

**ESSAYS ON PRICE DISCRIMINATION IN IMPERFECTLY
COMPETITIVE MARKETS**

by

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Abstract

This thesis aims at a theoretical study of price discrimination in imperfectly competitive markets and will address such issues as the viability of price discrimination when markets are oligopolistic rather than monopolistic. More specifically, it seeks to determine what effects the introduction of competition to a monopoly market will have on a firm's ability to price discriminate. Also examined is price discrimination in the presence of collusion among consumers, as well as the connection between a monopolist's linear pricing and resale among consumers.

The first paper is a version of an article published in the *Canadian Journal of Economics* and examines a simple two stage game in which firms compete in prices by a chosen pricing instrument. The instruments considered include a simple uniform pricing technology and a promotional pricing technology in the form of an advertised discount coupon. Consumers are separated by types, informed and uninformed. Therefore, a motive for price competition exists for the purpose of separating between the two types of consumers. It is shown that the sustainability of an asymmetric choice of pricing instrument between the two firms will prevail in a duopoly market in equilibrium. Consequently, the coexistence of two different pricing schemes is viable even when firms are symmetric in all other respects. In light of this finding, justification may be established for the observed differences in pricing strategies in markets today.

The second paper considers a model of homogeneous good Bertrand competition with asymmetric information. Consumers differ in a unidimensional type space and are grouped exogenously as either loyal or bargain-hunting. It is found that in such an asymmetric information environment, the equilibrium is unique and exists in mixed strategies. Furthermore, firms will compete via nonlinear price schedules similar to that of a monopolist to the extent that a "flattening out" effect takes place. More importantly, competition occurs in the reservation utility of the marginal consumer by use of pseudo-incentive compatible direct mechanisms.

Lastly, the third paper examines the standard nonlinear pricing model where we introduce a set of consumers who may act cooperatively in a pairwise manner (colluders). We model collusion by introducing a third party whose objective is to maximize the sum of utilities for any colluding pair. We assume that this third party is perfectly informed regarding the private information of the consumers. We find that in this framework, a multidimensional screening problem, due to the two dimensional private information held by a colluding pair, arises. We introduce a reduction method by which the representation of this multidimensionality may be suppressed. A sufficient condition for its existence is then characterized. This allows us to provide an equivalent representation which coincides with the standard nonlinear pricing model. Generalizations about asymmetric coalition sizes are then made possible.

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Chapter 1

Introduction

One of the major concerns of competition policy is the act of price discrimination. The pricing strategies which discriminate between consumers of different identities provide an edge for firms over consumers. This transfer of welfare is often undesirable. The current theory of price discrimination in oligopoly markets is well developed. However, the majority of the work lacks applicability due to the assumed symmetry in the models. In particular, the pricing strategies across firms in the same market are assumed to be symmetric if the firms themselves are symmetric. This assumption allows the theory to be developed with much simplicity, which, however desirable, is insufficient for the purpose of policy formulation. I intend to explore the area of price discrimination in oligopoly markets without making symmetric pricing strategies an exogenous assumption.

The analysis begins with Chapter 2, a version of which is published in the Canadian Journal of Economics, with a study on an oligopoly market where firms have a choice over the pricing strategies employed. It argues that the equilibrium outcome will not be a symmetric choice in pricing strategies as previous theoretical work has suggested (Butters (1977), Hallagan and Joerding (1985), Rosenthal (1980), Salop and Stiglitz (1977 and 1982), and Varian (1980)). Rather, asymmetric pricing strategies better define such a market environment. The theoretical results of this paper do help explain some of the casual empiricism that we observe on a daily basis. For example, the firm Wal-mart advertises with the slogan "Everyday low prices" while its competitors generally offer sales and coupons as a means to attract consumers. The findings also indicate that the adoption of different pricing strategies by symmetric firms is a form of collusion. This, in and of itself, has many policy implications which I hope to explore in my post-doctoral research.

An immediate question that arises from the results of Chapter 2 is whether the proposed equilibrium is robust with respect to a more generalized heterogeneity among consumers. This question is explored in Chapter 3, entitled "Loyal Consumers and Bargain Hunters: Price Competition with Asymmetric Information." This chapter analyzes the price competition between two identical firms when there exists asymmetric information such that the individual firms cannot observe the "type" of consumer they may be selling to. The literature on such pricing behavior in monopoly markets is rather substantial (Goldman, Leland, and Sibley (1984), Maskin and Riley (1984), Guesnerie and Laffont (1984) Laffont and Martimort (2002)), but relatively little has been said on the subject when strategic competition exists in such markets (Oren, Smith and Wilson (1983 and 1984), Hamilton and Thisse (1997), and Stole (1995)). A predominant part of the literature on competing nonlinear pricing includes a spatial dimension implying product differentiation. Therefore, the findings from this paper differs from past studies as perfectly homogeneous goods are considered. The results suggest that if firms have an existing stock of consumers for whom they do not compete in the market, then the pricing strategies will be similar to that of the monopolist. The only difference is that the actual prices will be randomized in such a way that profits from the stock of consumers they actually compete for, approach zero. Since nonlinear pricing is a form of price discrimination, the results suggest that when competition prevails, nonlinear pricing may not necessarily be undesirable. Since firms cannot differentiate between consumers, (i.e., those they compete for versus those they do not), the price, on average, decreases.

This paper also seeks to explain the observed pricing strategies of the cellular phone industry. The prices for the cellular phones seem to change on a random basis with no particular pattern involved. In other words, it is not surprising to find the same phone being sold for \$99 today and being offered free next week. By the same token, the reverse is also true. A phone which is free today may not be so next week.

The particular structure of the equilibrium derived in Chapter 3 hinges heavily on a common assumption adopted in the nonlinear pricing literature. That is, consumers must

not be able to cooperate or collude with one another. This assumption, while strong and simple contradicts the very behavior of consumers in today's market. For example, if large savings due to quantity discounts exist, a consumer may very well team up with other consumers so as to reap the benefits of such discounts. In fact, the very existence of buying groups in today's market serves as anecdotal evidence for this claim. As a result, Chapter 4 reverts our attention back to the standard nonlinear pricing model in a monopoly framework and addresses the question regarding the sustainability and desirability of price discrimination, or nonlinear pricing, in a world where consumers may collude.

The model presented in Chapter 4 which tries to encapsulate collusion among consumers suggests that a multidimensional screening problem, due to the two dimensional private information held by a colluding pair, arises. Hence, a reduction method by which the representation of this multidimensionality may be suppressed is introduced. A sufficient condition for its existence is then characterized. This allows us to provide an equivalent representation which coincides with the standard nonlinear pricing model. Generalizations about asymmetric coalition sizes are then made possible. Not surprisingly, we find that a monopolist is strictly worse off when consumers collude under perfect information. However, it is also shown that nonlinear pricing will still be desirable as the amount of resale among different "types" of consumers differentiate different groups of consumers from the perspective of the firm. Hence, the firm will wish to exercise price discrimination along this dimension.

Chapter 2

Coupons and “Everyday Low Prices”: Price Competition with Multiple Instruments¹

2.2 Introduction

Current theories of sales and equilibrium price dispersion models tend to focus on the sustainability of equilibrium in both oligopolistically and monopolistically competitive markets. As instructive as these models may be, the assumed symmetry of the equilibrium strategies in fact hampers the applicability of such theories. For example, it is not an uncommon phenomenon that firms of almost identical nature to adopt different, even polar extreme, strategies when engaged in competition. Therefore, a model seeking to explain divergent pricing behaviors is needed.

For example, Varian (1980) shows a price dispersion equilibrium in a monopolistically competitive market in which firms choose to vary their prices over time. The market, in his model, is assumed to be comprised of informed and uninformed consumers. Informed consumers know the exact distribution of prices at any one point in time and therefore, only purchase from the firm(s) offering the lowest price. Uninformed consumers, on the other hand, purchase from a random store. Because of the asymmetry in consumers, Varian shows that only a mixed strategy equilibrium exists which is taken to imply the randomization of prices through repeated play.

A more general representation of Varian’s model may be found in a study by Rosenthal (1980). Instead of assuming a 0-1 demand structure, he projects a similar equilibrium with a continuous demand function. The main result from his study is a rather counterintuitive idea; an unambiguous increase in price when the number of firms increases. A firm

¹A version of this chapter has been published. Kwong, Lester M.K. (2003) Coupons and ‘Everyday Low Prices’: Price Competition with Multiple Instruments. Canadian Journal of Economics. 36:443-62.

is assumed to sell in two markets, one in which it has monopoly power and the other in which competition exists. This parallels Varian's model if one takes the monopoly market to compromise uninformed consumers and competition to target informed consumers. Rosenthal then shows that as the number of competitors in the common market increases, the support of the equilibrium strategy remains constant while the changes in the equilibrium distribution of prices due to the increase in competition generate a higher expected market price. This result is driven solely by the assumption that the price to be offered in either market to be the same for all firms. The simultaneity and symmetry of his model effectively yields a mixed strategy Nash equilibrium much like the one derived by Varian.

Rosenthal's findings is largely due to the assumption that each firm is faced with two markets; one in which monopoly power may be exercised and one in which competition exists. The intuition behind the rise in price as a result of an increase in the number of firms is due to firms increasing its equilibrium distribution in response to the number of competitors. As the number of competitors increases, the probability of successfully charging the lowest price in the competitive market falls. As a result, firms will then have a tendency to forgo competition and extract profits from the monopoly market. In contrast to Varian's model, if one assumes that the number of uninformed consumers grows proportionally to the number of firms competing in the market, such a counterintuitive idea also results.

The above models considered may be thought of as a particular class of models within the promotional pricing literature. More specifically, they correspond to price dispersion models where a single price is offered by a firm at any point in time. However, it is not uncommon for firms to be faced with a set of pricing instruments. Therefore, if we restrict our attention only on uniform pricing equilibria, we may sometimes hamper our insight into a commonly observed phenomenon. We propose to develop a simple model in which firms simultaneously choose from a set of pricing instruments. The instruments that we will consider include uniform pricing and coupon pricing. The distinguishing feature between sales models and coupons models is that in the latter, a firm effectively offers two

different prices for the same good. Naturally, one wonders whether the proposed symmetric equilibrium of the sales models will hold when coupons are also available as a pricing instrument. The assumed symmetric equilibrium is one feature common to the models mentioned above. Such an assumption is generally regarded as being acceptable due to the simplified analysis. Furthermore, the existence of multiple asymmetric equilibria makes equilibrium selection difficult and perhaps, *ad hoc*. However, by imposing a symmetric equilibrium when multiple pricing instruments are available will be misleading, as will be seen.

As an example, the S.S. Kresge company, now known as Kmart, was established over a century ago. In the face of competition around in 1962, when Wal-mart opened its first store, Kmart launched a newspaper advertising program to entice shoppers to its stores. To this day, Kmart is still the leading print promotional retailer while Wal-mart's signature slogan "Everyday low prices" has made it a household name. The two firms' pricing strategies are clearly different since Kmart has operated on a promotional pricing strategy while Wal-mart has by its slogan committed itself to a somewhat uniform pricing strategy. By no means conclusive, this example illustrates instances in which different pricing strategies are adopted by firms of similar nature and scale competing in the same market.

This paper, while building on these earlier studies, differs in two ways; (i) An oligopolistic type market structure is adopted in place of the monopolistically competitive structure. This leads to a much more strategically competitive framework since positive profits may prevail even in a long-run equilibrium. (ii) No assumptions are made *ex ante* regarding the pricing instrument that any one firm will employ (in our case, it is presented as a simple choice between the two). In other words, we depart from common practice by not restricting our analysis to symmetric equilibria.

The remainder of this paper is organized as follows: Section 2 presents the model, and section 3 extends the model developed in section 2 by expanding the game to an arbitrary N number of firms. Finally, some concluding remarks will follow.

2.3 The Model

Consider the following game. There are two firms, 1 and 2. In the first stage, each firm simultaneously chooses to adopt a pricing instrument, either uniform or coupon. It is assumed that each firm is committed to the pricing instrument they have chosen in stage one. In stage two, after observing the choice made by each firm, firms compete by choosing their pricing strategy. A pricing strategy, here, is interpreted to be the prices offered in the market. For a firm using a uniform pricing instrument, this will be the choice of $p \in \mathbb{R}_+$. A firm using a coupon pricing instrument, on the other hand, will choose the pair $(p, d) \in \mathbb{R}_+^2$, where d denotes the discount from the coupon. The two firms are assumed to be homogeneous risk neutral firms producing a homogeneous product at a constant marginal cost c .

We assume that there are two types of consumers in the market, informed and uninformed, with a demand for one unit of some good, if and only if, the price, p , is less than or equal to their reservation price, r . Let the number of informed consumers be M and uninformed consumers be \mathcal{U} . Uninformed consumers are assumed to be distributed uniformly among the number of firms in the market; thus, their decision rule is simply to visit a firm, say firm j , and purchase one unit of the good if the observed price, $p_j \leq r$. Otherwise, they do not purchase the good at all. Informed consumers, on the other hand, know the exact price charged by each firm and purchase from the firm with the lowest price. If two firms charge the same lowest price, then we assume that the informed consumers are indifferent and for expositional simplicity, we assume that this group of consumers will be evenly distributed among those firms with the same lowest price in the market.¹

Given this formulation, it is clear that each firm is guaranteed $U = \frac{\mathcal{U}}{2}$ number of consumers in each period. We assume that the size of the market is common knowledge

¹One way to think about this setting, as Varian (1980) has pointed out, is to assume that informed consumers read the newspaper while uninformed ones do not. Alternatively, as one referee pointed out, informed consumers may be interpreted as those with a lot of free time (poor ones) while uninformed are those without (rich ones). If one grants the validity of this assumption, then the simplification can be justified.

	U_2	C_2
U_1	O_1	O_2
C_1	O_3	O_4

Figure 2.1: Payoff Matrix

among the firms, as is the distribution of informed and uninformed consumers. Firms also know the reservation price of the consumers, r ; thus, there are no uncertainties regarding the environment in which they operate.

Since actions by each firm in the first stage are observable, the solution to this game is determined by the equilibria of the subgames from stage two. Figure 2.1 shows the payoff matrix of the game where O_i denotes the payoffs from the subgames of stage two. Furthermore, U_i and C_i denotes the chosen pricing instrument by firm i where U_i and C_i denotes uniform and coupon, respectively. We begin first by determining the payoffs when both firms adopt the uniform pricing instrument. Correspondingly, this is O_1 in Figure 2.1.

2.3.1 Symmetric Uniform Pricing Equilibrium (O_1)

In this subgame, firms choose a uniform price so as to maximize expected profits. The profit function for this subgame, for firm i , can be characterized as:

$$\pi_{O_1,i}(p_i, p_j) = \begin{cases} (p_i - c)(M + U) & \text{if } p_i < p_j \\ (p_i - c)\left(\frac{M}{2} + U\right) & \text{if } p_i = p_j \\ (p_i - c)(U) & \text{if } p_i > p_j \end{cases} \quad (2.1)$$

for all $i \neq j$ and $i, j \in \{1, 2\}$. A pure strategy, in this subgame, will be given by $p \in \mathbb{R}_+$ while a mixed strategy is a randomization over some set $P \subset \mathbb{R}_+$ according to some distribution function denoted by $F_i^u(p_i)$ where the index u denotes the uniform pricing distribution function. We assume F_i^u is continuously differentiable and that f_i^u denotes the price density function.²

²The assumption of a continuously differentiable price distribution function is asserted with a proof that

Given this formulation, the objective for each firm can be expressed as:

$$\max_{f_i^u(p_i)} \int_{\underline{p}}^r \left((1 - F_j^u(p_i)) \pi_{O_1,i}^s + (F_j^u(p_i)) \pi_{O_1,i}^f \right) f_i^u(p_i) dp_i \quad (2.2)$$

subject to the constraints:

$$\int_{\underline{p}}^r f_i^u(p_i) dp_i = 1; f_i^u(p_i) \geq 0 \text{ for all } p_i \in \text{Supp}(f_i^u) \quad (2.3)$$

$$(1 - F_j^u(p_i)) \pi_{O_1,i}^s + (F_j^u(p_i)) \pi_{O_1,i}^f \geq (r - c) U \text{ for all } p_i \in \text{Supp}(f_i^u) \quad (2.4)$$

where $\text{Supp}(f_i^u)$ denotes the support of the price density function, $\pi_{O_1,i}^s$ are profits for firm i in the case where it successfully advertises the lowest price with probability $1 - F_j^u(p_i)$ and $\pi_{O_1,i}^f$ are profits from failing to do so with probability $F_j^u(p_i)$.³ In other words:

$$\pi_{O_1,i}^s(p_i) = (p_i - c)(M + U) \quad (2.5)$$

and

$$\pi_{O_1,i}^f(p_i) = (p_i - c)(U) \quad (2.6)$$

Furthermore, \underline{p} in (2.3) is the lower bound of the equilibrium support of prices. We prove the existence of this lower bound formally in Proposition 1. Note that (2.4) can be thought of as an individual rationality constraint where firm i by randomizing with $f_i^u(p_i)$ must generate as much profits as charging the reservation price and only attracting uninformed consumers. The solution to the mixed strategy equilibrium yields the equilibrium distribution function⁴ as:

$$F_i^u(p_i) = 1 - \frac{(r - p_i) U}{(p_i - c) M} \quad (2.7)$$

there does not exist any mass points in prices and that all prices are charged with positive density in the support of the equilibrium price density function. This is shown in the appendix.

³Note that the objective for each firm is stated in the form of an optimization problem. The solution, however, may be obtained easily by exploiting the equilibrium indifference condition for any particular firm. For the remainder of this paper, the solution to any objectives of this nature is obtained in a similar fashion.

⁴We prove this formally in Proposition 1.

Taking the derivative of the equilibrium price distribution function⁵ yields the equilibrium density of:

$$f_i^u(p_i) = \frac{(r - c)U}{(p_i - c)^2 M} \quad (2.8)$$

which is clearly non-negative. Expected profits for any firm using $f_i^u(p_i)$ can be represented as:

$$\begin{aligned} \mathbb{E}\{\pi_{O_1,i}(F^u(p))\} &= \int_{\underline{p}}^r \left((1 - F_j^u(p_i)) \pi_{O_1,i}^s + (F_j^u(p_i)) \pi_{O_1,i}^f \right) f_i^u(p_i) dp_i \\ &= (r - c)U \end{aligned} \quad (2.9)$$

for all $i \in \{1, 2\}$.

Proposition 1 *There exists a unique equilibrium in the subgame where both firms use the uniform pricing instrument. Furthermore, the equilibrium is in mixed strategies whereby both firms employ the distribution function as given by (2.7) over the region $[\underline{p}, r]$ where:*

$$\underline{p} = \frac{cM + rU}{M + U} \quad (2.10)$$

Proof: See appendix.

The intuition for Proposition 1 follows from the idea of Bertrand competition with a positive outside option. Firms would like to undercut price in order to capture the group of informed consumers. However, there exists a critical price, namely \underline{p} , such that undercutting is no longer profitable. Furthermore, reversion to forgoing competition for the uninformed consumers market is more profitable than having both firms set price at \underline{p} . Consequently, firms will randomize price according to the equilibrium distribution function given by (2.7).

2.3.2 Symmetric Coupon Pricing Equilibrium (O_4)

We now analyze the subgame where both firms utilize the coupon pricing instrument. This corresponds to O_4 in Figure 2.1. Firms issue coupons from which informed consumers can

⁵Note that this differs from Varian's model by the assumption of a fixed number of firms rather than a free-entry monopolistically competitive market in which profits are driven to zero. Consequently, we have imposed the individual rationality constraint. In all other respects, the model developed so far is identical to Varian's.

learn about the lowest effective purchase price and buy from the store offering it. Let d_i be the discount offered by firm i . Under this environment, a firm's pure strategy space is now the pair $(p_i, d_i) \in \mathbb{R}_+ \times \mathbb{R}_+$. The profit function under this environment is given by:

$$\pi_{O_4,i}(p_i, d_i, p_j, d_j) = \begin{cases} (p_i - c)U + (p_i - d_i - c)M & \text{if } p_i - d_i < p_j - d_j \\ (p_i - c)U + (p_i - d_i - c)\frac{M}{2} & \text{if } p_i - d_i = p_j - d_j \\ (p_i - c)U & \text{if } p_i - d_i > p_j - d_j \end{cases} \quad (2.11)$$

Proposition 2 *There exists a unique pure strategy equilibrium in the subgame where both firms use the coupon pricing instrument. More specifically, $p_i^* = r$ and $d_i^* = r - c$ for $i = 1, 2$.*

Proof: We begin by first proving that a necessary condition for profit maximization in a coupon pricing equilibrium is $p_i = r$ for all $i \in \{1, 2\}$.

Suppose firm i offers the following schedule (p_i, d_i) such that $p_i > r$. Then simply lowering price to $p'_i = r$ and the discount to $d'_i = d_i - (p_i - r)$ will increase profits. Suppose $p_i < r$ and d_i is the discount. Then the maximum profits the firm can earn, given p_j and d_j , $j \neq i$, are $\pi_{O_4,i} = (p_i - c)U + (p_i - d_i - c)M$. Suppose price increases to $p'_i = r$ and the discount changes to $d'_i = d_i - (p_i - r)$. Then profits in this case are $\pi'_{O_4,i} = (r - c)U + (r - d'_i - c)M$. Since $r - d'_i = p_i - d_i$ and that $r > p_i$, it is obvious that $\pi'_{O_4,i} > \pi_{O_4,i}$.

With $p^* = (r, r)$ established, let (p^*, d^*) be the equilibrium prices and discounts offered. Since in any equilibrium, $p_i^* = r$, competition between the two firms must occur in d_i . Therefore, this is simply a homogeneous good Bertrand game and hence, the proof is omitted. ■

Note the similarity between this game and the usual Bertrand price competition with homogeneous goods. Furthermore, expected profits under this symmetric coupon pricing equilibrium are:

$$\mathbb{E}\{\pi_{O_4,i}(p, d)\} = (r - c)U \quad (2.12)$$

for all $i \in \{1, 2\}$.

Comparing the expected profits from the coupon pricing equilibrium in (2.12) to those of the uniform pricing equilibrium in (2.9) reveals an interesting feature. The expected profits under both subgames are equivalent. However, in the symmetric uniform pricing equilibrium, equilibrium is unique and achieved through mixed strategies. In the symmetric coupon pricing equilibrium, on the other hand, equilibrium is obtained through a unique pure strategy. This seems to suggest that the availability of a more complex pricing instrument does not necessarily lead to a more profitable environment, *ex ante*. Essentially, the introduction of the coupon instrument enables firms to compete more vigorously in prices simply because doing so is not as “harmful” in the sense that they are guaranteed a profit of $(r - c)U$. In other words, the coupon pricing instrument allows firms to differentiate the two markets which further allows for more aggressive competition over informed consumers.

The analysis, thus far, also suggests that the welfare of informed consumers is higher in the coupon pricing equilibrium while that of uninformed consumers is lower. Due to intense competition in discounts for informed consumers, prices are driven to costs while uninformed consumers purchase at their reservation price. In the absence of such pricing instruments, firms compete in prices by forgoing profits captured from the surplus of uninformed consumers. Examination of the equilibrium distribution function under the uniform pricing equilibrium, (2.7), supports this argument. Namely, the frequency at which prices are randomized is determined by this trade-off in profits from pursuing the informed consumer market. Consequently, uninformed consumers are better off under the uniform pricing equilibrium since firms have no instruments to separate between the two types of consumers.

A corollary of this result is that when firms are no longer required to offer the same price in the two markets, the results of the Rosenthal (1980) and Varian (1980) models no longer hold. The ability for firms to price discriminate between types of consumers, through the usage of coupons, essentially characterizes a Bertrand game; thus, competition eliminates

all informational rents.

From our analysis above, if both firms choose the same price instruments, then profits are as defined in (2.9) and (2.12). Therefore, the solution to the game rests on the profitability of the adoption of asymmetric pricing instruments in equilibrium. We derive the equilibrium of the subgames when the two firms adopt different pricing instruments. This corresponds to O_2 and O_3 in Figure 2.1.

2.3.3 Asymmetric Pricing Instruments (O_2 and O_3)

By the symmetry of the problem, and so without loss of generality, let us assume that firm 1 prices with a coupon and firm 2 prices uniformly.⁶ The profit function for firm 1 is given by:

$$\pi_{O_3,1}(p_1, d_1, p_2) = \begin{cases} (p_1 - c)U + (p_1 - d_1 - c)M & \text{if } p_1 - d_1 < p_2 \\ (p_1 - c)U + (p_1 - d_1 - c)\frac{M}{2} & \text{if } p_1 - d_1 = p_2 \\ (p_1 - c)U & \text{if } p_1 - d_1 > p_2 \end{cases} \quad (2.13)$$

and the profit function for firm 2 is:

$$\pi_{O_3,2}(p_1, d_1, p_2) = \begin{cases} (p_2 - c)(U + M) & \text{if } p_2 < p_1 - d_1 \\ (p_2 - c)(U + \frac{M}{2}) & \text{if } p_2 = p_1 - d_1 \\ (p_2 - c)U & \text{if } p_2 > p_1 - d_1 \end{cases} \quad (2.14)$$

Lemma 1 *In the subgame where one firm chooses to employ a uniform pricing instrument while another uses a coupon pricing instrument, there does not exist a pure strategy equilibrium.*

Proof: See appendix.

The argument for the nonexistence of a pure strategy equilibrium in this subgame is analogous to the subgame where both firms use a uniform pricing instrument. The positive outside option makes setting price, or the discount, equal to cost unprofitable. Therefore, we look for a Nash equilibrium in mixed strategies in the coupons space for firm 1 and in the price space for firm 2.

⁶We also index the profit function by the subgame O_3 to imply the same for the subgame O_2 .

Lemma 2 *The lowest price any firm will offer the informed consumers is $\underline{p} = \frac{cM+rU}{M+U}$ and the highest, r .*

Proof: See appendix.

The intuition behind why firms will not offer any price lower than \underline{p} is because of the outside option. Note that for firm 2 using the uniform pricing instrument, charging a price equal to \underline{p} and succeeding yields profits exactly equal to setting price equal to the reservation price of the consumer and failing to capture any of the informed consumer market.⁷ In other words, instead of cutting price beyond \underline{p} , a firm will price at the reservation price and forgo the informed consumer market. The lack of an instrument to differentiate the two markets is the driving force behind the nonexistence of a pure strategy equilibrium. For firm 1, on the other hand, knowing that firm 2 will charge no lower than \underline{p} , will also charge no lower since any $p < \underline{p}$ simply undercuts its own profits.

Denote $p_1^e \equiv r - d_1$ as the effective price of an informed consumers if they purchase from firm 1. Let $f_1^a(p_1^e)$ be the effective price density function and $F_1^a(p_1^e)$ the distribution function for firm 1. Likewise, let $f_2^a(p_2)$ be the price density function and $F_2^a(p_2)$ the distribution function for firm 2. Note that the index a denotes the asymmetric pricing instrument equilibrium. From Lemma 2, we can formulate the optimization problem for firm 1 as:

$$\max_{f_1^a(p_1^e)} \int_{\underline{p}}^r \left((1 - F_2^a(p_1^e)) \pi_{O_{3,1}}^s + (F_2^a(p_1^e)) \pi_{O_{3,1}}^f \right) f_1^a(p_1^e) dp_1^e \quad (2.15)$$

subject to the constraints:

$$\int_{\underline{p}}^r f_1^a(p_1^e) dp_1^e = 1; f_1^a(p_1^e) \geq 0 \text{ for all } p_1^e \in [\underline{p}, r] \quad (2.16)$$

$$(1 - F_2^a(p_1^e)) \pi_{O_{3,1}}^s + (F_2^a(p_1^e)) \pi_{O_{3,1}}^f \geq \underline{\pi} \text{ for all } p_1^e \in [\underline{p}, r] \quad (2.17)$$

where $\underline{\pi} = (r - c)U + (\underline{p} - c)M$. (2.17) is firm 1's individual rationality constraint where the expected profits obtained by the randomization of prices in the support of $f_1^a(p_1^e)$ must

⁷Succeeding, here, may be interpreted as the case when $p_2 < p_1 - d_1$. Similarly, failure is taken to imply that $p_2 > p_1 - d_1$.

generate profits greater than π . Clearly, since firm 2's lower bound on prices is defined by \underline{p} , firm 1, by charging \underline{p} , will guarantee itself the whole of the informed consumer market. The probability of firm 2 charging the same price will have measure zero which makes this claim justifiable.

Firm 2's optimization problem is identical to that of the symmetric uniform pricing case and is restated here for completeness.

$$\max_{f_2^a(p_2)} \int_{\underline{p}}^r \left((1 - F_1^e(p_2)) \pi_{O_3,2}^s + (F_1^e(p_2)) \pi_{O_3,2}^f \right) f_2^a(p_2) dp_2 \quad (2.18)$$

subject to the constraints:

$$\int_{\underline{p}}^r f_2^a(p_2) dp_2 = 1; f_2^a(p_2) \geq 0 \text{ for all } p_2 \in [\underline{p}, r] \quad (2.19)$$

$$(1 - F_1^e(p_2)) \pi_{O_3,2}^s + (F_1^e(p_2)) \pi_{O_3,2}^f \geq (r - c) U \text{ for all } p_2 \in [\underline{p}, r] \quad (2.20)$$

where, again, (2.20) has the same interpretation as (2.4) in that the randomization of prices in the support of $f_2^a(p_2)$ must generate profits at least as great as resorting to the pure strategy of reservation price pricing and capturing only the portion of uninformed consumers for firm 2. The solution to the above optimization problems will specify the equilibrium distribution for the two firms.

Proposition 3 *There exists a unique equilibrium in the subgame where firm 1 uses the coupon pricing instrument and firm 2 uses the uniform pricing instrument. Furthermore, the equilibrium is in mixed strategies whereby firm 1 employs the distribution function:*

$$F_1^a(p_1^e) = 1 - \frac{(r - p_1^e) U}{(p_1^e - c) M} \quad (2.21)$$

and firm 2 uses the distribution function given by:

$$F_2^a(p_2) = \begin{cases} 1 & \text{if } p_2 \geq r \\ 1 - \frac{(r-c)U}{(p_2-c)(M+U)} & \text{if } p_2 \in (\underline{p}, r) \\ 0 & \text{if } p_2 \leq \underline{p} \end{cases} \quad (2.22)$$

for $p_1^e, p_2 \in [\underline{p}, r]$.

	U_2	C_2
U_1	$(r - c)U, (r - c)U$	$(r - c)U, (r - c)U + (\underline{p} - c)M$
C_1	$(r - c)U + (\underline{p} - c)M, (r - c)U$	$(r - c)U, (r - c)U$

Figure 2.2: Reduced payoff matrix

Proof: See appendix.

Clearly, Proposition 3 suggests that if asymmetric pricing instruments are chosen, then the firm pricing uniformly earns $(r - c)U$ while the firm pricing with coupons earns $(r - c)U + (\underline{p} - c)M$. More importantly, since $\underline{p} > c$, profits for the firm pricing with coupons is strictly greater than the firm pricing uniformly when the two firms have chosen different pricing instruments. This is due solely to the fact that coupons are superior to a uniform price in differentiating the two groups of consumers.

2.3.4 The First Stage

Having solved for the equilibrium of the four possible subgames in stage 2, we can easily obtain the subgame perfect equilibrium of the whole game. Because firms are committed to the pricing instrument chosen in stage 1, the payoffs are equivalent to the subgames in stage 2. The reduced payoff matrix of the whole game is shown in Figure 2.2.

Clearly then, this game has three pure strategy equilibria; two in which the two firms choose different pricing instruments in the first stage and one in which the two firms both choose the coupon pricing instrument.⁸ The existence of the asymmetric pricing instrument equilibria are possible only if firms are playing a weakly dominated strategy. While this may seem implausible we argue that these are perhaps the more probable equilibria. It is conceivable that coupon pricing is more costly than uniform pricing.⁹ Therefore, if we grant the assumption that a firm adopting the coupon pricing instrument must incur a cost of $\epsilon > 0$, then the remaining equilibria will be ones where firms have chosen different

⁸Of course, the proposed pure strategy equilibria are only pure “loosely speaking” since the firms are randomizing their pricing strategies in stage 2.

⁹For example, firms must incur the cost of advertising and distributing the coupon.

pricing instruments, provided that ϵ is not too large. The intuition follows from the fact that while coupon pricing is superior to uniform pricing, this is only contingent on the assumption that no other firms are pricing via coupons. Consequently, if some other firm is pricing with a coupon, then its opponents are indifferent. By imposing a small fixed cost to adopting the coupon pricing instrument, this eliminates the (C_1, C_2) equilibrium.¹⁰

This result also extends nicely when the choice of pricing instruments are no longer simultaneous in the first stage. If we imagined a model where an arbitrary firm is given a first-mover advantage, then an asymmetric choice of pricing instruments will also be an equilibrium with the first-mover choosing to adopt the coupons pricing instrument. Again, this is because the coupon pricing instrument weakly dominates the uniform pricing instrument.

2.4 Number of Firms

We now consider the model when we have an arbitrary $N < \infty$ number of homogeneous firms in the market. We begin this analysis with a reexamination of the benchmark cases. In the subgame with all N number of firms choosing the uniform pricing instrument, a unique mixed strategy equilibrium remains. The optimization problem for each firm under this environment can be slightly altered to accommodate the increased number of firms. We can restate the equivalence of (2.2) as follows. The objective facing each firm is:

$$\max_{f_i^u(p_i)} \int_{\underline{p}}^r \left((1 - F_j^u(p_i))^{N-1} \pi_{O_1,i}^s + \left(1 - (1 - (F_j^u(p_i)))^{N-1} \right) \pi_{O_1,i}^f \right) f_i^u(p_i) dp_i \quad (2.23)$$

subject to the constraints:

$$\int_{\underline{p}}^r f_i^u(p_i) dp_i = 1; f_i^u(p_i) \geq 0 \text{ for all } p_i \in [\underline{p}, r] \quad (2.24)$$

$$(1 - F_j^u(p_i))^{N-1} \pi_{O_1,i}^s + \left(1 - (1 - (F_j^u(p_i)))^{N-1} \right) \pi_{O_1,i}^f \geq (r - c) U \text{ for all } p_i \in [\underline{p}, r] \quad (2.25)$$

¹⁰If we grant that the (C_1, C_2) equilibrium may be eliminated by the argument of a small fixed cost, then an alternative equilibrium where firms randomize between the two instruments arise. This follows since the perturbed game will be a variation of the “battle of the sexes” game.

Which yields a well defined equilibrium distribution function for each firm as:

$$F_i^u(p_i) = 1 - \left(\frac{(r - p_i)U}{(p_i - c)M} \right)^{\frac{1}{N-1}} \text{ for all } i \in I \quad (2.26)$$

Proposition 4 *In the subgame where all N firms in the market are pricing uniformly, then (2.26) represents the equilibrium distribution for all N firms.*

Proof: The expected profit for any firm $i \in I = \{1, 2, \dots, N\}$ playing a pure strategy $p_i \in [p, r]$ yields expected profits of $(r - c)U$. By the homogeneity of firms assumed, there cannot exist any profitable unilateral deviations. ■

Alternatively, in the subgame where all N firms choose the coupon pricing instrument, the equilibrium is rather robust.

Proposition 5 *If there are $N > 2$ firms in the market all pricing by the usage of coupons, then there exists a set of pure strategy equilibria with the characteristic that all firms earn zero profits from informed consumers.*

Proof: Since the necessary condition for profit maximization, $p_i = r$ for all $i \in I$, is invariant to the number of firms, let $p_i^* = r$ for all $i \in I$. Let d^* be the equilibrium discount offered. Suppose any two firms $i, j \in I$ are such that $d_i^* = d_j^* = r - c$. Clearly, no profitable deviations exist for any firm $s \in I$, with $s \neq i, j$. Since undercutting is always profitable, equilibrium necessarily implies that all firms earn zero profits from informed consumers. Therefore, any permutation of the vector d^* with at least two of its members offering discounts of c constitutes a Nash equilibrium in pure strategies. ■

Our task now is to pursue the sustainability of asymmetric pricing instruments in equilibrium. From Proposition 5, if there exists more than two firms using a coupons pricing instrument, then we essentially get the Bertrand outcome in the informed consumer market. Therefore, it is only necessary that we focus on an asymmetric pricing strategy equilibrium in which only one firm employs the coupons pricing instrument. Assume that

firm 1 prices with a coupon and correspondingly, firms 2 to N price uniformly. Firm 1's problem can be stated as:

$$\max_{f_1^a(p_1^e)} \int_{\underline{p}}^r \left((1 - F_i^a(p_1^e))^{N-1} \pi_{O_{3,1}}^s + \left(1 - (1 - F_i^a(p_1^e))^{N-1} \right) \pi_{O_{3,1}}^f \right) f_1^a(p_1^e) dp_1^e \quad (2.27)$$

subject to the boundary constraints, nonnegativity constraints, and individual rationality constraints as given by (2.16) and (2.17). Likewise, for any firm $i \in I' = \{2, 3, \dots, N\}$, the optimization problem can be expressed as:

$$\begin{aligned} \max_{f_i^a(p_i)} \int_{\underline{p}}^r & [\pi_{O_{3,i}}^s (1 - F_1^e(p_i)) (1 - F_i^a(p_i))^{N-2} \\ & + \pi_{O_{3,i}}^f (F_1^e(p_i)) (1 - (1 - F_i^a(p_i))^{N-2})] f_i^a(p_i) dp_i \end{aligned} \quad (2.28)$$

also subject to the constraints given by (2.19) and (2.20), obviously substituting the generic index i for the value 2. The above two optimization problems yield the equilibrium distributions as:

$$F_i^a(p_i) = \begin{cases} 1 & \text{if } p_i \geq r \\ 1 - \left(\frac{(r-c)U}{(p_i-c)(M+U)} \right)^{\frac{1}{N-1}} & \text{if } p_i \in (\underline{p}, r) \\ 0 & \text{if } p_i \leq \underline{p} \end{cases} \quad (2.29)$$

for all $i \in I'$.

$$F_1^a(p_1^e) = \frac{(r-c)U - \pi_{O_{3,i}}^s (1 - F_i^a(p_1^e))^{\frac{N-2}{N-1}}}{\pi_{O_{3,i}}^f \left(1 - (1 - F_i^a(p_1^e))^{\frac{N-2}{N-1}} \right) - \pi_{O_{3,i}}^s (1 - F_i^a(p_1^e))^{\frac{N-2}{N-1}}} \quad (2.30)$$

where $\pi_{O_{3,i}}^s \equiv (p_1^e - c)(M + U)$ and $\pi_{O_{3,i}}^f \equiv (p_1^e - c)U$, for any $i \in I'$.

From the formulation of the optimization problem, the parallelism between the $N > 2$ and $N = 2$ case is obvious. Consequently, we state the following proposition without proof.

Proposition 6 *In a game with $N > 2$ number of firms, there exists $2^N - 1$ pure strategy equilibria. In particular, the equilibria will have at least one firm choosing the coupon pricing instrument. If we assume that a small $\epsilon > 0$ cost exists for the adoption of the*

*coupon pricing instrument, then there will exist N pure strategy equilibria with exactly one firm choosing the coupon pricing instrument.*¹¹

To extend the model with $N > 2$ number of firms, we, again, find that a similar set of equilibria exists. More specifically, an equilibrium where $N - 1$ firms choose to price uniformly while one firm chooses to price employing a coupon is sustainable. This suggests that when the pricing instruments are expanded to include coupons, the result of an increase in price due to an increase in the number of sellers, as proposed by Rosenthal (1980) may no longer hold. The obvious reason lies in the fact that coupons are an instrument for price discrimination. Consequently, under the simultaneous setting of this model, a Bertrand equilibrium may result in the market for informed consumers. In other words, we cannot discount the equilibria where two or more firms are offering coupons. This equilibrium, by the same argument as in the $N = 2$ case, is also attainable when firms move sequentially. Again, with the assumption of an ϵ cost for adopting the coupon pricing instrument, the firm with the first-mover advantage will choose to adopt the coupon pricing instrument while all remaining firms will purposely opt for a uniform pricing instrument to avoid the Bertrand equilibrium. As a result, we may attain the Rosenthal result by eliminating the set of equilibria with more than one firm offering coupons. The support of the equilibrium distributions remains unchanged, and the increase in price is a direct result of changes in the equilibrium distributions due to the lower probability of successfully capturing the whole of the informed consumer market.¹² This also suggests that a non-price discriminatory pricing instrument may be a tool for collusion.

2.5 Concluding Remarks

It has been shown in this paper that a simple game with multiple pricing instruments may result in an equilibrium where asymmetric pricing instruments are adopted by symmetric

¹¹In addition, there will also exist a class of mixed strategy equilibrium in the latter case where firms randomize between the two pricing instruments.

¹²Note that the support of the equilibrium distributions do not change only on the assumption that the size of the uninformed consumer market grows proportionally with the number of sellers in the market.

firms. The pricing instruments considered include a uniform pricing technology in addition to a coupon pricing technology captured through discount coupons. Although multiple equilibria exist, we argue that the pure strategy equilibria are more probable.

The analysis of this model when extended to include any arbitrary N number of firms yields a similar result indicating the robustness of the theory developed. Furthermore, this game, if played under a sequential move setting, thereby having a first-mover, an equilibrium with an asymmetric choice of pricing instrument is also possible. However, one must assume that firms are committed to their chosen pricing instrument. This is necessary since any one firm will rather price using coupons given no other firms are doing so. For any firm, the coupon pricing instrument weakly dominates the uniform pricing instrument since the flexibility provided by discount coupons guarantees profits of $(r - c)U$. Furthermore, if there exists a small fixed cost for adopting the coupon pricing instrument, then a follower has incentives to escape the Bertrand equilibrium by choosing the uniform pricing instrument.

The results of this paper captures the example provided at the beginning of this paper very nicely. Wal-mart, is committed to the uniform pricing by coining the slogan “Everyday low prices,” which clearly signals, not only to the consumers, but also to its competitors, its pricing instrument and strategy. In such an instance, one may think of the switching cost to be infinite, in terms of pricing instruments, because of reputational effects.

There is a considerable literature on the economics of sales, but little of it sheds light on the plausibility of the coexistence of different pricing schemes in the same market, especially when firms are symmetric by every account. This paper has shown that the resulting equilibrium will differ when each firm is allowed to offer multiple prices by making use of coupons. Consequently, the model developed here may serve as a foray into an area yet to be fully explored.

2.6 Appendix

Proof of Proposition 1: We begin this proof by first showing that no pure strategy equilibrium exists. Suppose $p^* = (p_1^*, p_2^*)$ is the vector of equilibrium prices in pure strategy. Assume that $p_i^* < p_j^*$ for $i, j \in \{1, 2\}$. Then it must be the case that $\pi_{O_1, j}(p^*) \geq \pi_{O_1, j}(p_i^*, p_i^*)$. Likewise, $\pi_{O_1, i}(p^*) \geq \pi_{O_1, i}(p_j^*, p_j^*)$. Profit maximization necessarily implies that $p_j^* = r$. Therefore, the two inequalities imply that $r \leq p_i^*$ contradicting the assumption that $p_i^* < p_j^*$. So suppose $p_1^* = p_2^* = \hat{p}$. Clearly, $\hat{p} = c$ cannot be an equilibrium strategy since profits from setting $p_i = r$ for any $i \in \{1, 2\}$ strictly dominate $\hat{p} = c$. Likewise, $\hat{p} = r$ cannot be an equilibrium since there exists $\epsilon > 0$ such that $\hat{p} - \epsilon$ yields profits which strictly dominate $\hat{p} = r$. Since each firm can be guaranteed a profit of $(r - c)U$, in order for there to be no upward deviation, it must be the case that $(\hat{p} - c)(\frac{M}{2} + U) \geq (r - c)U$. However, since $\hat{p} \in (c, r)$, then for some $\epsilon > 0$, $\hat{p} - \epsilon$ yields strictly greater profits and the deviating firm will capture all of the informed consumers.

Therefore, equilibrium, insofar as its existence, must be in mixed strategies. We now show that the support of the price density is given by $[\underline{p}, r]$. Clearly, any price above the reservation price yields zero profits and cannot be in the support of the density function in equilibrium. Note that each firm can guarantee itself a profit of $(r - c)U$ simply by reverting to charging the reservation price. The maximum profit that any firm can obtain is $(p - c)(M + U)$ in the case where the lowest price is advertised. Therefore, there must exist a critical price such that it is unprofitable to price below. This critical price, \underline{p} , can be found by determining the indifference condition $(r - c)U = (p - c)(M + U)$, where \underline{p} solves this equality.

Lastly, for uniqueness of the proposed mixed strategy equilibrium, by the indifference condition of an equilibrium mixed strategy, all pure strategies with positive support under $F_i^u(p_i)$ must yield the same profit. A simple substitution reveals that for any firm i , given its opponents' strategy of $F_i^u(p_i)$, expected profits from offering p_j are given by:

$$(1 - F_i^u(p_j))\pi_{O_1, j}^s + (F_i^u(p_j))\pi_{O_1, j}^f = (r - c)U$$

for all $p_j \in [\underline{p}, r]$. Clearly, no profitable deviations exist. For uniqueness, suppose there exists another equilibrium given by the pair $(\hat{F}_1(p_1), \hat{F}_2(p_2))$ which we do not assume to be symmetric. It must be the case that for any firm, 1 or 2, profits from offering p_i must be strictly greater than $(r - c)U$.¹³ Clearly, the range of prices with positive support in $(\hat{F}_1(p_1), \hat{F}_2(p_2))$ must be a strict subset of $[\underline{p}, r]$. In particular, the lower bound under the new distributions, \underline{p}' , must be strictly greater than \underline{p} . Therefore, a deviation by any firm with the inclusion of $\underline{p}' - \epsilon$ with $\epsilon > 0$ will yield strictly greater profits. ■

Proof of Lemma 1: By the same argument in Proposition 2, $p_1^* = r$ is a necessary condition for profit maximization. Suppose there exists a pure strategy equilibrium characterized by the triplet $(p_1^* = r, d_1^*, p_2^*)$. Let $c \leq p_2^* < r - d_1^*$. Then if $c < p_2^*$, the best response for firm 1 is to offer a discount $d_1' = r - p_2^* - \epsilon$ with $\epsilon > 0$. So suppose $c = p_2^* < r - d_1^*$; then the best response for firm 2 is to offer a price $p_2' = r$ since profits are strictly positive, rather than zero. Clearly, $c \leq p_2^* < r - d_1^*$ cannot be an equilibrium. Alternatively, suppose that $c \leq r - d_1^* < p_2^*$. The strategy $r - d_1^* < p_2^*$ is an equilibrium strategy if and only if $p_2^* = r$ since profits are maximal for firm 2. This necessarily implies that $r - d_1^* < \underline{p} = \frac{cM + rU}{M + U}$. Therefore, $r - d_1^* < \underline{p}$ cannot be a best response to $p_2^* = r$ since there exists $\epsilon > 0$ such that profits from offering $r - d_1^* = p_2^* - \epsilon$ are strictly better. This also rules out $c \leq p_2^* < r - d_1^*$ as an equilibrium strategy. Consider $r - d_1^* = p_2^*$. Since firm 2 can always guarantee itself a profit of $(r - c)U$ by reverting back to reservation price pricing, it must be the case that $(r - c)U \leq (p_2^* - c)(U + \frac{M}{2})$. Rewriting this expression, we have $p_2^* \geq \frac{2(r - c)U}{M + 2U} + c$. Therefore, a price cost margin exists. It is now obvious that there exists $\epsilon > 0$ such that $p_2^* - \epsilon$ is a best response for firm 1 since it leads to a higher level of profits. The above excludes all cases which suggest the nonexistence of a pure strategy equilibrium. ■

Proof of Lemma 2: For the upper bound, charging any price, for either firm, above the reservation price of the consumer yields zero profits. For firm 2, $p_2 = r$ yields strict positive profits and therefore, establishes the upper bound on r . For firm 1, as suggested

¹³This is necessarily true since no point masses exist in the equilibrium distribution, and the distribution defined by (2.7) is the unique continuous distribution yielding profits of $(r - c)U$ for all prices with positive support.

by Lemma 1, setting $d_1^* = \frac{2(r-c)U}{M+2U} + c - \epsilon$ with $\epsilon > 0$ yields strictly positive profits. Now suppose $p_2 < \underline{p}$. If $p_2 < r - d_1$, then profits for firm 2 are lower than charging $p_2' = r$ and not capturing any informed consumers. Therefore, $p_2 \geq \underline{p}$. Suppose $r - d_1 < \underline{p}$. Then it must be the case that firm 1 captures all of the informed consumers. Consequently, there exists $\epsilon > 0$ such that profits from offering $r - d_1 + \epsilon$ are strictly greater. ■

Proof of Proposition 3: We begin by first proving that the proposed mixed strategy is, indeed, an equilibrium. Note that the expected profits for firm 2 offering p_2 in pure strategy, given the strategy of firm 1, is:

$$\begin{aligned}\mathbb{E}\{\pi_{O_3,2}(r, F_1^a(p_1^e), p_2)\} &= (1 - F_1^a(p_2))\pi_{O_3,2}^s + (F_1^a(p_2))\pi_{O_3,2}^f \\ &= (r - c)U\end{aligned}$$

for all $p_2 \in [\underline{p}, r]$. Therefore, any deviations must lie outside of the region $[\underline{p}, r]$ and are by Lemma 2 ruled out.

Similarly, the expected profits for firm 1 offering p_1^e in pure strategy, given the strategy of firm 2, is:

$$\begin{aligned}\mathbb{E}\{\pi_{O_3,1}(r, p_1^e, F_2(p_2))\} &= (1 - F_2^a(p_1^e))\pi_{O_3,1}^s + (F_2^a(p_1^e))\pi_{O_3,1}^f \\ &= (r - c)U + \frac{(r - c)MU}{M + U} \\ &= (r - c)U + (\underline{p} - c)M\end{aligned}$$

for all $p_1^e \in [\underline{p}, r]$. Therefore, any deviations must lie outside of the region $[\underline{p}, r]$ and are, again, by Lemma 2 ruled out.

This completes the proof that the proposed mixed strategies are an equilibrium. For uniqueness, suppose there exists another pair of mixed strategies given by the pair $(\hat{F}_1(p_1^e), \hat{F}_2(p_2))$. Clearly, by the individual rationality constraint of firm 1, the strategy $\hat{F}_2(p_2)$ must generate an expected level of profits higher than $\underline{\pi} = (r - c)U + (\underline{p} - c)M$. The continuous portion of firm 2's distribution function is uniquely defined by the parameters of the model. Therefore, $\hat{F}_2(p_2)$ must either be a step function or a different set of point masses. The

expected profits for firm 1 can be rewritten as:

$$\begin{aligned}\mathbb{E}\{\pi_{O_3,1}\} &= \pi_{O_3,1}^s + \left(\pi_{O_3,1}^f - \pi_{O_3,2}^s\right) \widehat{F}_2 \\ &= (r - c)U + (p_1^e - c)M(1 - \widehat{F}_2)\end{aligned}$$

where for simplicity, it is assumed that the distribution is evaluated at p_1^e . Clearly, \widehat{F}_2 cannot be a step function if for all $p_1^e \in [\underline{p}, r]$, it is to generate the same profit. Therefore, we are left to show that the level of expected profits cannot be greater than $\underline{\pi}$. Suppose expected profits for firm 1 are $\pi' > \underline{\pi}$. This necessarily implies that the lower bound at which firm 2's distribution function remains positive is greater than \underline{p} . Let p' be the lower bound of this new distribution function. The region of prices firm 1 will offer must therefore be restricted to $[p', r]$ since offering anything lower than p' will simply undercut its own profit. Clearly, this cannot be a best response for firm 2 since offering $p' - \epsilon$, with $\epsilon > 0$, yields a strictly higher profit. This rules out all classes of distributions for firm 2 except for the one defined in (2.22) proving its uniqueness. ■

Lemma 3 *Every price in the range $[\underline{p}, r]$ is offered with positive density.*

Proof: Suppose there exists a $p \in [\underline{p}, r]$ such that it is offered with zero probability. Without loss of generality, assume this to be firm 1 and denote this price by p' . Then, the following must be true for firm 2:

$$\begin{aligned}&\pi_2^s(F_1(p_1), p')(1 - F_1(p')) + \pi_2^f(F_1(p_1), p')(F_1(p')) \\ &> \pi_2^s(F_1(p_1), p' - \epsilon)(1 - F_1(p' - \epsilon)) + \pi_2^f(F_1(p_1), p' - \epsilon)(F_1(p' - \epsilon))\end{aligned}$$

since there must exist an $\epsilon > 0$ such that $F_1(p') = F_1(p' - \epsilon)$. Clearly, $(p' - \epsilon)$ will be offered with zero probability for firm 2. Therefore, $(p' - \epsilon)$ will be offered with zero probability for firm 1, contradicting the assumption that p' is charged with zero probability. ■

Lemma 4 *There exist no point masses in the equilibrium distribution in the range (\underline{p}, r) with support $[\underline{p}, r]$ in equilibrium.*

Proof: Without loss of generality, suppose there exists n number of point masses denoted by the vector \tilde{p} for firm 1. Take any $\tilde{p}_i \in \tilde{p}$. By definition of a point mass:

$$\lim_{\epsilon \rightarrow 0} F_1(\tilde{p}_i + \epsilon) - \lim_{\epsilon \rightarrow 0} F_1(\tilde{p}_i - \epsilon) = \delta > 0$$

For firm 2, expected profits from offering $\tilde{p}_i + \epsilon$ in pure strategy yield profits of:

$$\begin{aligned} \mathbb{E}\{\pi_2(F_1(p_1), \tilde{p}_i + \epsilon)\} &= \pi_2^s(F_1(p_1), \tilde{p}_i + \epsilon)(1 - F_1(\tilde{p}_i + \epsilon)) \\ &\quad + \pi_2^f(F_1(p_1), \tilde{p}_i + \epsilon)(F_1(\tilde{p}_i + \epsilon)) \end{aligned}$$

Likewise, expected profits from offering $\tilde{p}_i - \epsilon$ in pure strategy yield profits of:

$$\begin{aligned} \mathbb{E}\{\pi_2(F_1(p_1), \tilde{p}_i - \epsilon)\} &= \pi_2^s(F_1(p_1^e), \tilde{p}_i - \epsilon)(1 - F_1(\tilde{p}_i - \epsilon)) \\ &\quad + \pi_2^f(F_1(p_1^e), \tilde{p}_i - \epsilon)(F_1(\tilde{p}_i - \epsilon)) \end{aligned}$$

By equilibrium condition, expected profits from offering any price as a pure strategy must yield the same profits. However:

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \{\mathbb{E}\{\pi_2(F_1(p_1), \tilde{p}_i + \epsilon)\}\} - \lim_{\epsilon \rightarrow 0} \{\mathbb{E}\{\pi_2(F_1(p_1), \tilde{p}_i - \epsilon)\}\} \quad (2.31) \\ &= -(\tilde{p}_i - c)M\delta - \lim_{\epsilon \rightarrow 0} \{\epsilon(F_1(\tilde{p}_i + \epsilon)) + \epsilon(F_1(\tilde{p}_i - \epsilon))\} \end{aligned}$$

The last term in (2.31) is nonnegative, and thus, the whole expression in (2.31) is non-positive. Clearly, this cannot be an equilibrium and thus, contradicts the assumption that $F_1(p_1)$ with n point masses is an equilibrium strategy. ■

The above two lemmas are sufficient for concluding that the equilibrium distribution functions will be continuous and strictly monotonic in the range $[\underline{p}, r]$ for both firms 1 and 2.

Chapter 3

Nonlinear Price Competition with Loyal Consumers

3.2 Introduction

With advances in information technology, consumer search has become easier than ever. More and more companies offer price comparison information on the internet so that consumers may compare the prices for a good offered in the market at any one point in time. Consumers may then find the lowest price and purchase from such firms. With the rise in the number of price comparison web-sites, search costs for consumers are drastically reduced.¹

However, in an empirical study by Baye, Morgan, and Scholten [5] using data obtained from such price comparison web-sites, they find little support for the law of one price. What is puzzling about this result is not that the law of one price does not hold but rather why firms in the market engage in such pricing tactics.²

One way to explain this finding, from a theoretical perspective, is to introduce some heterogeneity of preferences into the model. For example, one may assume that there exist two groups of consumers; loyal consumers and bargain hunters. One may interpret a bargain hunter as one who visits such price comparison sites whereas a loyal consumer is one who strictly prefers one firm over another.³ Then one may appeal to a class of price dispersion models such as those developed by Rosenthal [45] and by Varian [53] to explain

¹For a more detailed discussion of such markets, see Baye, Morgan, and Scholten [5].

²One simple way to explain this phenomenon is that a multiplicity of equilibria arises even in a simple $n \geq 3$ firm homogeneous good Bertrand game. However, at most one firm makes positive profits while others do not.

³Alternatively, one may think of loyal consumers as uninformed consumers while bargain hunters as informed ones. Under this interpretation, an informed consumer is one who is aware of the existence of such price comparison web-sites whereas an uninformed consumer does not.

such an observed phenomenon. The equilibrium in such models will generally be in mixed strategies where firms randomize in hopes of offering the lowest price in the market. In the event that a firm fails to offer the lowest price, then profits may still be captured from the group of loyal consumers.

One may translate the mixed strategy equilibrium as that of a stage game in a repeated setting.⁴ In this view, the observed price data in each period will simply be draws from some underlying distribution given by the parameters of the model. Anecdotal support for this type of pricing strategy may be found in a recent online news article which noted that “Amazon.com has been offering random discounts on a popular MP3 player, revealing a little-known marketing practice that is gaining popularity among e-tailers.”⁵ Furthermore, the article also observed, “Under some dynamic pricing models, price-conscious consumers could get lower prices, while people that demand high customer service could pay higher prices but receive longer warranties and express shipping for their items.” This seems to suggest that inherent in such price randomizing strategies is some sort of screening mechanism.

On this note, then, a model which seeks to explain competing screening mechanisms in an environment similar to that of the class of price dispersion models is desirable. Little is known about screening contracts in a competitive setting from a theoretical point of view. However, such mechanisms arise naturally if there is some underlying unobservable unidimensional parameter which may describe each consumer.⁶ For example, in a monopoly setting, if the willingness to pay by each consumer is unobservable, then it has been shown that there exists an optimal nonlinear pricing mechanism for the seller.⁷ Therefore, such pricing mechanisms are means for separating consumers by their tastes which, in turn, maximize profits for the monopolist.

⁴Of course, other equilibria will arise in a repeated game.

⁵See “MP3 player ‘sale’ exposes Amazon’s flexible prices,” CNET News.com, May 17, 2000.

⁶It is for theoretical simplicity and tractability that a unidimensional type space is considered. Extensions beyond the unidimensional case has been studied by Armstrong [2], Armstrong and Rochet [3], Laffont, Maskin and Rochet [27], McAfee and McMillan [33], and Rochet and Choné [42].

⁷See, for example, Goldman, Leland and Sibley [14], Maskin and Riley [32], and Mussa and Rosen [36].

However, the model applied in an oligopolistic setting brings about complications absent in monopoly markets. In particular, care must be taken in specifying the allowable mechanisms. For example, a pricing mechanism for one firm may include messages from consumers regarding the mechanisms the competitors are offering, in addition to the revelation of their private information.⁸ As a result, the structure of such mechanisms will have an inherent recursive nature in the sense that a firm's mechanism will depend on its competitor's mechanism.⁹

Aside from the issue of selecting the class of admissible mechanisms by the modeler, another problem arises when one considers competing screening mechanisms in a homogeneous good framework, namely, the zero profit Bertrand equilibrium. As a result, there have been limited attempts to examine such environments. Studies such as Hamilton and Thisse [18], Ivaldi and Martimort [20], Oren, Smith and Wilson [37, 38], Stole [49], and Rochet and Stole [44] have provided two differing methodologies to overcome the Bertrand equilibrium; namely, by introducing either a temporal or a spatial aspect. In the former approach, the structure of the game is essentially analogous to that of Cournot competition in which a firm first chooses capacity and then clears the market using nonlinear price schedules. A survey of this approach is provided by Wilson [54]. In the latter approach, one assumes firms compete in the spirit of Hotelling, where a linear city is considered. The general findings from such models suggest that if transport costs are sufficiently high (i.e., the market is uncovered), then firms are local monopolies. In the event that transport costs are sufficiently low, quantity distortions in the optimal mechanism disappear as firms engage in uniform cost markup pricing. A brief survey of this approach is provided by Rochet and Stole [43].

On the other hand, Mandy [31] simply considers a homogeneous good Bertrand game where firms compete via nonlinear price schedules. The equilibrium characterized in his

⁸One may interpret such messages as a price-match guarantee in which a consumer may report to a firm its competitor's price in the event that it is lower.

⁹This issue was addressed in Epstein and Peters [10] in which they examined the revelation principal with a language capable of describing mechanisms. In view of their findings, the model developed in this paper will be one in which the mechanisms a seller may offer is restricted to the class absent of this language.

study suggests that when firms may enter freely into the market, then their profits are driven to zero. However, the characterization of the equilibrium is incomplete and this leads to the possibility that consumers may be segmented by types, in terms of which firm they purchase from. Alternatively, when entry is restricted, firms may earn positive profits. The profitability of firms is solely determined by the structure of costs. However, much like the previous case, the exact price schedules offered by each firm may not be characterized.

Consequently, there are general questions that concern environments of price competition with asymmetric information. For instance, does the introduction of a second firm into the market disrupt the pricing behavior of a monopolist? In other words, will there be incentives for the second firm to charge, say, a uniform price so as to disrupt the incentive compatible direct mechanism that a monopolist may offer? Heuristically, such a strategy is feasible so long as there exists a large number of consumers on the low end who are compensated well enough by the uniform price. But then, one may predict the incumbent's mechanism to be suboptimal if such segmentation of the market is to be expected. Alternatively, one may propose the question regarding the distribution of consumers, in terms of types among the firms in equilibrium. For example, will one firm specialize in serving all high end consumers while another serves the low end consumers? More specifically, will consumers be partitioned according to their types in equilibrium? By the same token, will a monopolist predate or accommodate entry? Such questions, in our opinion, are of particular interest as competing mechanisms are often an observed phenomenon.

In order to provide an answer to the questions above, we will address the issue of competing mechanisms with a different approach. Our goal is to develop a model where the nonlinear pricing structure is embedded within the framework of Rosenthal [45] and Varian [53]. More specifically, our modelling strategy is to introduce loyal consumers into the market. Aside from the parallelism of this structure to the internet example above, the benefits from this assumed structure are obvious; one may examine price competition while avoiding the Bertrand paradox. In other words, a zero profit equilibrium will not result from undercutting. Furthermore, considerations of structures beyond the duopoly

case are much simpler unlike the Hotelling counterpart.

This paper proposes to address the question of nonlinear price competition in the framework described above. Section 2 will set up the basic framework. We then analyze the equilibrium in a duopoly setting in Section 3. In Section 4, some comparative statics are considered. Section 5 examines a simple extension to the model. More specifically, we will generalize the setting by considering a n -firm oligopoly market. Finally, some concluding remarks will follow in Section 6.

3.3 The Basic Framework

Assume a continuum of consumers in the unit interval $[0, 1]$ with unit density. Associated with each consumer is a taste parameter $\theta \in [\underline{\theta}, \bar{\theta}] \equiv \Theta \subset \mathbb{R}_+$ which is drawn from a continuously differentiable distribution function $F(\theta)$ with density $f(\theta)$. Nature draws a θ independently for each consumer according to $F(\theta)$ and the realization of θ is private information. Furthermore, consumers are assumed to fall into one of the two groups; loyal consumers and bargain hunters. A loyal consumer is here defined as one whose objective is to maximize utility by consuming the product from a preferred firm, not necessarily the one with the lowest prices. We assume loyal consumers are distributed evenly among all the firms in the market.¹⁰ In contrast, a bargain hunter is one who seeks to consume the product from a firm which maximizes her utility. Consequently, a bargain hunter is well informed regarding the prices of all firms in the market. In the event that a bargain hunter is indifferent to consumption from multiple firms, we assume that this type of consumer is uniformly distributed across such firms.¹¹

We further assume that all consumers only purchase from one firm. As a result, consumers are unable to split their total consumption between the two firms in the market.

¹⁰This symmetry assumption is for expositional simplicity. We address the issue of an asymmetric distribution of loyal consumers later in this paper.

¹¹Again, this is a simplifying assumption. It is conceivable that when bargain hunters are indifferent among multiple firms, there may still be an underlying “preferred” firm. For example, they may base their consumption decision on firm size or volume of advertisement in the event of a tie. However, as will become apparent, the equilibrium is invariant to the rule that resolves ties.

In the context of the competing mechanism literature, this, then, is a model of exclusive agency rather than a common agency.

Let $\alpha \in [0, 1]$ denote the proportion of consumers who are bargain hunters and $1 - \alpha$ denote the proportion of consumers who are loyal. We assume that the distribution of θ , $F(\theta)$, as well as the proportion of loyal consumers and bargain hunters, given by α , is common knowledge to all firms in the market. However, the actual type of each consumer remains unknown to the individual firms.

Utility for the consumer depends on his taste parameter, θ , as well as his total consumption of the good q .¹² Let $u : \Theta \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be the utility function for a consumer. We assume that the utility function satisfies the properties:

$$\frac{\partial u(\theta, q)}{\partial \theta} > 0; \frac{\partial u(\theta, q)}{\partial q} > 0; \frac{\partial^2 u(\theta, q)}{\partial q^2} < 0; \frac{\partial^2 u(\theta, q)}{\partial \theta \partial q} > 0$$

for all $\theta \in \Theta$ and $q \in \mathbb{R}_+$.

Due to the nature of the asymmetric information, firms in the market will have incentives to price nonlinearly. Firms are assumed to be homogeneous and produce with a constant marginal cost of c . Denote $T_i(q)$ as the price schedule offered by firm i where q denotes quantity.

Given this setup, the structure of the game is as follows. In period one, firms simultaneously choose $T_i(q)$. In period two, after observing the price schedules offered in the market, consumers make their consumption decision (i.e., bargain hunters choose the firm from which they will consume, as well as the consumption level, and loyal consumers, will, by construction, simply choose how much to consume from the firm they are loyal to.) In period three, market transactions take place and lastly, profits are realized. The timing of the game is summarized in Figure 3.1.

With the assumption that $\alpha \in (0, 1)$, each firm in the market is guaranteed $(1 - \alpha)/2$ number of consumers.¹³ The remaining α share is where competition between the two

¹²Note that the utility of a consumer only depends on the taste parameter but not on being a bargain hunter or a loyal consumer. As a result, consumers are loyal or bargain hunters by nature and have no "real" strategic advantages over loyal consumers.

¹³While each firm is guaranteed this fraction of consumers, identifying the consumer as being loyal or as

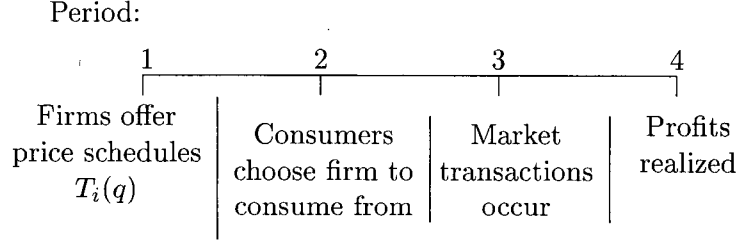


Figure 3.1: The timing of the game

firms takes place. Given the objective of a loyal consumer, we may obtain a consumption rule as a function of θ by her optimization problem. We define $q_i^L(\theta)$ as:

$$q_i^L(\theta) \equiv \arg \max_q u(\theta, q) - T_i(q) \quad (3.1)$$

where L denotes a loyal consumer and i represents the firm of the loyal consumer. We assume that a loyal consumer has outside utility normalized to zero so the solution to her optimization problem is given by:

$$\max\{\max_q u(\theta, q) - T_i(q), 0\} \quad (3.2)$$

A bargain hunter's consumption decision, on the other hand, involves two stages. Aside from the decision of how much to consume, she must also choose the firm to consume from. With the assumption that a bargain hunter also has outside utility equal to zero, a typical θ -type bargain hunter's optimization problem may be expressed as:

$$\max \left\{ \max_q u(\theta, q) - T_1(q), \max_q u(\theta, q) - T_2(q), 0 \right\} \quad (3.3)$$

This optimization problem may be solved by a two stage process. More specifically, for any firm $i \in \{1, 2\}$ in the market, a bargain hunter will have an optimal consumption rule. We denote this as $q_i(\theta)$ and define it as:

$$q_i(\theta) = \arg \max_q u(\theta, q) - T_i(q) \quad (3.4)$$

being a bargain hunter is not possible. As a result, the same price must be offered to all consumers who purchase from the same firm because of the lack of a more complicated screening mechanism.

Clearly then, the reduced form optimization problem, with (3.4), may be written as:

$$\max\left\{\max_{i \in \{1,2\}} \{u(\theta, q_i(\theta)) - T_i(q_i(\theta))\}, 0\right\} \quad (3.5)$$

The solution to this optimization problem, (3.5), may not be unique, but a solution necessarily exists for all $\theta \in \Theta$.

3.4 The Equilibrium

At this juncture, a word on the approach we take in deriving the equilibrium is in order. Our approach is to forgo the analysis of direct incentive compatible mechanisms and focus, instead, on indirect mechanisms. This departs from the common methodology in monopoly analysis where the revelation principle is invoked. In this setting, analyzing the mechanisms of the form $T_i(q)$ for $i = 1, 2$ rather than of the form $\langle T_i(\theta), q_i(\theta) \rangle$, allows us to reduce the dimensionality of the choice set of the firm. As a result, given any two price schedules, one can construct direct mechanisms which will yield the same outcome. In essence, an equilibrium may be defined by the direct mechanism dual to the indirect mechanism considered. This makes for comparison of the results from this setting to that of a monopoly easier.

Under this approach, then, given the two price schedules offered by the two firms, we may define each firm's respective segment of bargain hunters by $\Theta_i \subseteq \Theta$. More formally:

$$\Theta_i = \left\{ \theta \in \Theta \mid \max_{j \in \{1,2\}} \{u(\theta, q_j(\theta)) - T_j(q_j(\theta))\} = i \right\} \quad (3.6)$$

A graphical representation of the segmentation of Θ into Θ_1 and Θ_2 is given in an example in Figure 3.2 where $\Theta_1 \cap \Theta_2 \equiv \Theta_{1,2}$. We may interpret $\Theta_{1,2}$ as the θ -type bargain hunters who are indifferent to consuming from either firm.

Let $\Theta_{i,-j} \equiv \Theta \setminus \Theta_j$ be the θ -type bargain hunters who strictly prefers consuming from firm i . Note that by construction, the subset of Θ into each firm's respective market segment is solely determined by the price schedules offered by each firm. Consequently, an equilibrium may be defined by this segmentation. Therefore, given the two price schedules

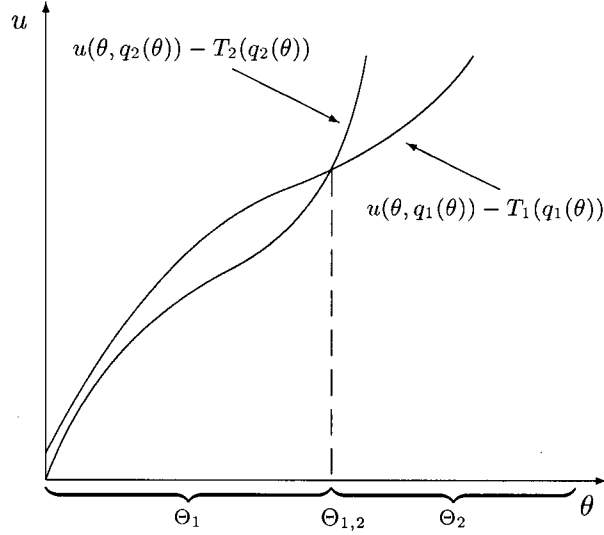


Figure 3.2: An example of Θ_1 and Θ_2 given T_1 and T_2

T_1 and T_2 , we may define a bargain hunter's optimal consumption rule as:

$$q^B(\theta) = \begin{cases} q_1(\theta), & \text{if } \theta \in \Theta_1 \\ q_2(\theta), & \text{if } \theta \in \Theta_2 \end{cases} \quad (3.7)$$

Define:

$$C^P = \{T : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid T \text{ is piecewise continuous}\} \quad (3.8)$$

Since C^P may be interpreted as the set of all possible tariffs offered by each firm, it is essentially the set of pure strategies. Alternatively, a mixed strategy is given by a measurable function $\Delta_i : C^P \rightarrow [0, 1]$ with the property that $\int_{T \in C^P} \Delta_i(x) dx = 1$ and $\Delta_i(x) \geq 0$ for all $x \in C^P$. We seek a Nash equilibrium in the price schedules offered by each firm.¹⁴

Given T_1 , T_2 , and thus, each firm's market segment of the bargain hunters, expected

¹⁴Note that we have not restricted our attention to purely nonlinear price schedules as the class of functions C^P does not eliminate the uniform price as a possibility.

profits for each firm may be expressed as:

$$\begin{aligned} \pi_i = & \alpha \int_{\Theta_{i,-j}} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta + \frac{\alpha}{2} \int_{\Theta_{i,j}} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta \\ & + \frac{1-\alpha}{2} \int_{\Theta} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta \end{aligned} \quad (3.9)$$

where $T_i(\theta) \equiv T_i(q_i^L(\theta))$.¹⁵ In a pure strategy equilibrium, each firm's choice of T_i maximizes (3.9).

Since for all $T_i \in C^P$, $q_i^L(\theta)$ is well defined, we therefore let \mathcal{C}^P be the set of all possible price schedules a firm may offer, evaluated at the induced demand of its loyal consumers, $q_i^L(\theta)$. More specifically:

$$\mathcal{C}^P = \{T \circ q^L : \Theta \rightarrow \mathbb{R}_+ \mid T \in C^P \text{ and } q^L = \arg \max_q u(\theta, q) - T(q)\} \quad (3.10)$$

It is clear that there exists a one to one mapping between C^P and \mathcal{C}^P . In essence, \mathcal{C}^P acts as our indirect mechanisms dual to direct mechanisms. Define \mathcal{T} to be the set of monetary transfer rules of all incentive compatible direct mechanisms. Then:

$$\mathcal{T} = \{T : \Theta \rightarrow \mathbb{R}_+ \mid \text{for all } \theta, \theta' \in \Theta, U(\theta, \theta) \geq U(\theta, \theta')\} \quad (3.11)$$

where $U(\theta, \theta') \equiv u(\theta, q(\theta')) - T(\theta')$. It is clear that $\mathcal{T} \subset \mathcal{C}^P$. Note that all incentive compatible direct mechanisms are well defined.¹⁶ In fact, for all $T \in \mathcal{T}$, they only differ in one dimension. More importantly, they differ by a degree of a constant given by the constant of integration (by the assumed quasilinear preference structure.)¹⁷ This may be interpreted as the reservation utility of the marginal consumer which we define to be \hat{u} . We characterize our first result in the following proposition.

Proposition 7 *Given $\alpha \in (0, 1)$, there exists no pure strategy equilibrium.*

¹⁵In the case where the distribution of loyal consumers to each firm is asymmetric or when the tie breaking rule is not even, then the profit function may be rewritten accordingly.

¹⁶See, for example, Maskin and Riley [32].

¹⁷See, for example, Laffont [26].

A formal proof of the proposition is provided in the appendix while a simple sketch is illustrated below. The intuition that one may obtain in the following argument will make the exposition of what follows much easier.

A pure strategy equilibrium, in this framework, will consist of a pair of strategies $(T_1, T_2) \in C^P \times C^P$. Since the set C^P is relatively large, we establish this proposition by first considering price schedules which, roughly speaking, intersect.¹⁸ We show that this is not possible relying on two specific deviations. More specifically, we consider a deviation for firm i by replicating firm j 's tariff over the quantity levels for which bargain hunters purchase from firm j . We establish this with the aid of Lemma 5, in the appendix, which claims that for all $\theta \in \Theta_i \neq \emptyset$ type consumers, whether bargain hunters or loyal consumers, will choose to consume the same quantity level from firm i . Furthermore, it is also shown that the price schedule offered by firm i must be uniformly higher than that of firm j 's over the set of quantity levels the bargain hunters are consuming at if $\Theta_i = \emptyset$. This implies that if both Θ_i and Θ_j are nonempty and not equal to Θ , then there must exist at least two quantity levels for which $T_1(q) \geq T_2(q)$ and $T_1(q') \leq T_2(q')$ with at least one inequality strict. In a loose sense, if bargain hunters are segmented in Θ to the firms they consume from, then the two price schedules must intersect.

This notion that the choice set for a bargain hunter is larger than that of the loyal consumer implies that if a firm, offering T_i as its price schedule, does not capture the $\Theta_{j,-i}$ bargain hunters, then by replicating T_j over such quantity levels (i.e., $q_j^L(\theta)$) will not change $q^B(\theta)$ for all $\theta \in \Theta_i$. Consequently, Lemma 5 provides ways to examine deviations, when analyzing an equilibrium, in a simple and systematic way.

The implications of Lemma 5 for our considered deviation are that for all $\theta \in \Theta_{j,-i} \neq \emptyset$, such θ -type bargain hunters purchase from firm j . Furthermore, $T_i(q_j^L(\theta)) > T_j(q_j^L(\theta))$ for all $\theta \in \Theta_{j,-i}$. So in equilibrium, the profits obtained from the proportion of loyal consumers who are consuming such quantity levels must be strictly greater than those obtained by replicating such portions of the price schedule.

¹⁸This is due to our restriction on piecewise continuous price schedules as defined in C^P .

In addition, we consider a deviation by firm i , and by firm j in replicating T_j and T_i , respectively, over all $q \in \mathbb{R}_+$. If both deviations are jointly unfeasible, then it cannot be the case that profits for a firm from selling only to θ -type loyal consumers are strictly greater than those from replicating its competitors tariff over such quantity levels which provides a sufficient contradiction.

This suggests that it must be the case that $\Theta_i \in \{\emptyset, \Theta\}$ in any pure strategy equilibrium. In other words, no equilibrium where bargain hunters are segmented between the two firms in pure strategies may be supported. We must then consider equilibria where such segmentation does not occur. We may further deduce that with the existence of a group of loyal consumers, profits must be bounded above zero. Therefore, motives to undercut an opponent's price schedule uniformly by some arbitrary ϵ always exist. As a result, an equilibrium where $\Theta_i = \Theta_j = \Theta$ occurs is not rationalizable. Immediately, then, we may reduce the set of pure strategies from \mathcal{C}^P to \mathcal{T} since equilibrium behavior necessarily implies an incentive compatible direct mechanism. In other words, an equilibrium in pure strategies necessarily implies that $T_i \circ q_i^L, T_j \circ q_j^L \in \mathcal{T}$ and that competition is for all of the θ -type individuals. As a result, the individual firm must offer a contract as prescribed by the monopolist, up to location.¹⁹ As a result, all competition must occur in \hat{u} , the reservation utility of the marginal consumer, since any functional changes in the optimal tariff will not be incentive compatible and will, therefore, be suboptimal.

Given this, the remaining proof for the nonexistence of a pure strategy equilibrium follows nicely from the discontinuities in the profit function. Because undercutting an opponent's tariff uniformly by an arbitrarily small amount will cause jumps in the profit function due to monopolization of the bargain hunting group of consumers, profits will be driven to zero. However, in such a process of undercutting, the inability to differentiate between bargain hunters and loyal consumers essentially eliminates all monopoly power. The loss of all monopoly power cannot be an equilibrium in this model, since an outside

¹⁹This follows since a direct incentive compatible individual rational mechanism is the optimal mechanism for a monopolist in this environment.

option with strictly positive profits exists. This corresponds to the case when a firm forgoes competition for bargain hunters and simply monopolizes loyal consumers.²⁰ Due to Proposition 7, equilibrium, insofar as it exists, will be in mixed strategies.

Notice that in specifying a mixed strategy equilibrium, one essentially has to consider all strategies within the set C^P , the set of all piecewise continuous price schedules, which makes characterization cumbersome. Therefore, in deriving the equilibrium, we rely on two lemmas (6 and 7) which we state and prove formally in the appendix. Here in the text, we provide a heuristic discussion of these results.

Lemma 6 shows that in any mixed strategy equilibrium, firms will not randomize with price schedules such that the resulting distribution of bargain hunters are segmented between the two firms. This follows from a proof that $\Theta_{s,k}$ is convex. In other words, whenever two price schedules, T^s and T^k , intersect, then $\Theta_s \subset \Theta_k$ or vice versa. The intuition behind this result is analogous to considerations of intersecting price schedules, in pure strategies, which result in $\Theta_{s,k}$ being nonconvex and have been ruled out in Proposition 7. The set of admissible equilibria, in terms of randomizable price schedules, are then reducible given Lemma 6. This turns out to be sufficient in a general characterization of the equilibrium which we establish in Lemma 7.

Mixed strategies which satisfy Lemma 7 have a very distinctive form. In particular, they only consist of price schedules $T \in \mathcal{T}$ with the “flattening” out property along θ -types where $T(\theta) - cq(\theta) \leq 0$. In other words, for all $i = 1, 2$, $T_i \circ q_i^L(\theta) \in \mathcal{T}$ for $\theta \in \Theta$ whenever $T_i(\theta) - cq_i^L(\theta) \geq 0$. For all $\theta \in \Theta$ such that $T_i(\theta) - cq_i^L(\theta) < 0$, $T_i(\theta) = c \cdot q_i^L(\theta)$. This is depicted in Figure 3.3.

Notice that incentive compatibility, in terms of a direct mechanism, is maintained for all θ -type consumers the firm extracts positive rents from. For the θ -type consumers who consume at a unit price of marginal cost, they will be consuming an efficient amount. But it is worth pointing out that the efficient amount will strictly correspond to the portions of the price schedule equalling marginal cost. Such consumers, by consuming on portions where

²⁰This is true even if only a single firm has loyal consumers.

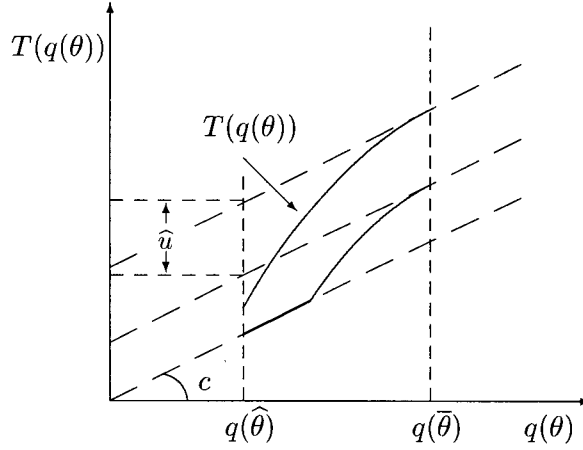


Figure 3.3: The “flattening” out property

the firm may extract strict positive rent from, are not feasible as the marginal consumer consuming a bundle at c is not willing to misrepresent. Therefore, it seems appropriate to call such price schedules pseudo-incentive compatible.

With the representation of the equilibrium made possible by Lemma 7 then, let $G_i(\hat{u}_i)$ be firm i 's equilibrium distribution function over \hat{u}_i with density $g_i(\hat{u}_i)$ and $S(G_i)$ denoting the support of this distribution.²¹ Let $\pi_i^M(\hat{u}_i)$ be monopoly profits from offering \hat{u}_i to the marginal consumer using pseudo-incentive compatible price schedules. In other words:

$$\pi_i^M(\hat{u}) = \int_{\Theta} (T_i(\theta) - cq_i^L(\theta) - \hat{u}) \mathbf{I}(T_i(\theta) - cq_i^L(\theta) - \hat{u} \geq 0) f(\theta) d\theta \quad (3.12)$$

where $\mathbf{I}(\cdot)$ is the indicator function. Then, the objective of the firm, given that it randomizes over pseudo-incentive compatible price schedules, faces the following maximization

²¹Essentially, a firm's mixed strategy is given by $\Delta_i(T_i)$ where T_i is pseudo-incentive compatible. But given that for any two pseudo-incentive compatible price schedules, they only differ in location and the length flattened (dependent on \hat{u}_i), we abuse notation and simply refer to a mixed strategy as $G_i(\hat{u}_i)$ to imply the randomization given by Δ_i .

problem:²²

$$\max_{g_i(\hat{u}_i)} \int_0^{\bar{u}} \left(\pi_i^s G_j(\hat{u}_i) + \pi_i^f (1 - G_j(\hat{u}_i)) \right) g_i(\hat{u}_i) du_i \quad (3.13)$$

subject to:

$$\int_0^{\bar{u}} g_i(\hat{u}_i) du_i = 1; g_i(\hat{u}_i) \geq 0 \text{ for all } \hat{u}_i \in S(G_i) \quad (3.14)$$

$$\pi_i^s(\hat{u}_i) G_j(\hat{u}_i) + \pi_i^f(\hat{u}_i) (1 - G_j(\hat{u}_i)) \geq \underline{\pi} \text{ for all } \hat{u}_i \in S(G_i) \quad (3.15)$$

where π_i^s and π_i^f represent firm i 's profits when it successfully offers the lowest nonlinear tariff and when it does not, respectively. More specifically, π_i^s and π_i^f as:

$$\pi_i^s(\hat{u}_i) = \frac{1 + \alpha}{2} \pi_i^M(\hat{u}_i) \quad (3.16)$$

$$\pi_i^f(\hat{u}_i) = \frac{1 - \alpha}{2} \pi_i^M(\hat{u}_i) \quad (3.17)$$

The constraints in (3.14) are the boundary conditions of the equilibrium density function and those in (3.15) may be interpreted as the firm's profit maximizing constraints. Note that $\underline{\pi}$ is the minimum profit the firm is guaranteed, which is equal to the expected profits from serving only the loyal group of consumers. In other words, $\underline{\pi} \equiv \pi_i^f(0)$. This is shown in Figure 3.4.

The solution to the optimization problem, (3.13), is obtained trivially since we may exploit the indifference condition of any two pure strategies within $S(G_i)$. By construction, equilibrium strategies will be symmetric, and in particular, in mixed strategies.²³

Proposition 8 *In equilibrium, both firms will randomize their price schedules according to the distribution function:*

$$G_i(\hat{u}_i) = \frac{\left(\frac{1-\alpha}{2}\right) (\pi_i^M(0) - \pi_i^M(\hat{u}_i))}{\alpha \pi_i^M(\hat{u}_i)} \quad (3.18)$$

²²We may express the optimization problem in this way since we have already restricted our attention to pseudo-incentive compatible price schedules. Therefore, a more formal way to express this is with the condition that $T_i \circ q_i^L \in \mathcal{T}$ over $\theta \in \Theta$ such that $T(\theta) - cq(\theta) \geq 0$ and $T_i = c \cdot q_i$ otherwise.

²³Note that the results, to this point, did not rely on the assumption that loyal consumers are distributed uniformly between the two firms. In the event that there are asymmetries, $\pi_i \neq \pi_j$, constraints (3.15)-(3.17) will have to be respecified to accommodate this change. Nevertheless, a similar equilibrium will emerge.

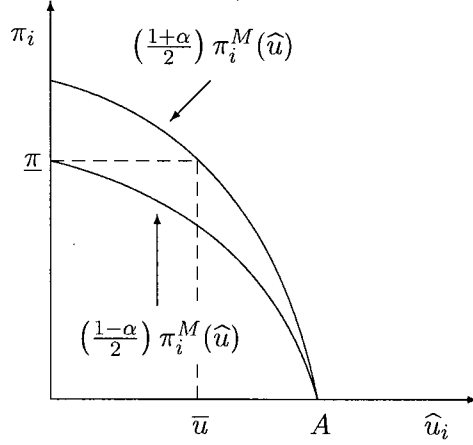


Figure 3.4: Characterization of \bar{u}

for $i \in \{1, 2\}$ over the support $S(G_i) = [0, \bar{u}]$ where \bar{u} satisfies the equality:

$$\left(\frac{1+\alpha}{2}\right) \pi_i^M(\bar{u}) = \left(\frac{1-\alpha}{2}\right) \pi_i^M(0) \quad (3.19)$$

Proof: See Appendix.

The interpretation of the equilibrium distribution, (3.18), is quite simple. In equilibrium, the frequency at which competition for bargain hunters occurs, with changes in \hat{u}_i , is determined by the gains of successfully capturing the α share of the market and by the losses due to increases in \hat{u}_i in the remaining $(1 - \alpha)/2$ share of the market over which the firm has monopoly power. In essence, firms smooth out profits over the support by choosing a frequency equal to the loss-benefit ratio.

Furthermore, the determination of the equilibrium support (3.19) comes from the notion that a firm will not wish to exhaust all profits since there is always an outside option of not pursuing bargain hunters because they may exercise monopoly power over loyal consumers. Therefore, the region over which firms will randomize their price schedules must not yield profits lower than those resulted from this outside option. This is depicted in Figure 3.4.

Another immediate result of Lemma 7 is given in the following corollary.

Corollary 1 *The equilibrium in Proposition 8 is the unique equilibrium of the game.*

From the analysis above, we derive a symmetric mixed strategy equilibrium in nonlinear price schedules. One of the interesting properties suggested by this equilibrium is that the price schedules offered by both firms will be pseudo-incentive compatible, in terms of a direct mechanism, over the whole of Θ . Consequently, the existence of loyal consumers is sufficient in shielding the equilibrium from segmentation. In other words, bargain hunters will always weakly prefer one firm's price schedule over another's, *ex post*. Furthermore, due to the degree of freedom in the price schedules given by \hat{u}_i , this is where competition for consumers exists. In essence, the problem considered here reduces down to competition in a single dimension.

3.5 Comparative Statics

The question regarding the equilibrium derived in Proposition 8 when α changes is one of particular interest. For example, if the size of the captive market increases, how will the equilibrium respond? Furthermore, analysis of the limiting game when α approaches unity may also be of interest. This corresponds to a game where no captive market or no loyal consumers exist. Correspondingly, this will be a game of pure Bertrand competition without a positive outside option.

Notice that the equilibrium support derived in Proposition 8 may be examined in a simple manner when considering changes in α . In particular, the lower bound of the support is invariant to α . As a result, one simply needs to consider the upper bound. As shown in Figure 3.4, \bar{u} is determined by a minimum profit condition given by (3.19). As α increases, the lower curve in Figure 3.4 (i.e., $((1 - \alpha)/2) \cdot \pi_i^M(\hat{u})$) pivots around the point A. Similarly, the upper curve (i.e., $((1 + \alpha)/2) \cdot \pi_i^M(\hat{u})$) pivots upwards around the same point. By the concavity of the profit function in \hat{u}_i , this is sufficient in concluding that the upper bound of the support, of the equilibrium distribution, increases when α increases.

With regards to the equilibrium distribution function itself, simple derivative with re-

spect to α shows that it is everywhere negative. This implies that as α increases, the region over which the firms compete in the reservation utility of the marginal consumer, increases, resulting in a pointwise lower distribution function. In a loose sense, the probabilities are reweighed according to the loss-benefit ratio which is everywhere lower.

This makes intuitive sense since increases in α imply that monopoly power for each firm is reduced (i.e., the number of loyal consumers each firm has is lower.) Consequently, the benefits from competition, in terms of the potential gains, increase, as well.

To examine the limiting case, note that by taking the limit of the definition of the equilibrium support, given by Proposition 8, one finds that:

$$\pi^M(\bar{u}) = \int_{\Theta} (T_i(\theta) - cq_i^L(\theta) - \hat{u}) \mathbf{I}(T_i(\theta) - cq_i^L(\theta) - \hat{u} \geq 0) f(\theta) d\theta = 0$$

Furthermore, taking the limit of the equilibrium distribution function, (3.18), we find that:

$$\lim_{\alpha \rightarrow 1} G_i(\hat{u}_i) = 0$$

This implies that as α approaches one, all the probabilities are transferred to the upper-bound of the equilibrium support. This is intuitively plausible. As the number of loyal consumers decreases, the gains from competition for the bargain hunters increase. Put in a different way, the loss from competition, due to the loss of monopolization over loyal consumers, decreases. As a result, firms compete more vigorously by placing higher probabilities on the upper-bound of the equilibrium support of the distribution function. In the limit, the flattening out process, as suggested by the mixed strategy equilibrium developed in the preceding section, implies that price is flattened out to marginal cost for all $\theta \in \Theta$. In fact, this is also the unique equilibrium of the limiting game. We state this formally in the following proposition.

Proposition 9 *If $\alpha = 1$, then the unique equilibrium is that for all $i \in \{1, 2\}$, $T_i = c \cdot q$.*

Proof: See Appendix.

3.6 An Extension

In this section of the paper, we extend the model by generalizing to an arbitrary n number of firms. However, an additional assumption regarding the size of the loyal consumer market needs to be addressed. In essence, one could make one of the following two assumptions:

Assumption 1 *The number of loyal consumers for each firm is independent of the number of firms.*

Assumption 2 *The number of loyal consumers is fixed and each firm captures $(1 - \alpha)/n$ number of loyal consumers.*

Under Assumption 1, the number of loyal consumers dedicated to each firm is fixed. Alternatively, under Assumption 2, the share of loyal consumers decreases with the number of firms. We derive a general n firm equilibrium below and, where appropriate, make a few remarks regarding the two assumptions.

The objective of the loyal consumer remains unchanged but the choice set of a bargain hunter increases. We maintain our notation and denote $q_i(\theta)$ as the induced demand for a θ -type bargain hunter facing the price schedule $T_i(q)$ from firm i . Similarly, we may define the segmentation of bargain hunters in terms of Θ given the set of price schedules offered by the n firms, $\{T_i\}_{i=1}^n$. Let $N = \{1, 2, \dots, n\}$ be the set of firms in the market. Then:

$$\Theta_i = \left\{ \theta \in \Theta \mid \max_{j \in N} \{u(\theta, q_j(\theta)) - T_j(q_j(\theta))\} = i \right\} \quad (3.20)$$

Similarly, a bargain hunter's optimal consumption rule is, therefore:

$$q^B(\theta) = q_i(\theta), \text{ if } \theta \in \Theta_i \quad (3.21)$$

Define Θ_i^j to be the θ -type bargain hunters who are indifferent to purchasing from exactly j number of firms with firm i being one of them. More formally:

$$\Theta_i^j = \{\theta \in \Theta_i \mid \exists! N' \subset N \text{ with } \#\{N'\} = j \text{ and } \theta \in \cap_{s \in N'} \Theta_s\} \quad (3.22)$$

Then, profits for firm i may be expressed as:

$$\pi_i = \sum_{j=1}^n \frac{\alpha}{j} \int_{\Theta_i^j} (T_i(\theta) - cq_i^L(\theta)) f(\theta) d\theta + \frac{1-\alpha}{n} \int_{\Theta} (T_i(\theta) - cq_i^L(\theta)) f(\theta) d\theta \quad (3.23)$$

The remaining analysis in this generalized extension is similar to that of the duopoly and we summarize the results in the following proposition.

Proposition 10 *For $n < \infty$, the equilibrium may be characterized by the following four properties:*

1. *No firm will employ a pure strategy.*
2. *Firms will randomize over strategies as prescribed by Lemma 7.*
3. *The support of the mixed strategy is given by $S(G_i) = [0, \bar{u}]$, for all $i \in N$, where \bar{u} solves:*

$$\left(\frac{1 - (n-1)\alpha}{n} \right) \pi_i^M(\bar{u}) = \left(\frac{1-\alpha}{n} \right) \pi_i^M(0)$$

where $\pi_i^M(\bar{u})$ is as defined in (3.12).

4. *Each firm randomizes according to the distribution function:*

$$G_i(\hat{u}_i) = \left(\frac{\left(\frac{1-\alpha}{n} \right) (\pi_i^M(0) - \pi_i^M(\hat{u}_i))}{\alpha \pi_i^M(\hat{u}_i)} \right)^{\frac{1}{n-1}}$$

over $S(G_i)$.

Proof: See Appendix.

As can be seen, one may construct a similar equilibrium to the case of the duopoly by taking on a n -firm oligopoly type market. Under Assumption 1, as the number of firms increases, the amount of monopoly power remains constant since the number of loyal consumers each firm has remains constant. Then, the support of the equilibrium distribution function remains unchanged and the distribution simply shifts to accommodate

the increased number of firms. This result is consistent with that of Rosenthal [45] in which an increase in the number of firms raises the expected price in the market.²⁴

Under Assumption 2, on the other hand, as the number of firms increases, the proportion of loyal consumers each firm will capture decreases. This is evident if one fixes the market demand to the unit interval and distributes $(1 - \alpha)/n$ number of loyal consumers to each firm. Consequently, since the support of the equilibrium distribution, $S(G_i)$, is determined by this allocation of loyal consumers to each firm, this will inevitably change the equilibrium support for each firm's strategies. This naturally leads to the question of what happens when n approaches infinity under Assumption 2.

It is intuitively clear that as n approaches infinity, the number of loyal consumers each firm has approaches zero. Similar to the case when α approaches one, by taking the limit of the equilibrium support, we find that the upper bound implies:

$$\pi_i^M(\bar{u}) = 0$$

This is consistent with the case when α approaches one as both cases suggest that each firm has no loyal consumers. Similarly, by taking the limit of the equilibrium distribution function, we find that:

$$\lim_{n \rightarrow \infty} G_i(\hat{u}_i) = 1$$

This implies that as n increases, the probabilities each firm place on the lower-bound of their equilibrium support increase. This contrasts the case when α approaches one since the probabilities, there, are transferred to the upper-bound. While the interpretation of α approaching one and n approaching infinity is similar, (i.e., the number of loyal consumers each firm has approaches zero,) the difference lies in the gains and the losses due to competition. As mentioned above, as α approaches one, the gains from competition for bargain hunters increase. On the other hand, as n approaches infinity, the number of loyal consumers each firm has decreases. However, the probability of being able to offer the

²⁴This counterintuitive result follows since the expected first order statistic of G_i with n firms is greater than that with n' firms if $n > n'$. This comparison is made possible since the support is invariant to the number of firms under Assumption 1.

lowest price schedule also decreases. Consequently, the probability of offering something in the upper-bound of the equilibrium support goes down and the probability of offering something in the lower-bound increases. Put differently, each firm will have a tendency to maximize profits by forgoing competition and as a result, the price schedules will approach that of a monopolist.

3.7 Conclusions

This paper has derived an equilibrium in a homogeneous good duopoly market with asymmetric information. Consumers are assumed to have private information regarding their types, which correspond to the realization of some unidimensional taste parameter given by $\theta \in \Theta$ and to the classification of loyal consumers and bargain hunters. The approach taken here differs from that in the competing spatial nonlinear pricing literature in that screening of the spatial dimension is not possible. This is especially true since the utility of the two groups of consumers in the model considered here does not depend on which group membership they possess.

The equilibrium derived is a symmetric mixed strategy equilibrium where firms randomize their price schedules in hopes of capturing all of the bargain hunters. An important factor in deriving this equilibrium is the assumption that firms must offer the same schedule to both groups of consumers.²⁵ This may be translated into the scenario in which the two groups are indistinguishable when transactions take place in the market and that firms are without additional instruments to screen this difference.

This equilibrium seems capable of describing the example stated in the introduction fairly well. When a consumer visits a price comparison web-site, she purchases from the firm offering the lowest price. Conversely, for someone who does not, loyalty to a particular firm dictates the purchasing decision. Ultimately, the firms are then unable to determine

²⁵See, for example, Rosenthal [45] and Varian [53]. Alternatively, if the two groups of consumers are distinguishable by the firm, then the equilibrium may be trivially determined as the optimal nonlinear price schedule, in terms of a monopolist's problem, is offered to loyal consumers while pure Bertrand competition for the bargain hunters will occur (i.e., $T = c \cdot q$ for all $\theta \in \Theta$).

consumer identity when transaction takes place.

Furthermore, it has been shown that both firms will offer a price schedule which partially coincides with an incentive compatible direct mechanism through a “flattening” out process. The implications of such pseudo-incentive compatible mechanisms are that consumption distortions disappear along the flattened portions. In relations to the spatial nonlinear price competition literature, the rate at which consumption inefficiencies erode is reduced. Clearly then, introduction of competition into the monopoly benchmark will not significantly disrupt the incentive compatibility of mechanisms offered.

While we have only analyzed the equilibrium of a static game, one may interpret the mixed strategy as the randomization of the price schedules each firm will offer in a repeated game. However, one explicit assumption must be made; namely, firms will not collude in the larger extended game.

Another important point to note is that the assumption of the exogenous grouping of consumers given by α suggests an equilibrium where positive profits are sustained. As shown above, without such an assumption, or if one takes α to be equal to 1, then we essentially derive the outcome of a Bertrand game where a zero profit equilibrium (with price equalling marginal cost) occurs.

A similar equilibrium is derived when extending the model to include an arbitrary n number of firms. Previous studies such as Kwong [23], Rosenthal [45] and Varian [53] have shown similar equilibria in markets with n number of firms. The difference in the results lies in the pricing of the goods. In our case, equilibrium is in price schedules which are pseudo-incentive compatible, in the direct mechanism sense, where in the others, a uniform price is considered. The general results of this model seem to be consistent with the price dispersion studies cited here.

3.8 Appendix

In this appendix, we provide the proofs of our results presented in the paper. In proving Proposition 7, the following Lemma will be shown to be fruitful.

Lemma 5 *Given T_1 and T_2 and thus, Θ_1 and Θ_2 :*

1. *for all $\theta \in \Theta_i$, $q^B(\theta) = q_i^L(\theta)$.*

Furthermore, if for some $i \in \{1, 2\}$:

2. *if $\Theta_i = \emptyset$, then for all $q^B(\theta) > 0$, $T_j < T_i$ with $j \neq i$.*

3. *if $\Theta_i \notin \{\emptyset, \Theta\}$ for all $i \in \{1, 2\}$, then there exists some $q, q' \in \mathbb{R}_+$ such that $T_1(q) \geq T_2(q)$ and $T_1(q') \leq T_2(q')$ with at least one inequality strict.*

Proof:

1. Suppose that for some $\theta \in \Theta_i$, $q^B(\theta) \neq q_i^L(\theta)$. This implies that there exists some other bundle $q' = q^B(\theta)$ or $q' = q_i^L(\theta)$ with the associated transfer $T_i(q')$ such that it yields higher utility for either the loyal consumer or the bargain hunter. Note that the associated transfer must be from firm i since $\theta \in \Theta_i$. Clearly then, such a choice of q' is available for both loyal consumers and bargain hunters. So if one finds it optimal to deviate, so must the other type, as well, thus contradicting the assumption that there exists some $\theta \in \Theta_i$ such that $q^B(\theta) \neq q_i^L(\theta)$.

2. Suppose $\Theta_i = \emptyset$ but that for some $\theta \in \Theta_j$ with $q^B(\theta) > 0$, $T_j > T_i$. Then a deviation to consuming from firm i exists which contradicts the assumption that $\Theta_i = \emptyset$.

3. If $\Theta_i, \Theta_j \notin \{\emptyset, \Theta\}$ then some bargain hunters consume from firm 1 and some from firm

2. If there does not exist some $q, q' \in \mathbb{R}_+$ such that $T_1(q) \geq T_2(q)$ and $T_1(q') \leq T_2(q')$ with at least one inequality strict, then from part 2, this implies that for some $i \in \{1, 2\}$, $\Theta_i = \emptyset$ or that $\Theta_i = \Theta_j$, which contradicts the assumption that $\Theta_i \notin \{\emptyset, \Theta\}$. ■

Proof of Proposition 7: To establish this proposition, we begin by first showing that the two firms will not, in equilibrium, offer two price schedules T_1 and T_2 such that $\Theta_i \notin \{\emptyset, \Theta\}$ for $i = 1, 2$.

Suppose given T_i and T_j , $\Theta_i, \Theta_j \notin \{\emptyset, \Theta\}$. Then firm i 's profits may be written as:

$$\begin{aligned} \pi_i = & \alpha \int_{\Theta_{i,-j}} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta + \frac{\alpha}{2} \int_{\Theta_{i,j}} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta \\ & + \frac{1-\alpha}{2} \int_{\Theta} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta \end{aligned} \quad (3.24)$$

Clearly then, if the pair (T_i, T_j) is an equilibrium, then firm i will not find it profitable to deviate by replicating T_j over the set $q_j^B(\theta) > 0$. Therefore, we derive the inequalities that for all $i \in \{1, 2\}$:

$$\frac{1-\alpha}{2} \int_{\Theta_{j,-i}} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta \geq \frac{1}{2} \int_{\Theta_{j,-i}} (T_j(\theta) - cq_j^L(\theta))f(\theta)d\theta \quad (3.25)$$

Alternatively, we may consider a deviation by firm i by replicating T_j over all $q \in \mathbb{R}_+$. Such a deviation is unprofitable if and only if:

$$\pi_i \geq \frac{1}{2} \int_{\Theta} (T_j(\theta) - cq_j^L(\theta))f(\theta)d\theta \quad (3.26)$$

(3.26) must also hold for firm $j \neq i$ if the pair (T_i, T_j) is, indeed, an equilibrium. Therefore, we have:

$$\begin{aligned} & \int_{\Theta_{i,-j}} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta + \int_{\Theta_{j,-i}} (T_j(\theta) - cq_j^L(\theta))f(\theta)d\theta \\ & - \int_{\Theta_{j,-i}} (T_i(\theta) - cq_i^L(\theta))f(\theta)d\theta - \int_{\Theta_{i,-j}} (T_j(\theta) - cq_j^L(\theta))f(\theta)d\theta \geq 0 \end{aligned} \quad (3.27)$$

Clearly, this cannot be true given (3.25) and thus, such a segmentation of Θ is not possible. Therefore, if an equilibrium exists, for all $i \in \{1, 2\}$, $\Theta_i \in \{\emptyset, \Theta\}$.

Now assume a pure strategy equilibrium exists and thus, $\Theta_i \in \{\Theta, \emptyset\}$ for $i = 1, 2$. Suppose that $\Theta_i = \Theta_j = \Theta$. Then $T_i = T_j$ and profits are such that $\pi_i = \pi_j$. As a result, it must be unprofitable for one firm to undercut by $\epsilon > 0$. This amounts to the condition:

$$\epsilon \geq \frac{\alpha}{2} \int_{\Theta} (T_i(q_i^B(\theta)) - cq_i^B(\theta))f(\theta)d\theta > 0 \quad (3.28)$$

which, by assumption, is not possible since $\alpha > 0$ and ϵ maybe arbitrarily chosen.

Therefore, without loss of generality, suppose that given T_i and T_j , $\Theta_i = \emptyset$ and $\Theta_j = \Theta$. Then all bargain hunters purchase from firm j . Clearly then profit maximization requires

that $T_i \circ q_i^L \in \mathcal{T}$. Similarly, profit maximization requires that $T_j \circ q_j^L = (T_i \circ q_i^L - \epsilon) \in \mathcal{T}$ for some $\epsilon > 0$.

The above implies that all competition between firms 1 and 2 must occur in \hat{u}_1 and \hat{u}_2 . Suppose these are equilibrium values and, without loss of generality, assume that $\hat{u}_1 < \hat{u}_2$.²⁶ By continuity, $\exists \epsilon > 0$ such that $\hat{u}_1 < \hat{u}_2 - \epsilon$ and that profits for firm 2 increase. Clearly then, a unilateral deviation for firm 2 exists, violating the notion of an equilibrium. Therefore, suppose that $\hat{u}_1 = \hat{u}_2$. Then each firm captures exactly half of the market. Again, by continuity, $\exists \epsilon > 0$ such that for firm i that increases \hat{u}_i to $u'_i = \hat{u}_i + \epsilon$ will capture the whole of the bargain hunters. This gain in profits is clearly greater than the loss due to the ϵ change so long as profits, given \hat{u}_i , are not zero. In the case that $\pi_i(\hat{u}_i) = 0$, this contradicts profit maximization since offering a contract with $\hat{u}_i = 0$ yields strictly positive profits. This may be accomplished by forgoing competition for bargain hunters and by monopolizing loyal consumers. Therefore, unilateral deviations exist for all values of \hat{u}_1 and \hat{u}_2 proving the nonexistence of a pure strategy equilibrium. Alternatively, one may think of a Bertrand price competition game with a strictly positive outside option. ■

We rely on the following two lemmas in the proof of Proposition 8.

Lemma 6 *In any mixed strategy equilibrium, if $S(G_1), S(G_2) \subset C^P$ are the equilibrium supports for firms 1 and 2, then no two pure strategies $T_s^i, T_k^j \in \{S(G_1) \cup S(G_2)\}$ are such that $\Theta_{s,k}$ is a countable set, for $s, k = 1, 2$ except when $\Theta_{s,k} = \emptyset$. Furthermore, $\Theta_{s,k}$ is a convex set.*

Proof: First note that if $\Theta_{s,k}$ is countable, then $T_s^i \in S(G_s)$ and $T_k^j \in S(G_k)$ intersect at least once at unique points over the relevant domain.²⁷ The proof of this, then, is analogous to that of Proposition 7. By definition of a mixed strategy equilibrium, all pure strategies in the support will yield the same level of expected profits. Therefore, consider any two

²⁶Roughly speaking, one could imagine that \hat{u}_i is so low such that for some $\theta \in \Theta$, negative profits are earned. Consequently, we consider price schedules which flatten out over $\theta \in \Theta$ whenever $T(\theta) - c \cdot q(\theta) < 0$ along the total cost curve.

²⁷By this, we mean over \mathbb{R}_+ such that $q_z^j(\theta) > 0$ for $z = 1, 2$.

price schedules offered by firms $s, k = 1, 2$ such that $\Theta_s \neq \Theta_k \notin \{\Theta, \emptyset\}$. Define $\Theta_s(T_s^i, T_k^j)$, with $s \neq k$ $T_s^i \in S(G_s)$ and $T_k^j \in S(G_k)$, to be the $\theta \in \Theta$ type bargain hunters that will purchase from firm s given T_s^i and T_k^j . $\Theta_{s,-k}(T_s^i, T_k^j)$ and $\Theta_{s,k}(T_s^i, T_k^j)$ are similarly defined. Therefore, consider the strategy T_s^i for firm s . Expected profits, given G_k is:

$$\mathbb{E}(\pi_s | T_s^i, G_k) = \int_{T_k^j \in S(G_k)} \pi_s(T_s^i, T_k^j) g_k(T_k^j) dT_k^j \quad (3.29)$$

A similar expression may be derived for another pure strategy $T_s^l \in S(G_s)$. Then, first consider the case when T_s^i and T_s^l are such that $\Theta_i(T_s^i, T_s^l) \notin \{\Theta, \emptyset\}$.²⁸ We can define an ordering over a partition, indexed by Λ , of Θ such that either $\Theta_s^\lambda(T_s^i, T_k^j) \subseteq \{\Theta_s^\lambda(T_s^l, T_k^j) \cup \emptyset\}$ or $\{\Theta_s^\lambda(T_s^i, T_k^j) \cup \emptyset\} \supseteq \Theta_s^\lambda(T_s^l, T_k^j)$ for all $T_k^j \in S(G_k)$ and $\bigcup_{\lambda \in \Lambda} \Theta_s^\lambda = \Theta$. Then consider the pure strategy \tilde{T}_s where:

$$\tilde{T}_s = \begin{cases} T_s^l & \text{if } \Theta_s^\lambda(T_s^i, T_k^j) \subseteq \{\Theta_s^\lambda(T_s^l, T_k^j) \cup \emptyset\} \\ T_s^i & \text{if } \{\Theta_s^\lambda(T_s^i, T_k^j) \cup \emptyset\} \supseteq \Theta_s^\lambda(T_s^l, T_k^j) \end{cases} \quad (3.30)$$

It is clear that \tilde{T}_s takes the lower envelope of the two functions T_s^i and T_s^l . Then, by the logic of the argument of Proposition 7, $\mathbb{E}(\pi_s | T_s^i, G_k) = \mathbb{E}(\pi_s | T_s^l, G_k) \geq \mathbb{E}(\pi_s | \tilde{T}_s, G_k)$ provides a contradiction, unless $\Theta_s^\lambda(T_s^i, T_k^j) \subset \Theta_s^\lambda(T_s^l, T_k^j)$, or vice versa, for all $\lambda \in \Lambda$ and for all $T_k^j \in S(G_k)$ which further implies that T_s^i and T_s^l are uniformly above or below, or partially overlap, T_k^j for all $T_k^j \in S(G_k)$.

For the remainder of the first part of the Lemma, we are left to show that if one firm randomizes price schedules, $T_s^i \in S(G_s)$, then the other firm will not randomize with any schedules, $T_k^j \in S(G_k)$ such that both $\Theta_s, \Theta_k \notin \{\Theta, \emptyset\}$, for all $T_k^j \in S(G_k)$ and for all $T_s^i \in S(G_s)$. Again, this may be established in a manner similar to that of Proposition 7. Without loss of generality, suppose $T_k^j \geq T_k^l$ over the relevant domain. Then for any T_s^i , $\Theta_k(T_k^j, T_s^i) \subseteq \Theta_k(T_k^l, T_s^i)$. Consider $\pi_k(T_k^j, T_s^i)$ and $\pi_k(T_k^l, T_s^i)$ and assume they are not pointwise equal in T_s^i . If there exists a $T_s^n \in S(G_s)$ such that $\pi_s(T_k^j, T_s^n) = \pi_s(T_k^l, T_s^n)$,

²⁸Since both T_s^i and T_s^l are offered by the same firm, we define $\Theta_i(T_s^i, T_s^l)$ as the set of $\theta \in \Theta$ who will prefer T_s^i over T_s^l if they are both available.

then consider a deviation to \widehat{T}_k such that $\widehat{T}_k = T_k^l$ over $\Theta_k(T_k^l, T_s^n)$ and $\widehat{T}_k = T_k^n$ elsewhere.²⁹ It follows that $\pi_k(\widehat{T}_k, T_s^n) > \pi_k(T_k^j, T_s^n) = \pi_k(T_k^l, T_s^n)$ at the point T_s^n . Furthermore, for all $T_s^v \in S(G_s)$ such that $\Theta_k(T_k^l, T_s^n) \subset \Theta_k(T_k^l, T_s^v)$, $\pi_k(\widehat{T}_k, T_s^n) \geq \pi_k(T_k^l, T_s^n)$ in T_s^n . Conversely, for all $T_s^v \in S(G_s)$ such that $\Theta_k(T_k^l, T_s^i) \supset \Theta_k(T_k^l, T_s^v)$, the inequality $\pi_s(\widehat{T}_k, T_s^n) \geq \pi_s(T_k^j, T_s^n)$ holds in T_s^n .³⁰ Therefore, a deviation in pure strategy exists. In the case where there does not exist any $T_s^i \in S(G_s)$ such that $\pi_k(T_k^j, T_s^i) = \pi_k(T_k^l, T_s^i)$, then choose $T_s^i \in S(G_s)$ such that $\int |\pi_k(T_k^j, T_s^i) - \pi_k(T_k^l, T_s^i)| d\theta$, over the relevant domain, is minimized. A similar argument now completes the proof.

The contradiction derived in the above proof requires both $\Theta_l, \Theta_s \notin \{\Theta, \emptyset\}$. Therefore, convexity of $\Theta_{s,k}$ immediately follows if we allow for either $\Theta_l \notin \{\Theta, \emptyset\}$ or $\Theta_s \notin \{\Theta, \emptyset\}$ since non-convex sets will generate Θ_k and Θ_s otherwise. ■

Lemma 7 *In any mixed strategy equilibrium, $T \in \mathcal{T}$ restricted to the domain $\widehat{\Theta} \subset \Theta$ such that for all $\theta \in \widehat{\Theta}$, $T(\theta) - cq(\theta) \geq 0$. Furthermore, $\Theta_{s,k} \equiv \{\Theta \setminus \widehat{\Theta}\}$, and $T(\Theta \setminus \widehat{\Theta}) = c \cdot q(\Theta \setminus \widehat{\Theta})$.*

Proof: Consider, first, the strategies such that for all $T^k, T^s \in S(G_i)$, $i = 1, 2$, with $\Theta_{k,s} = \emptyset$, they lie within the space \mathcal{C}^P . Define the set of such strategies as \mathcal{S} and assume $T \in \mathcal{S}$ but $T \notin \mathcal{T}$. Since such strategies are uniformly above or below one another, $\sup\{\mathcal{S}\} = \overline{T}$ is well defined.³¹ Let T^M be the optimal price schedule offered by the monopolist. Then, it is clear that for all $T \in \mathcal{S}$, $T \leq T^M$ for all $\theta \in \Theta$. Then, if $\overline{T} \notin \mathcal{T}$, expected profits can be improved by simply charging T^M so $\overline{T} \in \mathcal{T}$. It is then clear that for every $T \notin \mathcal{T}$ but in \mathcal{S} a deviation to some $T' \in \mathcal{T}$ exists by the idea that $\Theta_{k,s} = \emptyset$ for all $T \in \mathcal{S}$.³²

Now consider a strategy $T = T^M - k$ such that $\exists \theta \in \Theta$ for which $T(\theta) - cq(\theta) < 0$. It is evident that a deviation to offering T' where $T'(\widehat{\Theta}) = T(\widehat{\Theta})$ and $T'(\Theta \setminus \widehat{\Theta}) = cq(\Theta \setminus \widehat{\Theta})$

²⁹ Similarly, if $\pi_k(T_k^j, T_s^i)$ and $\pi_k(T_k^l, T_s^i)$ are pointwise equal in T_s^i , then consider any $T_s^i \in S(G_s)$.

³⁰ These inequalities follow since if $\Theta_k(T_k^l, T_s^i)$ is an increasing set in T_s^i , then $\Theta_k(\widehat{T}_k, T_s^i)$ is nondecreasing in T_s^i and vice versa. Consequently, $\pi_k(\widehat{T}_k, T_s^i)$ is nondecreasing in all directions of T_s^i if, in fact, $T_k^l, T_k^j \in S(G_k)$ is an equilibrium.

³¹ By this we mean the highest price schedule offered in equilibrium.

³² This follows since for all $T, T' \in \mathcal{T}$, $|T - T'| = k$, for all $\theta \in \Theta$, since maps within \mathcal{T} differs only by a constant of integration.

is profitable since incentive compatibility for all $\theta \in \hat{\Theta}$ is preserved and zero profits are earned by all $\theta \in \{\Theta \setminus \hat{\Theta}\}$ types.³³ ■

Proof of Proposition 8: We begin by proving that the support $S(G_i) = [0, \bar{u}]$ where \bar{u} satisfies (3.19). Suppose $\hat{u}_i < 0$, then the individual rationality constraint will not hold. Therefore, in equilibrium, it must be the case that $\hat{u}_i \geq 0$. Suppose $\hat{u}_i > \bar{u}$. Then the maximum profits a firm receives will be lower than if it simply monopolizes the loyal group of consumers and does not compete for bargain hunters.³⁴ This is clearly not profit maximizing behavior, thus establishing an upper-bound on the support of G_i in equilibrium for all $i \in \{1, 2\}$.

Then, in proving that the equilibrium distribution function follows (3.18), a mixed strategy must yield the same profits for all pure strategies in its support in order for it to be an equilibrium. Therefore, it is without loss of generality that we restrict our attention to $[0, \bar{u}]$. The expected profits for any $\hat{u}_i \in [0, \bar{u}]$ are given by:

$$\mathbb{E}(\pi_i | \hat{u}_i) = \pi_i^s G_j(\hat{u}_i) + \pi_i^f (1 - G_j(\hat{u}_i)) \quad (3.31)$$

By the firm's individual profit maximizing constraint, (3.15), expected profits must be at least equal to $\underline{\pi}$. Therefore:

$$\mathbb{E}(\pi_i | \hat{u}_i) \geq \underline{\pi} \quad (3.32)$$

In equilibrium, this condition must bind with equality since if $\mathbb{E}(\pi_i | \hat{u}_i) > \underline{\pi}$, then firm j may simply reduce \hat{u}_j until this condition binds with equality. Therefore, rearranging (3.31) yields the equilibrium distribution for firm j . Conversely, firm i 's equilibrium distribution function may be determined by the indifference condition of firm j . The symmetry of this problem makes the solution trivial.

We are left to show that no deviation from the equilibrium strategy is profitable. Therefore, suppose there exists an optimal deviation in pure strategy given by the price schedule

³³Note that incentive compatibility for all $\theta \in \hat{\Theta}$ is preserved since the price schedule is "flattened" out for lower θ -types at a higher price than along the unrestricted map $T \in \mathcal{T}$. Furthermore, we do not claim that incentive compatibility is preserved over $\theta \in \{\Theta \setminus \hat{\Theta}\}$ types. However, their behavior is irrelevant as zero profits are earned from such θ -types anyways.

³⁴Maximum, here, is interpreted as in the case that the firm does offer the lowest price schedule and captures all of the bargain hunters.

$\hat{T} \notin S(G_i) \equiv [0, \bar{u}]$ for firm i given firm $j \neq i$ uses (3.18). Then, by the assumption that $\hat{T} \notin S(G_i)$, the strategy \hat{T} must violate Lemma 6. For any $T_j \in S(G_i)$ we may construct $\Theta_{i,-j}(\hat{T}, T_j)$ and $\Theta_{j,-i}(\hat{T}, T_j)$ accordingly. Define $\pi_i(\Theta_{i,-j}(\hat{T}, T_j))$ and $\pi_i(\Theta_{j,-i}(\hat{T}, T_j))$ as the associated profits over $\Theta_{i,-j}(\hat{T}, T_j)$ and $\Theta_{j,-i}(\hat{T}, T_j)$ given \hat{T} and T_j for firm i , respectively. Notice that if $\Theta_{i,-j}(\hat{T}, T_j)$ is constant for all $T_j \in S(G_j)$ then it is uniformly above the maximal tariff for firm j or uniformly below the minimal tariff for firm i which cannot be the case since the monopolist's tariff $T^M \in S(G_j)$. Therefore, $\Theta_{i,-j}(\hat{T}, T_j)$ is a nonincreasing set in T_j as T_j approaches the upper bound of $S(G_j)$ from the lower bound. Define this sequence of strategies as T_j^n . Furthermore, $\Theta_{i,-j}(\hat{T}, T_j^n) \rightarrow \emptyset$ as $n \rightarrow \infty$ otherwise \hat{T} cannot be an optimal deviation. By the asymptotic property of $\Theta_{i,-j}(\hat{T}, T_j^n)$, it is clear that $\pi_i(\Theta_{j,-i}(\hat{T}, T_j^0)) < \pi_i(\Theta_{i,-j}(\hat{T}, T_j^0))$ and $\pi_i(\Theta_{j,-i}(\hat{T}, T_j^\infty)) > \pi_i(\Theta_{i,-j}(\hat{T}, T_j^\infty))$. Thus, there exists a finite N such that $\pi_i(\Theta_{j,-i}(\hat{T}, T_j^N)) = \pi_i(\Theta_{i,-j}(\hat{T}, T_j^N))$. Let $G(T_j^N) = \beta \in (0, 1)$. Then consider the following strategies:

$$\tilde{T}^1 = \begin{cases} \hat{T} & \text{if } \theta \in \Theta_i(\hat{T}, T_j^N) \\ T_j^N & \text{otherwise} \end{cases}$$

and

$$\tilde{T}^2 = \begin{cases} T_j^N & \text{if } \theta \in \Theta_i(\hat{T}, T_j^N) \\ \hat{T} & \text{otherwise} \end{cases}$$

If firm i plays the mixed strategy of \tilde{T}^1 with probability $1 - \beta$ and \tilde{T}^2 with probability β then expected profits are strictly greater than that of playing \hat{T} in pure strategies. This follows since $T_j^N \geq \hat{T}$ over $\Theta_i(\hat{T}, T_j^N)$ and $T_j^N \leq \hat{T}$ over $\Theta_j(\hat{T}, T_j^N)$. The nonexistence of the pure strategies \tilde{T}^1 and \tilde{T}^2 follows if $\hat{T} = T_j^N$ which is not allowed since $\hat{T} \notin S(G_i) = S(G_j)$. This contradicts the assumption that \hat{T} is an optimal deviation. \blacksquare

Proof of Proposition 9: For $\alpha = 1$, the proof in Proposition 7 for the nonexistence of two pure strategies which yield $\Theta_1 \neq \Theta_2$, both with positive measure, still holds. Thus, for all $i \in \{1, 2\}$, $\Theta_i \in \{\emptyset, \Theta\}$. So if $\Theta_i = \emptyset$, then from Condition 2 of Lemma 5, we know that for all $\theta \in \Theta_j$ such that $q_j^B(\theta) > 0$, $T_j < T_i$. Furthermore, $\pi_i = 0$, and $\pi_j \geq 0$. In

fact, $\pi_j = 0$ since if $\pi_j > 0$, then there exists some $\epsilon > 0$ such that firm i may deviate by setting $T_i = T_j - \epsilon$ and earn positive profits.

Now, note that if for some $\theta' \in \Theta$, $q_j^B(\theta') > 0$ and that $T_j(q_j^B(\theta')) - cq_j^B(\theta') > 0$, then there must exist some $\theta \neq \theta'$ such that $T_j(q_j^B(\theta)) - cq_j^B(\theta) < 0$. Since $T_j(q_j^B(\theta)) < T_i(q_i^B(\theta))$, by definition, firm j will not deviate by setting $T_j(q_j^B(\theta)) = T_i(q_i^B(\theta))$ if the profits from the permutated price schedule, \hat{T}_j , are lower. This is only possible if $\hat{T}_j(\hat{q}_j^B(\theta)) - c\hat{q}_j^B(\theta) \leq T_j(q_j^B(\theta)) - cq_j^B(\theta) < 0$ where \hat{q}_j^B is the induced demand from \hat{T}_j . Note that $T_j(\hat{q}_j^B(\theta)) < T_i(\hat{q}_j^B(\theta))$ since otherwise, a type θ individual will switch and firm j makes positive profits. So, given any increases in portions of the tariff by firm j where negative profits are earned, such θ -type individuals will always maintain on such portions, a uniform increase over that curve is always feasible. But that implies that setting $\hat{T}_j = c \cdot q$ is also possible over all q such that for all $\theta \in \Theta$, $T_j(q_j^B(\theta)) - cq_j^B(\theta) < 0$. But then, this is a contradiction since zero profits are earned from such θ -type consumers. Consequently, there cannot exist any θ -type from which firm j is making positive profits and thus, profits must be pointwise equal to zero over all q for which $q_j^B(\theta) > 0$. A simple Bertrand argument now completes the proof and is therefore, omitted. ■

The following lemma will aid in the proof of Proposition 10.

Lemma 8 *Given $\{T_i\}_{i=1}^n$ and the induced partition of Θ :*

1. *for all $\theta \in \Theta_i$, $q^B(\theta) = q_i^L(\theta)$.*

Furthermore, if for some $i \in N$;

2. *$\Theta_i = \emptyset$, then for all $q^B(\theta) > 0$, $T_i > \min\{T_j\}_{j=1}^n$.*

3. *$\Theta_i \notin \{\emptyset, \Theta\}$ for all $i \in N$, then there exists a vector $\{q_i\}_{i \in N} \in \mathbb{R}_+^n$, such that $T_i(q) \leq \min\{T_j\}_{j \neq i}$ for all $i \in N$, with at least one inequality strict.*

Proof : The proof is analogous to that of Lemma 5 and is thus omitted here for brevity. ■

Proof of Proposition 10:

1. We prove this first by establishing that no firms will employ pure strategies such that, in equilibrium, $\Theta_i \notin \{\emptyset, \Theta\}$. We proceed by induction.

Suppose given $\{T_i\}_{i \in N}$, there exists a firm, $j \in N$ such that $\Theta_j \notin \{\emptyset, \Theta\}$. Then there must exist $\Theta_{-j} \equiv \Theta \setminus \Theta_j$ such that for all $\theta \in \Theta_{-j}$, the bargain hunters will not purchase from firm j . Therefore, for all $\theta \in \Theta_{-j}$, the firm will only get loyal hunters. Clearly, no deviations exist for firm j if and only if:

$$\int_{\Theta_{-j}} (T_j(\theta) - cq_j^L(\theta)) f(\theta) d\theta \geq \max_{i \in N, i \neq j} \left\{ \int_{\Theta_{-j}} (T_i(\theta) - cq_i^L(\theta)) f(\theta) d\theta \right\} \quad (3.33)$$

Similarly, if there exists a firm s such that $\Theta_{-j} \subseteq \Theta_{-s}$, then $T_j = T_s$ over $\theta \in \Theta_{-j}$ by the above inequality. So, without loss of generality, assume there are $1 \leq m \leq n-1$ firms using T_j and $n-m$ firms using T_B where T_B captures all the bargain hunters over Θ_{-j} .³⁵

Define:

$$\pi_i(\Theta_{-j}) = \int_{\Theta_{-j}} (T_i(\theta) - cq_i^L(\theta)) f(\theta) d\theta$$

Then the following inequalities must hold:

$$\frac{\alpha}{n-m} \pi_B(\Theta_{-j}) \geq \frac{1-\alpha}{n} (\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})) \geq \frac{\alpha}{n-m+1} \pi_B(\Theta_{-j}) \quad (3.34)$$

From Proposition 7 we have already shown that this cannot hold for $n=2$. Therefore, suppose there does not exist a m such that $1 \leq m \leq n-1$ and (3.34) is true. This necessarily implies that either:

$$\frac{\alpha}{n-m} \pi_B(\Theta_{-j}) < \frac{1-\alpha}{n} (\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})) \quad (3.35)$$

is true or else:

$$\frac{\alpha}{n-m+1} \pi_B(\Theta_{-j}) > \frac{1-\alpha}{n} (\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})) \quad (3.36)$$

is true. Then, consider the case of $n+1$. Note that if (3.35) is true, then there exists a m such that:

$$m < n - \frac{\alpha n}{1-\alpha} \frac{\pi_B(\Theta_{-j})}{\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})} \quad (3.37)$$

³⁵This is without loss of generality as we may simply focus on a subset $\hat{\Theta}_{-j} \subseteq \Theta_{-j}$ where this statement is true.

Then if no deviations for the case of $n + 1$ is possible, then the inequality:

$$\frac{\alpha}{n - m + 1} \pi_B(\Theta_{-j}) \geq \frac{1 - \alpha}{n + 1} (\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j}))$$

must hold. But that implies that:

$$\frac{\alpha}{1 - \alpha} \frac{\pi_B(\Theta_{-j})}{\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})} > n$$

which would imply $m < 0$ for (3.37) to hold. Therefore, suppose (3.36) is true. Then this implies that there exists a m such that:

$$m > n + 1 - \frac{\alpha n}{1 - \alpha} \frac{\pi_B(\Theta_{-j})}{\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})} \quad (3.38)$$

is true. If no deviations for the case of $n + 1$ is possible, then the inequality:

$$n + 2 - (n + 1) \frac{\alpha n}{1 - \alpha} \frac{\pi_B(\Theta_{-j})}{\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})} \geq m$$

is true. This implies that:

$$1 > \frac{\alpha}{1 - \alpha} \frac{\pi_B(\Theta_{-j})}{\pi_j(\Theta_{-j}) - \pi_B(\Theta_{-j})}$$

But then this further implies that $m > n$ from (3.36) which is not possible since by assumption, $1 < m < n - 1$. Therefore, (3.34) for the case of $n + 1$ cannot hold. Consequently, by the principal of mathematical induction, $\Theta_j \notin \{\emptyset, \Theta\}$ cannot occur if $\{T_i\}_{i=1}^n$ are equilibrium price schedules offered by the n firms.

This suggests that for all $i \in N$, $T_i \circ q_i^L \in \mathcal{T}$. A simple Bertrand argument completes the proof.

2. The proof of Lemma 6 can be made invariant to the number of firms, provided $n < \infty$, simply by replacing T_k^j by T_{-s}^j and G_k by G_{-k} to denote a vector of price schedules offered by all n firms less firm s , and the vector of distribution functions by all n firms less firm s , respectively. Consequently, Corollary 7 follows immediately for the n firm case.

3 and 4. The proof of Proposition 8 can easily accommodate n firms. Therefore, the proof is omitted. ■

Chapter 4

Nonlinear Pricing with Collusive Consumers

4.2 Introduction

Consider the standard monopoly nonlinear pricing model developed by Maskin and Riley [32] and Mussa and Rosen [36]. Nonlinear pricing typically arises due to an assumed asymmetry of information given by a unidimensional parameter, taken to represent taste or willingness to pay, which is private to each consumer.¹ This theory, however, assumes that consumers act non-cooperatively. Since nonlinear pricing is defined by a nonconstant unit price of consumption, this assumption loses ground when collusion among consumers enters into the picture.

To illustrate this point, suppose an optimal price schedule, under the standard model, exhibits quantity discounts. It is then conceivable that two consumers may coordinate their efforts so as to purchase jointly to reap the benefits of the discount.² Collusion among consumers naturally arises if there are gains from cooperation in any economic environment. For example, in the auction literature, evidence suggests that collusion exists, in the form of bidding rings, in some procurement auctions.³ Casual observation suggests that households purchase as one entity rather than non-cooperatively. Conversely, buying groups or group purchasing organizations (GPOs) exist for the purpose of capturing quantity discounts, where available.

In this paper, we examine the nonlinear pricing problem while allowing for consumers to form coalitions under perfect information. For simplicity in exposition, we model collusion by introducing a third party whose objective is to maximize the sum of welfare of a

¹Extensions into environments of multidimensional private information are nontrivial. See, for example, a survey by Stole and Rochet [50].

²There also exist gains from cooperation if a price schedule exhibits quantity premiums. In general, so long as the price schedule is nonlinear, gains from cooperation exist.

³See, for example, Porter and Zona [40].

coalition's members. We find that the introduction of collusion among buyers inevitably creates a collusion incentive compatibility constraint which parallels an incentive compatibility constraint found in multidimensional screening problems. Due to the technical difficulties that arise, we propose a reduction method which mimics the collusion problem in a single dimension. Our results suggest that the collusion problem may then be translated into a similar problem with no collusion and thus existing theories from the standard nonlinear pricing problem may be applied.

While the literature on collusive consumers in a monopolistic screening environment is small, there are notable studies by Alger [1], Jeon and Menicucci [21] and Quesada [41]. A common feature among these papers is that the considered types space is discrete, and usually binomially distributed.⁴ In Alger, a model where consumers may engage in multiple purchases or joint purchases is considered. Under her restrictions on the type of admissible coalitions and mechanisms offered by the monopolist, she finds that with multiple purchases, strict quantity discounts will emerge while with joint purchases, quantity discounts are infeasible. The restriction on coalitions is that only consumers of the same type are able to collude. On the other hand, the mechanisms she considered are incomplete in the sense that a single consumer's quantity and payment allocation is independent on others' reports.

Jeon and Menicucci, on the other hand, consider a model where coalitions are formed under asymmetric information. Their main findings suggest that when coalitions are formed under asymmetric information, the associated transaction costs from maintaining incentive compatibility within the coalition will be greater than the expected gains. As a result, a monopolist may implement a mechanism which exploits this friction generated by the asymmetry of information in the formation of coalitions. More importantly, they find that a monopolist will not be worse off even with the possibility of collusion among buyers.

⁴Due to the dissimilarities between an analysis involving a discrete types space and that involving a continuous types space, generalizations of these papers are also desired. Even so, generalizations within a discrete types space are often technically challenging. In general, transitions from a simple 2-types setting to a 3-types setting are nontrivial. See, for example, the discussion in Jeon and Menicucci [21].

The main difference between Jeon and Menicucci's perspective and that of the present study concerns the assumed information structure within coalitions. This paper proposes an alternative answer to the question that arises as a theoretical corollary from their study; namely, the determination of the optimal pricing mechanism when coalitions are formed under perfect information.

Lastly, Quesada examines the issue of timing in coalition formation in a monopolistic screening problem under asymmetric information. She finds that if collusion is initiated after accepting or rejecting the monopolist's pricing mechanism then the dominant strategy implementation of the optimal contract without collusion is collusion-proof. Conversely, this will not be the case, if the collusion mechanism is offered before the decision is made whether or not to accept the monopolist's pricing mechanism. This latter result follows after allowing for a commitment to punish if collusion is rejected.

While the problem of collusion among consumers under perfect information is our main focus of investigation, another issue that this paper will address, albeit indirectly, is the viability of monopolistic price discrimination as an optimal pricing strategy in the face of resale. The concern is a valid one if collusion is taken to imply resale that causes both a reallocation of quantities and monetary transfers. Alternatively, one may interpret the third party as an additional agent who resells the good to coalition members.⁵

The remainder of this paper will be organized as follows. Section 4.3 will present the basic model as well as the equilibrium of the benchmark case in which no consumers collude. This will furnish a useful starting point for the later discussion. Section 4.4 will analyze the equilibrium and more specifically, the optimal pricing mechanism and collusion contract that it entails. Some extensions to the model will be discussed in Section 4.5, especially where the sizes of coalitions differ from those of the basic model presented in Section 4.3. Furthermore, a more general utility function is also considered in the extensions. Lastly, Section 4.6 will contain concluding remarks.

⁵However, under this alternative interpretation, the third party is only concerned about making zero rent.

4.3 The Model

There is a monopolist selling some good, which is produced at a constant marginal cost c , in the market to a continuum of consumers normalized with measure one. Consumers are differentiated in their tastes for which we assume to be summarized by a unidimensional parameter $\theta \in \Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$. We assume that θ is drawn independently and identically for all consumers from a continuously differentiable distribution function $F(\theta)$ with density $f(\theta)$. From here on, we refer to a consumer with a taste parameter θ as a θ type consumer.

A θ type consumer consuming q units for a total price of t derives net utility equal to $u(\theta, q) = \theta v(q) - t$ where $v(\cdot) \in C^2$ with $v(0) = 0$, $v' > 0$, and $v'' < 0$. As well, we normalize the outside utility for all $\theta \in \Theta$ type individuals to be zero.

The timing of the game is as follows. In period one, nature draws a $\theta \in \Theta$ independently and identically from $F(\theta)$ for all consumers in the market. This information is assumed to be private for each consumer.⁶ In period two, the monopolist offers a *pricing mechanism*, \mathcal{M} , to the consumers. We assume that the monopolist is committed to \mathcal{M} and that it is observable to all players in the game. In period three, consumers are randomly matched in a pairwise fashion. In period four, a third party creates a *collusion contract*, \mathcal{C} , with the objective of maximizing the sum of utility for any pair of individuals matched. We assume that the third party is perfectly informed regarding the private information held by all consumers in the market.⁷ Consumers may reject or accept \mathcal{C} . In the event that \mathcal{C} is accepted, then all market transactions are conducted through the third party. In other words, the third party chooses an optimal consumption bundle for some pair $(\theta_1, \theta_2) \in \Theta \times \Theta$ and redistributes it to the two consumers according to the specified contract \mathcal{C} . In the event that \mathcal{C} is rejected, then consumers interact with the monopolist independently through the

⁶Due to the presence of collusion and our assumption that coalitions are formed under perfect information, an alternative assumption is that each consumer is perfectly informed regarding the type of all consumers in the market, but that this information is unknown to the firm, may be made.

⁷Put differently, collusion occurs under perfect information. Furthermore, the use of this third party is not essential for our analysis. Instead, we may think of a coalition as a single individual acting in a way such that its objective is to maximize the sum of utility within the coalition. In other words, the third party maps out the Pareto frontier for the pair of consumers from collusion.

pricing mechanism \mathcal{M} .

4.3.1 The Pricing Mechanism

A pricing mechanism, \mathcal{M} , is here defined to be comprised of a pair $\langle t, q \rangle$ where $\mathcal{M} : \{\Theta \cup \emptyset\}^2 \rightarrow \mathbb{R}_+^2$. More specifically, $t(\theta_1, \theta_2)$ and $q(\theta_1, \theta_2)$ will denote the payment and quantity pair for a θ_1 type consumer when he is paired with a θ_2 type consumer if their report is, indeed, (θ_1, θ_2) . Since a consumer may not always find it profitable to remain in a coalition, we allow for reports of (θ, \emptyset) for a θ type individual. Thus, such a report will result in a payment and quantity pair given by $t(\theta, \emptyset)$ and $q(\theta, \emptyset)$.

4.3.2 The Collusion Contract

A collusion contract, \mathcal{C} , is here defined to be comprised of the quadruple $\langle p, x, r_1, r_2 \rangle$ where $p, x : \Theta \times \Theta \rightarrow \mathbb{R}_+$ and $r_1, r_2 : \Theta \times \Theta \rightarrow \{\Theta \cup \emptyset\}$. In essence, $p(\theta_1, \theta_2)$ and $x(\theta_1, \theta_2)$ denote the payment and quantity pair for a θ_1 type consumer when he is paired with a θ_2 type consumer. Conversely, $r_1(\theta_1, \theta_2)$ and $r_2(\theta_1, \theta_2)$ denote the pair of reports to \mathcal{M} from \mathcal{C} . For example, if $(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) = (\theta'_1, \theta'_2)$, then it implies that for a (θ_1, θ_2) pair of consumers, a report of (θ'_1, θ'_2) will be made from the collusion contract \mathcal{C} to the pricing mechanism \mathcal{M} .

Furthermore, there are two other constraints that \mathcal{C} must satisfy. The first is the balance of payment constraint given by:

$$p(\theta_1, \theta_2) + p(\theta_2, \theta_1) = t(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) + t(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)) \quad (4.1)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$. In essence, this constraint states that all money transfers must be accounted for. Alternatively, the aggregate net payment by any two colluding individuals must be equal to the aggregate payment to the monopolist. Secondly, we also have the balance of quantity constraint:

$$x(\theta_1, \theta_2) + x(\theta_2, \theta_1) = q(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) + q(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)) \quad (4.2)$$

with $x(\theta_1, \theta_2) \geq 0$ for all $(\theta_1, \theta_2) \in \Theta \times \Theta$. This implies that the aggregate quantity distributed to a colluding pair through \mathcal{C} must equal the total aggregate quantity obtained from the monopolist. In summary, a contract given by the quadruple $\langle p, x, r_1, r_2 \rangle$ such that it satisfies (4.1) and (4.2) is termed a collusion contract, \mathcal{C} .

4.3.3 The Benchmark Case

Our benchmark case consists of the above model in the absence of collusion. Therefore, the objective of the monopolist is to choose a direct mechanism $\mathcal{M}^b \equiv \langle t(\theta), q(\theta) \rangle$ such that its objective may be expressed as:

$$\max_{t(\theta), q(\theta)} \int_{\Theta} (t(\theta) - cq(\theta)) f(\theta) d\theta \quad (4.3)$$

subject to:

$$\theta v(q(\theta)) - t(\theta) \geq \theta v(q(\theta')) - t(\theta') \quad (4.4)$$

for all $\theta, \theta' \in \Theta$, and:

$$\theta v(q(\theta)) - t(\theta) \geq 0 \quad (4.5)$$

for all $\theta \in \Theta$. Constraint (4.4) is the incentive compatibility constraint and (4.5) is the individual rationality constraint. The most direct approach in solving this problem is to find an equivalent representation of (4.4). We characterize this representation in the following proposition.

Proposition 11 *Suppose the mechanism \mathcal{M}^b is compact valued.⁸ Then \mathcal{M}^b is incentive compatible if and only if:*

$$U(\theta) - U(\theta') = \int_{\theta'}^{\theta} v(q(s)) ds \quad (4.6)$$

for all $\theta, \theta' \in \Theta$ where:

$$U(\theta) = \theta v(q(\theta)) - t(\theta)$$

and $q(\theta)$ is nondecreasing.

⁸A mechanism \mathcal{M}^b is compact valued if the set $\{(t, q) : \exists \theta \in \Theta \text{ with } \mathcal{M}^b = \langle t(\theta), q(\theta) \rangle\}$ is compact. In summary, it guarantees the existence of $\max_x U(\theta, x)$, where $U(\theta, x) = \theta v(q(x)) - t(x)$, rather than only $\sup_x U(\theta, x)$.

Proof: Standard and hence omitted. ■

Define:

$$I(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

Then, given Proposition 11, the maximization problem of the monopolist may be stated in a equivalent optimization problem of:

$$\max_{q(\theta)} \int_{\Theta} (I(\theta)v(q(\theta)) - cq(\theta))f(\theta)d\theta$$

subject to (4.5), (4.6), and $q(\theta)$ being nondecreasing.

The approach is to generally ignore the monotonicity constraint and to maximize the objective function in a pointwise fashion. If the resulting quantity allocation rule $q(\theta)$ is monotonic, then it is the solution to the monopolist's problem. In the event that $q(\theta)$ is not monotonic, then an ironing procedure by employing optimal control theory in the optimization problem is required. In either case, a solution to this optimization problem exists and the techniques for solving such problems are well developed.⁹

From the benchmark case, it is interesting to note that under some regularity conditions, the optimal price schedule the monopolist offers will exhibit quantity discounts everywhere. In other words, T/q is decreasing in θ .¹⁰ In the context of collusive buyers, this obviously suggests that room for collusion exists.¹¹

To illustrate this point fully, consider two individuals of type $\theta_1, \theta_2 \in \Theta$. Furthermore, suppose there exists some θ_3 such that $q(\theta_1) + q(\theta_2) = q(\theta_3)$.¹² Since T/q is decreasing in θ , this implies that $T(\theta_3)/q(\theta_3) < \min_{i=1,2} \{T(\theta_i)/q(\theta_i)\}$. This is interesting since if the two consumers agreed to collude and purchase the same aggregate amount $q(\theta_3)$, then their total payment $T(\theta_3) < T(\theta_1) + T(\theta_2)$ and thus, collusion is Pareto improving from their perspectives.¹³

⁹See, for example Guesnerie and Laffont [15].

¹⁰This was first formalized by Maskin and Riley [32].

¹¹This is for illustrative purposes. It is sufficient to recognize that any nonlinearities in the optimal pricing mechanism exhibits room for collusion.

¹²It is not essential that we assume the existence of θ_3 . However, the logic that follows in the argument is more apparent.

¹³This argument is somewhat hasty since it assumes that *ex post* collusion, the aggregate amount consumed is unchanged. This may not necessarily be true as the optimal quantity pair may change in reflection

Given this observation, we now turn our attention to the optimal pricing mechanism and collusion contract when collusion exists.

4.4 The Equilibrium

Having examined the benchmark case, our discussion will now proceed to derive the optimal pricing mechanism and collusion contract. We begin by first studying the optimal collusion contract.

4.4.1 Optimal Collusion Contract

Let us first consider the objective of the third party. Since it is well informed about the type of each consumer, then for some pair $(\theta_1, \theta_2) \in \Theta \times \Theta$, its objective is to:

$$\max_{\mathcal{C}} \theta_1 v(x(\theta_1, \theta_2)) + \theta_2 v(x(\theta_2, \theta_1)) - t(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) - t(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)) \quad (4.7)$$

subject to:¹⁴

$$\theta_1 v(x(\theta_1, \theta_2)) - p(\theta_1, \theta_2) \geq \theta_1 v(q(\theta_1, \emptyset)) - t(\theta_1, \emptyset) \equiv S(\theta_1) \geq 0 \quad (4.8)$$

for all $\theta_1, \theta_2 \in \Theta$ and, of course, constraints (4.1) and (4.2). The implication of (4.8) is that \mathcal{C} will always implement a payment and quantity pair for all $\theta \in \Theta$ type consumers such that \mathcal{C} is always accepted regardless of whom they are matched with. For any \mathcal{C} that maximizes (4.7) subject to constraints (4.1), (4.2), and (4.8), given \mathcal{M} , is an optimal collusion contract.¹⁵ As well, this observation suggests that one may ignore the outside option $S(\theta)$ for all $\theta \in \Theta$ since rejecting the collusion contract \mathcal{C} will be off the equilibrium path.

of their collusive efforts. However, it is enough to recognize that even by choosing a suboptimal quantity, gains from cooperation may be realized.

¹⁴We may write the right hand side of (4.8) as such since it is without loss of generality to restrict our attention to the maps $t(\theta, \emptyset)$ and $x(\theta, \emptyset)$, offered by the monopolist, such that they are incentive compatible and individual rational.

¹⁵Put differently, a collusion contract is essentially a behavioral strategy for a pair of colluding consumers of type $(\theta_1, \theta_2) \in \Theta \times \Theta$ given a pricing mechanism \mathcal{M} .

4.4.2 Optimal Pricing Mechanism

We now consider the derivation of the optimal pricing mechanism. Since the pricing mechanism, \mathcal{M} , is offered to the consumers prior to \mathcal{C} , the monopolist, in choosing \mathcal{M} must take into account the introduction of \mathcal{C} in the later stages of the game. It is useful to refer to the *collusion-proofness principle* in the derivation of the optimal pricing mechanism. In the context of this paper, this principle states that it is without loss of generality to restrict our attention to pricing mechanisms \mathcal{M} such that the following conditions are satisfied:

1. There is truthful revelation from the optimal collusion contract \mathcal{C} to the pricing mechanism \mathcal{M} . In other words, $(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) = (\theta_1, \theta_2)$ for all $(\theta_1, \theta_2) \in \Theta \times \Theta$.
2. Suppose the pricing mechanism \mathcal{M} is optimal. Then the optimal collusion contract $\langle p(\theta_1, \theta_2), x(\theta_1, \theta_2), \theta_1, \theta_2 \rangle$ must be such that $p(\theta_1, \theta_2) = t(\theta_1, \theta_2)$ and $x(\theta_1, \theta_2) = q(\theta_1, \theta_2)$ for all $(\theta_1, \theta_2) \in \Theta \times \Theta$.

Heuristically, the collusion-proofness principle operates along the same line as the revelation principle. A collusion contract can essentially manipulate the report to the pricing mechanism, \mathcal{M} , and reallocate the quantity obtained by a payment scheme. In other words, there are no instruments available to the third party which are not available for the monopolist. Now suppose the monopolist finds it optimal to offer the pricing mechanism $\langle t^*, q^* \rangle$ and the third party offers the contract $\langle p^*, x^*, r_1^*, r_2^* \rangle$ such that there exists a pair $(\theta_1, \theta_2) \in \Theta \times \Theta$ where $r_1^* \neq \theta_1$ or $r_2^* \neq \theta_2$. Then for such pairs (θ_1, θ_2) , the monopolist can always implement the payment and quantity pair $t^*(r_1^*, r_2^*)$ and $q^*(r_1^*, r_2^*)$ without altering the result except in having truthful revelation from \mathcal{C} to \mathcal{M} . Furthermore, if $r_1^* = \theta_1$ and $r_2^* = \theta_2$ for all $(\theta_1, \theta_2) \in \Theta \times \Theta$, then it follows that $x^* = q^*$ and $p^* = t^*$ over the domain $\{\Theta \cup \emptyset\}^2$ is implementable. Due to the simplicity provided by the collusion-proofness principle, we first characterize the set of such pricing mechanisms.

Collusion-Proof Pricing Mechanisms

The objective of the monopolist, with the adoption of the collusion-proofness principle, may now be stated as:

$$\max_{\mathcal{M}} \int_{\Theta} \int_{\Theta} (t(\theta_1, \theta_2) + t(\theta_2, \theta_1) - c(q(\theta_1, \theta_2) + q(\theta_2, \theta_1))) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2 \quad (4.9)$$

subject to the constraints:

$$r_1(\theta_1, \theta_2) = \theta_1; \quad r_2(\theta_1, \theta_2) = \theta_2 \quad (4.10)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$:

$$\theta v(q(\theta, \emptyset)) - t(\theta, \emptyset) \geq \theta v(q(\theta', \emptyset)) - t(\theta', \emptyset) \quad (4.11)$$

for all $\theta, \theta' \in \Theta$:

$$\theta v(q(\theta, \emptyset)) - t(\theta, \emptyset) \geq 0 \quad (4.12)$$

for all $\theta \in \Theta$:

$$\theta_1 v(q(\theta_1, \theta_2)) - t(\theta_1, \theta_2) \geq S(\theta_1) \quad (4.13)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$:

$$\begin{aligned} & \theta_1 v(q(\theta_1, \theta_2)) + \theta_2 v(q(\theta_2, \theta_1)) - t(\theta_1, \theta_2) - t(\theta_2, \theta_1) \\ & \geq \theta_1 v(q(\theta'_1, \theta'_2)) + \theta_2 v(q(\theta'_2, \theta'_1)) - t(\theta'_1, \theta'_2) - t(\theta'_2, \theta'_1) \end{aligned} \quad (4.14)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$:

$$\theta_1 v'(q(\theta_1, \theta_2)) = \theta_2 v'(q(\theta_2, \theta_1)) \quad (4.15)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$ and:

$$x(\theta_1, \theta_2) = q(\theta_1, \theta_2); \quad p(\theta_1, \theta_2) = t(\theta_1, \theta_2) \quad (4.16)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$ and (4.1)-(4.2).

Notice that in this formulation, \mathcal{M} is collusion-proof due to constraints (4.10), (4.13), (4.14), (4.15), and (4.16).¹⁶ At this juncture, it is useful to make the following remarks.

¹⁶Constraints (4.10) and (4.16) follow by definition. Similarly, (4.13) appears in the original problem in the determination of the optimal collusion contract. Therefore, we are left to show that (4.14) and (4.15) imply an optimal collusion contract. This is shown in Proposition 19 in Appendix 4.7.2.

1. If constraints (4.10) and (4.16) hold, then (4.13), (4.14), and (4.15) completely characterize the optimal collusion contract, \mathcal{C} (i.e. the optimization of (4.7) subject to (4.8)).
2. Constraint (4.14) is essentially the collusion incentive compatibility constraint. While in a different context, it may also be interpreted as the incentive compatibility constraint to a particular class of multidimensional screening problems. This is particularly problematic as representation of this constraint is nontrivial, cumbersome and difficult to handle. Therefore, a simplification of the problem is in need before we can proceed with the analysis. We propose a γ -reduction to tackle this problem.

A γ -reduction

Consider the following. Let:

$$\gamma(\theta_1, \theta_2) = \{\gamma \in \mathbb{R} : (\theta_1, \theta_2) = \arg \max_{x, y} \gamma v(q(x, y) + q(y, x)) - t(x, y) - t(y, x)\} \quad (4.17)$$

In essence $\gamma(\theta_1, \theta_2)$ maps the two dimensional space $(\Theta \times \Theta)$ into a single dimension \mathbb{R} .¹⁷ We term the existence of (4.17) as a γ -reduction. However, there is a unique property regarding this reduction. Namely, given some pricing mechanism \mathcal{M} , for any pair (θ_1, θ_2) that finds it optimal to consume the payment and quantity pair as prescribed by \mathcal{M} through truthfully reporting, the fictitious $\gamma(\theta_1, \theta_2)$ type consumer will also find it optimal to consume the same payment and quantity pair along the pricing mechanism \mathcal{M} . What this allows us to accomplish is to reduce the dimensionality of the types space into a single dimension.¹⁸ This will prove to be fruitful in what follows.

Define $\hat{q}(\gamma) = q(\theta_1, \theta_2) + q(\theta_2, \theta_1)$ and $\hat{t}(\gamma) = t(\theta_1, \theta_2) + t(\theta_2, \theta_1)$. Let $\mathcal{M}^\gamma = \langle \hat{t}(\gamma), \hat{q}(\gamma) \rangle$ be termed the γ -pricing mechanism and Γ to be the target of the map $\gamma(\cdot)$ given some pricing mechanism \mathcal{M} . We have the following useful equivalence result.

¹⁷Note that this reduction is symmetric. In other words, if $\gamma = \gamma(\theta_1, \theta_2)$, then by the symmetry of the solution, $\gamma = \gamma(\theta_2, \theta_1)$. This follows since we are only dealing with aggregate levels.

¹⁸This is shown in Figure 4.1.

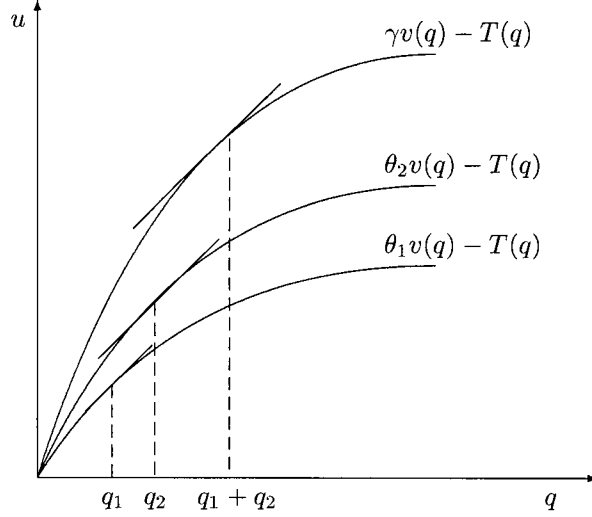


Figure 4.1: The γ -reduction

Proposition 12 *The pricing mechanism \mathcal{M} is collusion incentive compatible over $\Theta \times \Theta$ if and only if the γ -pricing mechanism \mathcal{M}^γ is incentive compatible over Γ .*

Proof: See Appendix 4.7.1.

The intuition behind Proposition 12 is simple. By construction, for any pair $(\theta_1, \theta_2) \in \Theta \times \Theta$, we find some $\gamma \in \mathbb{R}_+$ such that under the pricing mechanism \mathcal{M} they consume the same aggregate payment and quantity pair. Now if \mathcal{M} is collusion incentive compatible, then for any two pairs $(\theta_1, \theta_2) \neq (\theta'_1, \theta'_2)$ such that they consume different aggregate bundles, will map a different γ .¹⁹ Since γ and correspondingly γ' are not arbitrarily chosen but rather chosen such that given the pricing mechanism \mathcal{M} the corresponding choices are optimal, then equivalence in the relationship between collusion incentive compatibility of \mathcal{M} and incentive compatibility of \mathcal{M}^γ may be established.

However, it is obvious that this γ -reduction raises technical issues. Recall that the

¹⁹By this we mean that $q(\theta_1, \theta_2) + q(\theta_2, \theta_1) \neq q(\theta'_1, \theta'_2) + q(\theta'_2, \theta'_1)$ and similarly for t .

formulation of the γ -reduction is so that (4.14) may be restated in a simpler and tractable way. The ultimate goal is to find the optimal \mathcal{M} such that it maximizes the monopolist's objective subject to its constraints. However, if the γ -reduction is dependent on the specified pricing mechanism \mathcal{M} then γ itself is an endogenous choice. This, in itself, is problematic unless the γ -reduction is *robust* with respect to \mathcal{M} .

Definition 1 *A γ -reduction, given by the map $\gamma(\theta_1, \theta_2)$, is said to be **robust** if it satisfies (4.17) for every collusion-proof pricing mechanism and for all $(\theta_1, \theta_2) \in \Theta \times \Theta$.*

If one can find conditions under which a γ -reduction is robust, then one does not have to worry about the sensitivity of \mathcal{M} with respect to γ and vice versa. Furthermore, by defining robustness of the γ -reduction with respect to collusion-proof pricing mechanisms is without loss of generality since the ultimate goal is to derive the optimal pricing mechanism which, itself, must be collusion-proof. It is obvious that a robust γ -reduction cannot exist for all classes of utility functions. Thus, we state a sufficient condition for which a robust γ -reduction exists in the following proposition.

Proposition 13 *If $v'(\cdot)$ is a homogeneous function of degree k , then the γ -reduction is robust. Furthermore the γ -reduction is given by:*

$$\gamma(\theta_1, \theta_2) = \frac{\theta_1 \theta_2}{(\theta_1^{1/k} + \theta_2^{1/k})} \quad (4.18)$$

Proof: See Appendix 4.7.1.

The intuition for Proposition 13 is simple. For a γ -reduction to be robust the implication is that it must be independent of the chosen pricing mechanism \mathcal{M} . The reduction essentially creates a fictitious γ type individual who, ultimately, chooses the same aggregate payment and quantity pair as a coalition of type (θ_1, θ_2) . The objective of this coalition is clear; equalization of marginal utilities is necessary. Furthermore, marginal utility must be equated to the marginal cost of consuming that bundle. For this γ type individual to also find it optimal to consume this aggregate bundle, his marginal utility must also be equal

to the marginal cost of consuming this bundle. As a result, we know how each of these individuals will behave on the margin (i.e. (4.44)).²⁰

Clearly, this argument depends on the chosen pricing mechanism \mathcal{M} . However, it is only dependent on the quantity allocation rule, q , offered by the monopolist and not on the payment rule t . A homogeneous function, on the other hand, has the nice property that any two points on the function can be represented as a scalar multiple of each other. More importantly, this scalar multiple is uniquely defined if the two points are known. Since (4.44) provides us with three equations with three unknowns $(q(\theta_1, \theta_2), q(\theta_2, \theta_1), \gamma(\theta_1, \theta_2))$, we can determine what the appropriate γ -reduction is. More importantly, it will not depend on the chosen pricing mechanism \mathcal{M} as the points $q(\theta_1, \theta_2)$ and $q(\theta_2, \theta_1)$ are arbitrarily chosen.²¹

The simplification brought about by the existence of a robust γ -reduction is that it essentially solves the collusion problem. In other words, one only has to worry about how each “ γ type” individual behaves rather than how each (θ_1, θ_2) type coalition behaves.

Optimal Collusion-Proof Pricing Mechanism

With the aid of a robust γ -reduction, the determination of the optimal collusion-proof pricing mechanism is greatly simplified. Let $\Gamma = [\underline{\gamma}, \bar{\gamma}]$ where $\underline{\gamma} = \gamma(\underline{\theta}, \underline{\theta})$ and $\bar{\gamma} = \gamma(\bar{\theta}, \bar{\theta})$. We present the usual representation result for incentive compatibility of the γ -pricing mechanism \mathcal{M}^γ in the following proposition.

Proposition 14 *Suppose $v'(\cdot)$ is a homogeneous function of degree k and the γ -pricing mechanism \mathcal{M}^γ is compact valued.²² Then \mathcal{M}^γ is incentive compatible if and only if:*

$$U(\gamma) - U(\gamma') = \int_{\gamma'}^{\gamma} v(\hat{q}(s)) ds \quad (4.19)$$

²⁰It is also tempting to think that this γ -reduction is a solution to a more general multidimensional screening problem. However, the imposition of the no arbitrage condition, (4.15), is essential in the construction of this γ -reduction. Nevertheless, this technique seems promising to this natural, and perhaps more general, extension.

²¹As it turns out, $v'(\cdot)$ being a homogeneous function is also a necessary condition for the existence of a robust γ -reduction when we restrict our attention to strictly monotone pricing mechanisms \mathcal{M} in the sense that they are fully separating. This is shown in Proposition 20 in Appendix 4.7.2.

²²The interpretation of a compact valued mechanism is provided in footnote 8.

for all $\gamma, \gamma' \in \Gamma$ where:

$$U(\gamma) \equiv \gamma v(\widehat{q}(\gamma)) - \widehat{t}(\gamma) \quad (4.20)$$

and $\widehat{q}(\gamma)$ is nondecreasing.

Proof: Standard and hence omitted. ■

With the formulation of a robust γ -reduction and thus, our restriction on utility functions where $v'(\cdot)$ is a homogeneous function, the maximization of (4.9) subject to constraints (4.10)-(4.16) is greatly simplified. The existence of the γ -reduction satisfies constraints (4.10) and (4.16). Furthermore, incentive compatibility of \mathcal{M}^γ implies collusion incentive compatibility of \mathcal{M} due to Proposition 12 and so (4.14) is satisfied. Therefore, we are left with constraints (4.11), (4.12) and (4.13). Let us first consider constraints (4.11) and (4.12). Given a γ -pricing mechanism \mathcal{M}^γ offered by the monopolist, it must be the case that $q(\theta, \emptyset) = \widehat{q}(\theta)$ and $t(\theta, \emptyset) = \widehat{t}(\theta)$.²³ Therefore, for any choice of γ -pricing mechanism \mathcal{M}^γ , we have the following:

$$\theta_1 v(q(\theta_1, \theta_2)) - t(\theta_1, \theta_2) \geq \theta_1 v(\widehat{q}(\theta_1)) - \widehat{t}(\theta_1) \quad (4.21)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$. Notice that since the third party is perfectly informed about the type of each individual, there are no incentive issues when resolving $t(\theta_1, \theta_2)$. Therefore, there is one degree of freedom to satisfy (4.21) for any given pair $(\theta_1, \theta_2) \in \Theta \times \Theta$.²⁴

Thus, in the determination of the optimal collusion-proof pricing mechanism, we only need to deal with the remaining constraint:

$$\theta_1 v(q(\theta_1, \theta_2)) - t(\theta_1, \theta_2) \geq 0 \quad (4.22)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$.

²³This follows since $\theta = \gamma(\theta_1, \theta_2)$ is the best possible deviation from the collusion contract \mathcal{C} .

²⁴In essence, our determination of the optimal pricing mechanism will be silent on the individual transfers $t(\theta_1, \theta_2)$ and $t(\theta_2, \theta_1)$ with the exception of constraint (4.1). It is important to note that the choice of consuming the bundle $\widehat{q}(\theta_1)$ for a transfer of $\widehat{t}(\theta_1)$ is always weakly dominated by participating in the collusion contract \mathcal{C} . This follows since the allocation $\widehat{q}(\theta_1)$ with payment $\widehat{t}(\theta_1)$ can always be implemented by the third party.

Proposition 15 Suppose $v'(\cdot)$ is a homogeneous function of degree k . Then the solution to the optimization of (4.9) subject to (4.10)-(4.16) is determined, up to location, by the solution to the optimization of:

$$\max_{\hat{t}(\gamma), \hat{q}(\gamma)} \int_{\Gamma} (\hat{t}(\gamma) - c\hat{q}(\gamma))g(\gamma)d\gamma \quad (4.23)$$

subject to:

$$\gamma v(\hat{q}(\gamma)) - \hat{t}(\gamma) \geq \gamma v(\hat{q}(\gamma')) - \hat{t}(\gamma') \quad (4.24)$$

for all $\gamma, \gamma' \in \Gamma$ and:

$$\gamma v(\hat{q}(\gamma)) - \hat{t}(\gamma) \geq \underline{U} \quad (4.25)$$

for some $\underline{U} \in \mathbb{R}$.²⁵

Proof: See Appendix 4.7.1.

In essence, building on the preceding argument, we simply need to find a \underline{U} such that the optimal pricing mechanism is identified in the sense that constraint (4.22) is satisfied. Notice that the optimization problem stated in Proposition 15 is just the usual monopoly nonlinear pricing with one dimensional uncertainty (i.e., the benchmark case). Thus existing theories may be applied to solve this optimization problem. In other words, the objective function of the monopolist may be written as:

$$\max_{\hat{t}(\gamma), \hat{q}(\gamma)} \int_{\Gamma} (I(\gamma)v(\hat{q}(\gamma)) - c\hat{q}(\gamma) - \underline{U})g(\gamma)d\gamma \quad (4.26)$$

subject to:

$$\hat{t}(\gamma) = \gamma v(\hat{q}(\gamma)) - \int_{\underline{\gamma}}^{\gamma} v(\hat{q}(s))ds - \underline{U} \quad (4.27)$$

and $\hat{q}(\gamma)$ is nondecreasing where:²⁶

$$I(\gamma) = \gamma - \frac{1 - G(\gamma)}{g(\gamma)} \quad (4.28)$$

²⁵Note that $g(\gamma)$ is the density function of γ . This is derived in Lemma 11 in Appendix 4.7.2.

²⁶ $G(\gamma)$ is the distribution function of γ over Γ . In general, this computation is nontrivial. An example is provided in Appendix 4.7.3.

Now suppose the γ -pricing mechanism $\langle \hat{t}^*(\gamma), \hat{q}^*(\gamma) \rangle$ solves this maximization problem. Define $\hat{\gamma} = \max\{\gamma \in \Gamma : \inf\{\hat{q}^*(\gamma) > 0\}\}$. In other words, $\hat{\gamma}$ will denote the lowest γ type consumer being served.

Proposition 16 *Suppose the γ -pricing mechanism $\mathcal{M}^{\gamma*} = \langle \hat{t}^*(\gamma), \hat{q}^*(\gamma) \rangle$ maximizes (4.26) subject to (4.27) and to the monotonicity condition on $\hat{q}^*(\gamma)$. Then $\mathcal{M}^{\gamma*}$ is an optimal collusion-proof pricing mechanism if:*

$$\underline{U} = \hat{\gamma}v(\hat{q}(\hat{\gamma})) - \hat{\theta}_1v(q^*(\hat{\theta}_1, \hat{\theta}_2)) - \hat{\theta}_2v(q^*(\hat{\theta}_2, \hat{\theta}_1)) \quad (4.29)$$

where the pair $(\hat{\theta}_1, \hat{\theta}_2)$ maximizes $\theta_1v(q^*(\theta_1, \theta_2)) + \theta_2v(q^*(\theta_2, \theta_1))$ along the level curve $\gamma(\theta_1, \theta_2) = \hat{\gamma}$.

Proof: See Appendix 4.7.1.

The idea behind Proposition 16 is to determine the optimal \underline{U} in the sense that constraint (4.13), or equivalently (4.22), is satisfied in the optimization of the original problem. Notice that one, in general, cannot replace (4.13) with its counterpart under the γ -reduction such as:

$$\gamma v(\hat{q}(\gamma)) - \hat{t}(\gamma) \geq 0$$

for all $\gamma \in \Gamma$. This is due to the fact that $\gamma v(\hat{q}(\gamma))$ is generally different from $\theta_1v(q(\theta_1, \theta_2)) + \theta_2v(q(\theta_2, \theta_1))$. The fact that an optimal collusion-proof pricing mechanism \mathcal{M} must extract all surplus from the coalitions which consume the lowest bundle offered allows us to determine what \underline{U} is.

Not surprisingly, we also find that the monopolist is strictly worse off when consumers are able to collude. This follows since the ultimate pricing mechanism \mathcal{M} in this model must satisfy all the constraints of the benchmark model as well as the no arbitrage condition (4.15). Since (4.15) is not satisfied in the benchmark model, profits must be strictly lower. This finding is consistent with those in Jeon and Menicucci [21].

Remarks

In determining the solution to the original maximization problem of the monopolist, we proposed a method for which a robust γ -reduction is constructed. In summary, Propositions 15 and 16 allow us to map the multidimensional screening problem (i.e. the collusion incentive compatibility constraint) into a unidimensional one where existing theories may be applied. Without explicitly solving for the optimal collusion-proof pricing mechanism, some general results may still be obtained. For example, the robust γ -reduction essentially implies that the optimal pricing mechanism may be implemented by an indirect mechanism of the form $t(q) = \hat{t}(\gamma(q))$ where $\gamma(q) = \hat{q}^{-1}(\gamma)$.²⁷ This makes intuitive sense if one considers the objective of an optimal collusion contract \mathcal{C} . If such an indirect mechanism $t(q)$ does not exist, then consider some quantity level q such that for some reports (θ_1, θ_2) and (θ'_1, θ'_2) such that $\hat{q}(\gamma(\theta_1, \theta_2)) = \hat{q}(\gamma(\theta'_1, \theta'_2))$ but $\hat{t}(\gamma(\theta_1, \theta_2)) < \hat{t}(\gamma(\theta'_1, \theta'_2))$. Then the report of (θ'_1, θ'_2) for the pair (θ'_1, θ'_2) to the monopolist is always suboptimal since a report of (θ_1, θ_2) will always improve utility. Thus, the collusion incentive compatibility constraint essentially forces the optimal direct pricing mechanism to be implementable by an indirect mechanism in the form of a price schedule. Put differently, an optimal collusion contract will only consider the lower envelope of all price schedules offered by the monopolist.

In the determination of the optimal pricing mechanism, we have simply found the optimal \mathcal{M}^γ . In other words, we have found the γ -pricing mechanism $\hat{q}(\gamma)$ and $\hat{t}(\gamma)$. However (4.45) allows us to pin down $q(\theta_1, \theta_2)$ given \mathcal{M}^γ as:

$$\hat{q}(\gamma) = q(\theta_1, \theta_2) + q(\theta_2, \theta_1) = \frac{\theta_1^{1/k} + \theta_2^{1/k}}{\theta_1^{1/k}} q(\theta_2, \theta_1)$$

which leads to:

$$q(\theta_1, \theta_2) = \frac{\theta_2^{1/k}}{\theta_1^{1/k} + \theta_2^{1/k}} \hat{q}(\gamma)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$. However, the analysis has been silent regarding the individual transfers $t(\theta_1, \theta_2)$ and $t(\theta_2, \theta_1)$. The reason is there potentially exists an infinite number of

²⁷Invertibility is assured since incentive compatibility of the γ -pricing mechanism \mathcal{M}^γ places a monotonicity condition on the map $\hat{q}(\gamma)$.

transfers such that the γ -pricing mechanism \mathcal{M}^γ may be supported. This follows since we have the restriction $\hat{t}(\gamma) = t(\theta_1, \theta_2) + t(\theta_2, \theta_1)$, from (4.1), and $t(\theta_1, \theta_2) \leq \theta_1 v(q(\theta_1, \theta_2)) - (\theta_1 v(\hat{q}(\theta_1)) - \hat{t}(\theta_1))$ for all $(\theta_1, \theta_2) \in \Theta \times \Theta$ from (4.13). In other words, given some pair (θ_1, θ_2) such that either (4.13) is slack for either θ_1 or θ_2 there are gains to cooperation.²⁸ Therefore, the gain in surplus may be divided in a way such that both parties are willing to participate.²⁹

In addition, we find that the optimal pricing mechanism is dependent on the pair of report $(\theta_1, \theta_2) \in \Theta \times \Theta$ implying the effective use of complete contracts. In other words, an allocation and payment for a pair of individuals colluding will depend on the coalition's type (θ_1, θ_2) . As a result, each individual's utility is then dependent on whom one is colluding with.

Lastly, in relation to the question of quantity discounts of the optimal pricing mechanism, one essentially has to specify three key components. Firstly, the outside utility of a consumer in the market is essential. In this paper, we have assumed that the outside utility is normalized to zero. However, quantity discounts depend crucially on this assumption.³⁰ Secondly, the chosen distribution over Θ , $F(\theta)$, essentially determines the distribution of the γ -reduction. Since $I(\gamma)$, as stated in (4.28), identifies the optimal quantity rule $\hat{q}(\theta)$, quantity discounts will be exhibited only for appropriate choices. Lastly, the types space, Θ , will also partially play a role in determining whether the optimal pricing mechanism will exhibit quantity discounts everywhere.³¹ In summary, quantity discounts rely on the optimal indirect mechanism $t(q)$ to be both concave and having a nonnegative intercept,

²⁸Therefore, the collusion contract will reap the full benefits of this gain.

²⁹Notice, as well, that this argument does not rely on the actual structure of the optimal indirect mechanism $t(q)$. For example, if $t(q)$ is linear and passes through the origin, then there are no realization of gains. On the other hand, for any nonlinear price schedule $t(q)$, such gains will always exist. For quantity discounts, the argument is transparent. For quantity premiums, if $\theta_1 > \theta_2$, gains may be realized by asking the θ_1 type individual to purchase a little less than his optimal amount and in return, ask the θ_2 type individual to purchase a little more than his optimal amount. The goods may then be reallocated with a price such that both are willing to participate in the collusion contract \mathcal{C} . However, there exists a continuum of prices that makes this feasible. As a result, the optimal γ -pricing mechanism \mathcal{M}^γ determined here is unable to pin down the exact transfers $t(\theta_1, \theta_2)$ associated with the pricing mechanism \mathcal{M} .

³⁰See, for example, Kwong [24].

³¹See, for example, a brief discussion in Alger [1].

roughly speaking.³² It is easy to see that strict concavity of $t(q)$ alone is insufficient in generating quantity discounts since rays from the origin along $t(q)$ may not necessarily be monotone decreasing.³³

4.5 Extensions

In this section of the paper we consider several extensions to the model presented in the preceding section. Firstly, the determination of the optimal pricing mechanism a monopolist will offer when faced with asymmetric coalition sizes will be analyzed. Secondly, we will examine the implications of the above results to a more general utility function for each consumer.

4.5.1 Asymmetric Coalition Sizes

The assumption that all coalitions are comprised of a pair may be somewhat restrictive. Therefore, we consider a more general coalition formation structure. We simplify the consideration of coalition sizes by restricting our attention to finite sizes.³⁴

Let n denote the size of a coalition where $n \in \{1, \dots, N\}$ with $N < \infty$. Denote $p_s = \Pr(n = s)$ to be the probability that a coalition of size s is formed. Given a coalition of size n , let θ_{-i} denote the projection of the profile of types within the coalition $\theta \in \Theta^N$ less the i -th dimension. For example, if the profile of types for a coalition of size 3 is $\{\theta_1, \theta_2, \theta_3\}$ and suppose $N = 5$ then $\theta_{-1} = \{\theta_2, \theta_3, \emptyset, \emptyset\}$.

³²Note that an optimal price schedule will never have a negative intercept as this implies that for some quantity levels, consumption is subsidized.

³³We present a general result regarding quantity discounts of the optimal γ -pricing mechanism \mathcal{M}^γ in Proposition 21 in Appendix 4.7.2.

³⁴Note that a large number of consumers (a continuum) was assumed in the original formulation of the problem because it helps to simplify the problem. In particular, if the number of consumers is finite, then a problem regarding an “odd” or “even” number of consumers arises. If the number of consumers is odd, then there always exists an additional unpaired consumer and thus, in the construction of the optimal pricing mechanism, this may have an effect. In the case of considering different coalition sizes, we restrict our attention to coalitions such that no one is left unpaired. Conversely, it is not clear how one may construct a robust γ -reduction if the size of the coalition is infinite. A discussion regarding this effect on the determination of the optimal pricing mechanism is provided in Alger [1].

The objective of the third party is to:

$$\max_{\mathcal{C}} \sum_{i=1}^N (\theta_i v(x(\theta_i, \theta_{-i})) - t(r_1(\theta), \dots, r_n(\theta))) \quad (4.30)$$

subject to the balance of payment, balance of quantities, incentive compatibility, and individual rationality constraints.³⁵

We are, therefore, seeking for a collusion-proof pricing mechanism where $r_i(\theta) = \theta_i$ for all $\theta \in \Theta^N$. We have a generalization of Proposition 13 below.

Proposition 17 *If $v'(\cdot)$ is a homogeneous function of degree k , then the γ -reduction is robust. Furthermore, the γ -reduction is given by:*³⁶

$$\gamma_n(\theta) = \frac{\prod_{i=1}^n \theta_i}{\left(\sum_{i=1}^n \prod_{j \neq i} \theta_j^{1/k}\right)^k} \quad (4.31)$$

Proof: Straightforward given the proof of Proposition 13. ■

Given the appropriate γ -reduction, the objective of the monopolist is now simple. More importantly, we can write it in a very simple way. Namely:³⁷

$$\max_{\mathcal{M}^\gamma} \sum_{j=1}^N p_j \int_{\Gamma_j} (\hat{t}(\gamma) - cq(\gamma)) g_j(\gamma) d\gamma = \max_{\mathcal{M}^\gamma} \int_{\Gamma_j} (\hat{t}(\gamma) - cq(\gamma)) \sum_{j=1}^N p_j g_j(\gamma) d\gamma \quad (4.32)$$

subject to the incentive compatibility and individual rationality constraints.

Since the incentive compatibility and individual rationality constraints across any coalition size j are identical, the linearity of the problem in (4.32) is simple. More importantly, one can view γ as being drawn from a single distribution $G(\gamma)$ with density $g(\gamma) = \sum p_j g_j(\gamma)$ over $\Gamma = \cup \Gamma_j$.

Notice that in this framework, the problem is simplified by the existence of a robust γ -reduction not only because the collusion problem, for a given coalition size, is solved. Instead, one does not have to be concerned about the incentive constraints regarding a

³⁵These constraints are obviously generalized versions of (4.1), (4.2), (4.14), and, (4.13), respectively, and are suppressed here for simplicity.

³⁶We index the γ -reduction by n to denote a reduction for a coalition of size n .

³⁷We also let $g_j(\gamma)$ denote the density over Γ_j when the coalition is of size j .

coalition of size n pretending to be a coalition of size n' .³⁸ This follows since the optimal γ -pricing mechanism under the γ -reduction, \mathcal{M}^γ , is incentive compatible. Thus, any deviations from truthful revelation may be mapped to a potentially different γ .

4.5.2 Generalized Utility Function

We now consider a more general utility function for consumers. Suppose utility is given by the map $v : \mathbb{R}_+ \times \Theta \rightarrow \mathbb{R}_+$. More specifically, a consumer of type θ who consumes q units of a good for a payment of t receives net utility:

$$u(\theta, q) = v(\theta, q) - t \quad (4.33)$$

where $v \in C^2$, $v_\theta > 0$, $v_q > 0$, $v_{qq} < 0$, and $v_{\theta q} > 0$.³⁹

The main question then is whether a robust γ -reduction, much like the one derived in Section 4.4, exists in the framework of the basic model. As it turns out, a similar condition to that derived in Proposition 13 is sufficient. This finding is summarized in the following proposition.

Proposition 18 *Suppose $v_q(\theta, q)$ is a homogeneous function of degree k in q and $v_{\theta q} > 0$, then the γ -reduction is robust. Furthermore, the γ -reduction is defined implicitly by:*

$$v_q(\gamma(\theta_1, \theta_2), 1) = \frac{v_q(\theta_1, 1)v_q(\theta_2, 1)}{(v_q(\theta_1, 1)^{1/k} + v_q(\theta_2, 1)^{1/k})^k} \quad (4.34)$$

Proof: The proof of this is analogous to that of Proposition 13. More specifically, for any given pricing mechanism \mathcal{M} , we have the condition that:

$$v_q(\theta_1, q(\theta_1, \theta_2)) = v_q(\theta_2, q(\theta_2, \theta_1)) = v_q(\gamma, q(\theta_1, \theta_2) + q(\theta_2, \theta_1))$$

Now suppose v_q is a homogeneous function of degree k in q . Then the left-hand side of the above condition reduces to:

$$q(\theta_1, \theta_2) = \left(\frac{v_q(\theta_2, 1)}{v_q(\theta_1, 1)} \right)^{1/k} q(\theta_2, \theta_1)$$

³⁸For example, a more complete statement of the problem will include incentive constraints such as $\sum (\theta_i v(q(\theta_i, \theta_{-i}, \emptyset_{N-n})) - t(\theta_i, \theta_{-i}, \emptyset_{N-n})) \geq \sum (\theta_i v(q(\theta', \theta_{-i}, \emptyset_{N-n})) - t(\theta'))$ for all $\theta' \in \Theta^N$.

³⁹Note that we maintain the assumption of quasilinearity of the utility function.

This, combined with the right-hand side gives (4.34). Given the single crossing property, $v_{\theta q} > 0$ and that $v \in C^2$, this guarantees the monotonicity of v_q in θ thus invertibility. Furthermore, noting that (4.34) is independent of the chosen mechanism \mathcal{M} implies its robustness. ■

The above result suggests that homogeneity of the marginal utility function in q is sufficient in granting the existence of a robust γ -reduction which, as already shown, is suitable for solving the class of collusion problems where consumers are perfectly informed with whom they are colluding with. However, it is also important to note that requiring homogeneity of the marginal utility function in q is a very restrictive condition. This condition boils down to have the utility function v being multiplicatively separable in its arguments. To see this, if v_q is homogeneous of degree k in q , then this implies that:

$$v_q(\theta, q) = q^k v_q(\theta, 1) = q^k \tilde{v}(\theta)$$

where $\tilde{v}(\theta) \equiv v_q(\theta, 1)$. Hence, the utility function must be of the form $v(\theta, q) = \alpha(\theta)\beta(q) + \delta(\theta)$ where α , β and δ are functions.

This finding, however, should not come as a surprise given the earlier result as stated in Proposition 13. Homogeneity of the marginal utility function in q allows different points along the function to be determined as scalar multiples of some base point. Ultimately, this allows us to pin down the robust γ -reduction.

4.6 Concluding Remarks

We have studied a general monopoly pricing problem where consumers are privately informed about their taste parameter, while being able to form coalitions. We model the collusion problem by introducing a third party with an objective of maximizing the sum of utilities within a given coalition. We acknowledge that this third party is a modelling shortcut in that it is a reduced form of a more complex problem where individuals may be involved in a bargaining game.⁴⁰ This is evident given that the optimal pricing mecha-

⁴⁰See, for example, Laffont and Martimort [29].

nism constructed is silent on the individual transfers but addresses the aggregate levels of consumption and payment by a coalition only.

We find that introducing collusion into the monopolist's pricing problem raises a number of technical issues. Firstly, there exists a no arbitrage condition, (4.15), that distorts the optimal pricing mechanism a monopolist will offer in the absence of collusion. Secondly, truthful revelation from the collusion contract \mathcal{C} to the pricing mechanism \mathcal{M} , induced by invoking the collusion-proofness principle, turns into a collusion incentive compatibility constraint (4.14) which parallels a multidimensional incentive compatibility constraint. To simplify the collusion problem, we introduce the notion of a robust γ -reduction which essentially mimics the collusion problem but in a single dimension. A sufficient condition for the existence of this robust γ -reduction is characterized.

The results suggest that in the face of collusion, the problem is reducible into a problem resembling our benchmark case where no collusion occurs. Thus, representation of the collusion problem is drastically simplified since existing theories may then be applied. Our results also extend beyond the formation of coalitions into pairs. We show that the sufficient condition characterized is capable of extensions into arbitrarily large, but finite, coalition structures. This allows us to examine cases where coalition sizes may be asymmetric across groups.

In conclusion, we find that a price discriminating policy is potentially implementable even when consumers may collude and engage in resale. Consequently, limited arbitrage between consumers appears to be insufficient in disrupting the profitability and desirability of price discrimination for the monopolist.

4.7 Appendix

4.7.1 Proofs of Results

Proof of Proposition 12: Suppose \mathcal{M} is collusion incentive compatible over $\Theta \times \Theta$ but \mathcal{M}^γ is not incentive compatible over Γ . This implies that for all pairs $(\theta_1, \theta_2), (\theta'_1, \theta'_2) \in \Theta \times \Theta$, (4.14) holds. The γ -pricing mechanism \mathcal{M}^γ is not incentive compatible implies that there exists some $\gamma, \gamma' \in \Gamma$ such that:

$$\gamma v(\widehat{q}(\gamma)) - \widehat{t}(\gamma) < \gamma v(\widehat{q}(\gamma')) - \widehat{t}(\gamma') \quad (4.35)$$

Without loss of generality, let $\gamma = \gamma(\theta_1, \theta_2)$ and $\gamma' = \gamma(\theta'_1, \theta'_2)$. By definition $\widehat{q}(\gamma') = q(\theta'_1, \theta'_2) + q(\theta'_2, \theta'_1)$ and $\widehat{t}(\gamma') = t(\theta_1, \theta_2) + t(\theta_2, \theta_1)$. And thus, a contradiction is drawn since by definition:

$$(\theta_1, \theta_2) = \arg \max_{x, y} \gamma v(q(x, y) + q(y, x)) - t(x, y) - t(y, x) \quad (4.36)$$

Therefore, collusion incentive compatibility of \mathcal{M} implies incentive compatibility of \mathcal{M}^γ . Now suppose \mathcal{M}^γ is incentive compatible. Then for all $\gamma, \gamma' \in \Gamma$:

$$\gamma v(\widehat{q}(\gamma)) - \widehat{t}(\gamma) \geq \gamma v(\widehat{q}(\gamma')) - \widehat{t}(\gamma') \quad (4.37)$$

If \mathcal{M} is not collusion incentive compatible, then there exist some pairs $(\theta_1, \theta_2), (\theta'_1, \theta'_2) \in \Theta \times \Theta$ such that (4.14) does not hold. Without loss of generality, suppose a pair (θ_1, θ_2) finds it optimal to report (θ'_1, θ'_2) . By construction $\widehat{q}(\gamma') = q(\theta'_1, \theta'_2) + q(\theta'_2, \theta'_1)$ and $\widehat{t}(\gamma') = t(\theta'_1, \theta'_2) + t(\theta'_2, \theta'_1)$ which contradicts the assumption that \mathcal{M}^γ is incentive compatible. ■

Proof of Proposition 13: Consider a collusion-proof pricing mechanism \mathcal{M} . Since \mathcal{M} is a collusion-proof pricing mechanism, it implies that it must satisfy the objectives of the collusion contract \mathcal{C} . Now consider the maximization of (4.7) subject to its relevant constraints. The first order conditions are:

$$\theta_1 v'(x(\theta_1, \theta_2)) - \theta_2 v'(q(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) + q(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)) - x(\theta_1, \theta_2)) = 0 \quad (4.38)$$

$$\begin{aligned} \theta_1 v'(x(\theta_1, \theta_2)) & \frac{\partial(q(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) + q(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)))}{\partial r_1(\theta_1, \theta_2)} \\ & - \frac{\partial(t(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) + t(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)))}{\partial r_1(\theta_1, \theta_2)} = 0 \end{aligned} \quad (4.39)$$

$$\begin{aligned} \theta_1 v'(x(\theta_1, \theta_2)) & \frac{\partial(q(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) + q(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)))}{\partial r_2(\theta_1, \theta_2)} \\ & - \frac{\partial(t(r_1(\theta_1, \theta_2), r_2(\theta_1, \theta_2)) + t(r_2(\theta_1, \theta_2), r_1(\theta_1, \theta_2)))}{\partial r_2(\theta_1, \theta_2)} = 0 \end{aligned} \quad (4.40)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$. Furthermore, collusion-proofness implies that $r_1(\theta_1, \theta_2) = \theta_1$, $r_2(\theta_1, \theta_2) = \theta_2$, and $q(\theta_1, \theta_2) = x(\theta_1, \theta_2)$ for all $(\theta_1, \theta_2) \in \Theta \times \Theta$.

Now consider the γ -reduction for this pricing mechanism \mathcal{M} . From Proposition 12 we know that \mathcal{M}^γ is incentive compatible. Thus:

$$(\theta_1, \theta_2) = \arg \max_{x, y} \gamma(\theta_1, \theta_2) v(q(x, y) + q(y, x)) - t(x, y) - t(y, x) \quad (4.41)$$

The associated first order conditions imply:

$$\gamma v'(q(\theta_1, \theta_2) + q(\theta_2, \theta_1)) \frac{\partial(q(\theta_1, \theta_2) + q(\theta_2, \theta_1))}{\partial \theta_1} - \frac{\partial(t(\theta_1, \theta_2) + t(\theta_2, \theta_1))}{\partial \theta_1} = 0 \quad (4.42)$$

$$\gamma v'(q(\theta_1, \theta_2) + q(\theta_2, \theta_1)) \frac{\partial(q(\theta_1, \theta_2) + q(\theta_2, \theta_1))}{\partial \theta_2} - \frac{\partial(t(\theta_1, \theta_2) + t(\theta_2, \theta_1))}{\partial \theta_2} = 0 \quad (4.43)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$ along the γ -reduction. From (4.38)-(4.40), (4.42) and (4.43), we have that for all $(\theta_1, \theta_2) \in \Theta \times \Theta$ and for γ along the γ -reduction:⁴¹

$$\gamma v'(q(\theta_1, \theta_2) + q(\theta_2, \theta_1)) = \theta_1 v'(q(\theta_1, \theta_2)) = \theta_2 v'(q(\theta_2, \theta_1)) \quad (4.44)$$

Therefore, for any γ -reduction, (4.44) is derived for a collusion-proof pricing mechanism \mathcal{M} .

Now suppose v' is a homogeneous function of degree k . Then (4.44) implies that:

$$q(\theta_1, \theta_2) = \left(\frac{\theta_2}{\theta_1} \right)^{1/k} q(\theta_2, \theta_1) \quad (4.45)$$

⁴¹This results after noting the following two points. First, $\partial q(\theta_1, \theta_2)/\partial \theta_1 = \partial q(\theta_2, \theta_1)/\partial \theta_1 = 0$ is an impossibility. This is shown in Lemma 9 in Appendix 4.7.2. Second, if $\partial q(\theta_1, \theta_2)/\partial \theta_1 + \partial q(\theta_2, \theta_1)/\partial \theta_1 = 0$ and $\partial q(\theta_1, \theta_2)/\partial \theta_2 + \partial q(\theta_2, \theta_1)/\partial \theta_2 = 0$ then $q(\theta_1, \theta_2) + q(\theta_2, \theta_1)$ is not changing. Since \mathcal{M} may be arbitrarily chosen, simply pick one such that either $\partial q(\theta_1, \theta_2)/\partial \theta_1 + \partial q(\theta_2, \theta_1)/\partial \theta_1 \neq 0$ or $\partial q(\theta_1, \theta_2)/\partial \theta_2 + \partial q(\theta_2, \theta_1)/\partial \theta_2 \neq 0$.

More importantly:

$$\gamma v'(q(\theta_1, \theta_2) + q(\theta_2, \theta_1)) = \gamma \left(\frac{\theta_1^{1/k} + \theta_2^{1/k}}{\theta_2^{1/k}} \right)^k v'(q(\theta_2, \theta_1)) = \theta_2 v'(q(\theta_2, \theta_1)) \quad (4.46)$$

This further implies (4.18) for all $(\theta_1, \theta_2) \in \Theta \times \Theta$. Since $\gamma(\theta_1, \theta_2)$ as specified in (4.18) is independent of the pricing mechanism \mathcal{M} , it is robust. ■

Proof of Proposition 15: By definition $\hat{t}(\gamma(\theta_1, \theta_2)) = t(\theta_1, \theta_2) + t(\theta_2, \theta_1)$ and $\hat{q}(\gamma(\theta_1, \theta_2)) = q(\theta_1, \theta_2) + q(\theta_2, \theta_1)$. The objective function in (4.9) may be rewritten as:

$$\int_{\Theta} \int_{\Theta} (\hat{t}(\gamma(\theta_1, \theta_2)) - c\hat{q}(\gamma(\theta_1, \theta_2))) f(\theta_1) f(\theta_2) d\theta_1 d\theta_2$$

which is the expected value over γ since the γ -reduction given a homogeneous function k is robust. Therefore, (4.23) is established. It has been noted that the existence of a robust γ -reduction implies constraints (4.10) and (4.16). Incentive compatibility of \mathcal{M}^γ implies collusion incentive compatibility of \mathcal{M} . Therefore, we are left to check that (4.13) remains to be satisfied in this optimization. Since \underline{U} in (4.25) is a choice variable, for appropriate choice, it will necessarily satisfy (4.13) and be optimal. ■

Proof of Proposition 16: First note that $\underline{U} = U(\hat{\gamma}) = \hat{\gamma} v(\hat{q}^*(\hat{\gamma})) - \hat{t}^*(\hat{\gamma})$. Since \underline{U} is a constant and enters negatively into the objective function of the monopolist, an optimal pricing mechanism must minimize this constant. As well, optimality of the pricing mechanism \mathcal{M} will imply that all surplus is extracted from the coalition consuming this bundle. This implies that along the level curve of $\gamma(\theta_1, \theta_2) = \hat{\gamma}$.⁴²

$$\hat{t}^*(\hat{\gamma}) = t^*(\theta_1, \theta_2) + t^*(\theta_2, \theta_1) = \theta_1 v(q^*(\theta_1, \theta_2)) + \theta_2 v(q^*(\theta_2, \theta_1)) \quad (4.47)$$

Denote the pair $(\theta_1, \theta_2) \in \{\gamma(\theta_1, \theta_2) = \hat{\gamma}\}$ such that $\hat{t}^*(\hat{\gamma})$ is maximized as $(\hat{\theta}_1, \hat{\theta}_2)$. Then, it follows that (4.29) holds. Therefore, the optimal collusion incentive compatible

⁴²We also require that utility along the equilibrium path is increasing in θ for all $\theta \in \Theta$. This follows since for some $\theta_2 \in \Theta$, we require (4.13) to hold for all $\theta_1 \in \Theta_1$. The right hand side of (4.13) is given by $S(\theta_1)$. Thus, if $S(\theta_1)$ is increasing in θ_1 , then extracting full surplus from the pair $(\hat{\theta}_1, \hat{\theta}_2) \in \Theta \times \Theta$ (see below) will place enough structure on $t^*(\theta_1, \theta_2)$ such that (4.13) is satisfied for all $\theta_1 \in \Theta$ for any $\theta_2 \in \Theta$. In other words, if the right hand side of (4.13) is nondecreasing, then so, too, must the left hand side of (4.13) for θ_1 which minimizes (4.13) for a given θ_2 . That $S(\theta_1)$ is nondecreasing is shown in Lemma 12 in Appendix 4.7.2.

and collusion-proof pricing mechanism \mathcal{M} is identified since $\widehat{q}^*(\gamma) = q^*(\theta_1, \theta_2) + q^*(\theta_2, \theta_1)$ and $\widehat{t}^*(\gamma) = t^*(\theta_1, \theta_2) + t^*(\theta_2, \theta_1)$ for all $(\theta_1, \theta_2) \in \Theta \times \Theta$ along the γ -reduction. ■

4.7.2 Other Results

Proposition 19 *Given a pricing mechanism \mathcal{M} , if the collusion contract \mathcal{C} is such that:*

$$\theta_1 v'(x(\theta_1, \theta_2)) = \theta_2 v'(x(\theta_2, \theta_1)) \quad (4.48)$$

and:

$$\begin{aligned} & \theta_1 v(x(\theta_1, \theta_2)) + \theta_2 v(q(r_1, r_2) + q(r_2, r_1) - x(\theta_1, \theta_2)) - t(r_1, r_2) - t(r_2, r_1) \\ & \geq \theta_1 v(x(\theta_1, \theta_2)) + \theta_2 v(q(r'_1, r'_2) + q(r'_2, r'_1) - x(\theta_1, \theta_2)) - t(r'_1, r'_2) - t(r'_2, r'_1) \end{aligned} \quad (4.49)$$

for all $(\theta_1, \theta_2) \in \Theta \times \Theta$ and (4.1) and (4.13) are satisfied, then \mathcal{C} is an optimal collusion contract.

Proof: Suppose \mathcal{C} is an optimal collusion contract given \mathcal{M} . Then, this implies that there exists no redistribution of quantities between the two individuals given a report of (r_1, r_2) . This implies that:

$$\begin{aligned} 0 = \arg \max_y & \theta_1 v(x(\theta_1, \theta_2) - y) + \theta_2 v(q(r_1, r_2) + q(r_2, r_1) - x(\theta_1, \theta_2) + y) \\ & - t(r_1, r_2) - t(r_2, r_1) \end{aligned}$$

The first order conditions, along with (4.2), imply (4.48). Given (4.48), (4.49) is straightforward since if it does not hold, then there exists some pair $(\theta_1, \theta_2) \in \Theta \times \Theta$ such that a report of (r'_1, r'_2) is optimal. ■

Proposition 20 *For the class of strictly monotonic pricing mechanisms \mathcal{M} , if a γ -reduction is robust then $v'(\cdot)$ is a homogeneous function.*

Proof: Notice that if \mathcal{M} is strictly monotonic then either $\partial q(\theta_1, \theta_2)/\partial \theta_1 + \partial q(\theta_2, \theta_1)/\partial \theta_1 \neq 0$ or $\partial q(\theta_1, \theta_2)/\partial \theta_2 + \partial q(\theta_2, \theta_1)/\partial \theta_2 \neq 0$. Thus, (4.44) holds. Now consider two collusion-proof pricing mechanisms \mathcal{M} and $\widetilde{\mathcal{M}}$. From (4.44), we have:

$$\gamma(\theta_1, \theta_2) = \frac{\theta_1 v'(q(\theta_1, \theta_2))}{v'(q(\theta_1, \theta_2) + q(\theta_2, \theta_1))} = \frac{\theta_2 v'(q(\theta_2, \theta_1))}{v'(q(\theta_1, \theta_2) + q(\theta_2, \theta_1))} \quad (4.50)$$

for the pricing mechanism \mathcal{M} and:

$$\tilde{\gamma}(\theta_1, \theta_2) = \frac{\theta_1 v'(\tilde{q}(\theta_1, \theta_2))}{v'(\tilde{q}(\theta_1, \theta_2) + \tilde{q}(\theta_2, \theta_1))} = \frac{\theta_2 v'(\tilde{q}(\theta_2, \theta_1))}{v'(\tilde{q}(\theta_1, \theta_2) + \tilde{q}(\theta_2, \theta_1))} \quad (4.51)$$

for the pricing mechanism $\tilde{\mathcal{M}}$. Since \mathcal{M} and $\tilde{\mathcal{M}}$ may be arbitrarily chosen, let $\tilde{q}(\theta_1, \theta_2) = \alpha q(\theta_1, \theta_2)$ for all (θ_1, θ_2) and for some $\alpha > 0$. A γ -reduction being robust implies that for all $\alpha > 0$ and $(\theta_1, \theta_2) \in \Theta \times \Theta$:

$$\frac{\theta_1 v'(q(\theta_1, \theta_2))}{v'(q(\theta_1, \theta_2) + q(\theta_2, \theta_1))} = \frac{\theta_1 v'(\tilde{q}(\theta_1, \theta_2))}{v'(\tilde{q}(\theta_1, \theta_2) + \tilde{q}(\theta_2, \theta_1))} = \frac{\theta_1 v'(\alpha q(\theta_1, \theta_2))}{v'(\alpha(q(\theta_1, \theta_2) + q(\theta_2, \theta_1)))} \quad (4.52)$$

which holds only if $v'(\cdot)$ is a homogeneous function.⁴³ ■

Lemma 9 *Suppose \mathcal{M} is a collusion-proof pricing mechanism. For all $(\theta_i, \theta_j) \in \Theta \times \Theta$ such that $\min\{q(\theta_i, \theta_j), q(\theta_j, \theta_i)\} > 0$, if $q_{\theta_i}(\theta_i, \theta_j) = 0$ then $q_{\theta_i}(\theta_j, \theta_i) > 0$.*

Proof: Suppose for some $\theta_j \in \Theta$, $q_{\theta_i}(\theta_i, \theta_j) = 0$. This implies that there exists some $\theta'_i \in \Theta$ such that $q(\theta_i, \theta_j) = q(\theta'_i, \theta_j)$ with $\theta_i \neq \theta'_i$. Without loss of generality, assume that $\theta_i > \theta'_i$. Since \mathcal{M} is collusion-proof, there must not exist a reallocation of quantities that will improve the sum of utilities. This implies that:

$$q(\theta_i, \theta_j) = \arg \max_x \theta_i v(x) + \theta_j v(q(\theta_i, \theta_j) + q(\theta_j, \theta_i) - x)$$

and:

$$q(\theta'_i, \theta_j) = \arg \max_x \theta'_i v(x) + \theta_j v(q(\theta'_i, \theta_j) + q(\theta_j, \theta'_i) - x)$$

The respective first order conditions are:

$$\theta_i v'(q(\theta_i, \theta_j)) = \theta_j v'(q(\theta_j, \theta_i))$$

and:

$$\theta'_i v(q(\theta'_i, \theta_j)) = \theta_j v'(q(\theta_j, \theta'_i))$$

If $q(\theta_i, \theta_j) = q(\theta'_i, \theta_j)$ then $(\theta_i - \theta'_i)v'(q(\theta_i, \theta_j)) = \theta_j(v'(q(\theta_j, \theta_i)) - v'(q(\theta_j, \theta'_i)))$. More importantly, if $q(\theta_j, \theta_i) = q(\theta_j, \theta'_i)$ then this implies that $\theta_i = \theta'_i$ which is a contradiction to the assumption that $\theta_i > \theta'_i$. This is sufficient to conclude that $q_{\theta_i}(\theta_j, \theta_i) > 0$. ■

⁴³See Lemma 10 in Appendix 4.7.2.

Lemma 10 Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and that $f(x) > 0$ for all $x \in \mathbb{R}$. Then if:

$$\frac{f(x)}{f(y)} = \frac{f(\alpha x)}{f(\alpha y)} \quad (4.53)$$

for all $\alpha > 0$ and for all $x, y \in \mathbb{R}$ then f is a homogeneous function.

Proof: Fix y . Then let:

$$\beta(\alpha) = \frac{f(\alpha y)}{f(y)}$$

This implies that:

$$\beta(\alpha)f(x) = f(\alpha x)$$

Since $\beta(\alpha)$ is a constant, let $\beta(\alpha) = \alpha^k$. Then by definition, $f(\cdot)$ is a homogeneous function of degree k . ■

Lemma 11 Suppose $v'(\cdot)$ is a homogeneous function of degree k . Then the density of a robust γ -reduction is given by:

$$g(\gamma) = \int_{\Theta} f\left(\frac{\theta\gamma}{(\theta^{1/k} - \gamma^{1/k})^k}\right) \left| \left(\frac{\theta^{1/k}}{\theta^{1/k} - \gamma^{1/k}}\right)^{k+1} \right| f(\theta) d\theta \quad (4.54)$$

Proof: Rearranging (4.18) yields:

$$\theta_1 = \frac{\theta_2\gamma}{(\theta_2^{1/k} - \gamma^{1/k})^k} \quad (4.55)$$

Furthermore:

$$\frac{\partial \theta_1}{\partial \gamma} = \left(\frac{\theta_2^{1/k}}{\theta_2^{1/k} - \gamma^{1/k}} \right)^{k+1} \quad (4.56)$$

Thus:

$$g(\gamma, \theta) = f\left(\frac{\theta\gamma}{(\theta^{1/k} - \gamma^{1/k})^k}\right) \left| \left(\frac{\theta^{1/k}}{\theta^{1/k} - \gamma^{1/k}}\right)^{k+1} \right| f(\theta) \quad (4.57)$$

Integrating θ out from (4.57) yields (4.54).⁴⁴ ■

⁴⁴See, for example, Section 3-9 in Pfeiffer [39].

Lemma 12 Suppose $v'(\cdot)$ is a homogeneous function of degree k and that the γ -pricing mechanism \mathcal{M}^γ solves the maximization problem as stated in Proposition 15. Then $S(\theta)$ is nondecreasing.

Proof: Suppose \mathcal{M}^γ solves the maximization problem in Proposition 15. Then it is clear that $S(\theta) = \theta v(\hat{q}(\theta)) - \hat{t}(\theta)$. Furthermore, \mathcal{M}^γ is incentive compatible. Therefore, for $\theta > \theta'$, $\theta v(\hat{q}(\theta)) - \hat{t}(\theta) \geq \theta'(\hat{q}(\theta')) - \hat{t}(\theta')$; otherwise it contradicts the assumption of incentive compatibility of the γ -pricing mechanism \mathcal{M}^γ . ■

Proposition 21 Suppose $v'(\cdot)$ is homogeneous of degree k and the γ -pricing mechanism \mathcal{M}^γ solves the maximization problem as stated in Propositions 15 and 16. Then \mathcal{M}^γ exhibits quantity discounts everywhere if and only if:

$$\gamma v'(\hat{q}(\gamma))\hat{q}(\gamma) - \gamma v(\hat{q}(\gamma)) - \int_{\underline{\gamma}}^{\gamma} v(\hat{q}(s))ds - \underline{U} < 0 \quad (4.58)$$

for all $\gamma \in \Gamma$.

Proof: If \mathcal{M}^γ is the optimal γ -pricing mechanism, then quantity discounts suggest that:

$$\frac{d}{d\gamma} \left(\frac{\hat{t}(\gamma)}{\hat{q}(\gamma)} \right) < 0$$

for all $\gamma \in \Gamma$. By incentive compatibility of \mathcal{M}^γ , $\hat{t}(\gamma)$ necessarily follows (4.27) for some $\hat{q}(\gamma)$. Taking the derivative and noting that $\hat{q}(\gamma)$ is nondecreasing for all $\gamma \in \Gamma$ yield the desired result. ■

4.7.3 An Example of $G(\gamma)$

Suppose $\Theta = [0, 1]$ and that $F(\theta) = \theta$. Furthermore, let $v(q) = 2q^{1/2}$. Therefore, $v'(\cdot)$ is homogeneous of degree $-1/2$. From (4.18) we have:

$$\gamma(\theta_1, \theta_2) = (\theta_1^2 + \theta_2^2)^{1/2} \quad (4.59)$$

Since θ_1 and θ_2 are distributed uniformly, then γ is distributed over $\Gamma = [0, 2^{1/2}]$.⁴⁵ For the distribution function of γ , notice that we are simply calculating the area within a unit

⁴⁵This comes from the fact that $\gamma(0, 0) = 0$ and $\gamma(1, 1) = 2^{1/2}$.

square under a quarter circle with radius γ . Thus, for $\gamma < 1$, the distribution of γ is simply the area under the quarter circle given by $\pi\gamma^2/4$. When $\gamma > 1$, the problem no longer becomes so trivial. This is shown in Figure 4.2.

First note that area $A=B$. Furthermore the area of A may be computed by:

$$A = \frac{\eta}{360}\pi\gamma^2 - \frac{(\gamma^2 - 1)^{1/2}}{2} \quad (4.60)$$

where $\eta = \tan^{-1}((\gamma^2 - 1)^{1/2})$. Thus, the distribution function may be fully described by the equation:

$$G(\gamma) = \begin{cases} 1, & \text{if } \gamma \geq 2^{1/2} \\ \frac{1}{4}\pi\gamma^2 - 2\left(\frac{\tan^{-1}((\gamma^2-1)^{1/2})}{360}\pi\gamma^2 - \frac{(\gamma^2-1)^{1/2}}{2}\right), & \text{if } \gamma \in (1, 2^{1/2}) \\ \frac{1}{4}\pi\gamma^2, & \text{if } \gamma \in [0, 1] \\ 0, & \text{otherwise} \end{cases} \quad (4.61)$$

and the density function:

$$g(\gamma) = \begin{cases} 0, & \text{if } \gamma \geq 2^{1/2} \\ \frac{\pi\gamma}{2} - \frac{\pi\gamma \tan^{-1}((\gamma^2-1)^{1/2})}{90} - \frac{\pi\gamma}{180(\gamma^2-1)^{1/2}} + \frac{\gamma}{(\gamma^2-1)^{1/2}}, & \text{if } \gamma \in (1, 2^{1/2}) \\ \frac{\pi\gamma}{2}, & \text{if } \gamma \in [0, 1] \\ 0, & \text{otherwise} \end{cases} \quad (4.62)$$

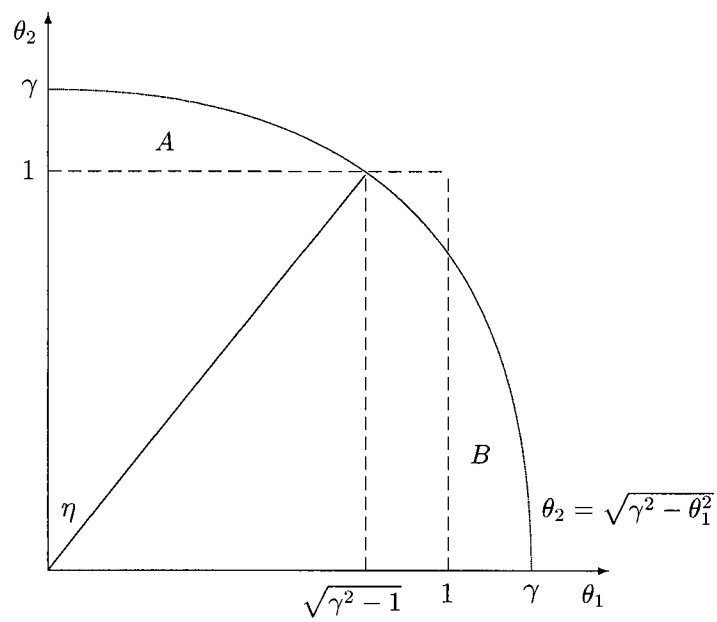


Figure 4.2: $G(\gamma)$ with $\gamma > 1$

Bibliography

- [1] Alger, I., "Buyer Strategies Limiting the Monopolist's Power: Multiple and Joint Purchases," *Rand Journal of Economics*, 1999, 30, 736-57.
- [2] Armstrong M., "Multiproduct Nonlinear Pricing," *Econometrica*, 1996, 64, 51-75.
- [3] Armstrong M. and J.-C. Rochet, "Multi-dimensional Screening: A User's Guide," *European Economic Review*, 1999, 43, 959-79
- [4] Banks, J. and S. Moorthy, "A Model of Price Promotions with Consumer Search," *International Journal of Industrial Organization*, 1999, 17, 371-98.
- [5] Baye, M., J. Morgan, and P. Scholten, "Price Dispersion in the Small and in the Large: Evidence from an Internet Price Comparison Site," *Mimeo*, Indiana University, 2002.
- [6] Butters, G. R., "Equilibrium Distributions of Sales and Advertising Prices," *Review of Economic Studies*, 1977, 44, 465-91.
- [7] Cabral, L. M.B. *Introduction to Industrial Organization*. Cambridge: MIT Press, 2000
- [8] Celik, G., "Mechanism Design with Collusive Supervision." *Mimeo*, University of British Columbia, 2003.
- [9] Elzinga, K. and D. Mills, "Price Wars Triggered by Entry," *International Journal of Industrial Organization*, 1999, 17, 179-98.
- [10] Epstein L. G. and M. Peters, "A Revelation Principal for Competing Mechanisms," *Journal of Economic Theory*, 1999, 99, 119-60.
- [11] Faure-Grimaud, A., J.-J. Laffont, and D. Martimort, "Collusion, Delegation and Supervision with Soft Information." *Mimeo*, 2000.
- [12] Folland, G. B. "Real Analysis: Modern Techniques and Their Applications," 2nd Ed., John Wiley & Sons, Inc., New York, NY, 1999.

- [13] Fudenberg, D. and J. Tirole. *Game Theory*. Cambridge: The MIT Press, 1991.
- [14] Goldman, M.B., H. E. Leland and D. S. Sibley, "Optimal Nonuniform Prices," *Review of Economic Studies*, 1984, 51, 305-19.
- [15] Guesnerie, R. and J.-J. Laffont, "A Complete Solution to a Class of Principal-Agent Problems with an Application to the Control of a Self-Managed Firm," *Journal of Public Economics*, 1984, 25, 329-69.
- [16] Guimarães, P., "Search Intensity in Oligopoly," *Journal of Industrial Economics*, 1996, 44, 415-426.
- [17] Hallagan, W. and W. Joerding, "Equilibrium Price Dispersion," *American Economic Review*, 1985, 75, 1191-94.
- [18] Hamilton, J. H., and J.-F. Thisse, "Nonlinear Pricing in Spatial Oligopoly," *Economic Design*, 1997, 2, 379-97.
- [19] Hendricks, K., R. Porter and G. Tan, "Bidding Rings and the Winner's Curse: The Case of Federal Offshore Oil and Gas Lease Auctions." *Mimeo*, 2003.
- [20] Ivaldi, M. and D. Martimort, "Competition Under Nonlinear Pricing," *Annales d'Économie et de Statistique*, 1994, 34, 71-114.
- [21] Jeon, D.-S. and D. Menicucci, "Optimal Second-degree Price Discrimination Under Collusion: On the Role of Asymmetric Information among Buyers," *Mimeo*, 2002.
- [22] Kaplansky, I., "Set Theory and Metric Spaces," 2nd Ed. Chelsea Publishing company, New York, NY, 1977.
- [23] Kwong, L. M.K., "Coupons and "Everyday Low Prices": Price Competition with Multiple Pricing Instruments," *Canadian Journal of Economics*, 2003, 36, 443-62.
- [24] Kwong, L. M.K., "On Quantity Discounts and Quantity Premiums in Monopoly Nonlinear Pricing." *Mimeo*, 2003.
- [25] Kihlstrom, R. E. and M. H. Riordan, "Advertising as a Signal," *Journal of Political Economy*, 1984, 92, 427-50.
- [26] Laffont, J.-J. *The Economics of Uncertainty and Information*. Cambridge: The MIT Press, 1989.
- [27] Laffont, J.-J., E. Maskin, and J.-C. Rochet, "Optimal Nonlinear Pricing with Two-Dimensional Characteristics," in: T. Groves, R. Radner, S. Reiter (Eds.), "Information, Incentives and Economic Mechanisms," University of Minnesota Press, Minneapolis, MN, 1987.

- [28] Laffont, J.-J. and D. Martimort, "Collusion and Delation." *Rand Journal of Economics*, 1998, 29, 280-305.
- [29] Laffont, J.-J. and D. Martimort, "Mechanism Design with Collusion and Correlation." *Econometrica*, 2000, 68, 309-42.
- [30] Laffont, J.-J. and D. Martimort, *The Theory of Incentives: The Principal-Agent Model*. Princeton University Press, Princeton, NJ, 2002.
- [31] Mandy, D.M., "Nonlinear Bertrand Competition," *Econometrica*, 1992, 60, 1293-1330.
- [32] Maskin, E. and J. Riley, "Monopoly with Incomplete Information," *Rand Journal of Economics*, 1984, 15, 171-96.
- [33] McAfee, R. P. and J. McMillan, "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, 1988, 46, 335-54.
- [34] McAfee, R. P., J. McMillan, and M. Whinston, "Commodity Bundling by a Monopolist." *Quarterly Journal of Economics*, 1989, 104, 371-83.
- [35] Mookherjee, D. and M. Tsumagari, "The Organization of Supplier Networks: Effects of Delegation and Intermediation." *Mimeo*, 2002.
- [36] Mussa, M. and S. Rosen, "Monopoly and Product Quality," *Journal of Economic Theory*, 1978, 18, 301-17
- [37] Oren, S., S. Smith, and R. Wilson, "Competitive Nonlinear Tariffs," *Journal of Economic Theory*, 1983, 29, 49-71.
- [38] Oren, S., S. Smith, and R. Wilson, "Pricing a Product Line," *Journal of Business*, 1984, 57, S73-S100.
- [39] Pfeiffer, P. E., *Concepts of Probability Theory*. New York: McGraw-Hill Book Company, 1965.
- [40] Porter, R. and D. Zona, "Detection of Bid Rigging in Procurement Auctions." *Journal of Political Economy*, 1993, 101, 518-38.
- [41] Quesada, L., "Modeling Collusion as an Informed Principal Problem." *Mimeo*, University of Wisconsin - Madison, 2003.
- [42] Rochet, J.-C. and P. Choné, "Ironing, Sweeping, and Multidimensional Screening," *Econometrica*, 1998, 66, 783-826.

- [43] Rochet, J.-C. and L. Stole, "The Economics of Multidimensional Screening," in: M. Dewatripont, L. P. Hansen, S. J. Turnovsky (Eds.), "Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress, Volume I," Cambridge University Press, Cambridge, MA, 2003.
- [44] Rochet J.-C. and L. Stole, "Nonlinear Pricing with Random Participation," *Review of Economic Studies*, 2002, 69, 277-311.
- [45] Rosenthal, R. W. "A Model in Which an Increase in the Number of Sellers Leads to a Higher Price," *Econometrica*, 1980, 48, 1575-79.
- [46] Rothschild, M. "Models of Market Organization with Imperfect Information: A Survey," *Journal of Political Economy*, 1973, 81, 1283-1308.
- [47] Salop, S. and J. Stiglitz, "Bargains and Ripoffs: A Model of Monopolistically Competitive Price Dispersion," *Review of Economic Studies*, 1977, 44, 493-510.
- [48] ———, "The Theory of Sales: A Simple Model of Equilibrium Price Dispersion with Identical Agents," *American Economic Review*, 1982, 72, 1121-30.
- [49] Stole, L., "Nonlinear Pricing and Oligopoly," *Journal of Economics and Management Strategy*, 1995, 4, 529-62.
- [50] Rochet, J.-C. and L. Stole, "The Economics of Multidimensional Screening," in *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress*, ed. by M. Dewatripont, L. P. Hansen and S. J. Turnovsky. Cambridge: Cambridge University Press, 2003.
- [51] Tirole, J. *The Theory of Industrial Organization*. Cambridge: The MIT Press, 1988.
- [52] Varian, H. R., "Price Discrimination," in *Handbook of Industrial Organization*, Vol 1, ed. by R. Schmalensee and R. Willig. Amsterdam: North Holland, 1989.
- [53] Varian, H. R., "A Model of Sales," *American Economic Review*, 1980, 70, 651-59.
- [54] Wilson, R., "Nonlinear Pricing," Oxford University Press, Oxford, 1993.