Investigation of the Force between Two Non-Commutative U(2) Monopoles

by

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Abstract

The force between two widely separated 't Hooft-Polyakov monopoles involves an ordinary Coulomb force as well as an attractive force with the same magnitude mediated by a scalar field. Manton arrived at this fact using an ansatz he discovered for a weakly accelerating monopole [1]. We study Manton's method, eliminate its ambiguities, interpret the ansatz as the external force law for a monopole, and compare it with another approach that uses the stress-energy tensor [2]. We find that the force between two monopoles in non-commutative spacetime does not alter from that in commutative spacetime to first order in the non-commutative parameter, θ, both by extending Manton's method and by finding the total energy of the system. We investigate Manton's method at $\mathcal{O}(\theta^2)$ but find it not very promising . We understand that the non-commutativity starts to affect dynamics only at $\mathcal{O}(\theta^2)$.

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Of course, the list does not stop here.

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Chapter 1

Introduction

Magnetic monopoles are interesting because they are solutions to all grand unified theories yet they have not been found in nature [3]. Non-commutative geometry is interesting because it seems to always come up when we look for quantum gravity solutions [4]. Non-commutative Yang-Mills theory, for instance, describes a limiting case of string theory [5] and is also the only gauge theory in which gauge transformations include translations and therefore can serve as a toy model for quantum gravity [6]. As well, it is the only known generalization of ordinary Yang-Mills theory that preserves maximal supersymmetry [7]. In this project, we were interested in magnetic monopoles in a non-commutative Yang-Mills theory. In particular, we wanted to find the force between two monopoles separated widely by a distance, s, in the U(2) perturbatively non-commutative gauge theory with a scalar field in flat space-time. We noticed that Manton has found the force to order $\frac{1}{s^2}$ between two monopoles in the SU(2) gauge theory in commutative space-time. He first discovered how the solution near one monopole changes under a weak acceleration, then uses the structure of the equations of motion to arrive at a solution for the region in between the two monopoles, and finally determines the force by equating the local and global solution where both are valid [1]. We were interested in extending this method for our problem.

We achieved the following:

1. We studied the Manton method in detail. We demonstrated the ambiguities of the method , and proposed the principle with which to find the correct solution. Using our understanding, we reinterpreted the method as the application of a constant external force law on either monopole. We then investigated the scope of the method by carrying out the method at dipole order and found that it works only for the lowest order force. We then studied Goldberg's way [2] to find the force between two monopoles using the stress-energy tensor on a static solution of the system. We found that its statement about the contributions to the force agrees with the force law above. We discussed the possibility of using Manton's global solution as the static solution to conclude that there is no higher order forces between two monopoles using Manton's global solution as the static solution.

2. For the non-commutative U(2) theory, we derived the analogue of Manton's ansatz for a single accelerating monopole. We showed using both the stress-energy tensor and the Manton method that the force between two non-commutative monopoles remains the same as that between two commutative monopoles to first order in the non-commutative paramter θ. We started to investigate the Manton method at second order in θ. We found that we can calculate the local accelerating monopole solution without difficulty with the help of the symmetry of the theory [8] [9], and showed a sample calculation. We studied how the non-commutativity hinders us from finding the global solutions in the same way Manton did.

This thesis was written in essentially the order described above. To make the report easier to follow, we chose to explain the background theories at different parts rather than all in the beginning. We included all calculations in the main text instead of appendices but made sure that before each long algebraic calculation a summary was given.

Chapter 2

Background: Single Monopole in Commutative Accelerated Yang-Mills Theory

Magnetic monopoles are classical solutions to field theories whose magnetic field far away from its center looks as though there is a magnetic charge at the center, that is,

$$B \rightarrow \frac{\hat{r}}{r^2} \text{ in the asymptotic region.}$$

They have not been detected in nature yet, but is predicted by all theories in which an internal symmetry group spontaneously breaks down to the U(1) group of Maxwell Electromagnetism [3]. In these grand unified theories, monopoles typically have such big masses that they are not likely to be produced by supernovae or current accelerators, but rather would have been produced copiously shortly after the Big Bang and would have hardly annihilated [10]. Their absence then is quite puzzling and should inform us about the very early universe. This is one of the main reasons why we study monopoles.

2.1 Monopoles in Maxwell Electromagnetism

Maxwell electromagnetism is built without magnetic charges. The divergence of the magnetic field being zero in this theory allows us to write it in the form of the curl of a vector potential field and to build the field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ for the Lagrangian formalism. The only way to build a "monopole" solution is to approximate a magnetic source by the end of a infinitely long and thin solenoid.

In this case, the vector potential would be undefined where the solenoid is. Explicitly, if we choose coordinates such that the solenoid is placed at the negative z-axis, then a vector potential (in spherical components) whose curl gives the monopole magnetic field would be

$$A^r = 0,\ A^\theta = 0,\ A^\phi = g\,(1 - \cos\theta)$$

where g is the apparent magnetic charge. We can see that on the negative z-axis, where $\theta = \pi$, the vector potential does not make sense since it points in all directions curling about the z-axis. This half-line where the vector potential is ill-defined is known as the Dirac String and is unavoidable however the vector potential is chosen.

The half-line singularily is only a mathematical defect if it cannot be detected by experiment. This is true classically but not quantum mechanically. If we perform a double slit diffraction experiment in which charged particles pass through two slits on a screen and are to be detected on a second screen, we would find that the interference pattern detected in the case where no solenoid is present in between the two different paths of the particles is different from the inference pattern when a solenoid is present [3]. This is because the probability density which determines the interference pattern is the square of the total wave function:

$$P = |\Psi_1 + \Psi_2|^2$$

where Ψ_1 and Ψ_2 are the wave functions of particles passing through the two different slits. The presence of a solenoid in between the two paths would contribute to a phase difference between Ψ_1 and Ψ_2, and the probability density would become:

$$P_{solenoid} = |\Psi_1 + e^{ie(4\pi g)}\Psi_2|^2$$

where e is the electric charge and $4\pi g$ is the magnetic flux through the solenoid. Dirac's statement is that the two interference patterns would be the same if the phase difference contributed by the solenoid is $2\pi ni$, which translates to the following relation between the electric and magnetic charge [11]:

$$g = \frac{N}{2e} \text{where N is an integer}$$

Other arguments would show that with the above relation, the solenoid could not be detected by any other conceivable experiments [3]. Therefore, the monopole is a genuine monopole which is not distinguishable experimentally from a monopole created by a single magnetic charge. The theory of monopole is thus started.

2.2 Monopoles in SU(2) Yang-Mills Theory

Monopoles takes a more elegant presence in Yang-Mills Theory, which is essentially a generalization of the classical field theory of electromagnetism with U(1) gauge symmetry to one with a larger gauge group SU(2). We will now see how U(1) electromagnetism can be embedded in this "bigger" theory, which can be seen as a prototype for grand unified theories, specifically, an SU(2) Yang-Mills theory with a scalar field in a Mexican-hat potential in four-dimenional Minkowski space-time, and how monopoles exist as non-singular classical solutions to it.

2.2.1 The Action and the Equations of Motion

The action of such theory looks like:

$$\begin{aligned} S &= \frac{1}{4}\int dx^4 \mathrm{Tr}\left[-G^{\mu\nu}(x)G_{\mu\nu}(x) + 2D^\mu\phi(x)D_\mu\phi(x) - \lambda\left(\phi(x)\phi(x) - c^2\right)\right] \\ \text{where } G^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu - ie[A^\mu, A\nu]; \qquad (2.1) \\ D^\mu\phi &= \partial^\mu\phi - ie[A^\mu, \phi] \qquad \text{for } \mu = 0,1,2,3 \qquad (2.2) \end{aligned}$$

where the fields are in the adjoint representation of SU(2), i.e., ϕ and A^μ are 2×2 Hermitian matrices and so can be written as linear combinations of the three Pauli matices:

$$\phi = \phi_a \frac{\sigma_\mathbf{a}}{2} \quad ; \quad A^j = A^j_a \frac{\sigma_\mathbf{a}}{2} \qquad \text{for } a = 1,2,3$$

Since the Pauli matrices satisfy the following identities:

$$\frac{\sigma_\mathbf{a}}{2}\frac{\sigma_\mathbf{b}}{2} = \frac{i}{2}\epsilon_{abc}\frac{\sigma_\mathbf{c}}{2} + \frac{1}{2}\delta_{ab}\frac{\mathbf{1}}{2} \quad ; \quad \mathrm{Tr}\frac{\sigma_\mathbf{c}}{2} = 0$$

we can treat them as the basis of the vector space $\mathbf{R^3}$ and represent the fields as vectors:

$$\phi = (\phi_1, \phi_2, \phi_3) \quad ; \quad \mathbf{A}^i = (A^i_1, A^i_2, A^i_3) \quad \text{where } i \text{ is the spatial index}$$

with the vector cross product corresponding to the commutator of the matrices and the dot product to the trace of products of matrices:

$$\begin{aligned} -i[A^\mu, \phi] &\mapsto \mathbf{A}^\mu \times \phi \\ Tr(G^{\mu\nu}G_{\mu\nu}) &\mapsto \mathbf{G}^{\mu\nu} \cdot \mathbf{G}_{\mu\nu} \end{aligned}$$

The action in this vector notation becomes

$$\begin{aligned} S &= \frac{1}{4}\int dx^4 \left[-\mathbf{G}^{\mu\nu}(x) \cdot \mathbf{G}_{\mu\nu}(x) + 2\mathbf{D}^\mu\phi(x) \cdot \mathbf{D}_\mu\phi(x) - \lambda\left(\phi(x) \cdot \phi(x) - c^2\right)\right] \\ \text{where } \mathbf{G}^{\mu\nu} &= \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu + e\mathbf{A}^\mu \times \mathbf{A}_\nu; \\ \mathbf{D}^\mu\phi &= \partial^\mu\phi + e\mathbf{A}^\mu \times \phi \end{aligned}$$

Varying the action with respect to the scalar field ϕ gives the first equation of motion:

$$\mathbf{D}^{\mu}\mathbf{D}_{\mu}\phi = -\lambda\left(\phi \cdot \phi - c^2\right)\phi \tag{2.3}$$

Varying with respect to the gauge field $\mathbf{A}^{\nu}$ gives the second:

$$\mathbf{D}_{\mu}\mathbf{G}^{\mu\nu} = -e\mathbf{D}^{\nu}\phi \times \phi \tag{2.4}$$

In this vector notation, the infinitesimal gauge transformations of the gauge field look like

$$\delta\mathbf{A}^{\mu} \;=\; \epsilon \times \mathbf{A}^{\mu} + \partial^{\mu}\epsilon$$

where ϵ is the infinitesimal gauge parameter, but those for the the scalar field and the field strength are simply infinitesimal rotations in $\mathbf{R}^3$

$$\begin{aligned} \delta\phi &= \epsilon \times \phi \\ \delta\mathbf{G}^{\mu\nu} &= \epsilon \times \mathbf{G}^{\mu\nu} \end{aligned}$$

Gauge invariant quantities are then invariants of this rotation, length of the vector fields which rotate in this internal $\mathbf{R}^3$ space under a gauge transformation.

2.2.2 The Asymptotic Condition and the Factorized Equations of Motion

We are looking for a finite energy configuration of fields that would give rise to some U(1) monopole magnetic field in some asymptotic region. Therefore, we need to impose some conditions on the fields such that the total energy is finite, define what it means to be in the asymptotic region, and also find a way to embed the U(1) electric and magnetic fields in this SU(2) theory such that the U(1) fields satisfy the Maxwell equations.

The Conditions at $r \to \infty$

The energy of a classical solution is given by the Hamiltonian which is related to the action in the usual way:

$$\mathcal{H} = \frac{1}{4}\int dx^4 \left[\mathbf{G}^{\mu\nu}(x)\cdot\mathbf{G}_{\mu\nu}(x) + 2\mathbf{D}^{\mu}\phi(x)\cdot\mathbf{D}_{\mu}\phi(x) + \lambda\left(\phi(x)\cdot\phi(x) - c^2\right)\right]$$

This is only finite if each term vanishes at infinity. Ignoring the first term for now, the last two terms vanishing at infinity implies the following boundary conditions for the scalar field and the gauge field:

$$\begin{aligned} |\phi| &\to c \text{ as r} \to \infty \\ |\mathbf{D}^{\mu}\phi| &\to 0 \text{ as r} \to \infty \end{aligned}$$

If we write ϕ as the product of its magnitude and a unit vector field (in SU(2) gauge space):

$$\phi = h(x)\hat{\phi}(x) \quad \text{where } |\hat{\phi}(x)|^2 = 1$$

the conditions above become conditions on $h(x)$ and $\hat{\phi}(x)$:

$$\begin{aligned} \mathbf{D}^{\mu}\hat{\phi} &= 0 \text{ as r} \to \infty \\ \partial^{\mu}h &= 0 \text{ as r} \to \infty \\ h &\to c \text{ as r} \to \infty \end{aligned}$$

The first two follow from the fact that $\mathbf{D}^{\mu}\phi$ can be separated into two perpendicular components and each needs to vanish:

$$\begin{aligned} \mathbf{D}^{\mu}\phi(x) &= (\partial^{\mu}h)\hat{\phi} + h\mathbf{D}^{\mu}\hat{\phi}(x) \\ &= (\partial^{\mu}h)\hat{\phi} + h\left(\partial^{\mu}\hat{\phi} + e\mathbf{A}^{\mu}\times\hat{\phi}\right) \\ \text{but } \partial^{\mu}(\hat{\phi}\cdot\hat{\phi}) &= \partial^{\mu}(1) = 0 \text{ implies } \hat{\phi}\cdot\partial^{\mu}\hat{\phi} = 0 \\ \text{and } (\mathbf{A}^{\mu}\times\hat{\phi})\cdot\hat{\phi} &= 0 \\ \text{therefore } \mathbf{D}^{\mu}\hat{\phi} &\perp \hat{\phi} \end{aligned}$$

The Asymptotic Condition

We want to define an asymptotic region between infinity and the core of the monopole where the above conditions may not all be satisfied but where the embedded U(1) magnetic field defined later on would satisfy the vacuum Maxwell equations. If we treat the right hand side of the equation of motion (Eq 2.4) as some "matter" current [2]

$$\mathbf{J}^\mu := -\mathbf{D}^\mu \phi \times \phi \tag{2.5}$$

then the matter current vanishes when

$$\mathbf{D}^\mu \hat{\phi} = 0 \tag{2.6}$$

This is the asymptotic condition that we will use in the next chapter [1]. Note that unlike at $r \to \infty$, $\partial^\mu h = 0$ is not imposed in the asymptotic region.

Factorization of Equations of Motion

We will now see how this condition gives rise to the definition of the U(1) field strength tensor that will define the magnetic field for the monopole.

The condition is true if the following relation between the gauge field and $\hat{\phi}$ is satisfied:

$$\mathbf{A}^\mu = \frac{1}{e}\partial^\mu \hat{\phi} \times \hat{\phi} + \lambda^\mu \hat{\phi}$$

because the second term of the covariant derivative of $\hat{\phi}$:

$$\mathbf{A}^\mu \times \hat{\phi} \;=\; (\frac{1}{e}\partial^\mu \hat{\phi} \times \hat{\phi} + \lambda^\mu \hat{\phi}) \times \hat{\phi} = (\partial^\mu \hat{\phi} \cdot \hat{\phi})\,\hat{\phi} - (\hat{\phi} \cdot \hat{\phi})\partial^\mu \hat{\phi} = -\partial^\mu \hat{\phi}$$

would cancel with its first term.

With this relation, the field strength tensor $\mathbf{G}^{\mu\nu}$ can be factorized into a unit vector field, $\hat{\phi}(x)$, and the magnitude of the field strength, a scalar in the gauge space, which also varies over space-time. Explicitly,

$$\begin{aligned}
\mathbf{G}^{\mu\nu} &= \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu + e\mathbf{A}^\mu \times \mathbf{A}_\nu; \\
\text{where } \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu &= \frac{2}{e}\partial^\nu \hat{\phi} \times \partial^\mu \hat{\phi} + (\partial^\mu \lambda^\nu - \partial^\nu \lambda^\mu)\,\hat{\phi} + (\lambda^\nu \partial^\mu \hat{\phi} - \lambda^\mu \partial^\nu \hat{\phi}) \\
\mathbf{A}^\mu \times \mathbf{A}^\nu &= \frac{1}{e}(\partial^\mu \hat{\phi} \times \hat{\phi}) \times (\partial^\nu \hat{\phi} \times \hat{\phi}) + (\partial^\mu \hat{\phi} \times \hat{\phi}) \times (\lambda^\nu \hat{\phi}) + (\lambda^\mu \hat{\phi}) \times (\partial^\nu \hat{\phi} \times \hat{\phi}) \\
&= [\partial^\mu \hat{\phi} \cdot (\partial^\nu \hat{\phi} \times \hat{\phi})]\,\hat{\phi} - [\lambda^\nu \partial^\mu \hat{\phi} - \lambda^\mu \partial^\nu \hat{\phi}] \\
&= \frac{1}{e}[(\partial^\mu \hat{\phi} \times \partial^\nu \hat{\phi}) \cdot \hat{\phi}]\,\hat{\phi} - [\lambda^\nu \partial^\mu \hat{\phi} - \lambda^\mu \partial^\nu \hat{\phi}]
\end{aligned}$$

But we already know that for any μ and ν

$$\partial^\mu \hat{\phi} \perp \hat{\phi} \text{ and } \partial^\nu \hat{\phi} \perp \hat{\phi} \quad \text{such that} \quad \partial^\mu \hat{\phi} \times \partial^\nu \hat{\phi} \parallel \hat{\phi}$$

which means

$$\frac{2}{e}\partial^\nu \hat{\phi} \times \partial^\mu \hat{\phi} = \frac{2}{e}[(\partial^\mu \hat{\phi} \times \partial^\nu \hat{\phi}) \cdot \hat{\phi}]\,\hat{\phi}.$$

Therefore, the SU(2) field strengh points in the direction of $\hat{\phi}$:

$$\mathbf{G}^{\mu\nu} = -\frac{1}{e}\left[(\partial^\mu \hat{\phi} \times \partial^\nu \hat{\phi}) \cdot \hat{\phi}\right]\,\hat{\phi} + [\partial^\mu \lambda^\nu - \partial^\nu \lambda^\mu]\,\hat{\phi}$$

We define $f^{\mu\nu}(x)$ to be the length of the SU(2) vector $\mathbf{G}^{\mu\nu}$ [1]:

$$f^{\mu\nu} = \mathbf{G}^{\mu\nu} \cdot \hat{\phi} = -\frac{1}{e}\left[(\partial^\mu \hat{\phi} \times \partial^\nu \hat{\phi}) \cdot \hat{\phi} + \partial^\mu \lambda^\nu - \partial^\nu \lambda^\mu\right]$$

and note that it is a gauge invariant quantity.

Now, the equation of motion and the Bianchi Identity for $\mathbf{G}^{\mu\nu}$ imply the free Maxwell equations for $f^{\mu\nu}$:

1. Equation of motion:

$$\begin{aligned}
\mathbf{D}_\mu \mathbf{G}^{\mu\nu} &= -e\mathbf{D}^\nu \phi \times \phi \\
\Longrightarrow (\partial_\mu f^{\mu\nu})\,\hat{\phi} + f^{\mu\nu}\mathbf{D}^\mu \hat{\phi} &= -e\left((\partial^\nu h)\,\hat{\phi} + h\mathbf{D}^\mu \hat{\phi}\right) \times h\hat{\phi} \\
\Longrightarrow \partial_\mu f^{\mu\nu} &= 0
\end{aligned} \tag{2.7}$$

2. Bianchi Identity:

$$\mathbf{D}_\mu \epsilon^{\mu\nu\alpha\beta} \mathbf{G}_{\alpha\beta} = 0 \implies \partial_\mu \epsilon^{\mu\nu\alpha\beta} f_{\alpha\beta} = 0 \tag{2.8}$$

We have embedded U(1) electromagnetism in this theory with $f^{\mu\nu}$ being the Maxwell field strength tensor. If $\hat{\phi}$ points in only one direction, for instance $\hat{\phi} = (0, 0, h)$, then the field strength takes the usual form:

$$f^{\mu\nu} = \partial^\mu \lambda^\nu - \partial^\nu \lambda^\mu$$

Unlike ordinary electromagnetism, the field strength contains also the term involving $\hat{\phi}$, and this will allow the monopole solution to be non-singular by giving rise to a topological charge. We will discuss this in section 2.2.4

Notice that the equations for $f^{\mu\nu}$ are decoupled from h. It is the other equation of motion Eq 2.3 that factorizes in the asymptotic region to give the equation of motion for h:

$$\begin{aligned}
\mathbf{D}_\mu \mathbf{D}^\mu \phi &= -\lambda(|\phi|^2 - c^2)\hat{\phi} \\
&= \mathbf{D}_\mu \Big((\partial^\mu h)\hat{\phi} + h(\partial^\mu \hat{\phi} + \mathbf{A}^\mu \times \hat{\phi}) \Big) \\
&= \mathbf{D}_\mu (\partial^\mu h \, \hat{\phi} + h \mathbf{D}_\mu \hat{\phi}) \\
&= (\partial_\mu \partial^\mu h) \;\; \hat{\phi} + (\partial^\mu h) \mathbf{D}_\mu \hat{\phi} \\
&= (\partial_\mu \partial^\mu h) \hat{\phi} \\
\implies \partial_\mu \partial^\mu h &= -\lambda(h^2 - c^2)
\end{aligned} \tag{2.9}$$

Although h and $f^{\mu\nu}$ seem to be independent of each other in the asymptotic region, they are not at the core of a monopole where the equations of motion are not decoupled. The relation they need to satisfy in the core is given through a first order ansatz in section 3.1.1.

Another important property of these asymptotic equations is their linearity in the U(1) fields $f^{\mu\nu}$ and h. In the next chapter (section 3.2.1), we will explain how Manton relies on this fact to find the solutions for the region between two monopoles.

2.2.3 Monopole Solution–both charges

Now that we have a U(1) field strength that satisfies the Maxwell equations, we can define a U(1) magnetic field in the usual way:

$$B^i = \frac{1}{2}\epsilon^{ijk} f^{jk}(\hat{\phi}, \lambda^l)$$

Since f^{jk} can be written in terms of only $\hat{\phi}$ and λ^l without involving h, restricting B^i to the monopole drop-off in the asymptotic region gives a condition for $\hat{\phi}$ and λ^l decoupled from h. Since the monopole magnetic field satisfies the free Maxwell equations which come from the equation of motion and the Bianchi Identity, any $\hat{\phi}$ and λ^l that produces the monopole field is automatically a solution to the equations of motion.

Since the SU(2) scalar h is related to the magnetic field B^i through the first order ansatz mentioned above which can be evaluated in the asymptotic reion also, the asymptotic profile of B^i in fact gives a condition on the scalar h. We will discuss this in section 3.1.1.

To solve for $\hat{\phi}$, we first try to find a relationship between the direction $\hat{\phi}$ is pointing at and the magnetic field B^i. It turns out that there is a solution for the choice $\lambda^l = 0$ (which Manton referred to as a gauge choice but is incorrect). For each gauge index (d=1,2 or 3), the gradient in real space of $\hat{\phi}_d$ is perpendicular to the magnetic field B^i: (Below we write the gauge indices explicitly as subscripts.)

$$\begin{aligned}
\left(\partial^i\hat{\phi}_d\right)\ B^i &= \epsilon^{ijk}\,\partial^i\hat{\phi}_d\,\left(-\frac{1}{2e}\epsilon_{abc}\partial^j\hat{\phi}_b\,\partial^k\hat{\phi}_c\,\hat{\phi}_a\right) \\
\text{but}\quad \epsilon^{ijk}\,\partial^i\hat{\phi}_d\,\partial^j\hat{\phi}_b\,\partial^k\hat{\phi}_c &= \begin{cases} 0 & \text{for } d=b,\ b=c \text{ or } c=d, \text{ by antisymmetry of } \epsilon^{ijk} \\ \pm\ \partial^i\hat{\phi}\cdot\left(\partial^j\hat{\phi}\times\partial^k\hat{\phi}\right) & \text{for } d\neq b\neq c \end{cases} \\
\text{and}\qquad & \left(\partial^j\hat{\phi}\times\partial^k\hat{\phi}\right)\parallel\hat{\phi}\ ,\ \ \partial^i\hat{\phi}\perp\hat{\phi}\ \text{ imply }\ \partial^i\hat{\phi}\cdot\left(\partial^j\hat{\phi}\times\partial^k\hat{\phi}\right)=0 \\
\text{therefore}\quad \left(\partial^i\hat{\phi}_d\right)\ B^i &= 0
\end{aligned} \tag{2.10}$$

This means that all the components of $\hat{\phi}$ is constant along the field lines of B. For a single monopole then, $\hat{\phi}$ is constant along the radial direction and so depends only on the spherical coordinate angles, θ and χ, where θ is the angle a vector makes with the +z-axis and χ is the azimuthal angle. The solution for a single charge monopole can be very simple in some fixed gauge:

$$\hat{\phi}_a = \pm\,\delta^i_a\,\frac{x^i}{r}$$

and we can check that its magnetic field is indeed the monopole field:

$$\begin{aligned}
B^i &= -\frac{1}{2e}\epsilon^{ijk}\epsilon_{abc}\partial^j\left(\pm\frac{x_b}{r}\right)\ \partial^k\left(\pm\frac{x_c}{r}\right)\ \left(\pm\frac{x_a}{r}\right) \\
&= -\frac{1}{2e}\epsilon^{ijk}\epsilon_{abc}\left(\frac{\delta^j_b}{r}-\frac{x_b x^j}{r^3}\right)\ \left(\frac{\delta^k_c}{r}-\frac{x_c x^k}{r^3}\right)\ \left(\pm\frac{x_a}{r}\right) \\
&= -\frac{1}{2e}\epsilon^{ijk}\epsilon_a{}^{jk}\left(\pm\frac{x_a}{r^3}\right) \\
&= \mp\frac{x^i}{er^3} \qquad \text{for } \hat{\phi}=\pm\hat{r}
\end{aligned}$$

Accordingly, the gauge field is:

$$\begin{aligned}
\mathbf{A}^i_a &= \frac{1}{e}\epsilon_{abc}\partial^i\hat{\phi}_b\hat{\phi}_c \\
&= \frac{1}{e}\epsilon_{abc}\left((\pm\,\delta^i_b\,\frac{1}{r})\,(\pm\frac{x_c}{r})-(\pm\frac{x_b x^i}{r^3})\,(\pm\frac{x_c}{r})\right) \\
&= \frac{1}{e}\epsilon_a{}^i{}_c\,\frac{x_c}{r^2} \qquad \text{for } \hat{\phi}=\pm\hat{r}
\end{aligned}$$

Now, for the monopole with a single negative charge, there is another solution. Instead of reflecting the solution for the positive charge monopole ($\hat{\phi}_\oplus = -\hat{r}$) about the origin in the SU(2) space (such that $\hat{\phi}_\ominus = +\hat{r}$), we can reflect $\hat{\phi}_\oplus$ about only one plane to obtain $\hat{\phi}_\ominus$; for instance,

$$\hat{\phi}_\ominus = \begin{pmatrix} \hat{\phi}_{\oplus 1} \\ -\hat{\phi}_{\oplus 2} \\ \hat{\phi}_{\oplus 3} \end{pmatrix}$$

Since $\hat{\phi}_\ominus$ has only one component with a relative minus sign, B^i, given by Eq 5.1, would be negative. This is the solution that comes about when we generalize the solution to multiple charges.

2.2.4 Topological Nature and Quantization of Charge

We now try to generalize the above solution to higher charge single monopoles.

First, the divergence of B does not depend on the term with λ^k, and is actually zero everywhere except at the origin:

$$\begin{aligned} \partial^i\, B^i &= \epsilon^{ijk}\left(-\frac{1}{2e}\epsilon_{abc}\partial^j\hat{\phi}_b\,\partial^k\hat{\phi}_c\,\partial^i\hat{\phi}_a + 2\partial^i\partial^j a^k\right) \\ &= \epsilon^{ijk}\left(-\frac{1}{2e}\epsilon_{abc}\partial^j\hat{\phi}_b\,\partial^k\hat{\phi}_c\,\partial^i\hat{\phi}_a\right) \\ &= 0 \qquad \text{by similar arguments as in Eq 2.10} \end{aligned}$$

By the divergence theorem, then, the magnetic flux through a surface enclosing the monopole core depends also only on $\hat{\phi}$. Now, for a monopole with a single positive charge, $\hat{\phi}$ maps the 2-sphere in physical space, which is parametrized by θ and χ, to a 2-sphere in gauge space once. Since $\hat{\phi}$ depends only on the angles and not on r, we can generalize to higher charges by choosing one angle, for instance χ, and defining $\hat{\phi}$ such that it has mapped a section of the 2-sphere in real space (described by $\chi = 0$ to some χ_0) to an entire 2-sphere in gauge space before χ

reaches 2π, such that

$$\hat{\phi} = (\sin\theta\cos N\chi, \sin\theta\cos N\chi, \cos\theta).$$

For $\chi > 2\pi/N$, the mapping of $\hat{\phi}$ to the 2-sphere starts anew. Now, for $\hat{\phi}$ to be single-valued, the values of $\hat{\phi}$ at $\chi = 0$ and $\chi = 2\pi$ have to be the same, and so N needs to be an integer, whether positive or negative. We can then check that this $\hat{\phi}$ gives the magnetic flux for N integral magnetic charges:

$$\int_{Surface} B^i da^i = \frac{4\pi N}{e}$$

Here, the Dirac quantization condition, $g \propto \frac{N}{e}$, is recovered. N is called the winding number or the topological charge of the solution.

Notice that the negatively charged monopoles are $\hat{\phi}$ with $-N$. This means that looking from above the x-y plane, if the vector $\hat{\phi}_{\oplus}$ (at all z values) rotates counterclockwise as χ increases, the vector $\hat{\phi}_{\ominus}$ (at all z values) would rotate clockwise as χ increases.

An important point is that there is no smooth gauge transformation that would take the solution from one N to another. Two solutions with different N's are said to be in different homotopy sectors. It is because the magnetic charge in this theory occurs as a topological charge that the situation of the Dirac string in the Maxwell theory can be avoided.

2.2.5 Solution at the Core of the Monopole and the BPS Limit

We have only looked at the solution in the asymptotic region so far, since it already captures the most important aspects of the monopole solution and is what will be discussed in the rest of this report.

The explicit solution at the core of a single charge monopole was found by Bogonolmy, Prasad and Sommerfield [12] [13] in what is known as the BPS limit following 't Hooft and Polyakov's ansatz [14] [15]. The outline is as follow.

Instead of using an asymptotic condition such that the SU(2) field strength tensor $\mathbf{G}^{\mu\nu}$ would factor under it, 't Hooft defined the U(1) electromagnetic field strength everywhere as the gauge-invariant expression [16]:

$$f^{\mu\nu}_{tHooft} = -\frac{1}{e}\left[(\mathbf{D}^{\mu}\hat{\phi}\times\mathbf{D}^{\nu}\hat{\phi})\cdot\hat{\phi}\right]+\mathbf{G}^{\mu\nu}\cdot\hat{\phi}$$

While in the asymptotic region, the first term vanishes because $\mathbf{D}^{\mu}\hat{\phi}=0$ and the second term factors as before, in the core, after being expanded, this U(1) field tensor still takes the form we had before in the asymptotic region:

$$f^{\mu\nu}_{tHooft} = -\frac{1}{e}\left[(\partial^{\mu}\hat{\phi}\times\partial^{\nu}\hat{\phi})\cdot\hat{\phi}+\partial^{\mu}\lambda^{\nu}-\partial^{\nu}\lambda^{\mu}\right]$$

with $\lambda^{\mu}=\mathbf{A}^{\mu}\cdot\hat{\phi}$.

Now, without the asymptotic condition, the Bianchi Identity of $\mathbf{G}^{\mu\nu}$ does not imply the Bianchi Identity for $f^{\mu\nu}_{tHooft}$ but rather

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^{\nu}f^{\rho\sigma}_{tHooft} = \epsilon_{\mu\nu\rho\sigma}\left(-\frac{1}{2e}\epsilon_{abc}\partial^{\nu}\hat{\phi}_b\,\partial^{\rho}\hat{\phi}_c\,\partial^{\sigma}\hat{\phi}_a\right)$$

Fortunately, the right hand side is identically zero because of its form as argued in Eq 2.10 and would integrate to a non-zero magnetic flux over a surface enclosing the monopole. The other two Maxwell equations

$$\partial_{\mu}f^{\mu\nu}_{tHooft} = 0$$

follow from the equation of motion of $\mathbf{G}^{\mu\nu}$.

't Hooft and Polyakov proposed an ansatz such that $f^{\mu\nu}_{'tHooft}$ would give a single charge monopole magnetic field far away from the monopole core:

$$\phi_a = \frac{x^a}{r}\,h(r) \tag{2.11}$$

$$A^j_a = \epsilon_{ajp}\frac{x^p}{r}\,W(r) \tag{2.12}$$

where h(r) and W(r) satisfy the following boundary conditions:

1. $h(r) \to c$ as $r \to \infty$ for the energy reason discussed before;
2. $W(r) \to \frac{1}{er}$ as $r \to \infty$ such that $\mathbf{A^i} = (1/e)\, \partial^i \hat{\phi} \times \hat{\phi}$ as $r \to \infty$, which is the asymptotic relationship between the gauge field and $\hat{\phi}$ derived before.

In the asymptotic region, the 't Hooft-Polyakov solution would reduce to the solution discussed in the previous sections.

The BPS Limit For the equations of motion to be satisfied, h(r) and W(r) need to satisfy a set of coupled "non-autonomous" differential equations [16] and has been solved only in the limit where the amplitude of the Mexican-hat potential $[\lambda(|\phi|^2 - c^2)]$ goes to zero, i.e., $\lambda \to 0$, while the value of $|\phi|$ at which this potential is minimum is retained such that $|\phi|$ still needs to approach c at infinity. This is known as the BPS limit [12]. The full solution of the single charge monopole in this limit is the ansatz in Eq 2.11 and Eq 2.12 with h and W solved:

$$\begin{aligned} h(r) &= \frac{c}{\tanh(c\,e\,r)} - \frac{1}{e\,r} \\ W(r) &= \frac{1}{e\,r} - \frac{c}{\sinh(c\,e\,r)} \end{aligned}$$

Note that these functions are smooth and do not diverge at the origin.

Chapter 3

Manton's Method to Find Force Between Two Commutative Monopoles

In electromagnetism, the force acting on an electrically charged particle by the electric and magnetic field is given by the Lorentz force law

$$\vec{F} \;=\; q(\vec{E} + \vec{v} \times \vec{B})$$

which is the equation of motion derived from varying the particle action

$$S_{particle} \;=\; \int \left(-\frac{1}{4} f^{\mu\nu} f_{\mu\nu} - J_\mu A^\mu \right) dx^4 + \int \sqrt{dx^\mu dx_\mu}$$

where J_μ is a conserved current, i.e., $\partial_\mu J^\mu = 0$, with its time component being the electric charge density and the spatial components being the electric current densities flowing in the three different spatial directions.

The magnetic monopole, on the other hand, is not a point particle but field configurations that extend over space, and there is no separate "particle" action for its dynamics. How do we find the force acting on it then? How do we find the force between two opposite charge monopoles and that between two same charge monopoles?

Canonically, we can find the force on an enclosed region, in which a monopole can situate, by calculating the momentum flux through its boundary surface using the stress-energy tensor. However, Manton arrived at the correct answers by his own method, and we outline it below:

Suppose two monopoles with same or opposite charges separated by a large distance, s, accelerate with a small acceleration, $\epsilon^2\vec{a}$, from rest due to the force each experiences. For this instant, Manton assumes the fields of the monopoles to be rigidly accelerating in opposite directions, and using this assumption simplifies the time-dependent equations of motion to equations that involve only spatial derivatives and terms with $\epsilon^2\vec{a}$. He then discovers a first order ansatz for each of the different charge monopoles to solve the modified equations up to $\mathcal{O}(\epsilon^2)$. Now, in the asymptotic region defined by Eq 2.6, the ansatz and its derivative for each monopole become equations for h, and $\hat{\phi}$; recall that h, and $\hat{\phi}$ determines the full solution since $\phi = h\hat{\phi}$ and $A^i = \partial^i\hat{\phi} \times \hat{\phi}$ after having chosen $\lambda^i = 0$. Manton then cleverly chooses a gauge in which the ansatzes are linear in terms one of the components of $\hat{\phi}$, Ψ, and in which $\hat{\phi}$ for different charge monopoles have the same dependence on Ψ such that a solution for Ψ in the region between two monopoles would imply a solution for $\hat{\phi}$ in the same region as well. He solves for h and Ψ for each monopole with its own charges and direction of acceleration for both $\mathcal{O}(\epsilon^0)$ and $\mathcal{O}(\epsilon^2)$. Finally, he uses the linearity in h and Ψ of the ansatz and builds the "global" solutions, h_{global}, and Ψ_{global}, for the region between the monopoles by adding the h and Ψ functions from the different monopoles, but with the freedom of adding any homogeneous solutions. He requires that to $\mathcal{O}(\epsilon^2)$, these global functions reduce to the solutions for each ansatz at each monopole, and determines the assumed acceleration $\epsilon^2\vec{a}$ in the matching process.

In this chapter, we study Manton's method in details, correct and clarify a few of his statements. We interpret the method as the application of an external force law on each monopole and find the limitations of this force law. We also study the canonical method mentioned above and look at statements made in the Manton method from that viewpoint.

3.1 Manton's Ansatz for a Single Accelerating Monopole

3.1.1 Derivation of the Accelerated Equation of Motion and Manton's First Order Ansatz

Manton starts by deriving the modification to the static equations of motion for the instant, $t = 0$, when a monopole starts to move. He assumes that the monopole accelerates "a little bit" rigidly from rest such that the scalar field and the gauge field only have time-dependence in terms of the Taylor-expanded spatial coordinates:

$$\phi(x^\nu) = \phi\left(x^i - \frac{1}{2}\left(\epsilon^2 a^i\right) t^2\right) \quad ; \quad \mathbf{A}^j(x^\nu) = \mathbf{A}^j\left(x^i - \frac{1}{2}\left(\epsilon^2 a^i\right) t^2\right)$$

where $\epsilon^2 a^i$ is the small acceleration. The time derivatives of these fields become non-zero:

$$\partial^0\phi - \epsilon^2 a^i t\, \partial^i\phi \quad ; \quad \partial^0\mathbf{A}^j = -\epsilon^2 a^i t\, \partial^i\mathbf{A}^j \tag{3.1}$$

Manton also makes the other term in the covariant time derivative depend on time in the same way. To accomplish that, he chooses a gauge in which $\mathbf{A}^0 = 0$ in the instantaneous rest frame of monopole such that at a small time t, $\mathbf{A}^0$ in the non-accelerated lab frame would be obtained by a Lorentz boost with the relative velocity $\vec{v} = -\epsilon^2 \vec{a} t$:

$$\mathbf{A}^0 = -\epsilon^2 a^i t \mathbf{A}^i$$

Combining the two terms, he writes the covariant time derivative of ϕ and of $\mathbf{G}^{j0}$ in terms of the covariant spatial derivatives:

$$\begin{aligned} \mathbf{D}^0 \phi = && -\epsilon^2 a^i t \ (\partial^i \phi + e\mathbf{A}^i \times \phi) && = -\epsilon^2 a^i t \, \mathbf{D}^i \phi \\ \mathbf{G}^{j0} = && -\epsilon^2 a^i t \ (\partial^j \mathbf{A}^i - \partial^i \mathbf{A}^j + e\mathbf{A}^j \times \mathbf{A}^i) && = -\epsilon^2 a^i t \ \mathbf{G}^{ji} \end{aligned}$$

Then, using these, he manipulates the equations of motion. He applies another covariant time derivative on these quantities, but keeps terms up to only $O(\epsilon^2)$:

$$\mathbf{D}_0 \mathbf{D}^0 \phi = \epsilon^2 a^i (\mathbf{D}^i \phi) + (\epsilon^2 a^j t \mathbf{A}^j) \times (-\epsilon^2 a^i t \mathbf{D}^i \phi) = \epsilon^2 a^i (\mathbf{D}^i \phi) + \mathcal{O}(\epsilon^4) \quad (3.3)$$

$$\mathbf{D}_0 \mathbf{G}^{0j} = -\epsilon^2 a^i \mathbf{G}^{ji} + (\epsilon^2 a^j t \mathbf{A}^j) \times (-\epsilon^2 a^i t \mathbf{G}^{ij}) = -\epsilon^2 a^i \mathbf{G}^{ji} + \mathcal{O}(\epsilon^4) \quad (3.4)$$

and he substitutes these in the equations of motion. The one equation of motion involving only ϕ, in terms of the acceleration $\epsilon^2 a^i$, looks like

$$\begin{aligned} \mathbf{D}_\mu \mathbf{D}^\mu \phi &= \mathbf{D}_i \mathbf{D}^i \phi + \mathbf{D}_0 \mathbf{D}^0 \phi \\ = \quad \mathbf{D}_i (\mathbf{D}^i + \epsilon^2 a^i) \phi &= \lambda \left(|\phi|^2 - c^2 \right) \end{aligned} \quad (3.5)$$

The time component of the other equation of motion becomes

$$\begin{aligned} \mathbf{D}_j \mathbf{G}^{j0} &= -e\mathbf{D}^0 \phi \times \phi \\ \Longrightarrow \quad -\epsilon^2 a^i t \, \mathbf{D}_j \mathbf{G}^{ji} &= \frac{1}{e} \epsilon^2 a^i t \, \mathbf{D}^j \phi \times \phi \end{aligned} \quad (3.6)$$

Factoring out $\epsilon^2 a^i t$, this equation reduces to simply one of the static equations of motion and is to be satisfied by the $\mathcal{O}(\epsilon^0)$ solution.

The spatial component of this second equation of motion can be written with the covariant time derivative replaced with terms with the acceleration as well:

$$\mathbf{D}_i \mathbf{G}^{ij} + \mathbf{D}_0 \mathbf{G}^{0j} = (\mathbf{D}^i + \epsilon^2 a^i) \mathbf{G}^{ij} = -e\mathbf{D}^j \phi \times \phi \quad (3.7)$$

Note that

1. to $\mathcal{O}(\epsilon^0)$, all three equations, Eq 3.5, 3.7, and 3.6, reduce to the static equations of motion, so the $\mathcal{O}(\epsilon^0)$ solution is also also the static monopole solution;

2. there are no explicit time derivatives in all three equations, and since the spatial derivatives of the fields with the accelerated coordinate dependence equal those of the fields with the static coordinate dependence, $\partial^i\left(\phi(\vec{x}+1/2\epsilon^2\vec{a}t^2)\right) = \partial^i(\phi(\vec{x}))$, changing the argument of the fields from the spatial coordinates to the accelerated coordinates is consistent with these equations. At $t=0$, the accelerated and the non-accelerated coordinate are the same, $(\vec{x}+1/2\epsilon^2\vec{a}t^2) = \vec{x}$, so we can now look for the solutions that have x as the argument.

The First Order Ansatz

Manton discovers a first order ansatz that solves the "perturbed" equations of motion(Eq 3.5, 3.7) in the BPS limit (section 2.2.5):

$$\mathbf{G}^{ij} = \pm\epsilon^{ijk}(\mathbf{D}^k + \epsilon^2 a^k)\phi \tag{3.8}$$

where the different signs correspond to the different charge of the monopoles, as explained below.

We check that it indeed solves the perturbed equations of motion. First, we substitute the ansatz in Eq. 3.5:

$$\begin{aligned}\mathbf{D}^i(\mathbf{D}^i\phi) + \epsilon^2 a^i(\mathbf{D}^i\phi) &= \mathbf{D}^i(\pm\frac{1}{2}\epsilon^{ijk}\mathbf{G}^{jk} - \epsilon^2 a^i\phi) + \epsilon^2 a^i(\mathbf{D}^i\phi)\\ &= \pm\frac{1}{2}\epsilon^{ijk}\mathbf{D}^i\mathbf{G}^{jk} = 0\end{aligned}$$

and see that it is satisfied using the Bianchi Identity. Second, we substitute the ansatz in Eq. 3.7, and find that this second equation is also satisfied:

$$\begin{aligned}
(\mathbf{D}^i + \epsilon^2 a^i)\,\mathbf{G}^{ij} &= \mathbf{D}^i[\pm\epsilon^{ijk}(\mathbf{D}^k + \epsilon^2 a^k)\phi] + \epsilon^2 a^i(\pm\epsilon^{ijk}\mathbf{D}^k\phi) + \mathcal{O}(\epsilon^4) \\
&= \pm\epsilon^{ijk}\mathbf{D}^i\mathbf{D}^k\phi + \mathcal{O}(\epsilon^4) \\
&= \pm\epsilon^{ijk}\left[\partial^i\partial^k\phi + e\mathbf{A}^i \times \partial^k\phi + \partial^i\,(e\mathbf{A}^k \times \phi) + e^2\mathbf{A}^i \times (\mathbf{A}^k \times \phi)\right] \\
&= \pm\epsilon^{ijk}\left[e(\partial^i\,\mathbf{A}^k) \times \phi) + e^2\mathbf{A}^k(\mathbf{A}^i \cdot \phi)\right] \\
\text{Now}\quad \frac{1}{2}\epsilon^{ijk}\mathbf{G}^{ik} \times \phi &= \epsilon^{ijk}(\partial^i\mathbf{A}^k) \times \phi + \frac{1}{2}\epsilon^{ijk}(e\mathbf{A}^i \times \mathbf{A}^k) \times \phi \\
&= \epsilon^{ijk}\left[(\partial^i\mathbf{A}^k) \times \phi + e\mathbf{A}^k(\mathbf{A}^i \cdot \phi)\right] \\
\text{so}\quad (\mathbf{D}^i + \epsilon^2 a^i)\mathbf{G}^{ij} &= \pm\frac{1}{2}\epsilon^{ijk}e\mathbf{G}^{ik} \times \phi \\
&= \pm\left[\mp e(\mathbf{D}^j + \epsilon^2 a^j)\phi \times \phi\right] = -e\mathbf{D}^j\phi \times \phi
\end{aligned}$$

The First Order Ansatz in the Asymptotic Region

Recall from the last chapter that we write $\phi = h\hat{\phi}$ and that the asymptotic condition $\mathbf{D}^k\hat{\phi} = 0$ allows the gauge field to be determined by $\hat{\phi}$ only if $\lambda^i = 0$, so solving for h and $\hat{\phi}$ will give the complete solution in the asymptotic region.

Recall also that in this region, the U(1) magnetic field $\vec{B}$ is given in terms of $\hat{\phi}$ through the asymptotic condition, and the monopole requirement that $\vec{B} \to \hat{r}/r^2$ has sufficed to give $\hat{\phi}$ to $\mathcal{O}(\epsilon^0)$. Manton's ansatz then provides the relation between $\vec{B}$ and h up to $\mathcal{O}(\epsilon^2)$, and so allows us to first solve for the $\mathcal{O}(\epsilon^0)$ h which is part of the static solution. It also allows us to solve for the $\mathcal{O}(\epsilon^2)$ corrections to both $\hat{\phi}$ and h for the accelerating monopole.

Manton's ansatz factorizes and reduces to the following relation between $\vec{B}$ and h in the asymptotic region:

$$\begin{aligned}
B^k\hat{\phi} = \frac{1}{2}\epsilon^{ijk}\mathbf{G}^{ij}\hat{\phi} &= \pm\left[(\partial^k h)\hat{\phi} + h\mathbf{D}^k\hat{\phi} + \epsilon^2 a^k h\hat{\phi}\right] \\
\Longrightarrow\quad B^k &= \pm\left(\partial^k h + \epsilon^2 a^k h\right) \quad \text{asymptotically} \qquad (3.9)
\end{aligned}$$

As expected, this ansatz is consistent with the original equation of motion (Eq 2.9):

$$\partial_\mu \partial^\mu h \quad = \quad \partial_k(\partial^k h + \epsilon^2 a^k h) = \partial_k B^k = 0$$

Now, the ansatz needs different relative signs for the opposite charge monopoles. This is because if oppositely charged monopoles are described by the same ansatz, h will switch sign as the charge and therefore B^k is switched; but we already know that under the charge inversion, one of the components of $\hat{\phi}$ also switches sign; this means that both $\oplus$ and $\ominus$ monopoles will have the same solution $\phi = h\hat{\phi}$ if both satisfy the same ansatz.

To avoid this ambiguity, we need the different ansatzes for different charge monopoles. We choose the sign convention that $B^k = +(\partial^k h + \epsilon^2 a^k h)$ for the $\oplus$ monopole. Consequently, h for both $\oplus$ and $\ominus$ monopoles are the same to $\mathcal{O}(\epsilon^0)$:

$$\begin{aligned} h^{(0)} \quad &= \quad -\frac{1}{r} + c \\ \text{such that} \quad \pm\, \partial^k h^{(0)} \quad &= \quad \pm \frac{x^k}{r^3} = B^{k(0)}_{\oplus/\ominus} \;\; \text{and} \;\; h^{(0)} \to c \;\; \text{as} \;\; r \to \infty \end{aligned}$$

Secondly, we want to derive the equation for $\hat{\phi}$ for the accelerating monopole which preferably would not depend on the unknown $h^{(\epsilon^2)}$. Expanding Eq 3.9 in orders of ϵ:

$$B^{k(0)}(\hat{\phi}) + B^{k(\epsilon^2)}(\hat{\phi}) \quad = \quad \pm \left[\partial^k(h^{(0)} + h^{(\epsilon^2)}) + \epsilon^2 a^k h^{(0)}\right]$$

we see that applying the curl operator to both sides would get rid of the term with $h^{(\epsilon^2)}$:

$$\epsilon^{ijk}\partial^j \left(B^{k(0)} + B^{k(\epsilon^2)}\right) \quad = \quad \pm\epsilon^{ijk}\left[\partial^j\partial^k \left(h^{(0)} + h^{(\epsilon^2)}\right) + \epsilon^2 a^k \, \partial^j h^{(0)}\right] = \pm\epsilon^{ijk}\left(\epsilon^2 a^k\right) \, B^{j(0)}$$

Substituting back the zeroth order magnetic field, we obtain an equation for $\hat{\phi}^{(\epsilon^2)}$ decoupled from $h^{(\epsilon^2)}$:

$$\nabla \times \vec{B}_{\oplus/\ominus}(\hat{\phi}) \quad = \quad \pm \, \frac{\hat{r} \times \epsilon^2 \vec{a}}{r^2} \qquad (3.10)$$

where $\epsilon^2\vec{a}$ points in the direction towards which the monopole in question is accelerating. We can see that when the acceleration is zero, the right hand side of the equation vanishes and the equation turns back into the static equation. When the acceleration is non-zero, the right hand side can be interpreted as the time derivative of the electric field as in the Maxwell equation, just that it is very small.

3.1.2 The Solution of a Single Accelerating Monopole

Solution of $\hat{\phi}$

In this section, we present the $\mathcal{O}(\epsilon^0)$ $\hat{\phi}$ solution in a different gauge and solve Eq 3.10 for $\mathcal{O}(\epsilon^2)$ correction to $\hat{\phi}$.

To simplify the problem, Manton chooses a gauge such that the solution would preserve the symmetry about the axis of separation of the monopoles. In the last chapter, we have defined $\hat{\phi}$ for opposite charges according to which direction $\hat{\phi}$ rotates as the azimuthal angle χ increases. For a system of two opposite charge monopoles on the z-axis, this choice of angle to define the monopole charge would break the convenient axial symmetry of the solution, since $\hat{\phi}$ near each monopole would be winding in different directions about the z-axis. Recall that gauge transformations of ϕ are simply its rotations in the internal $\mathbf{R}^3$ space, so we can choose another angle to define the winding. Manton chooses θ such that for a single monopole, if we look down on a plane defined by a constant χ, for instance the x-z plane, the oppositely charged monpoles would respectively have $\hat{\phi}$ rotating clockwise and counterclockwise as θ increases. This way, axial symmetry for the two-monopole system can be preserved provided that the corrections of $\hat{\phi}$ due to the acceleration also exhibit axial symmetry. We will see in section 3.2.1 how this gauge is crucial for solving for the two monopole system.

Magnetic field in terms of Ψ and Υ The next step is to write Eq 3.10 in terms of the two degrees of freedom of $\hat{\phi}$. Suppose due to the acceleration, $\hat{\phi}$ depends on the angles χ and θ differently than for a single static monopole, and is written in the form

$$\hat{\phi} = \begin{pmatrix} \sqrt{1-\Psi(\theta)^2}\Upsilon(\chi) \\ \sqrt{1-\Psi(\theta)^2}\sqrt{1-\Upsilon(\chi)^2} \\ \Psi(\theta) \end{pmatrix} \tag{3.11}$$

such that $|\hat{\phi}| = 1$ is still true. Note that for a single static monopole, $\Psi(\theta) = \mp\cos\theta$ and $\Upsilon(\chi) = \cos\chi$.

To see what Eq 3.10 means for $\Psi(\theta)$ and $\Upsilon(\chi)$, we first write the magnetic field explicitly for each gauge index and apply the real space gradient operator, denoted ∇_s, in spherical coordinates:

$$\vec{B} = -\epsilon_{abc}\frac{1}{2}\left(\nabla_s\hat{\phi}_b \times_s \nabla_s\hat{\phi}_c\right)\hat{\phi}_a \tag{3.12}$$

where the explicit lower index is the gauge index and the gradient of the components of $\hat{\phi}$ are as follow:

$$\begin{aligned}
\nabla_s\hat{\phi}_1 &= \left[\frac{-\Psi}{\sqrt{1-\Psi^2}}\frac{\partial\Psi}{\partial r}\Upsilon\right]\hat{r} + \left[\frac{1}{r}\frac{-\Psi}{\sqrt{1-\Psi^2}}\frac{\partial\Psi}{\partial\theta}\Upsilon\right]\hat{\theta} + \left[\sqrt{1-\Psi^2}\frac{1}{r\sin\theta}\left(\frac{\partial\Upsilon}{\partial\chi}\right)\right]\hat{\chi} \\
\nabla_s\hat{\phi}_2 &= \left[\frac{-\Psi}{\sqrt{1-\Psi^2}}\frac{\partial\Psi}{\partial r}\sqrt{1-\Upsilon^2}\right]\hat{r} + \left[\frac{1}{r}\frac{-\Psi}{\sqrt{1-\Psi^2}}\frac{\partial\Psi}{\partial\theta}\sqrt{1-\Upsilon^2}\right]\hat{\theta} \\
&\quad + \left[\sqrt{1-\Psi^2}\frac{1}{r\sin\theta}\left(\frac{-\Upsilon}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right)\right]\hat{\chi} \\
\nabla_s\hat{\phi}_3 &= \frac{\partial\Psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{\theta}
\end{aligned}$$

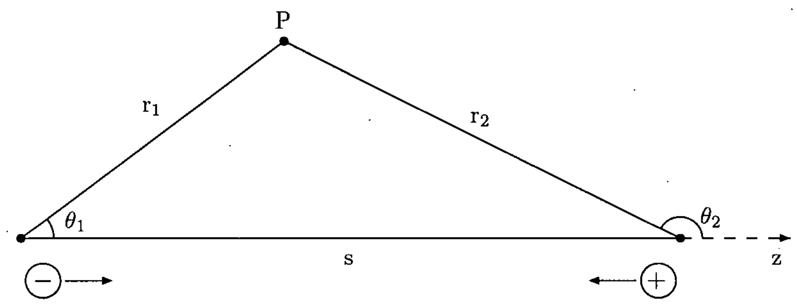

Figure 3.1: Two monopoles with different charges accelerating in opposite directions

Then, we obtain $\vec{B}$ explicitly in terms of $\Psi(\theta)$ and $\Upsilon(\chi)$:

$$\begin{aligned}
B &= -2\frac{1}{2}\left[\left(\nabla_s\hat{\phi}_2 \times_s \nabla_s\hat{\phi}_3\right)\hat{\phi}_1 + \left(\nabla_s\hat{\phi}_3 \times_s \nabla_s\hat{\phi}_1\right)\hat{\phi}_2 + \left(\nabla_s\hat{\phi}_1 \times_s \nabla_s\hat{\phi}_2\right)\hat{\phi}_3\right] \\
&= -\left(\frac{\sqrt{1-\Psi^2}}{r\sin\theta}\frac{\partial\Psi}{\partial r}\left[\frac{-\Upsilon}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right]\hat{\theta} - \frac{\sqrt{1-\Psi^2}}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\left[\frac{-\Upsilon}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right]\hat{r}\right)\left(\sqrt{1-\Psi^2}\,\Upsilon\right) \\
&\quad -\left(\frac{\sqrt{1-\Psi^2}}{r\sin\theta}\frac{\partial\Psi}{\partial r}\frac{\partial\Upsilon}{\partial\chi}(-\hat{\theta}) - \frac{\sqrt{1-\Psi^2}}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\frac{\partial\Upsilon}{\partial\chi}(-\hat{r})\right)\left(\sqrt{1-\Psi^2}\sqrt{1-\Upsilon^2}\right) \\
&\quad -\left(\frac{-\Psi}{r\sin\theta}\frac{\partial\Psi}{\partial r}\left[\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right](-\hat{\theta}) + \frac{-\Psi}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\left[\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right]\hat{r}\right)(\Psi) \\
&= -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\hat{\theta} + \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\hat{r}
\end{aligned}$$

We are now ready to solve Eq 3.10 for Ψ and Υ.

Solving for Ψ and Υ Suppose the $\ominus$ monopole is situated at $-\frac{s}{2}$ on the z-axis and accelerating with $(\epsilon^2 a)\ \hat{z}$ while the $\oplus$ monopole is situated at $+\frac{s}{2}$ on the z-axis and accelerating with $(-\epsilon^2 a)\ \hat{z}$, then the equations for them are exactly the same except in terms of coordinates with different origins:

$\ominus \longrightarrow$	$\oplus \longleftarrow$
$\vec{a}_\ominus = \epsilon^2 a(\hat{r}_1 \cos\theta_1 - \hat{\theta}_1 \sin\theta_1)$	$\vec{a}_\oplus = -\vec{a}_\ominus = \epsilon^2 a(-\hat{r}_2 \cos\theta_2 + \hat{\theta}_2 \sin\theta_2)$
$\nabla \times \vec{B}_\ominus \; = \; + \frac{\epsilon^2 \vec{a} \times \hat{r}_1}{r_1^2}$	$\nabla \times \vec{B}_\oplus \; = \; - \frac{\epsilon^2 (-\vec{a}) \times \hat{r}_2}{r_2^2}$

The coordinates subscripted 1 has its origin at the center of the monopole on the negative z-axis and the ones subscripted 2 at the center of the one on the positive z-axis. Consequently, the equations for $\Psi(\theta)$ and $\Upsilon(\chi)$ are the same for both monopoles as well:

$$\begin{aligned}
\nabla_s \times_s \vec{B} \;&=\; \hat{\chi}\,\frac{1}{r}\left[-\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)\right]\left[\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right] \\
&\quad +\hat{r}\;\frac{1}{r^2\sin^2\theta}\frac{\partial\Psi}{\partial r}\frac{\partial}{\partial\chi}\left(\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right) \\
&\quad +\hat{\theta}\;\frac{1}{r^3\sin^2\theta}\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial\chi}\left(\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right) \\
&=\; \hat{\chi}\,\frac{\epsilon^2 a\sin\theta}{r^2}
\end{aligned} \tag{3.13}$$

We will first show that $\Upsilon(\chi)$ remains unchanged from the static solution even when the monopole is accelerating. First, the inhomonogeneous term on the RHS has only a $\hat{\chi}$ component with coefficient that does not depend on the angle χ; therefore, the particular solution needs to give a $\hat{\chi}$ component on the LHS that is also independent of χ. Thus, for the particular solution, the possible χ dependence needs to be removed:

$$\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi} \;=\; constant$$

This is solved by $\Upsilon(\chi) = \cos(N\chi)$ but as explained above, $N = 1$ for both $\oplus$ and $\ominus$ monopoles for the chosen gauge. This means the particular solution of Υ has no $\mathcal{O}(\epsilon^2)$ correction. Also, this solution renders the $\hat{r}$ and $\hat{\theta}$ components of Eq 3.13 zero as needed, regardless of what the particular solution of the other function, $\Psi(\theta)$, would be.

For the homogeneous solution of $\Upsilon(\chi)$, assuming that $\Psi(\theta)$ depends on θ and perhaps r, we deduce the following equation for $\Upsilon(\chi)$ from the $\hat{\theta}$ and $\hat{r}$ components of Eq 3.13:

$$\frac{\partial}{\partial\chi}\left(\frac{-1}{\sqrt{1-\Upsilon^2}}\frac{\partial\Upsilon}{\partial\chi}\right) = 0$$

which implies that the term inside the bracket, say (I), is linear: $(I) = A\chi + B$. We already know that the $\mathcal{O}(\epsilon^0)$ $\Upsilon(\chi)$ solution gives $(I) = \textit{constant}$, and that the magnetic field is proportional to (I) from Eq 3.13, which means $(I) = A\chi$ would make $\vec{B}$ discontinuous at $\chi = 0$; therefore, the only admissible solution for $\Upsilon(\chi)$ is still just the static solution $\Upsilon(\chi) = \cos\chi$. We conclude that $\Upsilon(\chi)$ is not affected by the acceleration of the monopole.

We now simplify Eq 3.13 to an equation for $\Psi(\theta)$ only by putting in $(I) = 1$:

$$-r^2\frac{\partial^2\Psi}{\partial r^2} - \left(\frac{\partial^2\Psi}{\partial\theta^2} + \frac{-\cos\theta}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) = \epsilon^2 a\, r\,\sin\theta \tag{3.14}$$

and proceed to solve for Ψ. This equation can also be written in the following form which manifests its linearity in $\Psi(\theta)$:

$$-\nabla\times(\hat{\chi}\times\nabla\Psi) = \hat{\chi}\,\epsilon^2 a\, r\,\sin\theta \tag{3.15}$$

The equation's linearity in Ψ is crucial for Manton to build the global two monopole solution as will be discussed in section 3.2.1,

Particular solution of Ψ Since the first term in Eq 3.14 involves the second derivative in r of Ψ, if $\Psi \propto r$, then the first term vanishes. Now, if the θ dependence is $\frac{\sin^2\theta}{2}$, then

$$-\frac{\partial^2(\sin^2\theta/2)}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial(\sin^2\theta/2)}{\partial\theta} = \cos 2\theta + \cos^2\theta = \sin^2\theta$$

Combining, the correction of Ψ due to the inhomogeneous term is

$$\Psi^{(\epsilon^2)}_{part} = \frac{1}{2}\epsilon^2 a\, r\,\sin^2\theta \tag{3.16}$$

which gives the magnetic field correction

$$\vec{B}_{part} = +\hat{r}\frac{\epsilon^2 a\cos\theta}{r} - \hat{\theta}\frac{\epsilon^2 a\sin\theta}{2r}$$

Note that since $\vec{B}_{part}$ is proportional to $\epsilon^2 a/r$, it would be of $\mathcal{O}(\epsilon^3)$ when far away from the monopole $1/r$ is comparable to ϵ.

Homogeneous solution of Ψ We use separation of variables to find the homogeneous solution of Ψ:

$$\begin{aligned} \text{Let} \quad \Psi^{(\epsilon^2)}_{hom} &= R(r)\Theta(\theta) \\ \text{then} \quad -\frac{r^2R''}{R} &= \frac{\Theta''}{\Theta} - \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} = \lambda = const \end{aligned}$$

1. For the θ dependence,

$$\Theta'' - \frac{\cos\theta}{\sin\theta}\Theta' - \lambda\Theta = 0$$

Propose that $\Theta \sim \cos^k\theta\,\sin^l\theta$, then

$$\begin{aligned} (-2kl - l - \lambda)\cos^k\theta\sin^l\theta + k(k-1)\cos^{k-2}\theta\sin^{l+2}\theta & \\ + l(l-2)\cos^{k+2}\theta\sin^{l-2}\theta &= 0 \end{aligned}$$

Each term vanishing gives the conditions for k, l, and λ:

$$k = 0,1 \ ; \ l = 0,2 \ ; \quad \lambda = -2kl - l = \begin{cases} 0 & \text{for } k=0,\ l=0 \\ -2 & \text{for } k=0,\ l=2 \\ -6 & \text{for } k=1,\ l=2 \end{cases}$$

2. For the radial function,

$$r^2R'' - \lambda R = 0 \text{ where } -\lambda = 0,2,6$$

$$\text{Let } R(r) = r^n, \quad \text{this means } n(n-1) = -\lambda$$

$$\Longrightarrow n = \begin{cases} 0,1 & \text{for } -\lambda = 0 \\ -1,2 & \text{for } -\lambda = 2 \\ -2,3 & \text{for } -\lambda = 6 \end{cases}$$

Combining the angular and the radial parts:

$$\Psi^{(\epsilon^2)}_{hom} = R(r)\Theta(\theta) \sim \begin{cases} Ar + B & \text{for } -\lambda = 0 \\ \left(Cr^2 + \frac{D}{r}\right)\sin^2\theta & \text{for } -\lambda = 2 \\ \left(Er^3 + \frac{F}{r^2}\right)\cos\theta\sin^2\theta & \text{for } -\lambda = 6 \end{cases}$$

where A to F are constants of $\mathcal{O}(\epsilon^2)$.

Plugging these into the relation between B and Ψ (Eq 3.13), we obtain the following magnetic field correction:

$$\vec{B} \sim \begin{cases} -\hat{\theta}\frac{A}{r\sin\theta}; & \text{for } \Psi \sim Ar + B \\ C(\hat{r}\cos\theta - \hat{\theta}\sin\theta) & \\ \quad +D\left(\hat{r}\frac{2\cos\theta}{r^3} + \hat{\theta}\frac{\sin\theta}{r^3}\right); & \text{for } \Psi \sim \left(Cr^2 + \frac{D}{r}\right)\sin^2\theta \\ E\left[\hat{r}(2r - 3r\sin^2\theta) - \hat{\theta}3r\sin\theta\cos\theta\right] & \\ \quad +F\left[\hat{r}\frac{1}{r^4}(2 - 3\sin^2\theta) + \hat{\theta}\frac{2}{r^4}\sin\theta\cos\theta\right]; & \text{for } \Psi \sim \left(Er^3 + \frac{F}{r^2}\right)\cos\theta\sin^2\theta \end{cases}$$

However, only of one these magnetic fields is admissible and relevant. The first of these terms diverges at $\theta = 0$, so it is not allowed. The term with coefficient E is proportional to $\epsilon^2 ar$ which becomes of $\mathcal{O}(\epsilon^1)$ when $1/r$ is comparable to ϵ. This order was not mentioned by Manton and is trivial as will be shown in section 3.2.2. The terms with coefficients D and F are of $\mathcal{O}(\epsilon^3)$ for $1/r \sim \epsilon$ and is irrelevant in the determination of the acceleration $\epsilon^2\vec{a}$ as will be shown in section 3.2.1. The only term left is $\vec{B} \sim C(\hat{r}\cos\theta - \hat{\theta}\sin\theta) = C\hat{z}$, which comes from the following $\Psi^{(\epsilon^2)}_{hom}$:

$$\Psi^{(\epsilon^2)}_{hom\ \ominus} = -\frac{\sigma_1}{2}\epsilon^2 ar_1^2\sin^2\theta_1 \quad ; \quad \Psi^{(\epsilon^2)}_{hom\ \oplus} = \frac{\sigma_2}{2}\epsilon^2 ar_2^2\sin^2\theta_2 \tag{3.17}$$

where $-\sigma_1\epsilon^2 a$ and $\sigma_2\epsilon^2 a$ are simply Manton's names for the coefficient C for the different monopoles.

We have found the homogeneus solution of the magnetic field from $\Psi^{(\epsilon^2)}_{hom}$; however, if we were not interested in Ψ, we could have noticed that the homogenous solution of $\vec{B}$ in Eq 3.13 simply satisfies the vacuum Maxwell equations, and could have concluded that the solution in terms of a scalar potential U such that $\vec{B} = \nabla U$, is simply any linear combination of the multipole expansion terms with cylindrical symmetry:

$$U = \sum_l \left(Ar^l + \frac{B}{r^{l+1}} \right) P_l(\cos\theta)$$

where A, B are constants and $P_l(\cos\theta)$ is the Legendre Polynomial of $\cos\theta$ of order l. The homogenous magnetic field we obtained from $\Psi^{(\epsilon^2)}_{hom}$ simply corresponds to the term with the lowest l which would give rise to a non-zero magnetic field and has no $1/r$ dependence: $U = ArP_1(\cos\theta)$.

The complete solution for $\hat{\phi}$ for either monopole accelerating in its own direction is then

$$\hat{\phi} = \begin{pmatrix} \left(\Psi^{(0)} + \Psi^{(\epsilon^2)}_{part} + \Psi^{(\epsilon^2)}_{hom}\right) \cos\chi \\ \left(\Psi^{(0)} + \Psi^{(\epsilon^2)}_{part} + \Psi^{(\epsilon^2)}_{hom}\right) \sin\chi \\ \sqrt{1 - \left(\Psi^{(0)} + \Psi^{(\epsilon^2)}_{part} + \Psi^{(\epsilon^2)}_{hom}\right)} \end{pmatrix}$$

with the corresponding $\Psi^{(0)}$, $\Psi^{(\epsilon^2)}_{part}$, and $\Psi^{(\epsilon^2)}_{hom}$. Note that $\hat{\phi}$ for both monopoles to depend on Ψ in the same way for the chosen gauge. In section 3.2.1, we will see how Manton needs this to build the global two monopole solution.

Solution for h

Solving for $\hat{\phi}$ has given us both the particular and homogeneous solutions for the magnetic field due to the acceleration. Using these and the first order ansatz Eq 3.9, we can easily solve for h to $\mathcal{O}(\epsilon^2)$.

For the $\ominus$ monopole,

$$\vec{B}_\ominus = -(\nabla_s h_\ominus + \epsilon^2 \vec{a} h)$$

and from the last section

$$\vec{B}_\ominus = -\frac{\hat{r}_1}{er_1^2} + \hat{r}_1 \epsilon^2 a \frac{\cos\theta_1}{er_1} - \hat{\theta}_1 \frac{1}{2}\epsilon^2 a \frac{\sin\theta_1}{er_1} - \sigma_1 \epsilon^2 \vec{a}$$

The equation for h is then

$$\begin{aligned}\nabla h_\ominus &= \hat{r_1}\frac{\partial h_\ominus}{\partial r_1} + \hat{\theta_1}\frac{1}{r_1}\frac{\partial h_\ominus}{\partial \theta_1} \\ &= \frac{\hat{r}_1}{r_1^2} - \hat{\theta}_1\frac{\epsilon^2 a\ \sin\theta_1}{2r_1} + (\sigma_1 - c)\epsilon^2 a\left(\hat{r}_1\cos\theta_1 - \hat{\theta}_1\sin\theta_1\right) \end{aligned} \quad (3.18)$$

Solving for the $\hat{r}_1$component of the equation,

$$h_\ominus = c - \frac{1}{r_1} + (\sigma_1 - c)\epsilon^2 a r_1\cos\theta_1 + f(\theta_1)$$

The $\hat{\theta}_1$ component of the equation then determines $f(\theta_1)$ such that

$$h_\ominus = c - \frac{1}{r} + (\sigma_1 - c)\epsilon^2 a r_1\cos\theta_1 + \frac{1}{2}\epsilon^2 a\cos\theta_1 + k_1$$

For the $\oplus$ monopole, the $\mathcal{O}(\epsilon^0)$ magnetic field has a different sign from the $\ominus$ case, and the magnetic field correction obtained from $\Psi_{hom}^{(\epsilon^2)}$ also has a different sign because of the definition of the unknown coefficient σ_2:

$$\vec{B}_\oplus = +\frac{\hat{r}_2}{r_2^2} + \hat{r}_2\epsilon^2 a\frac{\cos\theta_2}{r_2} - \hat{\theta}_2\frac{1}{2}\epsilon^2 a\frac{\sin\theta_2}{r_2} + \sigma_2\epsilon^2\vec{a}$$

The first order ansatz has a different relative sign between B and h and the direction of the acceleration is reversed:

$$\vec{B}_\oplus = (\nabla_s h_\oplus - \epsilon^2\vec{a}h)$$

Thus, the equation and solution for $h_\oplus$ are

$$\begin{aligned}\nabla_s h_\oplus &= \frac{\hat{r}_2}{r_2^2} + \hat{\theta}_2\frac{1}{2}\epsilon^2 a\frac{\sin\theta_2}{r_2} + (\sigma_2 - c)\epsilon^2(\hat{r}_2\cos\theta_2 - \hat{\theta}_2\sin\theta_2) \\ \implies h_\oplus &= c - \frac{1}{r_2} + (\sigma_2 + c)\epsilon^2 a r_2\cos\theta_2 - \frac{1}{2}\epsilon^2 a\cos\theta_2 + k_2\end{aligned}$$

We now have the full solution in the asymptotic region for a single $\ominus$ monopole accelerating in the $+z$ direction and a $\oplus$ in the $-z$ direction.

Comparison with Fields of Accelerating Electric Charge

The magnetic field obtained above for an accelerating monopole is not analogous to the electric field for an accelerating point charge in normal electromagnetism. We will describe how and briefly why these fields are different, but show also that far away from the monopole where $r \sim s$, the term in the magnetic field that is relevant to how Manton obtains the acceleration, the $\mathcal{O}(\epsilon^0)$ term, is actually equal, up to $\mathcal{O}(1/s^2)$, to the Coulomb term in the electric field of an accelerating electric particle.

The differences between the fields result from the different ways we solve the two problems. For the magnetic monopole, we propose a time dependence for the solution, check that it is legitimate by evaluating the time derivatives of the fields with such time dependence in the equations of motion, and then solve these "half-static" equations, since they do not have time derivatives anymore, for the magnetic field, both the $1/r^2$ and $1/r$ terms. For the electric point charge, we simply solve the time dependent equations and let the time dependence of the fields come out of solving the equations:

$$\vec{E}_{electric}(\vec{x},t) \;=\; e\left[\frac{\hat{n}}{\gamma^2\left(1-\left(\vec{\beta}\times\hat{n}\right)^2\right)^{\frac{3}{2}} r^2}\right] + \frac{e}{c}\left[\frac{\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^3 r^2}\right]_{ret}$$

$$\text{where} \qquad \vec{\beta}(t)=\dot{\vec{x}}_0(t) \;\;,\;\; r(t)=|\vec{x}(t)-\vec{x}_0(t)| \;\text{ and }\; \vec{n}(t)=\frac{\vec{x}(t)-\vec{x}_0(t)}{r(t)}$$

The main difference between the fields is that the $1/r$ radiation fields above for the accelerating electric charge are in terms of quantities related to the path of the charge that are to be evaluated at an earlier time t_0 defined by:

$$|\vec{x}-\vec{x}_0(t_0)| \;=\; c\,(t-t_0) \quad \text{where } \vec{x}_0(t_0) \text{ is the path of the electric charge}$$

but the fields obtained by Manton for the accelerating monopole at $t=0$ depends on the motion of the monopole at the same instant.

The magnetic field for the accelerating monopole being not time-retarded causes a violation of special relativity: even if the monopole starts to accelerate only at $t = 0$, the $1/r$ term of its magnetic field, which is the analog of the radiation of the accelerating electric charge, takes no time to reach the other monopole. This is consistent with the fact that the assumption of the fields to be rigidly accelerating over all space also violates special relativity. However, the $\sim 1/r$ magnetic field for the accelerating magnetic monopole does not participate in the determination of the force between the monopoles (as described in section 3.2.1) and so we can ignore this problem.

Note that even if we take away the time-retardation of the radiation of the accelerating electric charge and chooses the charge to be constantly accelerating, $\vec{\beta}/c = \epsilon^2\vec{a}t$, the radiation terms of the magnetic monopole still has a different functional form. This is because the $\dot{E}$ term in the equation, $\vec{\nabla} \times \vec{B} = \dot{E}$, for the magnetic monopole problem comes directly from Manton's assumed time dependence of the fields, while the $\dot{B}$ term in the equation, $\vec{\nabla} \times \vec{E} = \dot{B}$, for the electric charge problem both affects and is affected by the radiation term in the electric field.

On the other hand, the static $1/r^2$ term in the magnetic field of the accelerating monopole, when compared to the analogous electric field, lacks the factors that depend on the velocity of the particle. However, since Manton's fields describe the instant when the monopole has zero velocity, the factors become zero, and so the $1/r^2$ fields for the electric charge and the monopoles are exactly analogous and are simply the respective static Coulomb fields. The facts that only the $1/r^2$ term of the magnetic monopole field is relevant in the determination of the acceleration and that this term is the same as the static monopole field are what make Manton's method gives the same result as the stress-energy tensor method described in section 3.3.

3.1.3 The First Order Ansatz as the External Force Law

We will now extract physical information from Manton's first order ansatz for the weakly accelerating monopole. Manton argues that the first order ansatz implies the Lorentz Force Law for a single monopole, but this is only half of the story: the first order ansatz informs us about the contribution of forces that can act on the monopole.

He argues that for a single accelerating monopole,since

$$\vec{B} = \pm\left[\nabla h + \epsilon^2 \vec{a}\left(c - \frac{1}{r}\right)\right] \quad \text{to } \mathcal{O}(\epsilon^2)$$

and ∇h needs to vanish at infinity for the monopole to have finite energy, $\vec{B}$ must be $\epsilon^2 ac$ at infinity. He claims that the Lorentz Force Law directly follows since c is the ratio between the mass and the charge of a single charge monopole:

$$\vec{B} = \epsilon^2 ac = \pm\frac{m(\epsilon^2 a)}{g} \tag{3.19}$$

I do not agree with the reason for $\nabla h = 0$ or that the Lorentz Force law necessarily holds at infinity. Rather, for the monopole to have finite energy, both ∇h and the magnetic field $\vec{B}$ need to drop to zero at infinity. Therefore, what Manton really assumes when he allows $\vec{B}$ to be non-zero at infinity but not ∇h is that the uniform "external" field that is left over even at infinity is comprised of only the magnetic field. If we choose the external field to include a gradient field of h, then these two types of field both contribute to the forces acting on a monopole and together satisfy the force law:

$$(\vec{B}_{ext} \mp \nabla h_{ext}) = \pm\frac{m(\epsilon^2 a)}{g}$$

For example, let us look at the solution to the first order ansatz in the asymptotic region near a $\ominus$ monopole:

$$\vec{B}_\ominus = -\frac{\hat{r}_1}{er_1^2} + \hat{r}_1\epsilon^2 a\frac{\cos\theta_1}{er_1} - \hat{\theta}_1\frac{1}{2}\epsilon^2 a\frac{\sin\theta_1}{er_1} - \sigma_1\epsilon^2\vec{a} = -\nabla h_\ominus - \epsilon^2\vec{a}c + \epsilon^2\vec{a}\frac{1}{r_1}.$$

At infinity where all the terms with r_1 in the denominator vanish, the undetermined external magnetic field reduces to $-\sigma_1\epsilon^2\vec{a}$, not $\epsilon^2\vec{a}c$, and ∇h reduces to $\epsilon^2\vec{a}(c-\sigma_1)$. Choosing the value of σ_1 would determine if the Lorentz force law is followed: if we force ∇h to go to zero at infinity, such that $\sigma_1 = c$, then the Lorentz Force Law (Eq 3.19) is satisfied; otherwise, if $\sigma_1 \neq c$ and $\nabla h \neq 0$ at infinity, the Lorentz Force Law is incorrect.

The interpretation of the first order ansatz as an external force law is valid not only at infinity, but also in the asymptotic region, since in this region, although the terms with $1/r^n$ has not dropped to zero, the ansatz still independently relates the constant magnetic and ∇h fields to the constant term $\epsilon^2\vec{a}c$.

3.2 Manton's Method to Determine the Acceleration between Two Monopoles

We now consider a system with two widely separated monopoles accelerating in opposite directions.

We know that at the core of each monopole, the first order ansatz needs to be satisfied such that the unfactorized non-linear equations of motion there are satisfied. In the asymptotic region close to the core of each monopole then, the solution is simply the asymptotic limit of the first order ansatz and we call this the "local" solution. Now, for the region between the monopoles, Manton discovers that he can easily obtain a global solution by "almost" superimposing the local solutions. He finds the acceleration by requiring this global solution to become the local solutions in regions close to the cores of the monopoles.

However, although Manton's result is correct, his method in fact does not give an unique answer. We will show two examples of global solutions which are built in the same manner that Manton's is built but which conclude a different acceleration between the monopoles. We then propose a rule to build the global solution such that it gives the right conclusion, and notice that under this rule Manton's procedure can be interpreted as the application of the external force law Eq 3.20, and the undetermined terms in each local solution as perturbative external fields produced by the opposite monopole. We then do an exerices to find out if the method can be used to give a higher order ($\mathcal{O}(1/(separation)^3)$ or above) force between the monopoles, conclude that it cannot, and examine the difference between the method at higher order and at the order for which it works.

3.2.1 The Global Solutions and the Matching Procedure

Manton's Way of Building the Global Solution

Manton discovers an easy way to build the global solution in between two monopoles.

First, recall from section 2.2.2 that in the region between two monopoles, the magnetic field $\vec{B}$ satisfies the vacuum Maxwell equations and h satisfies the Laplace equation. Both equations are linear differential equations.

Secondly, concentrate on the equation for $\vec{B}$. Recall from section 3.1.2 that $\vec{B}$ depends on $\hat{\phi}$ (Eq 3.12) and $\hat{\phi}$ is given in terms of the function $\Psi(\theta)$ and $\Upsilon(\chi)$ (Eq 3.11). In the gauge that Manton has chosen, the solutions for $\Upsilon(\chi)$ of the ansatzes for both $\oplus$ and $\ominus$ monopoles are the same, $\Upsilon = \cos\chi$, and given this, $\hat{\phi}$ depends on $\Psi(\theta)$ the same way for both monopoles and the magnetic field $\vec{B}$ depends linearly on $\Psi(\theta)$ the same way for both monopoles. Thus, we can write down the following function $\hat{\phi}_{global}$ which has $\Upsilon_{global} = \Upsilon_{\oplus/\ominus} = \cos\chi$ and depends on Ψ_{global} as the $\hat{\phi}$ for either monopole depends on Ψ:

$$\hat{\phi}_{global} = \begin{pmatrix} \sqrt{1-\Psi_{global}^2}\cos\chi \\ \sqrt{1-\Psi_{global}^2}\sin\chi \\ \Psi_{global} \end{pmatrix} \tag{3.20}$$

such that the global magnetic field $\vec{B}_{global}$ again depends linearly on Ψ_{global} through its dependence on $\hat{\phi}_{global}$. Then, the Maxwell equation for $\vec{B}_{global}$ would translate into a linear differential equation for Ψ_{global}.

Now, if we solve for Ψ_{global}, then $\hat{\phi}_{global}$ will be automatically determined by Eq 3.20 and will satisfy the equations of motion, and the global gauge field will in turn be determined in terms of $\hat{\phi}_{global}$. Therefore, for the asymptotic region between the two monopoles, solving for h_{global} and Ψ_{global} will give the full solution.

Finally, since the equations for h_{global} and Ψ_{global} in the region between the monopoles are both linear, the solution of h_{global} and Ψ_{global} can simply be the sum of the local $h_\ominus$, $h_\oplus$ and local $\Psi_\ominus$, $\Psi_\oplus$ functions, which satisfy the equations of motion by satisfying the respective first order linear ansatzes. Here, the sum of solutions means only the sum up to constants and homogeneous solutions of the local ansatzes so there are choices to make for the global solutions. Manton also requires the global solutions to

1. be symmetric under monopole exchange;
2. satisfy the appropriate boundary conditions at infinity;
3. reduce to the local solutions, $h_\ominus$, $h_\oplus$ and $\Psi_\ominus$, $\Psi_\oplus$, near each monopole.

Note that if the global function Ψ_{global} reduces to the local $\Psi_\oplus$ and $\Psi_\ominus$ near the different monopoles, then $\hat{\phi}_{global}$ will also automatically reduce to the local $\hat{\phi}$s.

Manton claims that in the process of matching the global solutions that satisfy the above requirements to the local solutions, the acceleration between the

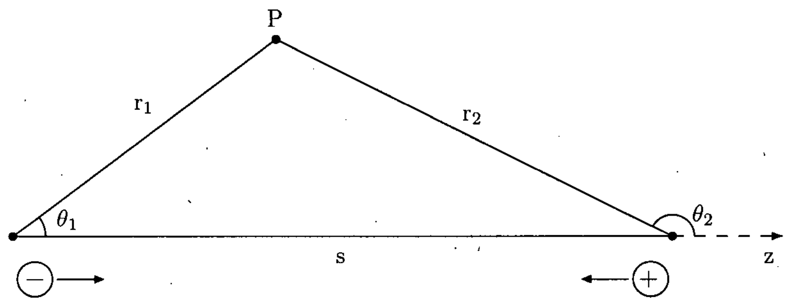

Figure 3.2: the two monopole system, the distances r_1, r_2 and the angles θ_1, θ_2.

monopoles is determined. He obtains the correct acceleration, and we will recount his choice of global functions in this section, but we will also see in next section that the acceleration is actually not uniquely determined.

The global solutions would also need to satisfy boundary conditions at infinity and be symmetric under monopole exchange. Finally, requiring the global functions to reduce to the respective local solution near each monopole, Manton claimed, would determine the acceleration of the monopoles uniquely. We can already see, however, that we have freedom to add constants or homogeneous solutions of the ansatz to the global solutions, and will discuss this in the section 3.2.2. Here, we recount Manton choices of global solutions and how he determined the acceleration. The notation used is such that $h^{(\epsilon^2)}_{\ominus hom}$ denotes the homogeneous solution of h to the $\ominus$ ansatz which is also of $\mathcal{O}(\epsilon^2)$.

Matching Ψ

In choosing Ψ_{global}, Manton must have noticed that when expanded near the opposite monopole, the $\mathcal{O}(\epsilon^0)$ solution of either local Ψ function would give rise to a function which is proportional to the homogeneous solution $\Psi^{(\epsilon^2)}_{hom}$ for the opposite monopole.

Explicitly, with r_1, r_2, θ_1, θ_2 defined in figure 3.2, near the $\oplus$ monopole, θ_1 is small, and

$$\Psi_{\ominus}^{(0)} - 1 = \cos(\theta_1) - 1 \approx \frac{1}{2}(\theta_1)^2 \approx \frac{1}{2}\sin^2\theta_1.$$

Then using the sine law and that r_1 is approximately the separation distance, s, in this region,

$$\Psi_{\ominus}^{(0)} - 1 \approx \frac{1}{2}\frac{r_2^2\sin^2(\theta_2)}{r_1^2} \approx \frac{1}{2}\frac{r_2^2\sin^2\theta_2}{s^2} + \mathcal{O}(\frac{1}{s^3}) \propto \Psi_{\oplus\ hom}$$

Similarly, near the $\ominus$ monopole, $(\pi - \theta_2)$ is small and $r_2 \sim s$, so

$$\Psi_{\oplus}^{(0)} - 1 = -\cos\theta_2 - 1 = \cos(\pi - \theta_2) - 1 \approx -\frac{1}{2}\frac{r_1^2\sin^2\theta_1}{s^2} + \mathcal{O}(\frac{1}{s^3}) \propto \Psi_{\ominus\ hom}$$

Therefore, if $\Psi_{global} + 1$ is the sum of the local Ψ functions without the homogeneous part, i.e.,

$$\begin{aligned}\Psi_{global} &= \Psi_{\ominus}^{(0)} + \Psi_{\oplus}^{(0)} - 1 + \Psi_{\ominus\ part}^{(\epsilon^2)} + \Psi_{\oplus\ part}^{(\epsilon^2)} \\ &= \cos\theta_1 - \cos\theta_2 - 1 + \frac{1}{2}\epsilon^2 a r_1 \sin^2\theta_1 + \frac{1}{2}\epsilon^2 a r_2 \sin^2\theta_2\end{aligned}$$

then Ψ_{global} would reduce to the local Ψ near each monopole up to $\mathcal{O}(\epsilon^2)$ provided that the coefficient of the local $\Psi_{hom}^{(\epsilon^2)}$ is matched with the coefficient of the term from the expansion of $\Psi^{(0)}$ of the opposite monopole. Thus, near the $\ominus$ monopole,

$$\begin{aligned}\Psi_{global} &\longrightarrow \cos\theta_1 - \frac{1}{2}\sigma_1\epsilon^2 a r_1^2 \sin^2\theta_1 + \frac{1}{2}\epsilon^2 a r_1 \sin^2\theta_1 + \mathcal{O}(\epsilon^3) = \Psi_{\ominus} \\ \text{if} &\qquad \sigma_1\epsilon^2 a = \frac{1}{s^2}\end{aligned}$$

Note that the radiation term from the opposite monopole, $\Psi_{\oplus\ part}$, is omitted here because it is of an irrelevant order in this geometric limit:

$$\Psi_{\oplus\ part} = \frac{1}{2}\epsilon^2 a r_2 \sin^2\theta_2 \approx \frac{1}{2}\epsilon^2 a\frac{r_1^2\sin^2\theta_1}{s} \sim \epsilon^3$$

Using exactly analogous arguments, for Ψ_{global} to reduce to $\Psi_{\oplus}$ near the $\oplus$ monopole

$$\sigma_2\epsilon^2 a = -\frac{1}{s^2}$$

Note that Ψ_{global} is also symmetric under monopole exchange and remains less than 1 to $\mathcal{O}(\epsilon^0)$ due to the added constant -1 such that $\hat{\phi}$ is a real unit vector; thus, it satisfies all of Manton's requirements for the global solutions.

We have now two equations involving the acceleration and two unknown coeffients σ_1 and σ_2. One more equation of these quantities would determine the acceleration.

Matching h

Manton chooses the following rather adhoc looking h_{global}, which includes only one of the local homogeneous solutions of h but with a term proportional to $1/s^2$ put in by hand:

$$\begin{aligned} h_{global} &= \left(h_\ominus^{(0)} + h_\oplus^{(0)} - c\right) - \frac{r_2\cos\theta_2}{es^2} + \left(h_{\ominus\ part}^{(\epsilon^2)} + h_{\oplus\ part}^{(\epsilon^2)}\right) + h_{\oplus\ hom}^{(\epsilon^2)} + const \\ &= \left(c - \frac{1}{er_1} - \frac{1}{er_2}\right) - \frac{r_2\cos\theta_2}{es^2} + \left(\frac{1}{2e}\epsilon^2 a\cos\theta_1 - \frac{1}{2e}\epsilon^2 a\cos\theta_2\right) \\ &\quad + \epsilon^2 a\left(\frac{\sigma_2}{e} + c\right) r_2\cos\theta_2 + const \end{aligned}$$

such that near the $\oplus$ monopole, the term $r_2\cos\theta_2/s^2$ cancels with the term from the expansion of $h_\ominus^{(0)}$ near this monopole, which equals

$$\begin{aligned} -\frac{1}{er_1} &= -\frac{1}{es} + \frac{r_2\cos\theta_2}{es^2} + O\left(\frac{1}{s^3}\right) \\ \text{since } r_1 &= s\sqrt{1 + \frac{r_2\cos\theta_2}{s} + \frac{r_2^2}{s^2}} \text{ by the cosine law} \end{aligned}$$

and the term $\epsilon^2 a\cos\theta_1/2e$ reduces to $\mathcal{O}(\epsilon^2)$ to simply a constant near the $\oplus$ monopole and can be absorbed by the constant in the global function.

Thus, h_{global} has been constructed to match $h_\oplus$ near the $\oplus$ monopole:

$$h_{global} \longrightarrow c - \frac{1}{er_2} - \frac{1}{2e}\epsilon^2 a\cos\theta_2 + \epsilon^2 a\left(\frac{\sigma_2}{e} + c\right) r_2\cos\theta_2 + const = h_\oplus$$

This h_{global} satisfies all of Manton's requirement for a global solution since the added term is a homogeneous solution of the equation of motion and the terms

$(\sigma^2/e+c)r_2\cos\theta_2$ could also be written as $(\sigma^2/e+c)r_1\cos\theta_1+const$ so that h_{global} is still symmetric under monopole exchange.

The acceleration is determined by the condition given when h_{global} is matched with $h_\ominus$ near the $\ominus$ monopole. Near the $\ominus$ monopole, the term that cancels the expansion term of $h_\ominus^{(0)}$ near the $\oplus$ monopole does not cancel but adds up with the expansion from $h_\oplus^{(0)}$ since

$$-\frac{1}{er_2} = -\frac{1}{es} - \frac{r_1\cos\theta_1}{s^2} + O(\frac{1}{s^3})\text{because}$$

and with $r_2\cos\theta_2$ with $r_1\cos\theta_1 - s$, h_{global} reduces to:

$$h_{global} \longrightarrow c - \frac{1}{er_1} - \frac{2r_1\cos\theta_1}{es^2} + \frac{1}{2e}\epsilon^2 a\cos\theta_1 + \epsilon^2 a\left(\frac{\sigma_2}{e}+c\right) r_1\cos\theta_1 + const$$

This only equals $h_\ominus$ if all the terms proportional to $r_1\cos\theta_1$ together form the local homogeneous h, $h_{\ominus\ hom}$. This implies the condition:

$$\epsilon^2 a\left(\frac{\sigma_2}{e}+c\right) - \frac{2}{es^2} = \epsilon^2 a\left(\frac{\sigma_1}{e}-c\right)$$

Then, substituting the values of $\sigma_1\epsilon^2 a$ and $\sigma_2\epsilon^2 a$ from before, Manton obtains twice the Coulomb attractive acceleration for a pair of opposite charge monopoles:

$$\epsilon^2 a = \frac{2}{ecs^2}$$

Force between two same charge monopoles

Manton finds the force between two same charge monopoles by the same procedures.

Suppose we switch the monopole on the +z-axis in the previous case to a $\ominus$ monopole, so that both monopole 1 (on -z-axis) and monopole 2 (on +z-axis) are $\ominus$. Then all we need to change in the steps above are the local functions near the new monopole 2, and the global functions accordingly.

Since monopole 2 has the same charge as monopole 1, but accelerates in the opposite direction, the local functions, call them $\Psi_{\ominus\ 2}$ and $h_{\ominus\ 2}$, are the same as the ones at monopole 1 except with the sign of $\epsilon^2 a$ changed and in its own coordinates:

$$\begin{aligned}\Psi_{\ominus\ 2} &= \Psi^{(0)}_{\ominus\ 1} - \left(\Psi^{(\epsilon^2)}_{\ominus\ 1\ hom} + \Psi^{(\epsilon^2)}_{\ominus\ 1\ part}\right) \\ &= \cos\theta_2 + \frac{1}{2}\sigma_2\epsilon^2 a r_2^2 \sin^2\theta_2 - \frac{1}{2}\epsilon^2 a r_2 \sin^2\theta_2\end{aligned}$$

Notice that the term involving σ_2 has the same sign as in $\Psi_\oplus$ in the previous opposite charge monopole system. Similarly,

$$h_{\ominus\ 2} = c - \frac{1}{er_2} - (\sigma_2 - c)\epsilon^2 a r_2 \cos\theta_2 - \frac{1}{2}\epsilon^2 a \cos\theta_2 + k_2$$

Now, as before, the requirement of the global function Ψ_{global} to reduce to the local functions near each monopole would give the unknown coeffients, σ_1 and σ_2, and the acceleration, $\epsilon^2 a$, in terms of the monopole separation s. Again, the particular solutions of Ψ do not participate in the matching.

Therefore, we only need to take note that in global function for the same charge system , $\Psi^{(0)}_{\ominus\ 2}$ has a different sign from $\Psi^{(0)}_{\oplus}$ in the opposite charge system, and that the homogeneous terms, the ones with coefficients σ_1 and σ_2, remain as before, to conclude that the matching procedure would give the same expression as before for $\sigma_2\epsilon^2 a$, and one with opposite sign from before for $\sigma_1\epsilon^2 a$. More explicitly,

$$\Psi_{global\ \ominus\ominus} = \cos\theta_1 + \cos\theta_2 \pm 1 + \Psi^{(\epsilon^2)}_{\ominus\ 1\ part} + \Psi^{(\epsilon^2)}_{\ominus\ 2\ part}$$

where the second term is now $+\cos\theta_2$ for the $\ominus$ monopole 2 and after expansion near the first monopole gives

$$\sigma_1\epsilon^2 a = -\frac{1}{s^2} \quad ; \quad \sigma_2\epsilon^2 a = -\frac{1}{s^2}$$

Now, when we construct the global h function in the same manner as before, we find that the only change is the sign change of the term involving σ_2:

$$\begin{aligned}h_{global} = {}& c - \frac{1}{er_1} - \frac{1}{er_2} - \frac{r_2\cos\theta_2}{es^2} + \frac{1}{2e}\epsilon^2 a\cos\theta_1 - \frac{1}{2e}\epsilon^2 a \cos\theta_2 \\ &+\epsilon^2 a\left(-\frac{\sigma_2}{e} + c\right) r_2\cos\theta_2 + const\end{aligned}$$

where the $\mathcal{O}(\epsilon^0)$ local h function for the new $\ominus$ monopole, $(c-1/er_2)$, is the same as the one for the $\oplus$ monopole in the previous case. The matching procedure is exactly the same and gives

$$\begin{aligned}
\epsilon^2 a\left(-\frac{\sigma_2}{e}+c\right)-\frac{2}{es^2} &= \epsilon^2 a\left(\frac{\sigma_1}{e}-c\right) \\
\Longrightarrow \quad 2\epsilon^2 ac &= \frac{2}{es^2}+\frac{(\sigma_2+\sigma_1)\epsilon^2 a}{e} \quad = \quad 0
\end{aligned}$$

Therefore, the force between two same charge monopoles vanishes.

3.2.2 Clarifications and Comments on Manton's Method

Other Consistent Solutions that Give Different Acceleration

We will now look at how the flexibility in choosing the global solution even according to Manton's requirements allows for global solutions that lead to different conclusions for the acceleration. We demonstrate this by the following two examples.

Example I The first example results in a zero acceleration between two monopoles with different charges. We choose the global function Ψ_{global} to include the homogenous solutions of the ansatzes, $\Psi^{\epsilon^2}_{\ominus hom}$ and $\Psi^{\epsilon^2}_{\oplus hom}$:

$$\begin{aligned}
\Psi_{global} &= \cos\theta_1-\cos\theta_2-1+\frac{1}{2}\epsilon^2 ar_1\sin^2\theta_1+\frac{1}{2}\epsilon^2 ar_2\sin^2\theta_2 \\
&\quad -\frac{1}{2}\sigma_1\epsilon^2 ar_1^2\sin^2\theta_1+\frac{1}{2}\sigma_2\epsilon^2 ar_2^2\sin^2\theta_2
\end{aligned}$$

This satisfies the equation of motion and are symmetric under monopole exchange. Recall that when expanded near the first monopole,

$$-\cos\theta_2-1 \quad \approx \quad -\frac{1}{2}\frac{r_1^2\sin^2\theta_1}{s^2}.$$

Now, from figure 3.2,

$$r_2\sin\theta_2 \;=\; r_1\sin\theta_1 \;\Longrightarrow\; \frac{1}{2}\sigma_2\epsilon^2 a\left(r_2^2\sin^2\theta_2\right) \;=\; \frac{1}{2}\sigma_2\epsilon^2 a\left(r_1^2\sin^2\theta_1\right)$$

Then near the $\ominus$ monopole, if this term with σ_2 cancels the expansion term from the $\oplus$ monopole, Ψ_{global} would reduce to the local function $\Psi_\ominus$, and the condition on σ_2 would be

$$\sigma_2\epsilon^2 a = \frac{1}{s^2}$$

Similarly, near teh $\oplus$ monopole, the homogeneous solution $\Psi^{(\epsilon^2)}_{\oplus\ hom}$ can cancel with the term from expanding $\Psi^{(0)}_\ominus = \cos\theta_1$:

$$\begin{aligned}\cos\theta_1 - 1 - \frac{1}{2}\sigma_1\epsilon^2 a r_1^2\sin^2\theta_1 \quad &\approx \quad -\frac{1}{2}\frac{r_2^2\sin^2\theta_2}{s^2} - \frac{1}{2}\sigma_1\epsilon^2 a r_2^2\sin^2\theta_2 = 0 \\ &\implies \quad \sigma_2\epsilon^2 a = -\frac{1}{s^2}.\end{aligned}$$

What we have chosen here is that the homogeneous solution for each monopole is used to cancel the effect of the $\mathcal{O}(\epsilon^0)$ solution of the same monopole near the other monopole.

Now, we can again build the global function h to include the homogeneous solutions of h:

$$\begin{aligned}h_{global} \quad = \quad & c - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{2}\epsilon^2 a\cos\theta_1 - \frac{1}{2}\epsilon^2 a\cos\theta_2 \\ & +(\sigma_1 - c)\epsilon^2 a r_1\cos\theta_1 + (\sigma_2 + c)\epsilon^2 a r_2\cos\theta_2 + k_1 + k_2\end{aligned}$$

and use them along with the terms $\mp c\epsilon^2 ar\cos\theta$ to cancel with the terms from the expansion of $-1/r$ near both monopoles. That is, near the $\ominus$ monopole,

$$(\sigma_2 + c)\epsilon^2 a r_1\cos\theta_1 - \frac{r_1 cos\theta_1}{s^2} \quad = \quad 0$$

and since $\sigma_2\epsilon^2 a = \frac{1}{s^2}$ from before,

$$\epsilon^2 ac = 0$$

In the same manner, the conditions near the $\oplus$ monopole also result in a zero acceleration:

$$(\sigma_1 - c)\epsilon^2 a r_2\cos\theta_2 + \frac{r_2 cos\theta_2}{s^2} \quad = \quad 0 \implies \epsilon^2 ac \quad = \quad 0$$

Example II Our second example does not yield information about the acceleration. If we choose the global function Ψ_{global} to include a term "put in by hand" similar to the one in Manton's h_{global}:

$$\begin{aligned} \Psi_{global} &= \cos\theta_1 - \cos\theta_2 + \frac{1}{2}\epsilon^2 a r_1 \sin^2\theta_1 + \frac{1}{2}\epsilon^2 a r_2 \sin^2\theta_2 \\ &\quad -\frac{1}{2}\sigma_1\epsilon^2 a r_1^2 \sin^2\theta_1 - 1 + \frac{1}{2}\frac{r_1^2\sin^2\theta_1}{s^2} \end{aligned}$$

then it is still symmetric under monopole exchange because $r_1^2\sin^2\theta_1 = r_2^2\sin^2\theta_2$. Near either monopole, the added term would cancel the $\mathcal{O}\left(\frac{1}{s^2}\right)$ contribution from the expansion of the corresponding $\cos\theta$, and therefore the only condition needed for Ψ_{global} to reduce to the local functions $\Psi_\oplus$ and $\Psi_\ominus$ is

$$\sigma_1 = -\sigma_2$$

Consequently, if we choose h_{global} to be Manton's h_{global}, which gives also only one condition between σ_1, σ_2 and the acceleration, there is not enough constraints to determine $\epsilon^2 ac$.

Thus, it is not true that coming up with symmetric solutions for h and Ψ in the region between the monopoles and matching them to the corresponding local functions in regions close to the monopoles would give a unique correct answer for the acceleration.

Another Requirement for the Global Solutions and the Matching Principle

Let me now propose that Manton's method provides the correct answer only when it obeys the following exchange principle, which is much like what we use in electromagnetism to determine the fields for a system with two widely separated sources.

What I called the exchange principle is the assumption that the ambiguity of the local solution near one monopole is due to the presence of the other monopole.

In other words, in the matching process, the local homogeneous solution of Ψ and h near one monopole should be "produced" by the expansion of the Ψ and h solution of the other monopole, and accordingly, the global functions should not include any of the homogeneous solutions. Manton's Ψ_{global} is the one prescribed by this principle but his h_{global} is not. We now show how the matching of h_{global} would be done under this principle.

First, similar to how Manton build Ψ_{global}, we build h_{global} for two opposite charge monopoles by adding the $\mathcal{O}(\epsilon^0)$ solutions of h and the $\mathcal{O}(\epsilon^2)$ particular solutions of h:

$$\begin{aligned} h'_{global} &= c - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{2}\epsilon^2 a \cos\theta_1 - \frac{1}{2}\epsilon^2 a \cos\theta_2 \\ &\quad -\epsilon^2 a c r_1 \cos\theta_1 + \epsilon^2 a c r_2 \cos\theta_2 + k_{global} \end{aligned}$$

Now, unlike the situation for Ψ, one term in the particular solution of h for each monopole has the same functional form, $\sim r\cos\theta$, as the term from the expansion of the solution of the other monopole. This term in the particular solution is of $\mathcal{O}(\epsilon^2)$ and so cannot be neglected in the matching process. The other term, $\sim \epsilon^2 a \cos\theta$, coes not participate in the matching as argued before. Therefore, near the $\ominus$ monopole,

$$\begin{aligned} h_{global} &\longrightarrow c - \frac{1}{r_1} - \left(\frac{1}{s} + \frac{r_1\cos\theta_1}{s^2}\right) + \frac{1}{2}\epsilon^2 a \cos\theta_1 - \frac{1}{2}\epsilon^2 a(-1) \\ &\quad -\epsilon^2 a c r_1 \cos\theta_1 + \epsilon^2 a c(-s + r_1\cos\theta_1) + k_{global} \end{aligned}$$

and for h_{global} to reduce to $h_\ominus$:

$$\begin{aligned} -\frac{1}{s} + \frac{1}{2}\epsilon^2 a + k_{global} &= k_1 \\ \left(-\frac{1}{s^2} + \epsilon^2 ac\right)(r_1\cos\theta_1) &= \sigma_1 \epsilon^2 a\,(r_1\cos\theta_1) \quad \Longrightarrow \quad \epsilon^2 ac = \frac{2}{s^2} \end{aligned}$$

Similarly, the limit near the $\oplus$ monopole gives

$$\begin{aligned} h_{global} &\longrightarrow c - \frac{1}{r_2} - \left(\frac{1}{s} - \frac{r_2\cos\theta_2}{s^2}\right) - \frac{1}{2}\epsilon^2 a \cos\theta_2 + \frac{1}{2}\epsilon^2 a(1) \\ &\quad +\epsilon^2 a c r_2 \cos\theta_2 - \epsilon^2 a c(s + r_2\cos\theta_2) + k_{global} \end{aligned}$$

which gives the matching conditions:

$$
\begin{aligned}
-\frac{1}{s}+\frac{1}{2}\epsilon^2 a + k_{global} &= k_2 = k_1 \\
\left(+\frac{1}{s^2}-\epsilon^2 ac\right)(r_2\cos\theta_2) &= \sigma_2\epsilon^2 a\,(r_2\cos\theta_2) \quad \Longrightarrow \quad \epsilon^2 ac \;=\; \frac{2}{s^2}
\end{aligned}
$$

For two same charge monopoles, the matching condition near monopole 1 does not change except that the value of σ_1 has been determined to be different by the matching the Ψ functions:

$$
-\frac{1}{s^2}+\epsilon^2 ac \;=\; \sigma_1\epsilon^2 a \quad \text{but} \quad \sigma_1\epsilon^2 a \;=\; -\frac{1}{s^2}
$$

which implies $\epsilon^2 ac = 0$. The matching condition near monopole 2is simply the negative of the one from the first monopole and so gives the same conclusion.

Notice for both systems, we obtain two conditions from matching h that agree with each other while Manton obtains one only.

Why ϵa is trivial It is very clear under this matching principle why the $\mathcal{O}(\epsilon)$ acceleration vanishes. Since the $\mathcal{O}(\epsilon^0)$ Ψ solutions, $\sim \cos\theta$, do not expand to give any order $1/s$ terms, the terms with σ_1 and σ_2 cannot be not "produced," i.e., $\sigma_1\epsilon a = \sigma_2\epsilon a = 0$. Then for h_{global}, the local particular solution for either monopole, $\sim \epsilon a c r\cos\theta$, which is to combine with the expansion from the $\mathcal{O}(\epsilon^0)$ h solution of the other monopole to produce the local homogeneous term, the term with the corresponding σ, has to be zero, because both the expansion term and the σ term are zero.

Interpretation of the Matching Process

With the exchange principle, we could have found the correct acceleration by matching the magnetic and ∇h fields instead of h and Ψ. In this case, we would only need to show that the global solution $\hat{\phi}$ exists but not solve for it explicitly.

Also, we can interpret this matching process as the application of the respective external force laws (Eq 3.20) on the monopoles.

First, according to the exchange principle and due to the linear dependence of the magnetic field $\vec{B}$ on Ψ, the global magnetic field is the superposition of the $\mathcal{O}(\epsilon^0)$ Coulomb fields and the $\mathcal{O}(\epsilon^2)$ particular solution to the first order ansatzes and does not include the undetermined local homogeneous solutions:

$$\vec{B}_{global} = -\frac{\hat{r}_1}{r_1^2} + \frac{\hat{r}_2}{r_2^2} + \vec{B}^{(\epsilon^2)}_{\ominus part} + \vec{B}^{(\epsilon^2)}_{\oplus part}$$

Note that this global magnetic field has similar contributions from individual monopoles as the electric field does for two separated electric charges except for the differences discussed in section 3.1.2; however, unlike in the two electric charge system, the superposition of fields here is only valid in the asymptotic region.

Similarly, the global ∇h field is the superposition of the ∇h fields from the different monopoles, which can be easily written in terms of the local magnetic fields from the first order ansatzes:

$$\begin{aligned} \nabla h_{global} &= \left[-\left(-\frac{\hat{r}_1}{r_1^2} + \vec{B}^{(\epsilon^2)}_{\ominus part}\right) - \epsilon^2\vec{a}\left(c - \frac{1}{r_1}\right)\right] \\ &\quad + \left[\frac{\hat{r}_2}{r_2^2} + \vec{B}^{(\epsilon^2)}_{\oplus part} + \epsilon^2\vec{a}\left(c - \frac{1}{r_2}\right)\right] \end{aligned} \tag{3.21}$$

That the constant terms $\epsilon^2\vec{a}c$ contributed by the different monopoles cancel each other will be important for our interpretation of the matching process. This cancellation is due to the monopoles accelerating in opposite directions and happens regardless of the charges of the monopoles.

Now, in the process of matching these global fields to the local fields near each monopole, the matching of the constant vectors is what gives the information about the acceleration.

The local magnetic field at each monopole contains only one constant term, the homogeneous solution to the ansatz, $\pm\sigma\epsilon^2\vec{a}$, which is to be equated to the expansion of the field from the other monopole under the exchange principle. To $\mathcal{O}(\epsilon^2)$, only the expansion of the static Coulomb field from the other monopole contributes. For instance, near the $\ominus$ monopole, the undetermined constant, $-\sigma_1\epsilon^2\vec{a}$, is matched to the following:

$$\vec{B}^{(0)}_{\oplus} = +\frac{\hat{r}_2}{r_2^2} \longrightarrow -\frac{\hat{z}}{s^2}$$

On the other hand, the local ∇h expressions contain the constant terms $\sigma\epsilon^2\vec{a}\mp\epsilon^2\vec{a}c$. For each monopole, this constant term, under the exchange principle, is to be given rise only by the expansion of the ∇h field from the other monopole because the constants that appeared in the global expression (Eq 3.21) cancelled each other. Again, to $\mathcal{O}(\epsilon^2)$, only the expansion of the $\mathcal{O}(\epsilon^0)$ field of the other monopole contributes, and this, depending on the charge of that other monopole, is simply plus or minus the contribution of the magnetic field from that monopole. For the $\ominus$ monopole, then, the constant terms $(\sigma_1\epsilon^2\vec{a} - \epsilon^2\vec{a}c)$ in the local ∇h is equated to the far-field limit of $\nabla h^{(0)}_{\oplus} = +\vec{B}^{(0)}_{\oplus}$.

We can now see that matching the global fields to the local ones under the exchange principle implies that the constant part of the first order ansatz at each monopole relates only the "external" fields produced by the other monopole to its acceleration. Thus, matching with the exchange principle and using the different first order ansatzes to determine the accelerations is like applying external force laws to the monopoles:

$$\begin{aligned}
\text{Two } \ominus \text{ monopoles:} \quad \pm(\epsilon^2\vec{a})\frac{m}{g} &= -\vec{B}_{ext} - \nabla h_{ext} \\
&= -\vec{B}_{ext} - (-\vec{B}_{ext}) \;=\; 0 \\
\ominus/\oplus \text{ monopoles:} \quad \pm\,\epsilon^2\vec{a}\frac{m}{g} &= \mp\vec{B}_{ext} - \nabla h_{ext} \\
&= \mp\vec{B}_{ext} - (\pm\vec{B}_{ext}) \;=\; \mp 2\vec{B}_{ext} \;=\; \hat{z}\frac{2}{s^2}
\end{aligned}$$

where the upper signs are for the monopole on the negative z-axis and the lower signs for the other one.

This way of finding the acceleration between two monopoles, then, has become similar to the way of finding the lowest order force between two widely separated local and spherically-symmetric electric sources by the multipole expansion in normal electromagnetism. There are a few differences:

1. the external force law (Eq 3.20) used for the monopole pair problem, unlike the Lorentz Force Law used in the electric problem, involves an extra ∇h force which is attractive regardless of the charges of the monopoles;
2. while in the electric problem, the mass of each electric source is free to vary with its total charge and so its acceleration under the external electric field from the other source varies accordingly, the mass of the monopoles is determined solely by the charge and the parameter c, and consequently, the acceleration of the monopoles is fixed once the external fields are known;
3. in the electric problem, the Lorentz Force Law can be applied at each point in either of the local charge distributions to give the induced multipole moments, but the external force law (Eq 3.20) for the magnetic monopoles is not to be applied pointwise (there is no pointlike magnetic sources to be acted on either) and does not allow us to find the deformation of the non-pointlike monopole under the influence of the external field.

3.2.3 Limitations of the Manton's Method

Apart from not being applicable as a local force law, Manton's ansatz also does not help us determine the force between two opposite charge monopoles above the lowest order. We will first show that the ansatz can actually be extended to the first order above lowest order in ϵ but then the matching procedure that works for the lowest order breaks down despite of the valid ansatz. Through this process, We will understand better how the matching process works for the lowest order. Manton's method, however, does work at all orders of ϵ for the same charge monopole pair. We will argue this at the end of this section.

Extension of First Order Ansatz to $\mathcal{O}(\epsilon^3)$ Because the $\mathcal{O}(\epsilon^1)$ acceleration is zero between two monopoles, Manton's ansatz for the accelerating monopole can easily be shown to work for an assumed acceleration of one higher order in ϵ, $\epsilon^3\vec{a}'$.

We can simply replace $\epsilon^2\vec{a}$ by $(\epsilon^2\vec{a}+\epsilon^3\vec{a}')$ in every step of the derivation (section 3.1.1) for the $\mathcal{O}(\epsilon^2)$ acceleration, find that each step remains valid because any terms with explicit time dependence (Eq 3.3, Eq 3.4) that would ruin the derivation are of the order of the square of the first non-trivial order, i.e., $\mathcal{O}((\epsilon^2)^2)$, and arrive at the following extended ansatz:

$$\vec{B} \;=\; \pm\left[\nabla h+(\epsilon^2\vec{a}+\epsilon^3\vec{a}')\left(c-\frac{1}{r}\right)\right]$$

This says that any $\mathcal{O}(\epsilon^3)$ constant external $\vec{B}$ and ∇h fields would contribute to an $\mathcal{O}(\epsilon^3)$ constant force on the monopole, $m(\epsilon^3\vec{a}')$, on top of the $\mathcal{O}(\epsilon^2)$ force.

Repeating Manton's Method at $\mathcal{O}(\epsilon^3)$ In electromagnetism, the first order force above the Coulomb order on a local charge distribution with a constant dipole density involves the gradient of the external electric field, $\vec{F}_{1/s^3}\sim\vec{p}\cdot\nabla\vec{E}$. The ansatz derived above involves only uniform external fields to $\mathcal{O}(\epsilon^3)$ and already signals that it may not work in a system where the gradient of the external fields is not uniform. Here, we show explicitly how matching the local and global solutions of the opposite charge monopole pair up to $\mathcal{O}(\epsilon^3)$ results in a questionable conclusion for $\epsilon^3\vec{a}'$ as well as an inconsistency. We will then see that Manton's procedure works at $\mathcal{O}(\epsilon^2)$ by "rescuing" the same inconsistent situation had we used only the static ansatz to build the global solution.

First, the equations for the local Ψ functions near both monopoles in the $\ominus/\oplus$

system includes the new acceleration:

$$\nabla \times \vec{B}(\Psi) = \hat{\chi}\,\frac{(\epsilon^2 a + \epsilon^3 a')\ \sin\theta}{r^2} \tag{3.22}$$

Solving, the local Ψ's near both monopoles contain a particular solution for that new term as well as the homogeneous solutions to this order:

$$\begin{aligned}
\Psi_{\ominus} &= \left[\cos\theta_1 + \frac{1}{2}\epsilon^2 a r_1 \sin^2\theta_1 - \frac{1}{2}\sigma_1\epsilon^2 a r_1^2 \sin^2\theta_1\right] \\
&\quad + \frac{1}{2}\epsilon^3 a' r_1 \sin^2\theta_1 - \frac{1}{2}\sigma_1'\epsilon^3 a' r_1^2 \sin^2\theta_1 - \rho_1\epsilon^3 a' r_1^3 \sin^2\theta_1\cos\theta_1 \\
\Psi_{\oplus} &= \left[-\cos\theta_2 + \frac{1}{2}\epsilon^2 a r_2 \sin^2\theta_2 + \frac{1}{2}\sigma_2\epsilon^2 a r_2^2 \sin^2\theta_2\right] \\
&\quad + \frac{1}{2}\epsilon^3 a' r_2 \sin^2\theta_2 + \frac{1}{2}\sigma_2'\epsilon^3 a' r_2^2 \sin^2\theta_2 + \rho_2\epsilon^3 a' r_2^3 \sin^2\theta_2\cos\theta_2
\end{aligned}$$

We write the global solution without the undetermined terms as prescribed by the exchange principle:

$$\begin{aligned}
\Psi_{global} &= \cos\theta_1 - \cos\theta_2 - 1 + \frac{1}{2}\epsilon^2 a r_1 \sin^2\theta_1 + \frac{1}{2}\epsilon^2 a r_2 \sin^2\theta_2 \\
&\quad + \frac{1}{2}\epsilon^3 a' r_1 \sin^2\theta_1 + \frac{1}{2}\epsilon^3 a' r_2 \sin^2\theta_2
\end{aligned}$$

Again, to match the global solution with the local solution near each monopole, we expand the terms "belonging" to the other monopole in the global solution to $\mathcal{O}(\epsilon^3)$ and equate the resulting terms with the ambiguities in the local solutions. This time, the expansion of the $\mathcal{O}(\epsilon^0)$ static parts of Ψ_{global},

$$\begin{aligned}
\cos\theta_1 &\longrightarrow 1 - \frac{1}{2}\frac{r_2^2\sin^2\theta_2}{s^2} + \frac{r_2^3\cos\theta_2\sin^2\theta_2}{s^3} + \mathcal{O}\left(\frac{1}{s^4}\right) \quad \text{near } \oplus \text{ monopole} \\
-\cos\theta_2 &\longrightarrow 1 - \frac{1}{2}\frac{r_1^2\sin^2\theta_1}{s^2} - \frac{r_1^3\cos\theta_1\sin^2\theta_1}{s^3} + \mathcal{O}\left(\frac{1}{s^4}\right) \quad \text{near } \ominus \text{ monopole}
\end{aligned}$$

gives $\mathcal{O}(\epsilon^3)$ terms that can "produce" the terms with coefficients $\rho_{1,2}$ in the opposite local solutions provided that

$$-\rho_1\epsilon^3 a' = -\frac{1}{s^3} \quad ; \quad \rho_2\epsilon^3 a' = \frac{1}{s^3}, \tag{3.23}$$

whereas the expansion of the $\mathcal{O}(\epsilon^2)$ particular solutions are proportional to the $\mathcal{O}(\epsilon^3)$ terms with coefficient $\sigma'_{1,2}$:

$$\begin{array}{lcll} \frac{1}{2}\epsilon^2 a r_1 \sin^2\theta_1 & \longrightarrow & \frac{1}{2}\epsilon^2 a \frac{r_2^2\sin^2\theta_2}{s} & \text{near } \oplus \text{ monopole} \\ \frac{1}{2}\epsilon^2 a r_2 \sin^2\theta_2 & \longrightarrow & \frac{1}{2}\epsilon^2 a \frac{r_1^2\sin^2\theta_1}{s} & \text{near } \ominus \text{ monopole} \end{array}$$

and therefore the matching conditions are

$$-\frac{\sigma'_1}{2}\epsilon^3 a' = \frac{1}{2}\frac{\epsilon^2 a}{s} \quad ; \quad \frac{\sigma'_2}{2}\epsilon^3 a' = \frac{1}{2}\frac{\epsilon^2 a}{s} \tag{3.24}$$

However, we are matching terms analogous to the radiation terms from an accelerating electric charge (section 3.1.2) to the local unknowns, and if, without the assumption that the fields of the monopole accelerate rigidly everywhere, these terms were retarded in time as the radiation in electromagnetism, then at the instant when the monopoles start to accelerate, their effect would not have reached the opposite monopoles to produce the undetermined homogeneous solutions there. Thus, the above condition seems to violate special relativity and is questionable. We will discover yet a more blatant break-down at $\mathcal{O}(\epsilon^3)$ of this method in the following.

To solve for the local h, we first find the magnetic field with the additional $\mathcal{O}(\epsilon^3)$ terms near each monopole from the local Ψ functions:

$$\begin{aligned} \vec{B}_\ominus &= \left[-\frac{\hat{r}_1}{r_1^2} + \hat{r}_1\epsilon^2 a\frac{\cos\theta_1}{r_1} - \hat{\theta}_1\frac{1}{2}\epsilon^2 a\frac{\sin\theta_1}{r_1} - \sigma_1\epsilon^2\vec{a}\right] + \hat{r}_1\epsilon^3 a'\frac{\cos\theta_1}{r_1} - \hat{\theta}_1\frac{1}{2}\epsilon^3 a'\frac{\sin\theta_1}{r_1} \\ &\quad -\sigma'_1\epsilon^3\vec{a'} - \rho_1\epsilon^3 a'\left(\hat{r}_1(2r_1 - 3r_1\sin^2\theta_1) - \hat{\theta}_1 3r_1\sin\theta_1\cos\theta_1\right) \\ \vec{B}_\oplus &= \left[+\frac{\hat{r}_2}{r_2^2} + \hat{r}_2\epsilon^2 a\frac{\cos\theta_2}{r_2} - \hat{\theta}_2\frac{1}{2}\epsilon^2 a\frac{\sin\theta_2}{r_2} + \sigma_2\epsilon^2\vec{a}\right] + \hat{r}_2\epsilon^3 a'\frac{\cos\theta_2}{r_2} - \hat{\theta}_2\frac{1}{2}\epsilon^3 a'\frac{\sin\theta_2}{r_2} \\ &\quad +\sigma'_2\epsilon^3\vec{a'} + \rho_2\epsilon^3 a'\left(\hat{r}_2(2r_2 - 3r_2\sin^2\theta_2) - \hat{\theta}_2 3r_2\sin\theta_2\cos\theta_2\right) \end{aligned}$$

Note that as before (section 3.2.2), matching $\vec{B}$ instead of Ψ gives the same equations for the unknown parameters but involves approximating the unit vector

$\hat{r}_1$ in terms of the unit vectors $\hat{r}_2$ and $\hat{\theta}_2$ near the second monopole and vice versa near the first one. For example, near the $\ominus$ monopole, in cylindrical coordinates,

$$\frac{\hat{r}_2}{r_2^2} \longrightarrow \left(-\hat{z} + \frac{r_1 \sin\theta_1}{s}\hat{\rho}\right)\left(\frac{1}{s^2} + \frac{2r_1\cos\theta_1}{s^3}\right) = -\hat{z}\frac{1}{s^2} + \hat{\rho}\frac{r_1\sin\theta_1}{s^3} - \hat{z}\frac{2r_1\cos\theta_1}{s^3}$$

while the undetermined terms with ρ_1 in the local $\ominus$ solution is proportional to the $\mathcal{O}(s^{-3})$ vector in the above expansion:

$$\rho_1\epsilon^3 a'\left(\hat{r}_1(2r_1 - 3r_1\sin^2\theta_1) - \hat{\theta}_1 3r_1\sin\theta_1\cos\theta_1\right) = \rho_1\epsilon^3 a'\left(-\hat{\rho}\, r_1\sin\theta_1 + \hat{z}2r_1\cos\theta_1\right)$$

The matching of these then gives the same condition for ρ_1 as before (Eq 3.23).

As well, notice that the local $\mathcal{O}(\epsilon^3)$ homogeneous $\vec{B}$ fields diverge at infinity, but since they are to be evaluated only near the monopoles, they are admissible.

We proceed to solve for h near both monopoles using the ansatz, which relates h to $\vec{B}$:

$$\begin{aligned}
h_\ominus &= \left[c - \frac{1}{r_1} + (\sigma_1 - c)\epsilon^2 a r_1\cos\theta_1 + \frac{1}{2}\epsilon^2 a\cos\theta_1 + k_1\right] \\
&\quad +(\sigma_1' - c)\epsilon^3 a' r_1\cos\theta_1 + \frac{1}{2}\epsilon^3 a'\cos\theta_1 + \rho_1\epsilon^3 a'(r_1^2 - \frac{3}{2}r_1^2\sin^2\theta_1) \\
h_\oplus &= \left[c - \frac{1}{r_2} + (\sigma_2 + c)\epsilon^2 a r_2\cos\theta_2 - \frac{1}{2}\epsilon^2 a\cos\theta_2 + k_2\right] \\
&\quad +(\sigma_2' + c)\epsilon^3 a' r_2\cos\theta_2 - \frac{1}{2}\epsilon^3 a'\cos\theta_2 + \rho_2\epsilon^3 a'(r_2^2 - \frac{3}{2}r_2^2\sin^2\theta_2)
\end{aligned}$$

Again, we expand the static solutions to $\mathcal{O}(\epsilon^3)$ for the matching:

$$\begin{aligned}
\frac{1}{r_1} &\longrightarrow \frac{1}{s} - \frac{r_2\cos\theta_2}{s^2} - \frac{r_2^2}{2s^3} + \frac{3}{2}\frac{r_2^2\cos^2\theta_2}{s^3} + \frac{3}{2}\frac{r_2^3\cos\theta_2}{s^4} - \frac{5}{2}\frac{r_2^3\cos^3\theta_2}{s^4} + \mathcal{O}\left(\frac{1}{s^5}\right) \\
\frac{1}{r_2} &\longrightarrow \frac{1}{s} + \frac{r_1\cos\theta_1}{s^2} - \frac{r_1^2}{2s^3} + \frac{3}{2}\frac{r_1^2\cos^2\theta_1}{s^3} - \frac{3}{2}\frac{r_1^3\cos\theta_1}{s^4} + \frac{5}{2}\frac{r_1^3\cos^3\theta_1}{s^4} + \mathcal{O}\left(\frac{1}{s^5}\right)
\end{aligned}$$

Predictably, this expansion does not give a $\mathcal{O}(\frac{1}{s^3})$ term proportional to $r\cos\theta$, which is what it gives at the lower order $\mathcal{O}(\frac{1}{s^2})$; hence the matching conditions for σ_1' and σ_2' are simply

$$\sigma_1'\epsilon^3 a' = \epsilon^3 a' c \quad ; \quad \sigma_2'\epsilon^3 a' = -\epsilon^3 a' c$$

and both imply the $\mathcal{O}(\epsilon^3)$ acceleration is given in terms of the $\mathcal{O}(\epsilon^2)$ one:

$$\epsilon^3 a' c = -\frac{\epsilon^2 a}{s} = -\frac{2}{s^3 c}$$

This result, however, is based on the questionable conditions, Eq 3.24.

On the other hand, the expansions of h above can give rise to the terms with ρ near the monopoles provided that

$$\rho_1 \epsilon^3 a' = \rho_2 \epsilon^3 a' = -\frac{1}{s^3}$$

This is in contradiction with the conditions obtained from matching Ψ (Eq 3.23).

The Scope of the External Force Law What this contradiction says is more transparent when we look at it in terms of the gauge invariant fields, $\vec{B}$ and ∇h.

First, we write down the first order ansatz accurate to $\mathcal{O}(\epsilon^3)$ but this time include also the set of magnetic multipole moments, $\vec{m}_{\vec{B},n}$, which are the homogeneous solutions for the perturbed equation Eq 3.22, and the multipole moments of ∇h, $\vec{m}_{\nabla h,n}$, which are determined by the ansatz in terms of the magnetic moments:

$$\begin{aligned}\left(\epsilon^2 \vec{a} + \epsilon^3 \vec{a}'\right) c - \left(\epsilon^2 \vec{a} + \epsilon^3 \vec{a}'\right) \frac{1}{r} &= \left(\vec{B}_{static} \pm \nabla h_{static}\right) + \left(\vec{B}_{rad} \pm \nabla h_{rad}\right) \\ &\quad + \sum_n \left(\vec{m}_{\vec{B},n} \pm \vec{m}_{\nabla h,n}\right)\end{aligned}$$

Now, both the static part of the fields and the analog of the radiation fields, which are the particular solutions of Eq 3.22, are totally determined and do not possibly lead to any contradiction. It is the fact that the undetermined multipole moments of both $\vec{B}$ and ∇h are to be matched under the exchange principle to the respective external fields that causes the contradiction: since the LHS of the ansatz contains no terms proportional to any moments above the lowest order of the multipole expansion, if the higher multipole moments of the external $\vec{B}$ and ∇h fields when equated with $\vec{m}_{\vec{B},n}$ and $\vec{m}_{\nabla h,n}$ respectively do not combine according to the RHS to vanish, then the ansatz has become false. For example, in the two different charge monopole system, although the ansatz is derivable for $\mathcal{O}(\epsilon^3)$, the $\mathcal{O}(\epsilon^3)$ external fields for each monopole in the $\ominus/\oplus$ system add up instead of cancel because of the difference in charge:

$$\begin{aligned} 0 &= \vec{m}_{\vec{B},2} \pm \vec{m}_{\nabla h,2} \qquad \text{near } \ominus/\oplus \text{ monopole} \\ &= \vec{B}_{ext} \pm \nabla h_{ext} \\ &= \vec{B}_{ext} \pm \left(\pm \vec{B}_{ext}\right) \;=\; 2\vec{B}_{ext} \;\neq\; 0 \end{aligned}$$

This failure of the ansatz means that unless there exists a solution other than Manton's ansatz for weakly rigidly accelerating monopoles for which there is no inconsistencies at $\mathcal{O}(\epsilon^3)$ when the two non-uniform external fields do not combine to zero, the assumption that the fields up to $\mathcal{O}(\epsilon^3)$ are rigidly accelerating under non-zero non-uniform total external field is incorrect. This is reasonable if the monopole were to behave similarly to a finite size ball of electric charge with spherically symmetric charge density under a non-uniform field: the ball would deform instead of accelerate rigidly.

The contradiction, however, does not imply any values for the $\mathcal{O}(s^{-3})$ acceleration; in particular, it does not imply that the $\mathcal{O}(s^{-3})$ force between two opposite charge monopoles is non-zero.

We can now also see that for the static solution of two opposite charge monopoles, for which neither monopole is accelerating, the static ansatz, which does not include the constant term $\epsilon^2\vec{a}c$, would be satistfied near each monopole, and the above contradiction for higher multipole moments would appear even for the lowest moment, the constant. The accelerated ansatz allows the two monopole solution to be consistent to $\mathcal{O}(\epsilon^2)$ by providing a "way out" for the lowest moment.

Finally, note that at $\mathcal{O}(\epsilon^4)$, if the acceleration at $\mathcal{O}(\epsilon^2)$ is non-zero, the ansatz is not satisfied even if there exists only uniform external fields at $\mathcal{O}(\epsilon^4)$.

On the other hand, for the two same charge monopole system, the contradiction above does not occur because near each monopole, the external $\vec{B}$ and ∇h moments arising from the fields of the other monopole already have the relationship required by the ansatz:

$$\vec{m}_{\vec{B},n} = \mp\vec{m}_{\nabla h,n} \text{ and } \vec{B}_{ext} = \mp\nabla h_{ext} \text{ near } \ominus/\oplus \text{ monopole}$$

Also, since the $\mathcal{O}(\epsilon^2)$ has been determined to be zero, the $\mathcal{O}(\epsilon^2)$ "radiation" terms that could potentially give a non-zero result for the acceleration at $\mathcal{O}(\epsilon^3)$ vanish, and so $\epsilon^3\vec{a}' = 0$. Now, since $\epsilon^2\vec{a} = 0$, the derivation of the ansatz is valid for $\mathcal{O}(\epsilon^4)$, and again, the matching at this order does not involve any inconsistencies and the lower order "radiation" terms being zero would lead to the $\mathcal{O}(\epsilon^4)$ acceleration being zero. We can do this at all orders of ϵ and conclude that the acceleration of monopoles in a two same charge monopole system vanishes to all orders of ϵ, i.e., the vanishing force between two same charge monopoles in the BPS limit is an exact result.

3.3 Finding the Force through Calculating the Momentum Flux

We look at another way to find the force between two monopoles proposed by Goldberg et al [2], which gives the result as Manton's procedure, and reinforce our interpretation of the first order ansatz as the uniform external force law. We also discuss the possibility of using Manton's two-monopole global solution without the $\mathcal{O}(\epsilon^2)$ terms as the static solution in Goldberg's method and the possibility of concluding that the only force between two monopoles is the $\mathcal{O}(1/s^2)$ force. In the process, we understand better what is essential in Manton's method.

Goldberg et al [2] find the force between two monopoles by calculating the rate of change of momentum of either monopole in a static two-monopole configuration. The momentum of the monopole here means the momentum of the fields within a ball that encloses all the "matter fields" $\mathbf{J}^{\nu}$ (as defined before in Eq 2.5) of the monopole; the surface of the ball, then, has to be in the asymptotic region where $\mathbf{J}^{\nu} = \mathbf{0}$.

The momentum current is the spatial component of the stress-energy tensor, which is the Noether current obtained from translational symmetry, and is conserved:

$$\partial_{\mu} T^{\mu i} \; = \; 0,$$

therefore, the rate of change of each space component of the momentum inside the ball equals its current, $\vec{p}^j$, integrated over the surface of the ball:

$$\begin{aligned} Force^j &= \int_{ball} \frac{\partial P^j}{\partial t} dV = \int_{ball} \nabla \cdot \vec{p}^j \; dV \\ &= \int_{\delta ball} \vec{p}^j \, dA \quad \text{where } \vec{p}^{\;j} = T^{ij} \text{ and } P^j = T^{0j} \end{aligned}$$

This integral is by definition the force on the enclosed monopole and what we need to evaluate.

3.3.1 Stress-Energy Tensor and Reduction to the Electric problems

First, since the boundary of the balls is in the asymptotic region , the magnetic field on it is given, without λ^i being set to zero, by:

$$B^i = \epsilon^{ijk} f^{jk} = -\frac{1}{2}\left((\partial^j \hat{\phi} \times \partial^k \hat{\phi}) \cdot \hat{\phi} \right) + \partial^j \lambda^k - \partial^k \lambda^j \tag{3.25}$$

Problem for U Now, we already know that $f^{\mu\nu}$ satisfies the vacuum Maxwell equations, i.e., $\vec{\nabla} \times \vec{B} = 0$, in the asymptotic region; therefore, we can write B^i

as the gradient of a scalar potential U:

$$\vec{B} = \vec{\nabla} U$$

The divergence of $\vec{B}$ being zero implies that U satisfies the Laplace equation:

$$\vec{\nabla}^2 U = 0$$

and the flux conditions on $\vec{B}$ for each monopole implies the flux conditions on U:

$$\int_{\delta ball} \vec{B} \cdot \vec{da} = \pm 1 \implies \int_{\delta ball} \vec{\nabla} U \cdot \vec{da} = \pm 1$$

Finally, along with the requirement that U approaches a constant at infinity since the magnetic tends to zero there, the problem for U up to the monopole order is exactly analogous to the problem for the electric potential, V, for two separated local electric charge distributions with the same or opposite total charges.

Stress Energy Tensor in terms of scalar potentials We can write the stress energy tensor in terms of U and h in the asymptotic region:

$$\begin{aligned}
T^{\mu\nu} &= \mathrm{Tr}\left[\frac{1}{4}g^{\mu\nu}F^{\rho\lambda}F_{\rho\lambda} - F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{1}{2}g^{\mu\nu}D^{\rho}\phi D_{\rho}\phi - D^{\mu}\phi D^{\nu}\phi\right] \\
&= \frac{1}{4}g^{\mu\nu}\left(f^{\rho\lambda}\hat{\phi} \cdot f_{\rho\lambda}\hat{\phi} - f^{\mu\rho}\hat{\phi} \cdot f^{\nu}{}_{\rho}\hat{\phi}\right) + \left(\frac{1}{2}g^{\mu\nu}\partial^{\rho}h\hat{\phi} \cdot \partial_{\rho}h\hat{\phi} - \partial^{\mu}h\hat{\phi} \cdot \partial^{\nu}h\hat{\phi}\right) \\
&= \left(\frac{1}{4}g^{\mu\nu}f^{ij}f_{ij} - f^{\mu i}f^{\nu}{}_{i}\right) + \left(\frac{1}{2}g^{\mu\nu}\partial^{\rho}h\partial_{\rho}h - \partial^{\mu}h\partial^{\nu}h\right) \\
&= \left(\frac{1}{2}g^{\mu\nu}\partial^{k}U\partial_{k}U - \delta^{\mu\nu}_{\mu,\nu\neq 0}\partial^{k}U\partial_{k}U + \partial^{\mu}U\partial^{\nu}U\right) + \left(\frac{1}{2}g^{\mu\nu}\partial^{\rho}h\partial_{\rho}h - \partial^{\mu}h\partial^{\nu}h\right)
\end{aligned}$$

Notice that for a static configuration of electric charges for which the magnetic field vanishes, the stress-energy tensor in terms of the electric potential, V, is:

$$\begin{aligned}
T^{\mu\nu}_{electric} &= \frac{1}{4}g^{\mu\nu}f^{0i}f_{0i} - f^{\mu 0}f^{\nu}{}_{0} - f^{\mu i}f^{\nu}{}_{i} \\
&= -\frac{1}{2}g^{\mu\nu}\partial^{k}V\partial_{k}V + \partial^{\mu}V\partial^{\nu}V - \delta^{\mu}_{0}\delta^{\nu}_{0}\partial^{k}V\partial_{k}V
\end{aligned}$$

We can compare the first bracket in $T^{\mu\nu}$ with $T^{\mu\nu}_{electic}$ for different values of μ and ν, and see that the dependence of the former on U is exactly the same as the dependence of the latter on V. Since the equations for U are also the same as those for V in the analogous electric problem, the bracket involving U in $T^{\mu\nu}$ would give the same force law up to the monopole order for the two magnetic monopole system as the force law for two ordinary Maxwell electric monopoles. The higher moments are not determined because we have only the flux conditions on U and not a charge distribution for the magnetic sources.

On the other hand, the second bracket in $T^{\mu\nu}$ involving h depends on h just as $-T^{\mu\nu}$ depends on V except for the irrelevant case $\mu\nu = 00$. We will now show that the problem for h can also be reduced to a static electric problem.

the problem for h We know that in the asymptotic region, the static first order ansatz can be factorized:

$$\vec{B} = \pm\vec{\nabla}h \qquad \text{for } \oplus / \ominus \text{ monopole;} \tag{3.26}$$

the equation of motion $D^i D_i \vec{\phi} = 0$ reduces to the Laplace equation for h:

$$\vec{\nabla}^2 h = 0$$

and unlike for U,the flux conditions for h at both monopole are the same, due to the change of sign in Eq 3.26 when the monopole charge is changed:

$$\int_{\delta ball} \vec{\nabla}h \cdot \vec{da} = \int_{\delta ball} \pm\vec{B} \cdot \vec{da} = \pm\pm 1 = 1 \tag{3.27}$$

Therefore, the problem for h, for both same and opposite charge monopole pairs, is analogous to the electric potential problem for two separate local electric sources with the same total charges.

The terms in the stress-energy tensor involving h would then give the force opposite to that between two same electric charges, i.e., Coulomb attraction, for both the same charge and opposite charge monopole pairs.

Adding the force contribution from both U and h, we obtain twice the Coulomb attraction between two opposite charge monopoles, and zero force between two same charge monopoles. This is the same statement as the one given by our interpretation of Manton's ansatz as the external force law, that for the two opposite charge monopole system, the external forces on each monopole add up while for the same charge monopole pair, the external forces on each monopole cancel.

3.3.2 Manton's $\mathcal{O}(\epsilon^0)$ Global Solution as the Static Solution

In order for the above result to be valid, we need to show there exists a static solution in terms of h, $\hat{\phi}$ and λ^i) that would give the required potential U. We already know that h has a solution since it simply satisfies the Laplace equation with boundary conditions; hence we need to show only that there are $\hat{\phi}$ and $\vec{\lambda}$ fields that would give a magnetic field $\vec{B}$ that satisfies Maxwell equations in the region between the monopoles as well as the flux conditions (Eq 3.26) at the monopoles, or equivalently, fields that give the potential U that satisfies the Laplace equation and the proper flux conditions.

While Goldberg shows the existence of the static solution by solving the second order static equations from scratch, we already use Manton's $\mathcal{O}(\epsilon^0)$ global solution as the static solution:

$$\Psi_{global} = \cos\theta_1 - \cos\theta_2 - 1 \quad ; \quad h_{global} = c - \frac{1}{r_1} - \frac{1}{r_2} \quad ; \quad \vec{\lambda}_{global} = 0$$

$$\text{and} \quad \phi_{global} = h_{global}\ \hat{\phi}_{global}(\Psi_{global}) \quad ; \quad A^i_{global} = \frac{1}{e}\, \partial^i \hat{\phi}_{global} \times \hat{\phi}_{global}$$

In this solution, however, higher multipole fields of $\mathcal{O}(\epsilon^0)$ that may be needed to solve the equations of motion to higher order in ϵ are omitted through Manton's choice that $\vec{B}$ depends only on $\hat{\phi}$, i..e., $\vec{\lambda} = 0$. This means that this solution, as a static solution for Goldberg's method, also does not determine the higher order force. We will first show that with the presence of these fields in Manton's global solution, the discussion in this chapter remains valid, and then briefly look at how these higher order fields could possibly be zero.

Effects of multipole terms in static solution on Manton's method The inclusion of any higher order fields in the $\mathcal{O}(\epsilon^0)$ global solution would not alter Manton's matching procedure or conclusion.

First, in terms of the gauge invariant $\vec{B}$ and ∇h fields, even if there is a higher order field in the global $\vec{B}$ and ∇h solutions, the external force law to $\mathcal{O}(\epsilon^2)$ would still use only the Coulomb terms in the global solution and the acceleration $\epsilon^2\vec{a}$ would still be determined to be the same. Now, in terms of the Ψ, h, and $\vec{\lambda}$ fields, the argument is more complicated. Having a higher order field from each monopole means that $\vec{\lambda}$ from each monopole is not zero, because $\vec{\lambda}$ vanishing implies that $\vec{B}$ depends only on the unit vector $\hat{\phi}$, but $\vec{B}$ being gauge invariant means that however $\hat{\phi}$ rotates in the SU(2) gauge space, provided that the change is continuous, $\vec{B}$ remains invariant, and so changing $\hat{\phi}$ (continuously) cannot add a higher order contribution to $\vec{B}$. Thus, the curl of $\vec{\lambda}$ term in Eq 3.25 needs to be non-zero $\mathcal{O}(\epsilon^0)$ to give rise to any higher multipole fields which, just as the Coulomb fields, are of $\mathcal{O}(\epsilon^0)$ in the global solution.

With $\vec{\lambda} \neq 0$, the local ansatz for an accelerating monopole is modified to:

$$\vec{B}_0(\Psi) + \nabla \times \vec{\lambda} = \nabla h + \epsilon^2 \vec{a} h \tag{3.28}$$

where $\vec{B}$ is still linear in Ψ but now also linear in $\vec{\lambda}$. Since the terms with λ cannot contribute without singularities (the Dirac string) to the monopole term of $\vec{B}$, the Coulomb order fields in the global $\vec{B}$ field still depends only on Ψ; thus, the $\mathcal{O}(\epsilon^2)$ homogeneous solutions of Ψ near each monopole, which is to be determined by the lowest order term in the expansion of the $\Psi^{(0)}$ solution from the other monopole, is still matched as before without any influence from $\vec{\lambda}$. On the other hand, the matching of λ is not described by Manton's method because near each monopole, the external fields to be matched, which corresponds to the far field limit of the magnetic dipole or above fields from the other monopole, is of $\mathcal{O}(1/s^3)$ and above. As for h, the added magnetic multipole fields do imply more terms for the $\mathcal{O}(\epsilon^0)$ solution of h, but the matching of the local and global h up to $\mathcal{O}(\epsilon^2)$ does not involve these extra terms. In conclusion, the global $\mathcal{O}(\epsilon^0)$ magnetic field containing higher multipole fields from each monopole, does not interfere with the procedure discussed in previous sections to obtain the $\mathcal{O}(1/s^2)$ force between monopoles.

Possibilities for the generalized ansatz The higher order fields can be non-zero or zero depending on how Manton's ansatz for a single accelerating monopole generalizes to higher order in ϵ.

For instance, at $\mathcal{O}(\epsilon^3)$, Manton's first order ansatz no longer holds true, and it is possible that the correct relation between the $\mathcal{O}(\epsilon^3)$ fields has a consistent solution only in the presence of some external dipole fields. Then, we would need to include dipole contributions in the $\mathcal{O}(\epsilon^0)$ global magnetic field, and there would be an $\mathcal{O}(1/s^3)$ force between the monopoles due to the coupling between the monopole charge and the added dipole field, just as the force equals $q \cdot \left[\vec{E}_{mon}(0) + \vec{E}_{dip}(0)\right]$ for an electric charge in the presence of an external electric field having both Coulomb and dipole contributions. However, it is also possible that extra degrees of freedom exist in the $\mathcal{O}(\epsilon^3)$ ansatz and no field needs to be added to the global magnetic field, just as Manton's $\mathcal{O}(\epsilon^2)$ ansatz contains the degree of freedom, $\epsilon^2\vec{a}$, which is determined by and does not impose any condition on the already determined $\mathcal{O}(\epsilon^0)$ global solution. Thus, if we can argue that the higher order local equations does not require higher order external fields, we can use Manton's global solution in Goldberg et al's method to conclude that the higher order forces are zero.

3.4 Conclusion for the Commutative Problem

Manton's first success is his idea of solving the time-dependent equations of motion for the instant when a monopole accelerates perturbatively from rest such that the time dependence of the solution can be specified and the time-dependent equations can be modified accordingly, and his discovery of the first order ansatz for this scenario. We interpret his ansatz in its factorized form in the asymptotic region as the lowest order external force law that says that both the magnetic field and the field ∇h contribute to the force on a monopole.

Manton's second success is his choice of gauge and his discovery that in this gauge the magnetic field $\vec{B}$ can be written as a linear function of one of the components, Ψ, of $\hat{\phi}$, which is defined by $\phi = h\hat{\phi}$, and the solutions of Ψ and h determine the full solution, ϕ and A^{μ}. Then, because the first order accelerated ansatzes as well as the equations of motion in the region between the monopoles are linear in $\vec{B}$ (and so in Ψ) and h, the solutions of both Ψ and h in the middle region are simply the solutions to the sum of the accelerated ansatzes, and the magnetic and ∇h fields are in turn simply superpositions up to homogenous solutions of the first order ansatz of those produced by both monopoles. Manton claims that requiring the global solutions of h and Ψ to reduce to the local ones near each monopole determines the acceleration between the monopoles.

On the other hand, we explore the ambiguities of the global solutions and find that they lead to ambiguity of the conclusion for the acceleration. We propose eliminating these ambiguities by a simple exchange priniple, which says that the global solutions should not include any homogeneous solutions to the accelerated ansatzes and that the homogeneous solutions at each monopole are to be determined by the far field limit of the solutions from the opposite monopole. This again suggests the interpretation of the Manton's ansatz as an external uniform force law at each monopole, with the external fields $\vec{B}_{ext}$ and ∇h_{ext} being simply the lowest order term in the multipole expansion of the fields from the opposite monopole.

We then discover that although Manton's ansatz is derivable for the next order in the small parameter ϵ quantifying the monopole acceleration, the ansatz cannot be satisfied near the monopoles within the two opposite charge monopole system. We speculate that this implies that the monopoles in such a system deform and is not accelerating rigidly at this order. We show that the acceleration between two same charge monopoles vanish to all orders of ϵ.

Goldberg et al arrives at the same conclusion for the force on a monopole in a two monopole system by calculating the momentum flux through a surface that encloses that monopole in a static two monopole configuration. More explicitly, they solve the static equations of motion for the two monopole system, substitute this static solution into the stress-energy tensor, and integrate the momentum currents given by the tensor over surface enclosing the monopole. Their success is in noticing that in the asymptotic region, the stress-energy tensoris composed of two pieces, each depends on a scalar potential as the U(1) electromagnetic stress-energy tensor depends on the electric scalar potential when only static electric fields are present; and in solving for the static two monopole configuration in terms of these two scalar potentials, U ($\vec{B} = \nabla U$), and h ($\phi = h\hat{\phi}$).

Goldberg's approach gives another perspective on the monopole force problem. First, the stress-energy tensor written in terms of the scalar potentials shows clearly the force contributions on a monopole in this theory and reinforces our interpretation of Manton's ansatz as the external force law.

Secondly, Goldberg et al's assumption that the static solution needs to satisfy only the flux condition, that the integral of the divergence of the magnetic field over a volume enclosing the monopole be proportional to the charge of the monopole, coincides with the ambiguity of the higher order multipole fields in

Manton's $\mathcal{O}(\epsilon^0)$ global solution, such that both Goldberg et al's and Manton's methods yield the same undetermined result for the higher order ($\mathcal{O}(1/s^3)$ and above) force between two monopoles, although Manton's method of using an ansatz seems to allow us to guess better at what happens at the next order of ϵ.

We are interested in finding out if it is possible to argue without solving completely for the higher order ansatz that the higher order forces between two opposite monopoles vanish, or otherwise if we can find the force to the dipole order by proposing a specific time dependence which accounts for the deformation of the monopole to this order and then modifying and solving the time-dependent equations of motion.

Chapter 4

Background: Non-Commutative U(N) Gauge Theory

In order to reach our goal of applying Manton's method on monopoles in non-commutative flat space, we need to know the formalism and the classical equations of motion of the non-commutative gauge theory. We do not cover the quantum aspects in this introduction [17] [18].

4.1 Operator Formalism to Star product Formalism

Non-commutative geometry on flat space-time can be described by coordinates that are not numbers but operators whose commutation relation is given by the non-commutative parameter $\theta^{\mu\nu}$:

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu}$$

where $\theta^{\mu\nu}$ is antisymmetric and constant under Lorentz transformation. The imaginary "i" is there because $\hat{x}^{\nu}$ is Hermitian and the commutato of Hermitian operator are anti-Hermitian.

We will consider only spatial non-commutativity, not space-time non-commutativity, which poses more complexities. We choose coordinates such that the first two coordinates do not commute:

$$\theta^{ij} = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.1}$$

To write down an action for a field theory in non-commutative geometry, we need to first define the derivative and the integral for the non-commutative coordinates. We want these linear operators to retain the properties they have in commutative geometry [7]:

$$\begin{aligned} \partial_i \hat{x}^j &= \delta_i^j \\ \partial_i \left(\hat{f}\hat{g}\right) &= (\partial_i \hat{f})\hat{g} + \hat{f}(\partial_i \hat{g}) \\ \int \mathrm{Tr}\ \partial_i \hat{f} &= 0 \ \text{ for } \hat{f} \neq 0\,,\ \hat{f} \longrightarrow 0 \ \text{ at infinity} \\ \int \mathrm{Tr}[\hat{f}, \hat{g}] &= 0 \end{aligned} \tag{4.2}$$

The following choice of derivative for the non-commutative coordinates, i,j,=1, or 2) satisfies all of the above

$$\partial_i \hat{f} = [\hat{d}, \hat{f}] = [-i(\theta^{-1})_{ji}\hat{x}^j, \hat{f}]$$

where $\theta^{ij}(\theta^{-1})_{ji} = 1$; and the integral is uniquely determined by the rules above [7].

Mixing of gauge space and real space The integral is written as $\int \mathrm{Tr}$ because in non-commutative U(N) gauge theories, the notions of integrating over real space and tracing over the gauge indices cannot be separated. First, note that functions of the non-commutative coordinates being operators does not prevent the incorporation of gauge symmetries into theories, although it can affect what gauge groups are allowed (this will be discussed in the next section). Secondly, in a U(N) gauge theory in non-commutative spatial background, an operator field, $\hat{\mathbf{f}}$, in the adjoint representation transforms formally as in the commutative theory (this will be derived in the another formalism in section 4.2):

$$\hat{\mathbf{f}} \longrightarrow \hat{\mathbf{U}}\hat{\mathbf{f}}\hat{\mathbf{U}}^\dagger \quad ; \quad \hat{\mathbf{U}}\hat{\mathbf{U}}^\dagger \;=\; \hat{\mathbf{1}}$$

except that the "unitary" matrices are now unitary operator matrices and the multiplication is an operator matrix multiplication. Note that because $\hat{\mathbf{f}}$ and $\hat{\mathbf{U}}$ are operators and do not commute even if $\hat{\mathbf{f}}$ and $\hat{\mathbf{U}}$ are not matrices, the transformation above is not trivial even if the gauge group is U(1).

Finally, a special property of the non-commutative gauge theory is that the set of translations in the non-commutative directions are also gauge transformations. With the above derivative, an infinitesimal translation $\delta\hat{f}$ becomes a commutator:

$$\begin{aligned} \hat{x}^i \to \hat{x}^i + a^i \implies \hat{f}(\hat{x}^i) &\to \hat{f}(\hat{x}^i) + a^i(\partial_i \hat{f}(\hat{x}^i)) \\ &= \hat{f}(\hat{x}^i) - i[a^i(\theta^{-1})_{ji}\hat{x}^j, \hat{f}(\hat{x}^i)] \end{aligned}$$

the exponential form of which is

$$\hat{f}(\hat{x}^i + \alpha^i) \;=\; e^{-i(\theta^{-1})_{ji}\alpha^i\hat{x}^j}\;\hat{f}(\hat{x})\;e^{i(\theta^{-1})_{ji}\alpha^i\hat{x}^j} \tag{4.3}$$

It remains to show that $\hat{U} = e^{-i(\theta^{-1})_{ji}\alpha^i\hat{x}^j}$ is unitary in the operator sense:

$$\begin{aligned} \hat{U}\hat{U}^\dagger \;=\; e^{-i(\theta^{-1})_{ji}\alpha^i\hat{x}^j}\;e^{i(\theta^{-1})_{lk}\alpha^k\hat{x}^l} &= e^{\frac{1}{2}[(\theta^{-1})_{ji}\alpha^i\hat{x}^j,(\theta^{-1})_{lk}\alpha^k\hat{x}^l]} \\ &= e^{\frac{i}{2}(\theta^{-1})_{ji}\alpha^i\alpha^k(\delta^j_k)} = \hat{1} \end{aligned}$$

This means that when we move in real space, we also move in the gauge space. Therefore integrating over real space is not orthogonal to tracing over the gauge indices: $\int$ and Tr are to be used together.

Basis for the non-commutative space Since the commutation relation between $\hat{x}^1$ and $\hat{x}^2$ is analogous to the commutator of $\hat{x}^i$,$\hat{p}^i$ in quantum mechanics, we can introduce creation and annihilation operators just as in quantum mechanics then:

$$c = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 - i\hat{x}^2) \quad ; \quad c^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 + i\hat{x}^2)$$
$$\text{s.t. } [c, c^\dagger] = 1 \tag{4.4}$$

Since the derivative operator can be written in terms of the coordinate operators, they can also be written in terms of c and $c^\dagger$.

We can then describe functions of $\hat{x}^1$ and $\hat{x}^2$ in terms of matrix elements $< m|f(\hat{x}^1, \hat{x}^2)|n >$ where $|n >$ for $n = 0$ to ∞ is the basis of the standard annihilation and creation operator Fock space:

$$c^\dagger \,|n >= \sqrt{n+1}\,|n+1> \quad ; \quad c = \sqrt{n}\,|n-1>$$
$$c^\dagger c\,|n >= n\,|n> \quad ; \quad < m|n >= \delta_{mn}$$

Integrating over the two non-commutative direction then amounts to Tracing over these states.

Derivation of the Star Product There is a way to write down non-commutative geometry without involving operators: Bayen et al [19] introduced a map between the operator-valued functions $\hat{f}(\hat{x})$ and number-valued functions $f(x)$ such that the operator product $\hat{f}(\hat{x})\hat{g}(\hat{x})$ would map to a product, called the star product, $f(x) * g(x)$, which reduces to the ordinary pointwise product $f(x)g(x)$ when the non-commutative parameter θ goes to zero.

We will need two maps. The first map $f(k)[\hat{f}]$ is defined as a formal fourrier transform of the operator function $f(\hat{x}$ to commutative momentum space:

$$f(k)[\hat{f}] = \frac{1}{2\pi}\int \mathrm{Tr}\, e^{-ik_i\hat{x}^i}\hat{f}(\hat{x})$$

The second map $\hat{f}(\hat{x})[f]$ is the formal inverse fourrier transform back to coordinate operator space, whose definition needs a prescription for the ordering of the non-commutative operators $\hat{x}^1$, $\hat{x}^2$. We will the standard Weyl-order definition:

$$\hat{f}(\hat{x})[f] = \frac{1}{2\pi}\int d^2k\, f(k)\, e^{ik_i\hat{x}^i}$$

An alternative formula may help to show what Weyl ordering is:

$$W[(\hat{x}^1)^m(\hat{x}^2)^r] = \frac{1}{2^m}\sum_{l=0}^{m}\frac{m!}{l!(m-l)!}(\hat{x}^1)^{m-l}(\hat{x}^2)^r(\hat{x}^1)^l$$

Once we have mapped the operator function to momentum space, we can inverse Fourrier transform it to ordinary coordinate space. Thus, an operator $\hat{O}(\hat{x})$ would map to $O(x)$ as follow:

$$O(x)[\hat{O}] = \frac{1}{(2\pi)^2}\int d^2k\, e^{-ik_ix^i}\int \mathrm{Tr}\, e^{ik_i\hat{x}^i}\hat{O}(\hat{x})$$

Now, the operator product, $\hat{O}(\hat{x}) = \hat{f}(\hat{x})\hat{g}(\hat{x})$, can be written in terms of $f(k')[\hat{f}]$ and $g(k'')[\hat{g}]$:

$$\begin{aligned}
\hat{O}(\hat{x}) &= \hat{f}(\hat{x})\hat{g}(\hat{x}) \\
&= \left(\frac{1}{\sqrt{2\pi}}\int d^2k'\, f(k')\, e^{ik'_i\hat{x}^i}\right)\left(\frac{1}{\sqrt{2\pi}}\int d^2k''\, g(k'')\, e^{ik''_j\hat{x}^j}\right) \\
&= \frac{1}{2\pi}\int d^2k'\int d^2k''\,\left(f(k')g(k'')e^{i(k'+k'')_i\hat{x}^i+\frac{1}{2}[ik'_i\hat{x}^j, ik_j\hat{x}^j]}\right) \\
&= \frac{1}{2\pi}\int d^2k'\int d^2k''\,\left(f(k')g(k'')e^{i(k'+k'')_i\hat{x}^i-\frac{i}{2}\theta^{ij}k'_ik_j}\right)
\end{aligned}$$

and maps to

$$\begin{aligned}
O(x)[\hat{O}] &= \int d^2k\, e^{ik\cdot x}\int \mathrm{Tr}\, e^{-ik\cdot\hat{x}}\hat{O}(\hat{x}) \\
&= \frac{1}{(2\pi)^n}\int d^2k'\int d^2k''\, e^{i(k'+k'')_ix^i}e^{-\frac{i\theta^{ij}}{2}k'_ik''_j}f(k')g(k'') \\
&= e^{\frac{i\theta^{ij}}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}}f(y^i)g(z^j)|_{y=z=x}
\end{aligned}$$

This $O(x) = f(x) * g(x)$ is the Moyal or star product of the functions $f(x)$ and $g(x)$.

Let us check that the commutator relation between the non-commutative coordinates still holds:

$$[\hat{x}^i, \hat{x}^j] \quad \rightarrow \quad x^i * x^j - x^j * x^i = i\theta^{ij}$$

and that the derivative operator in the operator formalism maps to ordinary derivatives in the star product formalism:

$$\begin{aligned} \hat{\partial}_i \hat{f} = [-i(\theta^{-1})_{ij}\hat{x}^j, \hat{f}(\hat{x})] \quad &\rightarrow \quad -i(\theta^{-1})_{ij} i\theta_{mn}\partial_m x^j \partial_n f \\ &= \quad (\theta^{-1})_{ij}\theta_{jn}\partial_n f = \partial_i f \end{aligned}$$

The properties of the derivative and integral in Eq 4.2 can also be checked to remain true.

The star product has the following properties:

1. associative: $(f * g) * h = f * (g * h)$
2. non-commutative: $f * g \neq g * f$
3. non-local: it involves all order derivatives of both functions f and g
4. $\int \mathrm{Tr}\ f * g = \int \mathrm{Tr}\ fg$ since the higher order terms of the star product expansion can all be written as total derivatives. This implies that functions can cycle inside the integral: $\int \mathrm{Tr}\ f * g * h = \int \mathrm{Tr}\ h * f * g$

The advantage of the star product formalism over the operator one is obvious in theories with perturbatively non-commutative background. In such theories, expanding the star product to the leading orders in θ will allow us to capture the main features of the theories and to distinguish effects due to the non-commutativity. We will use the star product formalism from now on.

4.2 The Action of Non-commutive U(N) Gauge Theory

We can build the action for a commutative U(N) gauge theory with a scalar field with minimal coupling by simply requiring it to contain terms quadratic in the first derivative of each of the scalar and gauge fields, and to be invariant under Lorentz and gauge transformations. We will first outline this and then argue that we can build the non-commutative action in the same way.

We first write all fields as Lorentz tensors, such that contracting the space-time indices of these fields will easily yield a Lorentz scalar in the end. Secondly, if the scalar field in the adjoint representation of the given U(N) gauge group and transforms as

$$\phi \;\to\; U\phi U^{\dagger}, \quad UU^{\dagger} = 1$$

we can build a first derivative of this scalar field that transforms in the same way, namely the covariant derivative $D^{\mu}\phi$ (as in Eq 2.2), such that the term quadratic in this derivative would transform like the term quadratic in the field itself, which is needed if the scalar field has a non-zero mass. The operation that would make the mass term gauge invariant would then also make the kinetic term gauge invariant, the operation being to take the Trace of the matrices in question.

Now, the covariant derivative calls for a gauge field (explicitly shown in the next section) that transforms like

$$A^{\mu} \;\longrightarrow\; UA^{\mu}U^{\dagger} + iU(\partial^{\mu}U^{\dagger})$$

and we again want a first derivative of this gauge field such that it transforms also like the scalar field, and such that the square of this derivative can be made gauge invariant by the same operation that made the other terms gauge invariant. We arrive at the normal expression for the field strength $F^{\mu\nu}$ (as in Eq 2.1). Finally, we assemble the integrand of the action by summing the Lorentz and gauge invariant "squares" of $D^\mu\phi$ and $F^{\mu\nu}$.

We can build the action for a non-commutative U(N) gauge theory with a scalar field in the adjoint representation by the same steps, except that the Trace operator is to be used with integration over the non-commutative directions (Sec 4.1) to make the action gauge-invariant. The step are the same because all the manipulations above do not depend on what type of non-commuting product acts between the matrices, be it the ordinary matrix product or the star matrix product, as long as it is still associative and satisfies the normal axios for a product, such as $\mathbf{1} \otimes \mathbf{f} = \mathbf{f}$. In particular, both the covariant derivative and the field strength in a non-commutative gauge field come about in the same way and have the same form as those in the (non-Abelian) commutative theory. As an examply, we will show explicitly the derivation of the non-commutative covariant derivative.

Derivation of noncommutative covariant derivative Given that the scalar field transforms in the adjoint representation as follow:

$$\phi \longrightarrow U * \phi * U^\dagger, \quad U * U^\dagger = 1$$

the covariant derivative is built such that it transforms similarly,

$$D^\mu\phi \longrightarrow U * D^\mu\phi * U^\dagger, \quad U * U^\dagger = 1$$

and the following quadratic expression to be used in the action also transforms similarly:

$$(D^\mu\phi)^\dagger * (D_\mu\phi) \longrightarrow U * (D^\mu\phi)^\dagger * (D_\mu\phi) * U^\dagger$$

Now, the space-time partial derivative of the scalar field does not satisfy this requirement:

$$\begin{aligned}\partial^\mu\phi \longrightarrow \partial^\mu(U*\phi*U^\dagger) &= U*(\partial^\mu\phi)*U^\dagger \\ &+ \left[(\partial^\mu U)*\phi*U^\dagger + U*\phi\,(\partial^\mu * U^\dagger)\right] \end{aligned} \tag{4.5}$$

but is to be included in the covariant derivative. The combination, $(\partial^\mu + T(A^\mu))\phi$, however, will transform as required if its extra term, $T(A^\mu)$, gauge transforms to give both the term $U*T*U^\dagger$ and terms to cancel the last two terms in the transformation of $\partial^\mu\phi$ in Eq 4.5, which can be rewritten as

$$\left[-U*(\partial^\mu U^\dagger)*(U*\phi*U^\dagger)+(U*\phi*U^\dagger)*U*(\partial^\mu U^\dagger)\right] \tag{4.6}$$

using *-unitary of U as well as the product rule for the derivative of star products:

$$(\partial^\mu U)*U^\dagger = \partial^\mu(U*U^\dagger) - U*(\partial^\mu U^\dagger) = -U*(\partial^\mu U^\dagger)$$

If the extra term $T(A^\mu)$ consists of the *-commutator of the scalar field and a field that transforms as follow:

$$A^\mu \longrightarrow U*A^\mu*U^\dagger + iU*(\partial^\mu U^\dagger)$$

then the terms produced by the gauge transformation of $T(A^\mu)$ will cancel with the terms in line 4.6. The covariant derivative therefore looks exactly the same as the familiar one in commutative gauge theories:

$$D^\mu\phi = \partial^\mu\phi - i[A*\phi - \phi*A]$$

The Action Since the non-commutative action is built in the same way the commutative action is built, it is simply the commutative action with ordinary matrix products replaced with star products:

$$S_{NC} = \frac{1}{4}\int dx^4 \mathrm{Tr}\left[F^{\mu\nu}(x)*F_{\mu\nu}(x) + 2D^\mu\phi(x)*D_\mu\phi(x) - \lambda(\phi(x)*\phi(x)-c^2)^2\right]$$

$$\text{where} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ie(A^\mu * A^\nu - A^\nu * A^\mu);$$
$$D^\mu * \phi = \partial^\mu \phi - ie(A^\mu * \phi - \phi * A^\mu) \quad \text{for } \mu, \nu = 0, 1, 2, 3$$

where the potential term is present such that in the limit $\theta \to 0$, this theory reduces to the commutative theory we studied before. As before, we will only consider the theory in the BPS limit– $\lambda \to 0$ but $(\phi(x) * \phi(x) - c^2) = 0$ at infinity–such that the last term in the action vanishes and does not contribute to the derivation of the equations of motion.

SU(N) not allowed Although the form of the gauge transformation is the same for non-commutative and (non-Abelian) commutative gauge theories, the *-gauge transformation does not allow SU(N) to be the gauge group of a non-commutative theory. The following decomposition of the infinitesimal U(2) *-gauge transformation clearly demonstrates this and the argument can be easily generalized for U(N) *-gauge transformations.

Let the infinitesimal *-unitary matrix be

$$U = \mathbf{1} - i(\alpha_0 \mathbf{t}_0 + \alpha_a \mathbf{t}_a); \quad a = 1, 2, 3$$

where $\mathbf{t}_0$ is the identity generator, $\mathbf{t}_a$ the generators for SU(2), and α_0, α_a are infinitesimal gauge parameters. Then, infinitesimally, the scalar field transforms as follow:

$$\begin{aligned} \phi &\longrightarrow \phi - i[(\alpha_0 \mathbf{t}_0 + \alpha_a \mathbf{t}_a) * (\phi_0 \mathbf{t}_0 + \phi_a \mathbf{t}_a) - (\phi_0 \mathbf{t}_0 + \phi_a \mathbf{t}_a) * (\alpha_0 \mathbf{t}_0 + \alpha_a \mathbf{t}_a) \\ &:= \phi - i\left[(\alpha_0 \mathbf{t}_0 + \alpha_a \mathbf{t}_a), (\phi_0 \mathbf{t}_0 + \phi_a \mathbf{t}_a)\right]_* \end{aligned}$$

Before expanding this, note that the *-commutator of two SU(2) fields produce a term that is not proportional to any of the SU(2) generators:

$$
\begin{aligned}
& [A_a(x)\,\mathbf{t}_a, B_b(x)\,\mathbf{t}_b]_* \\
= \; & A_a(x) * B_b(x) \left(\frac{1}{2}\epsilon_{abc}\mathbf{t}_c + \frac{1}{2}\delta_{ab}\mathbf{t}_0\right) - B_b(x) * A_a(x) \left(\frac{1}{2}\epsilon_{bac}\mathbf{t}_c + \frac{1}{2}\delta_{ab}\mathbf{t}_0\right) \\
= \; & \frac{(A_a * B_b + B_b * A_a)}{2}\epsilon_{abc}\mathbf{t}_c + \frac{(A_a * B_b - B_b * A_a)}{2}\delta_{ab}\mathbf{t}_0
\end{aligned}
$$

The *-gauge transformation above then expands to

$$
\begin{aligned}
\phi \quad \longrightarrow \quad & \phi_0\mathbf{t}_0 + \phi_a\mathbf{t}_a + \epsilon_{abc}\left(\frac{\alpha_b * \phi_c + \phi_c * \alpha_b}{2}\right)\mathbf{t}_a \\
& -i\left(\frac{[\alpha_0, \phi_a]_* + [\alpha_a, \phi_0]}{2}\right)\mathbf{t}_a - i\left(\frac{[\alpha_0, \phi_0]_* + [\alpha_a, \phi_a]_*}{2}\right)\mathbf{t}_0
\end{aligned}
$$

An important difference between this gauge transformation and an ordinary U(2) gauge transformation is that even when the infinitesimal form of the *-unitary matrix U involves only the SU(2) generators, i.e., $\alpha_0 = 0$, the transformation would still "create" a term that is proportional to the identity generator, which is not in the SU(2) space. In other words, SU(2) is not a close group under the *-gauge transformation and cannot be the gauge group for non-commutative gauge theories. Our problem will be set in a U(2) non-commutative theory.

4.2.1 Gauge Invariant Quantities

We already know that the gauge space and real physical space are not orthogonal in non-commutative gauge theories; we now check that simply taking the Trace of (without the integrating over space) any operator $O(x)$ that transforms like the adjoint scalar field indeed does not make it gauge invariant:

$$
\mathrm{Tr}\, O(x) \quad \longrightarrow \quad \mathrm{Tr}\,[U(x) * O(x) * U^\dagger(x)] \quad \neq \quad \mathrm{Tr}\, O(x) \quad \text{since} \mathrm{Tr}\,[A * B] \quad \neq \quad \mathrm{Tr}\,[B * A]
$$

The integrand of the action, for instance, is not gauge invariant, unlike in the commutative theory.

This means that if we do not want to study only quantities integrated over the non-commutative directions, we need to find a way to construct semi-local gauge-invariant operators. Gross et al [20] constructed gauge invairant operators in momentum space by attaching open Wilson lines to adjoint operators and then *-fourrier-transforming the combination.

First, the Wilson line is defined as the *-path-ordered exponential of the integral of the gauge field along a curve $\mathcal{C}$ starting at x:

$$W(x,\mathcal{C}) \;=\; \mathcal{P}_* \exp\left(ie\int_0^1 d\lambda\,\frac{ds^\mu}{d\lambda}\,A_\mu(x^\nu + s^\mu(\lambda))\right)$$

With the same argument as in the commutative theory, this Wilson line *-gauge transforms as follow:

$$W(x,\mathcal{C}) \;\longrightarrow\; U(x) * W(x,\mathcal{C}) * U^\dagger(x+l)$$

Now, $U^\dagger(x+l)$ is simply $U^\dagger(x)$ translated and can be written as the *-gauge transformed $U(x)$:

$$U^\dagger(x+l) \;=\; e^{ikx} * U^\dagger(x) * e^{-ikx}$$

where the non-zero components of the momentum k^μ is given by equation 4.3:

$$k_j \;=\; -(\theta^{-1})_{ji}l^i \;\Longrightarrow\; l^i \;=\; k_j\theta^{ij} \tag{4.7}$$

The combination $W(x,\mathcal{C}) * e^{ikx}$ then transforms as the adjoint sclar:

$$\begin{aligned} W(x,\mathcal{C}) * e^{ikx} \;&\longrightarrow\; U(x) * W(x,\mathcal{C}) * \left[U^\dagger(x+l) * e^{ikx}\right] \\ &=\; U(x) * W(x,\mathcal{C}) * \left[e^{ikx} * U^\dagger(x)\right] \end{aligned}$$

Therefore, for each operator $O(x)$ in the adjoint representation, we can define a corresponding operator $\tilde{O}(k)$ in momentum space by first attaching a Wilson line to it and then *-Fourrier transforming:

$$\tilde{O}(k) \;=\; \int dx^4\;\mathrm{Tr}\;O(x) * W(x,\mathcal{C}) * e^{ikx}$$

and it will be gauge invariant:

$$\tilde{O}(k) \longrightarrow \int dx^4 \; \mathrm{Tr}\, U(x) * O(x) * W(x, C) * e^{ikx} * U^{\dagger}(x) \;=\; \tilde{O}(k)$$

provided that the open Wilson line extend a vector $l^i(k)$ (Eq 4.7) from its starting point x.

Note that for an operator at momentum k, the Wilson line extends in a direction transverse to the momentum and to the commutative direction. Also, in the commutative limit, l^i reduces to 0 and the operator $\tilde{O}(k)$ reduces to the ordinary Fourier transform of the orginal operator $O(x)$. Finally, for operators at large k, the Wilson line is long and dominates such that all operators at large momentum would exhibit similar large k behaviour. [20]

4.2.2 Broken Lorentz and Rotational Invariance

Since the non-commutative tensor $\theta^{\mu\nu}$ is the same in any inertial frame, i.e. does not Lorentz Transform, the star products of two Lorentz tensors, and therefore the action, are only invariant under boosts in the commutative direction. The following simple example of the star product of two Lorentz scalars illustrates this:

$$\begin{aligned} f * g \;\overset{Lorentz}{\longrightarrow}\; (f * g)' &= \sum_{n=0}^{\infty} \frac{\left(\frac{i}{2}\frac{\partial}{\partial x'^{\mu}}\;\theta^{\mu\nu}\;\frac{\partial}{\partial y'^{\nu}}\right)^n}{n!} f(\Lambda^{-1}x')g(\Lambda^{-1}y')|_{x'=y'} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{i}{2}\left[\frac{\partial}{\partial x^{\rho}}\Lambda^{\rho}{}_{\mu}\right]\;\theta^{\mu\nu}\;\left[(\Lambda^{-1})_{\nu}{}^{\sigma}\;\frac{\partial}{\partial y^{\sigma}}\right]\right)^n}{n!} f(x)g(y)|_{x=y} \\ &= f * g \quad \text{only if} \quad \Lambda^{\rho}{}_{\mu}\,\theta^{\mu\nu}\,(\Lambda^{-1})_{\nu}{}^{\sigma} = \theta^{\rho\sigma} \end{aligned}$$

where Λ is the 4×4 linear Lorentz transformation matrix. If $\theta^{\mu\nu}$ is defined such that its only non-zero coponents are $\theta^{12} = [-\theta^{21} = \theta$, the last condition is satisfied only if Λ ($\Lambda \neq \mathbf{1}$) has non-trivial entries only in the 0 or 3 (time- or z-) components and therefore represents a boost in the commutative z-direction.

4.3 The Equations of Motion

The equations of motion in the non-commutative theories can be obtained by the normal variation procedure due to the properties of the star product.

We avoid varying the action twice, separately with respect to the gauge field and the scalar field, by rewriting the action. We write the scalar field as a extra space-time component of the gauge field and name the new 5-dimensionsal gauge field A'^{μ}, and require that all of its components are constant along the added spacial direction [21]:

$$A'^{4} = \phi \quad ; \quad \partial^{4} A'^{\mu} = 0$$

The covariant derivative of the scalar field can then be written as the fourth spatial component of the new field strength, $F'^{\mu\nu}$, defined by A'^{μ}:

$$D^{\mu} * \phi \quad = \quad \partial^{\mu} A^{4} - \partial^{4} A^{\mu} - ie \left[A^{\mu}, A^{4}\right]_{*} \quad = \quad F'^{\mu 4}$$

The action, with its potential term which does not affect the equations of motion in the BPS limit omitted, simplifies to:

$$S_{NC} \quad = \quad \frac{1}{4} \int dx^{4} \mathrm{Tr} \quad \left[F'^{\mu\nu}(x) * F'_{\mu\nu}(x)\right] \quad \text{for} \quad \mu, \nu = 0, 1, 2, 3, 4$$

Now, we vary the action with respect to the new gauge field:

$$\begin{aligned} \delta S &= \frac{1}{4} \int \mathrm{Tr} \quad \left[\delta F'^{\mu\nu} * F'_{\mu\nu} + F'^{\mu\nu} * \delta F'_{\mu\nu}\right] \\ \text{where} \quad \delta F'^{\mu\nu} &= \partial^{\mu}(\delta A^{\nu}) - \partial^{\nu}(\delta A^{\mu}) + [(\delta A^{\mu}), A^{\nu}]_{*} + [A^{\mu}, (\delta A^{\nu})]_{*} \end{aligned}$$

We assume the field strength drop to zero sufficiently fast at infinity; therefore, the factors inside the $\int \mathrm{Tr}$ can be cycled (by property 4 of the star product section 4.1). Integrating by parts, the variation becomes

$$\delta S \quad = \quad \frac{1}{2} \int \mathrm{Tr} \quad \left[\delta F'^{\mu\nu} * F'_{\mu\nu}\right] \quad = \quad - \int \mathrm{Tr} \quad \left[\delta A^{\nu} * \left(\partial^{\mu} F'_{\mu\nu} + \left[A^{\mu}, F'_{\mu\nu}\right]_{*}\right)\right]$$

Now, the star product between δA^ν and the other factor can be replaced by the ordinary local product (again by property 4 of the star product). So, for arbitrary δA^ν, the variation of the action vanishes only when

$$D_\mu * F'^{\mu\nu} = 0 \tag{4.8}$$

or equivalently in terms of the original scalar and gauge fields:

$$\text{for } \nu = 4: \quad D_\mu * (D^\mu * \phi) = 0; \tag{4.9}$$

$$\text{for } \nu = 0,1,2,3: \quad D_\mu * F^{\mu\nu} = ie \; [(D^\nu * \phi) \; , \; \phi]_* \tag{4.10}$$

For the U(2) non-commutative theory, each space-time component of the equation of motion has four components, one for each generator of the non-commutative U(2) gauge group. We will call the equation for $\mathbf{t_0}$, the identity generator, the U(1) sector and the ones for $\mathbf{t_a}$, the Pauli matices, the SU(2) sector.

4.3.1 Expansion of the Equation of Motion

We will be studying the problem of the force between two non-commutative monopoles in a perturbatively non-commutative theory. Therefore $\theta << 1$ and we can expand each sector of the equation of motion in the small parameter θ and study the equation order by order.

U(1) $\mathcal{O}(\theta^0)$ First, we expand the U(1) component of Eq 4.8 to $\mathcal{O}(\theta^2)$:

$$\begin{aligned}
\partial_\nu(\partial^\nu A_0^{'\mu} - \partial^\mu A_0^{'\nu}) &= \frac{\theta}{2}\left[\partial_\nu\left\{A_0^{'\mu}, A_0^{'\nu}\right\} + \partial_\nu\left\{A_a^{'\mu}, A_a^{'\nu}\right\} + \epsilon_{abc}\left\{A_a^{'\nu} A_b^{'\mu}, A_c^{'\nu}\right\}\right] \\
&+\frac{\theta}{2}\left[\left\{A_0^{'\nu}, (\partial^\nu A_0^{'\mu} - \partial^\mu A_0^{'\nu})\right\} + \left\{A_a^{'\nu}, (\partial^\nu A_a^{'\mu} - \partial^\mu A_a^{'\nu})\right\}\right] \\
&+\frac{\theta^2}{4}\left[\left\{A_0^{'\nu}, \left\{A_0^{'\mu}, A_0^{'\nu}\right\}\right\} + \left\{A_0^{'\nu}, \left\{A_a^{'\mu}, A_a^{'\nu}\right\}\right\}\right] \\
&+\frac{\theta^2}{4}\left[\left\{A_a^{'\nu}, \left\{A_0^{'\mu}, A_a^{'\nu}\right\}\right\} + \left\{A_a^{'\nu}, \left\{A_a^{'\mu}, A_0^{'\nu}\right\}\right\}\right] + \mathcal{O}(\theta^3) \\
\text{where} \quad \{f,g\} &= (\partial_1 f)(\partial_2 g) - (\partial_2 g)(\partial_1 f)
\end{aligned}$$

and terms with $\{f, g\}$ originate from the $\mathcal{O}(\theta)$ terms in the expansion of the *-commutator of f and g and therefore are antisymmetric under exchange of f and g.

To $\mathcal{O}(\theta^0)$, the RHS of Eq 4.11 is irrelevant, and expanding the gauge field A'^{μ} in orders of θ:

$$A'^{\mu} \;=\; A'^{\mu(0)} + A'^{\mu(\theta)} + A'^{\mu(\theta^2)}$$

we find that the equation for $A_0'^{\mu(0)}$ is totally decoupled from the SU(2) sector and is simply a sourceless U(1) electromagnetism equation (in 5 dimensions):

$$\partial_\nu \left(\partial^\nu A_0'^{\mu(0)} - \partial^\mu A_0'^{\nu(0)} \right) \;=\; \partial_\nu F_0'^{\nu\mu(0)} \;=\; 0$$

As in normal electromagnetism, the gauge field $A_0'^{\mu(0)}$ obviously has some gauge freedom, but here, we also have the freedom to choose the value of the gauge invariant quantity $F_0'^{\nu\mu(0)}$ without affecting the $\mathcal{O}(\theta^0)$ SU(2) sector.

To show the gauge freedom at different orders of θ of the gauge field and fields that transform in the adjoint, we expand the infinitesimal *-gauge transformation (Eq 4.7) of these fields to $\mathcal{O}(\theta^2)$, writting the extra terms present only in the transformation of the gauge field in square brackets:

$$\text{for} \quad U \;=\; \mathbf{1} - i\left(\alpha_0^{(0)} + \alpha_0^{(\theta)} + \alpha_0^{(\theta^2)}\right)\ \mathbf{t}_0 - i\left(\alpha_a + \alpha_a^{(\theta)} + \alpha_a^{(\theta^2)}\right)\ \mathbf{t}_a; \quad a = 1, 2, 3$$

$$
\begin{aligned}
f \quad \longrightarrow \quad & \mathbf{t}_a \left(f_a^{(0)} + \epsilon_{abc} \alpha_b^{(0)} f_c^{(0)} \right) \quad - \left[\mathbf{t}_a \partial^\mu \alpha_a^{(0)} \right] \\
& + \mathbf{t}_0 f_0^{(0)} \quad - \left[\mathbf{t}_0 \partial^\mu \alpha_0^{(0)} \right] \\
& + \mathbf{t}_a \left(\frac{\theta}{2} \left\{ \alpha_0^{(0)}, f_a^{(0)} \right\} + \frac{\theta}{2} \left\{ \alpha_a^{(0)}, f_0^{(0)} \right\} + \epsilon_{abc} \frac{\alpha_b^{(0)} f_c^{(\theta)} + \alpha_b^{(\theta)} f_c^{(0)}}{2} \right) \quad - \left[\mathbf{t}_a \partial^\mu \alpha_a^{(\theta)} \right] \\
& + \mathbf{t}_0 \left(\frac{\theta}{2} \left\{ \alpha_a^{(0)}, f_a^{(0)} \right\} + \frac{\theta}{2} \left\{ \alpha_0^{(0)}, f_0^{(0)} \right\} \right) \quad - \left[\mathbf{t}_0 \partial^\mu \alpha_0^{(\theta)} \right] \\
& + \mathbf{t}_a \epsilon_{abc} \left(\frac{\alpha_b^{(0)} f_c^{(\theta^2)} + \alpha_b^{(\theta^2)} f_c^{(0)} + 2 \alpha_b^{(\theta)} f_c^{(\theta)}}{2} - \frac{\theta^2}{4} \left\{ \left\{ \alpha_b^{(0)}, f_c^{(0)} \right\} \right\} \right) - \left[\mathbf{t}_a \partial^\mu \alpha_a^{(\theta^2)} \right] \\
& + \mathbf{t}_a \frac{\theta}{2} \left(\left\{ \alpha_0^{(0)}, f_a^{(\theta)} \right\} + \left\{ \alpha_0^{(\theta)}, f_a^{(0)} \right\} + \left\{ \alpha_a^{(0)}, f_0^{(\theta)} \right\} + \left\{ \alpha_a^{(\theta)}, f_0^{(0)} \right\} \right) \\
& + \mathbf{t}_0 \frac{\theta}{2} \left(\left\{ \alpha_a^{(0)}, f_a^{(\theta)} \right\} + \left\{ \alpha_a^{(\theta)}, f_a^{(0)} \right\} + \left\{ \alpha_0^{(0)}, f_0^{(\theta)} \right\} + \left\{ \alpha_0^{(\theta)}, f_0^{(0)} \right\} \right) \quad - \left[\mathbf{t}_0 \partial^\mu \alpha_0^{(\theta^2)} \right]
\end{aligned} \tag{4.11}
$$

Now, the second line of this transformation shows that no *-gauge transformation can alter the U(1) zeroth order adoint fields, $F_0^{'\nu\mu(0)}$ and $\phi_0^{(0)}$, whereas the U(1) zeroth order gauge field has the gauge freedom, $\partial^\mu \alpha_0^{(0)}$, where $\alpha_0^{(0)}$ is a free infinitesimal parameter. We will refer back to this equation when we discuss the gauge freedom of the higher order fields. For we non-commutative monopoles, we choose $F_0^{'\nu\mu(0)}$ and $\phi_0^{(0)}$ to vanish and the gauge in which $A_0^{\mu(0)}$ vanishes.

SU(2) $\mathcal{O}(\theta^0)$ Next, we expand the SU(2) component of the equation of motion:

$$\begin{aligned}
& \partial_\nu\left(\partial^\nu A_a^{'\mu}-\partial^\mu A_a^{'\nu}\right)+\epsilon_{abc}\left[\partial_\nu\left(A_b^{'\nu}A_c^{'\mu}\right)+A_{b\nu}^{'}\left(\partial^\mu A_c^{'\nu}-\partial^\nu A_c^{'\mu}\right)\right] \\
& +A_{a\nu}^{'}A_b^{'\nu}A_b^{'\mu}-A_a^{'\mu}A_{b\nu}^{'}A_b^{'\nu} \\
= \quad & \frac{\theta}{2}\left[\partial_\nu\left\{A_a^{'\nu},A_0^{'\mu}\right\}+\left\{A_0^{'\nu},A_a^{'\mu}\right\}\right] \\
& +\frac{\theta}{2}\left[\left\{A_{a\nu}^{'},\left(\partial^\mu A_0^{'\nu}-\partial^\nu A_0^{'\mu}\right)\right\}+\left\{A_{0\nu}^{'},\left(\partial^\mu A_a^{'\nu}-\partial^\nu A_a^{'\mu}\right)\right\}\right] \\
& +\frac{\theta}{2}\epsilon_{abc}\left[\left\{A_{0\nu}^{'},A_b^{'\mu}A_c^{'\nu}\right\}+A_{b\nu}^{'}\left(\left\{A_c^{'\nu},A_0^{'\mu}\right\}+\left\{A_0^{'\nu},A_c^{'\mu}\right\}\right)\right] \\
& +\frac{\theta^2}{4}\left[\left\{A_{a\nu}^{'},\left\{A_b^{'\mu},A_b^{'\nu}\right\}\right\}+\left\{A_{a\nu}^{'},\left\{A_0^{'\mu},A_0^{'\nu}\right\}\right\}\right] \\
& +\frac{\theta^2}{4}\left[\left\{A_{0\nu}^{'},\left\{A_0^{'\mu},A_a^{'\nu}\right\}\right\}+\left\{A_{0\nu}^{'},\left\{A_a^{'\mu},A_0^{'\nu}\right\}\right\}\right] \\
& +\frac{\theta^2}{4}\left[\left\{\left\{A_{b\nu}^{'},\left(A_b^{'\mu}A_a^{'\nu}-A_b^{'\nu}A_a^{'\mu}\right)\right\}\right\}\right] \\
& +\frac{\theta^2}{4}\left[\epsilon_{abc}\left(-2\partial_\nu\left\{\left\{A_b^{'\nu},A_c^{'\mu}\right\}\right\}+\left\{\left\{\left(\partial^\mu A_b^{'\nu}-\partial^\nu A_b^{'\mu}\right),A_{c\nu}^{'}\right\}\right\}\right)\right]
\end{aligned} \tag{4.12}$$

$$\text{where} \quad \{\{f,g\}\} \;=\; \left(\partial_1^2 f\right)\left(\partial_2^2 g\right)+\left(\partial_1^2 g\right)\left(\partial_2^2 f\right)-2\left(\partial_1\partial_2 f\right)\left(\partial_1\partial_2 g\right)$$

is symmetric under exchange of f and g and terms with these double brackets originate from the $mathcalO(\theta^2)$ terms in expansion of the *-anti-commutators.

At zeroth order in θ, this equation has an irrelevant RHS and is totally decoupled from the U(1) sector. In terms of the original fields, A^μ and ϕ, the equation is simply the SU(2) sector of the Eq 4.9 and Eq 4.10 with the *-product replaced by the ordinary product, which are obviously the equations of motion in the commutative SU(2) theory. Thus, the $\mathcal{O}(\theta^0)$ SU(2) field, $A_a^{'\mu(0)}$, is simply the solution to the commutative theory. According to the first line of Eq 4.11, its gauge freedom at this order is also exactly the same as in the commutative case.

U(1) $\mathcal{O}(\theta)$ We can use the choice for the U(1) $\mathcal{O}(\theta^0)$ fields, $A_0^{'\mu(0)}=0$, to simplify the U(1) equation, Eq 4.11. In fact, since the terms on the RHS of this equation are quadratic in the U(1) fields $A_0^{'\mu}$, they will contain at least one factor of the vanishing $A_0^{'\mu(0)}$ at $\mathcal{O}(\theta^2)$ when the θ expansion of $A_0^{'\mu}$ is put in. Thus, the U(1) equation accurate up to $\mathcal{O}(\theta^2)$ simplifies to:

$$\begin{aligned}\partial_\nu(\partial^\nu A_0^{'\mu} - \partial^\mu A_0^{'\nu}) &= \frac{\theta}{2}\left[\partial_\nu\left\{A_a^{'\mu}, A_a^{'\nu}\right\} + \epsilon_{abc}\left\{A_a^{'\nu}A_b^{'\mu}, A_c^{'\nu}\right\}\right] \\ &\quad +\frac{\theta}{2}\left[\left\{A_a^{'\nu}, (\partial^\nu A_a^{'\mu} - \partial^\mu A_a^{'\nu})\right\}\right] \end{aligned} \tag{4.13}$$

Note that the first order U(1) field, $A_0^{'\mu(\theta)}$, is determined independently of the first order SU(1) fields and depends solely on the zeroth order SU(2) fields, $A_a^{'\mu}$; and that the second order U(1) field, $A_0^{'\mu(\theta^2)}$, is determined only by the zeroth and first order SU(2) fields. In general, the U(1) component of the fields of any order in θ is determined independently of the SU(2) components at the same order, and is determined only by the lower order U(1) and SU(2) fields, which would have been determined already by lower order equations. This is because expanding the equation of motion, Eq 4.8, to an arbitrary order in θ only adds more terms with explicit θ dependence to Eq 4.13 but does not change its property that the terms that involve the highest order fields depend only on the U(1) component of the fields.

Another property of the U(1) equation is that to all orders of θ, it takes the form of the ordinary Maxwell equations, $A_0^{'\mu(\theta^n)}$ being the Maxwell gauge field, with a non-localized source comprising of the terms on the RHS, which involve lower order (lower than nth) fields and spread out over space-time.

We now look at the gauge freedom of the U(1) fields at this order. According to the fourth line of Eq 4.11, the transformation of the $\mathcal{O}(\theta)$ U(1) component of the field strength, the scalar field and fields that transform like them is governed only by the zeroth order gauge parameters $\alpha_0^{(0)}$ and $\alpha_a^{(0)}$. This means that these U(1) fields ($F_0^{\mu\nu(\theta)}$ etc.) have no gauge freedom at $\mathcal{O}(\theta)$ if we have completely fixed the gauge for the zeroth order fields. In general, at an arbitrary order n of θ, the U(1) component of fields that transform like the field strength has no gauge freedom that is not already determined by the lower order fields. This is because the infinitesimal transformation of these U(1) fields involve only *-commutators (Eq 4.7), the expansion of which already has an explicit factor of θ, and so no gauge parameter to $\mathcal{O}(\theta^n)$ can be involved. On the other hand, the U(1) component of the gauge field, however, has a new gauge freedom at each order of θ parametrized by $\alpha_0^{(\theta^n)}$: it transforms with the extra term $-\partial\alpha_0^{(\theta^n)}$.

In contrast, according to the third, fifth and sixth line of Eq 4.11, the SU(2) components of the field strength and the scalar field, and the SU(2) component of the gauge field, do have a new gauge freedom that depends on a new gauge parameter $\alpha_a^{(\theta^n)}$ at each order n of θ. Also, the terms in these lines arising from the star product expansion renders the gauge transformation of the SU(2) component of fields which transform like the field strength not simply a rotation in the SU(2) space, and therefore the magnitude of such SU(2) vectors are not gauge invariant unlike in the commutative theory.

SU(2) $\mathcal{O}(\theta)$ We use the choice that the zeroth order U(1) fields vanish again to simplify the SU(2) equation (Eq 4.12). To $\mathcal{O}(\theta)$, since all the terms on the RHS of this equation depend on the zeroth order U(1) fields $A_a^{'\mu(0)}$, they vanish, and the equation does not differ from the zeroth order SU(2) equation. This implies that both $A_a^{'\mu(0)}$ and $(A_a^{'\mu(0)} + A_a^{'\mu(\theta)})$ solve the same equation, and so are related by $\mathcal{O}(\theta)$ symmetry transformations of the theory. Now, the transformed solution, $(A_a^{'\mu(0)} + A_a^{'\mu(\theta)})$, is physically different from the original solution, $A_a^{'\mu(0)}$, only if the symmetry transformation is not a symmetry of the solution. However, these symmetry transformations actually simply change some choices we have had when solving for the zeroth order solution $A_a^{'\mu(\theta)}$, and so the transformed solution $A_a^{'\mu(0)} + A_a^{'\mu(\theta)}$ could really have been the zeroth order solution we have chosen. Therefore, we can choose $A_a^{'\mu(\theta)} = 0$ without any loss of information of the solution. For instance, for a monopole solution $A_a^{'\mu(0)}$, we have the freedom to choose its initial 4-position on the coordinate system, a first order correction $A_a^{'\mu(\theta)}$ which is an $\mathcal{O}(\epsilon)$ translation would simply change that choice, but that choice is arbitrary to begin with.

Also, as in the U(1) sector, the determination of the SU(2) fields, $A_a^{'\mu(\theta^n)}$ to each order n of θ is decoupled from the determination of the U(1) field, $A_0^{'\mu(\theta^n)}$ to the same order.

U(1) $\mathcal{O}(\theta^2)$ Now, using the result from above that the first order SU(2) fields can be set to zero, we can further simplify the $\mathcal{O}(\theta^2)$ U(1) equation (Eq 4.13):

$$\partial_\nu \left(\partial^\nu A_0^{'\mu(\theta^2)} - \partial^\mu A_0^{'\nu(\theta^2)} \right) = 0 \tag{4.14}$$

Interestingly, this has the same form as the zeroth order U(1) equation and is again totally decoupled from the SU(2) sector. Note however that this simplification is not a regular occurrence in even orders of θ and happens here only due to the triviality of the first order SU(2) solution. Both the third and fourth order U(1) fields depend on both the lower order SU(1) and U(1) fields.

SU(2) $\mathcal{O}(\theta^2)$ Finally, the $\mathcal{O}(\theta^2)$ SU(2) fields needs to satify the non-trivial Eq 4.12 and depend on both the lower order SU(2) and U(1) fields.

Chapter 5

First Order Force between Two Non-commutative Monopoles

The force between two non-commutative monopoles does not alter from the force between two commutative monopoles to first order in the non-commutative parameter θ. In fact, the effect of the non-commutativity in the dynamics is not seen to this order. We will show this both by the stress-energy tensor as well as by a slight extension of the Manton method.

5.1 Non-Commutative Monopoles

Magnetic monopoles in the commutative theory, as discussed in chapter 2 and 3, are defined by the asymptotic behavior of the U(1) magnetic field embedded in the SU(2) field strength tensor. In a non-commutative theory with small θ, the field strength is dominated by the lowest order term, i.e. simply the field strength of the commutative theory; therefore, the same embedded U(1) magnetic field can be used to define the non-commutative monopole. To $\mathcal{O}(\theta^2)$, the magnetic field is:

$$\begin{aligned}
B^i &= \frac{1}{2}\epsilon^{ijk}F^{jk} \\
&= \epsilon^{ijk}\left(\partial^j A_a^k - \frac{i}{2}\left(\left[A_a^j, A_0^k\right] + i\epsilon_{abc}A_b^j * A_c^k\right)\right)\mathbf{t_a} \\
&\quad +\epsilon^{ijk}\left(\partial^j A_0^k - \frac{i}{4}\left[A_a^j, A_a^k\right]\right)\mathbf{t_0} \\
&= \frac{1}{2}\epsilon^{ijk}F^{jk}_{commutative} + \epsilon^{ijk}\left(\partial^j A_0^{k(\theta)} + \frac{\theta}{4}\left\{A_a^{j(0)}, A_a^{k(0)}\right\}\right)\mathbf{t_0} \\
&\quad + \frac{1}{2}\epsilon^{ijk}\left(\theta\left\{A_a^{j(0)}, A_0^{k(\theta)}\right\} + 2\partial^j A_a^{k(\theta^2)} + \epsilon_{abc}\left[A_b^{j(\theta^2)}A_c^{k(0)} + A_b^{j(0)}A_c^{k(\theta^2)}\right]\right)\mathbf{t_a} \\
&\quad - \left(\frac{\theta^2}{8}\epsilon_{abc}\left\{\left\{A_b^{j(0)}, A_c^{k(0)}\right\}\right\}\right)\mathbf{t_a} + \mathcal{O}\left(\theta^3\right)
\end{aligned} \tag{5.1}$$

where several terms have vanished because $A_0^{i(0)} = A_a^{i(\theta)} = A_0^{i(\theta^2)} = 0$ as discussed in section 4.3.

Note that whereas the zeroth order commutative field strength is in the SU(2) sector and can be factorized, $F^{jk}_{commutative} = (f^{jk}\hat{\phi}_a)\,\mathbf{t_a}$, to give an embedded U(1) field strength which satisfies the Maxwell equations, the $\mathcal{O}(\theta)$ correction to the field strength is in the the U(1) sector, and the $\mathcal{O}(\theta^2)$ correction (as in hte last two lines of Eq 5.1), although in the SU(2) sector, cannot be factorized into a Maxwell U(1) field strength and a unit vector field. Thus, the higher order corrections to the field strength cannot be easily described by corrections to the embedded U(1) field strength $f^{\mu\nu}$ which we used to define the commutative monopole.

The definition of the non-commutative monopoles in terms of the $\mathcal{O}(\epsilon^0)$ embedded magnetic field and the fact that to $\mathcal{O}(\theta^0)$ the equation of motions are the same as those in the commutative SU(2) theory imply that any system of non-commutative monopoles is simply the solution of the analogous commutative SU(2) system (with trivial $\mathcal{O}(\theta^{(0)})$ U(1) fields) plus $\mathcal{O}(\theta)$ and above corrections for both the U(1) and SU(2) fields, which satisfy the equations of motion expanded to higher order. We will only need the trivial $\mathcal{O}(\theta)$ SU(2) correction in this chapter.

5.2 Force Correction from the Stress-Energy Tensor

We have established two facts: that to $\mathcal{O}(\theta^0)$, a system of two non-commutative monopoles equals the solution of two commutative monopoles, and that for any classical solution to the non-commutative theory, the $\mathcal{O}(\theta)$ SU(2) sector can be chosen to be zero as reasoned in section 4.3. These, along with the statement in this section that the $\mathcal{O}(\theta)$ correction to the non-commutative stress-energy tensor depends only on the $\mathcal{O}(\theta)$ SU(2) fields, and therefore vanishes, determines the $\mathcal{O}(\theta)$ correction to the force between two non-commutative monopoles to be zero.

We will also generalize that the forces within any system of non-commutative solitons [22] are the same, to $\mathcal{O}(\theta)$, as those within the system of commutative solitons to which the non-commutative ones reduce at zeroth order.

5.2.1 Non-Conservation of the Stress-Energy Tensor

There are more than one definitions of the stress-energy tensor in non-commutative gauge theories. For example, Yukawa and Ooguri obtain one by computing disk amplitudes in string theory in a large NS-NS two-form background field and taking the Seiberg-Witten limit [23] [24] [5]. This tensor is locally kinematically conserved, gauge invariant and vanishes as $\theta \to 0$.

To arrive at our statement about the $\mathcal{O}(\theta)$ correction to the force between monopoles, we will use the tensor obtained from the Noether procedure, because it seems more intrinsic to the theory. We will find that this tensor, interestingly, has very different properties than the one mentioned above: it is not locally conserved, not gauge-invariant, and reduces to the tensor for the commutative theory at the lowest order.

We derive the energy and momentum currents from translational invariance using Noether's theorem. The resulting current is neither locally conservative nor locally covariantly conservative. To obtain covariant conservation for the stress-energy tensor, $D_\mu * T^{\mu\nu} = 0$, we need to add a term which equals zero to the variation of the action, δS_{NC}, and carry out the derivation which is slightly different from the commutative case as in the usual way. We note that the final tensor and its conservation equation is not gauge invariant.The detailed derivation follows.

Derivation of the Stress-Energy Tensor We then add a term which equals zero to the "conservation" equation, (the algebra is a little dissimilar from the commutative case), to obtain a covariant conservation for the stress-energy tensor, $D_\mu * T^{\mu\nu} = 0$. We also note that the tensor is not gauge invariant and that its conservation equation is also only gauge covariant. The detailed derivation follows.

For simplicity, we switch back to the notation that the space-time indices go from 0 to 4, with $A^4 = \phi$ and $\partial_4 A^\mu = 0$ such that

$$S = \frac{1}{4}\int \mathrm{Tr}\, F^{\mu\nu} * F_{\mu\nu}\, dx^4$$

The transformation of the gauge field due to the translation has an ordering ambiguity and needs to be symmetrized so that in the operator formalism the transformation would be Weyl-ordered:

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + \epsilon^\mu(x) \\ \Longrightarrow \delta A^\mu &= \frac{1}{2}\left(\partial_\rho A^\mu * \epsilon^\rho + \epsilon^\rho * \partial_\rho A^\mu\right) \end{aligned}$$

The variation in the action is then

$$\delta S = \frac{1}{2}\int \mathrm{Tr}\, F^{\mu\nu} * \delta F_{\mu\nu}\, dx^4$$

We substitute δA^μ in this expression:

$$\begin{aligned}\delta S &= \frac{1}{2}\int dx^4 \mathrm{Tr}\quad F^{\mu\nu} * [(\partial_\mu \partial_\rho A_\nu) * \epsilon^\rho + (\partial_\rho A_\nu) * (\partial_\mu \epsilon^\rho) + (\partial_\rho A_\mu * \epsilon^\rho) * A_\nu + A_\mu * (\partial_\rho A_\nu * \epsilon^\rho)] \\ &+ \frac{1}{2}\int dx^4 \mathrm{Tr}\quad F^{\mu\nu} * [\epsilon^\rho * (\partial_\mu \partial_\rho A_\nu) + (\partial_\mu \epsilon^\rho) * (\partial_\rho A_\nu) + (\epsilon^\rho * \partial_\rho A_\mu) * A_\nu + A_\mu * (\epsilon^\rho * \partial_\rho A_\nu)]\end{aligned}$$

and rearrange the terms:

$$\begin{aligned}\delta S &= \frac{1}{2}\int dx^4 \mathrm{Tr}\quad F^{\mu\nu} * \{(\partial_\mu \partial_\rho A_\nu + \partial_\rho A_\mu * A_\nu + A_\mu * \partial_\rho A_\nu) * \epsilon^\rho + \partial_\rho A_\mu * [\epsilon^\rho, A_\nu]_*\} \\ &\quad + \frac{1}{2}\int dx^4 \mathrm{Tr}\quad \{\epsilon^\rho * (\partial_\mu \partial_\rho A_\nu + \partial_\rho A_\mu * A_\nu + A_\mu * \partial_\rho A_\nu) + [A_\mu, \epsilon^\rho]_* * \partial_\rho A_\nu\} * F^{\mu\nu} \\ &\quad + \frac{1}{2}\int dx^4 \mathrm{Tr}\quad \{(\partial_\rho A_\nu) * F^{\mu\nu} + F^{\mu\nu} * (\partial_\rho A_\nu)\} * \partial_\mu \epsilon^\rho \\ &= \frac{1}{2}\int dx^4 \mathrm{Tr}\quad \frac{1}{2}\partial_\rho (F^{\mu\nu} * F_{\mu\nu}) * \epsilon^\rho \; - \; \partial_\mu (\partial_\rho A_\nu * F^{\mu\nu} + F^{\mu\nu} * \partial_\rho A_\nu) * \epsilon^\rho \\ &\quad + \frac{1}{2}\int dx^4 \mathrm{Tr}\quad (\partial_\rho A_\nu * F^{\mu\nu} + F^{\mu\nu} * \partial_\rho A_\nu) * [A_\mu, \epsilon^\rho]_*\end{aligned}$$

We add $\frac{1}{2}\partial_\mu\left[\partial_\nu\left(F^{\mu\nu} * A_\rho\right) + \partial_\nu\left(A_\rho * F^{\mu\nu}\right)\right]$ to the integrand, which vanishes because of antisymmetry of the μ, ν indices in $F^{\mu\nu}$ and the symmetry in $\partial_\mu\, \partial_\nu$. Upon expansion, the first term in the expression is

$$\begin{aligned}\partial_\mu\, \partial_\nu\, (F^{\mu\nu} * A_\rho) &= \partial_\mu\; (\partial_\nu F^{\mu\nu} * A_\rho + F^{\mu\nu} * \partial_\nu A_\rho) \\ &= \partial_\mu\; (-[A_\nu, F^{\mu\nu}]_* * A_\rho + F^{\mu\nu} * \partial_\nu A_\rho)\end{aligned}$$

where the equation of motion $D_\mu * F^{\mu\nu} = 0$ has been used in the last step. These added terms combine with the terms in integrand (denoted as $s/2$ in the following) to form more combinations of $F^{\mu\nu}$:

$$
\begin{aligned}
s &= \frac{1}{2}\partial_\rho\left(F^{\mu\nu} * F_{\mu\nu}\right) * \epsilon^\rho \\
&\quad +\partial_\mu\left[\left(\partial_\nu A_\rho - \partial_\rho A_\nu + [A_\nu, A_\rho]_*\right) * F^{\mu\nu} - A_\nu * F^{\mu\nu} * A_\rho + F^{\mu\nu} * A_\rho * A_\nu\right] * \epsilon^\rho \\
&\quad +\partial_\mu\left[\left(F^{\mu\nu} * (\partial_\nu A_\rho - \partial_\rho A_\nu) + [A_\nu, A_\rho]_*\right) - A_\nu * A_\rho * F^{\mu\nu} + A_\rho * F^{\mu\nu} * A_\nu\right] * \epsilon^\rho \\
&\quad + \left(\partial_\rho A_\nu * F^{\mu\nu} + F^{\mu\nu} * \partial_\rho A_\nu\right) * [A_\mu, \epsilon^\rho]_* \\
&= \left(\frac{1}{2}\partial_\rho\left(F^{\mu\nu} * F_{\mu\nu}\right) + \partial_\mu\left(F^{\mu\nu} * F_{\mu\rho} + F_{\mu\rho} * F^{\mu\nu}\right)\right) * \epsilon^\rho \\
&\quad +\partial_\mu\left(-A_\nu * F^{\mu\nu} * A_\rho + A_\rho * F^{\mu\nu} * A_\nu + F^{\mu\nu} * A_\rho * A_\nu - A_\nu * A_\rho * F^{\mu\nu}\right) * \epsilon^\rho \\
&\quad + \left(\partial_\rho A_\nu * F^{\mu\nu} + F^{\mu\nu} * \partial_\rho A_\nu\right) * [A_\mu, \epsilon^\rho]_*
\end{aligned}
$$

Finally, the last two lines of the integrand regroup into
$[A_\mu, F^{\mu\nu} * F_{\rho\nu} + F_{\rho\nu} * F^{\mu\nu}]_* + (1/2)\,[A_\rho, F^{\mu\nu} * F_{\mu\nu}]_*$ in the following manner:

Inside the integral, the factor A^μ at the end of the last line can be "cycled" to the front and so the last line can be rewritten as

$$[A_\mu, (-\partial_\rho A_\nu) * F^{\mu\nu} + F^{\mu\nu} * (-\partial_\rho A_\nu)]_* * \epsilon^\rho \tag{5.2}$$

For the second line, we expand the derivative. The derivative on the factor A^ρ in all four terms gives the following commutator bracket

$$[A_\mu, (\partial_\nu A_\rho) * F^{\mu\nu} + F^{\mu\nu} * (\partial_\nu A_\rho)]_* * \epsilon^\rho; \tag{5.3}$$

the derivative on $F^{\mu\nu}$ gives

$$\left(A_\nu * [A_\mu, F^{\mu\nu}]_* * A_\rho - A^\rho * [A_\mu, F^{\mu\nu}]_*\right) * A_\nu * \epsilon^\rho \tag{5.4}$$

$$- \quad \left([A_\mu, F^{\mu\nu}]_* * A_\rho * A_\nu + A_\nu * A_\rho * [A_\mu, F^{\mu\nu}]_*\right) * \epsilon^\rho \tag{5.5}$$

which, after the interchange of some of the μ, ν indices, equal the following:

$$\left[A_\mu, \left(F^{\mu\nu} * [A_\nu, A_\rho]_* + [A_\nu, A_\rho]_* * F^{\mu\nu}\right)\right]_* * \epsilon^\rho \tag{5.6}$$

$$\left(-F^{\mu\nu} * A_\rho * A_\nu * A_\mu - A_\mu * A_\nu * A_\rho * F^{\mu\nu}\right) * \epsilon^\rho; \tag{5.7}$$

and the derivative on A_ν gives

$$[A_\mu, \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu]_* * \epsilon^\rho \tag{5.8}$$

$$+ \quad (A_\rho * [A^\mu, A^\nu]_* * \partial_\mu A_\nu - \partial_\mu A_\nu * [A^\mu, A^\nu]_* * A_\rho) * \epsilon^\rho \tag{5.9}$$

$$+ \quad ([A^\mu, A^\nu]_* * A_\rho * \partial_\mu A_\nu - \partial_\mu A_\nu * A_\rho * [A^\mu, A^\nu]_*) * \epsilon^\rho. \tag{5.10}$$

The terms 5.2, 5.3, and 5.6 combine to $[A_\mu, -F^{\mu\nu} * F_{\rho\nu} - F_{\rho\nu} * F^{\mu\nu}]_* * \epsilon^\rho$. The remaining terms, 5.4, 5.5, 5.7, 5.8, 5.9, and 5.10, combine to $(1/2)\,[A_\rho, F^{\mu\nu} * F_{\mu\nu}]_* * \epsilon^\rho$.

Therefore, since $\epsilon^\rho(x)$ is arbitrary, when we take away the star product between $\epsilon^\rho(x)$ and the other factors in the integrand, the covariant conservation law is obtained:

$$\begin{aligned} D_\mu * T^{\mu\nu} &= 0 && (5.11) \\ \text{where } T^{\mu\nu} &= +g^{\mu\nu}\frac{1}{4}F^{\alpha\beta} * F_{\alpha\beta} - \frac{1}{2}F^{\mu\rho} * F^{\nu}{}_{\rho} - \frac{1}{2}F^{\nu\rho} * F^{\mu}{}_{\rho} && (5.12) \end{aligned}$$

The difference in this derivation from that in the commutative theory, in which the tensor is gauge invariant as well as locally conserved,

$$\begin{aligned} \partial_\mu T^{\mu\nu}_{com} &= 0 && (5.13) \\ \text{where } T^{\mu\nu}_{com} &= \mathrm{Tr}\left(\frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\rho}F^{\nu}_{\rho}\right), \end{aligned}$$

is that in the commutative theory, the matrices in the integrand can cycled under the trace operator without involving the translation non-matrix parameter $\epsilon^\rho(x)$, but in the non-commutative theory, because $\epsilon^\rho(x)$ is a function of x and because it is related to the other factor by the star product, any cycling involves all factors inside the integral, including $\epsilon^\rho(x)$.

Note that $T^{\mu\nu}_{com}$ and therefore Eq 5.13 is invariant under the commutative gauge transformation whereas the non-commutative tensor $T^{\mu\nu}$ and its equation are not invariant under the non-commutative gauge transformation.

5.2.2 $\mathcal{O}(\theta)$ correction to the Force Between Two Non-Commutative Solitons

Since the stress-energy tensor is not locally conserved and not gauge invariant, it does not allow us to find the force between two non-commutative monopoles by calculating the momentum current flux through a surface enclosing a monopole as in the commutative theory (Sec 3.3): we need to either solve the problem of non-conservation or extract information from only the conserved and gauge invariant total energy and momenta. Because of the form of the stress-energy tensor, it is easy to arrive at a statement about the force at $\mathcal{O}(\theta)$ between solitons, but much more non-trivial to obtain one at $\mathcal{O}(\theta^2)$. Since our goal for this project is to extend Manton's method to the non-commutative theory, we have not pursued this method furthur at the non-trivial second order.

Conservation of global energy and momentum To show the conservation of global energy and momentum, we first integrate the covariant conservation equation over all space:

$$\int \mathrm{Tr}\, \partial_\mu T^{\mu\nu} dx^3 = -\int \mathrm{Tr}\, \left[A_\mu * T^{\mu\nu}\right]_* dx^3$$

and find that $\int \mathrm{Tr}\, \partial_\mu T^{\mu\nu} dx^3 = 0$ because the star-commutator vanishes inside the space integral. Rewriting in components,

$$\int \mathrm{Tr}\, \frac{\partial T^{0\nu}}{\partial t} - \sum_i \frac{\partial T^{i\nu}}{\partial x^i}\; dx^3 = 0$$

where the second term vanishes since $T^{i\nu} = 0$ at infinity for finite energy-momentum solitons, i.e., there is no current flowing in or out of the boundary of space, the energy and momentum charges, $T^{0\nu}$, integrated over all space is conserved in time:

$$\frac{\partial}{\partial t} \int \mathrm{Tr}\, T^{0\nu} dx^3 = 0$$

The force in terms of total energy The force between two non-commutative solitons, or monopoles, is the rate of change of the total energy of the system with respect to the separation distance s, $\partial E(s)/\partial s$, where $E(s)$ is the total energy for the solitons at separation s, $E(s) = \int \mathrm{Tr}\, T^{00}(s) dx^3$. Expanding E(s), we obtain the $\mathcal{O}(\theta)$ correction to the force along the axis of separation of the monopoles in terms of T^{00}:

$$\begin{aligned}
Force &= \frac{\partial}{\partial s}(E^{(0)}(s) + E^{(\theta)}(s) + E^{(\theta^2)}(s)) \\
&= \frac{\partial}{\partial s}\int \mathrm{Tr}\,\left[T^{00(0)}(s) + T^{00(\theta)}(s) + T^{00(\theta^2)}(s)\right]\; dx^3 \\
&= Force_{commutative} + \lim_{\Delta s \to 0} \frac{\int \mathrm{Tr}\,\left[T^{00(\theta)}(s+\Delta s) - T^{00(\theta)}(s)\right] dx^3}{\Delta s} \\
&\quad + \lim_{\Delta s \to 0} \frac{\int \mathrm{Tr}\,\left[T^{00(\theta^2)}(s+\Delta s) - T^{00(\theta^2)}(s)\right] dx^3}{\Delta s} \\
&= Force_{commutative} + Force^{(\theta)} + Force^{(\theta^2)}
\end{aligned}$$

No $\mathcal{O}(\theta)$ correction to total energy To $\mathcal{O}(\theta)$, the argument is simple: The force correction involves the difference of $T^{00(\theta)}(s)$ and $T^{00(\theta)}(s+\Delta s)$, which are the time-component of the $\mathcal{O}(\theta)$ stress-energy tensor for two solitons at separations s and $s + \Delta s$ respectively. For both separations, the $\mathcal{O}(\theta)$ stress-energy tensor depends only on the $\mathcal{O}(\theta)$ correction to the SU(2) components of the field strength, $F_a^{\mu\nu(\theta)}$, which in turn depends only on the $\mathcal{O}(\theta)$ correction to the SU(2) components of the gauge field, which can be chosen to vanish as argued in section 4.3. Thus, the $\mathcal{O}(\theta)$ force correction is zero.

Explicitly, the stress-energy tensor to $\mathcal{O}(\theta^2)$ is:

$$\begin{aligned}
\mathrm{Tr}T^{\mu\nu} &= \mathrm{Tr}\left(g^{\mu\nu}\frac{1}{4}F^{\rho\lambda}*F_{\rho\lambda}-\frac{1}{2}F^{\mu\rho}*F^{\nu}{}_{\rho}-\frac{1}{2}F^{\nu\rho}*F^{\mu}{}_{\rho}\right)\\
&= +\frac{1}{4}g^{\mu\nu}F^{\rho\lambda(0)}_{a}F_{\rho\lambda a}{}^{(0)}-\frac{1}{2}F^{\mu\rho(0)}_{a}F^{\nu}{}_{\rho a}{}^{(0)}-\frac{1}{2}F^{\nu\rho(0)}_{a}F^{\mu}{}_{\rho a}{}^{(0)}\\
&\quad +\frac{1}{2}g^{\mu\nu}F^{\rho\lambda(\theta)}_{a}F_{\rho\lambda a}{}^{(0)}-F^{\mu\rho(\theta)}_{a}F^{\nu}{}_{\rho a}{}^{(0)}-F^{\nu\rho(\theta)}_{a}F^{\mu}{}_{\rho a}{}^{(0)}\\
&\quad +\frac{1}{4}g^{\mu\nu}F^{\rho\lambda(\theta)}_{a}F_{\rho\lambda a}{}^{(\theta)}-\frac{1}{2}F^{\mu\rho(\theta)}_{a}F^{\nu}{}_{\rho a}{}^{(\theta)}-\frac{1}{2}F^{\nu\rho(\theta)}_{a}F^{\mu}{}_{\rho a}{}^{(\theta)}\\
&\quad +\frac{1}{2}g^{\mu\nu}F^{\rho\lambda(\theta^2)}_{a}F_{\rho\lambda a}{}^{(0)}-F^{\mu\rho(\theta^2)}_{a}F^{\nu}{}_{\rho a}{}^{(0)}-F^{\nu\rho(\theta^2)}_{a}F^{\mu}{}_{\rho a}{}^{(0)}\\
&\quad +\frac{1}{4}g^{\mu\nu}F^{\rho\lambda(\theta)}_{0}F_{\rho\lambda 0}{}^{(\theta)}-\frac{1}{2}F^{\mu\rho(\theta)}_{0}F^{\nu}{}_{\rho 0}{}^{(\theta)}-\frac{1}{2}F^{\nu\rho(\theta)}_{0}F^{\mu}{}_{\rho 0}{}^{(\theta)}\\
&\quad +\frac{\theta^2}{4}g^{\mu\nu}\left\{\left\{F^{\rho\lambda(0)}_{a},F_{\rho\lambda a}{}^{(0)}\right\}\right\}-\frac{\theta^2}{2}\left\{\left\{F^{\mu\rho(0)}_{a},F^{\nu}{}_{\rho a}{}^{(0)}\right\}\right\}-\frac{\theta^2}{2}\left\{\left\{F^{\nu\rho(0)}_{a}F^{\mu}{}_{\rho a}{}^{(0)}\right\}\right\}
\end{aligned} \tag{5.14}$$

where all the $\mathcal{O}(\theta^0)$ $F_0^{\mu\nu(0)}$ terms are omitted since the U(1) fields are all set to zero to this order). The $\mathcal{O}(\theta)$ correction is on the second line. These terms do not include derivative terms of the form $\{f,g\}$, which comes from commutators of star products because of the symmetricness of the tensor, and they vanish when $F_a^{\mu\nu(\theta)}$ vanish. (Note that any correction to the U(1) part of the field strength begins to contribute only at $\mathcal{O}(\theta^2)$.)

Now, recall from Eq 5.1 that the first order correction to the field strength for any system is only in the U(1) sector, i.e., $F_a^{\mu\nu(\theta)}=0$, because the SU(2) $\mathcal{O}(\theta)$ fields vanish by the equation of motion provided the zeroth order U(1) fields are chosen to be zero. This means to $\mathcal{O}(\theta)$, for any solution of a perturbative non-commutative U(N) gauge theory, the stress-energy tensor is the same as that for the commutative SU(N) solution to which the non-commutative solution reduces at zeroth order.

In particular then, for the cases in which two solitons [22][25] are at distances s and $s+\Delta s$ apart, the $\mathcal{O}(\theta)$ corrections of the total energy, $T^{00(\theta)}(s)$ and $T^{00(\theta)}(s+\Delta s)$, are zero; therefore, the force correction to this order vanishes.

This fact does not depend on the geometry of the system; for instance, it is true even if the solitons are separated by only a small distance or in the non-commutative direction. It also does not capture any effect of the non-commutative geometry.

the Problem in $\mathcal{O}(\theta^2)$ To calculate the force between two non-commutative monopoles to $\mathcal{O}(\theta^2)$ by calculating $T^{00(\theta^2)}(s)$, we need the solution of the fields it depends on: the $\mathcal{O}(\theta)$ U(1) and $\mathcal{O}(\theta^2)$ SU(2) fields for a two monopole system. We do not pursue this path.

5.3 Force correction Using the Manton Method

We now start investigating how the Manton method can be extended in the non-commutative theory. We will derive the first order ansatz for a single accelerating non-commutative monopole and check that the $\mathcal{O}(\theta)$ correction to the force between two non-commutative monopoles vanishes by this method.

5.3.1 First Order Ansatz for Single Accelerating Non-Commutative Monopole

In the non-commutative theory, Manton's first order ansatz with the product replaced by the star product solves the equations of motion under the assumption that the monopole is accelerated globally in the commutative direction. The argument also follows Manton's but with the product replaced by the star product.

First, as in the commutative case, we describe a non-commutative monopole accelerating rigidly from rest by a small amount by putting in the specific according time dependence in the solution:

$$\phi(x^\nu) = \phi(x^i - \frac{1}{2}\epsilon^2 a^i t^2) \quad ; \quad A^j(x^\nu) = A^j(x^i - \frac{1}{2}\epsilon^2 a^i t^2)$$

where $\epsilon^2 a^i$ is the small acceleration of the non-commutative monopole and is in the commutative direction (discussed below). Again, we choose the gauge in which the time component of the gauge field A^0 vanishes in the instantaneous rest frame of the monopole such that a Lorentz boost along the direction of motion back to the non-accelerating "lab" frame yields the following A^0 in the "lab" frame:

$$A^0 = -\epsilon^2 a^i t A^i$$

Since only Lorentz boost in the commutative direction is a symmetry of the action, and the derivation of the first order ansatz relies on the above expression for A^0, this method only works for acceleration in the commutative direction.

Then, the partial time derivative of the fields and the form of the time component of the gauge field allow the covariant time derivative of ϕ to be written in terms of the covariant spatial deriviatve of ϕ and the time component of the field strength tensor to in terms of its spatial components:

$$\begin{aligned} D^0 * \phi = & \quad -\epsilon^2 a^i t\,(\partial^i \phi - ie\ [A^i, \phi]_*) & = \epsilon^2 a^i t\ D^i * \phi \\ G^{j0} = & \ -\epsilon^2 a^i t\,(\partial^j A^i - \partial^i A^j - ie\ [A^j, A^i]_*) & = -\epsilon^2 a^i t\ G^{ji} \end{aligned}$$

Replacing ordinary products by star products in each step in the same derivation in the commutative theory (section 3.1.1), the time dependent equations of motion for the instant when the monopole starts accelerating from rest, as in the commutative case, can be written to $\mathcal{O}(\epsilon^2)$ with terms depending on the acceleration in place of terms with any time derivatives or explicit time dependence:

$$D_i * (D^i + \epsilon^2 a^i) * \phi = 0; \tag{5.15}$$

$$(\epsilon^2 a_i t)\ D_j * G^{ji} = (\epsilon^2 a_i t)\ ie\ \left[(D^i * \phi) * \phi - \phi * (D^i * \phi)\right]; \tag{5.16}$$

$$\left[D_i + \epsilon^2 a_i\right] * G^{ij} = ie\ \left[(D^j * \phi) * \phi - \phi * (D^j * \phi)\right] \tag{5.17}$$

where again the second equation does not give any information about the $\mathcal{O}(\epsilon^2)$ solution and is automatically satisfied by the static monopole solution.

Since the star product does not involve time derivatives, these rewritten equations of motion do not involve time derivatives; thus, the form of time dependence introduced in the argument of the fields are allowed as in the commutative tcase.

These equations of motions can be satisfied by an analogous ansatz to the one proposed by Manton for the perturbed commutative monopole:

$$G^{ij} = \pm\epsilon^{ijk}(D^k + \epsilon^2 a^k) * \phi \tag{5.18}$$

where besides the fields, the acceleration is also expanded in orders of θ:

$$\epsilon^2 a^k = \epsilon^2 a^{k(0)} + \epsilon^2 a^{k(\theta)} + \epsilon^2 a^{k(\theta^2)}$$

Note that Eq **??** is *-gauge covariant, so the uniform acceleration a^k, which does not *-gauge transform, can be determined in any gauge chosen.

This ansatz satisfies the equation of motion Eq 5.15 because the non-commutative Bianchi Identity:

$$\epsilon^{jki} D_i * G^{jk} = 0$$

depends only on the symmetry of the indices and holds independently of what kind of product acting on the factors. The ansatz also satisfies the equation of motion Eq 5.17. The proofs are exactly analogous to the ones shown in section 3.1.1 with ordinary products replaced with star products and the internal vector cross product replaced by the star commutator times $(-i)$:

$$\mathbf{A^i} \times \phi \ \longmapsto \ -i\ \left[A^i, \phi\right]_*$$

We now examine the U(1) and SU(2) sector of the ansatz to $\mathcal{O}(\theta^2)$.

U(1) component of the ansatz The component proportional to $\mathbf{t_0}$ of 5.18 is as follow:

$$\begin{aligned} &\mp\left(\partial^i\phi_0 + \frac{1}{2}\left(\left[A_0^i, \phi_0\right]_* + \left[A_a^i, \phi_a\right]_*\right)\right) \\ &+\epsilon^{ijk}\partial^j A_0^k + \frac{1}{4}\epsilon^{ijk}\left(\left[A_0^j, A_0^k\right]_* + \left[A_a^j, A_a^k\right]_*\right) = \epsilon^2 a^i \phi_0 \end{aligned} \tag{5.19}$$

Interestingly, if we expand the RHS to an arbitrary order n of θ,

$$\epsilon^2 a^{i(0)}\phi_0^{(\theta^n)} + \epsilon^2 a^{i(\theta)}\phi_0^{(\theta^{n-1})} ... + \epsilon^2 a^{i(\theta^n)}\phi_0^{(0)}$$

because the U(1) $\mathcal{O}(\theta^0)$ fields are chosen to be zero, $\phi_0^{(0)} = 0$, the term that depends on the $\mathcal{O}(\theta^n)$ correction to the acceleration, $\epsilon^2 a^{i(\theta^n)}$ vanishes. This means that at any order n, no matter what $\epsilon^2 a^{i(\theta^n)}$ is, the corrections to the U(1) fields are the same, i.e., the $\mathcal{O}(\theta^n)$ U(1) fields takes no part in determining the $\mathcal{O}(\theta^n)$ correction to the acceleration. In particular, the $\mathcal{O}(\theta^1)$ correction to the acceleration does not depend on the U(1) sector at all and the $\mathcal{O}(\theta^2)$ correction to the acceleration depends on the U(1) fields only up to $\mathcal{O}(\theta^1)$. These are the same as the statements obtained in section 5.2 by inspecting the stress-energy tensor.

5.3.2 $\mathcal{O}(\theta)$ Force Correction

We now know that only the SU(2) component of the first order ansatz (Eq 5.18), expanded as follow:

$$\begin{aligned} &\mp\left(\partial^i\phi_a - \frac{i}{2}\left(\left[A_0^i, \phi_a\right]_* + \left[A_a^i, \phi_0\right]_* + i\epsilon_{abc}\left[A_b^i * \phi_c + \phi_c * A_b^i\right]\right)\right) \\ &+\epsilon^{ijk}\partial^j A_a^k - \frac{i}{4}\epsilon^{ijk}\left(\left[A_a^j, A_0^k\right]_* + \left[A_0^j, A_a^k\right]_* + 2i\epsilon_{abc}A_b^j * A_c^k\right) \\ = \;& \pm\epsilon^2 a^i \phi_a \end{aligned} \tag{5.20}$$

can contain information about the $\mathcal{O}(\theta)$ correction to the acceleration. To $\mathcal{O}(\theta)$, since the $\mathcal{O}(\theta^0)$ U(1) fields vanish, this equation is simply the ansatz for the commutative theory with an extra $\mathcal{O}(\theta)$ modification to the assumed acceleration:

$$\mp\left(D^i\phi\right)_a + \frac{1}{2}\epsilon^{ijk}F_a^{jk} \;=\; \pm(\epsilon^2 a^i + \epsilon^2 a^{i(\theta)})\,\phi_a$$

where the fields are expanded in orders of the different small parameters ϵ and θ:

$$\begin{aligned} \phi_a &= \phi_a^{(0)} + \phi_a^{(\epsilon^2)} + \phi_a^{(\theta)} + \phi_a^{(\theta\epsilon^2)} \\ A_a^i &= A_a^{i(0)} + A_a^{i(\epsilon^2)} + A_a^{i(\theta)} + A_a^{i(\theta\epsilon^2)} \end{aligned}$$

Without the acceleration, Eq 6.1, is a linear fluctuation equation for ϕ_a and A^i_a, just like the second order equation of motion as discussed in section 4.3: it follows that $\phi^{(\theta)}_a = A^{i(\theta)}_a = 0$, but $\phi^{(\theta\epsilon^2)}_a$ and $A^{i(\theta\epsilon^2)}_a$ are still unknown.

Asymptotic condition to $\mathcal{O}(\theta)$ The asymptotic condition of the commutative theory can be extended to first order in θ as well. We can define the matter field J^μ to be what the covariant derivative of the field strength $D_\mu * F^{\mu\nu}$ equals to in Eq 4.10 such that at $\mathcal{O}(\theta^0)$, J^μ reduces to the matter field (Eq 2.6) defined for the commutative case:

$$\begin{aligned} J^\mu &= ie\,[D^\mu\phi, \phi]_* \\ &\doteq e\mathbf{t_a}(\epsilon_{abc}D^\mu\phi_c\,\phi_b) - e\mathbf{t_0}\theta\left\{D^\mu\phi^{(0)}_a, \phi^{(\theta)}_a\right\} + \mathcal{O}(\theta^2) \end{aligned} \tag{5.21}$$

We see that to first order in θ, the SU(2) component of J^μ depends on the SU(2) fields as the commutative matter field does. (We now switch to the vector notation for the SU(2) components of the fields as in section 2.2.1.) Thus, if we again write ϕ in terms of a magnitude field h times a unit vector field $\hat{\phi}$:

$$\phi = \left(h^{(0)} + h^{(\epsilon^2)} + h^{(\theta\epsilon^2)}\right)\left(\hat{\phi}^{(0)} + \hat{\phi}^{(\epsilon^2)} + \hat{\phi}^{(\theta\epsilon^2)}\right)$$

where $\hat{\phi}$ depends as before on a $\Psi(\theta)$ function that captures its dependence on the angle it makes with the z-axis (the commutative direction):

$$\hat{\phi} = \begin{pmatrix} \sqrt{1 - [\Psi(\theta)^{(0)} + \Psi(\theta)^{(\epsilon^2)} + \Psi(\theta)^{(\theta\epsilon^2)}]^2}\cos(\chi) \\ \sqrt{1 - [\Psi(\theta)^{(0)} + \Psi(\theta)^{(\epsilon^2)} + \Psi(\theta)^{(\theta\epsilon^2)}]^2}\sin(\chi) \\ \Psi(\theta)^{(0)} + \Psi(\theta)^{(\epsilon^2)} + \Psi(\theta)^{(\theta\epsilon^2)} \end{pmatrix}$$

such that $\hat{\phi}$ remains of unit length even with the $\mathcal{O}(\theta\epsilon^2)$ correction, we would obtain, by the same reasons as in the commutative case, the same asymptotic condition:

$$\mathbf{J}^\mu = 0 \implies \mathbf{D}^\mu\phi \times \phi = 0 \implies \mathbf{D}^\mu\hat{\phi} = 0;$$

the same relation between the gauge field and $\hat{\phi}$:

$$\mathbf{A}^{\mu} = \frac{1}{e}\partial^{\mu}\hat{\phi} \times \hat{\phi} + \lambda^{\mu}\hat{\phi};$$

and the same factorization of the SU(2) field strength into its magnitude $f^{\mu\nu}$ and the unit vector field $\hat{\phi}$:

$$\mathbf{F}^{\mu\nu} = f^{\mu\nu}\hat{\phi} = -\frac{1}{e}\left[(\partial^{\mu}\hat{\phi} \times \partial^{\nu}\hat{\phi}) \cdot \hat{\phi}\right] \hat{\phi} + [\partial^{\mu}\lambda^{\nu} - \partial^{\nu}a\lambda\mu]\ \hat{\phi}$$

as in the commutative case, except now all the fields include $\mathcal{O}(\theta\epsilon^2)$ corrections on top of the $\mathcal{O}(\epsilon^2)$ corrections. Thus, the non-commutative first order ansatz (Eq 5.18) can also be factorized in the asymptotic region:

$$B^k = \pm \left[\partial^k h + \left(\epsilon^2 a^k + \epsilon^2 a^{k(\theta)}\right)\ h\right]$$

The equation for $\Psi(\theta)$ would be modified to

$$\nabla \times \vec{B} = -\nabla \times (\hat{\chi} \times \nabla\Psi) = \pm\hat{\chi}\,\frac{\left(\epsilon^2 a^{(0)} + \epsilon^2 a^{(\theta)}\right)\ \sin\theta}{r^2}$$

For a system of two opposite charge monopoles separated in the commutative direction by a large distance s (fig 3.2) (s much bigger than the characteristic radius of the monopoles) accelerating from rest towards each other, we can solve the above equation with the respective signs and accelerations for the local magnetic field B (or the local Ψ) to $\mathcal{O}(\theta\epsilon^2)$ near each monopole:

$$\begin{aligned}
\vec{B}_{\ominus} &= \vec{B}^{(0)}_{\ominus} + \vec{B}^{(\epsilon^2)}_{\ominus\ part} + \vec{B}^{(\epsilon^2)}_{\ominus\ hom} + \vec{B}^{(\theta\epsilon^2)}_{\ominus\ part} + \vec{B}^{(\theta\epsilon^2)}_{\ominus\ hom} \\
&= -\frac{\hat{r}_1}{r_1^2} + \hat{r}_1\epsilon^2 a\frac{\cos\theta_1}{r_1} - \hat{\theta}_1\frac{1}{2}\epsilon^2 a\frac{\sin\theta_1}{r_1} - \sigma_1\epsilon^2\vec{a} \\
&\qquad +\hat{r}_1\epsilon^2 a^{(\theta)}\frac{\cos\theta_1}{r_1} - \hat{\theta}_1\frac{1}{2}\epsilon^2 a^{(\theta)}\frac{\sin\theta_1}{r_1} - \sigma_1'\epsilon^2\vec{a^{(\theta)}}
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
\vec{B}_{\oplus} &= \vec{B}^{(0)}_{\oplus} + \vec{B}^{(\epsilon^2)}_{\oplus\ part} + \vec{B}^{(\epsilon^2)}_{\oplus\ hom} + \vec{B}^{(\theta\epsilon^2)}_{\oplus\ part} + \vec{B}^{(\theta\epsilon^2)}_{\oplus\ hom} \\
&= +\frac{\hat{r}_2}{r_2^2} + \hat{r}_2\epsilon^2 a\frac{\cos\theta_2}{r_2} - \hat{\theta}_2\frac{1}{2}\epsilon^2 a\frac{\sin\theta_2}{r_2} + \sigma_2\epsilon^2\vec{a} \\
&\qquad +\hat{r}_2\epsilon^2 a^{(\theta)}\frac{\cos\theta_2}{r_2} - \hat{\theta}_2\frac{1}{2}\epsilon^2 a^{(\theta)}\frac{\sin\theta_2}{r_2} + \sigma_2'\epsilon^2\vec{a^{(\theta)}}
\end{aligned} \tag{5.23}$$

The ansatz for each monopole then gives ∇h near each monopole:

$$\begin{aligned}
\nabla h_{\ominus} &= -\left[\vec{B}^{(0)}_{\ominus} + \vec{B}^{(\epsilon^2)}_{\ominus\ part} + \vec{B}^{(\epsilon^2)}_{\ominus\ hom} + \vec{B}^{(\theta\epsilon^2)}_{\ominus\ part} - \sigma_1'\epsilon^2 a^{(\vec{\theta})}\right] \\
&\quad -(\epsilon^2\vec{a}^{(0)}c + \epsilon^2\vec{a}^{(\theta)}c) + \frac{\epsilon^2\vec{a}^{(0)} + \epsilon^2\vec{a}^{(\theta)}}{r_1} \\
\nabla h_{\oplus} &= +\left[\vec{B}^{(0)}_{\oplus} + \vec{B}^{(\epsilon^2)}_{\oplus\ part} + \vec{B}^{(\epsilon^2)}_{\oplus\ hom} + \vec{B}^{(\theta\epsilon^2)}_{\oplus\ part} + \sigma_2'\epsilon^2 a^{(\vec{\theta})}\right] \\
&\quad +(\epsilon^2\vec{a}^{(0)}c + \epsilon^2\vec{a}^{(\theta)}c) - \frac{\epsilon^2\vec{a}^{(0)} + \epsilon^2\vec{a}^{(\theta)}}{r_2}
\end{aligned}$$

Now, as in the commutative theory, since the solution of the system is in terms of the functions Ψ and h, and the equations for these functions are linear, we can write down the global solution for Ψ and h simply by adding the solutions near the monopoles but also applying the exchange principle that the undetermined homogeneous terms be determined by the expansion of the fields of the opposite monopole and not appear in the global solution:

$$\begin{aligned}
\Psi_{global} &= (\Psi_{\ominus} - \Psi_{\ominus\ hom}) + (\Psi_{\oplus} - \Psi_{\ominus\ hom}) + const \\
h_{global} &= (h_{\ominus} - h_{\ominus\ hom}) + (h_{\oplus} - h_{\ominus\ hom}) + const
\end{aligned}$$

The construction of these global solutions implies that the global magnetic and ∇h fields are also the sum of the fields near each monopole without the homogeneous terms:

$$\begin{aligned}
\vec{B}_{global} &= (\vec{B}_{\ominus} - \vec{B}_{\ominus\ hom}) + (\vec{B}_{\oplus} - \vec{B}_{\oplus\ hom}) \\
\nabla h_{global} &= (\nabla h_{\ominus} - \nabla h_{\ominus\ hom}) + (\nabla h_{\oplus} - \nabla h_{\oplus\ hom})
\end{aligned}$$

We now determine the $\mathcal{O}(\theta)$ constant homogeneous terms and the acceleration by requiring these global fields to reduce to the ones near the monopoles (Eq.5.22, Eq 5.23). As in the commutative case, this amounts to equating the undetermined terms at one monopole to terms from the multipole expansion of the static B and ∇h fields of the other monopole.

Specifically, near the $\ominus$ monopole, we need to expand the terms in the global magnetic field that originates from the opposite $\oplus$ monopole and is of a higher order than $\mathcal{O}(\epsilon^2)$. Now, although the particular solutions, $B^{(\epsilon^2)}_{\oplus part}$ and $B^{(\theta\epsilon^2)}_{\oplus part}$, are of at least $\mathcal{O}(\epsilon^3)$ when expanded, they should be time-retarded as argued in section 3.1.2 and should not affect the $\ominus$ monopole at the initial instant. Also, there is no relationship between the different small parameters θ and ϵ and we cannot compare the order $\mathcal{O}(\theta\epsilon^2)$ with $\mathcal{O}(\epsilon^3)$. Therefore, at first order in θ, the only term to expand would be the $\mathcal{O}(\theta)$ correction of the magnetic field from the $\oplus$ monopole, which is zero. This, then, determines the unknown $\mathcal{O}(\theta\epsilon^2)$ approximately uniform magnetic field near the $\ominus$ monopole, $-\sigma_1'\epsilon^2\vec{a}^{(\theta)}$, to be zero.

For ∇h_{global}, because the monopoles are accelerating in opposite directions, the terms $(\epsilon^2\vec{a}^{(0)} + \epsilon^2\vec{a}^{(\theta)})c$ from both monopole cancel and the expansion of the $\mathcal{O}(\theta\epsilon^0)$ ∇h term of the $\oplus$ monopole, which is zero, is to give rise to $\sigma_1'\epsilon^2\vec{a}^{(\theta)}(\sigma_1' - c)$ near the $\ominus$ monopole. Explicitly,

$$\begin{aligned}
\nabla h_{global} &= -(\vec{B}^{(0)}_{\ominus} + \vec{B}^{(\epsilon^2)}_{\ominus\ part}) + (\vec{B}^{(0)}_{\oplus} + \vec{B}^{(\epsilon^2)}_{\oplus\ part}) \\
&\quad -\vec{B}^{(\theta\epsilon^2)}_{\ominus\ part} + \vec{B}^{(\theta\epsilon^2)}_{\oplus\ part} \\
&\quad +(\epsilon^2\vec{a}^{(0)} + \epsilon^2\vec{a}^{(\theta)})(\frac{1}{r_1} - \frac{1}{r_2})
\end{aligned}$$

where the first line are the $\mathcal{O}(\theta^0)$ terms, and the last line would be irrelevant since it is of $\mathcal{O}(\epsilon^3)$ when expanded near either monopole. Again, the radiative terms $\vec{B}^{(\epsilon^2)}_{\ominus\ part}$ and $\vec{B}^{(\epsilon^2)}_{\oplus\ part}$ do not participate in the matching.

Near $\ominus$ monopole then, to $\mathcal{O}(\epsilon^2)$, the condition for the global ∇h field to reduce to $\nabla h_{\ominus}$ is

$$\sigma_1'\epsilon^2\vec{a}^{(\theta)} - \epsilon^2\vec{a}^{(\theta)}c = 0$$

We already know $\sigma_1'\epsilon^2\vec{a}^{(\theta)} = 0$, therefore $\epsilon^2\vec{a}^{(\theta)}c = 0$.

As in the commutative case, this matching procedure can be interpreted as applying a force law, which is the constant part of the factorzied first order ansatz (Eq 5.22), to the monopoles:

$$\vec{B}^{(\theta)}_{ext} = -\nabla h^{(\theta)}_{ext} - \epsilon^2 \vec{a}^{(\theta)} c$$

where the external fields near one monopole are the far field limit of the fields produced by the other monopole (the exchange principle), except in this case, the force law is accurate up to $\mathcal{O}(\theta\epsilon^2)$. Since both monopoles produce no non-vanishing $\mathcal{O}(\theta)$ fields, the $\mathcal{O}(\theta)$ external fields on both monopoles are zero, and the force correction to this order is zero.

Chapter 6

Preliminary investigation of the Manton Method to $\mathcal{O}(\theta^2)$

As shown before by the non-trivial $\mathcal{O}(\theta^2)$ corrections to the stress-energy tensor (Eq 5.14), the non-local property of the star product starts to affect the dynamics of a non-commutative gauge theory at $\mathcal{O}(\theta^2)$. In fact, these effects render most of the simplifications in and the interpretation of the commutative calculation (ch 3))not valid here. Our objective, then, is to employ only the general scheme of finding the local solution near an accelerating monopole and the global solution valid in between the two monopoles, and equating these solutions in a region both decribe, in order to determine the force between two monopoles up to $\mathcal{O}(\theta^2)$. This chapter reports on our preliminary efforts towards this goal.

We will look at the non-commutative first order ansatz to show which local fields need to be calculated, show a sample calculation of such field, and will conclude by discussing the difficulties in finding the global solutions.

6.1 The SU(2) Component of the First Order Ansatz

Because the $\mathcal{O}(\theta^2)$ U(1) sector of the first order ansatz for an accelerating non-commutative monopole (Eq 5.19) has no dependence on the $\mathcal{O}(\theta^2)$ correction to the acceleration $\epsilon^2 a^{(\theta^2)}$, the local $\mathcal{O}(\theta^2)$ SU(2) fields are the ones to be matched with the corresponding global solution to determine this acceleration correction and they satisfy the SU(2) component of the first order ansatz expanded to $\mathcal{O}(\theta^2)$:

$$\begin{aligned}
& \pm\left(\partial^i \phi_a + \epsilon_{abc} A^i_b \phi_c\right) + \epsilon^{ijk}\partial^j A^k_a + \frac{1}{2}\epsilon^{ijk}\epsilon_{abc}A^j_b A^k_c \\
= \; & \pm(\epsilon^2 a^{k(0)} + \epsilon^2 a^{k(\theta^2)})\phi_a \\
& \mp\frac{1}{2}\epsilon_{abc}\frac{\theta^2}{4}\left(\{\{A^i_b, \phi_c\}\} + \frac{1}{2}\epsilon^{ijk}\{\{A^j_b, A^k_c\}\}\right) \\
& \pm\frac{\theta}{2}\left(\{A^i_a, \phi_0\} + \{A^i_0, \phi_a\}\right) - \frac{\theta}{2}\epsilon^{ijk}\{A^j_0, A^k_a\} + \mathcal{O}(\theta^3)
\end{aligned} \tag{6.1}$$

where the fields are expanded up to $\mathcal{O}(\theta^2\epsilon^2)$ with vanishing $\mathcal{O}(\theta)$ terms:

$$\begin{aligned}
A^k &= A^{k(0)} + A^{k(\epsilon^2)} + A^{k(\theta\epsilon^2)}_a + A^{k(\theta^2)} + A^{k(\theta^2\epsilon^2)} \\
\phi &= \phi^{(0)} + \phi^{(\epsilon^2)} + \phi^{(\theta\epsilon^2)} + \phi^{(\theta^2)} + \phi^{(\theta^2\epsilon^2)}
\end{aligned}$$

Note that in this equation,

1. the $\mathcal{O}(\theta^2)$ and $\mathcal{O}(\theta^2\epsilon^2)$ fields are on only the LHS;
2. the last two lines are non-zero only at $\mathcal{O}(\theta^2)$ or above because the $\mathcal{O}(\theta^0)$ U(1) fields are chosen to vanish;
3. at $\mathcal{O}(\theta^2)$, there is no dependence on the $\mathcal{O}(\theta\epsilon^2)$ fields, because the terms with these fields also involve the SU(2) $\mathcal{O}(\theta)$ fields which vanish as explained in section 4.3;
4. there is dependence on all the lower order–$\mathcal{O}(\epsilon^2)$), $\mathcal{O}(\theta)$) and $\mathcal{O}(\theta\epsilon^2)$–U(1) fields and on the $\mathcal{O}(\theta^0)$ and $\mathcal{O}(\theta^2)$ SU(2) fields.

Therefore, before we look for the local $\mathcal{O}(\theta^2)$ SU(2) solution, we need to first solve for the local $\mathcal{O}(\theta)$ U(1) fields.

6.2 Calculation of the Local U(1) Solution

These local fields may also help in finding the $\mathcal{O}(\theta)$ global solution, as the local fields do in the commutative case, and this global solution may in turn be needed in the determination of the global $\mathcal{O}(\theta^2)$ SU(2) solution, to which the local SU(2) solution is to be matched. However, we have found no way in building the global solution from the local ones and will discuss the problem of doing it in sec 6.3.

The local $\mathcal{O}(\theta)$ U(1) fields involves both the static solution and its $\mathcal{O}(\theta\epsilon^2)$ correction which is due to the $\mathcal{O}(\theta^0)$ acceleration of the monopole. The static solution has been solved [26] [8]. We follow method used by Hata et al [8] [9] and calculate the $\mathcal{O}(\theta\epsilon^2)$ correction.

6.2.1 U(1) static Monopole Solution

The U(1) component of the equation for a static $\ominus/\oplus$ monopole is obtained by expanding the ansatz (Eq 5.19) to $\mathcal{O}(\theta\epsilon^0)$:

$$\epsilon^{ijk}\partial^j A_0^{k(\theta)} + \frac{1}{4}\epsilon^{ijk}\left\{A_a^{j(0)}, A_a^{k(0)}\right\} = \mp\left(\partial^i\phi_0^{(\theta)} + \frac{1}{2}\left\{A_a^{i(0)}, \phi_a^{(0)}\right\}\right) \quad (6.2)$$

(Note that the U(1) ansatz can see the acceleration indirectly through its dependence on the SU(2) fields.)

$\mathcal{O}(\theta)$ U(1) solution for the $\ominus$ monopole

We first look at the $\ominus$ case first and will argue that the solution for the $\oplus$ monopole is the same.

The equation above is a first order linear differential equation with inhomonogenous terms depending on the $\mathcal{O}(\theta^0)$ SU(2) fields, whose exact solutions in terms of the

$W(r)$ and $F(r)$ functions (Eq 2.11 and Eq 2.12) will be used following Hata et al [8]. Also, we now generalize the non-commutativity parameter θ back to the original form θ^{ij} such that symmetry in the choice of the commutative direction can help in determining the form of the solution. The inhomogeneous terms are explicitly

$$\begin{aligned}
\{A^i_a, \phi_a\} &= \theta_{ef}\partial_e\left(\epsilon_{aic}\frac{x_c}{r}W(r)\right)\partial_f\left(\frac{x_a}{r}F(r)\right) \\
&= \theta_{ef}\left(\epsilon_{aie}\frac{W}{r} + \epsilon_{aic}\frac{x_c}{r}W'\frac{x_e}{r} - \epsilon_{aic}x_c\frac{x_e}{r^3}W\right)\left(\frac{\delta_{af}}{r}F + \frac{x_a}{r}F'\frac{x_f}{r} - \frac{x_a}{r}F\frac{x_f}{r^3}\right) \\
&= \epsilon_{ief}\theta_{ef}\frac{WF}{r^2} + \epsilon_{ief}x_e\theta_{fc}x_c\left(-\frac{W'F}{r^3} - \frac{WF'}{r^3} + 2\frac{WF}{r^4}\right) \\
&= \epsilon_{ief}\theta_{ef}\frac{WF}{r^2} + \epsilon_{ief}x_e\theta_{fc}x_c\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{WF}{r^2}\right) \qquad (6.3)
\end{aligned}$$

and

$$\begin{aligned}
&\epsilon_{ijk}\left\{A^j_a, A^k_a\right\} \\
&= \epsilon_{ijk}\theta_{ef}\partial_e\left(\epsilon_{ajc}\frac{x_c}{r}W(r)\right)\partial_f\left(\epsilon_{akd}\frac{x_d}{r}W(r)\right) \\
&= \epsilon_{ijk}\theta_{ef}\left(\epsilon_{aje}\frac{W}{r} + \epsilon_{ajc}\frac{x_c}{r}W'\frac{x_e}{r} - \epsilon_{ajc}x_c\frac{x_e}{r^3}W\right)\left(\epsilon_{akf}\frac{W}{r} + \epsilon_{akd}\frac{x_d}{r}W'\frac{x_f}{r} - \epsilon_{ajd}x_d\frac{x_f}{r^3}W\right) \\
&= \epsilon_{ief}\theta_{ef}\frac{WW}{r^2} + \epsilon_{ief}x_e\theta_{fc}x_c\left(-2\frac{W'W}{r^3} + 2\frac{WW}{r^4}\right) \\
&= \epsilon_{ief}\theta_{ef}\frac{WW}{r^2} + \epsilon_{ief}x_e\theta_{fc}x_c\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{WW}{r^2}\right) \qquad (6.4)
\end{aligned}$$

These respect two symmetries:

1. A generalized rotation that rotates both the coordinates x^k and the non-commutative vector parameter θ^k, which is defined as $\theta^k = \frac{1}{2}\epsilon^{ijk}\theta^{jk}$ and points in the commutative direction;

2. space inversion

The generalized rotation symmetry is intuitive. Suppose θ^k points in the +z direction. If we rotate the coordinate system by an angle α about the +y axis, and at the same time we rotate the vector θ^k in the same way, θ^k would still point in the +z direction after the rotation. In fact, this is a symmetry of the equations of motion, and so the solution to it should be tensors with respect to this rotation. To $\mathcal{O}(\theta)$), the only independent tensors are

$$\begin{aligned} &\text{rank 0:} \quad \theta^i x^i \\ &\text{rank 1:} \quad \theta^k, (\theta^i x^i) x^k, \epsilon^{ijk}\theta^i x^j \end{aligned}$$

since others can be written as linear combinations of these using the following identity:

$$\epsilon_{bjk}\delta_{rc} = \epsilon_{cjk}\delta_{rb} + \epsilon_{bck}\delta_{jr} + \epsilon_{bjc}\delta_{kr} \tag{6.5}$$

Since the inhomogeneous terms (Eq 6.4 and Eq 6.3) satisfy space inversion symmetry, the terms $\partial^i\phi_a$ and $\epsilon^{ijk}\partial^j A_a^k$ also need to be unchanged under spatial inversion. Now, the derivative operator changes sign when space is inversed, so the fields also need to change sign, i.e., be odd under this discrete symmetry. Then, for the $\mathcal{O}(\theta)$ U(1) static equation, expanded below:

$$\epsilon^{ijk}\partial^j A_0^{k(\theta)} + \partial^i\phi_0^{(\theta)} = \epsilon_{ief}\theta_{ef}\left(-\frac{WW}{4r^2} - \frac{FW}{2r^2}\right) + \epsilon_{ief}x_e\theta_{fc}x_c\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{WW}{4r^2} - \frac{FW}{2r^2}\right) \tag{6.6}$$

The particular solution is given in terms of the odd tensor structures:

$$\begin{aligned} A_0^{i(\epsilon)} &= A(r)\epsilon^{ijk}\theta^j x^k = A(r)\theta^{ij}x^j \\ \phi_0^{(\epsilon)} &= B(r)2\theta^i x^i = B(r)\epsilon^{ijk}\theta^{jk}x^i \end{aligned}$$

such that their derivatives in Eq 6.2 are even tensor structures:

$$\begin{aligned} \partial^i\phi_0^{(\theta)} &= B(r)\epsilon^{ijk}\theta^{jk} + \epsilon^{ljk}\theta^{jk}x^l B'(r)\frac{x^i}{r} \\ \epsilon^{ijk}\partial^j A_0^{k(\theta)} &= \epsilon^{ijk}\theta^{kj}A(r) + \epsilon^{ijk}\theta^{kb}x^b A'(r)\frac{x^j}{r} \end{aligned}$$

We rewrite the tensor structure $\epsilon^{ljk}\theta^{jk}x^l x^i$ in terms of the ones in the inhomogeneous terms:

$$\epsilon^{ijk}x^j\theta^{kb}x^b = \frac{1}{2}(\epsilon^{jkl}\theta^{jk}x^l x^i - r^2\epsilon^{ijk}\theta^{jk})$$

to obtain the following ordinary differential equations in r for the coefficients of both tensor structures:

$$\begin{aligned}
\epsilon^{ijk}\theta^{jk} &: \ -A + B + rB' + \frac{1}{2}\frac{FW}{r^2} + \frac{1}{4}\frac{WW}{r^2} = 0 \\
\frac{1}{r}\epsilon^{ijk}x^j\theta^{kb}x^b &: \ A' + 2B' - \frac{1}{2}\left(\frac{FW}{r^2}\right)' - \frac{1}{4}\left(\frac{WW}{r^2}\right)' = 0
\end{aligned}$$

The particular solution of these equations are simple:

$$B(r) = 0 \quad ; \quad A(r) = \frac{1}{2}\frac{FW}{r^2} + \frac{1}{4}\frac{WW}{r^2}$$

We now show that the homogeneous solution is trivial. Assume A_{hom} and B_{hom} to be polynomials in r:

$$A(r) = A_n r^n \quad ; \quad B(r) = B_n r^n$$

such that

$$\begin{aligned}
\epsilon^{ijk}\theta^{jk} &: \quad -A_n + B_n + nB_n = 0 \quad \text{for all n} \\
\frac{1}{r}\epsilon^{ijk}x^j\theta^{kb}x^b &: \quad A_n + 2B_n = 0 \quad \text{for all n}\neq 0:
\end{aligned}$$

Combining these:

$$\text{for all n} \neq 0\text{:} \ B_n(2+1+n) = 0$$

Then B_{hom} is only non-zero for $n = -3$ or 0 and A_{hom} is given in terms of it:

$$B_n = \begin{cases} B_{-3} & \text{for } n = -3 \\ B_0 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \quad ; \quad A_n = \begin{cases} -2B_{-3} & \text{for } n = -3 \\ B_0 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

But the homogeneous solution with these coefficients is not admissble: the terms with B_{-3} is proportional to r^{-2} and blows up at the origin; while those with B_0 become infinite as $r \longrightarrow \infty$.

So the U(1) static $\ominus$ monopole is simply the particular solution:

$$\begin{aligned} A_0^{i(\theta)} &= \theta^{ij}x^j\left(\frac{1}{2}\frac{FW}{r^2}+\frac{1}{4}\frac{WW}{r^2}\right) \\ \phi_0^{(\theta)} &= 0 \end{aligned}$$

Aside Recall from section 4.3.1 that the $\mathcal{O}(\theta)$ U(1) field strength has no gauge freedom apart from that for the $\mathcal{O}(\theta^0)$ SU(2) fields. This implies the combination $\partial^j A_0^{k(\theta)} - \partial^k A_0^{j(\theta)}$ also has no gauge freedom. We can then interpret the solution as a Maxwell gauge field that defines a "magnetic" field $B_0^i = epsilon^{ijk}\partial^j A_0^{k(\theta)}$ and note that this magnetic field has a dipole term and a $1/r^4$ term that does not have the quadrupole angular dependence, which is due to the non-localized source for this "Maxwell" gauge field.

$\mathcal{O}(\theta)$ U(1) solutions for the $\oplus$ monopole

For the $\oplus$ monopole, we avoid solving the equation again by writing the inhomogeneous terms in terms of the ones for the $\ominus$ monopole. We already know how $\hat{\phi}^{(0)}$ and $\mathbf{A}^{i(0)}$ in the asymptotic region change when the monopole changes sign:

$$\hat{\phi}_\oplus = \begin{pmatrix} \phi_{\ominus 1} \\ \phi_{\ominus 2} \\ -\phi_{\ominus 3} \end{pmatrix};$$

$$\mathbf{A}^i_\oplus = \partial^i\hat{\phi}_\oplus \times \hat{\phi}_\oplus = \begin{pmatrix} (\partial^i\hat{\phi}_{\ominus 2})(-\hat{\phi}_{\ominus 3}) - (-\partial^i\hat{\phi}_{\ominus 3})\hat{\phi}_{\ominus 2} \\ (= \partial^i\hat{\phi}_{\ominus 3})\hat{\phi}_{\ominus 1} - (\partial^i\hat{\phi}_{\ominus 1})(-\hat{\phi}_{\ominus 3}) \\ (\partial^i\hat{\phi}_{\ominus 1})\hat{\phi}_{\ominus 2} - (\partial^i\hat{\phi}_{\ominus 2})\hat{\phi}_{\ominus 1} \end{pmatrix} = \begin{pmatrix} -A^i_{\ominus 1} \\ -A^i_{\ominus 2} \\ A^i_{\ominus 3} \end{pmatrix}$$

This leads to the sign change of one of the inhomogeneous terms:

$$\left\{A^{i(0)}_{\oplus a}, \phi^{(0)}_{\oplus a}\right\} = \theta^{ef}\partial^e \begin{pmatrix} -A^{i(0)}_{\ominus 1} \\ -A^{i(0)}_{\ominus 2} \\ A^{i(0)}_{\ominus 3} \end{pmatrix} \cdot \partial^f \begin{pmatrix} \phi^{(0)}_{\ominus 1} \\ \phi^{(0)}_{\ominus 2} \\ -\phi^{(0)}_{\ominus 3} \end{pmatrix} = -\left\{A^{i(0)}_{\ominus a}, \phi^{(0)}_{\ominus a}\right\}$$

$$\left\{A^{j(0)}_{\oplus a}, A^{k(0)}_{\oplus a}\right\} = \theta^{ef}\partial^e \begin{pmatrix} -A^{j(0)}_{\ominus 1} \\ -A^{j(0)}_{\ominus 2} \\ A^{j(0)}_{\ominus 3} \end{pmatrix} \cdot \partial^f \begin{pmatrix} -A^{k(0)}_{\ominus 1} \\ -A^{k(0)}_{\ominus 2} \\ A^{k(0)}_{\ominus 3} \end{pmatrix} = +\left\{A^{j(0)}_{\ominus a}, A^{k(0)}_{\ominus a}\right\}$$

In the asymptotic region then, the equation for the $\oplus$ monopole is the same as for the $\ominus$ monopole except the term $+\partial^i\phi^{(\theta)}_0$ has changed sign:

$$\epsilon^{ijk}\partial^j A^{k(\theta)}_{\oplus 0} + \frac{1}{4}\epsilon^{ijk}\left\{A^{j(0)}_{\ominus 0}, A^{k(0)}_{\ominus 0}\right\} = +\partial^i\phi^{(\theta)}_{\oplus 0} - \frac{1}{2}\left\{A^{i(0)}_{\ominus a}, \phi^{(0)}_{\ominus a}\right\}$$

However, $\phi^{(\theta)}_{\ominus 0}$ was zero, therefore the particular solution for the $\oplus$ monopole is just the same as the $\ominus$ one in the asymptotic region. Now, because of the similarity of the equations for the two different monopoles, we can deduce that the solutions are also the same at the monopole core, i.e.:

$$\begin{aligned} A^{i(\epsilon)}_{\oplus 0} &= \theta^{ij}x^j\left(\frac{1}{2}\frac{FW}{r^2} + \frac{1}{4}\frac{WW}{r^2}\right) \\ \phi^{(\epsilon)}_{\oplus 0} &= 0 \end{aligned}$$

The homogeneous solution vanishes for the same reasons as before. Therefore, although the oppositely charged monopoles have different $\mathrm{O}(\theta^0)$ SU(2) fields, they have the same $\mathrm{O}(\theta)$ U(1) corrections.

6.2.2 U(1) perturbed monopole

We now calculate by the same procedure as above how the U(1) fields change locally when a monopole is accelerating. As in the commutative case, we include the homogeneous solution with undetermined coefficients, for potentially absorbing expanded terms from the global solution such that the global solution would reduce to the local one near each monopole.

The equation to be solved now takes into account the accelerating $O(\theta^0)$ SU(2) fields as well:

$$\epsilon^{ijk}\partial^j A_0^{k(\theta\epsilon^2)} + \frac{1}{2}\epsilon^{ijk}\left\{A_a^{j(\epsilon^2)}, A_a^{k(0)}\right\} \quad = \quad \mp\left(\partial^i\phi_0^{(\theta\epsilon^2)} + \frac{1}{2}\left\{A_a^{i(\epsilon^2)}, \phi_a^{(0)}\right\} + \frac{1}{2}\left\{A_a^{i(0)}, \phi_a^{(\epsilon^2)}\right\}\right)$$

where $\phi_a^{(\epsilon^2)}$ and $A_a^{i(\epsilon^2)}$ are given in terms of the commutative solutions $h^{(\epsilon^2)}$ and $\Psi^{(\epsilon^2)}$ by Eq 6.8, Eq 6.9, and Eq 6.13.

The following section is the explicit calculation of the $\mathcal{O}(\theta\epsilon^2)$ corrections, $\phi_a^{(\theta\epsilon^2)}$ and $A_a^{i(\theta\epsilon^2)}$, to the U(1) fields for an accelerating non-commutative monopole. The result is given in Eq 6.19 and Eq 6.20.

Calculation of the Inhomogeneous terms

As in the static case, we can use the invariance under the generalized rotation, in which both the coordinates and the non-commutative vector θ^i is rotated, to find the form of the solutions. We do this by writing both the inhomogeneous terms and the solution in tensor structures built from θ^k and x^k. We will now show the explicit calculation of the inhomogenous terms, which involves mostly tensor multiplication, derivative and rewriting. The results are shown in Eq 6.11, Eq 6.12, Eq 6.15, Eq 6.16, Eq 6.17, and Eq 6.18.

Notation We will use the following notation:

$$\begin{aligned}
\theta^a &= \frac{1}{2}\epsilon^{ajk}\theta^{jk} \iff \epsilon^{ajk}\theta^a = \theta^{jk} \\
\theta^2 &= \theta^i\theta^i = \frac{1}{2}\theta^{ab}\theta^{ab} \\
\theta x &= \theta^a x^a = \frac{1}{2}\epsilon^{ajk}\theta^{jk}x^a
\end{aligned}$$

Calculation of $\left\{A_a^{i(0)}, \phi_a^{(\epsilon^2)}\right\}$ We will again first calculate this term for the $\ominus$ monopole and obtain the result for the $\oplus$ monopole by sign change arguments.

This term is explicitly the contraction of the first derivatives of the commutative SU(2) solution withe the non-commutative parameter:

$$\begin{aligned} \frac{1}{2}\left\{A_a^{i(0)}, \phi_a^{(\epsilon^2)}\right\} &= \frac{1}{2}\,\theta^{ef}\,\partial^e A_a^{i(0)}\,\partial^f \phi_a^{(\epsilon^2)} \\ \text{where } A_a^{i(0)} &= \epsilon_{aic}\frac{x^c}{r^2} \quad ; \quad \phi_a^{(\epsilon^2)} = h^{(0)}\,\delta\hat{\phi}_a + \delta h\,\hat{\phi}_a^{(0)} \end{aligned}$$

The field $\phi_a^{(\epsilon^2)}$ is still needs to be written the tensor structure form as follow.

For both $\ominus$ and $\oplus$ monopole,

$$\delta\hat{\phi} = \begin{pmatrix} -\frac{\delta\Psi\,\Psi}{\sqrt{1-\Psi^2}}\cos\chi \\ -\frac{\delta\Psi\,\Psi}{\sqrt{1-\Psi^2}}\sin\chi \\ \delta\Psi \end{pmatrix}$$

where $\Psi_\ominus = \cos\theta$ for the $\ominus$ one and according to Eq 3.16 and Eq 3.17,

$$\delta\Psi_\ominus = \left(\frac{\epsilon^2 ac}{2r} - \frac{\epsilon^2 a\sigma_1}{2r}\right)(x^2+y^2).$$

Then, $\delta\hat{\phi}$ can be written as the following using θ^{ij}/θ:

$$\delta\hat{\phi} = \left(\frac{\epsilon^2 ac}{2r} - \frac{\epsilon^2 a\sigma_1}{2r}\right)\begin{pmatrix} -xz \\ -xy \\ x^2+y^2 \end{pmatrix} = \frac{1}{\theta}\epsilon_{abc}\theta_{bk}x_k x_c\frac{1}{r^2}\left(\frac{1}{2}\epsilon^2 ar - \frac{1}{2}\epsilon^2 a\sigma_1 r^2\right) \tag{6.7}$$

such that one of the terms in $\phi_a^{(\epsilon^2)}$ looks like this:

$$h_\ominus\,\delta\hat{\phi}_{\ominus a} = \frac{1}{\theta}\epsilon_{abc}\theta_{bk}x_k x_c\frac{1}{r^2}\left(\frac{1}{2}\epsilon^2 ar - \frac{1}{2}\epsilon^2 a\sigma r^2\right)\left(c - \frac{1}{r}\right) \tag{6.8}$$

The other term involves $\delta h_\ominus$, which can be easily written in tensor form:

$$\begin{aligned} \delta h_\ominus\,\hat{\phi}_{\ominus a} = & \frac{1}{\theta}\epsilon_{abc}\theta_{bk}x_k x_c\frac{1}{r^2}\left(-\frac{1}{2}\epsilon^2 a - \epsilon^2 a(\sigma - c)r\right) \\ & + \frac{1}{2\theta}\epsilon_{abc}\theta_{bc}\frac{1}{r^2}\left(\frac{1}{2}\epsilon^2 ar^2 + \epsilon^2 a(\sigma - c)r^3\right) + k_1\frac{x_a}{r} \end{aligned} \tag{6.9}$$

Next, we need to calculate the derivatives of both terms in $\phi_a^{(\epsilon^2)}$. For the first term:

$$\begin{aligned}
\partial^f(h\delta\hat{\phi}_a) &= \frac{1}{\theta}\left(\epsilon_{abc}\theta_{bf}x_c + \epsilon_{abf}\theta_{bk}x_k\right)\frac{1}{r^2}\left(\frac{1}{2}\epsilon^2 ar - \frac{1}{2}\epsilon^2 a\sigma r^2\right)\left(c - \frac{1}{r}\right) \\
&+\frac{1}{\theta}\epsilon_{abc}\theta_{bk}x_k x_c\frac{1}{r^2}\left(\frac{1}{2}\epsilon^2 a - \epsilon^2 a\sigma r^2\right)\frac{x_f}{r}\left(c - \frac{1}{r}\right) \\
&+\frac{1}{\theta}\epsilon_{abc}\theta_{bk}x_k x_c\left(\frac{1}{2}\epsilon^2 ar - \frac{1}{2}\epsilon^2 a\sigma r^2\right)\left(\frac{-2cx_f}{r^4} + \frac{3x_f}{r^5}\right)
\end{aligned}$$

We eliminate the tensor structures that are not linearly independent of the others using the following identities, which are variations of the identity 6.5:

$$\begin{aligned}
\epsilon_{abc}\theta_{bf}x_c &= \epsilon_{fpq}\theta_{pa}x_q + 2\epsilon_{fab}\theta_{bk}x_k \\
\epsilon_{abc}\theta_{bk}x_k x_c x_f &= -\epsilon_{fbc}\theta_{ck}x_k x_b x_a + \epsilon_{abf}\theta_{bk}x_k r^2
\end{aligned}$$

such that

$$\begin{aligned}
\partial^f(h\delta\hat{\phi}_a) &= \frac{1}{\theta}\left(\epsilon_{fpq}\theta_{pa}x_q\right)\frac{1}{r^2}\left(\frac{1}{2}\epsilon^2 ar - \frac{1}{2}\epsilon^2 a\sigma r^2\right)\left(c - \frac{1}{r}\right) \\
&+\frac{1}{\theta}\epsilon_{fab}\theta_{bk}x_k\frac{2}{r^2}\left(\frac{1}{2}\epsilon^2 ar - \frac{1}{2}\epsilon^2 a\sigma r^2\right)\left(c - \frac{1}{r}\right) \\
&\qquad+\frac{1}{\theta}\epsilon_{fab}\theta_{bk}x_k r^2\left(-\frac{1}{2}\frac{\epsilon^2 ac}{r^3} - \frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r^3} + \frac{\epsilon^2 a}{r^4}\right) \\
&\frac{1}{\theta}\epsilon_{fbc}\theta_{bk}x_k x_c x_a\left(+\frac{1}{2}\frac{\epsilon^2 ac}{r^3} + \frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r^3} - \frac{\epsilon^2 a}{r^4}\right)
\end{aligned}$$

Similarly, we take the derivative of the other term in $\phi_a^{(\epsilon^2)}$ and use the above identities,

$$\begin{aligned}
\partial^f(\delta h\hat{\phi}_a) &= \frac{1}{\theta}[\epsilon_{fpq}\theta_{pa}x_q + 2\epsilon_{fab}\theta_{bk}x_k]\left(+\frac{1}{2}\frac{\epsilon^2 a}{r^2} - \frac{\epsilon^2 a(\sigma_1 - c)}{r}\right) \\
&+\frac{1}{\theta}[-\epsilon_{fbc}\theta_{ck}x_k x_b x_a + \epsilon_{abf}\theta_{bk}x_k r^2]\left(\frac{\epsilon^2 a}{r^4} + \frac{\epsilon^2 a(\sigma_1 - c)}{r^3}\right) \\
&+[\epsilon_{fpq}\theta_{pa}x_q + 2\epsilon_{fab}\theta_{bk}x_k]\left(\frac{1}{2}\frac{\epsilon^2 a(\sigma_1 - c)}{r}\right) \\
&+k_1\left(\frac{\delta_{af}}{r} - \frac{x_a x_f}{r^3}\right)
\end{aligned}$$

Finally, we add these two terms and obtain the following for the derivative of $\phi_a^{(\epsilon^2)}$:

$$\partial^f(\delta\phi) = \begin{array}{l|l} \frac{1}{\theta}\epsilon_{fpq}\theta_{pa}x_q\left(-\frac{1}{2}\epsilon^2 a\sigma_1 c - \frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r} + \frac{3}{2}\frac{\epsilon^2 ac}{r} - \frac{\epsilon^2 a}{r^2}\right) & \text{call the bracket A} \\ +\frac{1}{\theta}\epsilon_{fpq}\theta_{pq}x_a\left(\frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r} - \frac{1}{2}\frac{\epsilon^2 ac}{r}\right) & \text{call the bracket B} \\ +\frac{1}{\theta}\epsilon_{fap}\theta_{pq}x_q\left(-\epsilon^2 a\sigma_1 c + \frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r} + \frac{1}{2}\frac{\epsilon^2 ac}{r}\right) & \text{call the bracket C} \\ +\frac{1}{\theta}\epsilon_{fpq}x_p\theta_{qr}x_r x_a\frac{1}{r^2}\left(-\frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r} + \frac{3}{2}\frac{\epsilon^2 ac}{r} - 2\frac{\epsilon^2 a}{r^2}\right) & \text{call the bracket D} \\ +k_1\left(\frac{\delta_{af}}{r} - \frac{x_a x_f}{r^3}\right) & \end{array}$$

The derivative of the field that is present even when the monopole is not accelerating is much more readily obtained:

$$\partial^e A_a^{i(0)} = \epsilon_{aie}\frac{1}{r^2} - \epsilon_{aic}\frac{2x_c x_e}{r^4}$$

We can now combine the two derivatives to find the inhomogeneous term itself:

$$\begin{aligned} \theta_{ef}\ \partial^e A_a^{i(0)}\,\partial^f\phi_a^{(\epsilon^2)} = \quad & \frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,\epsilon_{fpq}\theta_{pa}x_q\,\frac{A}{r^2} && -\frac{2}{\theta}\theta_{ef}\left(\epsilon_{aic}x_c x_e\right)\epsilon_{fpq}\theta_{pa}x_q\,\frac{A}{r^4} \\ & +\frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,\epsilon_{fpq}\theta_{pq}x_a\,\frac{B}{r^2} && -\frac{2}{\theta}\theta_{ef}\left(\epsilon_{aic}x_c x_e\right)\epsilon_{fpq}\theta_{pq}x_a\,\frac{B}{r^4} \\ & +\frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,\epsilon_{fap}\theta_{pq}x_q\,\frac{C}{r^2} && -\frac{2}{\theta}\theta_{ef}\left(\epsilon_{aic}x_c x_e\right)\epsilon_{fap}\theta_{pq}x_q\,\frac{C}{r^4} \\ & +\frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,\epsilon_{fpq}x_p\theta_{qr}x_r x_a\,\frac{D}{r^4} && -\frac{2}{\theta}\theta_{ef}\left(\epsilon_{aic}x_c x_e\right)\epsilon_{fpq}x_p\theta_{qr}x_r x_a\,\frac{D}{r^6} \\ & +\frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,k_1\left(\frac{\delta_{af}}{r^3} - \frac{x_a x_f}{r^5}\right) && -\frac{2}{\theta}\theta_{ef}\left(\epsilon_{aic}x_c x_e\right)k_1\left(\frac{\delta_{af}}{r^5} - \frac{x_a x_f}{r^7}\right) \end{aligned} \tag{6.10}$$

We will label the left column I and the right one II and refer to the terms by its coefficient and the column label (e.g. the very first term is AI).

This expression looks long but in fact many of its terms vanish identically:

1. BI and BII vanish since

$$\theta_{ef}\ \epsilon_{fpq}\theta_{pq} = 2(\epsilon_{bef}\theta_b\theta_f) = 0$$

2. CII vanishes since

$$\theta_{ef}\left(\epsilon_{aic}x_c x_e\right)\epsilon_{fap}\theta_{pq}x_q = \theta_{ef}x_f x_e\theta_{iq}x_q - \theta_{ei}x_c x_e\theta_{cq}x_q = 0$$

3. DII vanishes since

$$\theta_{ef}\left(\epsilon_{aic}x_c x_e\right)x_a\,\epsilon_{fpq}x_p\theta_{qr}x_r = 0$$

We can disregard the terms kI and kII which are proportional to $1/r^3$ because it is subleading to the other terms in the far field limit.

The non-vanishing tensor structures are not independent of each other, but can be rewritten using identities derived from Eq 6.5 in terms of the three independent ones,$(\theta x)\theta_i$, $\theta^2 x_i$, and $(\theta x)^2 x_i$. The details are as follow.

The terms in column one are proportional to $\epsilon_{aie}\theta_{ef}$ which satisfies the identity:

$$\epsilon_{aie}\theta_{ef} = \epsilon_{aie}\epsilon_{efb}\theta_b = \delta_{af}\theta_i - \delta_{fi}\theta_a$$

Using this, we rewrite AI, CI and DI respectively as

$$\begin{aligned}
\mathbf{AI}: \quad \frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,\epsilon_{fpq}\theta_{pa}x_q\,\frac{A}{r^2} &= \frac{1}{\theta}((-2\theta_q x_q)\theta_i - \epsilon_{ipq}\theta_{pa}x_q\theta_a)\frac{A}{r^2} \\
&= \frac{1}{\theta}(-2\theta x\theta_i - \epsilon_{ipq}\left(\epsilon_{bpa}\theta_b\right)x_q\theta_a)\frac{A}{r^2} \\
&= \frac{1}{\theta}(-2\theta x\theta_i - (\theta_i x_a\theta_a - \theta_q x_q\theta_i))\frac{A}{r^2} \\
&= -\frac{2}{\theta}\,\theta x\,\theta_i\,\frac{A}{r^2}; \\
\mathbf{CI}: \quad \frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,\epsilon_{fap}\theta_{pq}x_q\,\frac{C}{r^2} &= \frac{1}{\theta}(\epsilon_{aap}\theta_i\theta_{pq}x_q - \epsilon_{iap}\theta_a\theta_{pq}x_q)\,\frac{C}{r^2} \\
&= \frac{1}{\theta}(-\epsilon_{iap}\theta_a(\epsilon_{pqb}\theta_b)x_q)\,\frac{C}{r^2} \\
&= \frac{1}{\theta}(-\theta_a\theta_a x_i + \theta_a\theta_i x_a)\,\frac{C}{r^2} \\
&= \frac{1}{\theta}(\theta x\theta_i - \theta^2 x_i)\,\frac{C}{r^2}; \\
\mathbf{DI}: \quad \frac{1}{\theta}\theta_{ef}\,\epsilon_{aie}\,\epsilon_{fpq}x_p\theta_{qr}x_r x_a\,\frac{D}{r^4} &= \frac{1}{\theta}(r^2\theta x\theta_i - (\theta x)^2 x_i)\,\frac{D}{r^4};
\end{aligned}$$

Similarly, we can rewrite AII:

$$\textbf{AII}: \quad -\frac{2}{\theta}\theta_{ef}\,\epsilon_{aic}x_c x_e\,\epsilon_{fpq}\theta_{pa}x_q\,\frac{A}{r^4} \;=\; \frac{2}{\theta}(r^2\theta x\theta_i - (\theta x)^2 x_i)\,\frac{A}{r^4}$$

The inhomogeneous terms for the $\ominus$ monopole is simply the sum of the above:

$$\begin{aligned}\frac{1}{2}\left\{A_a^{i(0)},\phi_a^{(\epsilon^2)}\right\} &= \frac{1}{2\theta}\theta x\theta_i\left(-\frac{\epsilon^2 a\sigma_1 c}{r^2}+2\frac{\epsilon^2 ac}{r^3}\right)\\ &+\frac{1}{2\theta}(\theta)^2 x_i\left(\frac{\epsilon^2 a\sigma_1 c}{r^2}-\frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r^3}-\frac{1}{2}\frac{\epsilon^2 ac}{r^3}\right)\\ &+\frac{1}{2\theta}(\theta x)^2 x_i\frac{1}{r^2}\left(\frac{\epsilon^2 a\sigma_1 c}{r^2}+\frac{3}{2}\frac{\epsilon^2 a\sigma_1}{r^3}-\frac{9}{2}\frac{\epsilon^2 ac}{r^3}\right)\end{aligned} \quad (6.11)$$

For the $\oplus$ monopole, the $\mathcal{O}(\theta^0\epsilon^2)$ SU(2) fields are different and all of the expresions $h_\oplus^{(0)}\delta\hat{\phi}_\oplus$, $(\delta h_\oplus)\hat{\phi}_\oplus^{(0)}$ and $A_{a\oplus}^{(0)}$ cannot be written in terms of tensor structures contructed using θ^{ij} and the coordinates. This is because these vectors have only their the first two components different from those for the $\ominus$ monopole, which are the tensors. To illustrate, since $\Psi_\oplus = -\cos\theta$

$$\delta\hat{\phi}_\oplus = \begin{pmatrix} -\frac{\delta\Psi\,\Psi}{\sqrt{1-\Psi^2}}\cos\chi \\ -\frac{\delta\Psi\,\Psi}{\sqrt{1-\Psi^2}}\sin\chi \\ \delta\Psi \end{pmatrix} \propto \begin{pmatrix} xz \\ yz \\ x^2+y^2 \end{pmatrix} \quad \text{while} \quad \delta\hat{\phi}_\ominus \propto \begin{pmatrix} -xz \\ -yz \\ x^2+y^2 \end{pmatrix}$$

and

$$(\delta h_\oplus)\hat{\phi}_\oplus \propto \begin{pmatrix} xz \\ yz \\ x^2+y^2 \end{pmatrix} \quad \text{while} \quad (\delta h_\ominus)\hat{\phi}_\ominus \propto \begin{pmatrix} -xz \\ -yz \\ x^2+y^2 \end{pmatrix}$$

Luckily, as in the static case, since the $\mathcal{O}(\epsilon^0\theta^0)$ gauge field, also has a relative sign change from the $\ominus$ monopole field in the first two components, and the entire term, that concerns the dot product of the (SU2) vectors $\partial\phi_a^{(\epsilon^2)}$ and $\partial A_a^{(0)}$, and still be can be still be written in tensor form:

$$\begin{aligned}
\frac{1}{2}\left\{A^{i(0)}_{a\oplus},\phi^{(\epsilon^2)}_{a\oplus}\right\} &= \frac{1}{2\theta}\theta x\theta_i\left(+\frac{\epsilon^2 a\sigma_2 c}{r^2}+2\frac{\epsilon^2 ac}{r^3}\right)\\
&+\frac{1}{2\theta}(\theta)^2 x_i\left(-\frac{\epsilon^2 a\sigma_2 c}{r^2}+\frac{1}{2}\frac{\epsilon^2 a\sigma_2}{r^3}-\frac{1}{2}\frac{\epsilon^2 ac}{r^3}\right)\\
&+\frac{1}{2\theta}(\theta x)^2 x_i\frac{1}{r^2}\left(-\frac{\epsilon^2 a\sigma_2 c}{r^2}-\frac{3}{2}\frac{\epsilon^2 a\sigma_2}{r^3}-\frac{9}{2}\frac{\epsilon^2 ac}{r^3}\right) \qquad (6.12)
\end{aligned}$$

Calculation for $\left\{A^{i(\epsilon^2)}_a,\phi^{(0)}_a\right\}$

The second inhomogeneous is calculated in the same way but involves the $\mathcal{O}(\epsilon^2)$ correction to the SU(2) gauge field:

$$\begin{aligned}
\frac{1}{2}\left\{A^{i(\epsilon^2)}_a,\phi^{(0)}_a\right\} &= \frac{1}{2}\,\theta^{ef}\,\partial^e A^{i(\epsilon^2)}_a\,\partial^f\phi^{(0)}_a\\
\text{where } A^{i(\epsilon^2)}_a &= \epsilon_{abc}\partial^i\delta\hat{\phi}_b\hat{\phi}_c+\epsilon_{abc}\partial^i\hat{\phi}_b\delta\hat{\phi}_c\\
\hat{\phi}^{(0)}_a &= \frac{x_a}{r} \quad \text{for } \ominus \text{ monopole}
\end{aligned}$$

We first need to write $A^{i(\epsilon^2)}_a$ in terms of the independent tensor structures

$$\begin{aligned}
A^{i(\epsilon^2)}_a &= \frac{1}{\theta}\epsilon_{abc}(\epsilon_{bpq}\theta_{pi}x_q+\epsilon_{bpi}\theta_{pq}x_q)\left(\frac{1}{2}\epsilon^2 a\frac{1}{r}-\frac{1}{2}\epsilon^2 a\sigma_1\right)\frac{x_c}{r}\\
&+\frac{1}{\theta}\epsilon_{abc}(\epsilon_{bpq}\theta_{pr}x_r x_q)x_i\left(-\frac{1}{2}\epsilon^2 a\frac{1}{r^3}\right)\frac{x_c}{r}\\
&+\frac{1}{\theta}\epsilon_{abc}\left(\frac{\delta_{ib}}{r}-\frac{x_b x_i}{r^3}\right)(\epsilon_{cpq}\theta_{pr}x_r x_q)\left(\frac{1}{2}\epsilon^2 a\frac{1}{r}-\frac{1}{2}\epsilon^2 a\sigma_1\right)\\
&= \frac{1}{\theta}(x_i\theta_{ab}x_b+2x_a\theta_{ib}x_b+r^2\theta_{ai})\left(-\frac{1}{2}\epsilon^2 a\frac{1}{r^2}+\frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r}\right). \qquad (6.13)
\end{aligned}$$

We then take the derivatives of the this and the scalar field $\phi^{(0)}_a$ and multiply these to obtain $\theta^{ef}\,\partial^e A_a^{i(\epsilon^2)}\,\partial^f\phi_a^{(0)}$:

$$\begin{aligned}
\partial^e A^i_{a(\epsilon^2)} &= \frac{1}{\theta}(\delta_{ie}\theta_{ab}x_b + \theta_{ae}x_i + 2(\delta_{ae}\theta_{ib}x_b + x_a\theta_{ie}))\left(-\frac{1}{2}\epsilon^2 a\frac{1}{r^2} + \frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r}\right)\\
&+\frac{1}{\theta}(x_i\theta_{ab}x_bx_e + 2x_a\theta_{ib}x_bx_e)\left(\epsilon^2 a\frac{1}{r^4} - \frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r^3}\right)\\
&+\frac{1}{\theta}\theta_{ai}x_e\left(\frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r}\right)\\
\partial^f\phi_a^{(0)} &= \delta_{af}\left(\frac{c}{r} - \frac{1}{r^2}\right) + x_ax_f\left(-\frac{c}{r^3} + \frac{2}{r^4}\right)\\
\theta_{ef}\,\partial^e A_a^{i(\epsilon^2)}\,\partial^f\phi_a^{(0)} &= (\theta_{if}\theta_{eb}x_b + \theta_{ef}\theta_{fe}x_i + 2\theta_{ef}\theta_{ie}x_f)\left(-\frac{1}{2}\epsilon^2 a\frac{1}{r^2} + \frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r}\right)\left(\frac{c}{r} - \frac{1}{r^2}\right)\\
&+\theta_{ef}x_i\theta_{fb}x_bx_e\left(\epsilon^2 a\frac{1}{r^4} - \frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r^3}\right)\left(\frac{c}{r} - \frac{1}{r^2}\right)\\
&+\theta_{ef}\theta_{fi}x_e\left(\frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r}\right)\left(\frac{c}{r} - \frac{1}{r^2}\right)\\
&+(\theta_{ef}x_ax_f\theta_{ae}x_i + 2r^2\theta_{ef}\theta_{ie}x_f)\left(-\frac{1}{2}\epsilon^2 a\frac{1}{r^2} + \frac{1}{2}\epsilon^2 a\sigma_1\frac{1}{r}\right)\left(-\frac{c}{r^3} + \frac{2}{r^4}\right)
\end{aligned}$$

Finally, we use the identity:

$$\theta^{if}\theta^{fb}x^b = (\theta x)\theta^i - \theta^2 x^i \tag{6.14}$$

to obtain the result for the $\ominus$ in terms of the three independent tensors:

$$\begin{aligned}
\frac{1}{2}\left\{A_{a\ominus}^{i(\epsilon^2)}, \phi_{a\ominus}^{(0)}\right\} &= \frac{1}{2\theta}\theta x\theta_i\left(+\frac{\epsilon^2 a\sigma_1 c}{r^2}\right)\\
&+\frac{1}{2\theta}(\theta)^2 x_i\left(-\frac{\epsilon^2 a\sigma_1 c}{r^2} - \frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r^3}\right)\\
&+\frac{1}{2\theta}(\theta x)^2 x_i\frac{1}{r^2}\left(-\frac{\epsilon^2 a\sigma_1 c}{r^2} + \frac{3}{2}\frac{\epsilon^2 a\sigma_1}{r^3}\right)
\end{aligned} \tag{6.15}$$

We use the sign change argument again to obtain the following expression for the $\oplus$ monopole:

$$\begin{aligned}
\frac{1}{2}\left\{A^{i(\epsilon^2)}_{a\oplus}, \phi^{(0)}_{a\oplus}\right\} &= \frac{1}{2\theta}\theta x\theta_i\left(-\frac{\epsilon^2 a\sigma_2 c}{r^2}\right) \\
&+\frac{1}{2\theta}(\theta)^2 x_i\left(+\frac{\epsilon^2 a\sigma_2 c}{r^2}+\frac{1}{2}\frac{\epsilon^2 a\sigma_2}{r^3}\right) \\
&+\frac{1}{2\theta}(\theta x)^2 x_i\frac{1}{r^2}\left(+\frac{\epsilon^2 a\sigma_2 c}{r^2}-\frac{3}{2}\frac{\epsilon^2 a\sigma_1}{r^3}\right)
\end{aligned} \tag{6.16}$$

It is important that the leading order terms ($\sim 1/s$) in the expansion of this inhomogeneous term cancels with the leading order terms in $\left\{A_0^{i(0)}, \phi_0^{(\epsilon^2)}\right\}$. This causes the U(1) $\mathcal{O}(\theta)$ solution to not change its leading order assymptotic behaviour, i.e., the dipole term in the U(1) static solution is not corrected when the monopole starts accelerating.

Result for $\epsilon^{ijk}\left\{A_0^{j(\epsilon^2)}, A_0^{k(0)}\right\}$

Similar calculation as above gives the final inhomogeneous term for the $\ominus$ monopole:

$$\begin{aligned}
\frac{1}{2}\epsilon^{ijk}\left\{A^{j(\epsilon^2)}_{a\ominus}, A^{k(0)}_{a\ominus}\right\} &= \frac{1}{2\theta}\theta x\theta_i\left(-\frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r^3}\right) \\
&+\frac{1}{2\theta}(\theta)^2 x_i\left(-\frac{1}{2}\frac{\epsilon^2 a\sigma_1}{r^3}\right) \\
&+\frac{1}{2\theta}(\theta x)^2 x_i\frac{1}{r^2}\left(3\frac{\epsilon^2 a\sigma_1}{r^3}\right)
\end{aligned} \tag{6.17}$$

For the $\oplus$ monopole, the expression is the same with σ_1 replaced by σ_2:

$$\begin{aligned}
\frac{1}{2}\epsilon^{ijk}\left\{A^{j(\epsilon^2)}_{a\oplus}, A^{k(0)}_{a\oplus}\right\} &= \frac{1}{2\theta}\theta x\theta_i\left(-\frac{1}{2}\frac{\epsilon^2 a\sigma_2}{r^3}\right) \\
&+(\theta)^2 x_i\left(-\frac{1}{2}\frac{\epsilon^2 a\sigma_2}{r^3}\right) \\
&+(\theta x)^2 x_i\frac{1}{r^2}\left(3\frac{\epsilon^2 a\sigma_2}{r^3}\right)
\end{aligned} \tag{6.18}$$

The particular solution

Adding up the inhomogeneous terms and substituting in the values $\sigma_1\epsilon^2 a = 1/s^2$, $\sigma_2\epsilon^2 a = -1/s^2$ and $\epsilon^2 ac = 2/s^2$, we obtain the equations for the U(1) corrections near each monopole in the two monopole system in fig 3.2:

$$\begin{aligned}\pm\partial^i\phi_0^{(\epsilon\theta)} + \epsilon^{ijk}\partial^j A_0^{k(\epsilon\theta)} &= +\frac{1}{\theta}\theta x\theta_i\left(\mp\frac{7}{4s^2r^3}\right) + \frac{1}{\theta}(\theta)^2 x_i\left(\pm\frac{5}{4s^2r^3}\right) \\ &\quad +\frac{1}{\theta}(\theta x)^2 x_i \frac{1}{r^2}\left(\pm\frac{3}{2s^2r^3}\right)\end{aligned}$$

where the signs on top are for the solution near the $\ominus$ monopole, the one below for the $\oplus$

Now, the solution contains a factor $\epsilon^2 a^{(\theta)}$ such that it is of $\mathcal{O}(\theta\epsilon^2)$, and since $\epsilon^2 a^{(\theta)}$ is odd under space inversions, the other factor of the solution, the tensor structure part, needs to be even such that the solution would be odd, as required by the equation. The proposed form of solution is then:

$$\begin{aligned}A_0^{i(\epsilon\theta)} &= A(r)\,\epsilon^2 a^{(\theta)}\,\frac{1}{\theta^2}(\theta x)\theta_{ij}x_j \\ \phi_0^{(\epsilon\theta)} &= \phi_1(r)\,\epsilon^2 a^{(\theta)}\,\frac{1}{\theta^2}\theta^2 + \phi_2(r)\epsilon^2 a^{(\theta)}\frac{1}{\theta}(\theta x)^2\end{aligned}$$

and the equations for the coefficient of the different tensors are:

$$\begin{aligned}\frac{1}{\theta}(\theta x)\theta_i :&\quad \pm 2\phi_2 - rA' - 3A = \mp\frac{7}{4s^2r^3} \\ \frac{1}{\theta}\theta^2 x_i :&\quad \pm\frac{1}{r}\phi_1' + A = \pm\frac{5}{4s^2r^3} \\ \frac{1}{r^2\theta}(\theta x)^2 x_i :&\quad \pm r\phi_2' + rA' = \pm\frac{3}{2s^2r^3}\end{aligned}$$

The particular solutions for the $\ominus/\oplus$ monopole then has the following coefficients:

$$\phi_1(r) = -\frac{7}{8s^2r}\ ;\ \phi_2(r) = -\frac{7}{8s^2r^3}\ ;\ A(r) = \pm\frac{3}{8s^2r^3}$$

(where the top sign is for $\ominus$)

The $\mathcal{O}(\theta)$ U(1) fields without the undetermined homogeneous terms near each monopole in a system of two non-commutative monopoles are then:

$$A_{0\ominus/\oplus}^{i(\epsilon^2)} = \theta^{ij}x^j\left(\frac{1}{2}\frac{c}{r^3} + \frac{1}{4}\frac{1}{r^4}\right) + \frac{1}{\theta}(\theta x)\theta_{ij}x_j\left(\pm\frac{3}{8s^2r^3}\right) \tag{6.19}$$

$$\phi_{0\ominus/\oplus}^{(\epsilon^2)} = 0 + \theta\left(-\frac{7}{8s^2r}\right) + \frac{1}{\theta}(\theta x)^2\left(-\frac{7}{8s^2r}\right) \tag{6.20}$$

Notice that the U(1) ϕ field becomes non-zero when the monopole starts to accelerate; it vanishes for the single static monopole.

Also, the particular solution does not change the asymptotic behaviour of the U(1) fields. In the region near the axis of the commutative direction (such that the vector from the monopole center to a point in this region makes a small angle with the axis of the commutative direction), as r becomes comparable to s, the static term of the gauge field is of the order s^{-3}, but the particular term contributes only at the next order, s^{-4}. This means that the particular solution may not be relevant in the deciding the acceleration, as in the commutative case.

The homogeneous solution

As in the commutative case, the undetermined homogeneous term is included to take into account the presence of the opposite monopole in the global solution. We have not found the global solution and do not exactly know which of the homogeneous solutions are relevant in the determination of the acceleration. But all of these solutions are well-known, so we will list them below.

Removing the inhomogeneous terms from the $\mathcal{O}(\theta)$ U(1) component of the first order ansatz (Eq 5.19) gives the Laplace equation for the U(1) component of ϕ:

$$\begin{aligned} \pm\ \partial^i \phi_0^{(\epsilon\theta)} &= \epsilon^{ijk}\, \partial^j A_0^{k(\epsilon\theta)} \\ \Longrightarrow\quad \pm\ \partial_i\partial^i \phi_0^{(\epsilon\theta)} &= \ = \ 0 \end{aligned}$$

With only the cylindrical symmetry, the solution is given by any linear combination of some polynomial of r multiplied by a Legendre Polynomial of $\cos\theta$ of some order l, $P_l(\cos\theta)$:

$$\phi_0^{(\epsilon\theta)} = \Sigma_l \left(Ar^l + \frac{B}{r^{l+1}} \right) P_l(\cos\theta)$$

For each order l, the $\phi_0^{(\epsilon\theta)}$ solution can be written in terms of the tensor structures, with the vector θ^k being the axis of the cylindrical symmetry. Then, the homogeneous solution of the gauge field $A_0^{i(\epsilon\theta)}$ can be written in tensor form interms of $\phi_0^{(\epsilon\theta)}$. The important point is that $A_0^{i(\epsilon\theta)}$ and $\phi_0^{(\epsilon\theta)}$ are related differently for different charge monopoles, and this may help in giving enough matching conditions to determine the acceleration.

6.3 Problems to be Solved: the Global Solutions

(Notation: in this section, $\mathcal{O}(\theta^n)$ represents both orders $\mathcal{O}(\theta^n)$ and $\mathcal{O}(\theta^n\epsilon^2)$)

To determine the acceleration between two monopoles using Manton's idea, we still need the $\mathcal{O}(\theta^2)$ local SU(2) solution (solution to Eq 6.1), and the global SU(2) solution, which depends on the global $\mathcal{O}(\theta)$ U(1) solution. The local SU(2) solution can be obtained by a calculation similar to the one for the local U(1) solution with no furthur difficulty. Instead of proceeding with the calculation, we now discuss the predictable problems in finding both the global $\mathcal{O}(\theta)$ U(1) and the global $\mathcal{O}(\theta^2)$ SU(2) solutions.

U(1) sector We immediately notice that the $\mathcal{O}(\theta)$ U(1) equation is linear in $\phi_0^{(\theta)}$ and $A_0^{k(\theta)}$ and an equation for the linear combinations $(\phi_{0\ominus}^{(\theta)}-\phi_{0\oplus}^{(\theta)})$ and $(A_{0\ominus}^{k(\theta)}+A_{0\oplus}^{k(\theta)})$ which involve the local fields at both monopoles can be obtained easily by summing the local equations:

$$\begin{aligned}\epsilon^{ijk}\partial^j\left(A_{0\ominus}^{k(\theta)}+A_{0\oplus}^{k(\theta)}\right)+\partial^i(\phi_{0\ominus}^{(\theta)}-\phi_{0\oplus}^{(\theta)}) &= -\frac{1}{4}\epsilon^{ijk}\left\{A_{s\ominus}^{j(0)},A_{s\ominus}^{k(0)}\right\}-\frac{1}{4}\epsilon^{ijk}\left\{A_{a\oplus}^{j(0)},A_{a\oplus}^{k(0)}\right\}\\ &\quad -\frac{1}{2}\left\{A_{a\ominus}^{i(0)},\phi_{a\ominus}^{(0)}\right\}+\frac{1}{2}\left\{A_{a\oplus}^{i(0)},\phi_{a\oplus}^{(0)}\right\}\end{aligned}$$

However, these linear combinations cannot be the global U(1) solutions. While global U(1) soluttions need to satisfy the second order differential equations of motion (Eq 4.8) which depends on the $\mathcal{O}(\theta^0)$ global SU(2) fields, the above linear combinations depend on some combination which is dictated by the star product of the local $\mathcal{O}(\theta^0)$ SU(2) fields and cannot be easily rewritten in terms of the $\mathcal{O}(\theta^0)$ global SU(2) fields. This non-ability to find the $\mathcal{O}(\theta^0)$ U(1) global solution by superimposing the local equations is due to the dependence of the U(1) equation on the non-linear SU(2) fields.

We note as well that the tensor structure method used for finding the local solutions cannot be used for any equation involving the $\mathcal{O}(\theta^0)$ global SU(2) fields simply because these fields cannot be put into the form of a tensor structure. For example, the first two components of ${\phi_a^{(0)}}_{global}$ contains a square root of the sum of two terms both with coordinate dependence:

$$ {\phi_a^{(0)}}_{global} = h_{global}\hat{\phi}_{a\ global} = h_{global}(r_1, r_2) \begin{pmatrix} \sqrt{1 - \left[\frac{z_1}{r_1} - \frac{z_2}{r_2} - 1 + \mathcal{O}(\epsilon^2)\right]^2} \cos\chi \\ \sqrt{1 - \left[\frac{z_1}{r_1} - \frac{z_2}{r^2} - 1 + \mathcal{O}(\epsilon^2)\right]^2} \cos\chi \\ \frac{z_1}{r^1} - \frac{z_2}{r_2} - 1 + \mathcal{O}(\epsilon^2) \end{pmatrix} $$

Unlike the local solutons, since the terms inside the square root, $\frac{z_1}{r^1}$ and $\frac{z_2}{r_2}$, are both large and depend on different coordinates, the square root cannot be expanded as tensor structures involving polynomials of the coordinates.

SU(2) sector Suppose we solved the U(1) component of the second order differential equation of motion for the global $\mathcal{O}(\theta)$ U(1) fields. Can we do anything other than solving the SU(2) second order equation of motion (Eq 4.12) to find the global $\mathcal{O}(\theta^2)$ SU(2) solution? We have already shown that the SU(2) component of the first order ansatz is non-linear, and so simply adding the SU(2) components of the ansatzes for the two monopoles will not give an equation for the global SU(2) fields. Can we extend Manton's way of finding the global solution through factorizing the SU(2) field strength and isolating parameters that satisfy linear equations such that the global solution of these parameters can be found by superposition? The following crude investigation shows that factorizing the SU(2) field strength to $\mathcal{O}(\theta^2)$ does not give a linear equation for the $\mathcal{O}(\theta^2)$ correction to the parameter Ψ (Eq 3.11) and that as in the U(1) sector, combinations of the ansatzes as candidates for the global equation most likely does not involve the lower order global fields and so is most likely inconsistent with the global second order equation of motion which does.

First, in this sector, the star product modifies the SU(2) component of both the field strength tensor and the asymptotic condition from the commutative case. This rendors Manton's way of finding the global solution not valid at $\mathcal{O}(\theta^2)$.

As shown in Eq 5.1, the SU(2) component of the field strength up to $\mathcal{O}(\theta^2)$ includes the "normal" commutative dependence on the "full" gauge field ($\mathbf{A}^\mu = \mathbf{A}^{\mu(0)} + \mathbf{A}^{\mu(\theta)} + \mathbf{A}^{\mu(\theta^2)}$), but also extra terms originating from the expansion of the star product that depend only on the lower order fields.

Furthurmore, the relation between the gauge field $\mathbf{A}^\mu$ and the scalar field ϕ in the asymptotic region is also changed from the commutative case because the asymptotic condition, that the matter field $\mathbf{J}^\mu$ (Eq 5.21) vanishes, also has extra terms due to the star product:

$$\begin{aligned} J^{\mu(\theta^2)} &= e\mathbf{t_a}\epsilon_{abc}\left(\phi_b^{(\theta^2)}(D^\mu\phi_c^{(0)}) + \phi_b^{(0)}(D^\mu\phi_c^{(\theta^2)}) - \frac{\theta^2}{4}\left\{\left\{\phi_b^{(0)}, (D^\mu\phi_c^{(0)})\right\}\right\}\right) \\ &\quad -e\mathbf{t_a}\theta\left(\left\{D^\mu\phi_a^{(0)}, \phi_0^{(\theta)}\right\} + \left\{D^\mu\phi_0^{(\theta)}, \phi_a^{(0)}\right\}\right) \end{aligned}$$

where all fields have corrections due to the acceleration when the monopole is accelerated, and terms with the $\mathcal{O}(\theta^0)$ U(1) fields and the $\mathcal{O}(\theta)$ SU(2) fields have been omitted. According to this expression, when we factorize the SU(2) components (in vector form) of ϕ:

$$\phi = \left(h^{(0)} + h^{(\theta^2)}\right)\left(\hat{\phi}^{(0)} + \hat{\phi}^{(\theta^2)}\right) \quad ; \quad |\hat{\phi}^{(0)} + \hat{\phi}^{(\theta^2)}| = 1$$

and write the zeroth order term of the SU(2) gauge field using the $\mathcal{O}(\theta^0)$ asymptotic condition, $\mathbf{A}^{\mu(0)} = \partial^\mu\hat{\phi}^{(0)} \times \hat{\phi}^{(0)}$, the relation between the full SU(2) gauge field $\mathbf{A}_a^i$ and the the full SU(2) scalar field ϕ, on top of the terms which are just the linear fluctuation of the relation $\mathbf{J}^\mu = \hat{\phi} \times \mathbf{D}^\mu \hat{\phi}$ that is satisfied in the commutative case (written in the first two lines of the next expression), involves an extra combination of many different vectors given in terms of $\hat{\phi}^{(0)}$ and its partial spatial derivatives:

$$
\begin{aligned}
J^{\mu(\theta^2)} \;=\;& \left(h^{(0)2}|\hat{\phi}^{(0)}|^2\right)\ \mathbf{A}^{\mu(\theta^2)} - \left(h^{(0)2}\mathbf{A}^{\mu(\theta^2)}\cdot\hat{\phi}^{(0)}\right)\ \hat{\phi}^{(0)} + \left(h^{(0)2}\right)\ \hat{\phi}^{(0)}\times\partial^\mu\hat{\phi}^{(\theta^2)} \\
&+ \left(h^{(0)}h^{(\theta^2)}|\hat{\phi}^{(0)}|^2 - h^{(0)2}\hat{\phi}^{(0)}\cdot\hat{\phi}^{(\theta^2)}\right)\ \hat{\phi}^{(0)}\times\partial^\mu\hat{\phi}^{(0)} \\
&------\text{above is the linear fluctuation of } \left(\hat{\phi}\times\mathbf{D}^\mu\hat{\phi}\right)---------- \\
&+ \theta^2\left(2\partial_1 h^{(0)}\partial_2^2\partial^\mu h^{(0)} - 2\partial_2^2 h^{(0)}\partial_1\partial^\mu h^{(0)}\right)\partial_1\hat{\phi}^{(0)}\times\hat{\phi}^{(0)} \\
&\qquad + \theta^2\left[-\partial_2 h^{(0)}\partial_1\partial_2\partial^\mu h^{(0)} + \partial_1\partial_2 h^{(0)}\partial_2\partial^\mu h^{(0)}\right]\ \partial_1\hat{\phi}^{(0)}\times\hat{\phi}^{(0)} \\
&+ \theta^2\left(2\partial_1 h^{(0)}\partial_2^2\partial^\mu h^{(0)} - 2\partial_2^2 h^{(0)}\partial_1\partial^\mu h^{(0)}\right)\hat{\phi}^{(0)}\times\partial_2\hat{\phi}^{(0)} \\
&\qquad + \theta^2\left[-\partial_1 h^{(0)}\partial_1\partial_2\partial^\mu h^{(0)} + \partial_1\partial_2 h^{(0)}\partial_1\partial^\mu h^{(0)}\right]\hat{\phi}^{(0)}\times\partial_2\hat{\phi}^{(0)} \\
&+ \theta^2\left(h^{(0)}\partial_2^2\partial^\mu h^{(0)} - \partial_2^2 h^{(0)}\partial^\mu h^{(0)}\right)\ \partial_1^2\hat{\phi}^{(0)}\times\hat{\phi}^{(0)} \\
&+ \theta^2\left(-h^{(0)}\partial_1^2\partial^\mu h^{(0)} + \partial_1^2 h^{(0)}\partial^\mu h^{(0)}\right)\ \partial_2^2\hat{\phi}^{(0)}\times\hat{\phi}^{(0)} \\
&+ \theta^2\left((4+[1])\partial_1 h^{(0)}\partial_2\partial^\mu h^{(0)} - (4+[1])\partial_2 h^{(0)}\partial_1\partial^\mu h^{(0)}\right)\ \partial_1\hat{\phi}^{(0)}\times\partial_2\hat{\phi}^{(0)} \\
&+ \theta^2\left(-2\partial_2 h^{(0)}\partial^\mu h^{(0)} + 2h^{(0)}\partial_2\partial^\mu h^{(0)}\right)\ \partial_1^2\hat{\phi}^{(0)}\times\partial_2\hat{\phi}^{(0)} \\
&+ \theta^2\left(2\partial_1 h^{(0)}\partial^\mu h^{(0)} - 2h^{(0)}\partial_1\partial^\mu h^{(0)}\right)\ \partial_1\hat{\phi}^{(0)}\times\partial_2^2\hat{\phi}^{(0)} \\
&+ \theta^2\left[\partial_1\partial_2 h^{(0)}\partial^\mu h^{(0)} - h^{(0)}\partial_1\partial_2\partial^\mu h^{(0)}\right]\ \partial_1\partial_2\hat{\phi}^{(0)}\times\hat{\phi}^{(0)} \\
&+ \theta^2\left[-\partial_2 h^{(0)}\partial^\mu h^{(0)} + h^{(0)}\partial_2\partial^\mu h^{(0)}\right]\ \partial_1\hat{\phi}^{(0)}\times\partial_1\partial_2\hat{\phi}^{(0)} \\
&+ \theta^2\left[+\partial_1 h^{(0)}\partial^\mu h^{(0)} - h^{(0)}\partial_1\partial^\mu h^{(0)}\right]\ \partial_1\partial_2\hat{\phi}^{(0)}\times\partial_2\hat{\phi}^{(0)} \\
&+ \theta\left\{\partial_1\partial^\mu h^{(0)}\partial_2\phi_0^{(\theta)} - \partial_2\partial^\mu h^{(0)}\partial_1\phi_0^{(\theta)}\right\}\ \hat{\phi}^{(0)} \\
&\qquad + \theta\left\{\partial_1\partial^\mu\phi_0^{(\theta)}\partial_2 h^{(0)} - \partial_2\partial^\mu\phi_0^{(\theta)}\partial_1 h^{(0)}\right\}\ \hat{\phi}^{(0)} \\
&\qquad + \theta^2\left\{\frac{1}{2}\partial_1\left\{\partial^\mu\hat{\phi}^{(0)}\times\hat{\phi}^{(0)}, h^{(0)}\hat{\phi}^{(0)}\right\}_{(\cdot)}\partial_2 h^{(0)}\right\}\ \hat{\phi}^{(0)} \\
&\qquad - \theta^2\left\{\frac{1}{2}\partial_2\left\{\partial^\mu\hat{\phi}^{(0)}\times\hat{\phi}^{(0)}, h^{(0)}\hat{\phi}^{(0)}\right\}_{(\cdot)}\partial_1 h^{(0)}\right\}\ \hat{\phi}^{(0)} \\
&+ \theta\left\{\partial^\mu h^{(0)}\partial_2\phi_0^{(\theta)} - h^{(0)}\partial_2\partial^\mu\phi_0^{(\theta)}\right\}\ \partial_1\hat{\phi}^{(0)} \\
&\qquad + \theta^2\left\{\frac{1}{2}\partial_2\left\{\partial^\mu\hat{\phi}^{(0)}\times\hat{\phi}^{(0)}, h^{(0)}\hat{\phi}^{(0)}\right\}_{(\cdot)} h^{(0)}\right\}\ \partial_1\hat{\phi}^{(0)} \\
&+ \theta\left\{\partial^\mu h^{(0)}\partial_1\phi_0^{(\theta)} - h^{(0)}\partial_1\partial^\mu\phi_0^{(\theta)}\right\}\ \partial_2\hat{\phi}^{(0)} \\
&\qquad + \theta^2\left\{\frac{1}{2}\partial_1\left\{\partial^\mu\hat{\phi}^{(0)}\times\hat{\phi}^{(0)}, h^{(0)}\hat{\phi}^{(0)}\right\}_{(\cdot)} h^{(0)}\right\}\ \partial_2\hat{\phi}^{(0)} \\
=\;& 0
\end{aligned}
$$

Here, the dependence of the $\mathcal{O}(\theta^2)$ fields are in the first two lines only.

Eliminating $\mathbf{A}_a^{\mu(\theta^2)}$ from the field strength with the above relation, we find that the field strength is not simply the "original" $\mathcal{O}(\theta^0)$ dependence on the entire $\hat{\phi}$ ($\hat{\phi} = \hat{\phi}^{(0)} + \hat{\phi}^{(\theta^2)}$ plus corrections from the star product that are independent of $\hat{\phi}^{(\theta^2)}$), i.e.:

$$\mathbf{F}_a^{\mu\nu} \neq f^{\mu\nu}_{commutative}(\hat{\phi})\ \hat{\phi} + Corr_*(\hat{\phi}^{(0)}, h^{(0)}, \phi_0^{(\theta)}, A_0^{\rho(\theta)})$$

but that even the terms with $\hat{\phi}^{(\theta^2)}$ are not the $\mathcal{O}(\theta^2)$ linear fluctuation of the original commutative form and has dependence on $h^{(\theta^2)}$ unless we could use some constraint to eliminate it as $\partial^\mu\hat{\phi}\cdot\hat{\phi} = 0$ does in commutative case:

$$\begin{aligned}\mathbf{F}_a^{\mu\nu} &= f^{\mu\nu}_{commutative}(\hat{\phi}^{(0)})\ \hat{\phi}^{(0)} + Corr_A^{\mu\nu}(\hat{\phi}^{(\theta^2)}, h^{(\theta^2)}, \phi_0^{(\theta)}, A_0^{\rho(\theta)}, \hat{\phi}^{(0)}, h^{(0)}) \\ &\quad + Corr_*(\hat{\phi}^{(0)}, h^{(0)}, \phi_0^{(\theta)}, A_0^{\rho(\theta)}) \end{aligned} \tag{6.21}$$

$\mathbf{F}_a^{\mu\nu}$'s change in the dependence on $\hat{\phi}$ renders the most important simplication in the calculation in the commutative theory not valid here,the simplication being the factorization of the SU(2) field strength and therefore of the first order ansatz into a magnitude and the unit vector $\hat{\phi}$, such that the curl of the factorized ansatz (Eq 3.13) gives a linear differential equation (Eq 3.14) for one component of $\hat{\phi}$ and the factorized ansatz itself gives a linear differential equation (Eq 3.18) of h, such that the global solutions could be obtained simply by solving the sum of these linear equations for the two monopoles. The global solution in the non-commutative theory to $\mathcal{O}(\theta^2)$ is not found in this "linear" way.

Factorization of the field strength to $\mathcal{O}(\theta^2)$ We look at what would happen if we still factorize the non-commutative SU(2) $\mathbf{F}^{\mu\nu}$ as an attempt to obtain the global equations for the correction parameters, $\Psi(\theta)^{(\theta^2)}$, $\Upsilon(\chi)^{(\theta^2)}$, the degrees of freedom of $\hat{\phi}$ (Eq 3.11), and $h^{(\theta^2)}$. There is advantage if we succeed: the global solution in the factorized form might allow us to use the SU(2) component of the factorized ansatz as a uniform external force law to determine the acceleration:

$$\vec{B}_{ext}^{(\theta^2)} = \pm \left(\vec{Dh}_{ext}^{(\theta^2)} + \epsilon^2 \vec{a}^{(\theta^2)} c \right)$$

where $\vec{Dh}$ is the magnitude of the SU(2) vector $\vec{\mathbf{D}}\phi$ up to $\mathcal{O}(\theta^2)$. Note, however, that this force law is not *-gauge invariant since the length of the SU(2) components of a field is not a *-gauge invariant quantity. We proceed to investigate inspite of this problem.

The SU(2) field strength would factorize as follow:

$$\mathbf{F}_a^{\mu\nu} = \sqrt{(f_{commutative}^{\mu\nu})^2 + |Corr_A^{\mu\nu}(\hat{\phi}^{(\theta^2)}, h^{(\theta^2)})|^2 + |Corr_*|^2} \; (\hat{\phi} + \vec{v}^{(\theta^2)})$$

and is not proportional to $\hat{\phi}$ anymore. The magnitude part has the following problmes:

1. it depends on $\hat{\phi}^{(\theta^2)}$ differently than $f_{commutative}^{\mu\nu}$ and so is most likely not linear in $\Psi^{(\theta^2)}$; this ruins the property that the U(1) embedded field strength is linear in Ψ as in commutative case;

2. it most likely involves more than one of the correction parameters (not just Ψ but maybe also h or Υ) unlike in the commutative case and so applying the curl to it would not give a decoupled equation for any of the parameters

Therefore, the factorization of the field strength does not help in finding the global SU(2) solutions in Manton's way.

Finally, any global equations obtained from combining the factorized local ansatz still involves only the local lower order fields, $\hat{\phi}_{\oplus/\ominus}^{(0)}$ and $h_{\oplus/\ominus}^{(0)}$, which are hardly likely to combine into the lower order global fields $\phi_{global}^{(0)}$ that appears in the second order equation of motion, which is definitely a global equation.

In conclusion, we have not found any equation for the $\mathcal{O}(\theta)$ global U(1) and the $\mathcal{O}(\theta^2)$ global SU(2) fields that is simpler than the second order equations of motion, which is what the first order ansatz tries to simplify in the first place.

6.4 Conclusion for the Non-Commutative Problem

We consider the perturbatively non-commutative U(2) gauge theory with a scalar field in the adjoint representation, in which the space-time non-commutative parameter θ defined by $[x^1, x^2] = i\theta$ is small. We employ the star product formalism such that the equations of motion can be expanded in θ and reduce to those in the commutative theory in the limit $\theta \to 0$. The U(2) gauge group means that all fields have a component proportional to the identity matrix, called the U(1) fields below, and three components proportional to the Pauli matrices, called the SU(2) fields below. The U(2) non-commutative monopole is defined to be the SU(2) commutative monopole (with trivial $\mathcal{O}(\theta^0)$ U(1) fields) with corrections of $\mathcal{O}(\theta)$ and higher. Our original goal is to find the $\mathcal{O}(\theta)$ term in the force between two non-commutative monopoles, but finding that it is trivial, we start investigating the problem at the $\mathcal{O}(\theta^2)$.

We show that the $\mathcal{O}(\theta)$ force correction is zero in the following two ways.

1. We first derive the non-locally conserved stress-energy tensor for the theory to show that at $\mathcal{O}(\theta)$, the tensor depends only on the SU(2) components of the $\mathcal{O}(\theta)$ corrections to the gauge field and scalar fields, both of which can be set to zero because the time dependent equations of motion at this order are simply the linear fluctuation of the equations of the commutative theory. This means that to this order, the total energy of any solution of the non-commutative theory does not change from the total energy of the $\mathcal{O}(\theta^0)$ solution to which the non-commutaive solution reduce when $\theta = 0$. Since the force between two non-commutative solitons separated by any distance s in any direction can be defined as the derivative of the total energy of the system with repect to the the separation distance, it is unchanged to

this order from the force between two commutative solitons in the same configuration. The force between two non-commutative monopoles is simply a particular case of this.

2. We derive the first order ansatz for a single non-commutative monopole weakly and rigidly accelerating in the commutative direction by replacing the ordinary product with the star product in Manton's derivation for the commutative ansatz and by assuming an additional $\mathcal{O}(\theta\epsilon^2)$ correction to the acceleration.

   To $\mathcal{O}(\theta)$, only the SU(2) components of the non-commutative ansatz are relevant to the determination of the force between the monopoles. We expand both the fields and the star products in this SU(2) ansatz in orders of θ, and find that to first order, it depends on the SU(2) components of the gauge field and scalar field just as the commutative ansatz does on the commutative SU(2) fields, but has a modified acceleration which includes the extra $\mathcal{O}(\theta\epsilon^2)$ correction.

   Because of the definition of the star product, the SU(2) fields to $\mathcal{O}(\theta)$ (only) can still be written as vectors in the SU(2) subspace of the U(2) gauge group of the theory. When we factorize the SU(2) scalar field into its magnitude h and the unit vector $\hat{\phi}$, we find that the asymptotic condition and the SU(2) field strength have the same dependence on $\hat{\phi}$ as in the commutative case, except that $\hat{\phi}$ has $\mathcal{O}(\theta)$ corrections. Therefore the magnetic field obtained from the SU(2) field strength still depends linearly on the third component of $\hat{\phi}$, Ψ, which also has $\mathcal{O}(\theta)$ corrections. We can then factorize the SU(2) ansatz and find that it remains linear in Ψ, but is in terms of the modified acceleration with the $\mathcal{O}(\theta\epsilon^2)$ correction.

   For the two monopole system, we can build the global SU(2) solutions again by adding the solutions of the local ansatzes, and note that both the local and global solutions now include $\mathcal{O}(\theta\epsilon^2)$ terms unlike in the commutative case. Then, by the exchange principle, because there is no $\mathcal{O}(\theta\epsilon^0)$ $\vec{B}$ and ∇h fields from either monopole to expand near the other monopole, the external fields near each monopole is zero , and the SU(2) ansatz as a uniform force law gives zero for the $\mathcal{O}(\theta\epsilon^2)$ acceleration.

We proceed to look for the $\mathcal{O}(\theta^2)$ force between two non-commutative monopoles by using Manton's general idea of solving for the local and global solutions of the system and equating them near each monopole.

We can easily solve the non-commutative first order ansatz for the local solutions up to $\mathcal{O}(\theta^2)$ by first assuming the solutions to be linear combinations of tensor structures which are products of the coordinates, x^i, and a more general non-commutative parameter, θ^{ij}, defined by $[x^i, x^j] = i\theta^{ij}$ [8] [9], and corresponds to different angular dependence; and then reducing the ansatz to one ordinary differential equation for each tensor structure and solving these.

Knowing the $\mathcal{O}(\theta^2)$ SU(2) solutions can be obtained similarly, we show explicitly only the calculation for the local $\mathcal{O}(\theta)$ and $\mathcal{O}(\theta\epsilon^2)$ U(1) solutions. The results are as follow:

1. The $\mathcal{O}(\theta)$ U(1) solution [8] consists of a vanishing U(1) $\mathcal{O}(\theta)$ scalar field and a U(1) $\mathcal{O}(\theta)$ gauge field that has a dipole potential as well a non-qradruple but $\sim 1/r^3$ term. We find it interesting that a "magnetic" field defined as the curl of the U(1) $\mathcal{O}(\theta)$ gauge field has no gauge freedom inherited from the star product gauge transformation once the gauge is fixed at the lower order, $\mathcal{O}(\theta^0)$.

2. For the accelerating monopole, the $\mathcal{O}(\theta)$ U(1) fields are independent of the $\mathcal{O}(\theta\epsilon^2)$ acceleration (although this has been determined tobe zero above) but have $\mathcal{O}\left(\theta/(s^2)\right)$ corrections due to the $\mathcal{O}(\theta^0)$ acceleration that do not alter

the asymptotic behavior of the U(1) gauge field. The $\mathcal{O}(\theta/(s^2 r))$ correction to the U(1) scalar field,however, is its only non-vanishing behavior at $\mathcal{O}(\theta)$.

We have not found a way to build the global solutions from the local solutions. In particular, we show how the $\mathcal{O}(\theta^2)$ expansion of the star product prevents us from building the $\mathcal{O}(\theta^2)$ SU(2) global solution using Manton's simplications.

We first expand the star product in the non-commutative field strength and find that its SU(2) components have extra terms (compared to the commutative field strength) that depend only on the $\mathcal{O}(\theta)$ and $\mathcal{O}(\theta^0)$ fields. If we then use the star-product extension of the asymptotic condition used in the commutative case to obtain a relation between the gauge field and the scalar field to $\mathcal{O}(\theta^2)$, we will find that the part of the SU(2) field strength that involves the $\mathcal{O}(\theta^2)$ fields is no longer a topological term in $\hat{\phi}$ as in the commutative case, and involves also the $\mathcal{O}(\theta^2)$ h field.

Then, if we define a magnetic field using the magnitude of the SU(2) field strength such that it reduces at $\mathcal{O}(\theta^0)$ to the magnetic field Manton uses in the commutative case, we will find that it does not depend linearly on Ψ (the third component of $\hat{\phi}$) as it does in the commutative case. Moreover, even without only the $\mathcal{O}(\theta^2)$ part of the asymptotic condition, we can see that to $\mathcal{O}(\theta^2)$, this "magnetic field" is not gauge-invariant, and does not even satisfy a linear equation, not to say the Maxwell's equations; thus, the superposition of magnetic field in the region between the monopoles, and the determination of the external fields near each monopole by multipole-expanding the fields from the opposite monopoles (that work in the commutative theory) are no longer valid to $\mathcal{O}(\theta^2)$. With the asymptotic condition extended to $\mathcal{O}(\theta^2)$, we can see also that there are no linear decoupled equations for Ψ and h, and so unlike in the commutative case, the solution in the region between monopole is not easily found by superimposing the local Ψ and h from the different monopoles.

We conclude that Manton's approach to find the commutative global solution does not work at $\mathcal{O}(\theta^2$. We think of three routes to proceed to look for the $\mathcal{O}(\theta^2$ force:

1. We can use the second order differential equations of motion as the equations for the global solution, and look for behaviours of the global solution that would give rise adn can be matched to the homogeneous terms in the local solutions near the monopoles. We can then use the matching conditions, if there are enough, to determine the force.

2. We can try to find the difference in total energy of the monopole pairs separated by distance s and $s + \delta s$ to $\mathcal{O}(\theta^2)$ using the stress-energy tensor. This involves again the solution to the second order equations of motions, but hopefully obtaining the difference between the total energies do not require solving the equations of motion entirely.

3. We can look for a way to keep track of some sort of flow of the covariantly conserved energy-momentum currents [27] and then find the flux of the momentum currents into a region enclosing a monopole.

None of these routes seems more promising than the other two.

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