THE CAPITAL ASSET PRICING MODEL AND THE PROBABILITY OF BANKRUPTCY: THEORY AND EMPIRICAL TESTS

by

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A B S T R A C T

Empirical evidence shows that the Capital Asset Pricing Model (CAPM) is misspecified. Securities of low systematic risk consistently earn more than predicted by the model, the reverse being true for securities of high systematic risk. Whilst the relationship between ex-post returns and systematic risk appears to be linear, the estimated regression coefficients are significantly different from their theoretic values. Various attempts to explain theoretically the causes of the misspecification have been explored, but fail to provide an adequate explanation of all the observed deficiencies.

The dissertation examines how the mechanism of bankruptcy affects the structure of returns for corporate financial assets. The hypothesis of the thesis is that the probability of bankruptcy across securities and across time is reflected in the residual return after abstracting from the market.

Using stochastic control theory, a two variable extended form of the continuous time analogue of the CAPM is derived. The second variable is associated with the probability of bankruptcy. The model provides a natural explanation of the deficiencies of the CAPM. A discrete time ex-post formulation of the model is used to test empirically the hypothesis.

This necessitates being able to measure the probability of bankruptcy. A model formulated in terms of a firm's ability to raise funds, either internally or externally, to cover fixed charges is developed, and the probability of bankruptcy estimated using the maximum likelihood methodologies of logit analysis and probit analysis. The ability of the model to predict bankruptcy

is tested on a secondary sample of bankrupt firms. Excellent results are obtained with the model predicting bankruptcy, for some firms, four or five years before the actual occurrence.

Using a pooling of time series and cross section data to estimate the coefficients of the regression equation representing the hypothesis, evidence is found indicating that bankruptcy is an explanatory factor of common stock returns.

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But even at the very grave
I trust the time shall come to be
When over malice, over wrong,
The good will win its victory.

(Boris Pasternak, January, 1959)

CHAPTER I

INTRODUCTION

One of the central issues in the theory of finance is the relationship between risk and return demanded by investors in securities. The Capital Asset Pricing Model (CAPM) provides such a theoretical relationship between the expected rate of return and the expected risk of an asset under conditions of market equilibrium.[1] Explicitly, the model states that the expected one period return for a security is a linear function of its systematic risk, which is a measure of the responsiveness of the security's return to changes in the return on the market as a whole.

In addition to providing insights into the functioning of capital markets, the CAPM holds considerable promise as an operational tool. Proposed applications have ranged from strategies for security selection, estimation of the cost of capital, measurement of investment performance, and establishing a structure of managerial fees. Before the model can be implemented with any degree of confidence, it is necessary to investigate the theoretical and empirical validity of the model.

One important assumption underlying the model is that of perfect capital markets,[2] in particular, the absence of costs associated with financial distress. The situation in which the firm's income before interest and taxes

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[1]Theoretically the model applies only to all equity financed firms.

[2]A perfect capital market is one where all participants are price takers, have equal and costless access to all information, and there are no transaction costs or taxes.

is less than its fixed commitments, for example interest payments on debt, is termed a condition of "financial distress." In such a case the firm must curtail dividends, or investment, or obtain a capital inflow (for example by selling stock) to meet its fixed commitments. The extreme situation of financial distress is when the firm is forced to declare bankruptcy because it is unable to meet the claims of its creditors.

In the theory of corporate finance the concepts of bankruptcy and financial distress are important factors in determining the optimal value of the firm. In the absence of corporate taxes and default, Modigliani and Miller[3] have shown that the value of the firm is independent of the mix of debt and equity, given the assumptions of perfect capital markets, homogeneity of expectations and the ability of the investor to replicate across all states of nature the income obtained by inverting in an unlevered firm. In a subsequent paper Modigliani and Miller[4] relax the assumption of no corporate tax, introducing a market imperfection. The firm receives a tax subsidy from the government for using debt and, as debt is riskless, it is advantageous for the firm to use as much debt as possible; capital structure is relevant to the value of the firm.

Such a conclusion has little practical appeal for corporate debt is not, in general, riskless. Increasing the level of debt increases the fixed charges and the probability that the firm will not be able to meet its financial

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[3] Modigliani, F. and Miller, M. "The Cost of Capital, Corporate Finance, and the Theory of Investment," *American Economic Review*, Vol. XLVIII, No. 3, (June, 1958), pp. 261-297.

[4] Modigliani, F. and Miller, M. "Corporate Income Taxes and the Cost of Capital: A Correction," *American Economic Review*, Vol. LIII (June, 1963), pp. 433-443.

obligations. Therefore, a second market imperfection, bankruptcy, must also be incorporated. By assuming there are no associated penalties or resource costs to bankruptcy and that investors have limited liability, Stiglitz[5] has shown that bankruptcy has no effect upon the value of the firm. In the case of no corporate tax, capital structure is irrelevant, and if there are corporate taxes, it is still advantageous for the firm to use as much as possible.[6] Bankruptcy is a "technicality;" the bankrupt firm can be replaced by a new firm and since no resources have been expended there are no economic losses. The assets of the bankrupt firm are still intact; the only difference is that the management (control) and ownership are in the hands of a different group of people (namely, the debt holders instead of the original equity holders).[7]

Within the same framework Stilglitz[8] has attempted to relax the assumption of homogeneity of expectations and has shown, given the additional assumption that short selling is not allowed, that bankruptcy does affect the value of the firm, even though there are no resource costs to bankruptcy. However, if the assumption of no short selling is relaxed, it can be shown that Stiglitz's conclusion is, in general, invalid. Short selling allows the

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[5] Stiglitz, J. E. "A Re-Examination of the Modigliani Miller Theorem," *American Economic Review*, Vol. LIX, No. 5 (December, 1969), pp. 784-793.

[6] For a further discussion see Modigliani, F. and Miller, M., "Reply to Heins and Sprenkle," *American Economic Review*, Vol. LIX, No. 4 (September, 1969), pp. 592-595.

[7] Debt has a fixed set of claims on the firm plus the right to "take over" the firm in the event the firm does not meet its obligations.

[8] Stiglitz, J. E. "Some Aspects of the Pure Theory of Corporate Finance: Bankruptcy and Take-Over," *Bell Journal of Economics and Management Science*, Vol. 3, No. 2 (Autumn, 1972), pp. 458-482.

investor the ability to replicate across all states of nature the income obtained by investing in an unlevered firm, given the assumptions of limited liability.

In reality, there are costs associated with financial distress. In the event of bankruptcy real resources are consumed. That is, bankruptcy proceedings involve costs -- legal fees, trustee fees, administration fees, turnover of employees due to uncertainty, and loss of customers due to uncertainty as to whether the firm will be able to fulfill contracts. Kraus and Litzenburger[9] have formally introduced the tax advantage of debt and bankruptcy penalties into a state preference framework. They have shown that the market value of a levered firm is equal to the unlevered market value, plus the corporate tax rate times the market value of the firm's debt, less the complement of the corporate tax rate times the present value of bankruptcy costs. Thus there is a trade-off between the effects of the two market imperfections: corporate tax and costs to bankruptcy, implying an optimal capital structure.

Under the more general concept of financial distress, it has been argued that there are additional costs associated with changes in financing and investment strategies.[10] If there is a real distinct possibility of bankruptcy, potential investors will demand a premium to compensate for the risks associated with bankruptcy. The levered firm might be in a poor bargaining position and have to offer higher returns to suppliers of additional capital.

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[9] Kraus, A. and Litzenberg, R. "A State-Preference Model of Optimal Financial Leverage," *Journal of Finance*, Vol. XXVIII, No. 3 (September, 1973), pp. 911-922.

[10] Robichek, A. and Myers, S. "Problems in the Theory of Optimal Capital Structure," *Journal of Financial and Quantitative Analysis*, Vol. 1 (June, 1966), pp. 1-35.

The underwriting costs of a common stock issue in such a situation might be very large, if not prohibitive. The existence of credit rationing or constraints on the investment policies of many institutions might further weaken the bargaining position of the firm by reducing competition among potential suppliers of funds. Loan contracts undertaken by the firm might impose constraints on the firm's financial and investment strategies thus limiting its freedom of action. This implies the existence of additional market imperfections besides that of bankruptcy. In a perfect market a firm should, at a price, be able to satisfy its capital requirements. If there are non-price restrictions inhibiting the firm's ability to raise capital, then these are market imperfections.

Hypothesis

In summary, the potential costs associated with financial distress are thought to be an important factor in determining the value of a firm. Assuming that investors are risk averse, they will demand a risk premium to bear the risks associated with financial distress. Unfortunately, an index of financial distress is not available. What is available, however, is the classification of firms as failed or non-failed. The difficulty of mathematilally modelling a concept like financial distress, which should be treated as a continuum, precludes its use in this thesis. Consequently, the extreme situation, bankruptcy, is the operational concept used. It is the sharp distinction between failure and non-failure that avails itself readily to mathematical analysis.

The hypothesis of the thesis is that differences in the probability of bankruptcy across securities and across time are reflected in the residual return after abstracting from the market.

The explicit objectives of the thesis are:

1. To analyze theoretically how the mechanism of bankruptcy affects the structure of returns for corporate financial assets.

2. To quantify the determinants of bankruptcy; to arrive at a figure which can be identified as the probability of bankruptcy.

3. To test empirically the hypothesis of the thesis.

From the theoretical analysis a two variable model is derived, the second variable being associated with the probability of bankruptcy. The ability to measure the probability of bankruptcy implies that the hypothesis of the thesis, as represented by the two variable model, can be empirically tested.

Importance

The empirical work of Beaver[11] and Westerfield[12] has offered some evidence which suggests that impending bankruptcy does appear to affect the structure of returns on common stocks. The behaviour of ex-post returns, after abstracting from the market, for common stocks of firms that eventually went bankrupt, are significantly different from those of healthy

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[11] Beaver, W. H., "Market Prices, Financial Ratios, and the Prediction of Failure," *Journal of Accounting Research*, Vol. 4 (Autumn, 1968), pp. 179-192.

[12] Westerfield, R., "The Assessment of Market Risk and Corporate Failure," University of Pennsylvania, Wharton School of Finance and Commerce, August 1970 (unpublished).

firms for the same time period. These findings, if correct, have important implications as to the significance of the effects of bankruptcy upon the structure of common stock returns. Fisher,[13] in an empirical study, advanced the hypothesis that the risk of default and marketability affect the risk premium on corporate bonds. All of these empirical studies suffer from the lack of any theoretic framework within which to investigate how the mechanism of bankruptcy affects the structure of returns on corporate financial assets.

A number of recent studies have concluded that the Capital Asset Pricing Model (CAPM) is misspecified.[14] It is found that the intercept term is non-stationary, consistently negative for securities with high systematic risk and positive for securities with low systematic risk. The lack of empirical fit can be attributed to the fact that either the model is correct and the difficulty is one of measurement, or that the model is incorrect and must be extended to include additional variables.

In the first case, measurement errors may result either in making the transition from an ex-ante to an ex-post formulation or because of errors in variables. The transformation of the ex-ante model to an ex-post formulation is based upon the assumption that the returns on any security can be represented by a market model; that is, the return on a security is a linear function of a market factor. Thus any test of the ex-post formu-

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[13] Fisher, L. "Determinants of Risk Premiums on Corporate Bonds," *Journal of Political Economy*, Vol. LXVII, No. 3 (June, 1959), pp. 217-237.

[14] Some recent studies are Black, F., Jensen, M. C. and Scholes, M., "The Capital Asset Pricing Model: Some Empirical Tests," published in *Studies in the Theory of Capital Markets*, edited by Jensen, M. (New York: Praeger, 1972); and Blume, M. and Friend, I., "A New Look at the Capital Asset Pricing Model," *Journal of Finance*, Vol. XXVII, No. 1 (March, 1973), pp. 19-34.

lation is a joint test of the CAPM and market model. Errors in variables might arise through measurement errors in the estimation of the individual beta factors or if the market factor is incorrectly specified; the market factor is supposed to measure the return on all assets and not simply the return on the New York Stock Exchange. Another problem which might vilify the results of any investigation is that of the skewness of the distributions of ex-post returns.

In the second case, a number of studies have attempted to relax the various assumptions underlying the model. Two variable models have been developed by Black[15] and Merton.[16] The second variable in the Black version, which has been termed the zero beta factor, arises from relaxing the assumption of investors being able to borrow and lend at the risk free rate of interest. The second variable in Merton's model is the result of relaxing the assumption of a constant investment opportunity set and reflects investors' attempts to hedge against such changes. Neither model provides an adequate explanation of all the observed deficiencies of the CAPM. There is nothing in the Black formulation to suggest that the second factor is non-stationary, and the Merton model does not explain why impending bankruptcy affects the residual return of common stocks after abstracting from the market.

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[15]Black, F. "Capital Market Equilibrium With Restricted Borrowing," *Journal of Business*, Vol. 45, No. 3 (July, 1972), pp. 444-455.

[16]Merton, R. C. "A Dynamic General Equilibrium Model of the Asset Market and its Application to the Pricing of the Capital Structure of the Firm," Massachusetts Institute of Technology, Sloan School of Management, December, 1970.

The primary focus of the thesis is to extend the formulations of the CAPM not from the viewpoint of restrictions upon the investor, but by considering the impact of bankruptcy upon the structure of returns. A two variable model is derived, the second variable being associated with the probability of bankruptcy. An essential step to empirically testing such a model is the development of an operational measure of the probability of bankruptcy for a firm.

Many of the studies dealing with the prediction of bankruptcy have concentrated upon the informational content of accounting numbers.[17] The hypothesis being that there is a difference in profile, as measured by accounting data, between failed and non-failed firms and that given these differences exist, models can be constructed, usually employing multiple discriminant analysis, to determine if a firm should belong to a group of firms having the characteristics of a failed firm or to a group of firms having the characteristics of a non-failed firm. The main consideration has been to classify a firm into one of these two groups. No attempt has been made to construct a theory of the determinants of bankruptcy or to measure the probability of a firm going bankrupt.

The approach proposed in the thesis is an extension of the previous studies in at least two ways. It identifies a set of variables that can be

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[17] See, for example, Beaver, W. H. "Financial Ratios as Predictors of Failure," *Empirical Research in Accounting: Selected Studies*, supplement to *Journal of Accounting Research* (1966), pp. 77-111; Altman, E. I. "Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy," *Journal of Finance*, Vol. XXIII, No. 4 (September, 1968), pp. 589-609; and Edmister, R. O. "An Empirical Test of Financial Ratio Analysis for Small Business Failure Prediction," *Journal of Financial and Quantitative Analysis*, Vol. 7 (March, 1972), pp. 1477-1493.

used in predicting bankruptcy and it introduces a new methodology to this field with which to measure the probability of bankruptcy, that of logit analysis and probit analysis.

To empirically test the hypothesis of the thesis, as represented by the derived two variable model, necessitates consideration of the methodology to be employed. The two variable model describes a relationship for a security at a specific time between its conditional expected return, the security's systematic risk, and the probability of bankruptcy. The traditional approach to the testing of the CAPM is to construct portfolios of securities so as to reduce measurement errors and to test the model using a small number of portfolios. For empirical testing of models which are extensions of the CAPM, elaborate forms of the same type of aggregation procedure have been utilized. However, these procedures make extensive use of estimated parameters and it is not clear how the aggregated effect of measurement errors in these parameters affects the final estimated coefficients. The thesis introduces a new methodology to the testing of two variable models, that of pooling time series and cross section data.

Organization of Thesis

A critical survey of the relevant financial literature and its importance to the thesis is given in Chapter II. The survey, which covers the topics of empirical studies in bankruptcy and the Capital Asset Pricing Model, stresses the underlying assumptions of the reviewed literature, the areas of deficiency and the contributions of the thesis.

Chapter III describes a model for the probability of bankruptcy in terms of ex-ante variables. To use the model for empirical estimation requires that the ex-ante variables be replaced by ex-post surrogates. A general formulation in terms of explanatory variables is developed. As the primary focus is upon prediction of bankruptcy, a second formulation using market values of appropriate corporate variables is constructed. The statistical methodology to estimate the coefficients of the proxy variables is described. The details of three different methods by which the model can be tested are given in the last part of the chapter.

In Chapter IV the theoretic analysis extending the Capital Asset Pricing Model to incorporate bankruptcy is given. The general framework detailing the financial assets available, their price dynamics, the nature of changes in the investment opportunity set, and the behaviorial assumptions for the individual are described. The general form of the equation of optimality is then derived. Due to the complexity of the general analysis, two models are presented. The first model assumes that the investment opportunity set is altered only by the event of bankruptcy. From this analysis a two variable model, which will be empirically tested, is derived. The second model assumes that the probability of bankruptcy changes stochastically over time.

Chapter V describes the empirical work of the thesis. The first part of the chapter describes the data, the results, and the testing of the model to estimate the probability of bankruptcy. The results of this work are used in the second part of the chapter, which describes the testing of the hypothesis of the thesis. The data, statistical methodology, and the results are presented.

Chapter VI summarizes the main findings of the thesis. A list of topics for further research that arise from the thesis is given.

CHAPTER II

A CRITICAL SURVEY OF THE RELEVANT FINANCIAL LITERATURE

In this chapter a review of the financial literature that is relevant to the thesis is given. The relationship of the reviewed literature to the thesis, its importance, the underlying assumptions and their implications are described. It is demonstrated that there are areas of deficiency in the literature and the contributions of the thesis in eradicating these deficiencies stated. Two topics are discussed, those of empirical studies on bankruptcy and the Capital Asset Pricing Model (CAPM).

The empirical studies on bankruptcy that are reviewed address three questions: are there costs to bankruptcy; does impending bankruptcy affect the price behaviour of corporate financial assets; and can bankruptcy be predicted. The questions of the existence of costs to bankruptcy and the effects of bankruptcy upon the structure of returns for corporate financial assets are of prime importance to the formulation of the hypothesis of the thesis. The last question, that of prediction of bankruptcy, is relevant to the testing of the hypothesis of the thesis, where it is necessary to measure the probability of a firm going bankrupt.

The second topic reviewed is the CAPM. The basic model and the empirical evidence indicating that it is misspecified is described. The implications of the misspecification and some of the theoretical attempts to explain its causes are reported. Incorporating some of the findings of the empirical studies on bankruptcy and the nature of the misspecification of the CAPM, the hypothesis of the thesis is presented. To test empirically the hypothesis a two variable model is used, the second variable being the probability of a firm going bankrupt. Before the hypothesis can be tested, it is necessary to determine the statistical methodology to use. Existing methods and their deficiencies are described and an alternative methodology advanced, that of pooling time series and cross section data.

Empirical Studies on Bankruptcy

The empirical studies on bankruptcy which are reviewed pertain to three areas: the existence of costs to bankruptcy, the effects of impending bankruptcy upon the price behaviour of corporate financial assets, and the prediction of bankruptcy. The first two topics combined with some of the findings from the empirical evidence on the misspecification of the CAPM contribute to the formulation of the hypothesis of the thesis. The last topic, that of prediction of bankruptcy, is relevant to the empirical testing of the hypothesis of the thesis, where it is necessary to measure the probability of a firm going bankrupt.

Costs to Bankruptcy

If there were no costs to bankruptcy then it would be a "technicality"; the bankrupt firm could be replaced by a new firm and since no resources have been expended there are no economic losses. However, if there are costs to bankruptcy then this is no longer true and it implies that capital structure will be relevant to the valuation of the firm.

Baxter[1] has considered the existence of costs to corporate bankruptcy

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[1] Baxter, N. D., "Leverage, Risk of Ruin, and the Cost of Capital," *Journal of Finance*, Vol. XXII, No. 3 (September, 1967), pp. 395-403.

and their effects upon capital costs. His thesis is that increased leverage enhances the probability of a firm going bankrupt and that the added administration and legal costs incurred during reorganization, as well as operating inefficiencies which are manifest due to the bankruptcy condition, reduces the value of the firm. Baxter analyzed the immediate effects of bankruptcy on the earnings and sales of a small sample of firms and found that they were adversely affected after the declaration of bankruptcy. This was interpreted that there are costs to bankruptcy and that excessive leverage would reduce the value of the firm. Consideration of only the immediate effects of bankruptcy can be potentially misleading for it is possible for a firm's operations either during or after reorganization to improve and for the original common stockholders to do exceedingly well, inclusive of discounting for their opportunity costs.

Altman,[2] in a more general analysis, considered not only the immediate but also the long term effects of bankruptcy. The findings of Baxter were confirmed, though some evidence was found that prohibited the general statement that investors always suffer because of bankruptcy. The empirical data showed that bankrupt firms' equity on average can be expected to fall in bankruptcy, though a number of bankruptcy reorganizations resulted in favorable overall performance.

The evidence of the existence of the costs to bankruptcy implies that capital structure will affect the valuation of the firm. Closely related to this is the question of how bankruptcy affects the mechanism of the structure of returns on corporate financial assets.

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[2] Altman, E. I., "Corporate Bankruptcy Potential, Stockholder Returns and Share Valuation," *Journal of Finance*, Vol. XXIV, No. 5 (December, 1969), pp. 887-900.

Bankruptcy and Stock Market Prices

The greater the probability of a firm going bankrupt the greater will be the ex-ante expected rate of return that risk-averse investors require. Each period investors will reassess the condition of the firm and adjust the market price of the common stock such that the ex-ante rate of return would continue to be commensurate with the higher risk. If at any time the probability of the firm going bankrupt is greater than expected, there will be a downward adjustment of the market price and the ex-post return will be less than the ex-ante expected rate of return. It is not possible to make any statement about the difference in ex-post returns for healthy firms and for those firms that fail. The direction and magnitude of any difference will depend upon the size and direction of change in the probability of the firm going bankrupt.

Beaver[3] considered what effects impending failure had upon the price behaviour of common stock prices. A crude market model was used. The residual between the ex-post return and the comparable Fisher Link Relative, which is an average rate of return on all firms listed on the New York Stock Exchange, was calculated on an annual basis for a group of failed firms up to five years prior to failure and for a group of non-failed firms. The results were that the median rates of return for failed firms were poorer than those of non-failed firms for five years prior to actual failure and the difference between the median values increased as failure approached. Beaver also derived tests to assess the failure predictive power of rates of return measures and several financial ratios that had been in a previous

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[3]Beaver, W. H., "Market Prices, Financial Ratios, and the Prediction of Failure," *Journal of Accounting Research*, Vol. 4 (Autumn, 1968), pp. 179-192.

study. Univariate tests showed that investors forecast failure sooner than any of the ratios used, with the average length of time from the year of the failure prediction to the date of failure being 2.45 years for the rate of return measure. This was interpreted by Beaver as recognition of the fact that the informational content of accounting numbers is not the only source investors use to detect the possibility of failure.

Westerfield,[4] using monthly data and the market model developed by Sharpe[5] examined the behaviour of residual returns, after abstracting from the market, for failed firms up to six years prior to failure. The market model parameters for the individual securities were estimated using data for the months 120 to 72 prior to failure. For the months 71 to 0 the residual of the realized return minus the ex-ante expected return were calculated. Using two performance measures, it was found that the market began to bid down the market price five years prior to failure, with a rapid deterioration occurring in the year subsequent to failure. This is in contrast to the findings of Beaver and suggests that forecasts of failure more than one year prior to when failure occurs, based upon indices of market performance, will be error prone. Westerfield also examined the relationship between the systematic risk measure and the rate of failure, the hypothesis being that firms whose common equity exhibit high systematic risk with market movements (high betas) experience a higher rate of failure than those assessed as low risk (low betas). Given the limitations of the

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[4] Westerfield, R., "The Assessment of Market Risk and Corporate Failure," University of Pennsylvania, Wharton School of Finance, August, 1970 (unpublished).

[5] Sharpe, W., "A Simplified Model for Portfolio Analysis," *Management Science*, No. 9 (January, 1963), pp. 277-293.

estimation technique and the small sample size, the results showed that on average high risk firms experience failure at a greater rate than low risk firms.

A deficiency of this study is its reliance upon the Sharpe[6] market model. The work of Black, Jensen, and Scholes[7] has shown that the realized return for high beta securities is consistently lower than that predicted by the market model used by Westerfield. This implies that the residual between the realized return and the estimated return will be biased downwards. The problem assumes greater importance when account is taken of Westerfield's findings that suggests that it is high beta securities that tend to fail more often than low beta securities. Thus the two performance measures used are biased towards indicating a deterioration earlier than when it actually occurs.

A common finding of these studies is that the market constantly underestimated the probability of a firm going bankrupt. From an ex-ante viewpoint the expected rate of return, conditional upon no bankruptcy, should increase if the probability of bankruptcy increases so to compensate risk averse investors for the increased risk, but this does not imply that the ex-post return should increase. If during a particular period the probability of a firm going bankrupt unexpected increases, then this will be reflected in a lower price at the end of the period than had been expected at the beginning of the period and the realized return will decrease. The

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[6] *Ibid.*

[7] Black, F., Jensen, M. C., and Scholes, M., "The Capital Asset Pricing Model: Some Empirical Tests," published in *Studies in the Theory of Capital Markets*, edited by Jensen, M. (New York: Praeger, 1972).

longer the time period the greater the potential seriousness this problem becomes for empirical studies. It does, however, stress the need to consider the ramifications of stochastic changes in the probability of bankruptcy in any theoretic investigations of the effects of bankruptcy upon the structure of returns for corporate financial assets.

Both studies offer empirical evidence which suggests that the price behaviour of common stocks is affected by impending bankruptcy. If this is correct, then it is pertinent to enquire what is the nexus between the market pricing process and corporate failure. The pursuit of this question is the primary focus of the thesis.

Prediction of Bankruptcy

To test empirically the hypothesis of the thesis a two variable model, which is an extended form of the CAPM, is used. The second variable is the probability of a firm going bankrupt and thus it is necessary to be able to estimate this quantity.

The primary focus of previous studies on the prediction of bankruptcy has been on the informational content of accounting statements and financial ratios. These studies have advanced the hypothesis that there is a difference in profile, as measured by accounting data, between failed and non-failed firms and this difference can be utilized as an aid to prediction; that is, it is possible using accounting data to allocate firms to one of two groups: failed and non-failed. This hypothesis was first used in a univariate form. Single financial ratios were tested for their predictive ability. However, univariate analysis can be potentially misleading for failure depends upon many different factors. Consequently, a multivariate approach to the prediction of bankruptcy has been developed.

Beaver[8] used a univariate approach to select from a sample of thirty financial ratios the one most able to correctly predict the failure status of a firm. Failure was defined to occur when a firm was unable to pay its financial obligation when they matured. Operationally, a firm was identified as failed when one of the following events occurred: bankruptcy, bond default, an overdrawn bank account, or non-payment of a preferred stock dividend.[9] Such a definition is very broad and is more in keeping with the concept of financial distress. To classify all those categories under one group, that of failure, will result in inefficient estimation, as not all the information is being utilized. Presumably, for a firm to default on its bond payments is a more serious event than for a firm to omit payment on a preferred dividend. By classifying these two events under the one group does not take this fact into account. Another problem is that many firms omit dividends for reasons other than that caused by impending failure and thus to classify these firms as failed results in a misclassification.

Beaver found that the ratio of cash flow to total debt was best at being able to correctly predict the failure status of firms and that this ability existed for at least five years before failure. Thus on the basis of a single financial ratio firms were allocated to one of two groups: failed and non-failed.

Univariate analysis is susceptible to faulty interpretation and is potentially misleading. For instance, a firm whose capital structure

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[8]Beaver, W. H., "Financial Ratios as Predictors of Failure," *Empirical Research in Accounting: Selected Studies*, supplement to *Journal of Accounting Research* (1966), pp. 77-111.

[9]*Ibid.*, p. 71.

contains a large proportion of debt may be regarded as a potential bankrupt. However, because of very low variability in its cash flow, the situation may not be considered serious. In an attempt to avoid this type of problem a multivariate approach considering several financial ratios has been developed. This involves being able to determine which financial ratios are important in detecting future bankruptcy, what weights should be attached to the selected ratios and how should the weights be objectively established. The methodology that is usually used is that of multiple discriminant analysis.

Multiple discriminant analysis (MDA) is a statistical technique used to classify an observation to one of two mutually exclusive groups.[10] The basis of the technique is to construct a discriminant function from a linear combination of explanatory variables, the weights being determined by minimizing the expected cost of misclassification.

To use MDA it is necessary to define the two mutually exclusive groups. For prediction of bankruptcy the two groups are defined to be bankrupt and non-bankrupt. Data are collected for the firms in the two groups; MDA then attempts to derive a linear combination of the characteristics (financial ratios) which discriminates between the two groups so as to minimize the cost of misclassifying a firm.

One of the advantages of MDA is that it is capable of considering an entire profile of characteristics common to the relevant firms, as well as the interaction of these properties, and to combine them in a single

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[10] MDA can be extended to the general case of many mutually exclusive groups. A good introduction to MDA is given in Anderson, T. W., *An Introduction to Multivariate Statistical Analysis* (New York: John Wiley & Sons, 1957).

discriminant function. There are, however, a number of potential drawbacks to MDA. It is necessary to make some specification about the conditional expected costs of misclassification and the parameters and form of the probability distributions that describe the properties of the different groups. In general, neither the costs of misclassification nor the prior probabilities are known. Without knowledge of the prior distributions it is not possible to calculate the probability of an observation belonging to a particular group. The primary focus of MDA is to allocate an observation to a particular group without measuring the probability of the observation belonging to that group.

Altman[11] developed a multivariate approach to the prediction of bankruptcy using MDA. A firm was classified as bankrupt if it filed a bankruptcy petition under Chapter X of the National Bankruptcy Act (U.S.A.). This definition avoids the problem of misclassification that was encountered in the Beaver study.[12] From an initial set of twenty-five financial ratios a discriminant function composed of five financial ratios characterizing liquidity, profitability, productivity, financial risk, and sales generating ability was determined. The discriminant function was able to correctly classify 94 per cent of the initial data sample and to achieve a high standard of success on two secondary samples. Thus on the basis of a linear combination of five financial ratios firms were allotted as either bankrupt or non-bankrupt.

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[11] Altman, E. I., "Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy," *Journal of Finance*, Vol. XXIII, No. 4 (September, 1968), pp. 589-609.

[12] Beaver, "Financial Ratios," *loc. cit.*

Using a different data sample, Deakin[13] replicated the Beaver[14] and the Altman[15] studies obtaining similar conclusions, but then proceeded to claim that the probability of group membership could be derived using a statistic having a chi-square distribution, the number of degrees of freedom equaling the number of variables used in the MDA.[16] This is wrong. The probability of assigning an observation to a particular group can only be calculated in this context with the knowledge of the prior distributions, but these are unknown. The statistic used by Deakin is probably[17] a test for the null hypothesis that the two group means are identical.[18]

Accounting data variables have been used by Fisher[19] in an empirical study of the determinants of the risk premiums on corporate bonds. It was hypothesized that the risk premium depended upon the marketability and the risk of default of the bond. The marketability of the bond was estimated by a single variable, the market value of all the firm's publicly traded bonds, and the risk of default was assumed to depend upon three variables: the coefficient of variation of the firm's net income, the period of solvency, and

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[13]Deakin, E. B., "A Discriminant Analysis of Predictors of Business Failure," *Journal of Accounting Research*, Vol. 10, No. 1 (Spring, 1972), pp. 167-179.

[14]Beaver, "Financial Ratios," *loc. cit.*

[15]Altman, "Financial Ratios," *loc. cit.*

[16]Deakin, *op. cit.*, p. 175.

[17]As Deakin fails to give clear definitions of the terms used in the statistic, it is difficult to make any positive statement.

[18]A description of this statistic is given in Anderson, *op. cit.*, p. 56.

[19]Fisher, L., "Determinants of Risk Premiums on Corporate Bonds," *Journal of Political Economy*, Vol. LXVII, No. 3 (June, 1959), pp. 217-237.

the ratio of the market value of equity to the par value of the firm's debt. From this formulation approximately 70 per cent of the variance of the dependent variable could be explained. There are, however, a number of deficiencies in the study. It is not clear that the marketability of a bond can be estimated by a single variable, or that the risk of default can be determined by a function of three variables. Without further investigation of the validity of these measures, the interpretation of the findings of the study are jeopardized.

Whilst this study breaks away from the strict use of financial ratios, it still relies upon the informational content of accounting numbers, a criticism that can be applied to all the studies pertaining to the prediction of failure. Accounting data reflects the consequences of past actions, whilst for prediction it is not the past but the future that is relevant. The focus of previous studies has been to allocate firms, on the basis of financial ratios, to one of two groups: failed or non-failed. No attempt has been made to offer either a theory of the determinants of failure or to measure the probability of a firm failing.

A contribution of the thesis is the development of a model to determine the probability of a firm going bankrupt. At any point in time this depends upon the firm's ability to raise funds, either internally or externally, to cover fixed charges. To use the model to empirically estimate the probability of bankruptcy requires that the ex-ante variables be replaced by ex-post surrogates. This necessitates consideration of the statistical methodology to utilize in estimating the coefficients of the ex-post surrogates. Multiple discriminant analysis can not be used, for it is primarily designed to allocate an observation to a particular group without measuring the probability of the observation belonging to that group. The thesis introduces a new methodology to estimate the coefficients and the probability of a firm going bankrupt, that of logit analysis and probit analysis.

To test empirically the hypothesis of the thesis, it is only necessary to be able to measure the probability of bankruptcy and not to explain its determinants; that is, the primary focus is upon predicting the probability of bankruptcy and not to advance a complete theory of its determinants. Consequently, two formulations of the model are derived. The first concentrates on being able to explain the determinants of bankruptcy, whilst the second is developed solely for its predictive ability using market values for appropriate corporate variables. Both formulations of the model will be used in the testing of the hypothesis.

Capital Asset Pricing Model

The second topic reviewed is the Capital Asset Pricing Model (CAPM). The theoretic foundations of the Model and the empirical evidence which indicates that it is misspecified are described. Some of the various attempts to explain the cause(s) of the misspecification are discussed and an alternative explanation, which forms the hypothesis of the thesis, is advanced.

To test empirically the hypothesis a two variable model is used, the second variable being the probability of a firm going bankrupt. Before the hypothesis can be tested, it is necessary to determine the methodology to utilize. Existing techniques and their deficiencies are described and an alternative methodology presented.

Foundations of the CAPM

The CAPM describes a linear relationship between the equilibrium expected return on an asset and its systematic risk, which is a measure of the asset's covariance with the market portfolio. The market portfolio is composed of an investment in every risky asset outstanding in proportion to its total value. The CAPM was originally formulated in a mean-variance context by Treynor,[20] Sharpe[21] and later clarified by Lintner[22] and Mossin.[23] In the development of the model it is assumed that: (a) all investors are single period expected utility of terminal wealth maximizers who choose among alternative portfolios on the basis of mean and variance; (b) all investors can borrow or lend an unlimited amount at an exogenously given risk free rate of interest and there are no restrictions on short sales of any assets; (c) all investors have identical subjective estimates of the means, variances, and covariances of return among all assets; (d) all assets are perfectly divisible and there are no transaction costs; (e) there are no taxes; (f) all investors are price takers; and (g) the quantities of assets are given.

The model may be stated in the mathematical form

$$E\ (\tilde{R}_j) = \beta_j\ [E\ (\tilde{R}_M)], \qquad [2.1]$$

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[20]Treynor, J., "Towards a Theory of Market Value of Risky Assets" (unpublished memorandum, 1961).

[21]Sharpe, W. F., "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance*, Vol. XIX, No. 3 (September, 1964), pp. 425-442.

[22]Lintner, J., "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, Vol. XLVII (February, 1965), pp. 13-37.

[23]Mossin, J., "Equilibrium in a Capital Asset Market, *Econometrica*, Vol. 34, No. 4 (October, 1966), pp. 768-783.

where, if $P_j(t)$ denotes the price of the j^{th} asset at the end of the period,

$$E(\tilde{R}_j) = \frac{E[\tilde{P}_j(t)] - P_j(t-1)}{P_j(t-1)} - r_F$$

= expected excess return on the j^{th} asset;

r_F = the riskless rate of interest;

$E[\tilde{R}_M]$ = expected excess return on a 'market portfolio' consisting of an investment in every asset outstanding in proportion to its value;

and $\beta_j = \text{cov}(\tilde{R}_j, \tilde{R}_M) / \text{var}(\tilde{R}_M)$

= the 'systematic' risk of the j^{th} asset.

The above relation states that the expected excess return on any asset is directly proportional to its systematic risk. If α_j is defined as

$$\alpha_j = E(\tilde{R}_j) - \beta_j E(\tilde{R}_M), \qquad [2.2]$$

then equation [2.1] implies that α for every asset is zero.

Empirical tests of the CAPM have been based upon ex-post data. The transformation of the ex-ante model to an ex-post formulation is based upon the assumption that the return on any security can be represented by a market model; that is, the return on a security is a linear function of a market factor. Thus any test of the ex-post formulation is a joint test of the CAPM and the market model. Using the market model originally proposed by Markowitz[24] and extended by Sharpe[25] and Fama[26] the ex-post formulation

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[24] Markowitz, H., *Portfolio Selection: Efficient Diversification of Investments*, Cowles Foundation Monograph No. 16 (New York: John Wiley and Sons, 1959).

[25] Sharpe, "A Simplified Model," *loc. cit.*

[26] Fama, E., "Risk, Return and Equilibrium: Some Clarifying Comments," *Journal of Finance*, Vol. XXIII, No. 4 (March, 1968), pp. 29-40.

of the CAPM, as represented by Equation [2.1] can be written

$$\tilde{R}_j = \beta_j \tilde{R}_M + \tilde{e}_j \qquad [2.3]$$

where $\tilde{e}_j$ is a normally distributed zero mean random variable. If assets are priced according to the CAPM then a joint test of the CAPM and the market model can be obtained by adding an intercept α_j to [2.3] and subscripting each of the variables by t, representing time, to obtain

$$\tilde{R}_{jt} = \alpha_j + \beta_j \tilde{R}_{Mt} + e_{jt}, \qquad [2.4]$$

which is a regression equation, the null hypothesis being that the intercept term $\{\alpha_j\}$, is zero for all assets.

<u>Cross Sectional Tests of the Model</u>

For cross sectional tests the procedure used is to estimate the cross sectional regression

$$\bar{R}_j = \nu_o + \nu_1 \hat{\beta}_j + e_j, \qquad [2.5]$$

where $\hat{\beta}_j$ is obtained from the regression of a time series of individual security returns on an index used as a proxy for the market portfolio. The null hypothesis is that $\nu_o = 0$, and $\nu_1 = (\bar{R}_M - \bar{r}_F)$, where $\bar{R}_M$ is the average return on the market index over the time period, and $\bar{r}_F$ is usually taken to be the yield to maturity of a government bond with the same maturity as the length of the time period under examination.

Evidence presented by Douglas,[27] who regressed the returns on a large cross sectional sample of common stocks on their own variance and on their covariance with an index constructed from the sample, found that the model

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[27] Douglas, G., "Risk in the Equity Markets: An Empirical Appraisal of Market Efficiency," *Yale Economic Essays*, Vol. 9 (Spring, 1969), pp. 3-45.

did not provide a complete description of security returns. For seven separate five year periods from 1926 to 1960, the average realized return was significantly positively related to the variance of the security's returns over time, but not to their covariance with the index of returns. These results appear to be in conflict with the relation given by [2.1] for the variance term should have a coefficient of zero.

Douglas also summarizes some unpublished results of Lintner's that also appear to be inconsistent with Equation [2.1]. Lintner estimates Equation [2.5], for a cross section of securities, adding an extra variable: the variance of the residuals from the time series regressions given by Equation [2.4]. This extra variance should have no explanatory power and thus its coefficient should be zero. In Lintner's tests it did not, the coefficient on the residual variance being positive and significant. Also, $\hat{\nu}_o$ was greater than zero and $\hat{\nu}_1$ much less than $(\bar{R}_M - \bar{r}_F)$.

Miller and Scholes[28] replicated the Lintner study on a different body of data obtaining the same general results. The source of the misspecification may arise not because the model is wrong, but due to the difficulty of measuring the different variables. For example, the biases introduced by (a) failure to account adequately for the riskless rate of interest, (b) possible non-linearity in the risk-return relation, or (c) distortions due to heteroscedasticity, may be the cause of the Douglas-Lintner finding. However, it was shown that these errors could not produce such results. It was also demonstrated that whilst measurement errors in the risk variable

[28]Miller, M. and Scholes, M., "Rates of Return in Relation to Risk: A Re-examination of Some Recent Findings," printed in *Studies in the Theory of Capital Markets*, edited by Jensen, M. (New York: Praeger, 1972).

$\{\beta_j\}$, and the correlation between the variance of the residuals from the time series regressions and the estimates of the risk value $\{\hat{\beta}_j\}$, could contribute substantially to the Douglas-Lintner results, they were not sufficient to account for all the observed deviations from the model. A problem which could account for such results is the presence of skewness in the probability distributions of ex-post returns. Miller and Scholes were able to show that skewness effects could cause serious difficulties and that combined with the measurement errors in the risk variable could in principle cause the Douglas-Lintner results. Whilst their analysis does not imply the complete rejection of the Douglas-Lintner results, it does show that they must be treated with caution in view of the econometric difficulties in testing the model.

<u>Time Series Tests of the Model</u>

Black, Jensen, and Scholes (B-J-S)[29] have tested the CAPM by using a time series procedure. The model can be tested by running a time series regression using the equation [2.4]; that is,

$$\tilde{R}_{jt} = \alpha_j + \beta_j \tilde{R}_{Mt} + \tilde{e}_{jt} , \qquad [2.4]$$

the null hypothesis being that the intercept term, α_j, is zero for all assets. Thus a direct test can be obtained by estimating [2.4] for a security over some time period and testing to see if α_j is significantly different from zero. Whilst this test is simple, it is inefficient in that it utilizes information on only a single security. To overcome this problem B-J-S

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[29]Black, F., Jensen, M., and Scholes, M., "The Capital Asset Pricing Model: Some Empirical Tests," printed in *Studies in the Theory of Capital Markets*, edited by Jensen, M. (New York: Praeger, 1972).

perform their tests on portfolio returns over the period 1931 to 1965, where the portfolios are constructed so as to maximize the dispersion of their systematic risk. They applied their tests to ten portfolios, which contained all securities on the New York Stock Exchange. The results indicated that the intercept term was directly related to the systematic risk level. For low risk securities the intercept term was positive, and for high risk securities it was negative. There was substantial indication that the intercept terms for the different portfolios were non-stationary, especially for securities whose level of systematic risk was different from unity. Similarly findings have also been obtained in a recent study by Blume and Friend.[30]

B-J-S go on to demonstrate that the process generating the return on individual securities can be described by a two variable model of the firm

$$\tilde{r}_{jt} = (1 - \beta_j)\, \tilde{r}_{Zt} + \beta_j\, \tilde{r}_{Mt} + \tilde{e}_{jt} , \qquad [2.6]$$

where $\tilde{r}_{Zt}$ represents the return on what they call the 'beta factor' and the other lower case r's indicate total returns. Using a grouping procedure which eliminates most of the difficulties associated with the biases introduced by measurement errors in the $\{\beta_j\}$ in cross sectional tests, an examination of the cross sectional relationships between risk and return for seventeen subperiods of lengths twenty-four months and four subperiods of length one hundred and five months was conducted. It appears that the relationship is highly linear, but both the intercepts and slopes fluctuate randomly from period to period and are often negative. B-J-S argue these

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[30] Blume, M. and Friend, I., "A New Look at the Capital Asset Pricing Model," *Journal of Finance*, Vol. XXVIII, No. 1 (March, 1973), pp. 19-34.

non-stationarities are consistent with the return generating mechanism described by [2.6], which implies that the intercept and slope in the cross sectional regressions will be $\bar{r}_Z$ and $(\bar{r}_M - \bar{r}_Z)$ respectively, where the bars denote sample means over the time period covered by the cross sections. Since $\bar{r}_Z$ will also be a random variable, equation [2.6] is consistent with the observed empirical results.

The evidence seems to indicate that the CAPM does not provide an adequate description of the process generating common stock returns. The documentation of non-stationarity and the existence of at least another factor imply that the model must be extended to include additional variables.[31]

Theoretical Extensions of the CAPM

The majority of the assumptions underlying the model violate to some degree the conditions observed in practice. A number of recent studies have attempted to relax various assumptions so as to incorporate some of the complexities of capital markets into the model. The results have indicated that the basic structure of the model is remarkably robust to violations of these assumptions.

The assumption that investors are single period expected utility of terminal wealth maximizers is very restrictive and may not be an accurate description of investors' behaviour. Fama[32] has investigated the conditions

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[31] An excellent survey article describing even more evidence about the misspecification of the CAPM is given in Jensen, M., "Capital Markets: Theory and Evidence," *Bell Journal of Economics and Management Science*, Vol. 3, No. 2 (Autumn, 1972), pp. 357-398.

[32] Fama, E., "Multiperiod Consumption--Investment Decisions," *American Economic Review*, Vol. LX, No. 1 (March, 1970), pp. 163-174.

for the validity of such an assumption. Arguing that investor's problem is more accurately stated as the maximization of the expected lifetime utility of consumption and terminal wealth, it is demonstrated that the single period CAPM can be justified in the context of a multiperiod problem if the investor behaves as if future consumption and investment opportunities are given and that tastes are not state dependent. Thus even though the investor must solve a multiperiod problem to arrive at the optimal current decisions, these decisions are indistinguishable from those of a risk averse single period expected utility of terminal wealth maximizer.

These findings have important implications for the conditions which must be satisfied if the CAPM is to be empirically tested. Previous empirical studies have utilized ex-post data extending over many time periods and have tacitly assumed the validity of the model in a multiperiod context. But tastes do change over time and thus one of the assumptions necessary to use the model over extended time periods is violated and hence the conclusions of such studies are jeopardized.

It is assumed in the CAPM that all assets are perfectly liquid; that is, all assets are marketable and there are no transaction costs. There are many assets for which this assumption is not applicable. For example, claims on labour income or social security payments are claims that can not be sold in capital markets. Thus it is pertinent to enquire what effect non-marketability of assets has upon the CAPM. For the special case of only two types of assets, perfectly liquid and perfectly non-liquid, Mayers[33] derives a simple expression between the expected rate of return on

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[33] Mayers, D., "Non-Marketable Assets and Capital Market Equilibrium Under Uncertainty," printed in *Studies in the Theory of Capital Markets*, edited by Jensen, M. (New York: Praeger, 1972).

any asset and its covariance risk in terms of market parameters and demonstrates that the basic implications of the model are not weakened in any major respect by the existence of non-marketable assets.

There are, however, many assets that can not be described as being either perfectly liquid or perfectly non-liquid, real estate, second hand automobiles, being possible examples. Marketability is not a dichotomous concept but is a continuum. To investigate the full effects of marketability upon the CAPM requires that its determinants be known and that the "degree" of marketability of an asset can be measured. Whilst the results of the Mayers' study are important, the study does not address itself to the more general and difficult problem of treating marketability as a continuum.

One of the assumptions of the CAPM is the existence of a riskless asset. In the presence of uncertainty about the level of future prices and as contracts are not denominated in real terms, the assumption of the existence of such an asset is tenuous. Black[34] has shown, under assumptions identical to those of the CAPM, that if a riskless asset, or borrowing or lending opportunities do not exist, then in equilibrium the portfolios of all investors consist of a linear combination of two basic portfolios, one being the market portfolio and the other a portfolio whose returns have zero covariance with the market portfolio and has minimum variance. This portfolio has been termed the "zero beta" portfolio. Black demonstrates that in equilibrium the expected return on any asset will be given by

$$E\,(\tilde{r}_j) = (1 - \beta_j)\,E\,(\tilde{r}_Z) + \beta_j\,E\,(\tilde{r}_M),$$

[34] Black, F., "Capital Market Equilibrium With Restricted Borrowing," *Journal of Business*, Vol. 45, No. 3 (July, 1972), pp. 445-455.

where E $(\tilde{r}_Z)$ is the expected return on the zero beta portfolio and the other variables are as previously defined. Whilst this is a two factor model and bears a close relationship to the model suggested by B-J-S,[35] there is nothing in the formulation to suggest that the second factor is not constant, and thus it can not adequately explain all the observed empirical deficiencies of the CAPM.

The CAPM, which is formulated in a discrete time framework, rests upon the assumption that there are no transaction costs. Whilst the formulation of a discrete time model is convenient for empirical work, its theoretic justification is questionable. The rational investor would always prefer to have the option to trade any instant of time--at no cost--then to be restricted to trading at discrete time intervals.

The usual reason given for the discrete time formulation is that transaction costs do exist. However, the approach is to take equal time spacings of non-specified length. If transaction costs are to be included, the interval between trading periods will become a variable depending upon the size of transaction costs, changes in the prices of securities, initial wealth distribution, and expectations. The inclusion of transaction costs also raises the question of the existence of an equilibrium.

Merton[36] has derived a continuous time version of the CAPM, although the symbols, which appear the same, have a different interpretation, being expressed in terms of instantaneous rates of return. Four extra assumptions

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[35] Black, Jensen, and Scholes, *loc. cit.*

[36] Merton, R. C., "A Dynamic General Equilibrium Model of the Asset Market and Its Application to the Pricing of the Capital Structure of the Firm," Massachusetts Institute of Technology, Sloan School of Management, December, 1970.

are required: trading takes place continuously, which follows directly from the assumption of no transaction costs; a constant investment opportunity set, that is, the means, variances and covariances that describe the characteristics of the different assets are constant; only local changes in the state variables of the process are allowed, which rules out Pareto-Levy or Poisson type processes; and investors act so as to maximize the expected utility of lifetime consumption and terminal wealth. Representing the price dynamics of securities by Weiner processes the continuous time analogue to CAPM is derived:

$$\alpha_j - r = \beta_j \ (\alpha_M - r),$$

where α_j is the instantaneous rate of return on the j^{th} asset; r is the instantaneous risk free rate of interest; α_M is the instantaneous rate of return on the market portfolio; and $\beta_j \equiv \sigma_{jM}/\sigma_M^2$; and σ_{jM} is the instantaneous covariance of the return on the j^{th} asset with the return on the market portfolio.

If any of the three conditions outlined by Fama[37] to justify the single period maximization of expected utility of terminal wealth assumption of the CAPM are violated, then the simple structure of the model will probably be destroyed. For example, if the investment opportunity set changes, then this will, in general, affect the consumption-investment decision of the investor. Merton,[38] in the context of a continuous time framework, demonstrates that changes in the investment opportunity set do

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[37] Fama, "Multiperiod Consumption," *loc. cit.*

[38] Merton, R. C., "An Intertemporal Capital Asset Pricing Model," Working Paper 588-72, Massachusetts Institute of Technology, Sloan School of Management, February, 1972.

affect the structure of the model. Under the assumption that these changes can be characterized by changes in a single instrumental variable--the riskless interest rate--a three variable model is derived. The third variable can be interpreted as the result of investors hedging against the effects of future unforeseen changes in the riskless interest rate. In equilibrium, the investor's portfolio will be a linear conbination of three portfolios: the riskless asset, the market portfolio, and a portfolio (or asset) which is perfectly negatively correlated with changes in the riskless interest rate. The model can be stated in the form:

$$\alpha_j = r_F + \lambda_1 (\alpha_M - r) + \lambda_2 (\alpha_N - r),$$

$$\lambda_1 = \frac{\beta_{jM} - \beta_{jN}\beta_{NM}}{1 - \rho^2_{NM}},$$

$$\lambda_2 = \frac{\beta_{jN} - \beta_{jM}\beta_{NM}}{1 - \rho^2_{NM}},$$

$$\beta_{jK} = \frac{\sigma_{jK}}{\sigma_{kK}},$$

σ_{jk} is the instantaneous covariance between the j^{th} and k^{th} assets; and ρ_{KM} is the instantaneous correlation between the returns on the market portfolio and the asset N, which is negatively correlated with changes in the riskless interest rate. Merton argues that the sign of λ_2 will be positive for low beta assets and negative for high beta assets. The sign for $(\alpha_N - r)$ is identical to that of $-\frac{\partial C}{\partial r}$, where C is the aggregate consumption function. Macro-economic theory usually assumes that aggregate saving is an increasing function of yields and therefore $\frac{\partial C}{\partial r} < 0$, which implies that $(\alpha_N - r)$ will be positive. Thus, Merton concludes that the model is at least consistent with empirical evidence.

Summary

Accumulating empirical evidence indicates that the CAPM does not provide an adequate description of the mechanism generating common stock returns. The time series work of Black, Jensen, and Scholes,[39] has shown that assets with high levels of systematic risk consistently earn less than that predicted by the model, whilst assets with low levels of systematic risk earn more than that predicted. Even though there appears to be a linear relationship between a security's ex-post return and its systematic risk, the relationship is non-stationary; over various time periods ex-post return and systematic risk have been inversely related. Black, Jensen and Scholes argue that the data indicate that the expected return on a security can be represented by a linear two factor model. The second factor, which they call the beta factor, is not explicitly identified.

The works of Westerfield[40] and Beaver[41] have shown that price behaviour of common stocks is affected by impending bankruptcy. From an ex-ante viewpoint, the expected rate of return, conditional upon no bankruptcy, should increase if there is an increase in the probability of bankruptcy to compensate risk averse investors for the extra risk. Within the framework

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[39] Black, Jensen, and Scholes, *loc. cit.*

[40] Westerfield, R., "The Assessment of Market Risk and Corporate Failure," University of Pennsylvania, Wharton School of Finance, August 1970 (unpublished).

[41] Beaver, "Market Prices," *loc. cit.*

of the CAPM this implies that the single explanatory variable, that of systematic risk, which is a measure of the covariance of the firm's return with the returns of all other securities, should increase. But this contradicts the assumption of a constant investment opportunity set. Thus the CAPM can not explain why price behaviour is affected by impending bankruptcy.

Various attempts have been made to explain these deficiencies. The effects of non-marketability of assets, the non-existence of a riskless asset, and restrictions upon the investor's ability to borrow or lend have been explored, though these fail to provide an adequate explanation of all the observed deficiencies. The effects of changes in the investment opportunity set have been shown to imply the existence of a three variable model which is at least theoretically consistent with the empirical findings of Black, Jensen, and Scholes,[42] though it does not provide an explanation of why impending bankruptcy affects the residual return behaviour, after abstracting from the market, of common stocks.

Hypothesis of the Thesis

If the probability of bankruptcy for a firm increases, then the expected return, conditional upon no bankruptcy, which risk averse investors require will increase to compensate for the extra risk. At any point in time the probability of bankruptcy for a firm is a function of its ability to raise funds, either internally or externally, to cover fixed charges. As conditions within the firm and the economy change over time, so will the firm's ability to raise funds, which implies that the probability of the firm going bankrupt will also vary across time. This will directly affect

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[42] Black, Jensen, and Scholes, *loc. cit.*

the expected rate of return which investors require on the firm's financial assets.

The thesis gives a theoretic explanation of the effects of bankruptcy upon the structure of corporate financial assets. The hypothesis of the thesis is that differences in the probability of bankruptcy across securities and across time are reflected in the residual return after abstracting from the market.

Empirical Testing of the Hypothesis

From the theoretical analysis a two variable model describing the structure of common stock returns is derived. The model is an extension of the CAPM and is of the form

$$\alpha_j = r_F + \lambda_j + \beta_j \ (\alpha_M - r_F - \bar{\chi}) \ ,$$

where α_j is the instantaneous conditional expected rate of return on the j^{th} asset; α_M is the instantaneous conditional expected rate of return on the market portfolio; r_F is the instantaneous risk free rate of interest; λ_j is the rate of probability of bankruptcy for the j^{th} asset; $\bar{\chi}$ is a weighted average of the $\{\lambda_j\}$; and $\beta_j \equiv \sigma_{jM}/\sigma_{MM}$, σ_{jM} being the instantaneous conditional covariance of the j^{th} asset with the market portfolio. To test empirically the hypothesis a discrete time, ex-post formulation of the model is used. However, before testing the hypothesis two preliminary steps are necessary. First, the probability of a firm going bankrupt over a given period needs to be estimated; and second, a choice of methodology to employ when testing the hypothesis must be made.

The probability of bankruptcy for a firm depends upon its ability to raise funds, either internally or externally, to cover fixed charges. A model describing the determinants of the firm's ability to raise funds is constructed. The primary focus is the prediction and estimation of the probability of bankruptcy, as opposed to constructing a full explanatory theory. The coefficients of the model are determined using logit analysis and probit analysis.

A methodology to test a two variable model has been developed by Black and Scholes.[43] In an attempt to examine the effects of dividends on common stock prices an ad hoc two variable extension of the CAPM has been advanced. The model is of the form

$$E\ (r_j) = \nu_0 + \nu_1\ \left(\frac{\delta_j - \delta_M}{\delta_M}\right) + \beta_j\ [E\ (r_M) - \nu_0]\ ,$$

where $E\ (r_j)$ is the expected return for the j^{th} asset; $E\ (r_M)$ is the expected return on the market portfolio; δ_j is the expected dividend yield for the j^{th} asset; δ_M is the expected dividend yield on the market portfolio; $\beta_j \equiv cov\ (r_j, r_M)\ /\ var\ (r_M)$; ν_0 is a constant to account for the existence of a beta factor; and ν_1 is a constant. The hypothesis is that the residual return on a security, after abstracting from the market factor, can be explained by the dividend yield. If the hypothesis is true, then the coefficient ν_1 should be non-zero and statistically significant.

A cross sectional analysis, which would utilize the information on all securities, is ruled out because of the econometric difficulties caused by errors in variables. Thus, a time series approach is used. The methodology is to construct a portfolio such that its expected return is ν_1 and

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[43] Black, F. and Scholes, M., "Divided Yields and Common Stock Returns: A New Methodology," Financial Note No. 19B, Massachusetts Institute of Technology, Sloan School of Management, August, 1971.

the portfolio to have minimum variance. To solve the equations representing the first order conditions requires knowledge of the beta coefficients, the expected dividend yields, and the variance-covariance matrix, all of which are unknown and must be estimated. To reduce measurement errors a method of aggregation is used to form a small number of portfolios. Securities are assigned to portfolios on the basis of their estimated beta coefficient and dividend yield. These portfolios are then treated as securities and their beta coefficients, expected dividend yield, and variance-covariance matrix estimated. The beta coefficient for a portfolio is determined by regressing its return on the market return, after subtracting the interest rate from both returns. Given these estimates, the final portfolio is constructed for different time periods.

There are a number of major deficiencies with this methodology. The estimation of the beta coefficients neglecting the dividend yield, implies that there is a missing variables problem, which will cause bias in the estimated coefficients. This will affect the estimation of the variance-covariance matrix, which uses the estimated beta coefficients. Without detailed knowledge of how the measurement errors of the different variables affect the final estimated coefficients, the applicability of the methodology is questionable.

The thesis introduces a new methodology to the testing of two variable models, that of pooling time series and cross section data. In the ex-post formulation of the model used to test the hypothesis of the thesis, the constant term and the coefficient multiplying the probability of bankruptcy are not firm specific. The time series data for all individual securities are combined to estimate these two coefficients, whilst simultaneously estimating the firm specific beta coefficients.

CHAPTER III

PROBABILITY OF BANKRUPTCY

Bankruptcy in a single period context occurs if, at the terminal point, the income of the firm is less than its fixed obligations. In a multi-period setting such a definition is not appropriate, for in an on-going firm income can be less than the obligations of the firm and yet the firm is not bankrupt; it can simply borrow more. At any point in time, the probability of a firm going bankrupt depends upon its ability to raise funds, either internally or externally, to cover fixed charges. A firm that fails to cover these fixed charges is said to be bankrupt. This definition of bankruptcy is the basic construct in formulating a model of the probability of bankruptcy.

The probability of a firm going bankrupt depends not only upon its current level of earnings but also its ability to raise funds, which is subsumed in its future earning power. However, such variables are ex-ante in nature and can not be directly observed. To use a model to empirically estimate the probability of bankruptcy requires that the ex-ante variables be replaced by ex-post surrogates. The construction of a model to estimate the probability of bankruptcy in terms of ex-post variables is described in the first part of the chapter. As the primary focus is upon the prediction of bankruptcy and not to advance a complete theory of its determinants, a second formulation utilizing market values for appropriate corporate variables is developed.

When the ex-ante determinants are replaced by ex-post surrogates, it is necessary to estimate the relative contribution of the different variables. The statistical methodology to estimate the coefficients of the proxy variables is described in the second part of the chapter.

As the probability of bankruptcy can not be observed, direct tests on the models are not possible. Thus, the main check on how well the models are specified must be on their predictive ability. The details of three different methods by which the models can be tested are given in the last part of the chapter.

Theory

In a single period model a definition of bankruptcy presents no problem; at the terminal point if the income of the firm is less than its fixed obligations, then a state of bankruptcy is declared. At the beginning of the period to estimate the probability of the firm going bankrupt at the terminal point is equivalent to estimating the probability of the firm's future income being less than the fixed obligations at the end of the period. In mathematical notation this may be expressed in the form

$$Pr\ (B) = Pr\ (FI - FC < 0), \qquad [3.1]$$

where Pr (B) is the probability of bankruptcy at the terminal point; FI is the firm's future income; and FC is the fixed charges at the terminal point. The term on the right hand side of the above expression is the probability of the firm's income net of all fixed charges being less than zero.

In a multi-period context a firm's income can be less than its fixed obligations without bankruptcy occurring; for the firm can simply borrow more. Apart from borrowing there are many other means by which a firm may be able to obtain extra sources of funds: issuance of equity, utilization of trade credit, selling of assets, or reduction of investment programs being possible examples. The ability of a firm to utilize these different sources depends extensively upon its size, the nature of its technology, future prospects, managerial ability, and the prevailing and expected economic conditions.

Donaldson[1] proposed three broad categories of funds a firm may utilize: uncommitted reserves, reduction of planned outflows, and liquidation of assets. Uncommitted reserves entails such factors as instant reserves (cash, very liquid assets), trade credit, negotiable reserves, addition of long term debt, and issuance of equity. Reduction of planned outflows involves the revising of existing commitments on outflows of funds; that is, the possible reduction of investment programs and general austerity measures. Liquidation of assets is either the selling of some of the firm's assets, or in the extreme case, the shutdown of the firm.

For the small firm the number of alternatives may not be as great. Its ability to obtain a commercial credit loan during a period of tight credit conditions may be very restricted. Due to the high issue costs, it may not have access to the equity markets.[2] Its capacity to conduct a general reduction of planned outflows may be very small, as might be its ability to engage in the liquidation of assets.

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[1]Donaldson, G., "Strategy for Financial Emergencies," *Harvard Business Review*, Vol. 47 (November-December, 1969), pp. 67-79.

[2]For an introductory discussion of some of the limiting factors see Duesenberry, J. S., "Criteria for Judging the Performance of Capital Markets," reprinted in *Elements of Investment*, edited by Wu, H. K. and Zakon, A. J. (New York: Holt, Rinehart and Winston, Inc., 1965).

An important source of funds for firms derives from the ability to borrow. But this ability depends upon the willingness of financial institutions to lend. Among the major financial institutions banks have been actively engaged in extending credit to businesses. Jaffee and Modigliani[3] have developed a simple model to determine the rationality and extent of credit rationing in a commercial loan market. The basis of the model is the derivation of the bank's supply curve for loans based upon the assumption that banks act to maximize expected profits and from considering the firm's demand function for a loan. It is shown that if a bank is a discriminating monopolist free to charge each customer a different rate, then credit rationing will not occur. However, if banks divide firms into a small number of risk classes and charge each class a different rate, then in general it will be optimal for the bank to ration credit. The exception to this being if the firm is classified as risk free, for then it is unprofitable for the bank to limit credit. The existence of credit rationing implies that for a firm not classified as risk free, there is a limit to the amount that it can borrow, which is dependent upon the banking structure and the state of the economy. Whilst the assumption is made that a firm borrows from a bank, the model is readily applicable to other types of financial institutions.

In a multi-period context the probability of a firm going bankrupt is determined by its ability to cover fixed charges either with its cash flow or by raising funds. Thus the probability of bankruptcy can be represented in the form

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[3]Jaffee, D. M. and Modigliani, F., "A Theory and Test of Credit Rationing," *American Economic Review*, Vol. LIX, No. 5 (December, 1969), pp. 850-872.

$$Pr\ (B_t) = PR\ (FI_t - FC_t + MB_t + AS_t < 0) \qquad [3.2]$$

where $Pr\ (B_t)$ is the probability of the event of bankruptcy occurring at the end of period t; FI_t is the firm's future income; FC_t is the fixed charges; MB_t is the maximum amount the firm could borrow; AS_t is all other alternative sources of funds; and the time suffix, t, is used to denote that the variables are valued at the end of the period t.

The first two terms on the right hand side of the above expression represent the firm's future income net of all fixed charges. The magnitude and characteristic of this term will depend upon the firm's financial structure and the type and state of the product and resource markets in which it deals; that is, the competitiveness of the markets, their cyclical behaviour, and external factors. For example, if there is economic recession the firm's product and resource markets may be affected, thus causing changes in its net cash flow. The firm's product diversification, and its technology will also influence its ability to stabilize its cash flows against cyclical behaviour and external factors.

The third term represents the maximum amount that the firm could borrow. This depends upon the banking structure of the economy, and the risk characteristics of the firm, as perceived by a bank. Over time, as economic conditions in the economy and the risk characteristics of the firm change, so will the amount of credit rationing and thus the borrowing power of the firm.

The last term represents the total of all other alternative sources of funds a firm may utilize. The nature of such sources, which Donaldson[4]

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[4]Donaldson, *loc. cit.*

describes in detail, consists of three broad categories: uncommitted reserves, reduction of planned outflows, and liquidation of assets. The availability of these sources depends upon the type of firm, its size, technology, and future prospects.

The firm's ability to raise funds is described by the maximum amount it could borrow and all other alternative sources. These quantities are not independent. A firm may be able to borrow using an asset as collateral, or it may issue a debenture with a negative pledge clause prohibiting it from pledging the asset to other creditors. It will not be able to do both. The different means by which a firm may be able to utilize alternative sources of funds are also not independent. If a firm issues debt, then this represents a claim against future earnings, which may inhibit its ability to issue equity.

The availability of the different sources by which a firm may be able to raise funds are dependent upon certain common factors: the existing financial structure of the firm, its operating characteristics, and the future prospects of the firm and the economy. If the future prospects for the firm are poor, then this may have a decremental effect upon its ability to borrow, to issue debt, or equity. This interdependence between the various sources prohibits unique empirical identification of the relative contribution of the underlying factors which determine a firm's ability to raise funds via particular sources.

Ex-Post Formulation

To use the model to empirically estimate the probability of bankruptcy requires the ex-ante variables be replaced by ex-post surrogates.

However, realized values of the maximum amount the firm could borrow or the total of all alternative sources are not readily observable and so proxy variables must be constructed. This requires that the underlying factors which contribute to the firm's ability to raise funds be identified and measured.

The variables determining the probability of bankruptcy, as stated in expression [3.2], are in terms of dollar amounts. As cross sectional data will be used, the variables are not adjusted for differences in size of firms and so will be dominated by scale effects. Very large scale effects among firms would be expected to lead to inefficient estimation of coefficients. To avoid this, the probability of bankruptcy can be written in the form

$$\Pr\ (B_t) = \Pr\ \left(\frac{FI_t - FC_t}{A_{t-1}} + \frac{MB_t + AS_t}{A_{t-1}} < 0\right), \qquad [3.3]$$

where A_{t-1} is the book value of the firm's assets at the start of period t. Thus the probability of a firm going bankrupt at the end of time period t depends upon its future cash flow net of all fixed charges per unit of assets and the total amount of funds that it could raise per unit of assets.

Whilst the ex-ante values are not observable, realized values of the firm's cash flow net of all fixed charges are readily available and can be used to form an ex-post surrogate. The ex-post data are regressed against time and then the estimated regression equation used to predict the future value of the firm's cash flow net of all fixed charges. This value is then divided by the book value of the firm's total assets and the resultant used as the ex-post surrogate.

Neither ex-ante nor realized values of the maximum amount that a firm could borrow are observable. In any particular period only that quantity of debt the firm actually borrowed from a bank can be readily determined, but such an amount need not necessarily be the maximum that the firm could have borrowed. Jaffee and Modigliani[5] have shown under simplistic assumptions that the maximum amount a firm can borrow to finance an investment project is given by

$$L = \frac{1}{1 + r_i} F^{-1} \left(\frac{r_i - \rho}{1 + r_i}\right),$$

where L is the maximum amount which the bank will lend the firm; r_i is the rate of interest the bank charges to firms assigned to the i^{th} risk class; F ( ) is the bank's subjective evaluation of the cumulative probability distribution of the outcome of the project; and ρ is the bank's opportunity rate. From an operational viewpoint the above equation can not be directly applied, as many of the terms can not be observed. However, it is still of value for it shows how the bank's opportunity rate and thus credit rationing affects the maximum amount a firm can borrow. Also, if credit rationing increases, then there will be a proportionally greater decrease in the maximum amount the firm can borrow, implying a non-linear relationship.

The ability of the firm to borrow will depend upon the amount of credit rationing in the economy, its current level of debt, and the optimal quantity of debt which it can utilize. The greater the amount of credit rationing, the less the risky firm will be able to borrow. The ability of the firm to borrow will be enhanced the larger the difference between the

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[5]Jaffee and Modigliani, *loc. cit.*

optimal level and the current level of debt. An approximate measure of this difference is the book value of net worth, which can be intuited as describing that part of the firm's assets not financed by debt. The proxy variable used for the ex-ante maximum amount the firm could borrow per unit asset for period t is

$$\left(\frac{\text{book value of net worth at t-1}}{A_{t-1}}\right) \exp\left(-CR_{t-1}\right),$$

where A_{t-1} is the book value of the firm's assets at time t-1; and CR_{t-1} is the amount of credit rationing at time t-1. The smaller the book value of net worth relative to the firm's total assets, the less the firm will be able to borrow. The functional form of dependence on credit rationing is used to account for the non-linear relationship between credit rationing and the amount a firm can borrow.

The final determinant of the probability of bankruptcy is the total of all other alternative sources of funds a firm may utilize. As Donaldson[6] outlined, this is dependent upon three broad categories: uncommitted reserves, reduction of planned outflows, and liquidation of assets. There are many factors which affect the aggregate total of funds that can be obtained from these different sources: operational efficiency, future prospects, business risk, financial risk being of prime importance. The dependence of the firm's ability to raise funds, either internally or externally, upon operational efficiency arises for two reasons. The greater the efficiency of the firm the more able it is to cope with reductions in planned outflows, or to undertake the liquidation of assets. For the potential

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[6] Donaldson, *loc. cit.*

investor the more efficient the firm then the more attractive it is as an investment proposition. The future prospects of the firm directly affect its ability to raise external sources of funds. If the firm is in an industry which is declining because of technological obsolescence, it will be very difficult to attract capital, as its future prospects will be bleak if it remains within the industry. Business risk measures the overall risk to the firm arising from the variability of its operating income, and pertains to its debt capacity. The more variable the cash flow, the greater the risk and thus restricts its ability to use debt financing. Financial risk arises from the firm's ability to cover fixed charges. The lower the financial risk, either because of low utilization of debt or stable cash flows, the more able it is to attract external financing. To measure the aggregate total of alternative sources, a linear function of these four attributes is used as a proxy variable.

Operational efficiency should be a measure of the productivity of the firm's assets, abstracting from tax or leverage factors. Various financial ratios have been used as proxy variables. Pinches and Mingo[7] use net income divided by total assets, whilst Beaver[8] suggests three other alternative ratios: net income to sales, net income to net worth, and net income to total debt. However, all of these measures are deficient, as they do not abstract from the effects of the firm's financial structure and thus are not

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[7]Pinches, G. E. and Mingo, K. A., "A Multivariate Analysis of Industrial Bond Ratings," *Journal of Finance*, Vol. XXVIII, No. 1 (March, 1973), pp. 1-18.

[8]Beaver, W. H., "Financial Ratios as Predictors of Failure," *Empirical Research in Accounting: Selected Studies*, supplement to *Journal of Accounting Research* (1966), pp. 77-111.

accurate measures of the utilization of the firm's assets. An alternation formulation by Altman[9] using earnings before interest and taxes divided by total assets avoids this deficiency. It abstracts from tax or leverage factors, and is a measure of the firm's earning power. This formulation is used in the thesis.

To attain the future prospects of the firm and thus its ability to attract capital requires measuring the profitability of the firm's future investment opportunities, their size and duration. All of these quantities are not directly observable. Miller and Modigliani[10] addressing themselves to the same problem, focused upon the most tractable component, the level of investment opportunities, as an overall measure of growth and future prospects. For an empirical estimator of the level of investment opportunities per unit asset, a linear five year growth rate of total assets is used. This measure is used in the thesis.

Business risk describes the risk to the firm that arises from the variability of its operating income, abstracting from tax or leverage factors. The debt capacity of the firm depends upon the variability of its cash flow and thus business risk: the more responsive the firm's cash flow to changes in the economy, the lower the optimal amount of debt which the firm can use. Van Horne[11] uses the coefficient of variation of operating

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[9] Altman, E. I., "Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy," *Journal of Finance*, Vol. XXIII, No. 4 (September, 1968), pp. 589-609.

[10] Miller, M. and Modigliani, F., "Some Estimates of the Cost of Capital to the Electric Utility Industry, 1954-1957," *American Economic Review*, Vol. LVI, No. 3 (June, 1966), pp. 333-391.

[11] Van Horne, J., *Financial Management and Policy* (New Jersey: Prentice Hall Inc., 1972).

income to measure business risk. However, this does not directly measure the responsiveness of the firm's cash flow to changes in the economy. An alternative formulation, and one that is used in the thesis, is to measure business risk by the absolute value of the proportional change in sales to the proportional change in gross national product; the more responsive sales to changes in the economy the greater the business risk.

Financial risk is a measure of the firm's ability to cover its fiscal charges. The greater its ability, the lower the financial risk and the more able it should be to attract external financing. To measure financial risk Altman[12] suggests two possible financial ratios: market of equity divided by the book value of total debt, and the book value of net worth divided by the book value of total debt. Both ratios are deficient, for they do not necessarily take account of all fixed charges which the firm must meet. The use of the book value of net worth does not measure the firm's ability to cover fixed charges. The ability of a firm to cover its fixed charges primarily depends upon its future cash flow, its variability, and the total of fixed charges which it covers. Norton[13] uses the coefficient of variation of the firm's past income over and above the amount of fixed charges. This measure is deficient for the firm's ability to meet fixed charges depends upon its future income as opposed to past income. The proxy variable used in the thesis is the difference between the firm's fixed charges and its future cash flow, the difference being divided by the standard deviation of the

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[12]Altman, *loc. cit.*

[13]Norton, J., "The Theory of Loan Credit in Relation to Corporation Economics," *Publications of the American Economic Association*, 3rd ser., Vol. V (1904), pp. 278-300.

future cash flow. The smaller the variable, the more able the firm to cover its fixed charges and the lower the financial risk. The firm's future cash flow is estimated by regressing realized values of its operating income against time and then the estimated regression equation used to predict the future value. The square root of the residual sum of squares is used as an estimate of the standard deviation of the firm's future cash flow.

Combining the proxy variables for the firm's future cash flow net of all fixed charges per unit assets, the maximum amount it could borrow per unit assets, and the total of all other alternative sources per unit assets, gives

$$\begin{aligned}\frac{FI_t - FC_t}{A_{t-1}} + \frac{MB_t}{A_{t-1}} + \frac{AS_t}{A_{t-1}} = \beta_0 + \beta_1 &\left(\frac{\text{estimated future cash flow net of all fixed charges}}{A_{t-1}}\right) \\ &+ \beta_2 \left(\frac{\text{book value of net worth at t-1}}{A_{t-1}}\right) \exp\left(-CR_{t-1}\right) \\ &+ \beta_3 \left(\frac{\text{earning before interest and tax at t-1}}{A_{t-1}}\right) \\ &+ \beta_4 (\text{five year linear growth rate in total assets}) \\ &+ \beta_5 \left(\left|\frac{\text{proportional change in sales}}{\text{proportional change in GNP}}\right|\right) \\ &+ \beta_6 \left(\frac{\text{fixed charges at t-1 - estimated future cash flow}}{\text{estimated standard deviation of future cash flow}}\right) \\ &+ \tilde{\varepsilon}, \end{aligned} \qquad [3.4]$$

where $\beta_0, \beta_1, \ldots, \beta_6$ are unknown coefficients; and $\tilde{\varepsilon}$ is a zero mean random variable error term, which is assumed to be of unit variance and uncorrelated between firms.

The coefficients in the above equation represent the relative contributions of the different underlying factors to the aggregate total of net funds available to the firm. The coefficients can not be estimated by regression, as the dependent variable is an ex-ante quantity which can not be measured or observed. Substituting Equation [3.4] into the expression [3.3] for the probability of bankruptcy gives

$$\Pr(B_t) = \Pr[\tilde{\varepsilon} < \beta_0 + \beta_1 \left(\frac{\text{estimated future cash flow net of all fixed charges}}{A_{t-1}}\right)$$

$$+\beta_2 \left(\frac{\text{book value of net worth at t-1}}{A_{t-1}}\right) \exp(-CR_{t-1})$$

$$+\beta_3 \left(\frac{\text{earnings before interest and taxes at t-1}}{A_{t-1}}\right)$$

$$+\beta_4 (\text{five year linear growth rate for total assets})$$

$$+\beta_5 \left(\left|\frac{\text{proportional change in sales}}{\text{proportional change in GNP}}\right|\right)$$

$$+\beta_6 \left(\frac{\text{fixed charges at t-1 - estimated future cash flow}}{\text{estimated standard deviation of future cash flow}}\right)], \quad [3.5]$$

where the coefficients have been redefined to include the minus sign. The probability of the event of the firm going bankrupt at the end of period t is the probability of the random variable error term minus the summation of the underlying factors which contribute to the net total of funds available to the firm being less than zero. Apart from the random error term, all the variables on the right hand side of the equation are ex-post and can be measured. The signs of the coefficients can be determined from theoretic considerations. Using a ceteris paribus argument, the greater the firm's future cash flow net of all fixed charges, and the amount which it could borrow, the lower the probability of bankruptcy. Thus the coefficients β_1 and β_2 should be negative. Similarly for the coefficients β_3 and β_4, as the greater the efficiency of the firm, the better its future prospects, the more able it is to raise extra sources of funds. The coefficients β_5 and β_6 should be positive; the greater the business risk, as measured by the variability of the firm's cash flows, and the larger the financial risk, the less able the firm to attract extra capital and the greater the probability of bankruptcy.

Predictive Model

A complete model for the probability of bankruptcy should describe all the interactions between the different factors. The firm's ability to borrow or to issue debt is dependent upon its debt capacity. But debt capacity is dependent upon the probability of bankruptcy and thus there is a circularity. The ability to use a particular source of funds is dependent upon the utilization of other sources. If a firm issues debt, this may have a decremental effect upon its ability to borrow from a bank or to issue equity. Due to the complex interaction of the underlying factors and the difficulty of measuring their magnitude and availability, a second formulation using market values for the appropriate variables is developed. The use of market values circumvents many of the difficulties of constructing proxy variables to measure such quantities as the maximum amount the firm can borrow and the total of all other alternative sources.

The probability of a firm going bankrupt depends upon its future income net of all fixed charges, the maximum amount it can borrow and all other alternative sources of funds. To use the model to empirically estimate the probability of bankruptcy requires that the ex-ante variables be replaced by ex-post surrogates.

For the firm's future cash flow net of all fixed charges the same proxy variable, as previously defined is used; that is, realized values of the firm's cash flow net of all fixed charges are regressed against time and then the estimated regression equation used to predict the future value of the firm's cash flow net of all fixed charges. This value is then divided by the book value of the firm's total assets and the resultant used as the ex-post surrogate.

The ability of the firm to borrow depends upon the amount of credit rationing in the economy, its current level of debt and the optimal level of debt which it can utilize. The firm's ability to borrow is enhanced the larger the difference between the optimal and current level of debt. To measure this difference requires that the debt capacity of the firm be known. However, debt capacity depends upon the probability of bankruptcy, implying that the explanatory variable is a function of the dependent variable. The market value of equity for a firm, which reflects the probability of bankruptcy, is a measure of its borrowing ability. For a given level of assets, the greater the market value of equity, the more able the firm to borrow. The proxy variable used to measure the ex-ante maximum amount the firm could borrow per unit of assets for period t is

$$\left(\frac{\text{market value of equity at t-1}}{A_{t-1}}\right) \exp\ (-CR_{t-1}),$$

where A_{t-1} is the book value of the firm's assets at time t-1; and CR_{t-1} is the amount of credit rationing at time t-1.

The total of all other alternative sources of funds for the firm depends upon three broad categories: uncommitted reserves, reduction of planned outflows, and the liquidation of assets. The ability of the firm to utilize these different sources primarily depends upon its operational efficiency, future prospects, business risk and financial risk. A variable which synthesizes these diverse quantities is the market value of equity. Using a ceteris paribus argument, the more efficiently the firm utilizes its assets, or the brighter its future prospects, the greater is the market value of its equity. Similarly, the lower the business and financial risk of the firm, the greater it's market value of equity. For a given level of assets, the greater the market value of equity the more able is the firm to generate and attract extra sources of funds.

Thus, the ex-post surrogates for the ex-ante net aggregate total of funds available to the firm, can be written:

$$\frac{FI_t - FC_t}{A_{t-1}} + \frac{MB_t}{A_{t-1}} + \frac{AS_t}{A_{t-1}} = \gamma_0 + \gamma_1 \left(\frac{\text{estimated future cash flow net of all fixed charges}}{A_{t-1}}\right)$$
$$+\gamma_2 \left(\frac{\text{market value of equity at t-1}}{A_{t-1}}\right) \exp\left(-CR_{t-1}\right)$$
$$+\gamma_3 \left(\frac{\text{market value of equity at t-1}}{A_{t-1}}\right)$$
$$+\tilde{\eta}, \qquad [3.6]$$

where $\gamma_0, \gamma_1, \gamma_2$, and γ_3 are unknown coefficients; and $\tilde{\eta}$ is a zero mean random variable error term, which is assumed to be of unit variance and uncorrelated between firms. Substituting Equation [3.6] into the expression [3.3] for the probability of bankruptcy gives

$$\Pr(B_t) = \Pr[\tilde{\eta} < \gamma_0 + \gamma_1 (\frac{\text{estimated future cash flow net of all fixed charges}}{A_{t-1}})$$

$$+ \gamma_2 (\frac{\text{market value of equity at t-1}}{A_{t-1}}) \exp(-CR_{t-1})$$

$$+ \gamma_3 (\frac{\text{market value of equity at t-1}}{A_{t-1}})], \qquad [3.7]$$

where the coefficients $\gamma_0, \gamma_1, \gamma_2$, and γ_3 have been redefined to absorb the minus sign. Apart from the random variable, all the terms on the right hand side of the above equation are ex-post and can be measured. The signs of the coefficients can be determined from theoretic considerations. Using a ceteris paribus argument, the greater the estimated future cash flow net of all fixed charges, the smaller is the probability of bankruptcy, and thus the coefficient γ_1 should be negative. The smaller the amount of credit rationing, or the greater the market value of equity, the more able the firm is to raise funds, either internally or externally, to cover fixed charges and the lower the probability of bankruptcy. Hence, the coefficients γ_2 and γ_3 should be negative.

Statistical Methodology

The probability of a firm going bankrupt has been formulated in terms of two models, as represented by Equations [3.5] and [3.7]. The general structure of these formulations is of the form

$$\Pr(B_{tj} \mid \underline{X}_{t-1,j}) = \Pr(\tilde{\varepsilon} < \underline{\alpha}' \underline{X}_{t-1,j}) \qquad [3.8]$$

where $\underline{X}_{t-1,j}$ is a vector of the j^{th} firm's attributes measured at time t-1; $\underline{\alpha}$ is a vector of unknown coefficients; $\Pr(B_{tj} \mid \underline{X}_{t-1,j})$ is the probability of the event that the j^{th} firm goes bankrupt during period t, given the vector of attributes $\underline{X}_{t-1,j}$; and $\tilde{\varepsilon}_j$ is a zero mean random variable error term which is assumed to have unit variance and to be independent among firms; that is, $E(\tilde{\varepsilon}_j) = 0$, var $(\tilde{\varepsilon}_j) = \sigma^2$, and cov $(\tilde{\varepsilon}_j, \tilde{\varepsilon}_k) = 0$, $j \neq k$, for all j and k. There are a number of special characteristics about Equation [3.8] which have important implications for statistical estimation. The dependent variable, the probability of a firm going bankrupt, can not be directly measured; that is, the ex-ante value or the realized values can not be observed. As the dependent variable is a probability, it is constrained to the interval zero-one. The prime focus is to derive some form of technique to estimate the probability of bankruptcy, constraining the estimate to a zero-one interval. The coefficients, $\underline{\alpha}$, which measure the relative contribution of the different attributes, are unknown. Thus, on both sides of the equation there are unknown quantities which, in general, are related in a non-linear manner.

Whilst the ex-ante probabilities of a firm going bankrupt can not be observed, at any point in time a firm is either bankrupt or not bankrupt. This suggests that collecting data for a random sample of bankrupt and non-bankrupt firms the coefficients, $\underline{\alpha}$, can be estimated by positing the model

$$(Z_{tj} \mid \underline{X}_{t-1,j}) = \underline{\alpha}' \, \underline{X}_{t-1,j} + \zeta \qquad [3.9]$$

where ζ is a random disturbance; and Z_{tj} is an indicator function defined by

$$Z_{tj} = \begin{cases} 1; \text{ if } j^{th} \text{ firm bankrupt at time } t; \\ 0, \text{ otherwise.} \end{cases}$$

Though the coefficients can be estimated by regression, they will not be efficient estimators. The systematic part of the right hand side, $\underline{\alpha}' \, \underline{X}_{t-1,j}$, may be larger than one or smaller than zero, whereas Z_{tj} takes only two values (0 and 1) which means that the disturbance term, ζ, given $\underline{X}_{t-1,j}$ can take only two values: $-\underline{\alpha}'\ \underline{X}_{t-1,j}$ and $1-\underline{\alpha}'\ \underline{X}_{t-1,j}$. If ζ is to have an expected value of zero for all values of $\underline{X}_{t-1,j}$, it must take the former value with probability $1-\underline{\alpha}'\ \underline{X}_{t-1,j}$ and the latter with probability $\underline{\alpha}'\ \underline{X}_{t-1,j}$. But $\underline{\alpha}'\ \underline{X}_{t-1,j}$ can be negative or larger than one. There is nothing in the estimation procedure to ensure that the estimated values of the dependent variable are constrained.[14]

The dependent variable of Equation [3.8] is an ex-ante probability which can not be observed, whilst the ex-post variables on the right hand side of the equation can readily be measured. Using this property, the coefficients, $\underline{\alpha}$, can be estimated using maximum likelihood. Consider a random sample of firms at time t and suppose that the first n' firms are bankrupt and the remainder n-n' non-bankrupt. The logarithmic likelihood function can then be written

$$\sum_{j=1}^{n'} \log \Pr(B_{tj}|\underline{X}_{t-1,j}) + \sum_{j=n'+1}^{n} \log\ [1-\Pr(B_{tj}|\underline{X}_{t-1,j})], \qquad [3.10]$$

where $\Pr(B_{tj}|\underline{X}_{t-1,j})$ is defined by Equation [3.8] and is thus a function of the parameters $\underline{\alpha}$. By differentiating [3.10] with respect to these parameters and equating the first derivation to zero, a set of non-linear equations are obtained and can be solved iteratively. For practical application the use of maximum likelihood requires that a particular form for the probability distribution be assumed; that is, the probability distribution of the random variable, $\tilde{\varepsilon}$, in Equation [3.8] must be specified.

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[14]For a more extensive discussion of the econometric problems, see Goldberger, A. S., *Economic Theory* (New York: John Wiley & Sons, 1964), pp. 248-255.

Two estimation procedures are used in the thesis: probit analysis and logit analysis.[15] The essential difference between the two procedures is the explicit form of the probability distributions. For probit analysis a normal probability distribution is assumed, whilst for logit analysis the distribution is logistic. The logistic distribution is very similar to the normal distribution, being slightly fatter in the tails and more centralized about the mean.[16]

Probit analysis[17] can be defined as follows: if the probability of a zero mean, unit variance normally distributed random variable being less than or equal to the scalar produce $\underline{\alpha}'\underline{X}$ is p, then the probit of $\underline{\alpha}'\underline{X}$ is $F^{-1}(p)$, where

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{1}{2}u^2\right) du,$$

$$\text{and} \qquad \begin{aligned} p &= \Pr(\tilde{Y} < \underline{\alpha}'\underline{X}) \\ &= F(\underline{\alpha}'\underline{X}), \end{aligned} \qquad [3.11]$$

given that $\tilde{Y} \sim N(0,1)$. The unknowns in the formulation are the set of parameters, $\underline{\alpha}$, and the probability p. Equation [3.11] can be substituted into expression [3.10], the logarithmic likelihood function, giving

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[15] An introductory discussion is given in Theil, H., *Principles of Econometrics* (New York: John Wiley & Sons, 1971), pp. 628-635.

[16] For a more detailed discussion see Winsor, C. P., "A Comparison of Certain Symmetrical Growth Curves," *Journal of the Washington Academy of Science*, Vol. 22, No. 4 (February, 1932), pp. 73-84.

[17] For a general discussion and applications of probit analysis see Finney, D. J., *Probit Analysis* (Cambridge: Cambridge University Press, 1971), 3rd edition; and Cragg, J. G., "Some Statistical Models for Limited Dependent Variables With Application to the Demand for Durable Goods," *Econometrica*, Vol. 39, No. 5 (September, 1971), pp. 829-844.

$$\sum_{j=1}^{n'} \log F(\underline{\alpha}'\underline{X}_{t-1,j}) + \sum_{j=n'+1}^{n} \log [1-F(\underline{\alpha}'\underline{X}_{t-1,j})], \qquad [3.12]$$

where

$$F(\underline{\alpha}'\underline{X}_{t-1,j}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\underline{\alpha}'\underline{X}_{t-1,j}} \exp(-\frac{1}{2} u^2)\, du,$$

and the parameters can be estimated by solving the set of first order conditions obtained by differentiating the likelihood function.

Logit analysis[18] can be defined in a similar manner. If the probability of a firm going bankrupt is equal to the probability, p, of a random variable, which has a logistic distribution, being less than or equal to the scalar product $\underline{\alpha}'\underline{X}$, that is,

$$\begin{aligned} p &= \Pr(\tilde{Z} < \underline{\alpha}'\underline{X}) \\ &= \frac{1}{1 + \exp(-\underline{\alpha}'\underline{X})}, \end{aligned} \qquad [3.13]$$

where $\tilde{Z}$ is a random variable having a logistic distribution, then the logit of $\underline{\alpha}'\underline{X}$ is

$$\log \left(\frac{p}{1-p}\right) = \underline{\alpha}'\underline{X}.$$

Again the unknowns in the formulation are the set of parameters, $\underline{\alpha}$, and the probability p. By substituting Equation [3.13] into expression [3.10], the parameters $\underline{\alpha}$ can be estimated by maximum likelihood.

To empirically estimate the parameters, $\underline{\alpha}$, the likelihood function

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[18]For an introductory discussion to logit analysis and for applications see Berkson, J., "Applications of Logistic Functions to Bio-Assay," *Journal of the American Statistical Association*, Vol. 39 (1944), pp. 357-365; and Baxter, N. D. and Cragg, J. G., "Corporate Choice Among Long-Term Financial Instruments," *The Review of Economics and Statistics*, Vol. LII, No. 3 (August, 1970), pp. 225-235.

must be constructed by taking a random sample of firms and then classifying the firms as bankrupt or not bankrupt. The procedure of using a random sample avoids selection bias. As the average probability of a firm going bankrupt is small, a very large random sample must be taken so as to obtain a representative collection of bankrupt firms. In practice, a common procedure is to collect data for all bankrupt firms over a specified time period and then to collect a random sample of non-bankrupt firms. It is necessary to determine how many firms should be included in the sample. Ideally, the number chosen should be the same as that obtained by taking a random sample of all firms and then classifying them as bankrupt or not bankrupt. Thus, to determine the required sample size entails estimating the average value of the probability of a firm going bankrupt.

Testing of the Model

As the probability of bankruptcy can not be observed, direct tests on the models are not possible. This implies that the magnitude of any bias or measurement error in the estimates can not be determined. Thus, the main check on how well the models are specified must be on their predictive ability. There are three methods by which the models can be tested.

From theoretic considerations the sign of the parameters can be determined and compared to those obtained from the empirical estimation. The number of estimated parameters with the correct sign provides insight into the specification of the model and the accuracy of the proxy variables at measuring the ex-ante quantities.

If the model is completely specified so as to measure all the different attributes of the firms in the data sample used to estimate the parameters, then it should be able to correctly identify the bankrupt and non-bankrupt firms in the sample. The classification ability provides information about the model's specification and the number of common determinants of bankruptcy.

The generality of the model and its overall independence of the peculiarities of the data sample used to estimate the parameters, can be tested by examining its predictive ability on a set of bankrupt firms not used in the original sample. By estimating the probability of bankruptcy over several time periods for firms in the new sample provides a demonstration of the model's predictive ability to discern a firm's path to bankruptcy.

Summary

In a multiperiod context a firm's income can be less than its obligations and yet it is not bankrupt; it can simply borrow more. The probability of a firm going bankrupt depends upon its ability to raise funds, either internally or externally, to cover fixed charges. A firm that fails to cover these charges is said to be bankrupt. From this definition, a model for the probability of bankruptcy is constructed in terms of ex-ante variables. To use the model to empirically estimate the probability of bankruptcy an ex-post formulation using proxy variables is developed. As the primary focus is upon the prediction of bankruptcy, as opposed to advancing a complete theory of the determinants of bankruptcy, a second formulation using market values of appropriate corporate variables is constructed. To empirically estimate the coefficients of the models, a statistical methodology employing probit analysis and logit analysis is used. Three different methods to check the predictive ability of the models are described.

CHAPTER IV

AN EXTENSION OF THE CAPITAL ASSET PRICING MODEL: BANKRUPTCY

The price behaviour of common stocks is affected by impending bankruptcy. As a firm progresses towards bankruptcy its ex-post returns, when compared to those of firms that did not go bankrupt during the same period appear to be significantly different in behaviour. Empirical evidence indicates that the CAPM does not provide an adequate description of the mechanism generating common stock returns. It is found that assets with high levels of systematic risk consistently earn less than that predicted by the model, the reverse being true for assets with low levels of systematic risk. Whilst there appears to be a linear relationship between ex-post returns and systematic risk, it is not constant with both the intercept and slope fluctuating randomly from period to period and are often negative. The data indicate that the expected return on a security can be represented by a linear two factor model, the second factor not being explicitly defined. Various attempts have been made to provide a theoretical explanation for the existence of a second factor. The effects of non-marketability of assets, changes in the investment opportunity set, the non-existence of a riskless asset, and restrictions upon the investor's ability to borrow or lend have been explored, though fail to provide an adequate explanation of all the observed deficiencies of the CAPM and why impending bankruptcy affects the residual return, after abstracting from the market, of common stocks.

A primary focus of the thesis is to extend the formulation of the CAPM not from the viewpoint of restrictions upon the investor, but by considering the impact of bankruptcy upon the structure of returns for corporate financial assets.

A model, formulated in continuous time, considers the investment-consumption decision of an individual acting to maximize the expected lifetime utility of consumption and terminal wealth. At each instant in time the individual must decide the portions of wealth to consume and to invest in financial assets. It is assumed that corporations issue both debt and equity as financial assets and that at each point in time there is a probability that the firm will go bankrupt the next instant. If bankruptcy occurs it is assumed that equity holders suffer a 100 per cent loss, whilst bondholders receive a non-negative liquidating premium.

When the investment opportunity set is altered only by the event of bankruptcy, a two variable model is derived which describes the expected return, conditional upon no bankruptcy, for a firm's common stock in terms of its systematic risk and a variable associated with the probability of the firm going bankrupt. For the general case when there are stochastic changes in the probability of a firm going bankrupt, additional terms arise reflecting investors' attempts to hedge against unexpected changes.

The foundations of the model are set out in the first part of the chapter. The major assumptions of the model, the nature of the financial assets available and their price dynamics are described. The general form of the equation of optimality and the system of equations describing the first order maximization conditions are derived. Due to the complexity of the general analysis, additional structure is injected into the analysis. In the second part of the chapter the equilibrium instantaneous expected rates of return, conditional upon no bankruptcy, for bonds and equity are derived given the assumption that the investment opportunity set is only change by bankruptcy. The final part of the chapter considers the effect of stochastic changes in the probability of bankruptcy upon the structure of returns. As much of the analysis is of a highly mathematical nature, an attempt has been made to relegate as much of the mathematics to Appendix A, whilst still maintaining continuity in the chapter.

Foundations of Model

For an investor to buy the bonds of a firm the expected rate of return[1] must compensate the investor for the risk that the firm will go bankrupt and for the risk of a capital loss which might result if there is an unexpected change in the general level of interest rates, or in the probability of bankruptcy. If bankruptcy occurs, it is assumed the firm is liquidated, the value of the firm's assets, net of tax minus the direct costs associated with bankruptcy, is distributed on a pro-rata basis to bondholders. Thus, the bondholder is subject to the risk of a direct loss if bankruptcy occurs. Intuitively, the expected rate of return on a firm's bonds that a potential investor requires will be a function of the risk free interest rate, the probability of bankruptcy, the expected loss if bankruptcy occurs, and the expected change in the general level of interest rates.[2]

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[1]Stiglitz uses the term 'nominal rate of interest.' Stiglitz, J., "A Re-Examination of the Modigliani-Miller Theorem," *American Economic Review*, Vol. 59, No. 2 (1969), pp. 786-793.

[2]For an introductory discussion see Fisher, L., "Determinants of Risk Premiums on Corporate Bonds," *Journal of Political Economy*, Vol. LXXVII, No. 3, (June, 1959), pp. 217-237.

To buy the shares of a firm, an investor buys a claim to a variable cash flow. From the traditional formulation of the capital asset pricing model, as given by Sharpe, Treynor, and Mossin,[3] the expected rate of return for a firm's equity depends upon the risk free interest rate, and its level of systematic risk, which is a measure of the covariance of the return on the firm's stock wih the return on the market portfolio. Such a formulation does not explicitly consider bankruptcy or changes in the investment opportunity set. If these factors are considered, then they will in general affect the expected rate of return which potential investors require.

In order to derive the equilibrium expected returns for a firm's bonds and equity, the demand functions for the different financial assets for an individual are obtained. It is assumed that at time t, there are n firms, each firm having a simple capital structure of one type of debt and one type of common stock. It is also assumed that there is a riskless asset, so that there are 2n+1 financial assets, which are assumed to be traded in a perfect capital market (with bankruptcy). It is important to note that the bonds of different firms are treated as different financial assets. If capital structure was irrelevant, then this would not be necessary. But, as capital is relevant then, as Stiglitz[4] correctly observes, the presence of bankruptcy creates a new security.

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[3]Sharpe, W., "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk,"*Journal of Finance*, Vol. XIX, No. 3 (September, 1964), pp. 425-442; Treynor, J., "Towards A Theory of Market Value of Risky Assets," unpublished memorandum (1961); Mossin, J., "Equilibrium In A Capital Asset Market," *Econometrica*, Vol. 34, No. 4 (October, 1966), pp. 468-783.

[4]Stiglitz, J., "Some Aspects of the Pure Theory of Corporate Finance: Bankruptcy and Take Overs," *Bell Journal of Economics and Management Science*, Vol. 3, No. 2 (Autumn, 1972), pp. 458-482.

It is assumed that the capital market is structured such that[5]

A1. all assets have limited liability;

A2. there are no transaction costs (excluding bankruptcy), personal taxes, or problems with indivisibilities of assets;

A3. there are sufficient number of investors with comparable wealth levels so that each investor can buy or sell as much of an asset without affecting the market price;

A4. the capital market is always in equilibrium; that is, there is no trading at non-equilibrium prices;

A5. there exists an exchange market for borrowing and lending at the same rate of interest;

A6. short sales of all assets, with full use of the proceeds, is allowed;

A7. there are homogeneous expectations among investors about the future prospects of each financial asset;

A8. bonds are, in general, risky. If a firm goes bankrupt, then it is liquidated; that is, it is assumed that the firm ceases to exist;

A9. trading in assets takes place continuously in time.

Assumptions A1 to A7 are the standard perfect market (excluding bankruptcy) assumptions, which have been extensively discussed in the financial literature.[6] Assumption A8 is a departure from the standard asset pricing model assumptions. It arises due to the conditions of the presence of bankruptcy. If a firm cannot meet its obligations and is unable

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[5] For an alternative discussion of these assumptions, see Merton, R.C., "An Intertemporal Capital Asset Pricing Model," Working Paper 588-72, Sloan School of Management, Massachusetts Institute of Technology (February, 1972).

[6] A good reference is Jensen, M., "Capital Markets: Theory and Evidence," *Bell Journal of Economics and Management Science*, Vol. 3, No. 2, (Autumn, 1972), pp. 458-482.

to secure extra finance, then it will be assumed that a state of bankruptcy is declared. It will also be assumed that the firm ceases to exist; that is, the firm is liquidated, the possiblity of the firm undergoing reorganization being neglected.

Assumption A9 follows directly from Assumption A2. If there are no costs, no indivisibilities, then investors should prefer to be able to revise their portfolios at any time. In reality there are transaction costs, indivisibilities, and it is for these reason that a discrete time formulation is advanced. Usually, the approach is to take equally spaced intervals of time which, though convenient from an empirical viewpoint, is theoretically unsatisfactory. The trading intervals will, in general, be stochastic and of a non-constant length, depending upon the types of securities available, the size and nature of price changes, transaction costs, and future expectations.[7]

It will be assumed that the individual acts in a manner to maximize the expected lifetime utility of consumption and terminal wealth; that is, the k^{th} individual acts so that

$$\text{Max } E_o\{\int_o^{T_k} U_k[C_k(s),s]ds + BF_k[W(T_k),T_k]\} \tag{4.1}$$

subject to an initial wealth constraint, where E_t represents the conditional expected value operator, conditional on the fact that all state variables at time t are known; $C_k(s)$ is the individual's instantaneous consumption at time s; $U_k[C(s),s]$ is the individual's utility function, which is assumed to be a

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[7]For a further discussion of this point, see pp. 46 of Merton, R. C., "A Dynamic General Equilibrium Model of the Asset Market and its Application to the Pricing of the Capital Structure of the Firm," Massachusetts Institute of Technology, Sloan School of Management, October, 1970.

strictly concave von Neumann- Morgenstern utility function; $BF_k[W(T_k), T_k]$ is a steady concave 'bequest' or utility of terminal wealth function; and T_k is the date of death of the k^{th} individual. It should be noted that implicit in the above formulation is the assumption of an additive utility function.[8]

If certain assumptions are made about the price dynamics of the stock and bond prices, then by the technique of stochastic control theory,[9] the optimal consumption and investment rules for the individual can be derived and thus the individual's demand functions for the different financial assets. Hence, by aggregating across individuals and using the equilibrium condition of zero excess demand, then the equilibrium instantaneous expected rates of return can be determined.

Such a problem has been considered by both Samuelson[10] and Merton,[11] the former treating the discrete time case and the latter the continuous time case. Both, however, treated capital structure as irrelevant and thus considered all equity firms.

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[8]For the case of multiplicative utility functions, see Pye, G., "Lifetime Portfolio Selection in Continuous Time for A Multiplicative Class of Utility Functions," *American Economic Review*, Vol. LXIII, No. 5 (December, 1973), pp. 1013-1016.

[9]A description of this technique is given in Bellman, F., *Dynamic Programming* (Princeton, N.J.: Princeton University Press, 1957).

[10]Samuelson, P. A., "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, Vol. LI, No. 3 (August, 1969), pp. 239-246.

[11]Merton, R. C., "Lifetime Portfolio Selection Uncertainty: The Continuous Case," *Review of Economics and Statistics*, Vol. LI, No. 3 (August, 1969), pp. 247-257.

Price Dynamics

It is proposed to represent the price movements of a firm's equity and bonds by stochastic differential equations. As it is not intended to present a rigorous derivation of the equations, the interested reader should refer to the paper by Itô,[12] in which the theory of stochastic differential equations was first advanced in 1951.[13] The application of stochastic differential equations to describe the price dynamics of bonds and common stocks has been extensively utilized by Merton[14] for the derivation of optimal consumption and investment rules.

The assumptions made about the price dynamics of the stock and bond prices are very important for they have direct bearing upon the derived form of the expressions for the equilibrium instantaneous expected rates of return. A complete description of the dynamics would require a specification of the supply side of the firm; that is, to relate the real assets and the production function of the firm to the price dynamics of the firm's stocks and bonds traded in the capital markets. For example, to specify the price dynamics for the firm's stocks requires some assumption about the dividend policy of the firm. If in the equity price equation dividends are treated as a random variable, then in order to have a closed system of equations it is necessary

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[12] Itô, K., "On Stochastic Differential Equations," *Memoirs of the American Mathematical Society*, No. 4 (1951), pp. 1-51.

[13] Other references are Itô, K., and McKean, H. P., *Diffusion Processes and Their Sample Paths* (New York: Academic Press, 1964); and Kushner, H. J., *Introduction to Stochastic Control* (New York: Holt, Rinehart & Winston, Inc., 1971).

[14] Merton, R. C., "Optimal Consumption and Portfolio Rules In A Continuous Time Model," *Journal of Economic Theory*, Vol. 3 (1971), pp. 373-413.

to specify an equation describing the dividend payments over time. However, unless some assumption is made about the form of the equation at the outset, such a description would imply a specification of the firm's behaviour in determining its investment and financial policies over some future time horizon.[15]

It will be assumed that the supply side of the firm is fixed and taken as given. Dividends can either be treated by assuming that the firm does not make actual dividend payments, but issues or repurchases its own shares in the market, or by assuming at the outset a form of an equation that describes dividend behaviour over time.

From assumption A9 trading takes place continuously in time and thus any representation of the price dynamics of a firm's stocks or bonds should also be in a continuous time framework. In practice coupons or dividends are paid on a discrete time basis. Discrete payments of either coupons or dividends represents a major theoretical problem in continuous time models, for it destroys the symmetry of the representation and it is no longer possible to have compact distributions.[16]

Symmetry is important for it offers a tremendous simplification for both the specification of the price equations and for the derivation of optimality conditions. If symmetry is not preserved then it would be necessary to identify the timing of the asymmetric events and to keep track of

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[15] This issue is discussed in Merton, R. C., "An Intertemporal Capital Asset Pricing Model," Working Paper No. 588-72, Massachusetts Institute of Technology, Sloan School of Management, February, 1972.

[16] A simple explanation of compact distributions is given in Samuelson, P. A., "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variance, and Higher Moments," *Review of Economic Studies*, Vol. 37 (October, 1970), pp. 537-542.

how far ahead in time these events will occur. Thus the whole specification process becomes far more complicated.

Compact distributions are important because for small time intervals the uncertainty neither "works out" (that is, zero variance) nor dominates the analysis (that is, infinite variance). Some properties of compact distributions and their usefulness are given in Samuelson.[17]

Given the assumption A2 of zero transaction costs, there is no reason why a firm should not pay a coupon or a dividend on a continuous time basis. One could always imagine the firm paying a continuous coupon or dividend to a trustee who would distribute the coupon or dividend on a discrete time basis in the name of the company, as in practice both coupon and dividends are paid on a discrete time basis.

If a firm goes bankrupt, then it is assumed that it is liquidated, the possibility of reorganization being neglected. The bondholder will receive, on a pro rata basis, the value of the firm, net of taxes, minus the direct costs associated with bankruptcy. It is assumed that the shareholder will receive nothing.

The assumption that the firm undergoes liquidation, and not reorganization avoids two difficult problems: the valuation of a firm's securities both whilst it is being reorganized and after reorganization; and a representation in continuous time of the price dynamics that is symmetric and of a compact distributional form during these periods. This is just one facet of the much broader problem that there are no theories of the firm that pertain

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[17] Samuelson, P. A. "General Proof That Diversification Pays," *Journal of Financial and Quantitative Analysis*, Vol. 2 (March, 1967), pp. 1-13.

to the state of bankruptcy and liquidation.

It is assumed that when an investor buys the bonds of a firm a subjective evaluation is made of not only the return to be expected if the firm does not go bankrupt, but also the return to be expected if the firm does go bankrupt; that is, the investor form a subjective probability distribution of the liquidation value of the firm (on a pro-rata basis) if bankruptcy occurred.

If bankruptcy does not occur it is assumed that the price of the bond at the end of the period can be represented by the equation

$$b_j(t+h) = b_j(t) \quad (1+r_j h) - g_j h + b_j(t)\gamma_j \sqrt{h}\, y_j(t), \tag{4.2}$$

$$j = 1, 2 \ldots, n,$$

where, for the j^{th} firm,

- $b_j(t+h)$ represents the price of a bond at time t+h;
- r_j represents the instantaneous conditional expected rate of return for the firm's bonds, conditional on the fact that the firm does not go bankrupt;
- g_j represents the instantaneous conditional coupon rate on the firm's bonds;
- γ_j^2 represents the instantaneous conditional variance;

and

- $y_j(t)$ represents a zero mean, unit variance, purely random process; that is, $y(t)$ and $y(t+s)$, $s > 0$ are independent and identically distributed gaussian random variables.

Implicit in the above formulations are a number of very important assumptions. First, the maturity of the bond has been neglected. It has been assumed that the bonds are perpetuities. An alternative approach would be to assume that all bonds had a long, but finite maturity such that all bonds matured after the individual's death.

There are two major reasons for making such an assumption about the bonds being perpetuities. If it is assumed that the bonds had finite maturity and matured within the lifetime of the individual, it would then be necessary to specify the new financing the firm undertakes. This will depend upon the investment opportunities which face the firm at that time and upon the general economic conditions. The basic problem is to construct a representation in continuous time to describe the price movements of the firm's bonds taking into account the point of discontinuity which might occur at the point in time when the bonds mature and the firm makes a new debt issue. Closely related to this is the question of symmetry. If a firm's bonds mature and the firm makes a new bond issue with different terms to those that have just matured, then it is necessary to keep track of this event in determining the investor's optimal contingent strategies. Hence, the whole specification process becomes far more complicated when symmetry is broken.

The second assumption about the formulation is the inclusion of a random element term. If it is not present then, given a general equilibrium state, the individual would know with certainty what the price of the bond would be at the end of the period. Unexpected changes, whether they be in the general level of interest rates, the probability of bankruptcy, or general economic conditions, affect the future price of the bond and thus the random element is added in an attempt to catch these effects. It should be noted that if the bond had a finite maturity, then it would be necessary for the variance term of the random element to be a function of the time to maturity. This is because at maturity the value of the bond, given that the firm has not defaulted, will be the principal value of the bond and will thus be independent of the influence of future expectations.

If bankruptcy occurs, it is assumed that the price of the bond at the end of the period can be represented by the expression

$$b_j(t+h) = A_j(t+h) - \theta_j(t+h), \tag{4.3}$$

$$j = 1, 2, \ldots, n,$$

where $A_j(t+h)$ represents the anticipated liquidation value per number of bonds outstanding for the j^{th} firm if bankruptcy occurred at time t+h; and $\theta_j(t+h)$ represents the direct costs associated with bankruptcy at time t+h, per number of bonds outstanding, for the j^{th} firm.

It is assumed that the investor forms a subjective evaluation of the liquidation value of the firm given that bankruptcy has occurred. The liquidation value of the firm will depend upon the expected general economic conditions prevalent at the end of the period, as well as upon the state, the type and marketability of the firm's assets. It is assumed that $A_j(t+h)$ represents the mean value of the investor's subjective evaluation of the firm given that bankruptcy has occurred. Before a liquidation premium can be paid to the bondholders, the direct costs associated with bankrupty -- legal fees, trustee fees, referee fees, administrative costs -- must be paid. These are represented by the term $\theta_j(t+h)$. Thus, the amount which the bondholder expects to receive, on a pro-rata basis, is thus

$$b_j(t+h) = \text{Max}[A_j(t+h) - \theta_j(t+h), 0] \tag{4.4}$$

For expositional simplicity, in the formulation of the general equations describing the price dynamics, Equation (4.3) is used, whilst it is formally recognized that Equation (4.4) is strictly correct.

Hence, the price dynamics of the j^{th} firm's bonds can be represented by

$$b_j(t+h) = \begin{cases} b_j(t)(1+r_j h) - g_j h + b_j(t)\,\gamma_j \sqrt{h}\, y_j(t); & \text{if no default,} \\ A_j(t+h) - \theta_j(t+h) & ;\ \text{if default,} \end{cases}$$

$$j = 1, 2, \ldots, n. \tag{4.5}$$

It is assumed that the event of bankruptcy can be described by the following type of stochastic process:[18]

$$\Pr\{j^{th} \text{ firm not going bankrupt in } (t,t+h]\} = 1-\lambda_j(t)h$$

and (4.6)

$$\Pr\{j^{th} \text{ firm going bankrupt in } (t,t+h]\} = \lambda_j(t)h,$$

$$j = 1, 2, \ldots, n,$$

where $\lambda_j(t)h$ can be interpreted as the probability of bankruptcy for firm j during the period $(t,t+h]$. Its determinants are extensively discussed in Chapter III.

It is assumed that the event of bankruptcy for one firm does not affect other firms. Conceptually, it is very simple to relax this assumption, but only at the cost of greatly increasing the complexity of the notation. The very small gain in generality of derived results does not warrant this cost.

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[18] A more rigorous deviation using Poisson probability distributions is given in Appendix A.

Define a random variable indicator function, $I_j(t,t+h)$ which can take only two values, zero and one, such that

$$\Pr[I_j(t,t+h) = 0] = 1 - \lambda_j(t)h, \tag{4.7}$$

and

$$\Pr[I_j(t,t+h) = 1] = \lambda_j(t)h,$$

$$j = 1, 2, \ldots, n.$$

The indicator function describes the status of the firm at the end of the period; that is, it indicates whether the firm has gone bankrupt or not. Equation (4.5) can be written in the form

$$b_j(t+h) = [b_j(t)(1+r_jh) - g_jh + b_j(t)\ \gamma_j\sqrt{h}\ y_j(t)][1 - I_j(t,t+h)]$$
$$+ [A_j(t+h) - \theta_j(t+h)]I_j(t,t+h)$$

If the event of bankruptcy does not occur to firm j in the $(t,t+h]$, then $I_j(t,t+h)$ is zero. If bankruptcy does occur then $I_j(t,t+h)$ equal one. In the limit, as h tends to zero, it can be shown that:[19]

$$db_j(t) = [b_j(t)r_j-g_j]dt + b_j(t)\gamma_jdZ_j - \{b_j(t) - [A_j(t+dt) - \theta_j(t+dt)]\}dq_j.$$
$$j = 1,2,\ldots,n, \tag{4.8}$$

where dq_j is a Poisson process characterizing the event of bankruptcy for the j^{th} firm; and dZ_j is a standard Gaussian-Wiener process.

The price dynamics of the j^{th} firm's equity will be affected by what happens to the firm's bonds, that is, if default occurs the value of equity,

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[19]Appendix A, Equation (A.2).

by assumption, will be zero. It is assumed that the price dynamics can be expressed in the form

$$p_j(t+h) = \begin{cases} p_j(t)(1+\alpha_j h) - f_j h + p_j(t)\sigma_j\sqrt{h}Y_{n+j}(t); & \text{if no default,} \\ 0 & ; \text{if default,} \end{cases} \tag{4.9}$$

$$j = 1, 2, \ldots, n$$

where

$p_j(t)$ represents the market price of a share for the j^{th} firm at time t;

α_j represents the instantaneous conditional expected rate of return for the j^{th} firm's equity, conditional on the fact that the firm does not go bankrupt;

f_j represents the instantaneous conditional dividend rate for the j^{th} firm;

σ_j^2 represents the instantaneous conditional variance for the j^{th} firm;

and

$Y_{n+j}(t)$ represents a zero mean, unit variance, purely random process, that is, $Y_{n+j}(t)$ and $Y_{n+j}(t+s)$, $s>0$, are independent and identically distributed gaussian random variables.

If it is assumed that there is no possibility of bankruptcy, then the equation becomes

$$\frac{\Delta p_j(t)}{p_j(t)} = (\alpha_j-\delta_j)h + \sigma_j\sqrt{h}\,Y_{n+j}(t),$$

where δ_j is the instantaneous dividend yield. Taking the limit as h tends to zero, then

$$\frac{dp_j(t)}{p_j(t)} = (\alpha_j - \delta_j)dt + \sigma_j dZ_{n+j}, \tag{4.10}$$

where dZ_{n+j} represents a standard Gaussian-Wiener process. If it is assumed that α_j, δ_j and σ_j are constant, then prices will be stationarily and log-normally distributed.[20]

In general, Equation (4.9) can be written in the form

$$p_j(t+h) = [p_j(t)(1+\alpha_j h) - f_j h + p_j(t)\sigma_j \sqrt{h}\, Y_{n+j}(t)][1 - I_j(t,t+h)].$$

If the firm does not go bankrupt in the interval (t,t+h] then $I_j(t,t+h)$ will equal zero. If it does go bankrupt, then $I_j(t,t+h)$ will equal one and the value of equity will be zero. In the limit as h tends to zero, it can be shown that:[21]

$$dp_j(t) = [p_j(t)\alpha_j - f_j]dt + p_j(t)\sigma_j dZ_{n+j} - p_j(t)dq_j. \tag{4.11}$$

It is instructive to consider what are the differences between the common stock and perpetuities of a firm and to enquire how these differences are reflected in the set of Equations (4.8) and (4.11) that are used in the model to describe the price behaviour of bonds and common stock. A bond-holder purchases a claim to a fixed series of payments, whilst the common stockholder purchases a claim to a variable cash flow. There is mutual interaction between the price of the two financial assets. For example, an unexpected change in the probability of bankruptcy will affect the price

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[20] For a more general derivation, see Merton, R. C. "Optimum Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory*, Vol. 3 (1971), pp. 373-413.

[21] Appendix A, Equation (A.3).

of both financial assets. In the event of bankruptcy, the bondholder has prior claim to the assets of the firm.

In the model the priority of claim of the bondholder in the event of bankruptcy is reflected in the difference between the nature of the coefficients of the dq_j term. The bondholder might receive a liquidation premium, whilst the common stockholder suffers a hundred per cent loss. The interaction between the two sets of assets, is represented by the correlation between the two Gaussian-Wiener processes dZ_j and dZ_{n+j}. What is not represented is the difference between the claims to a series of fixed and variable payments. An attempt can be made to rectify this by reinterpreting the instantaneous conditional expected dividend rate, and then to specify some type of process describing how the instantaneous conditional dividend rate might change over time.[22]

Over time changes in expectations occur as the information set available to the investor changes. That is, new information may cause a revision in the expectations about the behaviour of different financial assets. If the opportunity set is constantly changing, then this will probably affect the structure of returns. In an attempt to analyze the effects of changes in expectations, it is assumed that changes in the opportunity set can be described by the following set of stochastic differential equations:

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[22] It is assumed at the outset that it is possible to specify such an equation. It is recognized that this brushes over the major problem that to justify the specification of the equation would require a complete description of the supply side of the firm.

$$d\alpha_j = F_j(\alpha_j,t)dt + G_j(\alpha_j,t)dQ_j,$$

$$d\sigma_j = F_{n+j}(\sigma_j,t)dt + G_{n+j}(\sigma_j,t)dQ_{n+j},$$

$$dr_j = F_{2n+j}(r_j,t)dt + G_{2n+j}(r_j,t)dQ_{2n+j}, \tag{4.12}$$

and

$$d\gamma_j = F_{3n+j}(\gamma_j,t)dt + G_{3n+j}(\gamma_j,t)dQ_{3n+j},$$
$$j = 1, 2, \ldots, n,$$

where F( ), and G( ) are specified functions, and dQ represents a standard Gaussian-Wiener process. The first two equations describe the changes to the price dynamics of equity. The first equation shows how the instantaneous conditional expected return is affected and the second equation how the instantaneous conditonal standard deviation is affected. The last two equations describe the changes to the price dynamics of the bonds.

In a similar fashion, it is assumed the probability of bankruptcy is also stochastic over time, and the investor attempts to take into consideration the effects of such changes; that is

$$d\lambda_j = F_{4n+j}(\lambda_j,t)dt + G_{5n+j}(\lambda_j,t)dQ_{4n+j}, \tag{4.13}$$

$$j = 1, 2, \ldots, n$$

This implies that $\lambda_j(t)h$, as defined in Equation (4.6), should be reinterpreted as the mean value of the probability of bankruptcy for the j^{th} firm in the interval $(t, t+\Delta t]$.

Analogous to the above reasoning, the equation describing the changes in the instantaneous conditional expected dividend rate is assumed to be

$$df_j = F_{5n+j}(f_j,t)dt + G_{5n+j}(f_j,t)dQ_{5n+j}, \tag{4.14}$$

$$j = 1, 2, \ldots, n.$$

An instantaneous riskless asset means that at each instance of time, each investor knows with certainty the rate of return, r, over the next instant by holding the asset. However, the future values of r are not known with certainty. By convention the $(2n+1)^{th}$ financial asset is taken to be the instantaneous riskless asset. Hence, the price dynamics can be described by

$$\frac{dp_{2n+1}(t)}{p_{2n+1}(t)} = rdt, \tag{4.15}$$

and it is assumed that changes in the rate of return can be described by

$$dr = F_m(r,t)dt + G_m(r,t)dQ_m, \tag{4.16}$$

where $m = 6n+1$.

State Space Description and the Budget Constraint

Before proceeding to derive the budget constraint and the optimality equations, a brief digression on the implications of the fact that both the price behaviour of equity and bonds are influenced by whether or not a firm has defaulted will help to clarify the derivation of the equations.

If there are n firms in existence at time t, then at time t+h there are 2^n possible states, where a state is defined as a description of the n firms, listing those firms which have defaulted and those which are in existence at time t+h. Given the assumption that the event of bankruptcy by one firm has no affect upon the remaining firms, then it is only necessary to consider (n+1) of the possible states, for the other states have a probability of order h and make no contribution when a limiting process is used.

Consider three firms, A, B, and C. The eight states of the system are shown in Table 4.1. The probability that no firms default in the interval (t,t+h] equals

$$(1-\lambda_1 h)(1-\lambda_2 h)(1-\lambda_3 h)$$

$$= 1 - \sum_{j=1}^{3} \lambda_j h + 0(h),$$

where $\lambda_1 h$ equals the probability of bankruptcy for firm A in the interval (t,t+h]. The probability that one firm, say firm A, going bankrupt and the others do not, equals

$$\lambda_1 h(1 - \lambda_2 h)(1 - \lambda_3 h)$$

$$= \lambda_1 h + 0(h).$$

From Table 4.1, it is clear that only states 1, 2, 3 and 4 are important. Thus only those states where one or less bankruptcies occurred need to be considered. This greatly simplifies the analysis.

TABLE 4.1

THE PROBABILITY OF OCCURRENCE OF DIFFERENT STATES

STATES	FIRMS A	FIRMS B	FIRMS C	PROBABILITY OF OCCURRENCE
1	NB	NB	NB	$1-\sum_{j=1}^{3} \lambda_j h + 0(h)$
2	B	NB	NB	$\lambda_1 h + 0(h)$
3	NB	B	NB	$\lambda_2 h + 0(h)$
4	NB	NB	B	$\lambda_3 h + 0(h)$
5	B	B	NB	$0(h)$
6	B	NB	B	$0(h)$
7	NB	B	B	$0(h)$
8	B	B	B	$0(h)$

B = bankruptcy occurred in (t,t+h]

NB = bankruptcy did not occur in (t,t+h]

0(h) = of order h

To derive the budget equation it is necessary to examine the discrete time formulation of the model and then to take limits (that is, let h tend to zero) to obtain the continuous time formulation. Let

$W(t)$ represent the investor's tangible wealth at time t-;

$C(t)$ represent the investor's consumption rate at time t;

$y(t)$ represent the investor's wage income (may be stochastic) at time t;

and

$I(t)$ represent the investor's investment at time t+

where superscripts denoting the investor's identification have been dropped, except when required for clarity. The amount that an individual can invest at time t+ depends upon current tangible wealth, wage income, and planned consumption; that is,

$$I(t) = W(t) - [C(t) - y(t)]h. \tag{4.17}$$

Let

$N_j(t)$ represent the number of bonds of firm j purchased during period (t,t+h];

$N_{n+j}(t)$ represent the number of shares of firm j purchased during (t,t+h];

$N_{2n+1}(t)$ represent the number of shares of the instantaneous riskless asset purchased during (t,t+h];

and

$w_k(t)$ represent the fraction of the investment invested in the k^{th} asset during (t,t+h],

$$j = 1, 2, \ldots, n,$$

and

$$k = 1, 2, \ldots, 2n+1.$$

Hence, the individual's investment can be represented by

$$I(t) = \sum_{j=1}^{n} N_j(t)b_j(t) + \sum_{j=1}^{n} N_{n+j}(t)p_j(t) + N_{2n+1}p_{2n+1}(t).$$

The individual's wealth at the end of the period, t+h, will depend upon the price of the bonds and shares, the coupons and dividends received, and the state of the system; that is, upon which firms that went bankrupt in the interval (t,t+h].

Suppose that no firms went bankrupt in the interval (t,t+h], then the wealth at the end of the interval can be represented by

$$W(t+h) = \sum_{j=1}^{n} N_j(t)\ [b_j(t+h) + g_j h] + \sum_{j=1}^{n} N_{n+j}(t)\ [p_j(t+h) + f_j h]$$

$$+ N_{2n+1}(t)p_{2n+1}(t)$$

$$= I(t) \sum_{j=1}^{n} w_j(t)\ \left[\frac{b_j(t+h)+g_j h}{b_j(t)}\right] + I(t) \sum_{j=1}^{n} w_{n+j}(t)\ \left[\frac{p_j(t+h)+f_j h}{p_j(t)}\right]$$

$$+ I(t)w_{2n+1}(t)\ \frac{p_{2n+1}(t+h)}{p_{2n+1}(t)}$$

It will be assumed that all income is derived from investment in the financial assets; that is, y(t) = 0, which implies that

$$I(t) = W(t) - C(t)h.$$

Thus the change in wealth can be expressed in the form

$$W(t+h) - W(t) = \{W(t) - C(t)h\}\{\sum_{j=1}^{n} w_j(t)\left[\frac{b_j(t+h) + g_j h - b_j(t)}{b_j(t)} - rh\right] + \sum_{j=1}^{n} w_{n+j}(t)\left[\frac{p_j(t+h) + f_j h - p_j(t)}{p_j(t)} - rh\right] + rh\} - C(t)h,$$

where Equation (4.15) and the relation $\sum_{j=1}^{2n+1} w_j(t) = 1$ have been used. Substituting Equations (4.8) and (4.11) into the above expressions gives

$$W(t+h) - W(t) = \{W(t) - C(t)h\}\{\sum_{j=1}^{n} w_j(t)[(r_j - r)h + \gamma_j dZ_j] + \sum_{j=1}^{n} w_{n+j}(t)[(\alpha_j - r)h + \sigma_j dZ_{n+j}] + rh\} - C(t)h + O(h). \tag{4.18}$$

Let

$$\Delta W(t) \equiv W(t+h) - W(t),$$

then the expected value of $\Delta W(t)$ conditional on the fact that no bankruptcies occurred in the interval $(t,t+h]$ is

$$E_t[\Delta W(t)] = [W(t) - C(t)h]\left[\sum_{j=1}^{n} w_j(b)(r_j - r) + \sum_{j=1}^{n} w_{n+j}(\alpha_j - r) + r\right]h - C(t)h + O(h), \tag{4.19}$$

and

$$E_t[\Delta W(t)^2] = W(t)^2\left[\sum_{j=1}^{n}\sum_{j=1}^{n} w_j(t)\gamma_{ji}w_i(t) + 2\sum_{j=1}^{n}\sum_{j=1}^{n} w_j(t)\gamma_j\rho_{ji}\sigma_i w_{n+i}(t) + \sum_{j=1}^{n}\sum_{j=1}^{n} w_{n+j}(t)\sigma_{ji}w_{n+i}(t)\right] h + O(h), \tag{4.20}$$

when ρ_{ji} is the instantaneous conditional correlation between dZ_j and dZ_{n+i}.

Suppose that the j^{th} firm goes bankrupt in the interval $(t,t+h]$. This will affect the investors holding the j^{th} firm's bonds and equity, the bondholder might receive a liquidation premium, whilst the equity holder will suffer a hundred per cent loss. Hence, the wealth at the end of the interval, conditional on the fact that the j^{th} firm has gone bankrupt, is

$$W(t+h) = \{W(t) - C(t)h\}\{ \sum_{\substack{i=1 \\ i\neq j}}^{n} w_i(t)[\frac{b_i(t+h) + g_ih}{b_i(t)}] + w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)}]$$

$$+ \sum_{\substack{i=1 \\ i\neq j}}^{n} w_{n+i}(t)[\frac{p_i(t+h) + f_ih}{p_i(t)}] + w_{2n+1}(t) \frac{p_{2n+1}(t+h)}{p_{2n+1}(t)}\}$$

Substituting Equations (4.8), (4.11) and (4.15) into the above equations and using the relation $\sum_{j=1}^{2n+1} w_j(t) = 1$, gives

$$W(t+h) - W(t) = \{W(t) - C(t)h\}\{ \sum_{\substack{i=1 \\ i\neq j}}^{n} w_i(t)[(r_i-r)h + \gamma_i dZ_i] + w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)} -1]$$

$$+ \sum_{\substack{i=1 \\ i\neq j}}^{n} w_{n+i}(t)[(\alpha_i-r)h + \sigma_i dZ_{n+i}] - w_{n+j}(t) + rh\} + C(t)h + O(h).$$

$$= W(t)\{w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)} - 1] - w_{n+j}(t)\}$$

$$+ \{W(t) - C(t)h\}\{ \sum_{\substack{i=1 \\ i\neq j}}^{n} w_i(t)[(r_i-r)h + \gamma_i dZ_i]$$

$$+ \sum_{\substack{i=1 \\ i\neq j}}^{n} w_{n+i}(t)[(\alpha_i-r)h + \sigma_i dZ_{n+i}] + rh\}$$

$$- C(t)h\{1 + w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)} - 1] - w_{n+j}(t)\} + O(h) \tag{4.21}$$

The first term in the above expression, that is,

$$W(t)\{w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)} - 1] - w_{n+j}(t)\}$$

can be interpreted as the loss to the individual who invested in the j^{th} firm's securities, given that the firm goes bankrupt in the interval (t,t+h]. If the individual paid $b_j(t)$ for the bond at the beginning of the period, then at the end of the period the individual will receive a liquidating premium of $A_j(t+h) - \theta_j(t+h)$, which is less than the initial amount paid for the bond, given that $\theta_j(t+h)$ is greater than zero. Similarly, for equity, if the individual paid $p_j(t)$ at the beginning of the period, then the individual will suffer a hundred per cent loss at the end of the period. Note, however, that as short selling is allowed, the argument can be reversed to give the gains that occur when the firm goes bankrupt.

The Equation of Optimality: The Demand Functions for Assets

The individual is assumed to act in such a manner so as to maximize the expected lifetime utility of consumption and terminal wealth; that is, rewriting Equation (4.1)

$$\text{Max } E_0\{\int_0^T U[C(s),s]ds + BF[W(T),T]\}, \tag{4.1}$$

subject to an initial wealth constraint, the budget constraint, and $C(s) \geq 0$, and where the superscripts denoting the investor's identity have been dropped, except when required for clarity.

Define

$$J[W(t),\underline{\alpha},\underline{\sigma},\underline{r},\underline{\gamma},\underline{f},\underline{\lambda},r_F,t,s(t)] \equiv \underset{\{C,\underline{w}\}}{\text{Max}}\, E_t\{\int_t^T U[C(s),s]ds + BF[W(T),T]\} \tag{4.22}$$

where s(t) is a state vector which describes what firms are still in existence at time t; $\underline{\alpha},\underline{\sigma},\underline{r},\underline{\gamma}$, $\underline{f}$, and $\underline{\lambda}$ are vectors which describe the values at time t of

$\{\alpha_j\}, \{\sigma_j\}, \{r_j\}, \{\gamma_j\}, \{f_j\}$ and $\{\lambda_j\}$ respectively; and r_F is the value of the rate of return at time t on the instantaneous riskless asset. The function, J, is called the derived utility of wealth. Its arguments are the state variable which, at time t, an individual knows. The problem facing the individual is to choose values for $\{C, \underline{w}\}$ which maximizes the expression on the right hand side of (4.22); that is, the decision variables are the rate of consumption C, and the proportion of wealth to invest in the different financial assets $\{w_j\}$. This may be achieved using the Bellman principle of optimality.[23]

It can be shown[24] that the optimality conditions for an individual who acts according to Equation (4.1) in determining the consumption-investment contingent strategy at each point in time, are

$$
\begin{aligned}
0 = \underset{\{C,\underline{w}\}}{\mathrm{Max}} \quad & \Big(U[C(t),t] + J_t + \sum_{j=1}^{m} F_j J_j \\
& + J_W\{W(t)[\sum_{j=1}^{n} w_j(t)(r_j - r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_j - r) + r] - C(t)\} \\
& + \frac{1}{2} J_{WW}[\sum_{j=1}^{n} \sum_{i=1}^{n} w_j(t)\gamma_{ji} w_i(t) + 2 \sum_{j=1}^{n} \sum_{i=1}^{n} w_j(t)\gamma_j \rho_{ji} \hat{\sigma}_i w_{n+i}(t) \\
& + \sum_{j=1}^{n} \sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji} w_{n+i}(t)] W(t)^2 \\
& + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} G_i \nu_{ij} G_j J_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} W(t) w_j(t) \gamma_j \eta_{ij} G_i J_{iW} \\
& + \sum_{i=1}^{m} \sum_{j=1}^{n} W(t) w_{n+j}(t) \sigma_j \eta_{i,n+j} G_i J_{iW} \\
& + \sum_{j=1}^{n} \lambda_j(t)\{J[W_j, V(t), t, S_j] - J[W(t), V(t), t, S(t)]\}\Big),
\end{aligned}
\qquad (4.23)
$$

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[23] For a formal statement of this principle, see pp. 15 of Bellman, R. E. and Dreyfus, S. E., *Applied Dynamic Programming* (Princeton, N.J.: Princeton University Press, 1962).

[24] See Appendix A, Equation (A.17).

subject to the boundary condition $J[W(T),V(T),T,S(T)] = BF[W(T),T]$, and where subscripts on the function $J[W(t),V(t),t,S(t)]$ denote partial derivations; S_j is a state vector denoting that the j^{th} firm no longer exists; σ_{ij} is the instantaneous conditional covariance between dZ_i and dZ_j; ρ_{ji} is the instantaneous conditional correlation coefficient between dZ_i and dZ_{n+j}; ν_{ij} is the instantaneous conditional correlation coefficient between dQ_i and dQ_j; η_{ij} is the instantaneous conditional correlation coefficient between dQ_i and dZ_j; and W_j is defined[25] by

$$W_j(t) = W(t)\{1 + w_j(t)[\frac{A_j - \theta_j}{b_j(t)} - 1] - w_{n+j}(t)\}, \tag{4.24}$$

which can be interpreted as the new wealth position after the j^{th} firm has gone bankrupt.

Equation (4.23) is a partial differential equation which describes the behaviour of the derived utility function. As such, there are no stochastic elements in the equation; it is completely deterministic. The value of the derived utility function depends upon the consumption rate and the amount of investment in the different financial assets that are available. The optimal values of these decision variables that maximize the derived utility function at each point in time, can be determined by solving the set of equations which describe the first order conditions for a maximum. The assumptions about the form of the utility function and the bequest function ensure that a maximum, and not a minimum, is obtained.[26]

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[25]See Appendix A, Equations (A.15) to (A.20), where a detailed discussion is given for the reasons motivating this definition.

[26]A proof is given in Kushner, H. J., *Stochastic Stability and Control*, (New York: Academic Press, 1967).

The (2n+1) first order conditions are obtained by first differentiating Equation (4.23) with respect to the rate of consumption[27]

$$0 = U_c[C(t),t] - J_W; \tag{4.25}$$

then by differentiating (4.23) with respect to the amount of investment in equities;[28] that is, $\{w_{n+j}\}$:

$$0 = (\alpha_j - r)J_W + W(t)[\sum_{i=1}^{n} \sigma_{ji} w_{n+i}(t) + \sum_{i=1}^{n} \gamma_j \rho_{ij} \sigma_i w_i(t)]J_{WW} + \sum_{i=1}^{m} \sigma_j \eta_{i,n+j} G_i J_{iW} - \lambda_j J_W[W_j, V(t), t, S_j], \tag{4.26}$$

$j = i, 2, . . ., n;$

and finally, differentiating (4.23) with respect to the amount of investment in bonds;[29] that is, $\{w_j\}$:

$$0 = (r_j - r)J_W + W(t)[\sum_{i=1}^{n} \gamma_{ji} w_i(t) + \sum_{i=1}^{n} \gamma_j \rho_{ij} \sigma_i\, {}_{n+i}(t)]J_{WW} + \sum_{i=1}^{m} \gamma_j \eta_{ij} G_i J_{iW} - \lambda_j[1 - \frac{A_j - \theta_j}{b_j(t)}] J_W[W_j, V(t), t, S_j], \tag{4.27}$$

$j = 1, 2, . . ., n.$

Equation (4.25) is the intertemporal envelope condition: the marginal utility of current consumption equals the marginal derived utility of wealth. Equation (4.26) describes a system of equations for the investment in the commonstocks availableand Equation (4.27) describes a similar system for bonds.

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[27]See Appendix A, Equation (A.18)

[28]See Appendix A, Equation (A.19).

[29]See Appendix A, Equation (A.20).

The two systems are not independent. If a firm goes bankrupt, the event of bankruptcy will affect both the value of its bonds and equity. Events that affect the value of the firm, for example unexpected changes in the probability of bankruptcy, will be reflected in changes in both the value of the firm's bonds and equity. The lack of independence between the two systems implies that the two sets of equations must be solved simultaneously. A solution will not, however, be easy to obtain because the equations are non-linear. The non-linearity results from the presence of the terms $\{J[W_j, V(t), t, s_j]\}$ which is a consequence of the fact that bankruptcy causes a discontinuity in the wealth of the individual.

At this level of generality, little insight into the implications of the set of equations can be gained. It is proposed to add some further, and simplifying, assumptions to restrict the structure of the opportunity set.

Two models will be considered. The first is a simple model in which the investment opportunity set is assumed only to be altered by the event of bankruptcy; the probability of bankruptcy is assumed not to change stochastically over time. The equations that describe the bond price dynamics are also simplified. Whilst such a level of simplicity is unrealistic, it does afford penetrating insight into how the mechanism of bankruptcy affects the structure of returns. The second model relaxes the assumption that the probability of bankruptcy for a firm does not change stochastically over time. The framework is more realistic than that of the first model, but does not offer the same level of insight.

Bankruptcy and Structure of Returns

By reducing the level of generality of the formulation enables greater insight into the impact of bankruptcy upon the mechanism describing the structure of returns. It is assumed that the opportunity set characterized by $\{\alpha, \sigma, r, \gamma, f, \lambda, r_F\}$ is deterministic; that is, there are no stochastic changes in these parameters so that the individual knows with certainty their future values. It is further assumed that there is no stochastic element to the conditional equation describing the price dynamics of bonds, that is, Equation (4.5) becomes

$$b_j(t+h) = \begin{cases} b_j(t)(1+r_jh) - g_jh; \text{ if no default} \\ \\ A_j(t+h) - \theta_j(t+h); \text{ if default,} \end{cases} \tag{4.28}$$

$$j = 1, 2, \ldots, n$$

Whilst the absence of a stochastic element term is an oversimplification, it does imply that there will be no mutually interaction between bonds and common stocks apart from the direct effect of bankruptcy. Thus, there will be no interaction terms in the expressions for the equilibrium rates of return for bonds and common stock.

From Equation (4.23) the equation of optimality becomes

$$0 = \underset{\{c,\underline{w}\}}{\text{Max}} \left(U[C(t),t] + J_t + J_W\{W(t)[\sum_{j=1}^{n} w_j(t)(r_j - r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_r - r) + r] - C(t)\}\right.$$

$$\left. + \frac{1}{2} J_{WW}[\sum_{j=1}^{n}\sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji}\, w_{n+i}(t)]W(t)^2 + \sum_{j=1}^{n} \lambda_j\{J[W_j,t,S_j] - J[W,t,S(t)]\}\right) \tag{4.29}$$

and the first order conditions are, after simplification[30]

$$0 = U_c[C(t),t] - J_W, \tag{4.30}$$

$$0 = (\alpha_j - r - \frac{r_j - r}{L_j})J_W + W(t)[\sum_{i=1} \sigma_{ji} w_{n+i}(t)]J_{WW}, \tag{4.31}$$

and

$$0 = (r_j - r)J_W - \lambda_j L_j J_W(W_j, t), \tag{4.32}$$

$$j = 1, 2, \ldots, n,$$

where for expositional simplicity L_j is defined by

$$L_j = 1 - \frac{A_j - \theta_j}{b_j(t)},$$

which can be given an intuitive meaning. Suppose an individual purchased a bond for $b_j(t)$ and the firm went bankrupt. The individual would receive $(A_j - \theta_j)$ and thus suffer a loss of $[b_j - (A_j - \theta_j)]$. L_j is a percentage measure of that loss. If L_j equals one, the individual suffers a hundred per cent loss.

Equation (4.30) is the intertemporal envelope condition: marginal utility of consumption equals the marginal derived utility of wealth. Equation (4.31) describes a linear system of equations from which the demand functions for equity can be determined independently from the direct influence of the demand functions for bond. This independence is a consequence of the assumption of having no random element term in the bond equation. The system of Equations (4.32), that describe the demand functions for bonds are non-linear and thus, in general, it will be difficult to obtain an exact solution. The non-linearity results from the discontinuities in wealth that are caused by the event of bankruptcy.

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[30]See Appendix A, Equations (A.26), (A.27) and (A.28).

Consider first the demand functions for bonds, described by Equation (4.32). As the equations are non-linear, it is difficult to obtain an explicit solution. There are at least two alternatives. The first is to put more structure into the formulation by assuming a particular form for the individual's utility function and then attempt to solve the system of equations by a numerical iterative procedure. Whilst this method might produce a solution, it will be at a cost. It will be difficult to derive explicit forms for the general equilibrium rates of return, and even if they could be obtained, they will depend upon the specific assumed form for the utility functions for the individuals. The lack of generality and intractability of this method is a serious distraction to its utilization.

The second alternative is to make an approximation so as to obtain a linear system. The approximation evolves around the assumption that it is possible to expand the derivations of the derived utility function in a Taylor's series and to neglect quadrative and higher order terms;[31] that is,

$$
\begin{aligned}
&J_W\{W(t)[1 - w_j(t)L_j - w_{n+j}(t)],\ t\} \\
&= J_W[W(t),t] - W(t)[w_j(t)L_j + w_{n+j}(t)]J_{WW}[W(t),t], \\
&j = 1, 2, \ldots, n.
\end{aligned}
$$

For a quadratic utility function this approximation is exact, whilst for other cases of utility functions, for example the constant relative risk aversion class, the approximation can be very good, depending upon the numerical values of the parameters of the utility function.[32]

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[31] See Appendix A, Equation (A.24).

[32] For a full discussion, see Appendix A, Equations (A.52) to (A.62).

Using this approximation, Equation (4.32) can be expressed in the form

$$0 = (r_j - r - \lambda_j L_j) J_W + \lambda_j L_j [w_j(t) L_j + w_{n+j}(t)] W(t) J_{WW}, \qquad (4.33)$$

$$j = 1, 2, \ldots, n.$$

From this equation it can be shown[33] the equilibrium instantaneous conditional expected rates of return are

$$r_j = r + \lambda_j L_j + \{\pi - r - \gamma\}\left\{\frac{\lambda_j L_j [\bar{N}_j b_j(t) L_j + \bar{N}_{n+j} p_j(t)]}{\sum_{i=1}^{n} X_i(t) \lambda_i L_i [\bar{N}_i b_i(t) L_j + \bar{N}_{n+i} p_i(t)]}\right\}$$

$$j = 1, 2, \ldots, n, \qquad (4.34)$$

where $\bar{N}_j$ is the total number of bonds outstanding for the j^{th} firm; $\bar{N}_{n+j}$ is the total number of shares outstanding for the j^{th} firm; $X_i(t)$ is the proportion of the total market value of the j^{th} firm's bonds to the total market value of all bonds; π is the instantaneous conditional expected return on the bond market, defined by

$$\pi = \sum_{i=1}^{n} X_i(t) r_i; \qquad (4.35)$$

and γ can be interpreted as a weighted sum of the expected loss in the event of bankruptcy defined by

$$\gamma = \sum_{i=1}^{n} X_i(t) L_i \lambda_i. \qquad (4.36)$$

Equation (4.34) can be interpreted as the instantaneous conditional expected rate of return for a firm's bonds equals the sum of the risk free rate of return, the expected loss in the event of bankruptcy, and a market term. The second term, $\lambda_j L_j$, can be intuited as the expected loss if bankruptcy occurs.

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[33] See Appendix A, Equation (A.45).

It is composed of two terms: the rate of the probability of bankruptcy and the conditional expected loss, conditional upon the event of bankruptcy. The magnitude of this term will be dependent upon the expected net value of the firm's assets after the event of bankruptcy. It is possible for a trade-off to occur between these two terms: for example, the rate of probability of bankruptcy might be large whilst the expected loss in the event of bankruptcy might be small.

The second term, $\lambda_j L_j$, can be compared to the formulation of Fisher,[34] who hypothesized that the risk premium on a bond is a function of two terms: the probability of default and the marketability of a bond. This latter consideration is not relevant in the present context given the assumptions about the structure of the capital markets. Fisher did not, however, directly consider the impact upon the risk premium of the expected loss that might occur in the event of bankruptcy.

The third term can be interpreted as a market factor. It is composed of two elements: the first, $(\pi - r - \gamma)$, can be intuited as the instantaneous conditional expected risk premium on the bond market, and the second is a positive weighting factor.

It is instructive to examine the instantaneous conditional expected rate of return for a bond for the two cases of when an individual does not suffer a loss in the event of bankruptcy, $L_j = 0$, and when the probability of bankruptcy is zero. In both cases, the required rate of return is the risk free rate, as would be expected.

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[34] Fisher, L., "Determinants of Risk Premiums on Corporate Bonds," *Journal of Political Economy*, Vol. LXVII, No. 3 (June, 1959); pp. 217-237.

The demand functions for equity are described by Equation (4.31). This system of equations is linear and independent of the direct influence of the demand functions for bonds; that is, it does not contain terms like $\{w_j(t)\}$. This is a consequence of the assumption about the price dynamics of bonds described by Equation (4.28) which does not contain a random element term that would have precipitated interaction between the two sets of demand functions for bonds and equity.

From Equation (4.31) it can be shown[35] that the equilibrium instantaneous conditional expected rates of return can be expressed in the form

$$\alpha_j - r - \frac{r_j - r}{L_j} = \beta_j(\mu - r - \bar{\chi}), \qquad (4.37)$$

$$j = 1, 2, \ldots, n,$$

where μ is the instantaneous conditional expected return on the market, defined by

$$\mu = \sum_{j=1}^{n} Y_j(t)\alpha_j, \qquad (4.38)$$

$Y_j(t)$ being the proportion of the market value of the j^{th} firm's equity to the total market value of all equity; $\bar{\chi}$ is defined by

$$\bar{\chi} = \sum_{j=1}^{n} Y_j(t)\left(\frac{r_j - r}{L_j}\right); \qquad (4.39)$$

and β_j is the instantaneous conditional covariance of the return of the j^{th} firm's equity with the equity market, divided by the instantaneous conditional variance of the return on the market, defined by[36]

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[35]See Appendix A, Equation (A.39).

[36]From Equation (4.37) it can be shown that $\sum_{j=1}^{n} Y_j(t)\beta_j = 1$.

$$\beta_j = \frac{\sum_{i=1}^{n} Y_i(t)\sigma_{ji}}{\sum_{j=1}^{n}\sum_{i=1}^{n} Y_j(t)\sigma_{ji}Y_i(t)} \tag{4.40}$$

β_j is called the beta factor for the j^{th} firm. Greater insight into the significance of Equation (4.37) can be gained by eliminating the term $(\frac{r_j - r}{L_j})$. This can be achieved by using the expressions for the instantaneous conditional expected rate of return for the j^{th} firm's bonds. This gives[37]

$$\alpha_j - r - \lambda_j = \{\mu - r - \bar{\chi}\}\{\beta_j + \lambda_j [\frac{\bar{N}_j b_j(t) L_j + \bar{N}_{n+j} p_j(t)}{M(t) \sum_{j=1}^{n} \sum_{i=1}^{n} Y_j(t)\sigma_{ji}Y_i(t)}]\}, \tag{4.41}$$
$$j = 1, 2, \ldots, n,$$

where M(t) is the total market value of all equity. Using the definition of β_j given by Equation (4.40), Equation (4.41) can be written

$$\alpha_j - r - \lambda_j = \frac{(\mu - r - \bar{\chi})}{\sum_{j=1}^{n}\sum_{i=1}^{n} Y_j(t)\sigma_{ji}Y_i(t)} \{\sum_{i=1}^{n} \sigma_{ji}Y_i(t) + \lambda_j [\frac{\bar{N}_j b_j(t) L_j + \bar{N}_{n+j} p_j(t)}{\sum_{i=1}^{n} \bar{N}_{n+1} p_i(t)}]\}$$

If there are a large number of firms, the last term on the right hand side of the above expression can be neglected, as it is of order 2/n, where n is the number of firms. Hence, the expression for the instantaneous conditional expected rate of return for the j^{th} firm's equity can be written

$$\alpha_j - r - \lambda_j = \beta_j(\mu - r - \bar{\chi}) \tag{4.42}$$
$$j = 1, 2, \ldots, n.$$

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[37] It is not obvious from Equation (4.34) how Equation (4.41) is derived. As shown in Appendix A, Equation (A.46), an alternative form of Equation (4.34) can be developed. This alternative form has been used to eliminate $\left[\frac{r_j - r}{L_j}\right]$ from Equation (4.37).

The expression on the left hand side of the above equation can be interpreted as the instantaneous expected risk premium, as opposed to the instantaneous condition expected risk premium for the j^{th} firm. The term $\bar{\chi}$ can be identified as a weighted average of the rate of the probability of bankruptcy for all firms.[38] Hence, the expression $(\mu - r - \bar{\chi})$ can be interpreted as the instantaneous expected market risk premium, and thus Equation (4.42) may be written in the form

$$E(\dot{\tilde{R}}_j) - r = \beta_j [E(\dot{\tilde{R}}_M) - r], \tag{4.43}$$

where $E(\tilde{R}_j)$ is the instantaneous expected rate of return on the equity of the j^{th} firm; and $E(\tilde{R}_M)$ is the instantaneous expected rate of return on the equity market. This result is analogous to that derived by Merton,[39] and if the probability of bankruptcy for all firms is zero, the results are identical.

Both from a theoretical and empirical point of view the above results, whether they be expressed in the instantaneous conditional expected form of Equation (4.42) or the instantaneous expected form of Equation (4.43), are important. Theoretically, the results show that the continuous time analogy to the capital asset primary model is still valid for the case when bankruptcy is explicitly considered, provided the instantaneous expected rates of return are used and not the instantaneous expected rates of return conditional upon

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[38] From the definition of the instantaneous conditional expected return on the market, see (4.38), Equation (4.42) implies that

$$\bar{\chi} = \sum_{j=1}^{n} Y_j(t) \lambda_j .$$

This is not, however, a definitional identity. It is the result of the approximation made in deriving Equation (4.42) and, as such, is itself an approximation. See Appendix A, Equation (A.50) for a fuller discussion.

[39] Merton, R. C., "An Intertemporal Capital Asset Pricing Model," Working Paper 588-72, Massachusetts Institute of Technology, Sloan School of Management, February, 1972.

no bankruptcy. In the traditional capital asset pricing model, where capital structure is assumed to be irrelevant and bankruptcy is totally ignored, such a distinction is not necessary. But it is this distinction that makes the result important from an empirical viewpoint. In testing the capital asset pricing model, the assumption is made that it is possible to go from an ex-ante to an ex-post formulation and to use realized returns. However, from the way empirical tests are conducted, the realized returns are proxies for the expected return, conditional upon no bankruptcy; that is, they are proxies for the terms $\{\alpha_j\}$ and not $\{E(\tilde{R}_j)\}$. Thus, there is a basic misspecification error.

The effect of this error can be demonstrated, as shown in Figure 4.1. Merton[40] has shown that the traditional capital asset pricing model for continuous time is of the form

$$\alpha_j = r + \beta_j(\mu - r), \tag{4.44}$$

which is denoted in the figure by CAPM. If λ_j was constant and independent of the particular firm, that is, $\lambda_j = \lambda$, for all j, then Equation (4.42) becomes

$$\alpha_j = r + \lambda + \beta_j(\mu - r - \lambda). \tag{4.45}$$

This is represented in Figure 4.1 by the line denoted by CAPM'. The line is linear and flatter than the line CAPM due to the presence of the term λ. However, the rate of probability of bankruptcy does vary across firms and thus, in general, there will be a non-linear relationship between α and β. If it is assumed that as β increases, λ increases,[41] then a line of the form denoted by

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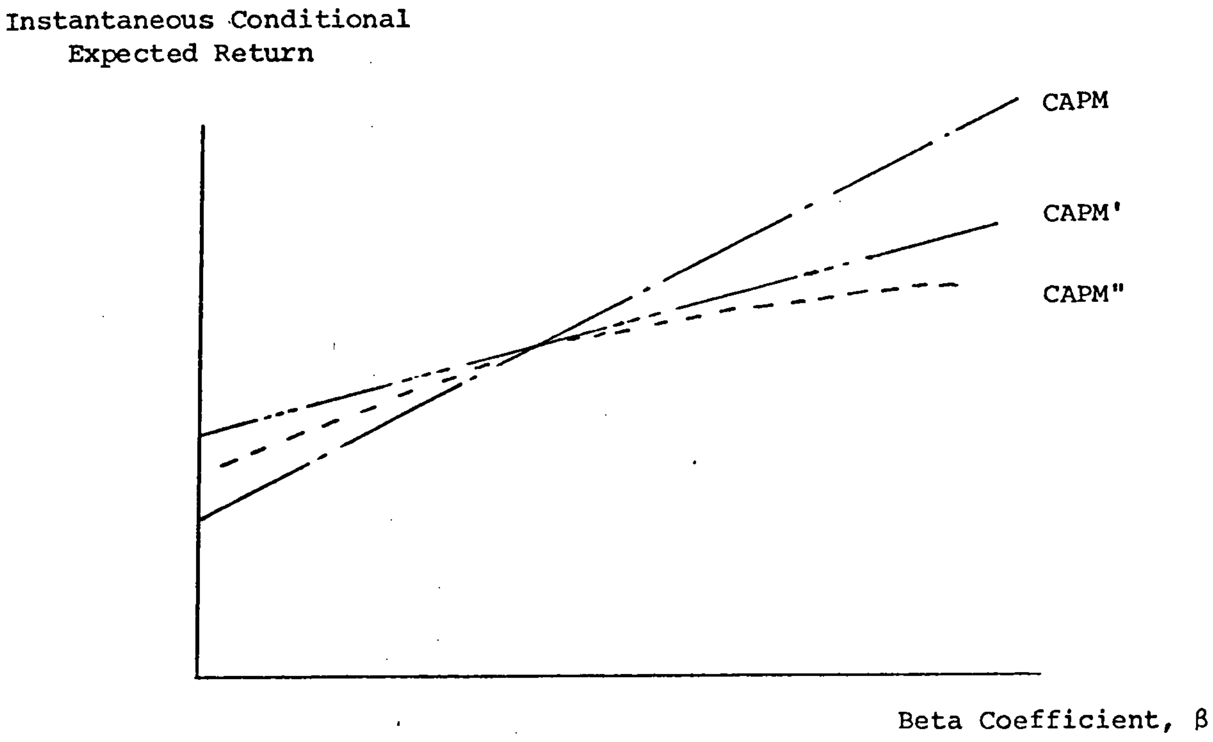
[40]Merton, *Ibid.*

[41]Westerfield has presented some evidence justifying this assumption. Westerfield, R., "The Assessment of Market Risk and Corporate Failure," University of Pennsylvania, Wharton School of Finance, August, 1970 (unpublished).

FIGURE 4.1

THE EFFECT OF BANKRUPTCY UPON THE CAPITAL MARKET LINE



CAPM denotes the equation $\alpha_j = r + \beta_j(\mu - r)$

CAPM' denotes the equation $\alpha_j = r + \lambda + \beta_j(\mu - r - \bar{\chi})$

CAPM" denotes the equation $\alpha_j = r + \lambda_j + \beta_j(\mu - r - \bar{\chi})$

CAPM" will be obtained.

Such a conclusion is very important when viewed in the light of recent empirical findings. The curve CAPM" is derived from the equation (4.42) which can be rewritten in the form

$$\alpha_j - r = (\beta_j - \beta_j \chi) + \beta_j (\mu - r), \quad j = 1, 2, \ldots, n. \tag{4.46}$$

If $\bar{\chi}$ is of the same order as λ, then for small values of $\beta (\beta < 1)$ the first term on the right hand side of the above equation will be positive, whilst for large values of $\beta (\beta > 1)$ it will be negative. Equation (4.46) describes a relation at a particular instant in time. There is no *a priori* reason to suppose that the various factors in this relation will remain constant over time. For example, the rate of the probability of bankruptcy may change because of a severe credit rationing. If this is so, then there is no reason for the curve CAPM" to be constant. Both conclusions are consistent with the time series results of Black, Jensen and Scholes[42] who found that the intercept term of the capital asset pricing model is non-stationary[43] and for a time series regressed over a 30 year period, the intercept was consistently negative for high risk portfolios ($\beta > 1$) and positive for low risk portfolios ($\beta < 1$). The misspecification of the model in light of the above discussion might also explain the negative

---

[42]Black, F., Jensen, M.C., and Scholes, M., "The Capital Asset Pricing Model: Some Empirical Tests," printed in *Studies in The Theory of Capital Markets* (Ed.) Jensen, M.C. (New York: Praeger, 1972).

[43]The time series regression was of the form $\tilde{R}_{jt} = \alpha_j + \beta_j \tilde{R}_{Mt} + \tilde{e}_{jt}$ where $\tilde{R}_{jt}$ is the ex-post excess return on the market portfolio over the same period, and e_{jt} a random error term.

relationship found over certain periods between average monthly returns and systematic risk.[44]

The model, as represented by Equation (4.42), forms the basis for the empirical testing of the hypothesis of the thesis. A discrete time formulation of the equation is used. The probability of bankruptcy is estimated utilizing the work developed in Chapter III.

Stochastic Changes in the Probability of Bankruptcy

In the model just considered knowledge of how the mechanism of bankruptcy affected the structure of returns was gained using a simple model in which the investment opportunity set did not change stochastically. Such an assumption is restrictive: the random arrival of new information and the reassessment of existing investment opportunities may cause the investment opportunity set to be altered with the implication that the expected rate of return required by potential investors will also change.

If the investment opportunity set is not constant, then this invalidates one of the conditions for the capital asset pricing model to be applicable for use in a multi-period context. Such a conclusion is hardly surprising, for the portfolio behavior of a rational investor would not be expected to be the same when there is a changing investment opportunity set instead of a constant one. Merton[45] has demonstrated that changes in the investment opportunity set do affect the structure of common stock returns. Under the assumption that all changes can be characterized by changes in a single instrumental variable --

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[44] See Black, Jensen, and Scholes, *loc. cit.*

[45] Merton, "An Intertemporal Capital Asset Pricing Model," *op. cit.*, p. 38.

the riskless interest rate -- a two factor model is derived. The second factor can be interpreted as the result of investors hedging against the effects of future unforeseen changes in the riskless interest rate.

Changes in the investment opportunity set can be caused by stochastic changes in the probability of a firm going bankrupt. As the firm's future income and its ability to borrow change over time, so will the probability of it going bankrupt. Some of these changes will be expected, and their significance will already be discounted in the price of the firm's financial assets. However, other changes will be unexpected and will affect the price of the firm's financial assets and the realized return of investors. The pertinent question to ask is how the stochastic nature of the changes in the rate of the probability of bankruptcy affect the structure of returns for financial assets? One method to analyze this problem is to represent the mechanism generating these stochastic changes by a specified process. It will be assumed that the mechanism can be represented in the form

$$d\lambda_j(t) = F_j(\lambda_j,t)dt + G_j(\lambda_j,t)dQ_j, \qquad (4.47)$$
$$j = 1, 2, \ldots, n,$$

where dQ represents a standard Gaussian-Wiener process. The above equation should be compared to Equation (4.13).

The equation describing the price dynamics of a firm's bonds will be assumed to contain a random element term, which reflects the uncertainty of price given that default has not occurred. This can be represented in the form, rewriting Equation (4.8)

$$db_j(t) = [b_j(t)r_j - g_j]dt + b_j(t)\gamma_j dZ_j - \{b_j(t) - [A_j(t+dt) - \theta_j(t+dt)]\}dq_j \qquad (4.8)$$
$$j = 1, 2, \ldots, n.$$

The inclusion of the random element term, dZ_j, describing the price dynamics of bonds will determine the degree of response upon the structure of expected rates of return caused by the stochastic nature of the changes in the rate of the probability of bankruptcy. It will also result in the demand functions for bonds and equity being directly correlated, as events will affect both types of assets.

The equation of optimality can be simply derived from the general case considered in Equation (4.23) and can be written in the form[46]

$$
\begin{aligned}
0 = \underset{\{c,\underline{w}\}}{\mathrm{Max}} \; (U[C(t),t] &+ J_t + \sum_{j=1}^{n} F_j J_j \\
&+ \{W(t)[\sum_{j=1}^{n} w_j(t)(r_j - r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_j - r) + r] - C(t)\}J_W \\
&+ \frac{1}{2}W(t)^2[\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_{ji}w_i(t) + 2\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_j\rho_{ji}\sigma_i w_{n+i}(t) \\
&\qquad \sum_{j=1}^{n}\sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji}w_{n+i}(t)]J_{WW} \\
&+ \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} G_i\nu_{ij}G_jJ_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} W(t)w_j(t)\gamma_j\eta_{ij}G_iJ_{iW} \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{n} W(t)w_{n+j}(t)\sigma_j\eta_{i,n+j}G_iJ_{iW} \\
&+ \sum_{j=1}^{n} \lambda_j\{J[W_j,\underline{\lambda}(t),t,S_j] - J[W(t),\underline{\lambda}(t),t,S(t)]\})
\end{aligned}
\tag{4.48}
$$

subject to the boundary condition $J[W(T),\underline{\lambda}(T),T,S(T)] = BF[W(T),T]$, and the set of first order maximization conditions are, after some manipulation,

$$o = U_c[C(t),t] - J_W, \tag{4.49}$$

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[46] See Appendix A, Equations (A.64), (A.65), (A.66) and (A.67).

$$0 = (\alpha_j - r - \frac{r_j - r}{L_j})J_W + W(t)[\sum_{i=1}^{n}(\sigma_{ji} - \frac{\gamma_j \rho_{ij} \sigma_i}{L_j})w_{n+i}(t)$$
$$+ \sum_{i=1}^{n}(\sigma_j \rho_{ji} \gamma_i - \frac{\gamma_{ji}}{L_j})w_i(t)]J_{WW} + \sum_{i=1}^{n}(\sigma_j \eta_{i,n+j} - \frac{\gamma_j \eta_{ij}}{L_j})J_{iW}, \tag{4.50}$$

and

$$0 = (r_j - r)J_W + W(t)[\sum_{i=1}^{n}\gamma_j \rho_{ij} \sigma_i w_{n+i}(t) + \sum_{i=1}^{n}\gamma_{ji} w_i(t)]J_{WW}$$
$$+ \sum_{i=1}^{n}\gamma_j \eta_{ij} G_i J_{iW} - \lambda_j(t) L_j J_W[W_j, \underline{\lambda}(t), t, S_j], \tag{4.51}$$

$j = 1, 2, \ldots, n.$

Equation (4.49) is the intertemporal envelope condition: marginal utility of consumption equals the marginal derived utility of wealth. Equation (4.50) describes a system of n linear equations in terms of the demand functions for bonds and equity. The direct dependence between the two sets of demand functions arises from the correlation of the price dynamics for the financial assets. This equation should be compared to Equation (4.31). Apart from the correlation terms, Equation (4.50) contains an extra set of terms, $\{J_{iW}\}$, that are a direct result of the stochastic nature of changes in the rate of the probability of bankruptcy. The significance of these terms will become very apparent when the individual demand functions are determined. Equation (4.51) describes a non-linear system of n equations in terms of the demand functions for bonds and equity, the non-linearity arising from the discontinuities in wealth that are caused by the event of bankruptcy. It also contains the extra set of terms $\{J_{iW}\}$, and should be compared to Equation (4.32). As in the last section,[47] it will be assumed that Equation (4.51)

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[47] See Equations (4.32) and (4.33).

can be approximated to give a linear system:[48]

$$0 = (r_j - r - \lambda_j(t)L_j)J_W + W(t)[\sum_{i=1}^{n} \gamma_{ji} w_i(t) + \sum_{i=1}^{n} \gamma_j \rho_{ij} w_{n+i}(t)]J_{WW}$$

$$+ \lambda_j(t)L_j[w_j(t)L_j + w_{n+j}(t)]W(t)J_{WW}$$

$$+ \sum_{i=1}^{n} \gamma_j \eta_{ij} G_i J_{iW}, \tag{4.52}$$

$$j = 1, 2, \ldots, n.$$

Hence, Equations (4.50) and (4.52) describe a system of 2n linear equations and thus it is possible to determine an explicit solution for the demand functions for bonds and equity. From the structure of these equations, it is clear that the form of such a solution will be involved, containing a large number of terms. The complexity of the solution arises because of the covariance terms and the terms that reflect the effects of the stochastic nature of the changes in the rate of the probability of bankruptcy.

It can be shown[49] that the demand functions for bonds for the k^{th} individual can be expressed in the following form, using matrix notation,

$$\underline{w}_1^k = H_1^k \underline{E}_1(\underline{C} - \underline{D}_2 \underline{D}_{12}^{-1} \underline{a}) + \underline{E}_1(\underline{D}_4 - \underline{D}_2 \underline{D}_{12}^{-1} \underline{D}_3) H_2^k, \tag{4.53}$$

and the demand functions for equities,

$$\underline{w}_2^k = H_1^k \underline{E}_2(\underline{a} - \underline{D}_1 \underline{D}_{21}^{-1} \underline{c}) + \underline{E}_2(\underline{D}_3 - \underline{D}_{21}^{-1} D_4) H_2^k, \tag{4.54}$$

where

$$H_1^k = -\frac{J_W^k}{J_{WW}^k};$$

$$\{\underline{H}_2^k\}_j = -\frac{J_{jW}^k}{J_{WW}^k},$$

---

[48]See Appendix A, Equation (A.68).

[49]See the whole of the last section of Appendix A.

$$\{\underline{w}_1^k\}_j = W(t)^k w_j(t)^k;$$

$$\{\underline{w}_2^k\}_j = W(t)^k w_{n+j}(t)^k;$$

$$\{\underline{a}\}_j = \alpha_j - r - \frac{r_j - r}{L_j};$$

$$\{\underline{C}\}_j = r_j - r - \lambda_j L_j;$$

$$\{\underline{D}_1\}_{ji} = \gamma_j \rho_{ij} \sigma_i - \frac{\gamma_{ji}}{L_j};$$

$$\{\underline{D}_2\}_{ji} = \lambda_j L_j + \gamma_j \rho_{ij} \sigma_i;$$

$$\{\underline{D}_3\}_{ji} = \sigma_j \eta_{i,n+j} G_i - \frac{\gamma_j \eta_{ij} G_i}{L_j};$$

$$\{\underline{D}_4\}_{ji} = \gamma_j \eta_{ij} G_i;$$

$$\{\underline{D}_{12}\}_{ji} = \sigma_{ji} - \frac{\gamma_j \rho_{ij} \sigma_i}{L_j};$$

$$\{\underline{D}_{21}\}_{ji} = \gamma_{ji} + \lambda_j L_j^2;$$

$$\underline{E}_1 = (\underline{D}_{21} - \underline{D}_2 \underline{D}_{12}^{-1} \underline{D}_1)^{-1};$$

and

$$\underline{E}_2 = (\underline{D}_{12} - \underline{D}_1 \underline{D}_{21}^{-1} \underline{D}_2)^{-1},$$

$$i,j = 1, 2, \ldots, n.$$

As the basic structure of the two demand functions represented by Equations (4.53) and (4.54) is identical, it will suffice to discuss just one. Consider the demand functions for equity, represented by Equation (4.54). The function is essentially composed of two parts: the demand that arises given a constant investment opportunity set, and the demand that arises from consideration of the effects of stochastic changes in the rate of probability of bankruptcy.

The first part, $H_1^k\underline{E}_2(\mathrm{a} - \underline{D}_1\underline{D}_{21}^{-1}\underline{C})$ can, perhaps, be more easily interpreted if written in a scalar form:

$$H_1^k\{\underline{E}_2(\underline{a}-\underline{D}_1\underline{D}_{21}^{-1}\underline{C})\}_j = H_1^k\Big[\sum_{i=1}^{n} {}_2E_{ji}\Big(\alpha_i - r - \frac{r_1 - r}{L_i}\Big) - \sum_{i=1}^{n} {}_2D_{ji}(r_i - r - \lambda_i L_i)\Big],$$

$j = 1, 2, \ldots, n$

where

$$\{\underline{E}_2\}_{ji} = {}_2E_{ji},$$

and

$$\{\underline{E}_2\underline{D}_1\underline{D}_{21}^{-1}\}_{ji} = {}_2D_{ji},$$

$i, j = 1, 2, \ldots, n.$

The first term in the above expression represents the demand for the j^{th} firm's equity based upon the instantaneous expected rate of return for the i^{th} firm's equity, and the second term represents a substitution term arising from the demand for the i^{th} firm's bond. The presence of such a term is to be expected, for bonds are substitutes for equity.[50] The factor H_1^k is strictly positive and is the usual preference factor reflecting the individual's desire between current and future consumption.

The second part, $\underline{E}_2(D_3 - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_4)\underline{H}_2^k$ can also be written in a scalar form:

$$\{\underline{E}_2(\underline{D}_3 - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_4)\underline{H}_2^k\} = \sum_{i=1}^{n} {}_3E_{ji}\ H_{2j}^k, \tag{4.55}$$

$j = 1, 2, \ldots, n,$

where

$$\{\underline{E}_2(\underline{D}_3 - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_4)\}_{ji} = {}_3E_{ji},$$

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[50] They could also be complements.

and

$$\{\underline{H}_2^k\}_j = H_{2j}^k,$$

$$i,j = 1, 2, \ldots, n.$$

The term can be directly attributed to the effects of the stochastic nature of the changes in the rate of the probability of bankruptcy, and can be interpreted as an attempt to hedge against such changes. It can be shown[51] that

$$H_{2j}^k = -\frac{\frac{\partial c^k}{\partial \lambda_j}}{\frac{\partial c^k}{\partial W^k}} \gtreqless 0$$

$$j = 1, 2, \ldots, n.$$

Thus if $\frac{\partial c^k}{\partial \lambda_j} < 0$ and ${}_3E_{jj} < 0$, then the investor will demand less of the j^{th} firm's equity. The form of expression (4.55) is important for it contains n preference terms of the individual, $\{H_{2j}^k\}$. This implies that if the equilibrium instantaneous conditional expected rates of return for bonds and equity are to be determined free of any preference terms, then the terms $\{H_{2j}^k\}$ and H_1^k must be eliminated.

Given the form of the demand function, it is clear that any attempt to derive the expressions for the equilibrium instantaneous conditional expected rates of return will be difficult due to the fact that not only are there correlation terms resulting from the presence of the bonds and common stock, but also because of the (n+1) preference terms.[52] Whilst it is possible to eliminate these preference terms, the resulting complexity and general lack of insight that results, does not justify the effort.

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[51] See Appendix A, Equation (A.22).

[52] See Appendix A, Equations (A.79) and (A.81).

Some insight can be gained by assuming that the stochastic changes in the rate of probability of bankruptcy for one firm acts as an instrumental variable, characterizing all the changes in the investment opportunity set. For convenience, call this firm the n^{th} firm. It can be shown[53] that the instantaneous conditional expected rate of return for the j^{th} firm's equity is of the form

$$\alpha_j - r - \frac{r_j - r}{L_j} - \varepsilon_j = \left(\frac{\sigma_{nM}\delta_{j\lambda} - \delta_{n\lambda}\sigma_{jH}}{Q}\right)(\mu - r - \bar{X} - \varepsilon_M)$$
$$+ \left(\frac{\sigma_{jM}\delta_{m\lambda} - \delta_{j\lambda}\sigma_M^2}{Q}\right)\left(\alpha_n - r - \frac{r_n - r}{L_n} - \varepsilon_M\right), \tag{4.56}$$
$$j = 1, 2, \ldots, n-1,$$

where

$$M(t) = \sum_{j=1}^{n} \bar{N}_{n+j} P_j(t);$$

$$Y_j(t) = \frac{\bar{N}_{n+j} P_j(t)}{M(t)};$$

$$\mu = \sum_{j=1}^{n} Y_j(t)\alpha_j;$$

$$\bar{X} = \sum_{j=1}^{n} Y_j(t)\left(\frac{r_j - r}{L_j}\right);$$

$$\xi_{ji} = \{\underline{D}_{12} - \underline{D}_1 \underline{D}_{21}^{-1} \underline{D}_2\}_{ji};$$

$$\varepsilon_{ji} = \{\underline{D}_1 \underline{D}_{21}^{-1}\}_{ji};$$

$$\sigma_{jM} = \sum_{i=1}^{n} \xi_{ji} Y_i(t);$$

$$\sigma_M^2 = \sum_{j=1}^{n} Y_j(t)\sigma_{jM};$$

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[53] See Appendix A, Equation (A.86).

$$\varepsilon_j = \sum_{j=1}^{n} \varepsilon_{ji}(r_i - r - \lambda_i L_i);$$

$$\delta_{j\lambda} = \sigma_j \eta_{n,n+j} G_n - \frac{\gamma_j \eta_{n,j\,i}}{L_j} + \sum_{i=1}^{n} \varepsilon_{ji}\gamma_i \eta_{n,i} G_n$$

$$\delta_{MY} = \sum_{j=1}^{n} Y_j(t)\delta_{j\lambda};$$

$$\varepsilon_M = \sum_{j=1}^{n} Y_j(t)\varepsilon_j;$$

and

$$Q = \sigma_{nM}\delta_{m\lambda} - \delta_{n\lambda}\sigma_M^{\;2},$$

$$j = 1, 2, \ldots, n.$$

A similar expression can be obtained for bonds.[54]

The complex nature of Equation (4.56) makes interpretation difficult. The left hand side of the equation can be intuited as the instantaneous expected rate of return. The first term on the right hand side can be interpreted as the instantaneous expected excess return on the market multiplied by a factor specific to the j^{th} firm. The second term arises from the stochastic nature of the changes in the investment opportunity set. Suppose $\sigma_{nM} = 0$, which can be interpreted as meaning that the n^{th} firm is uncorrelated with the market, the Equation (4.56) then simplifies to the form

$$\alpha_j - r - \frac{r_j - r}{L_j} - \varepsilon_j = \beta_j(\alpha - r - \bar{\chi} - \varepsilon_M) + [(\frac{\delta_{j\lambda}}{\delta_{n\lambda}}) - \beta_j(\frac{\delta_{n\lambda}}{\delta_{n\lambda}})]\;(\alpha_n - r - \frac{r_n - r}{L_n} - \varepsilon_n),$$

$$j = 1, 2, \ldots, n, \qquad (4.57)$$

where

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^{\;2}}$$

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[54] See Appendix A, Equation (A.91).

The above equation is in a form that can be more easily compared to the traditional capital asset pricing model given by Equation (4.44) and to the more general form which considers bankruptcy given a constant investment opportunity set, expressed by Equation (4.42). It is clear that the same general remarks that applied to Equation (4.42) apply to Equation (4.57), though perhaps with even greater emphasis: there is a non-linear, non-stationary relationship between α and β. There is, however, one major difference: Equation (4.57) contains an extra variable that arises from the stochastic nature of the changes in the investment opportunity set. The magnitude and sign of this variable will depend upon the particular security. If $(\alpha_n - r - \frac{r_n - r}{L_n} - \varepsilon_n)$ is positive and $\frac{\delta_{j\lambda}}{\delta_{n\lambda}}$ approximately equal to $\frac{\delta_{M\lambda}}{\delta_{n\lambda}}$, then for high beta securities the term will be negative, whilst positive for low beta securities. This offers an explanation of the empirical findings of Black, Jensen and Scholes,[55] who found that high beta stocks consistently earned less than that predicted by the CAPM, whilst the reverse being true for low beta stocks.

Summary

The primary focus of the chapter is to extend the formulation of the CAPM not from the viewpoint of restrictions upon the investor, but by considering the impact of bankruptcy upon the structure of returns for corporate financial assets.

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[55]Black, *et al.*, *op. cit.*

Two models are developed. In the first model it is assumed that the investment opportunity set is only altered by the event of bankruptcy. A simplified form of stochastic differential equations describing the price dynamics of bonds is used so as to abstract from interaction between bonds and common stock. For common stock the instantaneous conditional expected rate of return is a linear function of its systematic risk and a second variable which is associated with the probability of bankruptcy. The properties of the derived expression provide an explanation for recent empirical findings of the deficiency of the CAPM. The model is used as a basis for empirically testing the hypothesis of the thesis. The risk premium for bonds depends upon two variables: the first term is the product of the probability of bankruptcy and the liquidating dividend, and second term is a type of market factor.

The second model examines the effects of stochastic changes in the probability of bankruptcy upon the structure of returns. A general form of stochastic differential equation is used to describe the price dynamics of bonds, so that there is interaction between the two sets of financial assets: common stock and bonds. It is found that the demand functions for common stock contain an extra set of terms, reflecting investor's attempts to hedge against unexpected changes in the probabilities of bankruptcy for the different firms. Due to the complex nature of the demand functions, a simplifying assumption is made by using a single instrumental variable to characterize all the changes. It is found that the instantaneous conditional expected rate of return for common stock contains an extra term reflecting investors' attempts to hedge against unexpected changes.

CHAPTER V

EMPIRICAL RESULTS

In this chapter the empirical results of the thesis are presented. It describes the estimation of a model to determine the probability of a firm going bankrupt and the empirical testing of the hypothesis of the thesis using annual data.

If the probability of bankruptcy for a firm increases, then the expected return, conditional upon no bankruptcy, which risk averse investors require, will increase to compensate for the extra risk. At any point in time the probability of bankruptcy for a firm is a function of its ability to raise funds, either internally or externally, to cover fixed charges. As conditions within the firm and the economy change over time, so will the firm's ability to raise funds, and thus the probability of bankruptcy, which may affect the expected return which investors require on the firm's financial assets. The hypothesis of the thesis is that differences in the probability of bankruptcy across securities and across time are reflected in the residual return after abstracting from the market.

From the theoretical analysis given in Chapter IV, a two variable model describing the expected rate of return on a firm's common stock is derived. The model is of the form

$$\alpha_j - r = \Lambda_j + \beta_j(\alpha_M - r - \bar{\chi}),$$
$$j = 1, 2, \ldots, N,$$

where α_j is the instantaneous conditional expected rate of return on the j^{th}

asset; α_M is the instantaneous conditional expected rate of return on the market portfolio; r is the instantaneous risk free rate of interest; Λ_j is the rate of probability of bankruptcy for the j^{th} asset; $\bar{\chi}$ is a weighted average of the $\{\Lambda_j\}$ and $\beta_j = \sigma_{jM}/\sigma_{MM}$, σ_{jM} being the instantaneous conditional covariance of the j^{th} asset with the market portfolio. A discrete time formulation of the model is approximately given by

$$E(r_j) - r_F = \lambda_j + \beta_j[E(r_M) - r_F - \chi],$$

where $E(r_j)$ is the conditional expected rate of return on the j^{th} asset; $E(r_M)$ is the conditional expected rate of return on the market portfolio; λ_j is the probability of bankruptcy for the j^{th} asset for the period; χ is a weighted average of the $\{\lambda_j\}$; r_F is the risk free rate of interest; and $\beta_j \equiv \text{cov}(r_j,r_M)/\text{var}(r_M)$, $\text{cov}(r_j,r_M)$ being the conditional covariance of the j^{th} asset with the market portfolio. To test empirically the hypothesis an ex-post form of the model is used. This implies a transition from an ex-ante to an ex-post formulation using a market model. Thus any empirical test is a joint examination of the ex-ante formulation and the market model. The ex-post form of the model is

$$R_{jt} = \nu_o + \nu_1\lambda_{jt} + \beta_j(R_{Mt} - \chi_t) + u_{jt},$$

where R_{jt} is the realized excess return for the j^{th} asset during period t; R_{Mt} is the realized excess market return for period t; u_{jt} is a random disturbance term; and ν_o and ν_1 are constants. The hypothesis of the thesis is represented by testing if the coefficient, ν_1, is positive.

To test empirically the model, ex-post excess returns for the firm and market portfolio are required. It is also necessary to know how the probability of the firm going bankrupt over different periods. Once these data requirements are satisfied, the hypothesis can be tested. Evidence is found which supports the hypothesis of the thesis; that is, differences in the probability of bankruptcy across securities and across time are reflected in the residual return after abstracting from the market.

In the first part of the chapter the estimation of the probability of bankruptcy is presented. The data and statistical methodology used in the determination of the coefficients are described, as well as the means used to test the model's predictive ability. Finally, the empirical results are given. In the remaining part of the chapter the hypothesis of the thesis is empirically tested. The form of the regression equation representing the hypothesis and the statistical methodology used to estimate the regression coefficients are described. The results using aggregated portfolio data are given first and then results using individual security data presented.

Estimation of the Probability of Bankruptcy

The probability of bankruptcy for a firm is a function of its ability to raise funds, either internally or externally, to cover fixed charges. A firm that fails to cover these fixed charges is said to be bankrupt. From Chapter III, it is shown that the probability of bankruptcy can be represented by

$$\lambda_t = Pr(FNF_t + MB_t + AS_t < 0),$$

where λ_t is the probability of the firm going bankrupt in year t, given the state of the firm at year t-1; FNF_t is the firm's future cash flow net of all fixed charges at year t; MB_t is the maximum amount the firm can borrow for the year t; and AS_t represents all other sources of funds available to the firm at year t.[1] If the future cash flow net of all fixed charges plus the maximum amount the firm can borrow and all other alternative sources of funds is negative, then the firm is said to be bankrupt.

The variables determining the probability of bankruptcy, as stated in the above expression, are in terms of dollar amounts. As cross sectional data will be used, the variables are not adjusted for differences in the size of firms and so may be dominated by scale effects. Very large scale effects among firms would be expected to lead to inefficient estimation of coefficients. To avoid this, the probability of bankruptcy can be written in the form

$$\lambda_t = \Pr\left(\frac{FNF_t}{A_{t-1}} + \frac{MB_t}{A_{t-1}} + \frac{AS_t}{A_{t-1}} < 0\right),$$

where A_{t-1} is the book value of the firm's assets in the year t-1. Thus the probability at year t-1 of a firm going bankrupt in the year t depends upon its future cash flow net of all fixed charges per unit of assets and the total amount of funds that it could raise per unit of assets.

To estimate empirically the probability of bankruptcy requires that the ex-ante variables be replaced by ex-post surrogates. An estimate of the firm's future cash flow net of all fixed charges is obtained by regressing

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[1] In Chapter III a full discussion of the model for the probability of bankruptcy and statistical methodology is given.

net income against time over a five year period and then using the estimated regression equation to predict the next year's value. The estimate is then divided by the current book value of the firm's total assets. The operational definition of net income used is income after deducting all operating and non-operating income and expenses and minority interest but before preferred and common dividends.

The maximum amount a firm could borrow is estimated by a multiplicative function of the amount of credit rationing and that part of the firm not financed by debt, as measured by the book value of net worth. The function is divided by the asset size of the firm. The final form of the function being

$$\left(\frac{\text{book value of net worth at t-1}}{A_{t-1}}\right) \exp\left(-CR_{t-1}\right),$$

where A_{t-1} is the book value of the firm's total assets at time t-1; and CR_{t-1} is the amount of credit rationing at time t-1. Credit rationing is measured using a proxy variable developed by Jaffee.[2] In its simplest form it is the ratio of the amount of loans granted at the prime rate to the total amount of loans granted. Due to several data problems the data are smoothed and seasonally adjusted to have a mean of zero and a standard deviation of unity. To obtain a possibly more complete measure of credit rationing a principal component technique is used to combine four different series, the results being given in Jaffee.[3] The variable CR_{t-1} is measured

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[2]Jaffee, D. *Credit Rationing and the Commercial Loan Market* (New York: John Wiley & Sons, Inc., 1971).

[3]Jaffee, *Ibid.*, pp. 101-103.

using a linear average of the past four quarters of the first principal component.

It is assumed that all other alternative sources of funds can be measured by a linear function of efficiency, growth, business risk and financial risk. Efficiency is estimated by using earnings before interest and taxes divided by the book value of total assets; growth by a five year linear average growth rate in assets; business risk by the absolute value of the proportional change in sales to the proportional change in GNP; and financial risk by the difference between fixed charges and the firm's future cash flow, the difference being divided by the standard deviation of the firm's future cash flow. An estimate of the firm's future cash flow is obtained by regressing operating income against time over a five year period and then using the estimated regression equation to predict the next year's value. The standard deviation is estimated by using the residual sum of squares from the regression. The operational definition of operating income is net sales less cost of sales and operating expenses before deducting depreciation, amortization, interest, taxes, and dividends. For fixed charges it is all interest expense, the amortization of debt discount or premium and the amortization of expenses(that is, underwriting, brokerage fees, advertising costs, etc.).

The ex-post formulation of the model can be written

$$\lambda_t = \Pr\{\tilde{\varepsilon} < \beta_o + \beta_1\left(\frac{\text{estimated future cash flow net of all fixed charges for time t}}{A_{t-1}}\right)$$

$$+ \beta_2\left(\frac{\text{book value of net worth at t-1}}{A_{t-1}}\right)\exp(-CR_{t-1})$$

$$+ \beta_3 \left(\frac{\text{earnings before interest and taxes at t-1}}{A_{t-1}}\right)$$

$$+ \beta_4 (\text{five year linear growth rate for total assets})$$

$$+ \beta_5 \left(\left|\frac{\text{proportional change in sales}}{\text{proportional change in GNP}}\right|\right)$$

$$+ \beta_6 \left[\left(\frac{\text{fixed charges at t-1 - estimated future cash flow for time t}}{\text{estimated standard deviation of future cash flow}}\right)\right] \qquad (5.1)$$

where λ_t is the probability of the firm going bankrupt in year t, given the state of the firm at year t-1; $\tilde{\varepsilon}$ is a zero mean random variable error term, which is assumed to be of unit variance and uncorrelated between firms; β_o, $\beta_1, \ldots, \beta_6$ are coefficients which are to be estimated; and A_{t-1} is the book value of the firm's total assets at time t-1.

The prime focus is to be able to predict the probability of a firm going bankrupt and not to construct a complete general theory. Consequently, due to the complex interaction of the underlying factors and the difficulty of developing an accurate empirical representation of the determinants of bankruptcy, a second formulation using market values for the appropriate corporate variables is developed. The use of market values circumvents many of the difficulties of constructing proxy variables to measure such quantities as the maximum amount the firm could borrow and the total of all other alternative sources.

For the firm's future cash flow net of all fixed charges, the same proxy variable, as previously defined, is used; that is, realized values of the firm's cash flow net of all fixed charges are regressed against time over a five year period and then the estimated regression equation used to predict the next year's value of the firm's cash flow net of all fixed charges. This value is then divided by the book value of the firm's total assets and the resultant used as the ex-post surrogate.

The proxy variable used to measure the ex-ante maximum amount the firm could borrow per unit of assets for year t is

$$\left(\frac{\text{market value of equity at t-1}}{A_{t-1}}\right)\exp(-CR_{t-1}),$$

where A_{t-1} is the book value of the firm's assets at time t-1; and CR_{t-1} is the amount of credit rationing at time t-1.

The total of all other alternative sources of funds which the firm may utilize depends upon three broad categories: uncommitted reserves, reduction of planned outflows, and the liquidation of assets. A variable which synthesizes these diverse quantities is the market value of equity.

Thus, the second formulation of the model is of the form

$$\lambda_t = \Pr[\tilde{\varepsilon} < \gamma_o + \gamma_1\left(\frac{\text{estimated future cash flow net of all fixed charges for time t}}{A_{t-1}}\right)$$

$$+ \gamma_2\left(\frac{\text{market value of equity at time t-1}}{A_{t-1}}\right)\exp(-CR_{t-1})$$

$$+ \gamma_3\left(\frac{\text{market value of equity at time t-1}}{A_{t-1}}\right)], \qquad (5.2)$$

where A_{t-1} is the book value of the firm's total assets; $\tilde{\varepsilon}$ is a zero mean random variable error term, which is assumed to be of unit variance and uncorrelated between firms; and γ_o, γ_1, γ_2, γ_3 are coefficients which are to be estimated.

Statistical Methodology

The coefficients in Equations (5.1) and (5.2) cannot be estimated by regression as the dependent variable is unobservable. However, the coefficients can be estimated by using maximum likelihood. Consider a random sample of firms at time t and suppose the first n' firms are bankrupt and the remainder n-n' non-bankrupt. The logarithmic likelihood function can then be written

$$\sum_{j=1}^{n'} \log \Pr(B_{tj}|X_{t-1,j}) + \sum_{j=n'+1}^{n} \log[1 - \Pr(B_{tj}|X_{t-1,j})], \tag{5.3}$$

where $\Pr(B_{tj}|X_{t-1,j})$ is the probability of the j^{th} firm going bankrupt in year t, given a set of attributes measured at year t-1. By substituting either (5.1) or (5.2) into (5.3) and differentiating with respect to the coefficients, a set of non-linear equations representing the first order maximization conditions are obtained and can be solved iteratively.

Two estimation procedures are used in the thesis: probit analysis and logit analysis. The essential difference between the two procedures is the explicit form of the probability distributions. For probit analysis a normal probability distribution is assumed, whilst for logit analysis the distribution is logistic. A computer programme developed by Cragg[4] is used to determine the estimates of the coefficients for the two procedures.

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[4]Cragg, J. G. "Programs for Multiple Probit and Logit Analysis and Extensions to Them," mimeographed, University of British Columbia, 1968.

Data

To estimate the parameters of the model, the likelihood function must be constructed by taking a random sample of firms and then classifying the firms as bankrupt or not bankrupt. The procedure of using a random sample and then classifying the firms avoids selection bias. As the average probability of a firm going bankrupt is small,a very large random sample must be taken so as to obtain a representative collection of bankrupt firms. Due to the problem of collecting data for such a large sample an alternative procedure is used. Instead of taking a random sample of firms and then classifying the firms as bankrupt or non-bankrupt, data are collected for a universe of bankrupt firms and then for a universe of non-bankrupt firms.

For the 10 year period 1960-1969 a list of firms that declared bankruptcy under the Bankruptcy Act of 1938 in the U.S.A. is compiled. The four main data sources used in the compilation are: the Security and Exchange Commission Annual Reports, Moody's Industrial Manual, Wall Street Journal Index, and Dunn and Bradstreet. The criteria used to select the sample of bankrupt firms are: (a) at least three years of data be available; (b) the bankruptcy is not caused by fraud; and (c) the firm must have been actually offering a product or service for sale. No corporate shells entered the sample. A list of 34 firms used is given in Appendix B. The corporate data for these firms are obtainable in Moody's Industrial Manual. The average asset size of these firms is $11 million which is quite small, a reflection of the fact that large firms, like those that are traded on the New York Stock Exchange rarely go bankrupt, the obvious exception being Penn Central. If a large firm is in difficulties it is either acquired by another firm or a merger occurs.

To obtain a representative random sample of non-bankrupt firms, requires determining the number of firms to be selected. Ideally, the number chosen should be the same as that obtained by taking a random sample of all firms and then classifying them as bankrupt or not bankrupt. As the size of the sample of bankrupt firms is already known, then the size of the sample of non-bankrupt firms should be such that the number of non-bankrupt to bankrupt firms approximately equals the average probability of not going bankrupt to the average probability of bankruptcy. For the 10 year period the whole universe of bankrupt firms satisfying the selection criteria is used. Thus, to obtain a representative sample of non-bankrupt firms requires taking the whole universe, for the 10 year period, of non-bankrupt firms that satisfy the selection criteria from the same population which the bankrupt firms are selected.

As the average probability of bankruptcy is quite small, this implies that the size of the sample of non-bankrupt firms will be large. The data for the bankrupt firms is manually collected from Moody's Industrial Manual; thus to obtain data for a large sample of non-bankrupt firms using the same source would be prohibitively time consuming. To avoid this severe problem the non-bankrupt firms are sampled from the universe of firms contained on the Compustat File.

A characteristic that describes the population of bankrupt firms is that of asset size: all firms are less than \$200 million in size. This property is used to define a population of firms from which the non-bankrupt firms are selected. The sample of non-bankrupt is obtained by selecting, on a year by year basis for the 10 year period, all firms with asset size less than $200 million. The number of firms selected each year for both samples is shown in Table 5.1. For the year 1960 there is a null set of bankrupt firms, whilst there are 237 firms in the non-bankrupt set. This simply reflects the fact that, given the selection criteria, bankruptcy for the type of firms considered is a rare event. By taking a random sample of firms from the defined population over a 10 year period, there is no guarantee that for a particular year the set of bankrupt firms is not empty.

Predictive Ability

As the probability of bankruptcy cannot be observed, direct tests on the models are not possible. This implies that the magnitude of any bias or measurement error in the estimates cannot be determined. Thus, the main check on how well the models are specified must be their predictive ability. Three methods are used to test the models.

From theoretic considerations the signs of the parameters can be determined and compared to those obtained from empirical estimation. The number of estimated parameters with the correct sign provides insight into the specification of the model and the accuracy of the proxy variables at measuring the ex-ante quantities.

If the model is completely specified so as to measure all the different attributes of the firms in the sample used to estimate the parameters, then it should be able to correctly identify the bankrupt and non-bankrupt firms in the sample. The classification ability provides information about the model's specification and the number of common determinants of bankruptcy.

TABLE 5.1

NUMBER OF BANKRUPT AND NON-BANKRUPT FIRMS IN DATA SAMPLE

YEAR	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	TOTAL
BANKRUPT	0	5	8	3	2	3	4	3	3	3	34
NON-BANKRUPT	237	253	254	265	262	248	224	208	168	150	2303

The generality of the model and its overall independence of the peculiarities of the data sample used to estimate the parameters, can be tested by examining the predictve ability on a set of bankrupt firms not used in the original sample. A list of firms that went bankrupt in 1970, one year after the end of the period used for estimating the coefficients is compiled using the Security and Exchange Commission Annual Reports and a list of bankrupt firms given in an article by Altman.[5] By estimating the probability of bankruptcy over several time periods for firms in the new sample provides a demonstration of the model's predictive ability to discern a firm's path to bankruptcy.

Results

The probability of bankruptcy is estimated using the ex-post formulation given by Equation (5.1). The estimated coefficients are shown in Table 5.2. With th exception of growth and business risk, all the parameters have the correct sign. The variables representing the maximum amount the firm could borrow and efficiency, are the only two that are not statistically significant. The R-squared[6] values for logit analysis and probit analysis are high and, as expected, almost identical.

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[5]Altman, E. "Reply," *Journal of Finance*, Vol. XXVII, No. 3 (June, 1972), pp. 718-721.

[6]For the maximum likelihood analysis, R-squared is defined by

$$\text{R-squared} = \{1-\exp[2(L_w-L_\Omega)/T]\}/\{1-\exp[2(L_w-L_{max})/T\}$$

where L_w is the maximum of the logarithmic likelihood function using only a constant, L_Ω is the maximum using all variables and L_{max} is the maximum possible. T is the total number of observations.

TABLE 5.2

ESTIMATION OF COEFFICIENTS FOR A GENERAL MODEL

DESCRIPTION OF VARIABLES

	FUTURE CASH FLOW NET ALL FIXED CHARGES	MAXIMUM AMOUNT THAT CAN BE BORROWED	EFFICIENCY	GROWTH	BUSINESS RISK	FINANCIAL RISK	CONSTANT	R-SQUARED	LOGARITHM OF MAXIMUM LIKELIHOOD FUNCTION
Notation	β_1	β_2	β_3	β_4	β_5	β_6	β_o		
Expected Sign	-	-	-	-	+	+			
Logit Analysis	-10.313 (- 2.651)**	-0.824 (-1.405)	-6.901 (-1.885)	2.502 (4.126)*	-0.152 (-2.326)**	0.275 (2.435)**	-3.219 (-6.836)*	0.58	-76.092
Probit Analysis	-5.455 (-2.734)**	-0.438 (-1.617)	-3.069 (-1.599)	1.2307 (3.921)*	-0.0772 (-2.308)***	0.116 (2.014)***	-1.683 (-7.809)*	0.59	-75.286

(Figures in brackets are t-statistics

* statistically significant at 0.1%
** statistically significant at 2.0%
*** statistically significant at 5.0%

The ability of the model to classify correctly the bankrupt and non-bankrupt firms in the original data sample is shown in Table 5.3. A bankrupt firm is classified as non-bankrupt if its estimated probability of bankruptcy is less than the average value of the estimated probability of bankruptcy for the whole data sample. A non-bankrupt firm is classified as bankrupt if its estimated probability of bankruptcy is greater than the average value of the estimate probability of bankruptcy for the whole data sample. This criterion is used throughout. For the coefficients estimated using logit analysis, the model correctly identifies over 91 per cent of the non-bankrupt firms and over 94 per cent of the bankrupt firms. For probit analysis, the model correctly identifies over 90 per cent of the non-bankrupt firms and over 94 per cent of the bankrupt firms. The small discrepancies in the two sets of results arise from the differences between the logistic and normal probability distributions. The hypothesis that the model's classification ability is due to a purely random process can be rejected with a probability of over 99 per cent.

The model's ability to predict bankruptcy is shown in Table 5.4. The probability of bankruptcy is estimated over as many periods as available data permit, for a group of eight firms that declared bankruptcy in 1970. For Uniservices Incorporated the model predicts failure four years before the date of bankruptcy and for Bishop Industries three years. Bankruptcy is predicted two years in advance for G. F. Industries, and Roberts Company, and one year for Visual Electronics and Dolly Madison Incorporated. The model totally fails for Century Geophysical Incorporated, giving a probability of zero one year before bankruptcy. The failure of the model can be attri-

TABLE 5.3

CLASSIFICATION OF ORIGINAL DATA SAMPLE BY GENERAL MODEL

LOGIT ANALYSIS

PREDICTED OUTCOME	ACTUAL OUTCOME: BANKRUPT	NON-BANKRUPT	TOTALS	AVERAGE VALUE OF PROBABILITY
BANKRUPT	32	191	223	0.434
NON-BANKRUPT	2	2078	2080	0.009
T O T A L S	34	2269	2303	

Type one error = probability [of a firm bankrupt| non-bankrupt]
= 191/2269
= .084

Type two error = probability [of a firm non-bankrupt| bankrupt]
= 2/34
= 0.059

PROBIT ANALYSIS

PREDICTED OUTCOME	ACTUAL OUTCOME: BANKRUPT	NON-BANKRUPT	TOTALS	AVERAGE VALUE OF PROBABILITY
BANKRUPT	32	203	235	0.413
NON-BANKRUPT	2	2066	2068	0.009
T O T A L S	34	2269	2303	

Type one error = 0.098

Type two error = 0.059

TABLE 5.4

PREDICTIVE ABILITY OF GENERAL MODEL

NAME OF FIRM		1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	LAST DATE OF DATA	DATE OF BANK-RUPTCY
UNISERVICES INC.						0.0005 0.0001	0.151* 0.169*	0.737* 0.707*	0.468* 0.492*			Sept., 1968	1970
ROBERTS COMPANY		0.046 0.063	0.0007 0.0002	0.108 0.135	0.0009 0.0002	0.0014 0.0006	0.0014 0.0007	0.004 0.003	0.052* 0.067*			Nov., 1968	Feb., 1970
CENTURY GEOPHYSICAL INCORPORATED	0.1164 0.1133	0.010 0.008	0.106 0.108	0.147 0.191	0.045 0.058	0.0128 0.015	0.004 0.002	0.002 0.001	0.04 0.057	0.0 0.0		June, 1969	1970
VISUAL ELECTRONICS									0.002 0.0	0.044* 0.063*	0.998* 0.999*	March, 1970	June, 1970
G.F. INDUSTRIES							0.021 0.025	0.008 0.009	0.366* 0.343*	0.832* 0.740*		June, 1969	1970
DOLLY MADISON INC.							0.004 0.0034	0.012 0.015	0.003 0.001	0.03* 0.036*		Sept., 1969	June, 1970
FARRINGTON MANUFACTURING COMPANY	0.373 0.337	0.991 0.986	0.994 0.994	0.235 0.258	0.056 0.067	0.012 0.012	0.005 0.004	0.001 0.001	0.004 0.005	0.343* 0.330*		Dec., 1969	1970
BISHOP INDUSTRIES					0.0 0.0	0.015 0.016	0.003 0.002	0.017* 0.020*	0.982* 0.967*	0.968* 0.951*		Oct., 1969	Oct., 1970

Average value of probability of bankruptcy = 0.014

*indicates probability above average value.

buted to the incorrect signs of the growth and business risk coefficients, and the large variability of the firm's cash flow net of all fixed charges.

Alternative Model

The incorrect sign of two statistically significant coefficients, poor predictive ability, and high correlation among the variables, distracts from the appeal of the model. This precipitated development of a model of the form

$$\lambda_t = \Pr[\tilde{\varepsilon}<\beta_o+\beta_1(\frac{\text{estimated future cash flow net of all fixed charges for time } t}{A_{t-1}})$$

$$+ \beta_2(\frac{\text{book value of net worth at } t-1}{A_{t-1}})\exp(-CR_{t-1})$$

$$+ \beta_3(\frac{\text{fixed charges at } t-1 - \text{estimated future cash flow for time } t}{\text{estimated standard deviation of future cash flows}})], \tag{5.3}$$

where the coefficient β_3 measures the significance of financial risk. The estimated values of the coefficients are shown in Table 5.5. All the coefficients estimated over the 10 year period have the correct sign and are statistically significant. The R-squared values for both logit analysis and probit analysis are high.

The ability of the model to classify the original data is given in Table 5.6. For the coefficients calculated using logit analysis, the model correctly classifies over 89 per cent of the non-bankrupt and 91 per cent of the bankrupt firms. For probit analysis similar results are obtained. The hypothesis that the model's classification ability is due to a purely random process can be rejected with a probability of over 99 per cent.

TABLE 5.5

ESTIMATION OF COEFFICIENTS AND TEST FOR STATIONARITY
ALTERNATIVE MODEL

	METHODOLOGY	TIME	FUTURE CASH FLOW NET OF ALL FIXED CHARGES	MAXIMUM AMOUNT THAT CAN BE BORROWED	FINANCIAL RISK	CONSTANT	R-SQUARED	LOGARITHM OF MAXIMUM LIKELIHOOD FUNCTION ($\log_e L$)	STATIONARITY TEST
NOTATION			β_1	β_2	β_3	β_o			$-2[(\log_e L_1 + \log_e L_2) - \log_e L_3] \sim \chi^2_4$
EXPECTED SIGN			-	-	+				
	LOGIT ANALYSIS	1960-1969	-13.091	-1.084	0.330	-2.769	0.54	-84.490	5.68 Not significant at 10% level
			(- 4.182)*	(-1.973)*	(3.155)*	(-6.767)*			
		1960-1964	- 7.697	-3.372	0.389	-1.365	0.63	-37.227	
			(- 1.562)	(-2.183)	(2.654)	(-1.517)			
		1965-1969	-15.059	-0.026	0.268	-3.239	0.46	-46.422	
			(- 3.399)	(-0.038)	(1.827)	(-6.285)			
	PROBIT ANALYSIS	1960-1969	- 6.639	-0.549	0.1388	-1.470	0.53	-86.072	6.84 Not significant at 10% level
			(- 4.153)*	(-2.219)*	(2.702)*	(-8.031)*			
		1960-1964	- 3.684	-1.716	0.180	-0.774	0.64	-36.236	
			(-1.543)	(-2.483)	(2.3952)	(-1.908)			
		1965-1969	-8.087	-0.017	0.101	-1.729	0.46	-46.305	
			(-3.513)	(-0.05)	(1.486)	(-7.12)			

(figures in brackets are t-statistics)

*statistically significant at 5% or less.

TABLE 5.6

CLASSIFICATION OF ORIGINAL DATA SAMPLE: ALTERNATIVE MODEL

LOGIT ANALYSIS

PREDICTED OUTCOME	ACTUAL OUTCOME: BANKRUPT	NON-BANKRUPT	TOTALS	AVERAGE VALUE OF PROBABILITY
BANKRUPT	31	249	280	0.396
NON-BANKRUPT	3	2020	2023	0.009
TOTALS	34	2269	2303	

Type one error = probability [of a firm bankrupt|non-bankrupt]
= 249/2269
= 0.11

Type two error = probability [of a firm non-bankrupt|bankrupt]
= 3/34
= 0.089

PROBIT ANALYSIS

PREDICTED OUTCOME	ACTUAL OUTCOME: BANKRUPT	NON-BANKRUPT	TOTALS	AVERAGE VALUE OF PROBABILITY
BANKRUPT	31	261	292	0.372
NON-BANKRUPT	3	2008	2011	0.009
TOTALS	34	2269	2303	

Type one error = 0.115

Type two error = 0.089

The predictive ability of the model is demonstrated in Table 5.7. Failure is predicted five years in advance for Bishop Industries and four years for Uniservices Incorporated. For Roberts Company and G. F. Industries bankruptcy is predicted two years in advance and for the remaining firms, except Visual Electronics, a one year prediction is given. This includes Century Geophysical Incorporated for which the general model, as represented by Equation (5.1), failed by giving a probability of zero one year before bankruptcy. For Visual Electronics the model predicts a probability of approximately 90 per cent of it going bankrupt three months before it actually failed.

Due to the model's good classification and predictive ability, and the estimated coefficients having the correct sign, it is used in the second part of the empirical work to estimate the probability of bankruptcy in the testing of the hypothesis of the thesis.

Stationarity

The coefficients of the model are estimated over a 10 year period. As the model is to be used for estimation purposes, it is necessary to examine its stationarity; that is, over different sub-periods is the model still an accurate estimator of the probability of bankruptcy. To test for non-stationarity the data sample is split into two time periods (1960-1964) and (1965-1969). The coefficients of the model are estimated over the sub-periods and the logarithm of the maximum likelihood function determined. An asymptotic test for stationarity is given by[7]

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[7]For proof, see Chapter X of Mood, A. and Grayhill, F. *Introduction to the Theory of Statistics* (New York: McGraw-Hill, 1963).

TABLE 5.7

PREDICTIVE ABILITY OF ALTERNATIVE MODEL

NAME OF FIRM	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	LAST DATE OF DATA	DATE OF BANK-RUPTCY
UNISERVICES INC.						0.0007	0.116*	0.816*	0.637*			Sept.,	
						0.0003	0.134	0.753*	0.549*			1968	1970
ROBERTS COMPANY		0.031	0.002	0.049	0.013	0.005	0.0027	0.0025	0.019*			Nov.,	Feb.,
		0.043	0.001	0.062	0.013	0.003	0.0019	0.0016	0.022*			1968	1970
CENTURY GEOPHYSICAL INC.	0.576	0.168	0.018	0.463	0.052	0.065	0.008	0.004	0.005	0.0291*		June,	
	0.463	0.169	0.019	0.436	0.066	0.083	0.008	0.003	0.004	0.035*		1969	1970
VISUAL ELECTRONICS									0.0001	0.01	0.929*	March,	June,
									0.0	0.01	0.882*	1970	1970
G. F. INDUSTRIES							0.015	0.005	0.095*	0.468*		June,	
							0.018	0.005	0.109*	0.416*		1969	1970
DOLLY MADISON INC.							0.004	0.012	0.008	0.05*		Sept.,	June,
							0.004	0.013	0.008	0.06*		1969	1970
FARRINGTON MANUFACTURING COMPANY	0.774	0.987	0.99	0.464	0.069	0.012	0.012	0.008	0.002	0.244*		Dec.,	
	0.672	0.975	0.99	0.432	0.082	0.013	0.013	0.009	0.001	0.250*		1969	1970
BISHOP INDUSTRIES					0.0003	0.019*	0.012*	0.028*	0.911*	0.956*		Oct.,	Oct.,
					0.0	0.020*	0.012*	0.034*	0.855*	0.925*		1969	1970

Average value of probability of bankruptcy = 0.014

*Indicates probability above average value.

$$-2 \{[\log_e(L_1) + \log_e(L_2)] - \log_e(L)\} \sim \chi^2_k,$$

where L is the maximum likelihood function for the whole period; L_1 and L_2 are the maximum likelihood functions for the two sub-periods; and χ^2_k is a chi-square distributed random variable on k degrees of function, k being the number of coefficients to be estimated in the model.

The results for the stationarity test are shown in Table 5.5 While there is some variability in the coefficients, they all have the correct sign and are, in general, statistically significant. For logit analysis and probit analysis, there is no evidence of non-stationarity at the 10 per cent level.

Predictive Model

As the primary focus is the prediction and estimation of the probabilityof bankruptcy, a second formulation of the model using market values of the appropriate corporate variables is developed. This is represented by Equation (5.2).

The estimation of the coefficients and a test for stationarity are given in Table 5.8. For logit analysis all the coefficients have the correct sign, future cash flow net of all fixed charges and the constant being statistically significant. In probit analysis, the sign of the variable representing alternative sources of funds is incorrect, though the coefficient is statistically insignificant, as is the coefficient for the maximum amount that the firm could borrow. The R-squared is high for both logit analysis and probit analysis. In the test for stationarity there is fluctuation in the signs of the coefficients representing the maximum amount that the firm could borrow and alter-

TABLE 5.8

ESTIMATION OF COEFFICIENTS AND TEST FOR STATIONARITY: PREDICTIVE MODEL

METHODOLOGY	TIME	FUTURE CASH FLOW NET OF ALL FIXED CHARGES	MAXIMUM AMOUNT THAT CAN BE BORROWED	ALTERNATIVE SOURCES	CONSTANT	R-SQUARED	LOGARITHM OF MAXIMUM LIKELIHOOD FUNCTION	STATIONARITY TEST
NOTATION		Y_1	Y_2	Y_3	Y_0			
EXPECTED SIGN		-	-	-				
LOGIT ANALYSIS	1960-1969	-23.006	-0.270	-0.163	-3.848	0.48	-95.31	3.28
		(- 9.411)*	(-0.638)	(-0.293)	(-9.942)*			Not significant
	1960-1964	-23.425	-1.169	1.286	-4.120	0.54	-45.727	at 10% level
		(- 6.156)	(-1.068)	(1.139)	(-7.108)			
	1965-1969	-23.099	0.176	-0.979	-3.403	0.44	-47.95	
		(- 6.912)	(0.326)	(-1.219)	(-6.591)			
PROBIT ANALYSIS	1960-1969	-11.025	-0.121	+0.060	-2.044	0.49	-94.223	3.27
		(- 9.871)*	(-0.656)	(0.271)	(-12.599)*			Not significant
	1960-1964	-11.033	-0.481	0.589	- 2.133	0.54	-44.94	at 10% level
		(- 6.856)	(-1.042)	(1.282)	(- 9.39)			
	1965-1969	-11.295	0.079	-0.299	-1.845	0.44	-47.65	
		(- 6.864)	(0.313)	(-0.897)	(-8.16)			

(figures in brackets are t-statistics)

*statistically significant at the 0.1% level.

native sources of funds, though these coefficients are not statistically significant. There is no evidence of non-stationarity at the 10 per cent level.

The ability of the model to correctly classify the original data is shown in Table 5.9. For the coefficients estimated by logit analysis, the model correctly classifies over 91 per cent of the non-bankrupt and 85 per cent of the bankrupt firms. For probit analysis, over 90 per cent of the non-bankrupt and 88 per cent of the bankrupt firms are correctly identified. The hypothesis that the model's classification ability is due to a purely random process can be rejected with a probability of over 99 per cent.

The prediction ability of the model is demonstrated in Table 5.10. For Uniservices Incoproated bankruptcy is predicted four years in advance and for G. F. Industries and Bishop Industries a two year prediction is given. Bankruptcy is predicted one year in advance for Century Geophysical Incorporated, Dolly Madison Incorporated, and Farrington Manufacturing Company. For Visual Electronics the model predicts a 98 per cent chance of it failing three months before it went bankrupt. The probability of bankruptcy for Roberts Company does not exceed the average value, though for the last year for which data are available the probability of bankruptcy does increase. Comparing Table 5.10 to Table 5.7, it is seen that the predictive ability of the model does not exceed that of the alternative formulation using book values for the appropriate corporate variables. This is a reflection of the poor quality of the available market data.

TABLE 5.9

CLASSIFICATION OF ORIGINAL DATA SAMPLE: PREDICTIVE MODEL

LOGIT ANALYSIS

ACTUAL OUTCOME

PREDICTED OUTCOME	BANKRUPT	NON-BANKRUPT	TOTALS	AVERAGE VALUE OF PROBABILITY
BANKRUPT	29	201	230	0.391
NON-BANKRUPT	5	2068	2073	0.009
TOTALS	34	2269	2303	

Type one error = probability [of a firm bankrupt | non-bankrupt]
= 201/2269
= 0.089

Type two error = probability [of a firm non-bankrupt | bankrupt]
= 5/34
= .147

PROBIT ANALYSIS

ACTUAL OUTCOME

PREDICTED OUTCOME	BANKRUPT	NON-BANKRUPT	TOTALS	AVERAGE VALUE OF PROBABILITY
BANKRUPT	30	225	255	0.372
NON-BANKRUPT	4	2044	2267	0.009
TOTALS	34	2269	2303	

Type one error = 0.099

Type two error = 0.117

TABLE 5.10

PREDICTIVE ABILITY OF MODEL

NAME OF FIRM	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	1970	LAST DATE OF DATA	DATE OF BANK-RUPTCY
UNISERVIES INC.						0.001 0.001	0.067* 0.080*	0.973* 0.942*	0.906* 0.847*			Sept., 1968	1970
ROBERTS COMPANY		0.056 0.065	0.010 0.008	0.078 0.087	0.009 0.008	0.003 0.001	0.002 0.001	0.001 0.001	0.008 0.008			Nov., 1968	Feb., 1970
CENTURY GEOPHYSICAL INCORPORATED	0.152 0.179	0.041 0.077	0.008 0.013	0.927 0.867	0.112 0.119	0.071 0.089	0.001 0.001	0.0007 0.0004	0.003 0.003	0.012 0.014*		June, 1969	1970
VISUAL ELECTRONICS									0.001 0.001	0.006 0.007	0.987* 0.974*	March, 1970	June, 1970
G. F. INDUSTRIES							0.011 0.012	0.002 0.002	0.033* 0.072*	0.263* 0.33*		June, 1969	1970
DOLLY MADISON INCORPORATED							0.004 0.003	0.009 0.009	0.005 0.005	0.028* 0.033*		Sept., 1969	June, 1970
FARRINGTON MANUFACTURING COMPANY	0.398 0.591	0.982 0.983	0.993 0.992	0.44 0.484	0.012 0.024	0.004 0.004	0.006 0.009	0.004 0.010	0.004 0.013	0.09* 0.189*		Dec., 1969	1970
BISHOP INDUSTRIES					0.0004 0.0	0.006 0.006	0.005 0.004	0.012 0.014	0.974* 0.963*	0.994* 0.993		Oct., 1969	Oct., 1970

Average value of probability of bankruptcy = 0.014

* indicates that the probability is above the average value

Summary

To estimate the probability of a firm going bankrupt over a given period two models, one using book values and the other market values of the appropriate corporate variables, have been constructed. The models have been tested for non-stationarity and predictive ability. It is found that there is no evidence of non-stationarity. Both models have demonstrated good classification and predictive ability, being able to forecast bankruptcy, for some firms, four or five years before the actual occurrence.

Testing of Hypothesis

In Chapter IV it is shown that when the investment opportunity set is changed only by the event of bankruptcy, then in equilibrium the instantaneous conditional expected rate of return, conditional upon no bankruptcy, on common stock is represented by the expression

$$\alpha_j - r = \Lambda_j + \beta_j(\alpha_M - r - \bar{\chi}), \tag{5.4}$$

where α_j is the instantaneous conditional expected rate of return on the j^{th} asset; α_M is the instantaneous conditional expected rate of return on the market portfolio; r is the instantaneous risk free rate of interest; Λ_j is the rate of probability of bankruptcy for the j^{th} asset; $\bar{\chi}$ is a weighted average of the $\{\Lambda_j\}$ and $\beta_j \equiv \sigma_{jM}/\sigma_{MM}$, σ_{jM} being the instantaneous conditional covariance of the j^{th} asset with the market portfolio. As data are only available for discrete time intervals, a model formulated in continuous time cannot be tested directly. To formulate a discrete time analogy of the model involves integrating the conditional price distributions and using the equilibrium expression (5.4). A discrete time form of the model is approximately given by

$$E(r_j) - r_F = \lambda_j + \beta_j[E(r_M) - r_F - \chi], \tag{5.5}$$

where $E(r_j)$ is the conditional expected rate of return on the j^{th} asset; $E(r_M)$ is the conditional expected rate of return on the market portfolio; λ_j is the probability of the j^{th} firm going bankrupt during the period; χ is a weighted average of the $\{\lambda_j\}$; r_F is the risk free rate of interest; and $\beta_j \equiv \text{cov}(r_j,r_M)$, $\text{cov}(r_j,r_M)$ being the conditional covariance of the j^{th} asset with the market portfolio.

To test empirically the hypothesis an ex-post form of the model is used. This implies a transition from an ex-ante to an ex-post formulation using a market model. Thus any empirical test is a joint examination of the ex-ante formulation and the market model. The ex-post form of the model is

$$R_{jt} = \nu_o + \nu_1\lambda_{jt} + \beta_j(R_{Mt} - \chi_t) + u_{jt}, \tag{5.6}$$

where R_{jt} is the realized excess return for the j^{th} asset during period t; R_{Mt} is the realized excess market return for period t; ν_o and ν_1 are constants; and u_{jt} is a zero mean random disturbance term. It is assumed that R_M and u_j are normally distributed random variables and uncorrelated.

The hypothesis of the thesis is that differences in the probability of bankrupty across securities and across time are reflected in the residual return after abstracting from the market. Thus an empirical test of the hypothesis is represented by estimating the coefficients of Equation (5.6). If the coefficient, ν_1, of the probability of bankruptcy is positive, then this offers confirmation of the validity of the hypothesis.

Methodology

The constant term, ν_o, and the coefficient, ν_1, of the probability of bankruptcy are not firm dependent, in contrast to the beta coefficient, β_j, which is firm specific. This structural property of Equation (5.6) is utilized by the methodology employed to test the hypothesis.

Pooling of Time Series and Cross Sectional Data

The methodology used is that of pooling the time series and cross sectional data.[8] Time series data for individual securities are pooled together to estimate the two common coefficients, whilst simultaneously estimating the firm specific beta coefficients. Thus, for two securities the regression equation is of the form

$$\begin{bmatrix} R_{11} \\ \vdots \\ R_{1T} \\ R_{21} \\ \vdots \\ R_{2T} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_{11} & R_{M1} - \chi_1 & 0 \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_{1T} & R_{MT} - \chi_T & 0 \\ 1 & \lambda_{21} & 0 & R_{M1} - \chi_1 \\ \vdots & \vdots & \vdots & \\ 1 & \lambda_{2T} & 0 & R_{MT} - \chi_T \end{bmatrix} \begin{bmatrix} \nu_o \\ \nu_1 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} U_{11} \\ \vdots \\ U_{1T} \\ U_{21} \\ \vdots \\ U_{2T} \end{bmatrix}$$

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[8] A discussion of this methodology is given in Kuh, E. "The Validity of Cross-Sectionally Estimated Behaviour Equations in Time Series Applications," *Econometrica,* Vol. 27 (April, 1959), pp. 197-214; and Balestra, P. and Nerlove, M., "Pooling Cross Section and Time Series Data in the Estimation of a Dynamic Model: The Demand for Natural Gas," *Econometrica,* Vol. 34 (July, 1966), pp. 585-612.

where the suffix T denotes the number of time periods. The general N security case can be written in the matrix form

$$\underline{Y} = \underline{X}\,\underline{\Theta} + \underline{u}, \tag{5.7}$$

where $\underline{Y}$ is a (NxT) vector; $\underline{X}$ a [(NxT) x (N+2)] matrix; $\underline{\Theta}$ a (N+2) vector of coefficients; and $\underline{u}$ a (NxT) vector of random disturbance terms, N being the number of securities. It is assumed that

$$E(\tilde{u}_{jt}) = 0, \tag{5.8}$$

and

$$\operatorname{cov}(\tilde{u}_{jt}, \tilde{u}_{ks}) = \begin{cases} \sigma; & j = k,\ t = s, \\ 0; & \text{otherwise,} \end{cases} \tag{5.9}$$

$j, k = 1, 2, \ldots, N,$

$t, s = 1, 2, \ldots, T.$

The validity of the last assumption is tenuous. It implies zero correlation across time and between securities. If it is not satisfied, then it implies that the estimated coefficients whilst being unbiased and consistent, will not be minimum variance or, in general, asymptotically efficient. The coefficients, $\underline{\Theta}$, are estimated by ordinary least squares. For large data samples, the special structure of the $\underline{X}$ matrix can be utilized to reduce computational difficulties.

Aggregation

In order to be able to estimate the coefficients of Equation (5.6), it is necessary to know the probability of bankruptcy for every security over all time periods. As the probability of bankruptcy cannot be directly observed, it is estimated using the models developed in the first part of this chapter. This implies that there will be errors in the measurement of the variable, which will cause the estimations of the coefficients of Equation (5.6) to be biased. The effects of these measurement errors can be reduced by aggregation; that is, the relationship (5.6) can be aggregated over certain subsets of the data and the mean values of the variables used. As the coefficients ν_o and ν_1 are not firm specific, the same coefficients will still be appropriate for the aggregated relationship.

To increase the magnitude in the changes in the probability of bankruptcy over time, securities are assigned into portfolios on the basis of the value of bankruptcy for the previous year. This minimizes the within variance, whilst maximizing the between portfolio variance for the probability of bankruptcy. The number of portfolios is varied, enabling examination of possible aggregation effects. The first portfolio contains the securities which had the lowest probabilities, whilst the last portfolio contains the securities which had the largest probabilities. This assignment process is repeated on a year by year basis, so that, in general, the composition of each portfolio changes annually. The use of the previous year's probability value as a criterion for assignment avoids selection bias in the construction of the portfolios.

The average value of the probability of bankruptcy, and the average rate of return for securities in each portfolio are calculated on a yearly basis over the whole time period; that is, the average rate of return for the portfolio is defined to be

$$R_{pt} = \frac{1}{N_p} \sum_{j \varepsilon S_p} R_{jt}, \tag{5.10}$$

where S_p is the set of securities contained in the portfolio, and N_p the number of securities. The average probability of bankruptcy is defined to be

$$\lambda_{Pt} = \frac{1}{N_p} \sum_{j\varepsilon S_p} \lambda_{jt}. \tag{5.11}$$

Substituting Equation (5.6) into (5.10) gives

$$R_{pt} = \nu_o + \frac{1}{N_p} \sum_{j\varepsilon S_p} \nu_1 \lambda_{jt} + \frac{1}{N_p} \sum_{j\varepsilon S_p} \beta_j (R_{Mt} - \chi_t) + \frac{1}{N_p} \sum_{j\varepsilon S_p} u_{jt}. \tag{5.12}$$

As the coefficient ν_1 is not firm specific, then using Equation (5.11), Equation (5.12) becomes

$$R_{pt} = \nu_o + \nu_1 \lambda_{pt} + \beta_p (R_{Mt} - \chi_t) + u_{pt}, \tag{5.13}$$

where

$$\beta_p (R_{MT} - \chi_t) = \frac{1}{N_p} \sum_{j\varepsilon S_p} \beta_j (R_{Mt} - \chi_t),$$

and

$$u_{pt} = \frac{1}{N_p} \sum_{j\varepsilon S_p} u_{jt}.$$

that is, the coefficients ν_o and ν_1 are still appropriate for the aggregate relation, and are not weighted averages. The coefficients of (5.13) are estimated by pooling the time series and cross section aggregated data and then using ordinary least squares regression.

Data

To test empirically the model corporate data are required to estimate the probability of bankruptcy and price data for the rates of return. The data set consists of all firms common to the Compustat File and the University of Chicago Center for Research in Security Prices Monthly Price Relative File (CRSP), which contains monthly price, dividend and adjusted price and dividend information for all securities listed on the New York Stock Exchange (NYSE) in the period January, 1926 - June, 1970. A firm is included in the data set if it has continuous price data for the years 1959 to 1969, and continuous corporate data for the years 1955 to 1969. Using this criterion a total of 360 firms with 10 years of annual data are contained in the data set. The monthly returns on the market portfolio are defined as the returns which would have been earned on a portfolio consisting of an equal investment in every security listed on the NYSE at the beginning of each month. The risk free interest rate is defined as the 30 day rate on United States Treasury Bills.

For each security the annual excess rate of return and the annual excess market rate of return are calculated from the start of the firm's fiscal year for the 10 year period 1960-1969. The probability of bankruptcy is estimated using the two formulations represented by the expressions (5.2) and (5.3). The first expression utilizes market values of corporate variables, whilst the second uses book value. The values of the estimated coefficients are shown in Table 5.11. The coefficients estimated using probit analysis for the first model utilizing market values of corporate variables, are not used, as one of the coefficients had the wrong sign.

TABLE 5.11

VALUES OF COEFFICIENTS USED TO ESTIMATE THE PROBABILITY OF BANKRUPTCY

METHODOLOGY	VARIABLES			
MARKET VALUES OF CORPORATE VARIABLES	γ_o	γ_1	γ_2	γ_3
Logit	-3.848	-23.006	-0.270	-0.163
BOOK VALUES OF CORPORATE VARIABLES	β_o	β_1	β_2	β_3
Logit	-2.769	-13.091	-1.084	0.330
Probit	-1.470	- 6.639	-0.549	0.139

The use of the two formulations results in a noticeable difference in the estimated values of the probability of bankruptcy, on average the estimates from the model using market values of corporate variables [expression (5.2)] are greater and have more variability than those obtained from the alternative formulation using book values of corporate variables [expression (5.3)]. This difference in variability is reflected in the estimation of the regression coefficients in Equation (5.6) when testing the hypothesis. The average yearly values of the probability of bankruptcy are given in Table 5.12, which clearly demonstrates the difference in magnitude and variability.

Empirical Results

The validity of the hypothesis is tested by examining the estimated coefficients of Equation (5.6); if the coefficient, ν_1, is positive then this offers confirmation. The results are set out in two parts. The hypothesis is tested using aggregated security data, and then individual security data.

Use of Portfolios

The individual security data is aggregated into portfolios on the basis of the probability of bankruptcy for the past year, as previously explained. Using 360 securities, portfolios containing 12, 36 and 72 securities over a nine year period are constructed. For the probability of bankruptcy estimated by the market value formation of Equation (5.2), the results are shown in Table 5.13. The coefficient, ν_1, for the probability of bank-

TABLE 5.12

AVERAGE YEARLY VALUES OF THE PROBABILITY OF BANKRUPTCY

YEAR	MARKET VALUE FORMULATION	BOOK VALUE FORMULATION	
	LOGIT	LOGIT	PROBIT
1960	0.00589	0.00279	.00260
1961	0.00909	0.00384	.00374
1962	0.01034	0.00337	.00316
1963	0.00563	0.00293	.00273
1964	0.00276	0.00036	.00026
1965	0.00330	0.00039	.00033
1966	0.00443	0.00048	.00041
1967	0.00430	0.00071	.00064
1968	0.00673	0.00304	.00295
1969	0.00781	0.00276	.00274

TABLE 5.13

PORTFOLIO DATA: POOLING OF TIME SERIES AND CROSS SECTION

PROBABILITY OF BANKRUPTCY ESTIMATED
USING MARKET VALUES OF CORPORATE VARIABLES

NUMBER OF PORTFOLIOS	NUMBER OF SECURITIES IN EACH PORTFOLIO	NUMBER OF OBSERVATIONS	COEFFICIENT ν_o	COEFFICIENT ν_1	R-SQUARED
5	72	45	0.059 (44.141)*	4.601 (39.082)*	0.755
10	36	90	0.093 (16.393)*	2.593 (6.297)*	0.678
30	12	270	0.115 (17.695)*	1.175 (3.162)**	0.575

(Figures in brackets are t-statistics)

* statistically significant at the 0.1 per cent level

** statistically significant at the 0.2 per cent level ruptcy is positive, statistically significant and increases in magnitude as the level of aggregation increases. The constant term, ν_o, is positive and statistically significant. As expected, it decreases in magnitude when the level of aggregation increases. The value of R-squared increases as aggregation reduces the cross sectional nature of the data.

The probability of bankruptcy can be estimated by Equation (5.3) using book values for corporate variables. As the results are very similar for the logit and probit estimates, only the former are reported (see Table 5.14). The coefficient, ν_1, for the probability of bankruptcy is positive for two of the portfolios. In all cases it is not statistically significant. The constant term, ν_o, is positive, statistically significant, and decreases in magnitude as the level of aggregation increases. It is consistently larger than the constant term in Table 5.13. The value of R-squared increases as the level of aggregation increases, though it is uniformly lower than the values of R-squared given in Table 5.13.

The differences in Tables 5.13 and 5.14 result directly from the method of estimating the probability of bankruptcy. As shown in Table 5.12, the values obtained from the formulation given by Equation (5.2), which utilizes market values of corporate variables, are larger and have greater variability than those given by Equation (5.3) using book values. For many firms the book value formulation gives a zero value for the probability of bankruptcy, which causes severe econometric problems because of ill-conditioned matrices. Whilst aggregation mitigates this problem it does, however, reduce the overall variance. This will imply that to obtain reliable estimates of the coefficient, ν_1, will require large data samples.

TABLE 5.14

PORTFOLIO DATA: POOLING OF TIME SERIES AND CROSS SECTION

PROBABILITY OF BANKRUPTCY ESTIMATED
USING BOOK VALUES OF CORPORATE VARIABLES

NUMBER OF PORTFOLIOS	NUMBER OF SECURITIES IN EACH PORTFOLIO	NUMBER OF OBSERVATIONS	COEFFICIENT ν_o	COEFFICIENT ν_1	R-SQUARED
5	72	45	0.089 (13.039)*	0.924 (1.65)	0.732
10	36	90	0.114 (16.20)*	-0.266 (0.587)	0.651
30	12	270	0.121 (18.59)*	0.419 (1.342)	0.558

(figures in brackets are t-statistics)

*statistically significant at the 0.1 per cent level

The coefficient, ν_1, for the probability of bankruptcy is, in general positive and statistically significant. This provides some evidence of the validity of the hypothesis; that is, bankruptcy is a contributory factor to the structure of common stock returns.

If there is a missing variable which is not explained by either the probability of bankruptcy or the covariance with the market, then this will bias the estimates of the coefficients. If the data are aggregated, the pooling of time series and cross sectional data becomes more like an aggregated time series, and thus the variance of the missing variable will increase the t-statistics of the coefficients as well as causing bias. The situation will be exacerbated if the missing variable is non-stationary.

The constant term in a regression equation picks up individual effects not accounted for by the exogenous variables. The pooling of time series and cross-section data constrains the constant term to be the same for all securities (portfolios), and thus if there are individual security effects present, this will cause the estimated coefficients to be biased.

To examine these possibilities dummy variables could be introduced to represent security (portfolio) specific effects not explicitly accounted for by the exogenous variables. However, dummy variables will be highly correlated with the probability of bankruptcy. This arises because the within portfolio variance across time for the probability of bankruptcy will be minimized and thus the dummy variable will act as a proxy for it. The resulting multicollinearity will cause a loss of precision of estimation, as specific estimates may have very large sampling errors. Exploratory empirical work confirmed this and consequently this line of investigation was not pursued.

The limited size of the data set -- 360 securities with 10 years of annual data -- restricts the scope of methods to test the hypothesis using aggregated data. By forming portfolios on the basis of the past value of the probability of bankruptcy reduces the time span of data available by one year, and aggregation across securities decreases the size of the data set even further. Due to these limitations, further tests on the hypothesis are conducted using individual security data.

Individual Security Data

The use of individual security data facilitates the advantage of a large number of observations. Off-setting this is the disadvantage of the low discriminatory power of the exogenous variables due to errors in measurement. This is especially important when consideration is taken of the large errors in measurement when estimating the probability of bankruptcy. This implies that all the estimated regression coefficients will be biased. Thus the results using unaggregated data will not be expected to offer the same reliability as those obtained using aggregated portfolio data.

Five tests are conducted using unaggregated data. The first test is to draw a random sample of firms and to perform a regression pooling time series and cross sectional data. Three of the remaining tests examine various problems which may cause bias in the estimation of the regression coefficients. The final test considers the effect of changes in the probability of bankruptcy upon the structure of ex-post returns.

Random Sample

From the universe of 360 securities, a random sample of 100 is chosen,

TABLE 5.15

RANDOM SAMPLE: POOLING OF TIME SERIES AND CROSS SECTION DATA

DATA SET FOR PROBABILITY	METHODOLOGY FOR PROBABILITY	COEFFICIENT ν_o	COEFFICIENT ν_1	R-SQUARED	NUMBER OF OBSERVATIONS
BOOK VALUES	LOGIT	0.115 (10.541)*	-0.820 (-1.366)	0.303	1,000
	PROBIT	0.114 (10.504)*	-0.661 (-1.147)	0.303	1,000
MARKET VALUES	LOGIT	0.116 (10.142)*	-0.911 (-1.151)	0.304	1,000

(Figures in brackets are t-statistics)

*statistically significant at the 0.1 per cent level giving a total of 1,000 observations. The probability of bankruptcy is estimated using the market value formulation of Equation (5.2) and the book value formulation of Equation (5.3). The results of pooling time series and cross sectional data are shown in Table 5.15. The results are similar for the different methods of estimating the probability of bankruptcy. The coefficient, ν_1, for the probability of bankruptcy is negative, though statistically insignificant. The constant term, ν_o, is positive and statistically significant. The value of R-squared is quite large considering the strong cross sectional nature of the regression data.

The negativity of the coefficient, ν_1, and the lack of difference in the results when the probability of bankruptcy is estimated using market values of corporate variables or book values, reflects the effects of errors in the measurement of the variable and stands in contrast to the results observed in Tables (5.13) and (5.14). Aggregation of the data set into portfolios reduces the variance of the estimate of the probability of bankruptcy and results in more reliable estimation of the regression coefficients.

Effect of Asset Size

The coefficients used to estimate the probability of bankruptcy are determined using a data set of firms characterized by having an asset size of less than \$200 million. To test the hypothesis the asset size of the firms in the data set is not restricted. If there are scale effects in the coefficients used to estimate the probability of bankruptcy, then errors in its measurement will increase when applied to large firms. To examine this possibility, the universe of 360 firms is sorted by asset size into three groups and a pooling of time series and cross section data for the firms

TABLE 5.16

POOLING OF TIME SERIES AND CROSS DATA
ON GROUPS OF FIRMS SORTED BY ASSET SIZE

DATA SET FOR PROBABILITY	ASSET SIZE	COEFFICIENT ν_o	COEFFICIENT ν_1	R-SQUARED	NUMBER OF OBSERVATIONS
BOOK VALUES	\$(0 - 200)m.	0.117 (11.841)*	-0.998 (-2.712)**	0.358	1,270
	\$(200-500)m.	0.119 (9.654)*	-0.216 (-1.116)	0.287	1,010
	\$500m. and over	0.084 (10.691)*	-0.300 (-0.302)	0.281	1,320
MARKET VALUES	\$(0 - 200)m.	0.118 (11.645)*	-1.124 (-2.425)**	0.359	1,270
	\$(200-500)m.	0.119 (9.531)*	-0.147 (-0.621)	0.287	1,010
	\$500m. and over	0.076 (8.676)*	1.936 (1.767)	0.285	1,320

(Figures in brackets are t-statistics)

* statistically significant at the 0.1 per cent level.

** statistically significant at the 2.0 per cent level.

within each group conducted. As the results for the estimates of the probability of bankruptcy using the book value formulation of Equation (5.3) when its coefficients are estimated by either logit analysis or probit analysis are very similar, only the former are reported (see Table 5.16).

There does appear to be some evidence that the regression estimates are affected by asset size. The constant term, ν_o, for large firms is reduced in magnitude when compared to the value for small and intermediate size firms. The coefficient, ν_1, for the probability of bankruptcy, is negative and statistically significant for firms of asset size of less than $200 million.

<u>Adjustment of Time Period</u>

The probability of bankruptcy is estimated using corporate data that reflects the state of the firm at the beginning of its fiscal year. The annual rates of return data for a firm are calculated over the 12 month period starting at the beginning of the firm's fiscal year, and then the two data sets combined and used to estimate the regression coefficients. However, a firm's annual report is usually announced two or three months after the end of the fiscal year, and thus as the probability of bankruptcy is estimated using corporate data which, to the average investor, will only be available from a firm's annual report, it may not be used to refer to the 12 month period beginning at the start of the firm's fiscal year. Hence, the calculated annual rate of return and the estimate of the probability of bankruptcy, which are used as proxies for the values an investor ex-

TABLE 5.17

ADJUSTMENT OF TIME PERIOD

NUMBER OF MONTHS AFTER BEGINNING OF FISCAL YEAR	COEFFICIENT ν_o	COEFFICIENT ν_1	R-SQUARED	NUMBER OF OBSERVATIONS
0	0.117 (11.841)*	-0.998 (-2.712)**	0.358	1,270
2	0.104 (10.541)*	-0.800 (-1.504)	0.271	1,260
3	0.105 (I0.559)*	-0.873 (-1.631)	0.273	1,260
4	0.089 (9.401)*	-0.520 (-0.990)	0.302	1,260

(Figures in brackets are t-statistics)

*statistically significant at the 0.1 per cent level.

**statistically significant at the 2.0 per cent level.

pects, may not be synchronized. This possibility is examined by calculating the annual rate of return from a point two, three, and four months after the beginning of a firm's fiscal year.

The regression coefficients are estimated using a data set of firms with asset size less than $200 million. This choice of data set is motivated in view of the finding given in Table 5.16, which indicated some dependence of regression coefficients upon asset size. The size of the data set is reduced by one firm because of the extra price data requirements. As the findings are very similar for different methods of estimating the probability of bankruptcy, only those obtained from the formulation utilizing market values of corporate variables are reported. The results are shown in Table 5.17.

In Table 5.17 there does appear to be some difference between the results. The coefficient, ν_1, for the probability of bankruptcy, whilst negative, increases in value and becomes statistically insignificant when the rate of return data are shifted forward two months. The constant term, ν_o, is reduced in magnitude though remains statistically significant. The value of R squared is reduced. Further shifts in the rate of return data set produce only minor differences.

Cross Section Studies

In a regression with two independent variables if one is measured with error, then its estimated coefficient will be biased downwards, whilst the constant term will be biased upwards. If both variables are measured with error no comment can be made about the direction of the resulting biases. However, if an instrumental variable is substituted for one of the variables and the measurement errors in the instrumental variable are independent of the measurement errors in the remaining variable, then the coefficient of the remaining variable will be biased downwards. It is not possible to make any comment about the direction of bias in the constant term.

To explore the possible consequences of measurement errors in the variables, a series of cross section studies are conducted. The firm's beta coefficient is replaced by an instrumental variable. For individual securities an instrumental beta coefficient is estimated by regressing the ex-post, excess return against the realized return on the market using the past five years of monthly price data prior to the period of the cross sectional study. These estimates are substituted into the regression equations

$$R_j = \nu_o + \nu_1\lambda_j + \nu_2\hat{\beta}_j + u_j \qquad (5.14)$$
$$j = 1, 2, \ldots, N,$$

where R_j is the annual excess return for the j^{th} asset; λ_j is the probability of the j^{th} asset going bankrupt over a period of one year; $\hat{\beta}_j$ is the instrumental beta coefficient; u_j is a random disturbance term; and ν_o, ν_1 and ν_2 are regression coefficients. If the measurement errors in the beta coefficients are independent of the measurement errors in the probability of bankruptcy then the estimate of coefficient ν_1 will only be subject to a bias which underestimates the true value. The extra price data requirements reduce the data set to 318 firms. The probability of bankruptcy is estimated using Equation (5.2), which utilizes market values of corporate variables. Using annual data five cross section regressions for different periods of time are conducted. The results are given in Table 5.18.

TABLE 5.18

CROSS SECTION STUDY

TIME	COEFFICIENT ν_o	COEFFICIENT ν_1	COEFFICIENT[a] ν_2	ANNUAL EXCESS RETURN ON THE MARKET	R-SQUARED
1959-60	0.179 (4.417)*	-3.579 (-1.630)	-0.210 (-4.081)*	-0.061	0.111
1961-62	-0.016 (-0.419)	-0.139 (-0.555)	-0.110 (1.302)	-0.156	0.031
1963-64	0.155 (3.422)*	-0.750 (-0.851)	0.001 (-2.481)**	0.125	0.002
1965-66	-0.186 (-3.466)*	-0.930 (-1.182)	0.128 (5.843)*	-0.069	0.019
1967-68	0.245 (5.655)*	-0.386 (-0.887)	-0.083 (-7.438)*	0.237	0.015

(Figures in brackets are t-statistics)

*statistically significant at the 0.1 per cent level.

**statistically significant at the 2.0 per cent level.

[a] t-statistics are taken about the annual excess return on the market.

The coefficient, ν_1, for the probability of bankruptcy is consistently negative, though statistically insignificant from zero. The constant term, ν_o, is generally statistically significant and fluctuates in sign. The theoretic value of the coefficient ν_2 is the annual excess return on the market portfolio. When the estimated value is compared to its theoretic value it is found that, in general, they are different, the difference being statistically significant. Two of the estimates have the wrong sign.

Changes in the Probability of Bankruptcy

The method of portfolio construction used to increase the between portfolio variance of changes in the probability of bankruptcy and the positive results derived, suggests that similar results using individual security data should be obtained by considering the effects of changes in the probability of bankruptcy; that is, by considering a relationship of the form

$$R_{jt} = \nu_o + \nu_1\lambda_{jt} + \nu_2(\lambda_{jt}-\lambda_{jt-1}) + \beta_j(R_{Mt}-\chi_t) + u_{jt}, \qquad (5.15)$$

$$j = 1, 2, \ldots, N,$$

$$t = 2, 3, \ldots, T,$$

where R_{jt} is the annual excess rate of return for year t for the j^{th} asset; λ_{jt} is the probability of the j^{th} asset going bankrupt during year t; R_{Mt} is the market excess return; χ_t is a weighted average of the probabilities; u_{jt} is a random disturbance term; and ν_1, ν_1, ν_2, and $\{\beta_j\}$ are regression coefficients.

The relationship is examined by first taking a random sample of 100 firms from a universe of 360 firms and then by sorting the universe of firms by asset size into three groups. A regression pooling time series and cross section data is performed on each group. The probability of bankruptcy is

TABLE 5.19

DIFFERENCES IN THE PROBABILITY OF BANKRUPTCY

DATA SET	NUMBER OF OBSERVATIONS	COEFFICIENT ν_o	COEFFICIENT ν_1	COEFFICIENT ν_2	R-SQUARED
RANDOM	900	0.132 (10.497)*	0.433 (0.471)	-3.073 (-3.206)**	0.291
\$(0-200)m.	1,143	0.138 (12.287)*	-0.548 (-0.838)	-1.649 (-2.306)**	0.350
\$(200-500)m.	909	0.142 (10.39)*	0.300 (1.1375)	-0.841 (-3.111)**	0.271
\$500m. and over	1,188	0.078 (7.310)*	6.887 (4.932)*	-10.520 (5.637)*	0.282

(Figures in brackets are t-statistics)

* statistically significant at the 0.1 per cent level.

** statistically significant at the 2.0 per cent level.

estimated using Equation (5.2), which utilizes market values of corporate variables. The results are shown in Table 5.19.

The coefficient, ν_1, for the probability of bankruptcy is now positive, with the exception for the group of firms of asset size less than \$200 million. For firms of asset size \$500 million and over, the coefficient is statistically significant. The constant term, ν_o, is uniformly positive and statistically significant. The coefficient, ν_2, is consistently negative and statistically significant. The results should be compared to those given in Table 5.16. It is seen that the sign of the coefficient, ν_1, with one exception, is now reversed. For firms of asset size less than \$200 million, though the coefficient, ν_1, is still negative, it is no longer statistically significant.

Summary

The hypotheses of the thesis is that differences in the probability of bankruptcy across securities and across time are reflected in the residual return of common stocks after abstracting from the market. The hypothesis is tested using individual and aggregated security data. The use of individual security data provide a large data set with which to test the hypothesis. However, the biases that result from the errors in the measurement of the probability of bankruptcy present serious econometric problems. These problems can be mitigated by aggregating the data to reduce the variance of the estimates of the probability of bankruptcy. Securities are assigned into portfolios on the basis of past values of their probability of bankruptcy, and the aggregated portfolio data treated as representative securities. The use of aggregation does, however, reduce the size of the data set and prohibits extensive testing.

Evidence is found verifying the hypothesis of the thesis. Using aggregated portfolio data and estimating the probability of bankruptcy with market values of corporate variables, confirmation of the hypothesis is obtained. When the probability of bankruptcy is estimated using book values of corporate variables, the evidence is inconclusive. Use of individual security data does not give any clear indication of the validity of the hypotheses. The major difficulty appears to be due to the errors in the measurement of the probability of bankruptcy.

CHAPTER VI

SUMMARY

The purpose of this chapter is to summarize the main conclusions of the thesis and to describe the areas of further research that arise from it.

Conclusions

The impact of bankruptcy upon the structure of returns for corporate financial assets is investigated from a theoretical and empirical viewpoint. A model, formulated in continuous time, considers the investment-consumption decision of an individual acting to maximize the expected lifetime utility of consumption and terminal wealth. At each instant in time the individual must decide the portions of wealth to consume and to invest in financial assets. It is assumed that a firm issues both bonds and common stock as financial assets and that at each point in time there is a probability that the firm will go bankrupt the next instant. If bankruptcy occurs it is assumed that equity holders suffer a hundred per cent loss, and bondholders receive a non-negative liquidating dividend. From this formulation the equilibrium expected rates of return on the different financial assets--bonds and common stocks--are determined using stochastic control theory.

For bonds of infinite maturity, the equilibrium expected excess rate of return, conditional upon no bankruptcy, is a linear function of two variables: a market variable and a variable which is the product of the probability of bankruptcy and the expected loss if bankruptcy occurs. This result demonstrates, on a theoretical basis, some of the determinants of the risk premium on a bond and is directly amiable to empirical testing.

For common stocks a two variable expression for the equilibrium expected rate of return, conditional upon no bankruptcy, is derived. The expression is an extended form of the continuous time analogy to the Capital Asset Pricing Model (CAPM), the second variable being associated with the probability of bankruptcy. This result is important. It demonstrates that bankruptcy is a contributory factor in describing the structure of common stock returns. It provides a theoretic explanation of the recent empirical findings indicating that the CAPM is misspecified, and offers a natural interpretation of the "beta factor." This result is empirically tested. A discrete time formulation of the model is used. The hypothesis is that the probability of bankruptcy across securities and across time is reflected in the residual return after abstracting from the market.

To test the hypothesis it is necessary to be able to measure the probability of bankruptcy. Existing empirical work on bankruptcy has not addressed this problem, but has concentrated on constructing models to classify firms into one of two groups: bankrupt or not bankrupt. A model for the probability of bankruptcy in terms of a firm's ability to raise funds, either internally or externally, to cover fixed charges is constructed, and an ex-post formulation in terms of measurable quantities developed. The probability of bankruptcy is estimated using probit analysis and logit analysis. The ability of the model to predict bankruptcy is tested on a secondary sample of bankrupt firms. The results are very good with the model predicting bankruptcy, for some firms, four or five years before the actual occurrence.

The hypothesis is tested using individual and aggregated U.S.A. annual security data for the 10 year period 1960 to 1969. A new methodology to the testing of two variable extended forms of the CAPM, that of pooling time series and cross sectional data, is introduced. The time series data for all individual securities (portfolios) are combined to estimate the coefficients that are common to all securities (portfolios), whilst simultaneously estimating the firm (hypothesis) specific beta coefficients. Evidence is found verifying the hypothesis of the thesis; that is, bankruptcy is an explanatory factor of the structure of corporate financial assets.

Hence, the explicit objectives of the thesis: (a) to analyze theoretically how the mechanism of bankruptcy affects the structure of returns for corporate financial assets; (b) to quantify the determinants of bankruptcy, and to arrive at a figure which can be identified as the probability of bankruptcy; and (c) to test empirically the hypothesis of the thesis, are successfully achieved.

Future Research

The thesis demonstrates the need to develop a complete explanatory theory of the probability of bankruptcy. Such a theory needs to consider the factors that determine a firm's ability to raise funds and the interdependence between sources; that is, the effect of using one source on the ability to utilize other sources of funds. Development of such a theory entails consideration of the broader question of valuation. For example, a firm's ability to issue debt depends upon its debt capacity, but the concept of debt capacity is intrinsically related to the value of the firm and to the probability of bankruptcy. Thus there is circularity. Any analysis requires a description of the different attributes which determine the probability of bankruptcy and then a specification of the determinants of the attributes. This leads to a system of simultaneous equations which, in general, will be non-linear.

The development of a complete explanatory theory of the probability of bankruptcy is important not only from a theoretical viewpoint, but also because of the many practical applications to which it can be applied. For example, it can be utilized in business loan evaluation, and for internal management. For business loan evaluation -- consumer loans or commercial loans -- the same form of methodology can be utilized to determine the probability of an individual or firm defaulting on a loan. Similarly, it can be used for accounts receivable management to estimate the probability of a customer defaulting on payment. Other internal management uses involve determining the effects of different investment and financial mixes upon the probability of the firm going bankrupt.

A second area of research is to consider the effects of introducing the market imperfection of financial distress upon the Capital Asset Pricing Model. This necessitates defining an operational definition of financial distress and describing the consequences of financial distress upon the market value of the firm. From this basis the equilibrium rate of return can be determined.

A third area of research is the determination of the risk premium on a bond. The thesis provides a theoretic framework within which to analyze this problem and results which are directly amiable to empirical verification. Further development of this work needs to consider the effects of such factors as callability, maturity, marketability and term structure.

APPENDIX A

MATHEMATICAL DERIVATION OF THE RESULTS IN CHAPTER IV

The purpose of this appendix is to explain in greater detail the mathematical derivation of the results presented in Chapter IV. The appendix should be read in conjunction with Chapter IV, as in some cases results which have been given satisfactory explanation in the chapter are used in the appendix without further explanation.

The first part of the appendix describes the derivation of the stochastic differential equations, the equation of optimality, and the first order maximization conditions. A discussion of the difficulties in obtaining a solution to these equations and the approximations that are used to obtain a linear system is given.

In the second part the equilibrium instantaneous conditional expected rates of return, given the assumption of a constant investment opportunity set, are derived. The approximation used to derive a linear system of equations is examined in greater detail.

In the third and final part of the appendix, the general case of stochastic changes in the rate of the probability of bankruptcy is considered. The demand functions for the financial assets are derived. A special case is considered in which the stochastic changes in the rate of the probability of bankruptcy for one firm acts as an instrumental variable for all other changes. For this case, the instantaneous conditional expected rates of return are derived.

General Formulation

A derivation of the stochastic differential equations describing the price dynamics of the financial assets, the equation of optimality, and the first order maximization conditions is given. A discussion of the difficulties in obtaining a solution to these equations and the approximations that are used to obtain a linear system is presented.

Price Dynamics

An information derivation of these equations is presented in Chapter IV. It is proposed to give, on a slightly more rigorous basis, a derivation of the equations. The two approaches are, however, equivalent as the time interval tends to zero.

It will be assumed that the event of bankruptcy follows a Poisson process. A Poisson process is a continuous time process with a discrete state space; that is, one where there are discrete or discontinuous changes in the variables.[1] Let $N(t,t+h)$ denote the number of events in the time interval $(t,t+h]$.[2] Then the Poisson process is defined as follows:

$$\Pr[N(t,t+h) = 0] = 1 - \lambda(t)h + 0(h),$$

$$\Pr[N(t,t+h) = 1] = \lambda(t)h + 0(h),$$

and

$$\Pr[N(t,t+h) > 1] = 0(h),$$

---

[1] For a good introduction to Poisson processes, see Cox, D. R. and Miller, H. D., *Theory of Stochastic Processes* (London: Methuen & Co. Ltd., 1968).

[2] The bracket notation is defined as: if an element x belongs to the interval $(t,t+h]$, then $t < x \leq t+h$; that is, the point t is excluded from consideration.

where Pr( ) means the probability of; and $O(h)$ is the asymptotic order symbol defined by

$$f(h) \text{ is } O(h) \text{ if } \lim_{h \to o} [f(h)/h] = 0.$$

To a first order approximation, $\lambda(t)h$, can be interpreted as the probability of bankruptcy occurring in the interval $(t,t+h]$.[3] It is assumed that Poisson processes for different firms are independent; the event of bankruptcy for one firm does not affect other firms. Conceptually, it is very simple to relax this assumption, but at the cost of greatly increasing the complexity of notations. The very small gain in generality of results does not warrant the burden of using an even more complex form of notation.

It is perhaps not intuitively clear that the event of bankruptcy can be represented by such a process, as it is possible for $N(t,t+h)$ to equal, five, whilst the event of bankruptcy for a firm can only occur once in this model. However, the probability of $N(t,t+h)$ equalling five is of order h or less and thus in the limit as h tends to zero, is zero. As the whole formulation is in continuous time, and a limiting process is utilized, then the representation of the event of bankruptcy by a Poisson process is perfectly valid. The advantage of using a Poisson distribution lies in its continuity over the time domain.

From Equation (4.5) the price dynamics of the j^{th} firm's bonds can be described by

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[3]It is important to realize that $\lambda(t)$ is not a probability, but a probability rate; it is the probability per unit interval of time. The length of the interval is arbitrary; for example, it may be a day, a month, or a year.

$$b_j(t+h) = \begin{cases} b_j(t)(1+r_jh) - g_jh + b_j(t)\gamma_j\sqrt{h}Y_j(t); & \text{no default,} \\ \\ A_j(t+h) - \theta_j(t+h) & \text{; default,} \end{cases}$$

$$j = 1, 2, \ldots, n.$$

The above equation can be written in the form

$$b_j(t+h) = [b_j(t)(1+r_jh) - g_jh + b_j(t)\gamma_j\sqrt{h}Y_j(t)][1-N_j(t,t+h)] \\ + [A_j(t+h) - \theta_j(t+h)N_j(t,t+h). \tag{A.1}$$

If the event of bankruptcy does not occur to firm j in the interval $(t,t+h]$, then $N_j(t,t+h)$ is zero. If bankruptcy does occur then $N_j(t,t+h)$ equals one.

Define a stochastic process, $Z(t)$, by

$$Z(t+h) = Z(t) + Y(t)\sqrt{h},$$

where $Z(t)$ is a stochastic process with independent increments. The limit as h tends to zero of $Z(t+h)-Z(t)$ describes a Wiener process, or Brownian motion.[4] In the terminology of stochastic differential equations

$$dZ(t) \equiv Y(t)\sqrt{dt}.$$

In the limit as h tends to zero, Equation (A.1) can be written in the form:

$$db_j(t) = [b_j(t)r_j-g_j]dt + b_j(t)\gamma_jdZ_j - \{b_j(t)-[A_j(t)-\theta_j(t)]\}dq_j \tag{A.2}$$
$$j = 1, 2, \ldots, n,$$

where dq_j is a Poisson process characterizing the event of bankruptcy for the j^{th} firm.

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[4]For a general discussion, see Cox and Miller, *op. cit.*, pp. 205-208.

In a similar manner the price dynamics for equity can be derived. From Equation (4.9), it can be written in the form

$$p_j(t+h) = [p_j(t)(1+\alpha_j h) - f_j h + p_j(t)\sigma_j\sqrt{h}Y_{n+j}(t)][1-N_j(t,t+h)]$$

In the limit as h tends to zero, the above equation becomes

$$dp_j(t) = [p_j(t)\alpha_j - f_j]dt + p_j(t)\ _j dZ_{n+j} - p_j(t)dq_j, \qquad \text{(A.3)}$$

$$j = 1, 2, \ldots, n.$$

The Equation of Optimality: The Demand Functions for Assets

From Equation (4.22), the derived utility function is defined as

$$J[W(t),\underline{\alpha},\underline{\sigma},\underline{r},\underline{\gamma},\underline{f},\underline{\lambda},r_F,t,S(t)] \equiv \underset{\{c,\underline{w}\}}{\text{Max}}\ E_t\ \{\int_t^T U[C(s),S]ds + BF[W(T),T]\}, \qquad \text{(A.4)}$$

subject to a wealth constraint, budget constraint, and $C(s) \geq 0$. The derived utility function, J, can be written in a more compact and convenient form. Consider the set of equations describing how the opportunity set changes:[5]

$$\begin{aligned}
d\alpha_j &= F_j(\alpha_j,t)dt + G_j(\alpha_j,t)dQ_j,\\
d\sigma_j &= F_{n+j}(\sigma_j,t)dt + G_{n+j}(\sigma_j,t)dQ_{n+j},\\
dr_j &= F_{2n+j}(r_j,t)dt + G_{2n+j}(r_j,t)dQ_{2n+j},\\
d\gamma_j &= F_{3n+j}(\gamma_j,t)dt + G_{3n+j}(\gamma_j,t)dQ_{3n+j},\\
d\lambda_j &= F_{4n+j}(\lambda_j,t)dt + G_{4n+j}(\lambda_j,t)dQ_{4n+j},\\
df_j &= F_{5n+j}(f_j,t)dt + G_{5n+j}(f_j,t)dQ_{5n+j},
\end{aligned} \qquad \text{(A.5)}$$

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[5] See Equations (4.12), (4.13), (4.14) and (4.16).

and

$$dr_F = F_m(r_F,t)dt + G_m(r_F,t)dQ_m,$$
$$j = 1, 2, \ldots, n, \text{ and } m = 6n+1.$$

Define a (mxl) vector dV by

$$dV' = (d\alpha_1,\ldots, d\alpha_n,d\sigma_1,\ldots,d\sigma_n,dr_1,\ldots,dr_n,d\gamma_1,\ldots,d\gamma_n,d\lambda_1,\ldots, d\lambda_n,df_1,\ldots,df_n,dr_m) \tag{A.6}$$

Hence,

$$dV = Fdt + GdQ, \tag{A.7}$$

where F is a (mxl) vector such that $F' = (F_1,\ldots,F_m)$, dQ is a (mxl) vector such that $dQ' = (dQ_1,\ldots,dQ_m)$ and G is a (mxm) matrix with elements along the diagonal and zeros elsewhere. Given the definition (A.6), then by comparing (A.7) with (A.5) the elements of F and G can be identified.

Therefore, Equation (A.4) can be written in the form

$$J[W(t),V(t),t,S(t)] = \underset{\{c,\underline{w}\}}{\text{Max}}\ E_t\{\int_t^T U[C(x),x]dx + BF[W(T),T]\}, \tag{A.8}$$

subject to a wealth constraint, a budget constraint, and $C(x) \geq 0$. Thus,

$$J[W(t),V(t),t,S(t)] \geq E_t\{\int_t^{t+h} U[C(x),x]dx + J[W(t+h),V(t+h),t+h,S(t+h)]\},$$

where S(t+h) is a state description vector describing the system at time t+h. By using the mean value theorem for integrals, the first term on the right hand side of the above equation can be approximated to[6]

$$U[C(t),t]h + 0(h).$$

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[6]See Chapter 7 of Dreyfus, S. E., *Dynamic Programming and the Calculus of Variation* (New York: Academic Press, 1965).

Hence,

$$J[W(t),V(t),t,S(t)] \approx U[C(t),t]h + O(h) + E_t\{J[W(t+h),V(t+h),t+h,S(t+h)]\} \tag{A.9}$$

In order to evaluate the second term on the right hand side of Equation (A.9), a conditional expectation argument is used. At time (t+h) it is only necessary to consider (n+1) of the 2^n possible states of the system, the probability of occurrence of these states being known. Given a particular state, the expected value of the random variable, conditional upon the state is calculated and then the unconditional expected value determined. Mathematically,[7] the argument can be represented as: if (X,Y) is a two dimensional random variable, the conditional expectation of X for a given $Y=y_j$ is defined by

$$E(X|Y=y_j) = \sum_{i=1}^{\infty} x_i P_r(X=x_i|Y=y_j),$$

and the expected value of X is

$$E(X) = E_Y[E(X|Y = y_j)].$$

With probability $[1 - \sum_{i=1}^{n} \lambda_i h + O(h)]$ no defaults occur in the interval (t,t+h]. Thus, conditional upon the event, the budget constraint[8] is

$$W(t+h) - W(t) = \{W(t) - C(t)h\}\{\sum_{j=1}^{n} w_j(t)[(r_j-r)h + \gamma_j dZ_j]$$

$$+ \sum_{j=1}^{n} w_{n+j}(t)[(\alpha_j-r)h + \sigma_j dZ_{n+j}] + rh\} - C(t)h + O(h) \tag{A.10}$$

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[7]For an introductory discussion, see Meyer, P. L., *Introductory Probability and Statistical Application* (Massachusetts: Addison-Wesley, 1965).

[8]See Equation (4.18).

The expected value of the change in wealth, conditional upon no defaults, is[9]

$$E_t[W(t+h)-W(t)|\text{no defaults}] = [W(t)-C(t)h][\sum_{j=1}^{n} w_j(t)(r_j-r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_j-r)+r]h - C(t)h + O(h), \tag{A.11}$$

and

$$E_t\{[W(t+h)-W(t)]^2|\text{no defaults}\}$$
$$= [W(t)^2[\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_{ji}w_i(t) + 2\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_j\rho_{ji}\sigma_i\,{}_{n+i}(t)$$
$$+ \sum_{j=1}^{n}\sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji}w_{n+i}(t)]h + o(h). \tag{A.12}$$

Expanding the derived utility function $J[W(t+h),V(t+h),t+h,S(t+h)]$ aboutthe point $[W(t),V(t),t,S(t)]$, and taking expected values, conditional upon the event of no defaults, gives

$$J[W(t),V(t),t,S(t)] + J_t h + J_W\{W(t)[\sum_{j=1}^{n} w_j(t)(r_j-r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_j-r)+r] - C(t)\}h$$
$$+ \frac{1}{2}J_{WW}[\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_{ji}w_i(t) + 2\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_j\rho_{ji}\sigma_i w_{n+i}(t)$$
$$+ \sum_{j=1}^{n}\sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji}w_{n+i}(t)]W(t)^2 h + \sum_{j=1}^{n} J_j F_j h$$
$$+ \frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{m} G_j\nu_{ji}G_i J_{ji}\; h + \sum_{i=1}^{m}\sum_{j=1}^{n} W(t)w_j(t)\gamma_j\eta_{ij}G_i J_{iw}h$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} W(t)w_{n+j}(t)\sigma_j\eta_{i,n+j}G_i J_{iW}h + O(h), \tag{A.13}$$

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[9]See Equations (4.19) and (4.20).

where $S(t+h) = S(t)$, as no firms have defaulted; ν_{ij} is the instantaneous conditional correlation coefficient between dQ_i and dQ_j, conditional upon the fact that default has not occurred to either the i^{th} or j^{th} firm; η_{ij} is the instantaneous conditional correlation coefficient between dQ_i and dZ_j;

$$J_t = \frac{\partial}{\partial t} J[W(t), V(t), t, S(t)];$$

$$J_W = \frac{\partial}{\partial W} J[W(t), V(t), t, S(t)];$$

$$J_{WW} = \frac{\partial^2}{\partial W^2} J[W(t), V(t), t, S(t)];$$

$$J_j = \frac{\partial}{\partial V_j} J[W(t), V(t), t, S(t)];$$

$$J_{ij} = \frac{\partial^2}{\partial V_i \partial W_j} J[W(t), V(t), t, S(t)];$$

and

$$J_{jW} = \frac{\partial^2}{\partial V_j \partial W} J[W(t), V(t), t, S(t)].$$

With probability $\lambda_j(t)h + O(h)$ the event that the j^{th} firm goes bankrupt and no other firms go bankrupt in the interval $(t, t+h]$ occurs. The event of bankruptcy not only affects the j^{th} firm's bond price behaviour, but also its equity price behaviour. Conditional upon this event, the change in wealth is now of the form[10]

$$W(t+h) - W(t) = W(t)\{w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)} - 1] - w_{n+j}(t)\}$$
$$+ \{W(t) - C(t)h\}\{\sum_{\substack{i=1 \\ i\neq j}}^{n} w_i(t)[(r_i - r)h + \gamma_i dZ_i] + \sum_{\substack{i=1 \\ i\neq j}}^{n} w_{n+i}(t)[(\alpha_i - r)h + \sigma_i dZ_{n+i}] + rh\}$$

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[10] See Equation (4.21).

$$- C(t)h\{1 + w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)} - 1] - w_{n+j}(t)\} + O(h). \qquad (A.14)$$

The derived utility function, conditional on the event that the j^{th} firm went bankrupt, it will be of the form $J[W(t+h), V(t+h), t+h, S_j]$ where S_j is a state vector at time (t+h) denoting that the j^{th} firm no longer exists. As before, the derived utility function is expanded and the conditional expected values taken. However, unlike Equation (A.13) wealth $W(t+h)$ is not expanded about the point $W(t)$, but about the point

$$W(t)\{1 + w_j(t)[\frac{A_j(t+h) - \theta_j(t+h)}{b_j(t)} - 1] - w_{n+j}(t)\}$$
$$= W_j(t), \qquad (A.15)$$

which defines $W_j(t)$. The reason for this is to preserve the compactness of the change in wealth. The event of bankruptcy causes a discontinuity in the wealth function and thus it no longer becomes possible to represent the changes in wealth by a summation of compact distributions. The property of compactness is very important for it enables many of the terms in the Taylor expansion of the derived utility function to be neglected when a limiting process is used. It is possible to preserve such a property. The derived utility function is expanded in a Taylor's series and then its expected value is taken conditional upon the event that the j^{th} firm went bankrupt. Utilizing this conditional argument, the change in wealth that results solely from the bankruptcy of the j^{th} firm is known for a given investment in its bonds and equity. Thus any other changes in wealth resulting from the investment in the other financial assets can still be represented by the summation of compact distributions, but now centred around the wealth position $W_j(t)$, instead of $W(t)$. There are, however, a number of other important ramifications that result from the discontinuities that occur due to the event of bankruptcy, as will become quickly apparent.

The conditional expected value of the derived utility function $J[W(t+h), V(t+h), t+h, S_j]$ is thus

$$J[W_j(t),V(t),t,S_j] + \text{terms of order at most } h. \tag{A.16}$$

Substituting Equations (A.13) and (A.16) into Equation (A.9) and taking the unconditional expected value, gives

$$\begin{aligned}
&J[W(t),V(t),t,S(t)] \\
&= \underset{\{c,\underline{w}\}}{\mathrm{Max}}(U[C(t),t]h + 0(h) \\
&+ \sum_{j=1}^{n} \{\lambda_j(t)h + 0(h)\}\{J[W_j(t),V(t),t,S_j] + \text{terms of order at most } h\} \\
&+ \{1-\sum_{j=1}^{n}\lambda_j(t)h + 0(h)\}\{J[W(t),V(t),t,S] + J_t h + \sum_{j=1}^{m} J_j F_j h \\
&+ J_W\{W(t)[\sum_{j=1}^{n} w_j(t)(r_j-r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_j-r) + r] - C(t)\}h \\
&+ \frac{1}{2}J_{WW}[\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_{ji}w_i(t) + 2\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_j\rho_{ji}\sigma_i w_{n+i}(t) \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji}w_{n+i}(t)]\, W(t)^2 h \\
&+ \frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{m} G_j\nu_{ji}G_i J_{ji}h + \sum_{i=1}^{m}\sum_{j=1}^{n} W(t)w_j(t)\gamma_j\eta_{ij}G_i J_{iW}h \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n} W(t)w_{n+j}(t)\sigma_j\eta_{i,n+j}G_i J_{iW}h + 0(h)\}).
\end{aligned}$$

Simplifying, and taking the limit as h tends to zero, gives the fundamental partial differential equation for the derived utility function:

$$0 = \underset{\{c,\underline{w}\}}{\text{Max}} \Big(U[C(t),t] + J_t + \sum_{j=1}^{m} F_j J_j$$

$$+ J_W\{W(t)[\sum_{j=1}^{n} w_j(t)(r_j - r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_j - r) + r] - C(t)\}$$

$$+ \frac{1}{2} J_{WW}[\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_{ji}w_i(t) + 2\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_j\rho_{ji}\sigma_i w_{n+i}(t)$$

$$+ \sum_{j=1}^{n}\sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji}w_{n+i}(t)]W(t)^2$$

$$+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} G_i\nu_{ij}G_jJ_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} W(t)w_j(t)\gamma_j\eta_{ij}G_iJ_{iW}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n} W(t)w_{n+j}(t)\sigma_j\eta_{i,n+j}G_iJ_{iW}$$

$$+ \sum_{j=1}^{n} \lambda_j(t)\{J[W_j,V(t),t,S_j] - J[W(t),V(t),t,S(t)]\}\Big), \qquad \text{(A.17)}$$

subject to the boundary conditions $J[W(t),V(T),T,S(T)] = BF[W(T),T]$.

The 2n+1 first order maximization conditions are derived from Equation (A.17) by first differentiating with respect to the rate of consumption:

$$0 = U_C[C(t),t] - J_W \qquad \text{(A.18)}$$

where

$$U_C[C(t),t] = \frac{\partial}{\partial C}[C(t),t];$$

then by differentiating with respect to the proportion of wealth to invest in common stock, $\{w_{n+j}(t)\}$,

$$0 = J_W(\alpha_j - r) + J_{WW}\Big[\sum_{i=1}^{n} \sigma_{ji} w_{n+i}(t) + \sum_{i=1}^{n} \gamma_j \rho_{ji} \sigma_i w_{n+i}(t)\Big] W(t)$$

$$+ \sum_{i=1}^{n} \sigma_j \eta_{i,n+j} G_i J_{iW} - \lambda_j J_W[W_j, V(t), t, S_j],$$

$$j = 1, 2, \ldots, n; \tag{A.19}$$

and finally, differentiating with respect to the proportion of wealth to invest in bonds $\{w_j(t)\}$:

$$0 = (r_j - r) J_W + W(t)\Big[\sum_{i=1}^{n} \gamma_{ji} w_i(t) + \sum_{i=1}^{n} \gamma_j \rho_{ij} \sigma_i s_{n+i}(t)\Big] J_{WW}$$

$$+ \sum_{i=1}^{m} \gamma_j \eta_{ij} G_i J_{iW} - \lambda_j \Big[1 - \frac{A_j(t) - \theta_j(t)}{b_j(t)}\Big] J_W[W_j, V(t), t, S_j],$$

$$j = 1, 2, \ldots, n, \tag{A.20}$$

For expositional simplicity define

$$L_j(t) = 1 - \frac{A_j(t) - \theta_j(t)}{b_j(t)},$$

$$j = 1, 2, \ldots, n.$$

Substituting Equation (A.20) into (A.19) so as to eliminate $J_W[W_j, V(t), t, S_j]$ gives:

$$0 = \Big[(\alpha_j - r) - \frac{r_j - r}{L_j(t)}\Big] J_W + W(t)\Big[\sum_{i=1}^{n} \sigma_{ji} w_{n+i}(t) - \sum_{i=1}^{n} \frac{\gamma_j \rho_{ij} \sigma_i}{L_j(t)} w_{n+i}(t)$$

$$+ \sum_{i=1}^{n} \sigma_j \rho_{ji} \gamma_i w_i(t) - \sum_{i=1}^{n} \frac{\gamma_{ji} w_i(t)}{L_j(t)}\Big] J_{WW}$$

$$+ \sum_{i=1}^{m} \sigma_j \eta_{i,n+j} G_i J_{iW} - \sum_{i=1}^{m} \frac{\gamma_j \eta_{ij} G_i J_{iW}}{L_j(t)}. \tag{A.21}$$

At this point it is perhaps worth making a small digression to derive two inequalities that will be useful later. From Equation (A.18) the following relationships hold.

$$0 = U_{CC}[C(t),t]\frac{\partial C}{\partial W} - J_{WW}[W(t),V(t),t,S(t)],$$

$$0 = U_{CC}[C(t),t]\frac{\partial C}{\partial V_j} - J_{jW}[W(t),V(t),t,S(t)],$$

$$j = 1,\ 2,\ldots,n.$$

Hence, using Equation (A.18), we have the inequalities:

$$-\frac{J_W}{J_{WW}} = -\frac{U_C}{U_{CC}\frac{\partial C}{W}} > 0 \tag{A.22}$$

and

$$-\frac{J_{jW}}{J_{WW}} = -\frac{\frac{\partial C}{\partial V_j}}{\frac{\partial C}{\partial W}} \gtreqless 0,$$

$$j = 1,2,\ldots,n.$$

The demand functions for bonds can be derived from the set of Equations (A.19). In their stated form the equations are non-linear and thus in general it will be difficult to obtain an explicit solution. There are at least two alternatives. The first is to put more structure into the formulation by assuming a particular form for the individual's utility function and then attempt to solve the system of equations by a numerical iterative procedure. Whilst this method might produce a solution, it will be at a cost. It will be difficult to derive explicit forms for the general equilibrium rates of return, and even if these could be obtained, they will depend upon the specific assumed form for the utility functions for individuals. The lack of generality, and the complexity of this method is a serious distraction to its utilization.

The second alternative is to obtain an approximate solution by making two assumptions. First, the individual is indifferent as if there are n or (n-1) firms in existence at time t such that

$$J[W(t),V(t),t,S(t)] = J[W(t),V(t),t,S_j], \qquad j = 1,2,\ldots,n, \tag{A.23}$$

and all derivations are equal. If the number of firms in existence, n, is 'large', then such an assumption seems intuitively quite reasonable. The second assumption involves the ability to expand the derivations of the derived utility function and to neglect the quadratic and higher power expansion terms, that is

$$J_W[W(t)\{1 + w_j(t)[\frac{A_j-\theta_j}{b_j(t)} - 1] - w_{n+j}(t)\}, V(t), t, S(t)]$$

$$\simeq J_W[W(t),V(t),t,S(t)] - W(t)\{w_j(t)[1 - \frac{A_j-\theta_j}{b_j(t)}] + w_{n+j}(t)\}J_{WW}[W(t),V(t),t,S(t)]$$

$$j = 1, 2,\ldots,n. \tag{A.24}$$

The error introduced by making this assumption will be examined in the next section.

Given that these assumption hold, then substituting Equations (A.23) and (A.24) into (A.20) gives

$$0 = J_W(r_j-r) + J_{WW}[\sum_{i=1}^{n} \gamma_{ji}w_i(t) + \sum_{i=1}^{n} \gamma_j\rho_{ij}\sigma_i w_{n+i}(t)]W(t) + \sum_{i=1}^{n} \gamma_j\eta_{ij}G_iJ_{iW}$$

$$- \lambda_j(t)L_j(t)J_W + {}_j(t)L_j(t)[w_j(t)L_j(t) + w_{n+j}(t)]W(t)J_{WW} \tag{A.25}$$

$$j = 1, 2,\ldots,n.$$

Hence, Equations (A.21) and (A.25) give a set of simultaneous equations, linear in the demand functions for the financial assets $\{w_j\}$.

Constant Investment Opportunity Set

Suppose that the investment opportunity set characterized by $\{\underline{\alpha},\underline{\sigma},\underline{r},\underline{\gamma},\underline{\lambda},\underline{f},r_F\}$ is constant and that $\gamma_j \equiv 0$ for all j. The first order conditions, using Equations (A.18), (A.19), (A.20), (A.21) and (A.23) become

$$0 = U_C[C(t),t] - J_W, \tag{A.26}$$

$$0 = (\alpha_j - r - \frac{r_j - r}{L_j})J_W + W(t)[\sum_{i=1}^{n} \sigma_{ji} w_{n+i}(t)]J_{WW}, \tag{A.27}$$

and

$$0 = (r_j - r)J_W - \lambda_j L_j J_W(W_j,t), \tag{A.28}$$

$$j = 1, 2, \ldots, n.$$

Using the approximation described by expression (A.24), then Equation (A.28) becomes

$$0 = (r_j - r - \lambda_j L_j)J_W + \lambda_j L_j[w_j(t)L_j + w_{n+j}(t)]W(t)J_{WW}, \tag{A.29}$$

$$j = 1, 2, \ldots, n.$$

From Equation (A.27), the demand function for equity can be derived:

$$w_{n+i}(t)W(t) = (-\frac{J_W}{J_{WW}}) \sum_{j=1}^{n} \Lambda_{ij}(\alpha_j - r - \frac{r_j - r}{L_j}), \tag{A.30}$$

$$i = 1, 2, \ldots, n,$$

where $\{\Lambda_{ij}\}$ are the elements of the inverse of the instantaneous conditional variance-covariance matrix. From Equation (A.29) the demand function for bonds can be written in the form

$$[w_j(t)L_j + w_{n+j}(t)]W(t) = (-\frac{J_W}{J_{WW}})(\frac{r_j - r}{\lambda_j L_j} - 1) \tag{A.31}$$

$$j = 1, 2, \ldots, n.$$

To derive the expressions for the equilibrium instantaneous conditional expected rates of return, consider the demand functions for the k^{th} individual.

Equation (A.30) can be written in the form

$$\alpha_j - r - \frac{r_j - r}{L_j} = -\left[\frac{W^k(t)J^k_{WW}}{J^k_W}\right] \sum_{i=1}^{n} \sigma_{ji} w^k_{n+i}(t), \tag{A.32}$$

$$j = 1, 2, \ldots, n,$$

where the superscript k is used to identify the particular individual. Therefore, substituting for $\{w^k_{n+i}(t)\}$ and summing across all individuals, gives

$$\left(\alpha_j - r - \frac{r_j - r}{L_j}\right) \sum_{k=1}^{I} \left(-\frac{J^k_W}{J^k_{WW}}\right) = \sum_{i=1}^{n} \sum_{k=1}^{I} \sigma_{ji} p_i(t) N^k_{n+i}(t),$$

where I is the total number of investors; $N^k_{n+i}(t)$ is the optimal number of shares of the i^{th} firm that the k^{th} individual invests in. If the general equilibrium conditions are used, that is, all markets clear, then

$$\sum_{k=1}^{I} N^k_{n+i}(t) = \bar{N}_{n+i},$$

$$i = 1, 2, \ldots, n,$$

where $\bar{N}_{n+i}$ is the total number of shares outstanding for the i^{th} firm. Hence,

$$\left(\alpha_j - r - \frac{r_j - r}{L_j}\right) \sum_{k=1}^{I} \left(-\frac{J^k_W}{J_{WW}}\right) = \sum_{i=1}^{n} \sigma_{ji} p_i(t) \bar{N}_{n+i} \tag{A.33}$$

The above equation can be expressed in a form comparable to the traditional capital asset pricing model. Define M(t) to be the total market equilibrium value of all equities:

$$M(t) = \sum_{i=1}^{n} \bar{N}_{n+i} p_i(t); \tag{A.34}$$

and $Y_i(t)$ to be the percentage of the equilibrium value of the i^{th} firm's equity to the total market value:

$$Y_i(t) = \frac{\bar{N}_{n+i}p_i(t)}{M(t)} \tag{A.35}$$

$$i = 1, 2, \ldots, n.$$

Define the equilibrium instantaneous conditional expected rate of return on the equity market to be

$$\mu = \sum_{j=1}^{n} \alpha_j Y_j(t) \tag{A.36}$$

Multiplying Equation (A.33) by $Y_j(t)$, summing over all j and substituting Equations (A.34), (A.35) and (A.36) gives

$$[\mu - r - \sum_{j=1}^{n} Y_j(t)(\frac{r_j - r}{L_j})] \sum_{k=1}^{I} (-\frac{J_W^k}{J_{WW}^k}) = M(t) \sum_{i=1}^{n} \sum_{j=1}^{n} Y_i(t)\sigma_{ij}Y_j(t) \tag{A.37}$$

For expositional simplicity, define

$$\bar{\chi}(t) = \sum_{j=1}^{n} Y_j(t)(\frac{r_j - r}{L_j}). \tag{A.38}$$

Substituting Equation (A.37) into Equation (A.33) and using (A.38) gives

$$\alpha_j - r - \frac{r_j - r}{L_j} = \beta_j(\mu - r - \chi), \tag{A.39}$$

where

$$\beta_j = \frac{\sum_{i=1}^{n} \sigma_{ji}Y_i(t)}{\sum_{i=1}^{n} \sum_{j=1}^{n} Y_i(t)\sigma_{ij}Y_j(t)}$$

$$j = 1, 2, \ldots, n.$$

To derive the equilibrium instantaneous expected rates of returns for bonds, Equation (A.31) for the k^{th} individual can be written in the form

$$[N_i^k(t)b_i(t)L_i + N_{n+i}^k(t)p_i(t)] = (-\frac{J_W^k}{J_{WW}^k})(\frac{r_i - r}{\lambda_i L_i} - 1),$$

where $N_i^k(t)$ is the optimal number of bonds of the i^{th} firm that the k^{th} individual invests in. Summing over all individuals and using the general equilibrium condition that all markets clears; that is,

$$\sum_{k=1}^{I} N_i^k(t) = \bar{N}_i, \qquad i = 1, 2, \ldots, n,$$

where $\bar{N}_i$ is the total number of bonds outstanding for the i^{th} firm, gives the equation

$$\bar{N}_i b_i(t) L_i + N_{n+i} p_i(t) = \left(\frac{r_i - r}{\lambda_i L_i} - 1\right) \sum_{k=1}^{I} \left(- \frac{J_W^k}{J_{WW}^k}\right) \tag{A.40}$$

Define $M_B(t)$ to be the total market equilibrium value of all bonds:

$$M_B(t) = \sum_{i=1}^{n} N_i b_i(t); \tag{A.41}$$

and $X_i(t)$ to be the percentage of the equilibrium value of the i^{th} firm's bonds to the total market value of all bonds, that is,

$$X_i(t) = \frac{\bar{N}_i b_i(t)}{M_B(t)}, \qquad i = 1, 2, \ldots, n. \tag{A.42}$$

Define the equilibrium instantaneous conditional expected rate of return on the bond market to be

$$\pi(t) = \sum_{i=1}^{n} X_i(t) r_i \tag{A.43}$$

Multiplying Equation (A.40) by $X_i(t)$ and summing over all i gives

$$(\pi - r - \gamma) \sum_{k=1}^{I} \left(- \frac{J_W^k}{J_{WW}^k}\right) = \sum_{i=1}^{n} {}_i(t) \lambda_i L_i \; [\bar{N}_i b_i(t) L_i + N_{n+i} p_i(t)], \tag{A.44}$$

where $\gamma = \sum_{i=1}^{n} X_i(t)L_i\lambda_i$. Substituting Equation (A.44) into Equation (A.40) gives

$$r_j - r - \lambda_j L_j = \{\lambda_j L_j[N_j b_j(t)L_j + N_{n+j}p_j(t)]\}\{\frac{\pi - r - \gamma}{\sum_{i=1}^{n} X_i(t)\lambda_i L_i[\bar{N}_i b_i(t)L_i + \bar{N}_{n+i}p_i(t)]}\}$$

$$j = 1, 2, \ldots, n. \qquad \text{(A.45}$$

Equation (A.45) describesthe equilibrium instantaneous expected rate of return for the bonds of the j^{th} firm.

An alternative and useful derivation of Equation (A.45) is possible. Consider Equations (A.38), (A.39) and (A.40), thus by using these equations it is possible to eliminate the term $\sum_{k=1}^{I} (-\frac{J_W^K}{J_{WW}^K})$ from Equation (A.40); that is

$$r_j - r - \lambda_j L_j = \lambda_j L_j(\mu - r - \bar{\chi})[\frac{\bar{N}_j b_j(t)L_j + \bar{N}_{n+j}p_j(t)}{M(t)\Sigma\Sigma Y_i\sigma_{ij}Y_j}] \qquad \text{(A.46}$$

$$j = 1, 2, \ldots, n.$$

The usefulness of this alternative derivation is demonstrated when it is used to eliminate $(\frac{r_j - r}{L_j})$ from Equation (A.39), which describes the equilibrium instantaneous conditional expected rates of return for equity. Substituting Equation (A.46) into Equation (A.39) gives

$$\alpha_j - r - \lambda_j = (\mu - r - \bar{\chi})\{\beta_j + \lambda_j[\frac{\bar{N}_j b_j(t)L_j + \bar{N}_{n+j}p_j(t)}{M(t)\sum_{i=1}^{n}\sum_{j=1}^{n} Y_i\sigma_{ij}Y_j}]\}$$

Substituting the expression for β_j, gives

$$\alpha_j - r - \lambda_j = \frac{(\mu - r - \bar{\chi})}{\sum_{i=1}^{n}\sum_{j=1}^{n} Y_i\sigma_{ij}Y_j}\{\Sigma\,\sigma_{ji}Y_i + \lambda_j[\frac{\bar{N}_j b_j(t)L_j + N_{n+j}p_j(t)}{\sum_{i=1}^{n} N_{n+i}p_i(t)}]\}$$

If there are a large number of firms, the last term on the right hand side of the above expression can be neglected, as it is of order 2/n, where n is the number of firms. Hence, the expression for the instantaneous conditional expected rate of return can be written

$$\alpha_j - r - \lambda_j = \beta_j(\mu - r - \bar{\chi}), \tag{A.47}$$

$$j = 1, 2, \ldots, n.$$

Before proceeding to investigate the validity of the approximation of being able to expand the first derivatives of the derived utility function and neglect quadratic and higher order terms, it is worth reconsidering Equations (A.39), (A.36) and (A.47) in order to derive two identities. From Equation (A.36), the equilibrium instantaneous conditional expected rate of return on the equity market is defined to be

$$\mu = \sum_{j=1}^{n} \alpha_j Y_j(t)$$

Substituting Equation (A.39), gives

$$\mu = r \sum_{j=1}^{n} Y_j(t) + \sum_{j=1}^{n} Y_j(t)\left(\frac{r_j - r}{L_j}\right) + (\mu - r - \bar{\chi}) \sum_{j=1}^{n} Y_j(t)\beta_j,$$

which implies

$$\sum_{j=1}^{n} Y_j(t) = 1,$$

and

$$\sum_{j=1}^{n} Y_j(t)\beta_j = 1 \tag{A.48}$$

as one would expect, given the definitional forms of the various quantities. If Equation (A.47) is substituted into (A.36), then

$$\mu = r \sum_{j=1}^{n} Y_j(t) + \sum_{j=1}^{n} Y_j(t)\lambda_j + (\mu - r - \bar{\chi}) \Sigma\, Y_j(t)\,\beta_j,$$

which implies, using Equation (A.48), that

$$\bar{\chi} = \sum_{j=1}^{n} Y_j(t)\lambda_j \tag{A.49}$$

Note, however, that Equation (A.49) is not, unlike expression (A.48), a definitional identity. It is the result of the approximation made in deriving (A.47), a result which depends upon the ability to neglect terms of order (1/n). Equation (A.49) is obtained from a weighted summation of terms described by Equation (A.47) and thus neglects the summation of the terms that are considered to be of negligible significance. It is not, however, clear that the sum of these terms can be neglected.

Consider the left hand side of Equation (A.49). From Equations (A.38) and (A.36)

$$\begin{aligned} \bar{\chi} &= \sum_{j=1}^{n} Y_j(t)\left(\frac{r_j - r}{L_j}\right) \\ &= \sum_{j=1}^{n} Y_j(t)\lambda_j + (\mu - r - \bar{\chi}) \frac{\sum_{j=1}^{n} Y_j(t)\lambda_j [\bar{N}_j b_j(t) L_j + \bar{N}_{n+j} p_j(t)]}{M(t) \sum_{j=1}^{n} \sum_{i=1}^{n} Y_j(t)\sigma_{ji} Y_i(t)} \end{aligned} \tag{A.50}$$

Hence, Equation (A.49) neglects the last term of Equation (A.50), that is

$$(\mu - r - \bar{\chi}) \sum_{j=1}^{n} \left\{ \left[\frac{Y_j(t)\lambda_j}{\sum_{j=1}^{n} \sum_{i=1}^{n} Y_j(t)\sigma_{ji} Y_i(t)}\right] \left[\frac{\bar{N}_j b_j(t) L_j + \bar{N}_{n+j} p_j(t)}{M(t)}\right] \right\} \tag{A.51}$$

It is not clear that this term can be neglected, which is simply a reflection of the fact that the summation of terms, which are individually negligible, need not itself be negligible. This does not imply that Equation (A.47) is wrong, only that it is derived by neglecting a term of order (1/n).

Equations (A.39) and (A.45) were derived under the assumption that it was possible to expand in a Taylor's series the first derivative of the derived utility function and to neglect quadratic and higher order terms. Whilst this assumption is very convenient because of the resulting linearity, it would be of some comfort to determine the magnitude of the approximation. One approach to this equation is to put more structure into the formulation by assuming a particular form for the individual's utility function and to attempt to obtain an exact solution and then compare it to the solution obtained by assuming the validity of the approximation.

Assume a constant relative risk aversion utility function defined by

$$U[C(t),t] = \frac{C^{\nu}}{\nu} e^{-\rho t}, \tag{A.52}$$

where ρ and ν are positive constants and $\nu \leq 1$. The system of first order conditions can be written

$$0 = C^{\nu-1} e^{-\rho t} - J_W, \tag{A.53}$$

$$0 = (\alpha_j - r - \frac{r_j - r}{L_j}) J_W + W(t) [\sum_{i=1}^{n} \sigma_{ji} w_{n+i}(t)] J_{WW}, \tag{A.54}$$

$$0 = (r_j - r) J_W - \lambda_j L_j J_W(W_j, t), \tag{A.55}$$

$$j = 1, 2, \ldots, n,$$

and the equation of optimality

$$0 = \max_{\{c,\underline{w}\}} \{\frac{C^{\nu}}{\nu} \varepsilon^{-\rho t} + J_t + J_W\{ W(t) [\sum_{j=1}^{n} w_j(t)(r_j - r) + \sum_{j=1}^{n} w_{n+j}(\alpha_j - r) + r]$$

$$- C\} + \frac{1}{2} J_{WW} W(t)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} w_{n+i}(t) \sigma_{ij} w_{n+j}(t) \sum_{j=1}^{n} \lambda_j [J(W_j, t) - J(W,t]\} \tag{A.56}$$

subject to the boundary condition $J[W(T),T] = BF[W(T),T]$. To solve this system of equations take as a trial solution[11]

$$J[W(t),t] = B(t)e^{-\rho t}\frac{W^{\nu}}{\nu} \tag{A.57}$$

Substituting (A.57) into (A.54) and (A.55) gives

$$0 = (\alpha_j - r - \frac{r_j - r}{L_j}) + (\nu-1)\sum_{i=1}^{n}\sigma_{ji}w_{n+i}(t), \tag{A.58}$$

and

$$0 = (r_j - r) - \lambda_j L_j (1 - w_j L_j - w_{n+j})^{\nu-1} \tag{A.59}$$

$$j = 1, 2, \ldots, n.$$

Equation (A.59) can be written in the form

$$w_j L_j + w_{n+j} = 1 - (\frac{r_j - r}{\lambda_j L_j})^{\frac{1}{\nu-1}} \tag{A.60}$$

that facilitates direct comparison with Equation (A.31), which was derived assuming the validity of the approximation. Equation (A.31) can be written in the form

$$w_j L_j + w_{n+j} = (\frac{J_W}{WJ_{WW}})\ (1 - \frac{r_j - r}{\lambda_j L_j}); \tag{A.61}$$

and substituting (A.54) gives

$$w_j L_j + w_{n+j} = (\frac{1}{\nu-1})(1 - \frac{r_j - r}{\lambda_j L_j}) \tag{A.31'}$$

The pertinent question is how good an approximation is (A.31') to Equation (A.60)? The right hand side of (A.60) can be written as

$$1 - (1 + \frac{r_j - r - \lambda_j L_j}{\lambda_j L_j})^{\frac{1}{\nu-1}}$$

$$\simeq (\frac{1}{\nu-1})(1 - \frac{r_j - r}{\lambda_j L_j}) - \frac{1}{2}(\frac{1}{\nu-1})(\frac{1}{\nu-1} - 1)(1 - \frac{r_j - r}{\lambda_j L_j})^2 + \text{higher order terms.} \qquad (A.62)$$

Hence, the degree of approximation depends upon the ability to neglect the terms

$$- \frac{1}{2}(\frac{1}{\nu-1})(\frac{1}{\nu-1} - 1)(1 - \frac{r_j - r}{\lambda_j L_j})^2 + \text{higher order terms,}$$

but this will depend upon the value of ν. For small ν the approximation will be very good, but will drop off as ν increases in magnitude.

For one class of utility functions, the quadratic, the approximations will be exact, as can be shown by substituting in Equations (A.52), (A.53), (A.54) and (A.55).

This work offers some encouragement to use the approximation. It has been shown that for one type of utility function the approximation is exact, whilst for the class of constant relative risk aversion utilities functions the approximation can be very good, depending upon the value of the parameter of the function.

Stochastic Changes in the Rate of the Probability of Bankruptcy

Previously, it had been assumed that there were no stochastic changes in the investment opportunity set. The equation describing the bond price dynamics had been simplified so that there was no direct interaction between bonds and common stock apart from that of bankruptcy. Both assumptions are now relaxed so that effects of stochastic changes in the rate of the probability of bankruptcy upon the structure of returns can be analyzed.

The derived utility function will depend upon the individual's current wealth, W(t), a vector describing the current values of the rates of the probability of bankruptcy for the different firms, $\underline{\lambda}(t)$, a state vector describing which firms are currently in existence, $S(t)$, and time t. Thus the derived utility function may be written in the form $J[W(t), \underline{\lambda}(t), t, S(t)]$. To analyze the effects of stochastic changes in the rate of probability of bankruptcy it is necessary to assume a form that describes the stochastic nature of the mechanism generating the changes; for the j^{th} firm it is assumed that

$$d\lambda_j(t) = F_j(\lambda_j, t)dt + G_j(\lambda_j, t)dQ_j, \qquad j = 1, 2, \ldots, n, \tag{A.63}$$

which should be compared to the general system of equations described by Equation (A.5).

The equation of optimality can be derived from the general case described by Equation (A.17); that is

$$\begin{aligned}
0 = \underset{\{C,\underline{w}\}}{\text{Max}} \Big(& U[C(t),t] + J_t + \sum_{j=1}^{n} F_j J_j \\
&+ J_W\{W(t)[\sum_{j=1}^{n} w_j(t)(r_j - r) + \sum_{j=1}^{n} w_{n+j}(t)(\alpha_j - r) + r] - C(t)\} \\
&+ \frac{1}{2}J_{WW}[\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_{ji}w_i(t) + 2\sum_{j=1}^{n}\sum_{i=1}^{n} w_j(t)\gamma_j\rho_{ji}\sigma_i w_{n+i}(t) \\
&\quad + \sum_{j=1}^{n}\sum_{i=1}^{n} w_{n+j}(t)\sigma_{ji}w_{n+i}(t)]W(t)^2 \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} G_i\nu_{ij}G_jJ_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} W(t)w_j(t)\gamma_j\eta_{ij}G_iJ_{iW} \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} W(t)w_{n+j}(t)\sigma_j\eta_{i,n+j}G_iJ_{iW} \\
&+ \sum_{j=1}^{n}\lambda_j(t)\{J[W_j,\underline{\lambda}(t), t, S_j] - J[W(t), \underline{\lambda}(t), t, S(t)]\}\Big)
\end{aligned} \tag{A.64}$$

subject to the boundary condition $J[W(T),\lambda(T),T,S(T)] = BF[W(T),T]$. The set of first order maximization conditions are, after some manipulation,

$$0 = U_C[C(t),t] - J_W, \tag{A.65}$$

$$\begin{aligned} 0 = {} & (\alpha_j - r - \frac{r_j - r}{L_j})J_W + W(t)[\sum_{i=1}^{n} \sigma_{ji} w_{n+i}(t) - \sum_{i=1}^{n} \frac{\gamma_j \rho_{ij} \sigma_i}{L_j} w_{n+i}(t) \\ & + \sum_{i=1}^{n} \sigma_j \rho_{ji} \gamma_i w_i(t) - \sum_{i=1}^{n} \frac{\gamma_{ji} w_i(t)}{L_j}]\, J_{WW} \\ & + \sum_{i=1}^{n} \sigma_j \eta_{i,n+j} G_i J_{iW} - \sum_{i=1}^{n} \frac{\gamma_j \eta_{ij} G_i J_{iW}}{L_j}, \end{aligned} \tag{A.66}$$

and

$$\begin{aligned} 0 = {} & (r_j - r)J_W + W(t)[\sum_{i=1}^{n} \gamma_j \rho_{ij} \sigma_i w_{n+i}(t) + \sum_{i=1}^{n} \gamma_{ji} w_i(t)]J_{WW} \\ & + \sum_{i=1}^{n} \gamma_j \eta_{ij} \bar{G}_i J_{iW} - \lambda_j(t) L_j J_W[W_j, \underline{\lambda}(t), t, S_j\}, \end{aligned} \tag{A.67}$$

$j = 1, \ldots, n$.

Equation (A.65) is the intertemporal envelope condition. Equation (A.66) is derived by differentiating the equation of optimality by the proportion of wealth to invest in equity and Equation (A.67) is the equivalent equation for bonds. As in the previous section, this system of equations is non-linear. It is assumed that it is possible to replace the system of non-linear equations by an approximate linear system; that is, Equation (A.67) can be

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[12] See Equations (A.18), (A.20) and (A.21).

approximated by

$$o = (r_j - r - \lambda_j(t)L_j)J_W + W(t)[\sum_{i=1}^{n} \gamma_{ji}w_i(t) + \sum_{i=1}^{n} \gamma_j\rho_{ij}\sigma_i w_{n+i}(t)]J_{WW}$$

$$+ \lambda_j(t)L_j[w_j(t)L_j + w_{n+j}(t)]W(t)J_{WW}$$

$$+ \sum_{i=1}^{n} \gamma_j\eta_{ij}G_i J_{iW} \qquad \text{(A.68)}$$

$j = 1, 2, \ldots, n.$

Therefore, Equations (A66) and (A.68) describe two correlated linear systems of equations fromwhich it is possible to determine the demand functions for bonds and equity and then derive the equilibrium instantaneous conditional expected rates of return. Whilst such a derivation is conceptually simple, it is, unfortunately, mathematically very tedious. The complexity of the solution arises from all the covariance terms. It is, however, these terms which reflect the effects of bankruptcy upon the structure of returns.

Define the following (nxl) vectors

$$\{\underline{a}\}_j = \alpha_j - r - \frac{r_j - r}{L_j};$$

$$\{\underline{c}\}_j = r_j - r - \lambda_j L_j; \qquad \text{(A.69)}$$

$$\{\underline{w}_1\}_j = W(t)w_j(t);$$

$$\{\underline{w}_2\}_j = W(t)w_{n+j}(t);$$

and

$$\{\underline{U}\}_j = J_{jW};$$

$j = 1, 2, n,$

and the following (nxn) matrices

$$\{\underline{\Sigma}_1\}_{ij} = \gamma_{ij};$$

$$\{\underline{\Sigma}_2\}_{ij} = \sigma_{ij};$$

$$\{\underline{\Sigma}_3\}_{ij} = \gamma_i \rho_{ij} \sigma_i;$$

$$\{\underline{\Sigma}_4\}_{ij} = \gamma_i \eta_{ij} G_i;$$

$$\{\underline{\Sigma}_5\}_{ij} = \sigma_j \eta_{i,n+j} G_i;$$

$$\{\underline{L}\}_{ij} = L_j \delta_{ij};$$

and

$$\{\underline{\lambda}\}_{ij} = \lambda_i \delta_{ij}; \quad i,j = 1, 2, \ldots, n, \tag{A.70}$$

where

$$\delta_{ij} = \begin{cases} 1; & i=j, \\ 0; & i \neq j. \end{cases}$$

There, in matrix notation the set of equation described by (A.66) can be written in the form

$$0 = J_W \underline{a} + (\underline{\Sigma}_5 - \underline{L}^{-1}\underline{\Sigma}_4)\underline{U} + J_{WW}(\underline{\Sigma}_2\underline{w}_2 - \underline{L}^{-1}\underline{\Sigma}_3\underline{w}_2 + \underline{\Sigma}_3\underline{w}_1 - \underline{L}^{-1}\underline{\Sigma}_1\underline{w}_1), \tag{A.71}$$

and Equation (A.68) in the form

$$0 = J_W\underline{c} + \underline{\Sigma}_4\underline{U} + J_{WW}(\underline{\Sigma}_1\underline{w}_1 + \underline{\Sigma}_3\underline{w}_2 + \underline{\lambda}\,\underline{L}\,\underline{L}\,\underline{w}_1 + \underline{\lambda}\,\underline{L}\,\underline{w}_2). \tag{A.72}$$

This set of equations may be written in a more compact form:

$$H_1\underline{a} + \underline{D}_3\underline{H}_2 = \underline{D}_1\underline{w}_1 + \underline{D}_{12}\underline{w}_2, \tag{A.73}$$

and

$$\underline{H}_1\underline{c} + \underline{D}_4\underline{H}_2 = \underline{D}_{21}\underline{w}_1 + \underline{D}_2\underline{w}_2, \tag{A.74}$$

where

$$\underline{D}_1 = \underline{\Sigma}_3 - \underline{L}^{-1}\Sigma_1;$$

$$\underline{D}_{12} = \underline{\Sigma}_2 - \underline{L}^{-1}\Sigma_3;$$

$$\underline{D}_{21} = \underline{\Sigma}_1 + \underline{\lambda}\ \underline{L}\ \underline{L};$$

$$\underline{D}_2 = \underline{\lambda}\ \underline{L} + \underline{\Sigma}_3;$$

$$\underline{D}_3 = \underline{\Sigma}_3 - \underline{L}^{-1}\underline{\Sigma}_4;$$

$$\underline{D}_4 = \underline{\Sigma}_4;$$

$$H_1 = -\frac{J_W}{J_{WW}} > 0;$$

and

$$\{H_2\}_j = -\frac{J_W}{J_{WW}} \lesseqgtr 0,$$

$$j = 1, 2, \ldots, n.$$

Thus the demand functions for equity are

$$\underline{w}_2 = H_1\underline{E}_2(\underline{a} - \underline{D}_1\underline{D}_{21}^{-1}\underline{c}) + \underline{E}_2(\underline{D}_3 - \underline{D}_2\underline{D}_{21}^{-1}\underline{D}_4)\underline{H}_2, \tag{A.75}$$

and for bonds

$$\underline{w}_1 = H_1\underline{E}_1(\underline{c} - \underline{D}_2\underline{D}_{12}^{-1}\underline{a}) + \underline{E}_1(\underline{D}_4 - \underline{D}_2\underline{D}_{12}\underline{D}_3)\underline{H}_2 \tag{A.76}$$

where

$$\underline{E}_1 = (\underline{D}_{21} - \underline{D}_2\underline{D}_{12}^{-1}\underline{D}_1)^{-1},$$

and

$$\underline{E}_2 = (\underline{D}_{12} - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_2)^{-1}.$$

To derive the expression for the equilibrium instantaneous expected rate of return for equities, consider Equation (A.75) which can be written in the form

$$\underline{w}_2^k = H_1^k \underline{E}_2(\underline{a} - \underline{D}_1\underline{D}_{21}^{-1}\underline{c}) + \underline{E}_2(\underline{D}_3 - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_4)H_2^k \tag{A.77}$$

where the superscript k denotes the k^{th} individual. Summing over all individuals, the aggregate demand functions for equity can bederived and the instantaneous market equilibrium condition used. Let $\underline{ASE}$ be a (nx1) vector describing the aggregate supply for equity; that is, $\{\underline{ASE}\}_j = \bar{N}_{n+j}p_j(t)$, $j = 1, 2,\ldots,n$. Hence,

$$\underline{ASE} = A_1\underline{E}_2(\underline{a} - \underline{D}_1\underline{D}_{21}^{-1}\underline{c}) + \underline{E}_2(\underline{D}_3 - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_4)\underline{A}_2,$$

which implies that

$$\underline{a} = \frac{1}{A_1}\underline{E}_2^{-1}\underline{ASE} + \underline{D}_1\underline{D}_{21}^{-1}\underline{c} - (\underline{D}_3 - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_4)\frac{1}{A_1}\underline{A}_2, \tag{A.78}$$

where

$$A_1 = \sum_{k=1}^{I} H_1^k = \sum_{k=1}^{I} -\frac{J_W^k}{J_{WW}^k},$$

and

$$\{A_2\}_j = \sum_{k=1}^{I} H_{2j}^k = \sum_{k=1}^{I} -\frac{J_{jW}^k}{J_{WW}^k}.$$

For bonds a similar relation can be determined. From Equation (A.77) the demand functions for the k^{th} individual can be written in the form

$$\underline{w}_1^k = H_1^k\underline{E}_1(\underline{c} - \underline{D}_1\underline{D}_{21}^{-1}\underline{a}) + \underline{E}_1(\underline{D}_4 - \underline{D}_2\underline{D}_{12}^{-1}\underline{D}_3)\underline{H}_2^k, \tag{A.79}$$

and thus

$$\underline{c} = \frac{1}{A_1}\underline{E}_1^{-1}\underline{ASB} + \underline{D}_2\underline{D}_{12}^{-1}\underline{a} - (\underline{D}_4 - \underline{D}_4\underline{D}_{12}^{-1}\underline{D}_3)\frac{1}{A_1}\underline{A}_2 \tag{A.80}$$

where

$\{\underline{ASB}\}_j = \bar{N}_j b_j(t)$,

$j = 1, 2,\ldots,n.$

Equations (A.78) and (A.80) describe the equilibrium instantaneous conditional expected rates of return for equity and bonds, respectively. The equations are not, however, independent of the preference structure of individuals due to the presence of the terms A_1 and $\{\underline{A}_2\}_j$, $j = 1, 2, \ldots, n$. Whilst it is possible to eliminate these terms, the resulting complexity and general lack of insight that results does not warrant the effect. Some insight can be gained by assuming that, except for one firm, there are no stochastic changes in the rate of the probability of bankruptcy; this is equivalent to assuming that the stochastic changes in the rate of the probability of bankruptcy for this one firm act as an instrumental variable. For convenience, call this firm the n^{th} firm. For this case, A_2 is now a scalar quantity and $\underline{\Sigma}_4$ and $\underline{\Sigma}_5$ will be (nxl) vectors, which implies that $\underline{D}_3$ and $\underline{D}_4$ will be (nxl) vectors.

Let $M(t)$ represent the total market value of all equity

$$M(t) = \sum_{j=1}^{n} \bar{N}_{n+j} p_j(t);$$

and $Y_j(t)$ represent the proportion of the total value of the j^{th} firm's equity to the total market value of all equity; that is $Y_j(t) = \bar{N}_{n+j} p_j(t)/M(t)$. Define a (nxl) vector $\underline{Y}$ such that $\{\underline{Y}\}_j = Y_j(t), j = 1, 2, \ldots, n$. Thus, substituting into Equation (A.78) gives

$$\underline{a} = \frac{M}{A_1} \underline{E}_2^{-1} \underline{Y} - \frac{A_2}{A_1} \underline{D}_3 + \underline{D}_1 \underline{D}_{21}^{-1} (\underline{c} + \frac{A_2}{A_1} \underline{D}_4), \qquad \text{(A.81)}$$

which may be written in the scalar form

$$\alpha_j - r - \frac{r_j - r}{L_j} = \frac{M}{A_1} \sum_{i=1}^{n} \xi_{ji} Y_i(t) - \frac{A_2}{A_1} [\sigma_{j\lambda} - \frac{\gamma_{j\lambda}}{L_j} + \sum_{i=1}^{n} \varepsilon_{ji} \gamma_{i\lambda}]$$

$$+ \sum_{i=1}^{n} \varepsilon_{ji} (r_i - r - \lambda_i L_i),$$

where

$$\xi_{ji} = \{\underline{D}_{12} - \underline{D}_1\underline{D}_{21}^{-1}\underline{D}_2\}_{ji}$$

$$\varepsilon_{ji} = \{\underline{D}_1\underline{D}_{21}^{-1}\}_{ji},$$

$$\sigma_{j\lambda} = \sigma_j\eta_{n,n+j}G_n,$$

$$\gamma_{j\lambda} = \gamma_j\eta_{n,j}G_n.$$

$$j = 1, 2, \ldots, n.$$

Multiplying the above equation by $Y_j(t)$ and summing over j gives

$$\mu - r - \bar{\chi} = \frac{M}{A_1} \sum_{j=1}^{n} \sum_{i=1}^{n} Y_j(t)\xi_{ji}Y_i(t) - \frac{A_2}{A_1}(\sigma_{M\lambda} - \gamma_{M\lambda} + \sum_{j=1}^{n} \sum_{i=1}^{n} Y_j(t)\varepsilon_{ji}\gamma_{i\lambda})$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{n} Y_j(t)\varepsilon_{ji}(r_i - r - \lambda_i L_i), \tag{A.82}$$

where

$$\sigma_{m\lambda} = \sum_{j=1}^{n} Y_j(t)\sigma_{j\lambda},$$

and

$$\gamma_{M\lambda} = \sum_{j=1}^{h} Y_j(t) \frac{\gamma_{j\lambda}}{L_j}$$

The preference structure can be removed from Equation (A.82). The equity of the n^{th} firm must satisfy Equation (A.81), that is

$$\alpha_n - r - \frac{r_n - r}{L_n} = \frac{M}{A_1} \sum_{i=1}^{n} \xi_{ni}Y_i(t) - \frac{A_2}{A_1}(\sigma_{n\lambda} - \frac{\gamma_{n\lambda}}{L_n} + \sum_{i=1}^{n} \varepsilon_{ni}\gamma_{i\lambda})$$

$$+ \sum_{i=1}^{n} \varepsilon_{ni}(r_i - r - \lambda_i L_i),$$

which can be written in the form

$$\alpha_n - r - \frac{r_n - r}{L_n} - \varepsilon_n = \frac{M}{A_1}\sigma_{nM} - \frac{A_2}{A_1}\delta_{n\lambda}, \tag{A.83}$$

where

$$\varepsilon_n = \sum_{i=1}^{n} \varepsilon_{ni}(r_1 - r - \lambda_i L_i)$$

and

$$\delta_{n\lambda} = \sigma_{n\lambda} - \frac{\gamma_{n\lambda}}{L_n} + \sum_{i=1}^{n} \varepsilon_{ni}\gamma_{i\lambda}$$

From Equation (A.83) we have the relationship

$$\mu - r - \bar{\chi} - \varepsilon_M = \frac{M}{A_1}\sigma_M^2 - \frac{A_2}{A_1}\delta_{M\lambda}, \tag{A.84}$$

where

$$\varepsilon_M = \sum_{j=1}^{n}\sum_{i=1}^{n} Y_j(t)G_{ji}(r_i - r - \lambda_i L_i);$$

$$\delta_{M\lambda} = \sigma_{M\lambda} - \gamma_{M\lambda} + \sum_{j=1}^{n}\sum_{i=1}^{n} Y_j(t)\varepsilon_{ji}\gamma_{i\lambda};$$

and

$$\sigma_M^2 = \sum_{j=1}^{n}\sum_{i=1}^{n} Y_j(t)\xi_{ji}Y_i(t).$$

From Equations (A.83) and (A.84) it is possible to solve for $(\frac{M}{A_1})$ and $(\frac{A_2}{A_1})$ and the results substituted into Equation (A.81) to determine the instantaneous conditional expected rate of return

$$\alpha_k - r - \frac{r_k - r}{L_k} - \varepsilon_k = \left(\frac{\sigma_{kM}\delta_{M\lambda} - \delta_{k\lambda}\sigma_M^2}{Q}\right)\left(\alpha_n - r - \frac{r_n - r}{L_n} - \varepsilon_n\right)$$

$$\left(\frac{\sigma_{nM}\delta_{k\lambda} - \delta_{n\lambda}\sigma_{kM}}{Q}\right)(\mu - r - \bar{\chi} - \varepsilon_M), \tag{A.85}$$

$k = 1, 2, \ldots, n-1.$

where

$$Q = \sigma_{nM}\delta_{m\lambda} - \delta_{n\lambda}\sigma^2_M.$$

A similar result holds for the bond equation (A.80) which may be written in the scalar form, using Equation (A.41),

$$r_j - r - \lambda_j L_j = \frac{M_B}{A_1}\sum_{i=1}^{n} {}_1\xi_{ji}X_i(t) - \frac{A_2}{A_1}[\gamma_{j\lambda} - \sum_{i=1}^{n} {}_1\varepsilon_{ji}(\sigma_{i\lambda} - \frac{\gamma_{i\lambda}}{L_i})] + \sum_{i=1}^{n} {}_1\varepsilon_{ji}(\alpha_i - r - \frac{r_i - r}{L_i}), \tag{A.86}$$

$j = 1, 2, \ldots, n.$

Let

$${}_1\varepsilon_j = \sum_{i=1}^{n} {}_1\varepsilon_{ji}(\alpha_i - r - \frac{r_i - r}{L_i}),$$

and

$${}_1\varepsilon_{j\lambda} = \gamma_{j\lambda} - \sum_{i=1}^{n} {}_1\varepsilon_{ji}(\sigma_{i\lambda} - \frac{\gamma_{i\lambda}}{L_i}),$$

$j = 1, 2, \ldots, n,$

and thus Equation (A.86) may be written

$$r_j - r - \lambda_j L_j - {}_1\varepsilon_j = \frac{M_B}{A_1}\sum_{i=1}^{n} {}_1\varepsilon_{ji}X_i(t) - \frac{A_2}{A_1}\,{}_1\partial_{j\lambda}, \tag{A.87}$$

$j = 1, 2, \ldots, n.$

Multiplying Equation (A.87) by $X_j(t)$ and summing over j gives, using Equations (A.42) and (A.44)

$$\pi - r - \gamma - {}_1\varepsilon_M = \frac{M_B}{A_1}\gamma_M^2 - \frac{A_2}{A_1}\delta_{M\lambda}, \tag{A.88}$$

where

$$\gamma_M^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} X_j(t)\,{}_1\xi_{ji} X_i(t),$$

and

$$\delta_{M\lambda} = \sum_{j=1}^{n} X_j(t)\delta_{j\lambda}.$$

The instantaneous conditional expected rate of return for the n^{th} firm's bonds must also satisfy Equation (A.87); that is,

$$r_n - r - \lambda_n L_n - {}_1\varepsilon_n = \frac{M_B}{A_1}\gamma_{nM} - \frac{A_2}{A_1}\,{}_1\delta_{n\lambda}, \tag{A.89}$$

where

$$\gamma_{nM} = \sum_{i=1}^{n} {}_1\xi_{ni} X_i(t).$$

Equations (A.88) and (A.89) can be used to eliminate $(\frac{M_B}{A_1})$ and $(\frac{A_2}{A_1})$ in Equation (A.87) to give

$$r_j - r - \lambda_j L_j - {}_1\varepsilon_j = \left(\frac{{}_1\delta_n \gamma_{jM} - {}_1\delta_{j\lambda}\gamma_M^2}{Q_1}\right)(r_n - r - \lambda_n L_n - {}_1\varepsilon_n)$$
$$\left(\frac{{}_1\delta_{j\lambda}\gamma_{nM} - {}_1\delta_n \gamma_{jM}}{Q_1}\right)(\pi - r - \gamma - {}_1\varepsilon_M), \tag{A.90}$$

$j = 1, 2, \ldots, n-1,$

where

$$Q_1 = \gamma_{nM}\,{}_1\delta_{M\lambda} - {}_1\delta_{n\lambda}\gamma_M^2.$$

APPENDIX B

NAMES OF BANKRUPT FIRMS

NAME	DATE OF BANKRUPTCY
Atlas Sewing Center	1962
Avien Incorporated	1964
Barcalo Manufacturing Company	1965
Barchris Construction Corporation	1962
Betteringer Corporation	1961
Bishop Oil Company	1961
Bowl-Mor Company	1966
Buckner Industries Incorporated	1967
Davega Stores Corporation	1962
Dejay Stores Incorporated	1962
Dilbert's Quality Supermarkets Incorporated	1962
Erie Forge and Steel Corporation	1969
Fashion Tree	1968
Gilbert (A.C.) Company	1967
Goebel Brewing	1964
Grayson-Robinson Stores Incorporated	1962
Great Western Producers Incorporated	1965
Guidance Technology Incorporated	1962
International Oil and Gas Corporation	1965
Keystone Alloys Company	1966
Marrud Incorporated	1966

NAME	DATE OF BANKRUPTCY
McCandless Corporation	1968
Muskegon Motor Specialties Company	1961
National Video Corporation	1969
Okalta Oils	1961
Polycast Corporation	1966
Precision Radiation Instruments Incorporated	1963
Premier Albums Incorporated	1968
Puerto Rico Brewing Company Incorporated	1969
Trans-United Industries Incorporated	1963
United States Chemical Milling Corporation	1962
Vinco Corporation	1963
Webcor Incorporated	1967
Yuba Consolidated Industries	1961

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