INFORMATION IN STOCK PRICES AND TRADING VOLUME

by

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Abstract

In this thesis I develop two theoretical models to analyze how investors can infer private information from market prices and aggregate trading volume. In the first chapter I provide a closed form solution for a rational expectations equilibrium where all investors infer information about the state of the economy from (1) private signals, (2) the market price and (3) aggregate trading volume. The main result of this model is that trading volume reveals the relative quality of the aggregate private information in the economy. Investors use volume to decide how they should weight the market price relative to their own private signals when they update their beliefs. In the second chapter, I assume that investors make individual mistakes when they infer information from the price. I show that in a heterogeneous information economy, bounded rationality on the individual level is observationally equivalent to a psychological bias on the aggregate level. If the investors are not able to infer perfectly the true state of the economy from the price, then the aggregate demand corresponds to the demand of a representative agent who is "underconfident". The underconfidence of the representative agent causes the price to adjust to new information too slowly.

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Chapter 1

Introduction

In a financial market every investor is interested in the private information that other investors might possess. There are several ways to learn what other market participants know. In this thesis I examine how investors can infer private information from market prices and aggregate trading volume. I develop two theoretical models to address the following questions: (1) how can investors fully rationally infer information from prices and trading volume, and (2) how do the results of a world with perfectly rational investors change, if the rationality of investors is bounded, so that they are not fully able to infer all available private information?

Grossman (1976) was one of the first authors who showed how investors can extract information about the future payoff of a security from its price. In the first part of this thesis I extend the Grossman model to a case where investors extract information from both the price and the trading volume. This model provides a closed form solution for a rational expectations equilibrium where all investors infer information about the state of the economy from (1) private signals, (2) the market price and (3) aggregate trading volume.

Several empirical studies support the idea that trading volume contains information about future returns. For example, Llorente, Michaely, Saar, and Wang (LMSW, 2002) solve and test a model where trading volume predicts changes in the autocorrelation of returns.[1] However, the investors in LMSW's model rationally ignore trading volume when they update their beliefs, since volume does not contain any information beyond their own private signals and the market price. Therefore, in LMSW, trading volume provides information only to an outside observer of the economy, but not to investors within the economy.

My model shows how investors within the economy can learn from trading volume, and how volume information differs from information contained in the price. The main result of this model is that trading volume reveals the relative quality of the aggregate private information in the economy. Under low trading volume, the aggregate information is more precise compared to private signals than under high trading volume. Investors therefore use volume to decide how they should weight the market price relative to their own private signals when they update their beliefs. When trading volume is low, investors weight the market price more heavily. Conversely, when volume is high, investors weight their private signals more heavily.

In order to show how investors infer information from trading volume, I develop a model where a large number of small investors observe private noisy signals about a future dividend. In addition to their endowment of information, the investors are also endowed with private claims to a risky future labor income. The dividend and the labor income are correlated, so that investors have two motives for trading:

[1]For other evidence see also Gallant, Rossi, and Tauchen (1992), Campbell, Grossman, and Wang (1993), Conrad, Hameed, and Niden (1994), Lee and Swaminathan (2000), Gervais, Kaniel, and Mingelgrin (2001), and Connolly and Stivers (2003).

private information and risk sharing.

Private signals and labor endowments are identically distributed for all investors, so that all investors observe information of identical quality. Therefore, investors weight their signals equally when they update their beliefs. As a result, the equilibrium price depends on the average signal and the average exposure to the labor risk. Since investors are uncertain about the average labor risk, they are not able to fully infer the average dividend signal from the price.

In addition to the uncertainty about the aggregate dividend information and aggregate labor risk, the investors are also uncertain about the cross-investor correlation of the individual errors in their private dividend signals. The correlation of the individual signal errors is important, since this correlation determines the quality of the aggregate information relative to the private information. For example, if the signals are perfectly correlated, then the average signal contains the same information as the individual signals. However, if investor specific signal errors are uncorrelated, then investors know more in aggregate than they know individually. Investors therefore wish to know the correlation of the individual signals, in order to assess the precision of the average signal in the price.

In a symmetric economy where private information and labor risks are identically distributed for all investors, trading volume reveals the correlation of signals in the following way: since investors weight their signals and endowments identically when they calculate their demands, the number of shares that a given investor buys or sells depends only on the differences between his private signal and endowment and the signal and endowment of the average investor in the economy. Therefore, the individual trades are functions only of the investor specific components of signals and endowments. Hence, if these components are independent across investors, and if the number of investors in the economy is large, the per capita trading volume depends only on the distribution and not on the realization of these components. As a result, investors can infer the distribution of signals from trading volume. In particular, if investors are uncertain about the correlation of their private signals, then trading volume reveals this correlation.

In traditional models of heterogeneous information, such as Grossman and Stiglitz (1980), investors form a weighted average of their private signals and the market price when they update their beliefs. In these models, all investors know the optimal weights for the price and the signals, since these weights are independent of the state of the economy. In my model, investors are uncertain how they should weight the market price relative to their own private information. Observing trading volume removes this uncertainty. Under high trading volume, the quality of the aggregate signal in the price does not exceed the quality of their private signals. Since the price contains additional noise from the aggregate labor income shock, investors weight their own signals more heavily than the price when volume is high. However, under low trading volume, the quality of the aggregate information exceeds the quality of the individual signals. Therefore, investors weight the price more heavily when trading volume is low.

The idea that investors have a risk sharing and a private information motive for trading has been previously employed for example by Wang (1994) and LMSW (2002). However, in these models, there are only two agents that trade with each other. Therefore, trading volume does not provide any information for the investors beyond the information that they can infer from their own private signals and the market price. The technical difficulty that arises if investors are allowed to observe trading volume is that volume is a sum of absolute values and therefore not normally distributed. Asset pricing models with heterogeneously informed investors usually rely on the properties of the normal distribution in order to be tractable.

My model solves the problem that trading volume is not normally distributed by transforming a non-linear optimization problem into a problem that is linear conditional on the observation of trading volume. Several other authors have examined alternative approaches. For example, Bernardo and Judd (1996) show how to numerically solve a model where investors learn private information from trading volume. Their numerical approach has the advantage that it covers a large set of possible assumptions, however, a numerical approach does not provide the same clean economic intuition as an analytical solution.

Blume, Easley, and O'Hara (BEH, 1994) provide a closed-form solution for a model where investors learn from past prices and past trading volume. Similar to my model, the investors in BEH face two dimensions of uncertainty: the realization and the quality of other investors' signals. However, in order to solve their model, BEH have to assume that investors are not fully rational: even though investors know the price at which they trade, they employ this price in order to update their beliefs only after they have completed their trade. BEH (page 160) comment on the difficulty of solving a rational expectations equilibrium where investors learn from trading volume: "Alternatively, there could be nonrevealing equilibria in which traders condition on price and volume. However, as volume is a sum of absolute values it cannot be normally distributed. So although such an equilibrium might exist there seems to be no hope of constructing it, and hence no hope of using a contemporaneous data approach to study volume." As I show in this paper, the case for a non-revealing equilibrium where investors condition their demands on prices and on volume is not completely hopeless.

In the first part of this thesis I assume that all investors are fully rational, and that they are able to optimally analyze prices and volume. However, in reality prices aggregate information in a complicated way, and extracting this information is a difficult task. In the second part of this thesis I assume that the computational skills of the investors are limited: they make individual mistakes when they infer information from the price. I show that in a heterogeneous information economy where investors' rationality is bounded prices react too slowly to new information.

A large number of empirical studies document that prices potentially underreact to new information. For example, Ball and Brown (1968), Bernard and Thomas (1990) and others find that firms reporting unexpectedly high earnings outperform firms reporting unexpectedly low earnings. Givoly and Lakonishok (1979), and Stickel (1991) document similar drifts after analyst forecast revisions. Jegadeesh and Titman (1993) rank stocks according to past returns and find that past winners outperform past losers.

The debate whether these findings violate market efficiency is ongoing. Some authors argue that return continuations can be explained with changes in firm's risks. Other authors argue that prices underreact to new information because investors are psychologically biased. For example, Barberis, Shleifer, and Vishny (BSV, 1998) link underreaction to the conservatism bias. The conservatism bias has been identified in experiments by Edwards (1968). Individuals who are subject to this bias tend to underweight new information when they update their priors. BSV show that, if the representative agent in the economy suffers from the conservatism bias, then prices will adjust to new information slowly.[2]

[2]Other psychologically motivated explanations include overconfidence as for example in Daniel,

In this paper I show that even if the representative agents appears to be psychologically biased, this is not necessarily true for the individual investors. In a heterogeneous information economy where the computational skills of the investors are limited, so that they are not perfectly able to infer each other's information from the price, the aggregate demand is equivalent to the demand of a representative agent who underestimates the quality of his information. This underconfidence of the representative agent arises from the fact that the investors know more in aggregate than they know individually, and that they fail to infer perfectly the aggregate knowledge from the price. In that way, bounded rationality on the individual level is equivalent to a psychological bias on the aggregate level.

Several empirical studies support the notion that heterogeneous beliefs play an important role in the momentum phenomenon. For example, Verardo (2002) finds that profits from momentum strategies increase with the dispersion in analyst forecasts. Hong, Lim, and Stein (2000) find that momentum strategies work better among small stocks with low analyst coverage. Zhang (2005) finds that the post earnings announcement drift, the drift after analyst forecast revisions, and the profits from price momentum strategies all increase with various proxies for heterogeneous beliefs.

To see how heterogenous information causes return continuations consider the following example. There is a firm that will pay out an uncertain future dividend. The investors receive a signal about this dividend of the form "dividend + noise". Assume for the moment that this signal is public, and that all investors interpret the signal in the same way. If this signal becomes more precise as the dividend payout

Hirshleifer, Subrahmanyam (1998), and the disposition effect as for example in Frazzini (2005) and Grinblatt and Han (2005).

date approaches, the investors will increase the weight of the signal, as they update their beliefs. This updating process has two effects on the serial correlation of the returns. First, the "true" component of the signal produces a positive effect, since this value will be slowly incorporated into the price. Second, the noise component produces a negative effect, since, as the signal becomes more precise, the investors will reverse that part of their initial reaction to the signal that was due to the noise. If the economy is in a steady state, and the risk that the average investors has to bear does not change over time, these two effects will offset each other exactly. Hence, uncertain information does not produce serially correlated returns, if the investors are homogenously informed.

Assume now that the signal contains an investor specific noise component, so that the investors are heterogeneously informed. Assume that the rationality of the investors is bounded, and that the investors are not able to infer perfectly all the private information from the price. Since the individual signals are noisy, the investors will assign only small weights to these signals when they update their priors. However, the individual noise components will at least partially cancel each other out, when one aggregates the demands. Therefore, the aggregation of demands will reduce the negative noise effect on the correlation of returns, without reducing sufficiently the positive effect resulting from the true information component of the signals. The aggregate demand will be equivalent to the demand of a representative agent who is underconfident. As a result, prices adjust to new information too slowly.

Several other authors have developed heterogeneous information models to explain momentum. In these models, returns are positively autocorrelated because investors receive information sequentially. For example, in Hong and Stein (1999) information about a liquidating dividend spreads slowly through a group of "newswatchers". Since information spreads slowly, and since the newswatchers are not able to infer information from the price, the price adjusts slowly to each piece of new information. In Holden and Subrahmanyam (2002) some investors receive certain pieces of information before other investors. In their model, noise traders prevent the price from revealing all the information. Holden and Subrahmanyam state on page 4: "Thus, our consideration of the sequential nature of information acquisition is the key to generating positive serial correlation within a rational expectations model." In this paper, I show that the sequential information flow is not a necessary condition for return continuations. Instead, the key to momentum is simply the disagreement about future payoffs.

I show in this paper how heterogeneous beliefs combined with bounded rationality leads to aggregate underreaction to new information. If the price underreacts to new information, momentum traders will rationally chase trends, as for example in Hong and Stein (1999). I do not examine the effect of momentum traders on prices and returns in this paper. In the real world, we would expect that the actions of momentum traders diminish the abnormal profits from momentum strategies. It is therefore surprising that, for example, Grundy and Martin (2001) find that momentum strategies can produce risk adjusted returns of more than one percent per month. However, these momentum profits are profits before trading costs. The evidence for the profitability of momentum strategies after trading costs is less clear. For example, Lesmond, Schill, and Zhou (2004) find that momentum strategies are not profitable after trading costs. Korajczyk and Sadka (2004) construct a liquidity weighted momentum strategy in order to minimize trading costs. Taking price impact trading costs into account, they estimate that this strategy earns positive abnormal profits for an investment of up to \$5 billion.[3] In this paper, I do not address the question to which degree momentum traders should arbitrage momentum profits away. Instead, my focus is to show a compelling source of momentum.

[3]As of December 1999. At that time, the total market capitalization of the NYSE was \$11.7 trillion.

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Chapter 2

A Rational Expectations Equilibrium with Informative Trading Volume

2.1 Introduction

In this chapter I develop a rational expectations equilibrium where investors infer private information from the market price and from trading volume. All investors in this model observe private noisy signals about a future dividend and they are also endowed with claims to a risky future labor income. Investors have therefore two motives for trading: private information and risk sharing. Since investors are uncertain about the economy wide exposure to the labor risk, they are not able to fully infer the aggregate private information from the price.

In addition to the uncertainty about the aggregate dividend information and aggregate labor risk, the investors are also uncertain about the cross-investor correlation of the individual errors in their private dividend signals. The correlation of the individual signal errors is important, since this correlation determines the quality of the aggregate information relative to the private information. In the equilibrium, investors learn the correlation of signals from trading volume, and they learn the average signal from the price.

2.2 Setup of the model

The economy is populated by a countable set of investors. I will refer to an individual investor as investor i, $i = 1, 2, \ldots$. There are two time periods, $t = 0$ and $t = 1$. Figure 2.1 shows a picture of the time line. At time $t = 0$, investor i is endowed with

$$N_i = N + n_i$$

units of a non-traded asset. At time $t = 1$ the investors receive a payoff of Y for each unit N_i they are endowed with at time $t = 0$. The total non-traded income of investor i at time $t = 1$ is therefore given by $N_i Y$. I will refer to $N_i Y$ as the labor income of investor i, even though other interpretations are possible.

In addition to the labor income, the investors also receive income from their investments in the financial market. The financial market consists of two assets: a risk free bond and a risky firm. One dollar invested in the bond at time $t = 0$ pays one dollar at time $t = 1$. Investors can buy or sell an unlimited amount of the bond. Investors can trade shares of the firm at time $t = 0$ at the equilibrium price P. At time $t = 1$ the firm pays a liquidating dividend D for each share the investors hold at time $t = 0$ after trading. At time $t = 0$ the investors observe private noisy signals

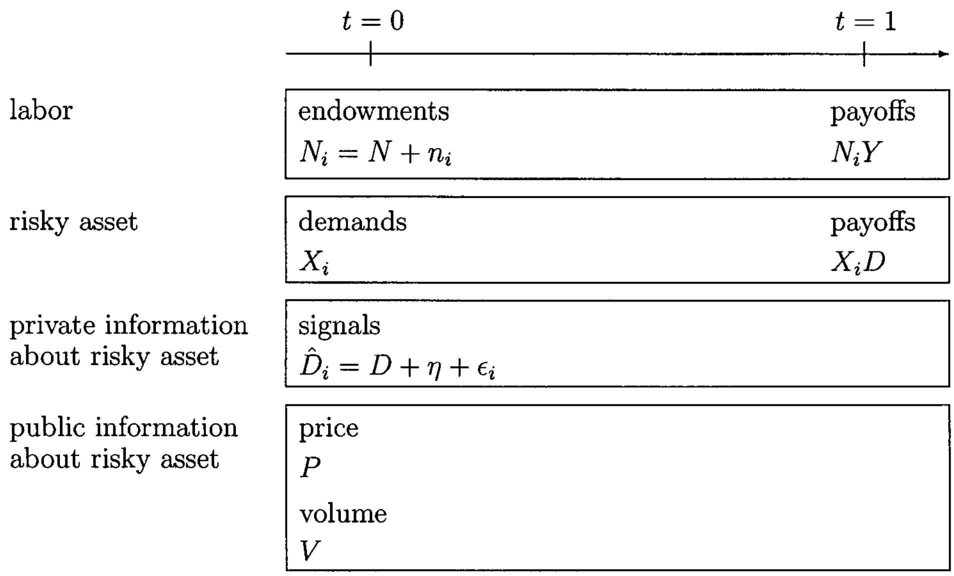


Figure 2.1: **Time Line.** This figure shows endowments, demands, payoffs, private information and public information.

about this dividend. The signal of investor i is given by

$$\hat{D}_i = D + \eta + \epsilon_i$$

where η and ϵ_i are error terms. The correlation of the ϵ_i across investors determines how much the investors disagree about the future payoff. There are two possible states of the world regarding this correlation. In state L, the correlation of signal errors is low. In this state, the ϵ_i are independent across investors. In state H the correlation of errors is high. In this state, all individual signals contain the same error term $\epsilon_i = \epsilon$, so that the errors are perfectly correlated across investors and all investors observe the same signal $D + \eta + \epsilon$. The investors might have some information about this correlation, however they do not know the realization of the correlation with certainty. For any given realization of the correlation state the random variables D, N, Y, η, $\{\epsilon_i\}_{i=0}^{\infty}$, $\{n_i\}_{i=0}^{\infty}$ are jointly normally distributed with mean zero and variances σ_D^2, σ_N^2, σ_Y^2, σ_η^2, σ_ϵ^2 and σ_n^2.[1] All variables are uncorrelated except for the correlation of the ϵ_i in state H, and except for D and Y, which are correlated with $\mathrm{Cov}[D,Y] = \sigma_{DY} > 0$.

Let X_i be the time $t = 0$ demand for the risky asset of investor i. Let

$$X = \lim_{h \to \infty} \frac{1}{h} \sum_{i=1}^{h} X_i$$

be the per capita demand, provided that this limit exists. An equilibrium is given by a price P that satisfies

$$X = \text{supply per capita} \tag{2.1}$$

with probability one. To simplify the notation I will set the supply equal to zero, and I assume that all investors own zero shares prior to the trading date $t = 0$. The assumption of zero supply means that all dividends that investors who hold long positions of the firm receive are paid by investors who hold short positions. This assumption will remove a constant from the equilibrium price, but it will not affect any of the results.

Since investors hold zero shares before trading, the number of shares that investor i trades is given by $|X_i|$. Let

$$V = \lim_{h \to \infty} \frac{1}{h} \sum_{i=1}^{h} |X_i|$$

[1]Since all variables are normally distributed, dividends and labor income can be negative. It is possible to choose means and variances so that the probability for negative payoffs will be arbitrarily small. However, since all investors know these distributions, the choice of the mean does not affect the trading volume. In order to simplify the notation I set therefore all the means equal to zero.

be the (double counted) per capita trading volume, provided that this limit exists.

Investor i chooses his demand X_i by maximizing

$$\mathrm{E}\Big[-e^{-\rho W_i}\Big|\mathcal{F}_i\Big]$$

where ρ is the coefficient of absolute risk aversion, W_i is the future wealth, and $\mathcal{F}_i$ is the information set of investor i. This information set is given by

$$\mathcal{F}_i = \{\hat{D}_i, N_i, P, V\}.$$

Investors can therefore condition their demand on their private dividend signals, their private labor endowments, the equilibrium price, and the equilibrium trading volume. Note that investors know N_i, their own exposure to the labor risk, but they do not observe N, the economy wide exposure to this risk. This assumption will prevent the equilibrium price from completely revealing all private information.

2.3 The equilibrium

Let

$$\hat{D} = \lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{h} \hat{D}_i$$

be the aggregate information about the future dividend. Then we have

$$\hat{D} = \begin{cases} D + \eta & \text{in state } L \\ D + \eta + \epsilon & \text{in state } H \end{cases}$$

If the correlation of signal errors ϵ_i is low (state L), then the individual signal errors cancel each other out in the average signal. Only in this case investors are able to learn information about the future dividend from the price. If the correlation of signals is high (state H), all investors observe the aggregate signal $\hat{D} = D + \eta + \epsilon$ directly, so that the price does not provide any additional information about D. Investors are not only uncertain about the realization of the average signal $\hat{D}$, but they are also uncertain wether the world is in state L or in state H. In order to find an equilibrium for this economy, I will first analyze the case where investors know the correlation of signals.

Lemma 1 (Correlation of signals is high). *Assume the world is in state H, and assume all investors know that the world is in state H. Then there exists an equilibrium price of the form*

$$P = \Phi_D^H (D + \eta + \epsilon) - \Phi_N^H N, \tag{2.2}$$

where Φ_D^H and Φ_N^H are constants. If the price is given by (2.2)*, then the coefficients are given by*

$$\Phi_D^H = \frac{\sigma_D^2}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \qquad \textit{and} \qquad \Phi_N^H = \rho \frac{\sigma_{DY}(\sigma_\eta^2 + \sigma_\epsilon^2)}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2},$$

and the demand of investor i is given by

$$X_i^H = -\frac{\sigma_{DY}}{\sigma_D^2} n_i,$$

and trading volume is given by

$$V^H = \sqrt{\frac{2}{\pi}} \frac{\sigma_{DY}}{\sigma_D^2} \sigma_n$$

Lemma (1) shows that, if all investors know that the world is in state H, the price P is a linear function of the aggregate signal $D + \eta + \epsilon$ and the exposure to the aggregate labor risk N. The price decreases with N, since the labor payoff Y and the dividend D are positively correlated.

Since all investors hold zero shares before they start trading, the demand X_i^H is equivalent to the number of shares that investor i buys or sells. Note that these equilibrium trades can also be written as

$$X_i^H = -\frac{\sigma_{DY}}{\sigma_D^2}\Big(N_i - N\Big)$$

In state H, investor i sells shares if his private labor risk exposure N_i is higher than the average risk exposure N. The equilibrium demands are independent of the dividend signals $\hat{D}_i$ since all investors observe the same signal $\hat{D}_i = D + \eta + \epsilon$.

Since the equilibrium demands depend only on investor specific components n_i, and since these components are independent across investors, the per capita trading volume is given by the unconditional expectation of the absolute number of shares that any given investor trades. Since the n_i are normally distributed, we have

$$V^H = \mathrm{E}|X_i| = \sqrt{\frac{2}{\pi}\mathrm{Var}[X_i]} = \sqrt{\frac{2}{\pi}}\frac{\sigma_{DY}}{\sigma_D^2}\sigma_n$$

As a result, volume is constant. Hence, if all investors know that the world is in state H, then investors ignore trading volume when they choose their demands.

Lemma 2 (Correlation of signals is low). *Assume the world is in state L, and assume all investors know that the world is in state L. Then there exists an equilibrium price of the form*

$$P = \Phi_D^L\big(D + \eta\big) - \Phi_N^L N, \tag{2.3}$$

where Φ_D^L and Φ_N^L are constants. If the price is given by (2.3), then we have

$$\Phi_D^L < \Phi_D^H < 1 \qquad \textit{and} \qquad \Phi_N^H > 0$$

and the demand of investor i is given by

$$X_i = \Psi_D \epsilon_i - \Psi_N n_i,$$

where $\Psi_D > 0$ *and* $\Psi_N > 0$, *and trading volume is give by*

$$V^L = \sqrt{\frac{2}{\pi}\Big(\Psi_D^2\sigma_\epsilon^2 + \Psi_N^2\sigma_n^2\Big)}$$

Lemma (2) shows that, if all investors know that the world is in state L, the price P is a linear function of the aggregate signal $D + \eta$ and the exposure to the aggregate labor risk N. The investor specific signal errors ϵ_i are not part of the price, since they cancel each other out in the aggregate demand.

Note that the equilibrium trades X_i^L can also be written as

$$X_i^L = \Psi_D\Big(\hat{D}_i - (D+\eta)\Big) - \Psi_N\Big(N_i - N\Big).$$

As opposed to state H, investors in state L do not only trade to share risk, but also because they are heterogeneously informed. In this case, the equilibrium demands depend on the differences between the individual signals $\hat{D}_i$ and the average signal $D+\eta$ and the individual labor endowments N_i and the average endowment N. Since investors are uncertain about the aggregate endowment N, the price does not fully reveal the aggregate dividend signal.

In both correlation states, L and H, the demands X_i depend only on the differences between individual and average signals and endowments, since information and labor risks are identically distributed across investors. If signals and endowments are identically distributed, then investors weight their signals and endowments identically when they form their demands. As a result, the price depends on the averages whereas the equilibrium demands depend on the differences of signals and endowments. Therefore, the price is a function only of the common components $\hat{D}$ and N, and the demands are functions only of the investor specific components ϵ_i (in state H) and n_i. As a result, for a given state of signal dispersion, equilibrium demands and the price are independent.

Theorem 1 (Informative trading volume). *If the equilibrium price is given by* (2.2) *in state L, and by* (2.3) *in state H, then we have*

$$V^L < V^H.$$

Hence, if investors are uncertain about the correlation of signal errors, then there exists an equilibrium where trading volume reveals this correlation.

Note that investors cannot learn the correlation of signals by comparing their private signals $\hat{D}_i$ to the price P, since the price depends partially on the unknown aggregate labor risk exposure N. However, as Theorem 1 shows, investors can infer the dispersion of beliefs in the economy from trading volume. Given this information, the investors use their private signals and the price to estimate the future dividend. In that way, observing trading volume helps the investors to separate two sources of uncertainty: uncertainty about the realization of the aggregate information, and uncertainty about the quality of the aggregate information.

Theorem 1 shows that trading volume in state H is higher than trading volume in state L. This result might seem surprising, since in state L investors have two reasons for trading, risk sharing and private information, whereas in state H investors only trade to share risk. To understand why volume increases with the correlation of signals, note that, given CARA utility and normal distribution, the general demand functions of the investors are given by

$$X_i = \frac{\mathrm{E}[D - P|\mathcal{F}_i] - \rho \mathrm{Cov}[D - P, N_i Y|\mathcal{F}_i]}{\rho \mathrm{Var}[D - P|\mathcal{F}_i]}. \tag{2.4}$$

The demand of investor i depends on the conditional expectation of the future investment payoff, the variance of this payoff, and the covariance of the investment payoff with the labor payoff. As Appendix 4 shows, in state L, the conditional expectation of the future dividend is a linear combination of the private dividend signal, the private labor risk, and the price:

$$E[D|\mathcal{F}_i] = \psi_D \hat{D}_i + \psi_N N_i + \psi_P P, \tag{2.5}$$

where $\psi_D > 0$, $\psi_N > 0$, and $\psi_P > 0$. Investors increase their expectations of D with their private labor risks N_i, since they use N_i to estimate the aggregate labor risk N in the price, and since the price decreases with this risk. As Appendix 4 shows, plugging (2.5) into (2.4) gives the state L equilibrium demands

$$X_i^L = \frac{\psi_D \epsilon_i + \psi_N n_i}{\rho \mathrm{Var}[D|\mathcal{F}_i]} - \frac{\sigma_{DY}}{\sigma_D^2} n_i \tag{2.6}$$

Hence, the equilibrium demands in state L can be written as the sum of two components: the first component is due to the fact that investors observe heterogeneous signals, and the second component is given by the equilibrium demand X_i^H in Lemma (1). By (2.6) we can write trading volume in state L as

$$V^L = \sqrt{\frac{2}{\pi}\left(\frac{\psi_D}{\rho \mathrm{Var}[D|\mathcal{F}_i]}\right)^2 \sigma_\epsilon^2 + \frac{2}{\pi}\left(\frac{\sigma_{DY}}{\sigma_D^2} - \frac{\psi_N}{\rho \mathrm{Var}[D|\mathcal{F}_i]}\right)^2 \sigma_n^2}.$$

As appendix 4 shows, we have

$$\frac{\sigma_{DY}}{\sigma_D^2} > \frac{\psi_N}{\rho \mathrm{Var}[D|\mathcal{F}_i]}.$$

Hence, private information affects trading volume in two opposing ways. On the one hand, private information increases volume since investors trade based on the differences of their dividend signals. On the other hand, private information reduces volume, since the fact that investors use their private labor risks to estimate the aggregate risk reduces the trading that is due to differences in labor risk endowments.

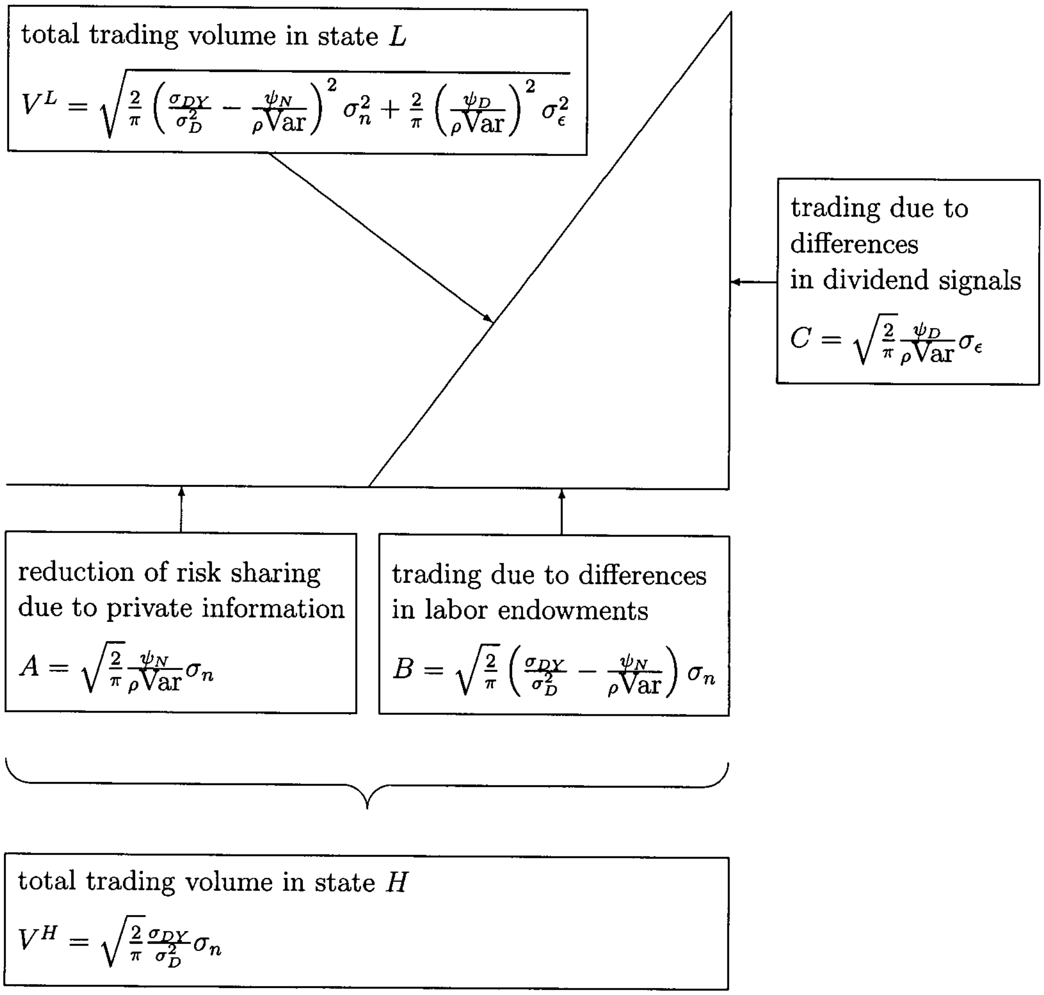


Figure 2.2: **Composition of Trading Volume.** If the world is in state H, so that investors only trade to share risk, then trading volume is given by V^H. The private signals in state L have two effects on V^H: (1) private information reduces V^H by A, since investors use their private labor endowments to estimate the aggregate endowment, and (2) the private signals induce the information trading C. The total trading volume under heterogeneous information is given by $V^L = \sqrt{B^2 + C^2}$. V^L is less than the sum of B and C, since these two trading motives partially cancel each other out for the average investor in the economy.

As Theorem 1 shows, the total effect of private information on trading volume is always negative. The reason for this negative effect is that for some investors in the economy the demand due to private signals will partially offset the demand due to risk sharing. Therefore, the number of shares that the average investor in the economy trades is less than the sum of his risk sharing demand and his private information demand. Figure 2.2 shows a geometric interpretation of the two components of trading volume volume under heterogeneous information and the relation of these components to trading volume under homogeneous information.

I have shown so far how trading volume depends on two extreme cases: the error terms ϵ_i are perfectly correlated or they are not correlated at all. In order to examine how trading volume depends on the correlation of error terms in general, assume the dividend signal of investor i is given by

$$\hat{D}_i = D + \eta + \sqrt{\omega}\,\epsilon_1 + \sqrt{1-\omega}\,\epsilon_{2i}, \qquad \sigma^2_{\epsilon_1} = \sigma^2_{\epsilon_{2i}} = \sigma^2_\epsilon \tag{2.7}$$

where $0 \leq \omega \leq 1$, ϵ_1 is a common error term and ϵ_{2i} are investor specific error terms, which are independent and identically distributed across investors. The case $\omega = 0$ corresponds to state L, and the case $\omega = 1$ corresponds to state H in the theorem. Since $\sigma^2_{\epsilon_1} = \sigma^2_{\epsilon_{2i}}$ the total variance of the signal does not depend on ω. Hence, the specification of the dividend signal in (2.7) allows to examine the effect of a change in the correlation of signal errors, independent of the effect of a change in the total error variance. Figure 2.3 shows the equilibrium trading volume, assuming that all investor know ω. As Figure 2.3 shows, trading volume decreases with the dispersion of signals.

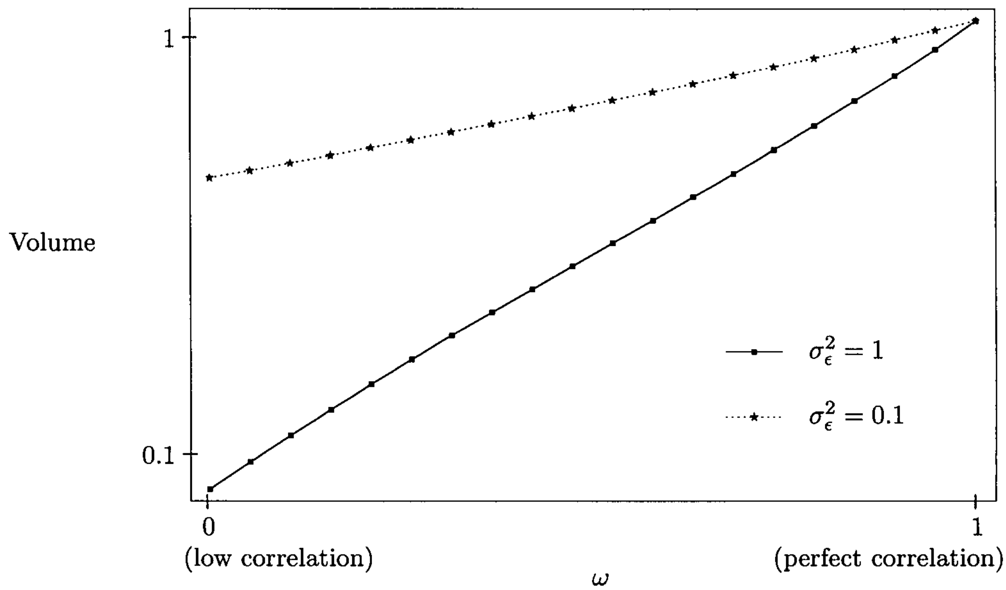


Figure 2.3: **Trading Volume and the Dispersion of Signals.** This figure shows how trading volume in a symmetric economy with a large number of small investors depends on the correlation of the investor specific signal errors, holding the total variance of the error terms constant. The private signal of investor i is given by $\hat{D}_i = D + \eta + \sqrt{\omega}\,\epsilon_1 + \sqrt{1-\omega}\,\epsilon_{2i}$, where η and ϵ_1 are common error terms and ϵ_i is an investor specific error term, and $\sigma^2_{\epsilon_1} = \sigma^2_{\epsilon_{2i}} = \sigma^2_\epsilon$. The total variance of the signal errors is given by $\sigma^2_\eta + \omega\sigma^2_{\epsilon_1} + (1-)\omega\sigma^2_{\epsilon_{2i}} = \sigma^2_\eta + \sigma^2_\epsilon$. The remaining parameters are given by $\rho = \sigma^2_D = \sigma_{DY} = \sigma^2_N = \sigma^2_n = 1$.

2.4 Properties of the Equilibrium

2.4.1 Updating of beliefs

Corollary 1. *If the equilibrium is given by the Theorem 1, the expected future dividend conditional on the information of investor i is given by*

$$E[D|\mathcal{F}_i] = \psi_D^H \hat{D}_i$$

if trading volume is high, and

$$E[D|\mathcal{F}_i] = \psi_D^L \hat{D}_i + \psi_N N_i + \psi_P P,$$

if trading volume is low, and we have $\psi_P > 0$, $\psi_N > 0$, *and* $0 < \psi_D^L < \psi_D^H$.

Corollary 1 shows that investors use the price to update their beliefs only when trading volume is low. Under high trading volume, the aggregate signal is identical to the individual signals. Since the uncertain aggregate labor risk N prevents the price from revealing the aggregate signal, investors completely ignore the price when they update their beliefs. However, under low trading volume, the quality of the aggregate information exceeds the quality of the private information. Therefore, if trading volume is low, investors reduce the weights on their private signals and weight the price more heavily. In this case, the investors also use their own labor risk N_i in order to estimate the aggregate risk N in the price.

2.4.2 Trading strategies

Corollary 2. *If the equilibrium is given by the Theorem 1, the demand of investor i is given by*

$$X_i^L = \Psi_D^L \hat{D}_i - \Psi_N^L N_i - \Psi_P^L P$$

$$X_i^H = \Psi_D^H \hat{D}_i - \Psi_N^H N_i - \Psi_P^H P$$

where all coefficients are greater than zero and we have $\Psi_P^H > \Psi_P^L$.

Corollary 2 shows that, after controlling for the signals $\hat{D}_i$ and the labor endowment N_i, investors always trade against the price. However, investors trade less aggressively against the price when trading volume is low than when trading volume is high. The reason for this behavior is that, under low trading volume, the price does not fully adjust to the aggregate information in the economy. Investors interpret therefore high prices partially as good news and low prices partially as bad news.

2.4.3 Volume predicts the future risk premium

I have so far assumed that the risky asset has a zero net supply. This assumption removes a risk premium from the return, since the average investor does not require a risk premium if he does not hold any shares. The following result shows the relation between volume and the risk premium, if the asset has a positive supply.

Corollary 3. *Assume the per capita supply of the risky asset is given by* $S > 0$, *and assume that all investors hold* S *shares prior to trading. Let* $P_{S=0}^s$, $s \in \{L, H\}$ *be the equilibrium price in Theorem 1. Then we have*

$$P^L = P_{S=0}^L - \Phi_S^L S \qquad \textit{and} \qquad P^H = P_{S=0}^H - \Phi_S^H S,$$

and

$$E[D - P|V^s] = \Phi_S^s S$$

and

$$E[D - P|V^H] > E[D - P|V^L] > 0.$$

Corollary 3 shows that the price decreases with the supply of the risky asset. The expected future return is given by the risk premium that investors require in order to hold the per capita supply S. Since the quality of the aggregate information decreases with trading volume, investors require a higher risk premium when trading volume is high. Hence, the expected future return increases with trading volume.

Chapter 3

Heterogeneous Beliefs and the Underconfident Representative Agent

3.1 Introduction

In the first part of this thesis I have assumed that all investors are fully rational, and that they are able to optimally analyze prices and volume. However, in reality prices aggregate information in a complicated way, and extracting this information is a difficult task. In this chapter I assume that the computational skills of the investors are limited: they make individual mistakes when they infer information from the price. I show that in a heterogeneous information economy where investors' rationality is bounded prices react too slowly to new information.

The remainder of this chapter is organized as follows. Section 3.2 describes the model for the base case where all investors are homogeneously informed. Section

3.3 adds heterogeneous beliefs. In section 3.4 I compare the aggregate demand to the demand of a representative agent. In section 3.5 I examine how the size of the momentum profits depends on the parameters of the model.

3.2 Homogeneous Information

There are two assets in the economy, a risk-free asset and a risky asset. Investors can trade these assets at the trading dates $t = 0, 1, 2, \ldots$. One dollar invested in the risk-free asset at time t pays $1 + r$ dollar at time $t + 1$. Let $R = \frac{1}{1+r}$ be the time t price of one dollar at time $t + 1$. Investors can buy or sell an unlimited amount of the risk-free asset.

Assumption 1 (Risky asset). *Investors can trade shares of a single risky asset at the equilibrium price P_t. At each trading date t, investors receive a dividend D_t for each share they are holding at time $t - 1$. The value D_t is given by the process*

$$D_t = D_{t-1} + d_t$$

where $D_0 = d_0$, $d_t \sim N(0, \sigma_d^2)$, and $E[d_t d_s] = 0$ for $t \neq s$.

The dividends are normally distributed with an unconditional mean of zero. The assumption of normal distribution allows to solve for the equilibrium price explicitly. Setting the unconditional expectation equal to zero simplifies the notation, but does not affect the results of the model.

Assume for the moment that the investors know the realizations of future dividends with certainty. Then the price of the asset is given by the present value of all future dividends

$$P_t^F = \frac{1}{r}\left[D_t + \sum_{s=1}^{\infty} R^{s-1} d_{t+s}\right]. \tag{3.1}$$

I will call P_t^F the full information value of the risky asset. Let $[t_1, t_2]$ be the time interval starting on trading day t_1 and ending on trading day t_2. Assume an investor buys one share of the asset at time t_1 and sell it at t_2. Let $T_1 = t_2 - t_1$ be the length of this interval. Then

$$\Delta_{t_1 t_2} = \sum_{s=1}^{T_1} R^s D_{t+s} + R^{T_1} P_{t_2} - P_{t_1} \tag{3.2}$$

is the cumulative dollar excess return that this investor receives. Plugging the full information value of (3.1) into (3.2), we get

$$\Delta_{t_1 t_2}^F = \sum_{s=1}^{T_1} R^s D_{t+s} + R^{T_1} P_{t_2}^F - P_{t_1}^F = 0. \tag{3.3}$$

Hence, under full information returns are serially uncorrelated.[1]

Next, assume investors do not know the realizations of future dividends. At each trading date, investors know only the realizations of all past dividends up to the current dividend D_t. In addition, the investors receive noisy public signals about each future dividend innovation. The time t signal about d_{t+s} is given by

$$\begin{aligned} \hat{d}_{t+s,t} &= d_{t+s} + \eta_{t+s,t} \\ \eta_{t+s,t} &= \sum_{j=1}^{s} \epsilon_{t+s,j}, \end{aligned}$$

where all $\epsilon_{t+s,j}$ are independent normally distributed random variables with mean zero and variance σ_ϵ^2. The first subscript of $\hat{d}_{t+s,t}$ and $\eta_{t+s,t}$ refers to the dividend

[1]Returns are only serially uncorrelated, if one calculates returns as in (3.2). Without taking dividends and the discount rate into account, we have, for example, $P_{t+1}^F - P_t^F = \sum_{s=1}^{\infty} \frac{d_{t+s}}{(1+r)^{s-1}}$, and

$$Cov[P_{t+1}^F - P_t^F, P_{t+2}^F - P_{t+1}^F] = \frac{\sigma_d^2}{r(1+r)(2+r)}.$$

innovation d_{t+s}, and the second subscript refers to the trading date at which the investor observes the signal. The time t information about d_{t+s} is given by the true dividend innovation d_{t+s} plus s independent noise terms. At each trading date, one noise term is removed from the signals, so that the signals become more precise, as the payout date of the corresponding dividend approaches. In that way, the investors know more about dividends that the firm will pay in the near future than dividends in the far future. For example, the time $t = 0$ signal about the dividend innovation at time $t = 3$ is given by

$$\hat{d}_{3,0} = d_3 + \eta_{3,0}, \qquad \eta_{3,0} = \epsilon_{3,1} + \epsilon_{3,2} + \epsilon_{3,3}.$$

At time $t = 1$, this signal becomes

$$\hat{d}_{3,1} = d_3 + \eta_{3,1}, \qquad \eta_{3,1} = \epsilon_{3,1} + \epsilon_{3,2}.$$

The information of the investors about d_3 is therefore more precise at $t = 1$ than at $t = 0$. Note that the signal $\hat{d}_{3,1}$ encompasses the information contained in the signal $\hat{d}_{3,0}$. An investor who observes $\hat{d}_{3,1}$ does not need $\hat{d}_{3,0}$ to forecast future dividends, since $\hat{d}_{3,0}$ is distributed as $\hat{d}_{3,1}$ plus noise. The time t information about all future dividends is therefore given by the vector

$$\mathbf{I}_t = \{D_t, \hat{d}_{t+1,t}, \hat{d}_{t+2,t}, \hat{d}_{t+3,t}, \ldots\}. \tag{3.4}$$

I omit past dividends in (3.4), since past dividends are irrelevant information for an investor who knows the current dividend. Given the information in (3.4), each investor has to decide how many shares to buy of the risky asset. The economy is populated by a countable set of investors. An individual investor is indexed by $i \in \{1, 2, 3, \ldots\}$. Let X_{it} be the number of shares that investor i holds at time t.

Assumption 2 (Investors). *Investor i chooses X_{it} by maximizing*

$$E_{it}\Big[-\exp\{-\rho W_{it+1}\}\Big],$$

where $E_{it}[\cdot]$ is the expectations operator conditional on the information of investor i at time t, $\rho > 0$ is the parameter of risk-aversion, and W_{it} is the wealth of investor i at time t.

For simplicity I assume that investors are myopic. This assumption allows to derive analytical results. It is straight forward to extend the model to the case where all investors maximize the utility of life-time consumption. I have analyzed this case numerically and found that the qualitative results match the results under a myopic investment policy.

If the time $t+1$ wealth of investors i is normally distributed, conditional on his information set at time t, the demand for the risky asset of investor i is given by

$$X_{it} = \frac{E_{it}[P_{t+1} + D_{t+1}] - (1+r)P_t}{\rho Var_{it}[P_{t+1} + D_{t+1}]}. \tag{3.5}$$

In order to find an equilibrium, we have to aggregate the demand of all investors. Let

$$X_t = \lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{h} X_{it} \tag{3.6}$$

be the average demand per investor.

Definition 1 (Equilibrium). *An equilibrium price is given by a real-valued price process P_t such that*

(a) X_t = average supply,

(b) *there exists real numbers* M, Φ_t, Φ_{Dt}, Φ_{dts} *such that* $|\Phi_t| < M$, $|\Phi_{Dt}| < M$, $|\Phi_{dts}| < M$, *and*

$$P_t = \Phi_t + \Phi_{Dt} D_t + \sum_{s=1}^{\infty} \Phi_{dts} \hat{d}_{t+s}.$$

In general there will exist more than one price process that satisfies condition (a).[2] Condition (b) rules out non-linear equilibria and equilibria that contain rational "bubble-components". The requirement that the equilibrium price depends only on the current dividend D_t and not on past dividends is not restrictive, since the investors do not need past dividends to forecast future dividends.

The supply of the risky asset will determine the risk premium. Since, in this model, the riskiness of the asset does not change over time, the risk premium will be a constant. Hence, the risk premium will not affect the correlation of returns. Therefore, in the remainder of the paper I will set the average supply equal to zero. This assumption will simplify the notation by removing the risk premium from the returns, but it will not affect any result of the model.

Lemma 3 (Equilibrium with noisy public signals). *If the information of the investors about future dividends is given by* (3.4), *then there exists a unique equilibrium price given by*

$$P_t = E_t\left[P_t^F\right] = \frac{1}{r}\left[D_t + \sum_{s=1}^{\infty} A_s R^{s-1}\left(d_{t+s} + \eta_{t+s,t}\right)\right],$$

where

$$A_s = \frac{Cov[d_{t+s}, \hat{d}_{t+s}]}{Var[\hat{d}_{t+s}]} = \frac{1}{1 + s\,\frac{\sigma_\epsilon^2}{\sigma_d^2}},$$

and P_t^F *is given by* (3.1).

[2]Whenever applicable, equations hold with probability one.

Lemma 3 shows that the equilibrium price equals the expectation of the full information value of the risky asset, conditional on the public information available at time t. The equilibrium price does not depend on the parameter of risk aversion, since all investors are homogenous, and the supply of the risky asset equals zero. Therefore, the demand of each investor equals zero in the equilibrium. The full information value of the asset is the present value of all future dividends. Since investors do not bear any risk in the equilibrium, the price must be given by the expected future payoff of the asset. If there would a positive supply of the risky asset, the price in Lemma 3 would be given by the expected fundamental value of the asset plus a constant depending on the risk aversion.

If the price is given by Lemma 3, we can write the price as

$$P_t = \underbrace{\left[P_t^F\right]}_{\text{full information value}} + \underbrace{\left[\frac{1}{r}\sum_{s=1}^{\infty}\left(A_s - 1\right)R^{s-1}d_{t+s}\right]}_{\text{uncertainty}} + \underbrace{\left[\frac{1}{r}\sum_{s=1}^{\infty}A_s R^{s-1}\eta_{t+s,t}\right]}_{\text{noise}}, \quad (3.7)$$

The price consists of three parts: the full information value of the asset given by (3.1), an uncertainty part, and the signal noise. From Lemma 3 we have $0 \leq A_s \leq 1$. The uncertainty part decreases therefore the effect of the dividends on the price, relative to the full information value. The weights of the dividends for the price in Lemma 3 are smaller than the weights for the full information value, since the investors receive uncertain signals about future dividends. The uncertainty increases with the ratio of signal noise variance to dividend variance $s\frac{\sigma_\varepsilon^2}{\sigma_d^2}$. If the investors observe undisturbed signals, the uncertainty effect in (3.7) disappears. Lemma 3 shows that A_s decreases monotonically with s. The uncertainty effect is therefore stronger for dividends in the far future than dividends in the near future. The uncertainty effect is stronger for dividends in the far future, since the investors receive more information about the dividends as the dividend payout date approaches.

Let P_t^U be the uncertainty part and P_t^N be the noise part of the price in (3.7), so that we have

$$P_t = P_t^F + P_t^U + P_t^N. \tag{3.8}$$

Recall from (3.2) that $\Delta_{t_1 t_2}$ is the cumulative excess return of the asset during the time interval $[t_1, t_2]$. If the price is given by the full information value P_t^F, this return is given by the fundamental return $\Delta_{t_1 t_2}^F$. If the price is given by (3.8), the uncertainty part P_t^U and the noise part P_t^N will cause the return to deviate from the full information return. These deviations are given by

$$\Delta_{t_1 t_2}^U = \frac{1}{(1+r)^{T_1}} P_{t_2}^U - P_{t_1}^U, \tag{3.9a}$$

$$\Delta_{t_1 t_2}^N = \frac{1}{(1+r)^{T_1}} P_{t_2}^N - P_{t_1}^N. \tag{3.9b}$$

The return $\Delta_{t_1 t_2}^U$ is the dollar excess return that is due to the dividend uncertainty, and $\Delta_{t_1 t_2}^N$ is the dollar excess return that is due to the noise. Plugging the price in (3.8) into the definition of the return in (3.2), we get

$$\Delta_{t_1 t_2} = \Delta_{t_1 t_2}^F + \Delta_{t_1 t_2}^U + \Delta_{t_1 t_2}^N.$$

The total return is the sum of the full information return, the uncertainty return, and the noise return. The full information return $\Delta_{t_1 t_2}^F$ equals zero by (3.3). Since the uncertainty return is not correlated with the noise return, we have

$$Cov[\Delta_{t_1 t_2}, \Delta_{t_2 t_3}] = \underbrace{Cov\left[\Delta_{t_1 t_2}^U, \Delta_{t_2 t_3}^U\right]}_{\text{uncertainty effect}} + \underbrace{Cov\left[\Delta_{t_1 t_2}^N, \Delta_{t_2 t_3}^N\right]}_{\text{noise effect}}. \tag{3.10}$$

Hence the covariance of the returns during the two time intervals $[t_1, t_2]$ and $[t_2, t_3]$ is the sum of two effects: the uncertainty effect and the noise effect.

Lemma 4 (Correlation of returns with noisy public information). *If the price is given by Lemma 3, then we have*

$$
\begin{aligned}
Cov\left[\Delta^U_{t_1t_2}, \Delta^U_{t_2t_3}\right] &> 0, \\
Cov\left[\Delta^N_{t_1t_2}, \Delta^N_{t_2t_3}\right] &< 0, \\
Cov[\Delta_{t_1t_2}, \Delta_{t_2t_3}] = Cov\left[\Delta^U_{t_1t_2}, \Delta^U_{t_2t_3}\right] + Cov\left[\Delta^N_{t_1t_2}, \Delta^N_{t_2t_3}\right] &= 0.
\end{aligned}
$$

Lemma 4 shows that the uncertainty effect is positive, the noise effect is negative, and the two effects offset each other exactly. The uncertainty effect is positive for the following reason. Assume the firm will pay a positive dividend at some trading date in the future. Since the price reacts to this information slowly, the positive future payoff will increase the return during the first period $[t_1, t_2]$ and during the second period $[t_2, t_3]$. In this way, the slow reaction of the price to news has a positive effect on the correlation of returns.

To see why the noise effect is negative, assume a signal associated with a particular future dividend contains a positive noise component. On each trading date, the investors increase the weight that they put on this signals when they update their beliefs. Therefore, the positive noise component will initially increase the price and produce a positive return, since the investors can not distinguish between noise and true information. However, at some future trading date, the investors will learn that this particular part of the signal was just noise. A positive noise component today will therefore decrease the price at some point in the future. In that way, the signal noise has a negative effect on the correlation of returns.

Lemma 4 shows that the uncertainty effect and the noise effect offset each other exactly. This result makes intuitively sense. Since the riskiness of the firm stays constant over time, changes of the price can only be due to changes in the information, and not to changes of the risk premium. But the price at any given time reflects all publicly available information. Future price changes are therefore caused only by the arrival of new information. Since new information is not correlated with today's information, past returns cannot have any predictive power for future returns.

3.3 Heterogenous Uncertainty about the Income of the Firm

I will now compare the economy with homogeneous beliefs of the previous section to an economy with heterogeneous beliefs. Assume Investor i receives at time t the information

$$\mathbf{I}_{it} = \{D_t, \hat{d}_{it+1,t}, \hat{d}_{it+2,t}, \hat{d}_{it+3,t}, \ldots\}, \tag{3.11}$$

where

$$\hat{d}_{it+s,t} = d_{t+s} + \sum_{j=1}^{s} \epsilon_{it+s,j},$$

and the noise terms $\epsilon_{it+s,j}$ are independent across the investors. The only difference between the homogeneous information structure in (3.11) and the information given in (3.4) is that now the investors observe signals with individual noise terms instead of common noise terms. There are two possible interpretations for the individual noise terms. One is that the investors receive private information about future dividends. The other interpretation is that the investors observe the same public information, but understand this information in different ways. In this case, d_{t+s} can be interpreted as the true information contained in the public signal, and the individual noise terms can be interpreted as the mistakes that the individual investors make when they analyze the public signal.

In either case, if the investors receive heterogenous signals they will have an incentive to use the price as an additional source of information. If the investors are perfectly rational, the result of Grossman (1976) will apply, and the price will fully reveal all information. In this case, the only possible equilibrium price is the full information value of the risky asset. The assumption of perfect rationality is strong, since it requires that the investors completely understand the underlying model of the economy, and are able to infer information from the price without an error. Instead, I will now assume that the rationality of the investors is bounded. When the investors use the price to infer information, they make individual mistakes. In addition, I will for simplicity assume that the investors only use the current price and not past prices as a source of information. This assumption is not as restrictive as it might seem, since, if all investors are rational, past prices are redundant information as the result in Lemma 5 will show.

Note that, since investors are myopic, their demands depend only on their beliefs about the next periods' payoff of the asset. Let

$$\Psi_{it} = P_t - E[P_t|\mathbf{I}_{it}]. \tag{3.12}$$

We can interpret Ψ_{it} as the unexpected part of the price P_t, after accounting for the information contained in $\mathbf{I}_{it}$. If $\mathbf{I}_{it}$ and P_t are jointly normally distributed, then it follows form the properties of the multivariate normal distribution that $\mathbf{I}_{it}$ and Ψ_{it} are independent, and that

$$E[D_{t+1} + P_{t+1}|\mathbf{I}_{it}, P_t] = E[D_{t+1} + P_{t+1}|\mathbf{I}_{it}] + \frac{Cov[D_{t+1} + P_{t+1}, P_t|\mathbf{I}_{it}]}{Var[P_t|\mathbf{I}_{it}]}\Psi_{it}. \tag{3.13}$$

Therefore, we can interpret $\mathbf{I}_{it}$ and Ψ_{it} in the following way: $\mathbf{I}_{it}$ contains the non-price related information that investor i at time t uses to forecast the future payoff, and Ψ_{it} contains the additional information that the investor obtains when he observes the price. Since $\mathbf{I}_{it}$ and Ψ_{it} are independent, the additional information in Ψ_{it} is strictly incremental to the non-price related information in $\mathbf{I}_{it}$. I will now assume that the investors make individual mistakes when they infer this additional information from the price. The information that investor i infers from the price at time t is given by

$$\hat{\Psi}_{2it} = \Psi_{2it} + \epsilon_{\Psi it} \tag{3.14}$$

where the individual noise terms $\epsilon_{\Psi it}$ are independent normally distributed with mean zero and variance $\sigma^2_{\epsilon_\Psi}$. The demand for the risky asset in (3.5) becomes then

$$X_{it} = \frac{E[P_{t+1} + D_{t+1}|\mathbf{I}_{it}, \hat{\Psi}_{2it}] - (1+r)P_t}{\rho Var[P_{t+1} + D_{t+1}|\mathbf{I}_{it}, \hat{\Psi}_{2it}]}. \tag{3.15}$$

Note that by (3.13) the demand function in (3.15) is fully rational, if $\sigma^2_{\epsilon_\Psi} = 0$.

Lemma 5 (Equilibrium with heterogeneous beliefs). *If the demand of the investors is given by* (3.15)*, then there exists an equilibrium price of the form*

$$P_t = \frac{1}{r}\left[D_t + \sum_{s=1}^{\infty} \Phi_{ds} R^{s-1} d_{t+s}\right]. \tag{3.16}$$

If the price is given by (3.16)*, then we must have*

$$\Phi_{d1} > \Phi_{d2} > \Phi_{d3} > \cdots,$$

and $A_s < \Phi_{ds} < 1$ *for* A_s *given by Lemma 3, and* $\lim_{\sigma^2_{\epsilon_\Psi} \to 0} P_t = P_t^F$.

Lemma 5 shows that the coefficients Φ_{ds} have the same properties as the coefficients A_s in the case of homogenous information in Lemma 3. However, as opposed to Lemma 3, the price in Lemma 5 does not contain any signal noise. The price does not contain any signal noise, because the price depends on the aggregate demand and therefore on the aggregate information in the economy. Since the investor specific noise terms in (3.11) cancel each other out in the aggregate demand, the equilibrium price cannot depend on these noise terms.

We can write the price in Lemma 5 as

$$P_t = \underbrace{\left[P_t^F \right]}_{\text{full information value}} + \underbrace{\left[\frac{1}{r} \sum_{s=1}^{\infty} \left(\Phi_{ds} - 1 \right) R^{s-1} d_{t+s} \right]}_{\text{uncertainty}}.$$

The price consists of two parts: the full information value of the asset given by (3.1) and a part that is due to the uncertainty about future dividends. By Lemma 5, the uncertainty part decreases the effect of the dividends on the price, relative to the full information value. Since $\Phi_{ds} > A_s$, the uncertainty effect in the heterogeneous information economy is smaller than in the comparable economy with homogeneous information. The uncertainty under heterogeneous information is small, since the investors know more on aggregate in the heterogeneous information economy than in the homogeneous information economy, and since the individual investors can access the aggregate information by observing the price. If the individual mistakes that the investors make when they learn from the price are small, the price will be close to the full information value by Lemma 5.

Lemma 6 (Correlation of returns with heterogenous beliefs). *If the price is given by Lemma 5, then we have*

$$Cov \left[\Delta_{t_1 t_2}, \Delta_{t_2 t_3} \right] = Cov \left[\Delta_{t_1 t_2}^U, \Delta_{t_2 t_3}^U \right] > 0,$$

where $\Delta_{t_i t_j}^U$ *is defined according to* (3.9a).

Lemma 6 shows that the returns of the time periods $[t_1, t_2]$ and $[t_2, t_3]$ are positively correlated. Recall from equation (3.10) that, if the investors receive homogenous information, the covariance of the returns is the sum of two effects: a positive uncertainty effect and a negative noise effect. In the case of heterogenous information, there is no noise component in the price, since the investor specific noise components cancel each other out in the aggregate demand. Therefore, the only remaining effect on the covariance is the positive effect generated by the uncertainty return $\Delta^U_{t_i t_j}$. This uncertainty return is due to the fact that the price reacts slower to future dividends than the full information price. In that way, the slow adjustment of the price to future dividends produces positively correlated returns.

Chordia and Shivakumar (2002) find that winners outperform losers not only during the post-formation period of a momentum trading strategy, but also during the pre-formation period. Chordia and Shivakumar interpret this finding as evidence that time varying expected returns cause momentum. Note, however, that the covariance in Lemma 6 works in both directions. If the returns during the formation period are positively correlated with the returns during the post-formation period, then the same will be true for the correlation between formation and pre-formation period. The findings of Chordia and Shivakumar are therefore consistent with the idea that aggregate underreaction to new information causes momentum.

3.4 The Underconfident Representative Agent

I will now construct a representative agent for the economy of the previous section. Recall that the heterogeneously informed investors of Lemma 5 observe the signals

$$\mathbf{I}_{it} = \{D_t, \hat{d}_{it+1,t}, \hat{d}_{it+2t}, \ldots\}, \qquad \hat{d}_{it+s,t} = d_{t+s} + \eta_{it+s,t}, \tag{3.17}$$

where the $\eta_{it+s,t}$ are individual noise terms. If the information of the investors is given by (3.17), then the aggregate demand does not contain any signal noise, since the individual noise terms cancel each other out. Since the aggregate demand does not depend on the signal noise, a representative agent must be able to observe the undisturbed signal

$$\mathbf{I}_t = \{D_t, d_{t+1}, d_{t+2}, \cdots\}. \tag{3.18}$$

As I will show in the following, the representative agent suffers from a psychological bias. He updates his priors too slowly, given the quality of his signals. Specifically, he calculates the distributions of future prices and dividends, as if he observed the noisy signal

$$\tilde{\mathbf{I}}_t = \{D_t, \tilde{d}_{t+1,t}, \tilde{d}_{t+2t}, \dots\}, \qquad \tilde{d}_{t+s,t} = d_{t+s} + \sqrt{Z} \cdot \eta_{t+s,t}, \tag{3.19}$$

where Z is a positive constant, and the noise terms $\eta_{t+s,t}$ in (3.19) have the same distribution as the $\eta_{it+s,t}$ in (3.17). I will assume that the psychological bias of the representative agent is restricted to the way he uses his information in (3.18). When he calculates the distributions of future prices, he fully rationally takes the correct functional form of the equilibrium price into account. The representative agent is therefore aware of the fact that the equilibrium price does not contain any of the noise terms η_{st}. However, he does not use the price as a source of information, since he is the only agent in the economy, and since the price can only contain information that he already knows. Even though he is aware that he has access to the full information in (3.18), his psychological bias drives him to update his priors according to (3.19).

Lemma 7 (Underconfident representative agent). *Let*

$$Z = 1 - \frac{1}{1+r}\left[\frac{Cov[D_{t+1} + P_{t+1}, P_t|\mathbf{I}_{it}]}{Var[P_t|\mathbf{I}_{it}] + \sigma^2_{\epsilon_\Psi}}\right].$$

Then the equilibrium price in the representative agent economy is equivalent to the price in the economy populated by individual investors of Lemma 5, and we have $Z \in [0, 1]$, $\lim_{\sigma^2_{\epsilon_\Psi} \to 0} Z = 0$, *and* $\lim_{\sigma^2_{\epsilon_\Psi} \to \infty} Z = 1$.

Note that the variable Z in Lemma 7 does not depend on i, the index for the individual investor, since covariances of normally distributed random variables only depend on the distribution of the signal and not on the realization of the signal. Part (a) of Lemma 7 shows that the aggregate demand in the heterogeneous information economy of the previous section is equivalent to the demand of a representative agent, who observes the undisturbed signal $\mathbf{I}_t = \{D_t, d_{t+1}, d_{t+2}, \cdots\}$, but updates his priors as if he observed the noisy signal $\tilde{\mathbf{I}}_t$ from equation (3.19). In that sense, the representative agent is underconfident. He underestimates the quality of his information. The coefficient $Z \in [0,1]$ measures the degree of underconfidence.

Recall from the previous section that $\mathbf{I}_{it}$ is the non-price related information of investor i at time t, and that $\hat{\Psi}_{2it}$ is the additional information that this investor obtains when he observes the price. Similarly to (3.13), it follows from the properties of the normal distribution that

$$E[D_{t+1} + P_{t+1}|\mathbf{I}_{it}, \hat{\Psi}_{it}] = E[D_{t+1} + P_{t+1}|\mathbf{I}_{it}] + (1+r)(1-Z)\hat{\Psi}_{it} \tag{3.20}$$

(see Appendix C). The function Z determines therefore the degree to which the beliefs of the investors depend on the information obtained from the price. If $\sigma^2_{\epsilon_\Psi} \to 0$, the investors are fully rational, and the price related information will affect the expected future payoff with the factor $1+r$. If $\sigma^2_{\epsilon_\Psi} \to \infty$, the investors are not able to infer any information from the price, and the price related information $\hat{\Psi}_{it}$ will not effect their beliefs at all.

By equation (3.19) the representative agent is underconfident, if

$$\mathrm{Var}[\sqrt{Z} \cdot \eta_{t+s,t}] = Z \cdot s\,\sigma^2_\epsilon > 0,$$

for $s = 1, 2, 3, \ldots$. The representative agent is therefore underconfident if and only if both of the following two conditions are satisfied: the individual investors have heterogeneously beliefs ($\sigma_\epsilon^2 > 0$), and the investors make individual mistakes when they learn from the price ($Z > 0$). Comparison of equations (3.19) and (3.20) shows that the measure Z ties the ability of the investors to learn from the price to the underconfidence of the representative agent. The better the investors are able to learn from the price, the higher is the influence of the information coming from the price on the private beliefs, and the lower is the aggregate underconfidence.

The relation of individual mistakes and aggregate underconfidence shows that bounded rationality on the individual level is equivalent to a psychological bias on the aggregate level. As a result of his underconfidence, the representative agent does not put enough weight on his signals, when he updates his beliefs. Thus, it seems as if the representative agent suffers from a conservatism bias, identified in experiments by Edwards (1968). Individuals who are subject to this bias tend to underweight new information when they update their priors. Barberis, Shleifer, and Vishny (1998) assume that a representative agent suffers from the conservatism bias to model the underreaction of the price to new information. Lemma 5 and Lemma 7 show that even if the representative agent is psychologically biased, this does not need to be the case for the individual investors. If the investors make individual mistakes when they infer information from the price, then the representative agent will be underconfident because the investors rationally reduce the weights that they put on the price related information when they update their beliefs. The aggregate conservatism bias and the resulting underreaction occurs therefore naturally in a world with heterogeneous information and bounded rationality.

3.5 Comparative Statics

Figure 3.1 shows how the dividend coefficients of the price in Lemma 5 depend on the parameters of the economy. The top curve shows the dividend coefficients for the case, where all investors know the realizations of all future dividends ($\sigma_\epsilon^2 = 0$). Since investors discount dividends in the far future more than they discount dividends in the near future, the dividend coefficients are higher for dividends in the near future than for dividends in the far future. The remaining curves of Figure 3.1 show how the the dispersion of beliefs and the ability to learn from the price affects the discounting. The fact that the dividend coefficients under heterogeneous information are smaller than under full information shows that the price underreacts to new information about future dividends. The price adjusts to news about future dividends slower if the dispersion of beliefs among investors is higher (high value of σ_ϵ^2) or if the investors are less able to infer information from the price (high value of $\sigma_{\epsilon_\Psi}^2$).

I will now demonstrate how the expected return of a momentum strategy depends on these parameters.

3.5.1 Momentum profits and length of the holding period

For the two consecutive time intervals $[t_1, t_2]$ and $[t_2, t_3]$, I will refer to the first interval as the formation period and to the second interval as the holding period. Consider the regression coefficient

$$\Psi = \frac{Cov\left[\Delta_{t_1 t_2}, \Delta_{t_2 t_3}\right]}{Var\left[\Delta_{t_1 t_2}\right]}, \tag{3.21}$$

where $\Delta_{t_i t_j}$ is the cumulative excess return for the period $[t_i, t_j]$, as defined in (3.2). The coefficient Ψ measures the ability to forecast future price differences based on

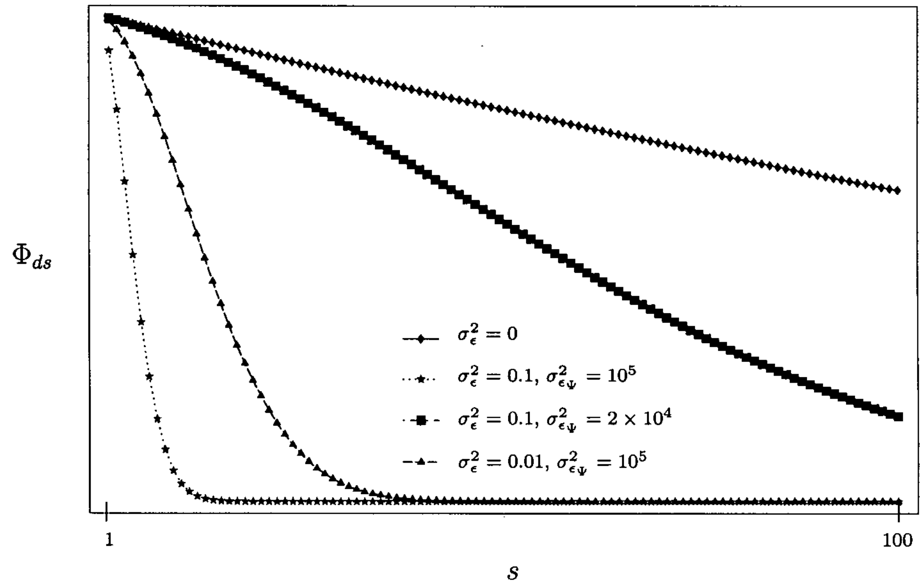


Figure 3.1: **Price Reaction to New Information.** *This figure shows the dividends coefficients of the price given by Lemma 5. The remaining parameters are given by* $\rho = \sigma_d^2 = 1$, $r = 0.004$.

the observations of past price differences. Since the unconditional expected returns are zero in this model, we have

$$E\left[\Delta_{t_2 t_3} \middle| \Delta_{t_1 t_2}\right] = \Psi \Delta_{t_1 t_2}. \tag{3.22}$$

Therefore, we can interpret Ψ as the cumulative expected dollar return for a buy-and-hold strategy with time horizon $T_2 = t_3 - t_2$, if the asset has increased by one dollar during the period $[t_1, t_2]$. Figure 3.2 shows how this momentum return increases monotonically with the length of the holding period. The dependence of the momentum return on the remaining parameters is less clear. Even though, as Figure 3.1 shows, the parameters of the economy affect the underreaction of the price to new information about future dividends in an intuitively obvious way, this effect is not as obvious for the momentum return. The dependence of the momentum return on the parameters is complicated, because the underreaction affects current and future prices, and the momentum return depends on the difference between

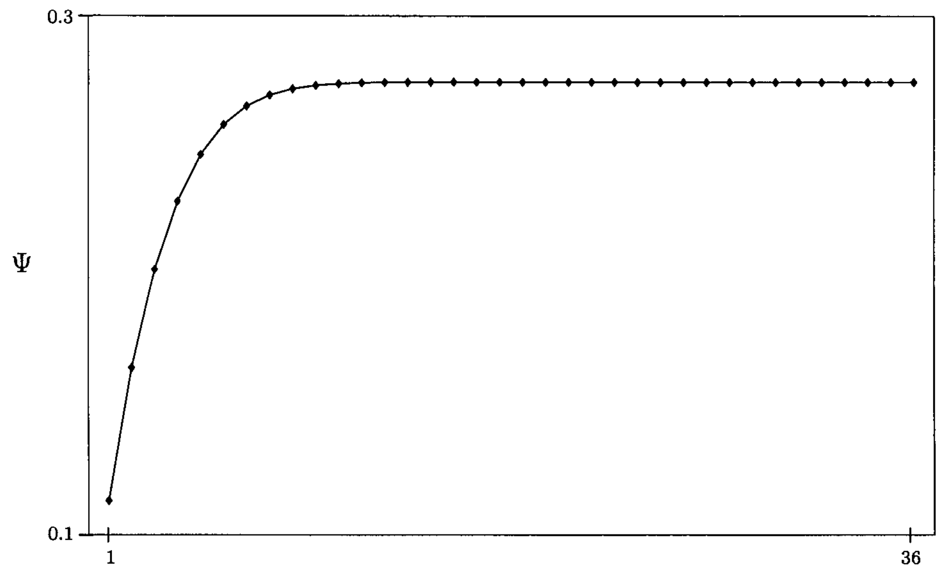


Figure 3.2: **Momentum Return.** *This figure shows how the momentum return* $\Psi = \frac{Cov[\Delta_{t_1t_2},\Delta_{t_2t_3}]}{Var[\Delta_{t_1t_2}]}$ *depends on length of the holding period* T_2*. The momentum return* $\Psi \cdot T_1$ *is the expected dollar return for a buy-and-hold strategy with time horizon* T_2*, if the asset has increased by one dollar during the formation period. The remaining parameters are given by* $T_1 = 6$, $\sigma_\epsilon^2 = 0.1$, $\sigma_{\epsilon_\Psi}^2 = 10^5$, $\sigma_d^2 = \rho = 1$, $r = 0.004$.

these prices. Therefore, instead of focussing on the dependence of the momentum return on the parameter values, I will now show how the behavior of the asset during the formation period affects the holding period return.

3.5.2 Momentum profits and reversals during the formation period

I will now divide the formation period $[t_1, t_2]$ into two separate time intervals $[t_{1a}, t_{1b}]$ and $[t_{1b}, t_2]$. Figure 3.3 shows a picture of the time line. In order to be able to

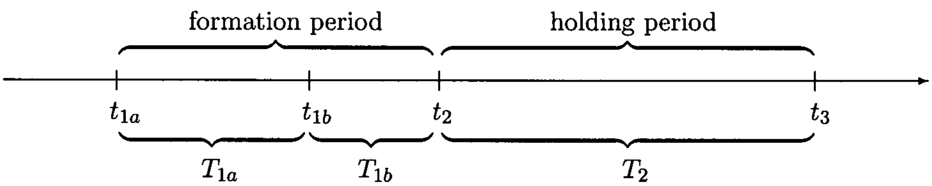


Figure 3.3: **Time Line.**

compare momentum returns for formation periods with different lengths, I multiply the expected holding period return Ψ with the length of the corresponding formation period. If the length of the formation period is T_1, then the resulting return $\Psi \cdot T_1$ is the expected dollar return for a buy-and-hold strategy with time horizon T_2, if the asset has increased by an average of one dollar per trading date during the formation period.

The top part of Figure 3.4 shows the expected returns from three separate regressions, where the dependant variable is either the return during the first part of the formation period, or the second part of the formation period, or the total formation period. For example, $\Psi_a T_{1a}$ in the top part of the Figure is the expected cumulative return for the holding period $[t_1, t_2]$, if the asset has had an average return of one dollar per trading date during the period $[t_{1a}, t_{1b}]$. Similarly, $\Psi_b T_{1b}$ is the expected holding period return for an average return of one dollar during the period $[t_{1a}, t_{1b}]$. The return ΨT_1 is the expected return, if the average return was one dollar during the period $[t_1, t_2]$.

The bottom part of Figure 3.4 shows coefficients Ψ_a and Ψ_b from

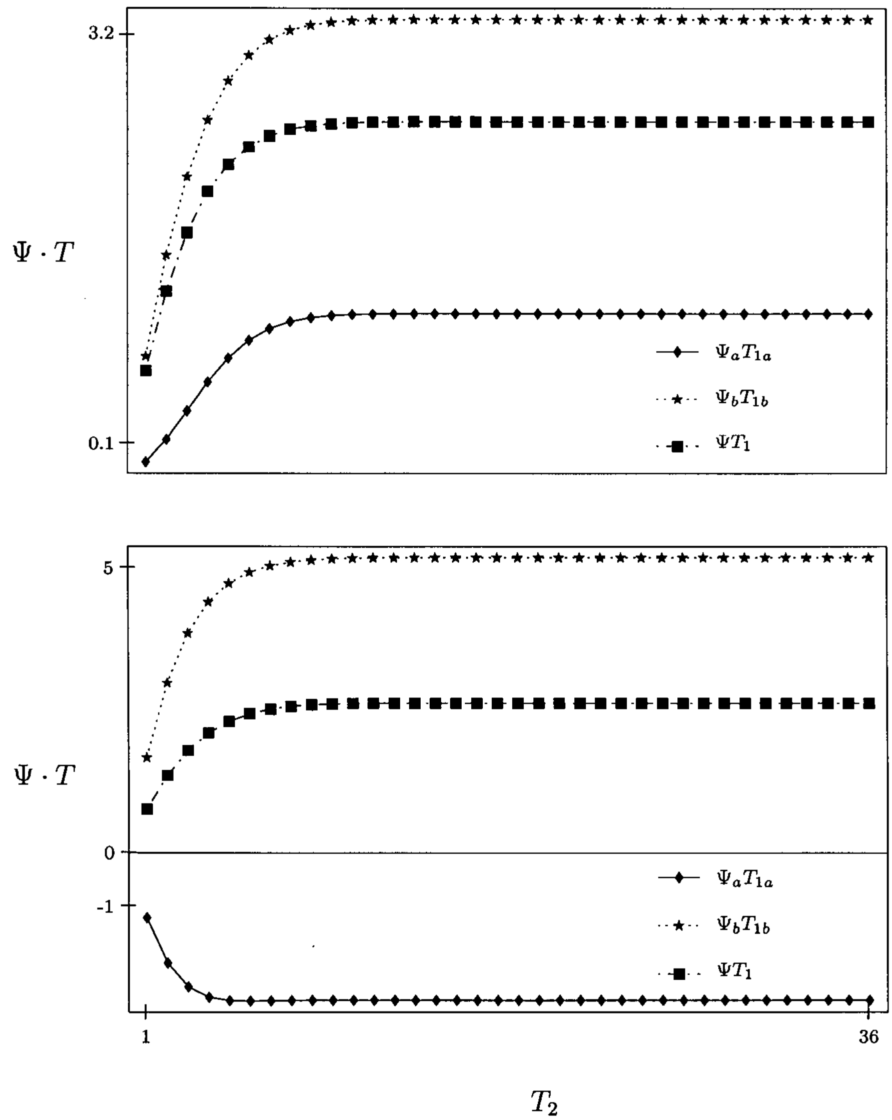
$$E\left[\Delta_{t_2 t_3} \middle| \Delta_{t_{1a} t_{1b}}, \Delta_{t_{1b} t_2}\right] = \Psi_a \Delta_{t_{1a} t_{1b}} + \Psi_b \Delta_{t_{1b} t_2}. \tag{3.23}$$

The sum $\Psi_a T_{1a} A + \Psi_b T_{1b} B$ can be interpreted as the expected dollar return for the holding period $[t_2, t_3]$, if the asset has increased by an average of A dollar per trading date during the first part of the formation period $[t_{1a}, t_{1b}]$, and an average of B dollar per trading date during the second part of the formation period $[t_{1a}, t_{1b}]$.

The top part of Figure 3.4 shows that taken separately, the returns during both parts of the formation period are positively correlated with the return during the holding period. However, if one calculates expected returns as in (3.23), the bottom part of Figure 3.4 shows that the return during the first part of the formation period is negatively correlated with the holding period return, and that the return during the second part of the formation period is positively correlated with the holding period return. Assume the return of the asset during the second part of the formation period $[t_{1b}, t_2]$ is positive. Then the expected holding period return will be higher, if the return at the beginning of the formation period $[t_{1a}, t_{1b}]$ is negative, than if the return at the beginning of the formation period is positive. In other words, the asset exhibits stronger momentum if the asset has experienced a recent reversal. In addition, it is also possible to predict future reversals. Assume, that the returns $\Delta_{t_{1a}t_{1b}}$ and $\Delta_{t_{1b}t_2}$ are both positive, but that the return at the beginning of the formation period $\Delta_{t_{1a}t_{1b}}$ is sufficiently higher than the return at the end of the formation period $\Delta_{t_{1b}t_2}$. In that case, as Figure 3.4 shows, the expected holding period return will be negative, even though the total return for the formation period $[t_1, t_2]$ is positive. As Figure 3.5 shows, for certain parameter values it is possible to predict the timing of the reversal during the holding period.

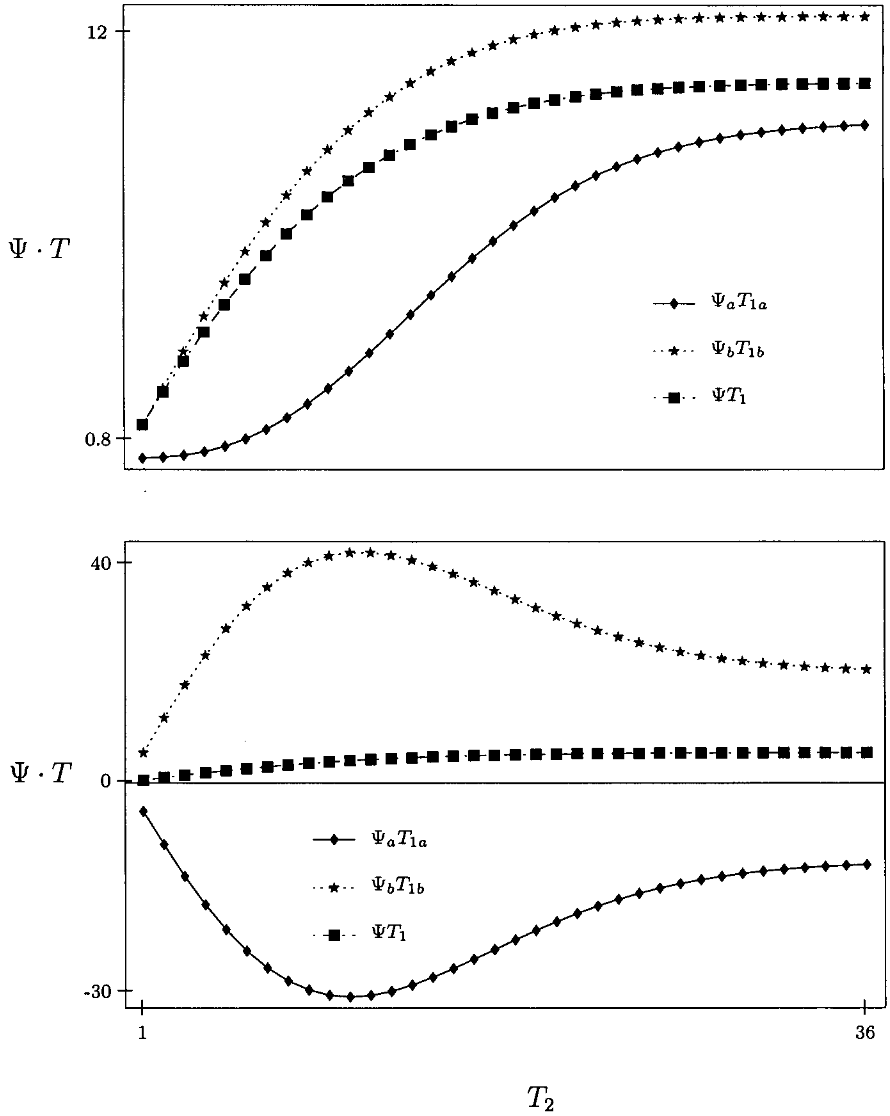
To understand the source of the predictive power of reversals note that future dividends produce trends in the price, if the price reacts to news about dividends slowly. Assume now that the price of the risky asset has increased in the recent past. Then it is not clear whether this trend has just started or whether it is already at the end. Dividing the formation period into two subperiods helps to distinguish between these two situations. If the price has decreased during the first part of the

Figure 3.4: **Momentum Return with Reversals during the Formation Period.** *In this figure, the formation period* $[t_1, t_2]$ *is divided into two time intervals* $[t_{1a}, t_{1b}]$ *and* $[t_{1b}, t_2]$*. The return* ΨT_1 *in the top and in the bottom part of the figure is the expected dollar return for a for the holding period* $[t_2, t_3]$*, if the asset has increased by an average of one dollar per trading date during the formation period* $[t_1, t_2]$*. The return* $\Psi_a T_{1a}$ *(*$\Psi_b T_{1b}$*) in the top part of the figure is the expected dollar return, if the asset has increased by an average of one dollar per trading date during the formation period* $[t_{1a}, t_{1b}]$ *(*$[t_{1b}, t_2]$*). In the bottom part of the figure, the sum* $\Psi_a T_{1a} A + \Psi_b T_{1b} B$ *is the expected dollar return, if the asset has increased by an average of* A *dollar per trading date during the first part of the formation period* $[t_{1a}, t_{1b}]$*, and an average of* B *dollar per trading date during the second part of the formation period* $[t_{1a}, t_{1b}]$*. The remaining parameters are given by* $T_{1a} = T_{1b} = 3$*,* $T_1 = T_{1a} + T_{1b}$*,* $\sigma_\epsilon^2 = 0.1$*,* $\sigma_{\epsilon_\Psi}^2 = 10^5$*,* $\sigma_d^2 = \rho = 1$*,* $r = 0.004$.



3.2
0.1
$\Psi \cdot T$
$\Psi_a T_{1a}$
$\Psi_b T_{1b}$
ΨT_1
5
0
-1
$\Psi \cdot T$
$\Psi_a T_{1a}$
$\Psi_b T_{1b}$
ΨT_1
1
36
T_2

Figure 3.5: **Momentum return with reversals during the formation period II.** *In this figure, the formation period* $[t_1, t_2]$ *is divided into two time intervals* $[t_{1a}, t_{1b}]$ *and* $[t_{1b}, t_2]$. *The return* ΨT_1 *in the top and in the bottom part of the figure is the expected dollar return for a for the holding period* $[t_2, t_3]$, *if the asset has increased by an average of one dollar per trading date during the formation period* $[t_1, t_2]$. *The return* $\Psi_a T_{1a}$ *(*$\Psi_b T_{1b}$*) in the top part of the figure is the expected dollar return, if the asset has increased by an average of one dollar per trading date during the formation period* $[t_{1a}, t_{1b}]$ *(*$[t_{1b}, t_2]$*). In the bottom part of the figure, the sum* $\Psi_a T_{1a} A + \Psi_b T_{1b} B$ *is the expected dollar return, if the asset has increased by an average of* A *dollar per trading date during the first part of the formation period* $[t_{1a}, t_{1b}]$, *and an average of* B *dollar per trading date during the second part of the formation period* $[t_{1a}, t_{1b}]$. *The remaining parameters are given by* $T_{1a} = T_{1b} = 3$, $T_1 = T_{1a} + T_{1b}$, $\sigma_\epsilon^2 = 0.01$, $\sigma_{\epsilon_\Psi}^2 = 10^5$, $\sigma_d^2 = \rho = 1$, $r = 0.004$.



12
0.8
$\Psi \cdot T$
$\Psi_a T_{1a}$
$\Psi_b T_{1b}$
ΨT_1
40
0
-30
1
36
T_2

formation period and increased during the second part, then it is more likely that the recent price trend is a new, and therefore the trend is more likely to continue.

Chapter 4

Conclusion

In this thesis I examine how investors can infer private information from market prices and aggregate trading volume. I develop two theoretical models to address the following questions: (1) how can investors fully rationally infer information from prices and trading volume, and (2) how do the results of a world with perfectly rational investors change, if the rationality of investors is bounded, so that they are not fully able to infer all available private information?

In the first part of this thesis, I provide a closed form solution for a rational expectations equilibrium where all investors infer information about the state of the economy from (1) private signals, (2) the market price and (3) aggregate trading volume. My model shows how investors can learn from trading volume, and how volume information differs from information contained in the price. The main result of this model is that trading volume reveals the relative quality of the aggregate private information in the economy. Under low trading volume, the aggregate information is more precise compared to private signals than under high trading volume. Investors therefore use volume to decide how they should weight the market price relative to their own private signals when they update their beliefs. When trading volume is low, investors weight the market price more heavily. Conversely, when trading volume is high, investors weight their private signals more heavily.

In this model, the price does not fully reveal all private information, since the price also depends on the unknown aggregate exposure to the labor risk. This combination of private information and unknown labor risk has two effects on the autocorrelation of returns. First, private information causes return continuations, since the price incorporates this information slowly. Second, the aggregate labor endowment causes return reversals, since this endowment influences the price temporarily without affecting the future payoff of the risky asset. The total effect in this model is always negative, so that returns are negatively autocorrelated, if investors are fully rational.

In the simplified world of the model in the first chapter of this thesis, the price can only be partially revealing, if the price contains an additional source of noise, such as the uncertain aggregate labor risk exposure. Without this additional noise, fully rational investors can infer all private information from the price. In real financial markets, however, prices aggregate information in a complicated way, and extracting this information is a difficult task. In the second part of this thesis I assume therefore that the computational skills of the investors are limited: they make individual mistakes when they infer information from the price. If investors make individual mistakes when they analyze prices, then they are not able to infer all private information from the price, even if there is no additional source of noise that disturbs the price. Therefore, in a world with bounded rationality, private information can have a positive effect on the autocorrelation of returns, without the negative noise effect that is required if all investors are fully rational. Hence, if the computational skills of the investors are limited, prices react to new information too slowly.

Appendix

Proof of Lemma 1

Assume the price is given by

$$P = \Phi_D^H(D + \eta + \epsilon) - \Phi_N^H N \tag{4.1}$$

Let $\mathcal{F}_i = \{\hat{D}, N_i, P, V\}$ be the information set of investor i. Then the demand of investor i is given by

$$X_i = \frac{\mathrm{E}[D - P|\mathcal{F}_i] - \rho\mathrm{Cov}[D - P, N_i Y|\mathcal{F}_i]}{\rho\mathrm{Var}[D - P|\mathcal{F}_i]}, \tag{4.2}$$

where

$$\mathrm{E}[D - P|\mathcal{F}_i] = \frac{\sigma_D^2}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2}\hat{D} - P \tag{4.3a}$$

$$\mathrm{Var}[D - P|\mathcal{F}_i] = \frac{\sigma_D^2(\sigma_\eta^2 + \sigma_\epsilon^2)}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \tag{4.3b}$$

$$\mathrm{Cov}[D - P, N_i Y|\mathcal{F}_i] = \frac{\sigma_{DY}(\sigma_\eta^2 + \sigma_\epsilon^2)}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2}N_i. \tag{4.3c}$$

Plugging the demand into the equilibrium condition in (2.1) and solving for the price and comparing coefficients with (4.1), we get Φ_D^H and Φ_N^H in Lemma 1. Plugging these coefficients into (4.1), and (4.1) into (4.2), we get the equilibrium demand

$$X_i = -\frac{\sigma_{DY}}{\sigma_D^2} n_i.$$

Since the n_i are independent across investors, we have

$$V^H = \lim_{h \to \infty} \frac{1}{h} \sum_{i=1}^{h} |X_i| = \mathrm{E}|X_i| = \sqrt{\frac{2}{\pi}} \frac{\sigma_{DY}\sigma_n}{\sigma_D^2} \tag{4.4}$$

Proof of Lemma 2

Assume the price is given by

$$P = \Phi_D^L(D + \eta) - \Phi_N^L N \tag{4.5}$$

Then the demand of investor i is given by (4.2). Let $\tilde{\mathcal{F}}_i = \{\hat{D}_i, N_i, V\}$ and $\mathcal{F}_i = \{\hat{D}_i, N_i, P, V\}$. To simplify the notation, I will write Φ_D for Φ_D^L and Φ_N for Φ_N^L in the following. Then we have

$$\mathrm{E}[D|\tilde{\mathcal{F}}_i] = \frac{\sigma_D^2}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \hat{D}_i \tag{4.6a}$$

$$\mathrm{Var}[D|\tilde{\mathcal{F}}_i] = \frac{\sigma_D^2(\sigma_\eta^2 + \sigma_\epsilon^2)}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \tag{4.6b}$$

$$\mathrm{Cov}[D, N_i Y|\tilde{\mathcal{F}}_i] = \frac{\sigma_{DY}(\sigma_\eta^2 + \sigma_\epsilon^2)}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} N_i \tag{4.6c}$$

$$\mathrm{E}[P|\tilde{\mathcal{F}}_i] = \Phi_D \frac{\sigma_D^2 + \sigma_\eta^2}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \hat{D}_i - \Phi_N \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} N_i \tag{4.6d}$$

$$\mathrm{Var}[P|\tilde{\mathcal{F}}_i] = \Phi_D^2 \frac{(\sigma_D^2 + \sigma_\eta^2)\sigma_\epsilon^2}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} + \Phi_N^2 \frac{\sigma_n^2 \sigma_N^2}{\sigma_N^2 + \sigma_n^2} \tag{4.6e}$$

$$\mathrm{Cov}[D, P|\tilde{\mathcal{F}}_i] = \Phi_D \frac{\sigma_D^2 \sigma_\epsilon^2}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \tag{4.6f}$$

Let

$$Z = \frac{\mathrm{Cov}[D, P|\tilde{\mathcal{F}}_i]}{\mathrm{Var}[P|\tilde{\mathcal{F}}_i]} \tag{4.7}$$

Then we have

$$\mathrm{E}[D - P|\mathcal{F}_i] = \mathrm{E}[D|\tilde{\mathcal{F}}_i] + Z\Big(P - \mathrm{E}[P|\tilde{\mathcal{F}}_i]\Big) - P \tag{4.8a}$$

$$\mathrm{Var}[D|\mathcal{F}_i] = \mathrm{Var}[D|\tilde{\mathcal{F}}_i] - Z\mathrm{Cov}[D, P|\tilde{\mathcal{F}}_i] \tag{4.8b}$$

$$\mathrm{Cov}[D, Y|\mathcal{F}_i] = \frac{\sigma_{DY}}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2}\Big[\sigma_\eta^2 + \sigma_\epsilon^2\Big(1 - Z\Phi_D\Big)\Big] \tag{4.8c}$$

Plugging (4.8) into the demand of investor i in (4.2) and plugging these demands into the equilibrium condition in (2.1) and solving for the price we get

$$\begin{aligned} P = \frac{1}{1-Z}\Bigg[&\frac{1}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2}\Big(\sigma_D^2 - (\sigma_D^2 + \sigma_\eta^2)Z\Phi_D\Big)(D + \eta) \\ &+ \left(Z\Phi_N\frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - \rho\frac{\sigma_{DY}}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2}\Big(\sigma_\eta^2 + \sigma_\epsilon^2\Big(1 - Z\Phi_D\Big)\Big)\right) N\Bigg] \end{aligned}$$

Comparing coefficients with (4.5), we get

$$\Phi_D = \frac{\sigma_D^2}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2(1 - Z)} \tag{4.9a}$$

$$\begin{aligned} \Phi_N &= \rho\frac{\sigma_{DY}}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2}\Big(\sigma_\eta^2 + \sigma_\epsilon^2\Big(1 - Z\Phi_D\Big)\Big)\frac{\sigma_N^2 + \sigma_n^2}{\sigma_N^2 + \sigma_n^2(1 - Z)} \\ &= \rho\sigma_{DY}\frac{\sigma_\eta^2 + \sigma_\epsilon^2(1 - Z)}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2(1 - Z)}\frac{\sigma_N^2 + \sigma_n^2}{\sigma_N^2 + \sigma_n^2(1 - Z)} \end{aligned} \tag{4.9b}$$

Note that an equilibrium exists, if there exists a real number Z, such that the denominators in (4.9) are not equal to zero, and such that Z satisfies (4.7), where Φ_D and Φ_N are given by (4.9). Let $g(Z)$ be the right hand side of (4.7). Then it follows from (4.9) that $g(0) > 0$, and we have

$$g(1) = \frac{\sigma_D^4\sigma_\epsilon^2(\sigma_D^2+\sigma_\eta^2)\sigma_N^2}{\sigma_D^4\sigma_\epsilon^2(\sigma_D^2+\sigma_\eta^2)\sigma_N^2 + \rho^2\sigma_\eta^4(\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2)\sigma_n^2(\sigma_N^2+\sigma_n^2)},$$

hence $0 < g(1) < 1$. Define $f(Z) = Z - g(Z)$. Then we have $f(0) < 0$ and $f(1) > 0$. Hence, since $f(Z)$ is continuous for $Z \in [0,1]$, it follows from the intermediate value theorem that there exists an equilibrium for some $Z \in [0,1]$.

Next I will show that in any (linear) equilibrium we must have $0 < Z < 1$. Assume the economy is in an equilibrium. From (4.6f) and (4.9a) we have

$$\mathrm{Cov}[D,P|\tilde{\mathcal{F}}_i] = \Phi_D^2\sigma_\epsilon^2\left(1 - \frac{\sigma_\epsilon^2 Z}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2}\right).$$

Using (4.7) we get

$$\mathrm{Cov}[D,P|\tilde{\mathcal{F}}_i]\left(1 + \frac{\Phi_D^2}{\mathrm{Var}[P|\tilde{\mathcal{F}}_i]}\frac{\sigma_\epsilon^4}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2}\right) = \Phi_D^2\sigma_\epsilon^2.$$

Using (4.6e) and (4.7) again we get

$$Z = \frac{\Phi_D^2\sigma_\epsilon^2}{\Phi_D^2\sigma_\epsilon^2 + \Phi_N^2\frac{\sigma_n^2\sigma_N^2}{\sigma_N^2+\sigma_n^2}}. \tag{4.10}$$

So, since $\Phi_D \neq 0$ by (4.9a), we have $0 < Z \leq 1$. Assume $Z = 1$. Then we have $\Phi_D = \frac{\sigma_D^2}{\sigma_D^2+\sigma_\eta^2}$ by (4.9a), and we must have $\Phi_N = 0$ by (4.10). But we can only have $\Phi_N = 0$ if $Z > 1$ by (4.9b). Hence $Z \in (0,1)$.

For the demand we have from (4.2) and (4.8)

$$X_i = \frac{1}{\rho\mathrm{Var}[D|\mathcal{F}_i]}\left(\tilde{\Psi}_D\hat{D}_i - \tilde{\Psi}_N N_i - \tilde{\Psi}_P P\right), \tag{4.11}$$

where

$$\tilde{\Psi}_D = \frac{1}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2}\left(\sigma_D^2 - (\sigma_D^2+\sigma_\eta^2)Z\Phi_D\right) \tag{4.12a}$$

$$\tilde{\Psi}_N = \rho\frac{\sigma_{DY}}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2}\left(\sigma_\eta^2 + \sigma_\epsilon^2\left(1 - Z\Phi_D\right)\right) - Z\Phi_N\frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2} \tag{4.12b}$$

$$\tilde{\Psi}_P = 1 - Z \tag{4.12c}$$

Plugging (4.5) into (4.11) we get

$$X_i = \frac{1}{\rho \mathrm{Var}[D|\mathcal{F}_i]} \Big[(\tilde{\Psi}_D - \tilde{\Psi}_P \Phi_D) D + \tilde{\Psi}_D \epsilon_i - (\tilde{\Psi}_N - \tilde{\Psi}_P \Phi_N) N - \tilde{\Psi}_N n_i \Big] \tag{4.13}$$

From (4.9a) and (4.12) we have

$$\begin{aligned}
\tilde{\Psi}_D - \tilde{\Psi}_P \Phi_D &= \frac{1}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \Big(\sigma_D^2 - (\sigma_D^2 + \sigma_\eta^2) Z \Phi_D \Big) - (1-Z)\Phi_D \\
&= \frac{1}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \left(\sigma_D^2 - \frac{\sigma_D^2(\sigma_D^2 + \sigma_\eta^2) Z}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2(1-Z)} \right) - (1-Z)\Phi_D \\
&= (1-Z)\Phi_D - (1-Z)\Phi_D \\
&= 0
\end{aligned} \tag{4.14}$$

Similarly, from (4.9b) and (4.12) we have

$$\begin{aligned}
\tilde{\Psi}_N - \tilde{\Psi}_P \Phi_N &= \rho \frac{\sigma_{DY}}{\sigma_D^2 + \sigma_\eta^2 + \sigma_\epsilon^2} \Big(\sigma_\eta^2 + \sigma_\epsilon^2 \Big(1 - Z\Phi_D \Big) \Big) - Z \Phi_N \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - (1-Z)\Phi_N \\
&= \Phi_N \left(\frac{\sigma_N^2 + \sigma_n^2(1-Z)}{\sigma_N^2 + \sigma_n^2} - Z \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - (1-Z) \right) \\
&= 0
\end{aligned} \tag{4.15}$$

From (4.8b) and (4.9a) we have $\mathrm{Var}[D|\mathcal{F}_i] = \sigma_D^2(1-\Phi_D)$. Hence we get from (4.13), (4.14), and (4.15)

$$X_i^L = \frac{1-Z}{\rho \sigma_D^2 (1-\Phi_D)} \Big(\Phi_D \epsilon_i - \Phi_N n_i \Big) = \Psi_D \epsilon_i - \Psi_N n_i. \tag{4.16}$$

Since the ϵ_i and n_i are independent across investors, we have

$$V^L = \sqrt{\frac{2}{\pi} \Big(\Psi_D^2 \sigma_\epsilon^2 + \Psi_N^2 \sigma_n^2 \Big)} \tag{4.17}$$

Proof of Theorem 1

By (4.4) and (4.17) we have $V^H > V^L$ if

$$\frac{\sigma_{DY}^2 \sigma_n^2}{\sigma_D^4} > \Psi_D^2 \sigma_\epsilon^2 + \Psi_N^2 \sigma_n^2. \tag{4.18}$$

From (4.8), (4.12b), and (4.16) we have

$$\begin{aligned} X_i^L &= \Psi_D \epsilon_i - \Psi_N n_i \\ &= \Psi_D \epsilon_i - \left(\frac{Z\Phi_N \frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2}}{\rho \mathrm{Var}[D|\mathcal{F}_i]} - \frac{\sigma_{DY}}{\sigma_D^2} \right) n_i \end{aligned}$$

Hence we have from (4.18) $V^H > V^L$ if

$$M = \Psi_D^2 \sigma_\epsilon^2 + \left(\left(\frac{Z\Phi_N \frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2}}{\rho \mathrm{Var}[D|\mathcal{F}_i]} \right)^2 - 2\frac{\sigma_{DY}}{\sigma_D^2} \frac{Z\Phi_N \frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2}}{\rho \mathrm{Var}[D|\mathcal{F}_i]} \right) \sigma_n^2 < 0$$

We have

$$\begin{aligned} &\rho \mathrm{Var}[D|\mathcal{F}_i]^2 M \\ &= \rho \mathrm{Var}[D|\mathcal{F}_i]^2 \Psi_D^2 \sigma_\epsilon^2 + Z\Phi_N \frac{\sigma_N^2 \sigma_n^2}{\sigma_N^2 + \sigma_n^2} \left(Z\Phi_N \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - 2\frac{\sigma_{DY}}{\sigma_D^2} \rho \mathrm{Var}[D|\mathcal{F}_i] \right). \end{aligned}$$

From (4.8b) and (4.9b) we have

$$\begin{aligned} Z\Phi_N \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - \frac{\sigma_{DY}}{\sigma_D^2} \rho \mathrm{Var}[D|\mathcal{F}_i] &= Z\Phi_N \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - \rho \sigma_{DY}(1 - \Phi_D) \\ &= Z\Phi_N \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - \Phi_N \frac{\sigma_N^2 + \sigma_n^2(1-Z)}{\sigma_N^2 + \sigma_n^2} \\ &= -(1-Z)\Phi_N \end{aligned}$$

Hence we get

$$\rho \mathrm{Var}[D|\mathcal{F}_i]^2 M = \tilde{\Psi}_D^2 \sigma_\epsilon^2 - Z(1-Z)\Phi_N^2 \frac{\sigma_N^2}{\sigma_N^2 + \sigma_n^2} - Z\Phi_N^2 \frac{\sigma_N^2 \sigma_n^2}{\sigma_N^2 + \sigma_n^2} \frac{\sigma_N^2 + \sigma_n^2(1-Z)}{\sigma_N^2 + \sigma_n^2}$$

From (4.14) and (4.10) we have

$$\begin{aligned}
\tilde{\Psi}_D^2\sigma_\epsilon^2 - Z(1-Z)\Phi_N^2\frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2} &= (1-Z)^2\Phi_D^2\sigma_\epsilon^2 - Z(1-Z)\Phi_N^2\frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2} \\
&= (1-Z)\left[\Phi_D^2\sigma_\epsilon^2 - Z\left(\Phi_D^2\sigma_\epsilon^2 + \Phi_N^2\frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2}\right)\right] \\
&= (1-Z)\left[\Phi_D^2\sigma_\epsilon^2 - \Phi_D^2\sigma_\epsilon^2\right] \\
&= 0.
\end{aligned}$$

Hence

$$M = -\frac{Z\Phi_N^2}{\rho \mathrm{Var}[D|\mathcal{F}_i]^2}\frac{\sigma_N^2\sigma_n^2}{\sigma_N^2+\sigma_n^2}\frac{\sigma_N^2+\sigma_n^2(1-Z)}{\sigma_N^2+\sigma_n^2} < 0,$$

so $V^H > V^L$.

Proof of Corollary 1

If signals are perfectly correlated, we have $E[D|\mathcal{F}_i] = \frac{\sigma_D^2}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2}\hat{D}_i$ by (4.3a). In this case, investors do not use the price to update their beliefs since they cannot directly observe the aggregate labor risk shock N. If the correlation of signals is low, we have

$$\begin{aligned}
\mathrm{E}[D|\mathcal{F}_i] &= \mathrm{E}[D|\tilde{\mathcal{F}}_i] + Z\Big(P - \mathrm{E}[P|\tilde{\mathcal{F}}_i]\Big) \\
&= \frac{1}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2}\Big(\sigma_D^2 - (\sigma_D^2+\sigma_\eta^2)Z\Phi_D^L\Big)\hat{D}_i + Z\Phi_N^L\frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2}N_i + ZP \\
&= (1-Z)\Phi_D^L\hat{D}_i + Z\Phi_N^L\frac{\sigma_N^2}{\sigma_N^2+\sigma_n^2}N_i + ZP
\end{aligned}$$

by (4.8a) and (4.14). So since $\Phi_N^L > 0$, $0 < \Phi_D^H < \Phi_D^L$, and $Z \in (0,1)$, we have $\psi_N^H > 0$, $\psi_P^h > 0$, and $0 < \psi_D^L < \psi_D^H$.

Proof of Corollary 2

Let $\text{Var}_L = \text{Var}[D|\mathcal{F}_i]$, if the correlation of signals is low, and let Var_H be the corresponding variance if the correlation of signals is high. Then we have $\Psi_P^H = \frac{1}{\rho \text{Var}_H}$ by (4.2). From (4.3b) and (4.8b) we have $\text{Var}_L = \text{Var}_H \frac{\sigma_\eta^2+\sigma_\epsilon^2(1-\Phi_D^L Z)}{\sigma_\eta^2+\sigma_\epsilon^2}$. So $\Psi_P^L = (1-Z)\frac{\sigma_\eta^2+\sigma_\epsilon^2}{\sigma_\eta^2+\sigma_\epsilon^2(1-\Phi_D^L Z)}\Psi_P^H$. So, since $\Phi_D^L \in (0,1)$ and $Z \in (0,1)$, we have $\Psi_P^L < \Psi_P^H$.

Proof of Corollary 3

Assume the price is given by

$$P = \Phi_D^L(D+\eta+\epsilon) - \Phi_N^L N - \Phi_S^L S$$

when the dispersion of signals is low, and by

$$P = \Phi_D^H(D+\eta) - \Phi_N^H N - \Phi_S^H S$$

when the dispersion of signals is high. Then it follows directly from Appendix A and B that

$$\begin{aligned}
\Phi_S^L &= \rho\frac{\sigma_D^2(\sigma_\eta^2+\sigma_\epsilon^2)}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2} \\
\Phi_S^H &= \frac{\rho}{1-Z}\frac{\sigma_D^2\left[\sigma_\eta^2+\sigma_\epsilon^2\left(1-Z\Phi_D\right)\right]}{\sigma_D^2+\sigma_\eta^2+\sigma_\epsilon^2}
\end{aligned}$$

hence $0 < \Phi_S^L < \Phi_S^H$.

Proof of Lemma 3

If an equilibrium exists, the price must be of the form

$$P_t = \Phi_t + \Phi_{Dt} + \sum_{s=1}^{\infty} \Phi_{dts} \hat{d}_{t+s} \tag{4.19}$$

by Definition 1. Then we have

$$D_{t+1} + P_{t+1} = \Phi_{t+1} + (1 + \Phi_{Dt}) D_{t+1} + \sum_{s=1}^{\infty} \Phi_{dt+1s} \left(d_{t+1+s} + \sum_{j=1}^{s} \epsilon_{t+1+s,j} \right). \tag{4.20}$$

The variance of the signals contained in the vector $\mathbf{I}_t$ is given by

$$Var\left[\hat{d}_{t+s,t}\right] = Var\left[d_{t+s} + \sum_{j=1}^{s} \epsilon_{t+s,j}\right] = \sigma_d^2 + s\,\sigma_\epsilon^2.$$

Let

$$A_s = \frac{\sigma_d^2}{\sigma_d^2 + s\,\sigma_\epsilon^2}. \tag{4.21}$$

Since all the elements of $\mathbf{I}_t$ are independent, we have

$$E\left[d_{t+1+s} \middle| \mathbf{I}_t\right] = A_{s+1}\, \hat{d}_{t+1+s,t}, \tag{4.22}$$

$$E\left[\sum_{j=1}^{s} \epsilon_{t+1+s,j} \middle| \mathbf{I}_t\right] = \frac{s\,\sigma_\epsilon^2}{\sigma_d^2} A_{s+1} \hat{d}_{t+1+s,t}. \tag{4.23}$$

So

$$\begin{aligned}
&E_{it}[D_{t+1} + P_{t+1}] \\
&= \Phi_{t+1} + (1 + \Phi_{Dt+1})(D_t + A_1 \hat{d}_{t+1,t}) + \sum_{s=1}^{\infty} \Phi_{dt+1s} \left[A_{s+1} + \frac{s\,\sigma_\epsilon^2}{\sigma_d^2} A_{s+1} \right] \hat{d}_{t+1+s,t} \\
&= \Phi_{t+1} + (1 + \Phi_{Dt+1})(D_t + A_1 \hat{d}_{t+1,t}) + \sum_{s=1}^{\infty} \Phi_{dt+1s} \frac{A_{s+1}}{A_s} \hat{d}_{t+1+s,t}.
\end{aligned}$$

Plugging this conditional expectation into (3.5), and (3.5) into the equilibrium condition in Definition 1, and solving for the price, we get

$$P_t = \frac{1}{1+r}\left(\Phi_{t+1} + (1+\Phi_{Dt+1})(D_t + A_1\hat{d}_{t+1,t}) + \sum_{s=1}^{\infty}\Phi_{dt+1s}\frac{A_{s+1}}{A_s}\hat{d}_{t+1+s,t}\right).$$

Comparison of coefficients with (4.19) gives

$$\begin{aligned}
\Phi_t &= \frac{1}{1+r}\Phi_{t+1},\\
\Phi_{Dt} &= \frac{1}{1+r}(1+\Phi_{Dt+1}),\\
\Phi_{dt+11} &= \frac{A_1}{1+r}(1+\Phi_{Dt+1}),\\
\Phi_{dts} &= \frac{1}{1+r}\frac{A_s}{A_{s-1}}\Phi_{dt+1s-1}.
\end{aligned}$$

If an equilibrium exists, then there exists a real number M such that $|\Phi_t| < M$, $|\Phi_{Dts}| < M$, and $|\Phi_{dts}| < M$ for all t, s, by condition (b) of Definition 1. So $\Phi_t = 0$, $\Phi_{Dt} = \frac{1}{r}$, and $\Phi_{dts} = \frac{1}{r}\frac{A_s}{(1+r)^{s-1}}$.

Proof of Lemma 4

I will prove the Lemma for $Cov\left[\Delta_{t,t+1}, \Delta_{t+1,t+2}\right]$. The case for general time intervals follows by induction. From (3.7) and (3.8) we have

$$P_t^U = \frac{1}{r}\sum_{s=1}^{\infty}\frac{A_s - 1}{(1+r)^{s-1}}d_{t+s}.$$

So

$$\Delta_{t,t+1}^U = \frac{1}{1+r}P_{t+1}^U - P_t^U = \frac{1}{r}\sum_{s=1}^{\infty}\frac{A_{s-1} - A_s}{(1+r)^{s-1}}d_{t+s}.$$

So

$$Cov\Big[\Delta^U_{t,t+1}, \Delta^U_{t+1,t+2}\Big] = \frac{\sigma_d^2}{r^2}\sum_{s=1}^{\infty}\frac{A_s - A_{s+1}}{(1+r)^s}\frac{A_{s-1}-A_s}{(1+r)^{s-1}} > 0,$$

since $A_s > A_{s+1}$. We have

$$P_t^N = \frac{1}{r}\sum_{s=1}^{\infty}\frac{A_s}{(1+r)^{s-1}}\eta_{t+s,t},$$

so

$$\Delta^N_{t_1 t_2} = \frac{1}{(1+r)^{T_1}}P^N_{t_2} - P^N_{t_1} = \frac{1}{r}\left[\sum_{s=2}^{\infty}\left(\frac{A_{s-1}-A_s}{(1+r)^{s-1}}\sum_{j=1}^{s-1}\epsilon_{t+s,j}\right) - \sum_{s=1}^{\infty}\frac{A_s}{(1+r)^{s-1}}\epsilon_{t+s,s}\right].$$

So we get

$$Cov\Big[\Delta^N_{t,t+1}, \Delta^N_{t+1,t+2}\Big] = \frac{\sigma_\epsilon^2}{r^2}\sum_{s=1}^{\infty}\frac{A_s - A_{s+1}}{(1+r)^{2s-1}}\Big[(A_{s-1}-A_s)(s-1) - A_s\Big].$$

Hence

$$\begin{aligned}
Cov\Big[\Delta_{t,t+1}, \Delta_{t+1,t+2}\Big] &= Cov\Big[\Delta_{t,t+1}, \Delta_{t+1,t+2}\Big] + Cov\Big[\Delta^N_{t,t+1}, \Delta^N_{t+1,t+2}\Big] \\
&= \frac{1}{r^2}\sum_{s=1}^{\infty}\frac{A_s - A_{s+1}}{(1+r)^{2s-1}}\Big[(A_{s-1}-A_s)\Big((s-1)\sigma_\epsilon^2 + \sigma_d^2\Big) - A_s\sigma_\epsilon^2\Big] \\
&= \frac{1}{r^2}\sum_{s=1}^{\infty}\frac{A_s - A_{s+1}}{(1+r)^{2s-1}}\Big[A_{s-1}\Big((s-1)\sigma_\epsilon^2 + \sigma_d^2\Big) - A_s\Big(s\sigma_\epsilon^2 + \sigma_d^2\Big)\Big] \\
&= \frac{1}{r^2}\sum_{s=1}^{\infty}\frac{A_s - A_{s+1}}{(1+r)^{2s-1}}\Big[A_{s-1}\frac{\sigma_d^2}{A_{s-1}} - A_s\frac{\sigma_d^2}{A_s}\Big] \\
&= 0.
\end{aligned}$$

So $Cov\Big[\Delta^N_{t,t+1}, \Delta^N_{t+1,t+2}\Big] < 0$.

Proof of Lemma 5

Assume the price is given by

$$P_t = \Phi_D D_t + \sum_{s=1}^{\infty} \tilde{\Phi}_{ds} d_{t+s}. \tag{4.24}$$

Since all elements of $\mathbf{I}_{it}$ are independent and since $Cov[d_t, d_s] = 0$ for $t \neq s$, we have

$$Var[d_{t+s}|\mathbf{I}_{it}] = s\,\sigma_\epsilon^2 A_s, \tag{4.25}$$

where A_s is given by (4.21), and we have $Cov[d_t, d_s|\mathbf{I}_{it}] = 0$ for $t \neq s$. So we get

$$\begin{aligned}
E[P_t|\mathbf{I}_{it}, D_t] &= \Phi_D D_t + \sum_{s=1}^{\infty} \tilde{\Phi}_{ds} E[d_{t+s}|\mathbf{I}_{it}], \\
Var[P_t + \epsilon_{Pit}|\mathbf{I}_{it}, D_t] &= \sum_{s=1}^{\infty} \tilde{\Phi}_{ds}^2 Var[d_{t+s}|\mathbf{I}_{it}] + \sigma_{\epsilon_P}^2, \\
E[D_{t+1} + P_{t+1}|\mathbf{I}_{it}, D_t] &= (1+\Phi_D)\Big(D_t + E[d_{t+1}|\mathbf{I}_{it}]\Big) + \sum_{s=2}^{\infty} \tilde{\Phi}_{ds-1} E[d_{t+s}|\mathbf{I}_{it}], \\
Cov[D_{t+1} + P_{t+1}, P_t + \epsilon_{Pit}|\mathbf{I}_{it}, D_t] &= (1+\Phi_D)\Phi_{d1} Var[d_{t+1}|\mathbf{I}_{it}] + \sum_{s=2}^{\infty} \tilde{\Phi}_{ds-1}\tilde{\Phi}_{ds} Var[d_{t+s}|\mathbf{I}_{it}].
\end{aligned}$$

Note that all of these conditional moments are finite, since $|\tilde{\Phi}_{ds}| < M$. Let

$$Y = \frac{Cov[D_{t+1} + P_{t+1}, P_t + \epsilon_{Pit}|\mathbf{I}_{it}]}{Var[P_t + \epsilon_{Pit}|\mathbf{I}_{it}]}. \tag{4.26}$$

We have

$$
\begin{aligned}
&E[D_{t+1} + P_{t+1} | \mathbf{I}_{it}, D_t, P_t + \epsilon_{Pit}] \\
&\qquad = E[D_{t+1} + P_{t+1} | \mathbf{I}_{it}, D_t] + Y\big(P_t + \epsilon_{Pit} - E[P_t | \mathbf{I}_{it}, D_t]\big) \\
&\qquad = \big(1 + \Phi_D(1 - Y)\big) D_t + \big(1 + \Phi_D - Y\Phi_{d1}\big) E[d_{t+1} | \mathbf{I}_{it}, D_t] \\
&\qquad\quad + \sum_{s=2}^{\infty} (\tilde{\Phi}_{ds-1} - Y\tilde{\Phi}_{ds}) E[d_{t+s} | \mathbf{I}_{it}] + Y(P_t + \epsilon_{Pit}). \qquad (4.27)
\end{aligned}
$$

Plugging the conditional expectation into (3.15), using (3.6), (4.22), and the equilibrium condition in Definition 1, we get

$$
P_t = \frac{1}{1+r-Y} \left[\big(1 + \Phi_D(1-Y)\big) D_t + \big(1 + \Phi_D - Y\tilde{\Phi}_{d1}\big) A_1 d_{t+1} + \sum_{s=2}^{\infty} (\tilde{\Phi}_{ds-1} - Y\tilde{\Phi}_{ds}) A_s d_{t+s} \right].
$$

Comparison of coefficients with (4.24) gives

$$
\Phi_D = \frac{1}{r} \qquad (4.28a)
$$

$$
\tilde{\Phi}_{d1} = \frac{1+r}{r} \frac{1}{Y + A_1^{-1}(1+r-Y)} \qquad (4.28b)
$$

$$
\tilde{\Phi}_{ds} = \frac{\tilde{\Phi}_{ds-1}}{Y + A_s^{-1}(1+r-Y)} \qquad (4.28c)
$$

Next I will show that $0 < Y < 1 + r$, if $\sigma^2_{\epsilon_\Psi} > 0$. From (4.28a) and (4.28b) we have

$$
1 + \Phi_D = \tilde{\Phi}_{d1}\big[Y + A_1^{-1}(1+r-Y)\big].
$$

So, using (4.28c) we get

$$
Cov[D_{t+1} + P_{t+1}, P_t + \epsilon_{Pit} | \mathbf{I}_{it}, D_t] = \sum_{s=1}^{\infty} [Y + A_s^{-1}(1+r-Y)] \tilde{\Phi}^2_{ds} Var[d_{t+s} | \mathbf{I}_{it}].
$$

Using (4.26) we get

$$Cov[D_{t+1}+P_{t+1}, P_t+\epsilon_{Pit}|\mathbf{I}_{it}, D_t] = \frac{(1+r)\sum_{s=1}^{\infty} A_s^{-1}\tilde{\Phi}_{ds}^2 Var[d_{t+s}|\mathbf{I}_{it}]}{1+Var[P_t+\epsilon_{Pit}|\mathbf{I}_{it}, D_t]^{-1}\sum_{s=1}^{\infty}(A_s^{-1}-1)\tilde{\Phi}_{ds}^2 Var[d_{t+s}|\mathbf{I}_{it}]}$$

So, applying (4.26) again, we get

$$Y = \frac{(1+r)\sum_{s=1}^{\infty} A_s^{-1}\tilde{\Phi}_{ds}^2 Var[d_{t+s}|\mathbf{I}_{it}]}{\sum_{s=1}^{\infty} A_s^{-1}\tilde{\Phi}_{ds}^2 Var[d_{t+s}|\mathbf{I}_{it}] + \sigma_{\epsilon_\Psi}^2} \tag{4.29}$$

So $0 < Y < 1+r$, since $A_s > 0$. Let

$$B_s = \frac{1+r}{Y + A_s^{-1}(1+r-Y)},$$

and

$$\Phi_{ds} = \prod_{j=1}^{s} B_j.$$

From (4.28c) we have

$$\tilde{\Phi}_{ds} = \frac{1+r}{r}\prod_{j=1}^{s}\frac{B_j}{1+r} = \frac{1}{r}\Phi_{ds}R^{s-1}. \tag{4.30}$$

Since $0 < Y < 1+r$ we have $A_s < B_s < 1$. So $A_s < \Phi_{ds} < 1$, and $\Phi_{ds} > \Phi_{ds+1}$, and $\lim_{s\to\infty}\Phi_{ds} = 0$. From (4.29) we have $\lim_{\sigma_{\epsilon_\Psi}^2} Y = 1+r$. Hence $\lim_{\sigma_{\epsilon_\Psi}^2\to 0} B_s = \lim_{\sigma_{\epsilon_\Psi}^2\to 0}\Phi_{ds} = 1$.

I will now show that an equilibrium exists. Let

$$\begin{aligned} L &= \frac{1}{r}, \\ U_s &= \frac{1}{r(1+r)^{s-1}}. \end{aligned}$$

Then we have from (4.30) $L < \tilde{\Phi}_{ds} < U_s$. Let $\mathcal{B}$ be the set of all infinite sequences of real numbers $\mathbf{x} = (x_0, x_1, \ldots)$, such that $x_s \in [L, U_s]$. For $\mathbf{\Phi} = (\Phi_{I0}, \Phi_{I1}, \ldots) \in \mathcal{B}$ let $f_s(\mathbf{\Phi})$ be the right hand side of (4.30). Let $\mathbf{f}(\mathbf{\Phi}) = (f_0, f_1, \ldots)$. Then $\mathbf{f} \in \mathcal{B}$.

With each element $\mathbf{x} \in \mathcal{B}$ associate the norm $||\mathbf{x}|| = \sup_{x_s \in \mathbf{x}} |x_s|$. Then $\mathbf{f}(\mathbf{\Phi})$ is continuous. Since $\lim_{s \to \infty}(U_s - L) = 0$, it is easy to show that $\mathcal{B}$ is compact.[1] So $\mathcal{B}$ is complete. Also $\mathcal{B}$ is clearly convex. Hence it follows from Schauder's fix point theorem that there exist a $\mathbf{\Phi}^* \in \mathcal{B}$, such that $\mathbf{f}(\mathbf{\Phi}^*) = \mathbf{\Phi}^*$.

Proof of Lemma 6

If the price is given by Lemma 5, then we have $P_t = P_t^F + P_t^U$, where

$$P_t^U = \frac{1}{r} \sum_{s=1}^{\infty} \left(\Phi_{ds} - R^{s-1}\right) d_{t+s}.$$

So

$$\Delta_{t,t+1}^U = \frac{1}{1+r} P_{t+1}^U - P_t^U = \frac{1}{r} \sum_{s=1}^{\infty} \frac{A_{s-1} - A_s}{(1+r)^{s-1}} d_{t+s}.$$

So

$$Cov\left[\Delta_{t,t+1}^U, \Delta_{t+1,t+2}^U\right] = \frac{\sigma_d^2}{r^2} \sum_{s=1}^{\infty} \frac{A_s - A_{s+1}}{(1+r)^s} \frac{A_{s-1} - A_s}{(1+r)^{s-1}} > 0,$$

since $A_s > A_{s+1}$.

Proof of Lemma 7

Assume the price is given by

$$P_t = \Phi_D D_t + \sum_{s=1}^{\infty} \Phi_{ds} d_{t+s}. \tag{4.31}$$

[1]The proof that $\mathcal{B}$ is compact is a simple extension of the proof that every k-cell is compact. See for example Rudin, 1976, page 39, Theorem 2.40.

For Z given by Lemma 7, Y given by (4.29), and A_s given by (4.21), we have

$$\begin{aligned} Var\left[\sqrt{Z}\sum_{j=s}^{\infty}\epsilon_{t-s,j}\right] &= s\,\sigma_\epsilon^2 Z \\ &= s\,\sigma_\epsilon^2\left(1-\frac{Y}{1+r}\right) \\ &= \sigma_i^2\frac{Y-(1+r)}{1+r}(1-A_s^{-1}) \\ &= \sigma_i^2\left(\frac{Y+A_s^{-1}(1+r-Y)}{1+r}-1\right). \end{aligned}$$

Let

$$\tilde{A}_s = \frac{\sigma_i^2}{\sigma_i^2 + s\,\sigma_\epsilon^2 Z}.$$

Then we have

$$\tilde{A}_s = \frac{1+r}{Y+A_s^{-1}(1+r-Y)}. \tag{4.32}$$

Under the assumptions of Lemma 7, the conditional expectation for the representative agent is given by

$$E[D_{t+1}+P_{t+1}|\hat{\mathbf{I}}_{it}] = (1+\Phi_D)(D_t+\tilde{A}_1 d_{t+1}) + \sum_{s=2}^{\infty}\Phi_{ds-1}\tilde{A}_s d_{t+s}.$$

So we get

$$P_t = \frac{1}{1+r}\left[(1+\Phi_D)(D_t+\tilde{A}_1 d_{t+1}) + \sum_{s=2}^{\infty}\Phi_{ds-1}\tilde{A}_s d_{t+s}\right].$$

Comparison of coefficients gives $\Phi_D = \frac{1}{r}$, and

$$\Phi_{d1} = \frac{1}{r}\tilde{A}_1 \tag{4.33a}$$

$$\Phi_{ds} = \frac{\tilde{A}_s}{1+r}\Phi_{ds-1}. \tag{4.33b}$$

Now the result follows from (4.28), (4.32), and (4.33).