DYNAMIC INVESTMENT MODELS WITH DOWNSIDE RISK CONTROL

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Abstract

Mean-variance analysis has been broadly used in the theory and practice of portfolio management. However, the continuous analogy is not fully studied either academically or in practice. This thesis provides a similar efficient frontier to Markowitz (1952) and a general solution using martingale method employed in Cox and Huang (1989). Comparisons between the expected utility approach and the mean-variance analysis have been made.

Traditional utility maximization cannot be used for explicit risk control of downside losses. An adjusted investment objective function by the worst case outcome is incorporated in the investment model. The problem can be divided into two subproblems as in Cox and Huang (1989). Closed form solution is derived for geometric Brownian motion and HARA utility setting. An interesting result is that the investor's decision is governed by a single "security" – a call option on a dynamic mutual fund.

A similar strategy, *Risk Neutral Excess Return*(RNER), to *Portfolio Insurance* is discussed. With geometric Brownian motion, the RNER strategy has a payoff structure similar to a straddle option strategy. Compare to the strategic asset allocation methods, such as Buy and Hold, Fixed Mix, and Portfolio Insurance , the new approach appears to be superior under a popular risk measure, *Value at Risk*(VaR).

A new objective function is defined for applying stochastic programming to financial investment under uncertainty. Incomplete market conditions are considered in implementing this model. The risk neutral probability is fully studied using stochastic programming techniques.

Table of Contents

List of Tables

List of Figures

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Dedication

To Ping, Evelyn and Edward.

Introduction

Dynamic investment is a decision process for dividing the total investment fund among the major asset classes such as equities, bonds, cash, options, etc. Generally, these decisions can be classified in two paradigms, "static" and "dynamic", each with trade-off between risk and return. The mean-variance analysis, as developed notably by Markowitz (1952, 1959) and Tobin (1958, 1965) is a typical static model. With the securities modeled by the means, variances, and covariances of their rates of return, the rational investor focuses on the efficient frontier (that is the subset of portfolios that achieve a maximum mean for a given variance), and then he makes a final choice that depends on his preferences toward risk. This approach is widely utilized because it is straightforward to implement. Although this model leads to some important consequences such as the separation or mutual fund theorem and the capital asset pricing model (Sharpe 1964), the model's simplicity is its major shortcoming (such as, one number represents the risk, both upward and downward deviations are considered to be risk, etc.) The second main approach in the portfolio management literature is to model the securities as Markovian processes and then solve the problem of maximizing the expected utility of the outcomes. The technology employed is dynamic programming. Merton (1969, 1971) used diffusion process models. The stochastic control methodology often reduces the problem to an intractable partial differential equation of which explicit solutions are obtained only for the simplest cases. For example, Merton's (1971) simplest case involves a formidable equation that is one of the few examples in Fleming and Rishel (1975). As a result of the difficulty in solving the stochastic control problem, there has recently been considerable interest in the application of stochastic calculus to developing investment models, especially in the context of option pricing. This interest was generated by the argument in Harrison and Kreps (1979) that if the model eliminates arbitrage opportunities, then there exists a probability measure under which the discounted securities price process is a martingale. Not only is this new methodology useful for pricing contingent claims, but also it provides a new approach to solving the optimal portfolio problem. Karatzas et al (1986, 1987), Harrison and Pliska (1981), Pliska (1986), Karatzas (1989), Cox and Huang (1989), Basak (1995), and Grossman and Zhou (1996), are examples of using this tool in forming continuous time investment portfolios by separating the whole problem into two subproblems. First one identifies the subspace of the attainable wealth and chooses the one that is the best among those which satisfy a martingale constraint, and then one determines the trading strategies by replicating the optimal wealth. The first part is a variation problem which is equivalent to a huge nonlinear optimization problem in discrete models (wealth is discretized as a random variable). In the version of the continuous model, the replication model can be reduced to a parabolic differential equation with the assumption that wealth is a function of the state variables.

Stochastic programming can be efficiently used to deal with models of investment under uncertainty when there are constraints, scenarios and a general objective function. In developing investment model using stochastic calculus, if the first step of identifying the "best" portfolio seems to be not straightforward, then the replication of this optimal portfolio will be much more difficult. The partial differential equation approach relies on the assumption that securities and portfolio value follow Markov decision processes. The stochastic programming provides a useful tool for approximating the problem to a discrete case which can take many constraints in forming investment strategies. Multiperiod stochastic programming is a useful tool for implementing investment models and it has made major improvements to the practice of investment management. Bradley and Crane (1972) and Kusy and Ziemba (1986) describe stochastic linear programs for bank asset/liability management. Cariño et al. (1994), Cariño et al. (1998a) and Cariño and Ziemba (1998b) discuss the Russell-Yausuda Kasai asset/liability management model. Mulvey and Vladimirou (1991) discuss a multiperiod stochastic network model for asset allocation, and Zenios (1993) describes stochastic programming models for fixed-income asset/liability management. Edirisinghe, Naik and Uppal (1993) applied a stochastic programming model for option replication with transaction costs and trading constraints by minimizing the initial costs of an European call option. See also Ziemba and Mulvey (1998) for a survey of additional applications. A major advantage of stochastic programming is its versatility of implementing replication model, i.e., the derivation of the hedging portfolio of a contingent claim. Making use of martingale analysis, a utility maximization in discrete time can be solved through a solution of nonsmooth problem that identifies the optimal portfolio and a stochastic programming that yields the optimal portfolio policies.

This thesis will focus on (1) developing the theory and application of investment models with downside control in continuous time with the assumption of Markov diffusion processes for asset prices, (2) a heuristic method combining a nonlinear optimization and a stochastic linear programming for solving dynamic investment models with downside control.

Part I focuses on the incorporation of downside risk control in the investment model in the continuous time framework. With the assumptions that the market is complete and that the market asset prices are modeled by Markov diffusion processes, the martingale analysis and the partial differential equation are two major mathematical tool used for deriving the optimal strategies.

Chapter 2 extends Markowitz's static mean-variance analysis to its analog of the continuous time version. The mean-variance analysis as an important criterion has not been properly incorporated in the continuous time models. The lack of this analysis in continuous time finance has downgraded its practicality in developing dynamic investment models. Applying martingale analysis, the efficient frontier and the optimal portfolio are derived assuming the absence of arbitrage and the existence of a riskless asset. If the price processes jointly have a Markovian structure, the optimal policies are obtained by solving a partial differential equation with boundary conditions. Furthermore, a close form solution is derived if asset prices jointly follow a multidimensional geometric Brownian motion.

Chapter 3 defines a new risk measure such that a utility maximizer will trade off the overall expected utility with the worst case outcome. Although the standard theory has been much developed on the utility maximization, adding the downside risk control to the problem is a new idea which has just recently started; see Basak (1995), Grossman and Zhou (1996). In addition to the characterization of risk aversion via utilities, investors are also concerned about the actual potential dollar loss in their portfolio. For example, they may ask what is the probability that wealth will fall below some amount, and what is the worst case payoff of the portfolio. Adding these requirements will eventually change investor's optimal policies of a standard utility maximization. Portfolio Insurance, which was a quite popular investment strategy in 1980's, provided a tool for investors to actively manage their portfolio. In the literature, the related topics have been studied extensively. Black and Jones (1987) discuss a simple, flexible approach to portfolio insurance for pension plans. Browne (1997) discusses an investment model for survival and growth with a liability stream and relates the problem to portfolio insurance. Browne (1999) studies an investment problem that maximizes the probability of reaching a given wealth level by a finite horizon and relates the problem to pricing of a digital option. Basak (1995) and

Grossman and Zhou (1996) analyzed the equilibrium security prices with the presence of portfolio insurance. Considering the downside risk control which is captured by the additional amount of reward besides the utility received from the actual wealth realization, investors set the objective function to be concave increasing functions of two variables: the realized wealth and the worst case outcome wealth. The investor wants to improve on the levels of both variables, but these two variables are in conflict with each other. For a given economic situation, one can not improve the overall wealth without having to reduce the worst case outcome. This approach provides an explicit control of the downside losses. Investors will adjust the optimal wealth and the worst possible outcome wealth according to the markets' situation. Hence, the investor's optimal strategy related to these two variables is sensitive to market conditions. The optimal strategy is equivalent to an option strategy on a dynamic mutual fund. The economic interpretation is that the marginal utilities of the two variable trade off each other. The result shows the optimal investment strategy is the hedging portfolio of the option on the optimal portfolio without the downside risk control. Finally, a closed form solution is given when asset prices jointly follow a geometric Brownian motion and the utility function is HARA (Hyperbolic Absolutely Risk Aversion). The Black and Scholes formula is employed for the solution of this specific case. This research result shows that if there exists such an financial intermediary, an investor will only need to invest an option and the investment strategy can be fully characterized by the price of this option. Unlike the results in Merton's pioneering paper, portfolio weights are no longer proportional to the wealth even in the standard market setting.

Chapter 4 defines and solves an investment model that is similar to Constant Proportional Portfolio Insurance. The Risk Neutral Excess Return strategy is a portfolio policy such that the discounted portfolio weight to the initial wealth is proportional to the changing risk neutral excess rate in the same asset. As a comparison to the CPPI strategy, this strategy focuses on the ratio of the changing discounted dollar amount in the risky assets to the initial portfolio value and sets it equal to the multiple of the risk neutral excess return plus some constant level. As CPPI does, the RNER strategy yields a portfolio return that protects against downside losses. Furthermore, the return of this strategy is a convex function of the market return, which shows that RNER strategy performs very well when the market ends on both tails, compared to typical asset allocation strategies, Buy and Hold, Fixed Mix, and Portfolio Insurance.

In Part II, discrete time models with market trading friction are studied. The continuous time model was easy to deal with because of the assumed nice structure on the asset prices. The key role played by *risk neutral probabilities*[1] can be easily identified and then the static first step which identifies the optimal wealth can be easily implemented for an unconstrained market. However, market is constrained and market friction exists everywhere. Trading continuously and costlessly are not realistic assumptions. Perfect hedging is not possible. How do we solve a model subject to those constraints?

Chapter 5 discusses the optimal solution in a constrained market setting. Using martingale analysis, the optimal terminal wealth can be identified by solving a static nonlinear optimization. However, investors are really interested in how to find the optimal portfolio policy over time. A suggestion for the investor is to approximate an "dominant" portfolio which might not be replicable under the current market situation by minimizing the downside deviation. We know that, if there is no arbitrage opportunity, there always exists a risk neutral probability by which the "dominant" portfolio can be identified. With this assumption, we suggest that investors follow a two-step strategy. At first, one solves the static problem as if market were unconstrained (In this case the optimal solutions for martingale analysis and the stochastic control approach coincide to each other). Secondly, one replicates the identified portfolio with all types of constraints, such as trading constraints, liquidity constraints, shorting costs, and transaction costs. As a side result, this method will provide the exact optimal solution when market is unconstrained. One important input for the stochastic programming is a scenario set which usually describes how the asset returns evolve. Farka's Lemma in linear programming can easily check for arbitrage opportunities of the chosen scenario set, but generating an arbitrage free scenario set is not an easy task for large scale computation. In this model, we assume the market is arbitrage free to get the risk neutral probability by solving a specific stochastic linear programming model. This will be a further investigation as a practical application to the theoretic model.

Chapter 6 presents a multiperiod stochastic linear programming model for asset liability management. In this model, we consider an incomplete market where trading constraints exist, such as transaction cost, holding constraints, etc. The investor's risk aversion is reflected by balancing the objective weights between the expected growth and the worst case payoff, which introduces a new objective function for the stochastic programming models. The IBM OSL package is used for solving this model. The sensitivity of the optimal portfolio with respect to the risk aversion is carefully studied.

[1]A risk neutral probability is a probability measure under which security prices discounted at the riskless rate are martingales

Part I: Continuous Time Models

Chapter 1

Financial Market and Portfolio Dynamics

In this chapter, some concepts, definitions, and assumptions, are introduced which will be used throughout the discussion. The basic framework of models will be established in this chapter unless specified elsewhere.

1.1 Financial Market and Asset Price Model

A *financial market* is defined as $n+1$ traded assets denoted by their price processes. Among these assets, one is a (locally) riskless asset earning prevailing interest rate $r(t)$ and the others follow a multidimensional diffusion process, i.e.,

$$\begin{aligned} dB(t) &= r(t)B(t)dt, \qquad B(0) = 1, \\ dS_i(t) &= b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dz_j(t), \quad \forall i = 1, 2, \cdots, n. \end{aligned} \tag{1.1.1}$$

where $z(t) = (z_1(t), \cdots, z_n(t))^\top$ is an n-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. The prices of these assets are positive and real-valued, $S(t) = (S_1(t), \cdots, S_n(t))^\top \in \Re^{+n}$ and there are no dividends paid for all assets. $b(t) = (b_1(t), \cdots, b_n(t))^\top$ is called the instantaneous mean vector and $\sigma(t) = (\sigma_{ij})_{n\times n}$, the volatility matrix. $\mathbf{F} = \{\mathcal{F}_t \subseteq \mathcal{F}; t \in [0,T]\}$ is a filtration generated by $(B(t), S(t))$. The requirement that the coefficients $r(t), b(t), \sigma(t)$ be adapted to $\mathbf{F}$, essentially makes them functions of the asset prices path$\{B(u), S(u), 0 \le u \le t\}$ up to time t to preclude anticipations of the future and allow for dependence on the past of the driving asset prices.

To apply the stochastic dynamic programming technique in a continuous-time model, the process of state variables must be chosen to be Markovian which is defined as

Definition 1.1.1. An $\mathcal{F}_t$ adapted stochastic process M_t is called a Markov process if for $\forall t, s \geq 0$ and a Borel measurable set Γ

$$\Pr[M_{t+s} \in \Gamma | \mathcal{F}_t] = \Pr[M_{t+s} \in \Gamma | M_t].$$

For example, an n- dimensional Brownian motion is Markovian. For Markov process M_t, the following lemma (see Karatzas and Shreve (1990)) is needed

Lemma 1.1.1. *For Markov process M_t, if the Markov property*

$$\Pr[M_{t+s} \in \Gamma | \mathcal{F}_t] = \Pr[M_s \in \Gamma]$$

is satisfied, then

$$E[f(M_{t+s})|\mathcal{F}_t] = g(M_t) \tag{1.1.2}$$

where $f(\cdot), g(\cdot)$ are Borel measurable functions.

A particular class of continuous-time Markovian processes called the Itô processes are defined as the solution to the stochastic differential equations

$$dM = b(t, M)dt + \sigma(t, M)dz.$$

To make Model (1.1.1) an Itô type, $(r(t), b(t), \sigma(t))$ are allowed to be functions (non-random) of $B(t)$ and $S(t)$. Throughout the discussion, assume that $b(t)$ and $\sigma(t)$ are Borel measurable and satisfy the Itô conditions and $\sigma(t)$ is a non-singular matrix.

A *portfolio* is a combination of assets, denoted by $\alpha(t)$ and $\theta(t) = (\theta_1(t), \cdots, \theta_n(t))^\top$. $(\alpha(t), \theta(t))$ is the vector of total number of shares held of each asset that forms the portfolio which satisfies the condition

$$\int_0^T ||\theta(t)\sigma(t)||^2 dt + \int_0^T |\theta(t)^\top (b(t) - r(t)S(t))| dt < \infty, \tag{1.1.3}$$

where $||\cdot||$ is the Euclidean $\Re^n$ norm. Assume $\alpha(t)$ and $\theta(t)$ are $\mathcal{F}_t$ measurable. If $(\alpha(t), \theta(t))$ satisfies the above conditions, then it is called an *admissible strategy.* The portfolio value generated by the strategy $(\alpha(t), \theta(t))$ is

$$W(t) = \alpha(t)B(t) + \theta(t)^\top S(t), \tag{1.1.4}$$

which is called a portfolio process. A *trading strategy* is a dynamic portfolio of assets to be held until "expiration date". A trading strategy is called "self-financed" if

$$W(t_2) = W(t_1) + \int_{t_1}^{t_2} \alpha(u)dB(u) + \theta(u)^\top dS(u), a.s. \forall t_1 \leq t_2 \leq T.$$

Assume all trading strategies are self-financed.[1]

A market is called *arbitrage free* if there does not exist a trading strategy such that, for a non-positive initial wealth, a non-negative final cash flow is positive with positive probability, i.e.,

$$W(T) \geq 0 \quad \text{and} \quad \Pr[W(T) > 0] > 0 \Rightarrow W(0) > 0.$$

The no arbitrage principle is assumed throughout the investment horizon.

1.2 Martingale Measure and Portfolio Dynamics

Martingale is a useful mathematical concept that is suitable for describing security prices and portfolio value process. The risk neutral probability measure derived upon this concept has been used in pricing derivative securities. The form mathematical definition is

Definition 1.2.1. An n- dimensional stochastic process is called a martingale with respect to a filtration $\mathcal{F}_t$ if

i) M_t is $\mathcal{F}_t$ measurable for all t,

ii) $E[|M_t|] < \infty$ for all t, and

iii) $E[M_{t+s}|\mathcal{F}_t] = M_t$ for all $s \geq 0$.

For example, a Brownian motion is a martingale.

It follows that the dynamics of the value $W(t)$ of the given portfolio (α, θ) at time t follows the stochastic differential equation

$$dW(t) = \alpha(t)dB(t) + \theta(t)^\top dS(t).$$

Harrison and Kreps (1979) have shown that the existence of an arbitrage-free market is equivalent to the existence of a probability measure under which all prices of securities discounted at the riskless rate are martingales. This probability measure is called the martingale measure (or risk neutral probability measure). A characterization of the risk neutral probability is as follows:

[1]For the present time we have assumed that the buying and selling prices are the same, but in practice there is a bid-ask spread. We will model this as transaction costs in Part II.

Definition 1.2.2. An *equivalent martingale measure* Q is a probability measure on $(\Omega, \mathcal{F})$ equivalent to P such that all prices in units of the riskless asset are martingales with respect to the σ - fields $\mathcal{F}_t$, *i.e.*

$$E^Q\left[\frac{S_i(s)}{B(s)} \middle| \mathcal{F}_t\right] = \frac{S_i(t)}{B(t)}, \quad \forall i = 1, 2, \cdots, s > t, n, \tag{1.2.1}$$

where $E^Q[\cdot]$ is the expectation operator taken under the probability measure Q.

For the setting of the market as in (1.1.1), arbitrage free is equivalent to the solvability of the equation

$$\sigma(t)\kappa(t) = b(t) - r(t)S(t).$$

and the condition

$$\int_0^T ||\kappa(t)||^2 dt < +\infty.$$

With a complete market setting–no transaction cost, no liquidity constraints, buying and selling assets at the same prices, and unlimited borrowing, we can derive

Theorem 1.2.1. *Assume κ satisfies the Novikov condition, $E\left[\exp\left\{\frac{1}{2}\int_0^T ||\kappa(s)||^2 ds\right\}\right])$ $< \infty$. For a complete market, an adapted stochastic process $X(t)$ is a portfolio process if and only if it is a discounted Q-martingale.*

Proof. With the setting of Model (1.1.1), there exists a unique equivalent martingale measure which has the explicit form (see Karatzas and Shreve 1990)

$$\frac{dQ}{dP} = \exp\left\{-\int_0^T \kappa(s)^\top dz(s) - \frac{1}{2}\int_0^T \| \kappa(s) \|^2 ds\right\},$$

where $\kappa(t) \equiv \sigma(t)^{-1}(b(t) - r(t)S(t))$, called the market price of risk. By the Novikov condition, $E^Q[\mathbf{1}_\Omega] = 1$, which implies that there is no arbitrage and Q is a probability measure. The exponential P - martingale generated by $\kappa(t)$ is

$$\eta(t) \equiv \exp\left\{-\int_0^t \kappa(s)^\top dz(s) - \frac{1}{2}\int_0^t \| \kappa(s) \|^2 ds\right\}.$$

Utilizing Levy's characterization for Brownian motion yields that

$$z^Q(t) \equiv z(t) + \int_0^t \kappa(s) ds$$

is a standard Brownian motion under the probability measure Q. This result is usually referred to as Girsanov's theorem.

Under the Brownian motion $z^Q(t)$, the asset prices and the portfolio value can be rewritten as

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma(t)dz^Q(t) \\ dW(t) &= r(t)W(t)dt + \theta^\top \sigma(t)dz^Q(t) \end{aligned} \tag{1.2.2}$$

which implies that all self-financed portfolio processes (including the primary assets) have the same drift rate $r(t)$ under the Q - Brownian motion, therefore the discounted portfolio process is a martingale.

Conversely, by the martingale representation theorem, $\exists\gamma(t)$, $\mathcal{F}_t$ measurable, and

$$de^{-\int_0^t r(s)ds}W(t) = \gamma(t)^\top dz^Q(t).$$

Hence,

$$dW(t) = r(t)W(t)dt + e^{\int_0^t r(s)ds}\gamma(t)^\top dz^Q(t).$$

and

$$\begin{cases} \theta(t) &= e^{\int_0^t r(s)ds}(\sigma^\top)^{-1}\gamma(t), \\ \alpha(t) &= W(t) - \theta(t)^\top S(t), \end{cases} \tag{1.2.3}$$

is a trading strategy that replicates the portfolio $W(t)$. □

1.3 Itô's formula

The fundamental tool for formal manipulation and solution of stochastic processes of the Itô type is Itô's formula:

Lemma 1.3.1. *Let $F(t, S_1, \cdots, S_n)$ be a C^2 function defined on $[0,T] \times \Re^n$ and take the stochastic integral*

$$S(t) = S(0) + \int_0^t b(u, S(u))du + \int_0^t \sigma(u, S(u))dz(u), \qquad \forall i = 1, \cdots, n;$$

then the time-independent random variable $Y(t) = F(t, S_1, \cdots, S_n)$ is a stochastic integral and its stochastic differential equation is

$$dY(t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}tr(\frac{\partial^2 F}{\partial S^2}dS). \tag{1.3.1}$$

Armed with Itô's formula, we are now able to formally differentiate most smooth functions of Brownian motions, and hence, integrate stochastic differential equations of Itô type. For example, if $b(t) = b \cdot S(t)$ and $\sigma(t) = \sigma \cdot S(t)$, applying (1.3.1) obtains

$$S(t) = S(0)e^{\sigma z(t) + (b - \frac{1}{2}\sigma^2)t}.$$

1.4 Partial Differential Equation

Although utility maximization can be dealt with using stochastic control methodologies as in Merton (1969, 1971), an elegant way of solving the utility maximization over time is the martingale analysis. For complete market, the investor's wealth process is a discounted Q- martingale. Since a martingale is uniquely determined by its terminal term, the problem can be solved by decomposing the problem into two different problems: the first problem identifies the optimal terminal wealth and the second problem determines the optimal policy that perfectly replicates the identified optimal wealth. So, the problem reduces to how to find the terminal wealth and how to find the optimal strategy over time. A static variational problem with the budget constraint can be set up for obtaining the optimal wealth, while the optimal strategy over time can be determined by solving a parabolic partial differential equation.

Let $W(t)$ be the investor's wealth process. An important step for setting up this model is how to choose a set of state variables which jointly form a vector Markovian process such that the wealth process is a deterministic function of these state variable at any moment. For example, for the case of geometric Brownian motion model, the state variable can be chosen to be the time and another mutual fund (fictitious, depending on the investor's utility function, see Merton 1990). Let $W(t) = F(t, X(t))$ where $X(t)$ is a vector of state variables and it is a Markovian process of Itô diffusion type

$$dX(t) = \phi(t, X(t))dt + \psi(t, X(t))dZ^Q(t).$$

By Itô's formula,

$$\begin{aligned} dW(t) =& \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{1}{2}\mathrm{tr}(\frac{\partial^2 F}{\partial X^2}dX) \\ =& (\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}\phi + \frac{1}{2}\mathrm{tr}(\frac{\partial^2 F}{\partial X^2}\psi\psi^\top))dt + \frac{\partial F}{\partial X}\psi dz^Q(t). \end{aligned} \tag{1.4.1}$$

Comparing (1.4.1) with (1.1.4), we obtain

Proposition 1.4.1. *The wealth process is determined by the stochastic differential equation*

$$-rF + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}\phi + \frac{1}{2}tr(\frac{\partial^2 F}{\partial X^2}\psi\psi^\top) = 0,$$

and the optimal strategy is

$$\begin{cases} \theta(t) = (\sigma^\top)^{-1} \cdot \psi^\top \cdot \frac{\partial F}{\partial X} \\ \alpha(t) = W(t) - \theta(t)^\top S(t) \end{cases} \tag{1.4.2}$$

1.4.1 The Black-Scholes Formula for European Call Options

In chapter 3, the B-S formula for European options is applied when solving for optimal investment strategy. As a review, this formula is presented as follows. The partial differential equation with associated boundary (Black-Scholes partial differential equation) that is satisfied by the price of the option $F(t, s)$ with strike price K and expiry date T is

$$\begin{cases} -rF + F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_s s = 0 \\ F(T, s) = [s - K]^+, \quad \text{and} \quad F(t, s) \sim s, \quad \text{as } s \to \infty. \end{cases} \tag{1.4.3}$$

The price of the option is

$$F(t, s) = sN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where $N(x)$ is the cumulative function of the standard normal random variable

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{1}{2}y^2} dy,$$

and

$$\begin{cases} d_1 = \frac{\ln(s/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t}. \end{cases}$$

If $t = T$, then

$$d_1 = \begin{cases} +\infty & \text{if } s \geq K \\ -\infty & \text{if } s \leq K. \end{cases}$$

1.5 The Investor's Objective Function

Traditionally, the investor's objective is maximizing the expected utility where the utility function is a non-decreasing concave function of wealth and/or consumption. Since the investor faces uncertainty while having to make his decision now, an appropriate downside risk control incorporated in his investment objective might be more convincing if he is a type who worries about the downside losses.

1.5.1 The Definition

While maximizing the expected utility, the investor may consider the worst possible outcome. An investment objective can be chosen to be

$$\max\ E[U(W,\ \inf_{\omega\in\Omega} W(\omega))],$$

where $U(\cdot,\cdot)$ is jointly concave and increasing in both variables. The investor has to balance the two competing variables: the overall wealth and the worst possible outcome. For example, with the log utility function, an investor with initial wealth \$100 is to decide whether to play a gamble that pays \$1 with probability 0.8 and −\$2 with probability 0.2. Without the downside control variable, the investor will simply accept this gamble. But if he were to consider the downside losses and set the objective function to be the convex weight of the two "utilities":

$$U(x,y) = \rho \ln x + (1-\rho)\ln y,$$

the situation will change. Obviously, $\rho = 1$ reduces to the traditional utility maximization problem and $\rho = 0$ represents the problem where the investor will not make any investment under uncertainty (put the money in the bank). As to the gamble above, the investor will take the gamble only if $\rho > 0.8375$. So, the choice of ρ represents the investor's control intensity of downside losses.

1.5.2 Risk Adjusted Perception

Usually, investors are maximizing the expected utility with the same probability as the one for characterizing the economic uncertainty. Considering the downside losses, an investor might adjust the probability measure to put more probability mass on the downside outcomes. Let $V(w)$ be the utility of wealth and let the objective function be

$$U(x,y) = \rho V(x) + (1-\rho)V(y).$$

The investor will reduce the probabilities of all possible outcomes by a ratio of ρ and add the total reduced probability mass to the worst possible outcome. Figure 1.1 depicts the relation between the original and adjusted probability measure used for the utility maximization problem.

In Figure 1.1, the total reduction of the probability mass from A to B are added to the worst point A to accommodate the control of the downside losses for making investment decisions.

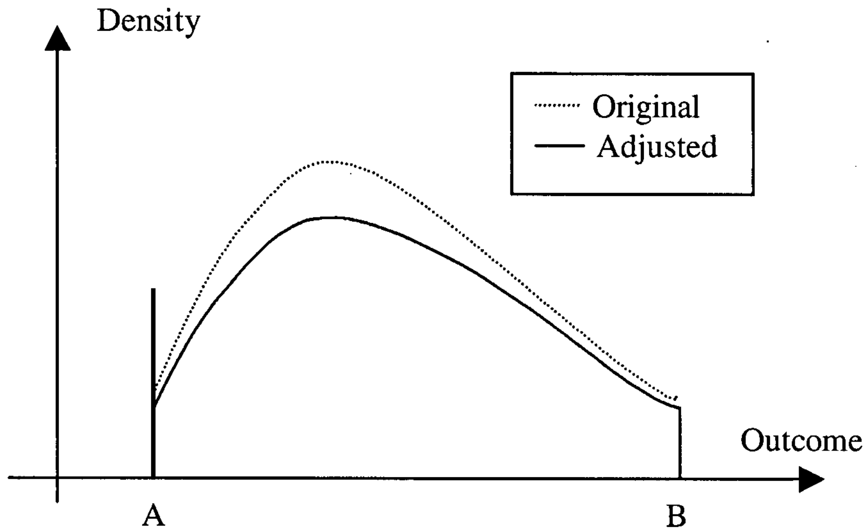

Figure 1.1: Shifting of Probability Mass

1.5.3 The Geometry of Economic Interpretation

In general, a downside-control investor will not maximize the expected utility with the objective probability measure but with an adjusted one. To achieve this purpose, one needs only to shift a portion of the probability mass at higher outcomes to that at lower outcomes. Consider a simple situation where there are only two possible outcomes of wealth, "good" and "bad", denoted by W_g and W_b. Let $\mathcal{W}$ be the set of possible outcomes which is assumed to be convex . The occurrence of outcome one is at probability p. The problem for an ordinary investor with utility function of wealth $U(W)$ is

$$\max_{(W_g,W_b)\in\mathcal{W}} \quad pU(W_g) + (1-p)U(W_b).$$

But a downside risk control type investor will maximize the expected utility with a shifted probability measure Q. If the investor weighs the "good" state with probability $q < p$, his investment problem is

$$\max_{(W_g,W_b)\in\mathcal{W}} \quad qU(W_g) + (1-q)U(W_b).$$

Since the marginal rate of substitution between this two approaches are changed, the optimal strategy is changed correspondingly. Figure 1.2 depicts the change of the optimal portfolio policy from Point A to Point B if downside risk control is considered.

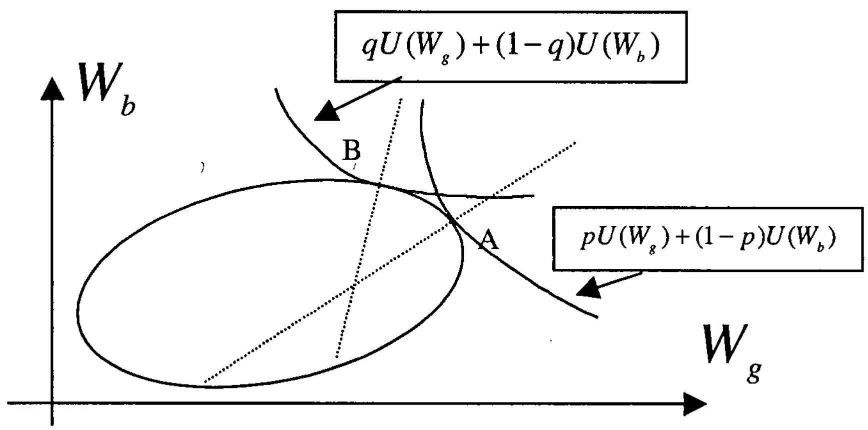

Figure 1.2: The Change of the Optimal Policy with Risk Control

Chapter 2

Continuous Time Mean-Variance Analysis

Markowitz (1952) mean-variance analysis blends elegance and simplicity. Compared to the expected utility models, it offers an intuitive explanation for diversification and a relatively simple computational procedure. However, most discussions of mean-variance analysis are restricted to static models. Hence, investors can only make decisions at the beginning and must wait for the results at the end of the horizon without adjusting the portfolio weights. This is awkward for mean-variance analysis compared to versatile dynamic (multiperiod or continuous time) models that maximize expected utility. Tobin (1958) showed that the mean-variance model is consistent with the von Neumann-Morgenstern postulates of rational behavior if the utility of wealth is quadratic. Since the quadratic utility function is increasing only up to some upper bound, it introduces a stochastic control problem with a constraint on the wealth level which is difficult to solve using standard stochastic control methodologies as used in Merton (1969, 1971). This chapter investigates and develops a dynamic version of mean-variance efficient frontier using martingale analysis and derives the optimal portfolio policies by solving the associated partial differential equations.

Mean-variance analysis, despite its importance as a practical investment criterion, see e.g. Grinold and Kahn (1995) and Ziemba and Mulvey (1998), has not been properly incorporated into continuous time models. The unavailability of this analysis in continuous time or multiperiod models has downgraded its practicality in developing dynamic investment models. This chapter considers continuous time models with a mean-variance criterion. Applying the martingale analysis as developed by Cox and Huang (1989), the efficient frontier and the optimal portfolio policies are derived assuming the absence of arbitrage and the existence of a riskless asset. If the price processes jointly have a Markovian structure, the optimal policies are obtained by solving a partial differential equation with associated boundary conditions. Furthermore, a closed form solution is derived if the asset prices jointly follow a multidimensional geometric Brownian motion.

One can ask how good is mean-variance analysis compared to the expected utility approach. The comparison provided in this chapter shows that the latter achieves a better performance if the outcome of the market state price is near its mean value. The expected utility approach has superior performance when the outcomes are in the tails of the state price which accommodates the investor's risk aversion. Hakansson (1971), Grauer (1981), and Kroll, Levy and Markowitz (1984) discussed and compared the optimal strategies obtained by the two criteria. Grauer and Hakasson (1993) compared the mean-variance and the quadratic utility approximation schemes for calculating optimal portfolios in the discrete time dynamic investment model. We investigate from another angle the possible advantages of mean-variance analysis over the expected utility approach. We calculate the probability that a mean-variance model outperforms an expected utility model in terms of the market state price. The investor determines the optimal target mean level for applying mean-variance analysis to maximize this probability. We provide a general method for calculating this probability and a closed form solution in the case of lognormal prices and logarithmic utility. A numerical example compares the two approaches.

2.1 The Efficient Frontier

2.1.1 The Mean-Variance Model and Its Variant Versions

The static mean-variance model may be formulated in two ways: minimizing variance subject to an expected wealth target or maximizing the expected wealth subject to a given level of risk, the variance. Both ways can trace out the efficient frontier by varying the corresponding parameters. Let $E[\cdot]$, $V[\cdot]$, and $V^2[\cdot]$ stand for the operators of expected value, standard deviation, and the variance of a random variable in question. The dynamic mean-variance model is

$$\begin{aligned} \min_{\theta(\cdot),W(\cdot)} \quad & V^2[W(T)] \\ s.t. \quad & E[W(T)] \geq \mu, \\ & dW(t) = r(t)W(t)dt + \theta(t)^\top \sigma(t)dz^Q(t), \quad 0 \leq t \leq T, \end{aligned} \tag{2.1.1}$$

where μ is the expected wealth target. The stochastic control problem above with an irregular objective function and a constraint on the target level of expected wealth present a significant obstacle for obtaining solutions through stochastic control methodology. However, the martingale approach can greatly reduce the complexity of the problem. The problem can be divided to two submodels: a static model for identifying the optimal attainable wealth and a replication model for obtaining the optimal portfolio policies. Consider the static problem

$$\begin{aligned} \min_{W} \ & V^2[W] \\ s.t. \ & E[W] \geq \mu \\ & E^Q[B^{-1}(T)W] = W_0, \end{aligned} \tag{2.1.2}$$

where W_0 is the initial wealth and $E^Q[\cdot]$ is the expectation operator under the risk neutral probability. The optimal portfolio values for models (2.1.1) and (2.1.2) coincide to each other using a replication argument.

Since the operator $V^2[\cdot]$ is in the objective function of Model (2.1.2), a Kuhn-Tucker solution procedure does not directly apply. However, a simpler version of Model (2.1.2) is given in Corollary 2.1.1.

Corollary 2.1.1. *Assuming the existence of a riskless asset, Model (2.1.2) is equivalent to the model*

$$\begin{aligned} \min_{W} \ & E[W^2] - \mu^2 \\ s.t. \ & E[W] = \mu \\ & E^Q[B^{-1}(T)W] = W_0. \end{aligned} \tag{2.1.3}$$

Proof. One needs only to show that, if W^* is optimal for (2.1.2), then $E[W^*] = \mu$. In fact, if $E[W^*] > \mu$, let

$$W^{**} = \frac{\mu}{E[W^*]}W^* + W_0(1 - \frac{\mu}{E[W^*]})B(T).$$

W^{**} is feasible to Model (2.1.2) since

$$\begin{aligned} E^Q[B(T)^{-1}W^{**}] &= \frac{\mu}{E[W^*]}E^Q[B(T)^{-1}W^*] + W_0(1 - \frac{\mu}{E[W^*]}) \\ &= \frac{\mu}{E[W^*]}W_0 + W_0(1 - \frac{\mu}{E[W^*]}) \\ &= W_0 \end{aligned}$$

and

$$\begin{aligned} E[W^{**}] &= \frac{\mu}{E[W^*]}E[W^*] + W_0(1 - \frac{\mu}{E[W^*]})E[B(T)] \\ &= \mu + W_0(1 - \frac{\mu}{E[W^*]})e^{\int_0^T r(t)dt} \\ &\geq \mu. \end{aligned}$$

However, W^{**} has a smaller variance than W^*, which contradicts the optimality of W^*. □

2.1.2 The Optimal Terminal Wealth

Denote $\xi = B(T)^{-1}\eta(T)$ as the contingent state price. The Lagrangian of Model (2.1.3) is

$$\mathcal{L}(W, \lambda, \rho) = E[W^2] - \mu^2 - \lambda(E[W] - \mu) - \rho(E[\xi W] - W_0),$$

where λ and ρ are the multipliers on the target expected wealth constraint and the martingale constraint, respectively. Here a normal representation of Lagarange form is used, but for the proof of a general case see Zhao, Haussmann and Ziemba (2000). The extended Kuhn-Tucker criteria are necessary and sufficient conditions due to the convexity of the objective function and the convexity of the feasible region. Theorem 2.1.2 characterizes the optimal value.

Theorem 2.1.2. *The optimal portfolio value is a linear function of the state price ξ with a negative slope if the target return is greater than the riskless rate. Furthermore, the optimal value is*

$$W = \frac{1}{2}\lambda + \frac{1}{2}\rho\xi \tag{2.1.4}$$

where the multipliers are given by

$$\begin{aligned} \lambda &= \frac{2\mu E[\xi^2] - 2W_0 E[\xi]}{V^2[\xi]} \\ \rho &= \frac{2W_0 - 2\mu E[\xi]}{V^2[\xi]}. \end{aligned} \tag{2.1.5}$$

Proof. Applying the Kuhn-Tucker criteria yields

$$\begin{aligned} 2W - \lambda - \rho\xi &= 0 \\ E[W] - \mu &= 0 \\ E[\xi W] - W_0 &= 0 \end{aligned}$$

which implies (2.1.4). Applying the operators $E[\cdot]$ and $E^Q[\cdot]$ to both sides of (2.1.4) yields

$$\begin{aligned} \lambda + \rho E[\xi] &= 2\mu \\ \lambda E[\xi] + \rho E[\xi^2] &= 2W_0. \end{aligned} \tag{2.1.6}$$

Solving (2.1.6) yields (2.1.5), which completes the proof of Theorem 2.1.2. □

The optimal portfolio is a linear function of the contingent state price, ξ, with non-positive slop. Hence, one can identify the optimal portfolio by obtaining the probability distribution of the contingent state price.

2.1.3 The Efficient Frontier

Let Ψ be the standard deviation of the optimal portfolio of Model (2.1.3). Substituting the optimal multipliers λ and ρ in the optimal wealth expression in (2.1.4) proves Theorem 2.1.3 as shown below.

Theorem 2.1.3. *The standard deviation of the optimal portfolio value is*

$$\Psi = \begin{cases} \frac{E[\xi]}{V[\xi]}(\mu - W_0(E[\xi])^{-1}) & \text{if } \mu \geq W_0(E[\xi])^{-1} \\ -\frac{E[\xi]}{V[\xi]}(\mu - W_0(E[\xi]))^{-1}) & \text{if } \mu < W_0(E[\xi])^{-1}. \end{cases} \tag{2.1.7}$$

Proof. By Theorem 2.1.2 and the two constraints in Model (2.1.3),

$$\begin{aligned} E[W^2] &= \frac{1}{2}\lambda E[W] + \frac{1}{2}\mu E[\xi W] \\ &= \frac{1}{2}\lambda\mu + \frac{1}{2}\rho W_0. \end{aligned} \tag{2.1.8}$$

Substituting λ and ρ of (2.1.4) into (2.1.8) yields

$$E[W^2] = \frac{(\mu E[\xi^2] - W_0 E[\xi])\mu}{V^2[\xi]} + \frac{(W_0 - \mu E[\xi])W_0}{V^2[\xi]}.$$

Hence

$$E[W^2] - \mu^2 = \frac{(E[\xi])^2}{V^2[\xi]}(\mu - W_0 E[\xi]^{-1})^2 \tag{2.1.9}$$

which completes the proof of Theorem 2.1.3.

Figure 2.1 depicts the feasible region of portfolio policies as shaded between the two lines in the mean-standard deviation space.

The upward linear segment is the efficient frontier and the downward linear segment the inefficient frontier; see Tobin (1958) and Ziemba et al (1974). The dynamic mean-variance model has a similar shape but with a different slope than the static mean-variance efficient frontier. The dynamic mean-variance efficient frontier is above that of the static mean-variance strategy since the set of self-financing strategy contains the set of static strategies as a subset. Furthermore, the mean and the standard deviation of the contingent state price uniquely determine the mean-variance efficient frontier.

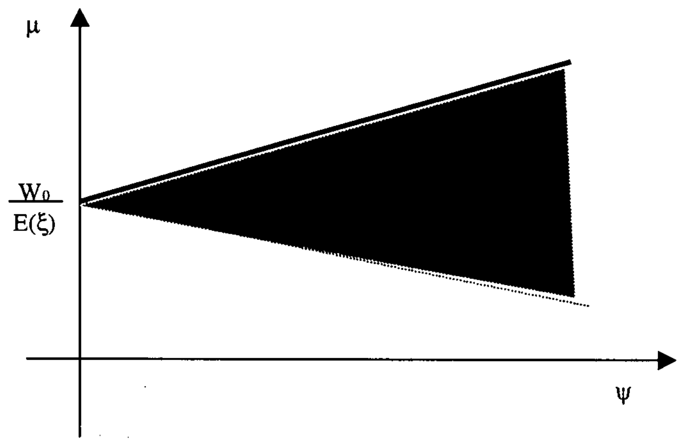

Figure 2.1: The Dynamic Mean-Variance Efficient Frontier

2.2 A Comparison with the Expected Utility Approach

2.2.1 The relation between the Terminal Portfolios

Much research in the literature has been focused on determining which of the expected utility approach and the mean-variance analysis is preferable in making sound investment decisions. The optimal portfolio generated from a utility maximization is not on the mean-variance efficient frontier except in a few special instances: either a "carefully" chosen quadratic utility function is used or the asset returns are joint normally distributed; see Samuelson (1970), and Ziemba and Vickson (1975) for other exceptions. However, investors and academic researchers do not accept these assumptions for practical use. For users of mean-variance analysis, the following question may be asked: what is the best choice of the target wealth such that the terminal portfolio value has the maximum probability of being higher than the portfolio value obtained from a utility maximization approach?

Let W_m and W_u be the terminal portfolio value for mean-variance analysis and an expected utility approach, respectively. W_m is given by (2.1.4) and (2.1.5). Using a martingale argument the optimal value for the growth optimal strategy is

$$W_u = U_x^{-1}(\lambda_u \xi), \tag{2.2.1}$$

where λ_u is the Lagrangian multiplier on the wealth constraint for given utility function $U(x)$ and $U_x^{-1}(\cdot)$ stands for the inverse function of the marginal utility of wealth;

see Cox and Huang (1989). Figure 2.2 depicts the relation of the two optimal portfolio values in terms of the state price ξ.

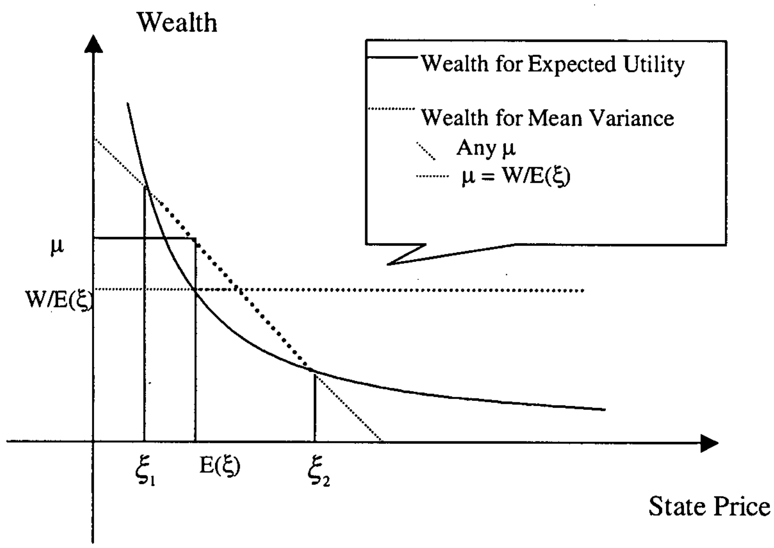

Figure 2.2: The Optimal Terminal Portfolio Values

Assume $U(x)$ is increasing and a Hara function, W_u is a convex function of ξ. On the other hand W_m is a straight line with negative slope. Therefore, there are two intersection points ξ_1 and ξ_2. Mean-variance analysis will be superior if the outcome of the state price ξ occurs around the mean value $E[\xi]$ of the state price, as represented by the bold dotted line segments, and be inferior if the outcome is beyond one of the tails, ξ_1 or ξ_2. Also, ρ changes as a function of μ. As μ increases, W_m shifts up, and at the same time becomes steeper. Hence, the effect of an increase of μ on the probability of outperforming the expected utility of wealth is non-monotonic.

2.2.2 Opportunities Superior to the Expected Utility Approach

By varying μ, investors can find the maximum probability that the mean-variance optimal portfolio will outperform the expected utility maximization portfolio. The maximum probability is given by solving

$$\max_{\mu} \quad \Pr\{\xi_1 \leq \xi \leq \xi_2 \mid \mu \geq W_0/E[\xi]\} \tag{2.2.2}$$

where ξ_1 and ξ_2, $\xi_1 < \xi_2$, are the two intersection points of W_u and W_m which satisfy the following transcendental equation,

$$-2U_x^{-1}(\lambda_u \xi) + \lambda + \rho\xi = 0. \tag{2.2.3}$$

Since W_u is a convex function of the contingent state price ξ and W_m is linear in ξ, W_u and W_m intersect at exactly two points for a given $\mu < \infty$. Since λ and ρ are functions of μ, so are ξ_1 and ξ_2. Both the mean-variance analysis and expected utility approaches are considered as standard approaches for constructing optimal investment strategies. For mean-variance optimizers, an interesting question is how to set the target wealth level such that the mean-variance criterion will be superior to the expected utility approach with maximum probability. With appropriate conditions, we can calculate the optimal value μ and, therefore, the maximum probability. Let $\phi(x)$ be the density function of ξ. Assuming that there is a solution to (2.2.2) and that both $\xi_1(\mu)$ and $\xi_2(\mu)$ are differentiable with respect to μ, then problem (2.2.2) becomes

$$\max_{\mu} \quad \int_{\xi_1}^{\xi_2} \phi(x)dx.$$

By the first order conditions, the optimal μ is given by

$$\phi(\xi_2(\mu))\xi_2'(\mu) - \phi(\xi_1(\mu))\xi_1'(\mu) = 0, \tag{2.2.4}$$

where "$\prime$" stands for the derivative.

2.2.3 A Numerical Example

Consider an investor having one dollar to invest between a riskless asset and a risky asset. The riskless interest rate for the period of August 2, 1999 to August 1, 2000 was about $r = 0.05$ per annum, i.e., the riskless asset price $B(t) = e^{rt}$. The S&P 500 is the risky asset. After scaling the initial index level to a dollar, Figure 2.3 depicts the price dynamics of the S&P 500 for this period (dividends are not considered for the calculation of the index return).

Assuming that the price $S(t)$ of the S&P 500 follows the geometric Brownian motion

$$dS(t) = bS(t)dt + \sigma S(t)dz(t)$$

with estimated $b = 0.101$ and $\sigma = 0.212$. Let the investment horizon be one year, so $T = 1$. The market price of risk $\kappa = (b - r)/\sigma = 0.24$ and, by definition, the state

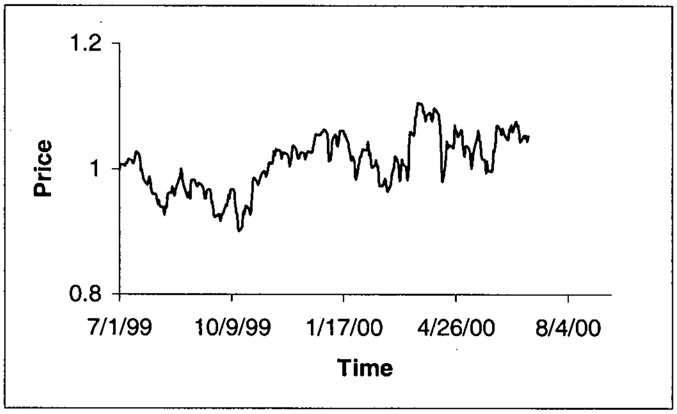

Figure 2.3: Scaled Index Level of S&P 500.

price ξ is

$$\xi = B(T)^{-1}\eta(T) = e^{-0.24z-0.08},$$

where z is a standard normal random variable. Hence, $E[\xi] = 0.95$ and $V^2[\xi] = 0.96$. By Equation (2.1.5),

$$\begin{cases} \lambda &= 35.69\mu - 35.42 \\ \rho &= -35.42\mu + 37.24. \end{cases}$$

The mean-variance optimal portfolio value is, by Equation (2.1.4),

$$W_m = (17.85\mu - 17.76) + (-17.76\mu + 18.62)\xi.$$

For the logarithmic utility, the optimal terminal wealth is (without loss of generality let $W_0 = 1$)

$$W_u = W_0\xi^{-1} = \xi^{-1}.$$

See Cox and Huang (1989) for a derivation of this. The intersection points, ξ_1 and ξ_2, are given by the quadratic equation

$$(-17.76\mu + 18.62)\xi_*^2 + (17.85\mu - 17.76)\xi_* - 1 = 0,$$

whose solutions are

$$\begin{cases} \xi_1(\mu) &= \frac{17.85\mu-17.76-\sqrt{240.94-562.99\mu+318.62\mu^2}}{2(17.76\mu-18.62)} \\ \xi_2(\mu) &= \frac{17.85\mu-17.76+\sqrt{240.94-562.99\mu+318.62\mu^2}}{2(17.76\mu-18.62)} \end{cases} \tag{2.2.5}$$

For given μ, the probability that the mean-variance model outperforms the growth optimal strategy is

$$\int_{\xi_1}^{\xi_2} \frac{1}{\sqrt{2\pi}\kappa} \exp\{-\frac{(\ln x + r + \frac{1}{2}\kappa^2)^2}{2\kappa^2}\} \cdot \frac{1}{x} dx.$$

Using the first order condition indicates that the numerical solution of the optimal μ is

$$\mu \approx 1.139,$$

which means that, for this specific investment environment, investors should set the target wealth to be about 13% higher than the initial wealth to maximize the probability of surpassing the growth optimal strategy (logarithmic utility). Then, the probability that the mean-variance model will beat the growth optimal strategy under the assumption of lognormal asset prices exceeds 70%. Figure 2.4 depicts the probabilities corresponding to different choices of μ.

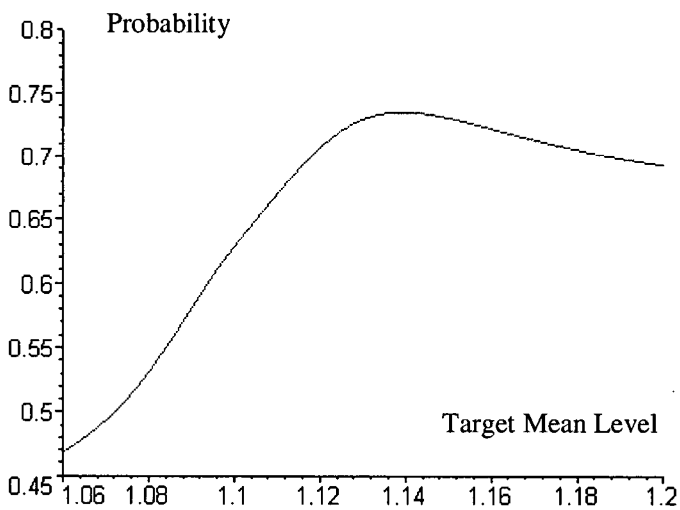

Figure 2.4: The Probability of Mean-Variance Superior to the Growth Optimal Strategy

Remark. Since the logarithmic utility has an expected portfolio return $E[\xi^{-1}] = e^{(r+\kappa^2)T}$ which is dominant in the long run (as $T \longrightarrow \infty$), the logarithmic utility will have a higher chance of beating the mean-variance analysis for the long investment horizon. This leads to the assertion that the logarithmic utility may have a high probability of beating a mean-variance model when the market investment environments are changed to a long investment horizon and/or a moderately high market price for risk (a high κ). See discussion on this in Hakansson and Ziemba (1995).

2.3 The Optimal Value Process and The Optimal Policy

The partial differential equation approach has been extensively used for the valuation of contingent claims based on the assumption of a Markovian structure for the underlying assets. Usually, a parabolic equation with boundary conditions has to be solved. We adopt this approach to calculate the optimal portfolio value for each time t which will derive the optimal portfolio policies by comparing the corresponding coefficients of the stochastic differential equations in wealth.

2.3.1 The Optimal Value Process

The assumption of completeness as in discussed in Chapter 1 reflects that the Brownian motion generates the filtration. Hence, any discounted martingale can be replicated with the market's primitive assets by the martingale representation theorem. Since a martingale is almost surely determined by the ending term of the martingale process, the optimal wealth $W(t)$ at time t must be

$$\begin{aligned} W(t) &= B(t)E^Q\left[B(T)^{-1}W\middle|\mathcal{F}_t\right] \\ &= B(t)\eta(t)^{-1}E\left[B(T)^{-1}\eta(T)W\middle|\mathcal{F}_t\right] \\ &= \xi(t)^{-1}E\left[\xi(T)W\middle|\mathcal{F}_t\right]. \end{aligned} \tag{2.3.1}$$

The vector process $(B(t), S(t), \xi(t))$ is a Markovian process by the definition of $\eta(t)$ and the asset price model setting in (1.1.1), but $\xi(t)$ is not a discounted martingale, therefore, it is not replicable. However $\zeta(t) = \xi(t)^{-1}$ is a discounted martingale, usually called an inflator process, since

$$d\zeta(t) = \zeta(t)[r(t)dt + \kappa(t)^\top dz^Q(t)]. \tag{2.3.2}$$

Since $r(t)$ and $\kappa(t)$ are deterministic functions of $B(t)$ and $S(t)$, $(B(t), S(t), \zeta(t))$ are jointly a Markovian process with the same filtration generated by the Brownian Motion.

Corollary 2.3.1. *With the setting of the underlying asset prices in (1.1.1), the optimal wealth at time t is a function of $(B(t), S(t), \zeta(t))$.*

2.3.2 The Optimal Portfolio Policy

Since $W(t)$ is a function of $(B(t), S(t), \zeta(t))$ at time t by Corollary 2.3.1, one needs only look for a function $F(t, B, S, \zeta)$ such that $W(t) = F(t, B(t), S(t), \zeta(t))$ satisfies the boundary condition. Assuming F is continuously differentiable in t and B and twice continuously differentiable in S and ζ, Ito's formula and the comparison of the differential equation with (1.2.2) yield

$$\begin{cases} F_t + r(t)B(t)F_B + b(t)F_S + r(t)\zeta(t)F_\zeta + \frac{1}{2}tr(F_{SS}\sigma(t)\sigma(t)^\top) \\ \qquad + \frac{1}{2}\zeta(t)^2\kappa(t)^\top\kappa(t)F_{\zeta\zeta} + \sigma(t)\kappa(t)F_{S\zeta} = r(t)F(t) \\ \sigma^\top F_S + F_\zeta\zeta\kappa = \sigma^\top\theta, \end{cases}$$

where $tr()$ is the trace function of a square matrix and $F_.$ and $F_{..}$ stand for partial derivatives. The $r(t), b(t), \sigma(t), \kappa(t), \theta$, and $F(t)$ are the stochastic processes and are deterministic function of asset prices as discussed in Chapter 1. Since the terminal value of the optimal portfolio $W = \frac{1}{2}\lambda + \frac{1}{2}\rho\zeta(T)^{-1}$ is implicitly a function of $r(T), b(T), \sigma(T)$ and $\zeta(T)$, the function $F(t, B, S, \zeta)$ is given by the partial differential equation stated in Theorem 2.3.2.

Theorem 2.3.2. *The optimal wealth $F(t, B, S, \zeta)$ is given by the solution to the following partial differential equation and the associated boundary condition.*

$$\begin{cases} F_t + r(t,B,S)BF_B + b(t,B,S)^\top F_S + r(t,B,S)\zeta F_\zeta \\ \qquad + \frac{1}{2}tr(F_{SS}\sigma(t,B,S)\sigma(t,B,S)^\top) + \frac{1}{2}\zeta^2\kappa^\top\kappa F_{\zeta\zeta} + F_{S\zeta}^\top\sigma(t,B,S)\kappa(t,B,S) = rF \\ F(T,B,S,\zeta) = \frac{1}{2}\lambda + \frac{1}{2}\rho\zeta^{-1}. \end{cases} \tag{2.3.3}$$

Therefore, the optimal portfolio policy is

$$\begin{cases} \theta(t) = F_S + F_\zeta\zeta(t)(\sigma(t)^\top)^{-1}\kappa(t) \\ \alpha(t) = B(t)^{-1}(W_t - \theta(t)^\top \cdot S(t)) \end{cases} \tag{2.3.4}$$

where $r, b,$ and σ are functions of (t, B, S).

To understand the optimal portfolio policy better, we provide an intuitive interpretation. Since $\zeta(t)$ is a discounted Q-martingale, let $\theta^\zeta(t)$ be the hedging portfolio of $\zeta(t)$ in the risky assets, i.e.

$$d\zeta(t) = r(t)\zeta(t)dt + \theta^\zeta(t)^\top\sigma(t)dz^Q(t). \tag{2.3.5}$$

Comparing (2.3.5) with (2.3.2) yields

$$\theta^{\zeta}(t) = \zeta(t)(\sigma(t)^{\top})^{-1}\kappa(t). \tag{2.3.6}$$

The optimal policy in (2.3.4) becomes

$$\begin{cases} \theta(t) = F_S + F_{\zeta} \cdot \theta^{\zeta}(t) \\ \alpha(t) = B(t)^{-1}(W(t) - \theta(t)^{\top} \cdot S(t)), \end{cases} \tag{2.3.7}$$

which can be interpreted as follows. If $\zeta(t)$ is considered to be a dynamic mutual fund, the optimal portfolio policy can be constructed by investing F_S units in the assets S and F_{ζ} units in the dynamic mutual fund ζ. To implement this strategy, one needs only to synthesize the dynamic mutual fund $\zeta(t)$ and combine the two hedging portfolios (i.e., the Δ strategies) to obtain the optimal investment portfolio. This interpretation will become more intuitive and clearer in light of the special case discussed in the next section.

2.4 A Special Case

In this section we consider a special case of the model in which the asset prices jointly follow a multivariate Brownian motion and the riskless rate is a constant throughout the investment horizon. This assumption implies that

$$r(t) = r, \quad \sigma(t) = \begin{pmatrix} S_1(t) & & & \\ & S_2(t) & & \\ & & \ddots & \\ & & & S_n(t) \end{pmatrix} \sigma, \quad b(t) = \begin{pmatrix} b_1 S_1(t) \\ b_2 S_2(t) \\ \vdots \\ b_n S_n(t) \end{pmatrix}$$

where r, σ, b are constant matrices. With these conditions, $\kappa(t)$ becomes a constant vector and

$$\begin{aligned} E[B(T)^{-1}\eta(T)] &= E[e^{-\kappa^{\top} z(T) - (r + \frac{1}{2}\kappa^{\top}\kappa)T}] \\ &= e^{-rT} \\ V^2[B(T)^{-1}\eta(T)] &= E[e^{-2\kappa^{\top} z(T) - (2r + \kappa^{\top}\kappa)T}] - e^{-2rT} \\ &= e^{-2rT}(e^{\kappa^{\top}\kappa T} - 1). \end{aligned} \tag{2.4.1}$$

Therefore,

$$\begin{aligned} \lambda &= \frac{2\mu e^{\kappa^{\top}\kappa T} - 2W_0 e^{rT}}{e^{\kappa^{\top}\kappa T} - 1} \\ \rho &= \frac{2W_0 e^{2rT} - 2\mu e^{rT}}{e^{\kappa^{\top}\kappa T} - 1}. \end{aligned} \tag{2.4.2}$$

2.4.1 The Closed Form Solution

Since λ and ρ are constants throughout the investment horizon, $W = W(T)$ is only a function of $\zeta(T)$. Then we try to look for a function $F(t, \zeta)$ of t and ζ such that it derives the optimal wealth process and the optimal portfolio policies. The partial differential equation (2.3.3) becomes

$$\begin{cases} F_t + r\zeta F_\zeta + \frac{1}{2}\zeta^2\kappa^\top\kappa F_{\zeta\zeta} = rF \\ F(T,\zeta) = \frac{1}{2}\lambda + \frac{1}{2}\rho\zeta^{-1}. \end{cases} \tag{2.4.3}$$

Solving this equation yields

Theorem 2.4.1. *The partial differential equation (2.4.3) has a closed form solution*

$$F(t,\zeta) = \frac{1}{2}\left(\lambda e^{-r(T-t)} + \rho\zeta^{-1}e^{(\kappa^\top\kappa-2r)(T-t)}\right). \tag{2.4.4}$$

The optimal portfolio policy at time t is

$$\begin{cases} \theta(t) = (\frac{1}{2}\lambda e^{-r(T-t)} - W(t))I_S^{-1}(\sigma^\top)^{-1}\kappa \\ \alpha(t) = B(t)^{-1}(W(t) - \theta(t)^\top S(t)), \end{cases} \tag{2.4.5}$$

where I_S is the diagonal matrix with asset prices, $S(t)$, as the entries.

Proof. The first and second order derivatives of $F(t, \zeta)$,

$$\begin{aligned} F_t &= \frac{1}{2}\lambda re^{-r(T-t)} - \frac{1}{2}\rho\zeta^{-1}(\kappa^\top\kappa - 2r)e^{(\kappa^\top\kappa-2r)(T-t)} \\ F_\xi &= -\frac{1}{2}\zeta^{-2}\rho e^{(\kappa^\top\kappa-2r)(T-t)} \\ F_{\xi\xi} &= \rho\zeta^{-3}e^{(\kappa^\top\kappa-2r)(T-t)}, \end{aligned}$$

satisfy Equation (2.4.3) and the boundary condition. So, $F(t, \zeta)$ is the solution to the partial differential equation. By Ito's formula,

$$dF(t,\zeta(t)) = rF(t,\zeta(t))dt - \frac{1}{2}\rho\zeta^{-1}e^{(\kappa^\top\kappa-2r)(T-t)}\kappa^\top dz^Q(t),$$

which derives the optimal strategy (2.3.7) by comparing with the wealth dynamics (1.2.2) and using (2.4.4). $\square$

2.4.2 Implementation of the Mean-Variance Optimal Strategy

In the static mean-variance model, the investor needs only to choose an appropriate target mean level to find the optimal portfolio strategy by minimizing the standard deviation. This process can be completed with the calculation of the first and the second moments and a quadratic optimizer. However, in dynamic investment analysis, the portfolio weights are changed according to the observed market asset prices. To illustrate the dynamic mean-variance analysis, we use the data of the previous example to compare the performances of the mean-variance analysis and the growth optimal portfolio. The target mean return level for the mean-variance analysis is chosen as 13.9% which maximizes the probability of outperforming the growth optimal strategy. Figure 2.5 describes the performances of the two strategies over time.

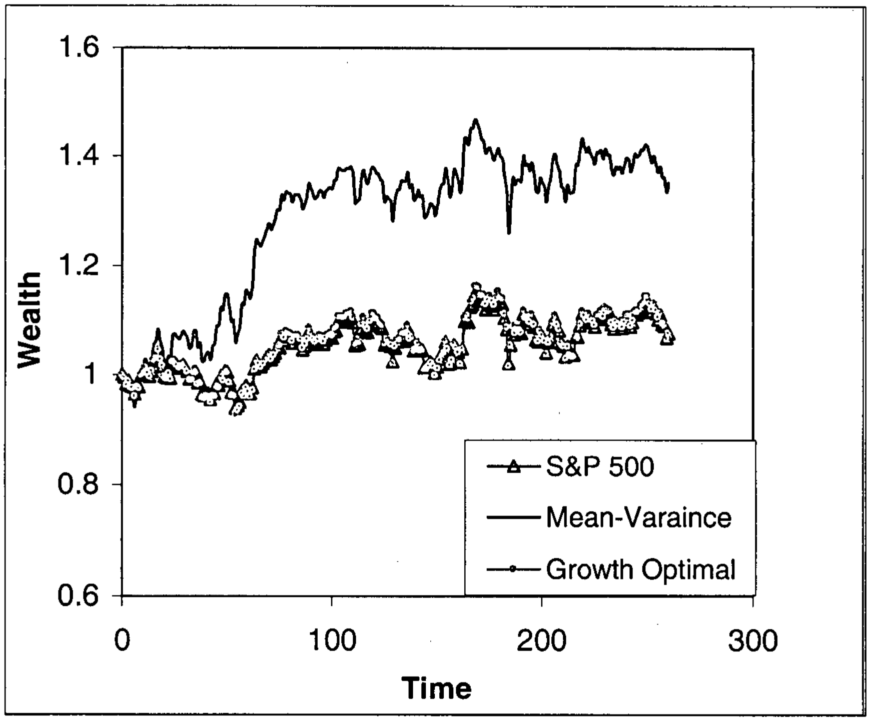

Figure 2.5: Performances over Time

While the growth optimal portfolio has a similar performance to the index for this specific data, the mean-variance analysis has a superior performance if we set the target mean level to be about 2% more than the mean return of S&P 500.

Chapter 3

A Model Using the Worst Possible Outcome

This chapter defines and solves the investment problem in a continuous time setting. The result is compared to the traditional expected utility maximization through an example.

3.1 Formulation and Solution

In standard utility maximization models, portfolio values are allowed to take zero or even negative values if the utility function on wealth is defined on the whole real line. This is not acceptable for some investors who have a liability stream to pay.

How do we efficiently control downside losses? Two possible ways are considered: Choosing a better risk averse utility function or changing the probability weights in forming the objective function. The first approach presents to us the traditional expected utility maximization with the objective probability measure, while the second approach is also an expected utility maximization problem but with different probability measure. The reassignment of probability to each possible outcomes draws the investor's attention to the bad-state outcomes.

Our focus is on the worst possible outcome of terminal wealth. By assumption, the wealth W at the end of the horizon is a random variable under probability measure P and it has a probability support Ξ which might be bounded or unbounded. Let

$$W^L = \begin{cases} \sup\left\{K; \Pr[W \geq K] = 1\right\}, & \text{if } \Xi \text{ is bounded} \\ -\infty & \text{if } \Xi \text{ is unbounded.} \end{cases} \tag{3.1.1}$$

We call W^L the *worst outcome of wealth* with respect to a given strategy. Investors develop optimal strategies to measure the overall performance (expected utility) while controlling the level of W^L. Increasing the level of W^L will automatically decrease the expected wealth, see Zhao and Ziemba (1999b). There is a trade-off between the overall wealth and the worst outcome. A natural way is to balance these two competing aspects is to maximize a convex combination of utilities received from both the actual outcome and the worst outcome of wealth. Hence, the investor's objective function for the optimization model, as defined in Section 5 of Chapter 1, is

$$V(x,y) = \rho U(x) + (1-\rho)U(y) \tag{3.1.2}$$

where x is the actual outcome of the wealth, y is the worst possible outcome, and $U(\cdot)$ is investor's utility function. This is equivalent to maximizing expected utility with a different probability measure, i.e. reducing the probabilities (density) of all states by the ratio ρ and adding the total reduced probability weights to that of the worst case outcome. The control intensity of downside losses increases as ρ decreases.

Let $\mathcal{A}$ be the set of all admissible strategies defined by condition (1.1.3). Let $\{W(t)\}$ denote a possible wealth process generated by an admissible strategy and

$$\mathcal{W} = \{\{W(t)\}|(\alpha(t),\theta(t)) \in \mathcal{A}\}.$$

The investor's dynamic optimization problem is

$$\begin{aligned} \max_{(\alpha(t),\theta(t))\in\mathcal{A}} \quad & E\left[\rho U(W(T)) + (1-\rho)U(W^L(T))\right] \\ s.t. \quad & dW(t) = (\alpha(t)r(t)B(t) + \theta(t)^\top b(t))dt + \theta(t)^\top \sigma(t)dz(t). \end{aligned} \tag{3.1.3}$$

If $U(\cdot)$ is an increasing function, an equivalent representation of Model (3.1.3) is

$$\begin{aligned} \sup_{K,(\alpha(t),\theta(t))} \quad & E\left[\rho U(W(T)) + (1-\rho)U(K)\right] \\ s.t. \quad & dW(t) = (\alpha(t)r(t)B(t) + \theta(t)^\top b(t))dt + \theta(t)^\top \sigma(t)dz(t), \\ & W(T) \geq K, \qquad P-a.s. \end{aligned} \tag{3.1.4}$$

where K is constant through time and $(\alpha(t),\theta(t)) \in \mathcal{A}$ is an admissible strategy. By Theorem 1.2.1, Model (3.1.4) can be divided into two problems, the martingale formulation identifying the optimal terminal wealth and a replication problem deriving the optimal portfolio strategy. The martingale formulation is

$$\begin{aligned} \sup_{W,K} \quad & E\left[\rho U(W) + (1-\rho)U(K)\right] \\ s.t. \quad & E^Q\left[\frac{W}{B(T)}\right] = W_0 \\ & W \geq K, \qquad P-a.s. \end{aligned} \tag{3.1.5}$$

where W is $\mathcal{F}_T$ measurable.

To solve Model (3.1.5), we provide the following Extended Kuhn-Tucker Criterion. For a general result see Zhao, Haussmann and Ziemba (2000).

Theorem 3.1.1. *(Extended Kuhn-Tucker Criterion). (W, K) is a pair of optimal solution to Model (3.1.5) if and only if the following conditions hold*

(*i*). $\rho U_x(W) - \lambda_0 \frac{\eta(T)}{B(T)} + \lambda = 0, \qquad P\text{ - }a.s.,$

(*ii*). $(1-\rho)U_x(K) - E[\lambda] = 0,$

(*iii*). $E\left[\frac{W\eta(T)}{B(T)}\right] - W_0 = 0,$

(*iv*). $\lambda(W - K) = 0, \qquad P\text{ - }a.s.,$

(*v*). $W \geq K, \lambda \geq 0, \qquad P\text{ - }a.s.$

where $U_x(\cdot)$ is the first order derivative, λ_0 is the Lagrange multiplier on the martingale constraint, and λ, a random variable, is the multiplier on the wealth level constraints.

Conditions (i), (iv) and (v) yield

$$\lambda = \left[\frac{\lambda_0 \eta(T)}{B(T)} - \rho U_x(K)\right]^+. \tag{3.1.6}$$

If $W > K$, then $\lambda = 0$ and $W = U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right)$. Here $U_x^{-1}(\cdot)$ is the inverse of the marginal utility $U_x(\cdot)$. Hence, the investor's optimal terminal portfolio value is

$$W = K + \left[U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right) - K\right]^+. \tag{3.1.7}$$

Since the portfolio value at any time $t \in [0, T]$ is the cost for hedging the terminal portfolio, by Theorem 1.2.1 its value at time t must be

$$\begin{aligned} W(t) &\equiv B(t)E^Q\left[\frac{W}{B(T)}\middle| \mathcal{F}_t\right] \\ &= B(t)E^Q\left[\left(\left[U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right) - K\right]^+ + K\right) B(T)^{-1}\middle| \mathcal{F}_t\right], \end{aligned} \tag{3.1.8}$$

which establishes Proposition 3.1.2.

Proposition 3.1.2. *The investor's optimal wealth is*

$$\begin{aligned} W(t) =& B(t)E^Q[K\,B(T)^{-1}|\mathcal{F}_t] + \\ & B(t)E^Q\left[\left[U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right) - K\right]^+ B(T)^{-1}\middle|\mathcal{F}_t\right] U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right), \end{aligned} \tag{3.1.9}$$

Proof. Since the optimal terminal wealth is

$$W = K + \left[U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right) - K\right]^+,$$

and $E^Q\left[\frac{W}{B(T)}\middle|\mathcal{F}_t\right]$ is a Q- martingale. By Theorem 1.2.1, $\frac{W(t)}{B(t)}$ is also a Q- martingale. But $\frac{W(t)}{B(t)}$ and $E^Q\left[\frac{W}{B(T)}\middle|\mathcal{F}_t\right]$ have identical terminal value and right-ended martingale is uniquely determined by the right-end term. So

$$W(t) = B(t)E^Q\left[\frac{W}{B(T)}\middle|\mathcal{F}_t\right]$$

which proves Proposition 3.1.2. □

The portfolio value may not be an explicit function of $(t, B(t), S(t))$ for general market parameters. $W(t)$ depends on $\eta(t)$ which is adapted to $\mathcal{F}_t$. Since $\eta(t)$ may depend on the past observations of asset prices $B(t)$ and $S(t)$ by its definition, the portfolio value $W(t)$ may not be expressible as a function of $(t, B(t), S(t))$ only. To make $W(t)$ a function of the state variables, we must enlarge the state space to accommodate past information when we apply the differential equation approach to derive the optimal trading strategies.

The proof of Proposition 3.1.2 implies that to calculate the portfolio value process one needs only to find the optimal target K and the multiplier λ_0. These quantities can be obtained from the Extended Kuhn-Tucker Criterion if $U(x)$ is strictly increasing and concave and has continuous derivatives. The following equations, which are used for calculating the optimal λ_0 and K, can be derived from the Extended Kuhn-Tucker Criterion.

$$\begin{cases} (1-\rho)U_x(K) - \rho E\left[\frac{\lambda_0\eta(T)}{\rho B(T)} - U_x(K)\right]^+ = 0 \\ E^Q\left[KB(T)^{-1}\right] + E^Q\left[\left[U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right) - K\right]^+ B(T)^{-1}\right] = W_0. \end{cases} \tag{3.1.10}$$

For given utility function $U(x)$, the value of the portfolio can be calculated directly through the calculation of a conditional expectation (3.1.8), after solving Equation (3.1.10) for λ_0 and K. The expectation term is similar to a standard expression for the valuation of contingent claims. If there is a way of transforming a stochastic process to a martingale, then methods for valuation of contingent claims can be applied to solve this problem. The second expression in Equation (3.1.10) can be interpreted to mean that investors allocate wealth to only two assets: the riskless asset and a call option on a mutual fund. For an optimal λ_0 and K, taking $U_x^{-1}(\frac{\lambda_0\eta(T)}{\rho B(T)})$ as the terminal value of a mutual fund which can be generated by the primary assets, the investor "invests" some amount in the option on the mutual fund with strike price K and the balance in the riskless asset that will guarantee the amount K needed for exercising these options. Interestingly, Equation (3.1.10) can be rewritten as

$$U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right) + E^Q\left[\left[K - U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right)\right]^+ B(T)^{-1}\right] = 0$$

which represents a standard utility maximizing portfolio plus a put option. This is exactly the portfolio insurance strategy. We discuss this issue in the next section. Since $\frac{\lambda_0\eta(T)}{\rho B(T)} = U_x\left(U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right)\right)$ and $U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right)$ is the terminal wealth when there is no requirement on the worst possible outcome wealth, the first equation in (3.1.10) implies that, at optimality, the expected marginal utility on K is equal to the decrement of the marginal utility induced by increasing wealth to the level K. Mathematically, this relation can be expressed as

$$U_x(K) = \rho E\left[U_x(L)\right] \tag{3.1.11}$$

where $L = \min\left\{U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right), K\right\}$. This relation represents a trade-off between the expected wealth and the worst possible outcome of wealth. The economic interpretation is that investors can only increase the expected value of wealth by reducing the worst possible outcome wealth, which characterize the potential losses. The optimal K is the wealth cutoff at which the marginal utility is equal to the average marginal utility below that point multiplied by the risk control intensity ρ.

Having established the diffusion process for the wealth, we next derive the optimal trading strategy which is equivalent to calculating the conditional expectation (3.1.9) for any t. This is usually a difficult task if parameters for asset prices are state-dependent, because in this case it is generally impossible to analytically compute the conditional density function under the risk neutral probability. However, the problem can be transformed to the solution of a partial differential equation with boundary conditions to which numerical methods might apply.

If we can construct a stochastic process $v(t)$ that is a discounted martingale and has a terminal value equal to $U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right)$, then a partial differential equation can be derived. We can apply Theorem 1.2.1 with $Y = U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right)$ and

$E^Q[B(T)^{-1}Y] = W_0 < \infty$. Let $v(t) = B(t)E^Q[U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right)B(T)^{-1}]|\mathcal{F}_t)$. Then $v(t)$ is a discounted martingale. By Theorem 1.2.1, the replicating portfolio value is given by the differential equation

$$dv(t) = r(t)v(t)dt + \tilde{\theta}(t)^\top \sigma(t)dz^Q(t),$$

where $\tilde{\theta}(t)$ is the solution of the utility maximization without the constraint $W \geq K$. $\tilde{\theta}$ may not only depend on the current asset prices, but also depend on past observations. If $\tilde{\theta}(t)$ is a function of $(t, B(t), S(t), v(t))$ which means the optimal strategy is Markovian, then $(B(t), S(t), v(t))$ are jointly Markovian as assumed. The portfolio value can be represented as a function of $(t, B(t), S(t), v(t))$ as argued in Chapter 1.

Denote $\xi(t) = \begin{pmatrix} S(t) \\ v(t) \end{pmatrix}$ and $\hat{\sigma}(t) = \begin{pmatrix} \mathbf{I} \\ \tilde{\theta}(t)^\top \end{pmatrix} \sigma(t)$. $\xi(t)$ is an $(n+1)$-vector and $\hat{\sigma}(t)$ is an $(n+1) \times n$ matrix.

$$d\xi(t) = \begin{pmatrix} dS(t) \\ dv(t) \end{pmatrix} = r(t)\xi(t)dt + \hat{\sigma}(t)dz^Q(t).$$

Let $W(t) = F(t, B(t), \xi(t))$ be the portfolio value at time t, where $F(t, x, y)$ is a real-valued function on $\Re^+ \times \Re \times \Re^{n+1}$. By Ito's formula, the optimal strategy is characterized as in Proposition 3.1.3.

Proposition 3.1.3. *Assume that the optimal portfolio value $W(t) = F(t, B(t), \xi(t))$ where F has continuous partial derivatives. Then F satisfies the following partial differential equation with associated boundary conditions:*

$$\begin{cases} -r(t)F + F_t + r(t)F_x x + r(t)F_y^\top y + \frac{1}{2}tr(F_{yy}\hat{\sigma}(t)\hat{\sigma}(t)^\top) = 0 \\ F(0, 1, y) = W_0 \\ F(T, x, y) = K + [v - K]^+ \end{cases} \tag{3.1.12}$$

where $tr()$ stands for the trace function of a matrix and $y = (S, v)^\top$. The optimal strategy is

$$\begin{cases} \theta(t) = F_S + F_v\,\tilde{\theta}(t) \\ \alpha(t) = (W(t) - \theta(t)^\top S(t))/B(t). \end{cases} \tag{3.1.13}$$

The boundary condition for the above partial differential equation follows from (3.1.8).

Finding the optimal strategy for the original problem has been divided into two steps: (i) calculate the optimal strategy $\tilde{\theta}$ when there is no wealth level constraints;

and (ii) find the optimal wealth as a solution to a partial differential equation and derive the optimal strategy thereafter. This approach is similar to the method used in Cox and Huang (1989) where an auxiliary process is used for deriving the optimal strategy. Since directly observable processes are the primary assets and the portfolio value process, all other processes that are related to the derivation of the optimal trading strategies must be replicable using primary assets. To implement the dynamic optimal strategy one has to synthesize these processes on the side as if they were market-derived assets. This is especially important when we interpret the option investment strategy. Since the underlying asset of a call option is replicable using primary assets, we have to choose a suitable auxiliary process such that we can easily derive the optimal strategy.

3.2 Option Strategy Interpretation

One approach to deal with downside risk is the option strategy. An option strategy for investment is usually used for the purpose of hedging. Before making decisions, investors have to think about how much to invest in different projects while simultaneously achieving the risk control. Exchange traded options might not be suitable for specific investors. Not only might the strike price (the target) but also the investment horizon be different from the investor's concern. A more dynamic way is needed for creating such strategies. For an application of this idea, see Zhao and Ziemba (1999a), which considers how to create synthetic option strategies with transaction cost in a discrete time and discrete state space model.

An alternative, possibly more insightful, interpretation of the optimal solution to the problem matches the view stated above. Using a martingale approach, the portfolio value at each time is a discounted Q - martingale given by (3.1.9), which can be interpreted as a strategy of investing in the call option on the mutual fund. In light of the option pricing methods, we may be able to construct an optimal strategy by the following four steps:

(i). Find the optimal λ_0 and K.

(ii). Create the dynamic mutual fund using the $n+1$ available primary assets.

(iii). Calculate the hedging portfolio for a call option with strike price K on the mutual fund.

(iv). Transform the hedging portfolio into the portfolio of the primary assets.

Remark 1. The first step can be obtained by solving (3.1.10). The second step is difficult to solve because finding the mutual fund

$$v(t) = B(t)E^Q\left[B(T)^{-1}U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right)\middle|\mathcal{F}_t\right]$$

is an intractable problem if market's parameters and the investor's utility function are arbitrarily chosen. The third step is to apply option pricing methodologies and the derivation of the hedging strategies. The last step is to combine the two decomposed methods together to obtain the optimal portfolio of the primary assets.

Two ways of obtaining the mutual fund $v(t)$ are calculating directly the conditional expectation and solving a boundary problem. These are general approaches which are valid for broad settings of market parameters and the utility function. Since a martingale is uniquely determined by the terminal value of the martingale and the filtration, we might want to look for a process that has the same terminal value as $U_x^{-1}\left(\frac{\lambda_0\eta(T)}{B(T)}\right)$. Instead of going through a general approach, we might want to examine how the process $m(t) = U_x^{-1}(\frac{\lambda_0\eta(t)}{\rho B(t)})$ will move with respect to the uncertainty and to derive the process that we need. Applying Itô's formula yields

$$\begin{aligned} dm(t) &= \frac{1}{U_{xx}}d\left(\frac{\lambda_0\eta(t)}{\rho B(t)}\right) - \frac{U_{xxx}}{2U_{xx}^3}\left(d\left(\frac{\lambda_0\eta(t)}{\rho B(t)}\right)\right)^2 \\ &= \frac{1}{U_{xx}}(-U_x\kappa(t)^\top dz(t) - U_x r(t)dt) - \frac{U_{xxx}U_x^2||\kappa(t)||^2}{2U_{xx}^3}dt \\ &= \frac{U_x(-2r(t)U_{xx}^2 - ||\kappa(t)||^2 U_x U_{xxx})}{2U_{xx}^3}dt + \frac{-U_x}{U_{xx}}\kappa(t)^\top dz(t). \end{aligned} \tag{3.2.1}$$

From (3.2.1), one can observe that the mutual fund chosen by investors depends only on the risk aversion and the skewness but not the specific form of the utility function. This implies that all investors with the same risk aversion and skewness of the utility function will choose the same mutual fund to invest, but this does not necessarily show that all investors with such a utility function will adopt the same investment strategy.

If the drift term in (3.2.1) is $rm(t)$, then $m(t)$ is a discounted martingale and option valuation models can be used. However, $m(t)$ is not a martingale in general, but if we can transform $m(t)$ to a martingale process which has the same terminal value as $v(T)$, then the problem has been reduced to a valuation problem of an option. The following proposition deals with this issue.

Proposition 3.2.1. *(drift killing) If a stochastic process $X(t)$ follows the diffusion*

$$dX(t) = \mu(t)X(t)dt + \gamma(t)^\top dz^Q(t)$$

then $\exp\{-\int_0^t \mu(s)ds\}X(t)$ *will be a (local)* Q *- martingale. Furthermore, if* $\mu(t)$ *is deterministic, then* $\exp\{\int_t^T \mu(s)ds\}X(t)$ *is a (local)* Q *- martingale that has the same terminal value as* $X(T)$ *almost surely.*

Proof. Since $\mu(t)$ is adapted, $\exp\{-\int_0^t \mu(s)ds\}$ is also $\mathcal{F}_t$ adapted and

$$d\exp\{-\int_0^t \mu(s)ds\}X(t) = exp\{\int_0^t \mu(s)ds\}\gamma(t)dz^Q(s).$$

Hence, $\exp\{-\int_0^t \mu(s)ds\}X(t)$ is a (local) Q - martingale.

If $\mu(t)$ is deterministic, then $\exp\{-\int_0^t \mu(s)ds\}$ is also $\mathcal{F}_t$ adapted and

$$d\exp\{\int_t^T \mu(s)ds\}X(t) = \exp\{\int_t^T \mu(s)ds\}\gamma(t)dz^Q(s).$$

So, $\exp\{\int_t^T \mu(s)ds\}X(t)$ is a (local) Q- martingale with terminal value $X(T)$. □

It would be convenient if the drift rate in (3.2.1), $\frac{U_x(-2r(t)U_{xx}^2-||\kappa(t)||^2U_xU_{xxx})}{2U_{xx}^3}$, were a deterministic function of t, as in this case we can apply Proposition 3.2.1 to transform $m(t)$ to a martingale. For some types of utility functions, e.g., the HARA family, with the assumption of geometric Brownian motion, we can verify that the drift rate of $v(t)$ is a deterministic function of t. However, in more general settings, this will not be the case and other approaches must be found.

We have discussed how to generate a mutual fund in any setting using market's primary assets. The partial differential equation approach and the direct calculation of the conditional expectation are considered as general approaches. Another way which is sometimes more effective in some settings is using Proposition 3.2.1. The mutual fund has the same terminal value as $U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right)$, and the option valuation method can be used to generate the investment optimal portfolio. Having established the mutual fund, we now find the generating portfolio using primary assets. The existence and computation of this generating portfolio is given by Theorem 1.2.1. For future use, we state this result as

Proposition 3.2.2. *If the stochastic process* $\frac{X(t)}{B(t)}$ *is a* Q *- martingale, i.e.*

$$dX(t) = r(t)X(t)dt + \gamma(t)^\top dz^Q(t),$$

then the self-financing strategy $(\alpha(t), \theta(t))$ *generates* $X(t)$*, where*

$$\begin{cases} \alpha(t)B(t) + \theta(t)^\top S(t) = X(t) \\ \theta(t)^\top\sigma(t) = \gamma(t)^\top. \end{cases} \tag{3.2.2}$$

The next step is to compute the hedging portfolio of the call option on the auxiliary mutual fund. Probably this is the most difficult task in the whole process if market parameters are specified in a broad sense. The call option can be hedged by the mutual fund $v(t)$ and market's primary assets. Let $C(t, B(t), S(t), v(t))$ be the value of the option at time t. By Itô's formula

$$dC(t, B(t), \xi(t)) = (C_t + r(t)B(t)C_B + r(t)C_\xi \xi(t) + \frac{1}{2}tr(C_{\xi\xi}\hat{\sigma}(t)\hat{\sigma}(t)^\top)dt + C_\xi^\top \hat{\sigma}(t)dz^Q(t). \tag{3.2.3}$$

Since $C(t)$ should be also replicable by using $(B(t), S(t), v(t))$, let $(\alpha^v(t), \theta^S(t), \theta^v(t))$ be the hedging portfolio of the call option, then

$$d(C(t, B(t), S(t), v(t)) = r(t)C(t, B(t), v(t))dt + (\theta^S(t)^\top + \theta^v(t)\tilde{\theta}(t)^\top)\sigma(t)dz^Q(t). \tag{3.2.4}$$

Equating (3.2.3) and (3.2.4) yields

Proposition 3.2.3. *Let $dv(t) = r(t)v(t)dt + \tilde{\theta}(t)^\top\sigma(t)dz^Q(t)$, where $\tilde{\theta}(t)$ is a non-random function of $(t, B(t), S(t), v(t))$. The value of the call option is the solution to the boundary problem*

$$\begin{cases} C_t + r(t)B(t)C_B + r(t)C_\xi\xi(t) + \frac{1}{2}tr(C_{\xi\xi}\hat{\sigma}(t)\hat{\sigma}(t)^\top) - r(t)C = 0 \\ C(T, B(T), S(T), v(T)) = [v(T) - K]^+. \end{cases} \tag{3.2.5}$$

The number of shares of the mutual fund $\theta^v(t)$ and the number of units of risky assets $\theta^S(t)$ needed for the hedging portfolio are given by the following linear relation

$$\theta^S(t)^\top + \theta^v(t)\tilde{\theta}(t) = C_S^\top + C_v\tilde{\theta}(t)^\top.$$

For a broad setting of parameters and the utility function, we have to apply numerical methods to solve the partial differential equation. However, in some cases, a closed form solution can be obtained. We discuss this in the next section. The following theorem, which is implied by Propositions (3.2.2) and (3.2.3), summarizes how to convert an option hedging portfolio to the optimal investment portfolio of the primary assets.

Theorem 3.2.4. *Let the mutual fund prices $v(t)$ follow the stochastic process*

$$dv(t) = r(t)v(t)dt + \tilde{\theta}(t)^\top\sigma(t)dz^Q(t)$$

and $(\alpha^v(t), \theta^S(t), \theta^v(t))$ is the hedging portfolio for the call option on $v(t)$ with strike price K, then the optimal investment portfolio $(\alpha(t), \theta(t))$ is given by

$$\begin{cases} \theta(t) = \theta^S(t) + \theta^v(t)\tilde{\theta}(t) \\ \alpha(t) = (W(t) - \theta(t)^\top S(t))B(t)^{-1}, \end{cases}$$

where $W(t)$ is the wealth at time t.

Remark 2. Along with Propositions 3.2.1 and 3.2.2, Theorem 3.2.4 implies that an investment problem of utility maximization is equivalent to a problem of valuing an option whose underlying asset is the auxiliary mutual fund and whose strike price is the worst possible outcome wealth. One issue which will not be discussed here is how to compute the conditional expectation $\frac{V(t)}{B(t)} = E^Q\left[U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right)B(T)^{-1} \mid \mathcal{F}_t\right]$. This problem amounts to solving a differential equation that $\frac{V(t)}{B(t)}$ satisfies. Fortunately, as shown in the next section, we can apply Proposition 3.2.1 to the case where utility function is HARA and the asset prices follow a multi-dimensional geometric Brownian motion.

As to the evaluation of the other expression, $E\left[\frac{\lambda_0\eta(T)}{\rho B(T)} - U_x(K)\right]^+$, in equation (3.1.10), we can reduce it to an option pricing problem as well. By the definition of Q with $\lambda_0 > 0, U_x(K) > 0$,

$$E\left[\frac{\lambda_0\eta(T)}{\rho B(T)} - U_x(K)\right]^+ = \frac{\lambda_0 U_x(K)}{\rho}E^Q\left[\left[\frac{1}{U_x(K)} - \frac{\rho B(T)}{\lambda_0\eta(T)}\right]^+ B(T)^{-1}\right].$$

Since $d\eta(t) = -\kappa(t)\eta(t)dz(t)$,

$$\begin{aligned} d\left(\frac{\rho B(t)}{\lambda_0\eta(t)}\right) &= \frac{\rho B(t)}{\lambda_0\eta(t)}(r(t)dt + \kappa(t)dz(t) + ||\kappa(t)||^2dt) \\ &= \frac{\rho B(t)}{\lambda_0\eta(t)}(r(t)dt + \kappa(t)dz^Q(t)) \end{aligned}$$

which proves that $\frac{\rho B(t)}{\lambda_0\eta(t)}$ is a discounted Q- martingale. The expression $\frac{\lambda_0\eta(T)}{\rho B(T)}$ is the marginal utility at $U_x^{-1}\left(\frac{\lambda_0\eta(T)}{\rho B(T)}\right)$ which is the optimal wealth when the requirement of the worst possible outcome wealth is removed from the model. Let

$$\varphi(t) = B(t)E^Q\left[\frac{\rho}{\lambda_0\eta(T)}\middle| \mathcal{F}_t\right],$$

then $\varphi(t)$ is a discounted martingale. Hence,

$$E^Q\left[\left[\frac{1}{U_x(K)} - \frac{\rho B(T)}{\lambda_0\eta(T)}\right]^+ B(T)^{-1}\right]$$

is the price of a put option whose underlying process is $\varphi(t)$ and whose strike price is $\frac{1}{U_x(K)}$, the reciprocal of the minimum marginal utility at the worst possible outcome wealth. At optimality, $E\left[\frac{\lambda_0\eta(T)}{\rho B(T)} - U_x(K)\right]^+$ is λ_0 times the value of a put option times the marginal utility at the worst possible outcome wealth divided by the control intensity ρ, i.e. the value of this put option is positively proportional to the expected rate of gaining marginal utility from the absence of the requirement of the worst outcome wealth K in the model.

3.3 HARA Utility and GBM Prices

In this section, we consider a specialization of the model in which investors have the HARA utility function $U(x) = \frac{1}{\delta}x^\delta$, $0 < \delta < 1$. The assumption of geometric Brownian motion for prices implies that

$$r(t) = r, \quad \sigma(t) = \begin{pmatrix} S_1(t) & & & \\ & S_2(t) & & \\ & & \ddots & \\ & & & S_n(t) \end{pmatrix} \sigma, \quad b(t) = \begin{pmatrix} b_1 S_1(t) \\ b_2 S_2(t) \\ \vdots \\ b_n S_n(t) \end{pmatrix}$$

where r, σ, b are constant matrices. By (3.2.1),

$$\begin{aligned} dm(t) &= \frac{2(1-\delta)r + (2-\delta)||\kappa||^2}{2(1-\delta)^2} m(t)dt + \frac{1}{1-\delta} m(t)\kappa^\top dz(t) \\ &= \left(\frac{r}{1-\delta} + \frac{\delta||\kappa||^2}{2(1-\delta)^2}\right) m(t)dt + \frac{1}{1-\delta} m(t)\kappa^\top dz^Q(t). \end{aligned}$$

Let $v(t) = m(t) \cdot \exp\left\{\frac{\delta}{1-\delta}(r + \frac{1}{2(1-\delta)}||\kappa||^2)(T-t)\right\}$. By Proposition 3.2.1, $\frac{v(t)}{B(t)}$ is a Q - martingale, and

$$dv(t) = rv(t)dt + \frac{1}{1-\delta} v(t)\kappa^\top dz^Q(t). \tag{3.3.1}$$

So, $E^Q\left[[v(T) - K]^+ B(T)^{-1}\right]$ is the price of the call option on $v(t)$ with strike price K. Hence, by the Black - Scholes formula

$$E^Q[[v(T) - K]^+ B(T)^{-1}] = \left(\frac{\rho}{\lambda_0}\right)^{\frac{1}{1-\delta}} e^{\frac{\delta}{1-\delta}(r + \frac{1}{2(1-\delta)}||\kappa||^2)T} \cdot \Phi(\epsilon_1) - Ke^{-rT} \cdot \Phi(\epsilon_2) \tag{3.3.2}$$

where

$$\begin{cases} \epsilon_1 &= \frac{\frac{\delta}{1-\delta}(r+\frac{1}{2(1-\delta)}||\kappa||^2)T+\frac{1}{1-\delta}\ln\frac{\rho}{\lambda_0 K^{1-\delta}}+(r+\frac{1}{2(1-\delta)^2}||\kappa||^2)T}{\frac{1}{1-\delta}||\kappa||\sqrt{T}} \\ \epsilon_2 &= \epsilon_1 - \frac{1}{1-\delta}||\kappa||\sqrt{T} \end{cases}$$

and $\Phi(x)$ is the standard normal cumulative distribution function.

Similarly, $E^Q\left[\left[K^{1-\delta} - \frac{\rho B(T)}{\lambda_0\eta(T)}\right]^+ B(T)^{-1}\right]$ can be evaluated as the price of the put option on the "asset", $\frac{\rho B(t)}{\lambda_0\eta(t)}$, with strike price $K^{1-\delta}$. Since,

$$d\left(\frac{\rho B(t)}{\lambda_0\eta(t)}\right) = \frac{\rho B(t)}{\lambda_0\eta(t)}(rdt + \kappa^\top dz^Q(t)).$$

By the Black-Scholes formula,

$$E^Q\left[\left[K^{1-\delta} - \frac{\rho B(T)}{\lambda_0\eta(T)}\right]^+ B(T)^{-1}\right] = K^{1-\delta}\cdot e^{-rT}\cdot\Phi(-\epsilon_2') - \frac{\rho}{\lambda_0}\cdot\Phi(-\epsilon_1') \qquad (3.3.3)$$

where

$$\begin{cases} \epsilon_1' &= \frac{\ln\frac{\rho}{\lambda_0 K^{1-\delta}}+(r+\frac{1}{2}||\kappa||^2)T}{||\kappa||T} \\ \epsilon_2' &= \epsilon_1' - ||\kappa||\sqrt{T}. \end{cases}$$

We are now able to calculate the optimal λ_0 and K. Equation (3.1.10) becomes

$$\begin{cases} (1-\rho) - \lambda_0 E^Q\left[\left[K^{1-\delta} - \frac{\rho B(T)}{\lambda_0\eta(T)}\right]^+ B(T)^{-1}\right] = 0 \\ K\cdot e^{-rT} + E^Q\left[[v(T)-K]^+ B(T)^{-1}\right] = W_0. \end{cases} \qquad (3.3.4)$$

Substituting (3.3.2) and (3.3.3) into the above equation yields the optimal λ_0 and K.

The investor's wealth at time t is

$$\begin{aligned} W(t) &= B(t)E^Q[W_T B(T)^{-1} \mid \mathcal{F}_t] \\ &= B(t)\cdot K\cdot E^Q[B(T)^{-1}\mid\mathcal{F}_t] + B(t)\cdot E^Q[[v(T)-K]^+ B(T)^{-1}\mid\mathcal{F}_t] \\ &= K\cdot e^{r(t-T)} + C(t, B(t), v(t)), \end{aligned}$$

which is equivalent to holding a call option on the mutual fund and investing the balance in the riskless asset (two fund separation theory applies). Hence, the investors optimal strategy is to replicate this call option.

Theorem 3.3.1. *Let $\tilde{\theta}(t) = (\tilde{\theta}_1(t), \cdots, \tilde{\theta}_n(t))^\top$ be the replicating portfolio of the mutual fund, where $\tilde{\theta}_i^m(t)$ is the number of shares of the ith risky asset. Then*

$$\tilde{\theta}(t) = \frac{v(t)}{1-\delta} I_S^{-1}(\sigma\sigma^\top)^{-1}(b - r\mathbf{1}),$$

where I_S is the diagonal matrix with risky asset prices as the entries, and the optimal investment portfolio of the primary assets is

$$\theta(t) = \Phi(\epsilon(t)) \cdot \tilde{\theta}(t), \tag{3.3.5}$$

where

$$\epsilon(t) = \frac{\ln \frac{v(t)}{K} + (r + \frac{1}{2(1-\delta)}(b - r\mathbf{1})^\top(\sigma\sigma^\top)^{-1}(b - r\mathbf{1}))(T - t)}{\sqrt{(b - r\mathbf{1})^\top(\sigma\sigma^\top)^{-1}(b - r\mathbf{1})(T - t)}}.$$

Furthermore, the optimal terminal wealth is

$$W(T) = K + \left(\frac{\rho e^{rT}}{\lambda_0 \eta(T)}\right)^{\frac{1}{1-\delta}} \tag{3.3.6}$$

whereλ_0 and K is given by (3.3.4).

Proof. Since $\tilde{\theta}(t)$ is the replicating portfolio, then

$$dv(t) = rv(t)dt + \tilde{\theta}(t)^\top I_S \sigma dz^Q(t).$$

By Theorem 2,

$$\tilde{\theta}(t)^\top I_S \sigma = \frac{v(t)}{1-\delta}\kappa^\top$$

which yields

$$\begin{aligned}\tilde{\theta}(t) &= \frac{v(t)}{1-\delta} I_S^{-1}(\sigma^\top)^{-1}\kappa \\ &= \frac{v(t)}{1-\delta} I_S^{-1}(\sigma\sigma^\top)^{-1}(b - r\mathbf{1}).\end{aligned} \tag{3.3.7}$$

The number of units of the mutual fund needed to hedge this call option is equal to $\Phi(\epsilon(t))$ by the Black - Scholes formula. If $(\alpha(t), \theta(t))$ denotes the optimal strategy (in units of primary assets), then (3.3.5) is proved. □

Theorem 3.3.1 indicates that an investor will implement the optimal portfolio by first synthesizing the optimal portfolio in which the worst possible outcome wealth is not included (creating a mutual fund or an index), then calculating the hedging portfolio for a call option on this mutual fund. This two step procedure constructs the optimal investment strategy.

The following two results are implied by the previous discussion.

Corollary 3.3.2. *For an investor with a HARA utility function, there exists a mutual fund such that the optimal investment strategy is equivalent to buying a call option on this mutual fund.*

Corollary 3.3.3. *For an investor with a HARA utility function, the optimal strategy is to invest in a mutual fund and the riskless asset. Furthermore, if all risky assets jointly follow a multi-dimensional geometric Brownian motion, then this mutual fund must have used a fixed mix strategy when it is created.*

By Equation (3.2.1), we observe that, if investors have same risk aversion and same risk skewness, their investment strategy should be identical.

An illustrating Example. We now examine the sensitivity of the investment strategy as the downside risk control intensity changes. Consider a two-asset market, one riskless and one risky asset. The riskless asset evolves with an annual continuous compound rate $r = 0.05$, and the risky asset has an instantaneous mean rate $b = 0.10$ and instantaneous volatility $\sigma = 0.3$. With this data, the market price for risk

$$\eta(t) = e^{-0.167z(t)-0.014t}.$$

Suppose the investment horizon $T = 1$. For computational simplification, we use logarithmetic utility function as investor's objective function, though the theory and the computational method apply to broad utility functions. Then, the investor's objective function for the optimization model is

$$\rho \ln W(T) + (1-\rho)\ln K.$$

Hence the process $m(t) = \frac{\rho B(t)}{\lambda_0 \eta(t)}$ is a discounted Q-martingale as proved in (3.2.1). The optimal λ_0 and K satisfy [cf. (3.3.2)-(3.3.4)]

$$\begin{cases} 1-\rho-\lambda_0\left[Ke^{-rT}\Phi(-\epsilon_2) - \frac{\rho}{\lambda_0}\Phi(-\epsilon_1)\right] = 0 \\ Ke^{-rT} + \frac{\rho}{\lambda_0}\Phi(\epsilon_1) - Ke^{-rT}\Phi(\epsilon_2) = W_0 \end{cases}$$

which implies that

$$\lambda_0 = \frac{1}{W_0},$$

and K satisfies that

$$K = W_0 e^{rT}\left(\frac{1-\rho\Phi(\epsilon_1)}{1-\Phi(\epsilon_2)}\right).$$

Note that ϵ_1 and ϵ_1' are identical when $\delta \to 0$ and are dependent on K.

An investor may be interested in examining the expected return for the choice of different downside control intensities. So, a problem arises as how to choose an optimal ρ that will maximize one's performance under some risk measure accommodate the control of downside losses.

Value at Risk, a popular measure that has been used in practice, will be discussed in Chapter 4 in detail. It is the maximum loss with respect to market expected return for a small tolerance α in probability. α is usually chosen between 0.05 or 0.01. Hence, the VaR of W is

$$VaR(W) := E[W] - K_W$$

where K_W is the maximum such that

$$\Pr[W \leq K_W] \leq \alpha.$$

Table 3.3 describes the performances with varying downside control intensities and the tolerance in probability is chosen to be 0.05.

The Sharpe ratio, as a measure of performance, is the ratio of excess expected return over the standard deviation. However, this measure considers both outperformance and underperformance as risk. Here we define a new performance measure that uses VaR and quantifies only the downside losses as risk.

$$\tau := \frac{E[W(T)] - W_0 e^{rT}}{VaR(W(T))}$$

Figure 3.1 depicts the performance with varying control intensity. By observation, the optimal ρ which maximizes τ is about 0.83.

Table 3.1: The Optimal K, $E[W(T)]$ and VaR

ρ	K	$E[W(T)]$	VaR
0.8	1.0441492	1.054342	0.010193
0.81	1.0427573	1.054833	0.012075
0.82	1.0411401	1.055292	0.014152
0.83	1.0392648	1.057022	0.017757
0.84	1.0370947	1.056448	0.019353
0.85	1.0345873	1.056281	0.021694
0.86	1.0316926	1.059291	0.027598
0.87	1.028351	1.058207	0.029856
0.88	1.024491	1.061408	0.036917
0.89	1.0200249	1.060135	0.04011
0.9	1.014843	1.063291	0.048448
0.91	1.0088053	1.063271	0.054466
0.92	1.0017281	1.064663	0.062935
0.93	0.9933616	1.064165	0.070803
0.94	0.983353	1.068967	0.085614
0.95	0.9711765	1.06948	0.098303
0.96	0.9559845	1.070905	0.114921
0.97	0.9362649	1.073707	0.137442
0.98	0.9088308	1.074417	0.165587
0.99	0.8643086	1.076517	0.212208
1	0.4884128	1.08264	0.270748

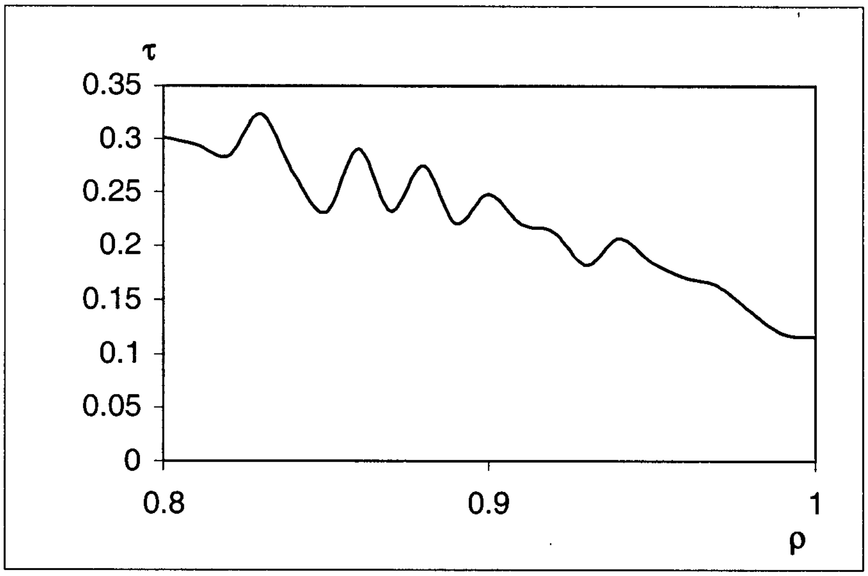

Figure 3.1: The Performance with Varying Control Intensity

Chapter 4

Risk Neutral Excess Return

Merton (1969, 1971) and Karatzas and Shreve (1986), among others, analyze this problem using complete market assumptions and a dynamic programming approach. A more elegant way of dealing with this type of problem is the use of martingale representation and non-smooth analysis for incorporating constraints on the state variables, for example, Harrison and Pliska (1981), and Cox and Huang (1989), Grossman and Zhou (1996) and Basak (1995). Using simple utility function assumptions and asset price processes, closed form solutions are available for fairly standard security markets. However, for investors with liability streams, explicit risk control approaches, such as the worst possible outcome, or VaR, etc., seem more appropriate. MacLean and Ziemba (1992, 1999) discuss the tradeoff between growth and security using the log utility capital growth model. Zhao and Ziemba (2000) introduce a reward function on the portfolio worst payoff that represents investors' risk attitude to downside losses in a discrete time model. The portfolio target is an endogenous choice variable determined by the risk aversion and investment opportunities. A problem is how to introduce liabilities for asset allocation. Mean-variance models are unable to model them in a natural way. However, portfolio insurance strategies provide natural ways for protecting downside losses. One may buy a put option on a reference portfolio with strike price K that matches one's liability, plus holding the portfolio. These strategies are called option based portfolio insurance. Another way is synthesizing a put option by creating a portfolio that has a "floor" using existing market securities. A representative of these strategies is Constant Proportional Portfolio Insurance; see Black and Perold. Black and Jones (1987) discuss a simple, flexible approach to portfolio insurance for pension plans. In this chapter, we devise a set of dynamic strategies, which will provide a similar payoff as a portfolio insurance strategy does. A reward function from achieving such a "floor" will be introduced to characterizes the investor's risk attitude. The investor may wish to maximize K as well as an overall potential payoff for all scenarios at the end of planning horizon.

4.1 Constant Proportional Portfolio Insurance

In asset allocation, if the risky assets increase in value, the proportions of the portfolio they comprise are also likely to increase. One must decide how to rebalance the portfolio in response to such changes. Dynamic strategies are explicit rules for doing so. A portfolio strategy for downside protection is portfolio insurance. Dybvig (1999) discusses asset allocation strategies linking to the spending rules in a way that preserves spending power in down markets but participates significantly in up markets. Black and Perold (1992) provide a theory of constant proportional portfolio insurance (CPPI), which maintains the portfolio's risk exposures to constant multiple of the excess of wealth over a floor. For expositional convenience, some of their results are reviewed here and all discussions in the remainder of this section are restricted to the two-asset case. At time t, the portfolio weight in dollar amount is

$$\pi(t) = m(W(t) - F(t)), \tag{4.1.1}$$

where m is a fixed multiplier and $F(t)$ is a deterministic function of t. To implement a CPPI strategy, the investor selects m and a quantity $F(t)$ below which the investor does not want the portfolio to fall. Assume $F(t) = Fe^{rt}$ grows at the riskless rate and is initially less than the total assets. From (4.1.1), one observes that a CPPI strategy sells stocks as they fall and buys stocks as they rise to maintain a constant proportion between the asset holdings and the "cushion" $W(t) - F(t)$. Let $W_{PI}(t)$ denote the portfolio value at time t, then, by Equation (1.1.4),

$$\pi(t) = \pi(0)e^{m\sigma dz^Q(t)+(r-\frac{1}{2}m^2\sigma^2)t} \tag{4.1.2}$$

and, therefore,

$$\begin{aligned} W_{PI}(t) &= \frac{1}{m}\pi(t) + F(t) \\ &= F(t) + (W(0) - F(0))e^{(1-m)(r+\frac{1}{2}m\sigma^2)t}\left(\frac{X(t)}{X(0)}\right)^m . \end{aligned} \tag{4.1.3}$$

A Buy and Hold (BH) strategy allocates wealth at the beginning of the planning horizon and holds the portfolio to the end with no transactions between periods except dividends are reinvested as received in the same asset at the then prevailing market price. A Fixed Mix (FM) strategy allocates wealth according to a preset investment policy such that each asset represents an identical proportion of the wealth at the beginning of each period. These two most simple strategies for asset allocation are special cases of portfolio insurance. A BH is CPPI with the multiplier equal to one and a floor equal to the value invested in the riskless asset, while an FM is CPPI with a zero floor. Let $W_{BH}(t)$ and $W_{FM}(t)$ represent the portfolio values for BH and FM strategies, then

$$W_{BH}(t) = F_0 e^{rt} + (W(0) - F_0)\left(\frac{X(t)}{X(0)}\right) \tag{4.1.4}$$

and

$$W_{FM}(t) = W(0)e^{(1-u)(r+\frac{1}{2}u\sigma^2)t}\left(\frac{X(t)}{X(0)}\right)^u, \tag{4.1.5}$$

where F_0 is the initial amount invested in the riskless asset for the Buy and Hold strategy, and u is the initial proportional weight in the risky asset for the Fixed Mix strategy.

4.2 The Risk Neutral Excess Return Strategy

While a CPPI strategy is characterized by the cushion between the total assets and the floor, are there any relations between the portfolio return and asset weights? We are looking for a strategy that relates portfolio weights to the change of the asset returns in a similar way. The rate of return of the risky asset can be viewed as a sum of riskless return and a risk neutral return

$$\frac{dX(t)}{X(t)} = rdt + \sigma dz^Q(t)$$

which implies that

$$\sigma z^Q(t) = \ln\left(\frac{X(t)}{X(0)}\right) + \frac{1}{2}\sigma^2 t - rt. \tag{4.2.1}$$

The quantity $\ln\left(\frac{X(t)}{X(0)}\right) + \frac{1}{2}\sigma^2 t$ is the risk compensated return of the risky asset at time t. The expected value of the risk compensated return is equal to the instantaneous rate of return for each asset under a risk neutral probability (see Harrison and Kreps (1979) for a definition of risk neutral probability). The difference between the risk compensated rate of return and the riskless return is called the *risk neutral excess return*, since its expected value under the risk neutral probability is 0. The term

$\frac{1}{2}\sigma^2 t$ can be interpreted as the risk premium for transferring a gamble. To control downside losses, we require that the discounted relative changes of the portfolio weight to the initial wealth be proportional to the changing risk neutral excess rate in the same asset. Our portfolio strategy $\pi(t)$ is chosen to satisfy the following differential equation

$$d\pi(t) = r\pi(t)dt + \alpha W(0)e^{rt}\,\sigma dz^Q(t) \qquad \alpha \geq 0 \tag{4.2.2}$$

which has the general solution

$$\frac{e^{-rt}\pi(t)}{W(0)} = \alpha(\sigma z^Q(t) + \beta). \tag{4.2.3}$$

As a comparison to the CPPI strategy, this strategy focuses on the ratio of the changing discounted dollar amount in the risky assets to the initial portfolio value and sets it equal to multiple of the risk neutral excess return plus some constant level. The multiple α and the constant level β is determined by the investor's risk aversion. We call such an approach a Risk Neutral Excess Return (RNER) strategy. The initial proportion of the portfolio in the risky asset equals $\alpha\beta$. This strategy is a portfolio insurance strategy as it is characterized by (4.2.3) as an approach of buying high and selling low to achieve a deterministic floor.

Substituting (4.2.3) in the portfolio dynamics, Equation (1.1.4), yields the portfolio value

$$\begin{aligned} e^{-rt}W(t) &= W(0) + W(0)\int_0^t (z^Q(s)\sigma + \beta)\alpha\sigma dz^Q(s) \\ &= W(0) + W(0)\int_0^t (z^Q(s)\alpha\sigma^2 + \alpha\beta\sigma)dz^Q(s) \\ &= W(0)\left(1 + \frac{1}{2}\alpha\sigma^2 z^Q(t)^2 - \frac{1}{2}\alpha\sigma^2 t + \alpha\beta\sigma z^Q(t)\right). \end{aligned} \tag{4.2.4}$$

Hence, the portfolio value is a quadratic function of $\sigma z^Q(t)$, the risk neutral excess return at time t. We will compare the performances of these four strategies under the Sharpe ratio and a measurement using VaR later in this chapter.

This strategy is applicable for any asset price models as long as the volatility matrix σ of the asset prices are known. In that case, investors can choose the α and β which may not be "optimal" in any sense.

4.3 The Optimization Model

In asset/liability management models in discrete time, the investment objective can be chosen to maximize the expected portfolio value less penalties for targets not met;

see Cariño, Ziemba et al (1994, 1998ab). This model is equivalent to a piecewise linear concave utility risk averse maximization problem if the penalty is a piecewise linear convex function of the shortfall. The continuous time versions of these models have not been well developed in the literature so far. Browne (1997) discusses an investment model for survival and growth with a liability stream and relates the problem to portfolio insurance. Also Browne (1999) studies problems that maximize the probability of reaching a giving wealth level by a finite horizon and relates the problem to the pricing of a digital option. We look for an alternative approach to achieve risk aversion without using a penalty function. In most asset allocation models, maximizing expected asset value is a primary objective, but the dispersion among scenarios brings large potential losses to the portfolio.

How do we control this risk? One way as in a mean-variance model is to adjust the expected value by a measure of dispersion. Technically, it is easy to handle, especially for the static model, but this lacks control power, because the measures chosen are usually based on the first two moments and the potential large losses still exist. The VaR approach has been implemented to address this issue. We utilize a new approach to measure risk – *reward on minimum subsistence.*

Definition 4.3.1. For an $\mathcal{F}_t$ measurable random variable Y, the worst payoff of Y is defined to be

$$Y^L = \sup\{k \in \Re; \Pr\{Y \geq k\} = 1\}, \tag{4.3.1}$$

that is, Y^L is the essential lower bound of Y. If Y is unbounded below, then Y^L is $-\infty$.

As an extension of utility based models, investors are also concerned about the minimum subsistence $W^L(T)$ of investment at the end of horizon. Assigning a reward to $W^L(T)$, characterized by a concave increasing function f, defined on $\Re \cup \{\infty\}$, determines the preference or risk preference between the expected return and the minimum subsistence in all scenarios. Investors are trying to push up the k as much as possible until they are satisfied with the level of the expected end of horizon wealth. Achieving a bigger k is at the cost of reducing the expected portfolio return. The dynamics of the portfolio value is derived as (1.1.4). The stochastic control model is

$$\begin{aligned} \max_{\pi(t)\in\mathcal{U}_l} \quad & E\left[W(T)\right] + f\left(W^L(T)\right) \\ s.t. \quad & e^{-rt}W(T) = W(0) + \int_0^t e^{-rs}\pi(s)'\sigma dz^Q(s), \forall t \in [0,T], \end{aligned} \tag{4.3.2}$$

where $\mathcal{U}_l$ is the set of preset admissible controls (strategies). Before discussing the solvability of the model, we discuss some of its properties.

Definition 4.3.2. A function $f(x)$ is called superior to $g(x)$, denoted by $f(x) \succeq g(x)$, if $f(x) \geq g(x)$ and $f'(x) \geq g'(x)$, where primes denote the first order derivatives.

Theorem 4.3.1. *Let $f(x)$ and $g(x)$ be concave increasing functions, and $W_f(T), W_f^L$ and $W_g(T), W_g^L$ be the corresponding optimal solutions to (4.3.2).*

(i) *If $f(x) \succeq g(x)$, then $W_f^L(T) \geq W_g^L(T)$, $E[W_f(T)] \leq E[W_g(T)]$ and $E[W_f(T)] + f(W_f^L(T)) \geq E[W_g(T)] + g(W_g^L(T))$.*

(ii) *If $h(x) = \eta f(x) + (1-\eta)g(x), 0 < \eta < 1$, then*

$$\begin{aligned} E[W_h(T)] + h(W_h(T)) \leq \eta(E[W_f(T)] + f(W_f^L(T))) \\ +(1-\eta)(E[W_g(T)] + g(W_g^L(T))). \end{aligned}$$

Proof. We suppress the time "T" in the proof. By optimality,

$$\begin{aligned} E[W_f] + g(W_f^L) \leq E[W_g] + g(W_g^L) \\ E[W_g] + f(W_g^L) \leq E[W_f] + f(W_f^L). \end{aligned} \tag{4.3.3}$$

Hence,

$$f(W_g^L) - f(W_f^L) \leq E[W_f] - E[W_g] \leq g(W_g^L) - g(W_f^L).$$

Since $f'(x) \geq g'(x), \forall x \in \Re$, *i.e.* $f(x) - g(x)$ is an increasing function, so

$$W_f^L \geq W_g^L.$$

Using (4.3.3) and the fact that $g(x)$ is an increasing function yields

$$E[W_f] \leq E[W_g].$$

If $f(x) \geq g(x)$ and $f'(x) \geq 0$, then

$$E[W_g] + g(W_g^L) \leq E[W_g] + f(W_g^L) \leq E[W_f] + f(W_f^L).$$

(ii). Let $h(x) = \eta f(x) + (1-\eta)g(x)$, then

$$h(W_h^L) = \eta f(W_h^L) + (1-\eta)g(W_h^L).$$

By optimality,

$$\begin{aligned} E[W_f] + f(W_f^L) \geq E[W_h] + f(W_h^L) \\ E[W_g] + g(W_g^L) \geq E[W_h] + g(W_h^L). \end{aligned}$$

So,

$$\eta(E[W_f] + f(W_f^L)) + (1-\eta)(E[W_g] + g(W_g^L)) \geq E[W_h] + h(W_h^L).$$

$\square$

Theorem 4.3.1 indicates how the two objectives, expected terminal wealth and the reward from the initial minimum subsistence are related when the function f changes. The optimal expected terminal value $E[W(T)]$, optimal certainty payoff $W^L(T)$, and the optimal value function $E[W(T)] + f(W^L(T))$ are decreasing, increasing and increasing, respectively, as the function f's superiority increases. The optimal value function $E[W(T)] + f(W^L(T))$ is convex in f in the sense that the optimal value function is defined over the set of concave increasing functions.

Solving (4.3.2) is not as easy, in general, as solving a stochastic control problem in which we can choose the utility function form. However, exogenously specifying the control level of the downside losses, Basak (1995) and Grossman and Zhou (1996) apply the martingale method along with the theory of nonsmooth analysis to study the equilibrium asset prices. Also, Zhao and Ziemba (2000) apply this method in discrete time to characterize the optimal investment portfolio in the context of utility maximization with the control for the downside losses. The problem presented here has an irregular objective function $E[W(T)] + f(W^L(T))$. $W^L(T)$ is the infimum of the probability support of $W(T)$. However, if we are primarily concerned about how to find a "good" strategy that meets our objective, then we can reduce the difficulty of the problem by restricting our control space. We shall choose the control set $\mathcal{U}_l$ to be the set of all RNER strategies as defined in last section.

Theorem 4.3.2. *For the standard complete market* (1.1.1), *there exists an optimal control* $\pi(\cdot) \in \mathcal{U}_l$ *that solves* (4.3.2). *Let* $\alpha = (\alpha_1, \cdots, \alpha_n)$ *and* $\beta = (\beta_1, \cdots, \beta_n)$ *be the optimal solutions, then*

$$\beta_i = \frac{1}{2}\sqrt{(b_i - r)^2T^2 + 4\sigma^2 T} - (b_i - r)T, \quad i = 1, 2, \cdots, n \tag{4.3.4}$$

and α *is given by*

$$(b - r\mathbf{1})T - f_x(W(0)e^{rT}(1 - \frac{1}{2}tr(\sigma\sigma' I_\alpha)T - \frac{1}{2}\beta' I_\alpha \beta)) \cdot \beta = 0, \tag{4.3.5}$$

where $tr()$ *is the trace function,* I_α *is the diagonal matrix with* α_i *as entries, and* $f_x(\cdot)$ *is the first order derivative of function* f. *Moreover, the optimal wealth and the worst possible outcome wealth are*

$$\begin{aligned} W(t) &= W(0)e^{rt}\left(1 + \frac{1}{2}z^Q(t)'\sigma' I_\alpha \sigma z^Q(t) - \frac{1}{2}tr(\sigma' I_\alpha \sigma)t + \beta' I_\alpha \sigma z^Q(t)\right) \\ W^L(T) &= W(0)e^{rT}\left(1 - \frac{1}{2}tr(\sigma' I_\alpha \sigma)T - \frac{1}{2}\beta' I_\alpha \beta\right). \end{aligned} \tag{4.3.6}$$

Equation (4.2.2) implies that there exist a constant vector $\beta = (\beta_1, \cdots, \beta_n)'$ and a diagonal matrix $I_\alpha = Diag(\alpha_1, \cdots, \alpha_n), \alpha_i \geq 0$ such that

$$e^{-rt}\pi(t) = W(0)I_\alpha(\sigma\tilde{B}(t) + \beta). \tag{4.3.7}$$

The wealth equation becomes

$$\begin{aligned} e^{-rt}W(t) &= W(0) + W(0)\int_0^t (\tilde{B}(s)'\sigma' + \beta')I_\alpha\sigma d\tilde{B}(s) \\ &= W(0) + W(0)\int_0^t (\tilde{B}(s)'\sigma' I_\alpha\sigma + \beta' I_\alpha\sigma)d\tilde{B}(s) \\ &= W(0)\left(1 + \frac{1}{2}\tilde{B}(t)'\sigma' I_\alpha\sigma\tilde{B}(t) - \frac{1}{2}tr(\sigma' I_\alpha\sigma)t + \beta' I_\alpha\sigma\tilde{B}(t)\right) \end{aligned} \tag{4.3.8}$$

where $tr()$ is the trace function. $W^L(T)$ can be determined from

$$\begin{aligned} e^{-rT}W^L(T) &= \inf_{y\in\Re^n}\left\{W(0)(1 + \frac{1}{2}y'\sigma' I_\alpha\sigma y - \frac{1}{2}tr(\sigma' I_\alpha\sigma)T + \beta' I_\alpha\sigma y)\right\} \\ &= \inf_{y\in\Re^n}\left\{W(0)(1 - \frac{1}{2}tr(\sigma' I_\alpha\sigma)T) + \frac{1}{2}(\sigma y + \beta)' I_\alpha(\sigma y + \beta) - \frac{1}{2}\beta' I_\alpha\beta)\right\} \\ &= W(0)(1 - \frac{1}{2}tr(\sigma' I_\alpha\sigma)T - \frac{1}{2}\beta' I_\alpha\beta). \end{aligned}$$

The expected value $E[W(T)]$ can be calculated as

$$\begin{aligned} e^{-rT}E[W(T)] &= W(0) + W(0)E\left[\int_0^T (\tilde{B}(t)'\sigma' + \beta')I_\alpha\sigma\theta dt\right] \\ &= W(0) + W(0)\int_0^T (\sigma\theta)'t + \beta')I_\alpha\sigma\theta dt \\ &= W(0)(1 + \frac{1}{2}(\sigma\theta)' I_\alpha(\sigma\theta)T^2 + \beta' I_\alpha(\sigma\theta)T) \\ &= W(0)\left(1 + \frac{1}{2}(b - r\mathbf{1})' I_\alpha(b - r\mathbf{1})T^2 + \beta' I_\alpha(b - r\mathbf{1})T\right). \end{aligned}$$

For a continuously differentiable function f, Model (4.3.2) becomes

$$\begin{aligned} \sup_{I_\alpha\in\Re^n\times\Re^n, \beta\in\Re^n}\Big\{ & W(0)e^{rT}(1 + \frac{1}{2}(b - r\mathbf{1})' I_\alpha(b - r\mathbf{1})T^2 + \beta' I_\alpha(b - r\mathbf{1})T) \\ & + f(W(0)e^{rT}(1 - \frac{1}{2}tr(\sigma' I_\alpha\sigma)T - \frac{1}{2}\beta' I_\alpha\beta))\Big\} \end{aligned} \tag{4.3.9}$$

The first order conditions are

$$\begin{cases} (b_i - r)T - f_x(A) \cdot \beta_i = 0, \\ (b_i - r)^2T^2 + 2\beta(b_i - r)T - f_x(A)(\sigma_i'\sigma_i T + \beta_i^2) = 0, \end{cases} \tag{4.3.10}$$

where $i = 1, \cdots, n$, $f_x(\cdot)$ is the first order derivative, and $A = W(0)e^{rT}(1 - \frac{1}{2}tr(\sigma\sigma' I_\alpha)T - \frac{1}{2}\beta' I_\alpha \beta)$. Solving (4.3.10) for α and β proves Theorem 4.3.2. □

It is observed from (4.3.4) that the optimal β_i's are independent of investors' risk aversion represented by the function f and are proportional to the risk premium of each risky asset, while the optimal α_i's determined by (4.3.5) are the discounted changes of the portfolio weight to the risk neutral excess return. Assuming lognormality, the asset prices can be represented as a function of the underlying Q - Brownian motion, therefore, the portfolio value and the worst possible outcome wealth given in Equation (4.3.6), are analytically written in the same way. One can calculate the corresponding means and standard deviations and other moments.

Theorem 4.3.2 provides a direct way of implementing this strategy. By assumption, σ is invertible, hence, there is a one to one corresponding relation between the realized asset prices, therefore the realized wealth, and the underlying state of the world. The strategy given in Theorem 4.3.2 can be dynamically implemented by observing current market prices. We can use any available asset price model to value and predict future asset returns, namely, appropriate probability distribution for asset future movements. In a lognormal world, this amounts to "calculating " the instantaneous mean rate vector b and the volatility matrix σ for a given riskless interest rate. Then we can implement this model by specifying the "continuous" rebalance points through the investment horizon.

Example. An investor has an endowment of \$1 and a certainty reward function $f(x) = x^\delta, 0 < \delta < 1$. There are two assets, one riskless (T-Bills) and one risky (the S&P 500). Market parameter estimates are $b = 0.12$, and $\sigma = 0.30$. Assume that these parameters are fixed and stationary through the planning horizon $T = 1$ (a year). The riskless rate for this period is $r = 0.06$. Setting $\delta = 0.18$ and by (4.3.4) and (4.3.5), the optimal α and β are

$$\alpha = 2.578 \quad \text{and} \quad \beta = 0.271.$$

Hence, the investor has about $\alpha\beta = 70\%$ of the wealth invested in the risky asset, and the balance in the riskless asset. After the initial investment, the portfolio has to be "continuously" adjusted by observing the stock price. Since the analytical solutions (Equation (4.3.6)) are available under the lognormal assumption, the mean and the volatility of the terminal wealth can be directly calculated. To provide a general methodology, we use Monte Carlo simulation instead. For this example, the actual portfolio mean is 11.14% with a standard deviation 32.32%. Assume there are 1000 rebalance points. Using (4.2.1) computes $\sigma z^Q(t)$ and the portfolio holdings $\pi(t)$ at time t. A sample distribution of 2000 paths is used for calculating the terminal portfolio value. The expected portfolio return is 11.65% with a standard deviation of 33.48%, while the stock market has an expected return of 12.93% with standard deviation 35.62%. The differences among the statistics are due to the sampling error and rebalances of the portfolio weights. The downside control ability is represented by a market realization as in Figure 4.1.

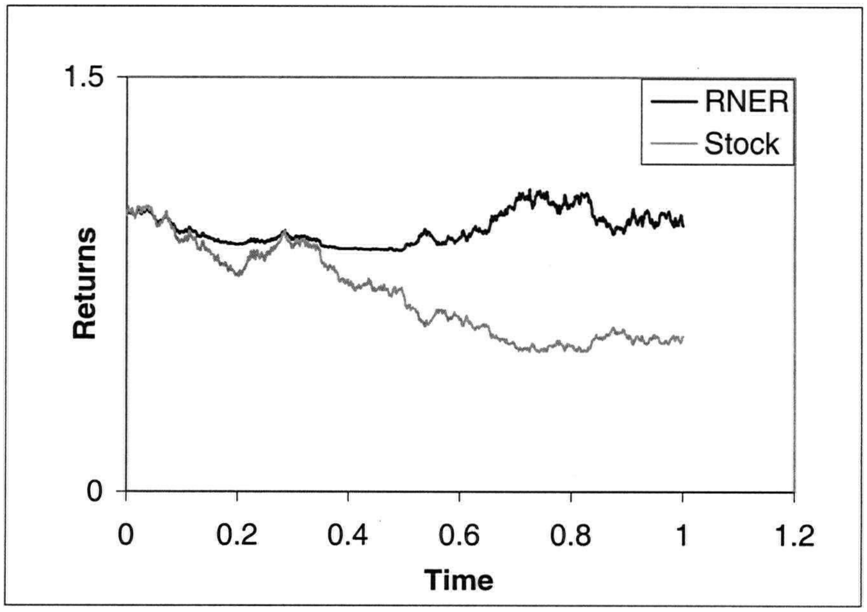

Figure 4.1: A Sample Path of the Portfolio Value Over Time

Based on a measure of performance defined in the next section using a popular risk measure, VaR, this downside risk control strategy, compared to standard strategies such as BH, FM, and CPPI, performs well. This analysis is presented in the next section.

4.4 Comparisons under VaR and the Sharpe Ratio Measures

Using a reward function for risk aversion makes it easy to calculate the optimal policy, but an appropriate function is difficult to find. We focus now on constructing return/risk efficient frontiers under the risk measures, Value at Risk (VaR) and the Sharpe ratio, for the four investment strategies discussed in section 2 of this chapter. Without loss of generality, our discussions are confined to the two-asset case.

The BH and FM strategies are two representative strategic asset allocation models. In these models, risk is defined to be the volatility (standard deviation) of the portfolio at the planning horizon. The optimal asset allocation is determined under a framework of the risk/return tradeoff that is to maximize the expected return given a certain level of risk. Performance is measured by the Sharpe ratio which is valid assuming that the portfolio return from these two strategies are normally distributed [1]. For a typical example, we show under the Sharpe ratio that both BH and FM outperform CPPI and RNER, but the results reverse if VaR is used as performance measurement. Leland (1999) shows that the β measure is not suitable when investors have a dynamic model with skewed investment portfolios, then a related measure must be used. The VaR has been popularly implemented as a measure of risk for investment management. A simple version of VaR is *the market loss that is not exceeded at a given confidence interval.* If W is the portfolio payoff at the end of planning horizon, then the VaR of W at confidence level $1 - \alpha$ is

$$VaR(W) = E[W] - sup\{K \in \Re; \Pr[W \leq K] \leq \alpha\}. \tag{4.4.1}$$

This measure provides a new approach to deal with downside risk. The probability that the portfolio will lose more than the VaR is α. Methods of calculating VaR based on the mean-variance approximation are not suitable when the asset prices follow a multivariate geometric Brownian motion, because the returns (arithmetic or geometric) of dynamic portfolios are generally neither normally nor lognormally distributed. Furthermore, for a positively skewed curve (the density function), the variance is mainly the over-performance in the right tail which should not be considered as "risk" at all. See Jorion (1996), Hull and White (1998), and Koedijk et al (1998) for discussions of VaR estimation. We analytically derive the VaRs for the four asset allocation strategies: BH, FM, CPPI and RNER. RNER is an alternative

[1] Hodges (1998) shows how to utilize a measure called the generalized Sharpe ratio for non-normally distributed assets. But in that theory an exponential utility function assumption is required for it to reduce to the Sharpe ratio in the normal distribution case - another suspect assumption.

strategy to CPPI with a similar mean but lower standard deviation. Under VaR, the RNER strategy outperforms both BH and FM strategies but is slightly inferior to CPPI.

4.4.1 Implementing VaR for Optimization Models

Let U be the utility function (strictly increasing and concave) and $W = g(x, \omega)$ be the terminal wealth with decision x and scenario (or state) ω. The investors' problem with VaR as an additional constraint can be formulated as

$$\begin{aligned} \max_{x} \quad & E[U(W)] \\ s.t. \quad & W = g(x, \omega) \\ & \Pr(W \geq W_0) \geq \alpha \end{aligned} \tag{4.4.2}$$

where $W_0 > 0$ is the target wealth and α is the security level. Although VaR has been a popular risk measure in measuring how the investor's portfolio incurs to the market uncertainty, research on implementing this risk measure is not popular due to difficulty of solution to a nonconvex problem by introducing this constraints.

Let $y(\omega)$ be a random variable, $0 \leq y(\omega) \leq 1$. The revised problem can be written as

$$\begin{aligned} \max_{x, y(\omega)} \quad & E[U(W \cdot z(\omega))] \\ s.t. \quad & W = g(x, \omega) \\ & y(\omega)(W - W_0) \geq 0 \\ & E[y(\omega)] \geq \alpha \\ & 0 \leq y(\omega) \leq 1 \qquad \forall \omega \in \Omega \end{aligned} \tag{4.4.3}$$

where

$$z(\omega) = \begin{cases} y(\omega) & \text{if } y(\omega) > 0 \\ 1 & \text{if } y(\omega) = 0 \end{cases} \tag{4.4.4}$$

Let x^* and $y(\omega)^*$ be the optimal solution to model (4.4.3) and W^* be the wealth at optimal.

Lemma 4.4.1. *For* $\forall \omega \in \Omega$, *if* $y(\omega)^* > 0$, *then* $y(\omega)^* = 1$.

Proof. If $y(\omega)^* < 1$ and $y(\omega)^* > 0$ for some $\omega \in \Omega$, then $W^* \geq W_0 > 0$, therefore $W^* > W^* \cdot z(\omega)^*$. Define

$$y(\omega)^{**} = \begin{cases} 1 & \text{if } y(\omega)^* > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.4.5}$$

Then $(x^*, y(\omega)^{**})$ is a feasible solution to Model (4.4.3). In this case $z(\omega)^{**} \equiv 1$, and therefore

$$E[U(W^* \cdot z(\omega)^{**})] = E[U(W^*)] > E[U(W^* \cdot z(\omega)^*] \tag{4.4.6}$$

which contradicts the optimality of $(x^*, y(\omega)^*)$.

Lemma 4.4.2.

$$P(W^* \geq W_0) \geq E[y(\omega)^*] \tag{4.4.7}$$

Proof. By Lemma 4.4.1 and the fact that

$$(y(\omega)^* > 0) \Longrightarrow (W \geq W_0) \tag{4.4.8}$$

we have

$$\begin{aligned} E[y(\omega)^*] &= \Pr(y(\omega)^* = 1) \\ &\leq \Pr(W \geq W_0) \end{aligned} \tag{4.4.9}$$

However, the inverse direction is not necessarily true.

Lemma 4.4.3. *At optimal, $z(\omega) \equiv 1$, for $\forall \omega \in \Omega$.*

Proof. Definition of $z(\omega)$ and Lemma 4.4.1.

Theorem 4.4.4. *If $(x^*, y(\omega)^*)$ is an optimal solution to model (4.4.3), then x^* is an optimal solution to Model (4.4.2). Conversely, if x^* is an optimal solution to Model (4.4.3), then there exists a $y(\omega)^*$ (may depend on x^*)such that $(x^*, y(\omega)^*)$ is an optimal solution to Model (5.1.1).*

Proof. If $(x^*, y(\omega)^*)$ is an optimal solution to Model (4.4.3), then by Lemma 4.4.2, $\Pr(W^* \geq W_0) \geq \alpha$, and therefore x^* is a feasible solution to Model (4.4.2). Now that x is any feasible solution to Model (4.4.2), then $(x, \mathbf{1}_\Omega)$ is a feasible solution to Model (4.4.3). This implies that

$$E[U(W)] = E[U(W \cdot z(\omega))] \leq E[U(W^* \cdot z(\omega)^*)] \tag{4.4.10}$$

By Lemma 4.4.3, $z(\omega)^* \equiv 1$ at optimal. Therefore, x^* is an optimal solution to Model (4.4.2).

Conversely, if x^* is the optimal solution to Model (4.4.2), then, we need only to show that $(x^*, \mathbf{1}_\Omega)$ is an optimal solution to Model (4.4.3). By the above argument, $(x^*, \mathbf{1}_\Omega)$ is feasible. Now that $(x^{**}, y(\omega)^*)$ is any optimal solution, x^{**} is an optimal solution to Model (4.4.2) by the first part of the theorem. Also we have,

$$E[U(W^{**} \cdot z(\omega)^*)] \leq E[U(W^{**})] = E[U(W^*)] \tag{4.4.11}$$

The first inequality is due to $0 \leq z(\omega)^* \leq 1$ and the second is due to the optimality of Model (4.4.2). It is implied that $(x^*, \mathbf{1}_\Omega)$ is optimal for Model (4.4.3). This completes the proof of Theorem 4.4.4. □

4.4.2 Calculation of the Mean and Volatility

Expected return and volatility are directly related to the calculation of the VaR and the Sharpe ratio, so we need to know how to compute them for a specific policy.

Buy and Hold

Buy and Hold is static in the sense that assets are allocated at the beginning period and held until the end of the horizon without transactions except dividend reinvestments. Let $W_{BH}(t)$ be the value of the portfolio at time t and u be the proportion of the total wealth allocated to the risky asset at the beginning, then the terminal portfolio value W_{BH} is

$$\begin{aligned} W_{BH} &= W(0)(1-u)e^{rT} + W(0)u \cdot \frac{X(T)}{X(0)} \\ &= W(0)((1-u)\, e^{rT} + u\, e^{\sigma z(T) + (b - \frac{1}{2}\sigma^2)T}). \end{aligned} \tag{4.4.12}$$

The expected final value $E[W_{BH}]$ and the volatility $V(W_{BH})$ are

$$\begin{aligned} E[W_{BH}] &= W(0)e^{rT}(1 + u(e^{(b-r)T} - 1)) \\ V[W_{BH}] &= uW(0)e^{bT}\sqrt{(e^{\sigma^2 T} - 1)}. \end{aligned} \tag{4.4.13}$$

The Fixed Mix

The Fixed Mix strategy requires one to rebalance the portfolio "continuously". Investors preset an appropriate portfolio mix u among the asset categories. Then, portfolio weights are balanced by selling assets with high past returns to buy assets with low past returns; the opposite of portfolio insurance. Let v be the proportion of wealth placed in the risky asset and $W_{FS}(t)$ the portfolio value at time t. The wealth Equation (1.1.4) becomes

$$dW_{FM}(t) = r\, W_{FM}(t)dt + vW_{FM}(t)\sigma dz^Q(t).$$

The terminal payoff W_{FM} is

$$\begin{aligned} W_{FM} &= W(0)e^{v\sigma z^Q(T)+(r-\frac{1}{2}v^2\sigma^2)T} \\ &= W(0)e^{v\sigma z(T)+v(b-r)T+(r-\frac{1}{2}v^2\sigma^2)T}. \end{aligned} \tag{4.4.14}$$

The expected value $E[W_{FM}]$ and the volatility $V[W_{FM}]$ of the terminal portfolio are

$$\begin{aligned} E[W_{FM}] &= W(0)e^{(vb+(1-v)r)T} \\ V[W_{FM}] &= W(0)e^{vbT+(1-v)rT}\sqrt{(e^{v^2\sigma^2T}-1)}. \end{aligned} \tag{4.4.15}$$

The CPPI

Let $\pi(t)$ be the wealth in dollar amount placed in the risky asset. The floor is given by Fe^{rt} at each time, i.e., the wealth in the risky asset is

$$\pi(t) = m(W(t) - Fe^{rt})$$

where m is a constant greater that 1. The portfolio $W_{PI}(t)$ is given by the differential equation

$$\begin{aligned} dW_{PI}(t) &= rW_{PI}(t)dt + \pi(t)\sigma dz^Q(t) \\ &= rW_{PI}(t)dt + m(W(t) - Fe^{rt})\sigma dz^Q(t) \end{aligned} \tag{4.4.16}$$

which implies that the terminal portfolio is

$$W_{PI} = Fe^{rT} + (W(0) - F)e^{m\sigma z(T)+m(b-r)T+(r-\frac{1}{2}m^2\sigma^2)T}. \tag{4.4.17}$$

The expected value and the volatility of the terminal portfolio are, respectively,

$$\begin{aligned} E[W_{PI}] &= Fe^{rT} + (W(0) - F)e^{(mb+(1-m)r)T}, \quad \text{and} \\ V[W_{PI}] &= (W(0) - F)e^{(mb+(1-m)r)T}\sqrt{e^{m^2\sigma^2T}-1}. \end{aligned} \tag{4.4.18}$$

The RNER

From Equation (4.3.8), the terminal wealth W_{NR} for the two asset case is,

$$W_{NR}(T) = e^{rT}W(0)(1 + \frac{1}{2}\alpha((\sigma z^Q(T))^2 - \sigma^2 T + 2\beta\sigma z^Q(T))). \tag{4.4.19}$$

Using the stochastic isometry for Itô's integral yields the expected terminal value and its volatility

$$\begin{aligned}E[W_{NR}] &= e^{rT}W(0)(1+\frac{1}{2}\alpha(b-r)^2T^2+\alpha\beta(b-r)T).\\ V[W_{NR}] &= e^{rT}W(0)\alpha\sqrt{\left(\frac{1}{2}\sigma^4T^2+\sigma^2((b-r)T+\beta)^2T\right)}.\end{aligned} \tag{4.4.20}$$

All terminal portfolio values of the four strategies are expressed in terms of normal random variables. Using (4.2.1) and (4.4.19) yields the portfolio return in terms of the gross asset returns

$$\begin{aligned}R_{BH} &= (1-u)e^{rT}+uR_X\\ R_{FM} &= e^{(1-v)(r+\frac{1}{2}v\sigma^2)T}\,(R_X)^v\\ R_{PI} &= \frac{F}{W(0)}e^{rT}+\left(1-\frac{F}{W(0)}\right)e^{(1-m)(r+\frac{1}{2}m\sigma^2)}(R_X)^m\\ R_{NR} &= e^{rT}(1+\frac{1}{2}\alpha((\ln R_X)^2+(2\beta-2rT+\sigma^2T)\ln R_X\\ &\quad+\frac{1}{4}\sigma^4T^2-r\sigma^2T^2-\sigma^2T+\sigma^2\beta+r^2T^2-2\beta rT)).\end{aligned} \tag{4.4.21}$$

We take the parameter data given in the previous example to analyze the performance of these strategies. Starting with the same initial portfolio, for example, the portfolio with 70% in the risky asset, the structure of the portfolio returns is given in Figure 4.2.

Both CPPI and RNER are designated to protect against downside losses while keeping the upward potential. Furthermore, an investor can sacrifice the gains in a flat market to achieve profits from a deeply down market by using a RNER strategy. Neither of the other three strategies can fulfill this task. Hence, a RNER strategy performs well in both directions but not for a flat market. One of interesting questions is how this strategy is compared to portfolio insurance. With this data, neither the CPPI nor the RNER dominate each other. RNER outperforms CPPI in a deeply downward market, while CPPI outperforms RNER in a highly upward market. The result is just the opposite when market is calm. A RNER strategy benefits from a slightly up market, while a CPPI benefits from a slightly down market. One could possibly think that a better strategy is composed of the four strategies taking use of their complementary properties. The following proposition about the curvature of the portfolio return with respect to the return of stock market is implied from (4.4.21).

Proposition 4.4.5. *For BH, FM, CPPI and RNER strategies, the returns of the terminal portfolios are linear, concave, convex and convex in terms of the return of the stock market, respectively.*

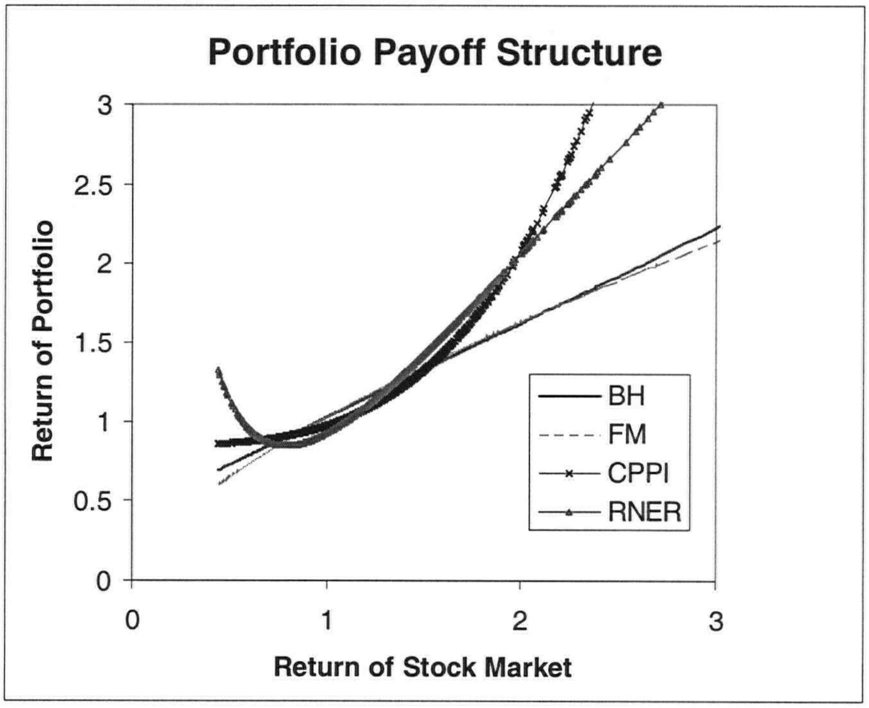

Figure 4.2: The Structure of the Portfolio Returns

The shape of the return function for a RNER strategy as in Figure 4.2 shows that portfolio performs better when the market has high and low tails returns. Compared to the CPPI portfolio insurance strategy, the RNER strategy will not only guarantee a floor but will also provide an upward return if the market falls even further. Hence, the payoff of this strategy is similar to that of a straddle option strategy.

4.4.3 Calculation of the VaR

While VaR is accepted by many practitioners, the calculation of an exact VaR has posed a formidable task for a given investment policy; see e.g. Hull and White (1998). Unlike mean-variance, there does not exist a uniform and analytic way of calculating the VaR, even for typical distributions, such as, the normal and lognormal. Monte Carlo simulation provides a statistical approximation. With the assumed distribution of the "uncertainty" (the simplest being normality), a large sample of the portfolio value is generated which then gives the VaR by finding the left tail cutoff value for the $1-\alpha$ confidence interval. A reduction of the calculating complexity can be approached as follows.

Proposition 4.4.6. *Let $VaR(X)$ denote the value at risk of a random variable X. If $f(x)$ is monotonic on $\Re$, then*

$$VaR(f(X)) = E[f(X)] - f(E[X] - VaR(X)). \tag{4.4.22}$$

Proposition 4.4.6 can be verified by (4.4.23).

$$\begin{aligned} VaR(f(X)) &= E[f(X)] - \sup\{K \in \Re; \Pr[f(X) \leq K] \leq \alpha\} \\ &= E[f(X)] - \sup\{f(f^{-1}(K)); \Pr[X \leq f^{-1}(K)] \leq \alpha, K \in \Re\} \\ &= E[f(X)] - f(\sup\{f^{-1}(K); \Pr[X \leq f^{-1}(K)] \leq \alpha, K \in \Re\}) \\ &= E[f(X)] - f(E[X] - VaR(X)). \end{aligned} \tag{4.4.23}$$

Proposition 4.4.6 reduces the computation of VaR of $f(X)$ given the distribution of X. This is equivalent to a portfolio mapping which maps a complicated portfolio to a simplest form such as a normal distribution. One can directly apply Proposition 4.4.6 to calculate the exact VaRs for BH, FM, and CPPI strategies.

If we have to resort to a numerical solution of $VaR(X)$, what would be the impact on the calculation of $VaR(f(X))$ caused by the approximation error? This introduces an interesting topic for numerical analysis. The ratio of the estimation errors between the portfolio value $f(X)$ and its building element X can be approximated using numerical analysis techniques. However, Proposition 4.4.6 does not always apply. For example, it does not for the strategy we developed here because the RNER is not monotonic. The new strategy results in a random payoff which is a quadratic function of a normal distribution. To calculate the "exact" VaR, we start from the definition to calculate the density function. Proposition 4.4.7 yields the exact formula for the density function of W_{NR}.

Proposition 4.4.7. *Let $f(x) = ax^2 + bx + c, a > 0$, and X a random variable with density function $\phi(x)$. Then, the density function $\phi_f(x)$ of $f(X)$ is*

$$\phi_f(x) = \frac{\phi(\frac{-b+q(x)}{2a}) + \phi(\frac{-b-q(x)}{2a})}{q(x)} \tag{4.4.24}$$

where $q(x) = \sqrt{4a(x-c)+b^2}, x > \frac{4ac-b^2}{4a}$.

By definition, the cumulative function of $f(X)$ is

$$\begin{aligned}
\Phi_f(x) &= \Pr[f(X) \le x] = \Pr[aX^2 + bX + c \le x] \\
&= \Pr[(X + \frac{b}{2a})^2 \le \frac{4a(x-c)+b^2}{4a^2}] \\
&= \begin{cases} \Pr[(X+\frac{b}{2a})^2 \le \frac{q(x)^2}{4a^2}] & \text{if } x > \frac{4ac-b^2}{4a}, \\ 0 & \text{if } x \le \frac{4ac-b^2}{4a} \end{cases} \\
&= \begin{cases} \Pr[\frac{-b-q(x)}{2a} \le X \le \frac{-b+q(x)}{2a}] & \text{if } x > \frac{4ac-b^2}{4a}, \\ 0 & \text{if } x \le \frac{4ac-b^2}{4a}. \end{cases}
\end{aligned} \tag{4.4.25}$$

Hence, the density function of $f(X)$ can be found by taking the derivative on $\Phi_f(x)$ with respect to x which verifies (4.4.24). Figure 4.3 shows the relations of the density functions of the portfolio returns determined by the four strategies.

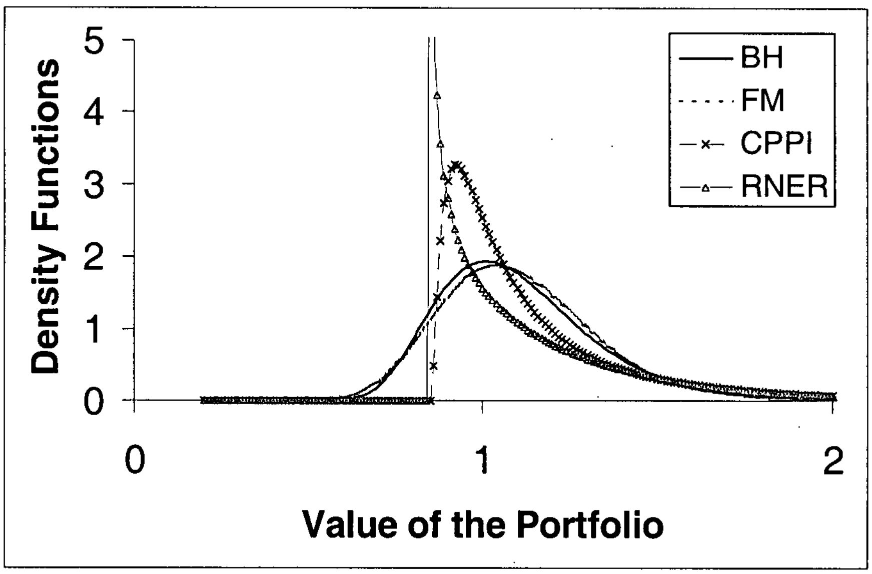

Figure 4.3: The Density Functions

Calculation of VaR "exactly" involves finding the left tail cutoff at the confidence level $1-\alpha$. This amounts to solving Equation (4.4.26) for K if X is a continuous random variable

$$\int_{-\infty}^{K} \phi_f(x)dx = \alpha. \tag{4.4.26}$$

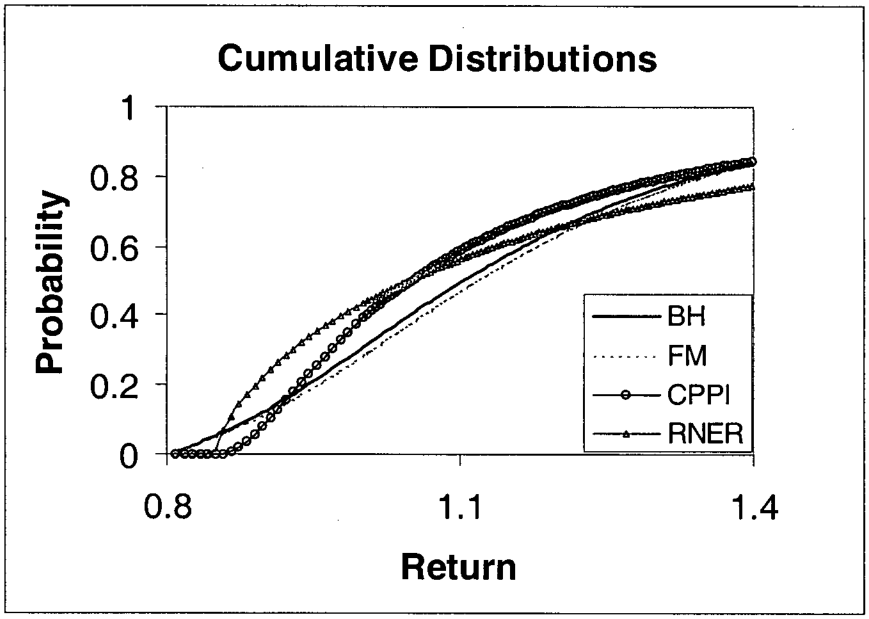

Figure 4.4: The Cumulative Distribution of the Four Strategies

Using a computer package, such as Maple or Matlab, which can manipulate the symbolic operations and provides efficient numerical solutions, the "exact" VaR can be obtained. The value at risk of X is

$$VaR(X) = E[X] - K. \tag{4.4.27}$$

To quantify the VaRs for these four strategies, it will be worthwhile to know about the tail movements of their return distributions. Figure 4.4 depicts the cumulative distributions of the four strategies.

From Figure 4.4, a 5% left tail cutoff is much higher for the RNER but slightly below that of the CPPI strategy.

4.4.4 The Efficient Portfolio for a Given VaR

Since the calculation of a given portfolio VaR proposes a major challenge for investors, the inverse problem of developing an optimal strategy given a VaR, v, is even more difficult. This problem requires the solution of the following constrained stochastic control model

$$
\begin{aligned}
&\max_{\pi(\cdot)\in\mathcal{U}_t} \; E[W(T)] \\
&s.t. \; e^{-rT}W(T) = W_0 + \int_0^T e^{-rt}\pi(t)\sigma dz^Q(t) \\
&\qquad \text{VaR} = v.
\end{aligned}
\tag{4.4.28}
$$

The constraint VaR $= v$ is equivalent to the following probability constraint by the definition of VaR,

$$\Pr[E[W(T)] - W(T) \geq v] = p.$$

Using equations (4.2.2), (4.4.26) and (4.4.28), we can obtain the portfolio VaR as a function of α and β for the downside control strategy. Then, solving the optimization Model (4.4.28) yields the optimal policy. This approach requires considerable computation. Basak (1999) discusses the investor's utility maximization problem incorporated with a risk management type constraint such as, VaR or limited expected losses (tail VaR) for a standard market. The objective of this model is to maximize the expected value which does not have the required differentiable smoothness as a general utility function. Furthermore, to solve the Model (4.4.28), we need to write the VaR in terms of α and β. Although we have developed Propositions 1 and 2 for analytically calculating the VaR, it would be very difficult to corporate the VaR through α and β in the optimization Model (4.4.28). Considering the difficulties, we provide a heuristic method instead:

- Simulate a large number (10000) of sample paths of the Brownian motion $z(t)$;
- Calculate the VaR as the p-quantile of the portfolio for each strategy;
- Solve the optimization model to obtain the optimal α and β.

Table 4.1 describes the efficient portfolios for given VaR.

This simulation strategy can be easily applied in practice. For example, one can use each time window of historical data as an outcome of the stock prices and apply the procedure introduced above. Eventually, one can obtain the optimal strategy using simulation with many sample paths.

4.4.5 Measures of Performance

It is very important to find a good performance measurement for evaluating different investment returns. Mathematically, this problem is equivalent to ranking the set of

Table 4.1: The Efficient Portfolio for Given VaR.

5% Lower Cutoff	Portfolio Return	Initial Risky Asset Weight	Standard Deviation
1	0.0758	0.1939	0.0924
0.95	0.0871	0.3507	0.1671
0.9	0.0984	0.5089	0.2399
0.85	0.1096	0.6676	0.3101
0.8	0.1210	0.8218	0.3905
0.75	0.1322	0.9805	0.4624
0.7	0.1435	1.1359	0.5394
0.65	0.1548	1.2935	0.6130
0.6	0.1661	1.4506	0.6874
0.55	0.1769	1.6176	0.7325
0.5	0.1887	1.7661	0.8342

all possible random variables defined on a probability space. There are some statistical measures, such as, mean, standard deviation, and upper and lower percentiles. However, the tradeoff between the risk and return has to be reflected in choosing any measure of performance. The Sharpe ratio is a measure that trades off volatility and expected payoffs. Here we define a measurement that deals with the control of downside losses as

$$M_{VaR} = \frac{E[W(T)] - W_0 e^{rT}}{\mathrm{VaR}(W(T))}, \tag{4.4.29}$$

which is the ratio of the excess return of the portfolio over the riskless asset to its VaR. This measure characterizes an investor's risk attitude towards the level of losses for trading an upward potential return, as compared to the Sharpe ratio which use standard deviation for characterizing the risk and return tradeoff. Using this measure, Table 4.2 describes the performances of these four strategies that start with the same initial portfolio (70% in the risky asset).

The statistics in Table 4.2 show that the RNER strategy has a high expected return only slightly below that for CPPI. For both the 5% lower tail cutoff and the VaR, the ranking order is CPPI, RNER, BH, and FM while a 95% upper tail cutoff yields RNER, CPPI, BH, and FM. For the Sharpe ratio, the ranking is FM, BH, RNER, and CPPI, while the ranking is CPPI, RNER, BH, and FM for the measure using VaR defined in Equation (4.4.29). In conclusion, both RNER and CPPI will receive higher standard deviations to achieve smaller losses than the BH and FM

Table 4.2: Performances of BH, FM, CPPI and RNER Strategies.

Statistics	Market	RNER	FM	BH	CPPI
Mean	0.1293	0.1165	0.1081	0.1090	0.1185
Standard Deviation	0.3562	0.3348	0.2415	0.2493	0.3935
5% Lower Cutoff	0.6539	0.8399	0.7638	0.7763	0.8756
95% Upper Cutoff	0.7978	0.8082	0.5498	0.5771	0.7534
VaR	0.4753	0.2766	0.3443	0.3328	0.2429
Sharpe Ratio	0.1893	0.1633	0.1915	0.1893	0.1439
Performance ratio using VaR	0.1419	0.1976	0.1343	0.1419	0.2332

strategies. An investor can tailor a portfolio using these four strategies that matches his risk attitudes.

4.5 A Summary of RNER

We have investigated the control of downside risk using a simple asset allocation strategy which requires portfolio dynamic shifts proportional to the change of the *risk neutral excess return.* Within a continuous time framework and the assumption of lognormality for stock prices, the RNER strategy can be used very efficiently. Investors can successfully achieve returns above some floor that meet their liability requirements and, at the same time, upward potentials can be achieved.

The performance measure using VaR trades off the volatility for downside losses. Both BH and FM have smaller volatility and larger downside losses than CPPI and RNER. Comparisons to BH, FM and CPPI with a typical example show that the RNER strategy is superior to those strategies in different ways, when the asset prices follow a multi-dimensional geometric Brownian motion. The RNER strategy has a better control of downside risk than BH and FM at the same time higher return. Compared to CPPI, RNER has more downside risk with slightly smaller mean return. However, the return structure of the RNER strategy shows that upward return favors the RNER strategy so that neither dominates the other.

The RNER strategy is related to Portfolio Insurance in the sense that both strategies "guarantee" a floor of return. However, this strategy requires a more intensive portfolio dynamic change which is reflected by the choice of large α, not like the Δ-strategy in traditional portfolio insurance. This strategy can be superior to a portfolio insurance strategy in the sense that it can actually benefit from both large declining and rising market movements. By examining the portfolio payoff function in terms of the stock prices, we found that the RNER strategy payoff structure is similar to a straddle option strategy.

Part II: Discrete Time Models

Chapter 5

Approximation for Incomplete Markets

This chapter considers a discrete time investment environment with general asset return processes characterized by scenarios. The scenarios of the asset returns are modeled as general discrete random variables which may or may not be serially correlated. The investor is assumed to be trading periodically to maximize the expected utility of terminal wealth with downside risk control characterized by the worst possible outcome wealth among all possible scenarios at the end of horizon. This discrete time stochastic control model is analytically intractable for arbitrary asset returns. Even with suitable assumptions on the asset return processes such as assuming normal distributions, the discrete time model is still not easy to solve because of its feature of a nonlinear multiperiod stochastic optimization model. We must look for other strategies to reduce the complexity of solution methods. Cox and Huang (1989) provided a martingale method for solving continuous time model in an unconstrained market. Pliska (1998) discussed how to solve multiperiod stochastic models using martingale method in a discrete time version of its continuous time analog. This chapter develops the approach along this line. The first task for martingale method is to identify a scenario set which excludes arbitrage opportunities, because martingale measures are derived on the assumption of arbitrage free market. In this chapter, a multiperiod stochastic linear programming model is developed for testing for the existence of arbitrage opportunities. The risk neutral probability is given by the dual solution of the stochastic linear programming model. Specifically, the risk neutral probability is equal to the optimal dual solutions times the risk free interest rate. Finding a risk neutral probability is the second task which is needed as input for the static model of identifying the optimal terminal wealth. An alternative method is provided for calculating the risk neutral probability knowing that the period-by-period conditional risk neutral probability at each decision node of a scenario can be characterized by a linear programming model. Upon obtaining the conditional risk neutral probability, the risk neutral probability for each scenario can be calculated by multiplying together the conditional risk neutral probabilities along the scenario. However, these two tasks are not easy to accomplish in actual investment practice. For an unconstrained market, the relation between the assumption of arbitrage free and the existence of risk neutral probabilities or equivalent martingale measures is characterized via a standard stochastic linear programming model with simple recourse. The risk neutral probability or equivalent martingale measure is a probability under which all asset returns have the same periodic conditional expectation equal to the riskless rate. Therefore, all portfolio value processes discounted at the riskless rate are martingales. This determines the set of attainable terminal wealth without downside control for an unconstrained market. The optimal portfolio value can be identified by solving a static large-scale nonlinear maximization problem subject to a martingale constraint on wealth.

After identifying the optimal terminal wealth the implementation of this model must be used to derive the one-period and scenario-wise optimal investment strategy. The theory is well developed but the computations need procedures as developed here. Based on the assumption that the wealth at each time is a function of the state variables in continuous time models, partial differential equations can be derived to solve for the optimal portfolio strategies. However, this method tends to obscure the role of the optimization methodology. Multi-period stochastic linear programming is a useful tool for implementing planning models under uncertainty and it has made major improvements to the practice of investment management. Edirisinghe, Naik and Uppal (1993) applied a stochastic programming model for option replication with transaction costs and trading constraints by minimizing the initial costs of an European call option. Cariño, Ziemba et al. (1994, 1998ab) successfully developed a planning model for a Japanese insurance company; see also Ziemba and Mulvey (1998) for a survey of additional applications. In this chapter, multiperiod stochastic programming is used for identifying the existence of market arbitrage opportunities as well as the implementation of the replication model. Trading periodically, we can obtain the optimal investment portfolio weights by replicating the terminal portfolio value scenario by scenario and minimizing the expected downside replicating error. It is proved that the replicating portfolio is exactly the optimal portfolio identified in the first step if the market is unconstrained. With trading constraints, liquidity constraints, shorting costs and transaction costs, etc., this optimal portfolio value is generally not perfectly replicable. Hence, the replicating portfolio is not exactly the optimal solution to the original problem in the presence of market frictions. Since the identified portfolio is no worse than any optimal portfolio value with policy constraints, we call this portfolio the "dominant" portfolio. The investor's goal is to find a portfolio which is the closest to the dominant portfolio in the sense of minimizing the expected downside deviation from the ideal portfolio and which satisfies market indispensable constraints. This ideal portfolio is perfectly replicable in an unconstrained market, but it is not in a constrained one. The discrepancy is characterized by the replicating downside error. Unlike standard utility maximization, we will incorporate the downside risk control into the investor's utility function as an additional endogenous decision variable as defined in Chapter 3. This variable is the worst possible outcome wealth among all possible scenarios. Because the investor is also concerned about the worst return while expecting an overall terminal wealth, one might want to integrate these two variables in the utility maximization. This specification represents sensitivity of the investor's risk aversion to market conditions. For a discussion of this idea, see Zhao, Haussmann and Ziemba (2000).

5.1 Utility Maximization for Unconstrained Markets

The Scenario Tree of Asset Returns. The asset returns are modeled as a vector stochastic process, $\mathbf{r}_t = (r_{t0}, r_{t1}, \cdots, r_{tn}), t = 1, \cdots, T$, which may be serially and cross assets correlated. The filtration generated by $\mathbf{r}_t$ consists of the σ-fields, $\mathcal{F}_t = \sigma(\mathbf{r}_1, \cdots, \mathbf{r}_t), t = 1, \cdots, T$. The market uncertainty is described as a scenario tree which specifies the information structure concerning the security returns revealed to the investor through time; see Figure 5.1.

Control of Downside Risk. Assume the investor has initial wealth W_0. Without injection and withdrawal of funds, the investor is assumed to trade periodically so as to maximize the utility of terminal wealth while controlling the downside risk. To incorporate the control of the downside risk, the investor utilizes a utility function $U(x, y)$ where x is the terminal wealth and y the worst possible outcome wealth among all possible scenarios. It is assumed that $U(x, y)$ is jointly concave and strictly increasing in both x and y. The worst possible outcome wealth is determined endogenously by the model, since the investor is allowed to have different wealth floors for different market conditions.

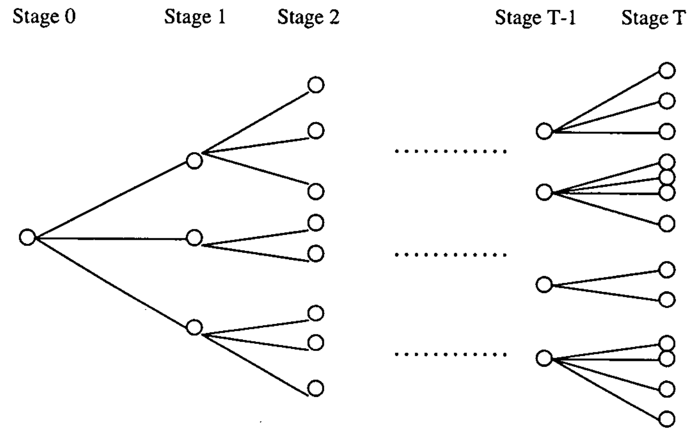

Figure 5.1: The Scenario Tree Over Time

The Portfolio Weights. Unlike some continuous time models where the portfolio is characterized as either the number of units of assets or proportional amount of the total wealth in each period, here the portfolio weights are defined to be the amount of wealth allocated in each asset. In this way, the replication model can be formulated as a problem of stochastic linear programming with simple recourse; for related definitions see Birge and Louveaux (1997). Let $\mathbf{x}_t = (x_{t0}, \cdots, x_{tn})^\top$ be the amount of wealth held in the riskless asset (x_{t0}) and other risky assets. Assume that $\mathbf{x}_t$ are $\mathcal{F}_t$-measurable for $t = 1, 2 \cdots, T-1$, so that $\mathbf{x}_t$ is non-anticipative. Each scenario is determined by a single path $\omega = (\omega_0, \cdots, \omega_T)$, where ω_t represents a single path of the information up to time t.

The Dominant Portfolio. Assume there are no trading constraints, no liquidity constraints, no shorting costs, and no transaction costs. An investor maximizes the utility of the terminal wealth and the downside risk control. The utility maximization problem is

$$\begin{aligned}
\max_{K,W,\mathbf{x}_t} \quad & E[\rho U(W) + (1-\rho)K] \\
\text{s.t.} \quad & \mathbf{x}_0^\top \cdot \mathbf{1} = W_0 \\
& (1+\mathbf{r}_t)^\top \mathbf{x}_{t-1} - \mathbf{1}^\top \cdot \mathbf{x}_t = 0, \quad \forall t = 1, 2, \cdots, T \\
& (\mathbf{1}+\mathbf{r}_T)^\top \cdot \mathbf{x}_{T-1} - W = 0 \\
& W - K \geq 0, \qquad \forall \omega \in \Omega. \\
& \mathbf{x}_t \text{ is } \mathcal{F}_t \text{ measurable.}
\end{aligned} \tag{5.1.1}$$

Harrison and Kreps (1979) have shown that an arbitrage-free market implies the existence of a probability measure such that all prices of securities discounted at the risk free rate are martingales. The above statement can be rephrased as that an arbitrage-free market allows for the existence of a probability measure Q under which all asset returns have the same conditional expected value as the riskless asset rate for $1 \leq t \leq T$, i.e., $E^Q[\mathbf{1} + \mathbf{r}_t | \mathcal{F}_{t-1}] = (1 + r_{t0}) \cdot \mathbf{1}$. If we assume the scenario tree shown in Figure 5.1 does not imply any arbitrage opportunities, then there exists such a risk neutral probability measure. Recalling the formal definition of arbitrage free in Chapter 1, Lemma 5.1.1 characterizes market exclusion of arbitrage.

Lemma 5.1.1. *Assume the existence of a riskless asset. A market is arbitrage free if and only if the following stochastic linear programming problem*

$$
\begin{array}{rlllll}
Z^* \equiv \min\limits_{W_0, \mathbf{x}_t} & W_0 & & & & \\
s.t. & W_0 & - \mathbf{1}^\top \mathbf{x}_0 & & & \geq 0 \\
 & & (1+\mathbf{r}_1)^\top \mathbf{x}_0 & - \mathbf{1}^\top \mathbf{x}_1 & & \geq 0 \\
 & & & \ddots & & \\
 & & & & (1+\mathbf{r}_T)^\top \mathbf{x}_{T-1} & \geq 0
\end{array}
\tag{5.1.2}
$$

has an optimal value Z^* equal to zero and all constraints are binding at optimality.

Proof. A portfolio W is represented by a sequence of $n+1$-dimensional vectors, $\mathbf{x}_0, \cdots, \mathbf{x}_{T-1}$. If a portfolio W satisfies $W \geq 0$ with $\Pr[W > 0] > 0$, then W is a feasible solution to Model (5.1.2), but not an optimal solution because $\Pr[W > 0] > 0$ implies that the constraints are not binding for all scenarios. Therefore the portfolio initial cost (the objective function of Model (5.1.2)) W_0 is greater than 0, which implies that the market is arbitrage free.

Conversely, we need to first show that the initial cost $W_0 \geq 0$ assuming no arbitrage if W is a feasible portfolio to Model (5.1.2). If $W_0 < 0$, then an arbitrage portfolio can be constructed by increasing the amount W_0 to W's riskless asset holding. Thus, $W_0 \geq 0$. Since the portfolio $\mathbf{x}_t = (0, \cdots, 0)^\top$ is a feasible portfolio, Z^* must be equal to 0. Hence, the optimal value of Model (5.1.2) is 0 and all constraints must be binding by the definition of arbitrage. □

Model (5.1.2) can be interpreted as the optimal strategy for the portfolio of payoff 0 in any scenario. It is implied that such a portfolio in the market will exactly have a zero initial cost in an unconstrained market.

Modeling the Risk Neutral Probability. Consider the dual of the Model (5.1.2). Let $q_t(\omega_t)$ be the multipliers for a scenario $\omega = (\omega_0, \cdots, \omega_T)^\top$ given the information up to t. Then

$$\sum_{\omega_t \in \Omega_t} q_t(\omega_t) \cdot (\mathbf{1} + \mathbf{r}_t(\omega_t|\mathcal{F}_{t-1})) = q_{t-1}(\omega_{t-1}) \cdot \mathbf{1}, \qquad 1 \leq t \leq T \tag{5.1.3}$$

where Ω_t is the set of possible scenarios up to time t for a given ω_{t-1} and $q_0(\omega_0) \equiv 1$.

Lemma 5.1.2. *The dual solutions of Model (5.1.2) determine the set of all risk neutral probabilities.*

Proof. Define Q as

$$Q : (\omega_0, \cdots, \omega_T) \longrightarrow \prod_{1 \leq t \leq T} (1 + r_{t0}(\omega_t|\mathcal{F}_{t-1})) \cdot q_T(\omega_T),$$

where $r_{00}(\omega_0) = 0$ and $r_{t0}(\omega_t|\mathcal{F}_{t-1})$ is constant for $\forall\ \omega_t \in \Omega_t$ given information $\mathcal{F}_{t-1}$, i.e., r_{t0} is predictable. Then Q is a probability measure defined on the set of scenarios Ω, and Q is a risk neutral probability on Ω. The absence of arbitrage implies the existence of a risk neutral probability. The uniqueness is guaranteed if and only if the market is complete, i.e., all contingent claims are replicable. Model (5.1.2) not only checks for the existence of arbitrage, but also calculates the risk neutral probability by solving its dual. Primal-dual algorithms can provide feasible primal and dual solutions simultaneously through the iterative process. □

The dual of Model (5.1.2) is a large scale linear system, but a decomposition method can be applied. Equation (5.1.3) can be rewritten as

$$\sum_{\omega_t \in \Omega_t} \rho_t(\omega_t|\mathcal{F}_{t-1})(1 + \mathbf{r}(\omega_t|\mathcal{F}_{t-1})) = (1 + r_{t0}) \cdot \mathbf{1},$$

where $\rho_t(\omega_t|\mathcal{F}_{t-1}) = (1 + r_{t0}) \cdot \frac{q_t(\omega_t)}{q_{t-1}(\omega_{t-1})}$ is the conditional risk neutral probability if $\sum_{\omega_t \in \Omega_t} \rho_t(\omega_t|\mathcal{F}t - 1) = 1$ for any node in the scenario tree at time t. Hence, the risk neutral probability can be calculated by multiplying together the conditional risk neutral probabilities along each scenario path. The task has been reduced to the following problem for each node in the scenario tree

$$\begin{cases} \sum\limits_{\omega_t \in \Omega_t} \rho_t(\omega_t|\mathcal{F}_{t-1})\mathbf{r}(\omega_t|\mathcal{F}_{t-1}) = r_{t0} \cdot \mathbf{1} \\ \sum\limits_{\omega_t \in \Omega_t} \rho_t(\omega_t|\mathcal{F}_{t-1}) = 1 \\ \rho_t(\omega_t|\mathcal{F}_{t-1}) \geq 0, t = 1, \cdots, T-1. \end{cases} \tag{5.1.4}$$

Hence, the large scale linear system amounts to solving a sequence of linear programming models. Therefore, the risk neutral probability can be characterized as in

Theorem 5.1.3. *The existence of a risk neutral probability is equivalent to the optimal objective value of the following linear programming model*

$$\begin{aligned}
\min \quad & Z_t \\
s.t. \quad & \sum_{\omega_t \in \Omega_t} \rho(\omega_t|\mathcal{F}_{t-1})\mathbf{r}_t(\omega_t|\mathcal{F}_{t-1}) = r_{t0}\mathbf{1} \\
& \sum_{\omega_t \in \Omega_t} \rho(\omega_t|\mathcal{F}_{t-1}) + Z = 1 \\
& \rho(\omega_t|\mathcal{F}_{t-1}) \geq 0, Z_t \geq 0, \qquad \forall \omega_t \in \Omega_t
\end{aligned} \tag{5.1.5}$$

having value 0 *for any* $1 \leq t \leq T-1$. *The risk neutral probability is given by*

$$Q(\omega) = \prod_{1 \leq t \leq T} \rho^*(\omega_t|\mathcal{F}_{t-1}), \qquad \forall \omega \in \Omega$$

where $\rho^*(\omega_t|\mathcal{F}_{t-1})$ *is the optimal solution of Model (5.1.5).*

Proof. If Model (5.1.5) has an optimal solution $\rho^*(W_t|\mathcal{F}_{t-1})$ and the optimal value $Z_t^* = 0$, then

$$\begin{cases} \sum_{\omega_t \in \Omega_t} \rho^*(\omega_t|\mathcal{F}_{t-1})\mathbf{r}_t(\omega_t|\mathcal{F}_{t-1}) = r_{t0}\mathbf{1} \\ \sum_{\omega_t \in \Omega_t} \rho^*(\omega_t|\mathcal{F}_{t-1}) = 1. \end{cases}$$

Let

$$Q(\omega) = \prod_{1 \leq t \leq T-1} \rho^*(\omega_t|\mathcal{F}_{t-1}),$$

then

$$\begin{aligned}
\sum_{\omega \in \Omega} Q(\omega) &= \sum_{\omega \in \Omega} \prod_{\omega_t \in \Omega_t} \rho^*(\omega_t|\mathcal{F}_{t-1}) \\
&= \prod_{\omega \in \Omega} \sum_{\omega_t \in \Omega_t} \rho^*(\omega_t|\mathcal{F}_{t-1}) \\
&= \prod_{\omega \in \Omega} 1 = 1.
\end{aligned}$$

Since $r_i = \prod_{1 \leq t \leq T} r_{ti}(\omega_t|\mathcal{F}_{t-1})$, for asset i,

$$\begin{aligned}
\sum_{\omega \in \Omega} Q(\omega) r_i(\omega) &= \sum_{\omega \in \Omega} \prod_{\omega_t \in \Omega_t} \rho^*(\omega_t|\mathcal{F}_{t-1}) r_{ti}(\omega_t|\mathcal{F}_{t-1}) \\
&= \prod_{\omega \in \Omega} \sum_{\omega_t \in \Omega_t} \rho^*(\omega_t|\mathcal{F}_{t-1}) \\
&= \prod_{\omega \in \Omega} 1 = 1.
\end{aligned}$$

So, Q is a risk neutral probability by definition. The converse of the theorem can be proved in the same way. Let $\mathcal{Q}$ denote the set of all risk neutral probabilities. $\square$

The Martingale Method. Let $W_t = (1+\mathbf{r}_t)^\top \mathbf{x}_{t-1}$ be the wealth at time t. Denote $E^Q[\cdot]$ as the operator of expectation under the risk neutral probability Q generated by the optimal dual solution. The conditional expectation of discounted wealth is

$$
\begin{aligned}
E^Q\left[\prod_{1\le i\le t}(1+r_{i0})^{-1}W_t \,\middle|\, \mathcal{F}_{t-1}\right] &= E^Q\left[\prod_{1\le i\le t}(1+r_{i0})^{-1}\cdot(1+\mathbf{r}_t)^\top \mathbf{x}_{t-1}\,\middle|\, \mathcal{F}_{t-1}\right] \\
&= \prod_{1\le i\le t-1}(1+r_{i0})^{-1}\cdot \mathbf{1}^\top \mathbf{x}_{t-1} \\
&= \prod_{1\le i\le t-1}(1+r_{i0})^{-1}\cdot W_{t-1}.
\end{aligned}
\tag{5.1.6}
$$

Theorem 5.1.4. *The discounted wealth process $\prod_{1\le i\le t}(1+r_{i0})^{-1}\cdot W_t$ under the risk neutral probability is a martingale. The optimal terminal portfolio value can be obtained by solving the following static concave programming problem*

$$
\begin{aligned}
\max_{K,W} \quad & E[U(W,K)] \\
s.t. \quad & E^Q\left[\prod_{1\le t\le T}(1+r_{t0})^{-1}W\right] = W_0, \quad \forall Q\in\mathcal{Q}, \\
& W-K\ge 0, \quad P-a.s.
\end{aligned}
\tag{5.1.7}
$$

where K is constant for $\forall\,\omega\in\Omega$.

Proof. Since the probability sample space Ω is finite, Equation (5.1.6) proves that the discounted wealth process is a martingale under any risk neutral probability. If (W,K) is an optimal solution to Model (5.1.7), then there exist a sequence of trading strategies $(\mathbf{x}_0,\cdots,\mathbf{x}_{T-1})$ such that Model (5.1.1) is solved. Consider the following stochastic linear programming problem that minimizes the expected discounted downside deviation

$$
\begin{array}{llcl}
\min\limits_{Z,\mathbf{x}_t} & E^Q[\prod\limits_{1\le t\le T}(1+r_{t0})^{-1}\cdot Z] & & \\
\text{s.t.} & -\mathbf{1}^\top\mathbf{x}_0 & \ge & -W_0 \\
& (1+\mathbf{r}_1)^\top\mathbf{x}_0 - \mathbf{1}^\top\mathbf{x}_1 & \ge & 0 \\
& \ddots & & \\
& (1+\mathbf{r}_T)^\top\mathbf{x}_{T-1}+Z & \ge & W \\
& Z & \ge & 0.
\end{array}
\tag{5.1.8}
$$

The dual of Model (5.1.8) is feasible and bounded, since $q_0 = 1, q_1(\omega_1), \cdots, q_T(\omega_T)$ is a feasible solution of the dual. Since the optimal (W, K) of Model (5.1.7) satisfies the martingale constraints, the optimal value of the dual model of Model (5.1.8) is 0. Hence, by strong duality, $E^Q[\prod_{1\le t\le T}(1+r_{t0})^{-1}Z] = 0$ and therefore, $Z = 0,\quad \forall\, (\omega_0, \cdots, \omega_T) \in \Omega$, which proves that $(W, K, \mathbf{x}_0, \cdots, \mathbf{x}_{T-1})$ is the optimal solution to Model (5.1.1). □

Computation of the Optimal Terminal Wealth. Assume that the utility function is twice continuously differentiable, then the optimal (W, K) can be obtained directly using Lagrange Multiplier Rules. Let

$$\eta_t(\omega_t) = \frac{q_t(\omega_t)}{p_t(\omega_t)}, \quad \forall\, \omega_t \in \mathcal{F}_t, \tag{5.1.9}$$

where $p_t(\omega_t)$ is the physical probability for scenario ω up to time t information. η_t is usually called the state price density or the Arrow-Debreu price per unit probability p_t of one dollar in state ω at time t.

Let λ_0 and $\lambda = \lambda(\omega)$ be the Lagrange multipliers on the constraints of Model (5.1.7), then its Lagrangian is

$$\mathcal{L}(W, K, \lambda_0, \lambda) = E[U(W, K)] - \lambda_0(E[\eta_T W] - W_0) + E[\lambda(W - K)].$$

The extended Kuhn-Tucker conditions (for proof, see Zhao, Haussmann and Ziemba (2000)) are

(i). $U_x(W, K) - \lambda_0\eta_T + \lambda = 0,\qquad$ P - a.s.,

(ii). $E[U_y(W, K)] - E[\lambda] = 0,$

(iii). $E[\eta_T W] - W_0 = 0,$

(iv). $\lambda(W - K) = 0,\qquad$ P - a.s.,

(v). $W \ge K, \lambda \ge 0,\qquad$ P - a.s.

where $U_x(\cdot,\cdot)$ and $U_y(\cdot,\cdot)$ are partial derivatives with respect to the first and second variables, respectively. The optimal W and K are related through

$$W = K + \left[U_x^{-1}(\lambda_0\eta_T, K) - K\right]^+, \tag{5.1.10}$$

where λ_0 and K are determined by

$$\begin{cases} E\, U_y\left(K + [U_x^{-1}(\lambda_0\eta_T, K) - K]^+, K\right) - E\,[\lambda_0\eta_T - U_x(K, K)]^+ = 0 \\ E\,[K\eta_T] + E\left[\eta_T\,[U_x^{-1}(\lambda_0\eta_T, K) - K]^+\right] = W_0, \end{cases} \tag{5.1.11}$$

where $U_x^{-1}(\cdot,\cdot)$ is the inverse function of $U_x(\cdot,\cdot)$ with respect to x. The relation between the optimal wealth and the state price density is depicted as in Figure 5.2 where η^* is the state price if W attains the minimum wealth K in which case $\lambda_0 \cdot \eta^* = K$.

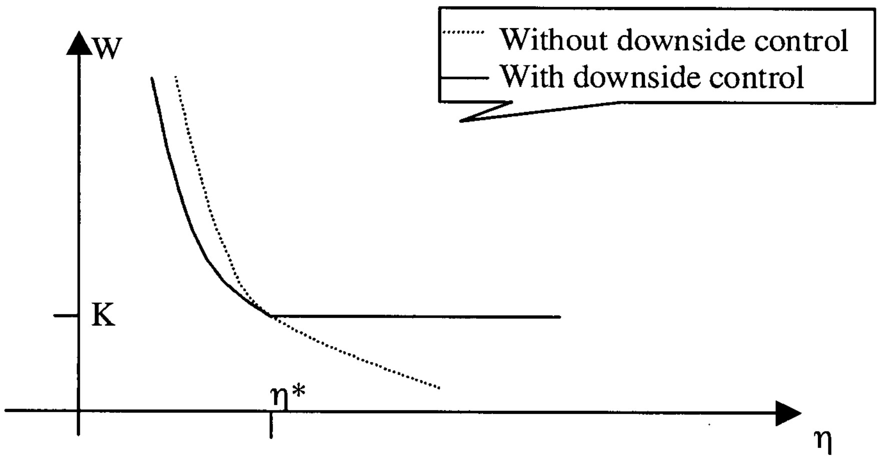

Figure 5.2: The Relation of Terminal Wealth and State Prices

This relation implies that the investor will become extremely risk averse if the state price exceeds η^* and therefore, the investor's wealth reaches the worst possible outcome wealth K. If the state price is below η^*, the investor will follow a traditional utility maximization as if there were no downside risk control. In that case, the downside risk control constraint is not binding. We state the above discussion as Theorem 5.1.5

Theorem 5.1.5. *The optimal value of Model (5.1.1) is given by Equation (5.1.10) and the optimal solution to Model (5.1.1) is given by the optimal solutions* $\mathbf{x}_0, \cdots, \mathbf{x}_{T-1}$ *to Model (5.1.8).*

Theorem 5.1.5 presents a method for solving the general investment model by decomposing the original problem to two subproblems: a static model and a replication model. The static (but usually large scale) nonlinear optimization model identifies the terminal wealth which satisfies the downside risk constraint, while the replication model is a multiperiod stochastic linear programming which requires powerful computational techniques. The IBM Optimization Routine Library is a useful package for solving such a large scale stochastic linear programming model.

The Option Interpretation. The second equation in (5.1.11) can be interpreted to mean that the investor allocates wealth to only two assets: the riskless asset and a call option with strike price K on a dynamic mutual fund with terminal value $U_x^{-1}(\lambda_0\eta_T, K)$. For an optimal λ_0 and K, the mutual fund is uniquely determined through the state price density η_T, which can be replicated using the market's $n+1$ primitive assets. The value of this option is the amount that the investor will "invest" in this option and the rest of the balance is invested in the riskless asset that will guarantee the amount K needed for exercising this option. The investor can complete his investment by following this synthetic strategy. For further results including a closed form solution, see Zhao, Haussmann and Ziemba (2000).

The Economic Interpretation of Utility Shifting. Also in (5.1.11), the first equation implies that, at optimality, the expected marginal utility on K is equal to the decrement of the marginal utility induced by increasing wealth to the level K. Mathematically, this relation can be expressed as

$$E[U_y(W, K)] = E[U_x(L, K)] - U_x(K, K)$$

where $L = min\{U_x^{-1}(\lambda_0\eta_T, K), K\}$. This relation represents a trade-off between the expected wealth and the worst possible outcome wealth. The economic interpretation is that, as a usual case, investors can only increase the expected value of wealth by reducing the worst possible outcome wealth, which characterizes the potential losses.

5.2 The Utility Maximization for Constrained Markets

Section 2 analyzed how to solve an investment problem with downside risk control in an unconstrained market. The static nonlinear optimization model is easy to solve after obtaining the risk neutral probability. This risk neutral probability is given by the dual solution of the stochastic linear programming Model (5.1.2) which checks for the existence of arbitrage opportunities. There are algorithms that can provide the primal and dual solution simultaneously. However, real markets have many constraints such as those related to trading, liquidity, transactions costs, etc. With these constraints added to the investor's problem (Model (5.1.1)), the new optimization model can not be easily decomposed into such two subproblems, because investor's wealth process is no longer a martingale process. The replication of such a terminal wealth is generally impossible. However, knowing that the optimal terminal wealth for an unconstrained market is superior to a terminal wealth subject to the market constraints, one can start to replicate such a portfolio while all constraints are satisfied and the downside replicating error is minimized. The error may not be zero as it is for an unconstrained market, but this is the best portfolio that is closest to the optimal portfolio with no constraints and, at the same time, satisfying the necessary constraints of the original problem.

Types of Constraints Considered. Trading constraints frequently require that a maximum amount is required when trades occur. These constraints are imposd to prevent arbitrage opportunities due to transactions. Let $\mathbf{y}_t = (y_{t1}, \cdots, y_{tn})^\top$ and $\mathbf{z}_t = (z_{t1}, \cdots, z_{tn})^\top$ represent the amounts bought and sold of the risky assets. The trading constraints are

$$\begin{aligned} \mathbf{0} \leq \mathbf{y}_t \leq \boldsymbol{\alpha}_t \\ \mathbf{0} \leq \mathbf{z}_t \leq \boldsymbol{\beta}_t, \end{aligned} \tag{5.2.1}$$

where $\boldsymbol{\alpha}_t = (\alpha_{t1}, \cdots, \alpha_{tn})^\top, \boldsymbol{\beta}_t = (\beta_{t1}, \cdots, \beta_{tn})^\top$ represent the buying and selling upper bounds of the amount for each asset traded in each period, respectively.

Usually, liquidity is defined as the ability to transact immediately and with negligibly small impact on the price of a security regardless the size of the transaction. One of the distinguishing features inherent in illiquid markets is a frequent inability to buy or sell an asset at its temporary equilibrium price. The reason is that not all the information available about the asset is fully reflected in its current return and hence the asset behavior becomes locally predictable, i.e., an excess demand will result in the increase of the stock return over the next time-step and likewise an excess supply will result in a decrease of the return. As a result, if a stock return is exceeding or going to exceed the riskless interest rate, the stock is unlikely to be available for purchase at its intrinsic price. Similarly, a falling market leads to an inability to sell the stock at its current intrinsic price. To accommodate these liquidity constraints, we specify a periodic holding constraint for each asset as

$$\boldsymbol{\gamma}_t \cdot \mathbf{x}_t^\top \mathbf{1} \leq \mathbf{x}_t \leq \boldsymbol{\Gamma}_t \cdot \mathbf{x}_t^\top \mathbf{1}, \tag{5.2.2}$$

where $\boldsymbol{\gamma}_t = (\gamma_{t1}, \cdots, \gamma_{tn})^\top$ and $\boldsymbol{\Gamma}_t = (\Gamma_{t1}, \cdots, \Gamma_{tn})^\top$ represent the limit percentages of the portfolio wealth held in each asset for period t.

The transaction costs are imposed for two purposes. The first is to pay brokerage fees and the second is to prevent frequent trades that might affect the equilibrium stock prices. For simplicity, we model these costs as proportional amounts to the transaction volume from asset to asset. Let $\boldsymbol{\theta}_t = (\theta_{t1}, \cdots, \theta_{tn})^\top$ be the proportional transaction costs for the risky assets. Buying and selling the same asset in each period is not optimal. At time t, the riskless asset has the amount

$$x_{t0} - (1 + r_{t0})x_{t-1,0} + (\mathbf{1} + \boldsymbol{\theta}_t)^\top \mathbf{y}_t - (\mathbf{1} - \boldsymbol{\theta}_t)^\top \mathbf{z}_t = 0. \tag{5.2.3}$$

The amount x_{ti} in the risky assets at time t is given by

$$x_{ti} - (1 + r_{ti})x_{t-1,i} - y_{ti} + z_{ti} = 0, \qquad \forall\, i = 1, 2, \cdots, n. \tag{5.2.4}$$

The Constrained Utility Maximization. The initial loading constraints are

$$\begin{aligned} x_{00} + (\mathbf{1} + \boldsymbol{\theta}_0)^\top \mathbf{y}_0 - (\mathbf{1} - \boldsymbol{\theta}_0)^\top \mathbf{z}_0 &= W_0, \\ x_{0i} - y_{0i} + z_{0i} &= 0. \end{aligned} \tag{5.2.5}$$

The terminal wealth in unit of riskless asset is

$$(1 + r_{T0}) \cdot x_{T-1,0} + \sum_{1 \le i \le n} (1 - \theta_{Ti})(1 + r_{Ti}) \cdot x_{T-1,i} - W = 0. \tag{5.2.6}$$

The constrained utility maximization model is

$$\begin{aligned} \max_{\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t, W, K} \quad & E[U(W, K)] \\ s.t. \quad & (5.2.1) - -(5.2.6) \\ & W - K \ge 0. \end{aligned} \tag{5.2.7}$$

The solution to Model (5.2.7), a multiperiod stochastic nonlinear programming model, is generally intractable. This is so large a model that even sophisticated software packages are not able to handle the solution of this model. Our aim is to decompose the problem into two problems as we did for the unconstrained market. The static model characterizes an dominant portfolio which is the same as that for the unconstrained market because it maximizes the expected utility when the market is unconstrained. The first model can be written as

$$\begin{aligned} \max_{W,K} \quad & E[U(W, K)] \\ & E^Q[B(T)^{-1} W] = W_0, \\ & W - K \ge 0, \end{aligned} \tag{5.2.8}$$

where K is nonrandom and W is $\mathcal{F}_T$ measurable. This problem can be solved as done in last section.

The second problem is the replication model for deriving the optimal strategy. This replication is not a perfect replication of the dominant portfolio identified in Model (5.2.8). It is not guaranteed that perfect replication can be achieved at the presence of market constraints. The replicating portfolio minimizes the expected downside deviation from the dominant portfolio which satisfies the downside risk control constraint. Let W_1 be the optimal solution to (5.2.8), then, the stochastic linear programming model is

$$\begin{aligned} \min_{x_t, y_t, z_t, Z} \quad & E^Q[\prod_{1 \le t \le T} (1 + r_{t0})^{-1} Z] \\ s.t. \quad & (5.2.1) - -(5.2.5) \\ & (1 + r_{T0}) \cdot x_{T-1,0} + \sum_{1 \le i \le n} (1 - \theta_{Ti})(1 + r_{Ti}) \cdot x_{T-1,i} + Z = W_1 \\ Z \ge 0 & \end{aligned} \tag{5.2.9}$$

Model (5.2.9) can be implemented by the IBM Optimization Solution Library.

The Deviation of Replication. One question about the approximate approach is how accurate the procedure can achieve. This leads to the comparison of the performances for both constrained and unconstrained approaches.

Let (W, K) be the optimal solution for the constrained market, (W_1, K_1) be the optimal solution for the unconstrained market and (W^*, K^*) be the solution obtained by the optimal replicating portfolio. Then

$$\begin{aligned} E[U(W, K)] \le E\, U(W_1, K_1), & \\ W_1 - W^* \le Z, \qquad P - a.s. & \\ K^* \ge \max\{W_1(\omega) - Z(\omega) : \forall\ \omega \in \Omega\}. & \end{aligned} \tag{5.2.10}$$

The difference of the optimal expected utilities between Model (5.2.7) and the constrained replication satisfies

$$\begin{aligned} E\, U(W, K) - E\, U(W^*, K^*) &\le E\, U(W_1, K_1) - E[U(W^*, K^*)] \\ &\le M \cdot E[W_1 - W^*]^+ + N \cdot (K_1 - K^*) \\ &\le M \cdot E[Z] + N \cdot (K_1 - K^*), \end{aligned} \tag{5.2.11}$$

where $M = U_x(K_1, K_1)$ and $N = E[U_y(W^*, K^*)]$, since $U(x, y)$ is jointly concave and strictly increasing in both x and y. Equation (5.2.11) indicates that the difference between the optimal solution and the replicating portfolio is bounded by a positive linear function of the expected downside deviation of the replication.

5.3 An Illustration

Suppose the investment opportunity set consists of the following five assets: a cash account earning a monthly risk free interest rate of 0.4%(for calculation simplicity, we assume a constant interest rate across time horizon) and four other risky assets, namely, the following major indices for stocks and bonds: the Dow Jones Industrial Average (DJIA), the Lehman Government Bond index (LEHM), the Nasdaq Composite (NSDQ), and the Standard & Poors 500 (S&P500). Denote the asset returns by $\mathbf{r_t} = (D_t, L_t, N_t, S_t)^\top$. Assume the conditional one-period returns of these four assets follow an identical and independent multivariate normal distribution, though it is not necessary for the method discussed here to assume this. Figure 5.3 depicts the monthly data from 12/07/1997 to 02/07/2000 of these assets.

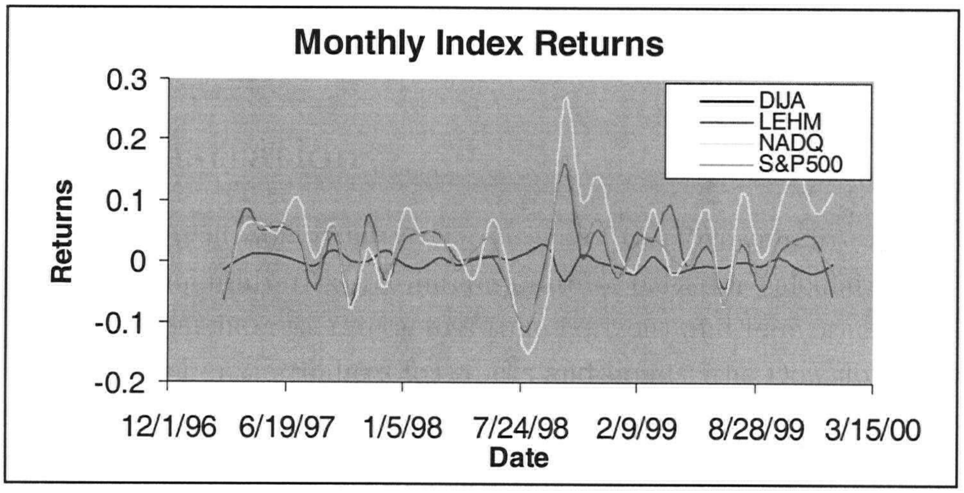

Figure 5.3: Monthly Index Returns (03/07/1997 - 02/07/2000)

It is estimated that

$$\begin{pmatrix} D_t \\ L_t \\ N_t \\ S_t \end{pmatrix} \to N\left(\begin{pmatrix} 0.0142 \\ -0.0006 \\ 0.0376 \\ 0.01777 \end{pmatrix}, \begin{pmatrix} 0.0029 & -0.0002 & 0.0028 & 0.0025 \\ -0.0002 & 0.0001 & -0.0003 & -0.0001 \\ 0.0028 & -0.0003 & 0.0062 & 0.0033 \\ 0.0025 & -0.0001 & 0.0033 & 0.0026 \end{pmatrix}\right). \quad (5.3.1)$$

Using a Cholesky matrix decomposition, we can write the joint distribution of these assets as

$$\begin{pmatrix} D_t \\ L_t \\ N_t \\ S_t \end{pmatrix} = \begin{pmatrix} 0.0142 \\ -0.0006 \\ 0.0376 \\ 0.01777 \end{pmatrix} + \begin{pmatrix} 0.0536 & 0 & 0 & 0 \\ -0.0028 & 0.0117 & 0 & 0 \\ 0.0522 & -0.0098 & 0.0579 & 0 \\ 0.0473 & 0.0040 & 0.0150 & 0.0126 \end{pmatrix} \cdot \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \\ \epsilon_{4t} \end{pmatrix} \tag{5.3.2}$$

where $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t})^{\top}$ are independent random variables with standard normal distributions.

Scenario Generation. One of the important issues in applied stochastic programming is how to generate an arbitrage free scenario set for model inputs. Generating an arbitrage free scenario is even more crucial in the model developed here because both the existence and the calculation of the risk neutral probability rely on the assumption of arbitrage free. For large-scale models, this becomes a very tedious task though the test procedure for the existence of arbitrage opportunities can be implemented by solving a large-scale stochastic linear programming problem. For simplicity, we implement the model with a four period investment horizon. At each node, we take a sample of size five as the possible scenarios for the next period. In this way, we can generate all possible 625 scenarios by random samples using a standard normal distribution. Now, we apply Model (5.1.1) to test for the existence of arbitrage opportunities. If the existence of arbitrage opportunities is tested positively, then we have to repeat the whole process.

The Calculation of the Risk Neutral Probability. Although the risk neutral probability can be obtained by solving the dual of Model (5.1.2), the problem can also be divided into a sequence of subproblems because of its separability to reduce the computational complexity. Thus, if we can calculate the periodic conditional risk neutral probability at each node, the risk neutral probability can be obtained by multiplying the periodic conditional probabilities along each scenario path. For this example, we need only to solve $1 + 5 + 25 + 125 = 156$ linear systems of size 5×5.

Although an identical return distribution of each asset across time is assumed, we choose different sample scenarios for different decision nodes of the same stage to accommodate the non-anticipativity which means that a decision is made based on the history of the information (realization) but not the future possible outcomes. By definition, the state price is given by the risk neutral probability, the physical probability, and the riskless interest rate. Therefore, the optimal wealth for a given utility function can be determined by the state prices upon obtaining the risk neutral probability. Let the investor's utility function have the form

$$u(x, y) = \ln x + \delta \ln y$$

where $\delta > 0$ is the goal weight between the terminal wealth and the worst possible outcome wealth. Assume $W_0 = \$100$, Figure 5.4 depicts the relation between the terminal wealth and the state price by varying the goal weight δ.

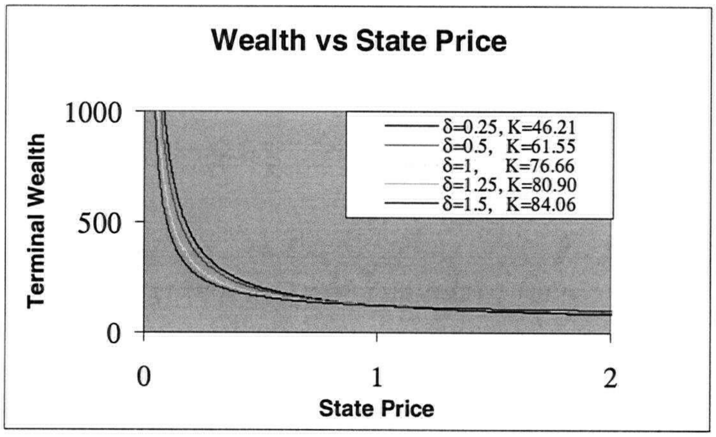

Figure 5.4: Relation between Wealth and State Prices for Varying δ

As δ increases, the investor becomes more risk averse by acquiring a larger worst possible outcome wealth K, and the overall wealth decreases simultaneously.

Replication for Constrained Market. Examining the sample distribution of the optimal terminal portfolio value, we found that both long and short positions are very large. For example, the following initial portfolio, short selling \$2394.37 of LEHM, buying \$2494.37 of S&P500, and 0 in the other three assets, is required for $\delta = 1$. The positions in succeeding periods are even larger. This position is so large that it is unlikely to be taken by any risk averse investor in practice. Although the optimal portfolio value is achievable in an unconstrained market, this huge position in risky assets implies that what may be realistic is such a market environment does not exist, therefore, the market should be constrained. Here, we provide a reasonable strategy by imposing constraints on constructing feasible policies. First, short sales and borrowing are not allowed. Secondly, asset holdings are constrained within 60% of the total wealth in each period for the riskier equity indices, DJIA, NSDQ and S&P500. Another type of constraint is the transaction cost which is modeled as a proportional amount (0.1%) of the transaction. By imposing these three types of constraints, we obtain a portfolio with expected quarterly return 5.14% with standard

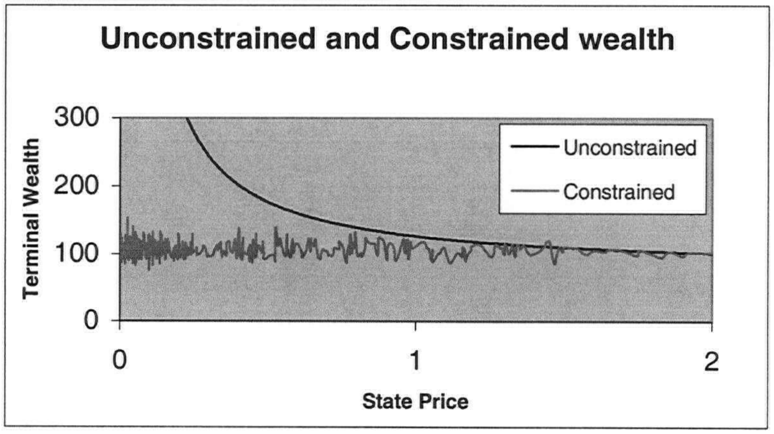

Figure 5.5: Wealth for Constrained and Unconstrained Market

deviation 9.92% as opposed to the expected return 3259.02% with standard deviation 13503.2% for the unconstrained market. Figure 5.5 depicts the wealth for both constrained and unconstrained market conditions.

If the results seem to be puzzling, the statistics for the performances under both constrained and unconstrained market conditions clarify this. The portfolio returns for both conditions have similar floors (worst possible outcome wealth), \$78.2279 for unconstrained and \$76.4991 for constrained. However, the 95% percentile returns are \$89.10 for the unconstrained market and \$91.35 for the constrained market, respectively. In the measure of Value at Risk with a 95% confidence interval, the losses are \$10.90 for the unconstrained condition and \$8.65 for the constrained condition. This statistical result indicates that the large portfolio returns in high state prices occur with tiny probabilities. Considering portfolio constraints, investors are interested in achieving a wealth return corresponding to a middle range of state prices.

5.4 Summary of the Approximation Method

The connections drawn in this chapter among arbitrage opportunities, risk neutral probability and stochastic linear program arise from analysis of duality relationships. A very efficient test for market arbitrage opportunity is characterized using a special instance of stochastic linear programming model. Also, the calculation of the risk neutral probability is reduced to the calculation of the periodic conditional risk neutral probabilities. Multiplying together the conditional risk neutral probabilities along a scenario is equal to the risk neutral probability of the scenario.

The investor's problem is formulated as a multiperiod stochastic nonlinear programming model. This model is intractable because of the non-linearity of the objective function and a multiperiod horizon. Utilizing martingale analysis, the problem is divided into two models. The first model is a static problem for identifying the scenario-wise optimal portfolio value for a given utility function. The second model is to replicate the identified portfolio by minimizing the downside expected replicating deviation. It is proved that the replicating portfolio is the optimal portfolio identified in the first model for the unconstrained market condition. However, with a constrained market, this replication might not be perfect. A numerical example is examined to investigate how a reasonable portfolio strategy can be obtained. We found that the replicating portfolio is greatly out of shape (both long and short positions are extremely large) for unconstrained markets. On the contrary, the replicating portfolio for the constrained market makes more sense in terms of portfolio return and the reality of its position in assets. It is evident that a 95% percentile is in favor of constrained market condition with this practical numerical example.

This chapter also shows the strong applicability of stochastic programming methodology for developing dynamic investment models. The method of replication is extremely useful in pricing contingent claims. We employed this idea to solve a complex investment model via two simpler models. Considering the identified portfolio as a "contingent claim", one can implement a replicating strategy by using finite resources subject to constraints. With a constrained market (this is always the case as we analyzed by data), the replicating error possibly exists for a constrained market which characterizes the upper bound of the investor's expected utility losses. The investor's expected utility loss can be characterized by the replicating deviation. With the analysis and statistical results of the numerical example, it is evident that including constraints can increase portfolio performance under the downside risk measure, Value at Risk. Due to its versatility of decomposing a complex problem into two simpler models that can be easily implemented, this method should be considered to be a very efficient way of solving practical models, both constrained and unconstrained.

Chapter 6

An Asset/Liability Model

Dynamic asset allocation concerns the selection of asset categories and the proportion of wealth placed in them over time. A problem is the potential decline of the investment portfolio below some critical limit. Therefore, instead of focusing only on expected return or mean-variance analysis, investors may wish to control the risk of downside losses. This can be done using option strategies in a multiperiod stochastic linear programming model that considers the distributions of the random returns and transaction costs. Synthetic option strategies provide an approach to managing an investment portfolio with downside risk control; see e.g. Arnott (1998), Boyle and Vorst (1992), Leland (1985), Leland and Rubinstein (1995), and Tilley and Latainer (1985). This strategy characterizes a payoff structure similar to a European call option on the initial portfolio. Suppose the investor's portfolio value is W_0 at time 0. After T periods, it is desired that the portfolio value W_T is worth at least some level K in any possible scenario. What can be done to achieve this goal? A simple answer is to buy or create a put option on the investment portfolio with a strike price equal to the target K. A shortcoming of the synthetic option strategy is that the investors' choice of target (the strike price) is exogenous to asset movements and adjustment of strategies. This K has to be "reasonably" chosen, since for a specific investor, different economic situations may influence the choice of the worst payoff K of the portfolio. For example, Value at Risk (VaR) which characterizes the downside losses within some level of confidence interval is a measure of potential change in value of a portfolio of financial instruments over a pre-set investment horizon. Using VaR, investors maximize the expected return given their exogenously set VaR at some risk level. It is preferable to determine K endogenously by the model, because investors then can have a larger feasible regions among all possible risk/return tradeoff given the level of the risk aversion. The investor may maximize K as an overall potential payoff for all scenarios at the end of planning horizon, i.e. what is the amount that should be insured for the whole planning horizon. In standard synthetic option strategies, the objective of maximizing the expected return is triggered purely by the *ex ante* choice of the target, no matter whether this choice is consistent with asset movements. Increasing K reduces the expected return, hence, the investor's objective should include K as a model variable. While a diversified portfolio is easy to construct, an option market may not be available to the options on this portfolio. Also, investors may buy index futures for hedging purposes and index options for downside risk control. Many traders adopt this strategy and try to manage the downside losses against a benchmark using optimal portfolio replication; see Dembo (1991).

Options and option-like strategies are very effective and versatile means for money managers to control risk. However, exchange-traded options have several major drawbacks for large institutional and corporate investors. Options with sufficient liquidity are limited to maturities of about three months. This makes the cost of long term protection extremely expensive, because it requires the purchase of a series of high priced short-term options. Hence, other strategies should be considered. An efficient way of doing this is to synthesize such a payoff structure using the current resources and considering the possible costs in the process of implementation.

In the finance literature, intertemporal decision making is frequently modeled as dynamic stochastic control problems over discrete or continuous time. The approach lies in finding the implementable optimal policies for each period as a function of the current or past states (observations) in order to meet the investment objective. Multiperiod stochastic programming can also provide a methodology for planning under uncertainty with respect to given constraints. Bradley and Crane (1972) and Kusy and Ziemba (1986) describe stochastic linear programs for bank asset/liability management. Cariño, Ziemba et al. (1994, 1998ab) discuss the Russell-Yasuda Kasai asset/liability management model. Mulvey and Vladimirou (1991) discuss a multiperiod stochastic network model for asset allocation, and Zenios (1993) describes stochastic programming models for fixed-income asset/liability management. Ziemba and Mulvey (1998) survey this field. One of the advantages of using a stochastic programming approach for financial planning is that it can handle "irregular" objective functions and complex constraints as well as general scenarios. Stochastic control models are much more difficult to solve and to include complex constraints, scenarios and objective functions. Now that more versatile computer packages for solving mathematical programming models are available, a stochastic programming model can be easily solved. The model developed here considers transaction costs, does not allow short sales in executing its trading strategy, and is subject to holding constraints in each risky asset. The objective function depends on all possible states at the horizon given the strategies used in each period with the planning target determined endogenously by the model. This modeling technique facilitates a new approach of allocating asset among cash, bonds and stocks with downside risk control.

6.1 The Stochastic Programming Model

6.1.1 Dynamic Replication with Portfolio Constraints

Buying put options has many drawbacks. In an orderly market, this is an executable strategy. However, if the market is not perfect, holding a put option and the stock portfolio has to bear the default risk as well as the risk from biased prices. Exchange-traded options may not be suitable for investors because their strike prices may deviate from their objective. Also, market options may not be written on the securities chosen, and purchasing a put option to protect a stock portfolio requires paying the market price of the option up-front, which may not reflect the true value of the options because a biased pricing formula might have been used.

Assume that the number of states of the world is finite, and that time evolves discretely taking the values $\{0, 1, \cdots, T\}$ and there is a filtration of information $(\mathcal{F}_t)_{0 \leq t \leq T}$. Let $\omega = (\omega_0, \cdots, \omega_T)$ denote a path of the information revealed through time, where ω_t is the information of the path ω revealed until time t. Since the investment in options does not require a further injection of funds we assume a self-financing strategy, so the opening value of the portfolio at time $t+1$ is the closing value at time t less transaction costs. The investor can move funds from asset to asset at each period incurring transaction costs. The presence of financial market frictions qualitatively changes the nature of the optimization faced by an investor. It requires one to either act or do nothing, an issue that does not arise in frictionless situations. Transaction costs changes decisions made under "perfect" conditions. See Davis and Norman (1990), Boyle and Vorst (1992), and Edirisinghe et al. (1993) for studies of replication with transaction costs.

Suppose the investment opportunities consist of n risky assets and a riskless asset. Consider the following notation.

W_0 : investor's initial wealth,

T : planning horizon,

ρ : continuously compound riskless rate of return equal in all periods,

r_t^i : continuously compound rate of return for risky asset i at time t, $i = 1, \cdots, n$ and $t = 0, 1, \cdots, T-1$,

α_t : amount allocated in the riskless asset at time t,

x_t^i : amount allocated to asset i at time t,

A_t^i : additional amount bought of asset i at time t,

D_t^i : additional amount sold of asset i at time t, and

θ_t^i : proportional transaction costs for purchases and sales of asset i at time t.

The initial portfolio satisfies

$$\alpha_0 + x_0^1 + \cdots + x_0^n = W_0. \tag{6.1.1}$$

At time $t+1$, investment in the riskless asset is α_{t+1}, where

$$\alpha_t e^{\rho} - \alpha_{t+1} - \sum_{i=1}^{n}(1+\theta_{t+1}^i)A_{t+1}^i + \sum_{i=1}^{n}(1-\theta_{t+1}^i)D_{t+1}^i = 0. \tag{6.1.2}$$

Buying and selling the same risky asset at the same time is not optimal in the presence of transaction costs. Hence, the following equations must hold for each risky asset at any time $t \in (0, T)$

$$x_t^i e^{r_{t+1}^i} - x_{t+1}^i + A_{t+1}^i - D_{t+1}^i = 0. \tag{6.1.3}$$

Divide the terminal portfolio payoff into two parts, a target K and the surplus z over the target. Then the terminal value of the replicating portfolio is characterized by

$$e^{\rho}\alpha_{T-1} + \sum_{i=1}^{n}(1-\theta_T^i)e^{r_T^i}x_{T-1}^i - z - K = 0. \tag{6.1.4}$$

Here K is deterministic and z is an $\mathcal{F}_T$ measurable variable. The portfolio $\mathbf{x}_t + 1$ not only depends on $\mathbf{x}_t$ but also $\mathbf{A}_t$ and $\mathbf{D}_t$. To improve the implementation ability of the model recommendations, we impose the portfolio turnover constraints to the stochastic program. Including constraints will also improve the management of the liquidity risk in practice. We illustrate this effect in a numerical example later in this chapter. Let $\mathbf{m}_t = (m_t^1, \cdots, m_t^n)$ be a vector of the upper bound of the proportion of the total value held in all risky assets at time t, i.e.

$$0 \leq \mathbf{x}_t \leq \mathbf{m}_t(\mathbf{x}_t' \cdot \mathbf{1} + \alpha_t). \tag{6.1.5}$$

For simplicity assume that $\mathbf{m}_t$ is deterministic through the horizon.

Although the randomness of the returns at each period are not explicitly expressed, we must keep in mind that constraints (6.1.2)-(6.1.5) represent statements for all scenarios that describe the market uncertainty.

6.1.2 A New Objective Function

In decision making, one must define the objective. For the option strategy, this is a minimum payoff in all scenarios via the put option. In asset/liability management, a penalty may be subtracted for targets not met as in the Russell-Yasuda model; see Cariño and Ziemba et al. (1994, 1998ab). Interdependence of the asset movement and the prescribed target is not considered in these models. In asset allocation models, maximizing expected asset value is a primary objective, but the dispersion among scenarios may yield large portfolio losses.

How can this risk be controlled? In mean-variance models, one adjusts the expected value by the variance measure of dispersion. The VaR approach addresses this issue. However this is typically based on a normal distribution assumption which is inconsistent with the evidence of fat tails in real asset markets. Hence, investors should choose their objective function so that the downside risk is considered. We utilize a new approach to measure risk, namely, the reward for the worst payoff.

Definition 6.1.1. Let $Y > 0$ be a random variable representing the terminal portfolio value. The worst payoff of Y is

$$Y^L = \sup\{K \in \Re; \Pr\{Y \geq K\} = 1\}.$$

The investors' payoff is characterized by the pair (z, Y^L), where z, the surplus, is defined by (6.1.4). The portfolio (α_t, x_t) is called self financing with transaction cost if Equations (6.1.1)-(6.1.5) are satisfied.

In Definition 6.1.1, we require a non-stochastic interest rate to assume that the probability of exceeding the worst Y^L is 1 . However, in some cases, such as models with uncertain short term interest rates and geometric Brownian motion, it may not have the perfect certainty of the worst case. One can then either relax the problem to a chance constraint or normalize all asset and portfolio values in unit of the riskless asset.

Assume that the investors' preference between the worst payoff W^L and the expected surplus over W^L is given by

$$E[z] + \mu W^L, \tag{6.1.6}$$

where the coefficient μ reflects risk aversion between a target and the expected surplus over this target. Investors are more likely to put wealth in the riskless asset to guarantee a minimum payoff with a large μ. The choice of the target and the expected surplus are discussed in the next section.

6.1.3 Formulation as a Recourse Problem

A stochastic programming with recourse formulation is a decision problem that maximizes the expected utility gained from the immediate decision at the current stage plus the expected utility that will be realized with constraints satisfied in the second stage. For our problem, the utility received at the first stage is μK and the second stage utility is measured by the expected surplus over the target K. The constraints that need to be satisfied are the allocation of wealth with the front load transaction costs at the first stage and the liability constraints at the second stage. Let $\mathbf{r}$ be the return vector of the risky assets, $\mathbf{x}$ the vector of the wealth portions invested in the risky assets, and $\boldsymbol{\theta}$ the transaction cost vector of risky assets. The dynamic recourse problem is to

$$\begin{aligned} \max_{K,\alpha,x_0} \quad & \mu K + Q_0(K, \alpha_0, \mathbf{x_0}) \\ \text{s.t.} \quad & \alpha_0 + \mathbf{x_0}'(1+\boldsymbol{\theta}) = W_0 \\ & \alpha_0, \mathbf{x_0} \geq 0 \end{aligned} \tag{6.1.7}$$

where

$$\begin{aligned} Q_t(K, \alpha_t, \mathbf{x}_t, \omega_t) = \max_{\mathbf{x}_{t+1} \geq 0} \quad & E\left[Q_{t+1}(K, \alpha_{t+1}, \mathbf{x}_{t+1}, \omega_{t+1}) | \mathcal{F}_t\right] \\ \text{s.t.} \quad & (6.1.2) - (6.1.5) \end{aligned}$$

and

$$\begin{aligned} Q_{T-1} = \max \quad & E\left[z(\omega_T) | \mathcal{F}_{T-1}\right] \\ \text{s.t.} \quad & z(\omega_T) = \alpha_{T-1} e^{\rho} + (1-\theta)\mathbf{x_{T-1}}' e^{\mathbf{r_T}(\omega_T)} - K \\ & z(\omega_T) \geq 0, \forall \omega = (\omega_1, \cdots, \omega_T) \in \Omega. \end{aligned} \tag{6.1.8}$$

It can be proved that the optimal solution K is equal to the worst payoff W^L of the terminal portfolio value. Since the risk aversion μ in this problem is a constant, the two-period recourse problem has the following simple solution.

Proposition 6.1.1. *If there are only two assets, one riskless and one risky, and one investment period, then there exists a μ_0 such that all investors with a $\mu < \mu_0$ will invest fully in the risky asset, and investors with $\mu > \mu_0$ will invest fully in the risk free asset. Investors with $\mu = \mu_0$ will be indifferent between the riskless and risky assets and any combination of them.*

If r follows a normal distribution, then $\mu_0 = e^{\overline{r}+\frac{1}{2}\sigma^2-\rho}$, where $\overline{r} + \frac{1}{2}\sigma^2 - \rho$ is the risk compensated rate of return. For a more general discussion of this concept see Zhao and Ziemba (2000a). However, if the number of assets is greater than 2 and the number of periods is greater than 1, then the result is not trivial.

6.1.4 The Multiperiod Stochastic Linear Programming Model

A multiperiod stochastic programming model considers the interdependency of uncertainty across the periods of the planning horizon in making decisions. A single period model with rollover cannot replace multiperiod models, because the uncertainties across periods are correlated and the future transaction costs may affect the initial decisions on portfolio construction, see Cariño, Myers, and Ziemba (1998) for calculation comparisons. The model objective is characterized by the immediate rewards after actions have been taken in each period. The first period reward is given by the selected target K and the risk aversion coefficient μ. The intermediate period rewards are zeros in this model, and the last period reward z is the excess return or surplus over the target K. The investor can control the portfolio by readjusting the weights subject to proportional transaction costs, assuming no injection and withdraw of funds. The downside risk control problem can be formulated as the multiperiod stochastic programming model

$$
\begin{array}{rllllllll}
\max\limits_{\mathbf{x}_0\geq 0,K} & E[\mu K+ & \max\limits_{\mathbf{x}_1\geq 0} E(0+ & \max\limits_{\mathbf{x}_2\geq 0} E(0+ & \cdots & & \max\limits_{\mathbf{x}_{T-1},z\geq 0} E(z)))] & & \\
\text{s.t.} & \alpha_0 + \mathbf{x}_0'(1+\boldsymbol{\theta}) & & & & & & & = W_0 \\
 & \ddots & \ddots & & & & & & \\
 & & & \alpha_t e^{\rho} - \Lambda_{t+1} & & & & & = 0 \\
 & & & x_t^i e^{\mathbf{r}_{t+1}^i} - \Gamma_{t+1}^i & & & & & = 0 \\
 & & & \ddots & \ddots & & & & \\
 & & & & \alpha_{T-2} e^{\rho} & & -\Lambda_{T-1} & & = 0. \\
 & & & & x_{T-2}^i e^{r_{T-1}^i} & & -\Gamma_{T-1}^i & & = 0 \\
 & K & & & & -\alpha_{T-1} e^{\rho} - & (1-\boldsymbol{\theta}) e^{\mathbf{r}_T\prime}\mathbf{x}_{T-1} + & z & = 0.
\end{array}
$$

where

$$\Lambda_{t+1} = \alpha_{t+1} - (1+\boldsymbol{\theta}_{t+1})'\mathbf{A}_{t+1} + (1-\boldsymbol{\theta}_{t+1})'\mathbf{D}_{t+1}, \text{ and}$$
$$\boldsymbol{\Gamma}_{t+1} = \mathbf{x}_{t+1} + \mathbf{A}_{t+1} - \mathbf{D}_{t+1}. \tag{6.1.9}$$

Λ_t is equal to the amount of wealth invested in the riskless asset at the start of period t after transaction costs, and $\boldsymbol{\Gamma}_t = (\Gamma_t^1, \cdots, \Gamma_t^n)$ are the amounts of the wealth invested in the risky assets. The first stage decision variables are the portfolio weights, α_0 and $\mathbf{x}_0 = (x_0^1, \cdots, x_0^n)'$, and the target K. Nonanticipativity is satisfied if $\mathbf{x}_t$ is $\mathcal{F}_t$-measurable. The optimal K depends upon the choice of μ, the risk aversion parameter. The second and third sets of constraints formulate the self-financing strategy. The last constraint represents that the terminal wealth is no less than the target K which is determined at stage 1.

6.2 Model Implications

We now discuss how the initial wealth W_0 and investor's risk aversion coefficient μ affect the optimal solution. Denote the optimal solution financed by $(\alpha_t(W_0,\mu), \mathbf{x}_t(W_0,\mu))$ by $(z(W_0,\mu), K(W_0,\mu))$ and the optimal objective by $J(W_0,\mu)$

Proposition 6.2.1. *For given $\mu \geq 1$,*

(a) $K(W_0,\mu)$ is nonnegative and bounded above by $\rho^T W_0$ if there are no arbitrage opportunities.

(b) $K(W_0,\mu)$ is increasing in μ, and $E[z(W_0,\mu)]$ is decreasing in μ.

Proof. (a) Assume for some $\mu \geq 1$, $K(W_0,\mu) < 0$. Then, $(z(W_0,\mu)+K(W_0,\mu), 0)$ is also a feasible solution and can be financed by the same optimal portfolio, but the value $E[z(W_0,\mu) + K(W_0,\mu)] > E[z(W_0,\mu)] + \mu K(W_0,\mu)$ since $\mu \geq 1$. Hence, $(z(W_0,\mu), K(W_0,\mu))$ cannot be an optimal pair with $K(W_0,\mu) < 0$, and $K(W_0,\mu) \geq 0$. Given the presence of a riskless asset, $K(W_0,\mu) \leq \rho^T W_0$ if there is no arbitrage opportunity,

(b) Let (z_1, K_1) and (z_2, K_2) be the optimal solutions for μ_1 and μ_2, respectively. By optimality

$$E[z_1] + \mu_2 K_1 \leq E[z_2] + \mu_2 K_2 \tag{6.2.1}$$

$$E[z_2] + \mu_1 K_2 \le E[z_1] + \mu_1 K_1. \tag{6.2.2}$$

Combining (6.2.1) and (6.2.2), yields

$$(\mu_2 - \mu_1)(K_2 - K_1) \ge 0.$$

So, $K(W_0, \mu)$ is increasing in μ. Similarly, it can be proved that $E[z(W_0, \mu)]$ is decreasing with μ.

Proposition 6.2.1 shows that, for a risk averse investor, the target K is always positive and is bounded above by the total return if all the fortune is invested in the cash instrument in an arbitrage free market. As the risk aversion coefficient μ increases, the investor is more risk averse and increases the target, therefore, decreases the expected surplus over the target.

Proposition 6.2.2. *The optimal value function $J(W_0, \mu)$ is linearly increasing in W_0 and convex increasing in μ.*

Proof. For $\forall\lambda > 0$, if (α_t, x_t) finances the optimal solution pair $(z(W_0, \mu), K(W_0, \mu))$ then $(\lambda\alpha_t, \lambda x_t)$ finances $(\lambda z(W_0, \mu), \lambda K(W_0, \mu))$, therefore, $J(\lambda W_0, \mu) \ge \lambda J(W_0, \mu)$. Since this inequality is also true for any (W_0, λ), it follows that $J(\lambda W_0, \mu) = \lambda J(W_0, \mu)$.

From (6.2.1) and (6.2.2), if $\mu_1 \le \mu_2$, then

$$J(W_0, \mu_1) \le E[z_1] + \mu_2 K_1 \le J(W_0, \mu_2) \tag{6.2.3}$$

which proves that $J(W_0, \mu)$ is increasing in μ.

Given $\lambda_1, \lambda_2 \ge 0, \lambda_1 + \lambda_2 = 1$, we want to prove that

$$J(W_0, \lambda_1\mu_1 + \lambda_2\mu_2) \le \lambda_1 J(W_0, \mu_1) + \lambda_2 J(W_0, \mu_2). \tag{6.2.4}$$

Let (z^*, K^*) be the optimal solution for $\lambda_1 + \lambda_2$, then

$$E[z_1] + \mu_1 K_1 \ge E[z^*] + \mu_1 K^*, \tag{6.2.5}$$

and

$$E[z_2] + \mu_2 K_2 \ge E[z^*] + \mu_2 K^*. \tag{6.2.6}$$

Combining (6.2.5) and (6.2.6) proves (6.2.4).

For given initial wealth W_0, if $(\mathbf{x}_0, \cdots, \mathbf{x}_{T-1}, z, K)$ is an optimal solution to model (6.1.9), then $\lambda(W_0, \mathbf{x}_0, \cdots, \mathbf{x}_{T-1}, z, K)$ will be a feasible solution to Model (6.1.9) for initial wealth equal to λW_0. Furthermore, it can be proved using Proposition 3 that $\lambda(W_0, \mathbf{x}_0, \cdots, \mathbf{x}_{T-1}, z, K)$ is also an optimal solution. This proves

Proposition 6.2.3. *The optimal strategy for the initial period in terms of the portfolio weights (proportions of the initial wealth) is independent of the initial wealth level.*

6.3 Asset Liability and Synthetic Put Option

Institutional investors are often obligated to future promised cash outflow as liabilities. For the model discussed here, we use the constant μ to represent investors' risk attitude between a target and its expected surplus. We can extend this to an arbitrary concave function that represents an investor's risk aversion. The liability constraint is not included in this model. What if the liability constraint is included? For problems when the liability constraint is binding, there is an intuitive way to interpret the optimal solution. Investors place part of their wealth in the risky and/or risk free asset and use the rest of their wealth to buy insurance on the initial portfolio that pays their liability. This can be done using a synthetic approach through option replication or by buying put options. The latter method can be implemented only if these options are market traded. The former method is more plausible if market factors are carefully manipulated. We now discuss the relation between the liability constrained and unconstrained model solutions.

How does an investor with risk aversion μ, endowment M, and a horizon liability L design his investment plan? The investor can find the optimal solution by solving the stochastic programming formulated in (8) with an additional liability constraint $K \geq L$. Derivatives can be used for the control of the downside losses (cf. Cariño and Turner 1998), but how much should be spent on the derivative asset to optimize the portfolio value with the liability being met? The following theorem illustrates the relation between these two solutions. We call the model without the liability constraint the *unconstrained problem*; and the one with a liability constraint the *constrained problem*.

Theorem 6.3.1. *Let $(z(W_0,\mu), K(W_0,\mu))$ be the optimal solution to the unconstrained problem with a starting wealth W_0 and the optimal self-financing portfolio is $(\alpha_t, \mathbf{x_t})$. If*

$$\frac{L}{K_{W_0}} = \frac{M}{W_0} := \beta \qquad (\geq 1) \tag{6.3.1}$$

then $(\beta z(W_0,\mu), L)$ is the optimal solution to the constrained problem with starting wealth M and the optimal self-financing portfolio is $(\beta\alpha_t, \beta\mathbf{x_t})$.

Proof. If $(z(W_0,\mu), K(W_0,\mu), \alpha_t(W_0,\mu), x_t(W_0,\mu))$ is an optimal solution to model (6.1.9) with initial wealth W_0, the unconstrained problem, then

$$\beta \cdot (z(M,\mu), K(M,\mu), \alpha_t(M,\mu), \mathbf{x}_t(M,\mu))$$

is a feasible solution to the constrained problem with the liability L and the starting wealth M. By Proposition 3, it is actually an optimal solution. The converse of the above argument is also true. This proves Theorem 6.3.1. $\square$

Investors can make their required insurance plan by solving the unconstrained problem to obtain the optimal target $K_{W_0} = K(W_0, \mu)$ and by solving (6.3.1) for W_0 which is then invested in the risk free and/or risky assets. Thus, the rest of the fund $M - W_0$ is used to buy insurance (put option) on his investment portfolio. In this way investors' liabilities at the horizon are guaranteed to be met. This also provides an approach for practitioners to implement derivative strategies.

The multiperiod stochastic programming model with an objective function given in this chapter recommends that the optimal investment strategy will allocate wealth to assets of superior performance through time to meet an optimal worst payoff K. Because a partial model objective is to increase K while expecting an overall good performance, investors will reallocate wealth in each intermediate period according to the suggested optimal solution as time goes along. This strategy is equivalent to synthesizing an option-like payoff with the worst payoff as the strike price and the optimal portfolio as the underlying asset.

Actual liabilities depend upon a number of factors, including interest rates at the horizon dates, cumulative inflation over the planning period, etc. Previous discussions in this section will not directly apply since L is a random variable in this case. A direct way of imposing constraints on K is required for the original formulation of the problem, which can be done by adding penalty constraints on targets missed. However, previous discussions can also apply if the return of the riskless asset reflects the randomness of the interest rate and inflation (The return of the riskless asset can be random through horizon but has no risk, i.e. the return is certain at the time of making investment decisions). One then needs to normalize all risky asset returns in terms of the returns of the riskless asset in order to apply the method discussed in this section.

6.4 Asset Returns and Scenario Generation

Generally, there are two ways of modeling future asset returns. The *adaptive expectations* approach depends only on past observations of the explanatory variables. Alternatively, a *rational expectations* model can be used using forecasts produced by conceptual macroeconomic models where expectations are used. The former approach is easy to deal with using standard assumptions and past data. The latter becomes a benchmark for the estimation of unobserved expectations, but it assumes that investors have "common" knowledge of the structure of the future events (e.g. coupons and yields for bonds, dividends and earnings for equities). This chapter does not focus on the evaluation of these strategies. We adopt the *adaptive expectations* approach, because modeling future events is as hard as choosing a data generating process that fits historical observations. To model the price interactions between assets, we use the *Vector Auto Regression* model for future asset returns:

$$\mathbf{r_t} = \mathbf{C} + \mathbf{D_1 r_{t-1}} + \mathbf{D_2 r_{t-2}} + \cdots + \mathbf{D_p r_{t-l}} + \epsilon_t. \tag{6.4.1}$$

Then,

$$E[\mathbf{r_t}|\mathcal{F}_{t-1}] = \mathbf{C} + \mathbf{D_1 r_{t-1}} + \mathbf{D_2 r_{t-2}} + \cdots + \mathbf{D_p r_{t-p}}$$

where $\mathbf{r_t}$ is the vector of logarithmic rates of return of the risky assets. $\boldsymbol{\epsilon_t}$ is the vector of random disturbances with mean zero which is identically and independently distributed across time periods. p is the number of lags used in the regression and $\mathbf{D_1}, \cdots, \mathbf{D_p}$ are time independent constant matrices which are estimated through statistical methods, such as maximum likelihood. While $\mathbf{D_1}, \cdots, \mathbf{D_p}$ characterize the persistence of future returns to the nearest past p realized returns, $\mathbf{C}$ is the vector of intercepts from auto regression.

The proper number of lags to use is not known *ex ante.* There are three criteria for determining how many to use. The first is to have enough so that $\mathbf{r_{t-p-1}}$ is insignificant in the regression. The second is to have enough so that the assumption that ϵ_t is independently and identically distributed is satisfied. The third is not to include unnecessary lags that would reduce the precision of the estimates.

We assume a second order stationary process for asset returns, where the first two moments of the process are independent of time t

$$\begin{aligned} E[\mathbf{r_t}] &= E[\mathbf{r_s}] \\ E[\mathbf{r_{t+h}}'\mathbf{r_{s+h}}] &= E[\mathbf{r_t}'\mathbf{r_s}] \end{aligned} \tag{6.4.2}$$

for $\forall s, t, 0 \leq s, t \leq T$.

Proposition 6.4.1. *Equation (6.4.1) is second-order stationary if* $\forall\lambda$, *such that*

$$|\mathbf{I_n}\lambda^l - \mathbf{D_1}\lambda^{l-1} - \cdots - \mathbf{D_p}| = 0$$

implies $||\lambda|| < 1$.

Proof. See Hamilton (1994).

The stationary process is based on the assumption that the returns and the volatility should be stationary in time. That is the effect that seasonal anomaly returns are negligible; for evidence on this see Fama (1998) and Keim and Ziemba (2000).

Generating good scenarios is an important aspect of any stochastic programming application. Mulvey (1996) describe a global scenario system, developed by Towers Perrin, based on a set of differential equations consistent with the underlying economic factors, such as, price and wage information, interest rates, growth rates, stock dividend yields, etc., for pension plans and insurance companies throughout the world. In the example below we utilize a vector auto-regression model where the current return structure is forecast by past returns. The residual of the past data will be used to model the disturbances of return in each period.

6.5 An Integrated Application

Developing financial planning models under uncertainty consists of modeling the investment asset returns and developing a good strategy within the framework of risk/return tradeoff. Suppose the investment opportunities consist of three assets: Cash, Bonds and Stocks. Assume a one-year horizon with quarterly portfolio reviews subject to transaction costs, i.e., a four period model. The cash return is $\rho = 0.0095$ identical for each quarter. The Salomon Brothers bond and S&P 500 indices are used as proxies for the bond and stock benchmarks, respectively. Figure 6.1 depicts quarterly returns of these two risky assets from January 1985 to December 1998.

The expected logarithmic rates of return are $ms = 0.04$ for Stocks and $mb = 0.019$ for Bonds which we take as the estimates of the unconditional expected rates of return. An appropriate vector auto regression model of order two is estimated as

$$\begin{aligned} s_t &= 0.037 - 0.193 s_{t-1} + 0.418 b_{t-1} - 0.172 s_{t-2} + 0.517 b_{t-2} + \epsilon_t \\ b_t &= 0.007 + 0.140 s_{t-1} + 0.175 b_{t-1} + 0.023 s_{t-2} + 0.122 b_{t-2} + \eta_t. \end{aligned} \tag{6.5.1}$$

The first lag coefficients of s_{t-1} and b_{t-1} are statistically significant at the 10% level. The model assumption of identical disturbances are checked by testing the auto correlation of residuals. The conditions of Proposition 5 are satisfied, therefore, Model (6.5.1) is second-order stationary as required. Uncertainty is characterized by the pair (ϵ_t, η_t). A random sampling approach is used to estimate the joint distribution of (ϵ_t, η_t), which eventually forms the total number of scenarios for solving the model. Since (ϵ_t, η_t) are identically and independently distributed, a large sample of

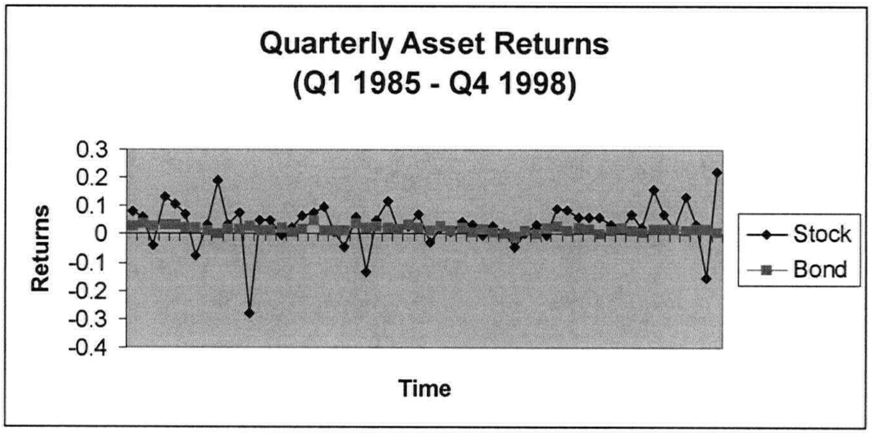

Figure 6.1: Historical Stock Returns

this pair can be generated for characterizing the disturbances in each period. We randomly selected 20 pairs of (ϵ_t, η_t) to estimate the empirical distribution of one period uncertainty. Model (6.5.1) is applied for generating the scenarios of the next period rate of return using the last two observations and the period disturbances re-sampled from the large sample. In this way, we have generated a large scenario tree with 160, 000 paths describing the possible outcomes of asset returns.

Transaction cost is imposed to prevent investors from buying and selling whenever the gain from transaction is less than the cost. At the same time, imposing transaction costs will restrict the portfolio weights from large asset turnovers from period to period. This dominant together with holding constraints will help construct and implement a more practical portfolio. The proportional transaction cost of $\theta_s =$ 0.5% for Stocks and $\theta_b = 0.1\%$ for Bonds are the same for all investment periods. The investment environment assumes no short selling and the holding proportions of wealth in Stocks and Bonds are bounded by 70%. So, a four-period model (one year planning horizon) with transaction costs and limited asset holding constraints is established. The results with varying risk aversion μ, such as initial portfolio weights, the optimal planning target, the expected portfolio terminal payoff, etc., are shown in Table 1 using the IBM OSL stochastic programming package.

As μ increases, i.e. risk aversion increases, investors move funds from stocks to bonds and/ or to cash. This results in a large K and achieves the purpose of downside risk control. Since the objective function is $E[z] + \mu K$, the expected value

Table 1: Risk Aversion, Portfolio Weights, Expected Payoff, Standard Deviation and Sharpe Ratio

Risk Aversion (μ)	Cash (α)	Stock (x^1)	Bond (x^2)	Target (K)	Expected Payoff	Standard Deviation	Sharpe Ratio
1.00	0.000	0.700	0.300	91.5	127.4	0.152	1.54
1.25	0.000	0.700	0.300	97.5	122.8	0.118	1.60
1.50	0.000	0.669	0.331	98.0	122.6	0.116	1.61
1.75	0.000	0.561	0.439	99.4	121.7	0.109	1.64
2.00	0.000	0.322	0.678	102.7	118.8	0.088	1.69
2.25	0.000	0.300	0.700	103.0	118.5	0.086	1.69
2.50	0.119	0.181	0.700	104.6	116.3	0.072	1.73
2.75	0.119	0.181	0.700	104.6	116.3	0.072	1.73
3.00	0.119	0.181	0.700	104.6	116.3	0.072	1.73
3.25	0.119	0.181	0.700	104.6	116.3	0.072	1.73
3.50	0.119	0.181	0.700	104.6	116.3	0.072	1.73
3.75	0.119	0.181	0.700	104.6	116.3	0.072	1.73
4.00	0.119	0.181	0.700	104.6	116.3	0.072	1.73
4.25	0.119	0.181	0.700	104.6	116.3	0.072	1.73
4.50	0.135	0.180	0.684	104.6	116.3	0.072	1.73
4.75	0.135	0.180	0.684	104.6	116.3	0.072	1.73
5.00	0.135	0.180	0.684	104.6	116.3	0.072	1.73
5.25	0.217	0.176	0.607	104.6	116.2	0.071	1.73
5.50	0.342	0.169	0.489	104.7	116.1	0.071	1.72
5.75	0.522	0.159	0.319	104.7	115.8	0.070	1.72
6.00	0.860	0.140	0.000	104.8	115.3	0.068	1.70

of the terminal portfolio can be calculated as

$$E[z+K] = J(W_0,\mu) - (\mu-1)K,$$

where $J(W_0,\mu)$ is the optimal objective value and K is the optimal worst payoff for a given pair (W_0,μ). The changing weights are strongly dependent on the correlation of the risky assets. Even an extremely risk averse investor should have been willing to invest certain amount in the risky assets, Stocks and/or Bonds. Figure 6.2 depicts the initial portfolio weights by varying the investor's risk aversion.

Kallberg and Ziemba (1983) examine the optimal composition of a static risky investment portfolio using alternative utility functions and parameter values, and derive similar results. In particular, assuming normally distributed returns, they show that investors with similar average risk aversion indices but different concave utility functions have similar optimal portfolio weights and expected utilities.

How does this compare to a static mean-variance model? The mean-variance efficient frontier is a ray through the market portfolio starting at the riskless asset, if the market is complete. With transaction costs and portfolio holding constraints, the efficient frontier is no longer a line but a convex curve. The results derived from this model are used to analyze the performance using the Sharpe ratio, and the comparison is made to the mean variance model. Using this scenario data which estimate

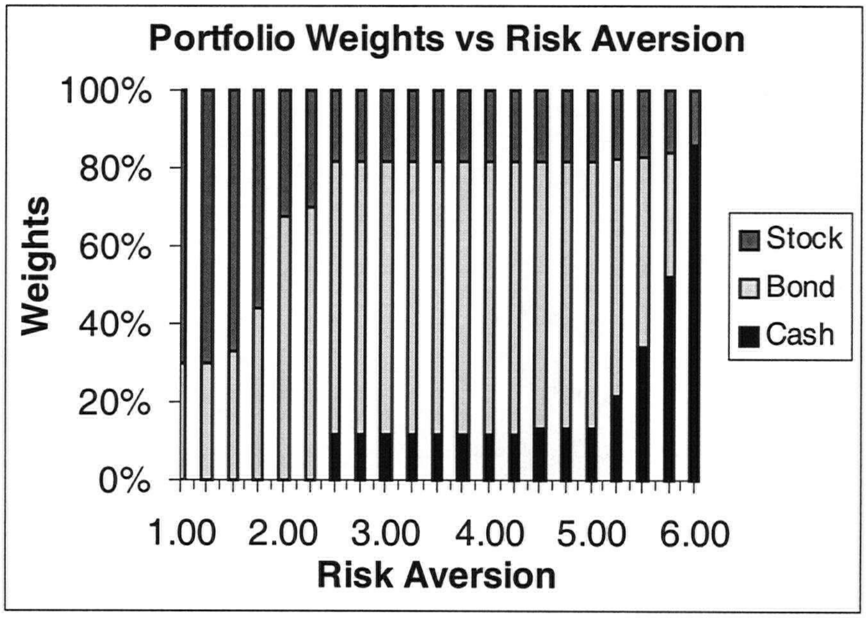

Figure 6.2: Optimal Weights vs Risk Aversion.

the expected annual rate of return as 36.8% for Stocks and 4.7% for Bonds, the corresponding standard deviations as 20.9% and 1.3%, and the correlation as -1.4% between Stocks and Bonds. To locate the global mean-variance efficient portfolio, one needs to solve a model that maximizes a Sharpe ratio subject to the trading constraints including transaction costs, holding constraints and the balance constraints for each period. For this scenario data, the global mean-variance efficient portfolio weights are 14.4% and 85.6% with the Sharpe ratio of 1.638. The low stock weight of 14.4% occurred because the bond was performing extremely well. In order to make comparisons between this model and the mean-variance analysis, we need to find the optimal expected portfolio return for the same standard deviation (same risk) when mean-variance analysis is used. As shown in Figure 6.3, the downside risk control model are dominant for most of the cases. This is because of the functioning of dynamic control utilizing the dynamic forecasts which are correlated across investment periods.

The effect of this downside risk control model is investigated under the performance measurement of the Sharpe ratio. Not only are the results preferable but also the optimal worst payoff makes the model more practical. The target K is the minimum terminal payoff. How is that convincing? Figure 6.4 gives a typical distribution

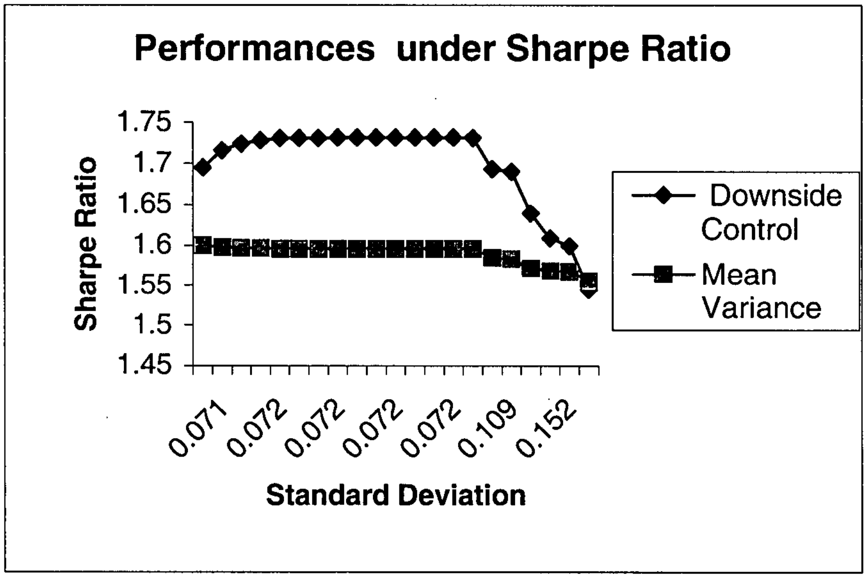

Figure 6.3: Comparisons under Sharpe Ratio for Varying μ

for $\mu = 2.5$ which is highly skewed to the right because of the effect of dynamic downside risk control. The floor of the terminal portfolio is about 4.6% more than the initial portfolio wealth with an expected rate of return 16.33% and a standard deviation 7.2%. If mean variance analysis is used, the expected return is 15.4% with the same standard deviation, but a floor above some desired level is no longer guaranteed. For the mean-variance model, the worst possible outcome wealth is about 5% below the initial wealth. These results reflect that investors periodically adjust the portfolio weights subject to any indispensable costs to acquire a bigger expected return and a desired wealth floor by utilizing the serial correlation among assets and cross correlation between assets.

Figure 6.5 shows the change of weights through the horizon for a typical scenario (Scenario 300 in the calculation). Initially, 11.9%, 18.1% and 70% are invested in Cash, Stocks and Bonds, respectively. In the second period, Stocks and Bonds hold 41.4% and 58.6% of the total wealth, and Cash has a zero weighting. In the third period, the optimal weighting is Stocks 70% and Bonds 30%. In the fourth period, all assets will be in Cash and Bonds. This stream of operations on allocating assets reflects how investment strategies capture the investment environment according to the forecast of the asset returns. The forecast model plays a very crucial role in multiperiod stochastic programming. If the forecast model were not able to anticipate

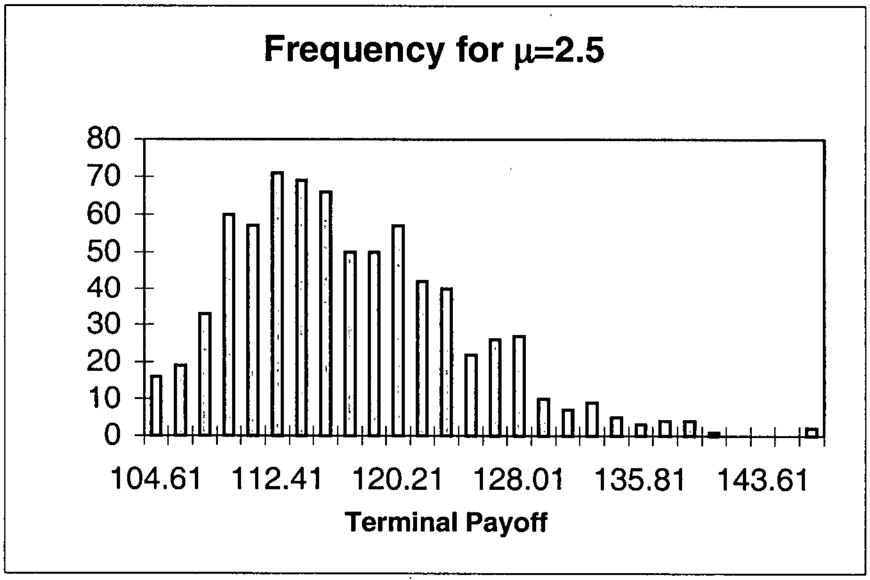

Figure 6.4: The Distribution of Terminal Portfolio Value

market swings, the optimal strategy would "optimize" error. The stream of wealth for this typical scenario is 99.8 $\rightarrow$ 103.3 $\rightarrow$ 106.7 $\rightarrow$ 112.3 $\rightarrow$ 113.9.

A problem that we do not explore here is how to test the performance of the model in reality, but see Kusy and Ziemba (1986) for such an example. Since the real-world realization may not be one of the scenarios that we have specified for the computational procedure, we must specify a policy that decides the adaptation of the real-world outcomes to the set of specified scenarios to implement a multiperiod stochastic programming model. Our focus here is to find the optimal strategy for a given set of scenarios. Modeling uncertainty through scenarios is the first major step for developing stochastic programming models.

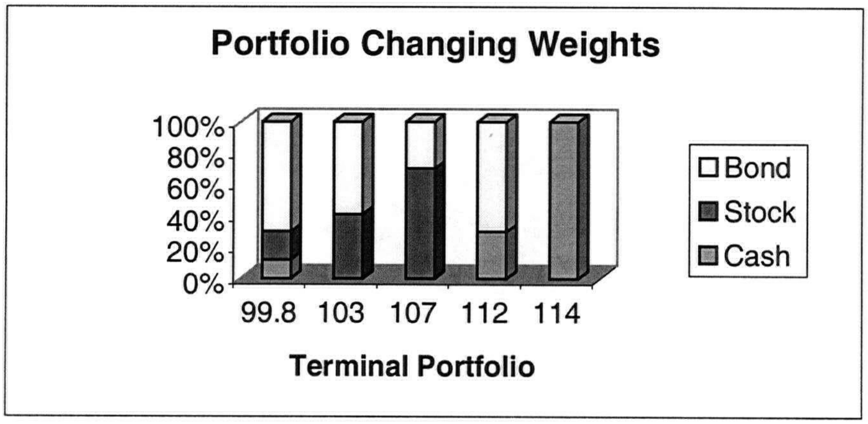

Figure 6.5: The Change of Weights for A Typical Scenario

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