COMPETITIVE SUPPLY CHAIN AND REVENUE MANAGEMENT:
FOUR ESSAYS

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Abstract

This dissertation includes four independent essays.

Essay one (chapter two) considers a two-echelon, two-supply chain (SC) system in which manufacturers supply a generic product to their exclusive retailers, who then use service level and retail price to compete for heterogeneous consumers. We question: how do varied consumer preferences get reflected not only in differentiated products/services, but through them to the choice of SC structure that delivers them? We find that SCs can strategically manipulate the product/service strategy and SC structure to hedge themselves from horizontal competition. The key finding is that in a market where consumers have stronger diminishing marginal utility on service, then less differentiated products/services will be observed, and only decentralized supply chains can be the market equilibrium. This is in contrast to the well-known result in marketing that choosing vertical integration is always a Nash equilibrium, and that choosing decentralization can only be a Nash equilibrium when product substitutability is high.

Essay two (chapter three) explores the classical revenue management problem in a competitive context, with both price and seat inventory competition. The main question is how should management make strategic marketing (pricing) and operational (seat allocation) decisions in such a competitive market? Do the conventional approaches (models and algorithms based on a monopoly market) give us the appropriate strategies? We find that in a market where price competition dominates, managers should set a lower price and safety protection level for full fare customers than in a monopoly or alliance market. In a market where seat inventory competition dominates, managers should set a higher price and safety protection level than a monopoly or alliance would. Interestingly, in a market where the two levels of competition are more evenly matched, managers should set a lower price and a higher safety protection level than a monopoly. We also explore the effect of the degree of competition and the market structure on the strategic decisions, and whether there is a first adopter advantage or second adopter disadvantage with revenue management.

Essay three aims to extend the understanding of the Newsvendor model to a competitive framework. In a market with both price and inventory competition, newsvendors can gain customers with price and secure the sales with availability. We find that the newsvendors
should adjust their inventory (safety stock or total inventory) and pricing strategies responsively to the nature of the competitive market. The profits of the newsvendors and their suppliers are also different under different competitive contexts. Both the Nash equilibrium strategy and the players’ profits are influenced by the demand correlation and variability, but in different ways under different competitive scenarios. These observations provide some theoretical basis for the strategic selection made by newsvendors operating in certain competitive markets.

Essay four (chapter five) explores the issue of competitors cooperating. It is a commonplace observation that even the most competitive firms often find it in their best interests to cooperate. An example of cooperation in operations management is when two supply chains agree in advance to transship or ‘pool’ surplus product for use by another. The alternative is to let their customers switch unsatisfied demand to a competitor. Which is preferable, and how does such a preference depend on the many parameters, prices, the nature of competition, the degree of competition, wholesale prices etc? To get answers, we study a stylized model under three market environments: a market with an exogenous retail price, an endogenous retail price, and with price competition. The summary answer is that strong price competition between substitutable goods should lead to caution in signing transshipment contracts. But with little price competition and particularly where retailers are free to set the transshipment price, then transshipment is probably the way to go. We also address the issue of an optimal transshipment price in each scenario, and compare the Nash equilibrium strategies between competing and transshipping.
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1 Introduction

1.1 Motivation

Operations Management is at the heart of most organizations. Whether it is a pair of jeans, sport shoes, midday snack, personal computer, airline ticket or office equipment, operations are central to make them available at the right place, time and price.

A critical activity of operations is to reflect and support the needs of markets. To successfully manage operations within a business and to be a winner in an industry, the demand at the consumer needs special consideration.

Traditional operations management has typically set the objective to be cost minimization subject to capacity or other types of constraints; factors that affect the demand side are less frequently considered.

Recently, in the stream of literature called “operations/marketing interfaces”, decisions such as pricing and efforts that can stimulate the market demand have been incorporated into the joint operational management and marketing strategy set; they are no longer isolated from operations.

However, the most important factor in a market context is still largely missing; competition.

Competition is the product of interacting decision makers targeting the same consumer population. Operations in a market context usually need competition to be considered. For some specific field of operations management such as supply chain (SC) management and revenue management, both closely linked to the market, competition is unavoidable.

Non-cooperative game theory has been widely used in economics in the last two decades for understanding competition, but it has not drawn the full attention of operations researchers. To clarify the nature of the strategic interaction between supply chains, or supply chain members, or revenue maximizing entities, non-cooperative game theory is an excellent analytical tool.

This dissertation is an effort of understanding competition between supply chains, supply chain members, and revenue maximizing entities. The main tool used is non-
cooperative game theory (usually, supermodular game theory). The dissertation focuses on analyzing operational/marketing strategies for business entities operating in a competitive market and contrasts the results achieved through conventional monopoly optimization approaches.

This dissertation attempts to add to research in the field of supply chain and revenue management, a consideration of both consumer behavior and market competition. The operational strategies should reflect the nature of the competitive market (this implicitly calls for empirical work to measure the market). The consideration of competition sometimes reverses the conclusions made by only studying a monopoly.

1.2 Research Problems and The Related Literature

Competition forms the basis of the marketing and economics literature. In recent years it has also attracted those working in Operations Research (Management Science). The following is a brief review of work related to the research problems in this dissertation. More detailed literature reviews are presented chapter by chapter as appropriate.

1. Competition in The Marketing Channel Literature

Ever since the first mention of "double marginalization" by Spengler (1950), the integrated channel has been referred to as a benchmark or ideal against which to measure the efficiency of a decentralized channel or the performance of a contract. McGuire and Staelin (1983) consider the case of two competing manufacturers selling their products through exclusive retailers. They show that the total channel profits with an independent retailer are higher than that with a vertically integrated retailer if and only if \( \theta \geq 0.43 \) (where \( \theta \) is the degree of product substitutability). Each manufacturer's profit is higher when both use decentralized channels than when both are vertically integrated if and only if \( \theta \geq 0.71 \). Looking at the Nash equilibrium in the channel structures, both manufacturers choosing vertical integration is a Nash equilibrium strategy for all values of \( \theta \). In addition, each manufacturer choosing a decentralized channel is also a Nash equilibrium in terms of channel profits when \( \theta \geq 0.77 \), and in terms of the manufacturer profits for \( \theta \geq 0.93 \). They assume that the manufacturers' profits are equal to the total channel profits when using vertical integration (which we think is not entirely reasonable,
so we only use total channel profits in our analysis, see chapter two).

This surprising result stimulated many follow-up papers in marketing. For example, Coughlan (1985), Moorthy (1988), Gupta and Loulou (1998) etc.

In this stream of papers, product substitutability ($\theta$) is treated very much as a “black box” with little discussion about its origins, and the competition is mainly limited to price. In a real world, the scope of competition for supply chains is much more than price. From the operations management point of view, substitutability can originate from product quality, service level (e.g., fill rate, delivery time), etc. Competition based on operational characteristics of supply chains, such as the fill rate, or product availability is essential. It is interesting to know how operations management should choose product strategies (e.g., product quality & service level) and the supply chain structure when targeting a competitive market with heterogeneous consumers. To put it in another way, how do varied consumer preferences get reflected in differentiated products and the choice of supply chain structure? Are there interwoven relationships among supply chain structure, product differentiation and heterogeneous markets? Chapter two considers this question as well as explaining some other issues.

2. **Competition Issues in the Inventory Management Literature**

“One of the fascinating but understudied topics in stochastic inventory theory is competition” (Porteus, 1990). Recent years have witnessed more and more literature on inventory management incorporating competition into their models. Here only the categories that are related to this dissertation are touched upon.

**Revenue Management.** Not much work considers Revenue Management in a competitive framework. Belobaba and Wilson (1997), Li and Oum (1999) and Netessine and Shumsky (2001) are among the few. The first uses a simulation approach. The latter two use game theory as the tool. The main analytical result is that under competition (duopoly), more seats are protected for high-fare passengers than under a monopoly. The competition in this case is because of consumers switching between airlines with the market price being assumed exogenous.

In the airline, hotel, and car rental industries etc., where Revenue Management is commonly used, consumers are usually price sensitive. So a natural research problem is;
how should firms that use Revenue Management and that operate in a market with both price and seat inventory competition make operational (seat inventory) and marketing (price) decisions? Chapter three addresses this and many more other questions, such as: do the conventional approaches (models and algorithms based on a monopoly market) give us the appropriate strategies? How does the degree of competition affect the results? What is the effect of market structure? Is there a first mover advantage?

*Inventory management for substitutable products.* Parlar (1988) was the first to analyze a substitutable product inventory management problem with a game theoretic tool. In his paper, two decision makers are assumed to know the substitution rates and the demand densities for both products when making simultaneous ordering decisions. The existence and uniqueness of the Nash solutions are obtained. The cooperative game version and maximin strategy version of the problem are also discussed. Lippman and McCardle (1997) study a more general version of the competitive newsboy problem. They consider different rules of splitting initial and excess demands. The general result is that competition leads to a higher industry inventory if there is perfect substitutability (i.e., the excess demand is fully reallocated). Mahajan and van Ryzin (1999) consider inventory competition among N firms that provide substitutable competing products and consumers choose firms dynamically based on availability. They show that equilibrium inventory levels exist under mild regularity conditions and find that the competitive equilibrium inventory levels are in general higher than the joint optimal, in agreement with Lippman and McCardle (1997). Rudi and Netessine (2000) also study the case with N products and assume that consumers substitute with a fixed probability. They focus on equilibrium analysis: existence, uniqueness and properties. They find that competition will not necessarily increase stock levels of all products. Anupindi and Bassock (1999) study the effect of stock centralization (pooling) on both manufacturer and retailers when consumers can search. They find that the retailers always benefit from centralization but the manufacturer may lose.

All these papers contribute to the understanding of inventory management for substitutable products. But the common characteristic of this stream of literature is that the substitutability is generated only by lack of availability. The main factor that drives
the consumer choice such as pricing and promotion are ignored and the market prices are assumed to be exogenous.

In a real market where newsvendors sell substitutable products, they can gain customers with price and secure the sale with availability. The natural research problems are: How should vendors in the competitive market make inventory and pricing decisions? How does demand correlation and variability affect these results? What are the implications for newsvendors' profits, suppliers' profit, and consumer welfare?

Chapter four aims to gain a thorough and clear understanding of newsvendors operating in different sorts of competitive markets. See chapter four for details.

*Inventory management with transshipment.* Transshipment is a traditional topic in the inventory management literature. Most of the research done assumes a centralized system with a single decision maker. Recently, this traditional topic has been viewed in a new angle and analyzed by a new tool. For example, Rudi et al. (2001) study the transshipment problem between two independent retailers (who are not involved in any type of competition). Their primary concern was to compare the inventory level set by independent decision makers with that set by a monopoly, and designed a transshipment price to adjust the “distorted” inventory level set by independent players. Anupindi et al. (1999 I & II) explore a ‘coopetition’ business world. Multiple retailers cooperatively agree on the allocation rule to share the transshipment profit, and then competitively decide on individual inventory levels. They focus on the existence and uniqueness of pure strategy Nash equilibrium, first best allocation mechanisms and criteria for implementation. Granot and Sosic (2000) find that the dual solution allocation rule in Anupindi et al. (1999) may induce incentive conflict on sharing surplus/demand, and study the effect of implementing various allocation rules. Seifert et al. (2001) consider a situation where both the retail channel and the manufacturer have market access and unilateral transshipment can happen from the retail store to the manufacturer’s virtual store. They find that the optimal base-stock level at the virtual store is lower when unilateral transshipment is allowed. Dong and Rudi (2001) study a transshipment setting with one manufacturer and N retailers that are owned by the same entity. They find that the manufacturer can extract most of the benefit from transshipment and that (numerically)
retailers are even worse off under transshipment when the manufacturer is a wholesale price setter.

All these examples from the literature relax the conventional assumption that transshipment is among retail outlets belonging to a centralized inventory system, and analyze the transshipment problem from a game theoretic view. But none of them really incorporates competition at the retailer level. It is interesting to ask: Is transshipment preferred by competitive retailers?

In the fifth chapter, we explore cooperation among competitors. From common sense, cooperation would seem to be always preferred by independent parties. But is this always true for competitors involved in ‘fierce’ competition? Incorporating this question into Operations Management, the research problem can be: When should competitors agree to cooperate, in particular for the case of transshipment in supply chains? How does the strategy depend on parameters such as profit margin, the nature of competition and the degree of competition? See chapter five for details.

3. Supermodular Game Theory Literature.

After defining all the above research problems, the key is to find an effective analytical tool to provide the solutions. It is my contention in this dissertation that supermodular game theory is available and is a useful tool to assist with some of these questions. Topkis (1998), Milgrom and Roberts (1990), etc. provide an excellent introduction for supermodular game theory. I have therefore cited the appropriate related theory in an appendix to this dissertation.

1.3 Methodology and Substantive Findings

Only the main findings are discussed here. See each chapter for more results.

Non-cooperative game theory is the main analytical tool of this dissertation.

1. For the second chapter (essay 1), we use multi-stage game theory. The key finding is, the choice of SC structure is very much dependent on consumers' preference. SCs (we assume a single authority can make the decision on SC structure by taking total SC profit as a criteria) can strategically manipulate the product/service strategy and SC structure to protect themselves in a competitive market. For instance, in a market where
consumers have weaker diminishing return on services/products, then more differentiated products/services and an integrated SCs form a market equilibrium. Less differentiated products and decentralized SCs are the market equilibrium if the opposite is true. Both integrated SCs and decentralized SCs can be the equilibria if the above criteria is in between.

This finding is in contrast to previous results that integrated SCs are always the equilibrium, and the decentralized SCs can only be the equilibrium if product substitutability is high.

This result provides a theoretical basis for the choice of SC structure for operations managers. Consumer preference determines the degree of product differentiation, both of them affects the strategy on SC structure.

2. The main finding of chapter three (essay 2) is that the nature of competition and the degree of competition affects the Revenue Management strategies, e.g., the optimal price and safety protection level for high fare customers. In a market where price competition dominates, mangers should set a lower retail price and safety protection level than a monopoly or alliance would. In addition, the ‘fiercer’ the price competition, the lower the retail price and safety protection level. In a market where seat inventory competition dominates, managers should set a higher retail price and safety protection level than a monopoly. In addition, the retail price and safety protection level increase with the spill rate. In a market where the two levels of competition are more evenly matched, managers should set a lower price and a higher safety protection level than a monopoly or an alliance. In addition the price and the safety protection level decreases with the degree of price competition and increases with the spill rate. The number of firms in a market affects the degree of competition and this in turn affects the Nash equilibrium strategies in a similar way. Finally, adopting Revenue Management can lead to a win-win situation.

These findings suggest that operations managers should be aware of the nature (and the degree) of market competition. Monopoly solutions are not the best response in a competitive market. The finding also suggests to government regulators that revenue management is worthwhile supporting.
3. In chapter four (essay 3), we find that newsvendors operating in a competitive market should adjust their (safety) inventory and pricing strategies responsively to the nature of competitive markets. These strategies are also influenced by demand correlation and variability, but in different ways under different competitive markets. (see chapter four for details). The profits of the newsvendors and their suppliers are also different under different competitive contexts. In general, newsvendors suffer from price competition but the manufacturer benefits. Newsvendors benefit from inventory competition but the manufacturer may either suffer or benefit.

These results provide some theoretical basis for strategic selection by newsvendors selling substitutable products.

4. In the fifth chapter (essay 4), we find that the strategy for transshipping or competing depends on parameters such as the transshipment price, profit margin and the degree of competition. Strong competition between substitutable goods should lead to caution in signing transshipment contract. But with little price competition and particularly where retailers are free to set the transshipment price, then transshipment is probably the way to go. We also design the optimal transshipment contract when there is no price competition and show that the non-price-competition transshipment price can make the newsvendors even worse off if they are actually price competitors. We also compare the retail prices and safety stock levels between competing and transshipping, including the cases of transshipping at one retailer's retail price.

These results can also apply to operations in different organizations in general. If the final products are highly substitutable and involved in price competition, it is better for the managers not to sign a contract on pooling the raw materials or sharing the capacity.

In summary, each of the four chapters has different research problems and different focuses. But the heart of the essays is similar:

What are appropriate operational strategies in a competitive market? What is the result of a decision based on a monopoly when in fact we are actually in a competitive world? How can we do better by recognizing the competitive environment?

There are many potential avenues for further research. For example, can we find heuristic policies that are simple to apply? Are there other fields in operations man-
agement where we need to be aware of the influence of market competition, in what ways? How might we adjust existing algorithms and methodologies based on monopoly assumptions such that they can perform better in a competitive context?

1.4 Structure of the Dissertation

The rest of the dissertation consists of chapter two to five in order. Each chapter is a separately organized paper, including a separate introduction, literature review, main body, illustrating tables and figures, and appendix. The bibliography and an appendix on supermodular game theory can be found at the end of the dissertation.
2 Supply Chain Structure under Price and Service Competition

Abstract: We address a two-echelon, two-supply chain (SC) system in which manufacturers supply a generic product to their exclusive retailers, who then use service level and retail price to compete for heterogeneous consumers. We question, how do varied consumer preferences get reflected not only in differentiated products/services, but through them to the choice of SC structure that delivers them? Would suppliers competing with both price and service in this market prefer to distribute via an integrated or decentralized SC? The main finding of this chapter is that the choice of service and SC structure is very much dependent on consumers' preference. SCs can strategically manipulate the product/service differentiation and SC structure to hedge themselves from horizontal competition. For instance, if consumers' utility function is less concave (weaker diminishing return in service), then providing well-differentiated services and choosing an integrated/coordinated SC is the Nash equilibrium strategy. If consumers' utility function is more concave (stronger diminishing return in service), then providing less-differentiated services and choosing a decentralized distribution system is the Nash equilibrium strategy. Otherwise the equilibrium strategy is to match the opponent. Competitors are more likely to choose decentralization in the long run. We also discover the issue of who gets to decide service levels is important for competitive SCs.

2.1 Introduction

A major theme of Supply Chain (SC hereafter) research has centered around the need to 'coordinate' chains; to gain the benefits that might arise from decentralization (e.g.: owner-operator motivation) without introducing any loss of efficiency from sub-optimization. The majority, although certainly not all, of such research has been with a single or monopoly chain. If we assume full knowledge is available to all parties and ignore potential benefits from decentralization such as a local motivation to work harder, a single centralized SC cannot be outperformed by a decentralized one. Indeed, authors often refer to the centralized case as a benchmark or ideal against which to test decentral-
ized coordinating schemes. The principal loss of efficiency of a decentralized SC is often termed 'double marginalization' (Spengler 1950) where the downstream stage (the retailer) uses the wholesale transfer price as an input cost rather than the entire chain that would use the marginal cost of production.

Consider two or more SCs operating in a competitive market, and the situation changes dramatically. For example, twenty years ago, McGuire and Staelin (1983) (MS hereafter) showed that for two SCs operating in a competitive market, the Nash equilibrium strategy can be either choosing vertical integration or choosing decentralization. Although choosing vertical integration is a safer strategy, choosing decentralization can benefit the SCs. In contrast, we find in this chapter that choosing decentralization is the only Nash equilibrium under certain consumer preference. Thus the assumption that SC coordination is a 'good thing' needs serious questioning when SCs operate in the real world of a competitive market. The intuition for the result of MS is that the distorted price decisions made by the retailers because they 'see' the wrong input prices actually lead to a concommittant 'cooling down' of price competition. The consequent gain from the reduction in competition more than compensates for the introduction of double marginalization when the competition is fierce, which is the case when the products are near substitutes.

The notion of product substitution plays a big role in the argument above. In MS and much other literature, product substitution is treated very much as a 'black box' with little discussion about its origins. Operations experience substitutability primarily from products that are differentiated in some way but which consumers are prepared to substitute for the product of their choice if the price is right. A simple example is when otherwise identical products come bundled with services such as warranties, delivery options, after sales service or product availability. Product differentiation in a heterogeneous market is valuable purely for its own sake because a better matching of products to consumer preferences will extract more of the consumer surplus and increase SC profits. In this respect it is similar to yield management, although not structured within a fixed capacity context. However, skillfully differentiating products can also 'cool down' a competitive market by appealing to heterogenous consumers: e.g. 'niche
markets', or offering enhanced levels of services for those willing to afford them.

Thus there are two alternative strategies. One is to decentralize, deliberately introducing double marginalization by distorting prices, but 'cooling down' competition. The second is to differentiate products. It is not clear how heterogeneous consumers drive product differentiation and indirectly affect the design of SCs. It is the purpose of this chapter to explore these interwoven themes of SC structure, product differentiation and heterogeneous markets.

We question, how do varied consumer preferences get reflected not only in differentiated products, but through them to the choice of SC structure that delivers them? Would supply chains competing with both price and service in this market prefer to be integrated or decentralized? There are two levels at which to ask such questions. The level of service could be taken as given, one can think of a 'mature' market, with the competition in the marketplace being solely via pricing. Alternatively, the very selection of a service level to bundle with a product might form the basis of competition, (we call it an emerging market).

We explore these issues with the use of a simple stylized model. An industry consists of two SCs, each with one manufacturer and one retailer, serving a market with an essentially generic product bundled with a 'service level' that might represent delivery times, warranties, maintenance contracts, packaging, availability, accessibility, provision of information, etc. The service level is assumed to be on a continuous scale from low to high and consumers exhibit preferences through a willingness to pay more for a higher level of service. A high level of service will cost more to provide.

Competition on service levels and profits between SCs we term 'horizontal competition'. Any efficiency loss within a SC due to decentralization we term 'vertical competition'. Note that in this stylized model we do not consider any possible gains that might come from other benefits of decentralization such as improved local information (we assume everyone equally and totally informed) or the incentive to work harder or smarter because of local responsibility.

We use a game theoretic methodology throughout the chapter. The multi-stage duopoly game we study includes three subgames; the first subgame is about SC struc-
ture; the second, a service level subgame and finally the pricing subgame. The second subgame is omitted in the fixed service level scenario. According to Fudenberg and Tirole (1991), a multi-stage game with observed actions requires that all players know the actions chosen at all previous stages when choosing their actions at the current stage, and that all players move ‘simultaneously’ in each stage. For a general review about oligopoly multi-stage games, see Shapiro (1989).

We have the following main results,

1. *With price competition only.* Both SCs choosing vertical integration is always an equilibrium structure regardless of the degree of product differentiation. If the service levels of the two SCs are similar (small product differentiation), then having both SCs choosing decentralization is also an equilibrium and will be preferred by both SCs. Moreover, SCs can be better off in the decentralized case even if not an equilibrium. In agreement with previous literature.

2. *The wholesale price with price competition only.* The wholesale price set by the manufacturer can substantially influence the optimal SC structure in the pricing subgame. If the wholesale price is set to maximize SC profit, decentralization is always better than integration.

3. *With both price and service competition.* Integration may not be an equilibrium, but decentralization may be the unique equilibrium. This is a key distinction from the price only case (and the seminal work of MS). Thus SCs might strategically manipulate both SC structure and service level to hedge the effects of horizontal competition. When consumers have stronger diminishing marginal utility on service, the right equilibrium strategy is to provide less differentiated products and only decentralized SC. When consumers have weaker diminishing marginal utility on service, well-differentiated products and choosing vertical integration will be the Nash equilibrium. Who makes the service decision matters and this affects the strategic behavior of SCs and their profits.

The key innovation of this chapter is to explicitly introduce a product characteristic (in our case service) which is differentially valued by consumers. Thus unlike the previous literature, we extend the scope of competition, the competition is not only limited to price, but also operations strategies. This allows us to trace the impact of consumer
choice on not only prices, market share; but to operational strategies (e.g., service level) and the actual choice of SC Structure itself. Despite being a highly stylized model, this paper tries to provide both a theoretical framework for understanding SC management and also some managerial insights.

After a brief literature review and model description, the paper has two main parts. The first part assumes a fixed service level, the second has the service levels as a decision variable.

2.2 Related literature

Recent papers that model horizontal competition in the SC literature include Parlar (1988), Lippman and McCardle (1997) and etc. Newsvendors compete because consumers would like to transfer to others (products are substitutable) when facing a stock out. Chen et al. (2001), Cachon (1999), Lariviere (1999) and etc., study the profit loss due to vertical decentralization in a SC and propose incentive coordinating mechanisms. There is no horizontal competition in this class of models. van Ryzin and Mahajan (1999) and Tsay and Agrawal (2000) incorporate both vertical and horizontal competition in their models, but the horizontal competition is among retailers selling the product of a single manufacturer. Boyaci and Gallego (2000) study a duopoly market with both vertical competition between echelons in a SC and horizontal competition between two SCs. There is no price competition, and market share is determined by service rate. They show numerically that although coordination (integration) is a dominant strategy for each SC, decentralization can make both SCs better off. Tsay and Agrawal (2000), point out "...the complexity of the demand perceived upstream explains why simpler, and typically deterministic, formulations are found in most existing multi-echelon analyses incorporating competition, including ours".

Many marketing researchers study the issue of SC integration/decentralization in a competitive market with a deterministic model. MS is in a similar spirit to us and studies two SCs with just price competition. They find that integration is always an equilibrium in a duopoly market and a structure with both SCs decentralized can be a Nash equilibrium when the degree of substitutability $\theta$ exceeds a critical number. The key
extensions of this chapter is to explicitly model consumer behavior and make the selection of service levels (or substitutability) endogenous. So we extend the scope of competition from price which is the basic of economics and marketing literature to competition in operations strategy, and see the consequence result on SC structure.

Iyer (1998) also addresses the relationship between market heterogeneity and SC structure, but his result is strongly based on spatial heterogeneity (different consumers are located at different places), because if the transportation cost is zero (no spatial differentiation), the SC structure becomes independent of consumers' heterogeneity in willingness to pay for product/service. That is, there is no way to differentiate SC integration from decentralization. In contrast, we approach the issue from another direction. We focus on the role that the heterogeneity of consumer willingness to pay plays in driving decisions on SC structure; using a continuous consumer type model to increase generality. We also use general utility and cost functions. When the travel cost is zero, his model becomes a particular case of ours.

Gupta and Loulou (1998) explore the SC structure problem where manufacturers invest in cost reduction. They find that when cost reduction is easier, the critical amount of substitutability $\theta$ for decentralization to be the equilibrium is reduced. They also analyze two examples of coordinated decision making in a supply chain. We compare our results with theirs when necessary.

The methodology of segmentation we use in this paper follows a traditional economic segmentation approach, first introduced by Hotelling (1929) and developed by numerous later works. Our segmentation model is similar to that of Moorthy (1988) and Bhargava and Choudhary (1999). In contrast to them, we use general utility and cost functions in our model whenever possible.

2.3 Introduction to the Model

Products are assumed essentially generic, differing only in the level of service bundled with them. To allow us to focus on the essential features we consider only a single level of service denoted by a continuous index $a_i \geq 0$ for product $i$. A larger $a_i$ indicates a "higher" level of service, more valued by a consumer, able to attract a higher price $p_i$.
and costing more to offer, \( c(a_i) \). We assume that \( c(a) \) is increasing and convex.

The chapter will consider just two products, one offered by each of two SCs. Apart possibly from asymptotic results, we suspect that a generalization to more than two holds little by way of interesting insights. Each SC will have a manufacturing stage followed by a retailing stage. When a single authority speaks for both stages or some coordination mechanisms enable each stage to make decision to maximize total SC profits, we shall call it an ‘integrated SC’. If the two stages act independently and maximize their own profits the SC is ‘decentralized’.

When the SC is decentralized, retailers purchase the intermediate product from the manufacturers at a wholesale price \( w_i \). For simplicity and without loss of generality, we assume that the marginal cost of production is zero. The results holds for the nonzero cost case. Throughout the chapter, product two will be taken as the product with a higher service level, \( a_2 > a_1 \) and will have \( p_2 > p_1 \). (if \( a_1 = a_2 \), Bertrand competition lead to zero profits for both SC. To avoid head to head competition, both SCs have incentive to differentiate themselves in service and price).

Consumers and the Market

Consumers are heterogeneous in that they value the level of service differently, having a utility function (consumer surplus) of the form \( U(\theta) = \theta u(a) - p \). Here \( u(a) \) is the utility derived from receiving service level \( a \) and \( p \) is the price paid. The heterogeneity of consumers is reflected in their “type” \( \theta \). A consumer with a larger \( \theta \) is prepared to pay more for service. For simplicity we take \( \theta \) as uniformly allocated on \([0,1]\). The more general case of \([0,b]\), where \( b > 1 \) is also investigated. We assume \( u(a) \) to be increasing and concave in \( a \). This kind of consumer surplus function is usually used in modeling product differentiation (Tirole 1988). We believe that the qualitative results of this paper do not depend in any material way on this particular form of utility function.

Without loss of generality we shall assume the total potential market for the product (with any level of service) to be 1. The actual demand for product \( i \) will be \( \lambda_i \). It is important to note that \( \lambda_1 + \lambda_2 \) will typically be less than 1. Consumers with a really low \( \theta \) will prefer no product to even the “cheap” one. Thus, we can derive the demand functions. Consumers of type \( \theta \geq \theta_1 \) will participate: i.e. \( U(\theta_1) = \theta_1 u(a_1) - p_1 = 0 \). The
critical customer who is indifferent between the two SCs is \( \theta_2 : \theta_2 u(a_1) - p_1 = \theta_2 u(a_2) - p_2 \).

So the two demand functions are \( \lambda_1 = \theta_2 - \theta_1 = \alpha p_2 - \beta p_1 \), \( \lambda_2 = 1 - \theta_2 = 1 - \alpha (p_2 - p_1) \)
where \( \alpha = \frac{1}{u(a_2) - u(a_1)} \), and \( \beta = \frac{1}{u(a_2) - u(a_1)} + \frac{1}{u(a_1)} \). The total demand served is \( \lambda_1 + \lambda_2 = 1 - (\beta - \alpha)p_1 \).

The Supply Chain Structure

We consider four types of SC structures. II means both SCs are integrated, DD both are decentralized. Similarly we define the mixed structures ID and DI. At any time the superscripts I and D will be used on any appropriate variable. Thus \( p^{ID}_1 (a^{ID}_i) \) would be the price (service level) of product \( i \) under ID. In the first part of the paper we do not need to further specify \( u(a) \) and \( c(a) \); but in the second part we shall specify a series of utility and cost functions \( u(a) = \sqrt[3]{a}, c(a) = ca, (n \geq 2) \) to obtain more explicit results.\(^1\)

We shall frequently simplify \( u_i = u(a_i) \).

For convenience we summarize our notation:

\( p_i, a_i, w_i \) — retail price, service level and wholesale price of SC \( i \).
\( c_i = c(a_i) \) — cost of providing service level \( a_i \).
\( \lambda_i \) — demand for product \( i \).
\( \pi^M_i, \pi^R_i, \pi_i \) — profits of manufacturer, retailer and SC respectively.

The sequence of events is as follows:

1. Supply Chain Structure Subgame. Players choose SC structures (I or D) simultaneously. (We use total SC profit criteria to compare efficiency of different structures. Otherwise, the profit of manufacturer (retailer) in decentralized SC is not comparable to the profit of an integrated SC.)

2. Service Level Subgame. Players make service level decisions simultaneously. In the case of decentralized SCs we need to consider who makes the service decision.

3. Wholesale Pricing Subgame. Manufacturers set a wholesale price \( w \). (Omitted for integrated SCs)

4. Retail Pricing Subgame. Retailers simultaneously set a retail price \( p \).

We assume that manufacturers and retailers are profit maximizers, and all players

\(^1\) Or \( u(a) = a, c(a) = ca^n \). They provide the same insights since by redefining the decision variable one form can be transformed into another.
know all information. The multistage game will be analyzed by backward induction, assuming a nested structure. We solve the Nash equilibrium of the retailer pricing subgame by taking \( w, a \) and supply chain structures as given. For decentralized SCs the wholesale pricing subgame assumes that all subsequent events are solved and prior events given. With the same logic, we study the service level subgame and analyze the simultaneous SC structure selection game. We apply the Nash equilibrium solution concept in each subgame and a subgame perfect equilibrium solution concept to the whole multi-stage game.

**Double Marginalization**

Double marginalization has been referred to above and we include a simple description for reference. A single SC with service level \( a \), price \( p \), and utility \( u = u(a) \) has an expected demand \( \lambda = 1 - p/u \). An integrated monopolist will solve:

\[
\pi^I = \max_{p} (p - c)(1 - p/u) = (\lambda')^2 u \text{ with } p^I = (u + c)/2, \text{ and } \lambda^I = (u - c)/(2u).
\]

For the decentralized case the retailer’s problem is,

\[
\pi^R = \max_{p} (p - c - w)\lambda = \max_{p} (p - c - w)(1 - p/u) \text{ with } p^D = p^I + w/2, \lambda^D = \lambda^I - w/(2u).
\]

Then the manufacturer determines a wholesale price \( \pi^M = \max_{w} \lambda^M = \max_{w} w(\lambda^I - \frac{w}{2u}) \). The unique wholesale price is \( w = \lambda' u \), and the optimal profits for the retailer, manufacturer, and total SC are respectively: \( \pi^R = \pi^I/4, \pi^M = \pi^I/2, \pi^D = \pi^M + \pi^R = 3\pi^I/4 \).

Decentralization damages the total SC profit by a quarter. This is termed “double marginalization” and occurs when the retailer reacts to the ‘wrong’ input cost of \( w \) instead of \( c \).

**Service Differentiation**

If the monopolist offers two SCs with differentiated services, he solves the following problem instead,

\[
\pi^I = \max_{p_1, p_2} (\alpha p_2 - \beta p_1)(p_1 - c_1) + (1 - \alpha(p_2 - p_1))(p_2 - c_2) \tag{2.1}
\]

\[
\frac{\partial \pi}{\partial p_1} = -\beta(p_1 - c_1) + \alpha p_2 - \beta p_1 + \alpha(p_2 - c_2) = 0
\]

\[
\frac{\partial \pi}{\partial p_2} = \alpha(p_1 - c_1) + (1 - \alpha(p_2 - p_1)) - \alpha(p_2 - c_2) = 0
\]

\[
\frac{\partial^2 \pi}{\partial p_1^2} = -2\beta, \frac{\partial^2 \pi}{\partial p_2^2} = -2\alpha
\]
\[
\frac{\partial^2 \pi}{\partial p_1 \partial p_2} = 2\alpha, \quad \frac{\partial^2 \pi}{\partial p_2 \partial p_1} = 2\alpha
\]

The Hessian Matrix is negative definite. So we have a unique optimal solutions,
\[
p_I^* = \frac{c_1}{2} + \frac{1}{2(\beta - \alpha)} \cdot \frac{u_2 - c_1}{2}, \quad p_2^* = \frac{c_2}{2} + \frac{\beta}{2\alpha(\beta - \alpha)} \cdot \frac{u_2 - c_2}{2}
\]
\[
p_I^* - c_1 = \frac{1}{2(\beta - \alpha)} \cdot \frac{u_1 - c_1}{2}, \quad p_2^* - c_2 = \frac{\beta}{2\alpha(\beta - \alpha)} \cdot \frac{u_2 - c_2}{2}
\]
\[
\lambda_I^* = \frac{\alpha c_2 - \beta c_1}{2} = \frac{u_1 c_2 - u_2 c_1}{2(u_2 - u_1) u_1}, \quad \lambda_2^* = \frac{1 - \alpha (c_2 - c_1)}{2} = \frac{u_2 - u_1}{2(u_2 - u_1)}
\]

We need \( \alpha c_2 > \beta c_1 \), \( 1 > \alpha (c_2 - c_1) \) to guarantee a positive demand. Which is equivalent to \( 1 > \frac{c_2 - c_1}{u_2 - u_1} \Rightarrow u_2 - u_1 > c_2 - c_1, \frac{c_2}{u_2} > \frac{c_1}{u_1} \).

The intuition for these constraints are: 1. The monopolist will benefit from offering the higher service \( SC \) if the increase in consumer valuation \( u_2 - u_1 \) is greater than the increase in cost \( c_2 - c_1 \) so that he can charge more without it costing too much. 2. The monopolist will benefit from offering a lower \( SC \) if \( \frac{c_2}{u_2} > \frac{c_1}{u_1} \), which means it is better for the monopolist to invest in a new lower \( SC \) than only offering a higher \( SC \).

Substituting the optimal price into (2.1), \( \pi^* = \frac{\alpha c_2 - \beta c_1}{2} \cdot \left( \frac{1}{2(\beta - \alpha)} - \frac{c_1}{2} \right) + \frac{1 - \alpha (c_2 - c_1)}{2} \cdot \frac{\beta}{2\alpha(\beta - \alpha)} - \frac{c_1}{2} \) = \( \frac{u_1 c_2 - u_2 c_1}{2(u_2 - u_1) u_1} + \frac{u_2 - u_1 - (c_2 - c_1)}{2(u_2 - u_1)} \cdot \frac{u_2 - c_2}{2} \)

And the optimal service levels are determined by \( \pi^{**} = \max_{a_1, a_2} \pi^* \).

Recall that if the monopolist uses a single \( SC \), then \( \pi^{**} = \max_a \frac{(u(a) - c)^2}{4u(a)} \).

To make further comparison, we need to specify cost function and the utility function. Let \( u(a_i) = \sqrt{a_i}, c(a_i) = ca_i \), the second order conditions are satisfied. Also \( \frac{c_2}{c_1} > \frac{u_2}{u_1} \) for all values, but \( u_2 - u_1 > c_2 - c_1 \) requires that \( c(\sqrt{u_2} + \sqrt{u_1}) < 1 \). For this pair of functions the optimal solution for a single \( SC \) is, \( a_1^* = \left( \frac{1}{3} \right)^2, p_1^* = \frac{2}{9c}, \pi_1^* = \frac{1}{27c}, \lambda_1^* = \frac{1}{3} \)

The optimal solutions for dual \( SCs \) are, \( a_1^* = \left( \frac{1}{50} \right)^2, a_2^* = \left( \frac{2}{50} \right)^2, p_1^* = \frac{3}{25c}, p_2^* = \frac{7}{25c}, \lambda_1^* = \frac{1}{5}, \lambda_2^* = \frac{1}{5}, \pi_1^* = \frac{2}{125c}, \pi_2^* = \frac{3}{125c}, \pi_1^* + \pi_2^* = \frac{1}{25c} \)

We observe,

1. A monopolist makes more profit by offering dual differentiated \( SCs \) to different market segments, i.e., \( \pi_1^{**} + \pi_2^{**} = \frac{1}{25c} > \pi_1^{**} = \frac{1}{27c} \)

2. More consumers participate when dual differentiated \( SCs \) are offered. i.e., \( \lambda_1 + \lambda_2 = \frac{2}{5} > \lambda = \frac{1}{3} \)

3. More consumer welfare and total welfare. With double \( SCs \) the consumer welfare is \( \frac{1}{50c} \), and social welfare is \( \frac{3}{50c} \). With a single \( SC \), the consumer welfare is \( \frac{1}{54c} \), and social welfare is \( \frac{3}{54c} \).
welfare is $\frac{3}{54c}$.

In more detail, compared with a single SC, $\frac{1}{15}$ consumers participate by paying $\frac{3}{25c}$ instead of not participating at all, $\frac{2}{15}$ consumers pay $\frac{3}{25c}$ compared to paying $\frac{2}{9c}$, and $\frac{1}{5}$ consumers pay $\frac{7}{25c}$ instead of paying $\frac{2}{9c}$.

SC differentiation increases not only the firm rent but also consumer welfare. Table 2.1-2.2 gives the results (with $c = 1$, all values time $10^{-3}$ except market share, the same for all tables if not specified).

Table 2.1. Comparison between double SC and single SC (equilibrium)

<table>
<thead>
<tr>
<th>Monopoly single SC</th>
<th>Monopoly dual SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{a}$</td>
<td>$p$</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>89.6</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>74.6</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>64.1</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>56.1</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>50.1</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>45.1</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>41.1</td>
</tr>
</tbody>
</table>

Table 2.2. Comparison between double SC and single SC (profits and welfare)

| Single SC | | | Double SC | | | |
|-----------|-----------|-----------|-----------|-----------|-----------|
| $\pi$ | consumer welfare | social welfare | $\pi$ | consumer welfare | social welfare |
| $\sqrt{a}$ | 71.554 | 35.79 | 107.35 | 75.16 | 37.58 | 112.74 |
| $\sqrt{a}$ | 96.017 | 48.01 | 114.03 | 99.54 | 49.77 | 149.31 |
| $\sqrt{a}$ | 114.0 | 57.01 | 171.01 | 117.34 | 58.67 | 176.01 |
| $\sqrt{a}$ | 127.9 | 63.96 | 191.86 | 130.95 | 65.48 | 196.43 |
| $\sqrt{a}$ | 138.9 | 69.44 | 208.34 | 141.74 | 70.87 | 212.61 |
| $\sqrt{a}$ | 147.9 | 73.94 | 221.84 | 150.52 | 75.26 | 225.78 |
| $\sqrt{a}$ | 155.4 | 77.69 | 233.09 | 157.8 | 78.92 | 236.76 |
2.4 Price Competition with Fixed Service Levels

In this section we study the multi-stage game assuming given fixed service levels. This can be interpreted as a mature market with service levels already well established.

**Both Supply Chains Integrated (II)**

Two integrated SCs play a simultaneous pricing game,

\[
\pi_1^{II} = \max_{p_1} (p_1 - c_1)\lambda_1 = \max_{p_1} (p_1 - c_1)(\alpha p_2 - \beta p_1) \tag{2.2}
\]

\[
\pi_2^{II} = \max_{p_2} (p_2 - c_2)\lambda_2 = \max_{p_2} (p_2 - c_2)(1 - \alpha(p_2 - p_1)) \tag{2.3}
\]

The reaction functions are \(\lambda_1 - \beta(p_1 - c_1) = 0\), \(\lambda_2 - \alpha(p_2 - c_2) = 0\). The two SCs follow a strategic complementary pattern, with convex iso-profit curves.

Equations (2.2)-(2.3) are strictly concave in \(p_1, p_2\) respectively, and the reaction functions are linear. Compact, nonempty, convex strategy spaces, with continuous payoff functions and quasiconcave in strategies, implies there exists a pure strategy equilibrium (Fudenberg and Tirole 1991). As the reaction functions are linear, there exists a unique Nash equilibrium.

**Proposition 2.1** There exists a unique Nash Equilibrium strategy in the retail pricing subgame. Moreover, \(p_1^{II} = \frac{1 + \alpha c_2 + 2\beta c_1}{4\beta - \alpha} = \frac{(u_2 - u_1)w_1 + u_1c_2 + 2u_2c_1}{4u_2 - u_1}, p_2^{II} = \frac{\beta(1 + \alpha c_2 + 2\beta c_1)}{4\beta - \alpha} = \frac{u_2(2(u_2 - u_1) + 2c_2 + c_1)}{4u_2 - u_1}; \lambda_1^{II} = \frac{(1 + \alpha c_2 - (2\beta - \alpha)c_1)}{4\beta - \alpha} = \frac{u_2((u_2 - u_1)w_1 + u_1c_2 - (2u_2 - u_1)c_1)}{(4u_2 - u_1)(u_2 - u_1)}, \lambda_2^{II} = \frac{2\beta + \alpha c_1 - (2\beta - \alpha)c_2}{4\beta - \alpha} = \frac{2u_2(u_2 - u_1) + u_1c_2 - (2u_2 - u_1)c_2}{(4u_2 - u_1)(u_2 - u_1)}, \) and

\[
\pi_1^{II} = \frac{(\lambda_1^{II})^2}{\beta} = \frac{\beta(1 + \alpha c_2 - (2\beta - \alpha)c_1)^2}{(4\beta - \alpha)^2} = \frac{(u_1(u_2 - u_1) + c_2u_1 - (2u_2 - u_1)c_1)^2 u_2}{(4u_2 - u_1)^2(u_2 - u_1)u_1} \tag{2.4}
\]

\[
\pi_2^{II} = \frac{(\lambda_2^{II})^2}{\alpha} = \frac{(2\beta + \alpha c_1 - (2\beta - \alpha)c_2)^2}{\alpha(4\beta - \alpha)^2} = \frac{(2u_2(u_2 - u_1) + c_1u_2 - (2u_2 - u_1)c_2)^2}{(4u_2 - u_1)^2(u_2 - u_1)} \tag{2.5}
\]

Note we need \(\frac{(2\beta - \alpha)c_1}{\alpha} < c_2 < \frac{2\beta + \alpha c_1}{(2\beta - \alpha)\alpha}\), i.e., \(\frac{2c_1}{u_1} - 1 < \frac{c_2 - c_1}{u_2 - u_1} < 2 - \frac{c_2}{u_2}\) to guarantee a positive demand.

**Decentralized supply chains**
Rather than work through all the other three cases: ID, DI and DD, we summarize the results in Table 2.3 and give the detailed derivations in Appendix 1.

Table 2.3. Pricing equilibrium under each market structure

<table>
<thead>
<tr>
<th></th>
<th>II</th>
<th>DI</th>
<th>ID</th>
<th>DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>p_1^P</td>
<td>p_2^P</td>
<td>p_1^P + \frac{\lambda_1^P}{2\gamma_1}</td>
<td>p_1^P + \frac{\lambda_1^P}{2\gamma_1} + \frac{\beta \lambda_4^P}{\alpha_4}</td>
</tr>
<tr>
<td>\lambda</td>
<td>\lambda_1^P</td>
<td>\lambda_2^P</td>
<td>\lambda_1^P + \frac{\lambda_2^P}{2\gamma_1}</td>
<td>\lambda_1^P + \frac{\lambda_2^P}{2\gamma_1} + \frac{\beta \lambda_4^P}{\alpha_4}</td>
</tr>
<tr>
<td>w</td>
<td>-</td>
<td>-</td>
<td>\frac{\lambda_1^P - \lambda_2^P}{2\sigma_4}</td>
<td>\frac{\lambda_1^P - \lambda_2^P}{2\sigma_4} + \frac{\beta \lambda_4^P}{\alpha_4}</td>
</tr>
<tr>
<td>\pi</td>
<td>\pi_1^P</td>
<td>\pi_2^P</td>
<td>\pi_1^P \frac{\lambda_1^P}{2\gamma_1} + \frac{\lambda_2^P}{2\gamma_1}</td>
<td>\pi_1^P \frac{\lambda_1^P}{2\gamma_1} + \frac{\lambda_2^P}{2\gamma_1} + \frac{\beta \lambda_4^P}{\alpha_4}</td>
</tr>
<tr>
<td>\pi^M</td>
<td>-</td>
<td>-</td>
<td>\frac{\pi_1^P - \pi_2^P}{2\gamma_1}</td>
<td>\frac{\pi_1^P - \pi_2^P}{2\gamma_1} + \frac{\beta \lambda_4^P}{\alpha_4}</td>
</tr>
<tr>
<td>\pi^R</td>
<td>-</td>
<td>-</td>
<td>\frac{\pi_1^P}{2}</td>
<td>\frac{\pi_1^P}{2} + \frac{\beta \lambda_4^P}{\alpha_4}</td>
</tr>
</tbody>
</table>

To keep Table 2.3 from being too cluttered the following terms and constants have been defined. \( \gamma_1 = 2\beta - \alpha, \gamma_2 = 3\beta - \alpha, \gamma_3 = 4\beta - \alpha, \gamma_4 = 8\beta - 3\alpha, \rho_1 = 1 + \frac{\beta \lambda_1^P}{2\gamma_1}, \rho_2 = 1 + \frac{\alpha \lambda_1^P}{2\gamma_1}, \tau = \frac{\gamma_2}{4\gamma_1 - \alpha \beta}, \tau_1 = \frac{\gamma_2 \lambda_4^P}{2\gamma_1}, \tau_2 = \frac{\gamma_4 \lambda_4^P}{4\gamma_1}, \delta_1 = \frac{2\gamma_2 \lambda_4^P + \gamma_4 \lambda_4^P}{4\gamma_1 - \alpha \beta}, \delta_2 = \frac{2\gamma_2 \lambda_4^P + \gamma_4 \lambda_4^P}{4\gamma_1 - \alpha \beta}, \sigma = 1 - \frac{\alpha \beta}{4\gamma_1} \).

Many observations can be drawn from the pricing games under four SC structures. Unilaterally decentralizing a SC always increases your own and opponent’s price, this is because of double marginalization and strategic complementarity. It also decreases your own demand by half and increases your opponent’s demand by less than half \( \lambda_2^D - \lambda_2^H = \alpha \lambda_1^H / 2\gamma_1 < \lambda_1^H / 2, \lambda_1^D - \lambda_1^H = \beta \lambda_2^H / 2\gamma_1 < \lambda_2^H / 2 \). So the total industry demand decreases. Unilateral decentralization always damages the SC itself, and benefits the opponent \( \pi_1^D / \pi_2^D = \pi_1^H / \pi_2^H = \gamma_2 / 2\gamma_1 \). So the total industry demand decreases. No SC would unilaterally decentralize in order to differentiate itself from an integrated opponent even though they face different market segments. This is in agreement with McGuire and Staelin (1983), but different from Gupta and Loulou (1998). This is because Gupta and Loulou’s model gives decentralization an additional advantage of reducing R&D investment, because committing to a higher price and inducing the other SC to a higher retail price, has a positive effect on its own demand. So in their model, the profits of a unilaterally decentralized SC may not always fall.

From unilateral decentralization to bilateral decentralization, retailer prices always increase \( p_1^{DD} - p_1^D = \delta_1 - \lambda_2^H / 2\gamma_1 > 0 \). Demands also always decrease \( \lambda_1^{DD} / \lambda_1^D = \frac{2\gamma_2 \lambda_4^P + \gamma_4 \lambda_4^P}{4\gamma_1 - \alpha \beta} \).
τ/2 < 1). According to Gupta and Loulou (1998), vertical decentralization in a competitive market has two effects: (i) a negative effect, so that an increase in own price decreases own demand, and (ii) a positive effect where the strategic complementarity also increases the opponent’s price, increasing the decentralized SC’s demand. We can see from II to ID (or DI), or from ID (or DI) to DD, the negative effect always dominates the positive effect, so decentralization always causes a demand loss. The loss is 1/2 from II to DI (or ID) and 1 − τ/2 < 1/2 from DI (or ID) to DD. The wholesale price also always increases (σ < 1), because the wholesale price is also strategically complementary. The final result is that SC profits may go up (τ > 1, which is equivalent to \( u_1 > 0.6u_2 \)). So when the two service levels are not too different, both SCs would like to stay decentralized. Actually in the DD structure, a manufacturer’s profit can be even greater than the total SC’s in ID or DI as long as \( u_1 > 0.87u_2 \). So the manufacturers may also prefer bilateral decentralization. This result mirrors McGuire and Staelin (1983), Gupta and Loulou (1998) and others. The following corollary concludes the above.

**Corollary 2.1** Unilateral decentralization always damages that SC. When service levels are not too dissimilar (\( u_1 > 0.6u_2 \)), ID and DI structures decentralize to DD (e.g., integrated SC would like to decentralize too if the other is decentralized). The manufacturer will prefer DD if \( u_1 > 0.87u_2 \).

**Proof.** SC one also has an incentive to decentralize given that SC two is already decentralized iff \( \tau_1 > 1 \). This is a necessary and sufficient condition for \( \pi^{DD}_1 > \pi^{ID}_1 \), for \( i = 1, 2 \).

What service level can satisfy the above inequality? Note that \( \tau_1 > 1 \) is equivalent to \( 64u_2^4 - 192u_2^3u_1 + 177u_2^2u_1^2 - 64u_2u_1^3 + 8u_1^4 < 0 \). Dividing the inequality by \( u_2^4 \) and let \( \frac{u_1}{u_2} = x \), we have \( 64 - 192x + 177x^2 - 64x^3 + 8x^4 < 0 \).

Since \( \frac{u_1}{u_2} = x \in [0, 1] \), as can be solved approximately that the inequality holds when \( x > 0.6 \).

Similarly, both manufacturer one and two prefer to decentralize if \( \tau_2 > 1 \). Note that when this happens, each manufacturer’s profit \( \pi^{DD}_1, \pi^{DD}_2 \) is greater than the total SC profit \( \pi^{ID}_1, \pi^{ID}_2 \).
The inequality is equivalent to $-128u_2^4 + 320u_2^3u_1 - 273u_2^2 + 96u_2u_1^3 - 12u_1^4 > 0$, i.e., $-128 + 320x - 273x^2 + 96x^3 - 12x^4 > 0$. where $x = \frac{u_1}{u_2}$. It turns out that when $x = \frac{u_1}{u_2} > 0.87$ (approximately), the inequality holds. So manufacturers also prefer both SC decentralized if $u_1 > 0.87u_2$.

Implication for Total Industry Profits

What does unilateral or bilateral decentralization bring to the total industry profit?

From table appendix 1, under structure DI, we have

$$
\pi_1^{DI} = \pi_1^{II} \left( \frac{3\beta - \alpha}{2(2\beta - \alpha)} \right), \quad \pi_2^{DI} = \pi_2^{II} \left( 1 + \frac{\lambda_2^{II} \alpha}{2\lambda_2^{II} (2\beta - \alpha)} \right)^2
$$

the total industry profit is,

$$
\pi_1^{DI} + \pi_2^{DI} = \pi_1^{II} + \pi_2^{II} + \frac{\lambda_2^{II}}{2\beta - \alpha} \left( \lambda_2^{II} \left( \frac{\lambda_2^{II}(-7\beta + 4\beta^2 + 2\alpha^2)}{4(2\beta - \alpha)\beta} \right) \right)
$$

Whether the unilateral decentralization of SC one benefits the whole industry depends on the sign of the last term, i.e., whether $\lambda_2^{II} - \frac{\lambda_2^{II}(-7\beta + 4\beta^2 + 2\alpha^2)}{4(2\beta - \alpha)\beta} > 0$, or $(3\beta^2 - \frac{5\alpha}{4} - \frac{\alpha^2}{2} + c_1(\beta^2 + \frac{\alpha^2}{2} - 3\beta\alpha)(2\beta - \alpha) > \alpha c_2(5\beta^2 + 3\alpha^2 - \frac{23\beta\alpha}{4})$. As $\frac{3\beta^2 - \frac{5\alpha}{4} - \frac{\alpha^2}{2}}{5\beta^2 + 3\alpha^2 - \frac{23\beta\alpha}{4}} > 0$, there exist values of $c_2 \geq c_1 \geq 0$ that satisfy this condition. A similar analysis applies to structure ID.

Although decentralizing will damage a SC's own profitability, this may bring more profit to the whole industry. Hence some form of strategic coordination between the two SCs can make both better off. Another question follows naturally; what happens to the total industry profit under DD compared to ID or DI? Note that if $\tau_1 > 1$, then

$$
\pi_1^{DD} + \pi_2^{DD} = \pi_1^{ID} \tau_1 + \pi_2^{II} \tau_1 > \pi_1^{ID} + \pi_2^{II} > \pi_1^{ID} + \pi_2^{II} > \pi_1^{ID} + \pi_2^{II}.
$$

and similarly $\pi_1^{DD} + \pi_2^{DD} > \pi_1^{DI} + \pi_2^{II}$. $\pi_1^{DD} + \pi_2^{DD} > \pi_1^{I} + \pi_2^{II}$.

Corollary 2.2 There are circumstances such that the unilateral decentralization of either SC makes the total industry more profitable. Furthermore, if $u_1 > 0.6u_2$, then DD is preferred by the total industry.

The Supply Chain Structure Game under Only Price Competition

Table 2.4 gives the strategic form of the SC structure game with fixed service levels. The parameters are as defined in Table 2.3.

Table 2.4 Supply Chain structure game (price competition only)
We use the total SC profit as the criterion to analyze the structure game. Thus $\pi_1^H > \pi_1^{DI}, \pi_2^H > \pi_2^{ID}$, so $II$ is always a Nash equilibrium. Recall that $u_1 > 0.6u_2$ implies $\tau_1 > 1$. $u_1 > 0.87u_2$ implies $\pi_1^{MDD} > \pi_1^{ID}$ ($\pi_2^{MDD} > \pi_2^{DI}$). If $\min\{\rho_1^2, \rho_2^2\} > 1/\tau_1$ (which is automatically true if $\tau_1 > 1$, can be true even if $\tau_1 > 1$ is violated), then $\pi_i^{DD} > \pi_i^H$.

We have:

**Corollary 2.3** $II$ is always a subgame perfect Nash equilibrium regardless of the degree of product differentiation. $DD$ is also an equilibrium if the degree of product differentiation is small. When $DD$ is an equilibrium, it always Pareto-dominates $II$ (both SCs are better off). It can Pareto-dominate $II$ even if it is not an equilibrium. Under certain service levels, even a single manufacturer's profit in the $DD$ setting can be greater than the total supply chain profit under the $DI$ or $ID$ setting.

Recall that A Pareto-dominates B if everyone prefers A to B. The result above echoes that of MS and other related literature. However our result is based on a heterogeneous market and asymmetric equilibrium.

If we consider only a one shot game, we might be tempted to predict more $II$ in the market. Thus players may feel safer integrating even when $DD$ can also be an equilibrium, because if the other player integrates, the player who remains decentralized will be worse off. $DD$ is however more likely to be an equilibrium in a repeated game (where this static multi-stage game is one stage game of the repeated game), because of the future value. Since $DD$ can bring more profits to both supply chains, there are incentives for tacit collusion such that both players choose decentralization.

**Sensitivity Analysis Concerning Heterogeneity and Wholesale Prices**

We would like to better understand the relationship between heterogeneity and supply chain structure. How does the degree of market heterogeneity affect our results?
We replace our assumption on consumer preferences to let $\theta$ be uniformly allocated between $[0, b]$, instead of $[0, 1]$. We can see that the previous work holds essentially the same. That is, II is a Nash SC structure, and DD can be a Nash equilibrium SC structure when $u(a_1) > 0.6u(a_2)$. We have the following results (for details see appendix 7),

**Proposition 2.2** 1. Both retail prices and wholesale prices increase with $b$. 2. SC two's demand and profit under each structure increase with $b$. 3. For SC one, under each structure, if the demand increases with $b$, then so does the profit. Otherwise the profit increases with $b$ if $b$ is sufficiently large. 4. SC two’s incentive to remain in equilibrium II or DD increases with $b$. SC one’s incentive to stay II or DD increases with $b$ if the demand does. Otherwise it increases with $b$ if $b$ is sufficiently large.

So a market with greater heterogeneity is always good news to a high service SC manager, they can set a higher price, serve more consumers, and earn more profits, but the low SC managers should be careful.

Coughlan (1985) found the use of mixed structures with the smallest firm being decentralized in a study of the international semiconductor industry. Her explanation involves cost effects, such as economies of scale among middleman. We suggest that there may also be the factor of market heterogeneity, based on our findings, but this needs further empirical study to understand.

We have show above that the Nash equilibrium SC structure may be either II or DD. In the DD case we are assuming that the wholesale price is set optimally by manufacturers. But are there ways of setting the wholesale price such that decentralization is always a dominant strategy? It turns out that if the wholesale price is set to maximize the total SC profit or total industry profits, decentralization is always the optimal strategy. Of course a side payment would be needed to ensure the manufacturers participation but this is not a coordinating two part tariff that mirrors integration behavior (i.e., the wholesale price equals to the marginal cost). It is not our intention to further explore contract design here, but we might observe that a contract design that focuses on coordinating the manufacturer and the retailer into behaving as though integrated may not be valuable. Optimal contract designs need to consider the competitive environment and not be focused so much on monopolies.
Proposition 2.3 There are ranges for wholesale prices such that decentralization is preferred to integration. Specifically, when \( w_1^* = \frac{\alpha_7 \lambda_{II}^I}{4 \beta^2 \gamma_1} \) and \( w_2^* = \frac{\gamma_3 \lambda_{II}^I}{4 \beta \gamma_1} \), the profits of decentralized SCs in ID and DI are maximized. Such optimal wholesale price can be sustained by a side payment from the retailer to manufacturer.

Proof. We take SC one as the example, SC two's case is similar. From (A2.1) and (A2.3)

\[
\frac{\partial \pi^{DI}_1}{\partial w_1} = \frac{\alpha \lambda_{II}^I}{4 \beta - \alpha} - \frac{4 \beta (2 \beta - \alpha) w_1}{(4 \beta - \alpha)^2} \implies w_1^* = \frac{\alpha (4 \beta - \alpha) \lambda_{II}^I}{4 \beta^2 (2 \beta - \alpha)} = \frac{\alpha_7 \lambda_{II}^I}{4 \beta^2 \gamma_1}
\]

And \( \lambda_{II}^I \big|_{w_1^*} = (1 - \frac{\alpha}{4 \beta}) \lambda_{II}^I \)

\[
\frac{\partial^2 \pi^{DI}_1}{\partial w_1^2} = -\frac{4 \beta^2 (2 \beta - \alpha)}{(4 \beta - \alpha)^2} < 0 \implies \pi^{DI}_1 \text{ is concave in } w_1.
\]

Let \( w_1 = \frac{\alpha (4 \beta - \alpha) \lambda_{II}^I}{2 \beta^2 (2 \beta - \alpha)} = 2w_1^* \) be the wholesale price such that \( \pi^{II}_1 = \pi^{DI}_1 \). For all \( 0 < w_1 < 2w_1^* \), SC one would prefer to decentralize. But as \( w_1^{DI} = \frac{\lambda_{II}^I (2 \beta - \alpha)}{2 \beta^2 (2 \beta - \alpha)} > 2w_1^* \), a manufacturer in SC one using an optimal wholesale price for herself will reduce SC one’s profit when SC two is integrated.

Proposition 2.4 Decentralization (DD) will always dominate integration (II) by using wholesale prices \( w_1^{**} = \frac{\alpha (4 \beta^2 (2 \beta - \alpha) + \alpha \lambda_{II}^I)}{(16 \beta^2 - 12 \alpha \beta + \alpha^2) \beta} \), and \( w_2^{**} = \frac{4 \beta \lambda_{II}^I (2 \beta - \alpha) + \alpha^2 \lambda_{II}^I}{(16 \beta^2 - 12 \alpha \beta + \alpha^2) \beta} \).

Proof. \( \pi^{DD}_1 = (\frac{\lambda_{II}^I}{\beta} + \frac{2 \beta w_1}{4 \beta - \alpha} + \frac{\alpha w_2}{4 \beta - \alpha}) (\lambda_{II}^I + \frac{\beta w_1}{4 \beta - \alpha} - \frac{(2 \beta - \alpha) \beta w_1}{4 \beta - \alpha}) \)

\[
\frac{\partial \pi^{DD}_1}{\partial w_1} = 0 \implies w_1 = \frac{(4 \beta - \alpha) \alpha \lambda_{II}^I + \alpha^2 \beta w_2}{4 \beta^2 (2 \beta - \alpha)}
\]

\[
\frac{\partial^2 \pi^{DD}_1}{\partial w_1^2} = -\frac{4 \beta^2 (2 \beta - \alpha)}{(4 \beta - \alpha)^2} < 0 \implies \pi^{DD}_1 \text{ is concave in } w_1.
\]

\[
\pi^{DD}_1 = (\frac{\lambda_{II}^I}{\alpha} + \frac{2 \beta w_2}{4 \beta - \alpha} + \frac{\beta w_1}{4 \beta - \alpha}) (\lambda_{II}^I + \frac{\beta w_1}{4 \beta - \alpha} - \frac{(2 \beta - \alpha) \beta w_1}{4 \beta - \alpha})
\]

\[
\frac{\partial \pi^{DD}_1}{\partial w_2} = 0 \implies w_2 = \frac{(4 \beta - \alpha) \alpha \lambda_{II}^I + \alpha^2 \beta w_1}{4 \beta (2 \beta - \alpha)}
\]

\[
\frac{\partial^2 \pi^{DD}_1}{\partial w_2^2} = -\frac{4 \beta^2 (2 \beta - \alpha) \alpha}{(4 \beta - \alpha)^2} < 0 \implies \pi^{DD}_2 \text{ is concave in } w_2.
\]

By solving the two equations involving \( w_1, w_2 \), we have

\[
w_1^{**} = \frac{\alpha (4 \beta^2 (2 \beta - \alpha) + \alpha \lambda_{II}^I)}{(16 \beta^2 - 12 \alpha \beta + \alpha^2) \beta}, \quad w_2^{**} = \frac{4 \beta \lambda_{II}^I (2 \beta - \alpha) + \alpha^2 \lambda_{II}^I}{(16 \beta^2 - 12 \alpha \beta + \alpha^2) \beta}
\]

By simple but tedious algebra, we can show that using the above optimal wholesale prices, \( \pi^{DD}_1 > \pi^{II}_1, \pi^{DD}_2 > \pi^{II}_2 \) always.

So if the wholesale price is set properly, decentralization will not damage the SC, and is an opportunity for SCs to capture more profits. This result echoes Coughlan and Wernerfelt (1989) and Bonanno and Vickers (1988).
We have shown that, with manufacturers' optimal wholesale prices, there exists a certain range of cost and utility functions such that the total industry is better off under unilateral decentralization. Next, we would like to see what the results would be if the wholesale price is set to maximize the total industry profits. Take DI for an example.

\[
\pi_1^{DI} + \pi_2^{DI} = \pi_1^I + \pi_2^I + \frac{(\alpha \lambda_1^I + 2\beta \lambda_2^I)w_1}{4\beta - \alpha} - \frac{\beta^2(4\beta - 3\alpha)w_1^2}{(4\beta - \alpha)^2}
\]

By a similar analysis, we also have the following observations:

1. The industry with mixed structure under wholesale price \( w_1 \) is better than the industry with two integrated SCs. To see this,

\[
\left(\pi_1^{DI} + \pi_2^{DI}\right)|_{w_1} = \pi_1^I + \pi_2^I + \frac{(\alpha \lambda_1^I + 2\beta \lambda_2^I)^2}{4\beta^2(4\beta - 3\alpha)} > \pi_1^I + \pi_2^I
\]

So with wholesale price \( w_1 \), the total industry can gain without any other requirements on cost function or service levels. But we still need a side payment to the manufacturer because \( w_1 \) is not his optimum.

2. Let \( \bar{w}_1 = \frac{(\alpha \lambda_1^I + 2\beta \lambda_2^I)(4\beta - \alpha)}{2\beta(4\beta - 3\alpha)} \) be the wholesale price such that \( \pi_1^I + \pi_2^I = \pi_1^{DI} + \pi_2^{DI} \).

Note whenever \( w_1 < \bar{w}_1 \), decentralization can be even more profitable to the total industry. If \( w_1^{DI} = \frac{\lambda_1^I(4\beta - 3\alpha)}{2\alpha(2\beta - \alpha)} < \bar{w}_1 \), then manufacturer 1's optimal wholesale price can make the total industry more profitable, which gives us the same requirement for \( c_1 \) and \( c_2 \) as we discussed above.

The same analysis applies to decentralization of only SC two. \( \pi_1^{ID} + \pi_2^{ID} = \pi_1^I + \pi_2^I + \frac{2\alpha \lambda_1^I + \alpha \lambda_2^I}{4\beta - \alpha}w_2 - \frac{\alpha \beta(4\beta - 3\alpha)w_2^2}{(4\beta - \alpha)^2} \)

By a similar analysis, we also have two critical numbers of \( w_2 \),

1. If \( w_2 = \frac{(2\alpha \lambda_1^I + \alpha \lambda_2^I)(4\beta - \alpha)}{2\alpha(4\beta - 3\alpha)} \), then \( \pi_1^{ID} + \pi_2^{ID} \) is maximized. With this wholesale price, the total industry is better off.

2. If \( \bar{w}_2 = \frac{(2\alpha \lambda_1^I + \alpha \lambda_2^I)(4\beta - \alpha)}{\alpha(4\beta - 3\alpha)} \), then \( \pi_1^I + \pi_2^I = \pi_1^{ID} + \pi_2^{ID} \). Whenever \( w_2 < \bar{w}_2 \), decentralization can be even more profitable to the total industry. If \( w_2^{ID} = \frac{\lambda_2^I(4\beta - 3\alpha)}{2\alpha(2\beta - \alpha)} < \bar{w}_1 \), manufacturer 1's optimal wholesale price can make the total industry more profitable.

**Corollary 2.4** Under wholesale prices \( \bar{w}_1 = \frac{(\alpha \lambda_1^I + 2\beta \lambda_2^I)(4\beta - \alpha)}{2\beta^2(4\beta - 3\alpha)} \) and \( \bar{w}_2 = \frac{(2\alpha \lambda_1^I + \alpha \lambda_2^I)(4\beta - \alpha)}{2\alpha(4\beta - 3\alpha)} \)
total industry profits under market structure DI and ID are maximized. Moreover,
\[ \pi^{DI}(\bar{w}_1) > \pi^{II}, \pi^{ID}(\bar{w}_1) > \pi^{II}. \]

So far we have only considered fixed service levels, analyzing only the pricing subgame. Now we study the impact of having the choice of service level as a decision variable.

2.5 Supply Chains with Price and Service Competition

Now, given the pricing equilibrium results, we consider the setting of service levels. The two SCs simultaneously make their choice of service level \(a_i\). With price setting there was no ambiguity about who set the prices, but the situation is different with service levels. We defer this discussion until after the case of two integrated SCs.

For service level game, we use a particular class of utility and cost functions in most instances, which enables us to see how the consumer preference affects the decisions. The particular choice of utility/cost functions we take is \(u(a) = \sqrt{a}\) and \(c(a) = ca\) or \(u(a) = a\) and \(c(a) = ca^n\), since they provide the same insight. We use whichever is simplest for exposition. There is no meaning for \(n = 1\) or \(n \to +\infty\) because two differentiated supply chains cannot exit. We study the cases for \(2 < n < 9\) and higher \(n\) represents a more concave utility function. These instances have provided all the varieties of results on equilibrium SC structure we expect. We conjecture that the results for \(n \geq 10\) would provide similar results as \(n = 9\).

Two Integrated Supply Chains with Service Level Decisions

In the II setting, two integrated SCs maximize:

\[
\max_{a_1} \pi^{II}_1 \quad \text{and} \quad \max_{a_2} \pi^{II}_2 \quad \text{where} \quad 0 \leq a_1 \leq a_2 - \varepsilon
\]

where \(\varepsilon\) is a small number.

Using \(u(a_i) = a_i\), \(c(a_i) = ca^2\) in equations (2.4)-(2.5), we have:

\[
\pi^{II}_1 = \frac{(ca_2-ca_1+1)^2a_1a_2(a_2-a_1)}{(4a_2-a_1)^2}, \quad \pi^{II}_2 = \frac{(2-2ca_2-ca_1)^2a_2^2(a_2-a_1)}{(4a_2-a_1)^2}
\]

The two functions are quasiconcave (see appendix 2), so we can find a unique pure strategy Nash equilibrium by solving the game.

\[^2\]A function is more concave than the other if it is a concave transformation of the other. The economic explanation is that a more concave utility function represents a stronger diminishing return or stronger diminishing marginal utility on service.

\[^3\]We assume \(a\) is chosen from an interval of the real line.
The two reaction functions are,

\[-8a_x^2 + 6a_1a_2 - 4a_1^2 - 22ca_1a_2^2 + 5ca_1^2a_2 + 2ca_1^3 + 24ca_2^3 = 0\]

\[4ca_2^3 + 4a_2^2 - 19ca_1a_2^2 + 17ca_2^2a_2 - 2ca_1^3 - 7a_1a_2 = 0\]

giving solutions

\[a_1 = 0.1994/c, a_2 = 0.4098/c, p_1 = 0.075/c, p_1 = 0.2267/c,\]

\[\lambda_1 = 0.3445, \lambda_2 = 0.2792, \pi_1 = 0.01215/c, \pi_2 = 0.01641/c\]

Figure 2.1 - 2.3 show iso-profit and reaction curves of the two SCs. SC one's profit increases with \(a_2\), SC two's profit decreases with \(a_1\).

Using the family of utility functions we get Table 2.5. We can see that the more utility the service level offers (the bigger the \(n\)), the less the equilibrium service level, the lower the equilibrium price and the higher total profit.

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</tbody>
</table>

**Decentralized supply chains under Service Level Decisions**

For the DI, ID and DD setting we need to specify which agent has the choice of service level \(a\). We refer to this as where 'market power' is lodged. Thus if the manufacturer has power, he is able to optimize \(\pi_M\) over \(a\) and faces a competitive retailer market. 'Retailer power' is similarly defined. If we assume that either by cooperation or intervention by a over-arching authority, the choice of service level \(a\) is made to maximize SC profits, we shall term this 'equal market power'.

With 'manufacturer power' a product is differentiated at source either by design features or offering warranties etc. 'Retailer power' might reflect an otherwise generic
product being differentiated at the 'street-front' level, for example by product availability or service contract.

There is also the issue of who pays the cost of service $c(a)$, the agent with market power or not? This turns out not to affect the analysis other than to adjust the wholesale price up or down by $c(a)$. For simplicity we have made the retailer to always incur the service cost $c(a)$.

The pricing equilibrium results showed the profit functions of retailer and manufacturer to be fractions of the total SC profits. So no matter which profit function we are maximizing, the others will be positive. Following most economic literature (Varian 1992), we assume that the opportunity profit of a firm in a competitive market (outside the market we consider) is zero. So we do not need to consider the participation constraint of the party without market power in our formulation.

We first study the service subgame under equal market power. In the DI setting, both SCs set $a$ to maximize SC profits. From appendix 1, the simultaneous service subgame is,

$$
\pi^{DI^*}_1 = \max_{a_1} \pi^{DI}_1 = \max_{a_1} \pi^{II}_1 \frac{u_2 - u_1}{2(u_2 - u_1)} \quad \pi^{DI^*}_2 = \max_{a_2} \pi^{DI}_2 = \max_{a_2} \pi^{II}_2 \left(1 + \frac{u_1 \lambda^{II}_2}{2(u_2 - u_1) \lambda^{II}_2} \right)^2
$$

The ID setting is,

$$
\pi^{ID^*}_1 = \max_{a_1} \pi^{ID}_1 = \max_{a_1} \pi^{II}_1 \left(1 + \frac{u_2 \lambda^{II}_1}{2(u_2 - u_1) \lambda^{II}_1} \right)^2 \quad \pi^{ID^*}_2 = \max_{a_2} \pi^{ID}_2 = \max_{a_2} \pi^{II}_2 \frac{u_2 - u_1}{2(u_2 - u_1)}
$$

In the DD setting the problem is,

$$
\max_{a_1} \pi^{DD}_1 = \max_{a_1} \pi^{II}_1 \frac{u_2}{4(u_2 - u_1)^2 - u_1 u_2} \quad \max_{a_2} \pi^{DD}_2 = \max_{a_2} \pi^{II}_2 \frac{u_2}{4(u_2 - u_1)^2 - u_1 u_2}
$$

All these objective functions are continuous and differentiable on the nonempty, convex and compact set, and we find there exist unique pure strategy Nash equilibria by solving each game given each pair of the functions. We summarize the equilibrium results for each setting in Table 2.6.

Table 2.6. Service equilibrium results under different structure ($u(a_i) = \sqrt{a_i}$, $c(a_i) = ca_i$).

<table>
<thead>
<tr>
<th>Setting</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>39.76/c^2</td>
<td>167.94/c^2</td>
<td>75/c</td>
<td>226.7/c</td>
<td>0.3445</td>
<td>0.2792</td>
<td>12.15/c</td>
<td>16.41/c</td>
<td>28.56/c</td>
</tr>
<tr>
<td>DI</td>
<td>46.26/c^2</td>
<td>171.48/c^2</td>
<td>105.9/c</td>
<td>238.2/c</td>
<td>0.1722</td>
<td>0.3353</td>
<td>10.28/c</td>
<td>22.37/c</td>
<td>32.65/c</td>
</tr>
<tr>
<td>ID</td>
<td>44.10/c^2</td>
<td>162.97/c^2</td>
<td>88.28/c</td>
<td>254.6/c</td>
<td>0.4384</td>
<td>0.1412</td>
<td>19.37/c</td>
<td>12.94/c</td>
<td>32.28/c</td>
</tr>
<tr>
<td>DD</td>
<td>61.01/c^2</td>
<td>155.87/c^2</td>
<td>138.3/c</td>
<td>256.9/c</td>
<td>0.2421</td>
<td>0.1978</td>
<td>18.72/c</td>
<td>19.99/c</td>
<td>38.71/c</td>
</tr>
</tbody>
</table>
For the other pairs of utility and cost functions, see appendix 3.

In the same industry (with the same utility and cost function), decentralizing SC one increases both SCs’ service levels and prices, benefiting SC two and the total industry. Interestingly, the damage to SC one itself becomes smaller when \( n \) increases, and when \( n \geq 6 \), decentralization benefits SC one. In industries where service levels are more valued, the lower SC is more likely to decentralize. Decentralizing SC two decreases its service level and market share, increases its price, SC one’s and total industry profit. In the DD setting, both SCs’ prices reach the highest level, service levels become closer to each other, and firms and total industry profits are maximized.

The Effect of Market Power

Next we explore the question of how market power affects service level equilibrium. We use a general utility/cost function but assume that the equilibrium exists and the objective functions are well behaved. Of course market power is not an issue for an integrated SC. Let \( r_i(a_j) \) be the optimal service level of SC \( i \) given a certain level of service of SC \( j \). The superscript M, R represents manufacturer power and retailer power respectively, and equal power otherwise. We have the following proposition.

**Proposition 2.5** Supply chain one behaves more aggressively, and supply chain two less aggressively under manufacturer market power. Supply chain one behaves less aggressively, and supply chain two more aggressively under retailer market power. That is, \( r_1^M(a_2) > r_1(a_2) > r_1^R(a_2) \), \( r_2^M(a_1) < r_2(a_1) < r_2^R(a_1) \).

**Proof.** We take the DD setting as an example, the proofs for DI and ID are similar.

From appendix 1, with manufacturer market power,

\[
\pi_1^{MD} = \pi_1^{DD} \frac{4u_2-u_1}{2(u_2-u_1)} = \pi_1^{DD} \frac{4u_2-u_1}{2(u_2-u_1)}
\]

\[
\pi_2^{MD} = \pi_2^{DD} \frac{4u_2-u_1}{2(u_2-u_1)} = \pi_2^{DD} \frac{4u_2-u_1}{2(u_2-u_1)}
\]

And

\[
\frac{\partial \pi_1^{MD}}{\partial a_1} \mid_{r_1^{DD}(a_2), a_2} = \left( 4u_2-u_1 \right) \frac{\partial \pi_1^{DD}}{\partial a_1} + \pi_1^{DD} \frac{u_2}{2(u_2-u_1)^2} \right) \mid_{r_1^{DD}(a_2), a_2} > 0
\]

\[
\frac{\partial \pi_2^{MD}}{\partial a_2} \mid_{a_1, r_2^{DD}(a_1)} = \left( 4u_2-u_1 \right) \frac{\partial \pi_2^{DD}}{\partial a_2} + \pi_2^{DD} \frac{u_1}{2(u_2-u_1)^2} \right) \mid_{a_1, r_2^{DD}(a_1)} < 0
\]

So given a certain service level \( a_2 \), \( r_1^{DD}(a_2) < r_1^{MD}(a_2) \); and given a certain level \( a_1 \), \( r_2^{DD}(a_1) > r_2^{MD}(a_1) \).
With retailer market power,

\[ \pi_1^{RD} = \pi_1^{ID} \frac{4(2u_2-u_1)^4}{(2u_2-u_1)^2-u_1u_2} = \pi_1^{DD} \frac{2u_2-u_1}{2(3u_2-u_1)} \]

\[ \pi_2^{RD} = \pi_2^{ID} \frac{4(2u_2-u_1)^4}{(2u_2-u_1)^2-u_1u_2} = \pi_2^{DD} \frac{2u_2-u_1}{2(3u_2-u_1)} \]

and

\[ \frac{\partial \pi_1^{RD}}{\partial a_1}(a_2) = \left( \frac{2u_2-u_1}{2(3u_2-u_1)} \frac{\partial \pi_1^{DD}}{\partial a_1} + \pi_1^{DD} \frac{-u_2u_1'}{2(3u_2-u_1)} \right) \pi_1^{DD}(a_2), a_2 < 0 \]

\[ \frac{\partial \pi_1^{RD}}{\partial a_2}(a_2) = \left( \frac{2u_2-u_1}{2(3u_2-u_1)} \frac{\partial \pi_1^{DD}}{\partial a_1} + \pi_1^{DD} \frac{-u_2u_1'}{2(3u_2-u_1)} \right) \pi_1^{DD}(a_1) > 0 \]

So given a certain \( a_2 \), \( r_1^{RD}(a_2) < r_1^{DD}(a_2) \), and given a certain \( a_1 \), \( r_2^{RD}(a_1) > r_2^{DD}(a_1) \). ■

In mixed structures (ID or DI), if service levels follow a strategic complementary pattern, then the overall service level with manufacturer power will be the highest in DI, and the lowest in ID. The result is opposite with retailer market power.

From the following Table 2.7, we can see that the equilibrium results are consistent with the above analysis.

Table 2.7. DI setting with different market power \((u(a_i) = \sqrt{a_i}, c(a_i) = ca_i)\).

<table>
<thead>
<tr>
<th>power</th>
<th>( a_1 )</th>
<th>( \pi_1^{MDI} )</th>
<th>( \pi_1^{RDI} )</th>
<th>( \pi_1^{DI} )</th>
<th>( a_2 )</th>
<th>( \pi_2^{MDI} )</th>
<th>( \pi_2^{RDI} )</th>
<th>( \pi_2^{DI} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>equal</td>
<td>46.27/c^2</td>
<td>7.209/c</td>
<td>3.067/c</td>
<td>10.28/c</td>
<td>171.48/c^2</td>
<td>–</td>
<td>–</td>
<td>22.37/c</td>
</tr>
<tr>
<td>manufacturer</td>
<td>50.09/c^2</td>
<td>7.343/c</td>
<td>3.107/c</td>
<td>10.45/c</td>
<td>176.06/c^2</td>
<td>–</td>
<td>–</td>
<td>21.71/c</td>
</tr>
<tr>
<td>retailer</td>
<td>38.61/c^2</td>
<td>6.896/c</td>
<td>2.969/c</td>
<td>9.865/c</td>
<td>162.33/c^2</td>
<td>–</td>
<td>–</td>
<td>23.78/c</td>
</tr>
</tbody>
</table>

(see appendix 4 for ID and DD)

The unilaterally decentralized SC prefers manufacturer power, and its opponent prefers retailer power. Both SCs and SC members prefer retailer power in DD.

When a SC decentralize unilaterally, the SC managers may prefer to choose manufacturer power, because it can reduce the efficiency loss that decentralization brings. They may avoid subcontracting with a larger retailer, as this makes the opponent even more profitable. But when both SCs decentralize, retailer power is preferred. The reason is that under retailer power, the two service levels are more differentiated by proposition 2.5, so the horizontal competition is even weaker. This would predict that a manufacturer entering a new competitive market would look for a larger retailer to resell their products. The intuition behind this is that by choosing retailers who have market power, their function as a buffer against competition (enabling less demand loss) is more signif-
icant. So in a market with a DD structure, strategic agreement on retailer power will 
make both SCs and all SC members better off.

The Effect of Supply Chain Structure

How does the SC structure affect the service level decision? For example, a new 
entrant into an integrated market may want to also integrate or decentralize. What 
impact does this choice have on the service decision? Are the results different for the 
lower service or upper service SC?

Our results are analytical and based on a general function, and we also assume the 
existence of service equilibrium and well-behaved objective functions.

**Proposition 2.6** 1. Under equal or manufacturer power, SC one will set a higher service level if it decides to decentralize no matter whether the other firm is integrated or decentralized. That is, \( r_1^{M\,}\,(a_2) < r_1^{DI\,}\,(a_2) \), \( r_1^{M\,}\,(a_2) < r_1^{MD\,}\,(a_2) \), \( r_1^{I\,}\,(a_2) < r_1^{DI\,}\,(a_2) \), \( r_1^{ID\,}\,(a_2) < r_1^{DD\,}\,(a_2) \). SC two will set a lower service level if it decides to decentralize no matter the structure of the other firm. That is, \( r_2^{MD\,}\,(a_1) < r_2^{M\,}\,(a_1) \), \( r_2^{MD\,}\,(a_1) < r_2^{I\,}\,(a_1) \), \( r_2^{DD\,}\,(a_1) < r_2^{DI\,}\,(a_1) \). 2. Under retailer power, if the other firm is integrated, the SC structure has no effect on the service level. If the other firm is decentralized, then SC one sets a higher service but SC two sets a lower service if it decides to also decentralize. That is, \( r_1^{R\,}\,(a_2) = r_1^{R\,}\,(a_2) \), \( r_1^{R\,}\,(a_2) < r_1^{R\,}\,(a_2) \), \( r_2^{R\,}\,(a_1) = r_2^{R\,}\,(a_1) \), \( r_2^{R\,}\,(a_1) < r_2^{R\,}\,(a_1) \).

**Proof.** We only take the case of equal market power, and the DI and II setting as an example. The other proofs are similar. In the DI setting, from appendix 1,

\[
\begin{align*}
\pi_{1}^{DI} &= \max_{\pi_{1}^{DI}} \pi_{1}^{DI} = \max_{a_1} \pi_{1}^{I} \frac{3\beta - \alpha}{4\beta - 2\alpha} = \max_{a_1} \pi_{1}^{II} \frac{3u_{2} - u_{1}}{4u_{2} - 2u_{1}}, \\
\pi_{2}^{DI} &= \max_{\pi_{2}^{DI}} \pi_{2}^{DI} = \max_{a_2} \pi_{2}^{I} (1 + \frac{\lambda_{1\,}^{I\,}u_{1}}{2\lambda_{2\,}^{I\,}(2\beta - \alpha)})^2 = \max_{a_2} \pi_{2}^{II} (1 + \frac{\lambda_{1\,}^{II\,}u_{1}}{2\lambda_{2\,}^{II\,}(2u_{2} - u_{1})})^2.
\end{align*}
\]

The reaction functions come from the following first order conditions,

\[
\begin{align*}
\frac{\partial \pi_{1}^{II}}{\partial a_1} &= \frac{\partial \pi_{1}^{I\,} \frac{3u_{2} - u_{1}}{4u_{2} - 2u_{1}} + \pi_{1}^{II} \frac{2u_{2}u_{1}^{I\,}}{(4u_{2} - 2u_{1})^2}}{0}, \\
\frac{\partial \pi_{2}^{II}}{\partial a_2} &= \frac{\partial \pi_{2}^{II} (1 + \frac{\lambda_{1\,}^{II\,}u_{1}}{2\lambda_{2\,}^{II\,}(2u_{2} - u_{1})})^2}{0} + \frac{\partial \frac{\lambda_{1\,}^{II\,}u_{1}}{2\lambda_{2\,}^{II\,}(2u_{2} - u_{1})}}{\partial a_2} \pi_{2}^{II} = 0.
\end{align*}
\]

Let \( r_1^{II\,}\,(a_2) \) be the optimal service level of SC one in the II setting given a certain level of \( a_2 \). \( r_1^{DI\,}\,(a_2) \) is similarly defined in the DI setting,

\[
\frac{\partial r_1^{DI\,}}{\partial a_2} |_{r_1^{II\,}\,(a_2),a_2} = \pi_{1}^{II} \frac{2u_{2}u_{1}^{I\,}}{(4u_{2} - 2u_{1})^2} |_{r_1^{II\,}\,(a_2),a_2} > 0.
\]
So $r_1^H(a_2) < r_1^D(a_2)$.

Note that when SC two decentralizes, it will increase price and decrease the service level. So the consumers of SC two may be worse off at that time. Etgar (1978)'s empirical result partly agrees with our result. He uses data from the property and casualty insurance industry to test his hypothesis, and finds that the forward vertical integration of competitive suppliers can help suppliers provide higher levels of service and differentiate their products from their competitors. This is true for higher service SCs (decentralization decreases service level), but not necessarily true for lower service SCs (decentralization usually increases service level). The reasons for the partial discrepancy may be that they did not consider different market segments in their empirical study; the joint effect may bias results toward higher level SCs. Or it may be that with more than 2500 insurance companies and 250,000 insurance distributors, our stylized duopoly market is inappropriate.

**Horizontal Competition Versus Vertical Competition**

We have shown that in the pricing game, horizontal competition damages both SCs, but adding vertical competition counters this profit loss. What happens with both price and service competition? We give an example of (M) monopoly double SCs, (II) duopoly integrated SCs and (DD) duopoly decentralized SCs. We use the pair of utility and cost function $u(a_i) = \sqrt{a_i}$, $c(a_i) = ca_i$. The results are in Table 2.8. (c.w. means consumer welfare, t.w. means total welfare).

<table>
<thead>
<tr>
<th></th>
<th>$\pi$</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>c.w.</th>
<th>t.w.</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>40 c</td>
<td>16 c</td>
<td>24 c</td>
<td>40 c</td>
<td>160 c</td>
<td>120 c</td>
<td>280 c</td>
<td>0.2</td>
<td>0.2</td>
<td>70 c</td>
<td>70 c</td>
</tr>
<tr>
<td>II</td>
<td>28.55 c</td>
<td>12.15 c</td>
<td>16.4 c</td>
<td>39.76 c</td>
<td>167.94 c</td>
<td>75.0 c</td>
<td>226.7 c</td>
<td>0.3445</td>
<td>0.2792</td>
<td>47 c</td>
<td>75.55 c</td>
</tr>
<tr>
<td>DD</td>
<td>38.71 c</td>
<td>18.72 c</td>
<td>19.99 c</td>
<td>61.01 c</td>
<td>155.87 c</td>
<td>138.3 c</td>
<td>256.9 c</td>
<td>0.2421</td>
<td>0.1978</td>
<td>26.8 c</td>
<td>65.51 c</td>
</tr>
</tbody>
</table>

We can see that there is a dramatic profit loss due to horizontal competition by comparing M with II. Adding vertical competition to the competing SC can recapture this loss nearly back to the industry optimal. In the II setting, service levels become more differentiated, prices still largely decrease. In the DD setting, service levels become less differentiated, but SCs can still charge higher prices. So DD can afford less differentiated
products without intensifying price competition. SC one gains more with DD. Other choices of utility and cost functions give consistent results. As consumers become more value service (i.e., \( n \) becomes larger), the consumer welfare, the total industry profit and the social welfare increase under all structures. See appendix 8 for \( n > 2 \).

The Supply Chain Structure Game

With price competition only, the industry equilibrium will converge to II or DD. II is always an equilibrium, but DD can be an equilibrium when the degree of product differentiation is not great. Our next goal is, what will the industry equilibrium structure be under both service and price competition? To compare profits, we need closed form solutions. So we will use specific utility and cost functions throughout this section. This also allow us to see how consumer preference drives the Nash equilibrium SC structure.

By using \( u(a_i) = \sqrt{a_i}, c(a_i) = c a_i \), we have the strategic form game (in tables 2.9-2.11) with different market power. Observe that, only II is the equilibrium supply chain structure, but DD can make both SCs better off regardless of market power. A SC’s incentive to stay in II or DD becomes less with manufacturer power, but stronger with retailer power. So II can be the unique market structure equilibrium regardless of market power, but market power will affect the supply chain’s incentive to stay in the equilibrium.

Table 2.9-1.11. Strategic form game (equal(E), manufacturer(M) and retailer(R) power respectively)

<table>
<thead>
<tr>
<th>E</th>
<th>SC two</th>
<th></th>
<th>M</th>
<th>SC two</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>D</td>
<td></td>
<td>I</td>
</tr>
<tr>
<td>SC</td>
<td>I</td>
<td>12.15/c, 16.40/c</td>
<td>19.37/c, 12.94/c</td>
<td>SC</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>10.28/c, 22.37/c</td>
<td>18.72/c, 19.99/c</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>SC two</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>I</td>
<td>D</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SC</td>
<td>I</td>
<td>12.15/c, 16.40/c</td>
<td>19.94/c, 12.55/c</td>
<td></td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>9.865/c, 23.78/c</td>
<td>18.85/c, 20.59/c</td>
<td></td>
</tr>
</tbody>
</table>

Can we claim that II is always an equilibrium under both price and service competition?

By using a series of utility and cost functions as shown in appendix 5, we find that
when \( n \leq 3 \), only II is the equilibrium. When \( 4 \leq n \leq 5 \), both II and DD are an equilibriu. When \( n \geq 6 \), only DD is an equilibrium. So we have the following observation:

**Observation 2.1** If consumers' utility function are less concave (have weaker diminishing return in service, i.e., \( n \) is small), then SCs choose well-differentiated services, and only II is the subgame perfect Nash equilibrium SC structure. If consumers' utility function are more concave (have stronger diminishing return in service, i.e., \( n \) is large), then SCs choose less-differentiated services, and only DD is the equilibrium. If the above criteria is in between, then the equilibrium service differentiation is in between, and II and DD can both be equilibria at the same time.

In a market that providing a larger service level will not bring a higher return on consumers' utility, then SC two will not invest on a higher service level. So the market ends up with less differentiated products. Then price competition becomes stronger, and then SCs might introduce double marginalization to 'cool down' the head to head competition. In a market that providing a larger service level will induce a higher consumer utility, then SC two will invest on a higher service level and SCs can easily differentiate in service. Price competition is weak and the negative effect of double marginalization is stronger, so no SCs want to decentralize. If the differentiation is in between, both II and DD can be equilibria. This is a key distinction from the price competition only game studied by previous literature. By explicitly modeling consumer heterogeneity and product differentiation, we are able to understand SC structure more deeply.

Does market power affect the SC structure? Using \( n = 5, 9 \), we analyze the SC structure game under manufacturer and retailer power (appendix 6), and observe,

**Observation 2.2** With manufacturer power, the result coincides with observation 1. But with retailer power, the equilibrium SC structure changes less as \( n \) increases, or alternatively is less sensitive to consumer preference.

The reason for this observation is the following. In DI or ID structure, the integrated SC gains higher profit under retailer power. So there is less incentive for the integrated SC in DI and ID to decentralize to DD.

We have observed that DD can provide better industry profits than II in the pricing game. With this class of functions, DD also brings higher profits no matter whether it is
an equilibrium SC structure or not. We suspect that DD can be sustained as a subgame perfect equilibrium even though it is not an equilibrium SC structure if the multi-stage game we have studied is played repeatedly. If two players both value the future, tacit collusion can enable an equilibrium outcome which makes both players better off.

2.6 Some Managerial Implications

"no longer will companies compete against other companies, but total supply chains will compete against other supply chains" (Deloitte Consulting 1999).

The primary objective of this research is to develop a basic theory of supply chains in a competitive and heterogeneous market. We have used a stylized multi-echelon, multi-supply chain model that incorporates both consumer behavior and individual firm behavior. In contrast to previous literature, we find that the choice of supply chain structure is very much dependent on consumers’ preference. Supply chain managers can manipulate product & service strategy and supply chain structures to mitigate competition. For example, if the market has consumers with stronger diminishing return on service, then setting a less differentiated service level and choosing decentralized SC would be the only Nash equilibrium strategy. If the market has consumers with weaker diminishing return on service, then we will observe more differentiated service levels and only integrated SCs in the market equilibrium. Otherwise, the degree of differentiation will be in between and the optimal strategy is to match the opponent (so both SC choosing integration simultaneously, both SC choosing decentralization simultaneously are all Nash equilibrium strategies). For managers who are operating a supply chain with fixed service levels, choosing integration is a safer strategy. But if they can set the wholesale price in favor of total supply chain profit, decentralization can actually benefit the supply chain.

The issue of who gets to decide service levels is important for competitive supply chains. In the DD structure, allowing the retailers to decide service levels can make both the total supply chain and the members better off. With retailer power, equilibrium SC structure is less sensitive to consumer preference and the effect of product differentiation. We also find it is beneficial for competitive supply chains to cooperate in structure design. For example, strategically decentralizing one supply chain can make both better off than
both integrated. Also, in the long run, tacit collusion with competitors and all SCs decentralized may lead to a win-win situation.

Several analytical extensions of this research are possible. For example, consider a nonuniform allocation of the consumer willingness to pay. Also, we have only considered simultaneous games. Stackelberg leadership in the subgame is another possible extension.

Also, this paper provides some hypotheses for further empirical studies. For instance, in industries where consumers have stronger diminishing marginal utility on service, we will more likely observe less differentiated products/service and decentralized supply chains in operation. In industries where consumers have weaker diminishing marginal utility on service, the likelihood of choosing more differentiated product and using integrated supply chains is larger. If we can observe mixed market structures, then the lower service supply chain is more likely to be the one decentralized.

**Figures**

Iso-profit curves and reaction curves

![Figure 1 SC 1's iso-profit curve](image1)

![Figure 2.2 SC 2's Iso-profit curve](image2)
2.7 Appendix

1. Proof of Result in Pricing Game

*II setting*

\[ \alpha p_2 - \beta p_1 - \beta (p_1 - c_1) = \lambda_1 - \beta (p_1 - c_1) = 0 \]

\[ 1 - \alpha (p_2 - p_1) - \alpha (p_2 - c_2) = \lambda_2 - \alpha (p_2 - c_2) = 0 \]

Observe that the reaction of two SCs in pricing subgame follows a strategic complementary pattern, and the iso-profit curves are convex.

The second order conditions and the strategic interactions are as follows,

\[ \frac{\partial^2 \pi_1}{\partial p_1^2} = -2\beta, \quad \frac{\partial^2 \pi_1}{\partial p_1 \partial p_2} = \alpha, \quad \frac{\partial^2 \pi_2}{\partial p_2^2} = -2\alpha, \quad \frac{\partial^2 \pi_2}{\partial p_2 \partial p_1} = \alpha, \quad \frac{\partial^2 \pi_2}{\partial p_1} = \frac{\partial^2 \pi_1}{\partial p_1} = \frac{1}{2} \]

The Nash equilibrium strategies can be solved from the reaction functions. The market shares and profits can be obtained by substituting the Nash equilibrium prices.

*DI setting*

The game becomes,

\[ \pi_1^{RI} = \max_{p_1} (p_1 - c_1 - w_1) \lambda_1 = \max_{p_1} (p_1 - c_1 - w_1)(\alpha p_2 - \beta p_1) \]

\[ \pi_2^{DI} = \max_{p_2} (p_2 - c_2) \lambda_2 = \max_{p_2} (p_2 - c_2)(1 - \alpha (p_2 - p_1)) \]

The reaction functions are,

\[ \lambda_1 - \beta (p_1 - c_1 - w_1) = 0, \quad \lambda_2 - \alpha (p_2 - c_2) = 0 \]

The second order conditions are, \[ \frac{\partial^2 \pi_1^{RI}}{\partial p_1^2} = -2\beta, \quad \frac{\partial^2 \pi_2^{DI}}{\partial p_2^2} = -2\alpha \]
So the functions are strictly concave, and the reactions are contraction mapping, which guarantees a unique Nash Equilibrium. The equilibrium prices are

\[
p_1^{DI} = \frac{1 + \alpha c_2 + 2\beta(c_1 + w_1)}{4\beta - \alpha} = p_1^{II} + \frac{2\beta w_1}{4\beta - \alpha} \quad (A2.1)
\]

\[
p_2^{DI} = \frac{\beta((c_1 + w_1)\alpha + 2 + 2\alpha c_2)}{\alpha(4\beta - \alpha)} = p_2^{II} + \frac{\beta w_1}{4\beta - \alpha} \quad (A2.2)
\]

We can then get,

\[
\lambda_1^{DI} = \beta(p_1^{DI} - c_1 - w_1) = \lambda_1^{II} - \frac{\beta(2\beta - \alpha)w_1}{4\beta - \alpha} \quad (A2.3)
\]

\[
\lambda_2^{DI} = \alpha(p_2^{DI} - c_2) = \lambda_2^{II} + \frac{\alpha\beta w_1}{4\beta - \alpha} \quad (A2.4)
\]

Then the manufacturer's problem is,

\[
\pi_1^{M^{DI}} = \max_{w_1} \lambda_1^{DI} = \max_{w_1} w_1(\lambda_1^{II} - \frac{\beta(2\beta - \alpha)w_1}{4\beta - \alpha})
\]

\[
\frac{\partial \pi_1^{M^{DI}}}{\partial w_1} = 0 \Rightarrow w_1^{DI} = \frac{(4\beta - \alpha)\lambda_1^{II}}{2(2\beta - \alpha)^2} = \frac{2\lambda_1^{II}}{\alpha}, \quad \frac{\partial^2 \pi_1^{M^{DI}}}{\partial w_1^2} = -\frac{2\beta(2\beta - \alpha)}{4\beta - \alpha} < 0.
\]

observe: 1. \( \frac{\lambda_1^{II}}{2\beta} < w_1^{DI} < \frac{3\lambda_1^{II}}{2\beta} \), i.e., \( \frac{\beta}{2}\lambda_1^{II} - c_1 < w_1^{DI} < \frac{3\beta}{2}\lambda_1^{II} - c_1 \). The optimal wholesale price has a lower and upper bound in terms of the profit margin of II setting. 2. \( \frac{\partial w_1^{DI}}{\partial \lambda_1^{II}} = \frac{4\beta - \alpha}{2\beta(2\beta - \alpha)} > 0 \), the higher the demand \( \lambda_1^{II} \), the more the manufacturer tends to set the wholesale price.

Substitute \( w_1^{DI} \) into above functions (A2.1)-(A2.4) and profit functions, we get the required results.

\[
\pi_1^{M^{DI}} = \frac{w_1^{DI}}{2} = \frac{(\lambda_1^{II})^2(4\beta - \alpha)}{4(2\beta - \alpha)}, \quad \pi_1^{R^{DI}} = (p_1^{DI} - c_1 - w_1)\lambda_1^{II} = \frac{(\lambda_1^{II})^2}{4\beta} = \frac{\pi_1^{II}}{4}
\]

\[
\pi_1^{DI} = \pi_1^{II} \frac{3\beta - \alpha}{2(2\beta - \alpha)} = \pi_1^{II} \frac{3\beta - \alpha}{2(2\beta - \alpha)}
\]

\[
\pi_2^{DI} = \frac{(\lambda_2^{DI})^2}{\alpha} = \frac{(\lambda_2^{II} + \frac{\lambda_1^{II}(\alpha)}{4\beta})^2}{\alpha} = \frac{\pi_2^{II}}{4} \left(1 + \frac{\lambda_1^{II}(\alpha)}{2\lambda_2^{II}(2\beta - \alpha)}\right)^2 = \frac{\pi_2^{II}}{4} \left(1 + \frac{\lambda_1^{II}w_1}{2(\lambda_2^{II} - w_1)\lambda_2^{II}}\right)^2
\]

**ID Setting**

\[
\pi_1^{ID} = \max_{p_1} (p_1 - c_1)\lambda_1 = \max_{p_1} (p_1 - c_1)(\alpha p_2 - \beta p_1),
\]

\[
\pi_2^{ID} = \max_{p_2} (p_2 - c_2 - w_2)\lambda_2 = \max_{p_2} (p_2 - c_2 - w_2)(1 - \alpha(p_2 - p_1)),
\]

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where $w_2$ is the wholesale price of manufacturer two.

The reaction functions are

$$\lambda_1 - \beta(p_1 - c_1) = 0, \lambda_2 - \alpha(p_2 - c_2 - w_2) = 0$$

The second order conditions are satisfied and we have a unique Nash Equilibrium by solving the two linear response functions.

$$p_1^{ID} = \frac{\beta(c_1 + 2 + 2(c_2 + w_2))}{\alpha(4\beta - \alpha)} = p_2^{II} + \frac{2\beta w_2}{4\beta - \alpha}$$

$$p_1^{ID} = \frac{1 + \alpha(c_2 + w_2) + 2\beta c_1}{4\beta - \alpha} = p_2^{II} + \frac{\alpha w_2}{4\beta - \alpha}$$

And the demand functions are

$$\lambda_1^{ID} = \beta(p_1^{ID} - c_1) = \lambda_1^{II} + \frac{\beta w_2}{4\beta - \alpha},$$

$$\lambda_2^{ID} = \alpha(p_2^{ID} - c_2 - w_2) = \lambda_2^{II} - \frac{(2\beta - \alpha)\alpha w_2}{4\beta - \alpha}$$

Note that $\lambda_1^{II} < \lambda_1^{ID} < \frac{\beta w_2}{3}, \lambda_2^{II} - \alpha w_2 < \lambda_2^{ID} < \lambda_2^{II} - \frac{\alpha w_2}{3}.$

Manufacturer two's problem is,

$$\pi_2^{MID} = \max_{w_2} w_2\lambda_2 = \max_{w_2} w_2(\lambda_2^{ID} - (\lambda_2^{II} - \frac{(2\beta - \alpha)\alpha w_2}{4\beta - \alpha})$$

We have a unique optimal solution

$$w_2^{ID} = \frac{\lambda_2^{II}(4\beta - \alpha)}{2(2\beta - \alpha)\alpha}$$

Observe that 1. $\frac{\lambda_1^{ID} - (2\beta - \alpha)\alpha w_2}{4\beta - \alpha} < \frac{\lambda_1^{II} - c_1}{2} < \frac{\lambda_1^{ID} - (2\beta - \alpha)\alpha w_2}{4\beta - \alpha}$ (lower bound and upper bound of optimal wholesale price) 2. $\frac{\partial\pi_2^{ID}}{\partial\lambda_2^{II}} = \frac{4\beta - \alpha}{2(2\beta - \alpha)\alpha} = \frac{1}{2\alpha(1-\eta)} > 0.$ A higher $\lambda_2^{II}$ leads to a higher wholesale price.

Substitute the wholesale price into the above, we have,

$$p_1^{ID} = p_1^{II} + \frac{\lambda_1^{II}}{2(2\beta - \alpha)} = p_1^{II} + \frac{\lambda_1^{II}(u_2 - u_1)u_1}{2(u_2 - u_1)},$$

$$p_2^{ID} = p_2^{II} + \frac{\beta\lambda_2^{II}}{\alpha(2\beta - \alpha)} = p_2^{II} + \frac{(u_2 - u_1)u_2\lambda_2^{II}}{2u_2 - u_1}$$

$$p_1^{ID} - c_1 = p_1^{II} - c_1 + \frac{\lambda_1^{II}}{2(2\beta - \alpha)} = \frac{\lambda_1^{II}}{\beta} + \frac{\lambda_1^{II}}{2(2\beta - \alpha)}$$

$$p_2^{ID} - c_2 - w_2^{ID} = \lambda_2^{II}(3\beta - \alpha) - \frac{\lambda_1^{II}(4\beta - \alpha)}{2(2\beta - \alpha)\alpha} = \frac{\lambda_2^{II}}{2\alpha},$$

$$p_2^{ID} - c_2 = p_2^{II} - c_2 + \frac{2\beta w_2}{4\beta - \alpha} = \frac{\lambda_2^{II}}{2\alpha}$$

$$\lambda_2^{ID} = \lambda_2^{II} - \frac{(2\beta - \alpha)\alpha w_2}{4\beta - \alpha} = \frac{\lambda_2^{II}}{2}, \lambda_1^{ID} = \lambda_1^{II} + \frac{\beta\lambda_2^{II}}{2(2\beta - \alpha)}$$

So the equilibrium profits are,
\[ \pi_2^{ID} = w_2 \lambda_2^{ID} = \frac{4\beta - \alpha}{4(2\beta - \alpha)} \pi_2^{II} = \pi_2^{II} \frac{1}{4(1 - \eta)} , \quad \pi_2^{R} = (p_2^{ID} - c_2 - w_2) \lambda_2^{ID} = \frac{\pi_2^{II}}{4} \]

\[ \pi_1^{ID} = \frac{(\lambda_1^{ID})^2}{\beta} = \pi_1^{II} (1 + \frac{\beta \pi_1^{II}}{2(2\beta - \alpha) \lambda_1^{II}})^2 = \pi_1^{II} (1 + \frac{w_2 \lambda_1^{II}}{2(w_2 - u_1) \lambda_1^{II}})^2 \]

\[ \pi_2^{ID} = \pi_2^{M^D} + \pi_2^{R} = \frac{\pi_1^{II}(3\beta - \alpha)}{4\beta - 2\alpha} = \frac{\pi_1^{II}(3w_2 - u_1)}{4w_2 - 2u_1} \]

By comparison, we have

\[ \frac{3}{4} \pi_2^{II} < \pi_2^{ID} < \pi_2^{II} , \quad \pi_1^{II} < \pi_1^{ID} < \pi_1^{II}(1 + \frac{\lambda_1^{II}}{2\lambda_1^{II}})^2 , \quad \pi_2^{M^D} < \frac{3\pi_1^{II}}{4}. \]

**DD Setting**

\[ \pi_1^R = \max(p_1 - c_1 - w_1) \lambda_1 = \max(p_1 - c_1 - w_1) (\alpha p_2 - \beta p_1) \]

\[ \pi_2^R = \max(p_2 - c_2 - w_2) \lambda_2 = \max(p_2 - c_2 - w_2)(1 - \alpha(p_2 - p_1)) \]

The equilibrium prices and demands are,

\[ p_1^{DD} = p_1^{II} + \frac{\alpha w_2}{4\beta - \alpha} + \frac{2\beta w_1}{4\beta - \alpha} = p_1^{ID} + \frac{2\beta w_1}{4\beta - \alpha} \tag{A2.5} \]

\[ p_2^{DD} = p_2^{II} + \frac{2\beta w_2}{4\beta - \alpha} + \frac{\beta w_1}{4\beta - \alpha} = p_2^{DI} + \frac{2\beta w_2}{4\beta - \alpha} \tag{A2.6} \]

\[ \lambda_1^{DD} = \lambda_1^{II} + \frac{\alpha \beta w_2}{4\beta - \alpha} - \frac{(2\beta - \alpha) w_1}{4\beta - \alpha} = \lambda_1^{ID} - \frac{\beta(2\beta - \alpha) w_1}{4\beta - \alpha} \tag{A2.7} \]

\[ \lambda_2^{DD} = \lambda_2^{II} + \frac{\alpha \beta w_1}{4\beta - \alpha} - \frac{\alpha(2\beta - \alpha) w_2}{4\beta - \alpha} = \lambda_2^{DI} - \frac{\alpha(2\beta - \alpha) w_2}{4\beta - \alpha} \tag{A2.8} \]

Then the manufacturers' wholesale price game is, \( \pi_1^M = w_1 \lambda_1^{DD} , \quad \pi_2^M = w_2 \lambda_2^{DD} \)

Both of the functions are strictly concave and we have a unique Nash equilibrium wholesale price equilibrium,

\[ w_1^{DP} = \frac{(4\beta - \alpha) \lambda_1^{ID}}{2(2\beta - \alpha)} \bigg|_{w_2^{DP}} , \quad w_2^{DP} = \frac{(4\beta - \alpha) \lambda_2^{DI}}{2(2\beta - \alpha)} \bigg|_{w_1^{DP}} \tag{A2.9} \]

Substitute the A2.9 into functions (A2.7)-(A2.8), we have

\[ \lambda_1^{DD} = \frac{\lambda_1^{ID}}{2} \bigg|_{w_2^{DP}} , \quad \lambda_2^{DD} = \frac{1}{2} \lambda_2^{DI} \bigg|_{w_1^{DP}} \tag{A2.10} \]

The profit functions become,

\[ \pi_1^{R^{DD}} = \lambda_1^{DD}(p_1^{DD} - c_1 - w_1) = \frac{(\lambda_1^{ID})^2}{4\beta} \bigg|_{w_2^{DP}} \tag{A2.11} \]
\[ \pi_{1}^{MDD} = \lambda_{1}^{DD} w_{1}^{DD} = \frac{4\beta - \alpha}{4(2\beta - \alpha)} \left( \frac{\lambda_{1}^{ID}}{\beta} \right)^{2} |_{w_{2}^{DD}} \]  
(A2.12)

\[ \pi_{1}^{DD} = \pi_{1}^{RDD} + \pi_{1}^{MDD} = \frac{3\beta - \alpha}{2(2\beta - \alpha)} \left( \frac{\lambda_{1}^{ID}}{\beta} \right)^{2} |_{w_{2}^{DD}} \]  
(A2.13)

Similarly for SC two.

The wholesale prices in (A2.9) can be written as

\[ w_{1}^{DD} = \frac{\lambda_{1}^{ID}(4\beta - \alpha)}{2(2\beta - \alpha)\beta} + \frac{\alpha w_{2}^{DD}}{2(2\beta - \alpha)}; \quad w_{2}^{DD} = \frac{\lambda_{1}^{ID}(4\beta - \alpha)}{2(2\beta - \alpha)\alpha} + \frac{\beta w_{2}^{PP}}{2(2\beta - \alpha)}. \]

Solving the above equations, we can get the required results for equilibrium wholesale prices.

\[ w_{1}^{PP} = \frac{(4\beta - \alpha)\lambda_{1}^{I} + \lambda_{1}^{I}(4\beta - \alpha)}{2(2\beta - \alpha)\alpha + \lambda_{1}^{I}(4\beta - \alpha)^{2}}, \quad w_{2}^{PP} = \frac{(2\alpha - \beta)\lambda_{1}^{I} + \lambda_{1}^{I}(2\alpha - \beta)^{2}}{4(2\beta - \alpha)\alpha + \lambda_{1}^{I}(2\beta - \alpha)^{2}}. \]

We can then get \( p_{1}^{PP}, p_{2}^{PP} \) by substituting this result into (A2.5)-(A2.6).

\[ p_{1}^{PP} = p_{1}^{I} + \frac{2(3\beta - \alpha)\lambda_{1}^{I} + (8\beta - 3\alpha)\lambda_{1}^{I}}{4(2\beta - \alpha)^{2} - \alpha \beta} \]

\[ p_{2}^{PP} = p_{2}^{I} + \frac{2(8\beta - 3\alpha)\lambda_{1}^{I} + \beta(8\beta - 3\alpha)\lambda_{1}^{I}}{2(2\beta - \alpha)^{2} - \alpha \beta} \]

The equilibrium wholesale prices can also be expressed in terms of the \( w_{1}^{PI}, w_{2}^{PI} \).

\[ w_{1}^{PP} = \frac{4(2\beta - \alpha)^{2}w_{1}^{PI} + \lambda_{1}^{I}(4\beta - \alpha)}{4(2\beta - \alpha)^{2} - \alpha \beta}, \quad w_{2}^{PP} = \frac{4(2\beta - \alpha)^{2}w_{2}^{PI} + \lambda_{1}^{I}(4\beta - \alpha)}{4(2\beta - \alpha)^{2} - \alpha \beta} \]  
(A2.14)

Substitute (A2.14) into the demand functions in (A2.10) we have,

\[ \frac{\lambda_{1}^{ID}}{2} |_{w_{2}^{w_{i}^{PP}}} = \lambda_{1}^{I} + \frac{\alpha \omega_{2}^{ID}}{4(2\beta - \alpha)^{2} - \alpha \beta} = \left( \frac{\alpha \omega_{2}^{ID}}{4(2\beta - \alpha)^{2} - \alpha \beta} \right) \frac{4(2\beta - \alpha)^{2}}{4(2\beta - \alpha)^{2} - \alpha \beta} = \frac{\tau \lambda_{1}^{ID}}{w_{2}^{w_{i}^{PP}}} \]  
(A2.15)

Then \[ \lambda_{1}^{ID} = \frac{\tau \lambda_{1}^{ID}}{w_{2}^{w_{i}^{PP}}} = \frac{\tau}{2} \left( \lambda_{1}^{I} + \frac{\alpha \omega_{2}^{ID}}{4(2\beta - \alpha)^{2} - \alpha \beta} \right) = \frac{\tau \lambda_{1}^{ID}}{2} \]  

Similarly,

\[ \lambda_{2}^{ID} = \frac{\tau \lambda_{2}^{ID}}{w_{1}^{w_{i}^{PP}}} = \frac{\tau}{2} \left( \lambda_{2}^{I} + \frac{\beta \omega_{1}^{ID}}{4(2\beta - \alpha)^{2} - \alpha \beta} \right) = \frac{\tau \lambda_{2}^{ID}}{2} \]

Substitute (A2.15) into (A2.11)-(A2.13), we have

\[ \pi_{1}^{PP} = \frac{(\lambda_{1}^{ID})^{2}}{\beta} |_{w_{2}^{w_{i}^{PP}}} = \frac{2(2\beta - \alpha)^{2} \lambda_{1}^{ID}^{2}}{4(2\beta - \alpha)^{2} - \alpha \beta} |_{w_{1}^{w_{i}^{PP}}} = \pi_{1}^{ID} \frac{2(3\beta - \alpha)}{4(2\beta - \alpha) - \alpha} = \pi_{1}^{ID} \tau_{1} \]
\[ \pi_2^{DD} = \frac{(\lambda_2^{DL})^2}{\alpha} |w_1 = w_2^{DL} 4^{3\beta - 2\alpha} = \frac{\tau^2(\lambda_2^{DL})^2}{\alpha} |w_1 = w_2^{DL} 4^{3\beta - 2\alpha} = \pi_2^{DL} = \pi_2^{DL} \]

Following the same logic,

\[ \pi_1^{MDD} = \frac{(\lambda_1^{DL})^2}{\beta} |w_2^{DL} 4^{3\beta - 2\alpha} = \frac{\tau^2(\lambda_1^{DL})^2}{\beta} |w_2^{DL} 4^{3\beta - 2\alpha} = \pi_1^{DL} \]
\[ \pi_1^{MDD} = \frac{(\lambda_1^{DL})^2}{\alpha} |w_1^{DL} 4^{3\beta - 2\alpha} = \frac{\tau^2(\lambda_1^{DL})^2}{\alpha} |w_1^{DL} 4^{3\beta - 2\alpha} = \pi_2^{DL} \]

Also \[ \pi_1^{RDD} = \frac{(\lambda_1^{DL})^2}{4\alpha} |w_2^{DL} = \frac{\tau^2(\lambda_1^{DL})^2}{4\alpha} |w_2^{DL} = \pi_2^{DL} \]

2. Proof of Quasiconcavity

\[ \pi_1^{II} = \frac{(a_2 - a_1 + 1)^2(a_2 - a_1)a_2a_1}{(4a_2 - a_1)^2} \text{ is quasiconcave in } a_1. \]

Function \( f(\cdot) \) is quasiconvex on a convex set \( \Gamma \) if (1) \( f(\cdot) \) is concave on \( \Gamma \), \( g(\cdot) > 0 \) on \( \Gamma \) and (2) \( g(\cdot) \) is convex on \( \Gamma \), \( f(\cdot) \geq 0 \) on \( \Gamma \).(Mangasarian 1969).

\( (ca_2 - ca_1 + 1)^2a_2a_1 \) is concave in \( a_1 \) because \( \frac{\partial^2((ca_2 - ca_1 + 1)^2a_2a_1)}{\partial a_1^2} = -2ca_2(2ca_2 - 3ca_1 + 2) < 0 \)
\[ \frac{(4a_2 - a_1)^2}{(a_2 - a_1)^2} \text{ is convex in } a_1 \text{ because } \frac{\partial^2((4a_2 - a_1)^2)}{\partial a_1^2} = \frac{18a_1^2}{(a_2 - a_1)^2} > 0. \]

Similarly \( \pi_2^{II} = \frac{(2 - 2ca_2 - ca_1)^2(a_2 - a_1)a_2^2}{(4a_2 - a_1)^2} \) is quasiconcave in \( a_2 \), because \( \frac{1}{(2 - 2ca_2 - ca_1)^2} \) is convex in \( a_2 \), and \( \frac{(a_2 - a_1)a_2^2}{(4a_2 - a_1)^2} \) is concave in \( a_2 \).

3. Equilibrium Result of Other Pairs of Utility and Cost Functions

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<th>( \pi )</th>
<th>( \pi_1 )</th>
<th>( \pi_2 )</th>
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<td>72.07</td>
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<table>
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<tr>
<th>DD Setting</th>
<th>$\pi$</th>
<th>$\pi^*_1$</th>
<th>$\pi^*_2$</th>
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<th>$p_2$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
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<tbody>
<tr>
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<td>40.22</td>
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<tr>
<td>$\sqrt{a_i}, ca_i$</td>
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<td>42.83</td>
<td>49.29</td>
<td>163.21</td>
<td>287.18</td>
<td>27.50</td>
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<td>57.96</td>
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<td>290.36</td>
<td>19.24</td>
<td>99.46</td>
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<td>171.74</td>
<td>292.29</td>
<td>9.84</td>
<td>79.57</td>
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<td>0.3296</td>
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<tr>
<td>$\sqrt{a_i}, ca_i$</td>
<td>131.16</td>
<td>56.47</td>
<td>74.69</td>
<td>173.06</td>
<td>292.81</td>
<td>7.12</td>
<td>72.23</td>
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<tr>
<td>$\sqrt{a_i}, ca_i$</td>
<td>136.40</td>
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<td>78.80</td>
<td>173.98</td>
<td>293.20</td>
<td>5.12</td>
<td>65.63</td>
<td>0.3411</td>
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</tr>
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</table>

4. **Effect of Market Power** \( u(a_i) = \sqrt{a_i}, c(a_i) = ca_i \)

### ID Setting

<table>
<thead>
<tr>
<th>power</th>
<th>$a_1$</th>
<th>$\pi_{1}^{MID}$</th>
<th>$\pi_{1}^{RID}$</th>
<th>$\pi_{2}^{ID}$</th>
<th>$a_2$</th>
<th>$\pi_{2}^{MID}$</th>
<th>$\pi_{2}^{RID}$</th>
<th>$\pi_{2}^{ID}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>equal</td>
<td>44.10/$c^2$</td>
<td>–</td>
<td>–</td>
<td>19.37/$c$</td>
<td>162.97/$c^2$</td>
<td>9.079/$c$</td>
<td>3.861/$c$</td>
<td>12.94/$c$</td>
</tr>
<tr>
<td>retailer</td>
<td>46.48/$c^2$</td>
<td>–</td>
<td>–</td>
<td>19.94/$c$</td>
<td>175.14/$c^2$</td>
<td>8.799/$c$</td>
<td>3.749/$c$</td>
<td>12.55/$c$</td>
</tr>
</tbody>
</table>

### DD Setting
### 5. Strategic Forms of Supply Chain Structure Game

<table>
<thead>
<tr>
<th>Power</th>
<th>(a_1)</th>
<th>(\pi_1^{DD})</th>
<th>(\pi_1^{RDD})</th>
<th>(\pi_2^{DD})</th>
<th>(\pi_2^{RDD})</th>
<th>(\pi_2^{MDD})</th>
<th>(\pi_2^{DD})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equal</td>
<td>61.01/c^2</td>
<td>13.30/c</td>
<td>5.418/c</td>
<td>18.72/c</td>
<td>155.87/c^2</td>
<td>14.20/c</td>
<td>5.785/c</td>
</tr>
<tr>
<td>Manufacturer</td>
<td>63.60/c^2</td>
<td>13.25/c</td>
<td>5.35/c</td>
<td>18.60/c</td>
<td>153.51/c^2</td>
<td>14.04/c</td>
<td>5.68/c</td>
</tr>
<tr>
<td>Retailer</td>
<td>54.66/c^2</td>
<td>13.32/c</td>
<td>5.528/c</td>
<td>18.85/c</td>
<td>161.85/c^2</td>
<td>14.55/c</td>
<td>6.038/c</td>
</tr>
</tbody>
</table>

\[ u(a_i) = \sqrt{\alpha_i}, \quad c(a_i) = ca_i \]

**SC two**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC</td>
<td>19.06,28.87</td>
<td>32.68,24.20</td>
</tr>
<tr>
<td>one</td>
<td>17.21,39.12</td>
<td>33.74,37.42</td>
</tr>
</tbody>
</table>

**SC one**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC</td>
<td>22.97,42.96</td>
<td>43.84,38.61</td>
</tr>
<tr>
<td>one</td>
<td>22.47,55.47</td>
<td>48.52,57.96</td>
</tr>
</tbody>
</table>

In the first game, only II is the Nash equilibrium. DD dominates II.

In the second game, both II and DD are equilibria. DD Pareto dominates II.

\[ u(a_i) = \sqrt{\alpha_i}, \quad c(a_i) = ca_i \]

**SC two**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC</td>
<td>23.23,47.95</td>
<td>46.34,43.85</td>
</tr>
<tr>
<td>one</td>
<td>23.28,60.04</td>
<td>52.27,64.61</td>
</tr>
</tbody>
</table>

**SC one**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC</td>
<td>23.23,47.95</td>
<td>46.34,43.85</td>
</tr>
<tr>
<td>one</td>
<td>23.28,60.04</td>
<td>52.27,64.61</td>
</tr>
</tbody>
</table>

In the third game, both II and DD are equilibria, and DD dominates II.

In the fourth game, only DD is the equilibrium.

\[ u(a_i) = \sqrt{\alpha_i}, \quad c(a_i) = ca_i \]

**Player two**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>player</td>
<td>23.10,52.50</td>
<td>47.96,48.38</td>
</tr>
<tr>
<td>one</td>
<td>23.58,63.54</td>
<td>54.74,70.10</td>
</tr>
</tbody>
</table>

**Player two**

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>player</td>
<td>22.79,56.90</td>
<td>49.22,51.04</td>
</tr>
<tr>
<td>one</td>
<td>23.49,66.70</td>
<td>56.47,74.69</td>
</tr>
</tbody>
</table>
For $n \geq 7$, only DD is the equilibrium.

6. Effect of Market Power on SC Structure

Manufacturer power

\[ u(a_i) = \sqrt{a_i}, \ c(a_i) = ca_i \]

<table>
<thead>
<tr>
<th>SC two</th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>player</td>
<td>player</td>
<td>player</td>
</tr>
<tr>
<td>I</td>
<td>22.38</td>
<td>49.93</td>
</tr>
<tr>
<td>D</td>
<td>61.37</td>
<td>55.84</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>SC two</th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>one</td>
<td>one</td>
<td>one</td>
</tr>
<tr>
<td>D</td>
<td>23.21</td>
<td>57.60</td>
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<td></td>
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<table>
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<tr>
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<th>D</th>
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<tr>
<td>one</td>
<td>one</td>
<td>one</td>
</tr>
<tr>
<td>D</td>
<td>22.94</td>
<td>48.32</td>
</tr>
<tr>
<td></td>
<td>53.15</td>
<td>57.00</td>
</tr>
</tbody>
</table>

In the first game, both II and DD are equilibria. In the second game, only DD is an equilibrium.

Retailer Power

\[ u(a_i) = \sqrt{a_i}, \ c(a_i) = ca_i \]

<table>
<thead>
<tr>
<th>SC two</th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>player</td>
<td>player</td>
<td>player</td>
</tr>
<tr>
<td>I</td>
<td>22.97</td>
<td>44.93</td>
</tr>
<tr>
<td>D</td>
<td>42.96</td>
<td>37.54</td>
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<table>
<thead>
<tr>
<th>SC two</th>
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<tr>
<td>one</td>
<td>one</td>
<td>one</td>
</tr>
<tr>
<td>D</td>
<td>22.94</td>
<td>48.32</td>
</tr>
<tr>
<td></td>
<td>53.15</td>
<td>57.00</td>
</tr>
</tbody>
</table>

In the third game, only II is equilibrium. In the fourth game, both II and DD are equilibria.

7. Development of results with $\theta$ bounded on $[0, b]$ instead of $[0, 1]$

Now the expected demand is

\[ \lambda_1 = \frac{\alpha p_2 - \beta p_1}{b}, \lambda_2 = \frac{b - \alpha (p_2 - p_1)}{b} \]
We use $\lambda_1 = \alpha p_2 - \beta p_1$, $\lambda_2 = b - \alpha(p_2 - p_1)$ for mathematical convenience.

The II Setting

The expected profit is,

$$\pi_1^{II} = \max_{p_1} \frac{(\alpha p_2 - \beta p_1)(p_1 - c)}{b}$$

$$\pi_2^{II} = \max_{p_2} \frac{b - \alpha(p_2 - p_1)(p_2 - c)}{b}$$

The first order conditions are

$$\lambda_1 = \alpha p_2 - \beta p_1 = \beta(p_1 - c_1),$$

$$\lambda_2 = b - \alpha(p_2 - p_1) = \alpha(p_2 - c_2)$$

and $p_1^{II} = \frac{b + \alpha c_2 + 2\beta c_1}{4\beta - \alpha}$, $p_2^{II} = \frac{2\alpha + \alpha c_1 - (2\beta - \alpha)c_2}{4\beta - \alpha}$

$$p_1^{II} - c_1 = \frac{b + \alpha c_2 - (2\beta - \alpha)c_1}{4\beta - \alpha},$$

$$p_2^{II} - c_2 = \frac{2\beta + \alpha c_1 - (2\beta - \alpha)c_2}{4\beta - \alpha}$$

$$\lambda_1^{II} = \frac{\beta + \alpha c_1 - (2\beta - \alpha)c_1}{4\beta - \alpha},$$

$$\lambda_2^{II} = \frac{2\beta + \alpha c_1 - (2\beta - \alpha)c_2}{4\beta - \alpha}$$

so $\lambda_1^{II} = \frac{\lambda_1^{II}}{b\beta}$, $\lambda_2^{II} = \frac{\lambda_2^{II}}{b\alpha}$

$$\pi_1^{II} = \frac{(\lambda_1^{II})^2}{b\alpha}, \pi_2^{II} = \frac{(\lambda_2^{II})^2}{b\beta}$$

From above we know $\frac{\partial \pi_1^{II}}{\partial b} > 0$, $\frac{\partial (p_1^{II} - c)}{\partial b} > 0$, $\frac{\partial \lambda_1^{II}}{\partial b} > 0$, $\frac{\partial \lambda_2^{II}}{\partial b} > 0$, and $\frac{\partial \lambda_1^{II}}{\partial b} > 0$ if $\alpha c_2 < (2\beta - \alpha)c_1$.

$$\frac{\partial \lambda_1^{II}}{\partial b} = \frac{\lambda_1^{II}(2b + \alpha c_2 - \lambda_1^{II})}{b^2} = \frac{\lambda_1^{II}[b + \alpha c_1 + (2\beta - \alpha)c_2]}{(4\beta - \alpha)b^2} > 0$$

if $b + (2\beta - \alpha)c_1 - \alpha c_2 > 0$.

If $\lambda_1^{II}$ increases with $b$ then $\pi_1^{II}$ also increases with $b$. Otherwise $\pi_1^{II}$ can increase with $b$ when $b$ is large.

$$\frac{\partial \pi_1^{II}}{\partial b} = \frac{\lambda_1^{II}(2b + \alpha c_2 - \lambda_1^{II})}{(2\beta - \alpha)b^2} = \frac{\lambda_1^{II}[b + \alpha c_1 + (2\beta - \alpha)c_2]}{(4\beta - \alpha)b^2}$$

and $\frac{2\beta + \alpha c_1 - (2\beta - \alpha)c_2}{b} > 0$ always.

The DI Setting

$$\pi_1^{DI} = \max_{p_1} \frac{(\alpha p_2 - \beta p_1)(p_1 - c_1 - w)}{b}$$

$$\pi_2^{DI} = \max_{p_2} \frac{b - \alpha(p_2 - p_1)(p_2 - c_2)}{b}$$

The first order conditions are

$$\lambda_1 = \alpha p_2 - \beta p_1 = \beta(p_1 - c_1 - w),$$
\[ \lambda_2 = b - \alpha(p_2 - p_1) = \alpha(p_2 - c_2) \]

and

\[ p_1^{DI} = p_1^{II} + \frac{b\delta w_1}{4\beta - \alpha}, p_2^{DI} = p_2^{II} + \frac{b\delta w_1}{4\beta - \alpha} \]

The profit margins and demands are

\[ p_1^{DI} - c_1 = p_1^{II} - c_1 + \frac{b\delta w_1}{4\beta - \alpha}, \]
\[ p_1^{DI} - c_1 - w_1 = p_1^{II} - c_1 - \frac{(2\beta - \alpha)w_1}{4\beta - \alpha} \]
\[ p_2^{DI} - c_2 = p_2^{II} - c_2 + \frac{\delta w_1}{4\beta - \alpha} \]
\[ \lambda_1^{DI} = \lambda_1^{II} - \frac{b(2\beta - \alpha)w_1}{4\beta - \alpha}, \lambda_2^{DI} = \lambda_2^{II} + \frac{\alpha \delta w_1}{4\beta - \alpha} \]
\[ \overline{\lambda_1^{DI}} = \frac{\lambda_1^{DI}}{b}, \overline{\lambda_2^{DI}} = \frac{\lambda_2^{DI}}{b} \]

Then the optimization problem of manufacturer one is,

\[ \pi_1^{M^{DI}} = \frac{1}{b}(\lambda_1^{II} - \frac{b(2\beta - \alpha)w_1}{4\beta - \alpha})w_1 \]

the optimal solution is \( w_1^{DI} = \frac{\lambda_1^{II}(4\beta - \alpha)}{2(2\beta - \alpha)} \)

Observe that the wholesale price increases with \( b \), similarly to \( \lambda_1^{II} \).

Substitute the wholesale price into the demand and profit margin functions,

\[ p_1^{DI} = p_1^{II} + \frac{\lambda_1^{II}}{2\beta}, p_2^{DI} = p_2^{II} + \frac{\lambda_1^{II}}{2(2\beta - \alpha)} \]
\[ p_1^{DI} - c_1 - w_1 = \frac{\lambda_1^{II}}{2\beta}, p_2^{DI} - c_2 = \frac{\lambda_1^{II}}{\alpha} + \frac{\lambda_1^{II}}{2(2\beta - \alpha)} \]
\[ \lambda_1^{DI} = \frac{\lambda_1^{II}}{2\beta}, \lambda_2^{DI} = \frac{\lambda_1^{II}}{\alpha} + \frac{\alpha \lambda_1^{II}}{2(2\beta - \alpha)} \]

We can see now that the retail prices, wholesale price and profit margin of both SCs increase with \( b \).

\[ \overline{\lambda_1^{DI}} = \frac{\lambda_1^{DI}}{b} = \frac{\lambda_1^{II}}{2b}, \overline{\lambda_2^{DI}} = \frac{\lambda_2^{DI}}{b} = \frac{\lambda_1^{II}}{\alpha} + \frac{\alpha \lambda_1^{II}}{2b(2\beta - \alpha)} \]
\[ \frac{\partial \overline{\lambda_1^{DI}}}{\partial b} = \frac{\partial (\lambda_1^{II})}{\partial b} = \frac{1}{2} \left. \frac{\partial \lambda_1^{II}}{\partial b} \right| \]

So \( \frac{\partial \overline{\lambda_1^{DI}}}{\partial b} > 0 \) if \( \alpha c_2 < (2\beta - \alpha)c_1 \), and \( \frac{\partial \overline{\lambda_1^{DI}}}{\partial b} < 0 \) if \( \alpha c_2 > (2\beta - \alpha)c_1 \).

\[ \frac{\partial \overline{\lambda_2^{DI}}}{\partial b} = \left. \frac{\partial (\lambda_1^{II} + \frac{\alpha \lambda_1^{II}}{2(2\beta - \alpha)})}{\partial b} \right| = -\frac{\alpha b c_2 (2\beta - \alpha) + \alpha c_2 [2(2\beta - \alpha)^2 - \alpha b]}{2b^2 (2\beta - \alpha) (4\beta - \alpha)} > 0 \]

So the demand of the SC two in the DI setting increases with the degree of heterogeneity. While the demand of the SC one increases with \( b \) if \( \alpha c_2 < (2\beta - \alpha)c_1 \).
The retailer's optimal profit is

\[ \pi_1^{DI} = \frac{(\lambda^{II})^2}{4\beta\alpha} = \pi_1^{II} \]

Manufacturer's optimal profit is

\[ \pi_1^{MDI} = \frac{(4\beta-\alpha)(\lambda^{II})^2}{4\beta(2\beta-\alpha)} = \pi_1^{II}(4\beta-\alpha) \cdot \frac{4(2\beta-\alpha)}{4\beta(2\beta-\alpha)} \]

The total SC profit is

\[ \pi_1^{D1} = \frac{\pi_1^{II}(3\beta-\alpha)}{2(2\beta-\alpha)} \]

\[ \pi_2^{D1} = \frac{1}{2a}(\lambda_2^{II} + \frac{a\lambda_1^{II}}{2(2\beta-a)})^2 = \frac{\lambda_2^{D1}(4\beta-\alpha)}{4\alpha} \]

\[ \frac{\partial \pi_1^{D1}}{\partial b} = \frac{(3\beta-\alpha)}{2(2\beta-a)} \frac{\partial \pi_1^{II}}{\partial b} > 0 \text{ if } b + (2\beta-\alpha)c_1 - \alpha c_2 > 0. \text{ If } \lambda_1^{D1} \text{ increases with } b \text{ then } \pi_1^{D1} \text{ also increases with } b. \text{ Otherwise } \pi_1^{D1} \text{ can increase with } b \text{ when } b \text{ is large.} \]

\[ \frac{\partial \pi_2^{D1}}{\partial b} = \frac{\lambda_2^{D1} \frac{\partial \lambda_1^{D1}}{\partial b} + \frac{\partial \lambda_2^{D1}}{\partial b}}{\alpha} > 0 \text{ since } \frac{\partial \lambda_2^{D1}}{\partial b} > 0. \text{ So SC two's profit always increases with } b. \]

The ID Setting

\[ \pi_1^{ID} = \max_{p_1} \left( \alpha p_2 - \beta p_1 \right) (p_1 - c_1) \]

\[ \pi_2^{ID} = \max_{p_2} \left( b - \alpha (p_2 - p_1) \right) (p_2 - c_2 - w_2) \]

The first order conditions give us

\[ \lambda_1 = \alpha p_2 - \beta p_1 = \beta(p_1 - c_1) \]

\[ \lambda_2 = b - \alpha(p_2 - p_1) = \alpha(p_2 - c_2) \]

and

\[ p_1^{ID} = p_1^{II} + \frac{\alpha w_2}{4\beta-\alpha}, p_2^{ID} = p_2^{II} + \frac{2\beta w_2}{4\beta-\alpha} \]

The profit margins and demand functions are

\[ p_1^{ID} - c_1 = p_1^{II} - c_1 + \frac{\alpha w_2}{4\beta-\alpha} \]

\[ p_2^{ID} - c_2 = p_2^{II} - c_2 + \frac{2\beta w_2}{4\beta-\alpha} \]

\[ p_2^{ID} - c_2 - w_2 = p_2^{II} - c_2 - \frac{(2\beta-\alpha)w_2}{4\beta-\alpha} \]

\[ \lambda_1^{ID} = \lambda_1^{II} + \frac{\alpha w_2}{4\beta-\alpha}, \lambda_2^{ID} = \lambda_2^{II} - \frac{\alpha(2\beta-\alpha)w_2}{4\beta-\alpha} \]

\[ \frac{\lambda_1^{ID}}{\lambda_1^{ID}} = \frac{\lambda_1^{D}}{\lambda_1^{D}}, \frac{\lambda_2^{ID}}{\lambda_2^{ID}} = \frac{\lambda_2^{D}}{\lambda_2^{D}} \]
Then the optimization problem of manufacturer one is,

$$\pi_2^{MID} = \frac{1}{b} (\lambda_2^I - \alpha \frac{2\beta - \alpha}{4\beta - \alpha} w_2) w_2$$

The optimal wholesale price is $w_2^{IP} = \frac{\lambda_2^I (4\beta - \alpha)}{2\alpha (2\beta - \alpha)}$.

Observe that the wholesale price increases with $b$, similarly to $\lambda_2^I$.

Substitute this into price, demand and profit margin functions,

we have $p_1^{ID} = p_1^{II} + \frac{\lambda_1^I}{2(2\beta - \alpha)}$, $p_2^{ID} = p_2^{II} + \frac{\lambda_2^I \beta}{\alpha (2\beta - \alpha)}$

The profit margins and demands are

$$p_1^{ID} - c_1 = p_1^{II} - c_1 + \frac{\lambda_1^I}{2(2\beta - \alpha)}$$
$$p_2^{ID} - c_2 = p_2^{II} - c_2 + \frac{\lambda_2^I \beta}{\alpha (2\beta - \alpha)}$$
$$p_2^{ID} - c_2 - w_2 = \frac{\lambda_2^I}{2\alpha}$$
$$\lambda_1^{ID} = \lambda_1^{II} + \frac{\beta \lambda_1^I}{2(2\beta - \alpha)}, \lambda_2^{ID} = \frac{\lambda_2^I}{2}$$

We can see that the retail prices and the profit margins are all increasing with $b$.

$$\lambda_1^{ID} = \frac{\lambda_1^{ID}}{b} = \frac{\lambda_1^{II} + \frac{\beta \lambda_1^I}{2b(2\beta - \alpha)}}{b}$$
$$\lambda_2^{ID} = \frac{\lambda_2^{ID}}{b} = \frac{\lambda_2^I}{2b}$$

$$\frac{\partial \lambda_1^{ID}}{\partial b} = \frac{1}{2} \frac{\partial \lambda_1^{ID}}{\partial b} > 0.$$  
$$\frac{\partial \lambda_1^{ID}}{\partial b} = \frac{\partial \left(\frac{\lambda_1^{II}}{b} + \frac{\beta \lambda_1^I}{2b(2\beta - \alpha)}\right)}{\partial b} = \frac{1}{b} \left[\frac{\beta}{4\beta - \alpha} + \frac{2\beta}{4\beta - \alpha} - \frac{\beta}{2(2\beta - \alpha)}\right] b - \lambda_1^{II} - \frac{\beta \lambda_1^I}{2(2\beta - \alpha)}$$

$$= \frac{1}{b} \frac{\beta(2\beta - \alpha) \alpha c_2 + \beta c_1 [2(2\beta - \alpha)^2 - \alpha \beta]}{2(2\beta - \alpha)(4\beta - \alpha)}$$

So $\frac{\partial \lambda_1^{ID}}{\partial b} > 0$ if $c_1 [2(2\beta - \alpha)^2 - \alpha \beta] > (2\beta - \alpha) \alpha c_2$.

$\frac{\partial \lambda_2^{ID}}{\partial b} < 0$ if $c_1 [2(2\beta - \alpha)^2 - \alpha \beta] < (2\beta - \alpha) \alpha c_2$.

The retailer’s optimal profit is

$$\pi_2^{RID} = \frac{(\lambda_1^{II})^2}{4ab} = \frac{\pi_2^{II}}{4}$$

The manufacturer’s optimal profit is

$$\pi_2^{MID} = \frac{(4\beta - \alpha)(\lambda_1^{II})^2}{4ab(2\beta - \alpha)} = \frac{\pi_2^{II} (4\beta - \alpha)}{4(2\beta - \alpha)}$$

The total SC profit is

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\[ \pi^T_2 = \frac{\pi^T_1(3\beta - \alpha)}{2(2\beta - \alpha)} \]

\[ \pi^T_1 = \frac{1}{\beta^2}(\lambda^H_1 + \frac{\beta \lambda^H}{2(2\beta - \alpha)})^2 \]

\[ \frac{\partial \pi^T_1}{\partial b} = \frac{(3\beta - \alpha)}{2(2\beta - \alpha)} \frac{\lambda^H_1}{\beta} > 0. \] SC two’s profit always increases with \( b \).

\[ \frac{\partial \pi^T_1}{\partial b} = \frac{\lambda^H_1(2\beta - \alpha - \alpha^2b - \alpha^2b^2)}{\beta^2} + \lambda^H_1 > 0 \] if \( 2^2 \beta^2 - 9\alpha^2b + 2a^2 > 0 \) \( \lambda^H_1 > 0 \) \( \lambda^H_1 \) increases with \( b \) then \( \pi^T_1 \) also increases with \( b \). Otherwise \( \pi^T_1 \) can increase with \( b \) when \( b \) is large.

**The DD Setting**

\[ \pi^{RD}_1 = \max_{p_1} \left( \frac{(\alpha p_2 - \beta p_1)(p_1 - c_1 - w_1)}{b} \right) \]

\[ \pi^{RD}_2 = \max_{p_2} \left( \frac{b - \alpha(p_2 - p_1)(p_2 - c_2 - w_2)}{b} \right) \]

The first order conditions are

\[ \lambda_1 = \alpha p_2 - \beta p_1 = \beta(p_1 - c_1 - w_1), \]

\[ \lambda_2 = b - \alpha(p_2 - p_1) = \alpha(p_2 - c_2 - w_2) \]

and \( p^{DD}_1 = p^{ID}_1 + \frac{2\beta w_1}{4\beta - \alpha}, p^{DD}_2 = p^{ID}_2 + \frac{2\beta w_2}{4\beta - \alpha} \)

The profit margins and demand functions are

\[ p^{DD}_1 - c_1 = p^{ID}_1 - c_1 + \frac{2\beta w_1}{4\beta - \alpha} \]

\[ p^{DD}_2 - c_2 = p^{ID}_2 - c_2 + \frac{2\beta w_2}{4\beta - \alpha} \]

\[ p^{DD}_1 - c_1 - w_1 = p^{ID}_1 - c_1 - \frac{(2\beta - \alpha)w_1}{4\beta - \alpha} \]

\[ p^{DD}_2 - c_2 - w_2 = p^{ID}_2 - c_2 - \frac{(2\beta - \alpha)w_2}{4\beta - \alpha} \]

\[ \lambda^{DD}_1 = \lambda^{ID}_1 - \frac{(2\beta - \alpha)w_1}{4\beta - \alpha}, \lambda^{DD}_2 = \lambda^{ID}_2 - \frac{(2\beta - \alpha)w_2}{4\beta - \alpha} \]

\[ \lambda^{DD}_1 = \frac{\lambda^{ID}_1}{b}, \lambda^{DD}_2 = \frac{\lambda^{ID}_2}{b} \]

Then the manufacturer’s wholesale pricing game is,

\[ \pi^{MDD}_1 = \max_{w_1} \left( \frac{1}{b} \left( \lambda^{ID}_1 - \frac{(2\beta - \alpha)w_1}{4\beta - \alpha} \right) \right) w_1 \]

\[ \pi^{MDD}_2 = \max_{w_2} \left( \frac{1}{b} \left( \lambda^{ID}_2 - \frac{(2\beta - \alpha)w_2}{4\beta - \alpha} \right) \right) w_2 \]
The wholesale equilibrium is \( w_1^{DD} = \frac{\lambda_1^D(4\beta - \alpha)}{2\alpha(2\beta - \alpha)}|w_2^{PD} \), \( w_2^{DD} = \frac{\lambda_2^D(4\beta - \alpha)}{2\alpha(2\beta - \alpha)}|w_1^{PD} \).

Observe that the wholesale price increases with \( b \), similar to \( \lambda_1^D \) and \( \lambda_2^D \).

Since \( b \) is hidden in \( \lambda_1^D \) and \( \lambda_2^D \), we can derive the same expression as \( \theta \in [0, 1] \).

\[ p_1^{DD} = (p_1^D + \frac{\lambda_1^D}{2\beta - \alpha})|w_2^{PD} \], \( p_2^{DD} = p_2^D + \frac{\beta \lambda_2^D}{(2\beta - \alpha)\alpha}|w_1^{PD} \)

The profit margins and demands are

\[ p_1^{DD} - c_1 - w_1 = \frac{\lambda_1^D}{2\beta} |w_2^{PD} \], \( p_2^{DD} - c_2 - w_2 = \frac{\lambda_2^D}{2\alpha} |w_1^{PD} \)

\[ \lambda_1^{DD} = \frac{\lambda_1^D}{2} |w_2^{PD} \], \( \lambda_2^{DD} = \frac{\lambda_2^D}{2} |w_1^{PD} \)

\[ \overline{\lambda_1^{DD}} = \frac{\lambda_1^D}{b}, \overline{\lambda_2^{DD}} = \frac{\lambda_2^D}{b} \]

Note that all the retail prices and the profit margins and wholesale price increases with \( b \).

The retailers’ optimal profits are

\[ \pi_1^{R^{DD}} = \frac{(\overline{\lambda_1^{DD}})^2}{4\beta b} |w_2^{PD} |, \pi_2^{R^{DD}} = \frac{(\overline{\lambda_2^{DD}})^2}{4\alpha b} |w_1^{PD} | \]

The manufacturers’ optimal profits are

\[ \pi_1^{M^{DD}} = \frac{\pi_1^{D(4\beta - \alpha)}}{4(2\beta - \alpha)} |w_2^{PD} |, \pi_2^{M^{DD}} = \frac{\pi_2^{D(4\beta - \alpha)}}{4(2\beta - \alpha)} |w_1^{PD} | \]

Total SC profits are

\[ \pi_1^{DD} = \frac{\pi_1^{D(3\beta - \alpha)}}{2(2\beta - \alpha)} |w_2^{PD} |, \pi_2^{DD} = \frac{\pi_2^{D(3\beta - \alpha)}}{2(2\beta - \alpha)} |w_1^{PD} | \]

By the same method, solving \( w_1^{PD}, w_2^{PD} \) and substituting them into the demand functions, we have,

\[ \overline{\lambda_1^{DD}} = \frac{\lambda_1^D}{b} = \frac{\lambda_1^D}{2b} |w_2^{PD} | = \frac{\lambda_1^D}{b} \frac{2(2\beta - \alpha)^2}{4(2\beta - \alpha)^2 - \alpha \beta} \]

\[ \overline{\lambda_2^{DD}} = \frac{\lambda_2^D}{b} = \frac{\lambda_2^D}{2b} |w_1^{PD} | = \frac{\lambda_2^D}{b} \frac{2(2\beta - \alpha)^2}{4(2\beta - \alpha)^2 - \alpha \beta} \]

\[ \frac{\partial \overline{\lambda_1^{DD}}}{\partial b} = \frac{\partial \lambda_1^D}{\partial b} \frac{2(2\beta - \alpha)^2}{4(2\beta - \alpha)^2 - \alpha \beta} \]

So \( \frac{\partial \overline{\lambda_1^{DD}}}{\partial b} > 0 \) if \( c_1 [2(2\beta - \alpha)^2 - \alpha \beta] > (2\beta - \alpha)\alpha c_2 \).

And \( \frac{\partial \overline{\lambda_2^{DD}}}{\partial b} = \frac{\partial \lambda_2^D}{\partial b} \frac{2(2\beta - \alpha)^2}{4(2\beta - \alpha)^2 - \alpha \beta} > 0 \) always.

\[ \pi_1^{R^{DD}} = \tau_3 \pi_1^{ID}, \pi_2^{R^{DD}} = \tau_3 \pi_1^{ID} \]
\[ \pi_{1}^{MDD} = \tau_{2} \pi_{1}^{ID}, \pi_{2}^{MDD} = \tau_{2} \pi_{2}^{DI} \]

\[ \pi_{1}^{DD} = \tau_{1} \pi_{1}^{ID}, \pi_{2}^{DD} = \tau_{1} \pi_{2}^{DI} \]

so \[ \frac{\partial \pi_{1}^{DD}}{\partial b} = \tau_{1} \frac{\partial \pi_{1}^{ID}}{\partial b} > 0 \] if \[ \frac{\partial \pi_{1}^{ID}}{\partial b} > 0 \].

\[ \frac{\partial \pi_{2}^{DD}}{\partial b} = \tau_{1} \frac{\partial \pi_{2}^{DI}}{\partial b} > 0 \] always.

So all the derivations of the original model hold quantitatively.

The reason that II and DD can both be Nash SC structures is because II \[ \pi_{1}^{I} > \pi_{1}^{D} \] and II \[ \pi_{2}^{II} > \pi_{2}^{ID} \]. Under certain service levels \( \pi_{1}^{DD} \pi_{1}^{ID} > 0 \), \( \pi_{1}^{DD} \pi_{2}^{ID} > 0 \). Note that these results are always true regardless of the exact value of \( b \). Next we are going to see how \( b \) affects these differences, i.e., if there are other advantages of staying in ID or DI, in which market (in terms of heterogeneity) it is easier to observe deviation from the equilibrium structure?

Since \( \pi_{1}^{II} \pi_{1}^{DI} \pi_{1}^{II} > 0 \), we have \[ \frac{\partial (\pi_{1}^{II} - \pi_{1}^{DI})}{\partial b} = \frac{\beta \alpha}{\beta - 2 \alpha \alpha} \frac{\partial \pi_{1}^{II}}{\partial b} \].

So \( \frac{\partial (\pi_{1}^{II} - \pi_{1}^{DI})}{\partial b} > 0 \) if \( b > \alpha c_{2} - (2 \beta - \alpha) c_{1} \). The same observation as \( \frac{\partial \pi_{1}^{II}}{\partial b} \) applies.

\[ \pi_{2}^{II} \pi_{2}^{ID} \pi_{2}^{II} > 0 \], we have \[ \frac{\partial (\pi_{2}^{II} - \pi_{2}^{ID})}{\partial b} = \frac{\beta \alpha}{\beta - 2 \alpha \alpha} \frac{\partial \pi_{2}^{II}}{\partial b} \].

Since \( \frac{\partial \pi_{2}^{II}}{\partial b} > 0 \) always, \( \frac{\partial (\pi_{2}^{II} - \pi_{2}^{ID})}{\partial b} > 0 \) always. So increasing \( b \) can increase the profit difference of SC two SC two between II and ID.

If DD is a Nash, then \( \tau_{1} > 0 \). Since \( \pi_{1}^{DD} \pi_{1}^{ID} = (\tau_{1} - 1) \pi_{1}^{ID} \), we have \[ \frac{\partial (\pi_{1}^{DD} - \pi_{1}^{ID})}{\partial b} = \frac{\tau_{1} - 1 \pi_{1}^{ID}}{\partial b} \].

The necessary and sufficient condition for \( \frac{\partial (\pi_{1}^{DD} - \pi_{1}^{ID})}{\partial b} > 0 \) is \( \frac{\partial \pi_{1}^{ID}}{\partial b} > 0 \), which is equivalent to \( 2b(3 \beta - \alpha) > \alpha (2 \beta - \alpha) c_{2} - c_{1} (2 \beta - \alpha)^{2} - \alpha \beta \). The same observation as \( \frac{\partial \pi_{1}^{II}}{\partial b} \) applies.

\[ \pi_{2}^{DD} \pi_{2}^{ID} \pi_{2}^{DD} > 0 \] and \[ \frac{\partial (\pi_{2}^{DD} - \pi_{2}^{ID})}{\partial b} = \frac{(\tau_{1} - 1) \pi_{2}^{ID}}{\partial b} \]. Since \( \frac{\partial \pi_{2}^{ID}}{\partial b} > 0 \) always, \( \frac{\partial (\pi_{2}^{DD} - \pi_{2}^{ID})}{\partial b} > 0 \) always. So the difference between \( \pi_{2}^{DD} \pi_{2}^{ID} \) always increases with \( b \).

8. Efficiency loss and gain

\[ u(a_{i}) = \sqrt[3]{a_{i}}, c(a_{i}) = ca_{i} \]
\[
\begin{array}{cccccccccc}
\pi & \pi_1 & \pi_2 & a_1 & a_2 & p_1 & p_2 & \lambda_1 & \lambda_2 & \text{c. w.} & \text{c. w.} \\
\hline
\text{M} & 75.16 & 29.98 & 45.17 & 33.46 & 127.40 & 177.85 & 315.29 & 0.2077 & 0.2404 & 37.58 & 112.74 \\
\text{H} & 47.93 & 19.06 & 28.87 & 24.79 & 137.35 & 73.95 & 217.83 & 0.3877 & 0.3587 & 95.66 & 143.58 \\
\text{DD} & 71.16 & 33.74 & 37.42 & 40.22 & 131.47 & 155.23 & 279.03 & 0.2933 & 0.2536 & 56.58 & 127.74 \\
\end{array}
\]

\[
u(a_i) = \sqrt{a_i}, \ c(a_i) = c a_i
\]

\[
\begin{array}{cccccccccc}
\pi & \pi_1 & \pi_2 & a_1 & a_2 & p_1 & p_2 & \lambda_1 & \lambda_2 & \text{c. w.} & \text{c. w.} \\
\hline
\text{M} & 99.54 & 39.68 & 59.87 & 28.42 & 105.84 & 33.46 & 315.29 & 0.2068 & 0.2674 & 58.67 & 176.01 \\
\text{H} & 58.87 & 21.89 & 36.98 & 15.84 & 115.23 & 70.95 & 207.02 & 0.3969 & 0.4328 & 159.29 & 225.22 \\
\text{DD} & 92.12 & 42.83 & 49.29 & 27.50 & 131.47 & 155.23 & 279.03 & 0.3274 & 0.3036 & 98.47 & 204.95 \\
\end{array}
\]

\[
u(a_i) = \sqrt{a_i}, \ c(a_i) = c a_i
\]

\[
\begin{array}{cccccccccc}
\pi & \pi_1 & \pi_2 & a_1 & a_2 & p_1 & p_2 & \lambda_1 & \lambda_2 & \text{c. w.} & \text{c. w.} \\
\hline
\text{M} & 117.34 & 46.75 & 70.59 & 24.65 & 90.66 & 210.54 & 354.68 & 0.2068 & 0.2674 & 58.67 & 176.01 \\
\text{H} & 65.93 & 22.97 & 42.96 & 10.28 & 98.95 & 68.15 & 198.21 & 0.3969 & 0.4328 & 159.29 & 225.22 \\
\text{DD} & 106.48 & 48.52 & 57.96 & 19.24 & 99.46 & 164.45 & 290.36 & 0.3274 & 0.3036 & 98.47 & 204.95 \\
\end{array}
\]

\[
u(a_i) = \sqrt{a_i}, \ c(a_i) = c a_i
\]

\[
\begin{array}{cccccccccc}
\pi & \pi_1 & \pi_2 & a_1 & a_2 & p_1 & p_2 & \lambda_1 & \lambda_2 & \text{c. w.} & \text{c. w.} \\
\hline
\text{M} & 130.95 & 52.10 & 78.85 & 21.70 & 79.38 & 274.92 & 367.47 & 0.2058 & 0.2737 & 65.48 & 196.43 \\
\text{H} & 71.18 & 23.23 & 47.95 & 6.67 & 86.32 & 65.84 & 191.54 & 0.3926 & 0.4557 & 180.06 & 251.24 \\
\text{DD} & 116.88 & 52.27 & 64.61 & 13.73 & 88.65 & 170.10 & 291.75 & 0.3342 & 0.3181 & 113.16 & 230.04 \\
\end{array}
\]

\[
u(a_i) = \sqrt{a_i}, \ c(a_i) = c a_i
\]

\[
\begin{array}{cccccccccc}
\pi & \pi_1 & \pi_2 & a_1 & a_2 & p_1 & p_2 & \lambda_1 & \lambda_2 & \text{c. w.} & \text{c. w.} \\
\hline
\text{M} & 141.74 & 56.44 & 89.25 & 19.48 & 70.66 & 294.60 & 377.76 & 0.2052 & 0.2777 & 70.87 & 212.61 \\
\text{H} & 75.60 & 23.10 & 52.50 & 4.267 & 76.24 & 64.07 & 187.02 & 0.3863 & 0.4740 & 195.96 & 271.56 \\
\text{DD} & 124.84 & 54.74 & 70.10 & 9.84 & 79.57 & 171.74 & 292.29 & 0.3381 & 0.3296 & 124.94 & 249.78 \\
\end{array}
\]

\[
u(a_i) = \sqrt{a_i}, \ c(a_i) = c a_i
\]
\[
\begin{array}{cccccccccccc}
& \pi & \pi_1 & \pi_2 & \alpha_1 & \alpha_2 & p_1 & p_2 & \lambda_1 & \lambda_2 & \text{c. w.} & \text{t. w.} \\
\text{m} & 150.50 & 59.94 & 90.59 & 17.64 & 63.72 & 310.66 & 386.27 & 0.2045 & 0.2808 & 75.26 & 225.78 \\
\text{II} & 79.69 & 22.79 & 56.90 & 2.67 & 67.92 & 62.80 & 184.22 & 0.3791 & 0.4892 & 208.17 & 287.86 \\
\text{DD} & 131.16 & 56.47 & 74.69 & 7.12 & 72.23 & 173.06 & 292.81 & 0.3403 & 0.3386 & 134.58 & 265.74 \\
\end{array}
\]

\[
u(a_i) = \sqrt{\alpha_i}, \quad c(a_i) = c\alpha_i
\]

\[
\begin{array}{cccccccccccc}
& \pi & \pi_1 & \pi_2 & \alpha_1 & \alpha_2 & p_1 & p_2 & \lambda_1 & \lambda_2 & \text{c. w.} & \text{t. w.} \\
\text{m} & 157.84 & 62.84 & 95.00 & 16.12 & 58.05 & 324.13 & 393.46 & 0.2040 & 0.2832 & 78.92 & 236.76 \\
\text{II} & 83.75 & 22.38 & 61.37 & 1.61 & 60.79 & 61.92 & 182.96 & 0.3712 & 0.5023 & 217.38 & 301.13 \\
\text{DD} & 136.40 & 57.60 & 78.80 & 5.12 & 65.63 & 173.98 & 293.20 & 0.3411 & 0.3463 & 142.40 & 278.80 \\
\end{array}
\]
Strategic Revenue Management under Price and Seat Inventory Competition

Abstract: this chapter explores the classical revenue management problem in a competitive context, with both price and seat inventory competition. We question: how should management make strategic marketing (pricing) and operational (seat allocation) decisions in such a competitive market? Do the conventional approaches (models and algorithms based on monopoly market) give us the appropriate strategies? We apply a noncooperative game theoretic methodology (usually, supermodular game). We find that in a market where price competition dominates, managers should set a lower retail price and safety protection level for full fare customers than in a monopoly or alliance market. In a market that seat inventory competition dominates, managers should set a higher retail price and safety protection level than a monopoly or alliance would. Interestingly, in a market where the two levels of competition are more evenly matched, managers should set a lower price and a higher safety protection level than a monopoly. Along with these main questions, we also explore other questions such as how the degree of competition and the market structure affects the strategic decisions, whether there is a first adopter advantage or a second adopter disadvantage with revenue management.

3.1 Introduction

The deregulation of the airline industry in 1978 brought increased competition and new challenges. This has encouraged innovation across all aspects of airline operations. One such innovation has been the widespread adoption of Yield Management, recently known by the more general name of Revenue Management (RM). RM focuses on both the strategic and tactical management of the allocation of aircraft seat capacity among different
fare classes to enhance revenue. Thus in order to offer coach class seats at a considerable
discount without the lose of those passengers prepared to pay a full fare, restrictions to
access the discount fare are introduced that segment the market. Common segmenting
techniques have been non-refundable tickets and availability windows such as requiring
a Saturday night; but all such techniques require that a certain number of seats be
reserved’ or ‘protected’ for those passengers prepared to pay a full price. These two de-
cisions, the number of seats ‘protected’ and the price at which they are offered form the
context and substance of this paper. RM has stimulated considerable academic research
interest (see McGill and van Ryzin 1999 for an extensive review of the literature). Most
of the literature to date assumes a monopoly market, in large part ignoring the impact
of competition. However, to study a major product of deregulation such as Revenue
Management, in isolation from the main driver of deregulation, that is competition; is
like studying fish out of water: there is much to study but the context is largely missing.
Competition here encompasses both price competition and also seat inventory supply
competition, whereby each airline reacts to the overflow of passengers from competitors.
As an example, take the direct flight from Vancouver, BC to Hong Kong. An online ticket
agency (http://www.travelocity.com/ for instance), will ask whether your time is flexible
or fixed. If you answer ‘flexible’, you will find many carriers (in this case 14) offering
all kinds of prices and time restriction combinations. If you specify a date (Aug. 19-21,
2002 say), you will find fewer carriers (in this case only Air Canada and Cathay Pacific)
offering the tickets with much higher prices. The airline market is competing both in
terms of prices, and seat protection levels.

This chapter investigates the impact of price and seat inventory competition on Rev-
enue Management. We ask: how should competitors in this market make pricing and
protection level decisions? How do these decisions differ from the decisions made by a
monopoly or an alliance? How does the degree of competition affect the results? How do the results differ when either price or seat inventory competition alone dominates? What is the effect of market structure? What could we say about consumers' welfare? Is there a first mover advantage or a second mover disadvantage in implementing RM?

Although RM with both price and seat inventory competition has been little explored, some work has been done considering RM under only seat inventory competition among carriers with fixed fares for each class. Belobaba and Wilson (1997) used a simulation approach to analyze a single non-stop market with just two airlines competing. They find that there is always a "first mover" advantage for the airline that initiates revenue management (that is, sets a fare class booking limit) in a competitive market; and the airline without RM can be hurt by this. When both carriers use a RM technique, both of them see a revenue increase. Li and Oum (1999) and Netessine and Shumsky (2001) analyze seat allocation problems with only seat inventory competition between two airlines with two fare classes. They find that under competition (duopoly), more seats are protected for higher-fare passengers than under a monopoly. Netessine and Shumsky use a very general model and consider various cases of consumer overflow or spill. They show in some cases a pure strategic equilibrium exists (or is unique). They also use numerical examples to study the effect of some factors such as the ratio of high fare to low fare and demand correlation on the booking limits of duopoly and monopoly. There is also a stream of literature on competitive inventory management. The competition arises from consumers switching among firms when product is not available. This 'newsvendor game', as it has been called, is studied by Parlar (1988), Lippman and McCardle (1997), Mahajan and van Ryzin (1999), etc. The main result is that competition leads to a higher industry inventories.

In contrast, this paper incorporates pricing and price competition into a classical seat
allocation model. The demand distribution of a carrier is affected by the selling prices and seat inventory decisions of all the carriers. Carriers need to consider the effect of any passenger overflow or spill when making decisions on both price and protection levels. Passenger overflow or ‘spill’ refers to that proportion of unsatisfied demand from one airline that will seek a seat on a competitive airline in the same fare class. Over the years, the level of sophistication both in practice and the literature about the classical seat allocation has grown considerably. As we widen the scope of study considerably by adding both seat and price competition effects, we return to a simplified model, which was first introduced by Littlewood\(^4\) (1972), in order to be able to gain some managerial insights without suffering from intractable mathematics. We consider just two airlines offering two direct flights which are substitutable in the eyes of consumers. Protection levels and prices are set once only with no dynamic updating or revision. There are two fare classes for otherwise identical seats and service; a lower class (discount) and a higher class (full fare). We assume the price of the lower class to be exogenous at \(p_0\) and that the airlines can sell all the seats at this price. This assumption could reflect early discount booking; low fare customers are accepted until the booking limit is reached. The model is an extension of Littlewood (1972) in that we incorporate both price and inventory competition for high fare class. Our two airlines need to set protection levels and prices for the full fare. The demand for this class is random and is influenced by the prices of both airlines and by the potential ‘spill’ from the competitor. There is no spill between classes, either down or up grading. Unused full fare seats that had been protected have a zero revenue.

Two factors control the ‘degree’ of competition. Passengers treat the two products as

\(^4\)He proposed that an airline should continue to sell discount seat as long as the following condition held: 
\[ r \geq R(1 - P) \]
Where \(r\) (\(R\)) is the mean revenue from low (high) yield passenger, \(P\) is the maximum risk that the acceptance of a low yield passenger will result in the subsequent rejection of a high yield passenger.
partially substitutable, reflected in the substitution factor $\theta$. In addition a fraction ($\alpha_j$) of passengers unable to purchase a full fare on carrier $j$ of their first choice may switch to another airline if seats are available. Like most static yield management models (Brumelle et al. 1990), we assume the following: 1. We ignore dynamic effects and treat discount sales 'as if' they were made at a single time. 2. Discount class demands occur before full fare class demands. 3. There are no cancellations.

Conventional economic sense tells us that with price competition only, the price in a duopoly market should be lower than with a monopoly. Recent Revenue management literature with passenger overflow gives us the result that more seats are protected for high fare passenger under duopoly. We find in this paper that those results only tell part of the story. In a market with both price and seat inventory competition (consumer overflow), both price and seat inventory level for the random demand (safety protection level) should be lower under a duopoly than a monopoly if price competition dominates; both price and safety protection level should be higher under a duopoly than a monopoly if seat inventory competition dominates; in a market where both levels of competition coexist and are without much difference, strategic decisions should have a lower price and higher safety protection level. Additional findings are:

1. A higher equilibrium price implies a higher safety protection level but a lower total protection level. A duopoly market ends up with a higher total protection level than a monopoly. 2. Equilibrium strategies (both price and safety protection levels) decrease with the substitution factor $\theta$, but increase with the spill rate $\alpha$. Total protection levels increase with both. 3. In a market with only price competition, equilibrium strategies decrease with the number of firms. This result is reversed in a market with only seat inventory competition. The two effects mitigate each other with both levels of competition. 4. There is first mover advantage in implementing RM, but the second
In addition to contributing to the Revenue Management literature, this chapter will also assist research incorporating inventory and price competition into classical inventory models. The chapter commences in the simplest case and progresses to the more complex.

3.2 General Model

Carriers in this duopoly market engage in a pricing game and a seat allocation game simultaneously. Their common problem is to set price $p_i \geq p_0$ and total protection level $y_i \leq c_i$ for full fare seats to maximize revenue, where the total capacity of flight $i$ is $c_i$.

We define the demand for full fare seats as $\tilde{D}_i$ and the total demand after a spill from airline $j$ to airline $i$ as $\tilde{D}_i = \tilde{D}_i + \alpha_j(\tilde{D}_j - y_j)^+$ where $0 \leq \alpha_j \leq 1$.

In addition we define $\tilde{D}_i = \tilde{D}_i^0 + L_i$, where $L_i = L_i(p_i, p_j)$. $\tilde{D}_i^0$ is the random component of demand without the pricing factor, and we assume $\tilde{D}_i^0 \geq 0$.

We define $z_i$ by $y_i = z_i + L_i$, a transformation of the decision variable $y_i$ representing seats reserved for the random component of demand; in inventory terms, 'safety stock'. We call $z_i$ the safety protection level.

Defining $\tilde{D}_i^{0T} = \tilde{D}_i^0 + \alpha_j(\tilde{D}_j^0 - z_j)^+$ helps to simplify the notation. As we make many use of supermodularity, some principle results are cited in the appendix for reference. Any reference such as to Lemma A1 refers to the appendix at the end of the dissertation. We also give the derivation of some differentiation formulas in the appendix of this chapter.

We shall use $f_X(x)$ as the density function of any random variable and assume $f_X(x) > 0$ and finite for $0 \leq x \leq c_i$. If we assume sufficient differentiability conditions on $f(x)$, we can ensure the requirement of theorems (Milgrom and Roberts 1990) in the appendix. Instead of using conditions required by Milgrom and Roberts, the supermodularity results can be derived from weaker conditions and the work of Topkis (1998). We use both
approaches in establishing supermodularity. $r_X(x) = f_X(x)/\Pr(X > x)$ is the failure rate function and it is convenient to assume this increasing, that is either the marginal distribution $\tilde{D}_i^0$ or $\tilde{D}_i^{OT}$ have IFR.

We introduce other notation when necessary.

The timing of events is as follows: 1. Each airline simultaneously decides on the price and the protection level for full fare customers, 2. unprotected seats sell out fully at the discount fare. 3. The full fare demands are realized, and revenues are collected.

The Nash equilibrium is main the solution concept.

The objective function of carrier $i$ is, $E \pi_i = (c_i - y_i)p_0 + p_iE \min\{\tilde{D}_i^{yr}, y_i\}$. After transformation $E \pi_i = (c_i - L_i - z_i)p_0 + p_i(L_i + E \min\{\tilde{D}_i^0 + \alpha_j(\tilde{D}_j^0 - z_j)^+, z_i\})$

Using models common in the economics literature (Shubik and Levitan 1980), we take $L_i = a - (b + \theta)p_i + \theta p_j$. In the conclusion of the paper, we mention an alternate pricing model. We make no assumptions about independence of $(\tilde{D}_1^0, \tilde{D}_2^0)$.

### 3.3 RM without Competition

We start by considering a simple and somewhat extreme case of RM with a price sensitive demand but with no competition. i.e., $\alpha_j = 0, \theta = 0$. Each carrier solves the joint optimization problem as if it were a local monopoly. We show that because each carrier’s problem is supermodular an equilibrium exists and is the same as for a monopoly. The carrier $i$’s problem is,

$$\max_{z_i, p_i} E \pi_i = \max_{z_i, p_i} (c_i - a + bp_i - z_i)p_0 + p_i(a - bp_i + E \min\{\tilde{D}_i^0, z_i\})$$

We have the following results,

3-1. $E \pi_i$ is a supermodular function in $(p_i, z_i)$ for $i = 1, 2$. This is because of Lemma A1-2 and,

$$\frac{\partial E \pi_i}{\partial z_i} = -p_0 + p_i \Pr(\tilde{D}_i^0 > z_i), \frac{\partial^2 E \pi_i}{\partial z_i \partial p_i} = \Pr(\tilde{D}_i^0 > z_i) \geq 0.$$
An optimal price and protection level exists (Lemma A3) (supermodular is not needed for the existence here actually) and are characterized by the following equations (we assume the solutions are interior):

\[-p_0 + p_i \Pr(\tilde{D}_i^0 > z_i) = 0\]
\[b p_0 + a - 2 b p_i + E \min\{\tilde{D}_i^0, z_i\} = 0\]

Alternatively, by Lemma A2-1 and A2-3, even without the differentiability, the function is also supermodular in \((p_i, z_i)\). The function includes three terms: \((c_i - a + b p_i)p_0 + p_i(a - b p_i)\) only has \(p_i\), so it is supermodular in \(p_i\) (Lemma A2-1); \(-z_i p_0\) only includes \(z_i\), so it is supermodular in \(z_i\) (Lemma A2-1); \(p_i E \min\{\tilde{D}_i^0, z_i\}\) includes both \(p_i\) and \(z_i\), and is the product of two parts, each increases with \(p_i\) or \(z_i\), according to Lemma A2-3, it is supermodular in \((p_i, z_i)\). Finally, the sum of supermodular function is still supermodular.

3-2. The Hessian matrix is,

\[
\begin{vmatrix}
-2b & \Pr(\tilde{D}_i^0 > z_i) \\
\Pr(\tilde{D}_i^0 > z_i) & -p_i f_{\tilde{D}_i^0}(z_i)
\end{vmatrix}
\]

If \(2bp_i f_{\tilde{D}_i^0}(z_i) - [\Pr(\tilde{D}_i^0 > z_i)]^2 = f_{\tilde{D}_i^0}(z_i)[2bp_i - \Pr(\tilde{D}_i^0 > z_i)] > 0\), (where \(r_{\tilde{D}_i^0}(z_i) = f_{\tilde{D}_i^0}(z_i)/\Pr(\tilde{D}_i^0 > z_i)\) is the failure rate), then the optimal is unique. Actually, as explained in Proposition 3.2, with an increasing failure rate, the optimal will be unique. Petruzzi and Dada (1999) study this monopoly problem sequentially rather than simultaneously optimizing \(p\) and \(z\). They show that under certain assumptions on the distribution function, the optimal price and inventory are unique.

**Comparison with a Monopoly or an Alliance**

If the two airlines are operated by a monopoly or they form an alliance, the problem becomes,

\[
\max_{z_i, p_i} E \pi = \max_{z_i, p_i} \sum_{i=1,2} [(c_i - a + b p_i - z_i)p_0 + p_i(a - b p_i + E \min\{\tilde{D}_i^0, z_i\})]
\]

The optimal solutions (if interior) are not surprisingly the same as the duopoly case.
Example 3.1 Assume $D_i$ is an exponential r.v. with parameter $\lambda = 1$. Then $E \min\{D_i, z_i\} = \int_0^{z_i} F(x) dx = 1 - \exp(-z_i)$, and $p_i = \frac{a + bp_i + 1 - \exp(-z_i)}{2b}$, $\exp(-z_i) = p_0/p_i$, $E\pi_i = (c_i - a + bp_i - z_i)p_0 + p_i(a - bp_i + 1 - \exp(-z_i))$

If $a = 2, b = 0.8, p_0 = 1, c_i = 2$, then for both duopoly and monopoly $z_i = 0.729, p_i = 2.074, E\pi_i = 2.711$. $E\pi = 5.422$.

3.4 RM with Only Price Competition

We consider the case with $\theta > 0$, $\alpha_i = 0$. This describes a market where demand is sensitive to the competitor's price, $L_i = a - (b + \theta)p_i + \theta p_j$, with a degree of substitution reflected in $\theta$. By supermodularity we shall demonstrate the existence of an equilibrium, obtain sufficient conditions for uniqueness, show that both price and protection levels for random demand ($z_i$) decrease with the competition $\theta$ and both are lower than for a monopoly, but the total protection level ($y_i$) is opposite.

The simultaneous game is,

$$\max_{p_i, z_i} E\pi_i$$

$$E\pi_i = (c_i - L_i - z_i)p_0 + p_i(L_i + E \min\{D_i^0, z_i\}) \quad \text{(3.1)}$$

where $L_i = a - (b + \theta)p_i + \theta p_j$.

4-1. Function (3.1) is supermodular in $(p_i, z_i)$ for $i = 1, 2$, by Lemma A1-2 because

$$\frac{\partial^2 E\pi_i}{\partial z_i} = -p_0 + p_i \Pr(D_i^0 > z_i), \quad \frac{\partial^2 E\pi_i}{\partial z_i \partial p_i} = \Pr(D_i^0 > z_i) \geq 0.$$

Applying Lemma A2-1, A2-3, and Using that the sum of supermodular function is supermodular, we have the same result.

4-2. In addition the game itself is a supermodular game. According to Definition A3, we need to prove increasing differences in $(p_i, p_j)$, $(z_i, p_j)$, $(p_i, z_j)$ and $(z_i, z_j)$. This is true because

$$\frac{\partial^2 E\pi_i}{\partial z_i \partial p_j} = 0, \quad \frac{\partial^2 E\pi_i}{\partial p_i \partial p_j} = \theta > 0, \quad \frac{\partial^2 E\pi_i}{\partial z_i \partial z_j} = 0, \quad \frac{\partial^2 E\pi_i}{\partial p_i \partial z_j} = 0.$$
Applying Lemma A2-1, A2-3, and A1-1, we can find increasing difference of the function in \((p_i, p_j), (z_i, p_j), (p_i, z_j)\) and \((z_i, z_j)\) without differentiability.

**Proposition 3.1** There exists a pure strategy equilibrium for this price competition only game. If interior, the equilibria are characterized by the first order conditions:

\[
-p_0 + p_i \Pr(\tilde{D}_{i}^{0} > z_i) = 0 \tag{3.2}
\]

\[
(b + \theta)p_0 + a - 2(b + \theta)p_i + \theta p_j + E \min \{\tilde{D}_{i}^{0}, z_i\} = 0 \tag{3.3}
\]

**Proof.** The above results establish the supermodularity of the game. By Lemma A4, there exist a pure strategy equilibrium. \(\blacksquare\)

So an equilibrium exists, but is it unique?

**Proposition 3.2** The equilibrium is unique.

**Proof.** Step 1. First we show that the best response, the solution of (3.2)-(3.3), is unique given any \(p_j\), and satisfies \(2(b + \theta) - \Pr(\tilde{D}_{i}^{0} > z_i)/[p_i r\tilde{D}_{i}^{0}(z_i)] = M(p_i) > 0\).

From (3.3) define \(J(p_i) = (b + \theta)p_0 + a - 2(b + \theta)p_i + \theta p_j + E \min \{\tilde{D}_{i}^{0}, z_i(p_i)\}\), where \(z_i(p_i)\) is uniquely solved from (3.2). Note that \(dJ(p_i)/dp_i = -M(p_i)\).

So \(dJ(p_i)/dp_i\) is decreasing in \(p_i\) and \(\lim_{p_i \to -\infty} dJ(p_i)/dp_i < 0\). \(J(p_i)\) starts positive, possibly increasing but then decreasing until \(J(p_i)\) is negative with a single solution for \(J(p_i^*) = 0\) and we must have \(dJ(p_i^*)/dp_i < 0\).

Step 2. Given that the best response is unique, we prove a unique equilibrium.

The best response function of player \(i\) to any \(p_j\) satisfies \(J(p_i) = 0\), i.e.,

\[
p_i = \frac{a + (b + \theta)p_0 + \theta p_j + E \min \{\tilde{D}_{i}^{0}, z_i(p_i)\}}{2(b + \theta)}
\]

In the \((p_i, p_j)\) quadrant, when \(p_i = p_0\), \(p_j = [(b + \theta)p_0 - a - E \min \{\tilde{D}_{i}^{0}, z_i\}] / \theta < 0\), the curve has a slope \(dp_j/dp_i = M(p_i)/\theta > 0\), which is increasing. So this curve (c1) is increasing and convex.
Similarly, the best response of player $j$ to any $p_i$ satisfies $p_j = \frac{a + (b + \theta)p_0 + \theta p_i + E \min\{\tilde{D}_{j, i}, z_j\}}{2(b + \theta)}$, when $p_j = p_0$, $p_i = [(b + \theta)p_0 - a - E \min\{\tilde{D}_{i, j}, z_j\}] / \theta < 0$. This curve (c2) has a decreasing and positive slope, $dp_j / dp_i = \theta / M(p_j)$. So it is increasing and concave.

The difference between c2 and c1 is also concave, starting positive and with a negative slope $\theta / 2(b + \theta) - 2(b + \theta) / \theta < 0$ as $p_i$ approaches infinity, so there exist a unique intersection between c2 and c1. Thus the equilibrium is unique.

**Equilibrium Properties**

Letting the equilibrium be $(z^*_i, z^*_j, p^*_i, p^*_j)$ we explore some of its properties.

**Proposition 3.3** 1. In equilibrium the safety protection level increases with the equilibrium price. 2. Carriers reserve a lower safety protection level and a lower price if the competition becomes fiercer. 3. In symmetric equilibrium, $dE_{\pi_i} / d\theta < 0$.

**Proof.**

1. From function (3.2), in equilibrium we have

$$
\frac{dz_i}{dp_i} = \frac{1}{p_i r_{z_i}(z_i)} = \frac{p_i (p_i r_{z_i}(z_i) > 0)}{p_i r_{z_i}(z_i)} 
$$

2. $E_{\pi_i}$ has increasing differences in $(z_i, -\theta)$ and $(p_i, -\theta)$ for $i = 1, 2$. Because

$$
\partial^2 E_{\pi_i} / \partial z_i \partial \theta = 0, \quad \partial^2 E_{\pi_i} / \partial p_i \partial \theta = p_j + p_0 - 2p_i = p_0 - p_i < 0 \quad \text{(if symmetric then} \quad p_j = p_i). \quad \text{According to Lemma A5, the equilibrium point increases with} -\theta, \quad \text{i.e., decreases with} \ \theta.
$$

3. $\frac{dE_{z_i}}{d\theta} = \frac{\partial E_{z_i}}{\partial p_j} \frac{dp_j}{d\theta} + \frac{\partial E_{z_i}}{\partial \theta} = \theta (p^*_i - p_0) \frac{dp_j}{d\theta} + (p^*_i - p_0)(p^*_j - p^*_i) < 0$. ■

The above results are intuitive. The seats protected for random demand always increase with the price. Both $p_i$ and $z_i$ decrease when competition is fiercer, and strong price competition damages the profits. How about the total protection level $y_i$?

**Proposition 3.4** 1. The total protection level decreases with the equilibrium price. 2. For identical carriers $dy^*_i / d\theta > 0$.

**Proof.** Given $p^*_i$, as $y^*_i = z^*_i + L^*_i$, $dy^*_i / dp^*_i = dz^*_i / dp^*_i - (b + \theta) = 1 / p_i r_{z_i}(z_i^*) - (b + \theta)$. 68
Define \( \overline{p}_i : 1/(b+\theta) = \overline{p}_i r_{\overline{p}_i}(z_i(\overline{p}_i)) \) where \( z_i(\overline{p}_i) \) is solved from (3.2), then \( d y^*_i / d p^*_i < 0 \) if \( p^*_i > \overline{p}_i \).

Given \( p^*_i \) define \( J (p_i) \) as in Proposition 3.2, and recall that \( dJ (p_i)/dp_i \) is decreasing in \( p_i \) and \( \lim_{p_i \to \infty} dJ (p_i)/dp_i = -2(b + \theta) < 0 \), with \( J (p_i) \) starting positive, possibly increasing but then decreasing until a unique \( J (p^*_i) = 0 \). So it is sufficient to show \( J (\overline{p}_i) > 0 \).

\[
J (\overline{p}_i) = [a - (b + \theta) \overline{p}_i + \theta p^*_i] + E \min\{\overline{D}^0_i, z_i(\overline{p}_i)\} - (b + \theta) \overline{p}_i[1 - Pr(\overline{D}^0_i > z_i(\overline{p}_i))].
\]

The first term is positive for a feasible \( \overline{p}_i \), and \( E \min\{\overline{D}^0_i, z_i(\overline{p}_i)\} \geq (b + \theta) \overline{p}_i(1 - Pr(\overline{D}^0_i > z_i(\overline{p}_i))) \).

For any IFR with \( r_{\overline{p}_i}(0) > 0 \) this holds as the LHS increases faster from \( z_i = 0 \) than the RHS. So \( \overline{p}_i < p^*_i \) and we have the required result.

2. We have \( dp^*_i / d\theta = p^*_i r_{p^*_i}(z^*_i) d z^*_i / d\theta \). With symmetric equilibrium \( p^*_i = p^*_j \) and

\[
\frac{dz^*_i}{d\theta} = \frac{d}{d\theta} - (b + \theta) \frac{dp^*_i}{d\theta} + \theta \frac{dp^*_i}{d\theta} = \frac{dz^*_i}{d\theta} (1 - b p^*_i r_{p^*_i}(z^*_i))
\]

Define \( \widehat{p}_i : 1 - b \widehat{p}_i r_{p^*_i}(z_i(\overline{p}_i)) = 0 \). Since \( dz^*_i / d\theta < 0 \) from Proposition 3.3, \( dy^*_i / d\theta > 0 \) if \( p^*_i > \widehat{p}_i \).

To show that \( p^*_i > \widehat{p}_i \) we use a similar methodology as above. A sufficient condition for \( J (\overline{p}_i) > 0 \) is,

\[
E \min\{\overline{D}^0_i, z_i(\overline{p}_i)\} > (b + \theta) \overline{p}_i \geq b \overline{p}_i = r_{p^*_i}(z_i(\overline{p}_i)).
\]

Two points for this Proposition. A higher equilibrium price 'signal' a lower total inventory. A market with fiercer price competition induces carriers to stock more in total.

**Comparison with Monopoly or Alliance**

What is the effect of collusion, where carriers make joint decisions on prices and protection levels? Do government regulators need to prohibit such collusion? The joint decision problem becomes
\[
\max E\pi = \max \sum_{i=1,2} \left[ (c_i - L_i - z_i)p_i + p_i(L_i + E\min\{\tilde{D}^0_i, z_i\}) \right].
\]

4-3. Because \(\frac{\partial^2 E\pi}{\partial x_1 \partial x_2} \geq 0\), where \(x_1, x_2 = p_1, p_2, z_1, z_2\) and \(x_1 \neq x_2\), the function is supermodular in \((p_1, p_2, z_1, z_2)\) (Lemma A1-2) and an optimal solution exists (Lemma A3). Interior optimal solutions are characterized by,
\[
p_0 = p_i \Pr(\tilde{D}^0_i > z_i)
\]
\[
(b + \theta)p_0 + a - 2(b + \theta)p_i + \theta p_j + E\min\{\tilde{D}^0_i, z_i\} + \theta(p_j - p_0) = 0
\]

By Lemma A2-1, A2-3, we also have the supermodularity without differentiability.

Comparing these with functions (3.2), (3.3), we have

**Proposition 3.5** In a market with only price competition, duopoly airlines charge a lower price and reserve a lower capacity for the random demand of full fare passengers than a monopoly or alliance would. But a consequence of Proposition 3.4 is that the total protection level in monopoly is lower.

**Proof.** From (3.2) and (3.3), for any given \(p_j\), a duopoly will set the optimal price by solving the following function
\[
p_i = \frac{a + (b + \theta)p_0 + \theta p_j + E\min\{\tilde{D}^0_i, z_i(p_i)\}}{2(b + \theta)} \quad \text{or} \quad 2(b + \theta)p_i - E\min\{\tilde{D}^0_i, z_i(p_i)\} = a + (b + \theta)p_0 + \theta p_j
\]

A monopoly will determine the optimal price for \(i\) from
\[
p_i = \frac{a + (b + \theta)p_0 + \theta p_j + E\min\{\tilde{D}^0_i, z_i(p_i)\} + \theta(p_j - p_0)}{2(b + \theta)} \quad \text{or} \quad 2(b + \theta)p_i - E\min\{\tilde{D}^0_i, z_i(p_i)\} = a + (b + \theta)p_0 + \theta p_j + \theta(p_j - p_0).
\]

Note that the LHS of both equations increases with \(p_i\). The monopoly’s equation has a larger RHS than the duopoly one. We then conclude that given any \(p_j\), \(p_i^M(p_j) > p_i^D(p_j)\). Similarly \(p_j^M(p_i) > p_j^D(p_i)\). All of them are up-ward sloping (e.g., \(p_i^M(p_j)/dp_j > 0\)), so the intersection of \(p_i^M(p_j)\) and \(p_j^M(p_i)\) is larger than that of \(p_i^D(p_j)\) and \(p_j^D(p_i)\).

In proposition 3.4 we have shown that \(y_i\) increases with \(\theta\). Since the total protection level under monopoly is equivalent to a duopoly with \(\theta = 0\) (no competition), we can
conclude that the total protection level in the duopoly is higher than that in the monopoly.

Although full fare prices and safety protection levels are lower in the duopoly market, the total protection levels are not. So low fare customers are worse off under duopoly because they are left with fewer seats, but full fare customers will be happier because they pay less and more seats are reserved for them in total. This may help government regulators; they can protect the welfare of certain group of customers by allowing or preventing collusion.

**Example 3.2** Let $D^0_i$ be exponentially distributed with $\lambda_i = 1$, then the symmetric duopoly equilibrium has, $p = \frac{a + (b + \theta)p_0 + 1 - \exp(-z)}{(2b + \theta)}$, $\exp(-z) = p_0/p$, and $E\pi = \frac{(c - a + bp - z)p_0 + p(a - bp + 1 - \exp(-z))}{2b}$.

The monopoly has an optimal symmetric solution $p = \frac{a + b}{{p}_0 + 1 - \exp(-z)}$, $\exp(-z) = p_0/p$. Figure 3.5 illustrates Propositions 3.3-3.5.

### 3.5 RM with Only Seat Inventory Competition

In this section we analyze the case with $\alpha_i > 0$, $\theta = 0$. This represents a market where the demand for one airline is not sensitive to the other airline’s price, $L_i = a - bp_i$, but consumers would like to change to another airline when facing a stockout in their full fare class. Using a supermodular game methodology, we will show the existence of an equilibrium, obtain sufficient conditions for uniqueness, and prove that both the price and safety seat protection level increases with the spill rate from their competitor. Prices and safety protection levels are also higher than for a monopoly or an alliance.

Two carriers set price and protection levels simultaneously,

$$\max_{p_i, z_i} E\pi_i$$
\[ \pi_i = (c_i - L_i - z_i)p_0 + p_i(L_i + E \min\{\tilde{D}_i^0 + \alpha_j(\tilde{D}_j^0 - z_j)^+, z_i\}) \quad (3.4) \]

where \( L_i = a - bp_i \).

We have the following results:

5-1. Function (3.4) is supermodular in \((z_i, p_i)\). This follows from applying Lemma A1-2, and

\[ \frac{\partial \pi_i}{\partial z_i} = -p_0 + p_i \Pr(\tilde{D}_i^{0\uparrow} > z_i), \text{ recall } \tilde{D}_i^{0\uparrow} = \tilde{D}_i^0 + \alpha_j(\tilde{D}_j^0 - z_j)^+. \]

\[ \frac{\partial^2 \pi_i}{\partial z_i \partial p_i} = \Pr(\tilde{D}_i^{0\uparrow} > z_i). \]

Or directly from Lemma A2-3, A2-1 and the sum of supermodular function is also supermodular.

5-2. Function (3.4) is supermodular in \((z_i, -z_j)\). This function includes two parts, the first part is a real-valued function in \(z_i\) which is in the subset of \(R^1\), so it is supermodular in \(z_i\) (Lemma A2-1). Since \(\tilde{D}_i^0 + \alpha_j(\tilde{D}_j^0 - z_j)^+\) is increasing on \(-z_j, z_i\), \(z_i\) is increasing in \(z_i\), the second part is supermodular in \((z_i, -z_j)\) (Lemma A2-2). Since the sum of supermodular functions is supermodular, function (3.4) is supermodular in \((z_i, -z_j)\), which is equivalent to having increasing differences (Lemma A1-1). Alternatively, \( \frac{\partial^2 \pi_i}{\partial z_i \partial z_j} = \alpha_j p_i f\tilde{D}_j^{0\uparrow}|_{\tilde{D}_j^0 > z_j}(z_i) \Pr(\tilde{D}_j^0 > z_j) \geq 0. \)

5-3. Function (3.4) has increasing differences in \((p_i, -z_j)\), \((p_i, -p_j)\) and \((z_i, -p_j)\).

Because

\[ \frac{\partial \pi_i}{\partial p_i} = bp_0 + a - 2bp_i + E \min\{\tilde{D}_i^0 + \alpha_j(\tilde{D}_j^0 - z_j)^+, z_i\}, \]

\[ -\frac{\partial^2 \pi_i}{\partial p_i \partial z_j} = \alpha_j \Pr(\tilde{D}_i^{0\uparrow} < z_i, \tilde{D}_j^0 > z_j) \geq 0, \]

\[ -\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} = 0, -\frac{\partial^2 \pi_i}{\partial z_i \partial p_j} = 0. \]

This can also be pursued by Lemma A2-3, A2-1, preservation of supermodularity upon summation and lemma A1-1.

So the game is supermodular.
5-4. Function (3.4) is also supermodular (equivalent to having increasing differences according to Lemma A1-2) in \((z_i, \alpha_j)\) and \((p_i, \alpha_j)\) given any \(z_j, p_j\). Because

\[
\frac{\partial^2 E_{ii}}{\partial z_i \partial \alpha_j} = p_i \frac{\partial \Pr(\tilde{D}^{qt}_i > z_i)}{\partial \alpha_j} = p_i \int_{z_j}^{+\infty} (Y - z_j) f_{\tilde{D}^{qt}_i = Y}(z_i - \alpha_j(Y - z_j)) f_{\tilde{D}^{qt}_j}(Y) dY > 0,
\]

\[
\frac{\partial^2 E_{pi}}{\partial p_i \partial \alpha_j} = \Pr(\tilde{D}^{qt}_j < z_i, \tilde{D}^{qt}_j > z_j) E[\tilde{D}^{qt}_j - z_j | \tilde{D}^{qt}_i < z_i, \tilde{D}^{qt}_j > z_j] \geq 0.
\]

Or by Lemma A2-3, A2-1 and sum of supermodular function is supermodular, function (3.4) is supermodular in \((p_i, \alpha_j)\). By Lemma A2-2, A2-1, preservation of summation, function (3.4) is supermodular in \((z_i, \alpha_j)\).

Then we have the following results:

**Proposition 3.6** There exists a pure equilibrium in safety protection level and price under seat competition only, characterized for \(i = 1, 2\) by (if interior)

\[
p_0 = p_i \Pr(\tilde{D}^{qt}_i + \alpha_j(\tilde{D}^{qt}_j - z_j)^+ > z_i) \quad (3.5)
\]

\[
bp_0 + a - 2bp_i + E \min\{\tilde{D}^{qt}_i, z_i\} = 0 \quad (3.6)
\]

**Proof.** This is a supermodular game in \((z_i, -z_j, p_i, -p_j)\) by results 5-1 to 5-3.

According to Lemma A4, the pure strategy equilibrium exists. ■

How about the uniqueness of the equilibrium?

**Proposition 3.7** With identical carriers, there exists a unique equilibrium. Otherwise, \(r_{\tilde{D}^{qt}}(0) > 1/(1 - \alpha_j)p_0\) and \(b > 1/2\) are sufficient for the uniqueness.

**Proof.** Using a similar method to proposition 3.2, we can show that given any \((p_j, z_j)\), the solutions of (3.5)-(3.6) (the best response of player \(i\)) are unique and make \(2bp_0 r_{\tilde{D}^{qt}}(z_i) / \Pr(\tilde{D}^{qt}_i > z_i) - \Pr(\tilde{D}^{qt}_i > z_i) > 0\).

Solving \(p_i = p_0 / \Pr(\tilde{D}^{qt}_i > z_i)\) uniquely from (3.5) and substitute it into (3.6) we have,
Then we can get,

\[
\frac{d z_i}{d z_j} = -\frac{\alpha_j \Pr(D^{0^T}_i < z_i, D^{0}_j > z_j) + 2b \Pr(D^{0^T}_i > z_i, D^{0}_j > z_j)}{\Pr(D^{0^T}_i > z_i) - \Pr(D^{0^T}_j > z_i)} < 0.
\]

Similarly, \( \frac{d z_j}{d z_i} < 0 \). If there are two equilibrium \((z_i, z_j), (z'_i, z'_j)\), and assume \( z_i > z'_i \) without loss of generality, then we must have \( z_j < z'_j \). This is a contradiction for identical carriers. So there is a unique symmetric equilibrium. Since the largest and smallest Nash equilibria in a symmetric game are also symmetric, we have a unique Nash equilibrium for this symmetric game.

For arbitrary carriers, let \( z_i = h(p_i, z_j) \) be the explicit function by solving \( z_i \) from function (3.5), let \( p_i = g(z_i, z_j) \) be the explicit function by solving \( p_i \) from (3.6), the following is sufficient for a contraction mapping,

\[
C_1 = \left| \frac{\partial h(p_i, z_j)}{\partial p_i} \right| + \left| \frac{\partial h(p_i, z_j)}{\partial z_j} \right| < 1, \quad C_2 = \left| \frac{\partial g(z_i, z_j)}{\partial z_i} \right| + \left| \frac{\partial g(z_i, z_j)}{\partial z_j} \right| < 1.
\]

Note that \( \Pr(D^{0^T}_i > z_i) / p_{0T}^{D^{0^T}_i} = \Pr(D^{0^T}_i > z_i) / p_i f_{D^{0^T}_i} > 1 \) and \( 1 > \Pr(D^{0^T}_i > z_i) / p_i f_{D^{0^T}_i} \) implies \( 2b / \Pr(D^{0^T}_i > z_i) > 1 \) and \( \alpha_j \Pr(D^{0^T}_i > z_i) / p_{0T}^{D^{0^T}_i} < 1 \) by Appendix 5, and \( \alpha_j < 1 \). A simplification of the functions gives the required result.

Equilibrium Properties

Letting the unique equilibrium be \((z^*_i, z^*_j, p^*_i, p^*_j)\), we next derive some equilibrium properties.

**Corollary 3.1**

1. \( dp^*_i / d\alpha_j \geq 0, dz^*_i / d\alpha_j \geq 0, dz^*_j / d\alpha_j \leq 0, dp^*_j / d\alpha_j \leq 0 \). Furthermore \( dE \pi^*_i / d\alpha_j > 0, dE \pi^*_j / d\alpha_j < 0 \). 2. In equilibrium, safety protection level increases with
the price.

Proof. 1. The supermodular game, $E\pi_i$ has increasing differences in $(p_i, \alpha_j)$ and $(z_i, \alpha_j)$, and $E\pi_j$ has increasing differences in $(-p_j, \alpha_j)$ and $(-z_j, \alpha_j)$. So using Lemma A5 we get the first four inequalities. Finally,

$$\frac{dE\pi_i^*}{d\alpha_j} = \frac{\partial E\pi_i^*}{\partial z_j} + p_i^* \text{Pr}(\tilde{D}_i^{0T} < z_i^*; \tilde{D}_j^0 > z_j^*) E(\tilde{D}_j^0 - z_j^*) E(\tilde{D}_i^{0T} < z_i^*; \tilde{D}_j^0 > z_j^*) \geq 0$$

where $\partial E\pi_i^* / \partial z_j = -\alpha_j p_i^* \text{Pr}(\tilde{D}_i^{0T} < z_i^*; \tilde{D}_j^0 > z_j^*) \leq 0$.

$$\frac{dE\pi_j^*}{d\alpha_i} = \frac{\partial E\pi_j^*}{\partial z_i} \frac{d\alpha_i}{d\alpha_j} < 0.$$

2. From function (3.5), for equilibrium $(p_i, z_i)$,

$$\frac{dp_i}{dz_i} = \frac{1}{p_i r_{D_i^0T}(z_i)} \geq 0.$$

The above result is intuitive; in equilibrium, the higher the spill rate from the competitor then the higher the safety protection level $(z_i)$ and price set by firms. But firm $i$'s price and safety protection level $(z_i)$ decreases with own spill rate $(\alpha_i)$ to the competitor $j$. The final result depends on both of the effects. Intuitively, the impact of $\alpha_j$ on carrier $i$ is direct, the impact of $\alpha_i$ on carrier $i$ is indirect through the reaction to its competitor. If $z_i = h_i(p_i, z_j)$ and $p_i = g_i(z_i, z_j)$ are best response functions then $|\partial h_i(p_i, z_j) / \partial z_j| = \alpha_j f_{D_i^0T}^{(z_i, z_j)}(z_i) \text{Pr}(\tilde{D}_j^0 > z_j) / f_{D_i^0T}(z_i) < 1$ always and $|\partial g_i(z_i, z_j) / \partial z_j| = \alpha_j \text{Pr}(\tilde{D}_i^{0T} < z_i, \tilde{D}_j^0 > z_j) / 2b < 1$ (upon uniqueness). When $\alpha_{ji} = \alpha_{ij} = \alpha$, the negative (indirect) effect of $\alpha$ is less than its positive (direct) effect, so the equilibrium point should still increase with $\alpha$. Figure 3.8, 3.9 illustrate these. Also see Corollary 3.3.

Corollary 3.2 In equilibrium total protection level decreases with the equilibrium price.

Proof. $y_i^* = z_i^* + L_i^*$, given $p_i^*$, $dy_i^*/dp_i^* = dz_i^*/dp_i^* = b = (1 - b p_i^* r_{D_i^0T}(z_i^*) / p_i^* r_{D_i^0T}(z_i^*))$. Define $\hat{p}_i : 1/b = \hat{p}_i r_{D_i^0T}(z_i(\hat{p}_i))$. Where $z_i(\hat{p}_i)$ is derived from function (3.5). The rest of the proof is similar to Proposition 3.4 (the random variable now is $\tilde{D}_i^{0T}$). We have $p_i^* > \hat{p}_i$ so $dy_i^*/dp_i^* < 0.$
Observe that a higher equilibrium price implies a lower total protection level, but a higher safety protection level. This result is the same as for the price competition only case. So in a market with only price observable, we can deduce information about safety and total protection levels.

The effect of spill rate $\alpha_j$ on the total protection level is more complex. The numerical results are shown in Figure 3.8.

**Comparison with a Monopoly or Alliance**

If the decisions are made by a single entity, a monopoly, alliance or collusion, how do the duopoly results compare and what can we say about consumer welfare?

If the two flights are operated by a monopoly or alliance, the problem becomes,

$$\max_{z_i, p_i} \left( (c_i - L_i - z_i) p_0 + p_i (L_i + E \min \{ \tilde{D}_i^0 + \alpha_j (\tilde{D}_j^0 - z_j)^+, z_i \}) \right)$$

where $L_i = a - b p_i$.

We have the following results,

5-6. The objective function is supermodular in $(p_1, -p_2, z_1, -z_2)$. The optimal solution exists.

By Lemma A1-2 and because

$$\frac{\partial E_\pi}{\partial z_i} = -p_0 + p_i \Pr(\tilde{D}_i^{0T} > z_i) - \alpha_i p_j \Pr(\tilde{D}_j^{0T} < z_j, \tilde{D}_i^0 > z_i),$$

$$\frac{\partial E_\pi}{\partial p_i} = b p_0 + a - 2 b p_i + E \min \{ \tilde{D}_i^{0T}, z_i \}, \quad \frac{\partial^2 E_\pi}{\partial z_i \partial p_i} = \Pr(\tilde{D}_i^{0T} > z_i) \geq 0,$$

$$-\frac{\partial^2 E_\pi}{\partial z_i \partial p_j} = \alpha_j \Pr(\tilde{D}_j^{0T} < z_j, \tilde{D}_i^0 > z_i) \geq 0,$$

$$-\frac{\partial^2 E_\pi}{\partial z_j \partial p_j} = \alpha_j \Pr(\tilde{D}_i^{0T} < z_i, \tilde{D}_j^0 > z_j) \geq 0,$$

$$\frac{\partial^2 E_\pi}{\partial z_i \partial p_j} = \Pr(\tilde{D}_j^{0T} > z_j) \geq 0, \quad \frac{\partial^2 E_\pi}{\partial p_i \partial p_j} = 0,$$

For $(z_i, -z_j)$, the logic used in result 5-2 applies. The sum of supermodular functions is supermodular.

Without the differentiability requirement, similar to results 5-1 to 5-3, we can get the supermodularity.

5-7. The optimal protection levels are characterized by
The optimal prices are characterized by

\[-p_0 + p_i \Pr(\tilde{D}_i^{0T} > z_i) - \alpha_i p_j \Pr(\tilde{D}_j^{0T} < z_j, \tilde{D}_i^0 > z_i) = 0 \quad (3.7)\]

Proposition 3.8 Assume identical carriers. With only seat inventory competition, a duopoly charges higher prices and sets higher safety protection levels for full fare customers than a monopoly does.

Proof. From (3.5)-(3.6), for a given \(z_j\), the duopoly \(i\) makes decisions on safety protection level and price according to the following functions,

\[p_0 = p_i \Pr(\tilde{D}_i^{0T} > z_i), \quad p_i = \frac{b p_0 + a + E \min\{\tilde{D}_i^{0T}, z_i\}}{2b}.\]

Let the corresponding explicit functions be \(z_i = h^D(p_i), p_i = g^D(z_i)\). Note that \(dh^D(p_i)/dp_i \geq 0, dg^D(z_i)/dz_i \geq 0\). So in the \((p, z)\) quadrant, \(h^D(p_i)\) is upward sloping with a decreasing slope and starts at \((p_0, 0)\). \(g^D(z_i)\) is also upward sloping with an increasing slope and starts at \(((bp_0 + a)/2b, 0)\) with \((bp_0 + a)/2b > p_0\) \((L_i > 0\) when \(p_i = p_0\)). (Figure 3.1). From (3.7)-(3.8), for a given \(z_j\), the monopoly determines a protection level \(i\) and price \(i\) by

\[p_0 = p_i \Pr(\tilde{D}_i^{0T} > z_i) - \alpha_i p_j \Pr(\tilde{D}_j^{0T} < z_j, \tilde{D}_i^0 > z_i), \quad p_i = \frac{b p_0 + a + E \min\{\tilde{D}_i^{0T}, z_i\}}{2b}.\]

Let the corresponding explicit functions be \(z_i = h^M(p_i), p_i = g^M(z_i)\). Note that we have \(h^M(p_i) \leq h^D(p_i)\) for any \(p_i\), \(g^M(z_i) = g^D(z_i)\) for any \(z_i\), \(dh^M(p_i)/dp_i \geq 0, dg^M(p_i)/dz_i \geq 0\). So in the \((p, z)\) quadrant, both \(h^M(p_i)\) and \(h^D(p_i)\) are upward sloping and start at the same point, and the former is always lower than the latter.

Let \(z_i^D(z_j), p_i^D(z_j)\) be unique intersection of (3.5)-(3.6) for a given \(z_j\). Similarly for a
monopoly. No matter if the monopoly solution is unique or not, we have $z_i^M(z_j) \leq z_i^D(z_j)$, $p_i^M(z_j) \leq p_i^D(z_j)$.

Similarly, $z_j^M(z_i) \leq z_j^D(z_i)$, $p_j^M(z_i) \leq p_j^D(z_i)$.

With identical carriers, we have the required result. ■

Seat inventory competition, in contrast to the case of only price competition, has a duopoly setting a higher price and safety protection level than the monopoly. Thus different competitive environments should drive a different RM design. Under seat inventory competition, full fare customers experience a higher safety stock under duopoly, but they also pay more. They may be better off under a duopoly in the sense that full fare customers usually value availability above money.

The effect of spill rate on total protection levels for the monopoly solution is also ambiguous. Numerically from Figure 3.8, we find the total protection levels of both duopoly and monopoly increase with spill rate; the duopoly one increasing faster. In this case we would expect a higher total protection level under duopoly than the monopoly.

### 3.6 RM with Both Price and Seat Inventory Competition

What would the results be if we consider a more complex case with both levels of competition? In this case we are unable to apply a similar methodology as the joint game is not supermodular. We therefore use a more traditional methodology to establish the existence of the equilibrium. We obtain some sufficient conditions for uniqueness, investigate the properties of the equilibrium and compare the market equilibrium between duopoly and monopoly. We find that there are three cases to consider.

Two carriers simultaneously set prices and protection levels to maximize their own profits,
\[
\max_{z_i, p_i} E\pi_i
\]

\[
E\pi_i = (c_i - L_i - z_i)p_0 + p_i(L_i + E \min\{D_0^i + \alpha_j(D_j^0 - z_j)^+, z_i\}) \tag{3.9}
\]

where \(L_i = a - (b + \theta)p_i + \theta p_j\).

6-1. Unlike the previous sections, the game is not supermodular.

\[
\frac{\partial^2 E\pi_i}{\partial p_i \partial z_i} = \Pr(\tilde{D}_i^{0T} > z_i) \geq 0, \quad \frac{\partial^2 E\pi_i}{\partial p_i \partial p_j} = \theta,
\]

\[
\frac{\partial^2 E\pi_i}{\partial z_i \partial z_j} = -\alpha_j \Pr(\tilde{D}_j^{0T} < z_i, \tilde{D}_j^0 > z_j) \leq 0, \quad \frac{\partial^2 E\pi_i}{\partial z_i \partial p_j} = 0,
\]

\[
\frac{\partial^2 E\pi_i}{\partial p_j \partial z_j} = -p_i \alpha_j \int_{\tilde{D}_j^0 > z_j} (z_i) \Pr(\tilde{D}_j^0 > z_j) \leq 0.
\]

So function (3.9) is supermodular in \((z_i, p_i)\), has increasing differences in \((z_i, p_j)\) and \((p_i, p_j)\), and has increasing differences in \((z_i, -z_j)\), \((p_i, -z_j)\). But \(E\pi_j\) is not supermodular in \((p_j, -z_j)\).

6-2. Function (3.9) has increasing differences in \((z_i, \alpha_j)\) and \((p_i, \alpha_j)\).

\[
\frac{\partial^2 E\pi_i}{\partial \alpha_i \partial \alpha_j} = p_i \int_{z_j}^{+\infty} (Y - z_j) f_{\tilde{D}_0^i \tilde{D}_0^j} (z_i - \alpha_j (Y - z_j)) f_{\tilde{D}_0^j} (Y) dY > 0, \quad \frac{\partial^2 E\pi_i}{\partial p_i \partial \alpha_j} = \Pr(\tilde{D}_i^{0T} < z_i, \tilde{D}_j^0 > z_j) E[\tilde{D}_j^0 < z_i, \tilde{D}_j^0 > z_j] \geq 0. \text{ Note that } \frac{\partial^2 E\pi_i}{\partial \alpha_i \partial \alpha_j} = 0, \quad \frac{\partial^2 E\pi_i}{\partial p_j \partial \alpha_i} = 0.
\]

6-3. If we assume the symmetric case \((p_i = p_j)\), then \((p_i, -\theta)\) have increasing differences.

\[
-\frac{\partial^2 E\pi_i}{\partial p_i \partial \theta} = -p_0 + 2p_i - p_j > 0.
\]

Since the game is not supermodular, we go back to the traditional methodology.

**Proposition 3.9** Under a reasonably mild condition, there exists a pure strategy equilibrium in both safety protection level and price, characterized by (if interior):

\[
-p_0 + p_i \Pr(\tilde{D}_i^{0T} > z_i) = 0 \tag{3.10}
\]

\[
(b + \theta)p_0 + a - 2(b + \theta)p_i + \theta p_j + E \min\{\tilde{D}_i^{0T}, z_i\} = 0 \tag{3.11}
\]
Proof. 1. If the strategy spaces are nonempty compact convex subsets of an Euclidean space and payoff functions are continuous and quasiconcave, there exists a pure strategy Nash equilibrium (Fudenberg and Tirole, 1991).

Function (3.9) is jointly concave in \((p_i, z_i)\) where \(p_i \geq p_i(z_i)\). Because,

\[
\frac{\partial^2 E_{i}}{\partial p_i^2} = -2(b + \theta), \quad \frac{\partial^2 E_{i}}{\partial p_i \partial z_i} = \Pr(\overline{D_i^{QT}} > z_i), \quad \text{and} \quad \frac{\partial^2 E_{i}}{\partial z_i^2} = -p_i \beta_i^{QT}(z_i).
\]

The value of the determinant is \(2(b + \theta)p_i \beta_i^{QT}(z_i) - [\Pr(\overline{D_i^{QT}} > z_i)]^2 = \beta_i^{QT}(z_i) [2(b + \theta)p_i - \frac{\Pr(\overline{D_i^{QT}} > z_i)}{\beta_i^{QT}(z_i)}].\)

Define \(p_i(z_i) : 2(b + \theta)p_i - \frac{\Pr(\overline{D_i^{QT}} > z_i)}{\beta_i^{QT}(z_i)} = 0\). Then it is easy to see that the set \(0 \leq z_i \leq c_i; \quad p_i(z_i) \leq p_i \leq \overline{p}_i < \infty\) is non-empty and compact and the function is concave on it (where \(\overline{p}_i\) is sufficiently large upper bound of \(p_i\)). \(p_i(z_i)\) is also convex when \(r'/r^2\) is decreasing where \(r = r_{\beta_i^{QT}}(z_i)\) (\(r'/r^2\) decreasing is a relatively mild assumption, satisfied at least by the normal, Gamma, Weibull, negative exponential and uniform).

2. In addition we will show that given any \((p_j, z_j)\) the best responses \((p^*_i, z^*_i)\) given by (3.10)-(3.11) have \(p^*_i > p_i(z_i)\).

Solving for \(z_i = z_i(p_i)\) (uniquely) from (3.10), and substitute into (3.11): \(J(p_i) = (b + \theta)p_0 + a - 2(b + \theta)p_i + \theta p_j + E \min\{\overline{D_i^{QT}}, z_i(p_i)\}\)

As in proposition 3.2, \(\partial J(p_i)/\partial p_i\) is decreasing with \(p_i\) (IFR of \(\overline{D_i^{QT}}\)); \(J(p_i)\) starts at a positive value, either decreasing indefinitely or first increasing then decreasing indefinitely. Thus there is only one root for \(J(p^*_i) = 0\) and \(\partial J(p^*_i)/\partial p_i = -2(b + \theta) + \Pr(\overline{D_i^{QT}} > z^*_i)/[p^*_i \beta_i^{QT}(z^*_i)] < 0\), hence \(p^*_i > p_i(z_i)\) and the equilibrium points cannot exist with \(p^*_i < p_i(z_i)\). •

What can we say about the uniqueness?

Proposition 3.10 \(r_{\beta_i^{QT}}(0) > 1/(1 - \alpha_j)p_0, \quad 2b + \theta > 1\) is sufficient for uniqueness.

Proof. Let \(z_i = h(p_i, z_j), p_i = g(p_j, z_i, z_j)\) be the explicit functions of (3.10) and (3.11) respectively.

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If $C_1 < 1, C_2 < 1$, then the reaction functions are contraction mappings and we can get

$$\frac{\Pr(\tilde{D}_{ij}^{\theta T} > z_i)}{\Pr(\tilde{D}_{ij}^{\theta T} < z_i)} < 1, \frac{2(b+\theta)}{\Pr(\tilde{D}_{ij}^{\theta T} > z_i)} > 1.$$

Thus $2(b+\theta)p_i\tilde{f}_{D_{ij}^{\theta T}}(z_i) - \frac{[\Pr(\tilde{D}_{ij}^{\theta T} > z_i)]^2}{2(b+\theta)} > 0$, and guarantees the strict concavity of function (3.9) in $(p_i, z_i)$.

By simplifying terms, we have the required result. •

**Equilibrium Properties**

Let $(z_i^*(p_j, z_j), p_i^*(p_j, z_j))$ be the best response functions for player $i$ given $(p_j, z_j)$.

Then,

**Proposition 3.11** $dz_i^*(p_j, z_j)/d\alpha_j > 0, dp_i^*(p_j, z_j)/d\alpha_j > 0, dz_i^*(p_j, z_j)/d\theta < 0, dp_i^*(p_j, z_j)/d\theta < 0$.

**Proof.** Given $(p_j, z_j)$, from (3.11) define

$$g(z_i) = p_i = \frac{(b+\theta)p_0 + a + \theta p_j + E \min\{\tilde{D}_{ij}^{\theta T}, z_i\}}{2(b+\theta)}.$$

From (3.10) define the explicit function $z_i = h(p_i)$. These are two upward sloping functions in $(p, z)$ quadrant. $g(z_i)$ starts at $[(b + \theta)p_0 + a + \theta p_j]/2(b + \theta), 0)$ with slope $dz_i/dp_i = 2(b + \theta)/\Pr(\tilde{D}_{ij}^{\theta T} > z_i)$, $h(p_i)$ starts at $(p_0, 0)$ with slope $dz_i/dp_i = \Pr(\tilde{D}_{ij}^{\theta T} > z_i)/p_i\tilde{f}_{D_{ij}^{\theta T}}(z_i)$, and we have $[(b + \theta)p_0 + a + \theta p_j]/2(b + \theta) > p_0$. The unique intersection is $(z_i^*(p_j, z_j), p_i^*(p_j, z_j))$.

Since $\partial g(z_i)/\partial \theta = -(a - b p_j + E \min\{\tilde{D}_{ij}^{\theta T}, z_i\})/2(b + \theta)^2 < 0$, $\partial h(p_i)/\partial \theta = 0$; $g(z_i)$ moves leftward with $\theta$, but $h(p_i)$ does not change. So the intersection moves left and down: a smaller $(p_i, z_i)$.

$$\partial h(z_i)/\partial \alpha_j = -\frac{\partial^2 E \pi_i}{\partial z_i \partial \alpha_j} < 0$$

because of $\partial^2 E \pi_i/\partial z_i \partial \alpha_j > 0$ and $\partial^2 E \pi_i/\partial z_i^2 = -p_i\tilde{f}_{D_{ij}^{\theta T}}(z_i) \leq 0$.

$$\partial g(p_i)/\partial \alpha_j = -\frac{E(\tilde{D}_{ij}^{\theta T} - z_j)\tilde{f}_{D_{ij}^{\theta T}} < z_i, \tilde{D}_{ij}^{\theta} > z_j \Pr(\tilde{D}_{ij}^{\theta T} < z_i, \tilde{D}_{ij}^{\theta} > z_j)}{2(b + \theta)} \geq 0.$$

So
Let $\alpha_{ij} = \alpha_{ji} = \alpha$, and assume identical carriers. Then in the symmetric equilibrium $(p_i^*, p_j^*, z_i^*, z_j^*)$, we have the following corollary.

**Corollary 3.3** In symmetric equilibrium, $\frac{dz^*_i}{d\alpha} > 0$, $\frac{dp^*_i}{d\alpha} > 0$, $\frac{dz^*_j}{d\theta} < 0$, $\frac{dp^*_j}{d\theta} < 0$.

**Proof.** From the above Proposition, given $p_j$, $z_j$, the best responses for $i$ satisfy $\frac{dr_{xi}(p_j, z_j)}{d\alpha} > 0$, $\frac{dr_{xi}(p_j, z_j)}{d\theta} > 0$, $\frac{dr_{pi}(p_j, z_j)}{d\theta} < 0$. For $i = 1, 2$. Then in the symmetric equilibrium we have the above result. ■

With both levels of competition, equilibrium decisions increase with the spill rate and decrease with product substitutability; the competitive environment influences strategic RM decisions. But how about profitability?

\[
\frac{dE^*}{d\alpha} = \frac{\partial E^*_i}{\partial z_j} \frac{\partial z^*_i}{\partial \alpha} + \frac{\partial E^*_i}{\partial p_j} \frac{\partial p^*_i}{\partial \alpha} + p_i^* \Pr\{\tilde{D}_i^{\theta T} < z_i^*, \tilde{D}_j^{\theta} > z_j^*\} E(\tilde{D}_i^{\theta T} - z_i^*) \Pr(\tilde{D}_i^{\theta T} < z_i^*, \tilde{D}_j^{\theta} > z_j^*) .
\]

Only the first term is negative, we suspect it is a second order effect and the equilibrium profit will increase with the spill rate. Figure 3.11 shows this.

With identical carriers,

\[
\frac{dE^*_i}{d\theta} = \frac{\partial E^*_i}{\partial z_j} \frac{\partial z^*_i}{\partial \theta} + \frac{\partial E^*_i}{\partial p_j} \frac{\partial p^*_i}{\partial \theta} + (p_i^* - p_0)(p_j^* - p_i^*) = \frac{\partial E^*_i}{\partial z_i} \frac{\partial z^*_i}{\partial \theta} + \frac{\partial E^*_i}{\partial p_j} \frac{\partial p^*_i}{\partial \theta} .
\]

The last term $\frac{\partial E^*_i}{d\theta}$ is cancelled out in symmetric equilibrium. The first term is positive, the second negative. So by adding seat competition, the negative effect of price competition can actually be mitigated. From the numerical example, when $\alpha$ is relatively high, increasing price competition in the duopoly can actually increase their profits, i.e., $dE^*_i/d\theta > 0$, Figure 3.12. The intuition is that increasing price competition can decrease $z_j$ and increase switching demand, and hence lead to a higher profit. Thus, although fierce price competition is damaging, slightly introducing price competition will benefit the industry.
Corollary 3.4 1. The total protection level decreases with equilibrium price; 2. For symmetric equilibrium $dy_i^*/d\theta > 0$.

Proof. $y_i^* = z_i^* + L_i^*$. So given $p_i^*$, $dy_i^*/dp_i^* = dz_i^*/dp_i^* - (b + \theta)$; in symmetric case $(p_i^* = p_j^*, z_i^* = z_j^*)$ $dy_i^*/d\theta = dz_i^*/d\theta - b dp_i^*/d\theta$.

From (3.10), we have

$$\frac{dz_i^*}{d\theta} = \frac{1}{pr_i pr_T(z_i^*)}.$$ 
$$\frac{dp_i^*}{d\theta} = \frac{dz_i^*}{d\theta} p_i^* r_{pr_T}(z_i^*) + \frac{dz_i^*}{d\theta} \alpha p_i \mathbb{I}_{D_i^0 > z_i^*} Pr(D_i^0 > z_i^*) - \frac{dz_i^*}{d\theta} \alpha p_i \mathbb{I}_{D_i^0 > z_i^*}.$$

Note that $dz_i^*/d\theta < 0$. Similar as Proposition 3.4, we have the required results. 

So similar to last two sections, a higher equilibrium price implies a lower total protection level; fiercer price competition induces competitors to protect more seats for full fare customers. So given the same spill level, stronger price competition between airlines is good news to full fare passengers. But low fare customers are more likely to be turned away.

Comparison with a Monopoly or Alliance

With only price competition, a duopoly sets a lower price and safety protection level for full fare customers compared to a monopoly, but with seat competition, a duopoly sets both higher. What happens with both levels of competition? A monopoly maximizes

$$\sum_{i=1,2} [(c_i - L_i - z_i) p_i + p_i (L_i + E \min \{D_i^0 + \alpha_j (D_j^0 - z_j) + z_i \})]$$

where $L_i = a - (b + \theta) p_i + \theta p_j$.

If the interior solutions exist, then they are characterized by the following,

$$-p_0 + p_i Pr(\tilde{D}_i^{0T} > z_i) - \alpha_j p_i Pr(\tilde{D}_i^{0T} < z_j, \tilde{D}_i^{0T} > z_i) = 0$$

(3.12)

$$(b + \theta)p_0 + a - 2(b + \theta) p_i + \theta p_j + E \min \{\tilde{D}_i^{0T}, z_i \} + \theta (p_j - p_0) = 0$$

(3.13)

By comparing (3.10)-(3.11) with (3.12)-(3.13), we have the following.

Proposition 3.12 Assume identical carriers, then one of the following can happen: 1.
$p_i^M < p_i^D, z_i^M < z_i^D$; 2. $p_i^M > p_i^D, z_i^M > z_i^D$; 3. $p_i^M > p_i^D, z_i^M < z_i^D$.

**Proof.** Given any $p_j, z_j$, from function (3.11) define $g^D(z_i) = \frac{(b+\theta)p_0+a+\theta p_j+E \min \{D^R, z_i\}}{2(b+\theta)}$.

From function (3.13), define $g^M(z_i) = \frac{(b+\theta)p_0+a+\theta p_j+E \min \{D^L, z_i\} + \theta(p_j-p_0)}{2(b+\theta)}$.

Note that $\frac{dg^D(z_i)}{dz_i} = \frac{dg^M(z_i)}{dz_i}$, so the two lines in the $(p, z)$ space are parallel and $g^M(z_i) > g^D(z_i)$.

From function (3.10) define the explicit function $z_i^D = h^D(p_i)$. From function (3.12), define $z_i^M = h^M(p_i)$. Using a similar analysis as in section 3.5, $h^M(p_i) < h^D(p_i)$ and they start at the same point $(p_0, 0)$. Let the intersection for the monopoly case be $z_i^M(p_j, z_j), p_i^M(p_j, z_j)$, and the unique intersection for the duopoly be $z_i^D(p_j, z_j), p_i^D(p_j, z_j)$. From Figure 3.2, $[z_i^M(p_j, z_j), p_i^M(p_j, z_j)]$ cannot exist outside the range of $F$, so we can have:

1. If $\alpha_j = \alpha_i$ is small, $(h^M(p_i)$ is close to $h^D(p_i))$, then the situation is similar to the scenario with price competition only, $z_i^M(p_j, z_j) > z_i^D(p_j, z_j), p_i^M(p_j, z_j) > p_i^D(p_j, z_j)$. Figure 3.2.

2. If $\theta$ is small $(g^M(z_i)$ is close to $g^D(z_i))$, then the situation is similar to the scenario with seat inventory competition only, $z_i^M(p_j, z_j) < z_i^D(p_j, z_j), p_i^M(p_j, z_j) < p_i^D(p_j, z_j)$. Figure 3.3.

3. Otherwise, we can have $z_i^M(p_j, z_j) < z_i^D(p_j, z_j), p_i^M(p_j, z_j) > p_i^D(p_j, z_j)$. Figure 3.4.

This analysis applies for player $j$. In the symmetric equilibrium, we have the required result. ■

This key result for strategic RM has not been addressed to date in the literature. A market with a weak price competition but a high customer willingness to switch to an alternative airline ends up with a higher price dispersion between the two fare classes, and higher safety protection levels. Full fare customers pay more but are more likely to get a seat. A market with strong price competition but a low customer willingness to switch will offer full fare customers a lower price but a higher probability of not getting
a seat. Interestingly, when the two factors affecting the degree of competition are not at extreme values, the welfare of the full fare customers is actually at its highest. They pay less than under a monopoly, but are more likely to be served. These results may be of interest to government regulators. They can protect different customers’ welfare by allowing or preventing collusion between the duopoly.

The following example further confirms the above analysis.

**Example 3.3** The profit function can also be written as

\[(c_i - L_i - z_i)p_i + p_i(L_i + E \min \{\tilde{D}_i^0, z_i\} + E \min \{\alpha_j(\tilde{D}_j^0 - z_j)^+, (z_i - \tilde{D}_i^0)^+\})\]

Assume \(\tilde{D}_i^0\) are identical, independent and exponentially distributed with \(\lambda = 1\).

\[E \min \{\tilde{D}_i^0, z_i\} = 1 - \exp(-z_i)\],

\[A = E \min \{\alpha_j(\tilde{D}_j^0 - z_j)^+, (z_i - \tilde{D}_i^0)^+\}\]

\[= \int_0^{z_i} (\int_{z_j}^{z_i - z_i + z_j} \alpha_j(x_j - z_j) \exp(-x_j) dx_j) \exp(-x_i) dx_i\]

\[+ \int_0^{z_i} (\int_{z_j}^{z_i - z_i + z_j} (z_i - x_i) \exp(-x_j) dx_j) \exp(-x_i) dx_i\]

\[= \frac{\alpha_j(\alpha_j \exp(-z_j) - \exp(-z_i + z_j \alpha_j) \alpha_j + \exp(-z_i - z_j) - \exp(-z_j))}{\alpha_j - 1} = \omega,\]

\[E \min \{\tilde{D}_i^0, z_i\} = 1 - \exp(-z_i)\],

\[\frac{\partial A}{\partial z_i} = \int_0^{z_i} (\int_{z_j}^{z_i - z_i + z_j} \exp(-x_j) dx_j) \exp(-x_i) dx_i\]

\[= \frac{\alpha_j(\alpha_j \exp(-z_i + z_j \alpha_j) - \exp(-z_i - z_j))}{\alpha_j - 1} = \eta,\]

\[\frac{\partial A}{\partial z_j} = - \int_0^{z_i} (\int_{z_j}^{z_i - z_i + z_j} \alpha_j \exp(-x_j) dx_j) \exp(-x_i) dx_i = -\omega\]

So for a duopoly, functions (3.10) and (3.11) become

\[-p_0 + p_i(\exp(-z_i) + \eta) = 0\]

\[p_i = \frac{(b + \theta)p_0 + a + \theta p_i + 1 - \exp(-z_i) + \omega}{2(b + \theta)}\]

For a monopoly, functions (3.12) and (3.13) become

\[-p_0 + p_i(\exp(-z_i) + \eta) - p_j \omega = 0\]

\[p_i = \frac{(b + \theta)p_0 + a + \theta p_i + 1 - \exp(-z_i) + \omega + \theta(p_j - p_0)}{2(b + \theta)}\]

We have the following numerical experiment.

**Table 3.1 Numerical Experiment (a = 2, b = 0.8, p_0 = 1)**
Given the same spill rate, duopoly market also ends up with higher total protection levels than the monopoly market. The reasoning is analogical to the scenario of price competition only. The effect of spill rate to total protection levels of the monopoly solutions is also ambiguous. In symmetric case, 
\[ \text{dy}^i_t / \text{d}a = dz^*_i / \text{d}a \{ 1 - [\Pr(D^T_i > z^*_i) - \alpha \Pr(D^T_i < z^*_i, D_j^T > z^*_j)]/2 \} - E[|D^T_i - z^*_i|^+ |D^T_i < z^*_i] \Pr(D^T_i < z^*_i)/2. \]
Where the first term is positive and the second term is negative. Similar as section 3.5, numerical results have that the total protection levels under both duopoly and monopoly increase with spill rate, but the duopoly one increases much faster.

### 3.7 The Effect of Market Structure

#### With Price Competition Only

How does the number of firms \( n \) in the market affect the equilibrium RM strategy? We revise the model by letting 
\[ L_i = a - b \pi_i + \sum_{j=1}^{n} \theta_j (\pi_j - \pi_i). \]
Note that the demand function allows the market grow with the number of firms, we do not restrict the market to be constant as the number of firm increases. To see the effect of \( n \) on RM strategy we assume symmetry, i.e., \( \theta_j = \theta \) and 
\[ L_i = a - b \pi_i + (n - 1) \theta (\pi_j - \pi_i). \]

Now \( E \pi_i \) has decreasing differences in \( (\pi_i, n) \) and \( (z_i, n) \) for \( i = 1, 2 \). So the supermodular game tells us that the equilibrium point decreases with \( n \). Firms are also worse off since 
\[ \frac{\partial E \pi_i}{\partial n} = \theta (\pi_j^* - \pi_i^*)(\pi_i^* - \pi_0) + \frac{\partial E \pi_i^*}{\partial \pi_j^*} \frac{\partial \pi_i^*}{\partial n} < 0 \]
in symmetric equilibrium.
So the more firms in the market, the lower should management set price and protection levels. Firms are also worse off.

**With Seat Inventory Competition Only**

We revise the model to incorporate seat competition and assume symmetry, \( \alpha_j \) are the same for all \( j \neq i \).

\[
\max_{z_i, p_i} (c_i - L_i - z_i)p_0 + p_i(L_i + E \min\{\bar{D}_i^0 + \sum_{j \neq i} \alpha_j(\bar{D}_j^0 - z_j)^+, z_i\})
\]

\[
= \max_{z_i, p_i} (c_i - L_i - z_i)p_0 + p_i(L_i + E \min\{\bar{D}_i^0 + (n - 1)\alpha_j(\bar{D}_j^0 - z_j)^+, z_i\})
\]

Let \( z_i = h(p_i) \) and \( p_i = g(z_i) \) be the best reply functions of player \( i \) given \( z_j \). Note that for each player \( i \), the best reply functions increase with \( n \). That is because of \( \frac{\partial^2 E_i}{\partial p_i \partial n} \geq 0 \) and \( \frac{\partial^2 E_i}{\partial z_i \partial n} \geq 0 \). So the symmetric equilibrium point also increases with \( n \). i.e., both the equilibrium prices and safety protection levels increase with the number of firms.

\[
\frac{\partial E_i}{\partial n} = p_i \Pr(\bar{D}_i^{0T} < z_i, \bar{D}_j^{0T} > z_j)E[\alpha_j(\bar{D}_j^0 - z_j)|\bar{D}_i^{0T} < z_i, \bar{D}_j^{0T} > z_j] + \frac{\partial E_i}{\partial z_j} \frac{\partial z_i}{\partial n}, \text{ the first term is the direct effect of } n, \text{ which is positive. The second indirect term is negative. We expect the profits of firms will increase with } n \text{ if the direct effect dominates.}
\]

**With Both Price and Seat Inventory Competition**

Still assuming symmetry, and we recall that under price competition only, the equilibrium price and protection levels decrease with \( n \), whereas under seat inventory competition only, the equilibrium price and protection levels increase with \( n \). How about with both?

Result 7-1. The effect of \( n \) on the equilibrium price and safety protection level is less than that under price competition only or seat inventory competition only.

In symmetric equilibrium, we have

\[
-p_0 + p \Pr(\bar{D}_i^{0T} > z) = 0 \Rightarrow z = h(p)
\]

\[
p = \frac{a + (b + (n - 1)\theta)p_0 + E \min\{\bar{D}_i^{0T}, z_i\}}{2b + (n - 1)\theta} \Rightarrow p = g(z)
\]

Observe that \( \frac{\partial h(p)}{\partial n} \geq 0 \), similar as the effect of \( \alpha_j \).
\[
\frac{dg(p)}{dn} = \frac{Pr(\tilde{D}_i^0 < z_i, \tilde{D}_j^0 > z_j)\alpha z_j E[\tilde{D}_i^0 - z_i | \tilde{D}_j^0 < z_i, \tilde{D}_j^0 > z_j] - \theta [a - bp_i + E \min \{D_i^0, z_i\}]}{(2b + \theta (n-1))^2}
\]

The first term is the effect of \( n \) on price because of seat competition, the second term is the effect of \( n \) on price because of price competition. The two terms can weaken each other.

### 3.8 Introduction of RM

Can we deduce any advantage of being a ‘first mover’ in the adoption of Revenue Management? To do this with the tools of equilibrium assumes that adoption is slow enough to achieve an equilibrium: one player having adopted, the other not. We again use supermodularity to show existence and uniqueness. We find that there is always a first mover advantage, that the second mover also benefits and that implementing Revenue Management is a dominant strategy.

**Price Competition Only**

In order to model a non-adopter, we make the reasonable assumption that the non-adopter sets a uniform price for the whole capacity higher than \( p_0 \).

**Scenario 1.** Neither airline uses Revenue Management (\( NN \) will denote this; \( N \) will mean ‘no’ and \( Y \) ‘yes’).

\[
\max_{p_i} p_i E \min \{\tilde{D}_i, c_i\} = p_i (L_i + E \min \{\tilde{D}_i^0, c_i - L_i\}) \text{ where } L_i = a - bp_i + \theta (p_j - p_i).
\]

**Proposition 3.13** There exists a unique pricing equilibrium in the NN game characterized by the following function

\[
\theta = 2(b + \theta)p_i + \theta p_j + E \min \{\tilde{D}_i^0, c_i - L_i\} + p_i (b + \theta) \Pr(\tilde{D}_i^0 > c_i - L_i) = 0 \quad (3.14)
\]

**Proof.** Step 1. Supermodular game guarantees the existence. According to Lemma A2-1, the objective function is supermodular in \( p_i \), \( \frac{\partial^2 E \pi_i}{\partial p_i \partial p_j} = \theta (1 - \Pr(\tilde{D}_i^0 > c_i - L_i) \quad (3.14)\)

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\( c_i - L_i \) + \( p_i \theta (b + \theta) f_{D_i} (c_i - L_i) \geq 0 \), so the function has increasing differences in \((p_i, p_j)\) and the game is supermodular.

Alternatively, we can rewrite the objective function as \( p_i (-b + \theta) p_i + E \min \{ D_i^0 + a + \theta p_j, c_i - (b + \theta) p_i \} \). According to Lemma A2-2, \( E \min \{ D_i^0 + a + \theta p_j, c_i - (b + \theta) p_i \} \) is supermodular in \((p_i, p_j)\). So \( p_i E \min \{ D_i^0 + a + \theta p_j, c_i - (b + \theta) p_i \} \) is also supermodular in \((p_i, p_j)\) according to Lemma A2-3. And \( -b + \theta \) \( p_i^2 \) is also supermodular according to Lemma A2-1.

Step 2. Uniqueness. \( \frac{\partial^2 E\pi_i}{\partial p_i^2} = -(b + \theta)[2-Pr(D_i^0 > c_i - L_i)] > 0 \). So the function is strictly concave in \( p_i \). It is sufficient to prove that the reaction functions are contraction mappings (Lemma A6). This is true since

\[
\frac{d E\pi_i}{d p_j} = \frac{\partial^2 E\pi_i}{\partial p_i \partial p_j} \quad \frac{\partial^2 E\pi_i}{\partial p_i^2} = \frac{\theta(1 - Pr(D_i^0 > c_i - L_i)) + p_i(b + \theta) f_{D_i} (c_i - L_i)}{(b + \theta)[2-Pr(D_i^0 > c_i - L_i)] + p_i(b + \theta) f_{D_i} (c_i - L_i)} \leq 1.
\]

Let the equilibrium be \((p_1^{NN}, p_2^{NN})\) and the profit be \( E\pi_i^{NN} \).

Scenario 2. Only airline one uses Revenue Management \((YN)\). Airline one makes a decision on both protection level and price, airline two decides on the uniform price at which to sell the whole capacity. The game is,

\[
\max_{z_1, p_1} (c_i - L_i - z_1)p_0 + p_1(L_1 + E \min \{ D_i^0, z_1 \})
\]

\[
\max_{p_2} p_2(L_2 + E \min \{ D_2^0, c_2 - L_2 \})
\]

As the game is supermodular by a similar method used in Proposition 3.1 and 3.13, we have

**Corollary 3.5** There exists a pure strategy equilibrium in this \(YN\) game.

It is intuitive that airline one always benefits, since given any \( p_2, z_1 = c_i - L_i \) is a feasible boundary solution.

\[
\max_{z_1} (c_i - L_i - z_1)p_0 + p_1(L_1 + E \min \{ D_i^0, z_1 \}) \geq p_1(L_1 + E \min \{ D_i^0, c_i - L_i \})
\]

How about airline two? Its objective function in the two scenarios is the same. Since

\[
\frac{\partial E\pi_2}{\partial p_1} = p_2 \theta (1 - Pr(D_2^0 > c_2 - L_2)) > 0,
\]

it benefits only if \( p_1^{NN} < p_1^{YN} \).
Proposition 3.14 \( P_1^{NN} \leq P_1^{YN} \). Implementing RM not only benefits the first mover but also benefits the competitor (if equilibrium is interior).

Proof. Step 1. We need to show that \( P_1^{YN} \) increases with \( p_0 \).

Given \( p_2 \), in the YN scenario, airline one makes a decision on \( p_1 \) and \( z_1 \) by functions \( (3.2)-(3.3) \). By taking derivatives of \( (3.2)-(3.3) \) w.r.t. \( p_0 \), we can solve

\[
\frac{dp_1}{dp_0} = \frac{1-(b+\theta)p_1r_{\beta,1}(z_1)}{-2p_1r_{\beta,1}(z_1)(b+\theta)+Pr(D_0^1 > z_1)}
\]

We have shown that in equilibrium the numerator is negative in Proposition 3.4 and the denominator is also negative in Proposition 3.2. So \( \frac{dp_1^{YN}}{dp_0} \geq 0 \).

Step 2. In the YN scenario when \( p_0 \) is very small, \( z_1 \) will be set to the largest possible value so \( z_1 = c_1 - L_1 \). Define \( p_0^* \) to be the value of \( p_0 \) such that the optimal solution of airline one is the boundary solution. i.e.,

\[
p_0^* = p_1 \Pr(c_1 - L_1 < \tilde{D}_0^p) \text{ and } a - 2(b+\theta)p_1 + \theta p_2 + (b+\theta)p_0^* + E \min\{c_1 - L_1, \tilde{D}_0^p\} = 0.
\]

This is equivalent to

\[
a - 2(b+\theta)p_1 + \theta p_2 + (b+\theta)p_1 \Pr(c_1 - L_1 < \tilde{D}_0^p) + E \min\{c_1 - L_1, \tilde{D}_0^p\} = 0
\]

This is exactly the reaction function of airline one in the NN scenario. Solving the game we have \( P_1^{YN} = P_1^{NN} \) and \( z_1^{YN} = c_1 - L_1^{NN} \).

Step 3. For any \( p_0 \geq p_0^* \), we have \( P_1^{YN} \geq P_1^{NN} \). So airline two is also better off.

Scenario 3. The game with both using Revenue Management (YY) has been analyzed in section 3.4. Following the same logic, airline two also benefits from implementation. So YY is the market equilibrium. We can also show that both players see a revenue increase from NN to YY by the same method as Proposition 3.14.

Corollary 3.6 With only price competition, there is always a first mover advantage and both using Revenue Management is the Pareto dominate market equilibrium.

With Seat Inventory Competition Only

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We then see how the RM affects the profitability of competitors in a market with only seat allocation competition. Similar with the above, we consider three scenarios.

**Scenario 1 NN.** No players uses RM, the game becomes,

\[
\max_{p_i} E\pi_i
\]

\[
E\pi_i = p_i E \min \{c_i, D_i + \alpha_j (\bar{D}_j - c_j)^+\} = p_i (L_i + E \min \{c_i - L_i, \bar{D}_i^0 + \alpha_j (\bar{D}_j^0 - c_j + L_j)^+\})
\]

where \(L_i = a - bp_i\).

**Proposition 3.15** This NN game is supermodular, so there exist pure strategy equilibria. Moreover, the equilibrium is unique. Which is characterized by

\[
a - 2bp_i + bp_i \Pr(c_i - L_i < \bar{D}_i^{0T}) + E \min \{c_i - L_i, \bar{D}_i^{0T}\} = 0 \quad (3.15)
\]

**Proof.** Step 1. The objective function is supermodular in \((p_i, -p_j)\). To show this, we only need to prove increasing difference. This is the case since

\[
-\frac{\partial^2 E\pi_i}{\partial p_i \partial p_j} = \alpha_j b \Pr(c_i - L_i > \bar{D}_i^{0T}, \bar{D}_j^0 > c_j - L_j) + \alpha_j b^2 p_i f_{\bar{D}_i^{0T}, \bar{D}_j^0 > c_j - L_j}(c_i - L_i) \Pr(\bar{D}_j^0 > c_j - L_j) \geq 0.
\]

where \(\bar{D}_i^{0T} = \bar{D}_i^0 + \alpha_j (\bar{D}_j^0 - c_j + L_j)^+\)

As this is a supermodular game, a pure strategy equilibrium exist.

Alternatively, the objective function can be written as \(p_i (a - bp_i) + p_i E \min \{c_i - a + bp_i, \bar{D}_i^0 + \alpha_j (\bar{D}_j^0 + a - bp_i - c_j)^+\}\). The first part is only function of \(p_i\), supermodularity is easily derived by Lemma A2-1. By Lemma A2-2 and A2-3, we can get the supermodularity in \((p_i, p_j)\) of the second part.

**Step 3. Uniqueness.** \(\frac{\partial^2 E\pi_i}{\partial p_i^2} = -b[2 - 2 \Pr(c_i - L_i < \bar{D}_i^{0T}) + bp_i f_{\bar{D}_i^{0T}}(c_i - L_i)] < 0\). So the function is strictly concave in \(p_i\). Consider the contraction mapping:

\[
\frac{\partial^i(p_i)}{\partial p_j} = \frac{a_j \Pr(c_i - L_i > \bar{D}_i^{0T}, \bar{D}_j^0 > c_j - L_j) + bp_i f_{\bar{D}_i^{0T}, \bar{D}_j^0 > c_j - L_j}(c_i - L_i) \Pr(\bar{D}_j^0 > c_j - L_j)}{2 \Pr(c_i - L_i > \bar{D}_i^{0T}) + bp_i f_{\bar{D}_i^{0T}}(c_i - L_i)} < 1
\]

**Scenario 2 YN.** If only firm one implements RM and sets two fare classes, the game becomes,
\[
\max_{z_1, p_1} (c_1 - L_1 - z_1)p_0 + p_1(L_1 + E\min\{\tilde{D}_2^0 + \alpha_2(\tilde{D}_2^0 - c_2 + L_2)^+, z_1}\))
\]
\[
\max_{p_2} p_2(L_2 + E\min\{c_2 - L_2, \tilde{D}_2^0 + \alpha_1(\tilde{D}_1^0 - z_1)^+\})
\]

We have the following results:

8-8. The function \(E\pi_1\) is supermodular in \((z_1, p_1)\) and have increasing differences in \((p_1, -p_2), (z_1, -p_2)\) because of the following

\[
\frac{\partial E\pi_1}{\partial z_1} = -p_0 + p_1 \Pr(\tilde{D}_1^{0T} > z_1)
\]
\[
\frac{\partial E\pi_1}{\partial p_1} = bp_0 + a - 2bp_1 + E\min\{\tilde{D}_1^0 + \alpha_2(\tilde{D}_2^0 - c_2 + L_2)^+, z_1\}
\]
\[
\frac{\partial^2 E\pi_1}{\partial z_1 \partial p_1} = \Pr(\tilde{D}_1^{0T} > z_1) \geq 0,
\]
\[
-\frac{\partial^2 E\pi_1}{\partial z_1 \partial p_2} = b\alpha_2 p_1 f_{\tilde{D}_1^{0T}|\tilde{D}_2^0 > c_2-L_2}(z_1) \Pr(\tilde{D}_2^0 > c_2 - L_2) \geq 0,
\]
\[
-\frac{\partial^2 E\pi_1}{\partial p_1 \partial p_2} = b\alpha_2 \Pr(\tilde{D}_1^{0T} < z_1, \tilde{D}_2^0 > c_2 - L_2) \geq 0.
\]

Alternatively, we can get the result by using Lemma A2-1 to A2-3.

8-9. Function \(E\pi_2\) is supermodular in \((-p_2)\) by Lemma A2-1 and has increasing differences in \((p_1, -p_2)\) and \((z_1, -p_2)\) because of

\[
\frac{\partial E\pi_2}{\partial p_2} = L_2 + E\min\{c_2 - L_2, \tilde{D}_2^{0T}\} - bp_2 \Pr(c_2 - L_2 > \tilde{D}_2^{0T}),
\]
\[
-\frac{\partial^2 E\pi_2}{\partial p_2 \partial p_1} = 0.
\]
\[
-\frac{\partial^2 E\pi_2}{\partial p_2 \partial z_1} = \alpha_1[\Pr(c_2 - L_2 > \tilde{D}_2^{0T}, \tilde{D}_1^0 > z_1) + bp_2 f_{\tilde{D}_1^{0T}|\tilde{D}_2^0 > z_1}(c_2 - L_2) \Pr(\tilde{D}_1^0 > z_1)] \geq 0
\]

Alternatively, by Lemma A2-1 to A2-3, \(E\pi_2\) is supermodular in \((p_1, -p_2)\) and \((z_1, -p_2)\).

Since the game is supermodular, we have

**Corollary 3.7** There exists a pure strategy equilibrium in this YN game.

**Proof.** This is the direct outcome of results 8-8 and 8-9. ■

How does using RM influence the profitability?

From the similar logic, the first mover always benefits (if the equilibrium is interior), because otherwise it will just set \(z_1 = c_1 - L_1\) and stay in the NN scenario. For firm two, he has total demand \(\tilde{D}_2^0 + \alpha_1(\tilde{D}_2^0 - c_1 + L_1)^+\) in the NN scenario and \(\tilde{D}_2^0 + \alpha_1(\tilde{D}_1^0 - z_1)^+\) in the YN scenario.
Proposition 3.16 Assuming IFR of r.v. $\tilde{D}_1^{0T}$, then $z_1^{YN} \leq c_1 - L_1^{NN}$, so the implementation of RM not only benefits firm one itself, but also its competitor (if the equilibrium is interior).

Proof. Step 1. Show that $z_1^{YN}$ decreases with $p_0$.

In YN scenario, given $p_2$, firm one makes a decision on $z_1$ and $p_1$ by functions similar to (3.5)-(3.6) with $\tilde{D}_1^{0T} = \tilde{D}_1^0 + \alpha_2(\tilde{D}_1^0 - c_2 + L_2)^+$. By taking derivatives of two functions w.r.t. $p_0$ we can get

$$\frac{dz_1}{dp_0} = \frac{b(2-\Pr(\tilde{D}_1^{0T} > z_1))}{[\Pr(\tilde{D}_1^{0T} > z_1)]^2 - 2bp_1f_{\tilde{D}_1^{0T}}(z_1)}.$$

Assuming IFR of $\tilde{D}_1^{0T}$, similar to proposition 3.7 we can show that the solutions $(z_1,p_1)$ are unique and make $[\Pr(\tilde{D}_1^{0T} > z_1)]^2 - 2bp_1f_{\tilde{D}_1^{0T}}(z_1)) < 0$. So $\frac{dz_1}{dp_0} < 0$.

Since the best response of firm two for $p_2$ does not change with $p_0$, at equilibrium $z_1^{YN}$ decreases with $p_0$.

Step 2. In YN scenario when $p_0$ is very small, by function (3.5) $z_1$ will be set to the largest possible value, i.e., $z_1 = c_1 - L_1$. Let $p_0$ be the value of $p_0$ such that the boundary solution $z_1 = c_1 - L_1$ is the optimal. So we have

$$p_0 = p_1 \Pr(c_1 - L_1 < \tilde{D}_1^{0T})$$

$$a - 2bp_1 + bp_0 + E \min\{c_1 - L_1, \tilde{D}_1^{0T}\} = 0.$$

Which are equivalent to

$$a - 2bp_1 + bp_1 \Pr(c_1 - L_1 < \tilde{D}_1^{0T}) + E \min\{c_1 - L_1, \tilde{D}_1^{0T}\} = 0.$$

This is exactly the reaction function of firm one in the NN scenario. Solving the game we have $p_1^{NN} = p_1^{YN}$ and $z_1^{YN} = c_1 - L_1^{YN} = c_1 - L_1^{NN}$.

Step 3. For any $p_0 \geq p_0$, we have $z_1^{YN} \leq c_1 - L_1^{NN}$.

The expected demand of firm two is larger in YN scenario, so firm two is also better off. $\blacksquare$

Scenario 3 YY. Both use RM, this scenario has been analyzed in section 3.5. Following
the above logic, we can find that after the follower implements RM, he will benefit also. Both players see revenue increase in YY than in NN. The intuition is that by setting the protection level, both firms can reduce the capacity waste, and also increase the expected demand of the competitor.

**Corollary 3.8** With only seat inventory competition, there is a first mover advantage in implementing RM. Both airlines implementing revenue management is the Pareto dominant market equilibrium.

So in a market with only seat allocation competition, implementing RM makes the industry also better off, leading to a win-win situation.

**With Both Price and Seat Inventory Competition**

We study three scenarios to see the effect of RM on profitability under two levels of competition. The games in scenario 1 and 2 have at least mixed strategy equilibria.

**Scenario 1. NN.** No firms use RM, the game is

\[
\max_{p_1} E \pi_i = p_1 E \min \{c_i, \tilde{D}_i + \alpha_j (\tilde{D}_j - c_j)^+\} = p_1 (E \min \{c_i - L_i, \tilde{D}_i^0 + \alpha_j (\tilde{D}_j^0 - c_j + L_j)^+\} + L_i)
\]

**Scenario 2. YN.** Only firm one uses RM, the game is

\[
\max_{z_1, p_1} (c_1 - L_1 - z_1)p_0 + p_1 (L_1 + E \min \{\tilde{D}_1^0 + \alpha_2 (\tilde{D}_2^0 - c_2 + L_2)^+, z_1\})
\]

\[
\max_{p_2} p_2 (E \min \{c_2 - L_2, \tilde{D}_2^0 + \alpha_1 (\tilde{D}_1^0 - z_1)^+\} + L_2)
\]

where \(L_i = a - (b + \theta)p_i + \theta p_j\).

**Scenario 3. YY.** This is the game we analyze in the main content of section 3.6.

What is the effect of implementing RM?

As before, firm one will benefit if using RM unilaterally because it can optimally choose \(z_1\) given player two fixes \(z_2 = c_2 - L_2\). For firm two, with both levels of competition, by setting \(p_2\) optimally given \(p_1\) and \(z_1\), its profit increases with \(p_1\), and decreases with \(z_1\).
\[
\frac{\partial E_{n_2}}{\partial p_1} = \theta p_2 (1 - \Pr(\tilde{D}_2^{gt} > c_2 - L_2)) > 0,
\]
\[
\frac{\partial E_{n_2}}{\partial z_1} = -\alpha_1 p_2 \Pr(\tilde{D}_2^{gt} < c_2 - L_2, \tilde{D}_1^0 > z_1) < 0.
\]

Combining the results of the last two subsections, we conjecture that firm two also benefits, and both firms see increases in their revenue in YY than NN.

3.9 Conclusions

This chapter places the classical revenue management problem in a competitive context. We ask the question: how should management make pricing and seat allocation decisions in such a competitive market? Do the conventional approaches (models and algorithms based on a monopoly market) give us the appropriate strategies?

We apply a noncooperative game theoretic methodology. We first make an equilibrium analysis under two extreme competitive markets; a market with price competition only and a market with seat inventory competition only. We prove the existence and uniqueness of equilibrium strategies, and also characterize properties of equilibrium points. We find that decisions on pricing and safety protection level, in a price competition only market, should be lower than with a monopoly, and they become even lower if the price competition is fiercer or the number of firms is larger. Both firms suffer when price competition becomes stronger. However, in a market with only seat allocation competition, airlines should set a higher price and safety protection level than the monopoly case, and these become higher if the consumer spill rate is larger or the number of firms is greater.

We then explore the same question using both levels of competition and find that the strategy needs to be adjusted according to the relative degree of the two kinds of competition. If price competition dominates (larger substitutability factor \(\theta\), smaller spill rate \(\alpha\)), then management should set a lower price and safety protection level than the monopoly. If inventory competition dominates (smaller substitutability factor \(\theta\), larger
spill rate \( \alpha \), then management should adjust price and safety protection level higher than the monopoly. Interestingly, if the two levels of competition are more evenly matched, then a lower price and higher safety protection level than the monopoly is suggested. So in this case the high fare customers’ welfare is maximized in the duopoly market. The negative effect of price competition on profitability can be mitigated by the positive effect of seat inventory competition. Counter to conventional results, profits from a duopoly may not always fall with degree of price competition \( (\theta) \). We find that with a high spill rate, slightly introducing price competition can benefit both competitors and the total industry.

Besides these key questions, we also address several other issues. A higher equilibrium price for a full fare customer implies that a lower total protection level is provided. Fiercer price competition in a full fare market induces competitors to set a higher total protection level, which is higher than what the monopoly would set. So the full fare customer is better off under a duopoly. The discounted customers are worse off because they are more likely to be turned down.

We also show that the adoption of RM always benefits the first mover, but the follower also benefits and both firms as adopters is a Pareto market equilibrium.

Our results may provide some managerial insights for RM managers. Most RM models or algorithms applied so far only consider a monopoly market, this paper illustrates the importance of incorporating competition, and adjusting prices and seat allocation to the market environment. Setting prices and protection levels according to a monopoly model is neither a best reply to competitors, nor a market equilibrium. Firms can always benefit by using the strategic best reply corresponding to their competitors’ strategies.

Our results on consumer welfare and total industry profits can also provides some insights for government regulation. For example, overall a RM technique is to be adva-
cated, and in addition slightly introducing price competition in a higher overflow market is beneficial to the industry. Government can also protect certain group of customers by allowing collusion. For example, in a market where price competition dominates, allowing collusion can increase low fare customers’ welfare because less seats will be protected.

An natural extension to this model is to incorporate into the pricing effect both a scale and location factor (Agrawal and Seshadri 2000), then \( A = S_i D_i^0 + L_i \), where \( S_i = S_i(p_i, p_j) \), \( L_i = L_i(p_i, p_j) \). Unfortunately the methodology of supermodular games does not work well for this model and a different approach will be needed.

3.10 Appendix Details for some of the Proofs

Let \( Y_i \) be random, \( y_i \) be real, and \( \varepsilon > 0 \). We have:

1. \( A = E \min \{Y, y\}, \frac{dA}{dy} = \Pr(Y > y) \) by simple algebra.

2. \( B = E \min \{Y_1 + (Y_2 + y_2)^+, y_1\} \), \( \frac{dB}{dy_2} = \Pr(Y_1 + (Y_2 + y_2)^+ < y_1, Y_2 + y_2 > 0) \)

**Proof.** \( \frac{dB}{dy_2} = \lim_{\varepsilon \to 0} E \min \{Y_1 + (Y_2 + y_2 + \varepsilon)^+, y_1\} - E \min \{Y_1 + (Y_2 + y_2)^+, y_1\} \)

Note that \( E \min \{Y_1 + (Y_2 + y_2 + \varepsilon)^+, y_1\} - E \min \{Y_1 + (Y_2 + y_2)^+, y_1\} \)

\( = 0 \) if \( Y_1 + (Y_2 + y_2)^+ > y_1 \)

\( = \varepsilon \) if \( Y_1 + (Y_2 + y_2 + \varepsilon)^+ < y_1 \) and \( Y_2 + y_2 > 0 \)

\( = 0 \) if \( Y_1 + (Y_2 + y_2 + \varepsilon)^+ < y_1 \) and \( Y_2 + y_2 + \varepsilon < 0 \)

\( = E(Y_2 + y_2 + \varepsilon) \) if \( \beta_1 = \{(Y_1, Y_2) : Y_1 + (Y_2 + y_2 + \varepsilon)^+ < y_1 \) and \( 0 > Y_2 + y_2 > -\varepsilon (\Rightarrow \varepsilon > Y_2 + y_2 + \varepsilon > 0)\} \)

\( = E[y_1 - (Y_1 + (Y_2 + y_2)^+)] \) if \( \beta_2 = \{(Y_1, Y_2) : Y_1 + (Y_2 + y_2)^+ < y_1 \) and \( Y_1 + (Y_2 + y_2 + \varepsilon)^+ > y_1(\Rightarrow \varepsilon > 0 < Y_1 + (Y_2 + y_2)^+) < \varepsilon\} \)

So \( \frac{dB}{dy_2} = \lim_{\varepsilon \to 0} \varepsilon \Pr(Y_1 + (Y_2 + y_2 + \varepsilon)^+ < y_1, Y_2 + y_2 > 0) + \lim_{\varepsilon \to 0} E(Y_2 + y_2 + \varepsilon(\beta_1) \Pr(\beta_1) \)

\( + \lim_{\varepsilon \to 0} E[y_1 - (Y_1 + (Y_2 + y_2)^+)](\beta_2) \Pr(\beta_2) = \Pr(Y_1 + (Y_2 + y_2)^+ < y_1, Y_2 + y_2 > 0) \)

2. \( C = \Pr(Y_1 + (Y_2 + y_2)^+ > y_1), \frac{dC}{dy_2} = f_{Y_1 + (Y_2 + y_2)^+}(y_2 + y_2 > 0(y_1) \Pr(Y_2 + y_2 > 0) \)

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Proof. $\frac{dC}{dy_2} = \lim_{\varepsilon \to 0} \frac{\Pr(Y_1 + (Y_2 + y_2 + \varepsilon)^+ > y_1) - \Pr(Y_1 + (Y_2 + y_2)^+ > y_1)}{\varepsilon}$

Note that $\Pr(Y_1 + (Y_2 + y_2 + \varepsilon)^+ > y_1) - \Pr(Y_1 + (Y_2 + y_2)^+ > y_1)$

= 0 if $Y_2 + y_2 + \varepsilon < 0$

= $\Pr(Y_1 + Y_2 + y_2 + \varepsilon > y_1) - \Pr(Y_1 + Y_2 + y_2 > y_1)$ if $Y_2 + y_2 > 0$

= $\Pr(Y_1 + Y_2 + y_2 + \varepsilon > y_1) - \Pr(Y_1 > y_1)$ if $0 > Y_2 + y_2 > -\varepsilon (\Rightarrow 0 < Y_2 + y_2 + \varepsilon < \varepsilon)$

So $\frac{dC}{dy_2} = \lim_{\varepsilon \to 0} \frac{f_{Y_1+Y_2+y_2}(y_1)\varepsilon \Pr(Y_2+y_2>0)}{\varepsilon}

+ \lim_{\varepsilon \to 0} \frac{[\Pr(Y_1+Y_2+y_2+\varepsilon>y_1)-\Pr(Y_1>y_1)]\Pr(0<Y_2+y_2+\varepsilon<\varepsilon)}{\varepsilon}

= f_{Y_1+Y_2+y_2}(y_1) \Pr(Y_2 + y_2 > 0)$

3. $D = \Pr(Y_1 + \alpha(Y_2 + y_2)^+ > y_1)$, $\frac{dD}{d\alpha} = \int_{-\infty}^{\infty} (\overline{Y}_2+y_2)f_{Y_1|Y_2=\overline{Y}_2}(y_1-\alpha(\overline{Y}_2+y_2))f_{\overline{Y}_2}(\overline{Y}_2)d\overline{Y}_2$

Proof. $D = \int_{-\infty}^{\infty} \Pr(Y_1 > y_1 - \alpha(\overline{Y}_2+y_2)|Y_2 = \overline{Y}_2)f_{\overline{Y}_2}(\overline{Y}_2)d\overline{Y}_2 + \int_{-\infty}^{y_2} \Pr(Y_1 > y_1|Y_2 = \overline{Y}_2)f_{\overline{Y}_2}(\overline{Y}_2)d\overline{Y}_2$

Taking derivatives, we have the result. 

4. $F = \E \min\{Y_1 + \alpha(Y_2 + y_2)^+, y_1\}$, $\frac{dF}{d\alpha} = \Pr(Y_1 + \alpha(Y_2 + y_2)^+ < y_1)\E[(Y_2 + y_2)^+|Y_1 + \alpha(Y_2 + y_2)^+ < y_1]$

Proof. $\frac{dF}{d\alpha} = \frac{E\min\{Y_1 + (\alpha+\varepsilon)(Y_2 + y_2)^+, y_1\} - \E\min\{Y_1 + \alpha(Y_2 + y_2)^+, y_1\}}{\varepsilon}$

Note that $E\min\{Y_1 + (\alpha+\varepsilon)(Y_2 + y_2)^+, y_1\} - E\min\{Y_1 + \alpha(Y_2 + y_2)^+, y_1\}$

= 0 if $Y_1 + \alpha(Y_2 + y_2)^+ > y_1$

= $\varepsilon E(Y_2 + y_2)^+ \iff \beta_3 = \{(Y_1, Y_2) : Y_1 + (\alpha+\varepsilon)(Y_2 + y_2)^+ < y_1\}$

= $E[y_1 - (Y_1 + \alpha(Y_2 + y_2)^+)] \iff \beta_4 = \{(Y_1, Y_2) : Y_1 + \alpha(Y_2 + y_2)^+ < y_1 < Y_1 + (\alpha + \varepsilon)(Y_2 + y_2)^+ \Rightarrow 0 < y_1 - [Y_1 + \alpha(Y_2 + y_2)^+] < \varepsilon(Y_2 + y_2)^+]}$

So $\frac{dF}{d\alpha} = \lim_{\varepsilon \to 0} \frac{\varepsilon E[(Y_2+y_2)^+|\beta_3] \Pr(\beta_3)}{\varepsilon} + \lim_{\varepsilon \to 0} \frac{E[y_1 - (Y_1 + \alpha(Y_2 + y_2)^+)|\beta_4] \Pr(\beta_4)}{\varepsilon}$

= $\Pr(Y_1 + \alpha(Y_2 + y_2)^+ < y_1)\E[(Y_2 + y_2)^+|Y_1 + \alpha(Y_2 + y_2)^+ < y_1]$ 

Similarly for $\varepsilon < 0$, we get identical left-hand derivatives.

5. $f_{\overline{Y}_2|\overline{Y}_2 > z_j}(z_i) \Pr(\overline{Y}_2 > z_j) < f_{\overline{Y}_2}(z_i)$

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Proof. \( \Pr(\bar{D}_i^{0T} > z_i) = \Pr(\bar{D}_i^{0T} > z_i | \bar{D}_j^0 > z_j) \Pr(\bar{D}_j^0 > z_j) + \Pr(\bar{D}_i^{0T} > z_i | \bar{D}_j^0 < z_j) \Pr(\bar{D}_j^0 < z_j) \)

Taking derivatives w.r.t. \( z_i \),

\[
f_{\bar{D}_i^{0T}}(z_i) = f_{\bar{D}_i^{0T}|\bar{D}_j^0 > z_j}(z_i) \Pr(\bar{D}_j^0 > z_j) + f_{\bar{D}_i^{0T}|\bar{D}_j^0 < z_j}(z_i) \Pr(\bar{D}_j^0 < z_j) > f_{\bar{D}_i^{0T}|\bar{D}_j^0 > z_j}(z_i) \Pr(\bar{D}_j^0 > z_j)
\]

Figures

(we use straight lines for simplicity)

Figure 3.1 For Proposition 3.8.
Figure 3.2 For Proposition 3.12.

Figure 3.3 For Proposition 3.12
Figure 3.4 For Proposition 3.12

Figure 3.5 to 3.7. Protection level, price and profit under duopoly and monopoly:

Price competition only
Figure 3.5 Protection level

Figure 3.6 Retail price

Figure 3.7 Profit

Figure 3.8 to 3.10. Protection level, price and profit under duopoly and monopoly: Seat inventory competition only

Figure 3.8 Protection level

Figure 3.9 Retail price
Figure 3.10 Profit

Figure 3.11 to 3.12 Profit as a function of theta and spill rate under duopoly: Price and seat inventory competition

Figure 3.11

Figure 3.12.
4 Understanding Competition between Newsvendors

Abstract: The Newsvendor model is a stylized archetype that serves as a basic building block in inventory/supply chain theory. A newsvendor stocks a perishable product (e.g., newspaper) in the face of unknown demand. When demand is realized, the newsvendor finds he/she has either stocked too many, investing unnecessary purchase costs or too few, either losing revenue or facing a lost sales penalty. Extensions of many types have been made to this stylized model such as including price-sensitive demand. Some extensions to include competition between newsvendors have been made but most results are based on the assumption of fixed prices. The centrality of this stylized model to inventory/supply chain theory is such that we need to considerably extend our understanding of this model within a competitive framework. It is the purpose of this chapter to add to this understanding. We find that the Newsvendors should adjust their inventory (safety stock or total inventory) and pricing strategies responsively to the nature of the competitive market. The profits of the Newsvendors and their suppliers are also different in different competitive contexts. Both the Nash equilibrium strategy and the players’ profits are influenced by demand correlation and variability, but in different ways under different competitive scenarios. These results provide some theoretical basis for the selection of strategies by newsvendors operating in competitive market.

4.1 Introduction

The Newsvendor model has long been used as a fundamental building block to analyze the single period single facility inventory problem (Lee and Nahmias 1993). The model describes a newsvendor stocking a perishable product (e.g., newspaper) in the face of an unknown demand. When the demand is realized, the newsvendor finds he/she has either stocked too many, investing unnecessary purchase costs or too few, either losing revenue
or facing a lost sales penalty. The simplicity of the model has provided valuable insights. It has however typically been only considered in the context of a monopoly or oligopoly with exogenous price rather than in a relatively general competitive framework.

For example, in a market where newsvendors sell substitutable products, consumers will select a vendor according to their preferred vendor and also the prices of other vendors. When product is unavailable at their first choice, they may naturally try others. So newsvendors typically face two types of competition: price competition and inventory competition. They will gain customers with price and secure the sale with availability. How should vendors in this competitive market make inventory and pricing decisions? How do different markets drive vendors to behave differently? How does demand correlation and variability affect these results? What is the implication for suppliers to the newsvendors? How about the consumers’ welfare? This chapter explores these questions.

There are two streams of related research. One stream is the price sensitive monopoly Newsvendor problem. Whitin (1955) is an early paper, but Petruzzi and Dada (1999) give an excellent review and extension. The central question is how the structure of optimal pricing and safety stock policies relate to the uncertainty of the demand. There are usually two ways of modeling demand random components: additive demand case \( D(p, \varepsilon) = L(p) + \varepsilon \) and multiplicative demand case \( D(p, \varepsilon) = L(p)\varepsilon \). Petruzzi and Dada (1999) provide a unified framework for the problems they considered.

The other stream of research is the competitive Newsvendor problem but with the price exogenously given; competition arising from consumers switching among firms when product is not available at their first choice. This ‘newsvendor game’, as it has been called, has been studied by Parlar (1988), Lippman and McCardle (1997), Mahajan and van Ryzin (1999), among others. The main result is that competition leads to higher industry inventories. (also refer to the literature review in chapter 1).
In this chapter competition is reflected both directly through retail product prices and indirectly through availability. Although the aggregate level of demand can be stimulated by prices, the allocation of demand among retailers is determined by a substitution factor $\theta$. This type of model is common in the marketing and economic literature and reflects either a degree of substitutability of the product or some other material feature of the retailer (e.g., accessibility). Either way, a higher $\theta$ makes a retailer's demand more sensitive to price differentials. In this sense the competition is active and affects the expected level of demand at a retailer.

There is also a passive level of competition. Should a retailer be unable to satisfy demand then a proportion, $\gamma$, of unsatisfied demand will be switched to a competitor. Thus this model is unlike the market share inventory competition (e.g., Boyaci and Gallego 2000) where the true level of demand is a function of inventories. There is no mechanism for retailers to be 'rewarded' for maintaining large inventories through increased demand; they simply (passively) get more of the demand, their own and others, when they do. So in our model price competition affects the mean level of demand and is determined before the random demand is known. Inventory competition has its effect subsequent to random demand being realized.

Bernstein and Federgruen (2002) study the equilibrium for retail industries with price and service competition. They focus on the existence and uniqueness of the stage game and the game with infinite horizon. There are several main differences between us. The first is the way of modeling demand. They assume that the consumers can perceive not only the prices but also the fill rates of all the retailers. The inventory competition in their model is ex ante; i.e., consumers make purchasing decisions by evaluating the prices and fill rates of all the retailers. So the expected demand function of a retailer is a function of all prices and fill rates. Different from them, inventory competition in our
model is ex post; i.e., after the random demand is realized, consumers switch to other retailers if not being satisfied by their first choice. Since we study competition between newsvendors (i.e., one shot game), consumers may not have time to learn the fill rate of all the retailers, we think this demand process is more reasonable for our problem. The second difference is that we not only establish the existence and uniqueness of the Nash equilibrium, but also analyze the difference in newsvendors’ strategic behavior, profits, and supplier’s profit under different competitive markets.

In this chapter we consider four types of markets: no competition, price competition, inventory competition, both price and inventory competition. We have no sense of leadership between retailers, or procurement decisions being in advance of pricing. Thus our tools are Nash equilibrium for inventory and retail prices, and our results concern comparisons between prices, inventories, retailer profits and manufacturer revenues.

We have the following main findings:

1. Newsvendors make different stocking and pricing strategies in different competitive scenarios.

2. Newsvendors suffer from price competition; but the manufacturer benefits. Newsvendors benefit from inventory competition; but the manufacturer may either suffer or benefit.

The following table illustrates the above two points.

<table>
<thead>
<tr>
<th>Retail price</th>
<th>Safety stock</th>
<th>Total inventory</th>
<th>Retailer’s profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inventory &gt; No &gt; Price</td>
<td>Inventory &gt; No &gt; Price</td>
<td>No &lt; Price</td>
<td>Inventory &gt; No &gt; Price</td>
</tr>
<tr>
<td>Inventory &gt; Both &gt; Price</td>
<td>Inventory &gt; Both &gt; Price</td>
<td>Inventory &lt; Both</td>
<td>Inventory &gt; Both &gt; Price</td>
</tr>
<tr>
<td>No &gt; Both or Both &gt; No</td>
<td>No &gt; Both or Both &gt; No</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: the comparison is among different types of competitions. ‘Inventory’ means Inventory competition. Result on Total inventory can be used for supplier’s profit.
3. Both Nash equilibrium strategies and players' profits are influenced by demand correlation and variability, but in different ways under different competitive scenarios and different profit margins.

The rest of the chapter is ordered as follows: Section 2 gives the model description and investigates the base case of no competition. Sections 3-5 consider price, inventory and both competition respectively. Finally, section 6 has some extensive numerical experiments.

4.2 The Model

To capture the nature of the problem without unnecessary complexity, we study a market with only two newsvendors buying products from one manufacturer. The manufacturer sets a unit cost $w$, then retailers procure inventories $y_i (i = 1, 2)$ from the manufacturer and then sell at price $p_i$. The market demand $\tilde{D}_i$ is uncertain and not realized until after the replenishment and pricing decisions.

We assume an additive demand model; $\tilde{D}_i = \tilde{D}_i^0 + L_i(p_i, p_j)$, consisting of a random component $\tilde{D}_i^0$ and a price sensitive deterministic component $L_i(p_i, p_j)$.

When consumers can switch a proportion $\gamma_i \in [0, 1]$ to vendor $i$ having not been satisfied by $j$ we set $\tilde{D}_i^+ = \tilde{D}_i + \gamma_i(\tilde{D}_j - y_j)^+$.

When we wish to consider only price sensitive rather than price competitive demand we replace $L_i(p_i, p_j)$ by $L_i(p_i)$.

The total demand of retailer $i$ thus includes both direct demand and the overflow (indirect) demand from the competitor.

We adopt a demand model common in the economic and marketing literature $L_i(p_i, p_j) = a - bp_i + \theta(p_j - p_i)$ (Shubik and Levitan 1980) and $L_i(p_i) = a - bp_i$ (Varian 1992). This demand function can be derived from consumers' utility maximization and captures con-
sumer heterogeneity and product differentiation.

It will be convenient to define \( Z_j \) ('safety stock') as the transformation of inventory level \( y_i : y_i = L_i + z_i \). \( \tilde{D}_i^{0T} = \tilde{D}_i^{0} + \gamma_i(\tilde{D}_i^{0} - z_j)^+ \) helps to simplify notation.

In general we take \( f_X(x) \) as the density function of a random variable \( X \), and assume \( f_X(x) > 0 \) and finite. We assume an increasing failure rate, \( r_X(x) = f_X(x)/\Pr(X > x) \), which is true for many commonly used distributions. There are no other assumptions (e.g. independence) on the joint random variables.

As a benchmark, we first consider newsvendors without any competition. Newsven­
dors maximize their profits by choosing an optimal price and inventory:

\[
\max_{y_i, p_i} -w y_i + p_i \min(D_i, y_i) = \max_{z_i, y_i} -w(z_i + a - b p_i) + p_i[a - b p_i + E \min(\tilde{D}_i^{0}, z_i)]
\]

An optimal safety stock and price exists. The optimal solution satisfies,

\[
\Pr(\tilde{D}_i^{0} > z_i) = \frac{w}{p_i}
\]

\[
p_i = \frac{a+b w+E \min(\tilde{D}_i^{0}, z_i)}{2 b}
\]

The method in step 1 of Proposition 4.2 (next section) shows that this solution is unique. Let this solution be: \( p_i = p_i(w), z_i = z_i(w), y_i = y_i(w) \).

The manufacturer’s problem is

\[
\max_w w[y_i(w) + y_j(w)]
\]

There exists an optimal \( w \) that maximizes the manufacturer’s profit. It is obvious that the manufacturer’s revenue increases with wholesale price and the retailers’ replenishment.

### 4.3 Newsvendors under Price Competition

In this scenario, we introduce only price competition. So \( \tilde{D}_i = \tilde{D}_i^{0} + a - b p_i + \theta(p_j - p_i) \). We establish the existence and uniqueness of Nash equilibrium, compare the Nash strategies and retailers’ profits with the case of no competition, and look at how the price
competition influence the supplier's profit and consumer welfare.

Each newsvendor chooses price and inventory level to maximize:

\[ E\pi_i = -wy_i + p_i E \min \{\bar{D}_i, y_i\} = -w(L_i + z_i) + p_i(L_i + E \min \{\bar{D}_i^0, z_i\}) \] (4.1)

where \( L_i = a - (b + \theta)p_i + \theta p_j \).

**Lemma 4.1** Function (4.1) is supermodular in \((p_i, z_i)\), having increasing differences in \((p_i, p_j), (z_i, p_j), (p_i, z_j), \) and \((z_i, z_j)\).

**Proof.** 
\[ E\pi_i = (p_i - w)L_i - wz_i + p_i E \min \{\bar{D}_i, z_i\}. \] By Lemmata A2-1 and A2-3; this is supermodular in \((p_i, z_i)\). Supermodularity is a result of being preserved by summation.

Similarly, (4.1) is supermodular in \((p_i, p_j), (z_i, p_j), (p_i, z_j), (z_i, z_j)\) (Definition A2). ■

**Proposition 4.1** There exists a pure Nash equilibrium strategy characterized by:

\[ \Pr(\bar{D}_i^0 > z_i) = \frac{w}{p_i} \] (4.2)

\[ p_i = \frac{a + (b + \theta)w + \theta p_j + E \min \{\bar{D}_i^0, z_i\}}{2(b + \theta)} \] (4.3)

**Proof.** By Lemma 4.1 and Definition A3, the game is supermodular, thus the pure Nash equilibrium exists (Lemma A4). (4.2)-(4.3) are first order conditions of function (4.1). ■

As it is hard to predict how the vendors will behave with multiple equilibria, the natural question is: Is the equilibrium unique?

**Proposition 4.2** The Nash equilibrium is unique.

**Proof.** First we show that given any \( p_j \), the best response, the solution of (4.2)-(4.3), is unique, and satisfies \( 2(b + \theta) - \Pr(\bar{D}_i^0 > z_i)/[p_j r_{\bar{D}_i^0}(z_i)] = M(p_i) > 0 \).
Uniquely solve $z_i(p_i)$ from (4.2) and substitute into (4.3) to give

$$a + (b + \theta)w + \theta p_j + E \min\{\tilde{D}_i^0, z_i(p_i)\} - p_i = 0 \quad (4.4)$$

Note that the LHS of (4.4) is concave in $p_i$, starts positive (when $p_i = w$) and goes to negative finally as $p_i$ increases. So there must exist a unique $p_i$ that solves (4.4), lying at a decreasing part of the curve, i.e., $Pr(\tilde{D}_i^0 > z_i)/[2(b + \theta)p_j \tilde{D}_i^0(z_j)] < 1$, or $M(p_i) > 0$.

This finishes the proof of unique best response of $p_i$ and $z_i$ given $p_j$.

Second we prove a unique equilibrium. Rearranging the terms of (4.4), the best response function of player $i$ to any $p_j$ satisfies,

$$p_i = \frac{a - (b + \theta)w + \theta p_j + E \min\{\tilde{D}_i^0, z_i(p_j)\}}{2(b + \theta)}.$$

In the $(p_i, p_j)$ space, when $p_i = w$, $p_j = [(b + \theta)w - a - E \min\{\tilde{D}_j^0, z_j\}] / \theta < 0$ and $dp_j/dp_i = M(p_i)/\theta > 0$, which is increasing in $p_i$. So this curve (1) is increasing and convex.

Similarly, the best response of player $j$ to any $p_i$ satisfies $p_j = \frac{a + (b + \theta)w + \theta p_i + E \min\{\tilde{D}_j^0, z_j(p_j)\}}{2(b + \theta)}$.

When $p_j = w$, we have $p_i = [(b + \theta)w - a - E \min\{\tilde{D}_i^0, z_i\}] / \theta < 0$, and $dp_j/dp_i = \theta/M(p_j) > 0$, which is decreasing in $p_i$. So this curve (2) is increasing and concave.

The difference between curve 2 and 1 is also concave, starting positive and with a negative slope $\theta/(2(b + \theta) - 2(b + \theta))/\theta < 0$ as $p_i$ approaches infinity, so there exist a unique intersection between them. Thus the equilibrium is unique. \[\square\]

Let the equilibrium be $(p_i^*, z_i^*)$. We analyze the equilibrium behavior of the newsvendors.

**Proposition 4.3** 1. A high price indicates a high equilibrium safety stock but a low total stock. 2. A high safety stock indicates a high equilibrium price and a high total stock. 3. $dp_i^*/d\theta < 0, dz_i^*/d\theta < 0$. 4. In symmetric equilibrium $dy_i^*/d\theta > 0$, $d\pi_i^*/d\theta < 0$.

**Proof.** 1. We will prove that given $p_i$, $dz_i^*/dp_i > 0$, $dy_i^*/dp_i < 0$ if $p_i > \overline{p}_i$ (defined
below). Further more, \( p_i^* > \bar{p}_i \). We use the IFR of \( r_{D_0} \) throughout the proof.

Directly from function (4.2) we have \( dz^*/dp_i = 1/p_i r_{\bar{D}_i}(z^*_i) > 0 \).

As \( y_i^* = z_i^* + L_i \), \( dy_i^*/dp_i = dz^*/dp_i - (b + \theta) = 1/p_i r_{\bar{D}_i}(z^*_i) - (b + \theta) \).

Define \( \bar{p}_i = 1/(b + \theta) \) where \( z_i(\bar{p}_i) \) is solved from (4.2), then \( dy_i^*/dp_i < 0 \) if \( p_i > \bar{p}_i \).

Now we show \( p_i^* > \bar{p}_i \). Recall that given \( p_i^* \), \( p_i^* \) is uniquely solved from function (4.4).

\( p_i^* > \bar{p}_i \) if
\[
\frac{a + (b + \theta) w + \theta p_i^* + E \min \{ D_0^0, z_i(\bar{p}_i) \}}{2(b + \theta)} - \bar{p}_i = \frac{a - (b + \theta) \bar{p}_i + \theta p_i^* + E \min \{ D_0^0, z_i(\bar{p}_i) \} - \bar{p}_i (1 - \Pr(D_0^0 > z_i(\bar{p}_i)))}{2(b + \theta)} > 0.
\]

It is sufficient to show \( E \min \{ D_0^0, z_i(\bar{p}_i) \} > (1 - \Pr(D_0^0 > z_i(\bar{p}_i)))/r_{\bar{D}_i}(z_i(\bar{p}_i)) \), which is true for \( D_0^0 > 0 \), IFR with \( r_{\bar{D}_i}(0) > 0 \).

2. Given \( z_i \), \( dp_i^*/dz_i = \Pr(D_0^0 > z_i)/[2(b + \theta)] > 0 \).

\( dy_i^*/dz_i = 1 - (b + \theta) dp_i^*/dz_i + \theta dp_i^*/dz_i = 1 - \Pr(D_0^0 > z_i)/2 + \frac{dp_i^*}{dp_i} \frac{dp_i^*}{dz_i} > 0 \).

3. Taking derivatives of function (4.4) w.r.t. \( \theta \), we have the best response \( dp_i/d\theta < 0 \) since
\[
\frac{dp_i}{d\theta} M(p_i) = \frac{-a - bp_j + E \min \{ D_0^0, z_i(\bar{p}_i) \}}{(b + \theta)}.
\]

Similarly we have \( dp_j/d\theta < 0 \). From Proposition 4.1 the best response functions are all increasing in \( (p_i, p_j) \) space. Both of them decrease with \( \theta \), then the Nash equilibrium prices decrease with \( \theta \) too. The result on \( z_i \) is by the chain rule: \( dz_i^*/d\theta = \frac{dz_i^*}{dp_i} \frac{dp_i}{d\theta} \).

4. For the total inventory level,
\[
\frac{dy_i^*}{d\theta} = \frac{dz_i^*}{d\theta} - (b + \theta) \frac{dp_i^*}{d\theta} + \theta \frac{dp_i^*}{d\theta} = \frac{dz_i^*}{d\theta} (1 - bp_i^* r_{\bar{D}_i}(z_i^*))
\]

because by symmetry \( \frac{dp_i^*}{d\theta} = \frac{dp_j^*}{d\theta} \). We omit the subscript.

Define \( \hat{p} : 1 - b\hat{p} r_{\bar{D}_i}(z(\hat{p})) = 0 \). Since \( dz^*/d\theta < 0 \) from above, \( dy^*/d\theta > 0 \) if \( p^* > \hat{p} \).

To show that \( p^* > \hat{p} \) we use a methodology similar to above. A sufficient condition is,
\[
E \min \{ D_0^0, z(\hat{p}) \} \geq (b + \theta) \hat{p} \geq b\hat{p} = \frac{1}{r_{D_0}(z(\hat{p}))}.
\]
This proposition makes three points. 1. Higher equilibrium price ‘signal’ higher fill rate but lower total inventory; higher equilibrium safety stock leads to higher total inventory but needs higher retail price to support. 2. Operating in a market with fiercer competition (\(\theta\) greater), retailers need to set a lower price, lower safety stock, but a higher total stock. Price competition drives the retail price down, this in turn promotes the deterministic demand and the stock for the deterministic demand, which can counter the reduction in safety stock. 3. Fiercer price competition damages retailers.

How does this compare with no competition at all? For this we use the special case of \(\theta = 0\), thus:

**Corollary 4.1** Price competitive newsvendors set a lower safety stock, lower retail price, but a higher total stock compared with the case without competition.

What is the impact of the introduction of price competition on the manufacturer (supplier)?

**Proposition 4.4** The manufacturer gains more profit when her downstream retailers are involved in price competition. This is true for both an endogenous or exogenous wholesale price.

**Proof.** We use the superscript \(P\) to denote the price competition scenario, and \(N\) for no competition. The manufacturer’s revenue is \(w[y_i(w) + y_j(w)]\). With an exogenous price \(w\), the manufacturer’s profit under price competition is larger since \(y_i^P(w) \geq y_i^N(w)\) for \(i = 1, 2\) by Corollary 4.1.

Now consider the endogenous wholesale price case. With price competition, the manufacturer’s problem is \(\max_w w[y_i^P(w) + y_j^P(w)]\). An optimal exists, say \(w^P\).
Without competition, the manufacturer solves \( \max_w w[y_i^N(w) + y_j^N(w)] \), with an optimal \( w^N \).

Then we have \( w^P[y_i^P(w^P) + y_j^P(w^P)] \geq w^N[y_i^P(w^N) + y_j^P(w^N)] \geq w^N[y_i^N(w^N) + y_j^N(w^N)] \). □

A price competitive market favors the manufacturer but hurts the newsvendors. How about the consumers? Suppose that the consumer surplus function is of the form \( U_i = R + \alpha \Pr(D_i^0 < z_i) - p_i, \alpha > 0 \). Where \( R \) is the utility gain from the product, \( \alpha \Pr(D_i^0 < z_i) \) is the component gained from the purchasing experience.

Comparing between the scenarios with/without price competition, we have \( U_i^P - U_i^N = \alpha \Pr(D_i^0 < z_i^P) - p_i^P - \alpha \Pr(D_i^0 < z_i^N) + p_i^N = (p_i^N - p_i^P) - \alpha w\left(\frac{1}{p_i^P} - \frac{1}{p_i^N}\right) \). So the difference depends on how the consumers value fill rate versus price. If \( \alpha \) is large, i.e., consumers value the service level (fill rate) much more than price, then the consumer will prefer a market without price competition.

4.4 Newsvendors under Inventory Competition

We now consider competition from having a larger inventory, where consumers take a proportion of their unsatisfied demand to the next retailer. The demand is \( D_i^T = D_i^0 + \alpha - bp_i + \gamma_i(D_j - y_j)^+ \), decreasing with own price, but increasing with the overflow from the competitor. After analyzing the existence and uniqueness of the Nash equilibrium, we investigate the equilibrium properties. We then compare newsvendors’ strategies, profits and the supplier’s profit with other scenarios.

Newsvendors set prices and safety stocks to maximize: \( E\pi_i = -wy_i + p_i E \min\{D_i^T, y_i\} \).

Or,

\[
E\pi_i = -w(L_i + z_i) + p_i(L_i + E \min\{D_i^0 + \gamma_i(D_j^0 - z_j)^+, z_i\})
\]

\( (4.5) \)

where \( L_i = a - bp_i \).
Lemma 4.2 Function (4.5) is supermodular in \((p_i, z_i)\), and has increasing differences in
\((p_i, -z_j), (z_i, -z_j), (p_i, \gamma_i), (z_i, \gamma_i), (p_i, -p_j), (z_i, -p_j), (p_i, -\gamma_j), (z_i, -\gamma_j)\).

Proof. Rewrite (4.5) as
\[
E\pi_i = (p_i - w) L_i - wz_i + p_i E\min\{\tilde{D}_i^0 + \gamma_i (\tilde{D}_j^0 - z_j)^+, z_i\}.
\]
Like Lemma 4.1, we have supermodularity in \((p_i, z_i)\). By Lemmata A2-3, A2-2, A1 we have
increasing differences in \((p_i, -z_j), (z_i, -z_j), (p_i, \gamma_i), (z_i, \gamma_i)\). By Definition A2 and Lemma
A1 we have increasing differences in the last four. ■

Proposition 4.5 There exists a pure Nash equilibrium strategy, characterized by

\[
\Pr(\tilde{D}_i^0 + \gamma_i (\tilde{D}_j^0 - z_j)^+ > z_i) = \frac{w}{p_i} \quad (4.6)
\]

\[
p_i = \frac{a + bw + E\min\{\tilde{D}_i^0 + \gamma_i (\tilde{D}_j^0 - z_j)^+, z_i\}}{2b} \quad (4.7)
\]

Proof. By Lemma 4.2 and Definition A3, the game is supermodular and a Nash equi­
librium exists (Lemma A4). Equations (4.6)-(4.7) are the first order conditions of (4.5).

■

Proposition 4.6 There exists a unique Nash equilibrium in the symmetric game.

Proof. To simplify the presentation, use \(\gamma_i = 1\). The same process holds for \(\gamma_i < 1\).
First we show that there exists a unique symmetric equilibrium. As in proposition 4.2,
given any \((p_j, z_j)\), the solutions of (4.6)-(4.7) (the best response of player \(i\)) are unique
and make
\[
2bp_i r_{\tilde{D}_i^{0T}}(z_i) - \Pr(\tilde{D}_i^{0T} > z_i) > 0.
\]

Solve \(p_i = w / \Pr(\tilde{D}_i^{0T} > z_i)\) uniquely from (4.6) and substitute into (4.7), then take
derivatives w.r.t. \(z_j\);

\[
\frac{dz_i}{dz_j} = -\frac{\Pr(\tilde{D}_i^{0T} < z_i, \tilde{D}_j^0 > z_j) + 2bw_i r_{\tilde{D}_i^{0T}}(z_i) \Pr(\tilde{D}_i^0 > z_i)}{2bw_i r_{\tilde{D}_i^{0T}}(z_i) \Pr(\tilde{D}_i^{0T} > z_i) - \Pr(\tilde{D}_i^{0T} > z_i)} < 0.
\]
similarly, $dz_j/dz_i < 0$. If there are two equilibrium $(z_i, z_j), (z'_i, z'_j)$, and assume $z_i > z'_i$ without loss of generality, then we must have $z_j < z'_j$, a contradiction. So there exist a unique symmetric equilibrium for this game.

Since the largest and smallest equilibria must be also symmetric, with unique symmetric equilibrium, we have a unique equilibrium for this game. □

If vendors are not identical, using a method by proposition 3.7 in chapter 3 we can find conditions such that the equilibrium is unique.

**Corollary 4.2** $r_{D^i_{qt}}(0) > 1/[(1 - \gamma_i)w_b], 2b > 1$ is sufficient for uniqueness.

**Proposition 4.7** 1. A high equilibrium price indicates a high equilibrium safety stock but a low total stock. 2. A high equilibrium safety stock indicates a high equilibrium price and a high total stock.

**Proof.** 1. Given $p_i$, directly from function (4.6) we have $dz_i^*/dp_i = 1/p_i r_{D^i_{qT}}(z_i^*) > 0$.

As $y_i^* = z_i^* + L_i$, $dy_i^*/dp_i = dz_i^*/dp_i - b = 1/p_i r_{D^i_{qT}}(z_i^*) - b$.

Define $\bar{p}_i : 1/b = \bar{p}_i r_{D^i_{qT}}(z_i(\bar{p}_i))$ where $z_i(\bar{p}_i)$ is solved from (4.6), then $dy_i^*/dp_i < 0$ if $p_i > \bar{p}_i$.

Recall that $p_i^*$ is uniquely solved from function (4.7). $p_i^* > \bar{p}_i$ if

$$\frac{a+bw+E \min(D_i^{qT}, z_i(\bar{p}_i))}{2b} - \bar{p}_i = \frac{a+b \bar{p}_i + E \min(D_i^{qT}, z_i(\bar{p}_i)) - \bar{p}_i(1-Pr(D_i^{qT} > z_i(\bar{p}_i)))}{2b} > 0.$$ 

It is sufficient to show $E \min(D_i^{qT}, z_i(\bar{p}_i)) \geq (1-Pr(D_i^{qT} > z_i(\bar{p}_i)))/r_{D^i_{qt}}(z_i(\bar{p}_i))$, which is true for $D_i^{qT} > 0$, IFR with $r_{D^i_{qt}}(0) > 0$.

2. Given $z_i$, $dp_i^*/dz_i = Pr(D_i^{qT} > z_i)/2b > 0$.

$dy_i^*/dz_i = 1 - bdp_i^*/dz_i = 1 - Pr(D_i^{qT} > z_i)/2 > 0$. □

This result is the same as the price competitive market.

**Corollary 4.3** Retailer's safety stock, retail price (if they are unique) and profit in equilibrium increase with the competitor's spill rate ($\gamma_i$), decrease with his own spill rate...
Proof. We need to show \( dz_i^*/d\gamma_i \geq 0, \ dp_i^*/d\gamma_i \geq 0, \ dz_j^*/d\gamma_j \leq 0, \ dp_j^*/d\gamma_j \leq 0, \ d\pi_i^*/d\gamma_i \geq 0, \ d\pi_j^*/d\gamma_j \leq 0. \)

From Lemma 4.2, the game is supermodular and has increasing differences in \((., \gamma_i), (., -\gamma_j)\).

By Lemma A5, we have the results.

\[
\begin{align*}
    d\pi_i^*/d\gamma_i &= \partial\pi_i^*/\partial\gamma_i + (\partial\pi_i^*/\partial z_j)(dz_j^*/d\gamma_i) \geq 0. \\
    d\pi_j^*/d\gamma_j &= (\partial\pi_j^*/\partial z_j)(dz_j^*/d\gamma_j) \leq 0. \quad \blacksquare
\end{align*}
\]

If the equilibrium is not unique, then according to Lemma A5, the largest and smallest equilibria have this property.

**Proposition 4.8** Inventory competitive newsvendors set higher safety stocks and higher prices compared with no competition.

**Proof.** Recall that under no competition, the prices and safety stocks are set by \( \Pr(D_i^0 > z_i) = \frac{w}{p_i} \Rightarrow z_i = h(p_i) \) and \( p_i = \frac{a + bw + E \min \{D_i^0, z_i\}}{2b} = g(z_i). \)

Given any \( z_j \), in the \((z_i, p_i)\) quadrant, the curve defined by (4.6) is convex increasing and always to the right of \( z_i = h(p_i) \); the curve defined by (4.7) is always higher than \( p_i = g(z_i) \) and both of them are concave increasing. At \( z_i = 0, \ h^{-1}(0) = w < g(0) = (a + bw)/(2b). \) So there exists unique intersection between \( h(p_i) \) and \( g(z_i) \). Similarly there is a unique intersection between (4.6) and (4.7). The intersection of (4.6) and (4.7) is greater than the intersection of \( h(p_i) \) and \( g(z_i) \). See Figure 4.1. \( \blacksquare \)

Recall that \( y_i = z_i + a - bp_i \). The comparison of total stocks is ambiguous. The effect of inventory competition would induce higher safety stocks; even though higher retail prices drive expected demand and hence deterministic stock down. Total stocks may go up or down. Our numerical results show this; in fact even the demand correlation and variation affect the result. Section 4.6 has more discussion on this. Figures 4.26
to 4.34 show the relationship between total inventory and correlation under different profit margins and variations. Figures 4.35 to 4.37 show the relationship between total inventory and coefficient of variation under different profit margins.

How do the two types of competition compare?

**Corollary 4.4** Inventory competitive newsvendors set higher prices and safety stocks than price competitive newsvendors. (a direct result of Corollary 4.1 and Proposition 4.8.)

In terms of total stock; compared with price competitive newsvendors, inventory competitive newsvendors set a higher price, which will induce less inventory for deterministic demand; but they also set a higher safety stock, which will increase inventory for stochastic demand. The final comparison is ambiguous. Actually the numerical results show that both can occur, as can be seen in section 4.6 and figures also 4.26 to 4.37 in the appendix.

To compare all three scenarios, we use superscript \( P \) for price competition scenario, \( I \) for inventory competition scenario, and \( N \) for no competition. The following proposition refers to retailers’ profits.

**Proposition 4.9** In equilibrium, \( E\pi_i^I^* \geq E\pi_i^N^* \geq E\pi_i^P^* \)

**Proof.** \[ E\pi_i^P = -w(L_i + z_i) + p_i(L_i + E \min\{\tilde{D}_i^0 + \gamma_i(\tilde{D}_j^0 - z_j)^+, z_i\} \]

\[ = E\pi_i^N + p_iE \min\{(z_i - \tilde{D}_i^0)^+, \gamma_i(\tilde{D}_j^0 - z_j)^+\} \geq E\pi_i^N \]

So given \( z_j^* \), \( E\pi_i^I^* = \max_{z_i, \theta} E\pi_i^I \geq \max_{z_i, \theta} E\pi_i^N = E\pi_i^N^*. \)

The second inequality is by Proposition 4.3 and \( \theta = 0 \) for the no competition case.

Inventory competitive newsvendors are better off because they have higher realized demand as a result of consumer overflow in the system, they also gain more revenue per
Consider now the manufacturer. With an exogenous wholesale price, numerical experiments show that total inventory procurements in the inventory competition scenario can be higher than, lower than or between of the other two scenarios (Figures 4.31, 4.32). The results depend on the profit margin, the coefficient of variation and the correlation.

Under inventory competition, the revenue function of the manufacturer is supermodular in \( w \); an optimal wholesale price exists. Following the same arguments as Proposition 4.4, with endogenous \( w \), all of the above three results can occur.

Some results for consumer welfare, given the same surplus function as before, are shown numerically in section 4.6.

### 4.5 Newsvendors under Both Competitions

What results are available by exploring both types of competition? The demand of vendor \( i \) is 
\[
\bar{D}_i^r = \bar{D}_i^0 + a - b p_i + \theta (p_j - p_i) + \gamma_i (\bar{D}_j - y_j)^+. 
\]
This game is not supermodular. Using a method as in proposition 3.9 in chapter 3, it can be shown that the pure Nash equilibrium strategy exists, and is unique under some conditions. Here we explore some equilibrium properties and compare the equilibrium strategies, newsvendors' profits, manufacturer's revenue and consumer welfare.

Newsvendors choose \( p_i \) and \( z_i \) to maximize:

\[
\mathbb{E} \pi_i = -w(L_i + z_i) + p_i (L_i + E \min \{ \bar{D}_i^0 + \gamma_i (\bar{D}_j - y_j)^+, z_i \}). \tag{4.8}
\]

where \( L_i = a - b p_i + \theta (p_j - p_i) \).

**Proposition 4.10** Let \( r = r_{D^0} \), with \( r' > 0 \) and \( r'/r^2 \) decreasing, there exists a pure
strategy Nash equilibrium. $r(0) > 1 / (1 - \gamma_i) w$ and $2 b + \theta > 1$ are sufficient for uniqueness.

The equilibrium is solved from

$$\Pr(\tilde{D}_i^0 + \gamma_i(\tilde{D}_j^0 - z_j)^+ > z_i) = \frac{w}{p_i} \quad (4.9)$$

$$p_i = \frac{a + (b + \theta) w + \theta p_j + E \min\{\tilde{D}_{ij}^0, z_i\}}{2(b + \theta)} \quad (4.10)$$

Let $(p_i^*, z_i^*)$ denote this equilibrium.

Similar to Proposition 4.3 and Corollary 4.3, we have the following results.

**Corollary 4.5** 1. In symmetric equilibrium, $dp_i^*/d\theta < 0, dz_i^*/d\theta < 0; dp_i^*/d\gamma_i \geq 0, dz_i^*/d\gamma \geq 0; dp_i^*/d\gamma \leq 0$.

The following comparisons are then obvious. We use a superscript $B$ to denote the scenario with both competitions.

**Proposition 4.11** 1. In symmetric equilibrium, $p^B \leq p^I, z^B \leq z^I$. 2. In symmetric equilibrium, $p^B \geq p^P, z^B \geq z^P$. 3. All of the following three can happen: $p^B \geq p^N, z^B \geq z^N; p^B \leq p^N, z^B \geq z^N; p^B \leq p^N, z^B \leq z^N$.

**Proof.** 1. Directly from the above corollary and $\theta = 0$ for the Inventory competition case.

2. Directly from Corollary 4.5 and $\gamma = 0$ for the price competition only case.

3. In the $(z, p)$ space, the curve defined by (4.9) is always to the right of the curve defined by $z_i = h(p_i)$; but the curve defined by (4.10) can be lower than, higher than or close to the curve defined by $p_i = g(z_i)$. So intersection(s) between (4.9) and (4.10) can have three circumstances compared with the intersection of $z_i = h(p_i)$ and $p_i = g(z_i)$: higher and to the right, lower and to the right, lower and to the left. •
The above proposition gives us a general picture of the difference in safety stocks and retail prices under different market conditions. Newsvendors selling substitutable products need to adjust their strategies to be responsive to market condition and consumer behavior.

How about newsvendors' profit?

**Corollary 4.6** \( E\pi^B_i \geq E\pi^P_i \).

The proof of the inequality is similar to Proposition 4.9. However both \( E\pi^B_i \geq E\pi^N_i \) and \( E\pi^B_i \leq E\pi^N_i \) can occur (Figure 4.43). Because inventory competition is good news to retailers, but price competition damages both of them. The final effect depends on the relative strength of the two. As in chapter 3, both \( E\pi^B_i \geq E\pi^I_i \) and \( E\pi^B_i \leq E\pi^I_i \) can occur.

Can we learn anything about total inventory?

**Corollary 4.7** In symmetric equilibrium. 1. \( dy_i^*/d\theta \geq 0 \). 2. \( y_i^B \geq y_i^I \).

The proof is similar to Proposition 4.3, but with the random variable being \( \bar{D}_i^{gt} \).

As with inventory competition, we can have \( y_i^B \geq y_i^P \geq y_i^N \), \( y_i^P \geq y_i^B \geq y_i^N \), and \( y_i^P \geq y_i^N \geq y_i^B \).

For the manufacturer's revenue:

**Corollary 4.8** With either endogenous or exogenous wholesale price, we have \( \pi^B_M \geq \pi^I_M \).

The reasoning is the same as Proposition 4.4. But also all the three results can happen: \( \pi^B_M \geq \pi^P_M \geq \pi^N_M \), \( \pi^P_M \geq \pi^B_M \geq \pi^N_M \), and \( \pi^P_M \geq \pi^N_M \geq \pi^B_M \).

For consumers' welfare, in the case of \( p^B \leq p^N \), \( z^B \geq z^N \), we are sure that consumers gain under both competitions compared with no competition. The next section illustrates some numerical results for other cases.
4.6 Numerical Experiments

The goals of these numerical experiments are to illustrate the theoretical results above, and to investigate cases when theoretical results have not been achieved, especially regarding total inventory ($y_i$). We also investigate sensitivity with regard to demand correlation and variability.

We assume that $\tilde{D}_i^0$ are normally distributed. We truncate negative observations to guarantee non-negative demand. We use Monte Carlo integration to evaluate $E \min\{\tilde{D}_i^0, z_i\}$ and $E \min\{\tilde{D}_i^{grt}, z_i\}$, with 10,000 random samples. We use the following combinations of the parameters.

1. **Profit margin**: we choose three groups of parameters such that the approximate profit margin can be relatively small ($p/w \approx 1.5$), medium ($p/w \approx 2$) and very large ($p/w \approx 4$). The three groups are:

<table>
<thead>
<tr>
<th>$p/w$</th>
<th>$a$</th>
<th>$b$</th>
<th>$w$</th>
<th>$\theta$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.5$</td>
<td>570</td>
<td>20</td>
<td>18</td>
<td>10</td>
<td>150</td>
</tr>
<tr>
<td>$2$</td>
<td>480</td>
<td>10</td>
<td>18</td>
<td>5</td>
<td>150</td>
</tr>
<tr>
<td>$4$</td>
<td>480</td>
<td>5</td>
<td>18</td>
<td>2.5</td>
<td>150</td>
</tr>
</tbody>
</table>

2. **Coefficient of variation (CV)**. We study three cases: low variability ($CV = 0.2$), medium variability ($CV = 0.5$), and high variability ($CV = 1$).

3. **Correlation**. We choose $\rho = -1, -0.6, -0.2, 0, 0.2, 0.6, 1$.

We assume full overflow $\gamma_i = 1$ because the effect of spill rate is not our primary concern. Chapter 3 studies the effect of $\gamma$ both analytically and numerically.

For price competition and no competition scenarios, the correlation does not affect the results. So $3 \times 3 \times 2 = 18$ cases are run. For inventory competition $3 \times 3 \times 7 = 63$ cases are run. For both competition scenarios, we choose the medium profit margin group and
3 \times 7 = 21 cases are run, which is enough for our purpose. So in total we have 102 cases, and all cases give us unique and stable equilibria.

**Results on safety stocks** (Figures 4.2 to 4.13). The first nine figures show the relationship between correlation and safety stock. The latter three pictures show the relationship between \( CV \) and safety stock (similarly for figures on price, inventory and profit). For all experiments, the results support our theory. In addition both \( z^B \geq z^N \) and \( z^B \leq z^N \) can happen (Figure 4.5 to 4.7). That is, price competition induce newsvendors to reduce safety stocks, but inventory competition promotes newsvendors to increase safety stocks. The two effects can counter each other and the safety stock under both competition can be either higher or lower than the safety stock with no competition.

Price competition and no competition scenarios are not affected by correlation. When \( \rho = 1 \), \( z^I = z^N \) and \( z^B = z^P \), i.e.: when random demands are perfectly positively correlated, there is no inventory competition between retailers. When the profit margin is medium or high, safety stocks first increase then decrease with \( \rho \). This differs from numerical experiments without pricing, where safety stocks always decrease with \( \rho \). This is because correlation also affects the decision on retail price. When the profit margin is small, the effect of pricing is small and the safety stocks decrease with \( \rho \). The pattern of safety stocks changing with \( \rho \) also explains why \( z^B \leq z^N \) usually occurs at extreme values of \( \rho \).

With a low profit margin, safety stocks decrease with demand variability (Figure 4.11). Newsvendors under price competition are affected the most significantly; newsvendors under inventory competition are affected least. When the \( CV \) changes from 0.2 to 1, safety stocks decrease 78.4% under price competition; and decrease 11.2% under inventory competition. With a high profit margin; safety stocks increase with demand variability (Figure 4.13). Newsvendors under inventory competition are the most sensi-
tive to a change of CV (from 0.2 to 1), about 56%; newsvendors under price competition are the least, about 38.3%. With a medium profit margin (Figure 4.12), newsvendors under inventory competition set higher safety stocks (increase 26%) when demand is more variable; newsvendors under price competition set a lower safety stock (decrease 7%) when demand is more variable. Newsvendors under both competition and no competition are in the middle in terms of adjusting safety stock strategy according to demand variability.

**Results on retail prices** (Figures 4.14 to 4.25). For all numerical results, we have $p^I \geq p^N \geq p^P$ (e.g., Figure 4.14), $p^I \geq p^B \geq p^P$ (e.g., Figure 4.17). Since in our numerical example the price competition is relatively fierce, we only have $p^N \geq p^B$. When we use a lower substitution factor $\theta$, we have $p^N \leq p^B$. That is, price competition makes retailers mark down the price, but inventory competition makes retailers increase prices. The two effects can mitigate each other and the retail price under both competitions can be either higher or lower than that under no competition.

The pricing strategies of newsvendors under price competition and no competition are not affected by correlation. Inventory competitive newsvendors and newsvendors under both competitions need to adjust their retail price down when the demand correlation is high. With a high CV, the change of price is more obvious. For the inventory competition case, the price difference between $\rho = -1$ and $\rho = 1$ is more significant when the approximate profit margin is low (Figure 4.16), decreases 9.9%; but the difference is very small for large profit margin case (Figure 4.22), only decrease 3.0%. So for industries with a large profit margin and under inventory competition, demand correlation does not affect price much. But for industries with small profit margin, demand correlation does affect price a lot. The price difference between inventory competition and price competition scenarios is large. With $CV = 0.5$ and $\rho = 0$, the difference is 9.0% at low
profit margin (Figure 4.23), 16.2% at high profit margin (Figure 4.25). With higher CV, the difference is even larger.

In general, retail prices decrease with demand variability. The retail prices under a price competition scenario is more likely to be affected by the CV. In contrast, inventory competitive retailers do not adjust prices very much with the change of CV; similarly for retailers under both competitions. Take the medium profit margin group as an example; when CV changes from 0.2 to 1, retail prices under price competition decrease 4.7% (8.4% for low profit margin group); only 0.7% (3.0% for low profit margin group) under inventory competition; 1.2% under both competition; 4.1% under no competition (8.4% for low profit margin). So involving inventory competition helps to mitigate demand variability. With a higher profit margin, retail prices are more less sensitive to demand variability.

Results on total inventory (Figures 4.26 to 4.37). We have $y^p > y^N, y^B > y^I$ for all the cases. So with price competition, newsvendors increase their total procurement amounts. Which is good news for the manufacturer no matter whether the wholesale price is given or optimally chosen. The inequalities between $y^p$ and $y^I$, $y^N$ and $y^I$, $y^p$ and $y^B$, $y^N$ and $y^B$ can be in both directions.

Total inventories under no competition and price competition are not influenced by the correlation. In all cases, total inventories under inventory competition and both competitions first increase then decrease with correlation. So $y^p \leq y^I$ and $y^N \leq y^I$ happen when correlation is in a middle range (e.g., Figures 4.29, 4.30, 4.31). Under inventory competition or both competition, extremely negatively correlated demand induces a higher retail price, which make the total inventory relatively low, so $y^I \leq y^p$ and $y^I \leq y^N$ usually happens when demand correlation is very negative. The same observation holds for $y^B$. When the down stream retailers are involved in inventory competition
or both competitions, the manufacturer prefers that the demands in the market are less correlated.

With a medium or high profit margin, the total inventory increases with demand variability. This effect is more obvious under inventory competition and both competition scenarios. With a medium profit margin (Figure 4.36), for $CV = 0.2$ to $CV = 1$, $y^I$ increases 18.9%; $y^B$ increases 11.2%; $y^N$ increases 9.8%; $y^P$ increases 2.2%. With a high profit margin (Figure 4.37), changing $CV = 0.2$ to $CV = 1$, $y^I$ increases 32.07%. With a low profit margin (Figure 4.35), total inventory decreases with demand variability. This effect is more significant under price competition. Changing $CV = 0.2$ to $CV = 1$, $y^P$ decreases 30.1%. To conclude, the effect of demand variability on total inventory depends on the approximate profit margin. For a high profit margin industry, high demand variability increases the amount of procurement; this is especially true for the inventory competition market. For a low profit margin industry, high demand variability decreases the total procurement; with a price competitive market affected the most. All these explanations translate into similar statements regarding the the manufacturer’s revenue. High demand variability may not always bad news for the manufacturer.

**Results on newsvendors’ profits** (Figures 4.38 to 4.49). We have $E\pi^I \geq E\pi^N \geq E\pi^P$, $E\pi^I \geq E\pi^B \geq E\pi^P$ for all the cases. This supports the analytical results. So inventory competitive newsvendors are more profitable compared with newsvendors under other types of competitive market. Price competition always damages newsvendors although the manufacturer prefers it.

Retailers profits under inventory competition and both competitions decrease with the demand correlation. The more negatively correlated the demand is, the more benefit inventory competition can bring. Profits under the price and no competition scenario are not affected by correlation. So $E\pi^B \geq E\pi^N$ usually happens when $\rho$ is small; $E\pi^B \leq E\pi^N$
usually happens when $\rho$ is large. So if the demand correlation is small, engaging in both price and inventory competition can benefit the newsvendors compared to no strategic interaction at all. With a low profit margin (Figures 4.38 to 4.40), demand correlation affects profits more dramatically. For example, when $\rho$ changes from $-1$ to $1$, the profit decreases 62.9% for high $CV$ case, 42.1% for the medium $CV$ case, and 17.5% for small $CV$ case. For high profit margin (Figures 4.44 to 4.46), the corresponding numbers are, 2.2% for low $CV$, 5.6% for medium $CV$, 10.6% for high $CV$.

In all cases, retailers' profits decrease with demand variability. Price competitive newsvendors suffer the most. Inventory competitive newsvendors and newsvendors under both competitions suffer the least. Changing $CV$ from 0.2 to 1, causes all the profits to decrease. The following table shows the percentage profit reduction under different profit margins and types of competition.

<table>
<thead>
<tr>
<th></th>
<th>Price comp.</th>
<th>no comp.</th>
<th>inventory comp.</th>
<th>both</th>
</tr>
</thead>
<tbody>
<tr>
<td>low profit margin</td>
<td>56.09%</td>
<td>55.28%</td>
<td>39.43%</td>
<td>-</td>
</tr>
<tr>
<td>medium</td>
<td>31.47%</td>
<td>30.53%</td>
<td>20.89%</td>
<td>20.17%</td>
</tr>
<tr>
<td>high</td>
<td>14.18%</td>
<td>13.62%</td>
<td>9.25%</td>
<td>-</td>
</tr>
</tbody>
</table>

So demand variability is always bad news for the retailers. But the damage varies under different competitive markets and different industries (in terms of profit margin).

**Results on consumer welfare.** Recall that we assume a consumers' surplus function $U_i = R + \alpha \Pr(\bar{D}_i^0 < z_i) - p_i$, $\alpha > 0$ if they buy products from newsvendor $i$. To explore the consumers' welfare under different competitive markets, we study a market with relatively medium conditions, that is, $CV = 0.5$, $\rho = 0$. The following table shows consumer surplus under different market scenarios and with different profit margins.

<table>
<thead>
<tr>
<th></th>
<th>Price comp.</th>
<th>no comp.</th>
<th>inventory comp.</th>
<th>both</th>
</tr>
</thead>
<tbody>
<tr>
<td>low profit margin</td>
<td>56.09%</td>
<td>55.28%</td>
<td>39.43%</td>
<td>-</td>
</tr>
<tr>
<td>medium</td>
<td>31.47%</td>
<td>30.53%</td>
<td>20.89%</td>
<td>20.17%</td>
</tr>
<tr>
<td>high</td>
<td>14.18%</td>
<td>13.62%</td>
<td>9.25%</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.3 Percentage of profit reduction with demand variability

Table 4.4 Consumer utility
<table>
<thead>
<tr>
<th></th>
<th>low margin</th>
<th>medium margin</th>
<th>high margin</th>
</tr>
</thead>
<tbody>
<tr>
<td>no</td>
<td>0.3α - 25.69</td>
<td>0.54α - 39.23</td>
<td>0.75α - 70.99</td>
</tr>
<tr>
<td>price</td>
<td>0.26α - 24.01</td>
<td>0.49α - 34.81</td>
<td>0.71α - 60.2</td>
</tr>
<tr>
<td>inventory</td>
<td>0.46α - 26.4</td>
<td>0.65α - 40.06</td>
<td>0.80α - 71.82</td>
</tr>
<tr>
<td>both</td>
<td>-</td>
<td>0.60α - 35.52</td>
<td>-</td>
</tr>
</tbody>
</table>

For consumers who greatly value service level (fill rate), their preference order is \(\text{inventory} > \text{both} > \text{no} > \text{price}\). For consumers who do not value the service level at all, the order should be the opposite. For consumers in the middle, we can have many combinations, for example, \(\text{both} > \text{price} > \text{inventory} > \text{no}\), \(\text{both} > \text{inventory} > \text{price} > \text{no}\) etc.

### 4.7 Conclusions

The Newsvendor model is a critical building block in inventory/supply chain theory, but has primarily been considered within a monopoly context. Some recent work has looked at the case of competition, but most are based on fixed retail price. The aim of this chapter has been to simply extend our understanding of the Newsvendor model into a more general competitive environments.

Our model is not in full generality. Rather we trade some generality by restricting random demand to additive shocks only and a linear expected demand function. This restriction allows us to achieve considerably more analytical results.

In particular, using supermodularity, we show the existence of pure strategy Nash equilibrium for Newsvendor game, and demonstrate the equilibrium properties in all market contexts. We also find conditions for the uniqueness of equilibria.

We find that price competitive newsvendors will usually shrink their investment in safety stocks when the competition becomes fiercer; they will also reduce prices to pro-
mote sales, but they would stock more in total to cover the higher demands. This will benefit up-stream manufacturers no matter whether the wholesale price is exogenous or endogenous. But price competition is a negative externality to newsvendors: their profits go down because of price competition. Numerical results show that the manufacturer prefers a low demand variability environment if the profit margin \( \frac{p}{w} \) is low, and a high demand variability if the profit margin is high. Newsvendors always prefer low demand variability.

Inventory competitive newsvendors will keep the safety stock up by considering the overflow consumers from competitor; they also mark up their general selling prices. Inventory competition is a positive externality to newsvendors: their profits go up under inventory competition because their expected demands are increased, and their profit margins are also increased. The total inventory level can be either higher or lower than that under price competition and no competition. So the manufacturer can be either better off or worse off. Numerical experiments show that the manufacturer in such a competitive market will prefer medium demand correlation, and high demand variability if the profit margin is high. Retailers, not surprisingly, always prefer negative demand correlation and low demand variability.

Under both competition, newsvendors will set a higher safety stock and price than a price competition only scenario, a lower safety stock and price than inventory competition only scenario. In terms of profitability, Newsvendors will gain higher profits than a price competition only scenario, lower profits than a inventory competition only scenario, but either lower or higher profits than a no competition scenario. The manufacturer prefers the both competition scenario than the inventory competition only. Numerical experiments show that with negatively correlated demands, newsvendors will be more profitable under both competitions. Compared with price competition or no competition,
the total inventory can be either higher or lower; it is more likely to be higher with a medium correlation or large coefficient of variation. The effect of demand variability on newsvendors profits and manufacturer's revenue is the same as the inventory competition scenario.

Demand correlation and variability affects the Nash equilibrium strategies. In general, retail price decreases with correlation (if inventory competition is involved) and demand variability. How safety stocks and total inventories change with correlation and variability depend on the approximate profit margin (the approximate profitability level of the industry). Safety stock and total inventory decrease with variability if the profit margin is low, increase with variability if the profit margin is high. Inventory competitive newsvendors are less affected by the demand variability in terms of profitability. Price competitive newsvendors are more sensitive to demand variation but are not affected by demand correlation. Inventory competitive newsvendors usually lose their profits when demand correlation becomes larger.

In general our results suggests that newsvendors selling substitutable products should cooperate to cool down the price competition and boost inventory competition, which will be helpful to both of them. For the manufacturers, they may try to discourage inventory competition under certain demand correlations and encourage price competition.

Figures
Safety stocks under different profit margins, correlation and coefficient of variation

Figure 4.1 For Proposition 4.8

Figure 4.2

Figure 4.3

Figure 4.4

Figure 4.5
Figure 4.6

Figure 4.7

Figure 4.8

Figure 4.9

Figure 4.10

Figure 4.11
Retail price under different profit margins, correlation and coefficient of variation
Figure 4.18

Figure 4.19

Figure 4.20

Figure 4.21

Figure 4.22

Figure 4.23
Figure 4.24
Total inventory under different profit margins, correlation and coefficient of variation

Figure 4.25

Figure 4.26

Figure 4.27

Figure 4.28

Figure 4.29
Figure 4.30

Figure 4.31

Figure 4.32

Figure 4.33

Figure 4.34

Figure 4.35
Newsvendors' profits under different profit margins, correlation and coefficient of variation

Figure 4.36
Figure 4.37

Figure 4.38
Figure 4.39

Figure 4.40
Figure 4.41
Figure 4.48

Figure 4.49
5 When Competitors Agree to Cooperate: the case of Transshipment in Supply Chains

Abstract: It is a commonplace observation that even the most competitive firms often find it in their best interests to cooperate. Taking the special case of competitive supply chains, there is a long history of cooperation when it helps both parties. Long term examples are joint supply arrangements such as flower or vegetable markets/auctions and recently e-auctions for parts or components have sprung up in many industries. One simple example of cooperation is when two supply chains agree in advance to transship or 'pool' surplus product for use by another. The alternative is to let their customers switch unsatisfied demand to a competitor. Now which is preferable, and how does such a preference depend on the many parameters, prices, the nature of competition, the degree of competition, wholesale prices etc., simply when should competitive retailers transship?

To get answers we study a stylized model under three market environments: a market with an exogenous retail price, an endogenous retail price, and with price competition. The summary answer is that strong price competition between substitutable goods should lead to caution in signing transshipment contracts. But with little price competition and particularly where retailers are free to set the transshipment price, then transshipment is probably the way to go. However this simple summary is tempered by both the many parameters in play and the nature of the stylized model studied.

5.1 Introduction

Even the fiercest competitors find it to their advantage to occasionally cooperate with each other. Such cooperation might be formalized legally in their support of price legislation, on a voluntary industry level such as with product standards or bilaterally; firm
to firm. In the field of logistics and supply chains, cooperation between competitors has had a rich and varied history. Examples historically might be the emergence of joint supply operations such as flower or vegetable auctions/markets and in more recent times e-auctions for parts and components. There has not been really extensive work either empirically or theoretically evaluating the benefits of this behavior. One of the reasons for the paucity of theoretical evaluations is that the modelling gets intractable very quickly. This is because embedded within the effects of cooperation must be a competitive structure such as a competitive equilibrium. One particular manifestation of cooperation within a competitive supply chain is when competitors in advance freely enter into an agreement to transship surplus product to a competitor who is suffering a shortage of an identical or substitutable product. A retailer with surplus product might inventory it for the next period; when not perishable, they might discard it or they might sell (transship) it to another retailer in need. Alternatively they might reason that "a portion of the unsatisfied customers at my competitor might make the journey to my store and I will then be able to sell the item at full price". If they believed this portion of 'indirect' customers to be substantial they might be quite unwilling to enter into any such cooperative transshipment agreement. The 'journey' in question might be virtual, such as using third party search engines such as http://www.autobytel.com for automobiles or http://www.smadesigngroup.com/jewelryguide/jretailers.htm for jewelry and watches. Is it obvious which choice is preferred? And how is such a choice influenced by parameters such as the transshipment price, the product profit margins, the degree of uncertainty of demand, the degree to which consumers would switch unsatisfied demand to another retailer in addition to the way that demand is based on relative retailer prices; the demand functions. There is also the role of the supplier(s) to consider, if these too were independent agents.
To make a theoretical evaluation that is tractable we propose a rather stylized model; designed to capture the essential essence of the trade-offs involved but quite incapable of reflecting the richness which emerges in the many arrangements and deals made in practical supply chains. We hope however to be able to use this simplified model to sift out the key features which might repay consideration by a finer grained analysis. The first big simplification is that we assume a single period environment, perhaps reflecting a highly perishable product. We take only two retailers selling a highly substitutable model supplied by a common independent supplier. Expected product demand is modeled in the common way of a consumer being normally associated with one retailer but prepared to try another on the basis of the price differential. A major simplification is that although demand is randomly distributed, the random shock is considered additive to the expected demand. However we make no unreasonable assumptions about demand independence between retailers nor about the form of the uncertain distribution. We choose not to specify any market or channel leadership, thus utilizing the common Nash equilibrium as the basis for competition rather than a Stackelberg. In general it is not reasonable to assume that a pure Nash equilibrium either exists or is unique, so care needs to be taken not to expend energy on elaborate results about equilibria that do not even exist. Where possible we have proven existence and uniqueness but this is traditionally a very difficult area and we draw careful attention to where we still have gaps in these results.

Our comparative evaluation is between three price regimes and either with/without the two retailers agreeing in advance to transship. The transshipment case is also subdivided into the case where the transshipment price is either given (exogenous) or a decision variable by the retailers. Thus there are six main scenarios.

Table 5.1 Scenarios considered
Transshipment

<table>
<thead>
<tr>
<th></th>
<th>Transshipment</th>
<th>No Transshipment</th>
</tr>
</thead>
<tbody>
<tr>
<td>price exogenous</td>
<td>scenario 1-1</td>
<td>scenario 1-2</td>
</tr>
<tr>
<td>price endogenous</td>
<td>scenario 2-1</td>
<td>scenario 2-2</td>
</tr>
<tr>
<td>price competition</td>
<td>scenario 3-1</td>
<td>scenario 3-2</td>
</tr>
</tbody>
</table>

The first price regime, an exogenous price, sees the retail price as fixed and demand therefore not price sensitive. Making the price endogenous allows for a price sensitive demand with price as a decision variable, but there is no direct price competition. The third price regime considers the competitive case.

The consideration of transshipment is not new; since the work of Krishnan and Rao (1965), the transshipment problem has been studied for decades. However, most models assume a centralized monopoly, i.e., retail outlets participating in transshipment are owned by a single entity. So both the inventory decision and transshipment decision of each outlet are made by maximizing total profits. Examples are Lee (1987), Jonsson and Silver (1987), Robinson (1990), Archibald et al. (1997). Two works are most directly related to this chapter. Rudi et al. (2001) study the transshipment problem between two independent locations (they are decentralized decision makers but there is no competition). They find that local decision making will not maximize joint profits and they design transshipment prices which induce local decisions to be consistent with joint profit maximization. Anupindi and Bassok (1999) study a market where consumer can search between retailers, and retailers can either centralize stocks or hold stocks separately. They find retailers will always benefit from centralizing stock, but the manufacturer and the system as a whole will lose when the market search level is high.

Our key findings are:

1. Transshipment price, profit margins and the degree of price competition influence
the retailers' decisions on transshipping or competing.

2. Retailers would prefer NOT to sign a transshipment contract when they are involved in fierce price competition. This is in contrast to the results obtained from exogenous and endogenous retail price scenarios. Especially, with the transshipment price structure designed by Rudi et al. (2001), retailers benefit from transshipment under exogenous and endogenous retail price scenarios, but they may lose under a price competition scenario. So when designing the transshipment contract, retailers will be hurt if they ignore price competition. In a market with fierce price competition and consumer switching, cooperation to transship is not a dominant strategy. If the transshipment price is fixed, the manufacturer would not wish the retailers to transship.

3. However should the transshipment price be one retailer's retail price, the manufacturer would always prefer independent retailers to transship at the competitor's retail price. The retailers agree with the manufacturer if they are not involved in price competition, but are opposed to the manufacturer if they are price competitive.

After defining our basic model the rest of the chapter is organized around the six scenarios above. We defer practically all proofs to an appendix, and as our methodology relies heavily on the theory of supermodular games, some basic results are assembled for reference in the appendix of the dissertation.

5.2 The General Model

Two retailers procure inventories \( y_i \) \((i = 1, 2)\) from a single supplier at a wholesale cost \( w \), they then sell products to the market at a retail price \( p_i \) (r-price). The demand \( \tilde{D}_i \) is uncertain and realized after the procurement (pricing and replenishment decisions). Assume the joint demand distribution of \( \tilde{D}_i \) \((i = 1, 2)\) is known. A contract to transship is assumed to be of the following form: in advance of demands being known retailers agree
that should one have excess and the other a shortage then the maximum available/needed is transhipped under agreed price conditions. Otherwise excess inventory has no value and excess demand goes unsatisfied. This stylized retailer with a totally perishable product is often termed a ‘newsvendor’ and we shall also often refer to retailers in this way.

The situation where retailers have contracted in advance to transship will be termed the T Game. The sequence of events in this scenario is as follows: 1. Before the selling season, two retailers simultaneously choose stock levels and retail prices. 2. Demand is realized at each retail outlet. 3. If needed, transshipment occurs from retail outlet $i$ to retail outlet $j$ at a transshipment price (t-price) $\tau_i$ (exogenous). We do not assume $\tau_i$ equals $\tau_j$ unless explicitly stated.

If there has been no transshipment agreement, the sequences 1 and 2 are the same but we replace 3 with: should a consumer find no stock remaining at their first choice retailer, they will ALL try the other (for simplicity of presentation, but does not affect the spirit of the main result). This will be termed the C game because retailers compete in inventory. The main purpose of the chapter is to compare the T Game with the C Game, to see whether contracting to transship is a preferred strategy by competitive newsvendors.

We incorporate pricing by using $\bar{D}_i = D_i^0 + L_i$ where $L_i = a - bp_i$ for a market with an endogenous price and $L_i = a - bp_i + \theta(p_j - p_i)$ for a price competitive market. These demand functions are commonly used in the economic (Varian 1992) and marketing (Shubik and Levitan 1980) literature. For convenience we define safety stock $z_i := y_i - L_i$, which is the inventory not associated with the known portion of the demand $L_i$. It also has economic meaning: safety stock. We assume that any density function $f_X(x)$ of a random variable $X$ is $> 0$ and finite in the strategy set, and that $r_X(x) = \frac{f_X(x)}{P_r(X>x)}$, the failure rate function, is increasing. This is a reasonable assumption in many circumstances.
Retailer $i$'s problem in the T game is, max \( E\pi_i^T \),

\[
E\pi_i^T = -w y_i + p_i E \min\{\tilde{Y}_i, y_i\} + \tau_i E \min\{(y_i - \tilde{D}_i)^+, (\tilde{D}_i - y_i)^+\} + (p_i - \tau_j) E \min\{(y_j - \tilde{D}_j)^+, (\tilde{D}_i - y_i)^+\} 
\]  

(5.1)

In the C game, retailer $i$'s problem is max \( E\pi_i^C \),

\[
E\pi_i^C = -w y_i + p_i E \min\{\tilde{Y}_i, y_i\} + p_i E \min\{(y_i - \tilde{D}_i)^+, (\tilde{D}_i - y_i)^+\} 
\]  

(5.2)

We begin by studying a market with an exogenous price.

### 5.3 T Game versus C Game: Exogenous Retail Price

**Scenario 1-1: The T game**

The two retailers have contracted to transship at \( \tau_i, \tau_j \) ex ante. By conjecturing what will happen after demands are realized, each retailer simultaneously makes an inventory decision to maximize their own profit. Each retailer solves the problem given by function (5.1).

**Proposition 5.1** 1. Function (5.1) is supermodular in \( y_i \), having increasing differences in \( (y_i, -y_j) \). 2. There exists a unique inventory level Nash equilibrium in the T game, which is characterized by,

\[
\Pr(y_i > y_j) + \frac{\tau_i}{p_i} \Pr(y_i + y_j - \tilde{D}_j < \tilde{D}_i < y_i) - (1 - \frac{\tau_j}{p_i}) \Pr(y_i < \tilde{D}_i < y_i + y_j - \tilde{D}_j) = \frac{w}{p_i} 
\]  

(5.3)

**Proof.** Appendix proof 1.  

3-1. With identical retailers, the equilibrium inventory level increases with \( \tau_i \) and \( \tau_j \).
Because \( \frac{\partial^2 E_{x|y|}}{\partial y_i \partial \tau_i} \geq 0 \) and \( \frac{\partial^2 E_{x|y|}}{\partial y_j \partial \tau_j} \geq 0 \), the best reply functions increase with \( \tau_i \) and \( \tau_j \), i.e., \( \frac{\partial r_i(y_i)}{\partial \tau_i} \geq 0 \) and \( \frac{\partial r_j(y_j)}{\partial \tau_j} \geq 0 \) \((i = 1, 2)\). By symmetry, we have the required result.

3-2. Again with identical retailers, the equilibrium inventory decreases with \( w \), because \( \frac{\partial E_{x|y|}}{\partial y \partial w} \leq 0 \), and the best reply functions decrease with \( w \), i.e., \( \frac{\partial r_i(y_i)}{\partial w} \leq 0 \).

So two retailers, operating in a market with a given retail price, who have contracted to transship, will have a unique market equilibrium, increasing with the \( t \)-price but decreasing with the wholesale price. But how does the profit in the \( T \) game change with the wholesale price?

\[
\frac{\partial E_{x|y|}}{\partial w} = -y_i + \frac{\partial E_{x|y|}}{\partial y_j} \frac{\partial y_j}{\partial w}
\]

The first term is always negative. The second term is the effect of \( w \) caused by transshipment. \( \partial y_j / \partial w \) is also always negative. Let \( j(\tau) = \frac{\partial E_{x|y|}}{\partial y_j} = -\tau \mathbb{P}(y_j^T < \tilde{D}_j < y_i^T + y_j^T - \tilde{D}_i) + (p - \tau) \mathbb{P}(y_i^T + y_j^T < \tilde{D}_j < y_j^T) \). \( \frac{dj}{d\tau} \) is ambiguous in sign. But we have \( j(\tau = 0) > 0 \), \( j(\tau = p) < 0 \). So by the continuity of \( j(\tau) \), there exists a range of \( \tau \) (small value) such that \( \frac{\partial E_{x|y|}}{\partial y_j} \frac{\partial y_j}{\partial w} < 0 \), then the transshipment term strengthens the negative effect of \( w \) on profit. There is also a range of \( \tau \) (large value) such that \( \frac{\partial E_{x|y|}}{\partial y_j} \frac{\partial y_j}{\partial w} > 0 \), then the transshipment term weakens the negative effect of \( w \) on profits.

**Scenario 1-2: The \( C \) game**

Here, retailers compete in terms of inventory rather than contract to transship. When a retailer runs out, customers switch to a competitor. Thus a retailer will lose customer if the inventory is set too low. Each retailer makes inventory decisions simultaneously to maximize expected profits by solving problem (5.2).

**Proposition 5.2** 1. Function (5.2) is supermodular in \( y_i \), having increasing differences in \((y_i, -y_j)\). 2. There exists a unique inventory equilibrium in the \( C \) game, characterized by

\[
\mathbb{P}(\tilde{D}_i > y_i) + \mathbb{P}(y_i + y_j - \tilde{D}_j < \tilde{D}_i < y_i) = \frac{w}{p_i}
\]

(5.4)
Proof. Appendix proof 2.

It is also straightforward to observe some properties of the equilibrium. For example, the equilibrium increases with the r-price and decreases with the wholesale price \( w \). So retailers competing with inventory in a market with an exogenous retail price have a unique equilibrium, in which inventory is set higher if the retail price is high, but lower if the wholesale price is high.

But how do profits change with the wholesale price? \( \frac{\partial E \pi^C}{\partial w} = -y_i^C + \frac{\partial E \pi^C}{\partial y_i^C} \frac{\partial y_i^C}{\partial w} \)

We have \( \frac{\partial E \pi^C}{\partial y_i^C} = -p_i \Pr(y_j^C < \tilde{D}_j < y_i^C + y_j^C - \tilde{D}_i) < 0. \)

So \( \frac{\partial E \pi^C}{\partial y_i^C} \frac{\partial y_i^C}{\partial w} > 0. \) In contrast to the T game, the negative effect of \( w \) is weakened by the term introduced by inventory competition in the C game.

Comparing the T game and C game

We compare the market equilibrium and retailers’ profits under the above two games.

**Proposition 5.3** With \( 0 \leq \tau_i \leq p_i \), and \( 0 \leq \tau_j \leq p_j \), then \( y_i^T \leq y_i^C \).

**Proof.** By comparing (5.3) and (5.4) we have the result. From (5.3) we have

\[
\Pr(\tilde{D}_i > y_i) + \Pr(y_i + y_j - \tilde{D}_j < \tilde{D}_i < y_i) = \frac{w}{p_i} + (1 - \frac{\tau_i}{p_i}) \Pr(y_i + y_j - \tilde{D}_j < \tilde{D}_i < y_i) + (1 - \frac{\tau_j}{p_j}) \Pr(y_i < \tilde{D}_i < y_i + y_j - \tilde{D}_j)
\]

From (5.4) we have

\[
\Pr(\tilde{D}_i > y_i) + \Pr(y_i + y_j - \tilde{D}_j < \tilde{D}_i < y_i) = \frac{w}{p_i}
\]

The LHS is decreasing with \( y_i \), and the RHS \((1 - \frac{\tau_i}{p_i}) \Pr(y_i + y_j - \tilde{D}_j < \tilde{D}_i < y_i) + (1 - \frac{\tau_j}{p_j}) \Pr(y_i < \tilde{D}_i < y_i + y_j - \tilde{D}_j) \geq 0 \) for \( p_i - \tau_i \geq 0 \) and \( p_i - \tau_j \geq 0 \). So we have the required result. Note that when \( p_i - \tau_i \leq 0 \) and \( p_i - \tau_j \leq 0 \), then \( y_i^T \geq y_i^C \). 

For the classical newsvendor problem, recall that \( \Pr(\tilde{D}_i > y_i) = \frac{w}{p_i} \). So the inventory level of the classical newsvendor is lower than competitive newsvendors and can be both higher and lower than the transshipment newsvendors.

How about the manufacturer, whose problem is \( \max_w w(y_i + y_j) \)?

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**Proposition 5.4** The manufacturer's problem is also supermodular, and always prefers that newsvendors not transship.

The reasoning is similar to proposition 4.4 in chapter 4.

Anupindi and Bassok (1999) compared the manufacturer's profit when retailers use separate stocks as compared to centralized stocks, in a market where consumers can switch unfilled demand. They find that the manufacturer prefers that retailers use separate stocks when the proportion of consumers switching is high. In contrast, our setting has retailers as independent units cooperating via transshipment. We find that the manufacturer prefers that newsvendors compete with inventory rather than transship (this result is not affected by the assumption of full consumer switch).

How about the retailers' profits? Do they prefer to transship? It depends on the wholesale price and the t-prices. A numerical example confirms this. In addition the demand correlation also has an impact. For simplicity, assume that $r_i = r_j = r$, $p_i = p$, and identical retailers.

**Proposition 5.5** 1. Retail profits increase with $\tau$ when $\tau$ is small, and decrease with $\tau$ when $\tau$ is large. 2. When transshipment occurs at the retail price, retailers are indifferent between the two games. So there is a range of $\tau$ such that transshipment is preferred by retailers. 3. When $\tau = 0$, either game might be preferred. 4. If the retailers can cooperate to choose $\tau$, they will always prefer to transship.

**Proof.** Appendix proof 3. ■

So if the retailers can choose the t-price, they will transship. Otherwise, whether they wish to transship will depend on the prevailing t-price.

In the following examples, we illustrate the results.
Example 5.1 Assume $\tilde{D}_i$ is exponential r.v. with parameter $\lambda = 1$. Then $E \min\{\tilde{D}_i, y_i\} = \int_0^{y_i} F(x)dx = 1 - e^{-y_i}, A = E \min\{((\tilde{D}_j - y_j)^+, (y_i - \tilde{D}_i)^+)\} = \int_0^{y_i} (\int_{y_j}^{y_i+y_j-x_i}(x_j-y_j)e^{-x_j}dx_j)dx_i + \int_0^{y_i} (\int_{y_j+y_i-x_i}(y_i-x_i)e^{-x_i}dx_i)e^{-x_i}dx_i = e^{-y_j}(1 - e^{-y_j}y_i), \frac{\partial A}{\partial y_i} = \int_0^{y_i} (\int_{y_j+y_i-x_i}e^{-x_j}dx_j)e^{-x_i}dx_i = y_i e^{-y_i} = - \int_0^{y_i} (\int_{y_j}^{y_i+y_j-x_i}e^{-x_j}dx_j)e^{-x_i}dx_i = -e^{-y_j}(1 - e^{-y_j} - e^{-y_j}y_i).

Standardize $w = 1$ then (5.3) becomes $p(e^{-y_j} - e^{-y_j} + y_i e^{-y_j} + e^{-y_j}y_i) + \tau(e^{-y_j} - e^{-y_j}y_i) = 1$, and (5.4) becomes $(y_i e^{-y_j} + e^{-y_j})p = 1$. The profit functions (5.1) and (5.2) become $E_i\pi_i = [-y_i + p(1 - e^{-y_i} + e^{-y_i}(1 - e^{-y_i} - e^{-y_i}y_i))]$. The results are illustrated by Figures 5.6 to 5.9.

We have the following observations,

1. For a given $p$, the equilibrium stocks in the T game increase with $\tau$.

2. When $p$ (e.g. $p = 1.5$) is small, there is a greater range of $\tau$ such that transshipment is not preferred, retailers preferring to compete rather than cooperate. When $p$ is large (e.g. $p = 5$), transshipment is preferred even when $\tau = 0$.

3. When $\tau = 1$, i.e., transship at wholesale price, it is always profitable to transship.

Using the uniform distribution, we have the same conclusions.

Example 5.2 Assume $\tilde{D}_i \in [0,1]$ is uniformly distributed. If $y_i + y_j \leq 1$, then $E \min\{\tilde{D}_i, y_i\} = \int_0^{y_i} F(x)dx = y_i - \frac{y_i^2}{2}, A = E \min\{((\tilde{D}_j - y_j)^+, (y_i - \tilde{D}_i)^+)\} = \int_0^{y_i} (\int_{y_j}^{y_i+y_j-x_i}(x_j-y_j)dx_j)dx_i + \int_0^{y_i} (\int_{y_j+y_i-x_i}(y_i-x_i)dx_i)e^{-x_i}dx_i\frac{y_i+y_j-x_i}{2} - \frac{y_i^2}{2}, \frac{\partial A}{\partial y_i} = \int_0^{y_i} (\int_{y_j+y_i-x_i}e^{-x_j}dx_j)e^{-x_i}dx_i\frac{y_i+y_j-x_i}{2} - \frac{y_i^2}{2}, Then (5.3) becomes $p(1 - y - \frac{y_i^2}{2}) + \tau(y - y^2) = 1$, (5.4) becomes $p(1 - \frac{3}{2}y^2) = 1$. The profit functions become $E_i\pi_i = -y + p(y - \frac{2}{3}y^3)$. The numerical experiments are in Table 5.2.

Table 5.2. Uniform Distribution (exogenous retail price) ($y_i + y_j \leq 1$)
If \( y_i + y_j \geq 1 \) and \( y_i \leq 1 \) then \( E \min\{D_i, y_i\} = \int_0^{y_i} F(x)dx = y_i - \frac{y_i^2}{2} \), \( A = E \min\{(\bar{D}_j - y_j)^+, (y_i - \bar{D}_i)^+\} = \int_{y_j}^{y_i} \left( \int_0^{y_i+y_j-x_j} (x_j-y_j)dx_j \right) dx_j + \int_{y_j}^{y_i} \left( \int_{y_i+y_j-x_j} (y_i-x_i)dx_i \right) dx_j \).

The following table concludes the numerical experiments.

**Table 5.3. Uniform Distribution (exogenous retail price) and \((y_i + y_j \geq 1, y_i \leq 1)\)**

<table>
<thead>
<tr>
<th>( p )</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau )</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>( y^C )</td>
<td>0.71</td>
<td>0.82</td>
</tr>
<tr>
<td>( y^T )</td>
<td>0.53</td>
<td>0.56</td>
</tr>
<tr>
<td>( E\pi^C_i )</td>
<td>0.741</td>
<td>1.661</td>
</tr>
<tr>
<td>( E\pi^T_i )</td>
<td>0.762</td>
<td>0.770</td>
</tr>
</tbody>
</table>

When the profit margin is not very big, transshipping at a low price is not worthwhile.

So if the manufacturer can influence the \( p/w \) and \( \tau/w \) by setting a larger wholesale price, she can make the retailers less likely to transship. Contracting to transship is beneficial to the competitive retailers in most cases. The manufacturer will always prefer the retailers to compete rather than to transship.

We also did examples using a bivariate normal distribution, using Monte Carlo in-
integration to evaluate the expected profits. We select the pair of parameters \((p/w = 5, \tau = 2)\) where transshipment is always profitable under both exponential and uniform distributions. We use \(D_i \sim N(150, 30)\), to see how the demand correlation affects the comparison between the C game and the T game. We find that the profit difference between transshipment and competition becomes smaller when the correlation becomes larger. When the demand is almost perfectly positively correlated \((\rho = 0.8)\), retailers are indifferent between transshipment and competition. The difference in inventory levels between transshipment and competition also becomes smaller as the correlation coefficient becomes larger (Figure 5.10 & 5.11).

5.4 T Game Versus C Game: Endogenous Retail Price

To incorporate pricing, we assume \(D_i = D_i^0 + L_i\) where \(L_i = a - b p_i\). The stock decision has two parts, \(y_i = L_i + z_i\). The first part is for the deterministic part of demand and depends on the retail price; the second term is for the stochastic part of the demand (call it safety stock). We assume that the joint distribution of \(D_i^0 (i = 1, 2)\) is known.

Scenario 2-1: The T game

Price sensitive newsvendors who contract to transship, at a given transshipment price \(\tau_i\), need to solve;

\[
\max_{z_i, p_i} E \pi_i^T = -w(L_i + z_i) + p_i(L_i + E \min\{z_i, D_i^0\}) + \tau_i E \min\{(z_i - D_i^0)^+, (D_i^0 - z_j)^+\} + (p_i - \tau_i) E \min\{(z_j - D_j^0)^+, (D_i^0 - z_i)^+\}
\]  

(5.5)

This game is not supermodular, so the methodology of the previous section will not work to show the existence of the equilibrium. However, function (5.5) is continuous in \((z_i, z_j, p_i)\) and according to a theorem of Glicksberg (Fudenberg and Tirole 1991), there
exists a Nash equilibrium in mixed strategies. Under certain conditions, a pure strategy Nash equilibrium also exist. To show this we need two assumptions:

**Assumption 1.** Define function \( \Theta_{X+Y,X}(x+y,x) := \frac{d\left[1-\Pr(X+Y>x+y,X>x)\right]}{dx} \)

Define function \( \Xi_{X+Y,X}(x+y,x) := \frac{\Theta_{X+Y,X}(x+y,x)}{\Pr(X+Y>x+y,X>x)} \)

\[ = \lim_{x^{-}\to 0} \left[ \frac{\Pr(X+Y>x+y,X>x+Y) - \Pr(X+Y>x+y,X>x)}{\Pr(X+Y>x+y,X>x)} \right] \]

We assume that the distribution of the random variables makes \( \Xi_{X+Y,X}(x+y,x) \) increasing in \( x \). i.e., \( \frac{d\Xi_{X+Y,X}(x+y,x)}{dx} \geq 0 \). This assumption is similar to IFR. Recall IFR means

\[ \lim_{x^{-}\to 0} \left[ \frac{\Pr(X+Y>x+y)-\Pr(X+Y>x+y,X>x)}{\Pr(X+Y>x+y,X>x)} \right] \text{increases with } x. \]

**Assumption 2.** Identical transshipment price.

Then we have the following proposition.

**Proposition 5.6** If we restrict the strategy set to be a compact and convex subset of \( Q \) (defined in the proof), then there exists a pure strategy equilibrium in prices and safety stocks characterized by

\[-w+p_i \Pr(\tilde{D}_i^0 > z_i) + \tau_i \Pr(z_i + z_j - \tilde{D}_j^0 < \tilde{D}_i^0 < z_i) - (p_i - \tau_j) \Pr(z_i < \tilde{D}_i^0 < z_i + z_j - \tilde{D}_j^0) = 0 \]

(5.6)

\[ bw + E \min\{\tilde{D}_i^0, z_i\} + a - 2bp_i + E \min\{(z_j - \tilde{D}_j^0)^+, (\tilde{D}_i^0 - z_i)^+\} = 0 \]

(5.7)

**Proof.** See proof 4 in the appendix. □

Let an equilibrium point be \((p_i, p_j, z_i, z_j)\). Can we design a transshipment price in this scenario such that the two retailers' joint optimal can be sustained?

**Proposition 5.7** In symmetric game, design \( \tau = \frac{p_i^m q_1}{q_2 + q_1} \), where \( q_1 = \Pr(z_1^M + z_2^M - \tilde{D}_2^0 < \tilde{D}_1^0 < z_1^M) = \Pr(z_1^M + z_2^M - \tilde{D}_1^0 < \tilde{D}_2^0 < z_2^M), q_2 = \Pr(z_1^M < \tilde{D}_1^0 < z_1^M + z_2^M - \tilde{D}_2^0) \)
= Pr(z_2^M < \bar{D}_2^0 < z_1^M + z_2^M - \bar{D}_1^0), and z_1^M = z_2^M, p_1^M = p_2^M is the joint optimal (monopoly) safety stock and pricing decision. Then the two independent retailers' optimal profits can be obtained.

The proof is straightforward. By substituting the above t-price into the first order conditions (5.6)-(5.7), it gives the same solution as the system optimal. Note that these t-prices can help to reach the system optimal because here the independent pricing decisions are the optimal decisions for the system. This proof is part of the proof of proposition 5.14. By coordinating the safety stock decisions only, the independent retailers behave like a monopoly. This t-price is unique because both retail price and safety stock increase with t-price. When \( \tau = 0 \), \( z^T \leq z^M \) and \( p^T \leq p^M \). When \( \tau = p^M \), \( z^T \geq z^M \) and \( p^T \geq p^M \). The design of this t-price is similar to Rudi et. al. (2001) where retail price is exogenous.

Observations on how the wholesale price affects profits are more complicated because the best response retail price for each retailer increases with \( w \) but the best response safety stock decreases with \( w \).

\[
\frac{\partial E\pi^T}{\partial w} = -(z_1^T + L_1^T) + \frac{\partial E\pi^T}{\partial z_1} \frac{\partial z_1^T}{\partial w}
\]

The first term is the direct effect of \( w \), which is negative. The second term is the effect of \( w \) through inventory transshipment, which is ambiguous in sign.

How does the t-price affect the retailers' profits?

\[
\frac{\partial E\pi^T}{\partial \tau} = E \min\{(y_i^T - \bar{D}_i)^+, (\bar{D}_j - y_j^T)^+\} - E \min\{(y_j^T - \bar{D}_j)^+, (\bar{D}_i - y_i^T)^+\} + \frac{\partial E\pi^T}{\partial z_1} \frac{\partial z_1^T}{\partial \tau}.
\]

The arguments are the same as in last section.

**Scenario 2-1: The C game**

If the price sensitive newsvendors do not contract to transship, then their customers switch to the competitor. Knowing this, a newsvendor set price and inventory to maxi-
mize $E\pi^C_i$

$$E\pi^C_i = -w(L_i + z_i) + p_i(L_i + E \min\{z_i, \bar{D}_i^0\}) + p_i E \min\{(z_i - \bar{D}_i^0)^+, (\bar{D}_i^0 - z_j)^+\}$$  \hspace{1cm} \text{(5.8)}

**Proposition 5.8** There exists a pure strategy equilibrium in safety stock and price in the C game characterized by

$$-w + p_i \Pr(\bar{D}_i^0 > z_i) + p_i \Pr(z_i + z_j - \bar{D}_j^0 < \bar{D}_i^0 < z_i) = 0 \hspace{1cm} \text{(5.9)}$$

$$bw + E \min\{\bar{D}_i^0, z_i\} + a - 2bp_i + E \min\{(\bar{D}_j^0 - z_j)^+, (z_i - \bar{D}_i^0)^+\} = 0 \hspace{1cm} \text{(5.10)}$$

**Proof.** See proof 5 in the appendix. ■

Let the equilibrium points be $(p_i^C, p_j^C, z_i^C, z_j^C)$.

**Proposition 5.9** With identical retailers, the equilibrium is unique.

The proof is similar to proposition 3.7 in chapter 3, and omitted.

$$\frac{\partial E\pi^C_i}{\partial w} = - (z_i^C + L_i^C) + \frac{\partial E\pi^C_i}{\partial z_i} \frac{\partial z_i^C}{\partial w}$$

The sign of $\frac{\partial z_i^C}{\partial w}$ is unknown. Inventory competition may strengthen or weaken the negative effect of the wholesale price on profit.

*Comparing the T game to the C game*

In which setting will newsvendors set a higher price, safety stock and total inventory? In which setting are newsvendors' and manufacturer's more profitable; competition or transshipment?

To gain some insight but maintain tractable mathematics, we focus on the symmetric case, assuming that retailers are identical. The equilibrium is unique in the C game.

**Proposition 5.10** 1. When the transshipment price is less than or equal to the retail prices, both safety stock and price are higher under competition than transshipment. 2.
Both safety stock and retail price for the transshipment case increase in the transshipment price.

**Proof.** See proof 6 in the appendix.

So the price setting newsvendors with transshipment usually set a lower safety stock and retail price than what they would do under inventory competition. The safety stock and retail price under transshipment increases with t-price.

What happens if the newsvendors transship at one party's retail price? Here we focus on the properties of the equilibrium.

**Proposition 5.11** 1. If transshipment prices are set to equal the competitor's retail price then safety stocks and retail prices are lower under transshipment. 2. The opposite is true if transshipment prices are set equal to your own retail price.

**Proof.** See proof 7 in the appendix.

The prior choice of t-price plays an important role in this section.

With an exogenous r-price (section 2), proposition 5.3 tells us that the T game never ends up with a higher equilibrium stock, no matter how the t-price \(0 \leq t \leq p\) is chosen ex ante. But when prices are decision variables, things are different. If the t-price is your own retail price, then the equilibrium prices and safety stocks in the T game are actually higher than that of the C game. However, if the transshipment price is the competitor's retail price, then the market ends up with a lower safety stock and retail price in the T game. The intuition is that with the former pricing scheme, retailers have incentives to charge more, which lead to a higher equilibrium price and stock. It is the opposite with the latter scheme. If the two retailers are independent, transshipping at one party's price is possible.

But how about retailers' and manufacturer's profits? By similar methods, we have
that the equilibrium $E\pi_i^T$ increases with $\tau$ when $\tau$ is very small (near zero), and decreases with $\tau$ when $\tau$ is very large (near the retail price).

To compare retailers' profits and total inventory, we use numerical methods, using specific demand distributions.

**Example 5.3** $\tilde{D}_i^0$ is exponentially distributed with $\lambda = 1$. Then in the C game, $p = \frac{a+bw+1-e^{-2z}}{2b}e^{-2z}$, the equilibrium stock is given by $-w + p(e^{-z} + e^{-2z}) = 0$. $E\pi_i^C = (p - w)(a - bp) - wz + p(1 - e^{-2z} - e^{-2z}z)$. In the T game ($\tau_i = \tau$ is not pre-determined to be any party's retail price), $p = \frac{a+wb+1-e^{-2z}}{2b}e^{-2z}$, the equilibrium stock is given by $-w + p(1 + z)e^{-2z} + \tau(e^{-z} - e^{-2z}) = 0$, $E\pi_i^T = (p - w)(a - bp) - wz + p(1 - e^{-2z} - e^{-2z}z)$. If $\tau_i = p_i$, then $p = \frac{a+wb+1-e^{-z}-2e^{-2z}}{2b}$. If $\tau_i = p_j$, then $p = \frac{a+wb+1-e^{-z}}{2b}$. The expressions for stock equilibrium and profits are the same as for the C game. We have the results in Figure 5.12 to 5.19 and Table 5.4 ($a = 2, b = 0.6$).

<table>
<thead>
<tr>
<th>$w$</th>
<th>$\tau$</th>
<th>$z^C$</th>
<th>$p^C$</th>
<th>$y^C$</th>
<th>$z^T$</th>
<th>$p^T$</th>
<th>$y^T$</th>
<th>$E\pi_i^C$</th>
<th>$E\pi_i^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>$\tau_i = p_i$</td>
<td>2.12</td>
<td>2.66</td>
<td>2.52</td>
<td>2.14</td>
<td>2.73</td>
<td>2.50</td>
<td>2.605</td>
<td>$&gt; 2.60$</td>
</tr>
<tr>
<td>0.4</td>
<td>$\tau_i = p_j$</td>
<td>2.12</td>
<td>2.66</td>
<td>2.52</td>
<td>2.10</td>
<td>2.60</td>
<td>2.54</td>
<td>2.605</td>
<td>$&lt; 2.61$</td>
</tr>
<tr>
<td>0.8</td>
<td>$\tau_i = p_i$</td>
<td>1.54</td>
<td>2.80</td>
<td>1.86</td>
<td>1.57</td>
<td>2.89</td>
<td>1.84</td>
<td>1.88</td>
<td>$&gt; 1.87$</td>
</tr>
<tr>
<td>0.8</td>
<td>$\tau_i = p_j$</td>
<td>1.54</td>
<td>2.80</td>
<td>1.86</td>
<td>1.51</td>
<td>2.72</td>
<td>1.88</td>
<td>1.88</td>
<td>$= 1.88$</td>
</tr>
<tr>
<td>1.2</td>
<td>$\tau_i = p_i$</td>
<td>1.20</td>
<td>2.93</td>
<td>1.44</td>
<td>1.23</td>
<td>3.03</td>
<td>1.41</td>
<td>1.32</td>
<td>$&gt; 1.31$</td>
</tr>
<tr>
<td>1.2</td>
<td>$\tau_i = p_j$</td>
<td>1.20</td>
<td>2.93</td>
<td>1.44</td>
<td>1.17</td>
<td>2.84</td>
<td>1.47</td>
<td>1.32</td>
<td>$&lt; 1.33$</td>
</tr>
<tr>
<td>1.4</td>
<td>$\tau_i = p_i$</td>
<td>1.07</td>
<td>3.00</td>
<td>1.27</td>
<td>1.10</td>
<td>3.09</td>
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<td>1.09</td>
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<td>1.4</td>
<td>$\tau_i = p_j$</td>
<td>1.07</td>
<td>3.00</td>
<td>1.27</td>
<td>1.04</td>
<td>2.91</td>
<td>1.29</td>
<td>1.09</td>
<td>$= 1.09$</td>
</tr>
</tbody>
</table>

We have the following observations,

1. The results on equilibrium prices and safety stocks illustrate Propositions 5.10 and 5.11. Both safety stocks and prices increase with the t-price. When $0 \leq \tau < p_i(p_j)$ or
τ_i = p_j, the C game equilibrium point is greater than that of the T game. When τ_i = p_i, the C game equilibrium point is smaller than that of the T game. We also find that the equilibrium prices and safety stocks in the T game is greater if the t-price is higher than the r-price.

2. With a large profit margin (p/w ≈ 6), when the t-price is less than the r-price, transshipment is always profitable for retailers. With a medium to small profit margin (p/w ≈ 3, 2.5, 2), when the t-price is small, competition is profitable for retailers; when the t-price is around the wholesale price, transshipment is profitable. When the t-price is larger than the r-price, competition is beneficial for all profit margins.

When transshipment occurs at your own retail price, retailers lose under transshipment compared to competition. But if transshipment is at the competitor’s retail price, retailers gain under transshipment compared to competition.

3. For all profit margins, the total inventories increase with the t-price. When the t-price is less than the r-price, the total inventory in the C game is larger. So the manufacturer would prefer retailers to compete. The reverse is true when the t-price is higher than the r-price.

When transshipment occurs at your own retail price, the T game ends up with a lower total inventory in equilibrium. The manufacturer will like retailers to be under competition. When transshipment occurs at the competitor’s retail price, the T game ends up with a higher total inventory in equilibrium, and the manufacturer will prefer retailers to use transshipment. It is interesting to know that the comparison of retailers and manufacturer’s profits under the C and T games are the same when the ex ante t-price is set differently. Transshipping at the own r-price damages the retailer and the manufacturer. Transshipping at the competitor’s r-price favors the manufacturer and the retailer. But when transshipment price is a parameter, there is always an incentive
conflict between the manufacturer and the retailers in terms of transshipping or leaving the consumers to switch.

The following uniform distribution also gives the same results.

**Example 5.4** Assume $D_1^0$ is uniformly distributed between $[0, 1]$. In the case of $z_i + z_j \leq 1$, we have the optimal price $p = \frac{a + wb + z - \frac{2}{3} z^3}{2b}$ in both the C game and T game ($\tau_i = \tau$ is not equal to the r-price). The safety stock equilibrium is determined by $-w + p(1 - \frac{3}{2} z^2) = 0$ in the C game and $-w + p(1 - z - \frac{z^2}{2}) + \tau(z - z^2) = 0$ in the T game. In both cases the profits are $(p - w)(a - bp) - wz + p(z - \frac{2}{3} z^3)$. If $\tau_i = p_i$, then $p = \frac{a + wb + z - \frac{2}{3} z^3}{2b}$. If $\tau_i = p_j$, then $p = \frac{a + wb + z - \frac{2}{3} z^3}{2b}$. The expression for stock equilibrium and profits are the same as for the C game. The numerical results are in Table 5.5.

Table 5.5. Numerical Experiment-Uniform Distribution ($z_i + z_j \leq 1$) (endogenous retail price)

<table>
<thead>
<tr>
<th>w</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_i$</td>
<td>0</td>
<td>0.7</td>
</tr>
<tr>
<td>$zC$</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>$pC$</td>
<td>1.83</td>
<td></td>
</tr>
<tr>
<td>$zT$</td>
<td>0.16</td>
<td>0.24</td>
</tr>
<tr>
<td>$pT$</td>
<td>1.66</td>
<td>1.73</td>
</tr>
<tr>
<td>$E\pi_i^C$</td>
<td>0.052</td>
<td></td>
</tr>
<tr>
<td>$E\pi_i^T$</td>
<td>0.04</td>
<td>0.05</td>
</tr>
</tbody>
</table>

If $z_i + z_j \geq 1$, and $z_i \leq 1$, the optimal prices are $p = \frac{a + wb + 2z - 2z^2 + \frac{3}{2} z^3 - \frac{1}{6}}{2b}$ in both games ($\tau_i = \tau$ not equal to r-price). The safety stock equilibrium are determined by $-w + p(-2z + \frac{z^2}{2} + \frac{3}{2}) = 0$ for the C game and $-w + p(\frac{3}{2} - 3z + \frac{3}{2} z^2) + \tau(z - z^2) = 0$ in the T game. The profit functions are $(p - w)(a - bp) - wz + p(2z - 2z^2 + \frac{3}{2} z^3 - \frac{1}{6})$ in
both cases. If \( \tau_i = p_i \), then \( p = \frac{a+wb+3z-\frac{z^2}{2b}+\frac{a+wb+z^2}{3}}{2b} \). If \( \tau_i = p_j \), then \( p = \frac{a+wb+z^2}{2b} \). The expression for the stock equilibrium and profits are the same as C game. The numerical results are in table 5.6.

Table 5.6 Numerical Experiment-Uniform Distribution \((z_i + z_j > 1, z_i < 1)\) (endogenous retail price)

<table>
<thead>
<tr>
<th>( w )</th>
<th>1.4</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_i )</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>( z^C )</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>( p^C )</td>
<td>1.44</td>
<td></td>
</tr>
<tr>
<td>( E\pi^C_i )</td>
<td>0.546</td>
<td></td>
</tr>
</tbody>
</table>

5.5 T game versus C game: With Price Competition

In this section, we consider a market with the deterministic part of demand sensitive to both retailers' prices. To incorporate price competition, we introduce a substitutability factor \( \theta \). This substitutability need not be just in terms of the product itself (we can assume the products are essentially identical). It could also reflect characteristics of the two retailers beyond products, such as reputation, location, packaging, advertising or even the format (superstores vs. groceries). But, whatever the source, this preparedness to substitute is ex-ante, consumers will be prepared to visit the alternate store on the basis of price. Ex post, when demand reveals that the store of choice is short of product then either the consumer will switch or wait for the transshipment.

Scenario 3-1. The T game
The two retailers’ objective functions are,

\[ E\pi_i^T = -w(z_i + L_i) + p_i(L_i + E \min\{\tilde{D}_i^0, z_i\}) + \tau_iE \min\{(z_i - \tilde{D}_i^0)^+, (\tilde{D}_j^0 - z_j)^+\} \]

\[ + (p_i - \tau_j)E \min\{(z_j - \tilde{D}_j^0)^+, (\tilde{D}_i^0 - z_i)^+\} \]

Where \( L_i = a - (b + \theta)p_i + \theta p_j \).

By the Glicksberg’ theorem (Fudenberg and Tirole 1991), there exists a Nash equilibrium in mixed strategies. If we adopt a similar method to that used in section 5.4, under the two assumptions, and if we restrict the strategy set to be subset of \( Q = \{(z_i, p_i) : 2(b + \theta)(p_i - \tau_j) - \frac{\Pr(\tilde{D}_i^0 > z_i, \tilde{D}_j^0 + \tilde{D}_i^0 > z_i + z_j)}{\tilde{D}_i^0, \tilde{D}_j^0} \geq 0\} \), then the objective function is concave in \((z_i, p_i)\) and the pure strategy equilibrium exists, characterized by the following functions,

\[-w + p_i \Pr(\tilde{D}_i^0 > z_i) + \tau_i \Pr(z_i + z_j - \tilde{D}_j^0 < \tilde{D}_i^0 < z_i) - (p_i - \tau_j) \Pr(z_i < \tilde{D}_i^0 < z_i + z_j - \tilde{D}_j^0) = 0 \tag{5.11} \]

\[(b + \theta)w + a - 2(b + \theta)p_i + \theta p_j + E \min\{\tilde{D}_i^0, z_i\} + E \min\{(z_j - \tilde{D}_j^0)^+, (\tilde{D}_i^0 - z_i)^+\} = 0 \tag{5.12} \]

The effect of the wholesale price is ambiguous, because the best response retail price increases with the wholesale price and best response safety stocks decrease with the wholesale price. Note that the best response functions are non-decreasing in the t-price, so in the symmetric equilibrium, safety stocks and prices are non-decreasing in the t-price.

The wholesale price \( w \) affects retailers’ profits in a more complicated way since

\[ \frac{\partial E \pi_i^T}{\partial w} = -(z_i^T + L_i^T) + \frac{\partial E \pi_i^T}{\partial z_i} \frac{\partial z_i^T}{\partial w} + \frac{\partial E \pi_i^T}{\partial p_j} \frac{\partial p_j^T}{\partial w} \]

The first term is the direct effect of \( w \), which is negative. The second term is the effect of \( w \) through inventory transshipment, which is ambiguous in sign. The third term is the effect of \( w \) through price competition, which is also ambiguous in sign.

How does the t-price affect retailers’ profits?

\[ \frac{\partial E \pi_i^T}{\partial \tau} = E \min\{(z_i^T - \tilde{D}_i)^+, (\tilde{D}_j - z_j^T)^+\} - E \min\{(z_j^T - \tilde{D}_j)^+, (\tilde{D}_i - z_i^T)^+\} + \frac{\partial E \pi_i^T}{\partial z_i} \frac{\partial z_i^T}{\partial \tau} + \]

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In contrast to the last sections, the last term is due to the price competition, which makes a larger t-price more profitable.

Scenario 3-2: The C game

Under price competition, in the C game, the two retailers set prices and safety stocks simultaneously to maximize,

\[
E\pi^C_i = -w(z_i + L_i) + p_i(L_i + E \min\{\tilde{D}_i^0, z_i\}) + p_iE \min\{(z_i - \tilde{D}_i^0)^+, (\tilde{D}_j^0 - z_j)^+\} \\
= -w(z_i + L_i) + p_i(L_i + E \min\{\tilde{D}_i^{OT}, z_i\})
\]

where \(\tilde{D}_i^{OT} = \tilde{D}_i^0 + (\tilde{D}_j^0 - z_j)^+\), \(L_i = a - (b + \theta)p_i + \theta p_j\).

For this game, the mixed strategy equilibrium always exists. We also have the following Proposition.

**Proposition 5.12** Let \(r_i = r^{OT}_i\), under a reasonably mild condition \((r' / r^2\) decreasing), there exists a pure strategy equilibrium in price and safety stocks, which is characterized by

\[
-w + p_i Pr(\tilde{D}_i^{OT} > z_i) = 0 \quad (5.13)
\]

\[
(b + \theta)w + a - 2(b + \theta)p_i + \theta p_j + E \min\{\tilde{D}_i^{OT}, z_i\} = 0 \quad (5.14)
\]

Furthermore, there is a unique equilibrium in the symmetric game.

The proof is similar to proposition 3.9 in chapter 3, so it is omitted here.

Comparing the T game and the C game

We compare the two games and find some properties of the equilibrium. Comparisons 1-4 are based on a symmetric equilibrium and the uniqueness of the C game equilibrium.

**Proposition 5.13** 1. Equilibrium safety stocks and prices are lower under transshipment than competition. 2. If transshipment prices are set endogenously to equal the
competitor's retail price then equilibrium safety stocks and retail prices are also lower under transshipment. 3. However the opposite is true if transshipment prices are set equal to your own retail price. 4. Equilibrium retail prices and safety stocks are increasing in transshipment prices and 5. decreasing in $\theta$ for both games.

**Proof.** See proof 8 in the appendix. 

To compare the retailers' profits and the total inventories under different scenarios, we use some specific demand distributions.

**Example 5.5** Assume $D_i^0$ is exponentially distributed with $\lambda = 1$. In the C game $p = \frac{a+(b+\theta)w+1-e^{-2z}-e^{-2z}}{2b+\theta}$. In the T game, $p = \frac{a+(b+\theta)w+1-e^{-2z}}{2b+\theta}$ if $\tau$ is not set to be the $r$-price. $p = \frac{a+(b+\theta)w+1+e^{-z}-2e^{-2z}}{2b+\theta}$ if $\tau_i = p_i$. And $p = \frac{a+(b+\theta)w+1-e^{-z}}{2b+\theta}$ if $\tau_i = p_j$. The equilibrium stocks and profits are the same as those in example 5.3. The results are shown in Figure 5.20 - 5.28 and tables 5.7 and 5.8. ($a = 2, b = 0.6$).

**Table 5.7** Numerical Experiment-Exponential Distribution (with price competition $\theta = 0.2$)

<table>
<thead>
<tr>
<th>$w$</th>
<th>$\tau$</th>
<th>$z^C$</th>
<th>$p^C$</th>
<th>$y^C$</th>
<th>$z^T$</th>
<th>$p^T$</th>
<th>$y^T$</th>
<th>$E\pi_i^C$</th>
<th>$E\pi_i^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>$\tau_i = p_i$</td>
<td>2.00</td>
<td>2.33</td>
<td>2.60</td>
<td>2.02</td>
<td>2.39</td>
<td>2.57</td>
<td>2.56 &lt; 2.58</td>
<td>&lt; 2.58</td>
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<tr>
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<td>$\tau_i = p_j$</td>
<td>2.00</td>
<td>2.33</td>
<td>2.60</td>
<td>1.98</td>
<td>2.27</td>
<td>2.62</td>
<td>2.56 &gt; 2.54</td>
<td>&gt; 2.54</td>
</tr>
<tr>
<td>0.8</td>
<td>$\tau_i = p_i$</td>
<td>1.43</td>
<td>2.50</td>
<td>1.93</td>
<td>1.46</td>
<td>2.58</td>
<td>1.91</td>
<td>1.86 &lt; 1.87</td>
<td>&lt; 1.87</td>
</tr>
<tr>
<td>0.8</td>
<td>$\tau_i = p_j$</td>
<td>1.43</td>
<td>2.50</td>
<td>1.93</td>
<td>1.41</td>
<td>2.42</td>
<td>1.96</td>
<td>1.86 &gt; 1.84</td>
<td>&gt; 1.84</td>
</tr>
<tr>
<td>1.2</td>
<td>$\tau_i = p_i$</td>
<td>1.11</td>
<td>2.66</td>
<td>1.52</td>
<td>1.14</td>
<td>2.74</td>
<td>1.50</td>
<td>1.31 &lt; 1.32</td>
<td>&lt; 1.32</td>
</tr>
<tr>
<td>1.2</td>
<td>$\tau_i = p_j$</td>
<td>1.11</td>
<td>2.66</td>
<td>1.52</td>
<td>1.08</td>
<td>2.59</td>
<td>1.53</td>
<td>1.31 &gt; 1.29</td>
<td>&gt; 1.29</td>
</tr>
<tr>
<td>1.4</td>
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<td>2.75</td>
<td>1.34</td>
<td>1.01</td>
<td>2.82</td>
<td>1.32</td>
<td>1.08 &lt; 1.09</td>
<td>&lt; 1.09</td>
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<td>2.75</td>
<td>1.34</td>
<td>0.96</td>
<td>2.67</td>
<td>1.36</td>
<td>1.08 &gt; 1.06</td>
<td>&gt; 1.06</td>
</tr>
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</table>

**Table 5.8** Numerical Experiment-Exponential Distribution (with price competition $\theta = 1.6$)
<table>
<thead>
<tr>
<th>( w )</th>
<th>( \tau )</th>
<th>( z^C )</th>
<th>( p^C )</th>
<th>( y^C )</th>
<th>( z^T )</th>
<th>( p^T )</th>
<th>( y^T )</th>
<th>( E\pi_i^C )</th>
<th>( E\pi_i^T )</th>
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</thead>
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<tr>
<td>0.4</td>
<td>( \tau_i = p_i )</td>
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<td>1.98</td>
<td>2.67</td>
<td>1.88</td>
<td>2.03</td>
<td>2.66</td>
<td>2.38</td>
<td>&lt; 2.42</td>
</tr>
<tr>
<td>0.4</td>
<td>( \tau_i = p_j )</td>
<td>1.86</td>
<td>1.98</td>
<td>2.67</td>
<td>1.83</td>
<td>1.92</td>
<td>2.68</td>
<td>2.38</td>
<td>&gt; 2.34</td>
</tr>
<tr>
<td>0.8</td>
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<td>1.31</td>
<td>2.18</td>
<td>2.00</td>
<td>1.33</td>
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<td>1.98</td>
<td>1.72</td>
<td>&lt; 1.76</td>
</tr>
<tr>
<td>0.8</td>
<td>( \tau_i = p_j )</td>
<td>1.31</td>
<td>2.18</td>
<td>2.00</td>
<td>1.28</td>
<td>1.48</td>
<td>2.39</td>
<td>1.72</td>
<td>&gt; 1.68</td>
</tr>
<tr>
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<td>2.38</td>
<td>1.57</td>
<td>1.03</td>
<td>2.45</td>
<td>1.56</td>
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<td>&lt; 1.24</td>
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<tr>
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<td>1.00</td>
<td>2.38</td>
<td>1.57</td>
<td>.97</td>
<td>2.32</td>
<td>1.58</td>
<td>1.21</td>
<td>&gt; 1.18</td>
</tr>
<tr>
<td>1.4</td>
<td>( \tau_i = p_i )</td>
<td>0.88</td>
<td>2.48</td>
<td>1.39</td>
<td>.91</td>
<td>2.54</td>
<td>1.39</td>
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<td>0.88</td>
<td>2.48</td>
<td>1.39</td>
<td>.85</td>
<td>2.42</td>
<td>1.40</td>
<td>1.00</td>
<td>&gt; .97</td>
</tr>
</tbody>
</table>

We have the following observations,

1. Newsvendors under the C game have a better ability to deter price competition. So under fiercer price competition (in the example \( \theta = 0.5 \)) and a higher wholesale price, their equilibrium profits in the T game are less than that in the C game for most transshipment prices (Figure 5.23). Specifically, newsvendors will lose by transshipment even when the transshipment price is equal to the wholesale price (which always benefits retailers under transshipment in previous sections). Figure 5.28 compares how newsvendors' profits change with \( \theta \) (\( w = 1.4, \tau = 1.5 \)) under T game and C game. We can see that the profits in the T game drop more sharply with the degree of price competition. The same evidence holds for other cases.

2. The results on equilibrium prices and safety stocks echoes proposition 5.13.

3. Total inventory in the C game is always higher than that in the T game (\( 0 < \tau < p \)). So the manufacturer prefers that the newsvendor compete. So under a market with strong price competition, the incentive of the manufacturer and the retailers are more likely to coincide. That is, not to transship.
4. When the transshipment price is the competitor's retail price, the equilibrium total inventory is higher in the T game than in the C game. The manufacturer would prefer retailers to transship rather than compete. When the t-price is your own r-price, the equilibrium total inventory is higher in the C game than in the T game. The manufacturer would prefer competition in inventory between retailers.

5. The results on newsvendors' profits are opposite to the results obtained from the last section. When the transshipment price is your own retail price, both newsvendors are better off under transshipment (T game) compared with competition (C game). When the transshipment price is the competitor's retail price, both newsvendors are worse off under transshipment, compared with competition (C game). This is the opposite to the case without price competition. The intuition is that with the former scheme, newsvendors try to set a higher price and higher safety stock. The high price won't damage the demand much under price competition (prices are complementary) but will decrease demand a lot in the case without price competition. So the former transshipment scheme is beneficial in a market with price competition. With the latter scheme, newsvendors try to set a lower price and a lower safety stock. The low price won't increase demand much under price competition, but can increase demand a lot in the case without price competition. So the latter transshipment scheme is beneficial in a market without price competition.

To conclude, under price competition and transshipment, the manufacturer would prefer the retailers' to transship at the competitor's price, but retailers would ask for their own retail price.

The main observation above is the first point: newsvendors under inventory competition (C game) have a stronger ability to deter price competition than they are under transshipment (T game). This is also an important difference from last two sections.

Why do the results differ under/without price competition? Now we provide an
an analytical explanation. Note that the profits under the C game change with \( \theta \) as follows,

\[
\frac{\partial E\pi_i^C}{\partial \theta} = (p_i^C - p_i^C)(p_i^C - w) + \frac{\partial E\pi_i^C}{\partial z_j} \frac{\partial E\pi_i^C}{\partial \theta} + \frac{\partial E\pi_i^C}{\partial p_j} \frac{\partial E\pi_i^C}{\partial \theta} .
\]

In symmetric equilibrium the first term is zero. Since \( \partial E\pi_i^C / \partial p_j > 0, \partial p_i^C / \partial \theta < 0 \), the last term is always negative, the so-called ‘damage of price competition’ to profit. We know that \( \frac{\partial E\pi_i^C}{\partial \theta} < 0 \) and \( \frac{\partial E\pi_i^C}{\partial p_j} = -p_i^C \Pr(z_j^C < \tilde{D}_j^0 < z_i^C + z_j^C - \tilde{D}_i^0) < 0 \)

So \( (\partial E\pi_i^C / \partial z_j)(\partial z_j^C / \partial \theta) > 0 \). This mitigates the damage of price competition to profit. So the newsvendors’ profits under C game do not drop with \( \theta \) so fast as might be thought. In chapter 2, we find circumstance such that \( \partial E\pi_i^C / \partial \theta > 0 \). That is, in a market with inventory competition, introducing price competition is actually good for the retailers. Under the T game,

\[
\frac{\partial E\pi_i^T}{\partial \theta} = (p_i^T - p_i^T)(p_i^T - w) + \frac{\partial E\pi_i^T}{\partial z_j} \frac{\partial E\pi_i^T}{\partial \theta} + \frac{\partial E\pi_i^T}{\partial p_j} \frac{\partial E\pi_i^T}{\partial \theta} .
\]

In the symmetric case the first term is zero. \( (\partial E\pi_i^T / \partial p_j)(\partial p_j^T / \partial \theta) < 0 \) as in the C game case and \( \partial z_j^T / \partial \theta < 0 \). But the sign of \( \frac{\partial E\pi_i^T}{\partial z_j} \) is ambiguous,

\[
\frac{\partial E\pi_i^T}{\partial z_j} = -p_i^T \Pr(z_j^T < \tilde{D}_j^0 < z_i^T + z_j^T - \tilde{D}_i^0) + (p_i^T - \tau)[\Pr(z_j^T < \tilde{D}_j^0 < z_i^T + z_j^T - \tilde{D}_i^0) + \Pr(z_i^T + z_j^T - \tilde{D}_i^0 < \tilde{D}_j^0 < z_j^T)]
\]

The first term is the same as \( \partial E\pi_i^C / \partial z_j \), but evaluated at \( z_j^T \) and \( z_i^T \). We may have \( \partial E\pi_i^T / \partial z_j > 0 \) or less negative as \( \partial E\pi_i^C / \partial z_j \) because of the latter two terms.

So transshipment between newsvendors may not mitigate the damage of price competition as much as inventory competition can. Numerically we compare the retailers’ profits under the C game and the T game when the substitution factor \( \theta \) changes, finding that the retailers’ profits in the T game drops more sharply with \( \theta \). (Figure 5.28).

Can the t-price suggested in the earlier sections benefit the newsvendors? Can we find a t-price to obtain the monopoly level of retail price and safety stock?

According to proposition 5.7, we can design a transshipment price as before, in the symmetric case, \( q_2 = q_4, q_1 = q_3 \), the t-price is \( \tau_1 = \tau_2 = \frac{p^M q_1}{q_2 + q_1} \). Recall that \( p^M = p_i^M = 164 \).
The following proposition is also based on symmetric equilibrium.

**Proposition 5.14** There is no t-price that can induce the system optimal under price competition. Under price competition, newsvendors in the T game with a t-price that is optimal for the no-price-competition scenario can be worse off than the newsvendors in the C game. If this happens, then any t-price lower than this ‘optimal’ t-price makes the newsvendors in the T game even worse. A better t-price contract should be greater than this ‘optimal’ t-price.

**Proof.** See proof 9 in the appendix. ■

On the basis of the above analysis, we find it is better for retailers not to transship under price competition (numerical results show that it is especially true when \( \theta \) is high, i.e., when they are close substitutable in the eyes of consumers). However, transshipment can be beneficial if the retailers are not close substitutes for consumers. For example, gas stations in different locations.

Using the optimal t-price contract suggested by proposition 5.7 if the market involves price competition can make the newsvendors even worse. Rudi et. al. (2001) consider a market without pricing, they design a transshipment contract between independent retailers such that the profits can be system optimal. We show above that this optimal transshipment contract can still obtain the system optimal with endogenous pricing, but can make the retailers worse off if they are involved in price competition. From the analysis of proposition 5.14, a better t-price contract under price competition should be higher than this no-price-competition optimal t-price. Since the t-price does not appear in the first order condition for price, no t-price can sustain the system optimal. A better t-price contract should be a function of the degree of price competition \( \theta \) and solves,

\[
\max_{\tau} \sum_{i=1,2} E\pi_i^T(\tau)
\]
But in this case the t-price is dependent on both retail price and safety stock.

5.6 Conclusion

Competitive retailers can either cooperate in sharing stock by ‘back of house’ arrangements such as transshipment or stock pooling, or accept the fact that their customers will switch unfilled demand, ‘transshipping’ their own demand. This is even more the case in the new economy where switching demand is made easier and less dependent on geography. For jointly owned retail outlets it seems fairly obvious that transshipment should be beneficial, but should competitors arrange to do it also? In this chapter, we study this question under six market conditions:

<table>
<thead>
<tr>
<th>Price Condition</th>
<th>Inventory Competition</th>
<th>Inventory Pooling (Transshipment)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price Exogenous</td>
<td>Scenario 1-1</td>
<td>Scenario 1-2</td>
</tr>
<tr>
<td>Price Endogenous</td>
<td>Scenario 2-1</td>
<td>Scenario 2-2</td>
</tr>
<tr>
<td>Price Competition</td>
<td>Scenario 3-1</td>
<td>Scenario 3-2</td>
</tr>
</tbody>
</table>

We explore existence and uniqueness of pure strategy Nash equilibria and explore some of their properties. We find that with identical retailers, the safety stock levels (or stock levels for the first scenario) and retail prices increase with transshipment price, decrease with substitution factor \( \theta \). Under reasonable transshipment prices (between zero and the retail price), inventory pooling (transshipment) ends up with lower safety stock levels (stock levels for the first scenario) and retail prices than inventory competition scenario.

In scenarios with an endogenous price and price competition, we also explore the cases of transshipment at one party’s retail price. We find transshipping at own retail price ends up with higher retail prices and safety stocks. The opposite is true if transshipping at the competitor’s retail price.
We have also investigated the impact on profits. When retailers do not compete on price, then agreeing to transship is more likely to be preferred when the transshipment price is high and/or the wholesale price is low. If retailers can choose an optimal t-price, they should transship (pool).

However, when retailers do compete on price, the results are different. When price competition is fierce, better not to transship. Indeed, ignoring price competition and transshipping at the non-price-competition optimal t-price can make retailers even worse off than not transshipping. No transshipment price can sustain a system optimal (monopoly) in this scenario.

The manufacturer would prefer retailers not to transship in all scenarios except using the transshipment price at the competitor’s retail price.

The summary answer is that if your products are reasonably substitutable and you are competing on price, let your customers do the walking. However if your customer base does not indulge in significant price comparison shopping, even though they might be price sensitive, then cooperating with competitors (pooling) can make a lot of sense.

Figures

Figures associated with the proofs (The straight lines below are just for illustration)
Figure 5.1 For Propositions 5.10, 5.13

Figure 5.2 For Propositions 5.11, 5.13

Figure 5.3. For Propositions 5.11, 5.13

Figure 5.4. For Proposition 5.14

Figure 5.5 For Proposition 5.14

Relationship between profit and t-price (exogenous retail price)
Relationship between profit and t-price (endogenous retail price)
Relationship between profit and t-price (with price competition $\theta = 0.5$).

Figure 5.20
Profit ($\theta = 0.5$, $w = 0.4$)

Figure 5.21
Profit ($\theta = 0.5$, $w = 0.8$)

Figure 5.22
Profit ($\theta = 0.5$, $w = 1.2$)

Figure 5.23
Profit ($\theta = 0.5$, $w = 1.4$)
Figure 5.24

Figure 5.25

Figure 5.26

Figure 5.27

5.7 Appendix

Proofs.
1. Proposition 5.1.

**Proof.** Part 1. The first two terms of (5.1) are real valued functions in \( y_i \), hence (Lemma A2-1) supermodular in \( y_i \). The third term is the minimum of two functions, one is increasing in \( y_i \), the other increasing in \( -y_j \), thus (Lemma A2-2) supermodular in \( (y_i, -y_j) \). Similarly the fourth term is also the minimum of two functions, one decreasing in \( y_i \), the other is decreasing in \( -y_j \), hence (Lemma A2-2) supermodular in \( (y_i, -y_j) \). Since the sum of supermodular functions is supermodular, function (5.1) is supermodular in \( (y_i, -y_j) \), equivalent to increasing differences.

Part 2.

Step 1. Existence. By part 1 the pure Nash equilibrium exists (Lemma A4).

Step 2. Uniqueness. \( E \pi_i^T \) is strictly quasiconcave in \( y_i \) because

\[
\frac{\partial^2 E \pi_i^T}{\partial y_i \partial y_j} = -[p_i f_{\tilde{D}_i}(y_i) + \tau_i v_{ij}^{11} - \tau_i u_{i}^2 + (p_i - \tau_j) v_{ij}^{21} - (p_i - \tau_j) u_{i}^1] < 0.
\]

Also \( \frac{\partial^2 E \pi_i'^{NT}}{\partial y_i \partial y_j} = -[\tau_i v_{ij}^{11} + (p_i - \tau_j) v_{ij}^{21}] \).

Define \( r_i(y_j) := y_i^e(y_j) \), the best reply function of \( i \) given \( y_j \). It is a contraction mapping because

\[
\frac{\partial r_i(y_j)}{\partial y_j} = -\frac{\partial^2 E \pi_i'^{NT}}{\partial y_i \partial y_j} / \frac{\partial^2 E \pi_i^{NT}}{\partial y_i^2} = -\frac{\tau_i v_{ij}^{11} + (p_i - \tau_j) v_{ij}^{21}}{p_i f_{\tilde{D}_i}(y_i) + \tau_i v_{ij}^{11} - \tau_i u_{i}^2 + (p_i - \tau_j) v_{ij}^{21} - (p_i - \tau_j) u_{i}^1}.
\]

And \( \left| \frac{\partial r_i(y_j)}{\partial y_j} \right| < 1 \).

Where \( v_{ij}^{11} := f_{\tilde{D}_i + \tilde{D}_j} \Pr(\tilde{D}_i < y_i + y_j) \), \( v_{ij}^{21} := f_{\tilde{D}_i + \tilde{D}_j} \Pr(\tilde{D}_i > y_i + y_j) \).

\( u_i^1 := f_{\tilde{D}_i | \tilde{D}_i + \tilde{D}_j < y_i + y_j} (y_i) \Pr(\tilde{D}_i + \tilde{D}_j < y_i + y_j) \), \( u_i^2 := f_{\tilde{D}_i | \tilde{D}_i + \tilde{D}_j > y_i + y_j} (y_i) \Pr(\tilde{D}_i + \tilde{D}_j > y_i + y_j) \).

According to Lemma A6, the equilibrium is unique. ■

2. Proposition 5.2.

**Proof.** Part 1. Similar to previously proposition.

Part 2.

Step 2. Uniqueness. It is sufficient to show that $E\pi_i^C$ is strictly quasiconcave in $y_i$ and the reaction functions are contraction mappings (Lemma A6).

$$\frac{\partial^2 E\pi_i^C}{\partial y_i^2} = p_i[-f_{D_i+D_j<\mu} (y_i + y_j) \Pr(\tilde{D}_i < y_i) - f_{D_i\geq\mu} (y_i) \Pr(\tilde{D}_i + \tilde{D}_j > y_i + y_j)] < 0.$$

So $E\pi_i^C$ is strictly quasiconcave in $y_i$. Let $r_i^C(y_j) = y_i^*(y_j)$ be the reaction function of $i$ given $y_j$.

$$\frac{\partial^2 r_i^C(y_j)}{\partial y_j^2} = \frac{-\partial^2 E\pi_i^C}{\partial y_i^2} - \frac{\partial E\pi_i^C}{\partial y_i} \frac{\partial^2 r_i^C}{\partial y_j^2}.$$

Then we have $|\frac{\partial r_i^C(y_j)}{\partial y_j}| < 1$. This completes the proof. ■

3. Proposition 5.5.

**Proof.** 1. From function (5.1), at equilibrium we have $E\pi_i^T = E \min\{(y_i^T - \tilde{D}_i)^+, (\tilde{D}_i - y_j)^+\} - E \min\{(y_j^T - \tilde{D}_j)^+, (\tilde{D}_j - y_i^T)^+\} + \frac{\partial E\pi_i^T}{\partial y_j} \frac{\partial y_i^T}{\partial \tau}$.

$$\frac{\partial E\pi_i^T}{\partial y_j} = -\tau \Pr(y_j^T < \tilde{D}_j < y_i^T + y_j^T - \tilde{D}_i) + (p - \tau) \Pr(y_i^T + y_j^T - \tilde{D}_i < \tilde{D}_j < y_j^T)$$

and

$$\frac{\partial y_i^T}{\partial \tau} > 0$$

from equilibrium properties. According to the assumption of identical, $y_i^T = y_j^T$ at equilibrium, and $E \min\{(y_i^T - \tilde{D}_i)^+, (\tilde{D}_i - y_j^T)^+\} - E \min\{(y_j^T - \tilde{D}_j)^+, (\tilde{D}_j - y_i^T)^+\} = 0.$

Then we have $\frac{\partial E\pi_i^T}{\partial \tau} = -\tau \Pr(y_j^T < \tilde{D}_j < y_i^T + y_j^T - \tilde{D}_i) - (p - \tau) \Pr(y_i^T + y_j^T - \tilde{D}_i < \tilde{D}_j < y_j^T)$ and $\frac{\partial y_i^T}{\partial \tau} \leq 0$, with $\frac{\partial E\pi_i^T}{\partial \tau} |_{\tau=0} \geq 0$.

2. When $\tau = p$, function (5.1) and (5.2) are equivalent. Also $E\pi_i^T$ is decreasing at this point, so there exist a range of $\tau$ such that $E\pi_i^T > E\pi_i^C$.

3. When $\tau = 0$, at equilibrium

$$E\pi_i^T = -w y_i^T + p E \min\{\tilde{D}_i, y_i^T + (y_j^T - \tilde{D}_j)^+\} = -w y_i^T + p E[\tilde{D}_i] \Pr(\beta_1) + p E[y_i^T + (y_j^T - \tilde{D}_j)^+] \Pr(\beta_2)$$

(A5.1)

Where $\beta_1 = \{(\tilde{D}_i, \tilde{D}_j)|\tilde{D}_i < y_i^T + (y_j^T - \tilde{D}_j)^+\}$, $\beta_2 = \{(\tilde{D}_i, \tilde{D}_j)|\tilde{D}_i > y_i^T + (y_j^T - \tilde{D}_j)^+\}$
From function (A5.1), the equilibrium stocks should satisfy

\[-w + p \Pr(\tilde{D}_i > y_i^T + (y_j^T - \tilde{D}_j)^+) = 0\]  \hspace{1cm} (A5.2)

then we have \(\Pr(\beta_2) = w/p\). Substitute this into (A5.1) we have \(E\pi_i^T = (p-w)E[\tilde{D}_i|\beta_1] + wE[(y_j^T - \tilde{D}_j)^+|\beta_2] = w\{(u-1)E[\tilde{D}_i|\beta_1] + E[(y_j^T - \tilde{D}_j)^+|\beta_2]\} (where \(u = p/w\)).

Similarly we have \(E\pi_i^C = -wy_i^C + pE\min\{\tilde{D}_i + (\tilde{D}_j - y_j^C)^+, y_j^C\} = (p-w)E[\tilde{D}_i|\alpha_1] + (p-w)E[(\tilde{D}_j - y_j^C)^+|\alpha_1] = w\{(u-1)E[\tilde{D}_i|\alpha_1] + (u-1)E[(\tilde{D}_j - y_j^C)^+|\alpha_1]\}.

Where \(\alpha_1 = \{(\tilde{D}_i, \tilde{D}_j)|\tilde{D}_i + (\tilde{D}_j - y_j^C)^+ < y_j^C\}.

The comparison of \(E\pi_i^T\) and \(E\pi_i^C\) depends on the parameter \((u)\) and the distribution of \(D_i, D_j\).

4. If the two retailers can cooperate to set \(\tau\), they can always set \(\tau\) to induce the inventory levels to be the system optimal, and the total profit equal to the monopoly case. To see this, we have

\[E\pi_i^T + E\pi_j^T = -w(y_i + y_j) + p_i E\min\{\tilde{D}_i, y_i\} + p_j E\min\{\tilde{D}_j, y_j\} + p_i E\min\{(y_i - \tilde{D}_i)^+, (\tilde{D}_j - y_j)^+\} + p_i E\min\{(y_j - \tilde{D}_j)^+, (\tilde{D}_i - y_i)^+\}

The system optimal \(y_i\) and \(y_j\) are given by

\[\Pr(\tilde{D}_i > y_i) + \frac{p_i}{p_i} \Pr(y_i + y_j - \tilde{D}_i < \tilde{D}_i < y_i) - \Pr(y_i < \tilde{D}_i < y_i + y_j - \tilde{D}_j) = \frac{w}{p_i}\]

\[\Pr(\tilde{D}_j > y_j) + \frac{p_j}{p_j} \Pr(y_i + y_j - \tilde{D}_i < \tilde{D}_j < y_j) - \Pr(y_j < \tilde{D}_j < y_i + y_j - \tilde{D}_i) = \frac{w}{p_j}\]

Compare this with equation (5.3), if \(\tau_i = \tau_j = 0\), then \(y_i^M \geq y_i^T\); if \(\tau_i = p_j, \tau_j = p_i\), then \(y_i^M \leq y_i^T\). Also \(y_i^T\) is nondecreasing in \(\tau_i(\tau_j)\), so there exists a \(\tau_i(\tau_j)\) such that \(y_i^M = y_i^T\).

Design \(\tau_1 = \frac{p\tilde{D}_i - p\tilde{D}_j}{\tilde{D}_i - \tilde{D}_j}, \tau_2 = \frac{p\tilde{D}_i - p\tilde{D}_j}{\tilde{D}_i - \tilde{D}_j}, \) where \(q_1 = \Pr(y_1^M + y_2^M - \tilde{D}_2^0 < \tilde{D}_1^0 < y_1^M), q_2 = \Pr(y_1^M + y_2^M - \tilde{D}_2^0 < \tilde{D}_1^0 < y_1^M + y_2^M - \tilde{D}_2^0), q_3 = \Pr(y_1^M + y_2^M - \tilde{D}_2^0 < \tilde{D}_1^0 < y_2^M), q_4 = \Pr(y_2^M < \tilde{D}_2^0 < y_1^M + y_2^M - \tilde{D}_1^0),\) and \((y_1^M, y_2^M)\) is the system optimal decision. Then the first order conditions in (5.3) are equivalent to the monopoly system optimal solutions, and
the retailers obtain the same profits as a monopoly with transshipment. This pair of transshipment price is unique. In symmetric case, \( p = p_1 = p_2, q_1 = q_3 \) and \( q_2 = q_4 \), then \( \tau = \frac{p_1 - 1}{q_2 + q_1} \).

4. Proposition 5.6.

**Proof.** According to the Debreu theorem (Fudenberg and Tirole 1991), it is sufficient to show that function (5.5) is quasi-concave in \((p_i, z_i)\). Take the derivative of function (5.5), rearranging terms, we have,

\[
\frac{\partial E^T}{\partial z_i} = -w + (p_i - \tau_j) \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j) + \tau_i \Pr(z_i + z_j - \tilde{D}_j^0 < \tilde{D}_i^0 < z_i) + \tau_j \Pr(\tilde{D}_j^0 > z_i)
\]

\[
\frac{\partial^2 E^T}{\partial z_i^2} = (p_i - \tau_j) \frac{d \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j)}{dz_i} - \tau_j f_{\tilde{D}_i^0}(z_i) + \tau_i u_i^2 - \tau_i v_{ij}^1 < 0.
\]

because \( \frac{d \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j)}{dz_i} = -u_i^2 - v_{ij}^2 < 0 \) and \( -\tau_j f_{\tilde{D}_i^0}(z_i) + \tau_i u_i^2 - \tau_i v_{ij}^1 < 0 \)

since \( f_{\tilde{D}_i^0}(z_i) > u_i^2 \) and \( \tau_j = \tau_i \) (assumption 2). Note the formula for \( u_i^2, v_{ij}^1 \) and \( v_{ij}^2 \) are the same as those in proposition 5.1, but we use \( z_i \) and \( z_j \) instead of \( y_i \) and \( y_j \).

\[
\frac{\partial^2 E^T}{\partial p_i^2} = -2b < 0.
\]

\[
\frac{\partial^2 E^T}{\partial z_i \partial p_i} = \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j).
\]

We are left to show the Hessian Matrix is negative definite.

\[
\begin{vmatrix}
-2b & \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j) \\
\Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j) & \frac{(p_i - \tau_j) \frac{d \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j)}{dz_i}}{\partial z_i} - \tau_j f_{\tilde{D}_i^0}(z_i) + \tau_i u_i^2 - \tau_i v_{ij}^1
\end{vmatrix}
\]

The determinant is,

\[
U' = 2b(p_i - \tau_j)\frac{d \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j)}{dz_i} + 2b(\tau_j f_{\tilde{D}_i^0}(z_i) - \tau_i u_i^2 + \tau_i v_{ij}^1) - \Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j)
\]

\[
= 2b(\tau_j f_{\tilde{D}_i^0}(z_i) - \tau_i u_i^2 + \tau_i v_{ij}^1) + \Theta_{\tilde{D}_i^0 + \tilde{D}_j^0, \tilde{D}_i^0}(z_i + z_j, z_i)U.
\]

Where \( U = 2b(p_i - \tau_j) - \frac{\Pr(\tilde{D}_i^0 > z_i, \tilde{D}_i^0 + \tilde{D}_j^0 > z_i + z_j)}{\tilde{D}_i^0 + \tilde{D}_j^0, \tilde{D}_i^0}(z_i + z_j, z_i) \).

Observe that \( 2b(\tau_j f_{\tilde{D}_i^0}(z_i) - \tau_i u_i^2 + \tau_i v_{ij}^1) > 0 \), and \( \lim_{z_i \to +\infty} U = 2b(p_i - \tau_j) > 0 \).
Since $\frac{Pr(\tilde{D}_i^0 > z_i, \tilde{D}_j^0 + \tilde{D}_j^0 > z_i + z_j)}{\tilde{D}_i^0 + \tilde{D}_j^0}$ decreases with $z_i$ (assumption 1). So given any $p_i$, we can always find $z_i \in [z_i(p_i), +\infty)$ such that the determinant is positive. Where $z_i(p_i)$ is solved from $U = 0$.

Define set $Q = \{z_i, p_i : U \geq 0\}$. Note that if $\Xi'/\Xi^2$ decreasing, then $Q$ itself is convex.

Then if the strategy space is a compact and convex subset in $Q$, function (5.5) is continuous and strictly concave in $(p_i, z_i)$ and the pure strategy equilibrium exists. ■

5. Proposition 5.8.

Proof. Step 1. Function (5.8) is supermodular in $(z_i, p_i)$. By Lemma A1, we only need to check the cross differentiation.

$$\frac{\partial^2 E_{\pi j}^C}{\partial z_i \partial p_j} = Pr(\tilde{D}_i^0 > z_i) + Pr(z_i + z_j - \tilde{D}_j^0 < \tilde{D}_i^0 < z_i) \geq 0.$$  

Step 2. Function (5.8) has increasing differences in $(z_i, -p_j), (p_i, -p_j)$ and $(p_i, -z_j)$.

$$\frac{\partial^2 E_{\pi j}^{NT}}{\partial z_i \partial p_j} = 0, \quad \frac{\partial^2 E_{\pi j}^{NT}}{\partial p_i \partial p_j} = 0, \quad -\frac{\partial^2 E_{\pi j}^{C}}{\partial p_i \partial z_j} = Pr(z_j < \tilde{D}_j^0 < z_j + z_i + \tilde{D}_i^0) \geq 0.$$  

Function (5.8) is supermodular in $(z_i, -z_j)$ by a similar method to Lemma 1, and has increasing differences in $(z_i, -z_j).$ (Lemma A1)

Similarly function $E_{\pi j}$ is supermodular in $(-z_j, -p_j)$ and has increasing differences in $(z_i, -p_j), (p_i, -p_j)$ and $(z_i, -z_j)$.

Step 3. Continuity and compactness are also satisfied, so the game is supermodular and has a pure strategy equilibrium (Lemma A4). ■

6. Proposition 5.10.

Proof. 1. When $0 \leq \tau < p_i(p_j)$, the symmetric T game equilibrium can be found by solving (from (5.6) and (5.7)),

$$-w + p Pr(\tilde{D}_i^0 > z) - (p - \tau) Pr(z < \tilde{D}_i^0 < 2z - \tilde{D}_j^0) + \tau Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z) = 0$$  

$$\Rightarrow -w + p Pr(\tilde{D}_i^0 > z) + p Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z) = (p - \tau) Pr(z < \tilde{D}_i^0 < 2z - \tilde{D}_j^0) + (p - \tau) Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z)$$

and the optimal pricing is,
\[ p^T(z) = \frac{bw + E \min\{D_i^0, z\} + a + E \min\{(z - D_i^0)^+, (D_i^0 - z)^+\}}{2b} = \frac{bw + M + a + N}{2b}. \]

The C game equilibrium can be found by solving (from (5.9) and (5.10)),

\[-w + p \Pr(D_i^0 > z) + p \Pr(2z - D_j^0 < D_i^0 < z) = 0\]

\[ p^C(z) = \frac{bw + E \min\{D_i^0, z\} + a + E \min\{(z - D_i^0)^+, (D_i^0 - z)^+\}}{2b} = \frac{bw + M + a + N}{2b}. \]

Where \( M = E \min\{D_i^0, z\} = E \min\{D_j^0, z\} \),

\[ N = E \min\{(D_i^0 - z)^+, (z - D_i^0)^+\} = E \min\{(z - D_j^0)^+, (D_i^0 - z)^+\}. \]

Define the explicit function of (5.6), (5.7) under symmetry by \( z = h^T(p), p = g^T(z) \).

The explicit function of (5.9), (5.10) under symmetry by \( z = h^C(p), p = g^C(z) \).

We have

\[ \frac{dh^C(p)}{dp} \geq 0, \quad \frac{dh^T(p)}{dp} \geq 0, \]

\[ \frac{dg^C(z)}{dz} \geq 0, \quad \frac{dg^T(z)}{dz} \geq 0, \]

\[ \frac{d^2 g^C(z)}{dz^2} < 0, \quad \frac{d^2 g^T(z)}{dz^2} < 0, \]

In the \((p, z)\) space, both \( h^C(p) \) and \( h^T(p) \) start at \((w, 0)\). Also \( h^C(p) \geq h^T(p) \), \( g^C(z) = g^T(z) \) start at \[((a + bw)/(2b), 0)\). All of them are increasing.

Let \((z^C, p^C)\) be the unique intersection of \( z = h^C(p) \) and \( p = g^C(z) \); and \((z^T, p^T)\) be the intersection(s) of \( z = h^T(p) \) and \( p = g^T(z) \).

From Figure 5.1 and by the above analysis, we have the required result.

2. Similar to equilibrium property on t-price in last section and noting that \( dp^T_i / d\tau = (dp^T_i / dz)(dz^T_i / d\tau) \).

7. Proposition 5.11.

**Proof.** 1. When \( \tau_j = p_i \ (i = 1, 2) \) (I charge your retail price if I transship to you), the C game equilibrium is the same. For the T game, the problem is \( E\pi^T_i = -w(z_i + L_i) + p_i(L_i + E \min\{D_i^0, z_i\}) + p_jE \min\{(z_i - D_i^0)^+, (D_j^0 - y_j)^+\} \)

The first order conditions that determine safety stocks are,

\[-w + p_i \Pr(D_i^0 > z_i) + p_j \Pr(z_i + z_j - D_j^0 < D_i^0 < z_i) = 0\]

and the optimal price is \( p_T^T(z) = \frac{bw+a+E \min\{\bar{D}_i, z_i\}}{2b} \).

The symmetric equilibrium solves,
\[
-w + p \Pr(\bar{D}_i > z) + p \Pr(2z - \bar{D}_j < \bar{D}_i < z) = 0,
\]
\[
p^T(z) = \frac{bw+a+M}{2b}.
\]

In the \((p, z)\) space, \( h^C(p) = h^T(p) \) starts from \((w, 0)\). \( g^C(z) \geq g^T(z) \), both start at \(((a + bw)/(2b), 0)\). All are increasing, and hence the required result (Figure 5.2).

2. When \( \tau_i = p_i \) (I charge my retail price when transshipping to you), the T game becomes
\[
E\pi_i^T = -w(z_i + L_i) + p_i(L_i + E \min\{\bar{D}_i, z_i\}) + p_i E \min\{(z_i - \bar{D}_i)^+, (\bar{D}_j - y_j)^+\} + (p_i - p_j) E \min\{(\bar{D}_i^0 - z_i)^+, (y_j - \bar{D}_j)^+\}.
\]

The first order conditions for the safety stocks are,
\[
-w + p_i \Pr(\bar{D}_i^0 > z_i) + p_i \Pr(z_i + z_j - \bar{D}_j^0 < \bar{D}_i^0 < z_i) - (p_i - p_j) \Pr(z_i < \bar{D}_i^0 < z_i + z_j - \bar{D}_j^0) = 0
\]

The optimal price is \( p_T^T(z) = \frac{bw+a+E \min\{\bar{D}_i, z_i\} + E \min\{(z_i - \bar{D}_i^0)^+, (\bar{D}_j^0 - z_j)^+\} + E \min\{(\bar{D}_i^0 - z_i)^+, (z_j - \bar{D}_j^0)^+\}}{2b} \).

The C game equilibrium stays the same. The symmetric T game equilibrium solves,
\[
-w + p \Pr(\bar{D}_i^0 > z) + p \Pr(2z - \bar{D}_j^0 < \bar{D}_i^0 < z) = 0, \quad p^T(z) = \frac{bw+a+M+2N}{2b}
\]

In the \((p, z)\) space, \( h^C(p) = h^T(p) \) start from \((w, 0)\). \( g^C(z) \leq g^T(z) \), both start from \(((a + bw)/(2b), 0)\). All are increasing, and hence the required result (Figure 5.3).


**Proof.** 1. T game equilibrium can be find by solving (from (5.11) and (5.12)),
\[
-w + p \Pr(\bar{D}_i^0 > z) - (p - \tau) \Pr(z < \bar{D}_i^0 < 2z - \bar{D}_j^0) + \tau \Pr(2z - \bar{D}_j^0 < \bar{D}_i^0 < z) = 0
\]
\[
\Rightarrow -w + p \Pr(\bar{D}_i^0 > z) + p \Pr(2z - \bar{D}_j^0 < \bar{D}_i^0 < z) = (p - \tau) \Pr(z < \bar{D}_i^0 < 2z - \bar{D}_j^0) + (p - \tau) \Pr(2z - \bar{D}_j^0 < \bar{D}_i^0 < z)
\]
\[
p^T(z) = \frac{a+(b+\theta)w+M+2N}{2b} \frac{bw+a+E \min\{\bar{D}_i^0, z_i\} + E \min\{(z_i - \bar{D}_i^0)^+, (\bar{D}_j^0 - z_j)^+\}}{2b+\theta}
\]

In the C game, from (5.13)-(5.14) the symmetric equilibrium solves,
\[-w + p \Pr(\tilde{D}_i^0 > z) + p \Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z) = 0\]

\[p^C(z) = \frac{a + (b + \theta)w + E \min\{\tilde{D}_i^0, z\} + E \min\{(\tilde{D}_j^0 - z)^+, (z - \tilde{D}_i^0)^+\}}{2b + \theta} = \frac{a + (b + \theta)w + M + N}{2b + \theta}\]

Where \(M = E \min\{\tilde{D}_i^0, z\} = E \min\{\tilde{D}_j^0, z\}\).

\[N = E \min\{(\tilde{D}_j^0 - z)^+, (z - \tilde{D}_j^0)^+\} = E \min\{(z - \tilde{D}_i^0)^+, (\tilde{D}_i^0 - z)^+\}\]

Define the explicit function \(z = h^C(p), p = g^C(z)\) for the symmetric version of (5.11)-(5.12). \(z = h^T(p), p = g^T(z)\) for the symmetric version of (5.13)-(5.14).

We have

\[\frac{dh^C(p)}{dp} \geq 0, \quad \frac{dh^T(p)}{dp} \geq 0,\]
\[\frac{dg^C(z)}{dz} \geq 0, \quad \frac{dg^T(z)}{dz} \geq 0.\]

In the \((p, z)\) space, \(h^C(p)\) and \(h^T(p)\) starts at \((w, 0)\), and \(h^C(p) \geq h^T(p)\). We also have \(g^C(z) = g^T(z)\) and they start at \([a + (b + \theta)w]/(2b + \theta)\).

Let \((z^C, p^C)\) be the unique intersection of \(h^C(p)\) and \(g^C(z)\), and \((z^T, p^T)\) be the intersection(s) of \(h^T(p)\) and \(g^T(z)\).

Similar to Proposition 5.10-1, we have the required result.

2. When \(\tau_j = p_i\) \((i = 1, 2)\) (I charge your r-price when transship to you), the C game is the same. For the T game, the problem is \(E \pi^T_i = -w(z_i + L_i) + p_i(L_i + E \min\{\tilde{D}_i^0, z_i\}) + p_j E \min\{(z_i - \tilde{D}_i^0)^+, (\tilde{D}_j^0 - z_j)^+\}\)

The F.O.C.s that determine safety stocks and prices are,

\[-w + p_i \Pr(\tilde{D}_i^0 > z_i) + p_j \Pr(z_i + z_j - \tilde{D}_j^0 < \tilde{D}_i^0 < z_i) = 0\]
\[(b + \theta)w + a - 2(b + \theta)p_i + \theta p_j + E \min\{\tilde{D}_i^0, z_i\} = 0.\]

The symmetric T equilibrium solves,

\[-w + p \Pr(\tilde{D}_i^0 > z) + p \Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z) = 0,\]
\[p^T(z) = \frac{(b + \theta)w + a + M}{2b + \theta}.\]

In the \((p, z)\) space, \(h^C(p) = h^T(p), g^C(z) \geq g^T(z)\). And the required result (Figure 5.2).
3. When $T_i = p_i$ ($i = 1, 2$) (I charge my r-price when transship to you) the T game becomes

$$E_\pi^T = -w(z_i + L_i) + p_i(L_i + E \min\{\tilde{D}_i^0, z_i\}) + p_iE \min\{(z_i - \tilde{D}_i^0)^+, (\tilde{D}_j^0 - z_j)^+\} + (p_i - p_j)E \min\{(\tilde{D}_i^0 - z_i)^+, (z_j - \tilde{D}_j^0)^+\}.$$  

The F.O.C. for the safety stocks and prices are,

$$-w + p_i \Pr(\tilde{D}_i^0 > z_i) + p_j \Pr(z_i + z_j - \tilde{D}_j^0 < \tilde{D}_i^0 < z_i) - (p_i - p_j)\Pr(z_i < \tilde{D}_i^0 < z_i + z_j - \tilde{D}_j^0) = 0$$

$$(b + \theta)w + a - 2(b + \theta)p_i + \theta p_j + E \min\{\tilde{D}_i^0, z_i\} + E \min\{(z_i - \tilde{D}_i^0)^+, (\tilde{D}_j^0 - z_j)^+\} + E \min\{(\tilde{D}_i^0 - z_i)^+, (z_j - \tilde{D}_j^0)^+\} = 0$$

The symmetric equilibrium solves,

$$-w + p \Pr(\tilde{D}_i^0 > z) + p \Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z) = 0,$$

$$p_T(z) = \frac{a + (b + \theta)w + M + 2N}{2b + \theta}.$$  

The C game stays the same.

In the $(p, z)$ space, $h^C(p) = h^T(p)$, $g^C(z) \leq g^T(z)$. And the required result (Figure 5.3).

4. From 1, we have

$$-w + p \Pr(\tilde{D}_i^0 > z) - (p - \tau)\Pr(z < \tilde{D}_i^0 < 2z - \tilde{D}_j^0) + \tau \Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z) = 0$$

Taking derivative w.r.t. $\tau$, we have

$$\frac{d\bar{z}}{d\tau} = -\frac{\Pr(z < \tilde{D}_i^0 < 2z - \tilde{D}_j^0) + \Pr(2z - \tilde{D}_j^0 < \tilde{D}_i^0 < z)}{-pf\tilde{D}_i^0(z) + (p - \tau)u_{ij} + \tau u_{ij} - 2(p - \tau)v_{ij} - 2\pi u_{ij}} \geq 0.$$  

where

$$f_{\tilde{D}_i^0 \tilde{D}_i^0 \tilde{D}_i^0 < 2z}\Pr(\tilde{D}_i^0 \tilde{D}_i^0 \tilde{D}_i^0 < 2z) = u_{ij}^1,$$

$$f_{\tilde{D}_i^0 \tilde{D}_i^0 \tilde{D}_i^0 > 2z}\Pr(\tilde{D}_i^0 \tilde{D}_i^0 \tilde{D}_i^0 > 2z) = u_{ij}^2.$$  

So function $z = h(p)$ increases with $\tau$, and $p = g(z)$ does not change with $\tau$. So the intersection(s) (equilibrium point) increase with $\tau$ as well.

5. Check the response functions in 1-3, we have $\frac{dp}{d\theta} = \frac{dg(z)}{d\theta} \leq 0$. Function $p = g(z)$ decreases with $\theta$ and $z = h(p)$ does not change with $\theta$, so the intersection(s) (equilibrium
point) decrease with $\theta$ too. ■


**Proof.** We use an industry monopoly as a benchmark. He sets retail prices and safety stocks to maximize joint profit, he is indifferent about transship or not, i.e.,

$$E\pi^T = E\pi^C = -w(z_1 + L_1) + p_1(L_1 + E\min\{\tilde{D}_1^0, z_1\}) + p_1E\min\{(z_1 - \tilde{D}_1^0)^+, (\tilde{D}_2^0 - z_2)^+\} - w(z_2 + L_2) + p_2(L_2 + E\min\{\tilde{D}_2^0, z_2\}) + p_2E\min\{(z_2 - \tilde{D}_2^0)^+, (\tilde{D}_1^0 - z_1)^+\}$$

Where $L_i = a - (b + \theta)p_i + \theta p_j$ or $L_i = a - bp_i$

The monopoly optimal safety stocks and prices solve,

$$-w + p_1\Pr(\tilde{D}_1^0 > z_1) + p_1\Pr(z_1 + z_2 - \tilde{D}_2^0 < \tilde{D}_1^0 < z_1) - p_2\Pr(z_1 < \tilde{D}_1^0 < z_1 + z_2 - \tilde{D}_2^0) = 0$$

$$p^M_i(z) = \frac{a + (b + \theta)w + \theta p_2 + \theta(p_2 - w) + E\min\{\tilde{D}_1^0, z_1\} + E\min\{(z_1 - \tilde{D}_1^0)^+, (\tilde{D}_2^0 - z_2)^+\}}{2(b + \theta)}$$

In symmetric case, we have (no matter $L_i = a - (b + \theta)p_i + \theta p_j$ or $L_i = a - bp_i$),

$$-w + p\Pr(\tilde{D}_1^0 > z) + p\Pr(2z - \tilde{D}_2^0 < \tilde{D}_1^0 < z) - p\Pr(z < \tilde{D}_1^0 < 2z - \tilde{D}_2^0) = 0$$

$$p^M(z) = \frac{a + bw + E\min\{\tilde{D}_1^0, z\} + E\min\{(z - \tilde{D}_1^0)^+, (\tilde{D}_2^0 - z)^+\}}{2b}$$

Recall that in the T game,

$$-w + p\Pr(\tilde{D}_1^0 > z) + p\Pr(2z - \tilde{D}_2^0 < \tilde{D}_1^0 < z) - (p - \tau)\Pr(2z - \tilde{D}_2^0 < \tilde{D}_1^0 < z) + \Pr(z < \tilde{D}_1^0 < 2z - \tilde{D}_2^0) = 0$$

$$p^T(z) = \frac{a + (b + \theta)w + \tau + E\min\{\tilde{D}_1^0, z\} + E\min\{(z - \tilde{D}_1^0)^+, (\tilde{D}_2^0 - z)^+\}}{2b + \theta}$$

$$p^T(z) = \frac{a + bw + E\min\{\tilde{D}_1^0, z\} + E\min\{(z - \tilde{D}_1^0)^+, (\tilde{D}_2^0 - z)^+\}}{2b}$$

if no price competition.

In the C game,

$$-w + p\Pr(\tilde{D}_1^0 > z) + p\Pr(2z - \tilde{D}_2^0 < \tilde{D}_1^0 < z) = 0$$

$$p^C(z) = \frac{a + (b + \theta)w + E\min\{\tilde{D}_1^0, z\} + E\min\{(z - \tilde{D}_1^0)^+, (\tilde{D}_2^0 - z)^+\}}{2b + \theta}$$

$$p^C(z) = \frac{a + bw + E\min\{\tilde{D}_1^0, z\} + E\min\{(z - \tilde{D}_1^0)^+, (\tilde{D}_2^0 - z)^+\}}{2b}$$

if no price competition.

Let $p = g^i(z)$ and $z = h^i(p)$ be the explicit functions obtained by the above functions, where $i = M, T, C$. By some algebra, we have $dg^M(z)/dz > 0, dh^M(p)/dp > 0$; $dg^T(z)/dz > 0, dh^T(p)/dp > 0$; $dg^C(z)/dz > 0, dh^C(p)/dp > 0$. So all explicit functions
are upward-sloping. Let the intersection of $g^i(z)$ and $h^i(p)$ be $(p^i, z^i)$, $i = M, T, C$.

In the $(p, z)$ space, without price competition, we have

$$g^M(z) = g^C(z) = g^T(z); h^C(p) > h^T(p) \text{ and } h^C(p) > h^M(p);$$ although it is not straight forward to compare $h^T(p)$ and $h^M(p)$ under optimal t-price, we know $h^T(p)$ passes $(z^M, p^M)$ and $p^T = p^M$ and $z^T = z^M$. (Figure 5.4).

So the equilibrium point under C game is different from that of monopoly benchmark (which is the same as equilibrium point under T game when t-price is chosen optimally). Newsvendors are worse off in C game than in T game with optimal t-price.

With price competition, in the $(p, z)$ space, we have $g^M(z) > g^C(z) = g^T(z); h^C(p) > h^T(p) \text{ and } h^C(p) > h^M(p); h^T(p)$ under the no-price-competition optimal ('optimal' thereafter) t-price stays the same as in Figure 5.4. Clearly, $(p^T, z^T)$ moves away from $(p^M, z^M)$. (Figure 5.5).

So now the T game does not have the same equilibrium as the monopoly solution, and either the C game or the T game might more closely approximate the monopoly solution. So it is quite possible that newsvendors are better off in the C game. Because price competition drives the equilibrium points in the C game and the T game down-ward and left-ward, which makes the equilibrium point in the C game closer to the monopoly benchmark but the equilibrium point in the T game further away from the benchmark. Under certain circumstances, newsvendors are better off in the C game because their joint profit is higher and we assume symmetric newsvendors.

**Proof.** Also from Figure 5.5, there is no t-price that pivot the $h^T(p)$ can make the pricing and safety stock decisions in the T game the same as monopoly point.

We can also check that the in symmetric T game equilibrium, $dz/d\tau > 0$, i.e., the function $h^T(p)$ increases with $\tau$. So for any t-price smaller than the 'optimal', we have a $h^T(p)$ lower than the $h^T(p)$ in Figure 5.5; price competition drives the equilibrium point
in the T game even further away from the monopoly benchmark. So newsvendors are even better off in the C game compared to the T game but with a t-price lower than 'optimal' t-price. Any t-price greater than the 'optimal' can make the equilibrium point closer to the monopoly benchmark and can be helpful to retailers in the T game. ■
Cited Theories for Supermodularity

**Definition A1** (Topkis 1998) Let $X$ and $T$ be partially ordered sets and $f(x,t)$ a real-valued function on a subset $S$ of $X \times T$. If $f(x,t') - f(x,t'')$ is increasing in $x$ for all $t' > t''$, then $f(x,t)$ has increasing differences in $(x,t)$ on $S$. If $f(x)$ is twice differentiable on $R^n$, then $f(x)$ has increasing differences on $R^n$ if and only if $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$ for all $i,j$ and all $x$.

The real line $R^1$ with the usual ordering relation $\leq$ on the real numbers is a partially ordered set. The set $R^n = \{x = (x_1, \ldots, x_n) : x_i \in R^1 \text{ for } i = 1, \ldots, n\}$ with the ordering relation $\leq$ where $x' \leq x''$ in $R^n$ if $x_i' \leq x_i''$ in $R^1$ for $i = 1, \ldots, n$ is a partially ordered set.

**Definition A2** (Topkis 1998) Suppose that $f(x)$ is a real-valued function on a lattice $X$, if $f(x') + f(x'') \leq f(x' \lor x'') + f(x' \land x'')$ for all $x'$ and $x''$ in $X$, then $f(x)$ is supermodular on $X$.

The real line $R^1$ is a lattice with $x' \lor x'' = \max\{x', x''\}$, $x' \land x'' = \min\{x', x''\}$. For any positive integer $n$, $R^n$ is a lattice with $x' \lor x'' = (x_1' \lor x_1'', \ldots, x_n' \lor x_n'')$, $x' \land x'' = (x_1' \land x_1'', \ldots, x_n' \land x_n'')$ for $x'$ and $x''$ in $R^n$.

**Definition A3** (Topkis 1998) A noncooperative game $(N,S,\{f_i : i \in N\})$ is a supermodular game with $N$ players if the set $S$ of feasible joint strategies is a sublattice of $R^m$ (or of $\times_{i \in N} R^m$), the payoff function $f_i(y_i,x_{-i})$ is supermodular in $y_i$ on $S_i$ for each $x_{-i}$ in $S_{-i}$ for each player $i$, and $f_i(y_i,x_{-i})$ has increasing differences in $(y_i,x_{-i})$ on $S_i \times S_{-i}$ for each $i$.

(a subset $X$ of $R^n$ has the property that $x' = (x'_1, \ldots, x'_n)$ and $x'' = (x''_1, \ldots, x''_n)$ in $X$ imply that $(\max\{x'_1, x''_1\}, \ldots, \max\{x'_n, x''_n\})$ and $(\min\{x'_1, x''_1\}, \ldots, \min\{x'_n, x''_n\})$ are in $X$, then $X$ is a sublattice of $R^n$). The closed intervals in a lattice is a sublattice of $X$.

Specifically, (Milgrom and Roberts 1990 Theorem 4), suppose there are finitely many players, $\Gamma = (N,S_n,f_n,n \in N, \geq)$ is supermodular if: 1. $S_n$ is an interval in $R^{k_n}$. That
Lemma A1 1. (Topkis 1998 Theorem 2.6.1.) If \( X_\alpha \) is a lattice for each \( \alpha \) in a set \( A \), \( X \) is a sublattice of \( \times_{\alpha \in A} X_\alpha \), and \( f(x) \) is (strictly) supermodular on \( X \), then \( f(x) \) has (strictly) increasing differences on \( X \). 2. (Milgrom and Roberts 1990) Let \( I = [x, \bar{x}] \) be an interval in \( \mathbb{R}^n \). Suppose that \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is twice continuously differentiable on some open set containing \( I \). Then \( f \) is supermodular on \( I \) iff for all \( x \in I \) and all \( i \neq j \), \( \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0 \).

According to definition 1 and Lemma 1, to prove supermodularity of a twice continuously differentiable function on a closed interval is equivalent to proving increasing differences.

Lemma A2 (Topkis 1998 Example 2.6.2.) 1. Any real-valued function on a subset of \( \mathbb{R}^1 \) is both supermodular and submodular. 2. If \( f_i(z) \) is increasing (decreasing) on \( \mathbb{R}^1 \) for \( i = 1, \ldots, n \) then \( f(x) = \min\{f_i(x_i) : i = 1, \ldots, n\} \) is supermodular on \( \mathbb{R}^n \). 3. (Topkis 1998 Corollary 2.6.3.) If \( X \) is a lattice and \( f_i(x) \) is nonnegative, increasing (decreasing), and supermodular on \( X \) for \( i = 1, \ldots, k \), then \( f_1(x)f_2(x)\cdots f_k(x) \) is also nonnegative, increasing (decreasing), and supermodular on \( X \).

Lemma A3 (Topkis 1998 Corollary 2.7.1.) Suppose \( f(x) \) is supermodular on a nonempty lattice \( X \). If \( X \) is a compact sublattice of \( \mathbb{R}^n \) and \( f(x) \) is upper semicontinuous on \( X \), then \( \arg \max_{x \in X} f(x) \) is a nonempty compact and subcomplete sublattice of \( \mathbb{R}^n \).

Lemma A4 (Topkis 1998 Theorem 4.2.1.) If \( (N, S, \{f_i : i \in N\}) \) is a supermodular game, the set \( S \) of feasible joint strategies is nonempty and compact, and the payoff functions \( f_i(y_i, x_{-i}) \) are upper semicontinuous in \( y_i \) on \( S_i \) for each \( i \), then the set of equilibrium points is a nonempty complete lattice and a greatest and a least equilibrium.
point exist.

Specifically, (Milgrom and Roberts 1990 Theorem 5), according to Milgrom and Roberts version in definition 3, if the game \( \Gamma \) is supermodular, then there exists a pure Nash equilibrium. Moreover, there exists a largest and smallest pure Nash equilibria in the given order.

**Lemma A5** (Topkis 1998 Theorem 4.2.2.) Suppose that \( T \) is a partially ordered set and \( (N, S^t, \{ f^t_i : i \in N \}) \) is a collection of supermodular games parameterized by \( t \) in \( T \) where in game \( t \) the payoff function for each player \( i \) is \( f^t_i(x) \) and the set of feasible joint strategies is \( S^t \). The set \( S^t \) is nonempty and compact and increasing in \( t \) on \( T \). The payoff function \( f^t_i(y_i, x_{-i}) \) is upper semicontinuous in \( y_i \), and has increasing differences in \( (y_i, t) \). Then there exists a greatest equilibrium point and a least equilibrium point for each game \( t \), and the equilibrium point is increasing in \( t \).

Specifically, (Milgrom and Roberts 1990 Theorem 6), suppose that \( (N, S_n, f_n(x_n, x_{-n}, t), n \in N, \geq) \) is a family of supermodular games satisfying \( \frac{\partial^2 f_n}{\partial x_n \partial t} \geq 0 \) for all \( n \) and \( i \). Then the smallest and largest pure Nash equilibria are nondecreasing functions of \( t \).

**Lemma A6** (Friedman 1990 Theorem 3.4) Assumption 1. the strategy set is compact and convex for each player, 2. the payoff functions are continuous, and bounded, 3. the payoff functions are strictly quasiconcave w.r.t. their own strategies for all players. Let \( \Gamma = (N, S, f) \) (N players, strategy set \( S \) and payoff functions \( f \)) be a game of complete information that satisfies assumptions 1-3. If the best reply function is a contraction mapping, then \( \Gamma \) has exactly one equilibrium point.

The best-reply mapping for player \( i \) is a set-valued relationship associating each strategy combination \( s \in S \) with a subset of \( S_i \) according to the rule: \( r_i(s) = \{ s_i^* \in S | f_i(s_i^*, s_{-i}) = \max_{s_i \in S_i} f_i(s_i, s_{-i}) \} \).

When \( r_i(s) \) is a differentiable function, the contraction means \( \sum_{j=1}^{m} |\frac{\partial r_i(s)}{\partial s_j}| < 1 \).
References


Competitive Market. Faculty of Management, McGill University, Montreal.


