tag{C.8}$$

Combining equations (C.7) and (C.8) yields

$$\sin\alpha \sin\psi + \cos\alpha \cos\psi = \cos(\alpha - \psi) = 0 \tag{C.9}$$

so $\alpha = \psi + \pi/2$ or $\alpha = \psi + 3\pi/2$. However $\alpha = \psi + 3\pi/2$ does not satisfy equations (C.7) and (C.8) so we must have $\alpha = \psi + \pi/2$. Furthermore

$$\sin\beta \sin\alpha = -\sin\theta \cos\phi \Rightarrow \sin\gamma = -\cos\phi \tag{C.10}$$

$$\sin\beta \cos\gamma = \sin\theta \sin\phi \Rightarrow \cos\gamma = \sin\phi \tag{C.11}$$

Combining equations (C.10) and (C.11) yields

$$\sin\gamma \sin\Phi + \cos\gamma \cos\Phi = \cos(\Phi - \gamma) = 0 \tag{C.12}$$

so $\Phi = \gamma + \pi/2$ or $\Phi = \gamma + 3\pi/2$. However $\Phi = \gamma + 3\pi/2$ does not satisfy equations (C.10) and (C.11) so we must have $\Phi = \gamma + \pi/2$. Finally then we have the relations

$$\alpha = \psi + \pi/2 \tag{C.13}$$

$$\beta = \theta \tag{C.14}$$

$$\gamma = \Phi - \pi/2 \tag{C.15}$$

That is, Wigner's (α, β, γ) and Goldstein's (ψ, θ, Φ) represent the same rotation if equations (C.13), (C.14), and (C.15) hold.

Wigner writes

$$\mathfrak{D}^L(\alpha\beta\gamma)_{\mu m_L} = \sum_k (-1)^k \frac{[(L+\mu)!(L-\mu)!(L+m_L)!(L-m_L)!]}{(L-\mu-k)!(L+m_L-k)!\,k!\,(k+\mu-m_L)!}$$

$$\times\ e^{i\mu\alpha} \cos^{2L+m_L-\mu-2k}\tfrac{1}{2}\beta \cdot \sin^{2k+\mu-m_L}\tfrac{1}{2}\beta \cdot e^{im_L\gamma} \tag{C.16}$$

With the aid of equations (C.13), (C.14), and (C.15) we have

$$e^{i\mu\alpha} = e^{i\mu(\psi+\pi/2)} = e^{i\mu\psi}\, i^{\mu} \tag{C.17}$$

and

$$e^{im_L\gamma} = e^{im_L(\phi-\pi/2)} = e^{im_L\phi}\,(-i)^{m_L} \tag{C.18}$$

Now

$$(-1)^k (i)^{\mu} (-i)^{m_L} = (i)^{2k} (i)^{\mu} (i)^{-m_L} = i^{2k+\mu-m_L} \tag{C.19}$$

Combining equations (C.16), (C.17), (C.18), and (C.19) we have

$$\begin{aligned} \mathfrak{D}^L(\alpha\beta\gamma)_{\mu m_L} &= \sum_k i^{2k+\mu-m_L} \frac{[(L+\mu)!(L-\mu)!(L+m_L)!(L-m_L)!]}{(L-\mu-k)!(L+m_L-k)!\,k!\,(k+\mu-m_L)!} \\ &\quad \times e^{i\mu\psi} \cos^{2L+m_L-\mu-2k}\tfrac{1}{2}\theta \cdot \sin^{2k+\mu-m_L}\tfrac{1}{2}\theta \cdot e^{im_L\phi} \\ &= D^L(\psi,\theta,\phi)_{\mu m_L} \end{aligned} \tag{C.20}$$

as stated.