On Black Holes in BMN Plane Wave

by

Shieh Hsien-Hang

BSc. Simon Fraser University

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

(Department of Physics and Astronomy)

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

September 9, 2004

THE UNIVERSITY OF BRITISH COLUMBIA FACULTY OF GRADUATE STUDIES

Library Authorization

In presenting this thesis in partial fulfillment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Shieh Hsien-Hang
Name of Author *(please print)*

~~Sep. 28. 2004~~ 28th Sep. 2004
Date (dd/mm/yyyy)

Title of Thesis: On Black Holes In BMN Plane Wave.

Degree: MSc. Year: 2004

Department of Physics and Astronomy
The University of British Columbia
Vancouver, BC Canada

Abstract

In this thesis, we attempt to construct a black hole solution with BMN plane wave asymptotics. We find it to be a very complicated problem. Through explicit computations, we showed the Penrose limit of the Schwarzschild Anti de-Sitter spacetime do not result in a background with event horizon, in accord with the no go theorems reviewed which suggest the symmetries of plane wave spacetimes are not compatible with the existence of regular event horizon. The detailed boundary and light cone structure of the BMN spacetime are studied. It is made clear that the conformal boundary of the plane wave is not related to the boundary of Anti-de Sitter. In order to understand the concept of temperature and thermal state in the BMN background, we study the response of an Unruh monople detector following various trajectories. The detector response function shows the vacuum state natural to the BMN plane wave has very different thermal behavior from the Minkowski vacuum. In particular, observers following any Killing trajectory will not regard the plane wave vacuum as a thermal state. This result can be viewed as a semi-classical verification of the no-go theorems. We also review the solution generating technique, the null Melvin twist and the correspondence principle of the black string solutions so generated in the plane wave geometry.

Contents

Acknowledgements

I would like to thank Dr. G. Semenoff and Dr. D. Witt for supervising the work and for giving me much encouragement. I would also like to thank D. Brecher, M. Laidlaw and H. Ling for many useful discussions. The author is supported by NSERC PGS A-255165-2002.

Chapter 1

Introduction and Overview

We will begin by giving an introduction to the recent developements leading to the current problem under investgation. It has long been obeserved that there is a connection between certain gauge theories and theories of gravitation. This connection is made explicit by the AdS/CFT correspondence [1]. This correspondence is realized in a way that it can be interpretated as an example of the holography priciple which states there exists a dual description of gravitaional theory on a d dimensional spacetime as some field theory on the lower dimensional boundary of the spacetime. In the simplest example of the correspondence, the boundary field theory is the $\mathcal{N} = 4$ super Yang-Mills theory and the dual quantum gravity theory lives on $AdS_5 \times S^5$. The correspondence is found to be a weak-strong coupling duality. It connects the strongly coupled regime of the gauge field theory to the weakly coupled sector of the gravitaional theory and vise versa. This makes the correspondence difficult to check but all the more significant. The behavior of gauge field theories in their strongly coupled regime has been a long standing problem. With the help of the AdS/CFT correspondence, we are able to gain understanding of strongly coupled gauge theory by studying weakly coupled gravity where all interactions are under control. More interestingly, the AdS/CFT correspondence is conjectured to be valid beyond the gravity limit. That is, the boundary field theory will encode all the information of the bulk, including all orders of corrections in the string tension α', and string coupling constant g_s. This conjecture has passed many checks through calculations involving gauge theory operators whose behaviors are protected by symmetry all the way to the strongly coupled limit and compared with the corresponding supergtravity calculations.

An interesting aspect of the AdS/CFT correspondence is how the boundary field theory detects topology change in the bulk. Here by topology change we mean those with the boundary of the spacetimes left invariant. In the present case, we will consider two topologically distinct spacetimes that are asymptotically AdS. They are the pure AdS spacetime and the AdS-Schwarzschild black hole. It has been shown that there exists a thermal phase transition between the two spacetimes in the bulk gravitational theory [2]. On the field theory side this is translated into the de/confinement phase transtion of the Yang-Mills theory [3]. Most recently, it is found that there exist a particular limit of the $AdS \times S$ spacetime known as the Penrose limit [4] such that in the resultant plane wave spacetime the free string theory specturm can be solved exactly in the lightcone gauge [5]. Moreover the corresponding limit of the boundary CFT is also shown to be dual to the bulk string theory [6]. It led to the consideration of a plane wave/CFT correspondence. However, it should be noted that so far the boundary CFT is only understood as a limit of the original Yang-Mills theory, and a completely self-contained description of the field theory is yet to be found. In this thesis, we will consider the possiblity of an analogue of the thermal phase transition in the plane wave spacetimes. That is, we wish to find a black hole phase for the plane wave spacetimes resultant from the Penrose limit. We will find such task is rendered very difficult by the symmetry properties of this class of spacetimes. In particular the existence of an event horizon is not consistent with the symmetries. We are then led to consider the analogue of Unruh radiation in these backgrounds in the hope that it will shed some light at the quantum level of the above inconsistency and how to modify the symmetry requirements. Lets start with a more detailed description of the AdS/CFT correpondence.

1.1 AdS/CFT Correpondence: Motivation

Consider the pure SU(N) Yang-Mills theory with Lagrangian:

$$L = \frac{1}{g^2}Tr[(\partial A)^2 + A^2 + A^3 + ...] = \frac{1}{g^2}Tr[(\partial A)^2 + V(A)] \tag{1.1}$$

where $A(x)$ are traceless Hermitian matrices living in the adjoint representation of SU(N), and the traces are to ensure SU(N) gauge invariance of the action. The Feymann diagrams of the above theory can be represented with the double line notation due to 't Hooft [7]: where the $i, j.. = 1..N$ are the SU(N) index. We see from fig(1.1). each propagtor contributes a factor of g^2 while all the vertices carry a factor of $1/g^2$. Due to the trace on the vertices we need to identify the gauge index on the same single line, and there is a sum over $i = 1..N$ for each closed single loop. As a result, we get a factor of N from each closed single line loop in a diagram. To summarize, each Feymann diagram will have a prefactor:

$$(g^2)^{\text{no. of propagators}-\text{no. of vertices}} N^{\text{no. of closed single lines}} \tag{1.2}$$

It turns out the double line diagrams have a natural geometrical representaion. We can draw an arbitrary double line graph on a two dimensional boundaryless Riemann surface with appropriate number of holes and cycles (topology). The Feymann diagrams give a triangulation of the corrsponding Riemann surfaces. The power counting in (1.2) can now be rearranged into:

$$(1.2) = N^E (Ng^2)^{\text{no. of propagators}-\text{no. of vertices}} \tag{1.3}$$

where E is the Euler number of the Riemann surface $E = \text{no. of faces} - \text{no. of edges} + \text{no. of vertices} = 2 - 2h$ where h is the genus of the correponding surface fig(1.2). (the faces and edges refer to those in the triangulation) If we take the 't Hooft limit: $N \to \infty$, $g^2 N = \lambda$, the 't Hooft coupling, fixed, the Feymann diagram expansion is now seen as a genus expansion of the Riemann surfaces. Note due to the Hermiticity of the matrices, we need to include only orientable surfaces in the expansion. If we introduce additional matter fields transforming in the fundamentals of SU(N) into the theory, we will need to consider Riemann surfaces with boundary, as the fundamentals are represented by single lines in the double line notation. The partition function of the theory now has the form:

$$log(Z) = \sum_{E=2} N^E f_E(g^2 N) \tag{1.4}$$

We can focus on contributions from the planar diagrams (those can be drawn on a two sphere or a plane). Heuristically, if we consider diagrams with more and more loops and vertices (higher order in $O(\lambda)$), we would be filling up the Riemann surfaces with little triangles and the partition function is seen to approximate the summing of the world sheet configurations as in string perturbation theory. Let us make this more quantitative by looking at the duality between $AdS_5 \times S^5$ and $\mathcal{N}=4$ super Yang-Mills on $R^{1,3}$.

The duality between type IIB string theory on $AdS_5 \times S^5$ and gauge theory can be understood as a result of matching two equivalent descriptions of a stack of N D-3 branes in the appropriate limit. On one hand, a stack of D-3's in 10 dimensional Minkowski space can be discribed by the excitation of open strings ending on the branes, the closed string modes that propogate in the bulk off the branes, and their interactions. In this picture, strings are propagating in flat space with Dirichlet boundary conditions for the open strings at the loaction of the branes. At low energy, we can focus on the massless sectors of the string theory spectrum. This configuration can be described by an effective action with all the massive excitations integrated out.

$$S_{eff} = S_{brane} + S_{closedstring} + S_{int} \tag{1.5}$$

The closed string dynamics can be captured by the 10 dimensional supergravity action while the open strings are described by the $\mathcal{N}=4$ super Yang-Mills. It can be shown that if we only keep the leading order terms in α', the close and open string sectors decouple. The low energy physics of the N closely positioned D-3 branes is just that of free 10 dimensional supergravity plus a super Yang-Mills theory living on the four dimensional world volume of the D-3 branes.

On the other hand, from calculation of gravitons scattering off the D-3 branes and the Ramond-Ramond charges they carry, it can be deduced the presence of the stack of N D-3 branes will generate a geometry which at large distances from the branes becomes an extremal black 3-brane solution of 10d supergravity carrying N units of Ramond-Ramond charge [8].

$$\begin{aligned} ds^2 &= f^{-1/2}[-dt^2 + dx^2 + dy^2 + dz^2] + f^{1/2}[dr^2 + r^2 d\Omega_5^2] \\ F_5 &= (1+*)dtdxdydzdf^{-1} \\ f &= 1 + \frac{R^4}{r^4}, R^4 = 4_s\alpha'^2 N \end{aligned} \tag{1.6}$$

for coincident branes. The horizon of the black three brane is at $r=0$. An excitation with energy E near the horizon will become $E' = \lim_{r=0} f^{-1/4}E$ as viewed by an observer at $r=\infty$ ($f(r)$ approaches unity) due to the Tolman red shift factor. Therefore from the point of view of observers at infinity, all of the physics happening near the horizon of the D-3 branes are redshifted to low energy. To these observers, the low energy physics in the background generated by the stack of D-3 branes also contain two pieces: the first is the red shifted version of the near horizon behavior; the second is the low energy supergravity modes propagating in the bulk. In the low energy limit we are interested in, the two pieces also decouples. This can be seen as in this limit the bulk low energy modes would have wavelength much larger than the scale set by the curvature of the black three brane. These modes are completely delocalized compared to a length scale that is "near" horizon. We now have two equivalent descriptions of the low energy physics in a background generated by a stack of N D-3 branes. In both versions there is a piece which is just the low energy 10 dimensional supergravity modes. Since in both cases the supergravity modes decouple, we are then prompted to identify the other part from the two descriptions. We will relate the near horizon physics of (1.6) to the $\mathcal{N}=4$ super Yang-Mills theory living on the world volume of the D-3 branes. As the gravity modes decouple from the physics on the brane in the low energy limit, it is safe to assume the world volume of the D-3's to be the 1+3 Minkowski spacetime. Note because of the red shift, the low energy excitations from the near horizon region as observed at infinity can in principle include modes with arbitarily high energy as measured by a near horizon observer. The duality is conjectured to exetend beyond the supergravity limit and include higher string modes. The near horizon geometry of (1.6) is the $AdS_5 \times S^5$ as can be checked by taking $r << R$, and approximate $f^{1/2}$ by R^2/r^2:

$$ds^2 = \frac{r^2}{R^2}[-dt^2 + dx^2 + dy^2 + dz^2] + \frac{R^2}{r^2}dr^2 + d\Omega_5^2 \tag{1.7}$$

In (1.7), the AdS part of the metric is written in the Poincare coordinate, which, however, only covers part of the AdS spacetime [Appendix A]. It is easy to see by taking the limit $r \to \infty$ that the metric (1.7) becomes conformal to the flat Minkowski space $R^{1,3}$ (the radius of the internal five sphere shrinks to zero). To an observer living on the boundary of the AdS spacetime, the duality can be stated as:

The type IIB string theory on $AdS_5 \times S^5$ is dual to the $\mathcal{N}=4$ SU(N) super Yang-Mills on the conformal boundary $R^{1,3}$ of $AdS_5 \times S^5$.

1.2 AdS/CFT Correpondence: The Duality

In this section we will introduce the mapping between the Hilbert spaces of the boundary field theory and the bulk string theory. First we will look at the parameters in the two theories and see how to map them to each other [1]. In the field theory, we have the Yang-Mills coupling g_{YM} and N from the gauge group SU(N) (and the theta angle which do not play an impotant role in the discussion to come). On the string theory side, there is the string coupling g_s, the funadamental string length $l_s^2 = \alpha'$ and from the background, the radius of $AdS_5 \times S^5$, R, and the R-R five form charge $N = R^4/(4\pi g_s \alpha'^2)$. From calculations of how D_p-branes couple to Ramond-Ramond $(p+1)$-form potential, we can calculate the relationship between the Yang-Mills coupling and the D_p-brane tension $\tau_p \propto 1/g_s(\alpha')^{-\frac{p+1}{2}}$ [9]. It gives us the following relation:

$$g_{YM,p}^2 \propto \tau_p^{-1}\alpha'^{-2} = g_s \alpha'^{(p-3)/2} \tag{1.8}$$

For $p = 3$, we see $g_{YM}^2 = g_s$. If we use units in which $R = 1$, we find

$$\alpha' \sim 1/(g_s N)^{1/2} \sim 1/(g_{YM}^2 N)^{1/2} = 1/(\lambda)^{1/2} \tag{1.9}$$

, and the 10 dimensional Newton's constant

$$G_{10} \propto \alpha'^4 g_s^2 \sim \frac{1}{N^2} \tag{1.10}$$

The relationship (1.9) gives evidence for our heuristic identification of the large N 't Hooft expansion and the string world sheet genus expansion in fig(1.3). The parameter α' is inversely proprotional to the 't Hooft coupling λ of the Yang-Mills theory. This indicates to perform perturbative canculations on the field theory side, we need to turn on the stringy (α') corrections to the gravity computations. On the other hand, in order for the supergravity description in the bulk to be valid, we need the scale set by the curvature of the spacetime to be much larger than the string scale, that is:

$$1 << (\frac{R}{l_s})^4 \sim g_{YM}^2 N = \lambda \tag{1.11}$$

This says the dual field theory will give comparable result to the bulk supergravity calculation only if we include all the loop corrections (at the planar level). We have reached the conclusion that the AdS/CFT correspondence is a strong/weak coupling duality. As mentioned before, this makes the duality hard to check, but it also gives us a new avenue to the nonperturbative behaviors of gauge theories. Notice also, there are two limiting procedures involved. Take the gravity calculations, for example. We need to first dial down the string coupling constant to supress string interactions (the higher genus diagrams) then take the limit $\alpha \to 0$, which allow us to focus on the supergravity contributions. The corresponding limits on the field theory side is to first take g_{YM} to zero then take the large N limit in such a way that the 't Hooft coupling $\lambda = g_{YM}^2 N$ is fixed but large. This will give us access to properties of the field theory at the planar level but at large 't Hooft coupling.

Next, we look at the symmetries of the two theories. The $\mathcal{N} = 4$, SU(N) super Yang-Mills on $R^{1,3}$ is a conformal theory. It has the conformal group SO(2,4) as its symmetry group. SO(2,4) also is the isomotry group of AdS_5. The generators of SO(2,4) includes the usual Poincare algebra plus dilatation $D = x^\mu \partial_\mu$ and the special conformal generator K_μ, which induces the transformation:

$$x'^\mu = \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}$$

The scaling dimension $\triangle$ of an operator is defined by how it changes under dilataion.

$$x^\mu \to a x^\mu, O(x) \to O'(x) = a^\triangle O(ax)$$

From the commutation relations between the translation generator P_μ, D, and K_μ, [1] we see the translations are raising operators and the K_μ's are the lowering operators of the scaling dimension. In particular, the ones that are annhilated by K's are called primary operators. If we consider the Euclidean version of the theory, we can take the dilatation operator as the Hamiltonian. Since the theory is conformally invariant, the background spacetime can be replaced by $S^3 \times R^1$ fig(1.4). A state in the space correpondes to an insertion of an operator at the origin of $R^{1,3}$ and propagated with the dilatation generator. The "time" in this radial quantization acctually does not map to the global time coordinate of $AdS_5 \times S^5$. The correct combination in the Lorentzian sector is $H = \frac{1}{2}(P_0 + K_0) = \partial_t$, where t is the global time coordinate of AdS. However, the operator/ state correspondence has the advantage that the energy eigenvalues of a state is just the scaling dimension of the operator conjugate to it. In the $S^3 \times R^1$ picture it is natural to denote the states by the scaling dimensions of the operators. This helps simplify the identification of the bulk and boundary Hilbert space. Since the background is the flat space we can use isometry to show the Hilbert space according to the two quantization schemes are isomorphic to each other. The isometry group of the S^5 is SO(6). In the AdS/CFT picture it corespondes to th R-symmetry of the $\mathcal{N} = 4$ super Yang-Mills theory. It will become important when we discuss the planewave limit of the duality

The mapping between states in the bulk and the operators on the boundary (remember their scaling dimensions labels the states on the boundary) proceed as follows [10]:

$$< e^{\int_\partial \phi_0(x) O(x)} >_{CFT} = Z_{string}[\phi(x,z)|_{z=0} = \phi_O(x)] \tag{1.12}$$

where we have written the AdS metric in the form:

$$ds^2 = \frac{1}{z^2}(-dz^2 + dx_i^2) \tag{1.13}$$

These coordinates also only covers the Poincare patch, and the boundary is at $z = 0$ with the x_i's labeling it. In the following we will work in the Euclidean sector with $z \to iz$. (1.12) says the boundary conditions on the bulk fields are to be considered as sources for the boundary theory operators with the appropriate scaling dimension. We will consider here only scalar fields. The cases with higher spin particles can be generalized natrually by considering adding tensorial operators to the CFT generating functional. The string theory partition function is viewed as a functional of the boundary value of the bulk fields.

From the metric (1.13), it is easy to see the transformation $z \to az$, $x_i \to ax_i$ is an isometry. when restricted to the boundary, the above transformation is just the dilatation. If the operator has scaling dimension $\triangle$, in order for the exponent of the left hand side of (1.12) to be scaling invariant, the corresponding bulk field must have the behavior $\phi(z,x) \sim z^{\triangle'}\phi_0(x)$ with $\triangle' = 4 - \triangle$. If we solve the bulk scalar wave equation:

$$(z^2\partial_z^2 - 3zd_z + z^2\nabla_{transverse} - m^2)\phi(z,x) = 0$$

with the ansatz $z^{\triangle'}\phi_0(x)$, we find there are two possible values for $\triangle' = 4$.

$$\triangle'_{\pm} = 2 \pm (4 + m^2)^{1/2} \tag{1.14}$$

The positive root gives regular solutions on the boundary, while the negative one makes the solution diverge. Also since $\triangle'_+ + \triangle'_- = 4$, it makes sense to associate a bulk field with $\triangle'_-$ to the boundary operator with scaling dimension $\triangle'_+$. This is also motivated by the requirement that the bulk field is regular on the interior of the AdS space.

Having established the dictionary, the first thing to notice is (1.12) gives us a way of obtaining correlation functions of the field theory operators at large λ through perturbative supergravity calculations (the large N, large 't Hooft coupling limit is taken as outlined before). To do this, we need to obtain the general solutions of the bulk wave equation subject to certain boundary condition $\phi_0(x)$. In fact, we will relate the left hand side of (1.12) to supergravity partition function evaluated at on shell bulk fields with their boundary behavior as free variables.

$$< e^{\int_\partial \phi_0(x) O(x)} >_{CFT} = e_{saddle}^{-I_{SUGRA}[\phi(x,z)|_{z=0}=\phi_0(x)]} \tag{1.15}$$

The correlation functions are obtained by functional differentiation of both sides with respect to $\phi_0(x)$ evaluated at $\phi_0(x) = 0$. The bulk wave fuction can be derived perturbatively by propagating the boundary condintion in to the bulk with the help of the "boundry to bulk" propagators. These propagators satisfies the homogeneous wave equation every where in the bulk, but approaches a delta function source on the boundary.

$$\lim_{z_1 \to 0} G_{boundary/bulk}(z_1, x_1|0, x_2) = (\frac{z}{z_1^2 + (x_1 - x_2)^2})^{\triangle'_-} \to z_1^{\triangle'_-} \delta(x_1 - x_2) \tag{1.16}$$

For free scalar theories the bulk field is then

$$\phi(z_1, x_1) = \int_\partial dx_2 G_{boundary/bulk}(z_1, x_1|0, x_2)\phi_0(x_2) \tag{1.17}$$

In terms of these boundary to bulk propagators, the field theory correlation function can be represnted geometrically as in fig(1.5).

Despite the diffuculties in checking the AdS/CFT correspondence directly, many tests has been performed and given evidences for the validity of the duality. These tests usually involves calculating quantities in the boundary field theory with no dependence on the 't Hooft coupling (often the ones protected from quantum corrections by symmetries) and compare with the corresponding supergravity/string theory objects according to the prescription (1.12): The spectrum of Chiral operators are invariant under change of the coupling. It is shown the supergravity fields on AdS spacetime is in one to one correspondence with the chiral primary operators in the CFT. While single trace operators corresponds to single particle states, multiple trace operators are dual to multiple particle states [1]. The decendents of a primary operator O (those obtained from acting derivatives or supercharges on the primaries), correspondes to excited states of the bulk field dual to O. Although the full spectrum of type IIB string theory on $AdS_5 \times S^5$ has not been mapped out entirely, it is congectured the non-chiral primary operators will correspond to single string states with higher excitations. Gauging certain global symmetry on the field theory side results in quantum anomaly that comes only from the one loop diagrams, and thus can be trusted at the large λ regime. Such anomaly can be seen from the calculation of correlation functions of currents of the broken symmetry. Using the construction outlined above, the field theory and the supergravity calculation result in identical expressions for the anomaly in the large N limit [11]. Non-local operators such as Wilson loops in the boundary gauge theory of spacial shapes are protected by super symmetry. Calculations from both the bulk and boundary points of view are also shown to match [12]. Most Recently, certain limit of $AdS_5 \times S^5$ along a null geodesic around the equator of the five sphere is shown to result in a spacetime on which string theory spectrum can be solved exactly. This background is a maximally supersymmetric plane wave solution of type IIB supergravity. Coresponding limit on the field theory side are also shown to be dual to the string theory on the plane wave geometry. We will discuss this limit in detail in a later section. We will now turn to finite temperature aspects of the AdS/CFT duality.

1.3 Hawking Page Phase Transition and De/confinement Transition

Considering the $\mathcal{N} = 4$, SU(N) super Yang-Mills on the boundary of $AdS_5 \times S^5$, we have established a holographic picture for the AdS/CFT duality. If the duality is true to all orders of α' and N, we are to consider all backgrounds that are asymptotically AdS. This can be seen explicitly in (1.12) which asserts the generating functional of the CFT can be computed by summing over all the bulk fields including variations of the background metric for the string theory as long as they are subjected to certain boundary conditions. It is interesting to see how to describe topological changes in the bulk using the field theory language. There are two known solutions to Einstein's equations asymptotically AdS. The first one is the AdS spacetime (in this section, we willuse the conventions of [3]:

$$ds_1^2 = (1 + r^2/b^2)dt^2 + \frac{dr^2}{1 + r^2/b^2} + r^2 d\Omega_{n-1}^2 \tag{1.18}$$

where b is related to the cosmological constant $\Lambda = \frac{n}{b^2}$. We will be calculating the Euclideanized partition function, the global time coordinate has been Wick rotated. The second solutionis the Schwarzschild-AdS black hole:

$$ds_2^2 = (1 + r^2/b^2 - M/r^{n-2})dt^2 + \frac{dr^2}{(1 + r^2/b^2 - M/r^{n-2})} + r^2 d\Omega_{n-1}^2 \tag{1.19}$$

the temperature of the Euclidean black hole is determined by the requirement the manifold is smooth and complete at the horizon $r_0 : 1 + r_0^2/b^2 - M/r_0^{n-2} = 0$. It is calculated to be:

$$T_H = \frac{r_0^2 + b^2}{2\pi b^2 r_0}$$

The topology of AdS spacetime is $R^n \times S^1$, and the Schwarzschild-AdS has $R^2 \times S^{n-1}$. They are dipicted in fig(1.6). There is a significant difference in the two cases. In ds_1^2, the temporal circle S^1 is not contractble and we can set the period of it to any positive number. In ds_2^2, the temporal circle is contractible provided it has the correct period $\beta = \frac{2\pi}{T_H}$, while it is the spatial circle being not contractible. $r = r_0$ is a lower bound for the radial coordinate and is a boundary of the spacetime.

Hawking and Page [2] demonstrated that there is a gravitaional phase transition between AdS and AdS black hole. It is done by comparing the contributions to the partition function from the two geometries. Being solutions to the equations of motion, these are two saddle points for the Einstein-Hilbert Action with cosmological constant:

$$I_{gravity} = I_{E-H} + I_{H-G}$$

$$= \frac{-1}{16\pi G_{n+1}} \int d^{n+1}x(g)^{1/2}(R + \frac{n(n-1)}{2b^2}) + \int_{\partial} K \tag{1.20}$$

where I_{H-G} is the Hawking-Gibbon's boundary term which cancells the contribution from varying the Ricci scalar to the equations of motion [13]. K is the extrinsic curvature on the boundary and G_{n+1} is the n+1 dimensional Newton's constant. On asymptotically AdS spacetimes the boundary term acctually evaluate to zero, and we will ignore it from now on. When plugging in solutions of Einstein's equation, the bulk term become:

$$I_{E-H} = \frac{n}{8\pi G_{n+1}} \int d^{n+1}x(g)^{1/2} \tag{1.21}$$

In both metrics $(g)^{1/2} = r^{n-1}$, however, the domain of integration in the t, and r coordinates are different. For AdS they are $t \in [0, \beta]$, $r \in [0, +\infty]$, where β could be any positive number and for Schwarzschild-AdS $t \in [0, \beta_H]$, $r \in [r_0, +\infty]$, where β_H is constraint to be the inverse Hawking temperature. This is a direct consequence of the topology difference between the two spacetimes. Note the radial coordinate is integrated all the way to infinity. Both contributions to the partition function have the same divergence due to the infinite volume. In order to analyze the relative stability of the two saddle points, we need to regularize them. Following [2], we can subtract the AdS contribution from the black hole phase. To do so, however, we need to identify the two metrics on a large $r = R$ submanifold. Namely, we will need to match the circumference of the time circle on $r = R$

$$\beta_H(1 + \frac{R^2}{b^2} - \frac{M}{R^{n-2}})^{1/2} = \beta(1 + \frac{R^2}{b^2})^{1/2}$$

With the identification we found the differnce between the two solutions is:

$$\lim_{R\to\infty}(I_{E-H}^{S-AdS} - I_{E-H}^{AdS}) \propto \frac{b^2 r_0^{n-1} - r_0^{n+1}}{G_{n+1}(nr_0^2 + (n-2)b^2} \tag{1.22}$$

(1.21) turns zero at $r_0 = b$. From the usual thermodynamics relation $Z = e^{-I} = e^{-\beta F}$, we see that at small r_0 (low temperature), the AdS phase is energetically favorable, while at large r_0 (high temperature) S-AdS black hole is the dominant solution. There is a phase transition between AdS nad S-AdS black hole. With the regularized S-AdS contribution to the partition function, we can also compute other thermodynamic quantities such as the energy and entropy of the black hole.

We now consider phase transitions in the boundary field theory. It is shown in [10] at least in the large 't Hooft coupling limit the large N $\mathcal{N} = 4$ SU(N) super Yang-Mills theory on a compact manifold (here in particular we consider $S^1 \times S^3$), has a de/confinement phase transition. The transition is characterized by the behavior of the order parameter:

$$C_1 = \lim_{N\to\infty} \frac{F(\beta)}{N^2} \tag{1.23}$$

There exists a low temperature confining phase with $C_1 = 0$ (or $F(\beta > \beta_H) \sim O(1)$) and a high temperature deconfining phase $C_1 \sim O(1)$ (or $F(\beta < \beta_H) \sim O(N^2)$). $F(\beta)$ is the free energy of the theory and β is the period of the temperal circle. The large N limit is essential for the de/confinement transition to exist on a manifold with finite volume. On the other hand, it is also known that the gauge theory on the decompactified background $R^3 \times S^1$ is always in the deconfined phase. There is yet another order parameter for the de/confinement phase transition. Consider the expectation value of a Wilson line operator in the gauge theory:

$$C_2 =< W(C) >=< Tr[Pe^{\int_C A}] > \tag{1.24}$$

In particular, we will consider the case when the Wilson line wraps around the temperal circle of the backgound. Introducing a Wilson line operator in the theory can be interpreted as adding an external static charge which transform in the fundamental representation of SU(N) (a quark) to the system. The expectation of the Wilson line is the cost of free energy of introducing the quark.

$$< W(C) >\sim exp(-F(\beta)\beta) \tag{1.25}$$

In the confining phase, the field theory can be considered as a theory containing only glue balls at finite temperature. It takes an infinite amount of free energy to introduce an external charge, and therefore the expectation value of the Wilson line (C_2) is zero. In the deconfined phase the cost of free energy of such operation is finite and so is the order parameter C_2.

The authors of [14] have utilized these order parameters and demonstrated de/confinement phase transition in weakly coupled large N gauge field theories on compact manifolds. Their result also shows in the confining phase the gauge theory spectrum has the Hagedorn behavior. The Hagedorn behavior is characterized by the exponential growth of the density of states with respect to increasing energy: $\rho(E) = e^{\beta_H E}$ Therefore, when computing the thermal partition function of the system, we get a critical temperature $\beta = \beta_H$, above which the thermal partition function diverges and the thermal ensemble is ill-defined.

$$Z_\beta = \int \rho(E)e^{-\beta E} \tag{1.26}$$

Studying the order parameters indincates the de/confinement transition could be either first order at a temperature below T_H, or second order happening right at T_H.

It is well known the spectrum of single string states also exhibits Hagedorn behavior [15]. The Hagdorn temperature can be considered as the temperature at which perturbative vacuum of string theory becomes unstable. This can be seen from the fact that some superstring states with nonzero winding around the themal circle become tachyonic if the period of the thermal circle is smaller than certain critical value β_H [16]. It is argued there is corespondingly some kind of phase transition when the winding modes condense. The phase transition can again be first order below T_H or second order occuring at exactly the Hagdorn temperature. The former is not truely a Hagdorn transition in that the partition function is not dominated by contributions from states with arbitrily high energy, while the latter is. In the AdS/CFT correpondence, we are identifying the Hilbert space of the Yang-Mills theory with that of the string theory. It is natural to identify the Hagdorn behavior and the phase transitions on both sides of the picture as well. The winding modes can be heuristically identified with the temperal Wilson lines used as order parameter of the de/confinement phase transition. Going back to the strongly coupled field theory limit. According to AdS/CFT duality, this correspondes to the regime where supergravity results can be trusted. The super gravity descrition of the Wilson lines is proposed to be [17]:

$$< W(C) >_{CFT} \sim e^{-S_C} \tag{1.27}$$

where S_C is the minimal area of a surface in the bulk bordered by C on the boundary of the spacetime. S_C is calculated with respect to the appropriate bulk background geometry. The proposal is motivated by the picture that we can view (low energy) excitations of open strings stretched between a stack of N D-branes and one distantly seperated D-brane as states of very massive quarks, with the mass proportional to the seperation. (This is because open string ends carry Chan-Paton factors that transform in the fundamental of the SU(N)) If we take the quark to be static ($m \to \infty$), the string can be considered to be stretched between the bulk of AdS and its boundary. Since Wilson lines in gauge theory correspondes to insertion of such a massive quark, in view of the prescription (1.12), the expectation value of these operators on the boundary should be dual to (in the supergravity limit) the exponential of the area of the superstring world sheet with the boundary condition that it ends on C. fig(1.7). In the low temperature phase, according to (1.21), the bulk geometry is that of the pure AdS. However, it has the topology such that the temperal circle is not contractable. Thus it is not the border of any string world sheet, and $< W(C) >= 0$. When the temperature is high, the dominating state in the bulk is the Schwarzscild-AdS black hole background. The black hole geometry admits contractable temperal circles, and thus C is the boundary of some string world sheet configuration. $< W(C) >$ is nonzero. The above result suggests that we can associate the confining phase in the boundary filed therory to the bulk AdS geometry and the deconfined phase to the black hole background [3]. More importantly, the de/confinement phase transition is dual to the Hawking-Page phase transition. As we have seen in (1.21), the Hawking page phase transition is a first order one. It is also understood that it happens at a temperature lower than the energy scale where stringy corrections become important, that is, it happens below the Hagdorn temperature.

¿From the above analysis, we can make the following congecture: in the large N, weak 't Hooft coupling limit, $\mathcal{N} = 4$ SU(N) super Yang-Mills has de/confinement phase transition, which could be identified with the Hagdorn behavior in the free superstring theory. It is not clear, however, what is the phase beyond the Hagdorn temperature on the string theory side. (For more interesting speculations on this point see [14]) When the 't Hooft coupling is large, we can approximate bulk theory by classical supergravity. The de/confinement phase transtions in the gauge theory is mapped to the Hawking-Page phase transition. The phase transition is first order and is below the Hagdorn temperature. Notice that the information about the de/confinement phase transition in the strong coupling limit is only obtained through the help AdS/CFT, since we have no control over the perturbation theory when λ is large.

1.4 BMN Limit of AdS/CFT

In the previous sections we have reviewed the basic elements of the AdS/CFT correspondence. We have seen that it can teach us about the behavior of strongly coupled gauge field theory through tree level calculations on the supergravity side. More interestingly, the duality tells us how to relate the thermodynamics of string theory to the de/confinement phase transition in large N Yang-Mills theory. However, due to the fact that we do not know completely the spectrum of string theory on $AdS_5 \times S^5$, we are not able check the correspondence completely with weakly coupled super Yang-Mills even just at the planar level.

Recently, it is found by the authors of [6], that by taking the Penrose limit of the $AdS_5 \times S^5$, we are able to obtain a spacetime on which free string theory can be solved exactly in lightcone gauge [5]. The spacetime is a maximally super symmetric plane wave [18]. They also showed the corresponding limit on the field theory side contains states that can be mapped in a one to one manner to the string theory states on the plane wave background. The particular Penrose limit taken is to focus on the null geodesic that travels around the great circle of the S^5. We will give detailed account of the Penrose limit in the next chapter. The BMN plane wave spacetime has the metric:

$$ds^2 = -2dudv + \mu^2 r^2 du^2 + \sum_{i=1..8} dx^{i2} \tag{1.28}$$

with null five form field strength:

$$F_{u1234} = F_{u5678} = \frac{4}{g_s}\mu \tag{1.29}$$

To establish the duality, we need to know the relations

$$u = \frac{t}{2\mu}, \quad v = \mu\frac{t-\psi}{2R^2} \tag{1.30}$$

where t is the global time of AdS_5 and ψ is the azimuthal angle on the five sphere. The lightcone momenta are related to the conserved quantities of $AdS_5 \times S^5$ through [1]:

$$-\frac{p_u}{\mu} = \frac{i}{\mu}\partial_u = i(\partial_t + \partial_\psi) = \triangle - J$$

$$-\mu p_v = i\mu\partial_v = \frac{i}{R^2}(-\partial_\psi) = \frac{J}{R^2} \tag{1.31}$$

where R is the radius of $AdS_5 \times S^5$. $\triangle = i\partial_t$, is the energy and $J = -i\partial_\psi$ is the angular momentum around the five sphere in the original spacetime. The Penrose limit involves taking the limit $R \to \infty$. The BPS condition requires physical states to satisfy $\triangle \geq J$. As a result, only states with large five sphere angular momentum

$$J \sim R^2 \sim (g_s N)^{1/2} \sim (g_{YM}^2 N)^{1/2} \tag{1.32}$$

would survive the limit. If we are interested in free string theory spectrum, the large R limit can be taken as follows: first let $g_s \to 0$, with $g_s N \sim g_{YM}^2 N$ fixed, then take the large 't Hooft coupling limit with J/R^2 and $\triangle - J$ fixed. If we are to consider string interactions, the large R limit can also be realized as taking N to infinity, $J \sim N^{1/2}$, while keeping g_s, $\triangle - J$ fixed. In this case, we can trade in the usual perturbation parameters λ, $g_s(g_{YM})$ with

$$\lambda' \to \frac{g_{YM}^2 N}{J^2}, g_s' \to \frac{J^2}{N} \tag{1.33}$$

It is easy to see the relation $g_s \sim g_s(\mu\alpha_v')^2$ from (1.9). When considering states with fixed light cone momentum p_v, the parameter λ' governs the loop expansion in the Yang-Mills theory, while g_s' being the parameter in string loop perturbation.

On the field theory side, the operator $J = -i\partial_\psi$ corresponds to the R charge which rotates two of the six scalars in the theory. Remember that we can use isometry to relate the eigenvalues of the global time translation generator and the dilatation generator $i\partial_t \to D$. The relevant states live in a sector of the super Yang-Mills theory with large scaling dimensions and large R-charges. Here we will also consider mapping operators in this sector, the BMN operators to string states on the plane wave background.

The free string theory can be solved exactly on this background in the light cone gauge in the Green-Schwarz formulism [5]. The light cone Hamiltonian and its spectrum is given by:

$$\begin{aligned} -p_u = H_{lightcone} = \triangle - J &= \sum_n a_n^\dagger \cdot a_n(\mu^2 + \frac{n^2}{(\alpha p_v/2)^2})^{1/2} \\ &= \sum_n \mu a_n^\dagger \cdot a_n(1 + \frac{4\pi g_s N n^2}{J^2})^{1/2} \end{aligned} \tag{1.34}$$

We will now give the dictionary for the identification of field theory operators and free string states [6]. It is natural to consider first the operators with the lowest value of light cone momentum $\triangle - J$. It turns out there is a unique single trace operator $Tr[(Z)^J]$, $Z = \phi^1 + i\phi^2$, where we have taken $J = -i\partial_\psi$ to be the generator that rotates the two scalars ϕ^1, ϕ^2 in the gauge theory. The association is then:

$$\mathrm{Tr}[Z^J] \to |0, p_u > \tag{1.35}$$

where $|0, p_u>$ is the lowest string state with no creation operators acting on it. In general we can insert operators into the chain of Z's in the trace above to make new operators with higher values of $\triangle - J$, which would correspond to string states with higher light cone momentum. There is one subtlety, however, we need to impose the level matching condition on the field theory operator. This is done by summing over all the places of inserting the extra operators weighted by a phase factor proportional to the momentum. If the total momentum along the string is not zero, the cyclicity of the trace along with the phase factor will make the operator so constructed vanish. The mapping between the string world sheet creation operator and the inserted is found to be (for the bosonic part):

$$a^{\dagger i} \to D_i Z \text{ i=1..4}$$

$$a^{\dagger i} \to \phi^{i-2} \text{ i=5..8}$$

where D_i is the gauge theory covariant derivative, and ϕ's are the rest of the scalar fields. For example:

$$\sum_{l=1}^{J} Tr[(D_1 Z) Z^l \phi^3 Z^{J-l}] e^{2\pi i n l/J} \to a_n^{\dagger 1} a_{-n}^{\dagger 5} |0, p_u > \tag{1.36}$$

The above identification has been check by computing $\triangle - J$ in the Yang-Mills theory in expansion in λ' and showed agreement with (1.33).

With the free string spectrum constructed, the thermodynamical behavior of string theory on this background has also been considered. It is possible to compute the free energy explicitly. It is found there exists a Hagedorn temperature for strings on this background [19]. The Hagedorn temperature is an increasing function of μ. When μ is small the Hagedorn temperature approaches the flat space result:

$$T_H = \frac{1}{2\pi(2\alpha')^{1/2}} \tag{1.37}$$

In the large μ limit, the Hagedorn temperature is pushed to infinity, and we do not expect a phase transition to occur.

In this thesis, we will attempt to understand if there is an analogue of the Hawking-Page phase transition for the plane wave background and if it can be associated with the Hagedorn transition found in the finite μ string theory result. The first step will be to find a black hole phase in the plane wave spacetime. However, it has proven to be a difficult task.

Before we move on, we will list perturbations of bosonic fields in type IIB supergravity around the BMN pp-wave background and refer to [5] for detailed definitions and their derivations. We will present their equations of motion to linear order of perturbation. We will see that these classical equations can all be related to the massless scalar equation. The field content of IIB supergravity are a complex scalar (the dilaton plus an axion), a complex two form potential consisting of the NS-NS three form and R-R three form field strength:

$$dC_2 = H_3 + iF_3$$

and the R-R four form. (we will not be concerned with their Hodge duals). Using the notations in [] and use the light cone gauge:

Complex Scalar Perturbation ($\delta\Phi$):

$$\nabla^2 \delta\Phi = 0 \tag{1.38}$$

Complex Two Form Perturbation ($\delta b^{\pm}_{ij}, \delta b^{\pm}_{i'j'}, \delta b_{ij'}$ where the unprimed indicies goes from 1..4, and the primed index goes from 5..8. This is the decomposition according to the $SO(4) \times SO(4)'$ symmetry of the background. The superscript $\pm$ labels irreducible representations of $SO(4)$):

$$\nabla^2 \delta b_{ij'} = 0$$

$$(\nabla^2 \pm i4\mu p_v)\delta b^{\pm}_{ij} = 0$$
$$(\nabla^2 \pm i4\mu p_v)\delta b^{\pm}_{i'j'} = 0 \tag{1.39}$$

Graviton and Four Form Perturbation $(\delta h^{\pm}_{ij}, \delta h^{\pm}_{i'j'}, \delta h_{ij'}, \delta h, \delta a_{ij'}, \delta a_{iji'j'}, \delta a$. It turns out some of the graviton and four form equations of motions are mixed due to the nontrivial background.

$$\nabla^2 \delta a_{iji'j'} = 0$$
$$(\nabla^2 \mp i4\mu p_v)(\delta h_{ij'} \pm \delta a_{ij'}) = 0$$
$$\nabla^2 \delta h_{ij} = \nabla^2 \delta h_{i'j'} = 0$$
$$(\nabla^2 \mp i8\mu p_v)(\delta h \pm \delta a) = 0 \tag{1.40}$$

fig 1.1

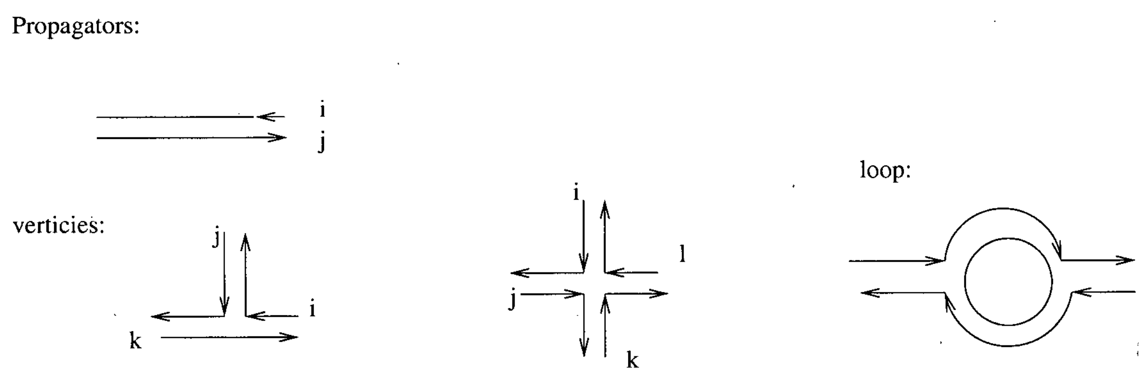


fig 1.2

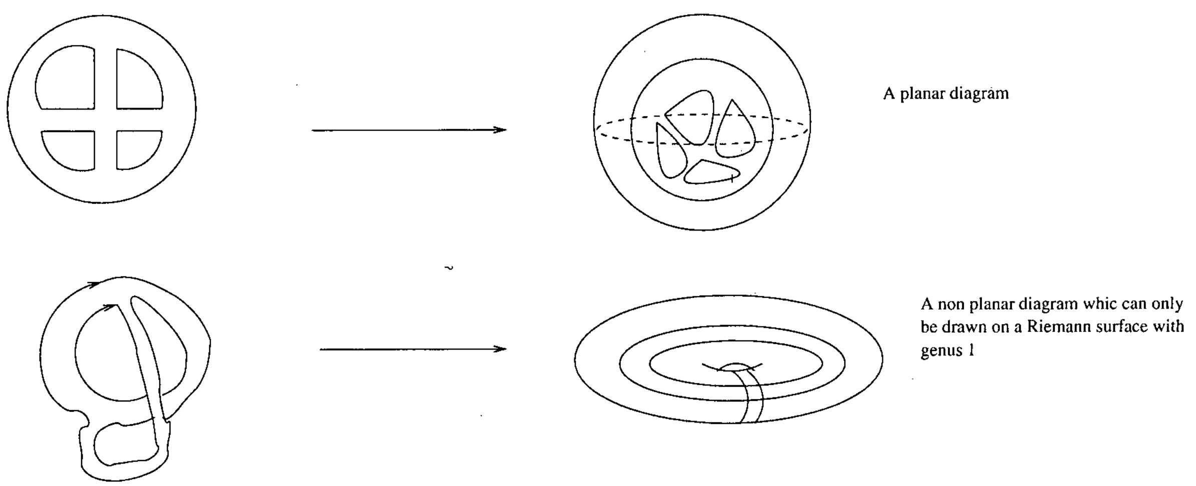


fig 1.3

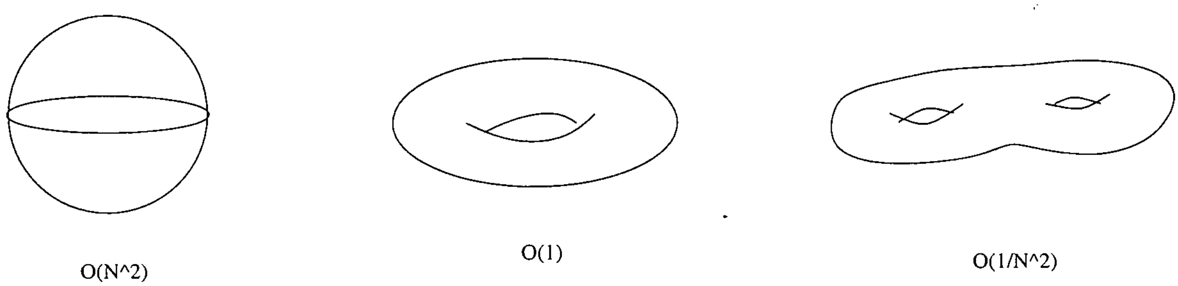
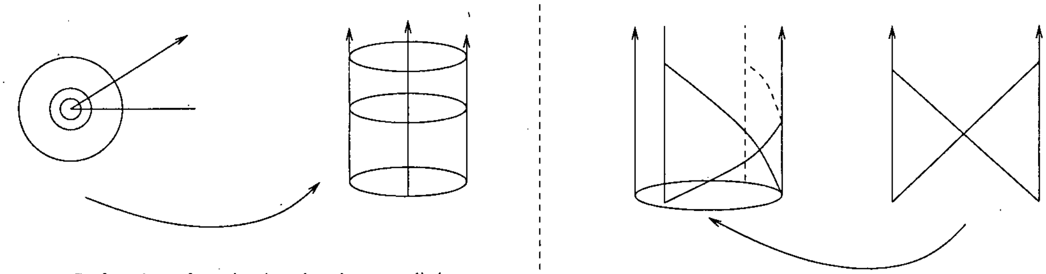


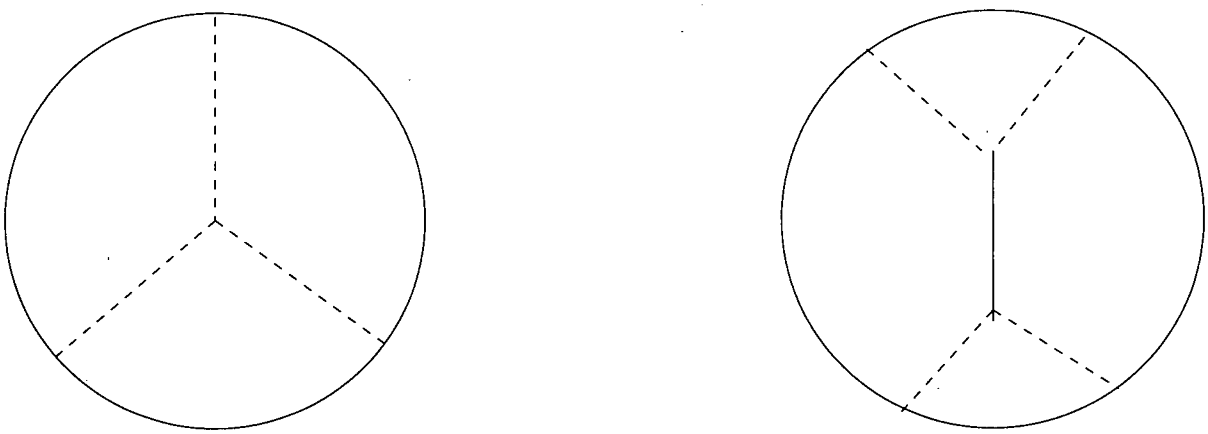
fig 1.4



Conformal transformation that takes planes to cylinder
The direction of the arrow points at the motion generated by the dilatation operator

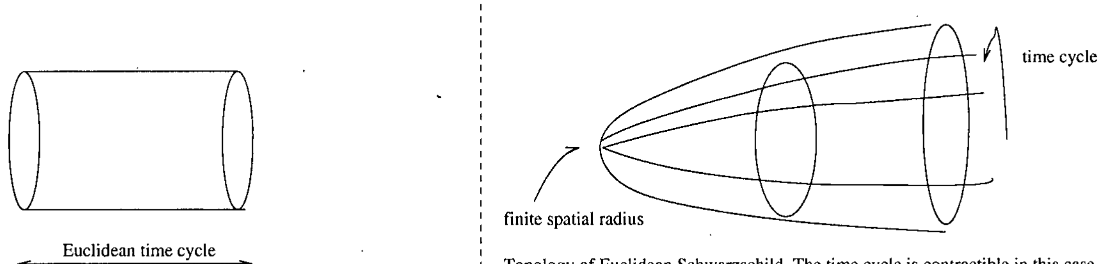
Penrose diagram for AdS spacetime. It can also be drawn on a cylinder but the arrow points in the direction of the global time

fig 1.5



Bulk calculations of three and four point correlation functions. The circle is the boundary of AdS
The dotted lines are boundary to bulk propagators. The solid line is bulk to bulk propagator

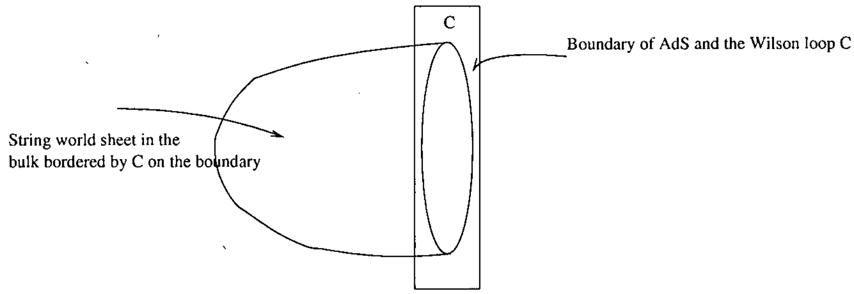
fig 1.6



Topology of Euclidean AdS The time direction can be identified with any positive period. The time cycle is not contractible.

Topology of Euclidean Schwarzschild. The time cycle is contractible in this case
The horizon is located at the tip, and it is of finit spatial radius. It is a boundary of the space. The spatial cycle is not contractible.

fig 1.7



The expectation value of a Wilson loop operator C on the boundary can be approximated by the exponential of the area of string world sheets bordered by C

Chapter 2

The Penrose Limit

2.1 Introduction

In this section we will review the limiting procedure known as the Penrose limit and some of its properties. Folowing the original work of Penrose [4] , the idea of Penrose limit of a spacetime starts out from the observation:
In a Lorentzian spacetime, in some neighborhood of a conjugate point free portion of a null geodesic, a coordinate system can be set up so that the metric takes the form:

$$g_{\mu\nu}dX^\mu dX^\nu = ds^2 = -2dUdV + AdV^2 + B_i dV dX^i + C_{ij}dX^i dX^j \tag{2.1}$$

where $\{A, B_i, C_{ij}\}$ are smooth functions of $\{U, V, X^i\}$. Following the above coordinate transformation another diffeomorphic transformation is made:

$$\begin{aligned} u &= U \\ \Omega^2 v &= V \\ \Omega x^i &= X^i \end{aligned} \tag{2.2}$$

The metric then takes the form:

$$g_{\mu\nu}dx^\mu dx^\nu = ds^2 = \Omega^2(-2dudv + \Omega^2 Adv^2 + \Omega B_i dv dx^i + C_{ij}dx^i dx^j) \tag{2.3}$$

Then the limit $\Omega \to 0$ is taken, and a well defined new metric $g'_{\mu\nu}$ can be defined with the conformal rescaling

$$g'_{\mu\nu} = \lim_{\Omega \to 0} \frac{1}{\Omega^2} g_{\mu\nu} \tag{2.4}$$

Because of the transformation (2.2), in the limit $\Omega \to 0$, the components survived can depend only on u. $g'_{\mu\nu}$ is noted to be the Rosen form of a pp-wave spacetime:

$$g'_{\mu\nu} = (-2dudv + C'_{ij}(u)dx^i dx^j) \tag{2.5}$$

The transformation (2.2) is the key that makes $\Omega \to 0$ is a well defined limit (with the rescaling). Physically, the Penrose limit is a generalization of the concept of a tangent space at a point p of a manifold. Here a neghborhood along a null geodesic is getting scaled up as the limit $\Omega \to 0$ is taken. While the neighborhood of a point is a flat space, the Penrose limit of any null geodesic in any spacetime is a pp-wave.

Similar sequence of operations (2.2), (2.4) can be taken on other objects defined on the spacetime, for example, gauge fields or Killing vectors. They will in general have different scaling laws in order for the limiting procedure to be well defined (in fact the scaling factor may vary from Killing vector to Killing vector). For a p-form potential, we need to impose certain gauge conditions.

$$A_{Ui_1 i_2..i_{p-1}} = A_{UVi_1..i_{p-2}} \tag{2.6}$$

and with the scaling

$$A' = \Omega^{-p} A \tag{2.7}$$

It is interesting to note that another Penrose limit of (2.5) in the $\frac{\partial}{\partial u}$ takes the metric back to itself and along $\frac{\partial}{\partial v}$ takes it back to Minkowski space.

We will now list several important properties of the Penrose limit: (I will here only state them will out proof. Rigorous treatment could be found in [18]

1. The Penrose limit of a solution of supergravity is also a solution of supergravity.

2. Under Penrose limit the dimension of the (super)symmetry algebra will not decrease. This does not say in the limit no two Killing vectors will become degenarate. it only says the total number of Killing vectors can not decrease. Also, even if the original spacetime do not have any supersymmetry, the resultant pp-wave after taking the Penrose limit will always preserve at least half of the maximal number of supersymmetry

3. If two null geodesics are related by some isometry, the Penrose along them will also be isometric to each other.

Another coordinate system usually used to describe pp-wave spacetimes is the Brinkman (or harmonic) coordinate. To get the metric from Rosen to Brinkman form, we need to perform the following transformation:

$$\begin{aligned} u' &= u \\ v' &= v + \frac{1}{2} M_{ij}(u) x'^i x'^j \\ x' &= Q^i_j x^j \end{aligned} \tag{2.8}$$

where Q satisfies

$$C_{ij} Q^i_k Q^j_l = \delta_{kl}$$
$$C_{ij}\left(\frac{dQ^i_j}{du} Q^k_l - \frac{dQ^i_k}{du} Q^j_l\right) = 0$$

while

$$M_{ij} = C_{kl} \frac{dQ^k_i}{du} Q^l_j$$

The resultant metric in Brinkman form is

$$g'_{\mu\nu} dx^\mu dx^\nu = -2du'dv' + (A_{ij}(u) x^i x^j) du'^2 +^{\prime i} dx'^j \tag{2.9}$$

where

$$A_{ij} = -\frac{d(C_{kl}\frac{dQ^l_j}{du})}{du} Q^k_i \tag{2.10}$$

it can be show Q^i_j always exist and is invertible. We will mostly be interested in the case when $C_{ij}(u)$ are diagonal,

$$C_{ij}(u') = a_i^2(u')\delta_{ij} \tag{2.11}$$

When this is the case, we have simply:

$$A_{ij}(u') = \frac{(a_i(u'))''}{a_i(u')}\delta_{ij} \tag{2.12}$$

where ' denotes differentiate with respect to u'.

As seen from the above, the resultant metric from Penrose limit will be flat space if and only if

$$(a_i(u'))'' = 0$$

If a_i's are of the form of hyperbolic (or trignometric) functions,

$$a_i(u) = A\sinh(\mu u) + B\cosh(\mu u)$$

$$a_i(u) = A\sin(\mu u) + B\cos(\mu u)$$

the A_{ij} is then a constant, diagonal matrix with positive (negative) eigenvalues. Notice when it is the case that a_i are trignometric, the metric in Rosen form could only covers a finite range of u until $a_i(u_0)$=0, and the metric becomes degenerate. However, when expressed a in Brinkman form the metric is analytically continued to all values of u. In the next section, we will apply Penrose limit to spacetimes of the form $AdS \times S$

2.2 Penrose limit of $AdS_p \times S^q$

The maximally supersymmetric planewave solutions to eleven dimensional and type IIB supergravity are [17]:

$$ds^2 = -2dudv - (\sum_{i=1}^{3} x^i + \frac{1}{4}\sum_{j=4}^{9} x^{j2})du^2 + \sum_{i=1}^{9} dx^{i2} \tag{2.13}$$

$$ds^2 = -2dudv - (\sum_{i=1}^{8} x^i)du^2 + \sum_{i=1}^{8} dx^{i2} \tag{2.14}$$

both of them can be obtained through Penrose limit of spacetimes of the type $AdS_p \times S^q$. We will start by reviewing how this comes about.

The type IIB supergravity solution is derived via the Penrose limit along a null geodesic in $AdS_5 \times S^5$. The null line goes around the great circle of the 5-sphere while stays at the spatial origin of AdS. The metric of $AdS_5 \times S^5$ in the covering space is:

$$ds^2 = R^2(-\cosh^2(\rho)dt^2 + d\rho^2 + \sinh^2(\rho)d\Omega_3^2 + \cos^2(\theta)d\psi^2 + d\theta^2 + \sin^2(\theta)d\Omega_3'^2) \tag{2.15}$$

where we have chosen the radius of AdS to be identical as that of the sphere. The Penrose limit in this case can be simply performed as a Taylor expension. Let

$$u = t$$
$$v = R^2(t - \psi) \tag{2.16}$$

Note (2.16) is not the usual choice for light cone coordinates. It, however, has the advantage in interpretation that the time in the resultant metric could be identified with global time of AdS. The usual choice:$u = \frac{t+\psi}{2}$ will also work and the gives an identical result in the limit, but u is periodically identified with the periodicity of ψ. Performing also the rescaling $R\rho = x$, $R\theta = y$, we get in the limit $R \to \infty$ (take $\Omega = 1/R$),

$$R^2(-(1+x^2/R^2)du^2+dx^2/R^2+(x^2/R^2)d\Omega_3^2+(1-y^2/R^2)(du^-dv/R^2)^2+dy^2/R^2+(y^2/R^2)d\Omega_3^2)$$
$$= -2dudv - (x^2+y^2)du^2 + dx^2 + x^2 d\Omega_3^2 + dy^2 + y^2 d\Omega_3^2 \tag{2.17}$$

In type IIB supergravity, besides the metric field (2.15), there is also the R-R five forms:

$$F_{S^5} = 4\frac{R^4}{g_s}d\Omega_5,\ F_{AdS_5} = *F_5$$

their Penrose limit is taken as prescribed in (2.6), (2.7), and we get:

$$F_{u1234} = F_{u5678} = \frac{4}{g_s}$$

To get to thesolution presented in (1.27), (1.28), we need the redefinition $u \to \mu u$, $v \to \frac{v}{\mu}$. The procedure presented above, however, only works for certain particular trajectories, since it depends on the fact at small ρ and θ the Taylor expension of $\cosh(\rho)$ and $\cos(\theta)$ do not contain terms with first order in $1/R$. It would not have worked if we are expanding about a finite value of ρ and θ or if the metric had other functional dependence. We will come back to this when we are discussing the Penrose limit in Schwarzchild-AdS black hole. From here on we will revert back to the more canonical procedure introduced in the previous section and always perform a coordinate transformation adapted to the null geodesic we choose.

The general form of the metric for $AdS_n \times S^{D-n}$ is:

$$ds^2 = R^2(-dt^2 + \sin(t)^2(\frac{dr^2}{1+r^2} + r^2 d\Omega_n^2) + a^2[d\psi^2 + \sin^2(\psi)d\Omega_{D-n-1}^2]) \tag{2.18}$$

where a is the ratio of the radius of the sphere to that of AdS. We have written the AdS part in a non static coordinate. We will again consider the Penrose limit along a null line going around the equator of the sphere. Note that despite the same motion on the sphere, t is not the same as the global time used in the previous derivation. We are following two different classes of null geodesics. We found that only the present class allows us to put the metric in the form (2.1). The coordinate transformation:

$$u = a\psi + t$$

$$v = R^2(t - a\psi)$$

takes the metric to:

$$ds^2 = R^2(-\frac{1}{R^2}dudv + \sin^2(\frac{u + v/R^2}{2})(\frac{dr^2}{1+r^2} + r^2 d\Omega_n^2) + a^2\sin^2(\frac{u+v/R^2}{2})d\Omega_{D-n-1}^2 \quad (2.19)$$

We also make the change in $d\Omega$:

$$d\Omega_n^2 = d\phi^2 + \sin^2(\phi)d\Omega_{n-1}^2 = \frac{1}{R^2}d\phi'^2 + \sin^2(\phi'/R^2)d\Omega_{n-1}^2 \quad (2.20)$$

Taking the limit $R \to \infty$, we get:

$$ds^2 = -dudv + \sin^2(u/2)\sum_{i=1}^{n-2} dx^{i2} + a^2\sin^2(u/2)\sum_{j=1}^{D-n-1} dx^{j2}$$

This is a pp-wave in the Rosen form. The C_{ij}'s are diagonal and are trigonometric functions. This coordinate system only covers $u = (0, 2\pi)$. We now perform the transformation (2.8) to get to the Brinkman form. Explicitly:

$$u' = u/2$$

$$v' = v - \frac{1}{2}\sum_{i=1}^{n-2}(\frac{\frac{d\sin(u/2)}{du}}{\sin(u/2)})x^{i2} - \frac{1}{2}\sum_{i=1}^{D-n-1}(\frac{\frac{d\sin(u/2)}{du}}{\sin(u/2)})x^{j2}$$

$$x'^j = \frac{x_j}{\sin^2(u/2)}x'^j = \frac{x_j}{a\sin^2(u/2)} \quad (2.21)$$

The metric in Brinkman form is:

$$-2du'dv' - (\sum_{i=1}^{n-2} x'^{i2} + \frac{1}{a}\sum_{j=1}^{D-n-1} x'^{j2})du'^2 + \sum dx'^{i2} \quad (2.22)$$

We have thus derived both (2.13) and (2.14). The difference in the eigenvalues are seen clearly here to come from the difference in the radius of AdS and the sphere.

Note in this section we forgo the rescaling of the metric $g^{\Omega}{}'_{ij} = \Omega^{-2}g_{ij} = R^2 g_{ij}$. This is justified as we have an over all R^2 in front of the metric, and it makes the limiting procedure well defined without the rescaling. However, if we look at all the dimensionful parameters in the space, they will still be rescaled according to the same rule as if we calculated their counterparts using the metric $g_{ij}^{\prime\Omega} = \Omega^{-2}g_{ij}$.

In the rest of this section we will give a heuristic argument for the fact that only the flat space and the metric with the above two forms can result from the Penrose limit of $AdS_p \times S^q$. Firstly, we note that AdS and the sphere are both homogeneous and isotropic and the product space is homogeneous. Every null geodesic with motion on the sphere can be related to the class considered in the previous example. Recalling property 3. of the Penrose limit in the previous section: the Perose limit of two null geodesics related by an isometry will also be isometric. We see (2.22) is a generic result for null lines with motion in the sphere.

We are left to check the null geodesics with only motion in the AdS. Since there is no motion along the sphere, Penrose limit acts like a uniform scaling:

$$d\Omega_n^2 = d\psi^2 + \sin^2(\psi)d\Omega_{n-1}^2 \to dy^2/R^2 + \frac{y^2}{R^2}d\Omega_{n-1}^2$$

where $y = R^2\psi$ and $R \to \infty$ limit is taken. We saw explicitly it gives the Eulidean flat metric. We can now focus only on the AdS part. The argument using isometry still applies, we could take any null geodesic in AdS and examine its behavior in the limit. But there is a more elegant proof. We note that Penrose limit preserves null tensors made of the Riemann tensor and its derivatives. It is seen as the diffeomorphism transformations (2.2) do not change the nullness of the tensor field. The scaling changes the Riemann tensor and its derivatives only by an overall factor Ω^p. More specifically, this tells us that the Penrose limit (along any null geodesic) of a Ricci flat and conformal flat space will also have zero Ricci tensor and Weyl tensors. The AdS space, which is conformally flat, satisfies

$$R_{ij} = \Lambda g_{ij}$$

and the scaling (2.4) takes one AdS space to another one with different radius. By the scale invariance of th Ricci tensor we have

$$R_{ij}^{\Omega} = \Lambda\Omega^2 g_{ij}^{\prime\Omega} = \Lambda^{\prime\Omega} g_{ij}^{\prime\,\Omega}$$

Therefore we see the cosmological constant Λ^{Ω} becomes zero in the limit and so do the Ricii tensors. The resultant spacetime has vanishing Ricii tensor and Weyl tensor, which implies zero Riemann tensor. Under Penrose limit (again, along any null geodesic), AdS becomes isometric to the flat space. The same analysis obviously applies to dS spacetime as well.

Putting things together we have learned that for null geodesics in $AdS_n \times S^{D-n}$ with no motion along the sphere, the Penrose limit results in D dimensional flat space. For null geodesics with motion in the sphere, the Penrose limit gives results of the form (2.22). The eigenvules of A_{ij} in the g_{uu} component depends on the relative radius and dimensionality of AdS and the sphere.

2.3 Penrose Limit of Schwarzchild-AdS Black Hole

One natural extension to the BMN/CFT correspondence is to examine its finite temperature version. As reviewed in the introduction, the picture in the AdS/CFT correspondence is such that the high temperature phase of the CFT is dual to the Schwarzchild-AdS (S-AdS) black hole solution in the bulk. Further, the Hawking-Page phase transition between Schwarzchild-AdS and pure AdS is mapped to the de/confinement phase transition in the dual CFT. It has been realized string theory on the BMN spacetime has a Hagedorn temperature [19] beyond which perturbative description of string theory breaks down. It is therefore interesting to see if we can associate a geometrical interpretation to the above picture. That is, to find a black hole phase in the BMN spacetime. A natural starting point would be to take the Penrose limit of the Schwarzchild-AdS solution and examine if the idea of temperature, horizon would survive the limiting process.

We will consider the Schwarzschild black hole in $AdS_5 \times S^5$, which has the metric:

$$-h'(r')dt^2 + \frac{1}{h(r')}dr'^2 + r'^2 d\Omega_3^2 + R^2[d\psi^2 + \sin^2(\psi)d\Omega_4'^2] \tag{2.23}$$

where

$$h'(r) = 1 + \frac{r^2}{R^2} - \frac{M}{r^2}$$

we can factor out the R^2 by

$$r = r'/R$$
$$t = t'/R$$

the metric then looks like:

$$R^2[-h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega_3^2 + d\psi^2 + \sin^2(\psi)d\Omega_4'^2] \tag{2.24}$$

with

$$h(r) = (1 + r^2 - \frac{M}{R^2 r^2})$$

In this form we again have the property that the rescaling part of Penrose limit takes one S-AdS to another with a different radius. In other words, with arbitrary overall scaling, we have only one dimensionless parameter left M/R^2. We have less symmetry in the metric and the calssification of Penrose limit along different trajectories is more difficult. We will start out with the in falling null geodesic with no angular momentum. The motion is in the t-r plane. To get the metric into the form (2.1), we need to consider the following coordinate transformation:

$$t = \int^u \frac{E}{h(Eu')}du' + v$$
$$r = Eu$$

where E is an integration constant. The five sphere will scale as in (2.20) and gives a flat Eclidean metric, while the S-AdS part becomes:

$$-2\Omega^2 E du dv + -\Omega^4 h(Eu)dv^2 + (Eu)^2(\Omega^2 d\psi^2 + \sin^2(\Omega\psi)d\Omega_2^2$$

(I have put in the scaling factors Ω) In the limit $\Omega \to 0$, it is clear the metric $\Omega^{-2}ds^2$ is just 10 dimensional Minkowski.

Next, we consider a in falling null geodesic which also travels around the great circle of the five sphere: $(t(\tau),\ r(\tau),\ \psi(\tau))$. The condition of it being null and geodetic gives its tangent vector to be:

$$(\dot{t}, \dot{r}, \dot{\psi}) \to (\frac{A}{-2R^2 h(r)}, \frac{1}{2R^2}(A^2 - B^2 h(r))^{1/2}, \frac{B}{2R^2}) \tag{2.25}$$

where A, and B are integration constants with $A \to$ "energy", $B \to$ "angular momentum" of the trajectory. We now need a coördinate transformation adapted to the null line and take the metric to the form (2.1). The following coordinate transformation accomplishes this:

$$U = 2R^2 \int^r \frac{1}{(A^2 - B^2 h(r'))^{1/2}} dr' = f(r)$$

$$V = \frac{2R^2}{B}[\psi - \frac{B}{2R^2}(U + W/R)]$$

$$W = (-2R^2)\frac{A}{B^2}[t + A \int^r \frac{1}{h(r')(A^2 - B^2 h(r'))^{1/2}} dr'] \tag{2.26}$$

To see how it works, it is acctually clearer if we look at th Jacobian of the transformation.

$$\frac{\partial(r,t,\psi)}{\partial(U,V,W)} = \begin{pmatrix} (A^2 - B^2 h(r)) & 0 & 0 \\ -\frac{A}{h(r)} & 0 & \frac{B^2}{-2A} \\ B/2 & B/2 & B/2 \end{pmatrix} \tag{2.27}$$

The first column is determined up to an over all constant by the requirement that u being the affine parameter along the null geodesic. It is just the transformation to the Eddington-Finkelstein patch. The other two colunms are chosen so that $g_{UV} = const$ and $g_{UW} = 0$. The choice is not unique, however, the resultant spacetime is.

As usual, we need to follow with a transformation (2.2) that makes the limit well defined. Here we revert back to introducing an extra scaling parameter Ω and rescaling the metric by Ω^{-2} for easier interpretation. The metric turns into:

$$R^2\Omega^2[-2B^2 du dv + B^2\Omega^2 dv^2 + 2B^2\Omega dw dv + (B^2 - \frac{B^4}{A^2}h(r))dw^2 + r^2 d\Omega_3^2 + \sin^2(B(v\Omega^2 + w\Omega + u))d\Omega_4^2] \tag{2.28}$$

In the limit $\Omega \to 0$, the rescaled metric $\Omega^{-2} g_{ij}$:

$$R^2[-2B^2 du dv + (B^2 - \frac{B^4}{A^2}h(r(u)))dw^2 + r(u)^2 \sum_{i=1}^{3} dx^{2i} + \sin^2(Bu)\sum_{j=1}^{4} dx^{2j}] \tag{2.29}$$

This is a pp-wave in its Rosen form, to write it in the Brinkman form, we will need:

$$u' = u$$

$$v' = v - \frac{1}{2}\frac{\partial_u (B^2 - \frac{B^4}{A^2}h(r(u)))^{1/2}}{(B^2 - \frac{B^4}{A^2}h(r(u)))^{1/2}} w^2 + \frac{1}{2}\frac{\partial_u r(u)}{r(u)} \sum_{i=1}^{3} x^{i2} + \frac{1}{2}\frac{\partial_u \sin(Bu)}{\sin(Bu)} \sum_{j=1}^{4} x^{j2}$$

$$w' = \frac{w}{(B^2 - \frac{B^4}{A^2}h(r(u)))^{1/2}}$$

$$x'^i = \frac{x^i}{r(u)}$$

$$x'^j = \frac{x^j}{\sin(Bu)} \tag{2.30}$$

and we get:

$$limit_{\Omega\to 0}(\Omega^{-2}ds^2) = -2du'dv' - B^2[(1-\frac{3\alpha}{r(u')^4})w'^2 + (1+\frac{\alpha}{r(u')^4})x'^{i2} + x'^{j2}]du'^2 + dx'^{i2} + dx'^{j2} \tag{2.31}$$

where $\alpha = \frac{M}{R^2}$. I have absorbed the R^2

We could also consider the in falling null geodesics with angular momentum around the three sphere of the S-AdS. The form of the metric is more complicated but do not give new qualitative features. We will concentrate on the non-rotating case from here on.

Some properties of the spacetime are in order. First of all, it does not have an event horizon. We will present a more general proof of this statement in a following chapter. Here we will give a heuristic argument. The main reason lies in the fact that under Penrose limit only information close to and along the null geodesic is retained. However, along the in falling null lines the presence of the Horizon is never felt. In other words the event horizon is not even an observer dependent horizon for these trajectories. When we focus only the geometry along the in falling null line, the property of the horizon as the boundary of the "visible" universe is lost. Second, we noticed at points corresponding to $r = 0$ in the original singularity, the metric becomes singular. In fact, it can be shown [20] for generic plane wave written in Brinkman form:

$$ds^2 = -2dudv - A_{ij}(u)x^i x^j du^2 + dx^{i2}$$

$A_{ij}(u) = R_{iuju}(u)$. Therefore, the singularity is a true curvature singularity, In this sense, we see that Penrose limit preserves the singularity of the original black hole in the form of a cosmological singularity.

Now we come to interprate the scaling of the various dimensionfull quantities. By rescaling we mean to recalculate them with respect to the metric $\Omega^{-2}ds^2$. In particular we would like to find out how the black hole temperature is changed in the limit. First we observe

$$R \to R^\Omega = R/\Omega, M \to M^\Omega = M/\Omega^2 \tag{2.32}$$

To check for consistency, we note $\alpha = M/R^2$ is not scaled (it is dimensionless to begin with).

The Hawking temperature for S-AdS has the form:

$$T_H = \frac{r_+^2 + R^2}{2\pi R^2 r_+}$$

where r_+ is the outter horizon and satisfies:

$$h'(r_+) = 0 \to r_+ = (\frac{(R^2 + MR^2)^{1/2} - R^2}{2})^{1/2}$$

¿From (2.31) it is easy to see $r_+ \propto \frac{1}{\Omega}$ as expected on dimensional grounds. Putting these together we find $T_H^\Omega = \Omega T_H$Thus when the Penrose limit is taken, the Hawking temperature goes to zero. A word of caution, we have been taking a quantity in the original S-AdS spacetime which can be associated with a geometrical object namely , the surface gravity of the event horizon. After the limit is taken it is no longer clear what this quantity really means, even though we can compute how it varies with the change of scale. In particular we have seen the spacetime obtained in the limit do not possess event horizon (which we will prove in the a following chapter).

We will now consider another class of null geodesics in S-AdS. We choose to look at the null geodesics that travels around the three sphere of the AdS part. It turns out we can only perform the Penrose limit at certain radius. The null and geodetic condition gives:

$$(\dot{r}, \dot{t}, \dot{\psi}) = (0, -A\frac{1}{rh(r)^{1/2}}, A\frac{1}{r^2}) \tag{2.33}$$

Here A corresponds to the angular momentum of the trajectory. We will seek for coordinate transformations that takes the metric to (2.1). The additional difficulty comes in as in the tangent vector we consider the orbits at fixed r while the metric has explicit dependence on the radius. We can circumvent the obsticle in two ways. First, one can perform a linearized coordinate transformation near a particular r_0 and Taylor expand the metric around this radius as in (2.17). We will get to the Brinkman form directly. However in order for the rescaling part of Penrose limit to be well defined we will need $\frac{dg_{UU}}{dr}|_{r=r_0} = 0$. Again, it is easier to first look at the Jacobian of the transformation. The one we will use is:

$$\frac{\partial(r,t,\psi)}{\partial(U,V,W)} = \begin{pmatrix} 0 & 0 & 1 \\ -A\frac{1}{r_0 h(r_0)^{1/2}} & 0 & 0 \\ A\frac{1}{r_0^2} & 1 & 0 \end{pmatrix} \tag{2.34}$$

The first column is again dictated by the requirement u being the affine parameter along the null line.

$$\begin{aligned} g_{UU} &= \frac{A^2}{r_0^2}[-\frac{h(r)}{h(r_0)} + \frac{r^2}{r_0^2}] \\ &= 0 + \frac{A^2}{r_0^2}[-\frac{h'(r_0)}{h(r_0)} + \frac{2r_0}{r_0^2}](r-r_0) + \frac{A^2}{r_0^2}[-\frac{h''(r_0)}{h(r_0)} + \frac{2}{r_0^2}](r-r_0)^2 \end{aligned}$$

With our condition on r_0, $-\frac{h'(r)}{h(r_0)} + \frac{2r}{r_0^2} = 0$, and set $\Omega w = (r - r_0)$ the metric turns into:

$$\Omega^{-2}ds^2 = R^2[\frac{A^2}{r_0^2}\frac{8M}{R^2 r_0^4 h(r_0)}w^2du^2 + 2Adudv + \frac{1}{h(r_0)}dw^2 + r_0^2\sin(\frac{A}{r_0^2}u)(d\phi^2 + \phi^2 d\zeta^2)] \tag{2.35}$$

The five sphere part is handled as before. The resultant pp-wave metric in Brinkman form is:

$$ds^2 = -2dudv + [\frac{A^2}{r_0^2}\frac{8M}{R^2 r_0^4}w^2 + (\frac{A}{r_0^2})^2[\sum_{i=1}^{2} x^{i2}]]du^2 + dw^2 + \sum_{i=1}^{7} dx^i \tag{2.36}$$

I have done some trivial rescaling to make the result cleaner.

The second way to deal with the situation is to take the coordinate transformation:

$$\frac{\partial(r,t,\phi)}{\partial(U,V,W)} = \begin{pmatrix} 0 & 0 & 1 \\ -A\frac{1}{rh(r)^{1/2}} & 0 & -Au(\frac{1}{rh(r)^{1/2}})' \\ A\frac{1}{r^2} & 1 & Au(\frac{1}{r^2})' \end{pmatrix} \tag{2.37}$$

$g_{UU} = 0$ automatically. We have thrown the problem to the requirement $g_{UW} = 0$. In particular, we will need to Taylor expand around r_0 where $g_{UW} = d(g_{UW})/dr = 0$ These conditions reads:

$$2h(r_0) = h'(r_0)r_0$$

$$h''(r_0)r_0 = h(r_0)h'(r_0)$$

They are satisfied only for fine tuned values of M, R. When this is the case the metric becomes ($\Omega w = r - r_0$):

$$-2\Omega^2 Adudv + \frac{1}{h(r_0)}\Omega^2 dw^2 + r_0^2\Omega^2 \sin^2(\frac{A}{r_0^2}u)(d\phi^2 + \phi^2 d\zeta^2)$$

The final form of the metric is:

$$-2dudv + (\frac{A}{r_0^2})^2 \sum_{i=1}^{2} x^{i2}du^2 + dw^2 + \sum_{i=1}^{7} dx^{i2} \tag{2.38}$$

For the above two cases, it is possible to take the value r_0 close to but not exactly on the event horizon. Also, it is clear either of the above plane wave spacetimes contain event horizons

If, instead, we consider orbits going around the five sphere, it can be shown the conditions for r_0 involves $h'(r_0) = 0$, which is not satisfied at any radius. We need to consider those orbits as special cases of (2.25). From the explicit construction, we learned the Penrose limit does not not retain enough global feature of the original spacetime to allow us to find the thermalized phase in the BMN/CFT correspondence.

Chapter 3

Geometrical Properties of BMN Planewave

In this chapter we will examine the geometrical properties of the BMN spacetime. In particular we will look in detail its causal structure, the Penrose diagram and how to obtain the Killing vectors from those of $AdS_5 \times S^5$ through the Penrose limit.

3.1 The Conformal Structure

We start by studying the conformal structure of the BMN plane wave. In particular, we would like to learn about the form and location of its conformal boundary. Knowledge about the conformal boundary would help us gain a clearer geometrical interpretation of the BMN/CFT duality as holography. It is easiest to start from the Rosen form of the metric:

$$ds^2 = -2du'dv' + \sin^2(\mu u') \sum_{i=1}^{8} dx'^{i2} \tag{3.1}$$

Remember the metric becomes degenerate at $u' = 0$, $u' = \pi/\mu$. A coordinate transformation

$$\begin{aligned} u &= -\cot(\mu u') \\ v &= v'/\mu \\ x^i &= x'^i \end{aligned} \tag{3.2}$$

will bring the metric to:

$$ds^2 = \frac{1}{1+u^2}(-2dudv + \sum_{i=1}^{8} dx^{i2}) = \frac{1}{1+u^2}(-2dudv + dr'^2 + r'^2 d\Omega_7^2) \tag{3.3}$$

which explicitly shows the conformal flatness of the BMN metric. Notice the coordinate transformation is only valid at the values of u' where the metric is well defined. Comparing to metric in the Brinkman, where u could take any real value, the conformal flatness is only a local one. We will continue following the standard techneque of obtaining the Penrose diagram and try to map the metric into an Einstein static universe (ESU). In particular, we will find, since the flat space is conformal to the ESU, we will have two conformal factors. One relating BMN to the flat space, the other relates the flat space to the ESU. The conformal boundary of a space is defined by where the conformal factor diverges. We would have to pay attention to both contributions in order to learn the true boundary of the BMN spacetime.

This will be accomplished through the following sequence of transfortmation [21]:

$$\begin{aligned} u &= r\cos(\theta) + t \\ r' &= r sin(\theta) \\ v &= \frac{r\cos(\theta) - t}{2} \end{aligned} \tag{3.4}$$

this puts the flat space metric into the physical coordinate:

$$ds^2 = \frac{1}{1+u^2}(-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\Omega_7^2) \tag{3.5}$$

The last part will give the familiar conformal diagram of the flat space.

$$\begin{aligned} \tan(\frac{\psi+\zeta}{2}) &= r + t \\ \tan(\frac{-\psi+\zeta}{2}) &= r - t \end{aligned} \tag{3.6}$$

and we finally obtain:

$$ds^2 = \frac{(1+(r+t)^2)(1+(r-t)^2)}{4(1+u^2)}[-d\psi^2 + d\zeta^2 + \sin^2(\zeta)(d\theta^2 + \sin^2(\theta)d\Omega_7^2)] \tag{3.7}$$

The expression in the square bracket is the ESU metric, which has the topology $R \times S^9$. The sphere has been parametrized with

$$\psi \in (-\infty, +\infty), \zeta \in [0,\pi], \theta \in [0,\pi]$$

We now take a closer look at the conformal factor. Written in the $\{\psi, \zeta, \theta\}$ coordinates the prefactor

$$(1+(r+t)^2)(1+(r-t)^2) = \frac{1}{(\cos(\psi)+\cos(\zeta))^2} \tag{3.8}$$

its divergence marks the conformal boundary of the flat space at $\pm\psi + \zeta = \pi$, with all other coordinates arbitrary. This is a cone fig(3.1) (remembering we are in 10 dimensions). The factor

$$\begin{aligned} \frac{1}{1+u^2} &= \frac{1}{1+(\cos^2(\theta/2)\tan(\frac{\psi+\zeta}{2}) - \sin^2(\theta/2)\tan(\frac{\zeta-\psi}{2}))^2} \\ &= \frac{1}{1+(\frac{\sin(\psi)+\cos(\theta)\sin(\zeta)}{\cos(\psi)+\cos(\zeta)})^2} \end{aligned} \tag{3.9}$$

goes to zero at again $\pm\psi + \zeta = \pi$ except at the lines $\{\theta = \pi, \psi + \zeta = \pi\}$, $\{\theta = 0, -\psi + \zeta = \pi\}$, and the "point" $\psi = 0$, $\zeta = \pi$, $\theta =$ arbitrary. This indicates the boundary of the region where the coordinate transformation (3.2) is valid. The interior of the cones is a whole Minkowski space and also the $u' \in (-\pi/\mu, \pi/\mu)$, strip of the BMN spacetime. Note at the points on the cone where the prefactor remains finite, the size of the of the seven sphere shrinks to zero.

Putting the two contributions together, we see the conformal factor of the $u' \in (0, \pi/\mu)$ strip of the BMN plane wave diverges only on one dimensional lines on the cones $\pm\psi + \zeta = \pi$ where (3.9) fail to cancel the infinities from (3.8). These are the true conformal boundaries of the BMN spacetime. ¿From the point of view of (3.9) as the boundary of the validity for the Rosen coordinates, we see that it is possible to analytically extend the region to include points beyond the strip $u' \in (-\pi/\mu, \pi/\mu)$. The whole picture is illustrated in fig(3.2). To make the visualiztion easier, we have reparametrize the embedding of the nine sphere in R^{10}

$$x = -\cos(\zeta) = \cos(\alpha)\cos(\beta)$$

$$y = -\cos(\theta)\sin(\zeta) = \cos(\alpha)\sin(\beta)$$

$$r = -r'\sin(\theta)\sin(\zeta) = r'\sin(\alpha)$$

where x, y, r (x', y', r') are the coordinates in the R^{10} (r (r') is the radial coordinate of a R^8 subspace). The metric then becoms:

$$\frac{1}{4[(\cos(\psi)+\cos(\zeta))^2 + (\sin(\psi)+\cos(\theta)\sin(\zeta))^2]}(-d\psi^2 + d\alpha^2 + \cos^2(\alpha)d\beta^2 + \sin^2(\alpha)d\Omega_7^2) \tag{3.10}$$

In these coordinates, the extension beyond the Rosen patch is more easily picturized and the conformal boundary is then just a spiral line on the surface of the ESU cylinder fig(3.2). It is also easy to see the boundary is a null line. The timelike boundary of $u' = \pm\infty$ are seen to be mapped to the timelike future and past of the ESU, which are two points $i^{\pm}$

When the metric is written in the Brinkman form

$$ds^2 = -2d\overline{u}d\overline{v} - \mu^2\overline{r}^2 d\overline{u}^2 + d\overline{r}^2 + \overline{r}^2 d\overline{\Omega}_7^2$$

there seems to be two distinct asymptotic regions namely, $\overline{v} \to \pm\infty$, $\overline{r} \to \infty$ (the timelike infinities $\overline{u} \to \pm\infty$ in the Brinkman coordinate are treated seperately as before). We would like to see how they are related to the one dimensional null boundary we just found. The key is to observe the coordinate transformation:

$$\overline{u} = -arccot(u) = -arccot([\tan(\frac{\psi+\zeta}{2})(\cos(\theta)+1) - \tan(\frac{-\psi+\zeta}{2})(\cos(\theta)-1)]/2)$$

$$\begin{aligned}\overline{v} &= v - \frac{1}{2}\frac{u}{1+u^2}r^2 \\ &= (\tan(\frac{\psi+\zeta}{2})(\cos(\theta)-1) - \tan(\frac{-\psi+\zeta}{2})(\cos(\theta)) + 1)/4 - \frac{1}{2}\frac{u}{1+u^2}r^2\end{aligned}$$

$$\overline{r} = r\sin(u) = \frac{\sin(\theta)}{2}[\tan(\frac{\psi+\zeta}{2}) + \tan(\frac{-\psi+\zeta}{2})]\sin(u) \tag{3.11}$$

We will focus on the piece of the boundary $\{\psi + \zeta = \pi,\ \theta = \pi\}$. Consider if we approach the boundary following a path $limit_{boundary}\tan(\frac{\psi+\zeta}{2})(\theta-\pi)^2 =$ finite. We see in particular u will be finite. Assume also, $\lim_{boundary}\overline{r} \propto limit_{boundary}\tan(\frac{\psi+\zeta}{2})(\theta-\pi) \to \infty$ Observing $v \propto \tan(\frac{\psi+\zeta}{2})$, we can then choose our path so that the divergent piece in $\overline{v}$ cancels leaving it finte. Approaching the conformal boundary in this fashion correspondes to taking $\overline{r} \to \infty$ while keeping all the other coordinates finite. Similarly, if we instead require $\lim_{boundary} \tan(\frac{\psi+\zeta}{2})(\theta-\pi) =$ finite, this will imply $\lim_{boundary} \tan(\frac{\psi+\zeta}{2})(\theta-\pi)^2 \to 0$. In this case, we will not be able to cancell the infinite part in $\overline{v}$. The result is to approach the $\overline{v} \to \pm\infty$ asymptotics with all the other coordinates kept finite.

More generally, if we choose to look at general pp-wave metric, the above analysis would not be applicable. This is because not all pp-wave spacetimes are (locally) conformally flat. To analyze the boundary behavior, we can instead use the ideal point construction [22]. We will return to this point in a later subsection.

We will close this section by examine if there is any relation between the conformal boundary of the BMN spacetime and that of $AdS_5 \times S^5$. Notice using (3.14), we can write the metric of $AdS_5 \times S^5$ in the following form:

$$\begin{aligned} ds^2 &= R^2[-\cosh^2(x/R)(du+\frac{dv}{R^2})^2+\frac{dx^2}{R^2}+\sinh(x/R)d\Omega_3^2+\cos^2(y/R)(du-\frac{dv}{R^2})^2+\frac{dy^2}{R^2} \\ &\quad +\sin^2(y/R)d\Omega_3'^2] \\ &= ds^2_{BMN}+R^2[(-\cosh^2(x/R)-1+\frac{x^2}{R^2})dt^2+(\cos^2(y/R)-1+\frac{y^2}{R^2})d\psi^2+(\sinh^2(x/R)-\frac{x^2}{R^2})d\Omega_3^2 \\ &\quad +(\sin^2(y/R)-\frac{y^2}{R^2})d\Omega_3'^2] \\ &= \cosh^2(x/R)[\frac{C}{\cosh^2(x/R)}ds^2_{ESU}+R^2[(-1+\frac{(1+\frac{x^2}{R^2})}{\cosh^2(x/R)})dt^2+\frac{1}{\cosh^2(x/R)}(\cos^2(y/R)-1+\frac{y^2}{R^2})d\psi^2 \\ &\quad +(\tanh^2(x/R)-\frac{x^2}{R^2\cosh^2(x/R)})d\Omega_3^2+\frac{1}{\cosh^2(x/R)}(\sin^2(y/R)-\frac{y^2}{R^2})d\Omega_3'^2] \end{aligned} \tag{3.12}$$

where C is the conformal factor thet relates the BMN metric to the ESU. To get to the conformal boundary of $AdS_5 \times S^5$, we would need to take $x \to \infty$, while keeping all the other coordinates finite. This limit correspondes to the $\overline{r} \to \infty$, $u, \overline{v}$ finte limit on the BMN side. Note that the divergence in C is due to the divergence in the factor that relates Minkowski space to the ESU, which goes like $\lim_{\theta\to\pi/2}\tan(\theta) \to \infty$. In comparasion to the exponential divergence of $\cosh^2(x/R)$ in the denominator, we see the metric (3.12) reduces to:

$$ds^2_{boundary} = -R^2dt^2 + R^2d\Omega_3^2$$

the usual $R \times S^3$ boundary of AdS_5. If instead we are interested in the $R \to \infty$ Penrose limit, it is easy to see we recover the BMN metric. From this point of view, it is clear the conformal boundary of the BMN spacetime is disconnected from the original boundary of $AdS_5 \times S^5$. These are two distinct limiting process of the same metric. The Penrose limit focuses on a portion of the spacetime that is very far from the original bounndary where the CFT lives on. It appears the holography interpretation of the BMN/CFT correspondence is much more subtle.

3.2 Isometries of the BMN Spacetime

In this section, we discuss the isometries of the BMN spacetime. We will also look at how to obtain the KIlling vectors through the Penrose limit of the isometries of $AdS_5 \times S^5$. Again, the BMN metric in the Brinkman form is:

$$-2dudv - \mu^2 \sum_{i=1}^{8} x^{i2} du^2 + dx^{i2} \tag{3.13}$$

Through the general discussion in the previous chapter, we know it has at least as many Killing vector as its ancestor $AdS_5 \times S^5$, whose isometry group is $SO(2,4) \otimes SO(6)$ with totally 30 generators. Looking at the metric alone, it turns out the Penrose will enhance the symmetry. The obvious ones by looking at the BMN metric are:

$$Z_u = \partial_u,$$

$$Z_v = \partial_v,$$

$$Z_{M_{ij}} = x^i \partial_j - x^j \partial_i \; i,j \in 1...8$$

while we also have rotations between the null direction v and the transverse components.

$$Z_i = -\cos(\mu u)\partial_i + \mu \sin(\mu u) x^i \partial_v \; i = 1...8,$$

$$Z_i' = -\mu \sin(\mu u)\partial_i - \mu^2 \cos(\mu u) x^i \partial_v \; i = 1...8,$$

The SO(8) symmetry generated by $Z_{M_{ij}}$ is the enhanced version of the original $SO(4) \otimes SO(4)$ symmetry each coming from the three spheres in AdS_5 and S^5. This only happens in the limit when the radius of the two components are matched. Even in the presnt case, when considering the presence of the five form fields needed to give gravitational support of the spacetime, the SO(8) symmetry is again broken to $SO(4) \otimes SO(4)$. We will only get the generators of $SO(4) \otimes SO(4)$ by considering the Penrose limit of the isometries of $AdS_5 \times S^5$, to which we will now turn to.

As discussed earlier, it is easy to see the $Z_{M_{ij}}$'s comes from rotations of the two three spheres. Penrose limit does nothing to them. For simplicity, we will now look at the Penrose limit of $AdS_2 \times S^2$, which gives

$$R^2(-\cosh^2(\rho)d\tau^2 + d\rho^2 + \cos^2(\theta)d\psi^2 + d\theta^2) \to -2dudv - (x^2 + y^2)du^2 + dx^2 + dy^2$$

where $\rho = Rx$, $\theta = Ry$. The isometry group of $AdS_2 \times S^2$ is $SO(2,1) \otimes SO(3)$, the 6 generators in the above coordinates are:

$$A_1 = \partial_\tau, \; B_1 = \partial_\psi$$

$$A_2 = \cos\tau\partial_\rho - \sin(\tau)\tanh(\rho)\partial_\tau, \; B_2 = \cos(\psi)\partial_\theta + \sin(\psi)\tan(\theta)\partial_\psi$$

$$A_3 = \sin(\tau)\partial_\rho + \cos(\tau)\tanh(\rho)\partial_\tau, \; B_3 = -\sin(\psi)\partial_\theta + \cos(\psi)\tan(\theta)\partial_\psi$$

The Penrose limit is done through first:

$$\begin{aligned}\tau = u + v\Omega^2 \; \rho = Rx \\ \psi = u - v\Omega^2\theta = Ry\end{aligned} \tag{3.14}$$

followed by the usual rescaling $ds^2 \to \Omega^{-2}ds^2$. The effect of the coordinate transformation (3.14) on the Killing vectors is (we will write down the leading terms in Ω^{-1})

$$A_1 = +\frac{1}{\Omega^2}\partial_v,\; B_1 = -\frac{1}{\Omega^2}\partial_v$$

$$A_2 = \frac{1}{\Omega}(\cos u\partial_x - \sin(u)x\partial_v),\; B_2 = \frac{1}{\Omega}(\cos(u)\partial_y + sin(u)y\partial_v)$$

$$A_3 = \frac{1}{\Omega}(\sin(u)\partial_x + \cos(u)x\partial_v),\; B_3 = \frac{1}{\Omega}(-\sin(u)\partial_y + \cos(u)y\partial_v)$$

In order for them to be well definedin the limit $\Omega \to 0$, we will need to perform the rescaling analogous to that for the metric.

$$\begin{aligned}A_1 \to \Omega^2 A_1 \; B_1 \to \Omega^2 B_1 \\ A_{2,3} \to \Omega A_{2,3} \; B_{2,3} \to \Omega B_{2,3}\end{aligned} \tag{3.15}$$

Comparing with the set of Killing vectors of the BMN plane wave, we see the Penrose limit preserves almost all of them. However, the two Killing vectors A_1, B_1 became degenerate under the limit, while there is no corresonding element for Z_u. We can rectify the situation by considering instead of A_1, B_1, their linear combinations

$$A^+ = A_1 + B_1 = \partial_\tau + \partial_\psi \to \partial_u$$

$$A^- = A_1 - B_1 = \partial_\tau - \partial_\psi \to \frac{1}{\Omega^2}\partial_v$$

The rescaling required no are

$$A^+ \to A^+,\; A^- \to \Omega^2 A^-$$

All of the Killing vectors of BMN plane wave are accounted for. We see now explicitly the rescaling factor differs from Killing vector to Killing vector, as aluded to in the previous chapter.

Using another coordinate chart which would lead to the Rosen form of th plane wave metric, we can obtain all the Killing vectors in the Rosen coordinates. However as shown in [18] we still need to use various linear combinations of the original Killinfg vectors in order for the limit to be well defined. For completeness we will write down all the Killing vectors in the Rosen coordinates:

$$Z_{u'} = \partial_{u'} + \frac{\mu^2}{2}\sum x'^{i2}\partial_{v'} + \sum \mu\cot(\mu u)x'^i\partial'_i,$$

$$Z_{v'} = \partial_{v'},$$

$$Z_{M'_{ij}} = x'^i\partial'_j - x'^j\partial'_i \; i,j \in 1...8$$

$$Z'_i = -\cot(\mu u)\partial'_i + x'^i\partial_v \; i = 1...8,$$

$$Z_i = \partial'_i \; i = 1...8,$$

The mysterious rotations in $v - x^i$ are now manefestively the translations in x'^i

3.3 Lightcone Structure in BMN

We have seen through explicit construction of the Penrose diagram, the BMN spacetime has a highly degenerate one dimensional null boundary. Here we will study carefully the causal structure of BMN. Through which we hope to shed some light on the nature of the one dimensional boundary. We will stick to the Brinkman form of the metric (3.13) in this section.

To begin, note that any point on a constant $\sum x^{i2} = r^2 = const$ plane can be shifted by the Killing vectors ∂_u, ∂_v, $Z_{M_{ij}}$ to the point $(0, 0, r_0, 0, 0...)$. And then by Z_i to the origin. Further since isometry does not change the causal structure, we can focus on first drawing the light cone at the origin of the spacetime, and use the Killing vectors to move it around. At the origin, consider the two class of causal curves:

$$\begin{aligned} u &= t \\ v &= \frac{\mu}{4}\sum A_i^2 \sin(2\mu t) + C(t) \\ x^i &= A_i \sin(\mu t) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} u &= t \\ v &= -\frac{\mu^2 \sum A_i^2}{2} t + \frac{\mu}{4}\sum A_i^2 \sin(2\mu t) + \mu \sum A_i^2 (1-\cos(\mu t))\sin(\mu t) + C(t) \\ x^i &= A_i(1-\cos(\mu t)) \end{aligned}$$

with $C(t)$ an increasing function and $C(0) = 0$. Both class of causal curves pass through the origin at $t = 0$. When $C(t)$ is a constant, the curves became null, while in general they are time like. The null curves in the first class are acctually null geodesics. Notice the curves in the first family go caustic at $u = t = \pi/\mu$, while the second does not. The importance of the second class will become clear momentarily. They are not geodetic. Let us look at the null curves. In both classes they mesh to form null surfaces. It is easier to describe them if we eliminate the integration constants A_i. The first family gives:

$$v = \frac{\mu}{2}\sum x^{i2} cot(\mu u) = h_1(u, r) \tag{3.17}$$

while the second family generates:

$$v = \frac{\sum x^{i2}}{(1-\cos(\mu u))^2}\left(-\frac{\mu^2}{2}t + \frac{\mu}{4}\sin(2\mu t) + \mu(1-\cos(\mu t))\sin(\mu t)\right) = h_2(u, r) \tag{3.18}$$

It is easy to see now a point (u_0, v_0, x_0^i) is causally connected to the origin (in the future direction) if and only if it satisfies

$$v_0 \geq \min\{h_1(r_0, u_0), h_2(r_0, u_0)\} \tag{3.19}$$

We will choose to look at the constant x^i slices. They are pictured in fig(3.3). There are several salient features to notice. First, In the limit x^i goes to zero, both surfaces coincides with $v = 0$ axis. Second, due to the divergence in the null direction v, the future of the origin contains every point on the plane $u = \pi/\mu$. More explicitly, any point with $u = \pi/\mu$, $r \neq 0$ will satisfy (3.19). However the future of a point is a closed set. Therefore we have to include the entire plane $u = \pi/\mu$. It is not difficult to see just as in the flat space case the future of the $u = c_0 = const$ surfaces consist of the half space $u > c_0$. As a result, the whole half space $u \geq \pi/\mu$ is in the future of the origin. Similarly, we can conclude the past of teh origin contains the half space $u \leq -\pi/\mu$.

We can now move the surfaces around by the Killing vectors. The translation in $u - v$ planes is trivial. We will focus on shifting to nonzero transverse positions (and only in one direction). This requires the Killing vector $-\cos(\mu u)\partial_i + \mu \sin(\mu u)\partial_v$, which generates:

$$\begin{aligned} u' &= u \\ v' &= -\frac{\mu}{4}\sin(2\mu u)c^2 + x^i \mu \sin(\mu u)c + v \\ x'^i &= -\cos(\mu u)c + x^i \end{aligned}$$

The origin is moved to $(0, ...x'^i = c, 0...)$ The surfaces are moved to

$$v' = -\mu c \sin(\mu u')(x'^i - c\cos(\mu u')/2) + \frac{\mu}{2}(x'^i - c\cos(\mu u'))^2 cot(\mu u')$$

and

$$v' = \frac{\mu}{4}\frac{x'^{i2} - c^2 + 2c\cos(\mu u')(c - x'^i)}{(1-\cos(\mu u'))^2} + \mu\frac{x'^i - c\cos(\mu u')}{(1-\cos(\mu u'))}(x'^i - c)\sin(\mu u') - \frac{\mu^2}{2}\frac{(x'^i - c\cos(\mu u'))^2}{(1-\cos(\mu u'))^2}t$$

The constant x'^i slices are shown in fig(3.4). Note when $x'^i = c$ the surface from the second class is just

$$v' = \frac{-\mu^2 c^2 u'}{2}$$

which can be obtained by simply solving the metric for null surfaces with constant $x'^i = c$, $-2dudv - \mu^2 c^2 du^2 = 0$. The qualitative feature does not change from the structure at the origin as expected. To summarize, fig(3.3), fig(3.4) illustrates the light cone structure of BMN. In particular, the future(past) of any point with $u = c_0$ will include the region $u \geq c_0 + \pi/\mu (u \leq c_0 - \pi/\mu)$

We will close by briefly review the ideal point construction of causal boundary for plane wave spacetimes. More formal explorations of the causal structure of pp-wave spacetimes using the ideal point technique can be found in [23], [24]. The ideal point construction on a spacetime M is to first complete every future(past) endless causal curves with a point of infinity, the ideal points. We will denote the set of ideal points by I. We then identify points in I if they have the same past and future. The set I with proper identification will be a representation of the "boundary at infinity" for M. It is usually convenient to map a point p in I as the future(past) $J^{\pm}(\gamma_p)$ of the curve γ_p to which p is attached to. For the general planewave spacetimes

$$-2dudv - f_{ij}(u)x^i x^j du^2 + \sum dx^{i2} \tag{3.20}$$

with $f(u)$ regular for all finit value of u the following can be proved [23]:
1. All causal curves are either with $u \to +\infty$ or u asymptotic to a finite value c_0 while v diverges to infinity.

2. The former has the entire spacetime as its past and future. On the other hand, all causal curves approaching the same value $u = c_0$ has the set $S = \{u \leq c_0\}$ as its past.

3. the future(past) of any point with $u = c_0$ will include the region $u \geq c_0 + \pi/\mu (u \leq c_0 - \pi/\mu)$ just as in the BMN case.

With these facts in mind it is easy to see we have to identify the future(past) ideal points associated with all curve asymptotes to $u = c_0$ in the future(past). We can then label the ideal points by the values of the coordinate u. Further, from fig(3.5). we can see the future of the future ideal point f_{c_0} is exactly the half space $u \geq c_0 + \pi/\mu$. Similarly, the past of the past ideal point $p_{c_0+\pi/\mu}$ is exactly $u \leq c_0$. The points f_{c_0} and $p_{c_0+\pi/\mu}$ have the same causal future and past and thus should be identified according to our program. The causal boundary of a general plane wave spacetime (3.21) is always a one dimensional null line. This method does not require conformal flatness and in particular applies to the 11 dimensional plane wave supergravity solution.

fig 3.1a

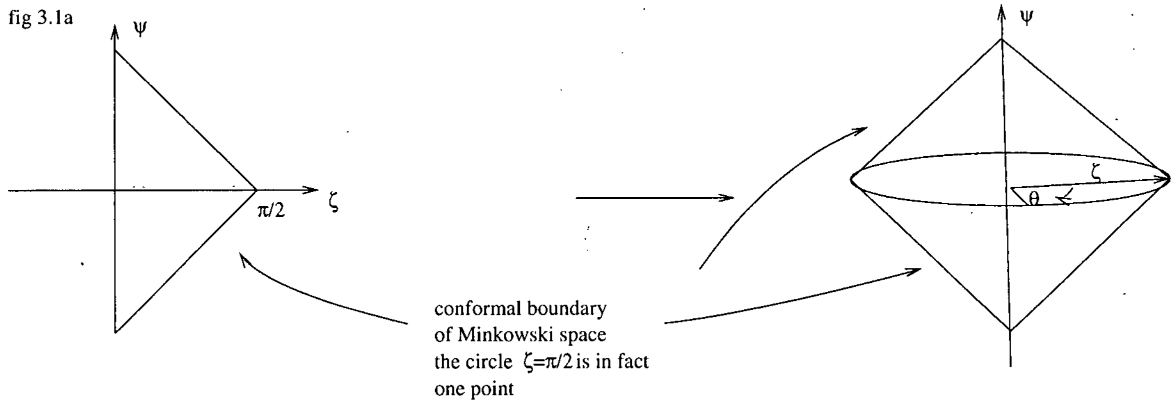


fig 3.1b

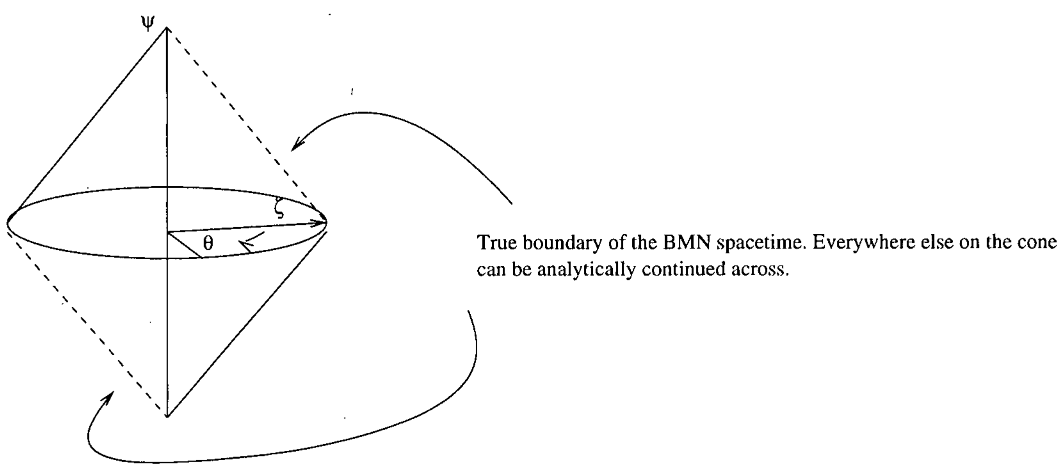


fig 3.2

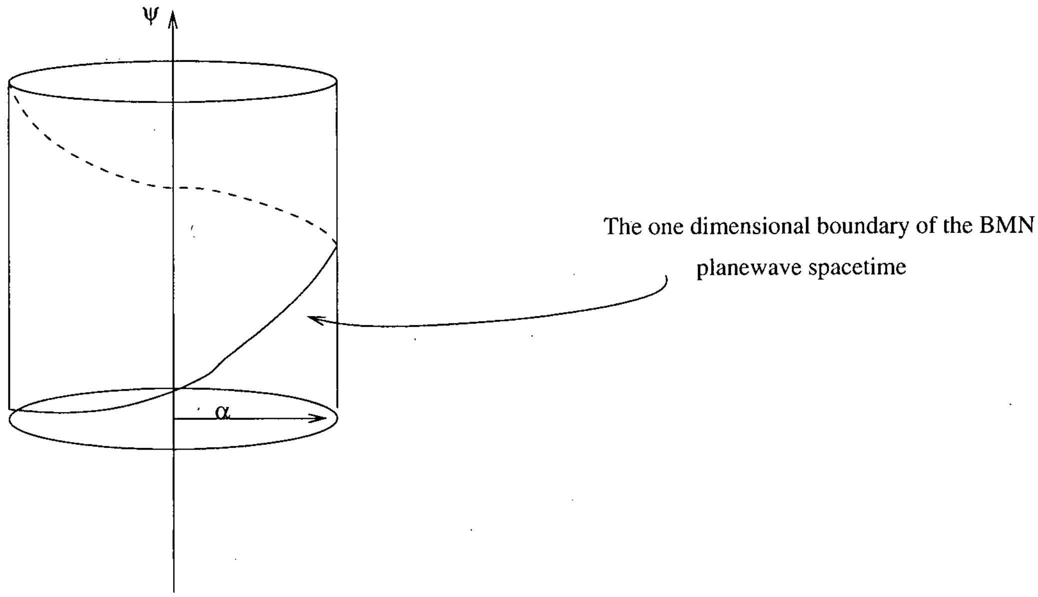
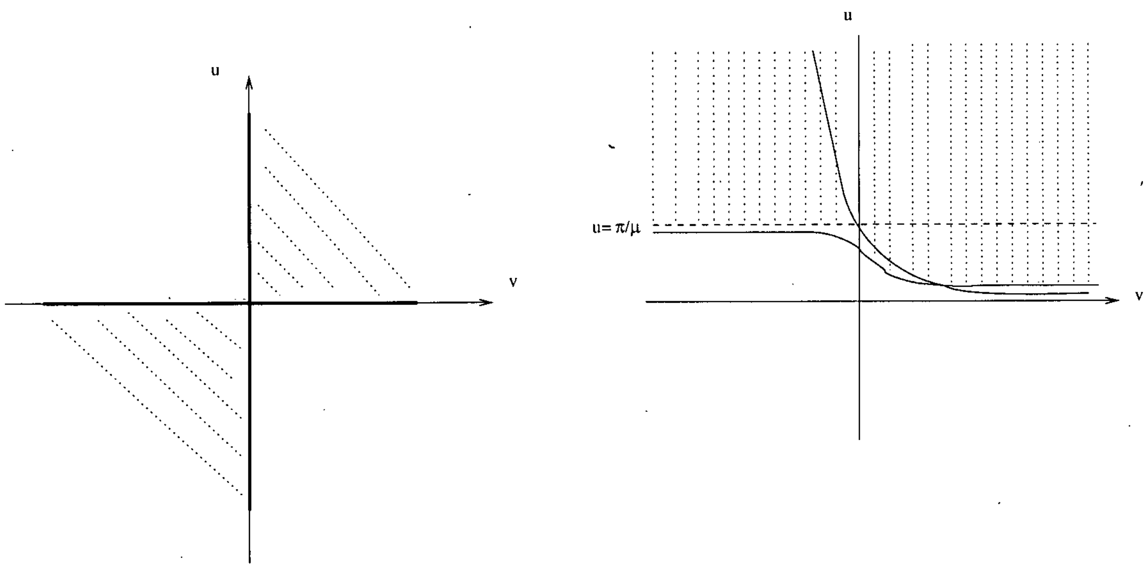
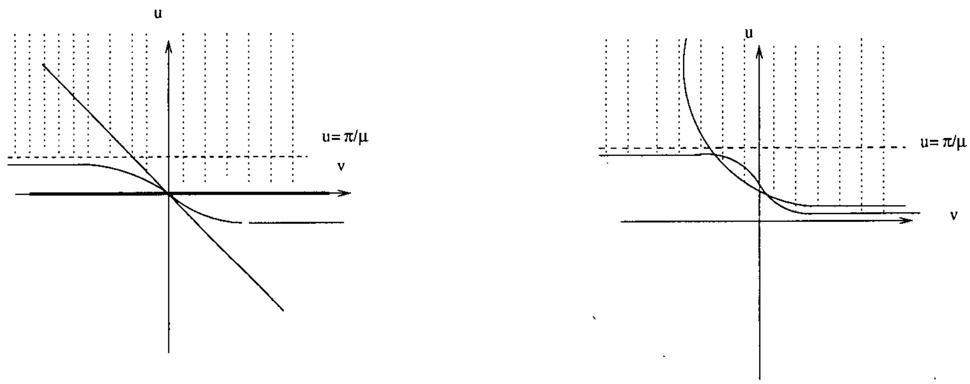


fig 3.3



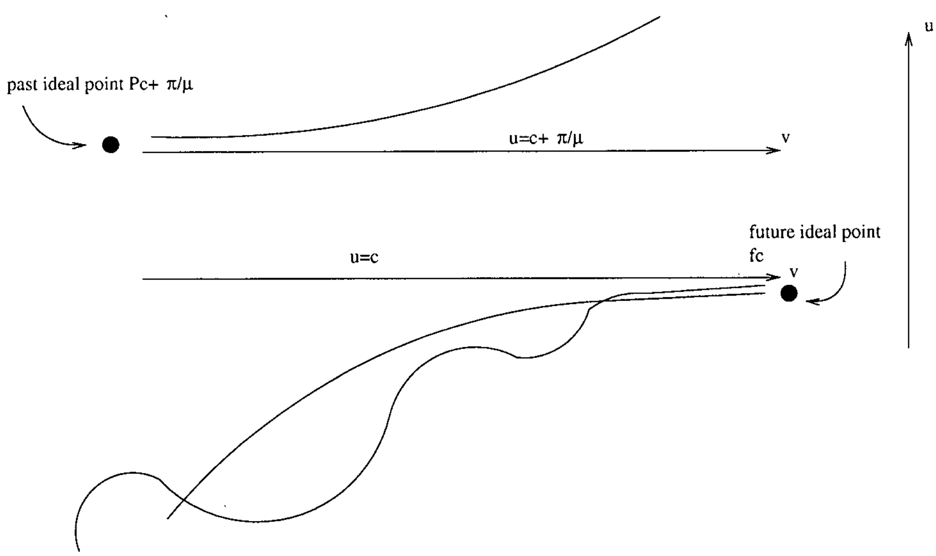
The light cone structure of the origin. The left hand side is the slicing at r=0. The graph on the right hand side is the slicing at nonzero r. The doted region is the future light cone of the origin. The important feature is that every point with $u > \pi/\mu$ is in the future of the origin.

fig 3.4



The light cone structure of a point at u=0, v=0, r=c off the center. The dotted region is the future light cone. The left hand side is the slice at r=c. The right hand side is the slicing at other values of teh transvese coordinates. Again we see the future of such a point contains every point u> π/μ

fig 3.5



The past ideal point fc is associated with the set of points with u<c. The future ideal point is associated with the set of points with u>c+ π/μ The two ideal points share the same causal future and past and they should be identified. This results in the one dimensional boundary of the spacetime.

Chapter 4

Theorems Regarding No Evenet Horizons in PP-Wave

We have been concerned with the problem of finding a black hole phase in the BMN/CFT correpondence. Black holes are classically defined as a region hiding behind the event horizon. In this chapter we will review two theorems which suggest there is no event horizon and thus no black holes in pp-wave spacetimes. In fact the second theorem gives indications that no regular event horizons exist for spacetimes with a null Killing vector.

4.1 No Event Horizons in PP-Wave

To begin, we will try to look carefully at the definition of an even horizon in a spacetime. For the case of asymptotically flat spacetimes, the event horizon is defined as the boundary of the causal past of the future null infinity (the scri). That is, signals sent from a certain region of the spacetime (behind the horizon) will never reach the asymptotic infinity. In a general spacetime, however, the idea of asymptotic infinity may not be well defined. In the case of pp-wave spacetimes,

$$ds^2 = -2dudv - H(u, r, \Omega)du^2 + dr^2 + r^2 d\Omega_n^2 \tag{4.1}$$

where Ω denotes the coordinates on the n-sphere, the authors of [25] propose we can adopt a working definition of (no) event horizon in pp-wave spacetimes:

A pp-wave spacetime does not have an event horizon if and only if for any given point p:$(u_0, v_0, r_0, \Omega_0)$ there is a future directed causal (null or timelike) curve that connects p to some point $(u_1 = u_0 + \epsilon, v_1, r_1, \Omega_0)$, where ϵ is a positive number and can be taken arbitrarily small, and v_1, r_1, can be taken to infinity.

The key is to replace the asymptotic infinity by the idea of "points arbitrarily far" in the transverse and the null directions. Notice since ∂_v is a null Killing vector, its integral curves are null geodesics. Therefore, we see that it is always possible to send light signals from any point of the spacetime to points with arbitarly large value of v. The event horizon can not stretch across the null direction even if it exists. We will now give the pecise statement of the theorem:

Theorem 1: PP-wave spacetimes (4.1) with $H(u, r, \Omega)$ nonsingular at finite value of its transverse coordinates do not admit an event horizon.

The proof proceed as follows. Given a point p_0:$(u_0, v_0, r_0, \Omega_0)$, we will seek the condition on v_1 such that p_0 can be causally connected to p_1:$(u_0 + \epsilon, v_1, r_1, \Omega_0)$, where r_1 can be taken arbitrarily large. We will find there is no upper bound for v_1. First assume H satisfies the nonsingular requirement, in particular, we have covered all the plane wave case (for which $H(u,r,\Omega) = h_{ij}(u)x^i x^j$) and the vaccum pp-wave solutions (we will come back to this later). Also, without loss of generality, we will assume $H(u,r,\Omega)$ is regular on a small interval about u_0. Therefore on the interval (r_0, r_1) we can assume $H(u,r,\Omega) \geq H_0$ =const. The proof will be technichally easy if we introduce the fiducial metric:

$$ds_f^2 = -2dudv - H_0 du^2 + dr^2 + r^2 d\Omega_n^2 \tag{4.2}$$

Note that on the interval of interest $ds^2 \geq ds_f^2$ and therefore a causal curve in the fiducial metric will also be causal in the pp-wave metric we are looking at. In other words, the light cone in the fiducial metric is smaller than that in the pp-wave. The task of looking for a causal curve connecting p_0, p_1 in pp-wave is reduced to finding the appropriate curves with ds_f^2. Notice that ds_f^2 is nothing but the flat Minkowski metric:

$$-dt^2 + dy^2 + dr^2 + r^2 d\Omega_n^2$$

(We can see this explicitly if we perform the coordinate transformation: $u = \frac{1}{H_0^{1/2}}(t-y)$, $v = (H_0)^{1/2} y$ if $H_0 > 0$ and $u = \frac{1}{(-H_0)^{1/2}}(t-y)$, $v = (-H_0)^{1/2} t$ if $H_0 < 0$. We do not have to do any thing if $H_0 = 0$.) Since we know any points in the flat space can be causally connected to points with arbitrarily large values of y, r. We can conclude we will be able to do the same for the pp-wave metric. Let us make this more quantitative: we are looking for a curve $C(\tau)$, $0 \leq \tau \leq 1$ in the fiducial metric satisfying:

$$C(0) = (u_0, v_0, r_0, \Omega_0)$$

$$C(1) = (u_0 + \epsilon, v_1, r_1, \Omega_0)$$

$$-2\dot{u}\dot{v} - H_0 \dot{u}^2 + \dot{r}^2 + r(\tau)^2 \dot{\Omega}^2 \leq 0$$

We can explicitly choose $C(\tau)$ to be

$$u(\tau) = u_0 + \tau\epsilon$$

$$v(\tau) = v_0 + \tau(v_1 - v_0)$$

$$r(\tau) = r_0 + \tau(r_1 - r_0)$$

$$\Omega(\tau) = \Omega_0$$

where the choice of v_1 is restricted by the causal condition:

$$-2\epsilon(v_1 - v_0) \leq H_0\epsilon^2 - (r_1 - r_0)^2$$

If we choose

$$v_1 - v_0 \geq \frac{-H_0\epsilon}{2} + \frac{(r_1 - r_0)^2}{\epsilon}$$

Therefore we have shown for any point p_0:$(u_0, v_0, r_0, \Omega_0)$, there exists a causal curve which connects p_0 to a point p_1 with large values of null and transverse radial coordinates.

An improtant assumption made above is the regularity of $H(u, r, \Omega)$ on the region $(u_0 - \epsilon, u_0 + \epsilon) \times [r_0, r_1]$. As mentioned earlier all the plane wave spacetimes will satisfy this condition. In particular, all the plane wave metric obtained in the Penrose limit of a spacetime (including the S-AdS geometry considered in section) falls into this category. The implication of theorem 1 is such that we can not obtain spacetimes with an event horizon by taking the Penrose limit of any spacetime. The Penrose limit does not retain enough global structure of the original metric. Here we will show the theorem applies to all the pp-wave metrics satisfying the vaccum Einstein equation as well. This follows from the observation that the vaccum Einstein's equation having the form:

$$\nabla_T H = 0 \tag{4.3}$$

where ∇_T is the harmonic operator in the flat transverse directions. $H(u, r, \Omega)$ have the usual decomposition into the spherical harmonics times r^l, or $r^{-(n-1+l)}$. The sigularities are at $r = 0$, or $r = \infty$. These are points at the boundary of our coordinate system, and therefore do not form obstructions for the causal curves used above.

In the most general case of non-vaccum solutions with arbitrary sources, $H(u, r, \Omega)$ can be virtually anything. There might be singularities in the transverse directions that interrupts the causal curves. However, as argued before the communication to large null coordinate v is unaffected as long as it remains a Killing vector. This suggests $u - v$ plane should be "blackened" together. A black string solution is more likely easier to be found.

4.2 Generalization of No Horizon Theorem

In the previous section, we have seen the metric (4.1) do not admit an event horizon. One important defining quality of (4.1) is that they have a covariantly constant null Killing vector ∂_v. (That it is covariantly constant can be checked straight forwardly by computing the Christoffel symbols) In this section we will explore the situation in a slightly generalized spacetime. In particular, we abandon the covariant constancy of ∂_v. The metric we will consider has the form:

$$e^{2A(r)}[-2dudv + H(r, y_i)du^2] + e^{2B(r)}[dr^2 + d\Omega_n^2] + \sum e^{2C(r)}dy_j^2 \tag{4.4}$$

The r dependence of the prefactor $e^{A(r)}$ modifies the covariant constancy of ∂_v, which remains a null Killing vector. We are interested in the particular case when $\overline{r}^2 = r^2 + \sum y_i^2$ is large the metric approaches the BMN metric (3.13). In this case, the Killing vector ∂_u will be asymptotically time like, and we will choose it to be the measure of coordinate time in the following. The authors of [26] exploit the seperation property of the Ricci tensor of (4.4), and established a solution generating technique. Specifically, The Ricci tensor of the (4.4) has the form:

$$R_{ij} = R_{ij}^0 + -\delta_{ui}\delta_{uj}\frac{1}{2}(e^{2A-2B}\nabla^2 H + e^{2A-2C}\nabla'^2 H + \overline{\partial}_i H\overline{\partial}_i(2A + (n-1)B + \sum_j C_j) \tag{4.5}$$

where ∇^2 denotes the flat Laplacian in the spherically symmetric R^{n+1}, and ∇'^2 is the Laplacian for the rest of the transverse directions. R_{ij}^0 is the Ricci tensor computed from (4.4) with H set to zero. This seperation is possible as a direct result of the null Killing direction ∂_v. The solution generating technique is the for a given (asymptotically flat) solution with $R^0_{ij} = T^0_{ij}$ it is possible to deform it by turning on additional matter source of th form $\triangle T_{ij} = \triangle T_{uu}$ and obtain a new solution with nonzero H. The new matter source could be modified to give the desired asymptotic behavior of $\lim_{\bar{r}\to\infty} H(\bar{r}) \sim -\bar{r}^2$. It also has to be compatible with the existing source T^0_{ij}. For form fields which contribute to the total source $T^0_{ij} + \triangle T_{uu}$, we have to check they satisfy self duality and the Bianchi identity. Various asymptotically BMN spacetimes have been derived using the solution generating technique outlined above [26]. However, only extremal horizons are retained in the resultant spacetime. This gives indication that even the requirement of having a null Killing vector is to restrictive for regular horizon to exist. The exact statement of the no-go theorem proved in [26] is:

Theorem 2. The solution generated by modifying the vacuum solution (4.4) (without the $Hdudu$ term) with only adding null matter source $T_{uu}(r, y_i)$, do not admit a regular $SO(n+1)$ invariant event horizon provided the source allows for the asymptotically timelike Killing vector ∂_u.

Note the requirement on the asymptotic time is what we would impose if we are looking for black holes in asymptotically BMN plane wave spacetime. The proof proceeds with first finding the most general asymptotically flat vacuum solutions with the form (4.4) with $H = 0$, that is setting $T^0_{uu} = 0$. As shown in [26], the solution with the boundary condition $\lim_{r\to\infty} A(r), B(r), C_j(r) \to 0$ can be found by directly solving the Einstein's equation:

$$A(r) = a\log(\frac{1-(r_0/r)^{n-1}}{1+(r_0/r)^{n-1}}) = a\log(f(r))$$

$$B(r) = -\frac{b}{n-1}\log[\frac{f(r)^{2a+\sum c_j}}{1-\frac{r_0}{r}^{2(n-1)}}]$$

$$C(r) = 1 - \frac{r_0}{r}^{2(n-1)} \tag{4.6}$$

with the constraint on the integration constants:

$$(a + \frac{1}{n}(a + \sum c_j))^2 + \frac{n-1}{n^2}(n(a^2 + \sum c_j^2) + (\sum c_j + a)^2) = 1 \tag{4.7}$$

Examining the near horizon behavior of the functions $A(r), B(r), C_j(r)$ we find:

$$e^{A(r)} \sim (r - r_0)^a$$

$$e^{B(r)} \sim (r - r_0)^{(1-2a-\sum c_j)/(n-1)}$$

$$e^{C_j(r)} \sim (r - r_0)^{c_j} \tag{4.8}$$

Note that for appropriate choice of the integration constants, the solution without $H(\bar{r})$ is just the Schwarzschild solution in flat space with horizon $r = r_0$. We will thus exam closely the effect of the deformation near r_0.

It turns out we do not have to solve for $H(\overline{r})$ explicitly. For the proof, we will leave it as an arbitrary function with the appropriate asymptotics. ($\lim_{\overline{r}\to\infty} H(\overline{r}) \sim -\overline{r}^2$.) The minimal requirement for the existence of a horizon at $r = r_0$ is such that it takes a null geodesic travelling in the u, v, r direction infinite coordinate time $\triangle u$ but finte proper time to reach the horizon. The requirement of finite proper time is to ensure that we can analytically continue across the horizon (that it is not the boundary of the spacetime). To truely establish the existence of the horizon we would need every null and timelike geodesics(that can reach the horizon) to satisfy this criterion. A null geodesic travelling in u, v, r statisfies:

$$\begin{aligned} \dot{u} &= \frac{E}{e^{2A(r)}} \\ \dot{v} &= e^{-2A(r)}(EH(\overline{r}) - G) \\ \dot{r} &= [e^{-2A-2B}(E^2H - 2GE)]^{1/2} \end{aligned} \tag{4.9}$$

where $\dot{}$ denotes differentiation with respect to the proper time τ. E, G are conserved quantities alongthe trajectory. Using (4.9) we can show:

$$\begin{aligned} \triangle\tau &= \int_{\infty}^{r_0^+} dr[e^{-2A-2B}(E^2H - 2GE)]^{-1/2} \\ \triangle u &= E\int_{\infty}^{r_0^+} dre^{-A+B}(E^2H - 2GE)^{-1/2} \end{aligned} \tag{4.10}$$

We are now in a position to analyze the behavior of the coordinate time and proper time near the assumed horizon $r = r_0$. Assume first $H \sim h(r - r_0)^g$ near the horizon. For $g \geq 0$, it is easy to see from (4.10) we will need:

$$a + \frac{(1 - 2a - \sum c_j)}{(n-1)} > -1,\ a - \frac{(1 - 2a - \sum c_j)}{(n-1)} \geq 1 \tag{4.11}$$

in order for the criterion to be met. Note the second condition can be written as:

$$a + \frac{1}{n}a + \frac{\sum c_j}{n} \geq 1 \tag{4.12}$$

which is exactly the expresstion in the first term of (4.7). This says the first term in (4.7) is greater than 1. Since the rest of (4.7) are just sums of squares, we see there is no choice of a, c_j such that both conditions (4.7) and (4.11) are fulfilled at the same time. The same condition also applies if H is logrithmically divergent. When H has power law divergence, or $g < 0$, the conditions are modified to:

$$a + (1 - 2a - \sum c_j)/(n-1) + |g| > -1,\ a - (1 - 2a - \sum c_j)/(n-1) - |g| \geq 1 \tag{4.13}$$

The second condition is more restrictive than the previous case. It is clear (4.7) can not be maintained, either. The same can be said if H has faster than power law divergence. Note in the case $H \to -\infty$ at the horizon, we have a repulsive singularity as in the case of negative mass Schwarzschild.

We can thus conclude any modification according to the the solution generating procedure with addition of only null matter will not result in a spacetime with event horizon (the new solution will of course have the correct BMN asymptotics if proper boundary condition are imposed). Attempts to bypass the no go theorem have been made in [26] by adding sources to other components of the Einstein's equation. This will modify the form of $A(r), B(r), C_j(r))$ depending on the explicit sources put in. No general statement has been made for such general deformations. In the specific cases studied, no solution with regular even horizon has been found. However, various extremal black hole solutions are generated with this program. This strongly suggests we need to abandon the null Killing vector requirement at least near the horizon in order to find the desired black hole solution in BMN.

Chapter 5

Unruh Effect in BMN PP-wave

In the last chapter we have learned that the symmetry of plane wave spacetimes is not compatible with the existence of a nonextremal black holes. The same conclusion may be true for spacetimes with a null Killing vector. Before we start thinking about how to modify the symmetries of plane wave spacetimes, it would be beneficial to examine closely the definition of temperature and thermal states in the plane wave background. The thermal properties, in particular the Hawking temperature of stationary black holes, are related to the geometrical properties of their event horizons. More generally, all that is required is a Killing horizon. In flat space, it is known that observers following the orbits of the boost Killing vector will see the Minkowski vacuum as a thermal state. These observers can only see part of the Minkowski spacetime, the Rindler patch, with the Killing horizon as a causal obstruct. The observed temperature can also be related geometrically to the surface gravity of the Killing horizon. The near horizon geometry of static black holes is the Rindler spacetime. We can gain insights into the relationship between Hawking ration and event horizon by studying Unruh radiation and the associated observer dependent horizon.

In the present, we study the results of putting an Unruh monopole detector in the BMN planewave. The Unruh detector has been one of the standard tool in analyzing thermal properties of a given quantum state. It is useful because in general curved spacetimes, the concept of particle in a state is not uniquely defined. It depends on the choice of time in carrying out canonical quantization. The hope is that we can determine which class of observers would see the vacuum state "natural" to the plane wave background as a thermal state. The meaning of "natural" will be explained later. If such trajectories exist, we will also need to relate the temperature observed to certain geometrical structure of the spacetime. Through this, we can learn more about the physical content of the no-go theorems in a semi-classical sense. We may also gain knowledge about how to modify the near horizon geometry in order to find explicit black hole solutions in BMN plane wave.

5.1 Unruh Monopole detector

We start by giving a short review of the idea of putting a monopole detector in a curved spacetime with a scalar field. An Unruh detector can be viewed as a very heavy scalar particle, whose kinematics can be treated classically [27] [28]. Suppose the detector is following a trajectory described by $x(\tau)$, where τ is the proper time of the world line, while there is another light scalar field $\phi[x(\tau)]$ interacting with the detector only on the trajectory. Mathematically this amounts to adding an interaction term $L_{int} = cm(\tau)\phi[x(\tau)]$ to the scalar field Lagrangian, where c is a small coupling constant and $m(\tau)$ is the detector's monopole moment operator. The detector can be regarded as a position dependent one point source for the scalar field.

The whole system can be discribed as a tensor product state as $|n_t> \otimes|E_i>$, where $|n_t>$ is the n particle state in the scalar filed Fock space and $|E_i>$ is the i^{th} excited state of the detector. Suppose now the scalar field is quantized with respect to some time like Killing vector ∂_t, and is in its vaccume state $|0_t>$. Assuming the detector starts out in $|E_i>$, the detector will in general not remain in its initial state due to interaction with the scalar field through L_{int}. If the detector undergoes a transition from $|E_i>$ to $|E_j>$ (while the scalar field goes from $|0_t>$ to $|n_t>$) and supposing c is small enough, we can approximate the amplitude of transition to first order in perturbation theory:

$$ic < E_j|\otimes < n_t| \int_{-\infty}^{+\infty} m(\tau)\phi[x(\tau)]d\tau|E_i > \otimes|0_t >$$

Let H_τ be the detector's Hamiltonian congugate to the proper time τ. Using

$$m(\tau) = e^{iH_\tau \tau} m(0) e^{-iH_\tau \tau}$$

and $H_\tau|E_i >= E_i|E_i >$, we can rewrite the transition amplitude as

$$ic < E_j|m(0)|E_i > \int_{-\infty}^{+\infty} d\tau e^{-i(E_i-E_j)\tau} < n_t|\phi[x(\tau)]|0_t > \tag{5.1}$$

We can then square the above expression and sum over all E_j and $|n_t>$ to obtain the total transition probability from $|0_t> \otimes|E_i>$ to all possible $|n_t> \otimes|E_j>$'s:

$$c^2 \sum_{E_j} | < E_j|m(0)|E_i > |^2 A_{i\to j}(\triangle E)$$

where $A_{i\to j}(\triangle E)$ is usually referd to as the "detector response function". It is given by:

$$A_{i\to j} = \int_{-\infty}^{+\infty} e^{-i(E_i-E_j)\triangle\tau} G^+(x(\tau), x(\tau'))\, d\tau\, d\tau' \tag{5.2}$$

, $G^+(x(\tau), x(\tau'))$ is the positive frequency Wightman function for the scalar field

$$< 0_t|\phi[x(\tau)]\phi[x(\tau')]|0_t >$$

, and $\triangle E = E_i - E_j$ is the change in the detector energy when particles are produced (absorbed) (the energy is conjugate to τ, and $\triangle\tau = \tau - \tau'$). Notice that all the details of the detector is in the prefactor $| < E_j|m(0)|E_i > |^2$, and the response function itself is independent of the internal structure of the detector $m(\tau)$.

If we are only measuring the amplitude of the detector making a transition from $|E_i>$ to $|E_j>$, we can forgo the sum over the detector energy and focus only on A_i. Effectively, it is to perform a Fourier resolution of the scalar propagator with respect to the energy conjugate to the proper time of the trajectory. From the scalar field point of view, the detector response function measures the probability of a particle with energy $\triangle E$ being produced at

τ and annihilated at τ', and the presence of the detector ensures energy conservation of the process through the term $exp(-i\triangle E(\tau-\tau'))$ Thus the detector response function relates the transition amplitude of the detector interacting with $\phi(x)$ to the propagation amplitude of $\phi(x)$.

5.2 Principle of detailed balance

We are interested in whether the detector following a certain trajectory would percieve the vacuum $|0_t>$ as a thermal state with a constant temperature. Thermal properties of a state can be examined through many ways. For example one can check if the two point function in the state of interest satisfies the KMS condition[42], or when the transition rate can be defined, it should be proportional to a Plank factor $\frac{1}{e^{\beta\triangle E}-1}$ if the state is thermal with temperature $1/\beta$.

Another way of defining temperature of a state is to couple the system of interest to an auxiliary one and analyse the occupation number for the states of the auxiliary system. In equilibrium we would expect the tansition propbability into and out of a state to be balanced:

$$Ni(E_i)P(E_i \to E_j) = N_j(E_j)P(E_j \to E_i)$$

, where $N(E)$ is the population number of a state of the auxiliary system with energy E, and $P(E_i \to E_j)$ is the amplitude of transition probability for the auxiliary system from state $|E_i>$ to $|E_j>$.

In our case the vacuum state $|0_t>$ of $\phi(x)$ is the one whose thermal properties we are interested in, and the detector serves as the auxiliary system which interacts with scalar field through the monople interaction term in the Lagrangian. In order to define temperature, we want the ratio of the population number of the detector energy eigenstates to be the Bose factor, that is:

$$\frac{N_i}{N_j} = \frac{P(E_j \to E_i)}{P(E_i \to E_j)} = e^{\beta(E_j - E_i)}$$

This says the detector's energy levels are thermally populated. And since the detector is in equilibrium with the scalar field vacuum, we can thus conclude that the vacumm state is a thermal state with temperatuer $1/\beta$ as seen by the monopole detector. This criterion for determining thermicity of a state is referred to as "principle of detailed balance". As we have seen in the previous section the transition amplitude of the detector is related to the propagation amplitude of the scalar field with appropriate energy, the ratio we will be computing would be the ratio of the detector response functions [29] [30]:

$$\frac{P(E_i \to E_j)}{P(E_j \to E_i)} = \frac{A_{i\to j}}{A_{j\to i}} \tag{5.3}$$

We will demonstrate the use of "principle of detailed balance" in this context in the next subsection.

5.3 Flat Space Example

We will choose to study a massless scalar field in flat $D = n+2$ Minkowski space to illustrate the features of the detector response function. We use the metric with (-, +, +, ...) signature:

$$ds^2 = -dt^2 + \sum_{i=1..n+1} dx^{i2} \tag{5.4}$$

We will consider a massless scalar field for simplicity. Solving the massless Klein-Gordon equation and performing canonical quantization with respect to the physical time , we get the usual plane wave expansion of the scalar field:

$$\phi(x) = \int_{-\infty}^{+\infty} \frac{dp^i}{(2E)^{1/2}} a_{p^i} e^{-iEt+ip^i x^i} + a^{\dagger}_{p^i} e^{iEt-ip^i x^i} \tag{5.5}$$

The normalization factor and the assignment of creation and annihialation operators are done with respect to the conserved Klein-Gordon inner product [31]:

$$(\, u^1_{p^i},\, u^2_{p^{i\prime}}) = -i \int_{\Sigma} u^1_{p^i} u *^2_{p^{i\prime}} - u^2_{p^i} u *^1_{p^{i\prime}}$$

,where Σ is the constant t Cauchy surface. In fact, since the product is conserved, we can choose to evaluate it on any spacelike surfaces.

The positive frequency Wightman function can be obtained by evaluating the p^0 integral of the scalar two point function along the prescribed contour (fig 5.1).

The resultant Wightman function is thus:

$$G^+(x(\tau), x(\tau')) = \frac{(-1)^n}{(4\pi^2[-2(\triangle u - i\epsilon)(\triangle v - i\epsilon) + \triangle x^{i2}])^{n/2}} \tag{5.6}$$

I have written the denominator in terms of light-cone coordinates:

$$u = \frac{t+x}{2^{1/2}}$$

$$v = \frac{t-x}{2^{1/2}} \tag{5.7}$$

The $i\epsilon$ in the denominator is needed to make the intergral spatial momentum leading to $G^+(x(\tau), x(\tau'))$ convergent. In other words, we should regard the Wightman function as the distribution obtained in the limit ϵ goes to zero. As a result, the poles of the Wightman function are shifted upward by $i\epsilon$.

In calculating the response function, we need to integrate over the proper time. In order for the integral to converge, the contour needs to be closed upward (downward) if $\triangle E < 0$ ($\triangle E > 0$). The form of the response function is thus determined by the pole structure of the two point function in the complex τ plane. It is easy to see for static observers ($x^i = const$) the Whightman function does not have poles in the complex plane and therefore the detector response function is zero. Physically, this means that static observers will not detect any particle production, and they would agree $|0_t>$ contains no particles.

We can look at this result from the point of view of energy conservation. Going back to the expresstion of the unsquared transition amplitude (5.1). We work in four dimesions in this example. For an observer following the inertial trajectory:

$$\begin{aligned} t &= t \\ x^i &= \delta_{1i} v t \end{aligned}$$

where $v < 1$ (the speed of light). Rewriting the above in terms of the proper time of the trajectory ($t = \tau(1-v^2)^{-1/2}$), and use

$$< n_t|\phi[x(\tau)]|0_t >= \int d^3k' \frac{1}{E^{1/2}} < 1_k|a^\dagger_{k'}|0_t > e^{iEt-ik'\cdot x} = \frac{1}{E^{1/2}} e^{iEt-ik\cdot x}$$

(5.1) becomes:

$$\frac{1}{E^{1/2}} \int_{-\infty}^{+\infty} d\tau e^{i(-\tau(E_i-E_j)+\tau(-k_1v+E)(1-v^2)^{-1/2})} \propto \frac{1}{E^{1/2}} \delta(-(E_i-E_j)+(-k_1v+E)(1-v^2)^{-1/2})$$

Suppose our detector starts out in its ground state, that is $E_j - E_i > 0$ and notice that $E > k_1 v$ for $v < 1$, the aregument of the delta function is always positive. The amplitude is always zero. We can see that for energetic reasons there is no particle detection.

Another very important class of trajectory is :

$$u(\tau) = (\alpha)e^{\frac{\tau}{\alpha}}$$

$$v(\tau) = (-\alpha)e^{\frac{-\tau}{\alpha}} \tag{5.9}$$

Observers following this class of trajectory have constatnt proper acceleration $1/\alpha$ and are referred to as Rindler observers. Substituting this into the detector response function we found the Wightman function is a function of $\triangle\tau$ only and has the form

$$G^+(x(\tau), x(\tau')) = \frac{(-1)^{n+1}}{(16\pi^2\alpha^2[\sinh^2(\triangle\tau/2\alpha - i\epsilon)])^{n/2}} \tag{5.10}$$

For this class of observers, we can define the rate of transition. The poles of the two point function are at

$$\tau = \tau' + 2im\pi + i\epsilon$$

where m is an integer. If we assume $\triangle E > 0$, and close the contour downward we get after the τ integration:

$$A_{i\to j} = \frac{(-i\triangle E)^{n-1}}{e^{2\pi\triangle E\alpha} - 1} \times \int_{-\infty}^{+\infty} d\tau'$$

The Planck factor

$$\frac{\triangle E^{n-1}}{e^{2\pi\triangle E\alpha} - 1}$$

shows that the detector energy levels are thermally populated while in interaction with the vaccum state $|0_t>$. In other words, the result of the interaction is as if the detector is static but inmersed in a thermal bath of scalar particles with temperature $\frac{1}{2\pi\alpha}$. The divergent integral is only a volumn factor. We can cure the divergence by adiabatically turning off the coupling, or in this case simply focus on the transition probability rate (per unit proper time) like we just did.

Notice the temperature is proportional to the proper acceleration of the trajectory. In the next subsection we will investigate the connection between this property to certain geometrical data, namely, the surface gravity, defined on the observer dependent horizon associated with this class of trajectory.

¿From the example above, we learned that when the two point function is a function of only $\triangle\tau$, we can define

$$G_1 = \int_{-\infty}^{+\infty} e^{-i\triangle E\triangle\tau} G^+(\triangle\tau)\, d\triangle\tau$$

as the rate of particle detection per unit proper time along the trajectory. This means the particle production as detected by our monopole detector is proper time translation invariant. Furthur, if G_1 has the form of Plankian distribution in $\triangle E$, we interperate the detector as in thermal equilibrium with the vaccum $|0_t>$.

However, it certainly not true that the Wightman function depends only on $\triangle\tau$ for all trajectories. When the Wightman function is not proper time translation invariant, we are not able to define G_1. We have to go back to using (5.1), which is in general divergent (with the τ' integration) unless we adiabatically turn off the detector. When this is the case, to determine the thermal properties, the principle of detailed balance has to be invoked.

The dependence on the proper time along the detector path only shows that the detector is not coupled homogeneousely with the vacuum state of the scalar field. The inhomogeneity may be due to the nature of the quantum fields in this background or plainly the trajectory. According to the principle of detailed balance, the detector could still be in thermal equilibrium if

$$\frac{A_{i\to j}}{A_{j\to i}} \propto e^{-\beta E} \tag{5.11}$$

where β is the inverse temperature. If we further require the proper acceleration is constant along the trajectory, we are able to associate certain properties of the detector's kenematics to the temperature. On the othe hand, If the proper acceleration along the paths is not constant, the vaccum $|0_t>$ as seen by the observer may not be a thermal state for all time but only for asymptotically early (late) proper time or the temperature registered in the spectrum of the detector is an averaged one. It is much harder to give geometrical interpretation to the temperature. This criterion implies the Plank distribution for the transition rate, whenever the Wightman function is a function of only $\triangle\tau$, and the integral over $(\tau+\tau')$ can be factored out. Note that while the numirator and the denominator are each a divergent expression, the ratio may still be well defined.

Let us see how principle of detailed balance works on the Rindler trajectory. When evaluated along (5.8), the poles of the integrand are at $\tau = \tau' + 2im\pi\alpha$. If $E_i - E_j > 0$ only the $m = -1, -2, ...$ poles contribute, the detector response function reads:

$$(-2\pi i) \times (\int_{-\infty}^{\infty} d\tau'(-i\triangle E)^{n-1}\frac{1}{e^{2\pi\triangle E\alpha} - 1}) \tag{5.12}$$

while if $E_i - E_j < 0$ the $m = 0, 1, 2, ...$ poles contribute (remembering the $m = 0$ is shifted to the upper half plane), and we get:

$$(2\pi i) \times (\int_{-\infty}^{\infty} d\tau'(i\triangle E)^{n-1}\frac{1}{1 - e^{-2\pi\triangle E\alpha}}) \tag{5.13}$$

The ratio of $A_{i\to j}$ over $A_{j\to i}$ is just $e^{-2\pi\alpha\triangle E}$, confirming the previous analysis. We have used

$$\frac{1}{1-x} = \sum_{n=0..\infty} x^n$$

in evaluating the residues. The positions of th poles and contours are dipicted in (fig 5.2).

Having considered the trajectories with constant proper acceleration, we turn our attention to a class of observers whoes proper acceleration is not constant but the ratio (5.10) still gives a Bose factor. For this particular family of path, it turns out the notion of temperature is an asymptotic one. The trajectory is:

$$\begin{aligned} u &= \arctan(\sinh(\tau/\alpha)) \\ v &= -\alpha^2/2\sinh(\tau/\alpha) \end{aligned} \tag{5.14}$$

They have proper acceleration:

$$\begin{aligned} a^u &= \alpha^2\frac{\tanh(\tau/\alpha)}{\cosh(\tau/\alpha)} \\ a^v &= \frac{-1}{2}\sinh(\tau/\alpha) \end{aligned}$$

and magnitude $\frac{1}{\alpha^2}\tanh^2(\tau)$ which tends to a constant value when $\tau \to \pm\infty$. The two point function is not proper time translation invariant, we need to resort to (5.10). The poles are at

$$\tau = \tau' + 2im\pi$$

just as the Rindler case. It turns out the expressions for the transition amplitudes are exactly the same as (5.11) and (5.12). It follows the ratio is just again $e^{-2\pi\alpha\triangle E}$, with temperature $\frac{1}{2\pi\alpha}$. Such notion of asymptotic temperature also happens in the case of black hole formation. In this case the observer is static (but accelerating) at infinity. Only at asymptotically late time will the observer percieve the presence of the black hole. The observer can then attribute the late time temperature to the horizon [34].

5.4 Geometrical Aspect of Rindler Temperature

Here, we would like to examine the connection between the thermal properties of a Rindler observer and the background geometry. One important question we would like to address is what are the criteria for thermal equilibrium. Before talking about it, we need to understand the some geometrical properties of the observer dependent horizons.
Some terminology, (We will use the definition according to [32])

Null surface: Here we define a null surface to a $D-1$ hypersurface with a null vector n^a orthognal to all of its tangent vectors and n^a is called the null generator of the null surface. For a Lorenzian signature spacetime n^a acctually lies in the null surface it generates.

Killing horizon: A Killing horizon is a null surface whose null generator coincides with a Killing vector.

Bifurcate Killing horizon: A bifurcae Killing horizon is a $D-2$ space like hypersurface where a Killing vector χ^a vanishes.

An important property of Killing horizons is the notion of surface gravity. By definition $\chi^a\chi_a = 0$, a constant, on a Killing horizon, it follows that $\nabla_c(\chi^a\chi_a)$ must be normal to the horizon. Since a Killing horizon is a null surface, χ^a is also normal to the horizon. One can thus define aproprtional constant κ as [33]:

$$\nabla^a(\chi^i\chi_i) = -2\kappa\chi^a \tag{5.15}$$

κ is called the surface gravity. Taking the Lie derivative along χ^a of the above equation and use the Killing equation:

$$\nabla_a\chi_b = -\nabla_b\chi_a \tag{5.16}$$

it follows κ has to be constant along the orbits of χ^a. It can be shown that [33]:

$$\kappa^2 = -1/2\nabla_a\chi_b\nabla^a\chi^b \tag{5.17}$$

,which can be related to the magnitude of the proper acceleration $a^2 = u^c\nabla_c u^a u^e\nabla_e u_a$ of the orbits of χ^a through:

$$\kappa^2 = \lim_{x\to horizon}(-\chi^a\chi_a)(a^c a_c) \tag{5.18}$$

where $u^a = (-\chi^c\chi_c)^{-1/2}\chi^a$ is the proper velocity of the orbit of χ^a. Note that as the horizon is approached $(-\chi^a\chi_a) \to 0$, $(a^c a_c) \to \infty$. The above expression for the surface gravity has the physical interpretation (at where χ^a is time like): it is the force needed at the position $-\chi^a\chi_a = 1$ to hold a test particle stationary just outside of the horizon. Here stationarity means following the orbits of the Killing vector. $-\chi^a\chi_a$ serves as a redshift factor. It is this last relation that relates the Rindler temperature to the surface gravity.

The special feature of a bifurcate Killing horizon is that κ is acctually constant over the horizon, and it can be proved without restrictions on the dimensionality or use of Eistein equation. (In contrast, the constancy of κ on a Killing horizon is only proved for four dimensions and requires the use of Einsein equation and the dominant energy condition [32].) To relate bifurcate Killing horizons to the Rindler observer, recall first a Killing vector field is the infinitesimal generator of an isometry transformation. An isometry on a connected manifold (where point can be connected by piecewise geodesic curves) is uniquely defined by its action on a point of the spacetime and its pull back action on the tangent space T_p. Suppose the Killing vector vanishes on a $(D-2)$ space like, static hypersurface S, that is, S is a fixed point for the isometry. The Killing vector is determined completely near S if we specify $F_{ab} = \nabla_a \chi_b$. Note by the Killing equation, F_{ab} is anti-symmetric.

The action of χ^a on T_p is given by the Lie derivative of v^a with respect to χ^a, where $v^a \in T_p$. Explicitly, it is related to F_{ab} through:

$$L_{\chi^a} v^a = F^a{}_b|_S v^b$$

where we have used the Killing equation and the fact that $\chi^a = 0$ on S. Notice that the infinitesimal action of the isometry on the tangent space is given by $F^a_{/,b}$. In a small neighborhood around S, we can take the metric to be

$$ds^2 = \pm dt^2 + dx^2 + h_{ij} dy^i dy^j$$

depending on the signature of the spacetime. h_{ij} is the metric of the spatial hypersurface S. Remembering χ^a vanishes identically on S, χ^a must commute with the generators of S. We can thus take the components $F^a_i = F^i_j = 0$. From the antisymmetry of F_{ab}, we conclude, at least in a neighborhood of S, F^a_b is the generator of a Lorentz boost to spacetimes with Lorentzian signature and a rotation to Euclidean signatured ones. This allows us to generalize the concept of Rindler temperature once we understand how it works in flat space. We would expect observers following the orbits of a Killing vector with a bifurcate Killing horizon to observe Rindler temperature.

We can now start to discuss the geometrical interpretation of Rindler temperature. Performing a coordinate transformation on the flat space metric: $(t, x \to \tau, N$, with $-\infty < \tau < +\infty$, and N nonnegative)

$$\begin{aligned} t &= N \sinh(\tau/N) \\ x &= N \cosh(\tau/N) \end{aligned}$$

The metric becomes:

$$ds^2 = -N^2 d\tau^2 + dN^2 + \sum dx^{i2} \tag{5.19}$$

this coordinate transformation is adapted to the integral curves of the Killing vector

$$\partial_\tau = x\partial_t + t\partial_x \tag{5.20}$$

With the coordinate limits on τ and N, we see that the new metric covers only the right wedge of the original Minkowski space. (fig.5.3)

We can perform yet another transformation to bring the Rindler spacetime to:

$$ds^2 = e^R(-dT^2 + dR^2) + dx^{i2} \tag{5.21}$$

with

$$\begin{aligned} T &= t \\ R &= ln(r) \end{aligned}$$

When the metric written in the Rindler coordinates τ, N, it is easy to see in the Euclidean sector $\tau \to i\tau$, the metic has a conical singularity unless τ is periodically identified with the correct period β (here $\beta = 2\pi$). After solving the field theory in the Euclidean sector we need to impose appropriate boundary conditions in the time direction (periodic for bosons and anti periodic for fermions). The two point function after Wick rotated back to the Lorentzian sector will then be periodic in imaginary time as in (5.9). When quantized with respect to τ, the two point function evaulated in the Minkowski vaccum $|0_t>$ has the properties of a thermal Greens function with temperature $1/\beta$.

Associated with an observer following any of the integral curves C_N (that is, with fixed N, and τ being the proper time) is the observer dependent horizon:

$$h_A : x + t = 0$$

which is the boundary of $\bigcup_{ponC_N} I_p^P$, where I_p^P is the causal past of the point p. There is also the past horizon :

$$h_B : x - t = 0$$

defined as the boundary of $\bigcup_{ponC_N} I_p^F$ (I^F denotes the causal future). The Killing vector ∂_τ is time like in the R, L region of the space time, and truns null on the horizons, and space like in U, D. The horizons $x + t = 0$, $x - t = 0$, are thus Killing horizons associated with ∂_τ. Notice that ∂_τ vanishes on the space like hypersurface $h_A \bigcap h_B$: $t = 0, x = 0$. S is a bifurcate Killing horizon.

An observer following C_N has proper acceleration $a^c a_c = 1/N^2$, and the norm of the Killing vector ∂_τ on C_N is $-x^2 + t^2 = -N^2$. According to the results from the previous subsection, we know the Rindler temperature observed along C_N is just $T_N = \frac{1}{N2\pi}$. From (5.17) we see the surface gravity is the normalized proper acceleration on the horizon. We can then associate the Rindler temperature with the surface gravity by

$$T = \frac{\kappa}{2\pi}$$

To check for consistancy, the surface gravity calculated with (5.19) gives $\kappa = 1$

Putting these together we get the general relation between the observed temperature and surface gravity:

$$T_N = \frac{\kappa}{2\pi(-\chi^a\chi_a)^{\frac{1}{2}}}$$

Again, the norm of the Killing vector $(\chi^a\chi_a)^{1/2}$ serves as the red shift factor. And we can associate the horizon with the temperature as measured at where the norm is 1. The constancy of temperature can also be relared to the constancy of the surface gravity over $h_A \bigcup h_B$.

5.5 Conformal Killing Horizon and Its Surface Gravity

In this section we will briefly discuss the idea of conformal Killing horizons and conformal invariance of Rindler temperature[35]. A conformal Killing vector ζ^a satisfies the equation [33]:

$$L_\zeta G_{ab} = 2\nabla_{(a}\zeta_{b)} = 2f(x)G_{ab} \tag{5.22}$$

$f(x)$ is a position dependent conformal factor. Note that if ζ^a is a conformal Killing vector of a spacetime (M,G_{ab}), then there exist a space time (M,$h(x)G_{ab}$) such that ζ^a is a true Killing vector.

Just like an ordinary Killing vector, if a null surface has its null generator being a conformal Killing vector, we will call it a conformal Killing horizon. We can analoguesly define the surface gravity of a conformal Killing horizon with $\kappa_1 =$ (5.14) or $\kappa_2 =$ (5.16). κ_1 is well defined since it really is just a property of null surfaces. κ_2 is also mathematically well defined as it uses only the kinematic quantities of a vector field. However the two are no longer equal to each other. It can be shown that they satisfy the relation [35]

$$\kappa_1 = \kappa_2 - f \tag{5.23}$$

Under a conformal transformation, a conformal Killing vector is still a conformal Killing vector, with possiblely changes in its conformal factor. Conformal Killing horizons also survives since they are null surfaces. It truns out κ_1 is also invariant under a conformal transformation: suppose 1. a conformal transformation takes G_{ab} to $G'_{ab} = h(x)G_{ab}$, 2. ζ is a conformal Killing vector of G_{ab} then

$$\begin{aligned}
\nabla_a(G'_{ij}\zeta^i\zeta^j)|_{horizon} &= \nabla_a(h(x)G_{ij}\zeta^i\zeta^j)|_{horizon} \\
&= -2h(x)\kappa_1 G_{aj}\zeta_j + \nabla_a(h(x))G_{ij}\zeta^i\zeta^j|_{horizon} \\
&= -2\kappa_1 G'_{aj}\zeta^j|_{horizon}
\end{aligned}$$

Note by the nature of the conformal transformation, ζ^j remains a conformal Killing vector for G'_{ab}. The above surface gravity is computed with respect to the "original" conformal Killing vector ζ of G'_{ab}. κ_2 will only be conformal invariant if the transformation has h(x) constant along the orbit of the conformal Killing vector ζ^a. More generally, κ_1 transforms for conformally related spacetimes in the following way. Assuming 1. $G'_{ab} \to h(x)G_{ab}$ 2. ζ is a conformal Killing vector of G_{ab}:

$$\begin{aligned}
\nabla'_a(G'_{ij}\zeta'^i\zeta'^j)|_{horizon} &= \frac{\partial x^c}{\partial x'^a}\nabla_c(h(x)G_{ij}\zeta^i\zeta^j)|_{horizon} \\
&= -2\frac{\partial x^c}{\partial x'^a}\kappa_1 h(x)\zeta_c + \frac{\partial x^c}{\partial x'^a}\nabla_c(h(x))G_{ij}\zeta^i\zeta^j|_{horizon} \\
&= -2\kappa_1 h(x)\zeta'_a|_{horizon}
\end{aligned}$$

The above is only valid when the confromal factor h(x) is non-zero. Otherwise $G_{ij}\zeta^i\zeta^j$ may not vanish on the horizon. Here the surface gravity is computed with the transformed conformal Killing vector ζ'. Notice while it is true $\kappa_1 h(x)$ is constant along the integral curves of ζ' on the horizon, it may vary from orbit to orbit.

In the next section, we will apply these considerations to the BMN pp-wave spacetime.

5.6 The Detector Response Function in the BMN Spacetime

Recall that the BMN metric is:

$$ds^2 = -2dudv - \mu^2(\sum_{i=1}^{8} x_i^2)du^2 + \sum dx_i^2. \tag{5.24}$$

The Ricci scalar of the spacetime is zero. Therefore, a massless scalar field in trhis background is also a conformally coupled field. We will consider this case in the following. ¿From the above, the massless scalar wave equation, $\partial_\mu((-g)^{1/2}g^{\mu\nu}\partial_\nu\phi(x)) = 0$, reads: $(g = detG_{\mu\nu} = -1)$

$$(-2\partial_u\partial_v - (\sum x_i^2)\partial_v^2 + \partial_i^2)\phi(x) = 0, \tag{5.25}$$

We can take the eigenfunctions in the light cone directions to be just the plane waves and substitute

$$f(u,v,x^i) = e^{ip_u u}e^{ip_v v}f'(x^i)$$

into the above wave equation. The remaining equation for the transverse directions have the structure of that of a harmonic oscillator:

$$(\partial_i^2 - \mu^2 r^2 p_v^2 + 2p_u p_v)f'(x^i) = 0$$

Let

$$x'^i = (\mu|p_v|)^{1/2}x^i$$

we then get

$$(-\partial_i'^2 + r'^2 - \frac{p_u}{\mu}sign(p_v))f'(x'^i) = 0$$

The general solutions are:

$$u_{p_v,n_i}(u,v,x^i) = N\prod_{i=1}^{8}(\frac{(\mu p_v/\pi)^{1/2}}{2^{n_i}n_i!})^{1/2}H_{n_i}((\mu p_v)^{1/2}x^i)e^{\frac{-\mu p_v x^{i2}}{2}}e^{ip_v v}e^{ip_u u}, \tag{5.26}$$

with

$$p_u = \frac{\mu sign(p_v)\sum(2n_i+1)}{2}, \tag{5.27}$$

and H_{n_i} are the Hermite polynomials. N is a normalization constant. Note the dispersion relation of p_u follows from the normalisibility of the transeverse eignfunctions, and effectively imposes a periodic boundary condition on the wave functions in u with period $2\pi/\mu$. This identification is closely related to the local conformal structure of the spacetime. The same situation is also seen in the solutions of wave equations on AdS spacetime.

We now move on to quatizing the theory. It is well known that the plane wave spacetimes do not possess Cauchy surfaces [36]. The best one can do is to choose a partial Cauchy surface such that the information of particle production in such spacetimes can be retained. That is, choosing surfaces such that the above chosen set of eigenfunctions is sufficient to analyze any resonable initial data specified on them. Such partial Cauchy surfaces do exist and are the $u = const$ slices [37], [20]. The only causal curves fail to intersect these surfaces are the ones going tangent to them. And they form a set of measure zero.

With this choice, we can construct the Hamiltonian with respect to u. The canonical quantization is carried out as the following:

$$\pi = \partial_v \phi,$$
$$[\phi, \pi]_{equal\, u} = i\delta(v - v')\delta(x^i - x'^i). \tag{5.28}$$

and let

$$\phi = \int_0^{+\infty} dp_v \sum_{n_i} a_{p_v,n_i} u_{p_v,n_i} + b_{-p_v,n_i} u_{-p_v,n_i}, \tag{5.29}$$

Taking $u = const$ slices as our partial Cauchy surfaces, we can define the conserved scalar product:

$$(u_{p_v,n_i}, u_{p'_v,n'_i})_{KG} = \int_\Sigma u_{p_v,n_i} \partial_v u *_{p'_v,n'_i} - u *_{p_v,n_i} \partial_v u_{p'_v,n'_i}$$

where Σ is the $u = const$ surfaces. We found, with respect to the Klein-Gordon product above, the normalization constant N can be fixed to be $1/2(\pi p_v)$. As usually is the case for quantum field theory in curved spacetime, the choice for positive energy modes is not fixed a priori. More explicitly, it is up to some convention that we can define the scalar product above, and this will affect the infinitesimal shifts in the positions of the poles in the two point function. Here we will pick up the overall sign such that when the $\mu \to 0$ limit is taken, the flat space result is recovered.

Performing canonical quantization, we can relate the commutators to the Klein-Gordon product as:

$$\begin{aligned} [a_{p_v,n_i}, b_{-p'_v,n'_i}] &= i(u_{p_v,n_i}, u_{-p'_v,n'_i})_{KG} \\ &= -sign(p_v)\delta(p_v + p'_v)\delta_{n_i,n'_i} \end{aligned}$$

and

$$[a_{p_v,n_i}, a_{p'_v,n'_i}] = [b_{-p_v,n_i}, b_{-p'_v,n'_i}] = 0$$

These relations establishes the negative p_v modes annihilates particles and the positive p_v modes creates. Therefore, the vaccum $|0_u>$is defined to be annihilated by all a_{-p_v,n_i}, with p_v positive. We can write $\phi(x)$ with the mode expansion as:

$$\phi(x) = \int_0^{+\infty} dp_v \sum_{n_i} a_{p_v,n_i} u_{p_v,n_i} + a^\dagger_{p_v,n_i} u*_{p_v,n_i} \tag{5.30}$$

We can calculate the positive frequency Wightman function through the Penrose limit of the two point function of $AdS_5 \times S^5$ [38], or using the fact that the BMN pp-wave is conformally related to the flat space. Here we will compute the two point function by direct mode summation using the eigenfunctions we found above.

$$< 0_u|\phi[x(\tau)]\phi[x(\tau')]|0_u >=$$

$$\int_0^{+\infty} dp_v \sum_{n_i} \prod_{i=1}^{8} \frac{(\mu/\pi)^{1/2}}{2^{n_i} n_i!}) p_v^3 H_{n_i}((\mu p_v)^{1/2} x^i) H_{n_i}((\mu p_v)^{1/2} x'^i) e^{\frac{-\mu p_v x^{i2}}{2} + \frac{-\mu p_v x'^{i2}}{2}} e^{-ip_v \triangle v} e^{-ip_u \triangle u}$$

where $\triangle u$ denotes $u - u'$, and $\triangle v$ denotes $v - v'$. The sums over the Hermite polynomials can be evaluated using the identity[39]:

$$\sum_{n=0}^{\infty} H_n(ax_i) H_n(ax_i')(z/2)^n/n! = \frac{1}{(1-z^2)^{1/2}} e^{a^2(\frac{2x_i x_i' - x_i^2 z^2 - x_i'^2 z^2}{1-z^2})}$$

if we take $e^{-i\triangle u} = z$. The above identity is valid for $z \neq 1$, and in order to apply it in our situation we will need to regularize the sum by adding a factor $e^{-\epsilon}$ and define

$$z = e^{-i(\triangle u - i\epsilon)}$$

After performing the sum over all 8 transverse directions while remembering $p_u = \frac{\mu \sum(2n_i+1)}{2}$ we get:

$$(\frac{z}{1-z^2})^4 \int_0^{+\infty} dp_v p_v^3 e^{-ip_v I}$$

where

$$I = \triangle p_v - \frac{\cos(\triangle u - i\epsilon)(x^2 + x'^2) - 2x \cdot x'}{4\sin(\triangle u - i\epsilon)}$$

Again, we will need to regularize the p_v integration by way of introducing a factor $e^{-\epsilon p_v}$. The positive frequency Wightman function is thus

$$\frac{\mu^4}{-(2\triangle v \sin(\mu\triangle u) - \mu(2x_i x_i' - \cos(\mu\triangle u)(x_i^2 + x_i'^2)))^4} \tag{5.31}$$

The $i\epsilon$ prescription is to replace

$$\begin{aligned} \triangle u &\to \triangle u - i\epsilon \\ \triangle v &\to \triangle v - i\epsilon \end{aligned} \tag{5.32}$$

from these formula. The prescription reqired to make the p_v integral convergent shifts the zeros of the denominator up by $i\epsilon$. This expression is previously obtained by [40]. Note that this propogator should not be used as the boundary to bulk propogator for the BMN plane wave, since we do not include modes that travel tangent to u=const planes in our complete set of eigenfunctions.

A few words can now be said about the scalar field arise from the string theory spectrum, the field equation is

$$((2\partial_u\partial_v - (\sum x_i^2)\partial_v^2 + \partial_i^2) - ic\mu\partial_v)\phi(x) = 0$$

The extra term will only modify the dispersion relation by changing the zero point energy

$$p_u = \frac{\mu sign(p_v)\sum(2n_i) + 1}{2} - \frac{c}{2}$$

Summing over all the transverse oscillators and integrate over p_v, and regularize the divergences as before, we found the two point function is

$$\frac{\mu^4}{-e^{ic\mu\triangle u/2}(2\triangle v\sin(\mu\triangle u)+\mu(2x_ix_i'-\cos(\mu\triangle u)(x_i^2+x_i'^2)))^4}$$

whuch only differ from the massless case by $e^{-ic\mu\triangle u/2}$. Since the multiplicative factor $e^{ic\mu\triangle u/2}$ does not introduce any new poles, we do not expect it to change the behavior of the detector response function. We will focus on the massless scalar field from here on.

Again, we can define the rate of particle detection per unit proper time only for the trajectory alnog which the two point function is a function of only $\triangle\tau$. In the following we will try to find a trajectory such that the Wightman depends only on $\triangle\tau$. For now, we will assume $\triangle E$ is positive (considering only particle production), and close the contour in the proper time integration downward.

Firstly, notice if the seperation of the two points is small the denominator is just the proper seperation between the two nearby points. This shows that if we are evaluating the Wightman function along a particular trajectory $u(\tau), v(\tau), x_i(\tau)$, the two point function will be proper time translation invariant (depends only on $\triangle\tau=\tau-\tau'$) if and only if the trajectory is an integral curve of some Killing vector of the spacetime. More explicitly, the denominator will be of the form (for two close by points on the trajectory)

$$G_{\mu\nu}\xi^\mu\xi^\nu(\tau-\tau')^2.$$

And we need $L_\xi G_{\mu\nu}\xi^\mu\xi^\nu = L_\xi G_{\mu\nu}=0$. L_ξ is the Lie derivative with respect to the tagent of the detector path (ξ^μ).

Our task now is to compute the form of our detector response function along the orbits of the Killing vectors. In the BMN spacetime, we also have the R-R 5 form $F_{+1234}=F_{+5678}=\mu$, which, breaks the apparent $SO(8)$ symmetry in the transverse directions. We will only have SO(4)×SO(4) isometry left. Again, the set of Killing vectors are:

$$Z_u = \partial_u,$$

$$Z_v = \partial_v,$$

$$Z_i = -\cos(\mu u)\partial_i + \mu\sin(\mu u)x^i\partial_v \; i=1...8,$$

$$Z_i' = -\mu\sin(\mu u)\partial_i - \mu^2\cos(\mu u)x^i\partial_v \; i=1...8,$$

$$Z_{M_{ij}} = x^i\partial_j - x^j\partial_i \; i,j\in 1...4 \text{ or } i,j\in 5...8$$

For our purposes, it is acctually safe to use the full SO(8) symmetry of the metric and not to restrict the i, j's in $Z_{M_{ij}}$

First of all, we notice that The BMN space time does not possess a bifurcate Killing horizon. The only candidate from the set of Killing vectors above are Z_i's, and Z_i''s. They would vanish on the $D-2$ dimensional hypersurface $x_i = 0$, $u = const$, where the constant depends on which Killing vector we are discussing. However, due to the null Killing vector ∂_v, these are null surfaces. If we push on to calculate F^a_b for them we find: (here we use the surface S:$u = 0$, $x^1 = 0$, for the Killing vector Z'^1, and write out only the u, v, x^1 part of the matrix)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu \\ -\mu & 0 & 0 \end{pmatrix}$$

To make the meaning of the generator more explicit, we perform the following coordinate transformation:

$$u = t + y$$

$$2v = t - y$$

The metric is transformed into:

$$ds^2 = -(1+\mu^2 r^2)dt^2 - 2\mu^2 r^2 dtdy + (1-\mu^2 r^2)dy^2 +^{i\ 2}$$

and $F^a_b|_S$ becomes:

$$\begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & -\mu \\ \mu & \mu & 0 \end{pmatrix}$$

which correspondes to rotation in the $y - x^1$ plane while boosting in the $t - y$ plane. The detector response function of a detector travelling in this fashion in flat space has been discussed in the literature. It is found that the ratio of (5.3) gives can be interpretated as having a temperature dependent on $\triangle E$ [41].

Coming back to the detector response function calculation. We note that the orbits of $cZ_u + c'Z_v + A_i Z_i(B_i Z'_i)$ are the geodesics of the spacetime:

$$\begin{aligned} u(\tau) &= c\tau + u_0 \\ v(\tau) &= c'\tau + (\mu/4)\sum_i A_i^2 \sin(2\mu c\tau + \phi_i) + v_0 \\ x_i(\tau) &= A_i \sin(\mu c\tau + \phi_i) \end{aligned}$$

where c, c', and A_i are integration constants.

The two point function evaluated along these trajectories has the simple form:

$$\frac{\mu^4}{(2c'\triangle\tau\sin(\mu c\triangle\tau))^4} \tag{5.33}$$

All the oscillations in the transverse directions cancels with the contribution from the v-direction. It is clear that the denominator has no complex zeros, and thus the $\triangle\tau$ integral vanishes. The detectors following the geodesics do not detect particle production.

The special case of $c'Z_v + A_i Z_i(B_i Z_i')$ has the orbit

$$\begin{aligned} u(\tau) &= u_0 \\ v(\tau) &= c'\tau - \sum A_i^2 \mu \sin(\mu u_0 + \phi_i) \cos(_0 + \phi_i)\tau^2/2 \\ x_i(\tau) &= -\cos(\mu u_0)\tau \end{aligned}$$

and the two point function is proportional to

$$\frac{1}{(\tau - \tau')^4}$$

It is obvious that no particle production is detected, either.

Next we check the class of orbits associated with $aZ_u + bZ_{M_{ij}}$. The integral curves goes around the center and is again a geodesic. And we find the two point function to be:

$$G^+(\triangle\tau) \propto \frac{1}{(\cos(b\triangle\tau) - \cos(a\mu\tau))^4}$$

We see again that there is no poles in the lower half complex $\triangle\tau$ plane.

Let us pause to complete the check of all geodesics (since we have checked most of them). If we only have motion in u, and the transverse directions r, θ. A timelike geodesic would look like:

$$u(\tau) = A\,\mathrm{arctanh}(\frac{\tau}{c^{1/2}})$$

$$r(\tau) = (c - \tau^2)^{1/2}$$

$$\theta(\tau) = B\,\mathrm{arctanh}(\frac{\tau}{c^{1/2}})$$

due to the form of $r(\tau)$, the two point function is not periodic in imaginary proper time. This class of geodiscs still do not bare thermal signature.

This is a result known for some time and is general for all plane wave spacetimes [36]. The reason for no cosmic particle production is due to the fact that we have a null Killing vector in the v direction. One can define a global frequency for the quantum fields associated with ∂_v and there will be no mixings of positive and negative frequency modes in the Bogoliuobov transformantion. And thus no particle productions along geodesics.

It turns out we only need to check one other class of trajectory, namely the ones that follows the Killing vector

$$Z = aZ_u + b_i Z_i + c_{jk} Z_{M_{jk}} + dZ_v$$

with typical orbits like (Here on we will focus on the case i=j=1, k=2 for clarity):

$$u(\tau) = a\tau + u_0$$

$$v(\tau) = d\tau + \frac{b\mu}{2(\mu a - c)} \sin(\mu(a-c)\tau) - \frac{b\mu}{2(\mu a + c)} \sin(\mu(a+c)\tau) + \frac{a(b\mu)^2}{c^2 - (a\mu)^2}(\tau/2 - \frac{\sin(2a\mu\tau)}{4a\mu})$$

$$x_1(\tau) = \sin(c\tau) + \frac{ba\mu\sin(a\mu\tau)}{c^2 - (a\mu)^2}$$

$$x_2(\tau) = -\cos(c\tau) - \frac{bc\cos(a\mu\tau)}{c^2 - (a\mu)^2}$$

Note that if we use Z_i' instead of Z_i, the trajectories only changes by a phase, and it is sufficient to check the case for only Z_i. we find the following expressions for the two point function:

$$\frac{\mu^4}{((\frac{ab^2}{c^2-(a\mu)^2} + d)\sin(\mu a\triangle\tau) + 2(\cos(c\triangle\tau) - \cos(a\mu\triangle\tau)))^4} \tag{5.34}$$

if $i = j$ or k, otherwise the terms completely decouple and we get back

$$\frac{\mu^4}{(2c'\triangle\tau\sin(\mu c\triangle\tau))^4} \tag{5.35}$$

For generic values if a, b, c, (5.33) does have poles in the lower half of the complex plane. However, in order for the response function to have a Plankian distribution in E, we need the two point function to be at least periodic in imaginary τ. The response function does not corresponds to a thermal distribution for this class of trajectory, either.

As seen from the above, we have checked all the cases such that the two point function depends only on $\triangle\tau$. We showed that none of them gives the desired Plankian behavior in E. In order for us to see if there is an analog of Rindler observers in BMN spacetime, we need to resort to a more general definition of thermal equilibrium. We will mostly confine our attention to detector trajectories that are restricted to the center of the BMN spacetime.

5.7 Trajectories with Thermal Signature

When the Wightman function is not proper time translation invariant, we are not able to define the rate of particle production per unit proper time. However, it is still possible the ratio of the transition amplitudes (5.10) yields the Bose factor $e^{-\beta\triangle E}$. As noted above while the response function itself may be divergent, the ratio may still be well defined. Since we have already ruled out trajectories for which the two point functions are proper time translation invariant we do not expect to see the two point function when evaluated on the trajectories considered in the following to represent a thermal Green's function with constant temperature for all τ. The temperature in the Bose factor is in some sense an averaged temperature, or since we are integrating over an infinite proper time interval, the result may very well represent the fact the integral is dominated by the late time behavior of the trajectory. We will now attempt to find trajectories that are "thermal" in view of the principle of detailed balance.

We will do all the calculations in four dimensional BMN for simplicity. The two point function is just the quartic root of the 10 dimensional case (with the sum over i goes from 1 to 2 now).

$$\frac{\mu}{-(2\triangle v\sin(\mu\triangle u) - \mu(2x_ix_i' - \cos(\mu\triangle u)(x_i^2 + x_i'^2)))}$$

where I have supressed the iϵ prescription in the denominator. It is the same as (5.31), and remember it shifts all the poles upward by iϵ.

We first notice the center ($x_i = 0$) is metrically flat, and all the Christoffel symbols are vanishing. We can compute what is the detector response function associated with an observer with constant proper acceleration confined to the center. It turns out the only class of observer with constant proper acceleration follows the path of the Rindler observers in flat space.

$$u(\tau) = (\alpha)e^{\frac{\tau}{\alpha}}$$
$$v(\tau) = (-\alpha)e^{\frac{-\tau}{\alpha}} \tag{5.36}$$

where $1/\alpha$ is the proper acceleration. In flat space, these observers will find themselves immersed in a hot bath of particles with temperature $\frac{1}{2\pi\alpha}$. In the present case, this trajectory is not the integral curve of a Killing vector anymore. Therefore, the Wightman function is not propertime translation invariant. The two point function does resemble that of a Rindler observer's as it is periodic in imaginary τ (or τ').

With the above trajectory, the two point function becomes

$$\frac{\mu}{(e^{-\tau'/\alpha} - e^{-\tau/\alpha})\sin(\mu(e^{\tau'/\alpha} - e^{\tau/\alpha}))}$$

, and it has poles at (viewing τ' as constant, we will carry out the τ integration first.):

$$\tau = \alpha ln(\frac{n\pi}{\mu} + e^{\tau'/\alpha}) + 2im\alpha\pi$$

where $n, m \in Z$. (fig 5.4) The flat space Rindler observer sees only the contribution from the $n = 0$ poles, as we can see if $\mu \to 0$, the additional poles goes off to infinity. This suggests when looking at the $\mu \sum(x_i^2)du^2$ term as a perturbation to the flat space metric, we have drastically changed the symmetry of the spacetime. In fact, this result is closely related to the conformal structure of the spacetime as we will elaborate below. If we are to find the analogue of a Rindler observer we need to adapt to the new symmetry properties of the BMN spacetime.

We apply principle of detailed balance to the present case. After the τ integration we have:
$\triangle E > 0$:

$$(-2\pi i)\frac{-i\triangle E}{e^{2\pi\triangle E\alpha} - 1} \times \int_{-\infty}^{+\infty} d\tau' +$$

$$\frac{1}{e^{2\pi\triangle E\alpha} - 1} \times -2i\pi \sum_n (-1)^n \int_{-\infty}^{+\infty} \frac{\mu}{n\alpha\pi} exp(\tau'/\alpha - i\triangle E(\alpha ln(n\pi/\mu + e^{\tau'/\alpha}) - \tau'))d\tau'$$

Where the first term is the contribution from the $n = 0$ poles, and it gives exactly the Plankian distribution with temperature $\frac{1}{2\alpha\pi}$ as seen by the Rindler observer in flat space. the divergent integral over τ' is just a volume factor. The contribution from the $n \neq 0$ poles

diverges exponentially with τ', which suggests the particle production rate is increasing as the detector follows the orbit. As a consistency check we see that the contribution from these extra poles are proportional to μ, and that we can again recover the flat space result by taking μ to zero.
As for $\triangle E < 0$, we get:

$$(2\pi i)\frac{i\triangle E}{1-e^{-2\pi\triangle E\alpha}} \times \int_{-\infty}^{+\infty} d\tau' +$$

$$\frac{1}{1-e^{-2\pi\triangle E\alpha}} \times 2i\pi \sum_n (-1)^n \int_{-\infty}^{+\infty} \frac{\mu}{n\alpha\pi} exp(\tau'/\alpha + i\triangle E(\alpha ln(n\pi/\mu + e^{\tau'/\alpha}) - \tau'))d\tau'$$

Clearly, the ratio of the two expressions does not have the form $e^{-\beta\triangle E}$. The detectors following constant acceleration trajectories at the center do not see the vacuum as a thermal state.

Let us look closer at the behavior of the transition amplitude. The exponential growth of the $n \neq 0$ poles can be understood through the following consideration. The BMN spacetime has $R = 0$, and our massless scalar field is automatically conformally coupled. It has been realized that the BMN spacetime is locally conformal to the flat space. This can be made explicit through the following coordinate transformations. First,

$$\begin{aligned} u' &= u \\ v' &= v + \frac{\mu}{4}\sin(2\mu u + 2\phi)\sum_{i=1}^{8} x'^{i2} \\ x'^i &= x^i/\cos(u+\phi) \end{aligned} \tag{5.37}$$

followed by,

$$\begin{aligned} u'' &= \tan(\mu u' + \phi) \\ v'' &= v'/\mu \\ x''^i &= x'^i \end{aligned} \tag{5.38}$$

The first one is the standard transformation from the Brinkman form to the Rosen form of the plane wave metric, after which the metric becomes:

$$ds^2 = -2du'dv' + \cos^2(\mu u' + \phi)\sum_{i=1}^{8} dx'^{i2}$$

ϕ here is a phase constant. The transformation is only valid for $u \in (\phi - \pi/2\mu, \phi + \pi/2\mu)$ and we need to use several patches to cover the whole spacetime. The second transformation takes the metric to :

$$ds^2 = \cos^2(\mu u' + \phi)(-2du''dv'' + \sum_{i=1}^{8} dx''^{i2}) = \frac{1}{(1+u''^2)}(-2du''dv'' + \sum_{i=1}^{8} dx''^{i2}) \tag{5.39}$$

After the coordinate transformation, the orbit (5.35) becomes:

$$u'' = \tan(\mu\alpha e^{\tau/\alpha}), v'' = (-\alpha)/\mu e^{\frac{-\tau}{\alpha}} \rightarrow u'' = \tan(-\frac{\alpha^2}{v''})$$

The relation between u'' and v'' shows that for u (the original coordinate) to increase from 0 ($\tau = -\infty$) to $u = +\infty$, the detector traverses through many cycles (from $u'' = -\infty$ to $+\infty$). (fig 5.4.) Each cycle contributes to a column ($n = i$ (i fixed integer), $m =$ arbitrary integer) of poles in the complex τ plane. This is seen by putting each cycle individually into the flat space two point function, and noting that each path has the form (5.13), which is thermal in th esense discussed above. The proper time it takes for the detector to travel through one cycle gets shorter and shorter as u grows large. As $\mu \rightarrow 0$, the second coordinate transformation become singular. However, it is clear that we only have one big cycle (the Rindler trajectory) left and the flat space picture is again recovered.

The Green's function respects the isometries of the spacetime, that is, it is invariant under a Killing transformation. Therefore the above analysis holds true also for trajectories that are related to (5.35) by some isometery. These related pathes will also have constant proper acceleration. It is interesting to note that we can use combinations of Z_i, Z_i' and $Z_{M_{ij}}$ to obtain off center trajectories from (5.35). We can repeat the same steps for the $x^i = const$ planes, which are also metrically flat. We found that due to the hyperbolic nature of the orbit we are considering, we still have the columns of poles for our two point functionand they are shifted compared to the center case, and the qualitative features are not altered.

With the above knowledge, we see that if we were to find a trajectory for which (5.10) is satisfied, we need to conform to the conformal structure of the BMN spacetime. We found there are families of trajectory for which the ratio (5.10) is thermal. And we have seen them before. Consider, (again no motion in the transverse directions is assumed):

$$u(\tau) = \frac{1}{\mu}\arctan(\sinh(c\tau))$$

$$v(\tau) = \frac{\mu}{2c^2}\sinh(c\tau) \tag{5.40}$$

Note the range of $u(\tau)$ is restricted to $[-\frac{\pi}{2\mu}, \frac{\pi}{2\mu}]$, fig(5.5). This is in conformity to the local conformal flatness of the spacetime The proper acceleration of the trajectory is:

$$\begin{aligned} a^u &= -\frac{c^2}{\mu}\frac{\tanh(c\tau)}{\cosh(c\tau)} \\ a^v &= \frac{\mu}{2}\sinh(c\tau) \end{aligned}$$

when $\tau \rightarrow \infty$ we found the magnitude of the proper acceleration approaches a constant c^2.

The Wightman function with (5.39) has the form:

$$\frac{1}{\frac{\mu}{2c^2}(\sinh(c\tau) - \sinh(c\tau'))\sin(\arctan(\sinh(c\tau)) - \arctan(\sinh(c\tau')))}$$

, which is not proper time translation invariant, but again periodic in imaginary proper time. The denominator has zeros at $\tau = \tau' + \frac{2i\pi m}{c}$ with them being a double zeros.

The detector response function evaluates to give (after integration over τ) $\triangle E > 0$:

$$(-2\pi i) \times \left(\int_{-\infty}^{\infty} d\tau' \frac{-i\triangle E}{\cosh(c\tau')g(\tau')} \frac{1}{e^{\frac{2\pi\triangle E}{c}} - 1}\right)$$

where

$$g(x) = \frac{d(\arctan(\sinh(c\tau)))}{d\tau}|_{\tau=x}$$

We see that the columns of poles in the complex plane sums to give the Planck distribution factors.

The $\triangle E < 0$ part can be similarly calculated to be:

$$(2\pi i) \times \left(\int_{-\infty}^{\infty} d\tau' \frac{i\triangle E}{\cosh(c\tau')g(\tau')} \frac{1}{1 - e^{\frac{-2\pi\triangle E}{c}}}\right)$$

It is easy to check the ratio of the emission and absorption amplitude is just

$$e^{-2\pi\triangle E/c}$$

Observers on this path observes a temperature of $\frac{c}{2\pi}$ after he follows through the trajectory.

Note that the temperature is proportional to the value of the late time proper acceleration. This indicates that even though there is explicit proper time dependence in the two point function, in the asymptotic past and future, an observer following the path will percieve the vacuum $|0_u>$ as a thermal state. The ratio used on (5.39) reflects the dominant contribution of the late time contribution to the response function. We can see this clearly if we look at the late time behavior of the two point function. Along the trajectory we have:

$$\frac{\mu}{-\frac{\mu}{c^2}(\sinh(c\tau) - \sinh(c\tau'))\sin(\arctan(\sinh(c\tau)) - \arctan(\sinh(c\tau')))}$$

when τ is large:

$$\sinh(c\tau) \to e^{c\tau}$$
$$\arctan(\sinh(c\tau)) \to \frac{\pi}{2} - e^{-c\tau}$$

and we get exactly the two point function of the flat space evaluated along the Rindler trajectory:

$$\frac{1}{\sinh^2(c\frac{\tau-\tau'}{2})} \tag{5.41}$$

which is a thermal Green's function.

To see the geometrical interpretation of this result, we will need to extend the trajectory to off the center and obtain a space filling congruence of trajectories with the same behavior. This can be done by either using the isometries Z_i, Z_i' or by explicitly finding the appropiate off-center paths. It should be noted the extension is not unique, and the observer dependent horizons may not be the same for different extensions. For example, to extend the Rindler horizon from the $t-x$ plane in the flat spacetime, we could have use the translational Killing vector ∂_y or the rotational one $y\partial_x - x\partial_y$. Of course the horizons would have to agree on the orginal two dimensional plane.

We choose the following two class of congruence as our extension. We will then adapt the coordinates to the parameters of the congruence. First,

$$\begin{aligned} u(\tau) &= \frac{1}{\mu}\arctan(\sinh(c\tau)) \\ v(\tau) &= \frac{\mu}{2}(N^2\sinh(c\tau) - \mu^2 x_i^2 \arctan(\sinh(c\tau))) \\ x_i &= const \end{aligned} \tag{5.42}$$

This class is not obtained through the isometries of the spacetime. The observer dependent horizon is the null surfaces $u = \frac{\pi}{2\mu}$ and $u = \frac{-\pi}{2\mu}$. The trajectories approaches the horizons at $\tau \to \infty$ or $N \to 0$. We will now perform a coordinate transformation to express the BMN metric in terms of N, τ. We found the metric becomes:

$$ds^2 = -c^2N^2d\tau^2 - 2Nc\tanh(c\tau)dNd\tau + 2\frac{\mu^2 c\arctan(\sinh(c\tau))x_i}{\cosh(c\tau)}dx_i d\tau + dx_i^2 \tag{5.43}$$

As $\tau \to +\infty$ the metric approaches

$$ds^2 = -c^2N^2d\tau^2 - 2cNdNd\tau + dx_i^2$$

with further coordinate transformation:

$$\begin{aligned} t &= \tau \\ r &= ln(N)/c \end{aligned}$$

and

$$\begin{aligned} T &= t - r \\ R &= r \end{aligned}$$

the metric turns into:

$$ds^2 = c^2e^{cR}(-dT^2 + dR^2) + dx_i^2$$

which is just the Rindler space(5.20).

There are a couple of subtle points. First the horizon is at $N = 0$ in the τ, N, x_i coordinates. However, in the T, R, x_i coordinates, it is at $T \to \infty$, $R \to \infty$, while in the usual coordinate transformation from flat space to Rindler, the horizon is mapped to $T = 0$, $R \to \infty$. This rembles the situation in the near horizon geometry of a static extremal black hole. There one of the Killing horizons forming the bifurcate Killing horizon is pushed to infinity. It is consistent with what we saw in the previous chapter it is easier to generate extremal black objects in backgrounds with the plane wave isometry. Second, in our congruence, there is a "caustic" at $\tau = 0$ where all curves on the same $x_i = const$ plane will intersect at $u = 0$, $v = 0$. Our coordinate transformation breaks down there.

The second class is obtained by translation of (5.39) with the Killing vector Z_i:

$$u = \frac{1}{\mu}\arctan(\sinh(c\tau))$$

$$v = \frac{\mu}{2}\frac{x_i^2}{2}\sin(2\arctan(\sinh(c\tau))) + \frac{\mu N^2}{2}\sinh(c\tau)$$
$$x'^i = -x_i\sin(\arctan(\sinh(c\tau))) \qquad (5.44)$$

Notice this time the congruence become "caustic" at $u = 0, v = 0, x^i = 0$, and the metric in the coordinate adapting to this family of curves will become degenarate at the origin.

$$ds^2 = -c^2N^2d\tau^2 + -cN\tanh(\tau)d + \sin^2(\arctan(\sinh(c\tau)))dx'^{i2} \qquad (5.45)$$

Again, as $\tau \to \infty$ we obtain (5.20), the Rindler space metric. This family of curves can actually be obtained through the inverse of the series of coordinate transformations (5.36), (5.37) with $\phi = \pi/2$ of the Rindler trajectories in (5.35). We will examine the relationship of the thermal behavior and the properties of conformal transformations in the next sub section.

From the above result, we can draw the conclusion that we have found a class of observer such that as they travel along the trajectory, they will see the spacetime as going from Rindler to BMN planewave and back to Rindler. And they would be able to associate the temperature their particle detector measures to the acceleration at asymptotic past and future. Even though the congruence becomes ill defined at $\tau = 0$, we can see that the individual trajectories approach inertial trajectories. The caustic seems to be a feature of trajectories with this asymptotic thermal behavior.

There is yet another class of observers for whom the ratio of the amplitude for particle production and absorption yields thermal result.

$$u(\tau) = \frac{1}{\mu}\sinh(c\tau)$$

$$v(\tau) = \frac{\mu}{2}(N^2\arctan(\sinh(c\tau)) - \mu^2x_i^2\sinh(c\tau))$$
$$x_i = const \qquad (5.46)$$

This class of observers have the same proper acceleration as the family considered above. The cancelation of the divergent factor in (5.10) is however more subtle. Restrict to the center again, we have the curves:

$$u(\tau) = \frac{1}{\mu}\sinh(c\tau)$$

$$v(\tau) = \frac{\mu}{2}(N^2 \arctan(\sinh(c\tau))) \tag{5.47}$$

Note it is just if we interchange u,and v in the previous class. However, this time u has infinity range, and do not seem to follow the local flatness of the spacetime. When evaluated on (5.45), the two point function has poles at

$$\tau = \frac{1}{c} arcsinh(\frac{n\pi}{\mu} + \sinh(c\tau')) + 2i\pi m/c$$

with $m, n \in Z$, and $n = 0$ being double poles and $n \neq 0$ single poles:
For $\triangle E > 0$:

$$(-2i\pi)((-i\triangle E) \sum_{m=1..\infty} e^{-\triangle E(2\pi m/c)} + \sum_{n\neq 0} e^{-\triangle E(2\pi m/c)} \frac{-1^n e^{-i\triangle E f(\tau')}}{\mu h(\tau')})$$

where

$$f(\tau') = \frac{1}{c}(\frac{n\pi}{\mu} + \sinh(c\tau')) - \tau'$$

$$h(\tau') = \cosh((\frac{n\pi}{\mu} + \sinh(c\tau')) + 2\pi m/c)(\arctan(\frac{n\pi}{\mu} + \sinh(c\tau')) - \arctan(\sinh(c\tau')))$$

And for $\triangle E < 0$:

$$(+2i\pi)((+i\triangle E) \sum_{m=0..\infty} e^{+\triangle E(2m\pi)/c} + \sum_{n\neq 0} e^{+\triangle E(2\pi m)/c} \frac{-1^n e^{-i\triangle E f(\tau')}}{\mu h(\tau')})$$

To see the cancelation we will need to make

$$\begin{aligned} \tau' &\rightarrow -\tau' \\ n &\rightarrow -n \end{aligned}$$

in the $\triangle E < 0$ part. Thanks to the sum over n, and observe both $f(\tau')$ and $h(\tau')$ are both odd under the transformation we can see the ratio becomes just

$$\frac{\sum_{m=1..\infty} e^{-\triangle E(2\pi m/c)}}{\sum_{m=0..\infty} e^{+\triangle E(2\pi m/c)}} = e^{-\frac{2\pi\triangle E}{c}}$$

If we look at the two point function, which is proportional to:

$$\frac{1}{(\arctan(\sinh(c\tau)) - \arctan(\sinh(c\tau')))\sin(\sinh(\tau) - \sinh(\tau'))}$$

we discover there is no suitable limit we can take to reduce it to the thermal Greens function of Rindler observers. It is primarily due to the term $\sin(\mu\triangle u)$ and the fact that the trajectory do not respect the conformal structure of the space time.

Also, after performing the coordinate transformation to τ, N, x_i, we get:

$$ds^2 = -c^2N^2d\tau^2 - cN\cosh(c\tau)\arctan(\sinh(c\tau))d + cx^i\sinh(c\tau)\cosh(c\tau)dx^i d\tau + \sum dx^{j2}$$

There is no obvious limit we can take to make the metric Rindler. This family of curves also goes caustic at $\tau = 0$. There is no observer dependent horizon associated with this class of orbits. They can reach arbtrarily large vaue of u, and from the analysis in chapter 3, they can see the whole spacetime.

5.8 Temperature Inherited Through Conformal Transformation

In the previous section we have looked at various trajectories in the BMN spacetime, such that an observer following them would see the vacuum $|0>$ as being thermally populated according to the priciple of detailed balance. However, the proper accelerations along the trajectories are not constant. As a result, we are not able to relate the temperature observed directly to the kinematical properties of the trajectory as before. The best we can do for the class of observer (5.41), (5.43) is to argue the asymptotic past and future behavior dominates the intergral in calculating the response function. The temperature can then be associated to the asymptotic value of the proper acceleration. As discussed above, this point of view is supported by the asymptotic form of the two point function which approaches a thermal Green's function and the behavior of the metric, which approaches the Rindler space metric.

Still we hope to understand the origin of the "thermal" properties of the detector response function. To this end, we again try to examine the conformal structure of the spacetime. First, let us take a closer look at the coordinatie transformation in (5.36), (5.37). In particular, we will consider the effect of the transformation on the family of trajectories:

$$\begin{aligned} u'' &= e^{c\tau} \\ v'' &= -\frac{1}{2c^2}e^{-c\tau} \\ x''^i &= 0 \end{aligned} \tag{5.48}$$

where the "ed coordinates refers to those in (5.38). It is important to note that here τ is not the proper time due to the conformal factor $\frac{1}{1+u''^2}$ when the metric is expressed in these new coordinates. It is easist if if we describe these curves by

$$u'' = -\frac{2c^2}{v''}$$

Performing the inverse of the coordinate transformation in (5.36), (5.37) with $\phi = \pi/2$, we get

$$u = \mathrm{arccot}(\frac{-2c^2}{v})$$

which is exactly the pathes (5.39). More generally, if we subject (5.39) to the coordinate transformation (5.36), (5.37) the result as we vary the value of ϕ (for clearness we will set $\mu = c = 1/2^{1/2}$ in this section):

$$u'' = \frac{v'' + \tan(\phi)}{1 - \tan(\phi)v''} \tag{5.49}$$

Ignoring the conformal factor in (5.38), when $\phi = 0$, the family of curves in (5.39) are "inertial" observers following $u'' = v''$ (or $x'' = 0$ in physical coordinates). As ϕ increase up to $\pi/2$, they became the "Rindler observers" in (5.47). Dispite appearances, they remain intergal curves of a Killing vector field for the Minkowski space. (And by the conformal relation they are conformal Killing vectors for the full BMN metric)

Again ignoring the conformal factor in (5.38), we can extend (5.47) to a space filling congruence by using the translational Killings vectors $\partial_{x''^i}$'s. It turns out this extension corresponds exactly to the extension of (5.39) with Z_i. This is simply because the $\partial_{x''^i}$'s are just Z_i's expressed in a different coordinate system. The observer dependent horizon associated with this congruence is $h_A : u'' = 0$, $h_B : v'' = 0$. The horizons are generated by the boost $\partial_\tau = u''\partial_{u''} - v''\partial_{v''}$. We know that we can relate the Rindler temperature observed by (5.14) to the surface gravity of the observer dependent horizon κ. $u''\partial_{u''} - v''\partial_{v''}$ is now not a Killing vector of the BMN spacetime but a conformal Killing vector. Nevertheless, since a conformal factor does not change the norm of a null vector, ∂_τ still turns null on $u'' = 0$, $v'' = 0$, that is, they are conformal Killing horizons. The picture of the horizons is more complicated in u, v, x^i coordinates. The $u'' = 0$ surface is mapped to $u = \frac{n\pi - \phi}{\mu}$ and the $v'' = 0$ surface is at $v = \frac{-\mu}{4}\sin(2\mu u)x^{i2}$

We can calculate the surface gravity of the conformal Killing horizon $u'' = 0$, $v'' = 0$ defined as

$$\nabla^a(\chi^i\chi_i) = -2\kappa_1\chi^a$$

As shown before, with this choice of definition for the surface gravity, we can relate it to the surface gravity calculated in the conformally related spacetime. All we need to do is to recall it from our computation of surface gravity of the Killing horizon in the Rindler space and multiply it by the conformal factor. $\kappa_1'' = 1 \cdot h(x)|_{horizon} = \frac{1}{1+u''^2}|_{horizon}$. In this case, h(x) is constant on the horizon. The surface gravity is also constant over the surface $h_A \cup h_B$ in the Minkowski space, and thus is constant on the corresponding horizon in the conformally transformed coordinates.

It would be tempting to using

$$T_N = \frac{\kappa''}{2\pi(-\chi^a\chi_a)^{1/2}} \tag{5.50}$$

again to relate the surface gravity so defined to the observed temperature ($\chi = \partial_\tau$ here). However, we have to be careful that $(-\chi^a\chi_a)$ is no longer constant along the trajectory (5.47) anymore. The norm of the conformal Killing vector $u''\partial_{u''} - v''\partial_{v''}$ is $N^2/(1 + u''^2)$ where N^2 is the norm claculated without the conformal factor. Notice that when $\phi = \pi/2$ the norm does approach its constant value in flat space when u approches $\frac{\pi}{2\mu}$ ($u'' = 0$), where the horizon is. This explains why we have the asymptotic Rindler behavior with (5.43) as the Horizon is approached in the far past and future. In those limits the temperature and surface gravity relation is just as in (5.49). The above is a direct result of the conformal factor $\frac{1}{1+u''^2}$ being nonconstant along the orbits of the conformal Killing vector. To be consistent, we can also consider the coordinate transformation with $\phi = 0$. As shown before, the orbits are "inertial" after the transformation. Observers following the trajectory in flat space will see the temperature being zero. Note the conformal factor also goes to zero if we look at the late time effect $\lim_{\tau\to\infty} u'' \to \infty$ as this time $u'' = \tan(\sinh(\tau))$. The relation $\kappa_1'' = \kappa_1 \cdot h(x)|_{horizon}$ is no longer valid. What we see here is that if we choose the wrong conformal transformation, the thermal properties inherited may not be explicit.

It is known that there exists the analogue of a Rindler observer in AdS spacetimes [Appendix. A]. In contrast to the observer dependent horizon in AdS, where the metric is (when expressed with explicit conformal flatness):

$$\begin{aligned} ds^2 &= R^2 \sec^2(\rho)(-d\tau^2 + d\rho^2 + \sin^2(\rho)d\Omega_n^2) \\ &= \frac{4\sec^2(\rho)R^2}{(1+(r+t)^2)(1+(r-t)^2)}(-dt^2 + dr^2 + r^2 d\Omega_n^2) \\ &= \frac{4R^2}{(1-(t^2-r^2))^2}(-dt^2 + dr^2 + r^2 d\Omega_n^2) \end{aligned} \tag{5.51}$$

where the transformation involves

$$r + t = \tan(\frac{\rho+\tau}{2})$$

$$r - t = -\tan(\frac{\tau-\rho}{2})$$

The singularity at $t^2 - r^2 = 1$ comes from the $\sec(\rho)$ and is the conformal boundary of AdS. We can see in the last expression the conformal factor is invariant along the orbits of the boost generator $r\partial_t + t\partial_r$. Thus the relation (5.48) shows the temperature for observers following the integarl curves of the boost is constant as in the Rindler case and can be related directly to their constant acceleration along the trajectory . Similarly, the same effect occur if we choose to consider the second definition of surface gravity, which is only conformally invariant when the conformal factor is constant along the conformal Killing vectors. The same analysis also applies to the Schwarzschild black holes in their maximally extended Kruscal coordinates. This also sets up the criterion for the conformal invariance of Hawking temperature.

As for the family of curves in (5.45), (consider only the ones confined to the center) they are transformed to under (5.36), (5.37) with $\phi = \pi/2$ (other values of ϕ does not generate qualitative differences this time)

$$u'' = -\cot(\sinh(\tau))$$

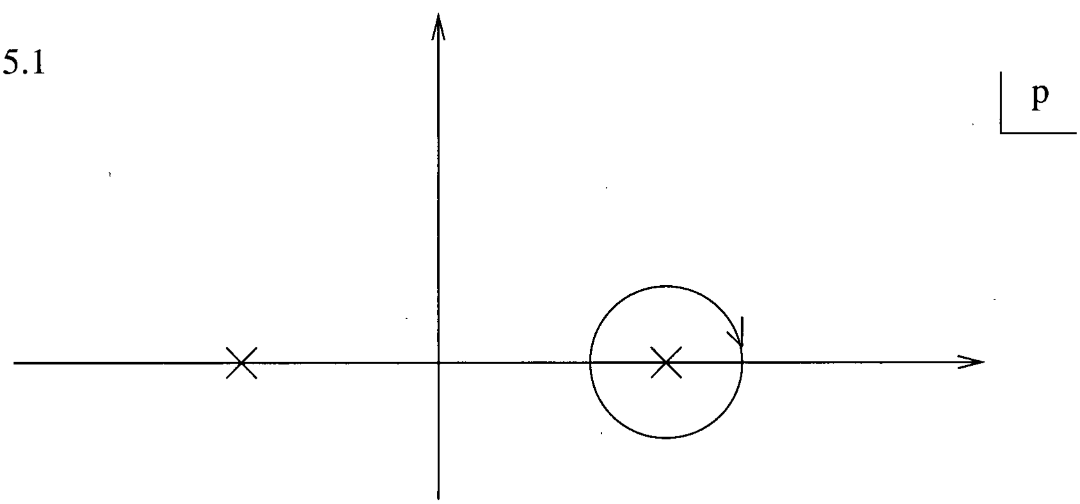
$$v'' = \arctan(\sinh(\tau)) \tag{5.52}$$

A picture of the orbit is dipicted in fig(5.6). We see again the multiple cycle structure as the trajectory with constant acceleration at the center. The diffenrence is that here the poles of the two point functions are arranged in a way that respects the symmetry required to make the inhomogeneous part of the ratio (5.3) cancell. It is easier to see in this picture that it does not have a well defined observer dependent horizon. This class of trajectory does not correspond to the integral curves of any of the conformal Killing vectors of Minkowski space, and thus not to orbits of conformal Killing vectors of the BMN spacetime, either. We are not able to relate it to the thermal signature of some true Killing trajectory.

In general, if the two point function is periodic in imaginary proper time of the trajectory, there is a chance that the observer can detect a thermal signature. According to principle of detailed balance, what is needed is not proper time translational invariance, but the poles of the two point function arranged in a proper way. In the example above, we can roughly relate the "thermalization" to the fact that each cycle in fig(5.6) are of the form (5.13), which do register thermal behavior. However, for the observer to see the vacuum as thermally populated, some subtle cancelation has to take place. From this point of view, we have a notion of equilibrium associated with the orbits (5.45). Among the "thermal" trajectories, some do in fact are integral curves of Killing vectors of in a conformally related spacetime. Through the relations of surface gravities of conformal Killing horizons in conformally related spacetimes, further geometrical meanings can be given to the thermal signature. Near a static black hole in asymptotically flat spacetime, the spacetime looks like the Rindler space. The event horizon corresponds to the the Killing horizon of the Rindler observers. For an observer following the generator of the Killing horizon in the near horizon region, the Minkowski vacuum is seen as a thermal state and used as the Hartle-Hawking vaccum. The Hawking temperature is defined as the red shifted Unruh temperaure as observed at spatial infinity. (Again for spacetimes not asymptotically flat, the temprature is measured at the where the norm of the Killing vector is unity). However, it is not clear whether this more general notion of thermal equilibrium is enough to make the plane wave vacuum state an analogue of the Hartle-Hawking vacuum. In the conventional sense, since the two point function in the planewave vacuum has explicit proper time dependence (even for (5.39)), we reach the conclusion that no temperature of stationary black holes can be defined with respect to the vacuum state of the BMN plane wave.

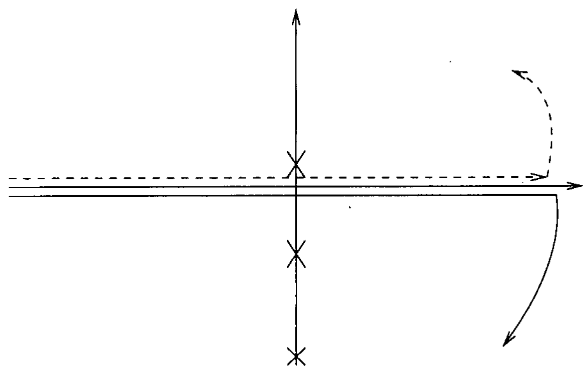
This is of course in accord with the no-go theorems, and can be viewed as a semi-classical verification of them.

fig 5.1



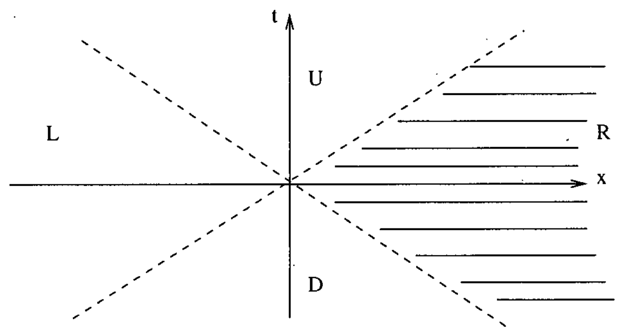
Contour used to get the positive frequency Wightman function

fig 5.2



The poles and contour used to evaluate the detector response function for a Rindler observer. The poles are shifted upward to define the two point function as a distribution. The contour closes up for particle absorption (the detector gain energy) and it should be closed downward for particle emission (the detector loses energy).

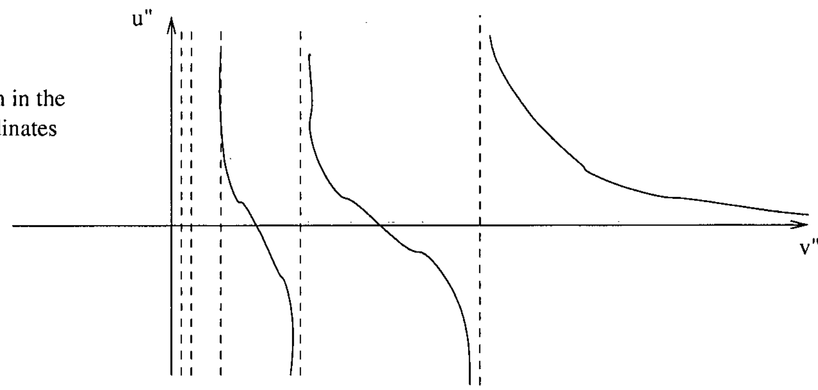
fig 5.3



The right Rindler wedge, the coordinate system 5.18 only covers this portion of the flat space.

fig 5.4

The orbit of (5.35) as seen in the conformally related coordinates u", v"



The pole structure of the two point function evaluated on (5.35) The distribution is one sided with respect to the imaginary axis

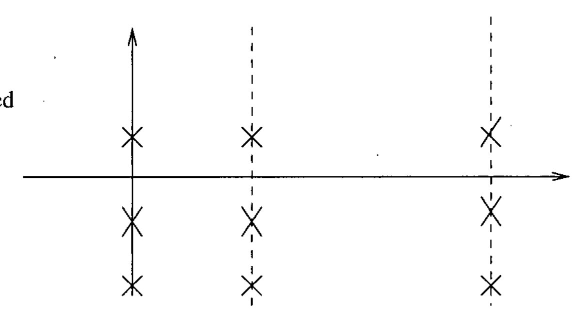
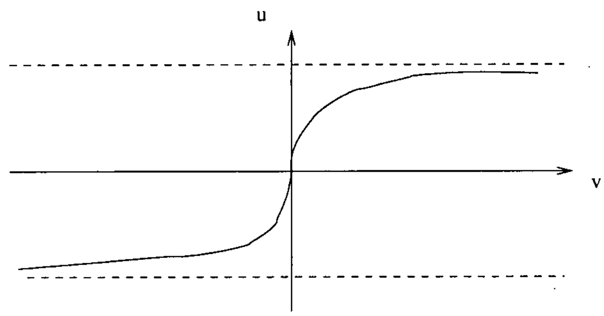
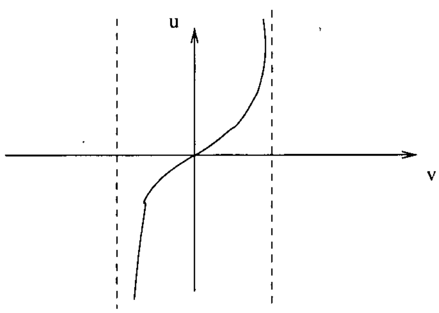


fig 5.5

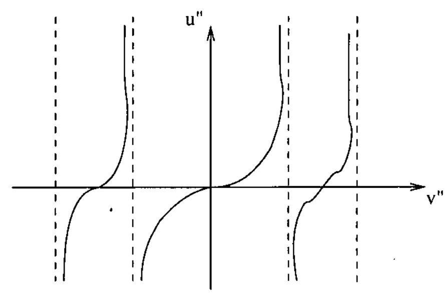


A trajectory along which observers sees the vacuum as thermally populated. We will see that for this trajectory the temperature observed does have a geometrical interpretation.

fig 5.6



Another orbit along which the vacuum state appears thermal
However, we do not have a good geometrical interpretation for the temperature.



The orbit on the left hand side as seen in an conformally related coordinate system. We see the multi-cycle structure as in fig 5.4
It turns out the pole distribution of the two point function is evenly distributed with respect to the imaginary axis.

Chapter 6

Black String Solution in BMN Plane Wave

We have seen in Chap. 3 that the symmetry properties of pp-wave spacetimes or more generally, spacetimes with a null Killing vector are not compatible with the existence of a regular event horizon. We have also seen that this conclusion can be understood quantum mechanically by examine whether it is possible for the near horizon geometry of a black hole to be plane wave. More specifically, in the last chapter, we learned the conformal structure of BMN spacetime is very different from that of the Rindler spacetime which we associate with the near horizon geometry of a stationary black hole. It seems that we have to abandon the null symmetry ∂_v at least near the event horizon in order for a black hole solution to be found. However due to the nonlinear nature of Einstein's equations, giving up certain symmetry requirements reduces its solvability. We can see immediately from Chap. 3, that the nice seperation property of the equations which allowed us to generate new solutions from a homogeneous solution will be lost. Many attempts has been made to find new solution generating techniques that will result in a black hole solution asymptotic to the BMN plane wave. For a nice review see [43]. In this chapter, we will first examine the possibility of having a small Schwarzschild black hole in the BMN spacetime. Then we will present a solution generating technique called the null Melvin twist due to [45], which succesfully generates various black string solutions asymptotically pp-wave. Unfortunately, these solutions are in the wrong background. Nevertheless, we will study its thermodynamical properties and in particular whether the correspondence principle applies in plane wave background. In [Appendix B], we will mention some other attempts in finding the black hole solution.

6.1 Compatibility

Despite the fact that we do not have an explicit solution for a black hole in pp-wave, we can still imagine having a small Schwarzschild black hole sitting in the pp-wave spacetime with some geometry that interpolates between them. Quantitatively we are considering the region of in the parameter space such that $1/M \gg \mu$, where M is the mass of the black hole. We are focusing in on the black hole and treating the five form as a perturbation in the Schwarzschild geometry. The five form perturbation would have to satisfy self-daulity and the Bianchi indentity $dF_5 = 0$ set by the black hole geometry. The influence of the plane wave background comes in the asymptotic boundary condition on the field strength. Here the boundary condition is such that they approach the constant five form when the transverse distance gets large. The same consideration has been applied to the analysis of stability of a small black hole in product space $AdS_p \times S^q$. Smallness is referred to as compared to the radius of the internal five sphere. When the radius of the black hole is smaller than the scale of the internal dimension, the metric (2.23) is no longer valid. We have to consider a black hole sitting in 10 dimensional spacetime and not in an effective 5 dimensional AdS. Uptodate, the explicit form of the metric of a small black hole in $AdS \times S$ has not been found. However, it is proved to be stable with respect to five form perturbation along the lines described above [44].

The authors of [26] considered the black string background:

$$ds^2 = (1 - M/r^6)dt^2 + dy^2 + \frac{1}{1 - M/r^6}dr^2 + r^2 d\Omega_7^2$$

$$= -2dudv + du^2 + (2M/r^6 - M^2/r^12)dv^2 + \frac{1}{1 - M/r^6}dr^2 + r^2 d\Omega_7^2 \qquad (6.1)$$

where $y = u - v$, $t = v$, as the background for the five form perturbation. The choice of a black string instead of a black hole background is motivated by symmetry. The black string solution has the same number of transverse directions as the BMN planewave and the Killing vector ∂_v is asymptotically null to be consistent with the planewave. The boundary condition on the five form perturbation will make it respect the null isometry at infinity as well. It is shown no five form with SO(8) symmetry in the transverse directions is compatible with this background. For $M \neq 0$, the Bianchi identity will force the five form to be zero. However, if th symmetry is broken into $SO(4) \times SO(4)$, there exists non trivial five form solution that satisfies self-duality, the Bianchi identity and the asymptotic boundary condition of BMN plane wave. It is also shown the energy momentum tensor of the five form solution is regular on the horizon suggesting the back reaction of the perturbation will not deter the existence of the event horizon.

The discussion do not give an explicit black string solution we seek. However, it shows it is possible for a black hole phase to exist in the plane wave background. We would need to be clever to find the proper matter support to break the null isometry while keeping the asymptotic behavior of the metric. The explicit solution may not be easy to construct though, as in the example of small black objects in $AdS_p \times S^q$.

6.2 Null Melvin Twist and Black Strings in Plane Wave Background

In this section we will disscuss the solution generating technique by the authors of [45] known as the null Melvin twist. In general null Melvin twist changes the asymptotically flat spacetimes to one that is asymptotically plane wave. It involves the follwing operations on a given solution of supergravity with at least one translational isometry:

1. Boost along one of the translational invariant directions, say y, with a boost parameter γ.

2. T-dualize in the direction of boosting.

3. Perform the twist along y with a twist parameter ω. (we will discribe the twisting procedure with explicit examples later) Notice that this will not change the isometry in ∂_y.

4. T-dual back along y.

5. Boost back to the original frame with boost parameter $-\gamma$. This will cancell the momentum generated in the solution from step 1.

6. Taking the limit $\omega \to zero$, $\gamma \to \infty$, with the following combination of parameters fixed.

$$\mu = \frac{1}{2}\omega e^{\gamma}$$

First consider using the above operations on the 10 dimensional flat space:

$$ds^2 = -dt^2 + dy^2 + dr^2 + r^2 d\Omega_7^2$$

obviously the first boost and T-duality does not change the solution in any way. The twist here is best described by first breaking up the transverse directions into four two planes parametrized by ρ_i, ϕ_i.

$$dr^2 + r^2 d\Omega_7^2 \to \sum d\rho_i^2 + \rho_i^2 d\phi_i^2 \tag{6.2}$$

The twist operation is to replace $d\phi_i$ by $d\phi_i + w_i dy$. It is just a rotation in the two planes with the amount of rotation dependent on the position along the y-axis.

$$\begin{aligned} x_i' &= \cos(w_i y)x_i - \sin(w_i y)z_i \\ z_i' &= \sin(w_i y)x_i + \cos(w_i y)z_i \end{aligned}$$

after the twist and T-dualize back along y, we get

$$ds^2 = -dt^2 + \frac{1}{1+\sum w_i^2\rho_i^2}dy^2 + \sum d\rho_i^2 + (\rho_i^2 - \frac{w_i^2\rho_i^2}{1+\sum w_i^2\rho_i^2})d\phi^2$$

$$e^{2\Phi} = \frac{1}{1+\sum w_i^2\rho_i^2},\ H = \sum_i \frac{w_i\rho_i^2}{1+\sum w_j^2\rho_j^2}d\rho_i \wedge dy \tag{6.3}$$

(6.3) is just the Melvin magnetic flux universe [46] with the flux turn on in each of the transverse two planes. w_i's can be regarded as the strength of the magnetic flux in the two planes. If we continue with the program setting the flux strength to be the same $w_i = w$ and take the double scaling limit as prescribed, we will obtain the 10-dimensional plane wave background with the same metric as the BMN pp-wave but is supported by NS-NS three form field strength instead of the R-R five form:

$$ds^2 = -(1+\mu r^2)dt^2 + \mu r^2 dtdy + (1-\mu r^2)dy^2 + dr^2 + r^2 d\Omega_7^2$$

$$e^{2\Phi} = 1,\ H = \frac{\mu}{2}r^2(dt+dy)\wedge d\sigma \tag{6.4}$$

To get to the familiar Brinkman form of the metric we only need to use $u = t+y$, $2v = t-y$. The appearence of the NS-NS three form is from the combination of twisting (which generates off diagonal metric elements) and T-duality (which relates off diagonal metric elements and NS-NS two form potential). Notice also the strength of the dilaton is scaled back to zero, due to the fact that we take $w \to 0$. The three form strength is null in this particular case. As expected, the procedure generates plane wave solutions from asymptotically flat ones. We can also obtain a plane wave background with the same metric as the BMN spacetime but is supported by R-R tow form potential by S-dual the solution (6.4).

Now we apply the null Melvin twist to the 10 dimensional Schwarzschild black string solution in flat space.

$$ds^2 = -f(r)dt^2 + dy^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_7^2 \tag{6.5}$$

The null Melvin twist generates the following black string geometry:

$$-\frac{f(r)(1+\mu^2r^2)}{k(r)}dt^2+\mu^2r^2\frac{f(r)}{k(r)}dtdy+(1-\frac{\mu^2r^2}{k(r)})dy^2+\frac{1}{f(r)}dr^2+r^2d\Omega_7^2+\frac{-\mu^2r^4(1-f(r))}{4k(r)}(\frac{1}{2}\sum d\phi)^2$$

$$e^{2\Phi} = \frac{1}{k(r)},\ H = \frac{\mu r^2}{2k(r)}(f(r)dt + dy) \wedge (\frac{1}{2}\sum d\phi) \tag{6.6}$$

where $f(r) = 1 - \frac{M}{r^6}$, $k(r) = 1 + \frac{\mu^2 M}{r^4}$. If we do not take the boosting steps, we will end up with

$$ds^2 = -f(r)dt^2 + \frac{1}{1+\mu^2r^2}dy^2 + \frac{1-f(r)}{f(r)}dr^2 + \sum_i d\rho_i^2 + (\rho_i^2 - \frac{(w\rho_i)^2}{1+w^2r^2})d\phi_i^2$$

$$e^{2\Phi} = \frac{1}{1+w^2r^2},\ H = \frac{w\rho_i^2}{1+w^2r^2}d\rho_i \wedge dy \tag{6.7}$$

(6.7) is a black string embedded in the Melvin flux universe. Comparing with (6.3), we notice the presence of the black string does not change the dilaton and the magnetic flux in back ground (this is not the case for the black string in pp-wave (6.4) and (6.6)). It is a little awkward to use two coordinate systems for the transverse directions. Explicitly, the transverse part of the metric is:

$$r^2d\chi^2 + \frac{r^2}{4}\cos^2(\chi)d\theta_1^2 + \frac{r^2\cos^2(\chi)}{4}(1 - \frac{\mu^2 M\cos(\chi)^2\cos(\psi_1)^2}{(r^4+\mu^2M)})d\psi_1^2 + (\frac{1}{4}r^2\cos(\chi)^2\cos(\psi_1)$$

$$-\frac{\mu^2Mr^2\cos(\psi_1)\cos(\chi)^4}{4(r^4+a^2M)})d\psi_1 d\phi_1 - (\frac{\mu^2Mr^2\sin(\chi)^2\cos(\chi)^2\cos(\psi_1)\cos(\psi_2)}{4(r^4+\mu^2M)})d\psi_1 d\psi_2$$

$$-\frac{\mu^2Mr^2\sin(\chi)^2\cos(\chi)^2\cos(\psi_1)}{4(r^4+\mu^2M)}d\psi_1 d\phi_2 + (\frac{1}{4}r^2\cos(\chi)^2 - \frac{\mu^2Mr^2\cos(\chi)^4}{4(r^4+\mu^2M)})d\phi_1^2 -$$

$$\frac{\mu^2Mr^2\sin(\chi)^2\cos(\chi)^2\cos(\psi_2)}{4(r^4+\mu^2M)}d\psi_2 d\phi_1 - \frac{\mu^2Mr^2\sin(\chi)^2\cos(\chi)^2}{4(r^4+\mu^2M)}d\phi_2 d\phi_1$$

$$+(\frac{1}{4}r^2\sin(\chi) - \frac{\mu^2Mr^2\sin(\chi)^4\cos(\psi_2)^2}{4(r^4+a^2M)})d\psi_2^2 + (\frac{1}{4}r^2\sin(\chi)\cos(\psi_2) - \frac{\mu^2Mr^2\cos(\psi_2)\sin(\chi)^4}{4(r^4+a^2M)})d\phi_2 d\psi_2$$

$$+(\frac{1}{4}r^2\sin(\chi)-\frac{\mu^2Mr^2\sin(\chi)^4}{4(r^4+a^2M)})d\phi_2^2+\frac{1}{4}\sin^2(\chi)(d\theta_2^2)$$

If we set $\mu=0$ the above is just the metric of the seven sphere writen in coordinates with explicit $U(1)_\chi\times SO(4)_1\times SO(4)_2$ symmetry. The solution (6.4) has regular event horizon located at $r=r_0=M^{1/6}$. The generator of the horizon is the time like Killing vector $\zeta=\partial_t$, whose norm is zero at the horizon. Despite the appearence of the cross terms G_{ty}, there is no ergosphere. This is because the cross term also vanishes at the horizon. Taking $r\to\infty$ we return to the plane wave solution (6.3) in all of the fileds (the metric, the dilaton and the NS-NS two form). All the expressions above are in the string frame.

Having identified the horizon and its generator, we can try to find other thermodynamic quantities of this background. These physical quantities have to be calculated from the Einstein frame metric.

$$G_{\mu\nu}^{Einstein}=e^{-1/2\Phi}G_{\mu\nu}^{string}$$

We can compute the surface gravity of the black string solution using (5.16):

$$\kappa^2=-\frac{1}{2}\nabla^a\zeta^b\nabla_a\zeta_b|_{r=r_0}$$

In order to relate this to the temperature, we have to be careful about the normalization of the generator of the horizon as we have learnt from the last chapter. In asymptotically flat space times, we take the temperature as measured by an observer at spatial infinity as $G_{tt}=1$ there. Here we have the same ambiguity as in asymptotically AdS spacetimes, where we choose the temperature to be as measured at where the norm of the generator is 1. The Hawking temperature computed this way is:

$$T_H=\frac{\kappa}{2\pi}=\frac{3}{2\pi}M^{-1/6} \tag{6.8}$$

We can also calculate the surface area of the event horizon per unit length of the black string. After some algebra, the area (of constant t, y, r) is:

$$A=\frac{\pi^4}{6}(k(r)-\mu^2r^2)^{1/2}r^7|_{r_0}=\frac{\pi^4}{6}M^{7/6} \tag{6.9}$$

The experience with asymptotically flat spacetimes will prompt us to conclude the entropy per unit length is:

$$S=\frac{\pi^4M^{7/6}}{24G_{10}} \tag{6.10}$$

Notice that both the temperature and the entropy defined this way are the same as that of the asymptotically flat black string (6.5) and are independent on μ. This suggests the null Melvin twist does not change the thermodynamics of the spacetime. In general, it can be proven that the area of the event horizon is invariant under the null Melvin twist [45]. However to really identify the surface gravity and area of the horizon to temperature and entropy of the black string, we will need to look more carefully into what is really meant by asymptotically plane wave. We will return to this problem later.

6.3 Correspondence Principle for Black Strings in Plane Wave Background

String theory has many successes in providing the microscopic interpretation of black hole entropy. However, most explicit calculations are restricted to counting states of extremal and near extremal black objects. The correspondence principle is due to the original idea of [47], which suggested that when the size of a black hole shrinks to the string scale and the curvature of at the horizon is strong, the black hole state can be approximated by a long single string with large mass. This correspndence is latter made quantitative in [48] where it is shown the entropy of a large class of black branes can be matched to the entropy of weakly coupled large mass string states with the possibility of presence of D-branes. Their calculation applies to black objects far from extremity, but the matching is only accurate up to a (dimensionless) constant of order of unity. This correspondence compliments the state counting technique and provides insight into the nature of black hole and thermodynamics of weakly coupled string theory.

Let us start by reviewing the correspondence principle in asymptotically flat space. We will consider the simplest case of a Schwarzschild black hole with no charge and angular momentum. The metric is:

$$ds^2 = -(1-(\frac{r_0}{r})^{n-2})dt^2 + \frac{1}{(1-(\frac{r_0}{r})^{n-2})}dr^2 + r^2 d\Omega_{n-2}^2 \tag{6.11}$$

for an $n+1$ dimensional spacetime. The mass of the black hole is given by $M_{BH} = \frac{r_0^{n-1}}{G_{n+1}}$, where G_{n+1} is the $n+1$ dimensional Newton's constant and $G_{n+1} = g_s^2 \alpha'^{\frac{n-1}{2}}$. The entropy of the black hole spacetime is :

$$S_{BH} \sim \frac{r_0^{n-1}}{G_{n+1}} = \frac{r_0^{n-1}}{g_s^2 \alpha'^{\frac{n-1}{2}}} \tag{6.12}$$

In writing down the above expressions, we have taken advantage of the fact that we have well defined thermodynamical relations for these quantities in classical theory of gravity in asymptyotically flat spacetimes. That is, the Hawking temperature, entropy and ADM mass of the black hole solution has the familiar relationship $dS = \frac{dE}{T}$. On the string side, the mass and entropy of a highly excited weakly coupled string state are:

$$M_{str} \sim (\frac{N}{\alpha'})^{1/2}$$

$$S_{str} \sim \alpha'^{1/2} M_{str} \sim \frac{1}{T_H^{flat}} M_{str} \tag{6.13}$$

where we have assumed the coupling is weak enough, and the string density of states exhibits the Hagedorn behavior (T_H^{flat} is flat space Hagedorn temperature). N is the excitation number of the state.

The precise statement of the correspondence priciple is [48]:

When the size of the horizon of a black hole (in string frame) becomes of the order of typical string size, the black hole state is described by a state of weakly coupled strings and D-branes with the same charge and angular momentum. The mass of the black hole is identified with the mass of the string state up to dimensionless prefactors of order of unity.

The condition on the black hole size (or equaivalently, on the curvature invariants at the horizon) is to say the matching happens when the stringy corrections become important at the horizon, or the classical description of geometry becomes ill-defined around the horizon.. As an immediate consequence of the correspondence principle, we should find the entropy of the black hole state to be well approximated by the entropy of the dual string state. Turning the argument around, if we set the expressions for th black hole and string state entropy to be equal, it is easy to see that the masses of the corresponding states will be equal (up to constants of order one) if the size of the horizon is approximately $\alpha'^{1/2} \sim T_H^{flat}$. In weakly coupled string theory, the scale of stringy corrections to geometry is generally set by T_H. For the cross over to happen, we thus expect the curvature near the event horizon to be also of this magnitude when the radius of the horizon comes down to the scale of crossover $r_0 \sim \alpha'^{1/2}$. In flat space, this is trivial (since there is no other dimensionful quantities in this background). The curvature constant $R^2 = R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} \sim 1/r_0^4$. The dimensionless quantity $R^2T_H^4$, is automatically of order unity. As for black strings in flat space we only need to note that when the string direction is compactified on a circle of radius r_c, we have effectively a black hole in 9 dimensional spacetime. The argument above goes through with the effective 9 dimensional Newton's constant $G_9 = G_{10}/r_c$. The compactification radius is to be of the order of the cross over radius $\alpha'^{1/2}$ to ensure the analysis does not suffer the Gregory-Laflamme instability (transition between black string and black holes) [49] in this and the T-dualized background [50].

Now we are in position to apply the correspondence principle to the solution (6.6) [50]. On the black string side, we know that the entropy and temperature are invariant under the null Melvin twist. If the usual thermodynamic relation $dE = TdS$ also survives, we have exactly the case of a ten dimensional black string in asymptotically flat spacetime. On the weakly coupled string theory part, we need to consider if the entropy of the strings has the Hagedorn behavior as before. This is done by taking advantage of the fact that the plane wave background (6.4) is a member of a large class of spacetimes which are related to the Minkowski space by T-duality, boosts and identifications (using the notations of [51]):

$$ds^2 = -f_1(r_i)dt^2 + \sum_{i=1}^{4} f_{2i}(r_i)dtd\phi_i + f_{3i}(r_i)d\phi_i^2 + dr_i^2 + e^{2\sigma_i}(dy + A_i((a_{+i}+c_{+i})d\phi_i + a_{+i}c_{-i}dt))^2$$

$$H_2 = -\frac{1}{2}c_{-i}F(r_i)r_i^2 d\phi_i \wedge dt + \frac{1}{2}F(r_i)r_i^2[(a_{+i} - c_{+i})d\phi_i + a_{+i}c_{-i}dt] \wedge dy$$

$$e^{2\Phi} = F(r_i),\ e^{2\sigma} = \frac{F(r_i)}{F'(r_i)}$$

$$A_i = \frac{1}{2}F'(r_i)r_i^2$$

$$f_1 = 1 + \frac{1}{4}(a_{+i}c_{-i}r_i^2)^2 FF',\ f_{2i} = c_{-i}[1 + \frac{1}{4}\sum(c_{+j}^2 - a_{+j}^2 - c_{-j}^2)r_j^2]FF'r_i^2$$

$$f_{3i} = (1 - \frac{1}{4}\sum c_{-j}^2 r_j^2)FF'r_i^2$$

$$F = \frac{1}{1+\frac{1}{4}\sum[(a_{+i}-c_{+i})^2 - c_{-i}^2]r_i^2}, \frac{1}{1+\frac{1}{4}\sum[(a_{+i}+c_{+i})^2 - c_{-i}^2]r_i^2} \tag{6.14}$$

The solutuon (6.4) can be obtained by setting all the a_{+i}, c_{+i}, c_{-i}'s the same with $c_- = 0$, $F = F' = 1$ and making the gauge choice $\phi_i \to \phi_i + \mu t - \mu y$. Since free string theory can be solved exactly in this class of background, the partition function can be calculated. It is found, with the parameter values chosen above, the string spectrum does have Hagedorn behavior [50], [19]. The Hagedorn temperature is this spacetime with pure NS-NS form fields is the same as that in flat space.

$$T_H^{(6.4)} = T_H^{flat} = \frac{1}{4\pi(\alpha')^{1/2}} \tag{6.15}$$

,which is independent of the flux strenth μ. We can then conclude the entropy of free strings on (6.4) is the same as that in Minkowski space. Setting the black string entropy and the entropy of free string to be equal, we again get the condition $r_0 = \alpha'^{1/2}$ for the correspondence to work. The curvature invariant R^2 of the black string in plane wave background with pure NS-NS form fields is also independent of μ and $R^2 \sim 1/r_0^4$ as in the asymptotically flat case. (The curvature is calculated in the string frame.) Therefore the correspondence principle is working in asymptotically plane wave spacetime.

We can also repeat the steps for the black string solution with R-R form fields obtained by S-dualizing (6.6). The string theory spectrum on the S-dualized version of (6.4) also has Hagedorn behavior with entropy $S_{str} = M_{str}/T_H$ [50]. In this case both the Hagedorn temperature and the curvature invariant R^2 (in string frame) have nontrivial dependence on μ.

$$\begin{aligned} T_H &\sim \frac{1}{\alpha'^{1/2}},\ \mu \text{ small} \\ &\sim \mu,\ \mu \text{ large} \end{aligned}$$

$$\begin{aligned} R^2 &\sim \frac{1}{r_0^4},\ \mu \text{ small} \\ &\sim \frac{\mu^2}{r_0^6},\ \mu \text{ large} \end{aligned}$$

Equating the entropy on both sides, we get the usual condition on the crossover radius $r_0 = 1/T_H$. With this and the expressions of T_H and R^2, we can check in both the small and large μ cases the quantity R^2/T_H^4 is of order unity. This gives evidence that the correspondence principle is working even in the case where the Hagedorn temperature has non-trivial dependence on μ.

We have been calling the solution (6.6) as being asymptotically plane wave as its metric approaches the plane wave form when the transverse distance is large. However the precise notion of a spacetime being asymptotically plane wave is still an open question. In particular, we would like to check if the causal structure of a solution like (6.6) has the peculiar property shown in chapter 3. And also we need to examine its global structure with the ideal point construction. These has proven to be computationally complicated due to the form of the metric. Related to this, there is a major assumption in the above analysis being somewhat unjustified. This is the assumption that in the asymptotically plane wave background the usual thermodynamic relation $dE = TdS$ is valid. Assuming the definitions of temperature and entropy are correct, we still need to find an appropriate definition for the ADM mass. This is complicated by the degenarated boundry of plane wave. The boundary can not be used as an usual Gauss surface for the momentum current. If we follow the standard procedure of saddle point approximation for the gravitational partition function, we find a further difficulty. Leaving the boundary term aside (it is not clear whether it should be included), we find the bulk part of the (Euclidean) action to be divergent. Explicitly, the bosonic part of type IIB supergarity action is [52]

$$S = \frac{1}{\kappa^2}\int d^{10}x(-g)^{1/2}e^{-2\Phi}(R - \frac{1}{12}H^2 + 4\partial_{mu}\Phi\partial^{\mu}\Phi) \tag{6.16}$$

Using the equations of motion:

$$\begin{aligned} R_{\mu\nu} + 2\nabla_{mu}\nabla_{\nu} - \frac{1}{4}H_{\mu\rho\omega}H^{\rho\omega}_{\nu} &= 0 \\ -\frac{1}{2}\nabla^{\omega}H_{\omega\mu\nu} + \nabla^{\omega}H_{\omega\mu\nu} &= 0 \\ -\frac{1}{2}\nabla^2\Phi + (\nabla\Phi)^2 - \frac{1}{24}H^2 &= 0 \end{aligned} \tag{6.17}$$

the on shell action can be written as:

$$S_{saddle} = \frac{1}{\kappa^2}\int d^{10}x(-g)^{1/2}e^{-2\Phi}(8(\nabla\Phi)^2 - 4\nabla^2\Phi) = \frac{1}{\kappa^2}\int d^{10}x(-g)^{1/2}e^{-2\Phi}\frac{1}{3}H^2 \tag{6.18}$$

To evaluate the action in the Euclidean sector, we choose to Wick rotate the Killing direction ∂_t. This acctually results in a complex metric. However, since ∂_t is the generator of the horizon, and the determinant of the metric after the rotation only changes its signature, this appears to be the correct choice.

The action evaluated on the Euclideanized version of (6.6), we have:

$$Vol_{\Omega_7}\int_0^{\infty} dr(1 + \frac{M\mu^2}{r^4})\frac{-16r^7\mu^2M(3r^2\mu^2M + 2M + r^6)}{(r^4 + \mu^2M)^3}$$

It is easy to see as $r \to \infty$ the expression is linearly divergent. Divergence of the bulk term is common in this type of calculations. We have seen this in chapter 1 when reproducing the

Hawking-Page phase transition. There the divergence is regulated by using the pure AdS space as a background since they share the same asymptotics and divergences. In this case it is more complicated as the absence of dilaton in the pure plane wave solution suggests its contribution to the action is zero. We are left with the problem of trying to find an appropriate background spacetime. Since the plane wave geometry (6.4) falls into the class of exactly solvable string theory background (6.14), a natural starting point is to search within this family. So far we have not been able to find a suitable choice.

Chapter 7

Summary and Open Questions

In this thesis we explore the possibility of finding a black hole solution in asymptotically BMN spacetime. This is motivated by the fact string theory in the plane wave exhibits the Hagedorn behavior [19]. If such solution is found, we would be able to understand more about the themodynamics of string theory on the BMN background and maybe relate the phase transition to behaviors of the dual BMN sector of the super Yang-Mills theory. Unfortunately, explicit black hole solution of this kind seems to be very difficult to derive. Results summarized in chapter 4 of the no horizon theorems in pp-wave [25], [26] suggest if the black hole solutions exist, near the event horizon, the null Killing isometry which characterizes the pp-wave family has to be modified. Giving up symmetry requirements makes directly solving the Einstein's equations a very difficult task. These thereoms also makes it clear that we are not able to access the black hole solutions by taking Penrose limits of black hole spacetimes. The Penrose limit does not keep enough global structure of the original spacetime. One might atempt to find other limiting proceedure under which the event horizons are retained. However, with the holography direction in the BMN/CFT less than clear, it would seem we co not have too many guide lines in constructing such generalizations of the Penrose limit. It is for this reason, we are led to consider carefully the concept of temperature and thermal states in the BMN spacetime.

With the knowledge that the Hawking temperature of stationary black holes can be viewed as the redshifted Unruh temperature from the near horizon geometry, we set out to see if we could find the analogue of a Rindler observerin the plane wave spacetime. We do this through the Unruh monopole detector following various trajectories in the BMN spacetime. We found observers following orbits of Killing vectors and geodesics do not observe the desired thermal spectrum. These observers do not see the vacuum of the BMN plane wave as a thermal state. Since the two point functions are not proper time translational invariant if evaluated on a non-Killing trajectory, we do not expect any observers in the BMN spacetime to regard the plane wave vaccum as a thermal state for all times. However, temperature itself is an averaged concept. We can still defined thermal equilibrium if we look at the behavior of the two point function averaged over the trajectory. We invoke the principle of detailed balance. As shown in chapter 5, we are able to find trajectories with thermal signature with this more general criterion for temperature. It is also argued if the "thermal trajectory" happens to be also the orbit of a true Killing vector of a conformally related spacetime, we can associate the temperature observed with certain geometrical quantities of the background, namely the surface gravity of conformal Killng horizon. In this case, thermal equilibrium can be connected with the constancy of the surface gravity over the conformal Killing horizon. Despite the generalized definition of thermal equilibrium, it is not clear if we can use it to define the temperature of a stationary black hole. This is because the state itself has explicit time dependence as viewed by such observers. Our result here is nontheless a semi-classical verification of the no go theorems.

Explicit black string solutions asymptotic to BMN are constructed using the null Melvin twist technique [45]. However, they are supproted by the wrong form fields. These solutions are supported by NS-NS or R-R three forms instead of R-R five forms which comes naturally from the Penrose limit of $AdS_5 \times S^5$. These black string geometries are still interesting on their own, though. It is shown the correspondence principle is working in this spacetime [50]. In particular, they can teach us about thermodynamics of black objects in a plane wave background. It is not clear whether these solutions still have the peculiar boundary and light cone structure of pp-wave spacetimes shown in chapter 3. Whether the boundary behavior of these black string solutions can be used to define what is meant by "asymptotically plane wave" remains an open question. Further, as seen in the end of chapter 6, the bulk part of the IIB supergravity is linearly divegent when evaluated on the black string solution (6.6). And we are not able to use the pure plane wave geometry (6.4) as a background to regulate the divergence. This suggest that there is no phase transtion between the two solutions. (Again this point relies heavily on the details of teh boundary behavior of these backgrounds, and we have not treated the possible boundray term in the action properly) More work is needed to find an appropriate solution to the regularize the gravitational action. A natural starting point seems to search among the class of exactly solvable string theory backgrounds in [51] to which the NS-NS supported plane wave is a member.

As described in section 6.1, It is consistent for an asymptotically BMN plane wave black hole solution with R-R five form field strength to exist. However, finding the explicit form of it may be a very challenging task.

Appendix A

Geometry of Anti-de Sitter Space

In this appendix we discuss the geometry of the AdS spacetime. It is easiest to consider the AdS_{n+1} space times as being a n+1 dimensional hyperboloid embedded in $R^{2,n}$. The convention for the signature of $R^{2,n}$ here is chosen to be (-,+,+,...,+,-). The hyperboloid is defined by

$$x_0^2 + x_{n+2}^2 - \sum_{i=1}^{n+1} x_i^2 = R^2 \tag{A.1}$$

We will call R the radius of the anti-de Sitter space. Note the hyperboloid contains closed time like curves (figA.1). We can parametrize the hyperboloid with the coordinates

$$\begin{aligned} x_0 &= R\cosh(\rho)\cos(t) \\ x_{n+2} &= R\cosh(\rho)\sin(t) \\ x_i &= R\sinh(\rho)\Omega_i \end{aligned} \tag{A.2}$$

where Ω_i parametrizes the n sphere of the spacetime and $\sum_{i=1}^{n} \Omega_i^2 = 1$. Notice that the entire hyperboloid is covered once if t goes through $[0, 2\pi]$ fig(A.1). The above parametrization lifts up the hyperboloid and describes the global covering of AdS_{n+1}. The closed time like curves are also "unwrapped " after the the lifting, τ now ranges $(-\infty, +\infty)$. To see the causal structure of the global AdS, we refer to the technique of drawing its Penrose diagram. The AdS spacetime is locally conformally flat, and its Penrose diagram can be drawn on the Einstein static universe (fig. 1.4). We need the coordinate transformation $\tan(\theta) = \sinh(\rho)$. The metric now takes the form:

$$ds^2 = \frac{R^2}{\cos^2(\theta)}(-dt^2 + d\theta^2 + \sin^2(\theta)d\Omega_{n-1}^2) = \frac{R^2}{\cos^2(\theta)}ds_{ESU}^2 \tag{A.3}$$

The conformal factor $\frac{R^2}{\cos^2(\theta)}$ diverges at $\theta = \pi/2$, and this marks the conformal boundary of the AdS spacetime. From fig.(1.4), we can see the global AdS covers only half of the Einstein static universe, and the embedding tells us the boundary is timelike. The time like boundry is related to the fact that asymptotically AdS spacetimes (spacetimes with the same conformal boundary structure as AdS) are not globally hyperbolic. They do not pocess complete cauchy surfaces. Informations can be sent in from the time like boundary. This is a physical interpretation of the boundary to bulk propagator found in (1.15). If we multiply the metric by any function of θ which goes to zero at $\theta = \pi/2$ and have the multiplicity to cancell(regularize) the divergence of the conformal factor, we see the boundary of AdS_{n+1} is $R_t \times S^{n-1}$. The freedom in choosing the regularization factor is related to the dilataion isometry of the bulk AdS spacetime and the conformal isometry of the boundry (compactified) Minkowski space.

Another useful coordinate system is given by the parametrization:

$$x_0 = \frac{1}{2u}(1 + u^2(R^2 + \sum x_i'^2 - \tau^2))$$

$$x_{n+2} = Ru\tau$$

$$x_i = Rux_i'$$

$$x_{n+1} = \frac{1}{2u}(1 - u^2(R^2 - \sum x_i'^2 + \tau^2)) \tag{A.4}$$

It covers only half of the hyperboloid. The metric now becomes:

$$ds^2 = R^2(\frac{du^2}{u^2} + u^2(-d\tau^2 + \sum dx_i'^2)) \tag{A.5}$$

This is the form we get from looking at the near horizon geometry of a black 3-brane solution in (1.6), the Poincare patch. Notice this patch of the AdS spacetime is its self conformally flat. The entire patch can be conformally mapped to the flat spacetime. To see this, simply define $U = -1/u$, and the metric is:

$$ds^2 = \frac{1}{U^2}(dU^2 - d\tau^2 + \sum x_i'^2) \tag{A.6}$$

The points with $U = 0$ ($u = \infty$) are the conformal boundary of this patch and is in fact part of the time like boundray of the global AdS spacetime. The points with $U = \infty$ ($u = 0$) marks where the coordinatization (A.5) breaks down (notice the metric is degenarate at these points). We can analytically continue across these "cosmological" boundaries and obtain the entire AdS hyperboloid. We can obtain this result directly from the metric (2.15) by pulling out appropriate conformal factors as was done in the end of chapter 5. Notice the coordinates are different for the flat space metric in (5.50) and (A.6). In (A.5), we also notice that the points $u = 0$ are the Killing horizon for the Killing vector ∂_τ. In fact, (A.6) has already been put in the form of Rindler spacetime as in (5.20). It is then obvious that observers following the orbits of ∂_τ are the analogue of the Rindler observers in AdS spacetime.

We can understand this from another point of view. Going back to the embedding coordinates $x_0, x_1 \ldots x_{n+2}$, and consider an observer following the hyperbolic line along the hyperboloid in th embedding space $_{2,n}$:

$$x_{n+2} = Z$$

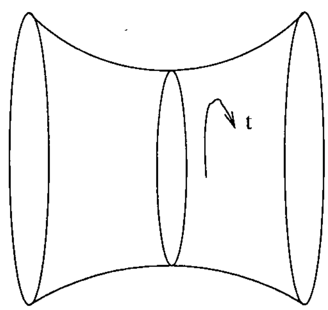
$$x_0^2 - x_1^2 = Z^2 - R^2 \tag{A.7}$$

with all other coordinates set to zero. This when projected onto the coordinates τ, u, x_i', is precisely the orbits of ∂_τ fig(A.2). In the embedding space, this trajectory follows that of an Rindler observer in $R^{2,n}$ with the familiar expression of temperature [53]:

$$T_{embedding} = \frac{a_{n+2}}{2\pi} \tag{A.8}$$

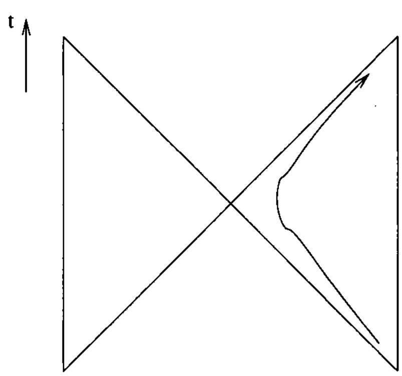
where a_{n+2} is the acceleration measured in the flat embedding spacetime. $a_{n+2} = (Z^2 - R^2)^{-1/2}$. Notice that there is a minimal value of Z for which the accelration is real. This corresponds to the point the trajectory turns from time like to space like, and thus does not cause real physical contradictions. This thermal behavior can also be checked by explicitly calculating the two point function in anti-de Sitter space and evaluate the detector responce function along the appropriate trajectory [53]. It is found, independent of the boundary conditions set for the wave equations, the detector response function contains the Planck factor along the orbits of ∂_τ. One can repeat this for de-Sitter space by considering Rindler observers in the embedding flat space. More generally this procedure works for Schwarzschild black holes in flat and AdS spacetimes with possiblely nonzero charges and momentum. The challenge is to find the proper embedding space. Thus this suggest we can map the Hawking radiation of a black hole geometry to the Unruh radiation in the embedding spacetime [54].

fig A.1



AdS space as a hyperboloid embedded in a hyher dimensional flat space
The time t is a periodic coordinate. There are closed timelike curves
We need to decompatify the time direction to get the global AdS space
which is causally well defined.

fig A.2



The path followed by an observer with constant acceleration in AdS.
This class of observer will measure a temperature analogue to the
Rindler temperature in flat space. The temperature is proportional to
the constant acceleration in the embedding space.

Appendix B

Other Attempts of Generating Black Hole Solutions in Plane Wave

One natural idea to consider is to glue together a black hole geometry with the plane wave spactime. We consider glueing together the BMN pp-wave (exterior) to the Schwarzschild black string (interior). The choice of black string is again motivated by symmetry. The metrics are:

Black string:

$$g^i_{\mu\nu}dx^\mu dx^\nu = -h(r)dt^2 + dy^2 + 1/h(r)dr^2 + r^2 d\omega_n^2$$

where $h(r)$ defines the position of the horizon. (For $d = n+3 = 10$ it is just $1 - M/r^6$). and $d\omega_n^2$ is the metric on a n-sphere.

BMN pp-wave:

$$g^o_{\mu\nu}dx^\mu dx^\nu = -2dudv + -\mu^2 r'^2 du^2 + dr'^2 + r'^2 d\Omega_n^2$$

We choose to match the metric on a $r = r' = const$ surface. The induced metric on the $d-1$ hypersurfaces are:

$$\gamma^i_{\mu\nu}dx^\mu dx^\nu = -h(r)dt^2 + dy^2 + r^2 d\Omega_n^2 \tag{B.1}$$

with the normal co-vector $n_\mu dx^\mu = 1/h^{1/2}(r)dr$ and

$$\gamma^o_{\mu\nu}dx^\mu dx^\nu = -2dudv + -\mu^2 r'^2 du^2 + r'^2 d\Omega_n^2 \tag{B.2}$$

with $n_\mu dx^\mu = dr$

The extrinsic curvatures are:

$$K^i_{\mu\nu}dx^\mu dx^\nu = -h^{1/2}h'(r)dt^2 + 0dy^2 + 2rh^{1/2}d\Omega_n^2 \tag{B.3}$$

$$K^o_{\mu\nu}dx^\mu dx^\nu = -2\mu^2 r' du^2 + 2r' d\Omega_n^2 \tag{B.4}$$

To glue the geometries together, we observe that on $r = const$, the induced metrics both inside and outside are just the 2-D Minkowski space cross a sphere. we make the following coordinate transformation on them.

(B.1):

$$T = h^{1/2}(r)t$$

$$Y = y$$

(B.2):

$$u = \frac{1}{\mu r'}(T - Y)$$

$$v = \mu r' Y$$

The induced metric on both sides became($r = r' = const$):

$$-dT^2 + dY^2 + r^2 d\Omega_n^2$$

There is a jump however in the extrinsic curvature for any finite r. This is most eeasily seen when apply the same coordinate transformation on (B.3), (B.4).

$$(B.3) \to -\frac{h'}{h^{1/2}}(r)dT^2 + 2h^{1/2}(r)d\Omega_n^2$$

$$(B.4) \to \frac{-2}{r^{1/2}}(dT^2 - 2dTdY + dY^2) + 2rd\Omega_n^2$$

According to the Israel junction condition [55], we can calculate the energy momentum tensor of domain wall as:

$$8\pi G_d S_{\mu\nu} = K^o_{\mu\nu} - K^i_{\mu\nu} - \gamma_{\mu\nu}(K^o - K^i)$$

where K^o and K^i are the trace of the respective extrinsic curvatures. The result for our case is:

$$8\pi G_d S_{\mu\nu} = [\frac{-2}{r^{1/2}} + \frac{2n}{r}(1 - h^{1/2})]dT^2 + \frac{4}{r^{1/2}}dTdY + [\frac{-2}{r^{1/2}} - \frac{2n(1 - h^{1/2})}{r} + \frac{h'}{h^{1/2}}]dY^2$$

$$+[2r(1 - h^{1/2}) - \frac{2n(1 - h^{1/2})}{r} + \frac{h'}{h^{1/2}}]d\Omega_n^2$$

In the inside coordinates:

$$8\pi G_d S^i_{\mu\nu} = h[\frac{-2}{r^{1/2}} + \frac{2n}{r}(1 - h^{1/2})]dt^2 + \frac{4h^{1/2}}{r^{1/2}}dtdy + [\frac{-2}{r^{1/2}} - \frac{2n(1 - h^{1/2})}{r} + \frac{h'}{h^{1/2}}]dy^2$$

$$+[2r(1 - h^{1/2}) - \frac{2n(1 - h^{1/2})}{r} + \frac{h'}{h^{1/2}}]d\Omega_n^2$$

In the outside coordinates:

$$8\pi G_d S^o_{\mu\nu} = (\mu^2 r^2)[\frac{-2}{r^{1/2}} + \frac{2n}{r}(1 - h^{1/2})]du^2 + \frac{4n}{r}(1 - h^{1/2})dudv + [\frac{1}{\mu^2 r^2}\frac{h'}{h^{1/2}}]dv^2$$

$$+[2r(1 - h^{1/2}) - \frac{2n(1 - h^{1/2})}{r} + \frac{h'}{h^{1/2}}]d\Omega_n^2$$

Unfortunately, we are not able to find suitable sources to account for the energy momentum of the domain wall. It would be more interesting if we were able to find consistent ways to break the symmetries so that we can match the plane wave geometry to a black hole.

Appendix C

References

1. Juan M. Maldacena, The Large N Limit of Superconformal Field Theories and Supergravity, Adv.Theor.Math.Phys. 2 (1998) 231-252 ; TASI 2003 Lectures on AdS/CFT, hep-th/0309246; O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Large N Field Theories, String Theory and Gravity, Phys.Rept. 323 (2000) 183-386

2. S.Hawking, D.Page, Thermodynamics Of Black Holes In Anti-de Sitter Space, Commun. Math. Phys. 87 (1983) 577

3. Edward Witten, Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories, Adv.Theor.Math.Phys. 2 (1998) 505-532

4. R. Penrose, Any Spacetime Has a Planewave as a Limit, in "Differential Geometry and Relativity pp 271-275, Reidel, Dordrecht, 1976

5. R.R.Metsaev, Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background, Nucl.Phys. B625 (2002) 70-96

6. David Berenstein, Juan Maldacena, Horatiu Nastase, Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills, JHEP 0204 (2002) 013

7. G. 't Hooft, A Planar Diagram Theories for Strong Interactions, Nucl. Phys. B72. (1974) 461

8. Donam Youm, Black Holes and Solitons in String Theory, Phys.Rept. 316 (1999) 1-232

9. C. Johnson, D-brane Primer, hep-th/0007170

10. Edward Witten, Anti De Sitter Space And Holography, Adv.Theor.Math.Phys. 2 (1998) 253-291

11. Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, Leonardo Rastelli, Correlation functions in the CFT(d)/AdS(d+1) correpondence, Nucl.Phys. B546 (1999) 96-118

12. J.K. Erickson, G.W. Semenoff, K. Zarembo, Wilson Loops in N=4 Supersymmetric Yang–Mills Theory, Nucl.Phys. B582 (2000) 155-175; Gordon W. Semenoff, Donovan Young, Wavy Wilson Line and AdS/CFT

13. S. W. Hawking, Gary. T. Horowitz, The Gravitational Hamiltonian, Action, Entropy, and Surface Terms, Class.Quant.Grav. 13 (1996) 1487-1498

14. Ofer Aharony, Joseph Marsano, Shiraz Minwalla, Kyriakos Papadodimas, Mark Van Raamsdonk,The Hagedorn/Deconfinement Phase Transition in Weakly Coupled Large N Gauge Theories, hep-th/0310285

15. R. Hagedorn, Statistical Thermodynamics of Strong Interactions at High Energies, Nuovo Cim. Suppl. 3 147 (1965)

16. J.J.Atick, E. Witten, The Hagedorn Transition And The Number Degrees Of Freedom Of String Theory, Nucl. Phys. B310, 291 (1988)

17. Juan M. Maldacena,Wilson loops in large N field theories, Phys.Rev.Lett. 80 (1998) 4859-4862

18. Matthias Blau, Jose Figueroa-O'Farrill, George Papadopoulos, Penrose limits, supergravity and brane dynamics, Class.Quant.Grav. 19 (2002) 4753

19. Gianluca Grignani, Marta Orselli, Gordon W. Semenoff, Diego Trancanelli, The superstring Hagedorn temperature in a pp-wave background, JHEP 0306 (2003) 006

20. Donald Marolf, Leopoldo A. Pando Zayas, On the Singularity Structure and Stability of Plane Waves, JHEP 0301 (2003) 076

21. David Berenstein, Horatiu Nastase, On lightcone string field theory from Super Yang-Mills and holography, hep-th/0205048

22. S.Hawking, G. Ellis, The large scale structure of spacetime, Cambridge: University Press, 1973

23. Donald Marolf, Simon F. Ross, Plane Waves: To infinity and beyond!, Class.Quant.Grav. 19 (2002) 6289-6302

24. Veronika E. Hubeny, Mukund Rangamani, Causal structures of pp-waves, JHEP 0212 (2002) 043

25. Veronika E Hubeny, Mukund Rangamani, No horizons in pp-waves, JHEP 0211 (2002) 021

26. James T. Liu, Leopoldo A. Pando Zayas, Diana Vaman, On Horizons and Plane Waves, Class.Quant.Grav. 20 (2003) 4343-4374

27. Birrell and Davies, Quantum fields in curved space, Cambridge; New York : Cambridge University Press, 1982.

28. W. Unruh, Notes on Black Hole Evaporation, Phys. Rev. D 14, 870V892 (1976)

29. Alexander Maloney, Andrew Strominger, Xi Yin, S-Brane Thermodynamics, JHEP 0310 (2003) 048

30. Marcus Spradlin, Andrew Strominger, Anastasia Volovich, Les Houches Lectures on De Sitter Space, hep-th/0110007

31. T. Jacobson,Introduction to Quantum Fields in Curved Spacetime and the Hawking Effect, gr-qc/0308048

32. R. Wald, Quantum field theory in curved spacetime and black hole thermodynamics, Chicago: University of Chicago Press, 1994.

33. R. Wald, General Relativity, Chicago: University of Chicago Press, 1984

34. T.Padmanabhan, Gravity and the Thermodynamics of Horizons, gr-qc/0311036

35. Ted Jacobson, Gungwon Kang, Conformal Invariance of Black Hole Temperature, Class.Quant.Grav. 10 (1993) L201-L206

36. R. Penrose, A Remarkable Property of Plane Waves In General Relativity, Rev. Mod. Phys, 37 215 (1965)

37. See for example, D. Brecher, J. P. Gregory, P. M. Saffin, String theory and the Classical Stability of Plane Waves, Phys.Rev. D67 (2003) 045014, and references therein.

38. Harald Dorn, Mario Salizzoni, Christoph Sieg, On the propagator of a scalar field on AdS x S and on the BMN plane wave, hep-th/0307229

39. Bateman Manuscript Project, Higher transcendental functions, Vol II, New York, McGraw-Hill, 1953-55

40. Samir D. Mathur, Ashish Saxena, Yogesh K. Srivastava, Scalar propagator in the pp-wave geometry obtained from $AdS_5 \times S^5$, Nucl.Phys. B640 (2002) 367-378

41. Jan Ivar Korsbakken, Jon Magne Leinaas, The Fulling-Unruh effect in general stationary accelerated frames, hep-th/0406080

42. See for example, T. Jacobson, Introduction to Black Hole Microscopy, hep-th/9510026, and references therein

43. Veronika E. Hubeny, Mukund Rangamani, Horizons and plane waves: A review, Mod.Phys.Lett. A18 (2003) 2699-2712

44. Gary T. Horowitz, Veronika E. Hubeny, Note on Small Black Holes in $AdS_p \times S^q$, JHEP 0006 (2000) 031

45. Eric G. Gimon, Akikazu Hashimoto, Veronika E. Hubeny, Oleg Lunin, Mukund Rangamani, Black strings in asymptotically plane wave geometries, JHEP 0308 (2003) 035

46. M. A. Melvin, Pure Magnetic and Electric Geons, Phys. Lett. 8 (1964) 65-70

47. L. Susskind, Some Speculations about Black Hole Entropy in String Theory, hep-th/9309145

48. Gary T. Horowitz, Joseph Polchinski, A Correspondence Principle for Black Holes and Strings, Phys.Rev. D55 (1997) 6189-6197

49. Ruth Gregory, Raymond Laflamme, The Instability of Charged Black Strings and p-Branes, Nucl.Phys. B428 (1994) 399-434

50. Akikazu Hashimoto, Leopoldo Pando Zayas, Correspondence Principle for Black Holes in Plane Waves, JHEP 0403 (2004) 014

51. J.G. Russo, A.A. Tseytlin, Exactly solvable string models of curved space-time backgrounds, Nucl.Phys. B449 (1995) 91-145

52. J. Polchinski. String Theory, Vol. 12, Cambridge, UK ; New York : Cambridge University Press, 1998. Michael B. Green, John H. Schwarz, Edward Witten, Superstring Theory, Vol. 12, Cambridge ; New York : Cambridge University Press, 1987.

53. S. Deser, Orit Levin, Accelerated Detectors and Temperature in (Anti) de Sitter Spaces, Class.Quant.Grav. 14 (1997) L163-L168

54. S. Deser, Orit Levin, Mapping Hawking into Unruh Thermal Properties, Phys.Rev. D59 (1999) 064004

55. See for example, Eric G. Gimon, Petr Horava, Over-Rotating Black Holes, Godel Holography and the Hypertube, hep-th/0405019