PROBING STUDENTS' THINKING WHEN INTRODUCED TO FORMAL COMBINATORICS THEORY IN GRADE 12

by

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Abstract

This research explores students' mathematical thinking when introduced to formal combinatorics theory. It identifies how students understand formal theory and modify their mathematical thinking and resolution strategies after having been introduced to combinatorics. This research is situated in the study of a Mathematics 12 class for the duration of a teaching unit on combinatorics, and of two groups of two students that solved specific combinatorial problems outside of class hours. Data includes videotapes of the classes and group sessions, copies of students' work and tests, students' answers to meta-cognitive questions and field notes.

I describe how students solved a specific combinatorial problem – the pathway problem – arguing that this description exemplifies how students shifted from resolution strategies based on counting and the use of different techniques such as drawings, graphs, lists, trees, amongst others, to the sole use of the taught algorithm. I argue that this shift followed both the emphasis given to the use of formulae during instruction and the students' lack of proficiency in the use of counting techniques. The latter is described in detail and points to the fact that students lacked practice and were not systematic.

Results from this study suggest that the shift from using counting techniques to using formulae was common throughout the unit. In particular, it was the case with the permutation and combination formulae. Nevertheless, in the case of permutations, some students still used repeated multiplication instead of the formulae. Students were confused as to which formula to choose between the permutation and combination formula. I illustrate how students saw combinations only as permutations without order and did not understand the impact of the division in the combination formulae. Students' understanding was limited and they had no other way to solve the problem than to apply a formula they did not understood.

Following these findings, I suggest teachers should not overlook the instruction of counting techniques and should make connections between these and the formulae, for instance in showing various methods for resolving problems using both methods. I also recommend teaching combinations by emphasizing the role of division in the formula and in computations when solving problems without using the formula.

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"And so Xanthippe, Socrates and Pythagoras play happily in the sunshine with the three little pigs, and not a further thought is given to the complicated calculations that have kept them awake all night."

Anno & Mori (1986, p. 41)

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Chapter 1: Introduction & Argument

1.1. Introduction

Combinatorics is a branch of discrete mathematics that studies enumerations of objects contained in sets. Counting how many objects are in a set is an important type of problem in combinatorics. Important concepts of combinatorics include permutation and combination. Despite often being perceived as a difficult subject, combinatorics – in itself or as a pre-requisite for probability and statistics – has increasing societal importance in today's scientific and democratic world, making the presence of this topic of mathematics in the curriculum particularly pertinent.

In Canada, the expectation for grades 9-12, as determined by the guidelines from the National Council of Teachers of Mathematics is for students to "develop an understanding of permutations and combinations as counting techniques" (NCTM Standards, 2003, p. 290) and to "understand meanings of operations and how they relate to one another"(NCTM Standards, 2003, p. 290). Actually the kind of combinatorics seen in Grade 12 is limited to counting problems. This explains why sometimes the subject is referred to as *counting* instead of combinatorics. From now I will use the term combinatorics in its restrictive sense of the kind seen in high school. Combinatorics is related to and is, to some extent, a prerequisite for learning statistics and probability. However, simple problems – such as finding the number of combinations of a three-number lock or predicting how many two-letters words can be written – are seen as early as in primary school. Nevertheless the formal study of combinatorics as a theory is left for the last year of high-school and usually precedes the study of probability. Its focus is on the permutation and combination formulae. At this stage combinatorics is complex and subtle. It is then considered a difficult subject, particularly because it demands more than rote application of formulae, making understanding crucial. Furthermore numbers can be large, adding to the difficulty for students.

In the combinatorics studied in high school, much can be deduced from the counting principle, a principle that most students grasp intuitively. This means that, at least initially, students can achieve much without knowing the formulae by using their problem-solving skills and various counting techniques. However, some scaffolding is needed when complexity arises. At this level students are introduced to formulae. With the instruction of a formal theory of combinatorics, students are presented with a vast field that has been systematized into categories of problems that can each be solved by the application of a specific formula. Their use, with the help of calculators, facilitates computation but also seems to make students forget about the rich and varied array of methods that can be used for solving combinatorial problems. These include counting, logical reasoning, lists, diagrams, trees, tables, and so on. It is this particularly crucial transition from a problem-solving methodology to a more formalised application of combinatorics theory that this research seeks to probe, by focussing on students' mathematical thinking and understanding within a case study in a Mathematics 12 class in a Canadian school. The purpose of this research is therefore to explore students' mathematical thinking when they are introduced to formal combinatorics theory; and to identify how students understand it and modify their mathematical thinking and resolution strategies after having been introduced to it.

1.2. Argument

The problematic of this research project is rooted in my teaching experience. I have taught combinatorics in state schools in Switzerland for five years. It is a subject that I like. The subject is rich and challenging, and it has many connections with other topics in mathematics and with real life. Moreover its less algorithmic nature makes it somewhat different from the other mathematics done in high school: it is more open and requires critical thinking. However this also means that it is not easy to teach. Students solving combinatorial problems really need to reason, to understand and to grasp the meaning of what they are doing, proceeding logically at each step to reach a solution. This puts some balance into the curriculum that, in my opinion, often over-emphasizes the procedural side of mathematics. As a teacher, I noticed every year when teaching combinatorics in high school that many students are able to solve basic problems on their own, using not formulae but different strategies such as lists, diagrams, counting or logical reasoning. When presented with formulae and formal theory, the students also managed to grasp the effectiveness of the formulae and use them in closely related problems. Yet students often got confused about which formula to apply or appeared to get stuck when a problem demanded more than just applying one formula on a specific situation. Part of the problem rests with the complex nature of the subject itself. Moreover, there are many interconnections between different parts of combinatorics, as somehow the structure of the subject is combinatorial in nature. It is a subject that not all math teachers feel confident with, as it is not as straightforward as other mathematical theories, and rapidly becomes rather tricky as complexity increases and numbers become very large.

From the perspective of the teacher, conveying the need to be precise, to apply logic and theory has to be balanced with the necessity of explaining that different paths are possible when solving particular problems, and that problems cannot be solved mechanically. Yet combinatorics is often presented and taught as a clean and closed mathematical theory composed of a small number of formulae – of which the permutation and combination formulae are the backbone – and linked to a series of related exercises. It is often taught in a straightforward, procedural way, making categories of exercises to be solved in a specific way, generally limited to the application of the formulae. This means that the capabilities and previous knowledge of the students are not taken into account.

Real difficulties seemed to arise for many of my students in Switzerland when they had to choose which formula to use. For most of them, it seemed that knowledge of the mathematical theory was disconnected from their problem-solving abilities. One year, I designed a decision-tree to help them choose which formula to use in which context. They were grateful for being given a trick to succeed in solving a range of problems but I was not convinced that doing so greatly improved the comprehension of many of my students. It was only an algorithm, one more rule that allowed students to circumvent the lack of understanding. Somehow, over-reliance on formulae and overlooking the understanding of the computation being done resulted in loss of meaning and, to some extent, to poor achievement.

Clear thinking is crucial to success in combinatorics. This means that when answers are incorrect, the teacher cannot always understand how the student wrestled with the problem. As a teacher, it is often difficult to probe the students' thinking when the way they are taught requires them to give answers to a problem, and not explain or justify how they proceeded and thought about it. I therefore often found it difficult to help the students because I was unable to identify how they were thinking, making their decisions and understanding.

In retrospect, it seemed to me that the difficulties that the students were encountering could broadly be divided into two types. Firstly, they sometimes used a formula that corresponded to a different type of combinatorial operations. This is one of the types of error identified by Batanero, Navarro-Pelayo and Godino (Batanero, Navarro-Pelayo & Godino, 1997). In their study, following a test given to 700 students exposed to different types of instruction, they concluded that to follow an algorithm often elicits meaning instead of strengthening the reasons for the mathematical steps taken. Secondly, I also noticed that another type of difficulty arose when the problem was slightly more complicated and was not a simple application of a formula. Then the students seemed unable to use and connect the two skills of problem-solving and theoretical knowledge that they apparently had mastered separately beforehand. Was it only a matter of cognitive overload or was it that the students expected to solve the problem very quickly – having in this case only to use one formula? The latter behaviour is what Schoenfeld described as students believing that "assigned mathematics problems [can be solved] in five minutes or less" (Schoenfeld, 1988, p. 151).

Carrying out this piece of research was for me a way to address these problems and probe how accurate my impressions as a teacher were on students' thinking and understanding of combinatorics. In this research, I therefore started from the idea that students' approach to the subject of combinatorics was fundamentally fragmented, due to the way formal theory is usually presented and taught to them. The teaching of combinatorics focuses on a number of formulae – usually comprising the one for simple permutations, permutations with similar object, permutations of k out of n objects, combinations of k out of n objects; but sometimes completed with the ones for permutations with repetition of k out of n objects, combinations with repetition of k out of n objects – without taking time to make multiple meaningful connections between them. The combinatorial theory taught categorizes types of problems and specifies which formula solves each category, as in the decision-tree I designed for my students. Yet this has the serious drawback of leaving students with a fragmented view of combinatorics, whereas the subject is in fact combinatorial in nature. It would be much more representative to think of the subject as a web with many ramifications and interconnections. I therefore focussed on the following research question:

Research question

How do students integrate combinatorics theory with their previous knowledge of counting strategies?

Hypothesis

In order to answer my research question, I have focussed on students as they integrate the new subject matter with their previous knowledge and know-how. I assume that because the students' approach is fragmented and over-relies on formula they are unable, when faced with more complex combinatorial problems, to continue to elicit meaningful connections and draw from the problem-solving strategies they used previously. *In particular, I argue that their thinking shifts from meaning-related problem-solving to algorithmic approaches in which the focus is on choosing the right formula.* In probing students' thinking and understanding, I am particularly interested in grasping *what prevents assigning meaning to the decision of choosing a formula* and how this influences students' ability to think through combinatorics.

In the next chapter I review the literature on the teaching and learning of combinatorics. Then I describe the methods I used and the context in which this study took place. Results will follow in three chapters and be discussed in the concluding chapter.

Chapter 2: Literature review

In this chapter, I further define the field of combinatorics, giving first some brief historical background to the subject itself. I then focus on the position of combinatorics in the school curriculum, and follow with a review of the literature on teaching, learning and understanding of combinatorics – a rather small and fragmented field. After looking at Skemp's concepts of understanding, I situate my research within the field.

2.1. The emergence of combinatorics

Combinatorics is part of the larger field of discrete mathematics. Discrete mathematics is the study of discrete – in opposition to continuous – mathematical structures that also includes elementary probability theory, logic, information theory, and so on. This is not a new field of mathematics since "[c]ombinatorics goes back to the 16th century when games of chance played an important role in the life of members of privileged classes" (Vilenkin, 1971, p. xi). Yet, as with the Pascal Triangle that was attributed not to its inventor but to its most famous proponent, the 'birth' of combinatorics is often attributed to Pascal and Fermat in the 17th century for their effort "in the pursuit of theoretical studies of combinatorial problems [that] provided approaches to the development of enumerative combinatorics as the study of methods of counting various combinations of elements of a finite set" (Abramovich & Pieper, 1996). One should also note that probability, the twin sister of combinatorics, was probably born at the same time as "problems arising from games of chance were the moving force behind the development of combinatorics and probability" (Vilenkin, 1971, p. xi). Thus it is not surprising to find combinatorics and probability side by side in almost every mathematics curriculum.

Jacob Bernoulli, Leibniz, Euler and Newton also developed combinatorics further, but the advent of computers brought back an interest in discrete mathematics in general and combinatorics in particular, making them popular fields of mathematics in the last decades. Computers are discrete machines which means that combinatorics, as well as other discrete mathematics topics, are part of many computer science degree curricula. This means that counting the number of operations is fundamental, and is done in part through combinatorics. Combinatorial problems very often result in manipulating large numbers, so computer and calculators made these computations possible.

2.2. Place, importance and relevance of combinatorics in the curriculum

Combinatorics has a peculiar place in mathematics curricula. The study of combinatorics as a theory – which consists of the permutation and combination formulae and is often complemented with other formulae – is done in high school, often in the last year. Nevertheless many problems that are combinatorial in nature are seen in earlier grades. Moreover, in British Columbia, it only entered the curriculum as a topic less than a dozen years ago. It was put in it with probability and statistics. This is revealing of one of the main reasons why combinatorics is taught: it is a fundamental – it is needed for the computation of sample space and the number of favourable events – pre-requisite for the teaching of probability. As a matter of fact many curricula follow the sequence combinatorics-probability-statistics. It is reflected in textbooks like Addison-Wesley's Mathematics 12 (Alexander & Kelly, 1999) and Mathpower 12 (Thompson, 2000) that typically follow the same sequence.

There are other reasons to study combinatorics. Kapur discussed combinatorics in relation to curriculum modernization with an objective of finding "a proper balance between abstraction and applications" (Kapur, 1970, p. 111). He mentioned eleven reasons why combinatorics is important and should be taught in school, in particular:

- it "does not depends on calculus" and so it can be done early in the curriculum (Kapur, 1970);

- "it can be used to train students in concepts of enumeration, [...] making conjecture, generalisation, [...] existence, systematic thinking" (Kapur, 1970, p. 114);
- it has many applications in other fields of study (physical sciences and engineering, biological sciences, social sciences, management sciences) as well as programming and recreational mathematics (Kapur, 1970);
- "students can appreciate the powers and limitations of mathematics as well as the powers and limitations of computers through combinatorial mathematics" (Kapur, 1970, p. 114).

Moreover, combinatorics plays an important role in computing science and is part of its curriculum. For instance combinatorics is used to compute the number of operations required by a program and hence its speed.

Considering the vaster field of discrete mathematics, Kenney and Bezuszka point out that this kind of mathematics "can be used to illustrate and emphasize effectively NCTM's four overall curriculum standards for all students. That is, discrete mathematics problems [:]

- require that many problem-solving strategies be applied to interesting real-world problem application;
- lend themselves well to situations in which students collaborate and develop verbal and written skills in the process of solving the problem;
- demand the sustained use of critical thinking and reasoning procedures in working towards a solution;
- promote mathematical connections within and across disciplines through a wide range of problem types"

(Kenney & Bezuszka, 1993, p.676). All of the above is also appropriate for combinatorics.

Finally, other important mathematical concepts or ideas are widely encountered throughout combinatorics, which makes it study rich and interesting from a mathematical perspective. There is, for instance, recursion. There is also the fact that numbers can rapidly become enormous: it is a way to challenge the concept of linearity which is so prevalent in people's way of thinking. Moreover, concerning mathematical procedures, combinatorics is a field that is ideally suited to mathematization and exercise representation, particularly lists, trees and graphs. Classification, decomposition into two sub-problems, as well as logical and numerical procedures are also seen in a meaningful context.

Finally, it is important to point out that many authors deplore the little place that combinatorics has in the curriculum despite its importance and relevance. Kapur wrote in 1970 that "combinatorial mathematics is an essential component of the mathematics of the discrete and as such it has an important role to play in school mathematics. This role has been little exploited so far" (Kapur, 1970, p. 114). In 2005 English complained that "despite its importance in the mathematics curriculum, combinatorics continues to remain neglected" (English, 2005, p. 121).

2.3. The teaching, learning and understanding of combinatorics

In a similar fashion, many authors deplore that there is also not much literature on the teaching and learning of combinatorics. For instance English asserts that "[r]esearch on children's combinatorial reasoning has not been prolific, despite its role in the development of early probability ideas" (English, 2005, p. 126), an assertion that is also valid for high school students. Yet all the authors I encountered in the literature until now were enthusiastic about combinatorics, mainly because it is as a rich and interesting field that has many implications and that can be made meaningful by making connections to real life contexts. Moreover it fits snugly with many of the *NCTM Standards & Principles*' objectives. One should, alas, recognize that there is no such thing as a free lunch and "the majority [of people] are absolutely perplexed before a plain combinatorial problem" (Dumont in Fischbein et al., 1970, p. 269). Combinatorics is difficult, particularly for students (Hadar & Hadass, 1981; Batanero et al., 1997) but also for their teachers (Burghes & Galbraith, 2000).

2.3.1. The teaching and learning of combinatorics

At the middle and high school level, the focus of the literature is generally on the content (Kapur, 1970), its inherent difficulties (Hadar & Hadass, 1981) and the presentation of a particular way of introducing combinatorial concepts (Abramovich & Pieper, 1996; Burghes & Galbraith, 2000; Sherell, Robertson and Sellers, 2005). I briefly develop each of these references below.

Kapur proposed a variety of combinatorial problems that could be done at the school level. Many do not require the use of combinatorial formulae but many are also very complex (Kapur, 1970). Hadar and Hadass described the difficulties faced by students when they try to solve a combinatorial problem. They used the *misaddressed letters problem* and its resolution as an example. They pointed to seven pitfalls, in particular: the understanding of what needs to be counted, the construction of a systematic method, the realisation of the counting plan (Hadar & Hadass, 1981). Abramovich and Pieper described how they used manipulatives and computers to visually explore permutations and combinations. Their focus was on "recursive reasonings as a means for approaching these ideas" (Abramovich & Pieper, 1996). Burghes and Galbraith proposed to use the British National Lottery to put combinatorics in context by giving real life examples and so motivate students. They gave several examples of how to compute the odds of winning (Burghes & Galbraith, 2000). Sherell, Robertson and Sellers simply used a computer to generate different permutations to help students study pattern and discover the formulae. "This allow[ed] the student to better visualize patterns and [could] help them derive the formulas that represent these patterns" (Sherell et al., 2005, p. 114).

2.3.2. The understanding of combinatorics

Dubois (1984) proposed a classification for combinatorial configurations. In doing so he showed how trying to systematise combinatorics is complex. He reckoned that his work does not suit pre-university students. Moreover he acknowledged that the pedagogical use of his work necessitated finding an "efficient systematic approach that would allow the student to easily find the appropriate method to solve a combinatorial problem" (Dubois, 1984, p 54, personal translation). This approach is exemplary of the very mathematical approach that is common when teaching mathematics and combinatorics in particular: the theory is systematized and polished before being presented to the students.

Nevertheless the theoretical work of Dubois was significant in making explicit that the different combinatorial operations – permutations and combinations – can be modelled in three different manners: selection, distribution and partition. This modelling influences the understanding of combinatorial situations and hence resolution strategies. Building on this, Batanero, Navarro-Pelayo and Godino (1997) analyzed Spanish textbooks and "found that combinatorial operations are usually defined using the idea of sampling [selection]", and that "exercises in these textbooks [...] refer either to sampling or to distribution problems" and that "situations of partition of a set into subsets are hardly employed in these exercises at all" (Batanero et al., 1997, p. 185). Probing further by testing more than 700 pupils doing a battery of combinatorial problems, they showed that modelling is an important but non-trivial process and that these three models are not equal in difficulty. Moreover, mastering one model does not transfer into another model. Then they concluded that "understanding a concept (e.g. combinations) cannot be reduced to simply being able to reproduce its definition. Concepts emerge from the system of practices carried out to solve problem-situations" (Batanero et al., 1997, p. 196). They recommended considering the "combinatorial model [..] as a didactic variable in organising elementary combinatorics teaching." (Batanero et al., 1997, p. 181).

Duckworth (1996) shared the idea that practice is as a path that leads to understanding. She described how a Grade 9 student knew that the solution to a permutation problem was 24 but was not able to make a full list. She explained that the formula his/her student used "represents his understanding instead of substituting for it" (Duckworth, 1996, p. 135). She further advocated that "looking for relationship among systems enhances our understanding" (Duckworth, 1996, p. 134) and described different ways to list permutations. This dichotomy between applying the formulae and understanding is also deplored by Abramovich and Pieper who stated that "student learning of combinatorics has often been limited to plugging numbers into formulas for permutations and combinations without developing any conceptual understanding of combinatorial ideas" (Abramovich & Pieper, 1996, p. 4).

English (2005) looked at some elementary ideas of combinatorics – mainly the use of the fundamental counting principle and graphic representation when solving basic combinatorial problems – and children's related reasoning. She looked at children younger than 10 year olds. "The majority of the children could solve the problems in a variety of ways and could represent the problems symbolically" but they also "lacked a complete understanding of the problem structure" (English 2005, p. 129). Drawing from her study and after a review of the field – where Batanero et al. have a prominent place – she advocated for the importance of combinatorics in the elementary curriculum, and she recommended to increase "children's access to powerful combinatorial ideas [by:] foster[ing] independent thinking, [...] encourag[ing] flexibility in approaches and representations, [...] focus[sing] on problem structures, [...] encourag[ing] sharing of solutions, and [...] provid[ing] problem-posing opportunities" (English, 2005, pp. 121-122). She added that problems that lead students to explore combinatorial ideas and processes without direct teacher instruction should be included in the curriculum.

At this point it is worth mentioning two books from renowned children's literature author Mitsumasa Anno: *Anno's Mysterious Multiplying Jar* (Anno & Anno, 1983) and *Anno's Three Little Pigs* (Anno & Mori, 1986). They are both written for a young audience – only counting and simple multiplications are used – but they are in line with English's stance by presenting complex combinatorial concepts. The Multiplying Jar describes the concept of factorial and the fact that with such an operation numbers can become very large. Anno's Three Little Pigs uses the ubiquitous story of the wolf and the three little pigs to show the concepts of permutation and combination, using a lot of beautiful drawings that include trees and lists. Yet the story is entertaining and allows many levels of readings by skipping the mathematics or taking time to explore it. This approach is refreshing because it is very visual, and dares to present complex ideas to a young audience. This contrasts with the approach of doing much of the combinatorics late in the curriculum and mostly through the use of formulae.

2.4. Instrumental and relational understanding

With the introduction of formulae, techniques for solving mathematical problems change. Sometimes understanding is not necessary and the rote application of a formula or algorithm is sufficient. Since I am interested in students' understanding of combinatorics related to its formulae, such a phenomenon has to be taken into account. Skemp's concepts of understanding are ideally suited for that and I used them as an analytical framework. I define them in the next paragraphs.

In his seminal paper *Relational Understanding and Instrumental Understanding* (Skemp, 1976), Skemp defined two different types of understanding: relational understanding was "knowing both what to do and why", whereas instrumental understanding was described as "rules without reasons" specifying that "possession of such a rule, and ability to use it, was what [many pupils and teachers] meant by understanding" (Skemp, 1976, p. 80). Skemp added that there were consequently two "effectively different subjects being taught under the same name, 'mathematics' " (Skemp, 1976, p.85): one reaching for students' relational understanding and the other reaching for students' instrumental understanding. Moreover, the use of rules as a way to understand and do mathematics is also problematic as Erlwanger showed with the example of Benny, a pupil who created his set of rules to do maths and managed to be quite successful in worksheets and tests despite many of his ideas and rules being wrong (Erlwanger, 1973).

Skemp advocated for relational understanding and listed some of its advantages; including particularly that "it is more adaptable to new tasks [...], easier to remember [and] it can be an effective goal in itself" (Skemp, 1976, pp 87-88). In contrast he pointed out that instrumental understanding "involves a multiplicity of rules rather than fewer principles of more general application" (Skemp, 1976, p. 83) but that it was also attractive because "within its own context, [it] is usually easier to understand" and the "rewards are more immediate and more apparent" (Skemp, 1976, pp. 86-87).

The concept of instrumental understanding rings a bell with many teachers who experienced students successfully knowing how to do mathematical procedures without understanding them. Moreover duality brings simplicity in the use of this concept. Nevertheless I would argue that it is too simple, particularly because it does not take into account that mathematical thinking is, as Mason (1985) wrote, a "dynamic process which [...] expands our understanding" (Mason, 1985, p. 158). From my perspective, students rely on both instrumental and relational of understanding, sometimes alternatively, sometimes in conjunction, and sometimes exclusively on one.

2.5. Responding to the literature

There are few studies that look at students understanding when they are taught combinatorics and its set of formulae. Only Batanero et al. (1997) looked specifically at students' understanding in a case where some students had been specifically taught the combinatorial formulae, but it was focussed on the effect of which combinatorial model the students used. Yet she pointed out that many students who were taught the formulae achieved better but also made new mistakes involving the wrong use of a formula. The reasons for these *new* mistakes were not explained. These are very common mistakes, in particular the confusion between permutations and combinations. It is this gap, in particular, that this piece of research aims to fill. How students integrate these formulae with their previous knowledge is a fundamental aspect of students' understanding. So a piece of research aimed at probing students' thinking when they are taught these formulae could bring new and welcome understanding on students learning of combinatorics. In order to do this, I draw additionally from Biryukov's notion of meta-cognition. Biryukov defines meta-cognition as "the notion of thinking about one's own thoughts. It includes the awareness about what one knows - 'metacognitive knowledge', what one can do – 'metacognitive skills' and what one knows about one's own cognitive abilities – 'metacognitive experience'" (Biryukov, 2004, p. 1). It is interesting to note that when studying meta-cognition in mathematics, Biryukov chose combinatorics "because of its non-algorithmic character", and because "[s]olving problems from this domain develops students' critical thinking abilities and thus it leads to activating their meta-cognitive skills" (Biryukov, 2004, p. 3).

In the next chapter, I lay out the methodology of this research project, describing the practical context and approaches used. This lays out the steps taken to respond to the need for further research by drawing on the example of combinatorics as taught and learned in Grade 12 in British Columbia.

Chapter 3: Method

This chapter is divided into four parts. In the first I explain my choice of a qualitative exploratory approach. I then describe the context in which this study took place. In the third part I lay out in detail the method I used. The final part deals with how I analysed the data.

3.1. A qualitative exploratory approach

The purpose of this research was to explore students' mathematical thinking when they were introduced to formal combinatorics and how they integrated it with their previous knowledge. This was done trying to answer the research question: How do students integrate combinatorics theory with their previous knowledge of counting strategies? Mathematical thinking is not superficial: it is complex, and since it is difficult to make it explicit, it is also elusive. This dictated the choice of depth over breadth. This, with the exploratory tone of this research, favoured the use of a qualitative method. As a matter of fact, mathematical thinking is a construct that is not easy to operationalize or quantify. Moreover I have already used quantitative methods for a research project in Switzerland (Perrin & Mendès, 2002) and I found that the quantitative methods used were limiting, particularly in their openness to let the unexpected emerge.

I was interested in students' thinking, something that is very personal and internal. I tracked students' thinking by collecting its external manifestations: comments, written computations and drawings. I also took into account interactions between students and with the teacher, class atmosphere, and body language. This was done in part through class observation. But this was not enough: thinking is elusive – many thoughts are not expressed – so I had to be more pro-active in having students externalise and express their thinking. This was done through the use of written meta-cognitive questions done in class and during problem-solving sessions done outside of class hours with two groups of two students. The following paragraphs map out more in detail how this was carried out.

3.2. Context of the research

Being interested in students learning combinatorial formulae implied working with Grade 12 students. The reason is simply that combinatorics is only found, as a topic per se, in the Grade 12 curriculum. It is interesting to note that combinatorics is taught only in the last year of high school, in British Columbia as well as in Geneva, Switzerland. Actually students encounter combinatorics several times much earlier on – during primary school – but not in a systematic manner. It is not labelled combinatorics and no theory or formulae are proposed. Doing research in a Grade 12 mathematics class also meant that the students were in their last year of high school and faced a mathematics provincial examination. So the additional work that I asked students to do was limited, and, as far as possible, I made sure it had a learning purpose and that it was beneficial to the students. In keeping with the rules governing research at the University of British Columbia, I submitted to the requirement of the Behavioural Research Ethics Board (see appendices I and J).

The choice of limiting this research to a single class followed the exploratory nature of the research and the determination to look for depth rather than breadth. As a matter of fact, probing students' thinking is no simple task if some depth of understanding is looked for. Keeping the focus on a single class for the duration of the teaching unit on combinatorics was not exhaustive but allowed a reasonable sample of different students' understanding and ways of thinking. Observing the whole unit made sense for tracking changes in how students think.

Mathematics 12 is a course where much is at stake. It is, for example, a prerequisite for some university programs in Canada. Moreover, a provincial exam sanctions and evaluates completion of the course. So finding a teacher who would not only accept the added work and stress of a research project going on in his/her class but also agree to change his/her schedule by teaching combinatorics earlier in the year than usual, could have been somewhat tricky. Having a fellow Masters student invite me in a very welcoming manner to conduct my research in the Mathematics 12 course he was teaching was a relief, as was the prospect of a smooth collaboration. So I did not hesitate and accepted the offer of working with Mr Cho (pseudonym) in his Mathematics 12 class. Using convenience sampling limits generalizability, but since this research was primarily exploratory, this was not a prime concern.

The research took place in a Pre-Kindergarten to Grade 12 independent, co-educational day school in Vancouver. The school was started in 1996 with about 300 students offering pre-school to Grade 8 programmes. Ten years later, the school has more than 850 students, 90 faculty members and offers programmes up to Grade 12. It is in an affluent neighbourhood of Vancouver and fees of more than twelve thousands dollars per annum restrict its access to students of high socio-economic status. All students wear uniforms.

Mr Cho is 40 and comes from Taiwan. He did his schooling there, completing an engineering degree. He worked as a sales and general manager in a computer company for seven years and then moved to Canada where he did a mathematics and physics degree followed by a B.Ed. He has been teaching maths and science for seven years. He is the Head of the mathematics department in his school and is now doing a Master in Education.

There were two Mathematics 12 course offered in the school. Mr Cho's class was composed of 25 students, in majority male and Asian [see table 3.1]. Of these 25 students,

Table 3.1: Race and gender composition of Mr Cho's Mathematics 12 class

Students	Caucasian	Asian	Total
M	4	12	16
F	1	8	9
Total	5	20	25

seven had already taken a Mathematics 12 class the year before and five attended a summer school preparatory class for Mathematics 12 [see table 3.2]. Mathematics 12 is a full year course, and the weekly load alternates between two and three blocks of 70 minutes each.

Table 3.2: Students' previous experience of combinatorics

No previous experience	Summer school	Mathematics 12
13	5	7

In the weeks preceding the unit on combinatorics, the teacher had taught exponentials and logarithms. A typical lesson consisted of Mr Cho progressing through the handout (see appendix B) he gave the students, presenting the concepts and formulae and then going through the exercises. Students followed what he was doing, asking questions from time to time. Sometimes Mr Cho gave the students an exercise or two to solve on their own and walked around the room checking students' work and answering individual students' questions. Students where quiet and focused most of the time.

3.3. Research design

The research was divided into two stages. Both lasted for the whole duration of the teaching unit on combinatorics. The first stage looked at the whole class: the macro level; whereas the second stage, the micro level, reached for more depth by focussing on two groups of two students who solved specific combinatorial problems.

Since the purpose of this research was to explore students' thinking and probe how it changes when they are introduced to formal combinatorics, the context in which students were introduced to formal combinatorics was very important. This context was the mathematics class. The dynamic of the whole class had to be taken into account, including teaching material, teacher's style, interaction between the students, and with the teacher, etc. Thinking might be an internal process, yet it is prompted and modified by interaction with other events, ideas and people.

Keeping track of the whole class for the duration of the teaching unit on combinatorics aimed at noticing an evolution or a change in students' thinking. Moreover the depth acquired through the problem-solving sessions in small groups was put into perspective by comparing it with information gathered from the whole class. The whole class observation revealed what was most common when students reasoned on combinatorics. Generalization was limited but comparing the macro and micro level at least enriched the findings by strengthening and contrasting the analysis.

3.3.1. Whole class observation

On the practical side, the data collected on the whole class was divided following four aspects of the research: class observations, a pre-test, meta-cognitive questions and discussions with the teacher, Mr Cho.

Classroom observations

The first aspect of the data collection on the whole class was to make regular visits to observe how the class was going. Instruction influences students' ways of thinking, so it is necessary to observe it in situ. A class is a complex system, and as such it "is not just the sum of its parts, but the product of the parts and their interactions" (Davis & Simmt, 2003, p. 138). This means that special attention has to be given to the interactions, and not only by looking at the parts separately. It is a way, following Davis and Simmt to "observe the 'thinking' of [the collective]" (Davis & Simmt, 2003, p. 144). I came to all the classes during the teaching unit on combinatorics [10 visits], collecting data through video-taping and field notes. This gave me an image of the whole class as well as a sense of the dynamics and interactions within it.

In class, Mr Cho used a tablet-PC and a data-projector instead of a traditional overhead projector or chalkboard. This allowed me to have access to what had been written on the board in the form of computer-files, so I didn't have to use the video to record what he was writing. This was really useful as it allowed me to focus my observation on the students, without losing track of what was being done by Mr Cho.

Design of the pre-test

The second aspect of the data collection on the whole class was to probe what the students already knew and what they could do prior to combinatorics instruction. For this purpose I designed a pre-test (see appendix A) that was done in class by the students before Mr Cho started teaching the unit on combinatorics. The pre-test contained three problems: the menu problem [fig. 3.1], the pathway problem [fig. 3.2] and the partition problem [fig. 3.3]. The reasons for choosing them are presented below.

Fig. 3.1: The first problem in the pre-test: the menu problem

A restaurant proposes a menu composed of four appetizers and five main dishes as well as two desserts.

a) How many different menus can be composed of one appetizer, one main dish and one dessert?

b) Now consider that the Chef is quite particular and does not allow guests to mix fish and meat. How many different menus can be composed if there are two appetizers that contain meat and two that contain fish; and the main dishes are: beef, chicken, lamb, salmon or halibut?

Fig. 3.2: The second problem in the pre-test: the pathway problem

How many different paths lead from A to B when the only possible moves are the ones going down or to the right?

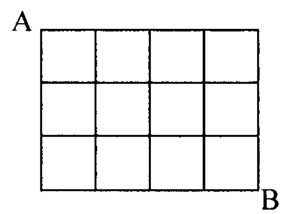


Fig. 3.3: The third problem in the pre-test: the partition problem (from Batanero et al., 1997, p. 197)

A boy has four different coloured toy cars (black, orange, red and grey) and he decides to give away the cars to his friends Peggy, John and Linda. In how many different ways can he distribute the toy cars? For example he could give all the cars to Linda.

The first problem, *the menu problem*, was chosen because it is about the fundamental counting principle, which, as it name implies, forms the basis of counting. As such it is the starting point of most combinatorics courses. Technically it is just a multiplication. It is so simple that most people use it without really thinking about it. So I wanted to know how many students were familiar with the concept and how many could explain it. When combinatorial problems start to be a bit more complex, choosing between the additive and multiplicative operations causes trouble to many students, so I also put a variation on the exercise, part b, that asks for the decomposition of the problem into two sub-problems and the addition of the two solutions. Solutions to both parts of the menu problem are given in figures 3.4 and 3.5.

Fig. 3.4: Solution to part a of the menu problem

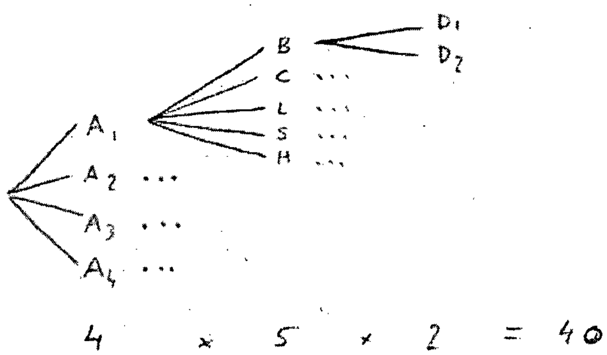
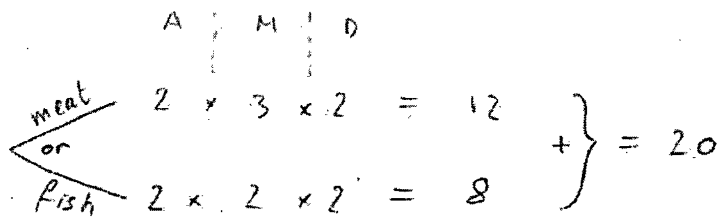


Fig. 3.5: Solution to part b of the menu problem



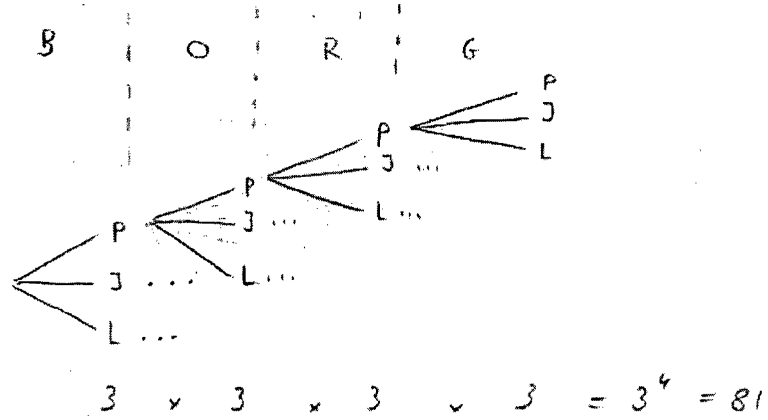
The second problem was *pathway problem*. It is a classic problem in high school combinatorics, it and often some of its variations appear in many textbooks – such as Addison-Wesley's Mathematics 12 (Alexander & Kelly, 1999) and Mathpower 12 (Thompson, 2000) – and exams. It is an example of the application of a combinatorial formula to a geometrical counting problem. As such it has somehow become part of some curricula. From a mathematical point of view it can be qualified as rich and beautiful. Its richness comes from all its possible variations as well as its relationship with counting, Pascal's triangle and the formulae for permutation with similar objects or the one for combinations. All these connections are already beautiful to a mathematical mind but it is also a very visual problem that is different from the more traditional combinatorial problems that revolve around permutation or selection of physical objects. I chose to ask this problem because I was expecting that students who had not been introduced to any of the two common methods that are taught to solve it (using Pascal's triangle or the formula for permutation with similar objects) would revert to counting. This would give an insight into which methods would be used and how proficient students would be. Different resolution strategies will be seen in chapter 4.

Finally I wanted to see how students would tackle a more difficult problem that can still be tackled without using a formula. I selected *the partition problem* (from Batanero et al., 1997). Using a list or a tree could be long and cumbersome – since there are 81 possibilities – but their use allows interesting combinatorial aspects of the problem to be seen. Moreover if one assigns people to the toy cars instead of distributing the toy cars to the people, this makes the problem much easier to solve: then it is a relatively straightforward application of the fundamental counting principle. Two resolutions of the partition problem are given in figures 3.6a and 3.6b.

Fig. 3.6a: Resolution of the partition problem if distributing cars to people.
There are four different partitions: 4-0-0, 3-1-0, 2-2-0 and 2-1-1. For each partition, one must compute the number of ways to select the toy cars but also select the people receiving some toy cars. This is complicated but counting techniques can be used at each step instead of the formulae.

Partition	Toy car selection	People selection	Product (use FCP)	total
4-0-0	1	3	1x3	3
3-1-0	4	${}_3P_2$	4x6	24
2-2-0	${}_4C_2$	3	6x3	18
2-1-1	${}_4C_2$ x ${}_2P_2$	3	(6x2)x3	36
TOTAL				**81**

Fig. 3.6b: Resolution of the partition problem if assigning people to the toy cars.
Each toy car can be given to one of the three people:



Meta-cognitive questions

I added a few meta-cognitive questions [fig. 3.7] to the three problems. The objective was to not only look at whether or not students managed to solve some combinatorial problems but also focus on the approaches the students took when solving these problems. I added the same meta-cognitive questions to each of the abovementioned problems in the pre-test.

Fig. 3.7: Meta-cognitive questions posed alongside each problem in the pre-test
State your thinking when solving the problem.
Try to explain how you approached the problem and what you thought about while you worked on it:

1. *Describe how you approached the problem and worked at solving it.*
2. *Which ways did you consider but did not use in the end?*
3. *How confident are you that you have found a correct answer?*

This exercise was:

easy		*ok*		*difficult*
1	2	3	4	5

Design of meta-cognitive prompts

The third aspect of the data collection relative to the whole class was to have the students regularly answer meta-cognitive prompts. Journals were first considered but the focus on combinatorics, with its short teaching unit, was too restrictive for full-size journals that require a rather larger time span to be efficient. The idea was to benefit from some of the advantages of journals, while keeping time constraints in perspective. By scaling it down to a few questions, only a bit more than five minutes of class time was needed to answer them.

I asked the students to write down what they had learned three times during the unit on combinatorics. I planned to do it at the end of each class every week to look for changes and to see what kind of progression there was.

An important aspect of these questions is that they provided data from all the students. As such it put all students on an equal footing and it counterbalanced the possible distortion in the whole-class observation resulting from the particular position students had or played relatively in the class dynamics. It was also a tool that could be used by a teacher in his/her regular class to have some feedback on his/her students in a format different from the ubiquitous quizzes.

Moreover, involving the students "to write what they have learned that day and what they still have questions about" also has the benefit of "[... putting] them in charge of clarifying their own thinking and diagnosing their own misunderstanding" (Silver, Kilpatrick & Schlesinger, 1990, p. 21). Actually, Mr Cho took the opportunity to tell his students that to think back about what they had learned was a good review technique and that they should do it by themselves regularly.

So at the end of the second class I asked them to write down what they had learned. When looking at their answers I thought that it would be better not to ask them this question at the end of a class because I had the impression that students were mostly recalling what had been done in the previous hour. So I chose, for the next time, to pose the question at the beginning of the following class. That allowed the students' minds to settle down a bit and organise the new knowledge learned. I slightly reworded the question into '*What are the important ideas or concepts that you learned this week?*' and I posed it at the beginning of the fifth class, the last class of the year before the Christmas break and at the beginning of the sixth class, the first after the Christmas break. This allowed me to see the effect of a two week break in students' recall of the subject. There were not many differences. This, added to the fact that the rest of the unit was on the binomial theorem which is not combinatorics per se but a theorem in which the nCr formula appears, made me decide to stop asking.

Moreover, the second time I used meta-cognitive prompts, I added another question: '*What is difficult when solving a combinatorial exercise?*' which gave interesting insights. The third time I also asked them to '*Explain the most important [concept]*' but students did not really answer it. Explaining concepts is hard and time consuming. It also seemed that it was out of their mind frame. They stuck to listing concepts and formulae that summarized what was important in the unit. It was disappointing but retrospectively the question was not clear enough. I should have asked them to define a specific concept and explain it with an example.

Teacher's point of view

The fourth aspect of the data collection was to have regular discussions with the teacher, Mr Cho. I collected data through field notes, except for one of our meetings when I audio-taped a discussion where I probed his opinions on the actual mathematical thinking of his students.

There were multiple reasons for these discussions. Firstly, as the teacher, Mr Cho played a central role in the classroom. His ideas about combinatorics, and about how students learn it, had an influence on his teaching. In a similar fashion, the way he taught and presented the subject had an influence on students' thinking. It was also interesting to share our observations and interpretations. These at times differed, as did our diagnostics, but in any case it allowed both of us to develop our understanding of students' thinking. This resulted in enriching and refining the data in a consistent manner. Moreover this process of triangulation also served in validating my interpretations. This was also a way for me to give back something to Mr. Cho for welcoming me into his class for this research project and give him access into how I conducted the research.

I expected that this component would be much stronger but lack of time and other constraints meant that many of our meetings were connected to organisational matters. I had expected that we would be able to talk informally briefly after each visit I made but this was not always possible because there was little time between Mr Cho's teaching blocks.

3.3.2. Group sessions

The purpose of the problem-solving group sessions was to gain more depth into students' thinking and understanding. Tracking the thoughts of the 25 students at the same time was not a realistic option. The choice of having students work in small groups was based on the necessity of looking at individuals for depth and providing means to have students externalise their thinking in an environment similar to doing maths in a class. Maintaining the group small allowed for increased focus on individuals. The choice of having the students do the maths in a group instead of being interviewed was twofold: firstly to limit my external interference, as the researcher or as the teacher, and secondly to 'accelerate' students' thinking by putting them in a relatively challenging problem-solving situation where they had to share and externalise their thinking.

I planned to have a session with each group once per week from the start of the teaching unit, but as I relied on volunteer students to participate it happened in a different manner. It took me more time than expected to find volunteers, and I could not organise a session during the first week. Moreover Mr Cho helped me in asking two more students to participate. So I had two groups of two students. The reason for having more than one group was to limit the risk of too much particularity.

On the practical side, I held the group sessions at the school outside of class hours, either at lunchtime, or in the morning when students had a free block of time. Each group did three sessions lasting around 30 minutes. These took place every week, except for group 2 that started one week later and did two sessions in the last week of the teaching unit. Group 1 was composed of Victoria and Xinlei (pseudonyms) who usually sat together in class. Group 2 was composed of Nick and Patrick (pseudonyms), who did not. Moreover Victoria and Xinlei were of similar abilities. On the other hand, Nick had difficulties but Yvan did not. Dynamics were different within the two groups. Finally, even if the goal was to have the students think and solve the problems on their own without the interference from a researcher or a teacher, the need arose to ask some questions in order to unblock the group when they were stuck or to probe further for the sake of clarity. I gave the problems on a sheet of paper (appendix D) and let the students work on their own for a few minutes before having them go to the board to solve it together.

A smartboard was used during five of the six group sessions. A traditional whiteboard was used in the other case. A smartboard is a white board linked to a computer and a data projector. Writing on the board is done with special magnetic pens: there is no ink but the computer 'reads' what has been written magnetically and projects it on the board. This technology allowed me to save what had been written or drawn on the board in an electronic document easily and conveniently. Using different colours for each student gave a relatively precise account of what had been written by which student; the video then helped me to consider when it had been written. This allowed me to focus the video recording on the students' interactions and comments. The data collected through the video and the smartboard thus complement each other, giving a rich and detailed account of what happened during each group session. Thus the data contain not only the end product but also the whole developing process, with the verbal and written components connected together. This was important because some computation made sense only when the student explained his or her rationale, and conversely the act of writing encouraged a student to reconsider comments previously made.

Criteria for choosing the problems for the group sessions

Probing how students integrate the new mathematical theory and concepts they are presented with is not an easy task. To achieve this I chose to investigate students trying to solve mathematical problems. The reason was for the students' activity to be the closest possible to the real situation of studying mathematics in a school environment, where problems and exercises are preponderant. Direct questioning and interviewing were not ruled out but given the role of complementary tool, in the form of specific question asked during the group sessions.

Challenging problems were preferred over routine exercises because students would therefore not be able to just plug numbers into a formula, use a wording cue, remember a similar exercise, and solve the problem rapidly. Moreover the classic and routine exercises were going to be seen in class, so I would have the opportunity to observe students solving them there. I was expecting that challenging students would oblige them to use the formulae more cautiously, to use them only after having carefully thought about their suitability.

Choosing good problems was not easy. First they had to:

- contain important and related mathematical concepts, specifically the ones that I was interested in (permutation, combination and some solving techniques);
- be interesting for the students to engage with;
- be challenging to have students use and display rich thinking;
- be open in order not to force students to use a particular method.

Secondly, I had to find problems that were at exactly the level where they would reveal how students integrated new knowledge. That was very difficult because the problems had to be at the frontier of their knowledge. It had to be close to what they had already done so they could use their new knowledge in a relatively familiar context. And, at the same time, it had to be new and challenging so as not to be a simple routine exercise they could solve algorithmically without much thinking.

There was an additional constraint. The problems had to be selected to fit in the curriculum followed by Mr Cho. Timing was of the essence. If Mr Cho taught or showed a concept in class, a good problem could quickly become a routine exercise. So even if, prior to the fieldwork, I had selected some possible problems for the group sessions, I did not choose all the problems nor did I initially plan when to pose them. There were two main reasons for this flexibility. They are developed below and followed by a vignette that illustrates the process as it took place.

Firstly, I was interested in how students integrate 'new' mathematical concepts they are presented with and, as such, a mathematical concept might be unheard of at some point and be known a short while after a presentation by a teacher or fellow student. The student might know the concept but not understand it, or only in an instrumental fashion. Similarly, a range of problems could easily become rote exercises when a solution had been presented.

Secondly, organisational constraints were such that I was going to have to select the problem according to when I could organise a group session rather than the other way around. I was not teaching the class, so even if I had an idea of what was going to be taught and in what sequence, thanks to Mr Cho giving me the handout he gave to all his students, I was in the dark concerning the exact dates and times. I had no control over it. Moreover a teaching sequence is rarely a rigid frame, changes are often made. Some changes were made and that influenced the choices I had to make. Furthermore, I was dependant on the availability and willingness of the volunteers to come to the group sessions.

Vignette: *I was looking for a problem to pose for the first session of group 1. I would have liked to give the group a problem that was similar to the one seen in class but with a twist, to challenge them a bit. Since the students had already seen simple permutations and* *worked a bit on them, I thought that a problem involving a permutation with similar objects was a good choice, right on the edge. The students had done some exercises counting the number of anagrams of a word with all the letters being different. So asking a similar question with a word with some letters being identical would challenge them but in a situation that was both familiar and in close relation to what they had previously done. Unfortunately the group session was scheduled for after the next class, and at that point Mr Cho would have covered the question on permutation with similar objects. The problem would have become an exercise. What would likely have happened is that the students would have recognized the kind of problem and so applied the ad hoc formula without any reference to why and how it works, rapidly computed the answer with their calculator and that would have been it. This would have not been very different from the kind of data that I was going to get from the quiz and test that Mr Cho had planned. I had found a problem that seemed to be very appropriate and yet I was not going to be able to use it properly, at the right moment. So I had to discard it and look for another problem.*

Problems done during the group sessions

In the end I used five problems: the golf balls problem [fig. 3.8], the weather forecast problem [fig. 3.9], the seating problem [figs. 3.10 and 3.11], the squares problem [fig. 3.12] and the misaddressed letter problem [fig. 3.13]. The reasons for choosing them are presented below.

Fig. 3.8: The golf balls problem
A company sells bags containing three coloured golf balls for Christmas.
How many different bags can be made if there are ten colours to be chosen from?

Fig. 3.9: The weather forecast problem (adapted from Paulos, 2000, p. ix)
The weather forecast for the weekend is a 50% chance of rain for Saturday and a 50% chance of rain for Sunday. Mr X said that it means that there is a 100% chance that it will rain this week-end.
Do you think Mr X is right? Justify your answer and try to explain Mr X's reasoning.

Fig. 3.10: The seating problem with three professors (Andreescu & Feng, 2003, p. 3)
Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students.
In how many ways can Professors Alpha, Beta, and Gamma choose their chairs?

Fig. 3.11:Tthe seating problem with four professors

Ten chairs in a row are to be occupied by six students and Professors Alpha, Beta, Gamma and Delta. These four professors arrive before the six students and decide to choose their chairs so that each professor will be between two students.
In how many ways can Professors Alpha, Beta, Gamma and Delta choose their chairs?

Fig. 3.12: The squares problem

In how many ways can you fill in three of the squares below so that no coloured square touches another one?

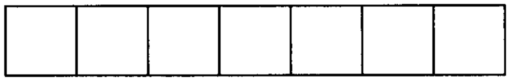


Fig. 3.13: The misaddressed letters problem

Someone writes n letters and writes the corresponding addresses on n envelopes. How many different ways are there of placing all the letters in the wrong envelopes?

The *golf balls problem* was chosen because it looked like a problem that could be solved by just using a formula. Yet it was not the case because students had not been introduced – and never were – to the formula for combinations with repetition. So students had to revert to mixing a combinatorial formula with problem solving skills or else use a counting technique. Below are some possible strategies to solve it [fig. 3.14].

Fig. 3.14: Some possible resolutions to the golf balls problem

Using the combinatorial formulae

$$\left.\begin{array}{ll} \textit{three different ones} = {}_{10}C_3 = & 120 \\ \textit{two same and another one} = {}_{10}P_2 = & 90 \\ \textit{three same} = & 10 \end{array}\right\} = 220$$

Or with lists, 2 versions, using only four colours to keep it shorter

AAA	*BBB*	*CCC*	*DDD*	**or**	*AAA*	*BBB*	*CCC*	*DDD*
					AAB	*BBC*	*CCD*	
AAB	*BBA*	*CCA*	*DDA*		*AAC*	*BBD*	*CDD*	
AAC	*BBC*	*CCB*	*DDB*		*AAD*	*BCC*		
AAD	*BBD*	*CCD*	*DDC*		*ABB*	*BCD*		
					ABC	*BDD*		
ABC	*ABD*	*ACD*	*BCD*		*ABD*			
					ACC			
					ACD			
					ADD			

The *weather forecast problem* was chosen because the catch resides in the application of the fundamental counting principle. A simple concept, but here it is hidden by putting the emphasis on probability. Many people forget there are actually four possible outcomes for the weekend weather and the probability of rain for the weekend is 75%. (There are four equiprobable possible situations: RR, RN, NR and NN, where R means rain and N no rain; with the first letter describing the weather on Saturday and the second the one on Sunday.)

The *seating problem* was chosen because it needs to be decomposed into two sub-problems. The first sub-problem consists of selecting the seats for the professors and the second is to seat the professors on the assigned seats. The solution is obtained by multiplying – according to the fundamental counting principle – the results to both sub-problems. I was interested in seeing if the students would be able to use the permutation formula and if so how, as it can be used to solve the second sub-problem. The first sub-problem is the same problem as the squares problem. I first gave the seating problem with three professors to group 1. In it the second sub-problem is permutation of only three objects and I found that too basic so I gave the version with four professors to group 2.

The *squares problem* was chosen because it is mathematically similar to the first sub-problem of the seating problem – it is actually exactly the same as the version with three professors – but within a different context. I wanted to probe if the students would recognise it or if they would transfer some of their experience from one session to the next. Moreover there are many ways to solve it, using various counting techniques or the combinatorial formulae they had learned. Below are some possible strategies for solving it [fig. 3.15].

Fig. 3.15: Some possible resolutions to the squares problem

With a list, using numbers referring to the place of the coloured squares

135	146	157	246	257	357
136	147		247		
137					

Or using the combinatorial formulae (rather tricky)

3 adjacent squares $= 7 - (3 - 1) = 5$

	2 adjacent squares & another	$= 6x5$	$= 30$
–	*3 adjacent squares (because counted twice)*		$= 5$
=	*3 squares with at least two adjacent*		$= 25$

	3 squares chosen randomly	$= {}_7C_3$	$= 35$
–	*3 squares with at least two adjacent*		$= 25$
=	*3 non adjacent squares*		$= 10$

The *misaddressed letter problem* was chosen because it is a very complex and challenging problem. Hadar and Hadass (1981) used this problem to give an example of the difficulties faced by students when trying to solve a combinatorial problem. But they did not include examples of students' work. I therefore wanted to see what students would do and how far they could go, and if they would face the difficulties described by Hadar and Hadass. I was positively surprised that with a bit of help at some crucial moments, they managed to solve it.

3.4. Analysis of the data

At the end of the fieldwork I had amassed a lot of data from different sources: video from the whole class (10 one-hour tapes), and from the group sessions (6 tapes of approximately 30 minutes each); computer files of the notes written on the board by the teacher during class and by the students during the group session (tablet-PC and smartboard); students' answers to the pre-test, to the end-of-unit test (see appendix C), and to the meta-cognitive prompts asking what they had learnt (see appendix H); audio recording of a discussion with Mr Cho (30 minutes); and field notes (see appendix E for two excerpts).

I did not transcribe the video and audiotapes in full but for all of them I did make a precise log (see appendix F for an excerpt) describing what took place. This was rich in selected quotes and indicated the mathematics done by the students. This allowed me to have a good idea of what took place in class and what the students did during the group sessions. I did go back several times to the video, watching it for confirmations or contradictions when the log was not clear enough. I read the logs several times, first to familiarise myself with them, and then to compare and put into perspective the different streams of data. I read parts of the logs from the group sessions and the whole class in conjunction, looking for similarities or differences in the way a subject was approached by the students or taught by Mr Cho. I did the same with the two groups to see if both groups did something similar or not. The field notes added a few pieces of information to the picture. Finally I transcribed in detail the excerpts that I decided to include in the written thesis.

For the pre-test I looked at the students' answers and classified them according to the resolution method adopted (see appendix G). This was put in parallel with students' success and with their previous experience of combinatorics. In the case of the end-of-unit test, I did not analyse all the questions. I did look at all the students' answers, but I did not analyse the ones that were too technical or that dealt with the binomial theorem. I analysed the questions using counting, looking at success rates, the methods and formulae used; and I selected examples that were typical or representative of peculiar ways of understanding.

For the meta-cognitive questions, I also reverted to counting and to selecting typical and particular answers. I then put the three sets of answers into perspective looking for changes or an evolution.

Finally, data from the logs, the tests and the meta-cognitive questions were put into perspective. It resulted in the emergence of trends that became the focus of each of the three following results chapters.

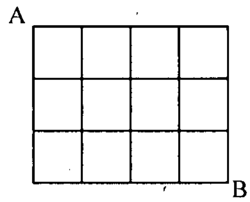
The last point I need to mention is that I ensured the anonymity of the participants by giving them pseudonyms. The pseudonyms were names for the teacher and the four students that participated in the group sessions but only consisted of a letter for the other students. The reason was to differentiate between the people I got to know relatively well and other students about which data is more limited and less detailed.

The results will be presented and discussed in the next three chapters. Chapter 4 describes how, following instruction about a particular combinatorial problem, students shifted their resolution strategies from counting techniques to an algorithmic method. Chapter 5 deals with students' lack of proficiency with counting techniques, which leads students to abandon their use. Chapter 6 expands on the shift towards students solely using formulae, focussing further on what happens when students learned formal combinatorics. The results and main ideas emerging in these three chapters are then discussed and put into perspective in the seventh and concluding chapter.

Chapter 4: The pathway problems: a shift from counting techniques to an algorithmic method of resolution

In this chapter I look longitudinally at one specific problem – the pathway problem [fig. 4.1] and some of its variations – and give an account on how students' approaches to this specific problem changed after having encountered the problem and been shown methods of resolution.

Fig. 4.1: The pathway problem (on a 3 by 4 regular grid)
How many different paths lead from A to B when the only possible moves are the ones going down or to the right?



This chapter is divided into three parts. The first looks at students' ways of solving the pathway problem before having received any instruction. The second gives an account of the teaching related to this problem as well as some students' encounters and progress when trying to solve it. Finally, the third part focuses on the effect of instruction, through results in the end-of-unit test and through what students remember after time took its toll.

4.1. Before instruction

The second problem of the pre-test – done before the unit on combinatorics started – was the basic pathway problem [fig. 4.1 above]. Students who had not already seen it used counting techniques or unusual (sometime even outlandish) strategies. At that point, it was a real mathematical problem for them, not a practice exercise. It demanded a strategy to keep track of the counted paths. As such it was challenging and student's ability to resolve the problem was limited.

Table 4.1: Results to the pre-test pathway problem from students with no previous experience

Students whose answers were correct		Students whose answers were wrong	
3		8	
Student	**Method used**	**Student**	**Method used**
I and O	drew and counted	Q	counting with list
Yvan	'add at crossing'	W and T	counting
		B, C and Nick	counted squares
		J	bizarre tree
		P	probability function

From the 11 students who did the pre-test and who had not experienced combinatorics in summer school or in a previous Mathematics 12 class, three solved the problem correctly. [See table 4.1 for results to the pre-test pathway problem from students with no previous experience.] Of these three, two counted all possible ways. They "drew it out" as student I explained [see fig. 4.2 and 4.3]. The third, Yvan (pseudonym), used Pascal's triangle but explained that he had "seen similar problems solved like this", but did not tell where or when.

Fig. 4.2: Student O's answer

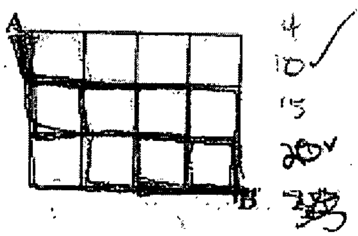
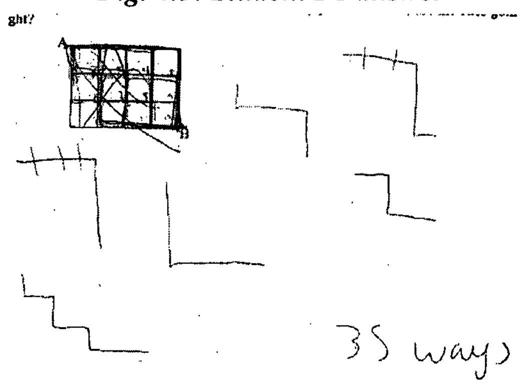
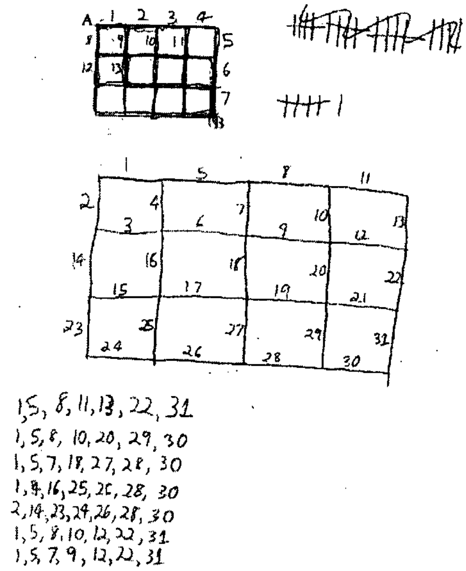


Fig. 4.3: Student I's answer



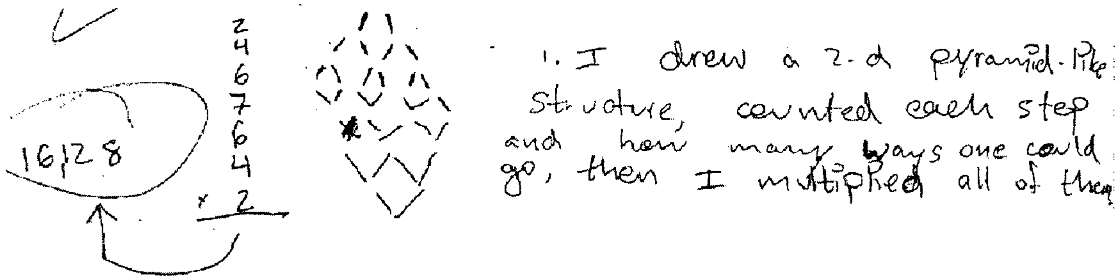
Of the eight students who did not solve the problem, counting was also used by three students, but in all cases unsuccessfully. Student Q's work was the more elaborate as he numbered all edges and wrote down systemically some paths as a succession of these numbers [fig. 4.4]. Moreover student P who said he used a 'probability function' [but he did

Fig. 4.4: Student Q's answer



not indicate which and how] wrote that he could have counted "every path by hand but it would [have] take[n] too long." Another student had a similar opinion. Counting was the most used technique, but students were not proficient and it resulted in low achievement. The methods used by the other unsuccessful students were rather perplexing. Student J's work is below [fig. 4.5] and the three others used computation involving either the number of squares in the grid or the sides of these squares.

Fig. 4.5: Student J's answers



Finally, the eight students for whom the problem was new and who ranked its difficulty considered this pathway problem relatively difficult, with an average of 3.9 on a scale from 1 to 5 with 1 being easy, 3 ok and 5 difficult.

Students who had no previous experience of the subject mainly used counting techniques and achievement was low. As a matter of fact, most students had difficulties being systematic in keeping track of which paths had already been counted; only one student used a list. This can explain the poor success rate.

4.2. During instruction

The following section is an overview of what Mr Cho taught on the pathway problem. This is put in parallel with what some students did and how they integrated the subject matter. The observed students acted differently from one another and demonstrated a variety of behaviours and ways of integrating new material in response to actual teaching.

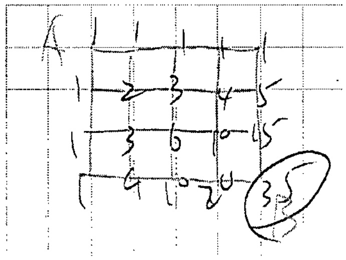
The teaching related to the pathway problem happened in three stages. The first was the class after the pre-test, the first of the unit on combinatorics. There were two other classes in which time was devoted to the pathway problem: after one week and after two weeks of teaching. Sections are organised following this chronological line.

4.2.1. First overview of the pathway problem

The first teaching related to the pathway problem happened in the class after the pre-test. Mr Cho made some comments about the pre-test. He then asked the students how they solved the problem and some answered that they used Pascal's triangle. These were the students who had already done some combinatorics beforehand. So Mr Cho drew a grid on the board [fig. 4.6] and showed one resolution method by adding the numbers at the crossings. It did it talking:

Mr Cho: *1-2-1… 1-3-…3-1… then you have what? 4-6-4 and then you have 1 here and you have 10-10-5, and you have 20-15, and you come over with what? 35. That's by Pascal's triangle.*

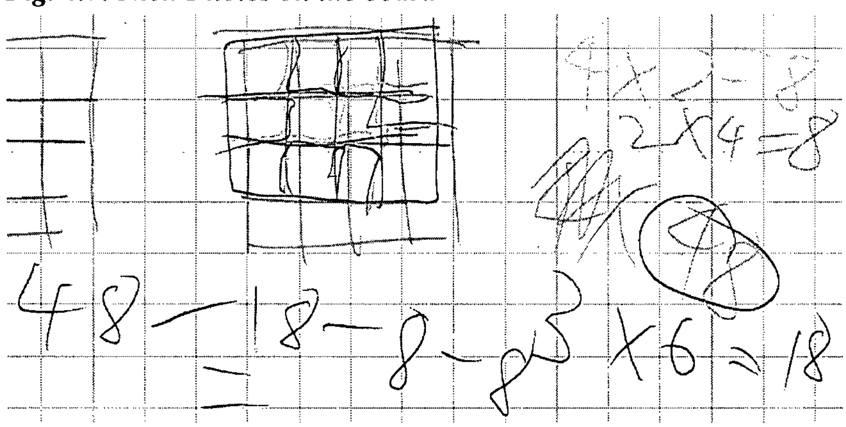
Fig. 4.6: Mr Cho's note on the board (excerpt)



Having put the pathway problem in the pre-test had made students encounter it earlier on and Mr Cho presented a resolution method for it earlier than it was usually the case when Mr Cho taught the course following his handout.

Mr Cho then reassured Nick (pseudonym) who did not know this method, telling him that they were going to learn it later. After that he asked student J if there was a different way to solve the problem. During the next five minutes student J, and after him Nick explained their methods. Nick wanted to present his method and since it required some drawings, Mr Cho invited him to do it on the board. Both methods were wrong. Student J used a tree-like diagram [see fig. 4.5 above] and Nick counted squares and sides of squares [fig. 4.7]. Mr Cho told them their methods were wrong but did not explicitly explain why and both students accepted Mr Cho's conclusion.

Fig. 4.7: Nick's notes on the board



This first teaching encounter with the pathway problem was limited, in time and in content. Only one way to solve the problem was presented, rapidly and without much explanation. Yet it was enough for some students to integrate it and be able to use it as I am going to show in a following section. Moreover alternative ways of solving similar problems were somehow cast aside.

4.2.2. First teaching on the pathway problem

The second time students saw the pathway problem in class was one week later. This time Mr Cho spent more time explaining how to solve it, showing two methods and giving students supplementary exercises to do. Mr Cho was previously focussed on permutations with similar objects and, after having presented the formulae, he was doing the related exercises from the handout one after the other. Two exercises in the handout were pathway problems: the fifth was a 3 by 3 grid [fig. 4.8] and the eighth consisted of two variations of the problem [fig. 4.9].

Fig. 4.8: Exercise 5 from the handout

Student A wants to visit student B. Roads are shown as lines on a grid. Only south and east travel directions can be used. The trip shown is described by the direction of each part of the trip: ESSESE How many different paths can A take to get to B? (20)

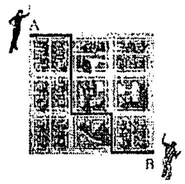


Fig. 4.9: Exercise 8 from the handout

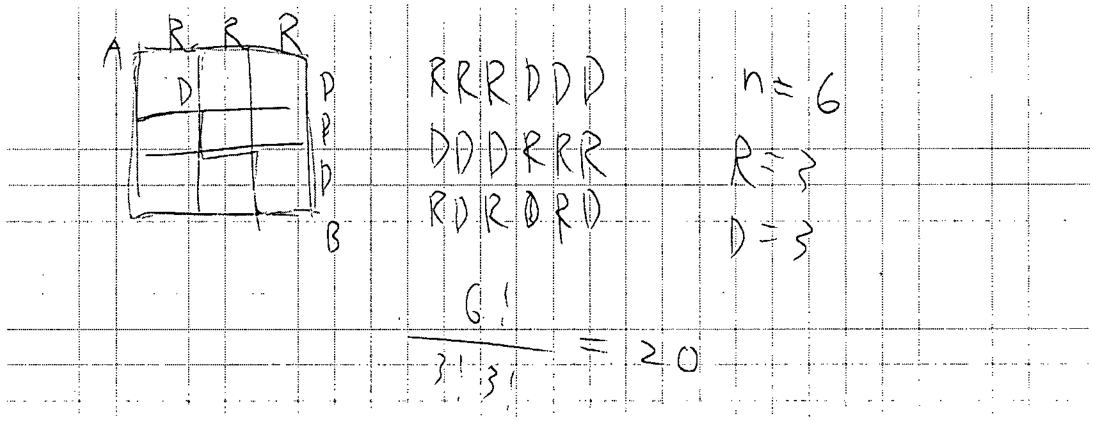
On each grid, how many different paths are there from A to B?

Teaching

Below are excerpts from the transcription of Mr Cho's lecture when he reached exercise 5.

Mr Cho: *And you see this question before. Next page, page 9, did you see this question before? ... Remember [inaudible] and we see lots of people do it this way. This one is a 3 by 3 which means you have 9... On your paper [pre-test] I do see what you do. And lots* *of people just do like the paths. They colour with the pens. You just do like that. You do the green. [He draw the green path.] Like that. If you are lucky like me with a computer then you can do different colours... Choose [a colour on the computer] and draw a line. But most people not that lucky and...*

Fig. 4.10: Mr Cho's note on the board (excerpt)



[Mr Cho does a digression for 2 minutes, about colours numbers.]

Mr Cho: *Ok guys! This one, you do last time... actually if I write... don't forget, they force you either going right or going? Down. So you see if I am going right, I can consider right, right, right, ... down, down, down [He write R R R D D D on the edges of the grid making one path from A to B, see fig. 4.10.] So you see you have right right right down down down. [He writes RRRDDD beside the grid.] One choice. Or I can do what? Down down down right right right. [he writes DDDRRR beside the grid.]. Or I can do what? [Some students laugh gently and mimic Mr Cho saying down down down but Mr Cho continues unperturbed.] Right down... right down... right down [He writes RDRDRD beside the grid.]. And you are going to write forever. [Some students say down down down again, still laughing.] Did you see any characteristic for this one?*

A student: *[inaudible] 6 factorial.*

Mr Cho: *Totally... each time [inaudible] which group you got? How many choices?*

Students in chorus: *6.*

Mr Cho: *6. You always have 6 different... So that means total number of choice is what?*

Students in chorus*: 6.*

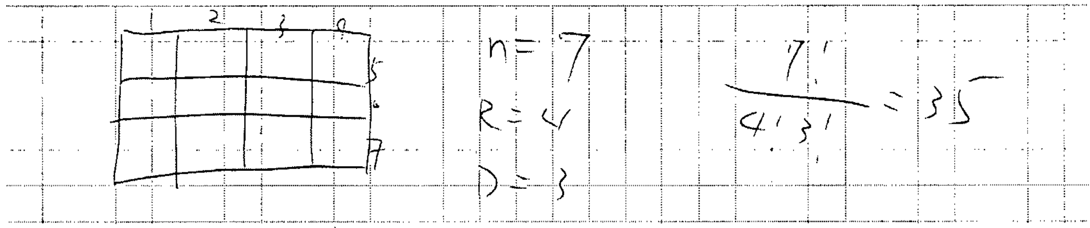
Mr Cho: *6. ... And then, even at this here, you can see how many right steps you always...*

A student: *3.*

Mr Cho: *Three. How many down steps do you have? 3 as well. This one is like... totally like... spelling a word as well. And you just [inaudible] divide by what? 3 factorial, 3 factorial. That's why you come out with... the answer is what? 20 steps [sic]. And you remember the question you do before [The question from the pre-test]? You have... This one is 3 and 4.*

[Mr Cho draws a 3 by 4 grid, see fig. 4.11, and solves the related pathway problem in the same manner.].

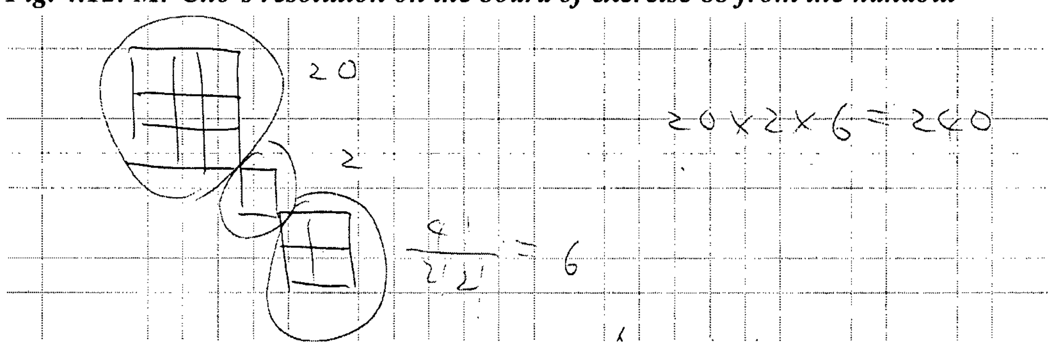
Fig. 4.11: Mr Cho's note on the board: a 3 by 4 grid and the solution to the related pathway problem



[A student asks a question unrelated to the course.]

Mr Cho: *Anyone has some kind of question? ... Ok. However something will be a bit different. We look at the last example. Ok I am gonna jump to the last example [exercise 8b]. What about the last example b? I am going to do the last example b. ... We do this one first... because this one is much easier than you think. [He draws the grid on the board.] Ok, for this one, all these parts... it's totally the same as what? The example we got. Just 20 [He circles the top grid and writes 20 beside it, see fig. 4.12.]. How about here [for the lone square in the middle]? How many ways you can choose?*

Fig. 4.12: Mr Cho's resolution on the board of exercise 8b from the handout



A few students: *2!*

Mr Cho: *2. How many here [for the bottom grid]?*

One student: *4.*

A few other students: *[Inaudible]*

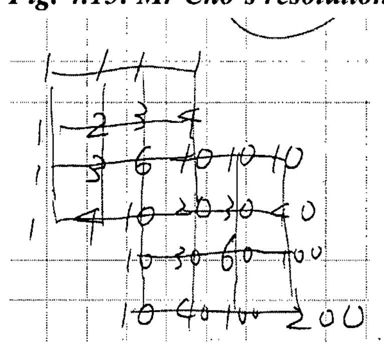
Mr Cho: *So you have what? 4 factorial divided by 2 factorial divided by what? 2 factorial so you have 24 divided by... 2 [equal] 12 divided by 2 [equal]... You've got what? 6! So what total choices do you have? 20 times 2 times what? 6. So you've got what? Mmm... 12... [He is calculating the answer.] Two hundred and forty.*

[Mr Cho takes a break for a few seconds and then starts again.]

Mr Cho: *But, part a [exercise 8a]... is this question. So that's why I am going to teach you a different way. Even if you are going to learn later, I'll still teach you here first. For this one is [inaudible] complicated than you think. Because we have some overlapping in-between. So that creates some problem. Because you come here you have right down right down... right... down... here it is already on the other one. And you can choose... Up to here you either can choose down or choose what? Right. So you have more choice happen[ing] here. So you're not able to... Once they are overlapping, you [will] still be able to do that but you need to do huge calculations. So [the]only way to do this one... I am going to teach you a much easier way. It's by counting with the numbers. How many go through? Ok. You need to watch this one closely. Because any time you see this kind of question, you are going to do this way. ... When you walk here you can consider you start with one. When you walk here, how many choices? 1.*

[He starts to write the numbers at each crossing, talking as he proceeds through the grid he has drawn on the board, see fig. 4.13.]

Fig. 4.13: Mr Cho's resolution on the board of exercise 8a from the handout



A few students: *1.*

Mr Cho: *1. Walk here, you still have how many choices?*

Students in chorus: *1.*

Mr Cho: *One. One. Same thing... you go here, one one... one [He completes the top sub-grid except for the 20 and then ask:] Ok most of people have problem [with] how to continue with this one. Just think about how many ways from here to here?*

A student: *1.*

Mr Cho: *1. That means [if the] solution is 10, you still consider is what?*

Some students: *10.*

Mr Cho: *10, you still have 10 ways to come here. Then you still have 10 to come this way. [He writes the 10s on the top and left of the bottom sub-grid.] And then 10 plus 10 you got what?*

Some students: *10.*

Mr Cho: *20. This one you got what? 30 and ... [He finishes the exercise]. So I give you one more example. Try to do... Guys, you try to do this one. [He draws an irregular grid on the board, see fig. board 4.14.]*

[All the students copy the grid in silence.]

Fig. 4.14: First supplementary exercise

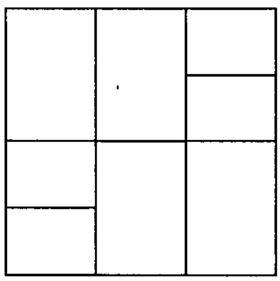
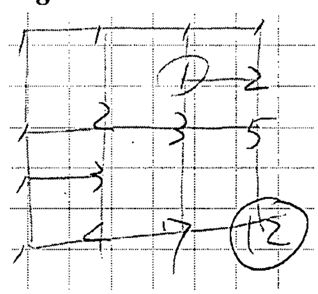


Fig. 4.15: Mr Cho's resolution

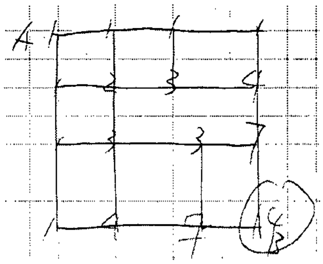


Mr Cho: *You need to try [inaudible] this one. ... This one is really fun to...*

[All the students work in silence during 2 minutes]

Mr Cho: *Every one got 12? I will go this one very slowly. [He draws the same grid again.] [Where] most of people got wrong is this one. Don't forget... you are only able to choose... going down or... right. So for going down, one step going down they still have what? One come[s] here. [He writes the 1 and circles it, see fig. 4.15.] Most of people got wrong is this one. So be careful for that one! [He continues and finishes the problem.] Before you go write down one more question [fig. 4.16] and see if you are able to do it.*

Fig. 4.16: Second supplementary exercise, with Mr Cho's resolution



[The students copied the grid and solve the exercise.]

The previous excerpts provide an example of Mr Cho's style and how the subject matter was taught. Mr Cho covered the pathway problem, giving two methods to solve it and two sorts of variations. Nevertheless, what was taught by Mr Cho is different from what students learned. Students are not empty vessels that absorb new knowledge that is simply presented to them. So it is of the utmost importance to look beyond what is taught and what comes out in tests; and look at students when they try to integrate the new material – concepts, algorithms, etc.

Four examples of student learning

During this class, rather than just filming the whole class, I also filmed individual students and what they were doing while Mr Cho was progressing along the handout. For practical reasons I only filmed seven students. To film individual students and what they are writing implies being close to them and can disturb students. So I only filmed students that were either seated in the front or rear row, always asking for their permission first. Focussing on what they were doing was revealing of some learning processes and of the way the students actually followed the course. Some were watching what Mr Cho was doing, some were not. Some had already learned one resolution method only having seen it once. One was able to use the same method in a variation of the problem despite the particular hurdle that this variation contained, whereas another student was blocked. Before reaching the hurdle and having to think about it to resolve it, students used the algorithmic methods automatically.

Below are descriptions of the way four students were following the course. In all the four following examples, students' learning was not linearly following Mr Cho's teaching.

Yvan

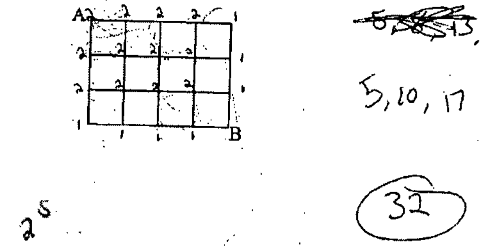
First there was Yvan who was doing other exercises well ahead in the handout. While Mr Cho was doing permutation with similar objects on page 8, Yvan was already at page 12, using the combination formula that had not yet been taught in class. Only from time to time did he stop his work to look at what was being done in class.

Student A

Student A was also ahead, but just by a few exercises. He did exercise 5 on his own and then went directly to the next pathway problem: exercise 8 [fig. 4.8 and 4.9 above]. He did both variations, part a and b, very rapidly, using Pascal's triangle. He wrote down each row of the Pascal's triangle diagonally on the grids in the same way that Mr Cho had shown one week earlier. The result was the same as what Mr Cho was going to do [fig. 4.13 above] about twenty minutes later, except for a mistake in the final answer: student A wrote 100 instead of the correct 200.

When doing exercise 8b [fig. 4.9 above], student A only took a break of less than a second when he finished the upper grid and had to continue with the lone square. Not only had student A learnt how to do it but he had also been able to transfer the method to a more complex case without difficulty. This contrasted with the answer he gave to the pre-test that, despite his previous experience in summer school, was wrong and used some sort of counting and multiples of 2 resulting in 2^5 [fig. 4.17]. It is difficult to know

Fig. 4.17: Student A's answer in the pre-test



how much student A learnt during summer school about this particular problem. I also can't tell for sure if Mr Cho's short explanation after the pre-test was enough to make that change happen or if student A had a discussion with other students who knew how to solve the pre-test problem. Nevertheless, something clearly happened between the pre-test and this class: student A had learnt an algorithm – the use of Pascal's triangle – to solve simple pathway problems and managed to adapt it to more complex variations of the problem.

Victoria

The two other students that I filmed in detail, Victoria and Xinlei, sat together. They also took part later in the group sessions in the first group. Victoria had done summer school and had learnt there how to solve that kind of problem, as she told me during the first group session, the day after this class. She had had no trouble solving the pathway problem as it appeared in the pre-test.

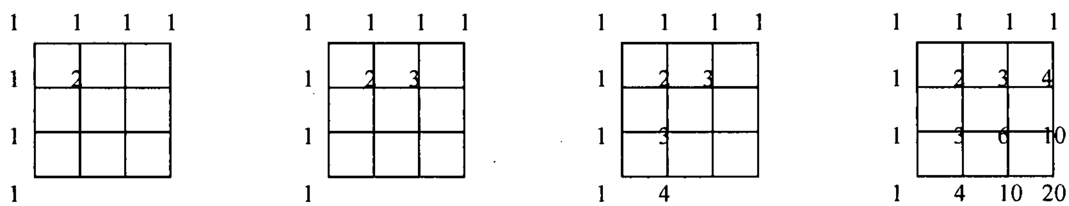
During the lecture she seemed inactive. In her handout, all the exercises – on permutations with similar objects - up to exercise 4 had already been done, and presumably the next ones were also done but I cannot tell for sure. She only appeared to be passive. But she was not: she was following what Mr Cho was doing. After Mr Cho had shown how to use the formula for permutation with similar objects to solve exercise 5 in the handout, she completed her notes by doing exercise 5 using Pascal's triangle. Thus she had, written on her notes, two methods for solving this problem. Later she would help Xinlei and do the supplementary exercises given by the teacher [fig 4.14 and 4.16] as well.

Xinlei

Xinlei was following Mr Cho's lecture, working out and taking notes alongside what Mr Cho was doing. But when Mr Cho arrived at exercise 5, Xinlei did it on his own; for a while not paying much attention to the comments and digressions Mr Cho was doing around this exercise. Later in the lecture, he followed more closely and did the supplementary exercises given by Mr Cho on the board [fig. 4.14 and 4.16 above] like every other student.

Xinlei looked at exercise 5 briefly and started to write numbers at the crossings on the picture from exercise 5. He started with all 1s one the top and the left hand side of the grid and put the 2 at the first crossing. Then he took two seconds to figure out and wrote the 3 on the right of the 2. Then he wrote the second 3 and the 4 beneath. His right hand, holding his pen, moved over the two numbers that had to be added, showing that he actually added the 1 and the 3 together to get the 4. Then he wrote the second 4 – presumably using symmetry – and completed the rest rapidly [see Fig. 4.18].

Fig. 4.18: Xinlei's steps when solving exercise 5



Then he drew a 3 by 3 grid besides the picture and wrote Pascal's triangle on the new grid. He once again started with writing all 1s, but this time he wrote each diagonal successively – 2, then 3-3, then 4-6-4, etc. All in all, it took him approximately 50 seconds to do exercise 5.

After doing exercise 5, while Mr Cho was still digressing about colours, Xinlei did exercise 6 and 7 [fig. 4.19 and 4.20], plugging the numbers from the exercises into the formula for permutation with similar objects and then using the calculator to find the answer. He was right both times and it took him approximately two minutes.

Fig. 4.19: Exercise 6 from the handout with answer

There are 3 blue flags, 3 white flags, and 2 flags. How many different signals can be constructed by making a vertical display of 8 flags? (560)

$$\frac{8!}{3!\,3!\,2!}$$

Fig. 4.20: Exercise 7 from the handout with answer

On a 5-question true-false test, two answers are T and three answers are F. How many different answer keys are possible? (10)

$$\frac{5!}{2!\,3!}$$

Then Xinlei started exercise 8, part a [see fig 4.9 above]. He wrote the numbers at each crossing of the top 3 by 3 grid, without much hand movement. But then he stopped. He had trouble finding what * [see fig. 4.21] should be. He thought for a few seconds and then his hand moved: he was trying to add, but there was nothing on the left of the *. So he asked Victoria: "Can you help me with this?" Victoria's answer was inaudible because Mr Cho's voice covered any answer she may have given. Nevertheless, after that, Xinlei completed the top and the left of the bottom grid with 10s and then computed the others. He was done and immediately started exercise 8b. He again used the same method. It took him 20 seconds of work with a three second pause when he had to find what to write at ** [fig. 4.22]. This exercise had been done automatically, except for finding **.

Fig. 4.21: Xinlei's partial resolution of exercise 8a

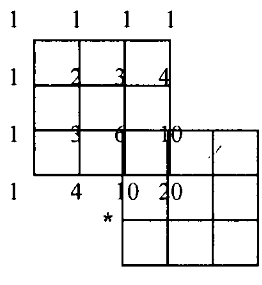
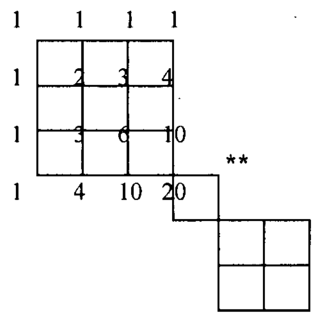


Fig. 4.22: Xinlei's partial resolution of exercise 8b



At that point Xinlei sat back and looked at what Mr Cho was doing. Since Mr Cho was showing how to solve exercise 5 with the formula for permutation with similar objects, Xinlei wrote down the formula, as it gave him a second method to solve exercise 5. Then he followed Mr Cho going through both parts of exercise 8, sometimes answering the prompts of Mr Cho.

Then Mr Cho drew an irregular grid [fig. 4.14 above] on the board, and asked the students to do "one more example" which turned out to be a real problem for Xinlei. He copied the grid and wrote the 1s, a 2 and the two 3s [fig. 4.23a], but not the circled 1 [fig. 4.23d]. The last 3, the one below the circled 1, was not written automatically: he had to think before [fig. 4.23b]. The missing 1 gave him even more trouble when trying to compute the 2 on its right. He spent some time thinking about it, moving his pencil above the top right square, trying to figure out how and what to compute. He did a number of larger pencil moves, two of which representing the two different possible paths to go to the crossing on the right of the circled one. He then said something [inaudible] to Victoria beside him. She laughed and answered something [inaudible] back. Then he had a quick glance at her handout page, where the exercise was certainly already done, and immediately wrote on his grid the 1 and 2 that caused him trouble [fig. 4.23c]. After that, finding the 5 took him a few seconds. Presumably, after having been challenged, Xinlei had to fold back to his previous knowledge about this kind of problem before resuming solving it. At that moment Victoria helped him, showing him with her pencil that going on top and then going over the circle 1 was "one way" and then showed the leftmost 2 and the 3 and the 5 on its right and said "this is 3 and 5." Then she computed and wrote 4 on the bottom and continued saying: "7 … 12." Xinlei wrote 7 and 12 and the exercise was over. At that point Mr Cho asked if everyone had got 12 and started correcting the problem [fig. 4.23d]. Xinlei again followed it answering to the prompts of Mr Cho. Xinlei solved the next supplementary exercise [fig. 4.24] without any difficulty.

Fig. 4.23: Xinlei's steps when solving the first supplementary exercise and Mr Cho's solution (d)

a ***b*** ***c*** ***d***

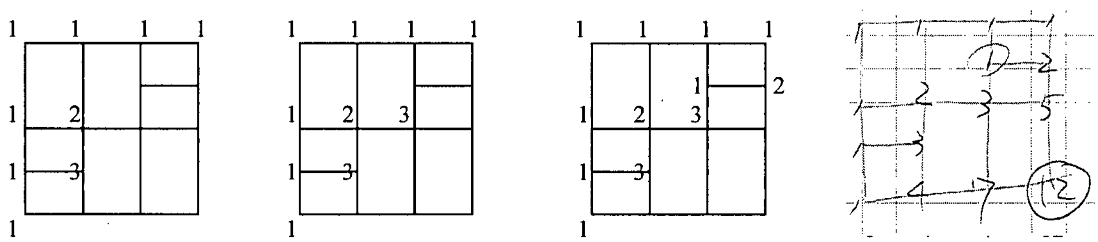
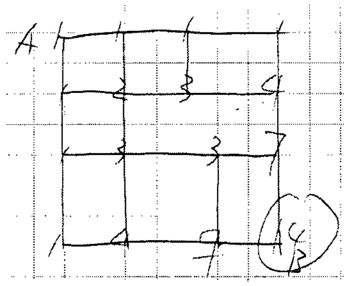


Fig. 4.24: Second supplementary exercise with Mr Cho's solution



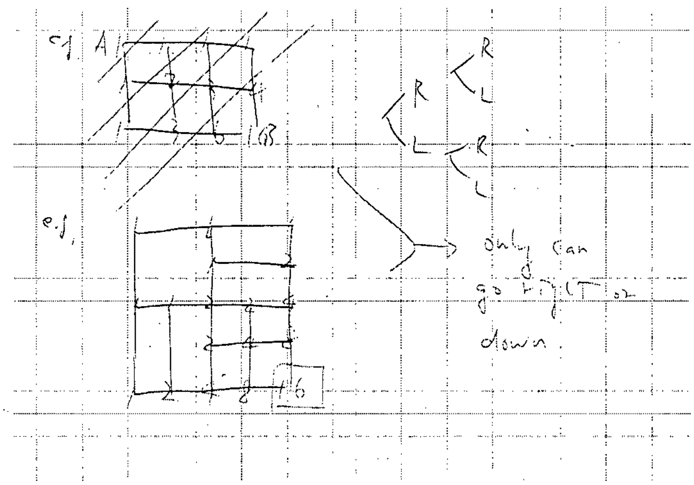
These four students displayed a variety of ways of following the course and integrating the mathematics related to the pathway problem and its variation. Rhythm of work and learning were very different between Yvan and Xinlei. Student A was a good example of how something can be quickly learned – having seen the method once was enough – and how such knowledge can be adapted in a similar but more difficult case – a variation of the pathway problem. But Xinlei showed that sometimes a variation is a real hurdle and asking for the solution was a simple shortcut to alleviate that hurdle and go back to a straightforward and algorithmic method of solving the problem. Knowing how to solve one problem does not necessarily gives the tools to solve a similar problem.

4.2.3. Second teaching on the pathway problem

The third time students saw the pathway problem in class was the first class after the Christmas holidays, which corresponded to one week of teaching after seeing it for the second time, and two weeks after seeing the problem and a resolution method when discussing the pre-test. At that point, students had already seen the permutation and combination formulae and were to be taught the binomial theorem, a theorem that is closely related to Pascal's triangle and the combination formula.

Mr Cho presented Pascal's triangle and showed that its coefficients can be computed with the combination formula and that they also correspond to the coefficients in the development of $(a+b)^n$ for n equal 2 and 3. Then he commented about the two pathway problems [fig. 4.25] that follow Pascal's triangle and preceded the binomial theorem in the handout. With the basic 3 by 2 grid, he made comments and drew lines to make the link with

Fig. 4.25: Mr Cho's notes on the board when teaching about the pathway problem



Pascal's triangle visible [top of fig. 4.25]. Then he had the students do the second problem [bottom of fig. 4.25], telling them: "This is review, you should be able to get the answer." After a while he corrected it and made some comments about the down-or-right only restriction: linking it to a tree and etymologically to the *bi*nomial since there were *two* choices. Then he started teaching the binomial theorem.

At this point Mr Cho did not come back to the pathway problem or any of its variations. The teaching on the pathway problem was over. It had covered the regular problem and two kinds of variations: irregular grids and composition of grids. Mr Cho showed two methods and told the students which method to use with which type of grid. I have mentioned that these variations of the pathway problems are well-known variations and appear in textbooks and in exams. This latter fact explains why such a problem has somehow become part of the curriculum and relatively substantial time is devoted to its instruction when it is still only a particular application problem; even if it is an interesting and beautiful one.

4.3. After instruction

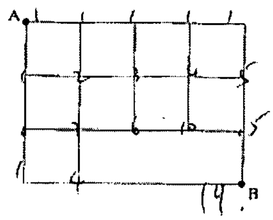
The effect of instruction on students on the pathway problem can be looked at from three perspectives. The first one is to look at the students' answers to a similar question in the end-of-unit test. The second is to probe students' understanding of the problem. Finally I look at how much was remembered after some time. This is done in the next three sections.

4.3.1. High achievement on the test

All 25 students did the-end-of-unit test. The 14th of the 20 multiple-choice questions was a pathway problem with an irregular grid [fig. 4.26]. All but three students got the correct

Fig. 4.26: Multiple-choice question 14 from the end-of-unit test, with key

How many different paths are there from A to B on the grid shown below if only moves to the right and down are allowed?



A. 16
B. 19
C. 35
D. 70

answer. They all used the 'add at crossings' method (which is appropriate since the grid is small and irregular) writing all the numbers at each crossing and nothing else. All the answers looked like figure 4.26 which is Mr Cho's answer in the test key. Only student H additionally wrote 'r→' and 'd ↓'.

The two students who were wrong and who gave a justification also used the same technique but were troubled by the non regularity of the grid; they answered 35 which is the result for the regular grid. Actually student Q added edges to the grid to make it regular [fig. 4.27], whereas student L added the numbers as if there were edges and crossing [fig. 4.28]. He did this even though he first tried something else, but it is difficult to know what exactly.

Fig. 4.27: Student Q's answer to question 14

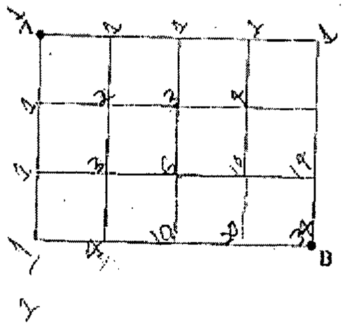
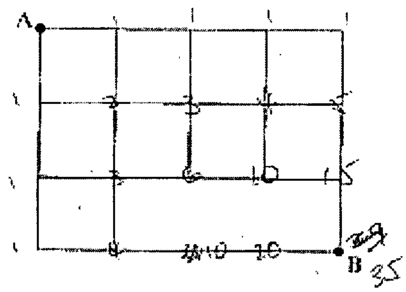


Fig. 4.28: Student L's answer to question 14



It is possible that he got it right in the first place. The data provides no evidence of why the third student was wrong.

Achievement in this exercise is noticeable. A large majority of the students managed to solve this exercise, thus displaying their instrumental understanding. But not much can actually be said about their relational understanding. Somehow students Q and L illustrate that both types of understanding are not always achieved simultaneously. Their answers show that they knew the principle of the algorithm (add at crossing) but also that they had difficulties deciding where and how to use this principle. Moreover student L's answer seems to contradict him having understood relationally the principle of the algorithm as the sequence 4-10-20 on the bottom of fig. 4.28 makes no sense mathematically.

4.3.2. Victoria's understanding of the pathway problem

Students' written answers to problems can be deceptive. They are also very often short and limited, in part due to the mathematical notations. As a matter of fact answers to combinatorial problems are often limited to a formula and the numerical solution. They are not reliable enough to assess what students understood and gained from instruction – as teachers we cannot decipher how and for what reasons students choose that specific formula. More depth is needed. Here follows an investigation of Victoria's understanding of the pathway problem. It gives some insight into how understanding can be less straightforward than written answers would suggest.

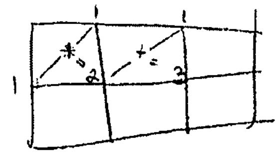
Victoria had already seen the pathway problem, during summer school. She had no trouble solving the pathway problem as it appeared in the pre-test. At that time, she also managed to explain how she solved it, although on an instrumental level [see fig. 4.29].

Fig. 4.29: Victoria's answer to the first meta-cognitive question from the pre-test

① - placed the # 1 on each intersection on the top line and left hand side line, then added the 2 diagonal #'s to get the corner one

i.e

1+1=2 →



However, when I had time to ask her why the algorithm worked, she could not give an answer directly. I had to probe deeper. Admittedly, answering that kind of question – why and not just how – is much more challenging. It is also something that students are, unfortunately, not often asked to do. I somehow had the impression that Victoria had not really understood my question, or maybe she could not understand my question as it was not framed as what is usually done in the context of school mathematics. I also have to point out that, in this case, there is also the difficulty that what needed to be shown might seem obvious.

Below follow some excerpts of the dialogue I had with Victoria.

Thomas Perrin: *My question is... why did you do that?*

Victoria: *Because... this summer with Mr [Cho]... I remember*

Thomas: *[...] you explained what you did [fig. 4.29]. It was really clear. [...] You explained you added up the two numbers. [...] But my question is why do you think this strategy works?*

Victoria: *Mm... it's just an easy way because if you think about it, there is only one way to go [showing the top line with her pen]. If you can only go... right or down. And if you look at here, it's two [pointing at the first crossing]. You can only come from two ways [she shows with her pencil the 2 ways]. And so from here [top left corner of the grid], if you* *were actually to draw it out [showing the three possible paths] there would be three ways. So… that's why I did it.*

[She then made a comment that this strategy was the one to use when the grid was irregular]

Thomas: *Ok… […] But if we go a bit further down, then we can have… 3 and 3 [pointing at the grid]. How do you make sure the answer is 6?*

Victoria: *Well… just relying on the pattern that is at the beginning, we just have to… assume it will happen for the rest of it.*

[…]

Thomas: *But here [pointing at the square were the two 3s add up to 6] why do you think we add instead of multiplying?*

Victoria: *If you got 3 ways from here and 3 ways from here [pointing at the specific edges on the grid] then you would add. Because… in order to get here you can only comes from this side or this side. So then you wouldn't multiply because… you're adding from this side and this side [pointing again].*

It is difficult to interpret how deep Victoria's understanding was. She certainly had a good command of the algorithm and it seemed justified to her as it followed a pattern that made sense for the first squares in the grid [when there are only 1, 2 or 3 paths]. Yet the reason – the mathematical proof – why the algorithm worked was not present. Nevertheless it was within reach and some probing allowed Victoria to express it. I am not suggesting that my questions made her realise the reason why the algorithm works, but I really had the impression that my questioning forced her to clarify her knowledge about this problem.

Her understanding was mostly instrumental, but some aspects were also relational. However, focussing so much on algorithms and procedure had the effect of putting the relational understanding 'somewhere afar', somewhere where it was not immediately accessible. One can suppose this to have some influence on students' further learning and capacities to adapt.

4.3.3. What is remembered from the pathway problem

Finally it is also worth having an idea of what and how much students remember from their instruction. In the case of the pathway problem I can extrapolate from the answers given to the pre-test by the 9 students who had already done some combinatorics, either in grade 12 or in summer school.

The success rate was indeed better for those with some experience of combinatorics (5 out of 9, 56%) than for those without (3 out of 10, 30%) but it was much lower than the one obtained at the end-of-unit test (22 out of 25, 88%). Time had taken its toll. More significantly, the methods used by the two groups were dramatically different.

The students who had not encountered such kinds of problem used different strategies involving counting – either appropriately or not – whereas the students who had had some instruction mostly used the 'add at crossing' method (5 out of 9), Pascal's triangle or a formula related to combinatorics (2 out of 9) [see table 4.2]. But the two students who used a formula were wrong. They did not use the permutation with similar objects formula that would have been correct but student E used the one for simple permutation (${}_4P_3$ actually) and student A answered 2^5, which could be derived from the FCP or the formula for permutation with repetition. Only one of the nine students still used a – wrong – counting strategy.

Table 4.2: Results to the pre-test pathway problem from students with previous experience

Students whose answers were correct		Students whose answers were wrong	
5		4	
Student	**Method used**	**Student**	**Method used**
D, G, K, R and U	'add at crossing'	M	Pascal's triangle
		E	used ${}_4P_3$
		A	used 2^5
		H	counting (squares)

Finally and not surprisingly, students who had already seen the problem rated its difficulty very differently if they managed to solve it or not: it was either easy (1 or 2 on the scale ranging from 1 to 5) or difficult (4 or 5). Somehow the change was more qualitative than quantitative: either one knew how to do it and it was easy or one did not know and the problem seemed out of reach.

4.4. Conclusion

It is interesting to put into perspective that the pathway problem and some of its variations were well covered in class, taking a fair amount of instructional time, and that it is only a specific case – albeit an interesting and beautiful case – of where combinatorial phenomenon are present. One can, and should, question if all this time was used to work on the mathematically interesting aspects and features of the problem and its link with the subject taught or if it was mere preparation for a possible and popular question in a test.

It is also interesting to note that for some students, seeing the 'add at crossing' method once seemed enough to make them relatively proficient in its use. Nonetheless, for some students, this use was limited to a specific case of the problem and adaptability to a novel situation was limited. Knowing how to solve one problem does not necessarily give the tools to solve a similar problem. One reason is that students' understanding was mostly instrumental, often limited to knowing how to use the algorithm. Relational understanding was present but not preponderant. Showing students how to solve the pathway problem or one of its variations with the 'add at crossing' method or using the formula for permutation with similar objects made the students shift from using counting techniques to the algorithmic 'add at crossing' method. It also transformed a challenging mathematical problem into a routine exercise. As such it was not surprising that the success rate at the end-of-unit test was so high. Yet one has again to ask what has really been achieved, and how has this topic been integrated by the students. This could be done by asking the students to give more justifications – the why questions – when they are solving problems. Having the students answer the meta-cognitive questions in the pre-test was of great help in assessing their understanding. It was certainly more reliable than a multiple choice question and was not as time consuming as it may seem. Moreover students might also gain from the practice by improving their justifications – a fundamental aspect of mathematics – and explanations skill as well as their meta-cognitive skills.

On a more technical side, the second method – the one using the formula for permutation with similar objects – was really not popular with students. One can see two reasons for that. Firstly, the 'add at crossing' algorithm is easier to remember than the formula for permutation with similar objects as it is a counting strategy that relies on addition instead of being a formula which uses notations that obscure meaning and refer to more complex mathematical concepts. Secondly, since all the problems that students were presented with could be solved using the 'add at crossing' method, there was no incentive to use the other method. That could have been avoided by giving an exercise with a 20 by 10 grid and by showing an alternate way to solve some of the problems using this second method.

To sum up, the effect of Mr Cho's instruction on students' achievement and ways of solving the problem was noticeable: students shifted from counting techniques to algorithmic methods and achievement went up significantly. A challenging problem became a routine exercise. Students seemed to have integrated the new material well, but a closer look at some students painted a picture where understanding was more instrumental than relational.

Similar shifts happened when students were introduced to the combinatorial formulae. But before describing them in detail in chapter 6, I will describe students' lack of proficiency with counting techniques in the next chapter because these further explain these shifts. This serves to add nuance to my argument by partly contradicting my hypothesis by indicating how students accurately made use of counting in certain circumstances.

Chapter 5: Students' understanding and use of the fundamental counting principle and their use of counting techniques

In this chapter I look at students' understanding and use of the fundamental counting principle (FCP) and their use of counting techniques. The reasons are multiple. The FCP is a fundamental aspect of combinatorics because it is embedded in most counting techniques and from it stem all the combinatorial formulae. The counting techniques are an alternate way of solving many combinatorial problems, and can still be used in conjunction with the formulae to solve other problems. Moreover, students used such techniques before being taught the formulae. As such they are an integral part of combinatorics.

This chapter is therefore divided into four parts. The first deals with students' understanding of the FCP and examines how they solved the menu problem which required its use. The second part then looks more in detail at students' lack of proficiency with trees and lists. This partly contradicts my hypothesis which supposed that students were proficient with counting techniques. The third part is a discussion on some possible reasons for this lack of proficiency. The last part is the conclusion.

5.1. The fundamental counting principle

In Grade 12 students learn what English (2005) calls combinatorial operations: the permutation and combination formulae. These formulae stem from of the *fundamental counting principle* (FCP):

> *If one item can be selected in m ways, and for each way a second item can be selected in n ways, then the two items can be selected in m·n ways.*

Another counting principle that is often neglected but is also important in combinatorics is the *addition principle for counting*:

If there are m ways of selecting one item from one group, and there are n ways of selecting one item from a second group, then there are m+n ways of selecting one item from the two groups.

In the next sections I focus on how students solved the *menu problem* [fig. 5.1] that was the first problem of the pre-test. The reason is that the solution to that problem requires the application of the fundamental counting principle, and provides evidence about students' knowledge and use of it.

Fig. 5.1: The menu problem

A restaurant proposes a menu composed of four appetizers and five main dishes as well as two desserts.

a) *How many different menus can be composed of one appetizer, one main dish and one dessert?*
b) *Now consider that the Chef is quite particular and does not allow guests to mix fish and meat. How many different menus can be composed if there are two appetizers that contain meat and two that contain fish; and the main dishes are: beef, chicken, lamb, salmon or halibut?*

In the next two sections, the students' answers are analyzed with two directions in mind: firstly the students' understanding of the FCP and their ability to solve the menu problem; and secondly an overview of the strategies used by the students to solve the problem. Finally a third section puts these different aspects in perspective.

5.1.1. Students' understanding of the FCP and ability to solve the problem

The first thing of note is that the meta-cognitive questions really helped to understand better what the students did, particularly the first: "*Describe how you approached the problem and worked at solving it.*" Too often the only answer given to such a problem is 4x5x2 = 40. This is right, but it also lets teachers only *suppose* and *assume* that the student knows why, but with no certainty. In this case the student can simply remember that in a combinatorial problem of this kind, one has to multiply (instrumental understanding) to get the right answer, without understanding why (relational understanding).

As a result of the meta-cognitive questions, the students' answers were much more developed than they usually are in a straightforward test, and hence allowed a better insight into how students solved the problem. Yet some uncertainties remained: the student who just explained that he multiplied the numbers did not show that his understanding was relational. This might show that multiplying was obvious for the students, or that they knew from previous experience that such types of problems called for multiplication.

The majority of the students were successful with both parts of the problem. Out of 21 students, only three students were not successful with part a of the problem. One of them seemed to have an understanding of the fundamental counting principle but made a computational error, whereas the two other students clearly did not use the fundamental counting principle: the first simply counted all possibilities one after the other [fig. 5.2] and the second even had difficulties in representing the situation. For part b, only 14 students

Fig. 5.2: Counting one by one

solved it, but I must note that three students knew how to do it but forgot the dessert and so did not get the right answer. Actually the wording of part b was somewhat misleading and such forgetting could have been prevented by adding the sentence "*There are still two desserts to choose from*" at the end. Moreover another made a computational error but his explanations clearly showed that he had understood how to solve such a kind of problem. So most students understood the fundamental counting principle or knew how to use it [see table 5.1].

Table 5.1: Student's success to and understanding of the menu problem

Part of the menu problem	Number of students who's answer were:		
	Correct and showed some relational understanding	wrong but showed that the FCP was understood	wrong without showing understanding of the FCP
a	18	1	2
b	14	5	2

They also found the problems relatively easy as they ranked them, on average, at 2.25 on a five point scale with 1 being easy, 3 ok and 5 difficult. Finally, I need to mention that the two students who did not grasp the fundamental counting principle had no previous experience in combinatorics.

5.1.2. Overview of the strategies used by the students

The most interesting feature that emerged was not whether the students managed to solve the problem or not, but how they solved it and how they represented the problem. Some students simply found the answer using multiplication and used a diagram only to check if they were right. Some students had to count all the possibilities and sometimes even to write them all down on a list or a tree. Lastly, some started a list or a diagram and found that the answer could be found using a multiplication. Figures 5.3 and 5.4 show two students' answers and the justification they gave when answering the meta-cognitive questions.

Fig. 5.3: A student started a list and then multiplied

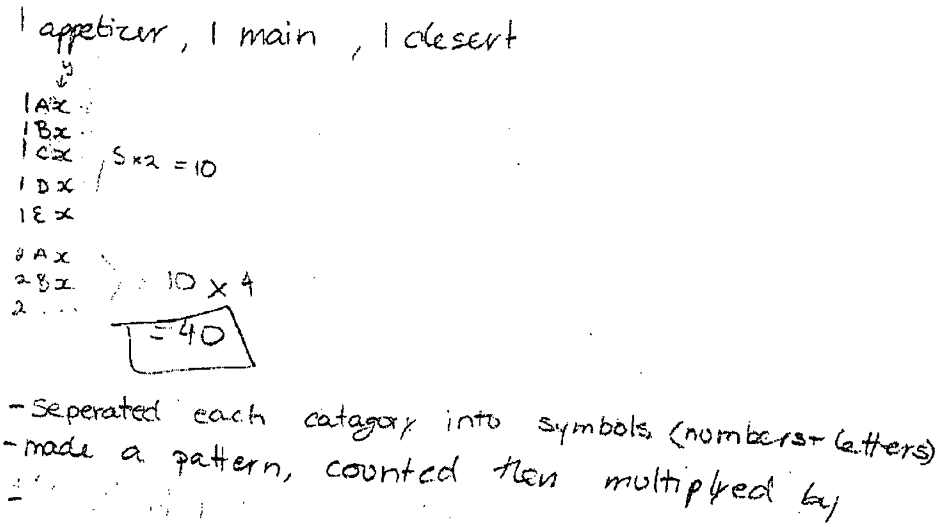
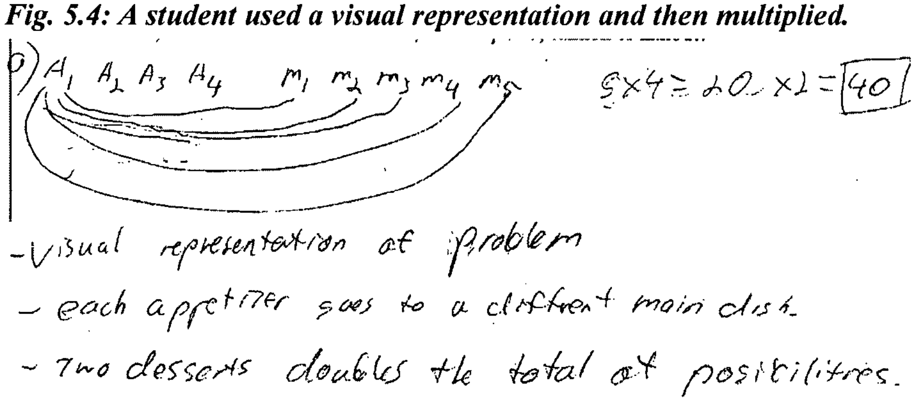
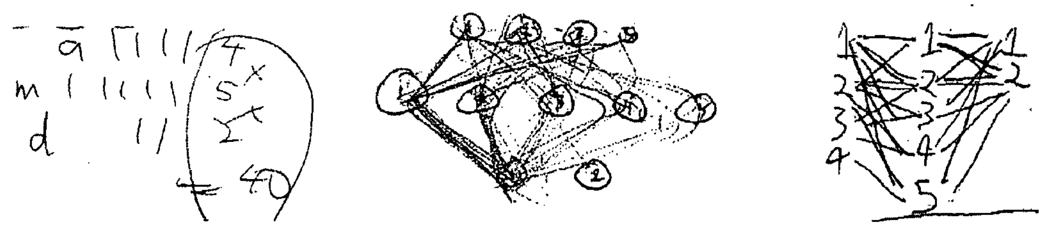


Fig. 5.4: A student used a visual representation and then multiplied.



The most common method consisted of drawing a graph – or something approaching a graph – by listing the possibilities for the appetizers, the main dishes and the desserts and then connecting them to compose menus. Mr Cho told me this was the method that they were taught in elementary school. 11 students used it and figures 5.2 and 5.5 shows such graphs, and only two did not find the correct answer. Such a representation is fine for a basic problem such as this one; but the graphs do not show the fundamental counting principle *explicitly* and tend to be messy, as one can see from figure 5.5, so such a method would be of limited use if the problem were more complex.

Fig. 5.5: Different versions of the most common method used: drawing some kind of graph



I was expecting that trees and lists would be used more often, or at least partially, as drawing the full tree or writing the full list is time consuming. So I was surprised by the fact that only five students used either a tree (two students) or a list (three students). Of these five students only two had a previous experience in combinatorics before, during summer school or a previous mathematics 12 class.

Finally the impact of instruction was tangible in the ways students solved the problem. For the students having already done a Mathematics 12 course or been prepared for it during summer school, the most common method was simply to multiply. Seven students did it that way, but only three students had to use a diagram first before computing the solution. In contrast the numbers were reversed for the students without previous experience of Grade 12 combinatorics: three simply multiplied whereas eight used a diagram [see table 5.2]. So experience results in the fundamental counting principle being more straightforwardly put to use, without having to revert to relational understanding to find it. It becomes somewhat *obvious* and doesn't need justification.

Table 5.2: Strategies used by students to solve the menu problem in relation to previous experience in combinatorics (Mathematics 12 or summer school)

strategy used	Students who solved the menu problem	
	without previous experience	with previous experience
	11	10
multiplication (no diagram)	3	7
diagram	8	3

Moreover previous experience seems to have increased the occurrence of a *formula syndrome*: four students mentioned the existence of a formula to solve the problem but only in stating that they had forgotten it or how to use it. Despite that all managed to solve the problems, reverting to different strategies. Of these four students, three had already done some combinatorics. From a teaching perspective, this certainly encourages taking into account different resolution strategies when teaching combinatorics and not only focussing on the formulae.

5.1.3. Discussion

All students with previous experience in combinatorics managed to solve the problem or show they had understood it. The two students who displayed no understanding of the fundamental counting principle had no previous experience. So instruction seems to have had a positive effect on the success rate – if only by often displaying the use of the FCP – in terms of using the FCP and knowing when to use it. As a matter of fact, previous experience in combinatorics seems to have made the problem obvious to the extent that justification was not considered necessary. Nevertheless, students' answers to the meta-cognitive questions and answers to part b of the menu-problem show that all students that answered correctly had some relational understanding of the FCP. For instance, many students rediscovered it again using multiples strategies: counting, trees, lists, but mostly graphs.

As seen in the section above, the students had a good command of the FCP. Yet their most common representation, graphs, does not use a tree structure, making it difficult to read, especially when there are more than two stages. In that case trees would be a better option. Lists are also a good option but the trees have the advantage of explicitly showing structure, whereas lists are only the end product.

Finally, I have to mention that these conclusions are conditional because the small sample size limits the validity of the findings, but it certainly gives insight into how students represent the fundamental counting principle.

5.2. Student's use of trees and lists

The next section will look at students' answers to the pre-test and end-of-unit test and deal with students' lack of proficiency when using trees or lists and the resulting poor achievement. Then the following section will look at the group sessions in search of some possible reasons for this lack of proficiency, and finally the last section takes on a more positive tone by looking at students' improvement in the use of lists done.

5.2.1. Lack of proficiency and poor achievement

During the pre-test, many students used some counting techniques. This brought about success when the question was simple, as in the menu problem, but when the problem started to be more complicated, like the pathway problem, most students displayed their limitations by not being able to keep count.

In the next paragraphs I look at three problems. In the two problems from the pre-test I focus on the different counting techniques used by the students and the trouble they had using them. In the problem from the end-of-unit test I focus on achievement and the use of trees.

Pre-test second problem: the pathway problem

When trying to solve the pathway problem [see chapter 4] by counting all the paths, the difficulty is to keep track of the paths counted. Student B's work shows how challenging it is [fig. 5.6]. It is no surprise that only two students managed to find the correct solution this way. Student Q used labelling and was systematic in making his list, yet this strategy was time consuming and he could not finish in time [fig. 5.7].

Fig. 5.6: Student B's work

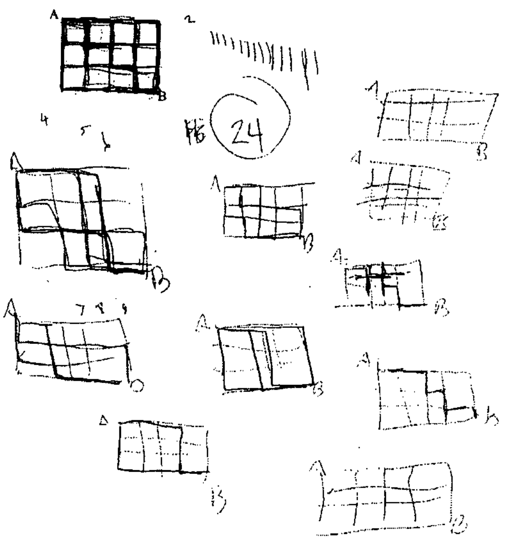
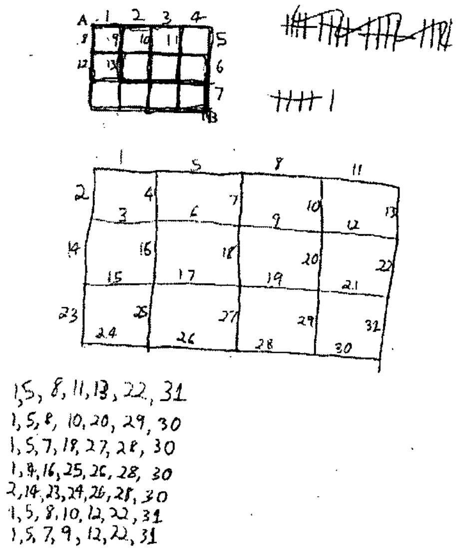


Fig. 5.7: Student Q's work



Pre-test third problem: the partition problem

Looking at students' answers to the third problem of the pre-test [fig. 5.8] it is interesting to note that more students used a counting strategy than with the pathway problem.

> ***Fig. 5.8: Third pre-test problem: partition***
> *A boy has four different coloured toy cars (black, orange, red and grey) and he decides to give away the cars to his friends Peggy, John and Linda. In how many different ways can he distribute the toy cars? For example he could give all cars to Linda.*

One reason is that students did not have an algorithm to straightforwardly apply as this problem was not typical; they had to use something else. Some students tried graphs [fig. 5.9] or had partial lists [fig. 5.10] but only four – students A, I, O and Y – out of the 17 students who tried solving this problem had lists that were more developed. No student used a tree, even partially.

Fig. 5.9: Student P's work

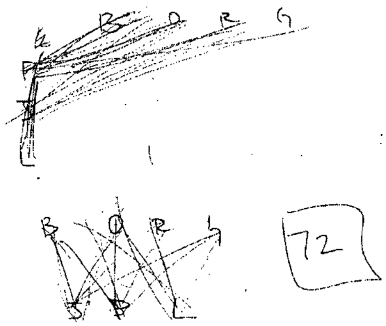
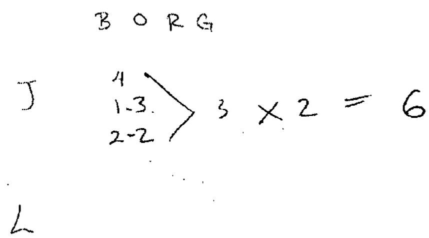


Fig. 5.10: Victoria's work



Student I and O tried to write the list in full, using grouping of letters to represent the toy cars to be given. Since the list was long they tried to use multiplication to shortcut the cumbersome process of listing. Whereas student O showed that she had a good command of listing – being systematic and finding all combinations with 3 or 2 letters [fig. 5.11] – student I's skills were put to the test by such a challenging problem: she only listed three out of six combinations of 2 letters [fig. 5.12]. She was also not systematic enough, thus forgetting the

Fig. 5.11: Student O's work

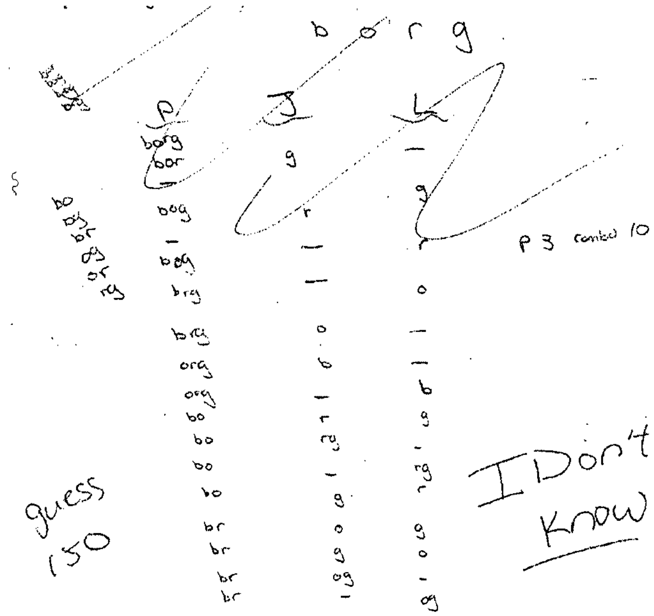
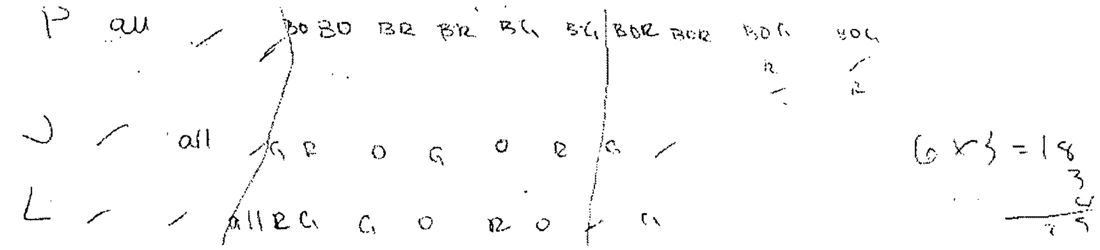


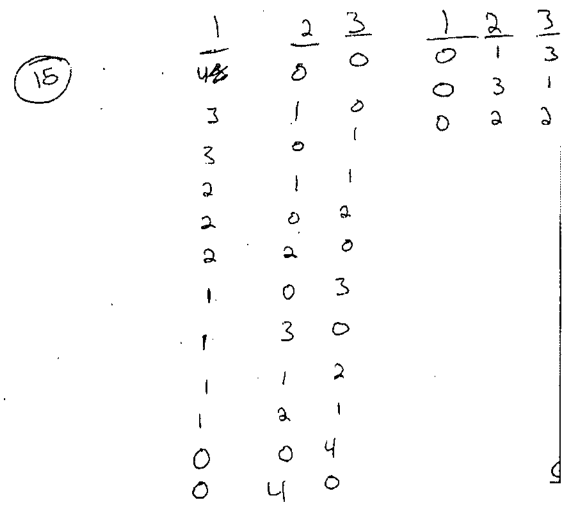
Fig. 5.12: Student I's work



possibility of giving two toy-cars to two people. Less important but also revealing is the fact she first gave 4 then 2 then 3 toy cars. Ordering was not used as a tool to prevent misses.

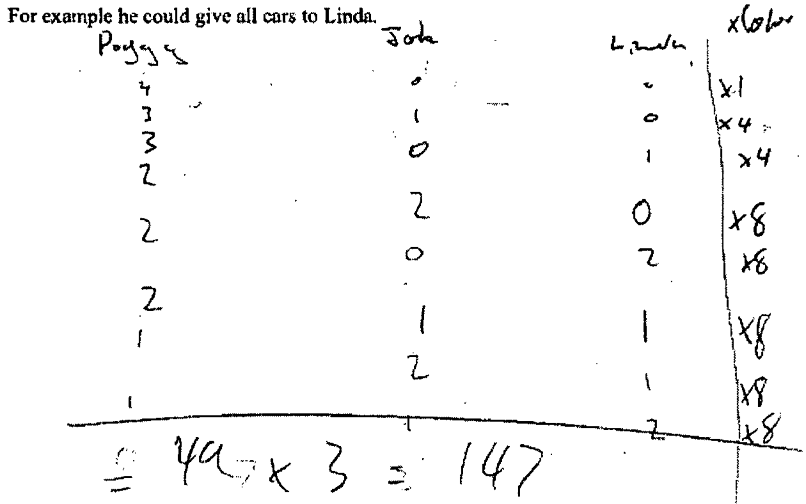
Student A and Y used another strategy, listing the possible numerical partitions: the number of toy-cars to be given to each person. Despite not being fully systematic, student A chose the first number in a decreasing manner but not the second, and he was successful in writing the list [fig. 5.13], but did not realize there was generally more than one possibility of

Fig. 5.13: Student A's answer



giving the toy cars for each numerical partition of the list. This is however something that Yvan did [fig. 5.14]. Correctly at first and then wrongly (the two first 8 should

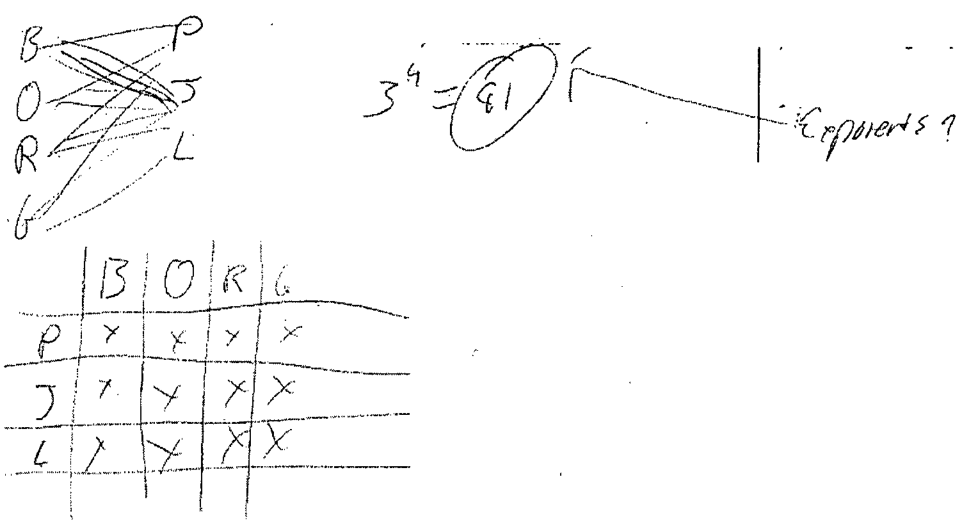
Fig. 5.14: Yvan's work



be 6 and the three last should be 12). Moreover multiplying everything by 3 ended up in counting some items more than once (the 2-2-0 and 1-2-1 possibilities). Checking that no item has been counted twice is not as obvious and as easy a task as it may seem and Yvan might have overlooked it. I also need to recognize that using this strategy made the resolution of the problem complex and difficult.

Finally only student M [fig. 5.15] who gave the correct answer to this problem used a graph and double-entry table. He seemed not to be sure of his answer though. So, lists and trees are not the only way to represent the problem in a way that leads to success.

Fig. 5.15: Student M's work



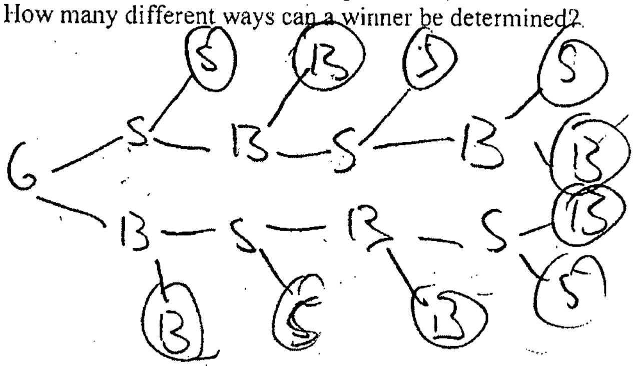
End-of-unit test problem

The multiple-choice question 20 [fig. 5.16] from the end-of-unit test gives an idea of students' achievement when some marks were at stake in a question from the test that required using a tree or a list. Looking at it also points to a new hurdle: the use of an inappropriate formula in lieu of a counting technique.

Fig. 5.16: Multiple-choice question 20 from the end-of-unit test and its solution

Sam and Bruce play a golf match. The first person to win 2 holes in a row or a total of 3 holes wins the match. How many different ways can a winner be determined?

A. 11
B. 10
C. 9
D. 8



All 25 students did the 20th multiple-choice question. As with many counting problems, no formula can be used in a simple and straightforward manner, and since the number of possibilities is limited it makes sense to use a counting technique like a tree or a list. It is not a very difficult problem yet only 14 (56%) students managed to get the correct answer. Moreover, seeing many correct answers with wrong justifications, I am convinced that the percentage would have been even lower if it would have been a written instead of a multiple-choice question. As a matter of fact seven students who answered correctly had a justification that clearly was wrong (5) or incomplete (2). Incorrect justifications included the use of various inappropriate combinatorial formulae [fig. 5.17 and 5.18], whereas the two incomplete ones were trees [fig. 5.19 and fig. 5.20].

Fig. 5.17: Student C's answer

$$\frac{5!}{2!5!}$$

Fig. 5.18: Student I's answer

$$3\,({}_3C_2)$$

Fig. 5.19: Student E's tree

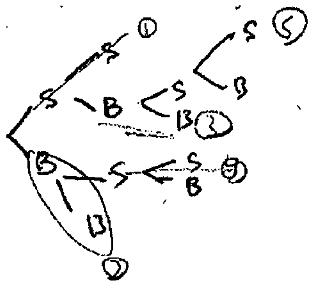
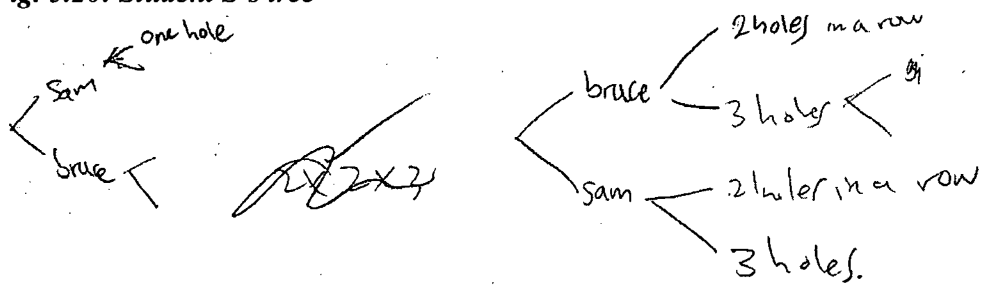


Fig. 5.20: Student S's tree

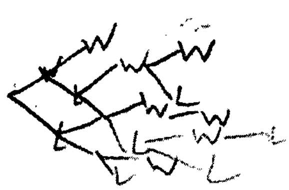


In a similar fashion, despite being the most commonly used technique to solve the problem, drawing a tree was not particularly successful as a strategy: only 6 out of the 11 students who used a tree got the correct answer [see table 5.3]. Moreover two of the successful students were presumably lucky: the trees they both drew show either mistakes [fig. 5.21] or limitations [fig. 5.20]. The situation is similar, if on a smaller scale, for students using a list: only two (8%) students used a list and only one got it right.

Table 5.3: Table of strategies students used when solving multiple-choice question 20 from the end-of-unit test

Method	Students that were		
	Right	Wrong	Total
Tree	6	5	**11**
List	1	1	**2**
nCr	1	3	**4**
5! / 3!2!	2	-	**2**
Other	1	-	**1**
no justification	3	2	**5**
Total	**14**	**11**	**25**

Fig. 5.21: Student N's tree



5.3. Discussion on the reasons for students' lack of proficiency with lists

Students' lack of proficiency with lists was also apparent during the group sessions. During these group sessions the students used lists many times. In their case two reasons explaining their lack of proficiency came to light. Firstly, they were not systematic, or at least, not systematic enough. Secondly they seemed to lack practice. These two reasons are developed in the following two sections.

5.3.1. Students shortcomings in using lists systematically

The lack of proficiency when writing a list comes in great part from the students not being systematic enough. Being systematic is of the utmost importance when counting because one must neither forget to count an item nor count it twice. Students managed to do basic and small lists with relative ease but slightly longer or more complicated lists challenged them.

Below are several examples of students' un-systematic listing when they solved combinatorial problems. In the first sub-section I focus on a list that was long. In the second and third section I focus on two particular types of lists that caused much trouble to these four students: lists where the order is not important, and lists where there is a specific constraint – in this case colouring squares that are *not* adjacent.

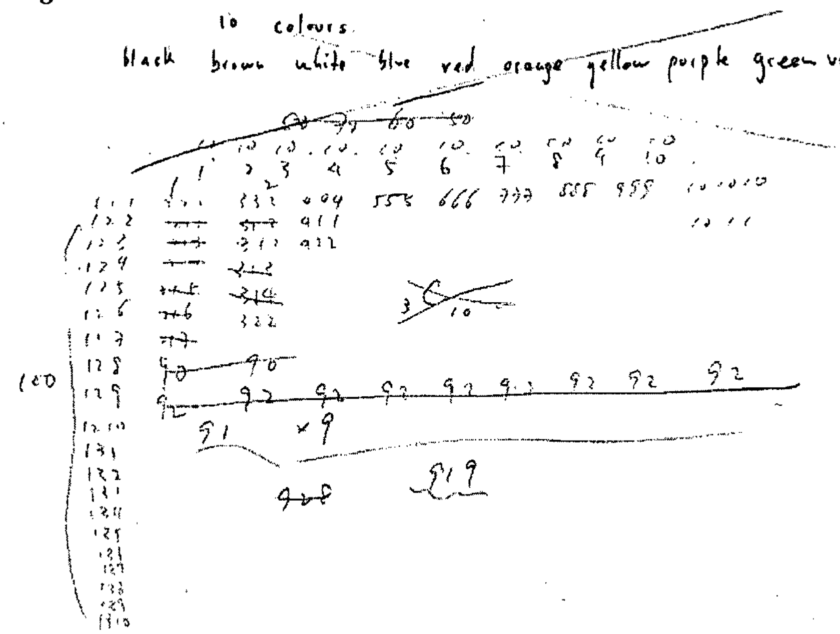
Long lists

The first example is Xinlei's work done at the beginning of a session when he was familiarising himself with the golf balls problem [fig. 5.22]. He wrote

Fig. 5.22: The golf balls problem
A company sells bags containing three coloured golf balls for Christmas.
How many different bags can be made if there are ten colours to be chosen from?

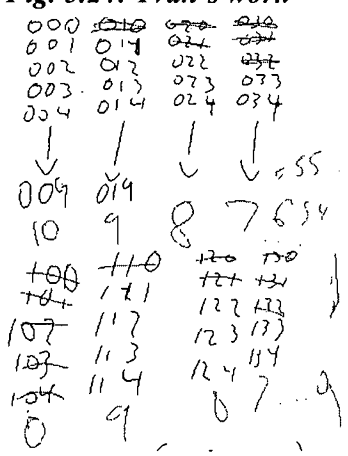
the list of all triples and crossed out all those that were repeated [fig. 5.23], because since the order did not matter they need not be counted twice. The troubling thing in this case is that the

Fig. 5.23: Xinlei's work



numerical order of the triples is not respected. That is a serious flaw. As a matter of fact he had trouble finding similar triples and crossed them out. In comparison, Yvan, doing the same exercise in group 2, was systematic and actually managed to solve the problem [fig. 5.24].

Fig. 5.24: Yvan's work



Lists where the order is not important

The second example is a transcript of both Xinlei and Victoria when they tried to solve the simplified variation of the problem with four colours instead of ten. Xinlei and Victoria had tried to solve the original golf balls problem [fig. 5.22 above] with formulae and then shifted to the list done by Xinlei [fig. 5.23 above]. Both methods were inconclusive, so I proposed to simplify the problem by having fewer colours. Xinlei chose to use four colours. Victoria wrote A, B, C and D to represent the four colours. She computed 4x3x2x1 and said: "24 is when you can use them only once." As she was hesitant, I asked them to make a list and she wrote the one in fig. 5.25b [fig. 5.25b is an excerpt of fig. 5.25a which was the end state of the smartboard page] She wrote the first four items in the list (AAA, ABA, ABC and ABB) then

Fig. 5.25a: Group 1 work

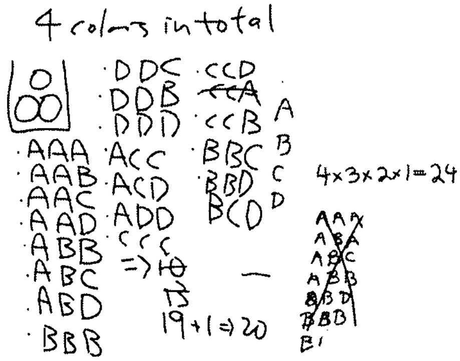
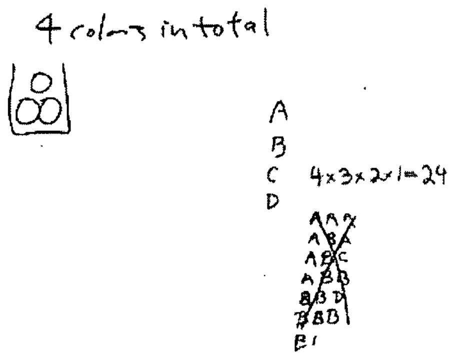


Fig. 5.25b: Victoria's first list



paused, wrote BBD then, after a comment from Xinlei, transformed it into ABD and paused again. She started to write BA then overwrote the A with B to make BBB and went to the next line and wrote B and started to write another letter but stopped. Xinlei proposed one more item but Victoria said they "did it wrong" pointing to the middle letters in all the items in the list. In other words, she first changed the middle letter but only one (from A to B but not to C and D) then went on to shift the third letter but not in alphabetical order and then wanted to play with the first letter. That is really unsystematic.

Xinlei crossed the list out and started anew and wrote the list on fig. 5.25c. He only took a short break before writing ABB, and when it was done he said: "so in total you have 1, 2 ,3, […] ten!" After a short while Victoria pointed out that BBB was missing, so he added

Fig. 5.25c: Xinlei's list

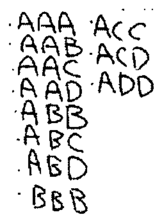


Fig. 5.25d: The missing item

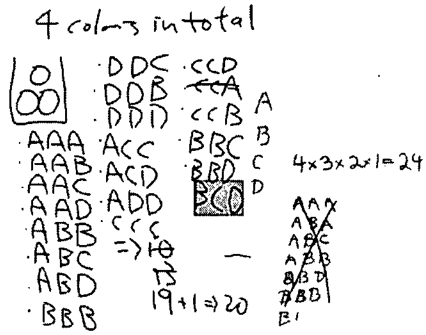
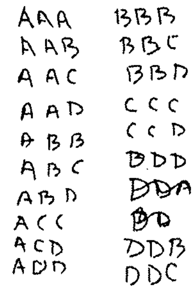


Fig. 5.25e: Third attempt



BBB, CCC, and DDD and said "So 13!" [fig. 5.25d] Victoria again pointed to a missing one: DDC. Xinlei completed the list by adding DDB and DDC. That made him realise CCD was also missing and so they looked for some more and completed the list, ending up with 19 (BCD was still missing) [fig. 5.25d]. They spent some more time checking if they had got them all. At that point I asked them "how confident are you?" Victoria answered: "not great!" and laughed. It took Xinlei a while to find that they were missing one: BCD. He then said he was "pretty sure that [was] it." So this time they were more systematic generating possibilities but they had no reliable way to check if the list was complete or not. The third attempt [fig. 5.25e] was better but they still had some difficulties.

Lists where there is a specific constraint

Likewise, listing caused trouble to the students when there was a constraint. The way group 1 solved the seating problem [fig. 5.26] is interesting because the students decomposed the problem into two sub-problems that were both solved using lists. The first sub-problem

> ***Fig. 5.26: The seating problem***
> *Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students.*
> *In how many ways can Professors Alpha, Beta, and Gamma choose their chairs?*

consisted of selecting the seats for the professors and the second was to seat the professors on the assigned seats. The list for the second sub-problem was a basic list resulting from a permutation, but the list for the first sub-problem was not so ordinary because there was the constraint of not having two adjacent seats occupied by professors. Students' achievement was very different for the two sub-problems.

Group 1 started by drawing strokes representing the seats and wrote down three possibilities [top of fig. 5.27], and the need to decompose the problem came rapidly. (Actually two possibilities were wrong because they had not yet realised they had misunderstood the problem, looking for two students between the professors instead of professors between two students.) Xinlei wrote a list of five "combinations of ABG" of the three seated professors Alpha, Beta and Gamma [fig. 5.27]. He did not write the five permutations in an assured

Fig. 5.27: Xinlei's list

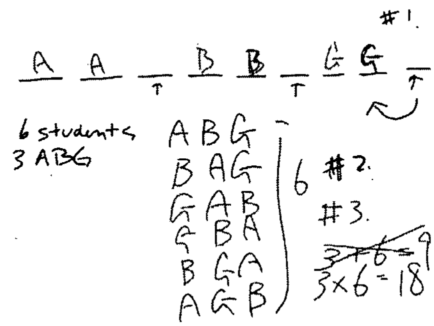
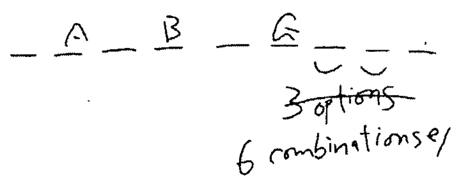


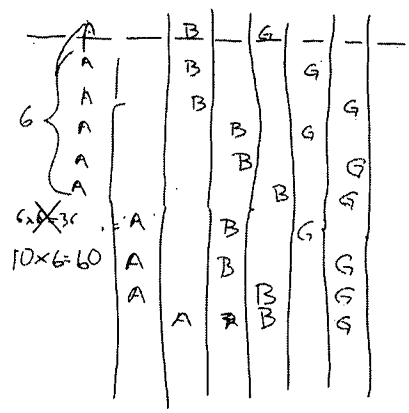
Fig. 5.28: Seat selections



manner; he paused between each or between two items in the list, generating each item from the ones above by permuting some letters. He was not following a systematic procedure, yet he knew there should be six possibilities in total. That was a *fact* he knew and could use at will (like 2+2 is known to be 4 and there is no need to compute); something that was remembered as a whole because seen previously – it was an example given in class when permutation had been introduced – and because the numbers were small enough to be remembered as is. Victoria helped him find the sixth and missing one [AGB in the bottom of the list in fig. 5.27].

Solving the first sub-problem, the one consisting of selecting the seats for the professors, was the difficult part of the problem. The difficulty comes from the constraint. At first Xinlei and Victoria from group 1 did only find 3 possibilities for selecting the seats for the professors [top of fig. 5.27, and fig. 5.28]. They had moved all the professors together keeping the same space between them. I had to intervene twice to tell them some were missing. Xinlei first noted that they could move B independently. He found two more to a total of five possibilities but wrote nothing. They both discussed it for a while and then started a new list [fig. 5.29]. This time systematically; but they stopped after the first 6. After my second comment, they both realised that they could *also* move A and completed the list to find the correct answer of 10 possibilities.

Fig. 5.29: Group 1 final list



Group 2 also had difficulties with this sub-problem - I recall that group 2 had the same problem but with one more professor. They first tried to make groups of one professor and one or two students [fig. 5.30]. Then they went back to listing a few possibilities. But this time, instead of moving all professors [top of fig. 5.30], they changed the relative positions of the professors [grey area in fig. 5.31]. After some more trial-and-error, they were able to grasp the problem and rewrite a list more systematically and find the correct answer. So, both groups had had difficulties with the particular constraint imposed on this list, and they also struggled to find a systematic way to generate possibilities by moving only one professor at a time in an orderly fashion.

Fig. 5.30: Group 2 first attempts

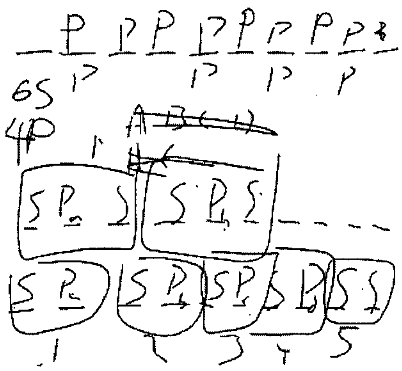
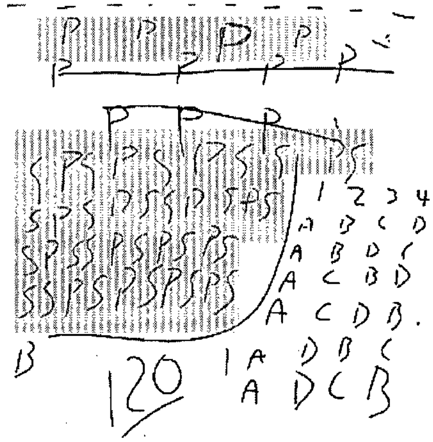


Fig. 5.31: Group 2 successful attempt

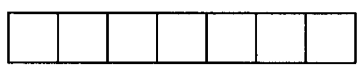


The problems that the students did in the pre-test – the menu problem excluded – and during the group sessions were not straightforward exercises. The sets of objects that had to be counted were sometimes large and not always obvious. When students tried to solve them, they were not systematic enough, which lead to numerous mistakes.

5.3.2. Students' use of lists improving with practice

In the two previous examples, both groups adopted more systematic procedures which led them to successfully solve the seating problem. They started with some trial and errors and I had to make some helping comments but, in the end, they managed to list all possibilities correctly. This suggests that practice helped students improve their listing skills. This was even more visible with the group sessions that took place on week later. It is interesting to compare what students in both groups did when they tried to solve the squares problem [fig. 5.32]. The squares problem is mathematically identical to the first of the sub-problems of

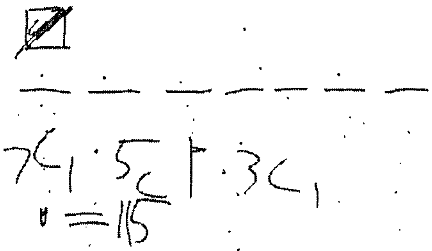
Fig. 5.32: The squares problem
In how many ways can you fill in three of the squares below so that no coloured square touches another one?



the seating problem. The students from the first group came rapidly to the conclusion that the square problem was similar to the seating problem and they solved the problem by listing all possibilities very systemically in approximately two minutes. There were no hesitations. Actually Xinlei said "It's the same problem as last time" when reading the problem, while Victoria admitted later she had to "think about how to solve it" before arriving at the same conclusion.

It took much longer for the second group to realise that the two problems were similar. Nick first tried to solve it using the combination formula [fig. 5.33] "from what we learned this morning". In the previous class, a few hours before, they had learned how to

Fig. 5.33: Nick's first answer

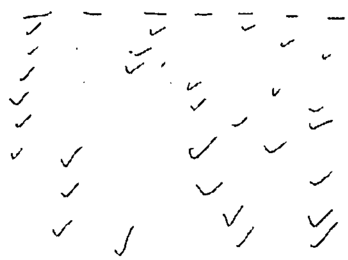


compute the different hands in poker, using the product of two to five combination formulae. It is important to note how the effect of the course influenced Nick's way of reasoning. The new material was still fresh in his memory, maybe still being processed and integrated, and as such available for use. 'Choosing' cards in a deck to compose a hand or 'choosing' where to sit a bunch of professors looks somewhat similar. Moreover the word 'choose' was used by Mr Cho when referring to the combination formulae – ${}_7C_1$ is often pronounced '7 choose 1'. Probably this word trick acted like an automatism and directed Nick in using the combination formula. This could have had no consequence in a simple exercise for practicing the use of formulae, but in the face of a challenging problem, eluding reflection on the appropriateness of the formula is not a strategy that might lead to success.

Yvan realised that Nick's answer had a flaw, but he did not realise the problem was similar to the seating problem. Moreover he was first reluctant to "try them all out" as suggested by Nick because of the supposedly 115 possibilities that would have taken "way too much time". Nick insisted that "that's what we did last time I think" and added "I don't think there are 115." Yvan asked if it was less or more and Nick answered "less... there is only like 7 spaces." At that point Yvan started to look for some possible solutions writing on the smartboard dots for the coloured squares [see fig. 5.33, above].

After starting anew with a better notation he worked it out very systemically from left to right, and despite some moments of hesitation, he found all possibilities one after the other [fig. 5.34]. So, regardless of the fact that they had done a similar problem before, Yvan

Fig. 5.34: Yvan's list



and Nick were not in an automatic mode that could have somewhat defined the other group's behaviour at the same stage. They were still thinking about the problem and wondering if their strategy was a good one. This active situation led them to notice a pattern emerging from the list: there were three possibilities with ticks on the first and third stroke, then two with ticks and on the first and fourth stroke, then one with ticks on the first and fifth stroke. Yvan completed the list, thinking about the pattern and when he wrote the last possibility, he said: "So there is only one". Nick confirmed "I think we got it all."

Both groups showed that they had dramatically improved their listing skills with a little practice. They were much more systematic solving the squares problem. This also gave them the confidence that their list was complete. Nevertheless integration of this new skill was very different in the two groups. One group managed to link the new problem with the previous one and rapidly remembered the counting technique to solve it. And so they did, in an instrumental way. On the other hand, the other group had to struggle. Having had a class on combinatorics and been taught new material a few hours before being posed the problem put them into another frame of mind that led them astray and slowed their progression but also put them in a more active state of mind allowing them to notice new mathematical elements in their work.

5.4. Conclusion

In this chapter I focused on students' command of the fundamental counting principle and other counting techniques. The reason was that these are related to combinatorics and can be viewed as pre-requisites for it. The combinatorial formulae are based on the FCP and counting techniques like trees and lists can be representations of these combinatorial formulae. As such they are – to some extent – used by teachers to explain and justify the combinatorial formulae. Moreover counting is an important skill that has multiple uses, not only in mathematics, but in other academic subjects – computing for instance.

The majority of the students knew the FCP and solved a relatively simple problem requiring its application. When solving the problem, some students simply applied the FCP and used a multiplication, whereas others had to draw diagrams or graphs to get the correct answer. The former and more straightforward use of the FCP was predominantly chosen by students who had already had some instruction on combinatorics. This suggests some possible effect of instruction and consequence on how students integrate a mathematical concept such as the FCP: the FCP seemed to become obvious and readily usable but this also had the effect of being presented and used without justification.

In contrast, students were not proficient with counting techniques when the set of objects to count was larger or more complex. Students showed shortcomings when it came to trees and lists. It was particularly true with lists that were not basic enumerations but that had some constraint or when the order did not matter. The most striking students' weakness was the fact that they were not systematic. Students were looking for items to complete the list without having a vision of the whole list and its structure. It resulted in many mistakes.

The situation looked grim at first, but students in both groups significantly improved their listing skills with a little practice. So a simple lack of practice might be an important factor in their lack of proficiency. This is encouraging and should persuade teachers not to skip teaching or reviewing trees and lists. This can be done by proposing a variety of problems that can – and sometimes need to – be solved by use of counting techniques, and also by showing alternate solutions using various methods. This would give students more flexibility when approaching other combinatorial problems as well as probability – by using a probability tree for instance.

Chapter 6: Students' understanding of the combinatorial formulae

In this chapter I look at how students integrate the formulae they learned during the unit on combinatorics. More specifically the first part of the chapter deals with the preponderance given to these formulae, as it is the context of instruction in which students' learning take place. The second part is dedicated to the ways students integrated the combinatorial formulae they were taught and particularly at how they used, misused and understood the fundamental counting principle, the factorial, and the permutation and combination formulae. Conclusions are drawn in the third and last part.

6.1. Preponderance of the formulae and students shifting to use them

Before looking at how students integrate the combinatorial formulae, it must be acknowledged that this teaching unit is based on them. But giving so much weight to formulae makes students abandon and disregard counting techniques when they learn the formulae. This is not without consequences. This shift is also influenced by the teaching style.

6.1.1. Students shift to using formulae

As seen in chapter 4 with the pathway problems, instruction can have a dramatic effect on relations to a particular range of problems. There was a shift in the strategies used by the students to solve this kind of problem: they mostly used counting strategies prior to instruction, but then they all adopted the algorithmic method of resolution taught. As a result, a range of problems became routine exercises. This happened with one kind of problem but the same shift to using mathematical formulae instead of counting techniques was prevalent in the whole unit.

Three reasons that could explain this shift emerged during this piece of research. Firstly, many students judged these counting techniques to be long and cumbersome, as some of them wrote in the pre-test when commenting on other strategies that they had considered employing [fig. 6.1]. Secondly, this perception was reinforced by the teaching style and the format of the test, as I will elaborate in the next section. Thirdly, since most students were not proficient in the use of counting techniques, as was seen in chapter 5, they had no reason to continue to use them.

Fig. 6.1: Student K's answer when asked if he considered another strategy to solve the partition problem in the pre-test.

2. Count all The possible ways but
takes too long

6.1.2. Preponderance of the formulae in the taught material

Somehow Mr Cho's point of view and teaching style – and for that, I reckon he is certainly representative of many teachers – influenced students' tendency to use a formula instead of a counting technique. As he told me during an interview, he did not consider that trees and lists had to be integrated in the teaching alongside the new material taught because they had been previously seen and consequently should be mastered. This has an impact: students do not see counting techniques as part of the subject but as something unrelated. Students cannot be expected to make and think alone about the connections between subjects that have been taught separately. Integration of these two aspects of combinatorics cannot be grasped.

Counting techniques' advantages and usefulness were not shown. Mr Cho compared some counting techniques to something basic and done in elementary school – failing to recognize that they also encompass more sophisticated strategies. This made it clear that what was to be learnt and used were the formulae. This, combined with the fact that they might be cumbersome and time consuming, suggested to students not to use counting strategies. This causes students no trouble as long as the problems they are going to encounter can be solved with the use of a formula – which was the case of all problems but one in the end-of-unit test – but that brings limitations to the use of what has been learnt, particularly when one thinks about students' difficulty to recall formulae over time.

6.1.3. Some consequences of the shift away from counting techniques

After a little instruction, the shift away from counting techniques had occurred and it resulted in students mostly using the combinatorial formulae. The only other method that was used was the repeated multiplication [fig. 6.2]. This brought limitations in students' abilities to solve problems and an inappropriate overuse of the formulae. In the next paragraph I will look at some inappropriate answers that appeared in the multiple choice question 20 from the end-of-unit test.

Fig. 6.2: Student U's correct answer to written question 4b using a repeated multiplication

4. A class has 30 students.

How many ways can an executive committee consisting of 3 people (president, vice-president, secretary) be selected from the class? **(1 mark)**

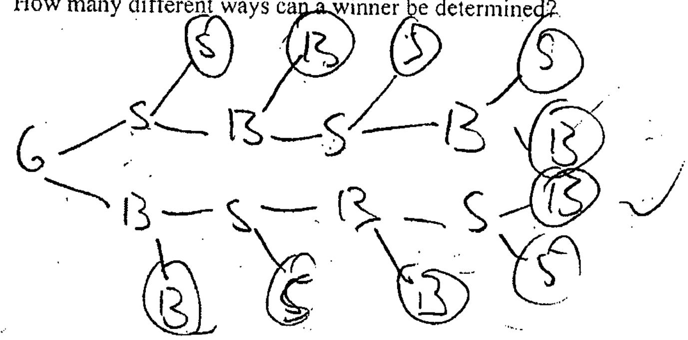
$\frac{30}{P} \cdot \frac{29}{VP} \cdot \frac{28}{S}$ $\boxed{24360}$ ✓

Looking at answers to the multiple-choice question 20 [fig. 6.3] from the end-of-unit test is revealing of an inappropriate use of the formulae. While this question is a typical

Fig. 6.3: Multiple-choice question 20 from the end-of-unit test, with answer

20. Sam and Bruce play a golf match. The first person to win 2 holes in a row or a total of 3 holes wins the match. How many different ways can a winner be determined?

A. 11
B. 10
C. 9
D. 8



one to be solved using a tree and not a formula because there are too many conditions, six (24%) students used a combinatorial formula when solving or trying to solve the problem. Even if three managed to answer correctly, all justifications were wrong and one can assume they were lucky. Four used the nCr formula in one manner or the other [fig. 6.4 a,b and c] and two used a formula that can either be a combination or a permutation with similar objects [fig. 6.4 c]. The formulae they used were highly inappropriate and in some respect show that their understanding was very limited. It is instrumental understanding: students know how to apply them in a set of clearly defined situations, but that is about it. One can argue that some

Fig. 6.4: Students' computations used when answering question 20 from the end-of-unit test

a	*b*	*c*	*d*
$^{3}C_{2}$	$3\,({}_{3}C_{2})$	18 holes $^{18}C_1 \; {}_{2}C_{2} \; {}_{14}C_{4}$	$\frac{5!}{2!3!}$

students were lost and using a formula was only a way to get out of a somewhat desperate situation. I certainly agree with that but that does not really explain why most used the combination – nCr – formula in this exercise where order clearly matters.

6.1.4. Discussion

When students faced the teaching of this unit, within which combinatorial formulae were given a preponderant place, they abandoned and disregarded other counting techniques. This was not without consequence, and had an influence on how they learned the subject, for instance over-relying on formulae or using them inappropriately. This context led to a more instrumental understanding of the combinatorial formulae, and hence some confusion. The following part of this chapter is dedicated to exploring students' understanding of the combinatorial formulae.

6.2. Students' use and understanding of the combinatorial formulae

In this part of the chapter I look at how students integrated the three combinatorial operations that are the factorials and the permutation and combination formulae. More specifically I look at how students used, misused and understood them. Each of these three combinatorial operations will be treated in a separate section because students' understanding varies from one to the other.

The backbone of the teaching unit on combinatorics is a sequence that starts with the fundamental counting principle (FCP) which is then followed by the three combinatorial operations that are the factorial (n!), the permutation formula (nPr) and the combination formula (nCr). This sequence follows the mathematical deductions that link these combinatorial operations, as each is developed from the previous one. The formulae for group permutation and permutation with similar objects complete this sequence.

Each step in this sequence is more complex and more difficult to understand. As seen in chapter 5, the FCP is already known by most of the students. Factorials are nothing more than a notation for a kind of repeated multiplication and are not difficult if the notation is acknowledged. The difficulties arise with the formulae: first with the permutation formulae and then with the combination formula which is the most complex. Students' understanding and command of the combinatorial operation followed the same pattern, decreasing as the subject becomes more complex. The FCP gave no trouble, and they passed the hurdle caused by the notation. Factorials became part of students' mathematical repertoire and did not give them trouble anymore. The permutation formula was relatively well understood: students displayed that they had both instrumental and relational understanding. The fact of also seeing variations like the formulae for group permutation and permutation with similar objects certainly helped them to have a thorough experience with the concept of permutation, which deepened students' grasp of it. On the other hand, their understanding of the combination formula was more instrumental than relational and limited to simple application of it in a series of specific cases. The students' difficulty in comprehending the concept of combination was apparent in the fact that many were confused and hesitant whereas to use the permutation or the combination formulae to solve a problem. Moreover some examples of the use of the combination formulae seemed to indicate that an important hurdle resided in not acknowledging the importance of the division that result from the fact that order does not matter. I illustrate this in the next sections.

6.2.1. The fundamental counting principle and factorials

The fundamental counting principle and factorials belong to what students had well understood and had good command of. This section is divided into two, starting with the FCP – which was already known by most students – and then looking at factorials – which were new but became part of the students' mathematical repertoire as the novelty passed.

Fundamental counting principle

As seen in chapter 5, most of the students knew the FCP already. Yet, as seen in the menu problem from the pre-test, some students had to revert to drawing diagrams and graphs to find the solution. Instruction, which is described below, seemed to have made this less necessary, as students shifted to use only the multiplication.

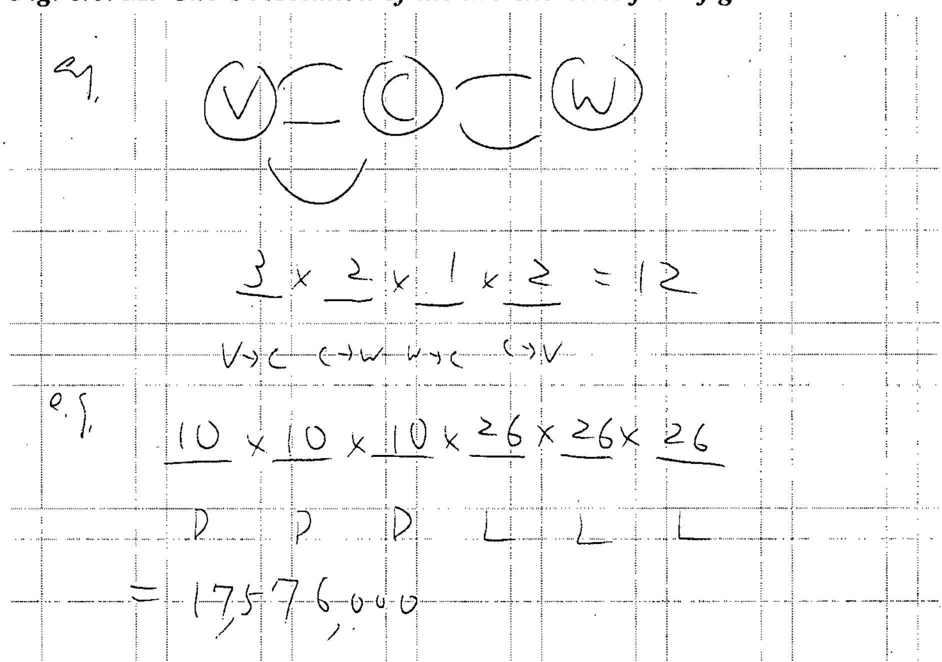
As I have mentioned before, after having commented on the pre-test problems, Mr Cho started the unit on combinatorics. He stated the FCP and moved to the exercises in the handout, doing them one after the other. The FCP was not explicitly explained; most of the explanations were context related as they came along within the exercises. Below in figures 6.5 and 6.6 are two exercises from the handout. Their resolution was done on the board by Mr Cho. The method of drawing strokes to represent each successive choice [fig. 6.6] is common and the students seemed to have integrated it and its use well. As a matter of fact, they used it many times later in the unit.

Fig. 6.5: Two exercises from the handout that are related to the FCP

e.g. John is planning to drive from Vancouver to Winnipeg via Calgary. There are three roads from Vancouver to Calgary and two roads from Calgary to Winnipeg. How many different "round-trip" routs are there from Vancouver to Winnipeg, passing through Calgary, if no road is used more than once? (12)

e.g. How many different license plates can be made that consist of three digits followed by three letters? (17576000)

Fig. 6.6: Mr Cho's resolution of the two exercises from fig. 6.5



Factorials

In the case of factorials, integration was less straightforward. They actually caused trouble to some students. The main reason was that students had to get accustomed to the new notation, and realise that the factorial formulae was not much more than a notation – it is actually only a particular repeated multiplication. The algebraic exercises [some of which are in fig. 6.7] that Mr Cho did after giving the definition of factorials gave the students some practical examples and some practice. In the following paragraphs, I lay out one example of the particular struggle of one of the student to make sense of factorials, and the specific notation used.

Fig. 6.7: Four exercises on factorials from the handout

(b) $\frac{7!}{3!}$ *(c)* $\frac{8!}{4!\,3!}$ *(d)* $\frac{(n+2)!}{n!}$ *(f)* $\frac{(n+2)!}{(n-1)!}$

Nick was struggling; it was going too fast for him – at one point he asked Mr Cho to slow down. The concept of factorial was not clear, so he asked to Mr Cho several questions. Below follows a transcript of some excerpts from two such occurrences.

[Mr Cho is doing exercise c in fig. 6.7.]

Mr Cho: *We have 8 times 7 times what? 6 times 5 times 4 factorial, divided by 4 factorial, 3 factorial.*

Nick: *Mr [Cho] what does the factorial sign really means? What is its significance?*

Mr Cho: *It means continuous times: 1 times 2 times 3... that's the definition.*

Nick: *But then if you... like... cancel it out, does it like... diminish the original...*

Mr Cho: *No, no, no, factorial is just one way of notation.*

[Nick continues to follow the course, but he looks as if he is struggling.]

[...]

[Later on Mr Cho is doing the exercise f in fig. 6.7.]

Mr Cho: *[...] So you have what?* n *plus two times* n *plus one times what?* n.

Nick: *Just in case we cancel out in the first step?*

Mr Cho: *You cannot cancel out in the first steps, you must develop. You must what? Factor out.*

Nick: *Sorry, does the factorial represent an infinite number?*

Mr Cho: *factorial means only a notation, remember... only notation.*

Nick: *So it means the numbers go on until infinity*

Mr Cho: *No, up to one. Did you see the definition?*

Nick: *I saw it, I saw it! [...]*

Mr Cho: *Factorial is just a notation, just a way to write a math notation.*

Nick: *I know, but like... When* n *is a positive integer and* n *is larger than one, then...when does it end?*

Mr Cho: *It ends after 1. ... Always ends after 1. What is integer? And always positive integer.*

Nick: *[after a few seconds and not sounding convinced] Ok.*

Mr Cho: *So, what does nine factorial means?*

Nick: *Nine minus... Nine times nine minus... Nine plus one... Oh no... O God! ...Nine times eight times... seven*

Mr Cho: *Yeah, time until what?*

Nick: *Until one.*

Mr Cho: *Yeah! That means factorial. ... Just mathematical notation.*

Nick: *Ok.*

These excerpts show that integration of a mathematical concept, even a relatively simple such as factorials, is not always a smooth and straightforward process. Later on, at the end of the course, when I asked the students to write down what they had learned during the first week of the unit on combinatorics, Nick wrote the following [fig. 6.8]:

Fig. 6.8: Nick's answer when asked about the difficulty of combinatorics

difficulty: what is the purpose of (!)

overcome: it is a symol representing a series of calculating

But the original difficulty faded, and four weeks later, after the Christmas Holidays break, when he was asked what they had learned so far in this unit, the factorial appeared too, but this time it was only an item in a list, along with other combinatorial concepts and formulae.

Moreover, in his answers to the end-of-unit test Nick showed that he had no difficulty with basic questions such the one in figure 6.7b but still had some trouble with the algebraic manipulation of the factorial. The concept of factorial was understood as well as its link to other concepts of combinatorics seen in class [fig. 6.9]. His shortcomings with algebraic manipulation of factorials certainly had some consequences on his computational skills but this relates to a rather technical matter and the use of the calculator makes the resolution of many problems possible without the use of these manipulations. The other students seemed to fare the same or better than Nick.

Fig. 6.9: Nick's answer when first asked about what he learned

fundamentally counting principle $\xrightarrow{\text{Similar}}$ factorial $\xrightarrow{\text{Similar}}$ permutations

Actually the factorial is a tool and a building block for the subsequent permutation and combination formulae. Its subsequent use reinforced it as being only "a symbol representing a series of calculation" as Nick put it and it somehow became part of the students' mathematical repertoire.

6.2.2. Permutations

Students' understanding of permutation was a mix of instrumental and relational understanding. The concept of permutation in itself – what a permutation is – was well understood, and it was translated by most students into a proficient and appropriate use of the formula, at least when the problems were simple. It was seen in a number of characteristic 'real life' situations like anagrams, putting objects in a line, and orderings books on a shelf [fig. 6.10] that allowed them to make sense of the concept but also to associate it with specific problems in which the permutation formula had to be used. A common representation was a

Fig. 6.10: Excerpt from student R's answer when asked what he learned

Cases that exist in permutations

↳ street
↳ book shelf
↳ job opportunities
↳ letter arrangement

list requiring the permutation of a number of letters. Students in both groups, for example, wrote down or started such a list when solving the seating problem [fig. 6.11, done by group 1. Group 2 had the same problem but with one more professor called Delta] and realised that the second sub-problem was only a permutation [grey area in fig. 6.12 and 6.13].

Fig. 6.11: The seating problem (Andreescu & Feng, 2003, p. 3)
Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta, and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students.
In how many ways can Professors Alpha, Beta, and Gamma choose their chairs?

Fig. 6.12: Group 1's list when trying to solve the seating problem

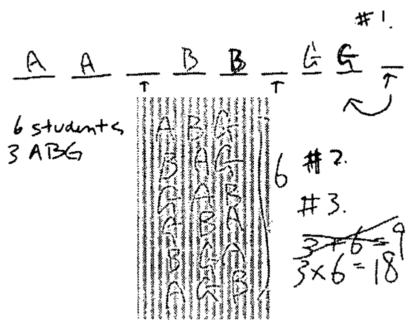
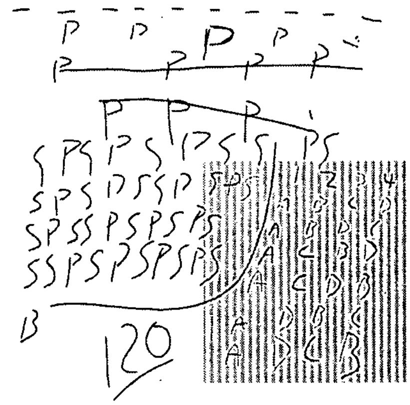
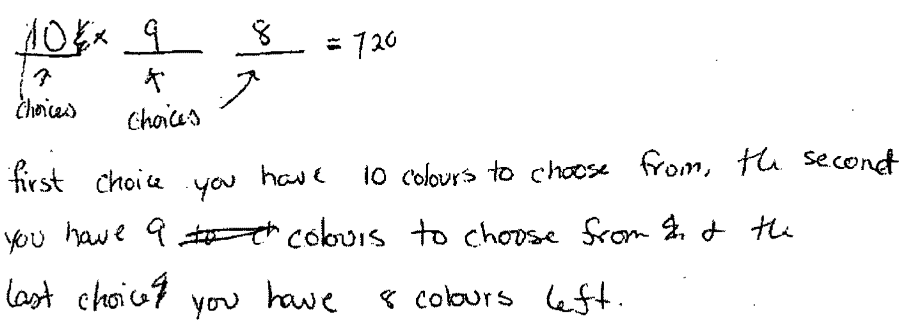


Fig. 6.13: Group 2's list when trying to solve the seating problem



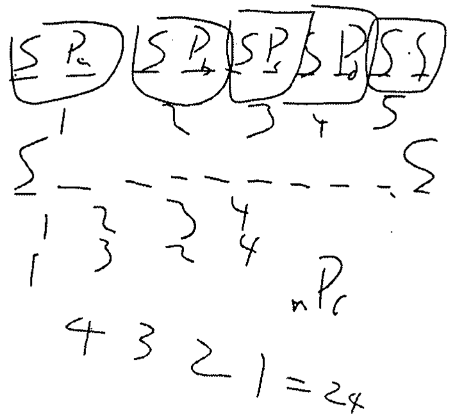
Another common representation of permutation is an ordered selection of objects. This representation, with the use of the FCP, allows the number of permutations to be counted by using repeated multiplications, a method which seemed to make sense to many students, as can be seen from Victoria's justification on figure 6.14. These two representations were often

Fig. 6.14: Victoria's justification for using a repeated multiplication



connected together and sometimes also with the permutation formula. As a matter of fact, when group 2 was trying to solve the seating problem and Yvan explained that the four professors could be permuted and wrote two such permutations [1234 and 1324 in fig. 6.15], Nick said "We could go like… 4 times 3 times 2 times 1" just before Yvan could say that it was "just the same thing as" and wrote down nPr [fig. 6.15]. Some students, like Yvan, had integrated that this method of counting was similar to using the permutation formula, but for some it was not always the case, which led to confusion as I will show in the subsequent section on combinations.

Fig. 6.15: Group 2 trying to solve the seating problem



Most students had this knowledge and know-how. This allowed most of them to solve basic problems that needed the sole application of the permutation formula, like the multiple-choice question 5 and 7 from the end-of-unit test [fig. 6.16 and 6.17]. Actually 72% of the students managed to find the correct answer to question 7 and 68% to question 5 [see table 6.1]. One can explain that the achievement rate was a bit lower in question 5 because the answers to choose from were not simplified [fig. 6.16].

Fig. 6.16: Multiple-choice question 5 from the end-of-unit test, with answer

A soccer coach must choose 3 out of 10 players to kick tie-breaking penalty shots. Assuming the coach must designate the order of the 3 players, determine the number of different arrangements she has available.

A. $\frac{10!}{7!}$ $\quad$ $_{10}P_3$ $\quad$ $\frac{10!}{7!}$

B. $\frac{10!}{3!}$

C. $\frac{10!}{3!7!}$

D. $\frac{10!}{3!3!4!}$

Fig. 6.17: Multiple-choice question 7 from the end-of-unit test, with answer

A man has 7 different pets and wishes to photograph them 3 at a time arranged in a line. How many different arrangements are possible?

A. 21
B. 35
C. 210
D. 840

$_7P_3$

Table 6.1: Students' answers to multiple-choice questions 5 and 7 from the end-of-unit test

Students' answers	Question 5		Question 7	
Correct	16	64%	18	72%
Wrong	9	36%	7	28%
Use of C instead of P	3	12%	4	16%

Most students gave no justification or used the permutation formula, but a few students used the FCP and the repeated multiplication, like student E [fig. 6.18], or both, like student G [fig. 6.19]. Finally, I have to add that some students were confused where to use the permutation or the combination formula. Three students used the combination formula and were wrong in question 5 and 7 and another student made the same mistake in question 7. Some like Xinlei managed to spot their mistake in time and correct it [fig. 6.20]. Xinlei used the word line in the question to help him decide that order did matter and that the permutation formula was required instead of the one for combination.

Fig. 6.18: Student E's justification to question 7

| | | | | | |

A B C D E F G

$\underline{7} \cdot \underline{6} \cdot \underline{5}$

Fig. 6.19: Student G's justification to question 7

$\underline{7} \cdot \underline{6} \cdot \underline{5} = \underline{210}$

$_7P_3 = \underline{210}$

Fig. 6.20: Xinlei's justification to question 7

A man has 7 different pets and wishes to photograph them 3 at a time arranged in a line. How many different arrangements are possible?

A. 21
B. 35
C. 210
D. 840

$\cancel{{}_7C_3}$ order

${}_7P_3$

This confusion between permutation and combination is an important feature of students learning combinatorics and I will develop it in the following section that deals with how students integrated the combination formula into their mathematical knowledge.

6.2.3. Combinations

The case of students learning about and integrating the formula for combination was more complex than for the permutation. Not only had the students to learn a new concept and another formula, they also had to make sense of them in relation to permutations seen earlier. At this point, most students got confused. The two main reasons of this confusion were that, firstly, combinations were seen as a permutation without order; and secondly, the concept and the formula were complex by themselves and not fully understood. These two reasons are developed successively in the following paragraphs, but before I just want to add that the confusion between permutation and combination is already present in the language. As a matter of fact, what is commonly referred as a combination in everyday language – as in a combination lock for instance – is actually a permutation in the mathematical terminology. This was perceptible several times in the group sessions when Xinlei used the word combination but was actually speaking of permutations.

Combinations as selections

Combinations were seen in class after permutations and were presented as a selection for which the order of selection of the objects was not important. Many students took the habit of saying 'choose' when reasoning about a problem and selecting objects. To choose is to select and the formula for combination – nCr – contains the letter C, the first letter of the word choose. Combinations were seen as selections. But so were permutations! So they had to be differentiated. Consequently, when students learned combinations, they simultaneously learned that it was order that made the distinction between the two. Despite that being known, students found it difficult to choose between the two when trying to solve a problem.

In effect, when students were asked after two weeks of class what were "the important ideas or concepts that [they had] learned this week", most students listed permutation and combination, and to a lesser extent some also mentioned the FCP, factorials, grouped permutation and permutation with similar objects, but also order [see table in fig. 6.2]. Below is the answer from student Q [fig. 6.21], which is typical, even if many students' answer were a bit sketchy and did not go straight to the point as he did.

Table 6.2: Students' answers when asked what they learned during the week

Students' answers	Students	
Permutation	18	75%
Combination	17	71%
Group permutation	7	29%
Order / no order	6	25%
Factorials	6	25%
FCP	4	17%
Permutation with similar objects	4	17%

Fig. 6.21: What student Q wrote he had learnt

1. We learned factorials, the idea of $_nP_r$ and $_nC_r$, and how to recognize the difference between the questions and what formulas to use.

Moreover to the question: '*What is difficult when solving a combinatorial exercise?*' the most common answer, given by seven students (29%) referred to the choice to be made between the permutation and the combination formulae. Five more students (21%) were not so restrictive, and found that choosing the method or formula to use in general was difficult [see table 6.3]. The answers from students C, D and O below give a sense of the difficulties that students faced [fig. 6.22, 6.23 and 6.24]. Student O's answer is particularly interesting because he revealed that despite knowing the difference between the two formulae it was not of much use when he had to solve an exercise. I also have to mention that this difficulty is crucial since the use of calculators transforms some combinatorial exercises into simply choosing the right formula and plugging the numbers.

Table 6.3: Students' answer when asked what is difficult when solving a combinatorial exercise

Difficulty when solving a combinatorial exercise	Students	
Choosing between permutation (nPr) and combination (nCr)	7	29%
Choosing which method or formula to use	5	21%

Fig. 6.22: What student C found difficult when solving a combinatorial exercise

Knowing which formula to use
like either nPr or nCr

Fig. 6.23: What student D found difficult when solving a combinatorial exercise

2. Recognizing what the problem is asking and what to use

Fig. 6.24: What student O found difficult when solving a combinatorial exercise

I can't really tell the difference when to use nPr or
nCr. I know what the difference is but can't tell
which one to use when given a question

This apprehension about choosing the right formula was justified as can be seen from the results to the written question 6a from the end-of-unit test [fig. 6.25]. 15 (60%) of the 25 students were right but of the ten who were wrong seven (28%) used the permutation instead of the combination formula [see table 6.4].

Fig. 6.25: Written question 6a from the end-of-unit test, with answer

How many groups of 3 chairs can be chosen from 7 chairs if the chairs are all different colours?

(2 marks)

$_7C_3 =$ [illegible] $= 35$ groups

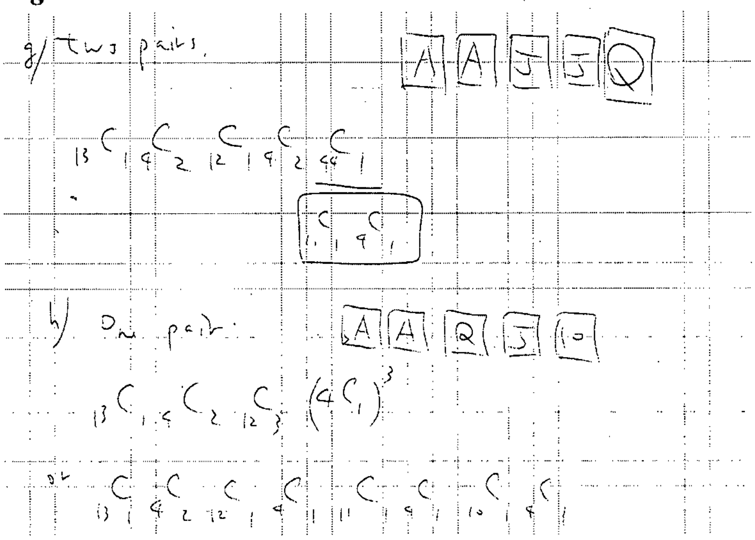
Table 6.4: Students' answers to written question 6a

Students' answers	Students	
Correct (used the combination formula, nCr)	15	60%
Wrong and used the permutation formula (nPr)	7	28%
Wrong and used another method	3	12%

The combination formula and the role of the division in it

Another difficulty that was not apparent to the students was why the fact of not taking the order into account implies a division in the combination formula, and its influence on the use of the combination formula. This mathematical fact is actually not understood nor acknowledged by most students and so they relied on the instrumental application of the formula to compute combination. Intriguingly, this came to light when Mr Cho made a mistake in computing the number of poker hands that have two pairs [fig. 6.26, top half. Mr Cho forgot to divide by 2 or use $_{13}C_2$ instead of $_{13}C_1 x _{12}C_1$]. No student realised that when multiplying combination formulae successively the lack of order might not necessarily have been taken into account. They did not react either when for the number of hands with only one pair he gave two different solutions that would have led to two different answers [fig. 6.26, bottom half; the first solution is right but the second is wrong and should have been divided by 3 factorial].

Fig. 6.26: Mr Cho notes on the board



These problems are undeniably complex and the difficulty is technical. Not understanding that the difference between the formulae is a division does not help differentiate the combination from the permutation formula. It results in a general confusion that was apparent when group 1 tried to solve the golf balls problem [fig. 6.27].

Fig. 6.27: The golf balls problem
A company sells bags containing three coloured golf balls for Christmas.
How many different bags can be made if there are ten colours to be chosen from?

Xinlei started and wrote on the board [left hand side of fig. 6.28] what he did on his own at the beginning of the session [fig. 6.29], despite having already crossed it out once and tried to list all possible bags. When he stopped after having written 720, Victoria asked him if it should be $_{10}C_3$ instead. Below follows the transcript of the ensuing discussion.

Fig. 6.28: Group 1 notes on the board

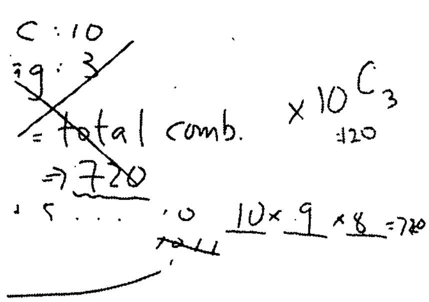
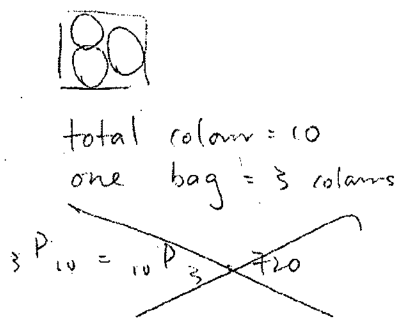


Fig. 6.29: Xinlei's work done at the beginning of the session



Xinlei: *No. Then we have to find out…*

Victoria: *But order doesn't count. Is permutation ordered? Like…*

Xinlei: *[a bit puzzled] yeah… but…*

Victoria: *Order doesn't count because… I just think it's a C but I am not sure*

[Xinlei gives the pen to Victoria]

Victoria: *[She writes* $_{10}C_3$ *on the right hand side of the board, see fig. 6.28] … choose 3.*

Xinlei: *Why is it C?*

Victoria: *'cause order wouldn't… count. … Would it matter?*

Xinlei: *What the C stand for?*

Victoria: *Combination.*

Xinlei: *Oh, is it? What's the [inaudible]?*

Victoria: *It's wrong but I got this answer the second time [referring to her own work done alone at the beginning of the session, see fig. 6.30].*

Xinlei: *How did you know it's wrong? What did you get for ten-C-three?*

Victoria: *It's equal to one-twenty [she writes it].*

Xinlei: *Are you sure? I got [inaudible]*

Victoria: *What?*

Xinlei: *[inaudible]*

Victoria: *How do you calculate it?*

Xinlei: *Yeah.*

Victoria: *I do it again… possibilities… times… equal seven-twenty [she draw three strokes and then writes 10, a multiplication sign and so on].*

Xinlei: *That's ordered, that's just the same as ordered*

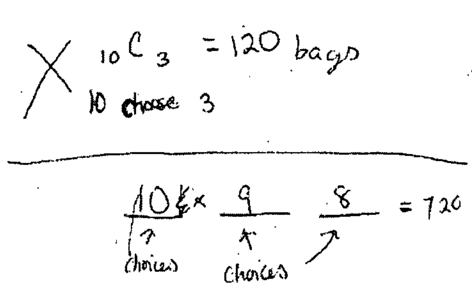
Victoria: *what?*

Xinlei: *That theory is the same as this one [pointing successively at 10x9x8 and the 720 he wrote previously on the left hand side of the board].*

Victoria: *Yeah… I don't know the difference between P and C, I got confused.*

[There is a short pause and then Xinlei starts making a list [fig. 6.28 bottom left] like he did alone beforehand]

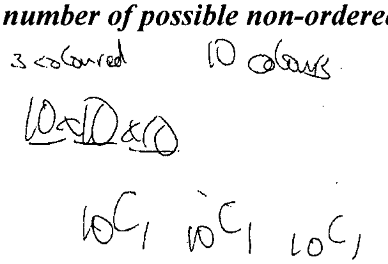
Fig. 6.30: Victoria's work done alone at the beginning of the session



Such confusion is certainly not the sole result of the lack of understanding of this concept of division. Victoria's confusion apparently also comes from the decision to be taken whether order matters or not. Nevertheless, one can reasonably think that this lack of understanding adds more to this confusion than alleviates it.

Another lack of understanding that needs to be pointed out is the fact that no students seemed aware that when making two (or more) selections without order, there is still the order in which the two selections have been done that needs to be considered - actually when looking at the formulae, this phenomenon translates into the fact that $_nC_1 x _{n-1}C_1$ is not equal to $_nC_2$ but equal to $_nP_2$. As a result, some students used a product of combination formulae thinking that the result will still take into account that order does not matter [fig. 6.31]. It is effectively the case in some situations – like counting possible poker hands or written question 3 from the end-of-unit test [Fig. 6.33 below], which explain why students used it – when the combination formulae are each used to compute sub-problems that are independent [in a somewhat similar sense that is used in probability]; but otherwise it is not the case and a division still needs to be applied.

Fig. 6.31: Nick's use of a product of combination formulae intended to compute the number of possible non-ordered bags of three coloured golf balls.



The fact that not taking the order into account implies a division in the combination formula is a rather technical aspect. Nevertheless it is at the core of what order means and implies mathematically. Since it is actually not understood nor acknowledged by most students, some instructional time should be spent dealing specifically with it. The next paragraph indicates some ways to emphasize the role played by the division in the combination formula.

Actually, when the combinatorial formula is defined, it is often relative to the permutation formula [fig. 6.32a]. As such, the division is apparent. Yet while this notation is generally not used in the resolution of problems, some students may know the definition but either use the one in figure 6.32c for their calculations or just use the pre-defined nCr function on their calculator. Giving more emphasis to this aspect of the combination formula

Fig. 6.32: The definition of the combination formula from Mr Chos' handbook

a	*b*	*c*
$_nC_r = \frac{_nP_r}{r!}$	$= \frac{\frac{n!}{(n-r)!}}{r!}$	$= \frac{n!}{(n-r)!r!}$

and showing the importance of the division and its role in relation to the concept of order, might be a teaching strategy to alleviate this hurdle. I suspect showing some resolution of problems that use this concept would be a means to have students encounter this concept practically. This would also give students another strategy to solve some problems involving combinations such as written question 3 from the end-of-unit test [fig. 6.33]. Students H and

Fig. 6.33: Written question 3a from the end-of-unit test, with answer

A toy box contains 4 different cars and 6 different trucks.

a) In how many ways can a collection of 5 toys be chosen if the collection must consist of 2 cars and 3 trucks? **(2 marks)**

$_4C_2 \; _6C_3 = 120$

L's answers [fig. 6.34 and 6.35] show that they had acknowledged the fact that order did not matter but were not able to translate that into their calculations. Yet they were actually not far from a correct answer; only a division by 2 (factorial) and by 3 factorial is missing. One could even consider having students solve a few basic combination problems before the formula

Fig 6.34: Student H's answer

Total = 10

${}_{10}C_2 \quad {}_{10}C_3 = \underline{\ \ }\,4 \quad \underline{\ \ }\,3 \quad \underline{\ \ }\,6 \quad \underline{\ \ }\,5 \quad \underline{\ \ }\,4$

$= 1440$

Fig. 6.35: Student L's answer

${}_2C_1\,{}_4C_2\,{}_6C_3 \quad \underline{4} \times \underline{3} \times \underline{6} \times \underline{5} \times \underline{4} = \boxed{1440 \text{ ways}}$

cars trucks

would even be introduced. This would require solving them with a list, and also approaching the problem algebraically with repeated multiplications in order to notice the effect of the division. A problem like the handshake problem [fig. 6.36] would be suitable. The idea is to have a problem that, should it be solved using the combination formula nCr, the r in it would be 2 or 3 in order to make it possible for the students to solve it without the formula.

Fig. 6.36: A variant of the handshake problem

A group of 5 friends meet. Each person shook the hand of all his or her friends. How many handshakes have taken place?

Finally one should also take some instructional time discussing that ${}_nC_1x_{n-1}C_1$ is not equal to ${}_nC_2$ but equal to ${}_nP_2$ and that when making two (or more) selections without order, there is still the order in which the two selections have been done that needs to be considered. Showing this inequality in class and its corresponding equality might be a way to help students prevent this particular misuse of the combination formula.

6.3. Conclusion

In this chapter I looked at how students integrated the formulae they learned during the unit on combinatorics. The first part of the chapter was devoted to the context of instruction with the emphasis put on these formulae. It was apparent in Mr Cho's teaching style and in the end-of-unit test, as well as in students' work. The latter stemmed from students' consideration that counting techniques are too long and too cumbersome and from their lack of proficiency in these counting techniques. It resulted in students abandoning and disregarding their use, and over-using formulae. This brought limitations in students' abilities to solve problems, particularly since there was a lot of confusion about which formula to apply in which situation.

The second part of this chapter looked at how students integrated the combinatorial formulae they were taught during the course. The focus was put successively on the factorials and the permutation and the combination formulae. Once the novelty and the difficulty in recognizing that factorials are nothing more than a notation for a kind of repeated multiplication were passed, factorials became part of students' mathematical language and repertoire. They were well understood.

In the case of permutations, students' understanding was a mix of instrumental and relational understanding. The concept of permutation in itself – what a permutation is – was well understood. They also a saw a number of characteristic problems – including some 'real life' situations – that allowed them to make sense of the concept but also to associate it with specific problems in which the permutation formula had to be used. Most students used the formula but some also used repeated multiplications that were equivalent to the formula.

Nevertheless, despite seemingly displaying a good understanding of permutations, confusion arose with the introduction of combinations. Students' understanding of combinations was more instrumental than relational, and even if they knew that the difference between a permutation and a combination was that order does matter or not, choosing the right formula – something that many problems turned out to be reduced to – caused much trouble to many of them. The two main sources of confusion were, firstly, that the combination was only seen as a permutation without order; and secondly, that the consequence that order does not matter has on the mathematical computations of combinations – technically a division by a factorial – was neither understood nor acknowledged by the students. Students had no relational understanding of the combination formula and so its use was of no help in showing its relation with, and lessening the confusion with, permutations.

I propose to give more emphasis to the importance of the division and its role in relation to the concept of order when teaching on combinations. This could be a means to have students encounter this concept practically and could also give students another strategy to solve problems involving combinations. In order to reduce some misuses of the combination formula, time should also be taken in making clear to students that, when making two or more selections without order, there is still the order in which these selections have been done that needs to be considered.

Chapter 7: Discussion and Conclusion

In this final chapter I discuss the research question and the hypothesis in the light of the results laid out in the past three chapters, making further connections with the literature discussed at the outset. I also make some explicit recommendations for the teaching of combinatorics in Grade 12. I end by discussing the strengths and limitations of this piece of research, indicating possible fruitful avenues for further scholarship.

7.1. Reflecting back on the hypothesis

Guided by the research question that focussed on how students integrate combinatorics theory with their previous knowledge of counting strategies, the purpose of this research was to explore students' mathematical thinking when they were introduced to formal combinatorics theory. Furthermore, its aim was to identify how students understand formal theory and modify their mathematical thinking and resolution strategies after having been introduced to it.

The hypothesis was that because the students' approach is fragmented and over-relies on formula they are unable, when faced with more complex combinatorial problems, to continue to elicit meaningful connections and draw from the problem-solving strategies they used previously. In particular, results indicated that their thinking shifts from meaning-related problem-solving to algorithmic approaches in which the focus is on choosing the right formula. In probing students' thinking and understanding, I was particularly interested in grasping what prevents assigning meaning to the decision of choosing a formula and how this influences students' ability to think through combinatorics.

This research offers a significant contribution to describing students' thinking and approaches in combinatorics, an important but neglected mathematical topic in the curriculum on which education research is scarce. In particular I pointed out students' lack of proficiency with counting techniques such as trees and lists. I also described how students integrated a ubiquitous combinatorial problem – the pathway problem – and how it shifted from a challenging problem into a routine exercise with students' understanding becoming instrumental and relational understanding being cast aside. Finally a key finding was to put in evidence reasons why most students have difficulty understanding combinations and are confused whether to use the permutation or the combination formula.

7.1.1. Lack of proficiency with counting techniques

The first thing to mention is that contrary to what I was expecting, students lacked proficiency with counting techniques such as trees and lists. This made the hypothesis appear in a different light. Lack of proficiency with counting techniques limited students in their choice of strategies and made the shift to using formulae more likely but also limited the possible connections made between these two different aspects of combinatorics. As a matter of fact, trees and lists were seldom used and the only counting strategy to which students reverted was the use of the fundamental counting principle and repeated multiplication – but not division, so students had no other way of solving problems than to use the formulae when computing combinations.

More particularly, students knew how to use counting techniques when problems were simple and basic, but they already found the procedures 'too long'. When the set of objects to count was larger or more complex, students were not proficient and showed worrying shortcomings, particularly in the use of lists. The most striking student weakness was the fact that they were not systematic. Students were looking for items to complete the list without having a vision of the whole list and its structure. It resulted in many mistakes. Moreover lists that were not basic enumerations but that had some kind of constraint – like when the order does not matter – gave students trouble. This lack of proficiency is of some concern because these counting techniques like trees and lists can be used as representations of these combinatorial formulae. As such they are – to some extent – used by teachers to explain and justify the combinatorial formulae. Moreover counting is an important skill that has multiple uses, not only in mathematics, but in other academic subjects as well as in everyday life.

This lack of proficiency with the basic techniques that are trees or lists is surprising and also worrying. It is surprising because they have – or should have – been done in previous grades, and are considered by many teachers to be known. It is worrying because they are a useful and fundamental part of combinatorics; they are relatively simple and visual strategies that have more chance of being remembered than arcane formulae. One has also to consider that counting techniques are often remembered – if partially – whereas formulae are not. Their applications are also broader than formulae that are context sensitive. Students can modify and complete what they have remembered about counting techniques: they can use it as a starting point. In stark opposition, the use of formulae is more of a black and white issue: either students remember a formula well and are proficient with it and that leads to the correct answer, or – in the majority of cases – the formula is incorrect, completely forgotten or applied in a case it should not be, leading to a wrong answer. The longer term goal of teaching combinatorics is also an issue – and that can only mean past the exam.

7.1.2. Shifts

There was effectively some shift following instruction in formal theory but it was not as marked and clearly delineated as originally expected. Actually it makes more sense to speak of shifts in the sense that there were more limited shifts, specific to a mathematical concept or formula and varied in their extent.

Clear shifts

Some shifts were rapid and clear-cut. The best example appeared in the pathway problem. It is a sort of ideal type. Before having been presented with a resolution algorithm, students usually used counting – technically using different techniques – when trying to solve the problem, but as soon as they had been presented with an algorithmic method they shifted to its sole use. Instrumental understanding was present but relational understanding was not, or at least was relegated to the background in students' minds. For students, the algorithm worked and the nice pattern seemed to be a good enough justification. It transformed a challenging mathematical problem into a routine exercise and consequently the success rate at the pathway problem in the end-of-unit test was very high. The rewards were immediate and apparent, as pointed out by Skemp (Skemp, 1976, p. 87). Nevertheless, the dichotomy between relational and instrumental understanding appeared to be not always defined by exclusion. One key finding in this research was to illustrate that the two can, in certain circumstances, also complement each other in a dynamic process. Specifically, when presented with some variations of the pathway problem, instrumental understanding was not enough to solve the new problems. It resulted in students either being blocked and in need of being provided with another algorithm – one more rule – or having to revert to relational understanding. Somehow one can wonder if the *raison d'être* of all these variations is to engage – and test – students' relational understanding because having been seen once the original problem does not have this effect anymore.

The shift was also clear in the use of the fundamental counting principle (FCP). Before instruction, most students used diagrams to help their counting, but after instruction they all used the multiplication – the FCP actually – without diagrams or justification. The FCP had been integrated into their mathematical repertoire and had become obvious and readily usable.

Partial shifts

The shift to the sole use of formula was less dramatic with the factorials and the permutation and combination formulae. Actually many students reverted to repeated multiplications (based on the FCP) to compute some permutation problems, but they also did it for some combination problems. This, however, was problematic, and was never a successful resolution strategy because they had no understanding that they had to use a division, an essential feature when computing combinations, since order does not matter.

Factorials, for instance, became a new tool in the students' repertoire after some troubles with mastering the notation had been overcome. Factorials are not much more than a notation and a tool in computing permutation and combination so, following instruction, their use was inevitably embedded in students' work when solving all combinatorial problems. Nonetheless, some students sometimes reverted to repeated multiplications. Somehow to write 10x9x8 made more sense to students than 10!/7! It was more comprehensible and limitations with algebraic manipulations of the factorials had no effect on computation.

The shift was also apparent with both the permutation and combination formulae, particularly in simple and typical problems. In these cases, many problems were reduced to routine exercises and most students relied on instrumental understanding to solve them. In contrast, some students reverted to the FCP and repeated multiplication. This generally happened when the problem was more challenging or could not be solved by the direct and single use of one of the known formulae. It also happened when students were confused about whether to use the permutation or the combination formula. So in some cases the shift was not complete or, more precisely, there was a shift away from using the formulae and a return to problem-solving strategies. Nevertheless, students who used problem-solving strategies were relatively successful when dealing with a permutation problem, but not at all when solving a combination problem. The latter resulted from students' limited and mostly instrumental understanding of combinations which in turn resulted from the way combinations were taught and presented. The particular case of students' understanding of the combination formula is developed in the next section.

7.1.3. The special case of the combination: the use of division was not integrated

In their study Batanero et al. (Batanero et al., 1997) identified that a common mistake students made was using one formula instead of another. In this case, many teachers would add that the major confusion is between permutation and combination formulae.

This research has proposed an explanation for this specific confusion. I have argued that when the role of the division is not made explicit and combinations are only seen as permutations without order. As such, they are only seen in opposition to permutation but not in themselves and consequently resolution of problems is only done using the ad-hoc formula, making students' understanding purely instrumental and limited to choosing the formula. This led to confusion with regard to when to use which formula when the situations were complex or not typical. Moreover, students who attempted to solve the problem using a problem-solving strategy using repeated multiplications – a method that was more meaningful – were doomed to failure since they did not know they had to use a division, and even less how. By not understanding the role of the division in the combinatorial formula, the only meaning students could assign to a combination was that of a permutation without order. It was correct but the constant reference to permutation might bring add more, rather than less, confusion.

7.2. Strengths & Limitations

This research was limited to one class and only four students took part in the group sessions. So the sample is small. Moreover convenience was used for its selection. This results in the main limitation of this research: limited generalizability. Nevertheless that was acknowledged from the start as this research was mainly exploratory in nature. The findings need to be corroborated – and in some case also developed and refined – by more research. For instance, in the case of the pathway problem I am pretty confident that much would have been the same if I had observed another class. But in the case of students' lack of proficiency with counting techniques, the picture might be more complex as there are many different counting techniques. Moreover, as Duckworth (Duckworth, 1996) showed, there are also many ways of being systematic. Finally, in the case of students' understanding and use of combination. I pointed at a possible cause for students' limited understanding. From this future research could aim at probing if emphasising the role of division in the combination formula and showing ways to use the division alongside problem-solving strategies would effectively enhance students understanding and know-how.

7.3. Recommendations for teaching practice

An expected outcome from this research was to be able to provide some recommendations concerning the teaching of combinatorics. Some findings point to ways of reconsidering teaching with a concern for the students and their understanding of the subject. Moreover insight into how instruction affects students' thinking should make teachers aware of the implications of some aspects of their teaching. Following this study I would recommend:

- teaching counting techniques in parallel to formulae;
- emphasizing the role of the division when teaching combinations;
- using meta-cognitive questions;
- using challenging problems.

The first two are specific to the teaching of combinatorics. The other two are more general and could also be applied to other topics within mathematics. All four recommendations are developed below.

7.3.1. Teaching counting techniques in parallel to formulae

Teachers should not take for granted that students master counting techniques such as trees and lists. Counting techniques should be seen in themselves before introducing the formulae. Instruction should not be limited to presenting simple permutations of three or four objects, but more complex cases should also be seen, in particular cases where there are constraints and in which order does not matter – such as in, respectively, the squares problem and the golf balls problem. Moreover students should practice using lists, trees or another counting techniques in solving problems. Teachers should be aware that the emphasis put on formulae in the unit on combinatorics induces a shift in which students abandon other strategies that they would have used before learning the formulae. This limits their capacities to solve problems that need more than the sole application of a formula in a routine exercise. It is the teacher's responsibility to help students make the connections between the new subject matter learned and counting techniques. As such teachers should first have students solve some basic problems only using counting techniques and later, after having introduced the combinatorial formulae, still show some alternative ways to solve problems using counting techniques or combining counting techniques and formulae. This would let students see that there are multiple ways of solving a combinatorial problem and that counting techniques are useful and are as much part of combinatorics as the formulae.

7.3.2. Emphasizing the role of the division when teaching combinations

As I showed in the previous chapter, combinations are only seen in opposition to permutations. This leads to confusion and limited understanding. Moreover combination problems are always solved using the formula and students have no alternate strategies. To remediate these shortcomings, I propose giving more emphasis on the teaching of combinations and in particular on the importance of the division in the combination formula and its role in relation to the concept of order. Practically, students should encounter combination problems before being shown the formula. This would be the chance to look at the role of division – so the problem should be limited to the selection of two and three objects – and to consider different strategies such as various counting techniques and repeated multiplications. Only then should the formula be introduced. Other resolutions strategies should however not be abandoned but should still be shown alongside those using the formula. Thereafter one should also point to the following fact in order to reduce some misuses of the combination formula: when making two or more selections without order, there is still the order in which these selections have been made that needs to be considered [for instance ${}_nC_1 \times {}_{n-1}C_1 = {}_nP_2$ and not ${}_nC_2$]. These recommendations imply that more time should be spent on the teaching of combinatorics with the objective of focusing on students' understanding. This might contrast with the actual place of combinatorics in many curricula, often reduced to being only necessary formulae used in probability.

7.3.3. Using meta-cognitive questions

Finally I want to mention that during this piece of research, I used meta-cognitive questions as well as challenging problems. I think that using both would enrich the teaching of mathematics; and not only in combinatorics but in other topics of mathematics too. Personally I plan to use them in the future when I resume teaching mathematics.

The meta-cognitive questions allowed probing into how students solved some problems in more depth. One goal of education is students' understanding and meta-cognitive questions are tools that allow teachers to probe it. These questions put some emphasis on the methods and strategies used and made students' answers richer, as they explained what they did in more detail. It expanded students' answers that too often are limited to a solution without much justification. The meta-cognitive questions should be separate questions directly following the original question. This makes explicit that they are important – one could also consider giving points for their completion in a test. One should allow and encourage students to use their own words. As a result, students are not restricted to mathematics notations that often restrain them. Moreover teachers should not be too picky; the goal is not to achieve mathematical excellence but to help students' understanding – because they "develop more sophisticated mathematical understandings as they attempt to communicate their reasoning" (Simon & Blume, 1996, in Kilpatrick, Martin & Shifter, 2003, p. 237) – and to make this understanding visible to teachers and other students. Another use of meta-cognitive questions could be to ask for mathematical justification of why their method works. Asking students to give justifications to their reasoning and resolution methods is similar to a proof. It is a fundamental aspect of mathematics that is too often put aside because it is deemed too difficult.

7.3.4. Using challenging problems

In respect to students' understanding and know-how, challenging problems were more revealing than basic and routine problems. Rapidly, many problems become exercises as students learnt how to solve them. Rules, algorithms and mimetism can and sometimes do replace understanding. Challenging problems are a way of engaging students in their new knowledge and putting it to the test. As such they should complement more traditional and routine problems. Obviously, when presenting students with challenging problems, expectations are not the same as with routine problems. One should not expect students to solve the problem in one go, particularly as it is relatively time consuming. Moreover students often need to be and should be helped: these moments are ideally suited for discussion because they deal with students' difficulties and relate to their understanding and use of the mathematics they learned. Moreover I was actually surprised that in the groups students managed – under my supervision and with a bit of help – to solve complicated problems like the misaddressed letters problem. This recommendation is in line with suggestions from Stigler and Hiebert (Stigler & Hiebert, 1999) to teach mathematics more like the Japanese do – problem-solving has a much larger place in the Japanese style of teaching.

7.4. Ideas for further research

The main limitation of this research is that results cannot be generalised because the sample used was too small. Nevertheless its findings bring a significant contribution to the field of mathematics education by pointing to shortcomings in students' understanding. It would be interesting and of practical use to carry out research that would look to confirm or negate the findings of this study. I propose below two areas worth further research.

7.4.1. Is students' lack of proficiency with counting techniques widespread?

Students' lack of proficiency with counting techniques surprised and worried me. It would be interesting to do further research, in particular in probing how deep and widespread students' lack of proficiency is. This would require a relatively large sample of students. It would also be interesting to investigate if practice and instruction specifically aimed at the use of different counting techniques could alleviate this lack of proficiency.

7.4.2. Would emphasising the role of the division when teaching combinations improve students' understanding?

Another aspect where research would be fruitful relates to the teaching and learning of combinations. In effect, following this study, I suggest emphasising the teaching on combinations – so they are more than just permutations without order – and focus on the division when solving combination problems and in particular on its role in the combination formula. It would be very interesting to carry out research looking at the effects of such instruction and investigate if students' understanding and achievement improves, and in particular if the confusion between permutation and combination fades.

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Counting Puzzles 1: Menu du Jour

Student's code number: Q Date:

A restaurant proposes a menu composed of four appetizers and five main dishes as well as two desserts.

a) How many different menus can be composed of one appetizer, one main dish and one dessert?
b) Now consider that the Chef is quite particular and does not allow guests to mix fish and meat. How many different menus can be composed if there are two appetizers that contain meat and two that contain fish; and the main dishes are: beef, chicken, lamb, salmon or halibut?

a) 40

b) 18 meat
8 fish

1, 2 > meat
3, 4 > fish

1, 2, 3 > meat
4, 5 > fish

- numbered meals
- tallied in beginning
- did rest in head
- a) added by multiple of 10 for each appetizer
- b) added by multiple of 6 for meat and 4 for fish.

State your thinking when solving the problem.
Try to explain how you approached the problem and what you thought about while you worked on it:

1. Describe how you approached the problem and worked at solving it?
2. Which ways did you consider but did not use in the end
3. How confident are you that you have found a correct answer?

1. Decided to lay it out in a tally form. then attatch them. and tally up for begining. Then, figured out pattern and worked it out in my head.
2. Did not consider any other possibilities.
3. I am fairly confident ~~Altho~~ although not completely confident that I have the correct answer.

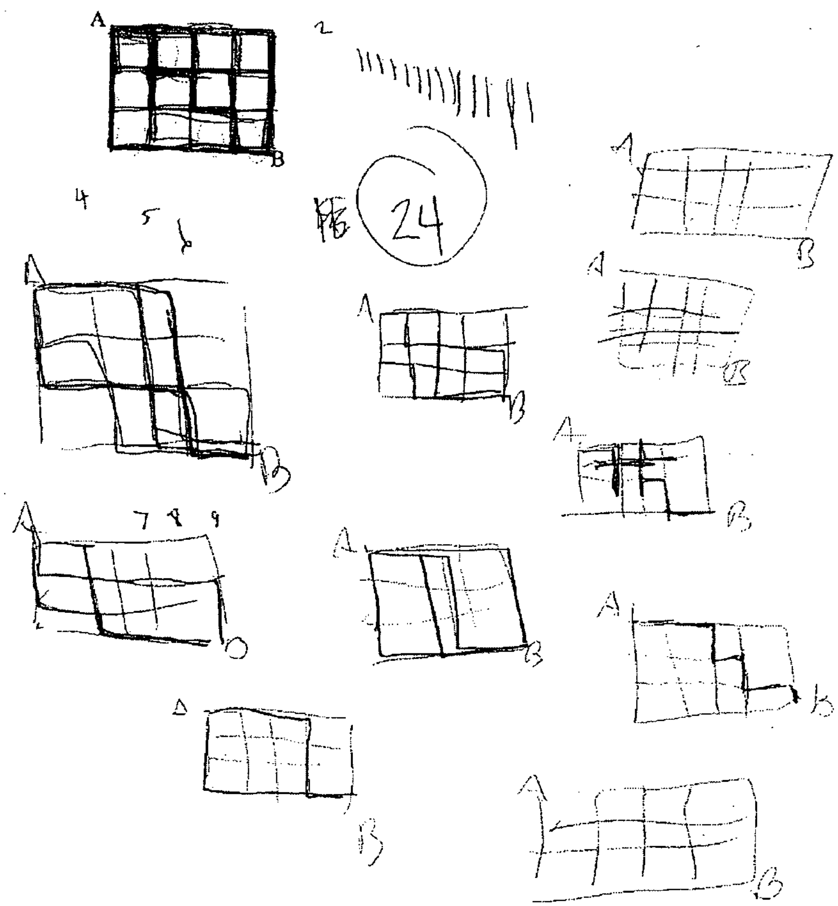
This exercise was:

easy		ok		difficult
1	2	3	4	5

Counting Puzzles 2: One way through city blocks

Student's code number: B   Date:

How many different paths lead from A to B when the only possible moves are the ones going down or to the right?



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State your thinking when solving the problem.
Try to explain how you approached the problem and what you thought about while you worked on it:

1. Describe how you approached the problem and worked at solving it?
2. Which ways did you consider but did not use in the end
3. How confident are you that you have found a correct answer?

1. guessing
drawing as many path is possible
↳ for each box, there are two ways to go, either right or ~~left~~ down so 12×2 = 24

3. not at all

This exercise was:

easy		ok		difficult
1	2	3	4	5

Counting Puzzles 3: Partition

Student's code number: R Date:

A boy has four different coloured toy cars (black, orange, red and grey) and he decide to give away the cars to his friends Peggy, John and Linda. In how many different ways can he distribute the toy cars? For example he could give all cars to Linda.

$$4!\,3! = 144$$

State your thinking when solving the problem.
Try to explain how you approached the problem and what you thought about while you worked on it:

1. Describe how you approached the problem and worked at solving it?
2. Which ways did you consider but did not use in the end
3. How confident are you that you have found a correct answer?

1) First I tried remembering how to use nCr and nPr and played around with my calculator.

2) nPr and nCr

3) → Not very confident
→ I guessed

This exercise was:

easy		ok		difficult
1	2	3	4	5

Appendix B

Excerpts from Mr Cho's handout

Unit 7 – Permutations and Combinations (Counting Techniques)

Fundamental Counting Principle

In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so on, until nth has k_n, the total number of possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdot \cdots \cdot k_n$$

e.g. A paint manufacturer wishes to manufacture several different paints. The categories include

Colour:	Red, Blue, White, Black, Green, Yellow
Type:	Latex, Oil
Texture:	Flat, Semigloss, High Gloss
Use:	Outdoor, Indoor

How many different kinds of paint can be made if a person can select on colour, one type, one texture, and one use? (72)

e.g. There are four blood types, A, B, AB, and O. Blood can also be Rh+ and Rh-. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labeled? (16)

* e.g. John is planning to drive from Vancouver to Winnipeg via Calgary. There are three roads from Vancouver to Calgary and two roads from Calgary to Winnipeg. How many different "round-trip" routs are there from Vancouver to Winnipeg, passing through Calgary, if no road is used more than once? (12)

Permutation

The arrangement of n objects in a specific **order** using r objects at a time is called a permutation of n objects taking r objects at a time. The is written as ${}_nP_r$, and the formula is

$${}_nP_r = \frac{n!}{(n-r)!}$$

e.g. ${}_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$

e.g. In how many ways can 3 desks be filled from amongst 10 students? (750) 720

e.g. How many different ways can a chairperson and an assistant chairperson be selected for a research project if there are seven scientists available? (42)

e.g. There are 10 different books. How many ways can 4 of these books be arranged on a shelf? (5040)

e.g. John and Tom invited four other people to sit on their bench. In how many ways can these six people be seated on this bench if:
(a) there are no restrictions. (720)

(b) John is seated at the left and Tom is seated at the right end. (24)

e.g. How many 3-letter permutations can be formed from the letters of the word CLARINET? (336)

Combination:

A selection of distinct objects without regard to order is called a combination, which means order is not important in the selecting process.

e.g. Given the letters A, B, C, and D, list the permutations and combinations for selecting two letters.

Combinations of r Objects Taken From n Distinct Objects:

The notation ${}_nC_r$ is used for the number of combinations of r objects taken from n distinct objects.

$$\begin{aligned}{}_nC_r &= \frac{{}_nP_r}{r!} \\ &= \frac{\frac{n!}{(n-r)!}}{r!} \\ &= \frac{n!}{(n-r)!r!}\end{aligned}$$

e.g. ${}_{16}C_3 = \frac{{}_{16}P_3}{3!} = \frac{\frac{16!}{(16-3)!}}{3!} = \frac{16!}{(16-3)!3!} = 560$

e.g. Determine the number of possible lottery tickets that can be created in 6/49 lottery where each ticket has six different numbers, in no particular order, chosen from the numbers 1 through 49 inclusive. (13983816)

Appendix C

Excerpts from Mr Cho's end-of unit test, with key

Math 12 – Test on Combinatorics

Date: __________ Class: __________ Name: Key

PART A: Multiple Choice (1.5 marks for each question)

1. When you play lotto 5-30, you must choose 5 different integers from 1 to 30. How many combinations are possible?

A. $\frac{30!}{5!25!}$

B. $\frac{30!}{25!}$

C. $25!$

D. $\frac{30!}{5!}$

$_{30}C_5$ $\quad \frac{30!}{25!5!}$

2. Determine the 4^{th} term in the expansion of $(x-2y)^5$.

A. $-80x^2y^3$

B. $-40x^3y^2$

C. $40x^3y^2$

D. $80x^2y^3$

$n=5 \quad k=3$

$_5C_3\, x^2(-2y)^3$

3. Determine the number of different arrangements of all the letters in APPLEPIE.

A. 3 360

B. 6 720

C. 40 312

D. 40 320

$\frac{8!}{3!2!}$

4. How many different pasta meals can be made from 4 choices of pasta and 2 choices of sauces, if only one pasta and one sauce is selected for each meal?

A. 4

B. 6

C. 8

D. 16

~~4~~ $4 \cdot 2$

3. A toy box contains 4 different cars and 6 different trucks.

a) In how many ways can a collection of 5 toys be chosen if the collection must consist of 2 cars and 3 trucks? **(2 marks)**

$${}_4C_2 \; {}_6C_3 = 120$$

b) In how many ways can a collection of 5 toys be chosen if the collection must consist of at least ~~3 cars~~? **(2 marks)**

$$\cancel{{}_{10}C_5 - {}_4C_0 \; {}_6C_5 = 246}.$$

$${}_4C_3 \; {}_6C_2 + {}_4C_4 \; {}_6C_1 = 66$$

4. A class has 30 students.

a) How many ways can a committee of 3 people be selected from the class? **(2 marks)**

$${}_{30}C_3 = 4060$$

b) How many ways can an executive committee consisting of 3 people (president, vice-president, secretary) be selected from the class? **(1 mark)**

$${}_{30}P_3 = 24360$$

c) If there are 10 boys and 20 girls in the class, how many ways can a committee of 3 people be selected from the class if the committee must contain 1 boy and 2 girls? **(1 mark)**

$${}_{10}C_1 \; {}_{20}C_2 = 1900$$

Appendix D

The misaddressed letter problem as posed during a group session, with Nick's work

Please write your name on the back: 1

g … s …

Someone writes n letters and writes the corresponding addresses on n envelopes. How many different ways are there of placing all the letters in the wrong envelopes?

n_l

n_e

5×5 — 1

6×6

$n^2 - 1$

Appendix E

Excerpts from the field notes

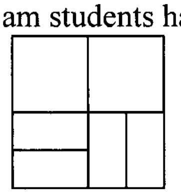
MA 3 Field notes

Thu 8 Dec 05

- [Nick] seems to have trouble when the factorial stops…
- When I started focusing on [student E], [Nick] had a question to [Mr Cho] in the whole class session → problem of focus on one thing: lose something else.
- [student E] work on her own (exercise in the brochure) still (time to time) keep in touch where [Mr Cho] is in the course
- [Mr Cho] does the exercise on the board, and asks some questions to the students
  - The students barely have time to do the exercise by themselves.
  - [Mr Cho] ask a Q but give the answer directly afterwards (1sec at most, and he already started to write the answer) → the teaching goes fast but what about the students

MA 10 Field notes

Tue 10 Jan 06

- Change of plan: [Mr Cho] whispers in my ear at the beginning of the course that he is not doing the quiz
- [Mr Cho]: 'did you work during the holidays?'
- meta Q: what learned before the holidays (10 min)
- a girl: 'Oh, I do not like that',
  then later: 'I think I passed this test' (front row near the wall)
- some students ask: 'what do you mean by most important' (→Q is badly worded)
  some others say: 'they're all important'
- [Mr Cho] does some housekeeping: (test on the 19th)
  mid year exam is soon and is worth 30% of this tern (10% of the year)
- **10am YM starts with Pascal's triangle**
- 2 ways of drawing:
  1. same as usually done
  2. the one I showed the day before [Ma 9]
- most students are focused on [Mr Cho] & board
- 10.15am students have to do an exercise on their own
  note:  **it's a good exercise (good to see how Pascal's triangle works)**
- 10.30am, All students turn the page at the same time: they are (closely) following
- N-1 = n: I have the impression that [Mr Cho] spent much time with (-1), which is a technicality (and instrumental) I have to admit that it is confusing and a source of mistake)

Appendix F

Excerpts from the video logs

07.33	Yvan	circle each poss (it contains Sat and Sun outcome) and tick(3) or cross(1) if it does rain thinks for a little while I am confident the answer is 50% and not 75	←conflict→...
	Nick	No, here you are assuming that (stops speaking) ... Ok so either this happen or that ...	
	Yvan	I know it should be 50%	←conflict is quite deeply inbedded
	Nick	By common sense	
		... both are thinking silently	
	T	So what kind of weather can happen during this weekend?	
08.17	Yvan	Na na na, there is a 75%	
	Nick	<?>	
	Yvan	yeah	
	Nick	why	← Nick play his role of making Yvan speak
	Yvan	because... I was looking at the wrong way I was looking at raining both day instead of just this weekend. In order to satisfy just weekend it can rain for one day and not the other the only way to have no rain, , if this comes true and this comes true (pointing at Sat/50% and then at Sun/50%), no rain on either day, there is a 50% chance and another 50% chance and the odds ??? consecutively	concept of True→ good command of prob
	Nick	Ok	
08.55	Yvan	- 25% chance... 1 over 2 times 1 over... write 1/2x1/2=1/4 (1/2 prob on 1st day) So 1 over 4 of neither being true in the end 3 remaining so 75% chance it will rain. - I was wrong because I looked at the whole weekend not just one day So half... (explain one more time) → ¼ →3/4	
	T	... combinatorics... 2x2=4 poss	
	Yvan	yeah...	Yvan seems to be slightly embarrassed that he was confused by such a simple problem
	T	let's go to the next problem	
	Nick	sure	
	T	interest in prob?	
	Yvan	play poker and blackjack try to figure out the actual probabilities.. getting one hand or then other hand... and I learned from that.	
	T	I was impressed next problem the same: read, try alone and..	

Appendix G

Excerpts of the breakdown of the data: the pathway problem in the pre-test

■ = student with previous experience of combi

	answer	method	try-outs/graph	method considered but not used	difficulty
A	32	2^5		counting them	4
B	24	12square, 2 paths around each square (L or R)→12x2	draw several paths on grids	-	-
C	48	12squares x 4sides (doubt about answer)	draw a few paths on the figure	-	4
D	35	Pascal's triangle	very good justification (seems to have it worked back)■	counting	2
E	24	4P3 (formula despite forgot it■)		counting	3
F	absent				
G	35	Pascal's triangle (taught by YM)	4!x3!	Permutations, Combinations, 'but I forgot'■	-
H	144	12 blocks, mirror effect of each option→12^2	some counting on the figure ???	-	4
I	35	'I just counted'	seems to have done some Pascal's triangle (row1 ok, row2 & 3 ??) ?? draw some paths (in the air)	'none'	3
J	16128	2x4x6x7x6x4x2 (each number is the sum of paths/side of squares when figure redrawn in pyramidal fashion)■	tree ???	counting	3
K	35	Pascal's triangle	justification: ok■	counting	2
L	absent				
M	-		Pascal's triangle done OUT of the figure	do each possibility by hand	5
N	14	strange counting of sides of squares■	Right+Down & D+R paths composed of 7 'moves' each, 7x7=49	-	5
O	35	only wrote down 4,10,20,35 beside the grid, & draw path	draw some paths on the figure	no	3
P	24	plugged number of intersection in the probability function of the calculator■	ticks on the edge of the grid	counting■	3

■ E

1. I put in formula (even though kind of forgot already).

■ G

Permutation / Combination, but i forgot!

Appendix H

Meta-cognitive prompts asking what students had learnt, with student K's answer

K

Please write your name on the back.
January 10th

1. **What are the important ideas or concepts that you learned during the two last weeks of class (before the Holiday break)?**
2. **Explain the most important one?**
3. **Is there something else that you find important to mention?**

1) Pascal Triangle
Permutation - Alike, different
Combination

2) Permutation = is 'in' order $_nP_r$
Combination = randomly select $_nC_r$
Pascal = add up sides to find x

3) No.

------------------------------PLEASE KEEP FOR YOUR FILES------------------------------

Please check the box indicating your decision:

☐	I CONSENT to participating in the *Probing students' thinking when introduced to combinatorics theory* project as described in the above form.
☐	I acknowledge that I have received a copy of this consent form for my own files.

Name (print) ____________________ Date:__________ [redacted]

Phone: ____________________ e-mail: ____________________

----------------DETACH CONSENT SLIP AND RETURN TO Thomas PERRIN----------------

Please check the box indicating your decision:

☐	I CONSENT to participating in the *Probing students' thinking when introduced to combinatorics theory* project as described in the above form.
☐	I acknowledge that I have received a copy of this consent form for my own files.

Name (print) ____________________ Date:__________ [redacted]

Phone: ____________________ e-mail: ____________________

------------------------PLEASE KEEP FOR YOUR FILES------------------------------

Consent: By signing this consent form I understand that my child's participation in this study is entirely voluntary and that she/he may refuse to participate or withdraw from the study at any time without jeopardy to her/his class standing, grades, or relationship with the school.

Please check the box indicating your decision:

☐	I CONSENT to my child's participation in the study activities described above in the form that will take place **during class time**, and if chosen, I CONSENT to my child's **voluntary** participation in the study the problem-solving sessions described above in the form that will take place **outside class time**.
☐	I DO NOT CONSENT to my child's participation in the study as described in the form.

☐	I acknowledge that I have received a copy of this consent form for my own files.

[redacted], mathematics class

Name of student (please print) ____________________ Date:____________

Signature of parent/guardian______________________________

--------------DETACH CONSENT SLIP AND RETURN TO Thomas PERRIN--------------

Consent: By signing this consent form I understand that my child's participation in this study is entirely voluntary and that she/he may refuse to participate or withdraw from the study at any time without jeopardy to her/his class standing, grades, or relationship with the school.

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☐	I DO NOT CONSENT to my child's participation in the study as described in the form.

☐	I acknowledge that I have received a copy of this consent form for my own files.

[redacted], mathematics class

Name of student (please print) ____________________ Date:____________

Signature of parent/guardian______________________________

-------------------------PLEASE KEEP FOR YOUR FILES-------------------------

Assent: By signing this assent form I understand that my participation in this study is entirely voluntary and that I may refuse to participate or withdraw from the study at any time without jeopardy to my class standing, grades, or relationship with the school.

Please check the box indicating your decision:

☐	I assent (**I say "yes"**) to my participation in the study activities described above in the form. I understand that my participation in the problem-solving sessions that will take place **outside class time** is **voluntary**. I understand the nature of my participation in this project. With my assent I acknowledge receiving a copy of the study information.
☐	I DO NOT assent (**I say "no"**) to my participation in the study as described in the form.

[redacted], mathematics class

Name of student (please print) ______________________ Date:____________

Signature______________________________

-------------DETACH CONSENT SLIP AND RETURN TO Thomas PERRIN-------------

Assent: By signing this assent form I understand that my participation in this study is entirely voluntary and that I may refuse to participate or withdraw from the study at any time without jeopardy to my class standing, grades, or relationship with the school.

Please check the box indicating your decision:

☐	I assent (**I say "yes"**) to my participation in the study activities described above in the form. I understand that my participation in the problem-solving sessions that will take place **outside class time** is **voluntary**. I understand the nature of my participation in this project. With my assent I acknowledge receiving a copy of the study information.
☐	I DO NOT assent (**I say "no"**) to my participation in the study as described in the form.

[redacted], mathematics class

Name of student (please print) ______________________ Date:____________

Signature______________________________