ESSAYS ON STRATEGIC TRADING, ASYMMETRIC INFORMATION, AND ASSET PRICING

by

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Abstract

This thesis presents three models of asset pricing involving non-competitive behavior and asymmetric information. In the first model, a risk averse investor with private information about dividends trades shares over an infinite time horizon with risk neutral uninformed agents. The informed investor trades strategically in equilibrium. The second model also involves an infinite time horizon, but all agents are risk averse and equally informed about dividends. Non-competitive behavior is exogenously specified; price takers trade shares with a strategic investor who accounts for the effects of her trades on the stock price. In this case, an endogenous information asymmetry arises in equilibrium. Closed form equilibria are derived for both models and implications for price dynamics are explored. While the first model constitutes a new extension of the multiperiod Kyle model of insider trading, the second model generates more interesting price dynamics. If the strategic investor manages a large mutual fund, significant risk premia and price volatility may arise in equilibrium. In fact, if mutual fund participation is sufficiently widespread, multiple equilibria may exist. The third model extends the multiperiod Kyle model to a case where the insider observes a noisy signal of the stock's terminal liquidation value. An equilibrium much like Kyle's is derived. Price tends toward value over time, and stock price volatility depends on both the drift and volatility of the insider's private signal. Like the Kyle model, the insider's trading activity leaves no detectable trace in trading volume, expected returns, or price volatility.

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Chapter 1

Introduction

The relationship between the price of a financial asset and its fundamental value is central to financial economics. Black (1986) attributes deviations of price from value to the presence of irrational *noise traders*; people who, from an objective point of view, would be better off not to trade. On average, noise traders should expect to lose money to people who are better informed about the asset's fundamental value. From the standpoint of rational investors, noise traders simply introduce random variations in the asset supply. Irrational noise traders are not the only possible cause of random supply variations. Spiegel (1998) argues that supply noise is also caused by random changes to the capital base of the economy as society creates and destroys assets. Whatever their cause might be, the presence of random supply shocks together with differentially informed rational investors can lead to rich models of asset pricing.

Perhaps the two most influential models of this type are Kyle (1985) and Wang (1993). In Wang's model, informed investors maintain an infinitely long–lived informational advantage over their uninformed counterparts. Trading in a stock and a riskless bond takes place continuously over an infinite time horizon. The stock's dividend dynamics are determined in part by a stochastic growth factor which can only be observed by the informed investors. Uninformed investors can only observe the stock price and the dividend process. The supply of stock is stochastic and *a priori* unobservable. Wang obtains a competitive equilibrium in which the informed can infer the supply from the stock price and dividend, while the uninformed can infer neither the supply nor the dividend growth factor. The resulting informational asymmetry is impounded in both investors' consumption–investment policies. A similar dynamic arises in a finite horizon context in the multiperiod Kyle model (1985), where an insider with exclusive knowledge of the terminal stock price executes trades through a risk neutral market maker. Supply noise, attributed to noise traders, prevents the market maker from observing the insider's trades. As Kyle puts it, "the noise traders in effect provide camouflage which enables the insider to make profits at their expense." In both models, the uninformed rational agents cannot distinguish between supply variations and informed agents' trades. Consequently, the informed may trade on their private information without revealing it to the uninformed. The Kyle model illustrates how information asymmetry and non–competitive behavior can jointly arise. The stock price moves in response to changes in the insider's shareholdings, causing her to trade strategically rather than competitively.

The purpose of this thesis is to investigate the relationship between non–competitive behavior and informational asymmetry from three different perspectives. Our first model, presented in Chapter 2, is a hybrid version of the Kyle and Wang models. A risk averse investor with private information about a stock's dividend stream trades shares with uninformed investors over an infinite time horizon. While this model has the same information structure as Wang's, it also resembles Kyle's in that the uninformed agents are risk neutral. By contrast to the Kyle model, the true value of the stock is not announced at some finite date. Instead, partial information about stock fundamentals is continually released to the uninformed via the dividend stream. Like the Kyle model, we find an equilibrium in which the informed investor trades strategically. Our first model can be viewed as a contribution to the literature on applications and extensions of the Kyle model. The model is unique in that the informed investor's total wealth has no impact on her equilibrium shareholdings, while the composition of her investment portfolio does. Such "leverage dependence" does not arise in other models of this type. Moreover, in contrast to Wang's model, the informed investor may rationally act as a price chaser while the uninformed agents act as contrarians.

Unfortunately, the first model is not compelling from an econometric standpoint. Since the uninformed agents are risk neutral, the excess stock return is simply a Brownian motion conditioned on public information. This property violates the most basic findings of empirical studies of stock price behavior. Substantial supply noise is also required in order to generate realistic levels of stock price volatility, or variability. Since supply noise variations are generally thought to be much smaller than stock price variations [Spiegel (1998)], this is an unattractive characteristic. Our second model, appearing in Chapter 3, generates more realistic implications for the time series behavior of stock prices. In this case, agents have identical information about dividends but have different beliefs about the extent to which their trades move the stock price. There are two types of rational agents in this economy; a risk averse strategist, who accounts for the impact of her trades on prices, and risk averse price takers. Despite the fact that there is no *a priori* information asymmetry, we find an equilibrium in which the strategist can infer the supply of stock from the stock price and dividend while price takers cannot. In contrast to the Kyle and Wang models, this information asymmetry does not arise as a result of privately held information about stock fundamentals. Instead, it arises as a consequence of agents' heterogenous beliefs about how their respective trades affect the stock price. As in the first model, the strategist is a price chaser and her equilibrium investment policy exhibits leverage dependence. However, unlike the first model, significant price variability can arise even when the supply noise is relatively small. As Spiegel (1998) points out, this is a desirable feature of any asset pricing model involving supply noise. Furthermore, risk premia and serial correlation in stock returns can be strongly influenced by the size of the strategist's shareholdings. Viewing the strategist as the manager of a large mutual fund, this behavior is consistent with anecdotal evidence reported in the popular press.

Our third model, appearing in Chapter 4, is closer in nature to the multiperiod Kyle model. However, it involves a weaker informational advantage on the part of the insider. Instead of assuming that the insider knows the terminal share value, we assume that she has exclusive knowledge of a noisy dynamic signal terminating at the terminal share value. Like Back (1992), the model accommodates relatively general distributions for the terminal share value. We show that, within an interesting parametric class of diffusion models for the private signal, an equilibrium virtually identical to Kyle's exists. We also obtain a closed form representation for the equilibrium price process. For example, if the signal follows a geometric Brownian motion, then so does the equilibrium share price. The arrival pattern of the insider's private information is of little consequence to the equilibrium price or to the market maker's perception of trading volume. This behavior is much different from that of the Admati–Pfleiderer (1988) model, in which the insider's informational advantage is short–lived. There, the insider acts on private information immediately, imparting an informational component to price changes. By contrast, the insider's equilibrium trading strategy in this model conveys no information to the market maker about shocks to the insider's signal process. This model differs substantially from Kyle's in the way that information is impounded in the insider's shareholdings. In the Kyle model, the insider can simply "look ahead" to the terminal share value at any prior point in time. In the model constructed here, the insider must form her trades based on successive realizations of the signal process. The fact that the resulting equilibrium should be so similar to that of the Kyle model is interesting and not at all obvious. The third model has been independently discovered by Back and Pedersen (1998), who arrive at the same equilibrium under somewhat different technical assumptions.[1]

[1]Back and Pedersen's model is slightly more general than ours in that they permit noise trades to have deterministic variability patterns. Our model requires noise trades to have constant variability. However, this additional generality is of minor consequence to the equilibrium. Back and Pedersen also assume that the insider's private signal is her expectation of the terminal share value. Our approach is informationally equivalent to theirs in that the insider's private signal can be expressed as a function of time and the expected terminal share value, and is monotone increasing in the latter variable. It should be noted that Back and Pedersen's work first came to my attention in June 1997, two weeks after I completed the first draft of Chapter 4. I immediately sent a copy of the draft to Kerry Back, who confirmed via email that my work appeared to be an independent discovery of the same equilibrium.

Chapter 2

An Infinite Horizon Counterpart to the Kyle Model

This chapter presents an infinite horizon, continuous–time asset pricing model involving non-competitive behavior and informational asymmetry. A risk averse investor with private information about a stock's dividend stream trades shares with risk neutral uninformed investors. Like the multiperiod Kyle model (1985), the informed investor trades strategically in equilibrium. However, unlike the Kyle model, the true value of the stock is not announced at some finite date. Instead, partial information about stock fundamentals is continually released to the uninformed via the dividend stream. This model has some features in common with the Kyle model. The strategic investor's equilibrium investment policy has continuously differentiable sample paths. The model also satisfies a "no–trade" theorem analogous to that of the Kyle model. However, there are also some important differences. For example, unlike the Kyle model, the strategic investor's equilibrium policy depends upon how much stock she currently holds. The larger her shareholdings, the more favorable market conditions must be in order for her to further increase her position. This behavior is largely due to the fact that in the present model, supply noise reverts to a zero mean, while the supply noise in the Kyle model follows a Brownian motion. The use of a mean reverting noise process follows Wang (1993), and is required in order to obtain a stationary equilibrium. In order to provide some intuition, we discuss this aspect of the strategic trader's equilibrium policy in some detail at this point.

Because of her market power, it is costly for the strategic trader to turn shares over at a high rate. However, her private information, having a Brownian component, can change rapidly. Consequently, her current shareholdings may not reflect the current state of her private information. She is unable to act sufficiently quickly on bad news to prevent it from partially eroding the value of her position. This problem worsens as her position in the stock increases relative to the total supply of shares. She must pay a large cumulative premium to unwind a large position if the market as a whole is relatively thin. To avoid this scenario, her shareholdings never deviate too far from the total supply of stock, which is stationary. While the insider trading strategy in the Kyle model is also sensitive to the supply of stock, this phenomenon does not arise there because the total supply is not stationary. Moreover, despite the similarity of our model to Wang (1993), the strategic investor may rationally act as a price chaser, in contrast to the contrarian behavior exhibited by Wang's informed investor. More accurately, the price chases her; when she buys stock the price tends to rise and when she sells it the price tends to fall. This is a direct consequence of the strategic trader's market power.

While our model extends the Kyle model in several directions, it unfortunately does not produce realistic stock price dynamics. The equilibrium excess stock return is simply a Brownian motion conditioned on public information. Furthermore, the equilibrium price variability is smaller than that of a simple benchmark model in which no informed trader is present. Consequently, substantial supply noise is required in order to generate realistic levels of stock price variability. As Spiegel (1998) points out, this is an undesirable characteristic of any asset pricing model. Despite these drawbacks, the first model establishes a useful context for our second model, presented in Chapter 3.

2.1 Model I

The following model is a hybrid version of Kyle (1985) and Wang (1993). A stock and a riskless bond are continuously available for trade throughout the time interval $[0, \infty[$. The riskless rate of interest is a constant $r > 0$, while the stock pays dividends at the rate D_t per unit time, where

$$\begin{aligned} dD_t &= (\Pi_t - kD_t)\,dt + \sigma_D dz_D(t), && (2.1)\\ d\Pi_t &= a_\Pi\left(\bar{\Pi} - \Pi_t\right)dt + \sigma_\Pi dz_\Pi(t). && (2.2) \end{aligned}$$

An information asymmetry arises because the dividend rate D is common knowledge, but only one investor, the *informed* investor, has exclusive knowledge of the dividend growth factor Π. The terms $a_\Pi(> 0)$, $\sigma_j(> 0)$, $k(\geq 0)$ and $\bar{\Pi}$ are constants, while $[z_D, z_\Pi]^\top$ is a two–dimensional Brownian motion. When $k > 0$, the dividend rate reverts to the "stochastic mean" Π_t. Otherwise the dividend process is nonstationary. Dividends may take on both positive and negative values in this somewhat stylized model. Following Wang, the total supply of stock at time t is $1 + \Theta_t$, where

$$d\Theta_t = -a_\Theta \Theta_t dt + \sigma_\Theta dz_\Theta(t). \tag{2.3}$$

The terms a_Θ and σ_Θ are positive constants, while z_Θ is a standard Brownian motion independent of z_D and z_Π. The factor $-\Theta$ is analogous to noise trader demands in the Kyle model.

The informed investor trades shares with uninformed risk neutral agents. In previous literature related to the Kyle model, the informed investor seeks to maximize the expected utility of accumulated wealth at the time the liquidation value of the stock is announced. Since there is no such announcement date in our model, we assume instead that the informed investor is risk averse over her (infinite) consumption stream $\{c_t\}$. The informed investor seeks to maximize

$$\mathbb{E}\left[\int_0^\infty -e^{-\rho s - \gamma c_s} ds\right], \tag{2.4}$$

where ρ and γ are positive constants. The informed investor's information filtration $\mathcal{I} = \{\mathcal{I}_t; t \geq 0\}$ is generated by D, Π, the stock price P, and her initial wealth and shareholdings. While a typical uninformed agent cannot observe the informed investor's shareholdings X or the supply noise Θ, she can deduce their difference $Y = X - \Theta$ since in aggregate, the uninformed hold the residual supply $1 + \Theta - X$. Each of the uninformed agents believes that the informed investor's shareholdings take the form $X = X^*$, where X^* is some $\mathcal{I}$–adapted state–contingent process. Thus, if $\mathcal{U}$ denotes the filtration generated by D and Y, we require

$$P_t = \mathbb{E}^* \left[\int_t^\infty e^{-r(s-t)} D_s ds \middle| \mathcal{U}_t \right], \tag{2.5}$$

where $\mathbb{E}^*$ denotes the uninformed agents' conditional expectation operator under the belief $X = X^*$.[1] Provided the state dependence of X^* is sufficiently simple, P is a $\mathcal{U}$–adapted semimartingale and the excess stock return $dQ_t = dP_t + D_t dt - rP_t dt$ takes the form $dQ_t = dQ_t^X$, where

$$dQ_t^X = a_Q dt + \lambda dX_t + b_{QD} dz_D(t) + b_{Q\Theta} dz_\Theta(t). \tag{2.6}$$

Here λ and the b_{Qj}s are nonzero constants, while a_Q is an $\mathcal{I}$–adapted process.[2] We call (2.6) (or, more briefly, Q^X) the *pricing rule*. It generalizes the linear pricing rule assumed in Kyle (1985). X must clearly be a semimartingale in order for the pricing rule (2.6) to be well defined. Given the pricing rule Q^X, the informed investor's self–financing budget constraint takes the form

$$dW_t = (rW_t - c_t)dt + X_{t-} dQ_t^X, \tag{2.7}$$

where her nominal wealth $W_t = B_t + X_t P_t$ is defined as the sum of her riskless bond holdings B_t and the market value of her shareholdings X_t. Here "X_{t-}" denotes the left–hand limit of X at time t.[3] As in Wang (1993), there is no exogenous lower bound on the informed investor's nominal wealth. She seeks a self–financing, utility maximizing consumption–investment policy

[1] Here and throughout the remainder of the chapter, (in)equalities involving random variables hold with probability one and information filtrations satisfy the usual conditions. For example, the filtration generated by a random process is the right–hand limit of the corresponding null–augmented natural filtration. [Karatzas and Shreve (1991, §2.7)].

[2] In other words, the excess return is an affine combination of the informed investor's incremental order dX and exogenous shock terms. Thus, the informed investor has partial control over the corresponding price innovation dP.

[3] Discrete–time dynamics analogous to (2.7) can be obtained as follows. Suppose that if the informed investor wishes to hold X_t shares at time t, she submits an order for $X_t - X_{t-\epsilon}$ shares at time $t - \epsilon$ and pays P_t per share

(c, X) among a class $\mathcal{A}$ of Q^X*–admissible* policies, where the notion of admissibility will be developed in the sequel. Given these primitives, we define an equilibrium as follows.

Definition 2.1 *The pricing rule* Q^X*, the uninformed belief* $X = X^*$*, and the* Q^X*–admissible policies* $\mathcal{A}$ *comprise an equilibrium if there is a consumption process* c^* *such that, with probability one,*

$$\begin{aligned} &i) \quad (c^*, X^*) \in \operatorname{argmax}_{(c,X)\in\mathcal{A}} \; \mathbb{E}\left[\int_0^\infty -e^{-\rho s - \gamma c_s} ds \,\middle|\, \mathfrak{I}_0\right], \\ &ii) \quad dQ_t^X\Big|_{X=X^*} = dP_t + D_t dt - rP_t dt\Big|_{X=X^*} \text{ for every } t \geq 0. \end{aligned}$$

The stock price P *appearing in ii) is defined by* (2.5).

Definition 2.1 is directly comparable to Kyle's definition of equilibrium (1985, p. 1318).[4] The Q^X–admissible policies $\mathcal{A}$ represent possible deviations from the informed investor's optimal policy. In equilibrium, the informed investor has no incentive to deviate from a policy confirming the uninformed agents' belief X^*. Like Kyle (1985) and Wang (1993), we do not require the bond market to clear in equilibrium. Some external mechanism (*i.e.* the government) maintains an infinitely liquid supply of riskless bonds.

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at time t. If she consumes at the rate $c_{t-\epsilon}$ over the time interval $[t-\epsilon, t[$, we must then have

$$B_t = (1 + r\epsilon)B_{t-\epsilon} + X_{t-\epsilon}D_{t-\epsilon}\epsilon - c_{t-\epsilon}\epsilon - P_t\left(X_t - X_{t-\epsilon}\right).$$

Upon rearrangement, it follows that

$$W_t - W_{t-\epsilon} = \left(rW_{t-\epsilon} - c_{t-\epsilon}\right)\epsilon + X_{t-\epsilon}\left(P_t - P_{t-\epsilon} + D_{t-\epsilon}\epsilon - rP_{t-\epsilon}\epsilon\right).$$

Taking the limit as $\epsilon \downarrow 0$ yields (2.7). Equation (2.7) differs slightly from its competitive counterpart, where the "$t-$" subscript is replaced by "t" [Wang (1993), Equation (4.9)]. This distinction is important; under the pricing rule (2.6), Q_t jumps ($Q_t \neq Q_{t-}$) precisely when the informed investor's shareholdings jump. The competitive self–financing condition involving $X_t dQ_t$ does not properly reflect the impact of such jumps on the informed investor's wealth.

[4]Using Kyle's terminology, the first condition corresponds to the profit maximization condition, while the second corresponds to the market efficiency condition.

2.2 Admissibility and a candidate pricing rule

The pricing rule (2.6) cannot be specified arbitrarily. Likewise, the choice of a pricing rule impacts the definition of the class $\mathcal{A}$ of admissible consumption–investment policies. In this section we examine the constraints involved in choosing a pricing rule and defining the class of admissible policies. Until indicated otherwise, z denotes the three–dimensional Brownian motion $[z_D, z_\Pi, z_\Theta]^\top$.

We begin by defining an admissible pricing rule.

Definition 2.2 *Q^X is an* admissible pricing rule *if there is a finite–dimensional $\mathcal{I}$–adapted semimartingale Ψ such that the Bellman equation*

$$\begin{aligned} J(W_t, \Psi_t) &= \max_{c,x} \left\{ -e^{-\gamma c} dt + e^{-\rho dt} \mathbb{E}\left[J(W_t + dW_t, \Psi_t + d\Psi_t) \,|\mathcal{I}_t \right] \right\} \qquad (2.8) \\ \text{s.t.} \quad & dW_t = (rW_t - c)dt + X_t dQ_t^X \\ & dX_t = x dt \end{aligned}$$

has a unique C^2 solution J.

The existence of an admissible pricing rule precludes the presence of arbitrage opportunities that might be exploited by the informed investor.[5] The Bellman equation (2.8) differs from those arising in standard price–taking portfolio choice problems. The choice variable x appearing in (2.8) relates to the *rate* at which the informed investor purchases stock rather than to her shareholdings directly. The somewhat heuristic statement of Definition 2.2 can be made rigorous by applying the Itô formula. However, the intuition behind the definition is straightforward. It simply asserts that the solution to the informed investor's Bellman equation, given that X has absolutely continuous sample paths, is unique.[6]

[5] Jarrow (1992) provides general conditions ruling out arbitrage in a discrete–time economy involving a non–price–taking investor.

[6] It is possible to generalize Definition 2.2 to accommodate sample paths involving Brownian components and jumps. However, this additional generality involves lengthy technical considerations and is of no consequence to the equilibrium constructed below.

Given the admissible pricing rule Q^X, we define the class $\mathcal{A}$ of Q^X–admissible policies as follows.

Definition 2.3 *The consumption–investment policy (c, X) is Q^X–admissible if it is an $\mathfrak{I}$–adapted semimartingale such that, for any fixed $t \geq 0$,*

$$\mathbb{E}\left[e^{-\rho\tau_k} J(W_{\tau_k}, \Psi_{\tau_k}) \,|\mathfrak{I}_t\right] \xrightarrow{\mathbb{P}} 0 \quad \text{as } k \longrightarrow \infty \tag{2.9}$$

for any nondecreasing sequence of bounded $\mathfrak{I}$–stopping times $\tau_k \longrightarrow \infty$. Here W is the wealth process (2.7) associated with (c, X).

(2.9) is a minor variation on Wang's transversality condition (1993, Equation (4.9)).[7] It ensures that Q^X–admissible consumption–investment policies satisfying the local optimality condition (2.8) are optimal over the entire class $\mathcal{A}$. As the definition suggests, every policy in $\mathcal{A}$ is associated with a unique wealth process W defined by the informed investor's initial wealth and the dynamics (2.7).

We now provide an example of a relatively simple admissible pricing rule, and examine its implications for admissible consumption–investment policies.

Lemma 2.1 *If b_Q is a constant nonzero row vector and $\lambda = \|b_Q\|^2\gamma\,(1 + 2a_\Theta/r)^{-1}$, then*

$$dQ_t^X = \lambda\,(a_\Theta X_t dt + dX_t) + b_Q dz_t \tag{2.10}$$

is an admissible pricing rule. The Bellman equation (2.8) has the solution

$$J(W, X) = -\frac{1}{r}\exp\left(1 - \frac{\rho}{r} - r\gamma W + \frac{r\gamma\lambda}{2}X^2\right).$$

Proof. See Appendix A.

[7] A bounded $\mathfrak{I}$–stopping time is a bounded, nonnegative random variable τ such that $\{\tau \leq t\} \in \mathfrak{I}_t$ for every $t \geq 0$ [Karatzas and Shreve (1991, §1.2)].

Thus, the state process Ψ appearing in Definition 2.2 is simply X, the informed investor's shareholdings.

The following Lemma demonstrates that there is a relatively large, easily described class of admissible policies corresponding to the pricing rule Q^X.

Lemma 2.2 *Let Q^X be the pricing rule defined in Lemma 2.1. Let (c, X) be an $\mathfrak{I}$-adapted semimartingale such that X has continuous sample paths of finite variation over finite time intervals, and let W be the associated wealth process. If i) there exists $\epsilon < \rho/r\gamma$ such that*

$$c_t + \frac{r\lambda}{2}X_t^2 \leq rW_t + \epsilon \quad \textit{for every } t \geq 0,$$

and ii) for every $T \geq 0$ there exists a finite valued, $\mathfrak{I}_0$-measurable random variable K_T such that

$$\max_{0 \leq t \leq T} X_t^2 \leq K_T \left(1 + \max_{0 \leq t \leq T} \|z_t\|^2\right),$$

then (c, X) is Q^X-admissible.

Proof. See Appendix A.

According to Lemma 2.2, as long as the informed investor's shareholdings X have continuous sample paths of finite variation and her consumption rate and shareholdings remain within certain limits, her consumption–investment policy is Q^X–admissible. Investment policies having continuous, finite variation sample paths are of particular importance. Like the Kyle model, optimal investment policies *must* have this property, as demonstrated below.

Lemma 2.3 (Verification Lemma) *Suppose Q^X and J are as defined in Lemma 2.1. Let $\mathcal{A}_t(w, x) \subseteq \mathcal{A}$ denote the Q^X-admissible strategies (c, X) such that $W_t = w$ and $X_t = x$ a.s. Then with probability one,*

$$J(w, x) = \sup_{(c,X) \in \mathcal{A}_t(w,x)} \mathbb{E}\left[\int_t^\infty -e^{-\rho(s-t)-\gamma c_s} ds \,\middle|\, \mathfrak{I}_t\right] \quad \forall (w, x) \in \mathbb{R}^2 \;\; \forall t \geq 0.$$

If $(c, X) \in \mathcal{A}$ is such that i) X has continuous sample paths of finite variation over finite time intervals and ii)

$$c_t = rW_t - \frac{r\lambda}{2}X_t^2 + \gamma^{-1}(\rho/r - 1) \quad \forall t \geq 0,$$

then (c, X) is a dynamically consistent optimal policy. Any admissible policy for which X has a nonzero martingale or jump component is strictly suboptimal.

Proof. See Appendix A.

In particular, the solution to the informed investor's Bellman equation coincides with her value function.[8] The informed investor's optimal policy is not uniquely determined by the pricing rule. According to Lemma 2.3, her optimal consumption rate is uniquely determined as a function of her wealth and shareholdings, but there are many possible optimal trading strategies. As Back (1992, Lemma 2) demonstrates, a similar property holds for the Kyle model; any insider trading strategy having continuous, finite variation sample paths and satisfying a terminal condition is optimal. In both models, the informed investor's shareholdings can only be pinned down by the equilibrium market clearing condition. We proceed with the construction of an equilibrium in the following section.

2.3 Constructing an equilibrium

In this section, we construct a stationary equilibrium; *i.e.* one in which the equilibrium stock price has no explicit dependence on time.[9] We begin by defining the *fundamental share value*

$$\Phi_t = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} D_s ds \middle| \mathfrak{I}_t\right] = m_D D_t + m_\Pi \Pi_t + m_1,$$

[8]Dynamic consistency, or the lack thereof, plays an important role in other models of strategic investment. Basak (1995) provides an example of a model in which a non–price–taking agent's optimal policy is time–inconsistent. Kihlstrom (1998) relates such time–inconsistency to the Coase conjecture.

[9]However, keep in mind that if $k = 0$, the dividend process itself is a nonstationary stochastic process.

where

$$m_D = \frac{1}{r+k}, \quad m_\Pi = \frac{m_D}{r+a_\Pi}, \quad m_1 = \frac{a_\Pi m_\Pi \bar{\Pi}}{r}.$$

We then define the function

$$\phi(\mu) = \frac{a_\Pi}{\sigma_D^{-2} + \sigma_\Theta^{-2}\mu^2}\left(\sqrt{1 + \frac{\left(\sigma_D^{-2} + \sigma_\Theta^{-2}\mu^2\right)\sigma_\Pi^2}{a_\Pi^2}} - 1\right).$$

Assumption 2.1 *The transcendental equation*

$$\left[m_D\sigma_D + m_\Pi\sigma_D^{-1}\phi(\mu)\right]^2 + \left[m_\Pi\sigma_\Theta^{-1}\mu\phi(\mu)\right]^2 = \left(1 + \frac{2a_\Theta}{r}\right)\frac{m_D + m_\Pi\phi(\mu)\left(\sigma_D^{-2} + \sigma_\Theta^{-2}\mu^2\right)}{\gamma\mu} \tag{2.11}$$

has a positive root μ_Δ*.*

A sufficient condition for (2.11) to have a positive root is $\gamma > \gamma_{\mathrm{crit}}$, where

$$\gamma_{\mathrm{crit}} = \sigma_\Theta^{-1}\left(1 + \frac{2a_\Theta}{r}\right)\frac{m_\Pi\sigma_\Pi}{(m_D\sigma_D)^2 + (m_\Pi\sigma_\Pi)^2}.$$

If $\gamma > \gamma_{\mathrm{crit}}$, the graphs of the left– and right–hand sides of (2.11) must cross at some positive value of μ. Thus, Assumption 2.1 holds if the variability σ_Θ^2 of the supply noise is sufficiently large.

Assumption 2.2 *The conditional distribution of* $[\Pi_0, \Theta_0]^\top$ *given* $\mathfrak{U}_0$ *is Gaussian with covariance matrix* Ω*, where*

$$\begin{aligned}
\Omega_{11} &= \phi(\mu_\Delta), \\
\Omega_{12} &= \frac{\mu_\Delta\Omega_{11}}{\left(\sigma_D^{-2} + \sigma_\Theta^{-2}\mu_\Delta^2\right)\Omega_{11} + a_\Pi + a_\Theta}, \\
\Omega_{22} &= \frac{\Omega_{12}}{a_\Theta}\left(\mu_\Delta - \frac{\sigma_D^{-2} + \sigma_\Theta^{-2}\mu_\Delta^2}{2}\Omega_{12}\right).
\end{aligned}$$

Notice that $\Omega_{11} > 0$ by definition, while straightforward algebra establishes that $\det \Omega > 0$. Therefore Ω is positive definite.

In what follows, define the row vector $b_Q = [b_{QD}, 0, b_{Q\Theta}]$, where

$$b_{QD} = m_D \sigma_D + m_\Pi \sigma_D^{-1} \phi(\mu_\Delta), \quad b_{Q\Theta} = -m_\Pi \sigma_\Theta^{-1} \mu_\Delta \phi(\mu_\Delta).$$

Let Q^X be the corresponding pricing rule (2.10) and let $\mathcal{A}$ denote the set of Q^X–admissible consumption–investment policies. We may now state the equilibrium.

Proposition 2.1 *Suppose that Assumptions* 2.1 *and* 2.2 *hold. Then there exists a belief* X^* *such that* $\left(Q^X, X^*, \mathcal{A}\right)$ *comprise an equilibrium.* X^* *satisfies*

$$\frac{dX^*}{dt} = -\mu_\Delta \Delta - a_\Theta X^*,$$

where $\Delta_t = \mathbb{E}^* \left[\Pi_t \,|\mathfrak{U}_t\right] - \Pi_t$*. The informed investor's optimal consumption rate* c^* *is given by Lemma* 2.3*, while the equilibrium stock price satisfies* $P_t = m_D D_t + m_\Pi \mathbb{E}^* \left[\Pi_t \,|\mathfrak{U}_t\right] + m_1$ *for every* $t \geq 0$.

Proof. See Appendix B.

2.4 Properties of the equilibrium

By definition of the fundamental share value Φ_t, Proposition 2.1 implies that we may write $\Delta_t = m_\Pi^{-1}(P_t - \Phi_t)$. Thus, Δ is a measure of mispricing relative to the fundamental share value. When Δ is positive the stock is overvalued and if Δ is negative it is undervalued. To implement the policy (c^*, X^*), the informed investor need only observe current levels of nominal wealth W, the mispricing error Δ, and her shareholdings. As Lemma 2.3 indicates, her nominal wealth and shareholdings alone suffice to describe her value function. In equilibrium, the rate with which the informed investor purchases stock is

$$\frac{dX^*}{dt} = -\mu_\Delta \Delta - a_\Theta X^*.$$

Since μ_Δ and a_Θ are both positive, dX^*/dt is decreasing in the mispricing error and in the informed investor's current shareholdings. Thus, if she has a long position or a small short position in overpriced stock ($\Delta_t > 0$), she will sell; *i.e.* $dX^*/dt < 0$. However, if she has a large short position she will purchase stock, even if it is moderately overpriced. To understand this behavior further, notice from Lemma 2.3 that the informed investor's value function J is decreasing in $|X|$. Thus, for a given amount of nominal wealth W, she is worse off the greater the proportion of wealth (long or short) invested in the stock. This reflects her market power; since $\lambda > 0$, her trades impart changes to the excess stock return that reduce her instantaneous profits. Thus, the larger her position in the stock, the more costly it is to unwind that position. Consequently, she may choose to reduce her position and forego short–term profits that might be obtained by increasing it further.[10] The informed investor's distaste for large positions in the stock market is also reflected in the fact that her consumption rate c^* is decreasing in $|X|$. As $|X|$ increases, investing in the bond becomes more attractive than immediate consumption. She draws upon her savings to unwind her position in the stock market should subsequent market conditions turn against her.

One respect in which Wang's model differs from the Kyle model relates to the informational role of prices. Equilibrium prices in Wang (1993) reveal a signal to the uninformed investors that they cannot observe *a priori.* The uninformed impound this information in their portfolio selection decisions. By contrast, the actions of uninformed agents in the Kyle model do not depend on *a priori* unobservable signals. The equilibria constructed here are similar to Wang's in that market participants expect prices to reveal an important signal to the uninformed, and in equilibrium, this expectation is confirmed. Defining $Q_t^X = \int_0^t dQ_s^X$, the equilibrium pricing rule can be expressed as

$$Q_t^X = \lambda(Y_t - Y_0) + \lambda a_\Theta \int_0^t Y_s ds + \zeta_t, \tag{2.12}$$

where $\zeta = b_{QD} z_D + (\lambda\sigma_\Theta + b_{Q\Theta}) z_\Theta$. Given that the uninformed agents have no *a priori* knowledge of ζ, how can they enforce this pricing rule? The device of an "artificial market" [Grossman (1981), Back (1993)] provides an explanation. In an artificial market in which ζ

[10]Strictly speaking, Lemma 2.3 states that the informed investor is indifferent over many different trading strategies. Her "choice" of this particular strategy is actually determined by the market clearing condition rather than by her own utility maximization considerations.

is exogenously revealed to uninformed agents, the pricing rule (2.12) and the informed policy described in Proposition 2.1 are mutual best responses; they comprise an equilibrium in the artificial market. They must then comprise an equilibrium in the actual market because ζ is revealed to the uninformed by equilibrium prices, dividends, and the residual supply $1 - Y$. By holding the belief X^*, the uninformed induce the informed investor to trade in such a way that ζ can be inferred from publicly observable signals.[11] This is similar to Wang's competitive equilibrium, where the equilibrium price reveals a linear combination of the private signal Π and the supply noise Θ to the uninformed.

The present model differs from Wang (1993) in that the supply noise Θ is not revealed to the informed investor. (2.12) implies that the informed investor can infer Y, and hence $\Theta = X - Y$, from observed excess stock returns if she knows the initial value Y_0. However, Y_0 cannot be inferred from the information available to her at time zero. Consequently, a form of informational diversity prevails. The informed investor observes Π but cannot infer the residual supply, while the uninformed observe the residual supply but cannot determine Π. By constrast, the informed investor in the Kyle and Wang models can infer the residual supply from public signals, resulting in an informational hierarchy.

Another important difference between the our model and Wang's is the following. In Wang's model, the informed investor's value function depends on her wealth W, the mispricing error Δ, and the supply noise Θ. By contrast, in our model the informed investor's value function J depends only on her nominal wealth and her shareholdings X. There is no apparent dependence on either the mispricing error Δ or the supply noise Θ. To explore this further, consider the definition of (nominal) wealth W. In both models, W is defined as the sum of the informed investor's bondholdings B and the market value XP of her shareholdings. In Wang's model, the informed investor's wealth is equivalent to W dollars in cash since she may costlessly liquidate her shareholdings at any time. Both the stock and the bond are perfectly liquid assets. However, this is not true in our model. It is impossible for the informed investor to liquidate her shareholdings without incurring a loss. Therefore the portion of nominal wealth

[11] This issue would not arise if the pricing rule only involved a linear combination of dt, dD, and dY, where the coefficients are $\mathfrak{U}$-adapted processes. By definition, such a pricing rule could be enforced by the uninformed agents regardless of the informed investor's behavior. However, we are unable to find a pricing rule of this form that admits an equilibrium. See Section 5.2.1 for further details.

attributed to her perfectly liquid bondholdings B should be distinguished from the remainder XP attributed to her somewhat illiquid shareholdings. It is more appropriate to interpret her value function, or welfare, as a function of her bondholdings B, the stock price P, and her shareholdings X; *i.e.* $J(W, X) = J(B + XP, X)$. From this viewpoint, the mispricing error Δ and the supply noise Θ do have an impact on her welfare. By virtue of the dynamics of X^*, the entire history of mispricing errors Δ_s and supply noise Θ_s, $s < t$, is impounded in the time t stock price.[12]

By definition of the informed investor's value function J, we have

$$\lambda X \frac{\partial J}{\partial W} + \frac{\partial J}{\partial X} = 0. \tag{2.13}$$

The first term in the sum is the informed investor's change in nominal wealth λX per unit share purchased, times her marginal utility of nominal wealth. The second term is her marginal disutility of holding additional shares. The informed investor can temporarily drive up the stock price by purchasing more shares, but this increases her exposure to dividend risk and incurs additional liquidity risk should she want to sell them at some future date. As (2.13) indicates, the welfare effects of these factors cancel. The nominal wealth benefit derived by purchasing shares perfectly compensates for the dividend and liquidity risk associated with holding additional shares. Equality (2.13) does not simply hold along the equilibrium path. If it failed to hold at some arbitrary pair (W, X), there would be an incentive for the informed investor to adjust her shareholdings there, contradicting the definition of her value function. Consequently, the informed investor is indifferent to marginal changes in her shareholdings in every state of the world. In particular, as time progresses, she is indifferent between trading now or waiting until some future time to trade, even if, by waiting, she deviates from the equilibrium path. Much the same behavior arises in the Kyle model. The notion that the insider is willing to refrain from trading until the last possible instant is a key intuition supporting Back's (1992) formulation and extension of Kyle (1985).

[12]This dependence also confounds any attempt to compare the equilibria on the basis of Pareto dominance. The interpretation of such a comparison would be questionable in any case if supply noise is attributed to noise trading. A noise trader cannot be assigned a meaningful notion of welfare [Cao (1998)].

Assumption 2.1 is illustrated in Figure 2.1, where graphs representing the left– and right–hand sides of equation (2.11) are shown.[13] These graphs, denoted respectively by $LHS(\mu)$ and $RHS(\mu)$, cross at $\mu = \mu_\Delta$. As γ decreases, $RHS(\mu)$ moves upwards. As illustrated in the figure, the graphs eventually separate as γ falls to zero. Thus, there are no stationary equilibria of the type described in Proposition 2.1 for small γ (or equivalently, small σ_Θ). While this does not rule out the possibility that a strategic equilibrium exists for small γ, it does suggest that a *stationary* strategic equilibrium does not exist unless γ is sufficiently large. The model exhibits somewhat interesting behavior when $\gamma < \gamma_{\text{crit}}$. Equation (2.11) may have two roots within this range. As shown in Figure 2.2, a graph of μ_Δ vs. γ typically has two branches. The upper branch approaches ∞ as $\gamma \uparrow \gamma_{\text{crit}}$, while the lower branch approaches zero as $\gamma \uparrow \infty$. As one moves rightwards along the lower branch, μ_Δ decreases. In other words, as the informed investor becomes more risk averse, she trades less aggressively on her private information. This relationship is reversed along the upper branch. As the informed investor's risk aversion coefficient increases to γ_{crit}, the aggressiveness of her trading increases without bound. Nonetheless, the variability of the stock price approaches the variability of the fundamental share value Φ_t as one moves rightwards along the upper branch.

The next result characterizes the distributions of some important state variables as seen by the uninformed agents. Let $\Pi_t^* = \mathbb{E}^*[\Pi_t|\mathcal{U}_t]$ and $\Theta_t^* = \mathbb{E}^*[\Theta_t|\mathcal{U}_t]$.

Proposition 2.2 *For every $t \geq 0$, $[\Delta_t, \Theta_t^* - \Theta_t]^\top$ is $N(0,\Omega)$-distributed, conditioned on $\mathcal{U}_t$. Moreover, there exists a $\mathcal{U}$-adapted Brownian motion $[\tilde{z}_D, \tilde{z}_Y]^\top$ such that*

$$\begin{aligned} dD_t &= (\Pi_t^* - kD_t)dt + \sigma_D d\tilde{z}_D(t), \\ d\Pi_t^* &= a_\Pi(\bar{\Pi} - \Pi_t^*)dt + \sigma_D^{-1}\phi(\mu_\Delta)d\tilde{z}_D(t) + \sigma_\Theta^{-1}\mu_\Delta\phi(\mu_\Delta)d\tilde{z}_Y(t), \\ dY_t &= -a_\Theta Y_t dt + \sigma_\Theta d\tilde{z}_Y(t). \end{aligned}$$

In particular, from the perspective of the uninformed agents, innovations to the difference $Y_t = X_t^* - \Theta_t$ have the same law as innovations to the supply noise $-\Theta_t$.

Proof. See Appendix B.

[13]The parameter values used in several of Wang's numerical examples (1993, §5) are also used here.

Propositions 2.1 and 2.2 imply that the equilibrium stock price is a two–factor process. Price innovations are positively correlated with dividend innovations and with innovations to Y. Since the uninformed investors' (conditional) estimation error for the state variables Π_t, Θ_t has a time invariant, nondegenerate Gaussian distribution, the same is true of the pricing error $P_t - \Phi_t = m_\Pi \Delta_t$. In other words, like Wang's uninformed investors, they cannot improve their estimate of the fundamental share value Φ_t. The informed investor maintains a long–lived informational advantage over them. The equivalence of the dynamics of Y and those of $-\Theta$ corresponds to the "no–trade" theorem prevailing in Kyle's model [Back (1992, Lemma 5)]. If the uninformed confine their attention to the residual supply $1 - Y$, the informed investor appears not to trade the stock. However, the informed investor's trades do have an impact on the dividend dynamics inferred by the uninformed. In the absence of an informed investor, the uninformed agents' conditional expectation of Π is independent of the residual supply $1 - Y = 1 + \Theta$. As Proposition 2.2 illustrates, this is no longer true in the presence of an informed investor.

2.5 Price stabilization and price chasing

Recalling our previous notation, the variability of the equilibrium stock price is $b_{QD}^2 + b_{Q\Theta}^2 = LHS(\mu_\Delta)$. It is straightforward to show that the price variability in the absence of an informed investor is $LHS(0)$, while $LHS(\infty) = \lim_{\mu \longrightarrow \infty} LHS(\mu)$ is the variability of the fundamental share value. For the parameter values corresponding to Figure 2.1, it is apparent that the stock price has a higher variability than that of the fundamental value, since $LHS(\mu_\Delta) > LHS(\infty)$. However, the price variability is still lower than it would be in the absence of an informed investor. Thus, using Wang's (1993) terminology, the informed investor stabilizes the stock price.[14]

In Wang's model, informed investors are typically contrarians while the uninformed are "price chasers." In other words, informed investors buy when the stock price falls while the uninformed

[14]Numerical experimentation suggests that $LHS(\mu)$ is monotone decreasing for a wide range of parameter values, which in turn suggests that the informed investor always stabilizes prices. However, we are unable to provide general conditions under which $LHS(\mu)$ is monotone decreasing, or to provide a counterexample.

buy when the stock price rises. In the present model, this relationship is reversed. The informed investor is a price chaser, while the uninformed agents are contrarians. Because of the informed investor's market power, prices tend to rise when she purchases stock and fall when she sells it. This is illustrated in Figure 2.3, where the steady-state correlation

$$\rho_{QX^*}(\tau) = \lim_{t \longrightarrow \infty} \left. \frac{\mathrm{Cov}(Q^X_{t+\tau} - Q^X_t, X_{t+\tau} - X_t)}{\sqrt{\mathrm{Var}(Q^X_{t+\tau} - Q^X_t)\mathrm{Var}(X_{t+\tau} - X_t)}} \right|_{X=X^*}$$

between changes in excess returns and changes in the informed investor's shareholdings is plotted for several values of the supply noise mean reversion rate a_Θ.[15] Notice that $\lim_{\tau \longrightarrow 0} \rho_{QX^*}(\tau) = 0$. Since the informed investor's shareholdings have zero instantaneous variability, there is zero correlation between dX^*_t and dQ^X_t over any infinitesimal time interval $[t, t + dt]$. Figure 2.3 indicates that the larger a_Θ is, the smaller the correlation between changes in stock returns and changes in the informed investor's shareholdings. In other words, the uninformed are less able to estimate the informed investor's trades as a_Θ increases. Another aspect of this behavior is the fact that the steady state variance $\lim_{t \longrightarrow \infty} \mathrm{Var}(X^*_{t+\tau} - X^*_t)$ increases rapidly as a_Θ increases. This is somewhat counterintuitive; one might expect the informed investor to make *smaller* trades as a_Θ increases. An increase in the supply noise mean reversion rate would seem to reduce the informed trader's ability to "hide" her trades from the uninformed. However, we find instead that as a_Θ increases, the uninformed agents' ability to filter out the supply noise declines rapidly. The informed investor trades more agressively to compensate for this decline. This phenomenon is also illustrated in Figure 2.4, where

$$\rho_{QY}(\tau) = \lim_{t \longrightarrow \infty} \left. \frac{\mathrm{Cov}(Q^X_{t+\tau} - Q^X_t, Y_{t+\tau} - Y_t)}{\sqrt{\mathrm{Var}(Q^X_{t+\tau} - Q^X_t)\mathrm{Var}(Y_{t+\tau} - Y_t)}} \right|_{X=X^*}$$

is plotted using the same parameter values. When a_Θ is small, Y is composed almost entirely of supply noise $-\Theta$, since the informed investor's trades are small relative to changes in Θ. As a_Θ grows ρ_{QY} also grows, since the uninformed agents' ability to filter out the supply noise decreases. For the parameter values chosen in this case, Y is still largely composed of supply

[15]The covariances appearing here can be computed in closed form [Karatzas and Shreve (1991, §5.6)]. Aside from a_Θ, which is allowed to vary, and σ_Θ, the model parameters coincide with those used in Figures 2.1 and 2.2. σ_Θ has been doubled to ensure that $\gamma_{\text{crit}} < \gamma$ for all the values of a_Θ appearing in Figure 2.3.

noise. However, the informed investor compensates for the uninformed agents' reduced filtering ability by increasing the size of her trades.

Finally, it is easy to prove that

$$\lim_{\tau \longrightarrow 0} \left. \frac{\operatorname{Cov}(Q^X_{t+\tau} - Q^X_t, Y_{t+\tau} - Y_t)}{\operatorname{Var}(Y_{t+\tau} - Y_t)} \right|_{X=X^*} = -\sigma_\Theta^{-1} b_{Q\Theta} \in]0, \lambda[.$$

This limit describes the sensitivity of excess return innovations to innovations dY_t in the residual supply. Therefore, using Kyle's (1985) terminology, it can be interpreted as the inverse of *market depth*. However, unlike the Kyle model, market depth is strictly larger than λ^{-1}, where λ is the sensitivity of excess stock returns to the informed investor's incremental order. The difference arises because in the Kyle model, the market maker only observes one signal; the residual supply. In the present model, the uninformed also observe the signal $\zeta = b_{QD} z_D + (\lambda \sigma_\Theta + b_{Q\Theta}) z_\Theta$, which is correlated with the residual supply.

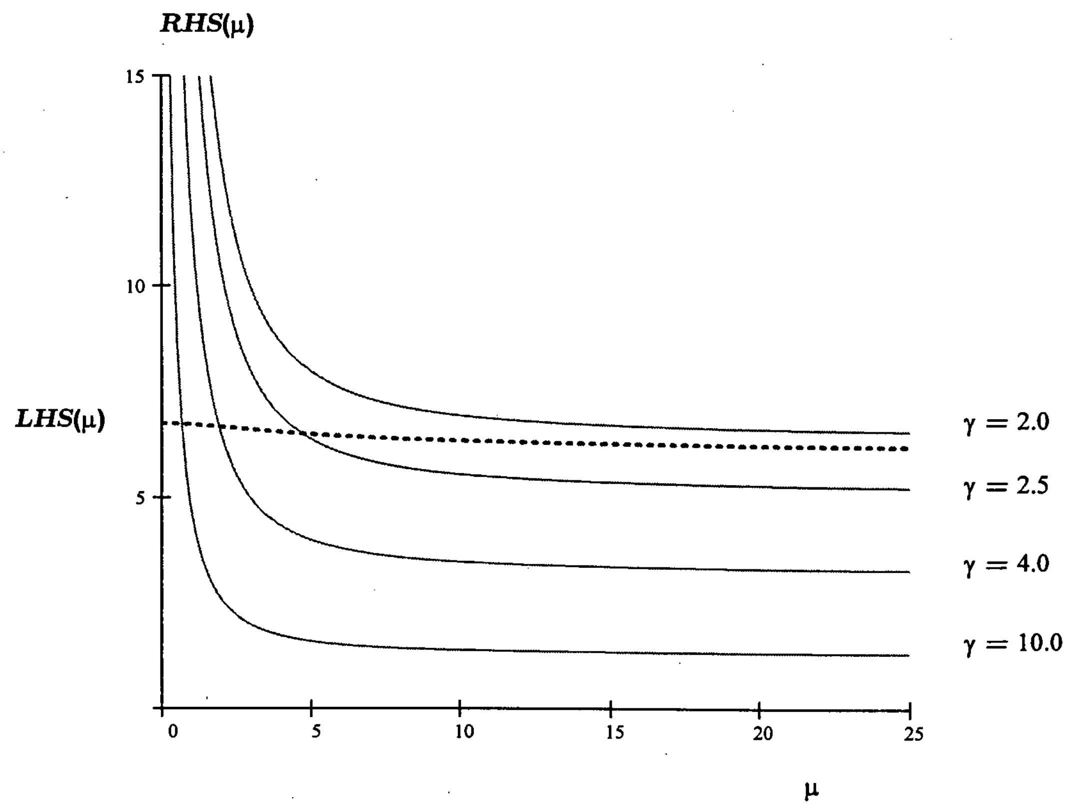

Figure 2.1 Model I - Equilibrium

$r = 0.05$, $k = 1.0$, $\sigma_D = 1.0$, $a_\Pi = 0.2$, $\sigma_\Pi = 0.6$, $a_\Theta = 0.4$, $\sigma_\Theta = 3.0$, $\gamma_{crit} = 2.112$

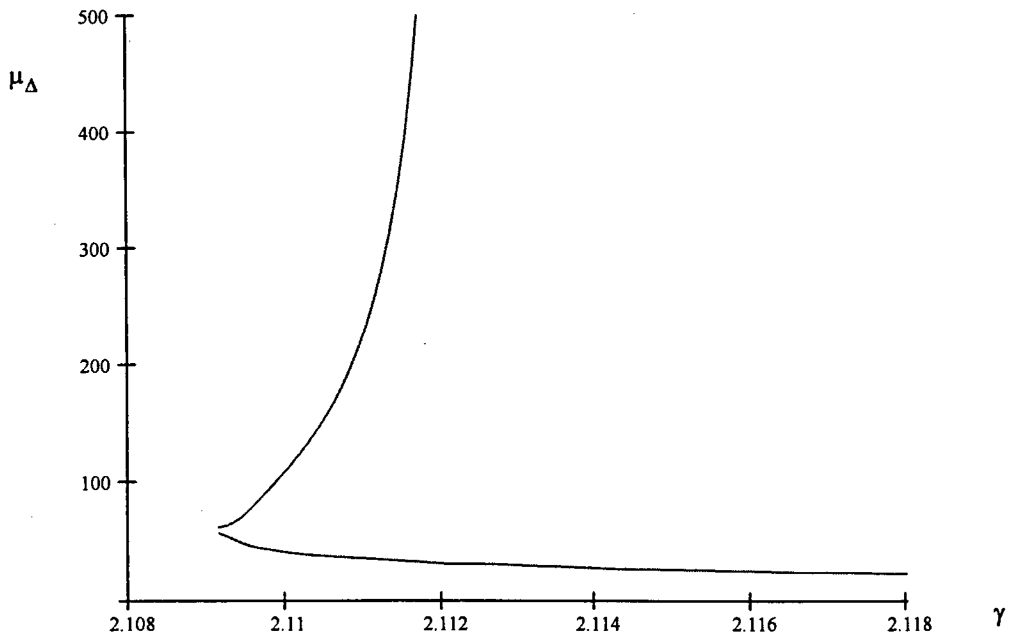

Figure 2.2 Model I - Multiple Equilibria

$r = 0.05$, $k = 1.0$, $\sigma_D = 1.0$, $a_\Pi = 0.2$, $\sigma_\Pi = 0.6$, $a_\Theta = 0.4$, $\sigma_\Theta = 3.0$, $\gamma_{crit} = 2.112$

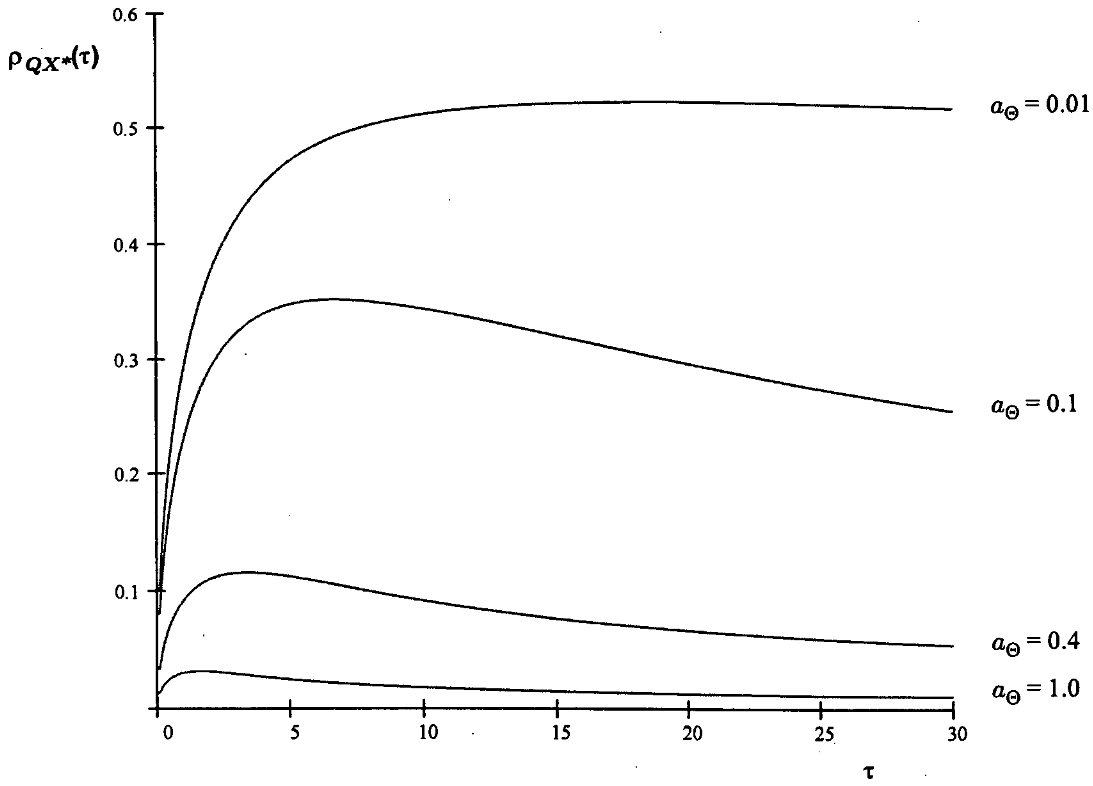

Figure 2.3 Model I - ρ_{QX^*}

$\gamma = 3.0$, $r = 0.05$, $k = 1.0$, $\sigma_D = 1.0$, $a_\Pi = 0.2$, $\sigma_\Pi = 0.6$, $\sigma_\Theta = 6.0$

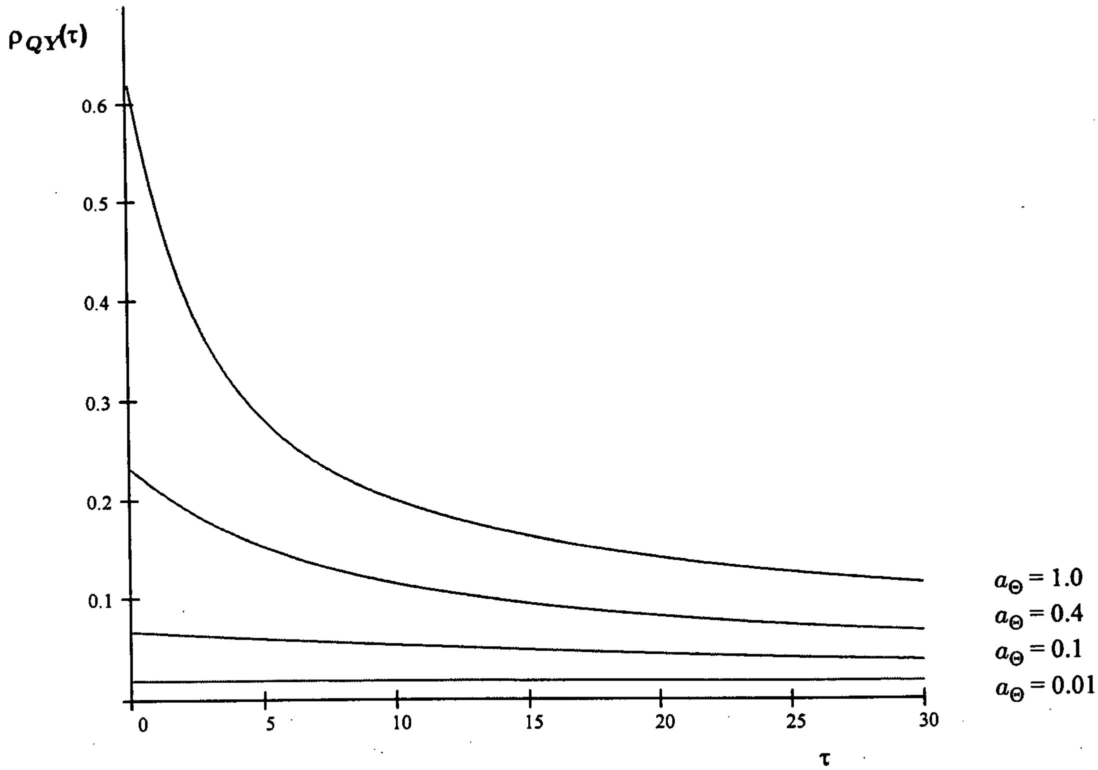

Figure 2.4 Model I - ρ_{QY}

$\gamma = 3.0$, $r = 0.05$, $k = 1.0$, $\sigma_D = 1.0$, $a_\Pi = 0.2$, $\sigma_\Pi = 0.6$, $\sigma_\Theta = 6.0$

Chapter 3

A Model of Endogenous Information Asymmetry

In this chapter, we present a model in which risk averse agents have identical information about dividends, but different beliefs about the extent to which their trades move the stock price. There are two types of agents in this economy; a strategist, who accounts for the impact of her trades on prices, and price takers. Despite the fact that there is no *a priori* information asymmetry, we find an equilibrium in which the strategist can infer the supply of stock from the stock price and dividend while price takers can only infer partial information about the supply. Thus, an endogenous informational asymmetry arises in equilibrium. Unlike the model presented in Chapter 2, the equilibrium stock price exhibits interesting properties involving risk premia, serial autocorrelation, and excess variability. Each of these quantities is strongly influenced by the size of the strategist's trades. If the strategist trades on behalf of a significant proportion of the investor population, numerical examples indicate that very large risk premia and price variability may arise in equilibrium. While we make no attempt to calibrate the model to actual data, these examples indicate that the price variability can be much larger than that of a conventional representative agent benchmark model. For example, if half the members of a moderately risk averse population delegate their investment decisions to a strategic fund manager, the expected risk premium of the stock can be twice the size of its counterpart in the benchmark model, while the stock price variability can be thirty times larger. If ninety percent of the population invests exclusively in the fund, the expected risk premium of the stock can be ten times larger than that of its benchmark counterpart, while the price variability can be three orders of magnitude larger. Since conventional representative agent models deliver risk premia and price variability that are unreasonably small in comparison to empirical observations, these findings are encouraging.

3.1 Model II

Subrahmanyam (1991, §2) extends the one–period Kyle model (1984) to a case where both the informed investor and the market maker are risk averse. In our second model, we also investigate an equilibrium arising between two types of risk averse agents. However, we depart somewhat from Subrahmanyam by assuming that both types of agents have the same *a priori* informational standing. Instead, one investor, the *strategist*, believes that her trades move prices while the remaining $N-1$ investors take prices as given. In the resulting equilibrium, the stock price reveals the supply noise Θ to the strategist but not to the price takers. The strategist impounds this private information in her equilibrium consumption–investment policy.[1]

Both the strategist and the price takers have time–additive preferences (2.4) over consumption, with respective preference parameters ρ, γ and ρ', γ'. The dividend rate satisfies (2.1), where $\Pi_t = \bar{\Pi}$ is a constant known to both investors; *i.e.*

$$dD_t = (\bar{\Pi} - kD_t)dt + \sigma_D dz_D(t).$$

All investors observe the dividend rate D_t at each date t. The per capita supply of stock is $1+\Theta_t$, where Θ satisfies (2.3) and is independent of D. Like our first model, we assume that the excess stock return is determined by a pricing rule Q^X of the form (2.6). Given this pricing rule and their respective self–financing budget constraints, both investors compute their optimal

[1] Instead of imposing a zero profit condition on the market maker, Subrahmanyam assumes that the market maker achieves a reservation level of expected utility in equilibrium. We instead seek a market clearing equilibrium in which each agent is a utility maximizer. A related problem is addressed in Cuoco and Cvitanić (1998, §8), who present a model in which the dependence of price on a large investor's shareholdings is exogenously specified. However, their model does not arise from equilibrium considerations, nor does it involve information asymmetry.

consumption–investment policies. The strategist faces the same type of optimization problem as the first model's informed investor. A price taker, on the other hand, arrives at his optimal policy (c', X') by treating the strategist's shareholdings X as an exogenous process. Thus, in order to determine his optimal policy, the price taker must hold a belief $X = X^*$ about the dynamics of X.

Given the pricing rule Q^X and her budget constraint, the strategist seeks a solution to the Bellman equation (2.8), where $\mathfrak{I}$ now denotes the information filtration generated by D, the stock price P, and the strategist's initial wealth and shareholdings. As before, we say that Q^X is an admissible pricing rule if (2.8) has a unique C^2 solution J for some $\mathfrak{I}$–adapted state process Ψ. In turn, the consumption–investment policy (c, X) is Q^X–admissible if it is an $\mathfrak{I}$–adapted semimartingale satisfying the transversality condition (2.9). Let $\omega = (N-1)/N$ denote the proportion of price takers among the population of investors. By analogy to our first model, we say that an equilibrium arises under the following conditions.

Definition 3.1 *The pricing rule Q^X, the price takers' belief $X = X^*$, and the Q^X–admissible policies $\mathcal{A}$ comprise an equilibrium if there is a consumption process c^* such that, with probability one,*

i) $\quad (c^*, X^*) \in \text{argmax}_{(c,X)\in\mathcal{A}} \ \mathbb{E}\left[\int_0^\infty -e^{-\rho s - \gamma c_s} ds \,\middle|\, \mathfrak{I}_0\right],$

ii) *each price taker optimally holds X' shares, where $\omega X' + (1-\omega)X^* = 1 + \Theta$.*

The market clearing condition *ii)* is identical to that of Wang (1993, Equation (4.16)). Each price taker holds the same belief $X = X^*$. In equilibrium, the strategist has no incentive to deviate from a policy (c^*, X^*) that confirms this belief and clears the stock market.

3.2 Constructing an equilibrium

In this section, we construct a stationary equilibrium. Unlike our first model, restrictions on the parameter values or the initial distribution of the underlying state variables are unnecessary. We begin by specifying the coefficients λ, b_{QD}, $b_{Q\Theta}$ appearing in the pricing rule (2.6).

Lemma 3.1 *Given $b \in \mathbb{R}$, define $b_Q(b) = [b_{QD}, b]^\top$ and $\lambda(b) = \|b_Q(b)\|^2 \gamma (1 + 2a_\Theta/r)^{-1}$, where $b_{QD} = \sigma_D/(r+k)$. Let*

$$\begin{aligned}
c_0(b) &= \frac{r^2\gamma'^2\|b_Q(b)\|^2}{2\omega^2 b}\left(\frac{b_{QD}^2}{b} - \frac{\omega}{\gamma'\sigma_\Theta}\left[1 + \frac{2a_\Theta}{r}\right]\right), \\
c_1(b) &= \frac{r\gamma'}{\omega b}\left(\|b_Q(b)\|^2\left[\frac{\lambda(b)}{(1-\omega)b} + \sigma_\Theta^{-1}\right] - \frac{\lambda(b)\omega}{2(1-\omega)\gamma'\sigma_\Theta}\left[1 + \frac{2a_\Theta}{r}\right]\right), \\
c_2(b) &= \frac{\lambda(b)}{(1-\omega)b}\left(\frac{\lambda(b)}{2(1-\omega)b} + \sigma_\Theta^{-1}\right).
\end{aligned}$$

Then there exists $b = b_{Q\Theta} < 0$ such that the quadratic $\psi(\xi; b) = c_2(b)\xi^2 + c_1(b)\xi + c_0(b)$ has a root $\Xi(b) < 0$ satisfying

$$\Xi(b) = \frac{(1-\omega)(r + a_\Theta)b}{(1-\omega)b + \sigma_\Theta\lambda(b)}.$$

Proof. See Appendix C.

This result can be motivated as follows. By analogy to our first model, define

$$Y = (1-\omega)X - \Theta, \tag{3.1}$$

where X is the strategist's shareholdings. $1 - Y$ is the residual supply of stock per capita, net of the strategist's shareholdings. The market clearing condition can be stated as $\omega X' = 1 - Y$. We seek an equilibrium in which Y has dynamics

$$dY_t = (\mu_1 + \mu_Y Y_t)\, dt - \sigma_\Theta dz_\Theta(t), \tag{3.2}$$

where the μ_is are constant scalars to be determined. Given b_{QD}, $b_{Q\Theta}$ as defined above, let $\lambda = \lambda(b_{Q\Theta})$, $\Xi = \Xi(b_{Q\Theta})$, and $b_Q = [b_{QD}, b_{Q\Theta}]$. Assuming that the pricing rule Q^X defined in Lemma 2.1 holds with $z = [z_D, z_\Theta]^\top$, the excess stock return corresponding to (3.2) is

$$dQ_t = \frac{\lambda}{1-\omega}\left[\mu_1 + (\mu_Y + a_\Theta)Y_t\right] dt + b_Q dz_t. \tag{3.3}$$

Since $b_{Q\Theta} < 0$, instantaneous supply shocks and price shocks are negatively correlated, as we should expect. Moreover, since Y and consequently z are observable to the price taker, his optimization problem is a conventional dynamic program in the state variable Y.[2] His Bellman equation yields a quadratic in Y, each of whose coefficients must vanish. The coefficient of Y^2 is $\psi(\Xi; b_{Q\Theta})$, which is zero by definition. Furthermore, if we set

$$\begin{aligned}\mu_1 &= \frac{(1-\omega)r\sigma_\Theta\Xi}{\omega b_{Q\Theta}} \frac{(\gamma' b_{QD})^2 (r+2a_\Theta)}{\gamma\omega\Xi^2 + \gamma'(1-\omega)(r+a_\Theta)(r+2a_\Theta)} > 0, \\ \mu_Y &= \Xi - a_\Theta < 0,\end{aligned}$$

the other two coefficients of the price taker's Bellman equation also vanish and he optimally holds X' shares, where $\omega X' = 1 - Y$. Thus, the stock market clears.[3] We now state an equilibrium.

Proposition 3.1 *Let $\Phi_t = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} D_s ds \middle| D_t\right]$ be the fundamental share value. Suppose the initial stock price P_0 satisfies $P_0 = \Phi_0 + p_0 + p_Y Y_0$, where*

$$p_Y = -\sigma_\Theta^{-1} b_{Q\Theta}, \qquad p_0 = \left(p_Y - \frac{\lambda}{1-\omega}\right)\frac{\mu_1}{r}.$$

Then there exists a belief X^* such that $\left(Q^X, X^*, \mathcal{A}\right)$ comprises an equilibrium. X^* satisfies

$$\frac{dX^*}{dt} = \frac{\mu_1}{1-\omega} + \mu_Y X^* - \frac{\mu_Y + a_\Theta}{1-\omega}\Theta, \tag{3.4}$$

where the constants μ_i are as defined above. The strategist's optimal consumption rate c^* is given by Lemma 2.3. Moreover, the equilibrium stock price satisfies $P_t = \Phi_t + p_0 + p_Y Y_t$ for every $t \geq 0$.

Proof. See Appendix C.

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[2]In fact, the price taker's optimization problem is of the same form as that faced by Wang's uninformed investor [Wang (1993, Appendix C)].

[3]Notice that since $\mu_Y + a_\Theta = \Xi < 0$, the expected excess stock return $\mathbb{E}_t dQ_t/dt$ increases as the residual supply $1 - Y$ increases. In equilibrium, the increase is just enough that the price takers, in aggregate, continue to hold the residual supply.

The proof of Proposition 3.1 relies on the fact that Lemmas 2.1, 2.2, and 2.3 remain true when $z = [z_D, z_\Theta]^\top$. The equilibrium share price is uniquely determined by the initial share price P_0, the pricing rule Q^X, and the strategist's optimal shareholdings X^*. Equations (3.1) and (3.4) imply that (3.2) holds in equilibrium.

3.3 Properties of the equilibrium

It is important to distinguish the pricing rule Q^X from the equilibrium share price $\Phi_t + p_0 + p_Y Y_t$. The strategist views the sensitivity of the price to an incremental order dX as λ, not p_Y. These two quantities generally differ. It is also important to notice that since $p_Y \neq 0$, both investors can infer Y_t from the time t stock price and dividend. The strategist can then deduce the supply noise $\Theta_t = (1 - \omega)X_t^* - Y_t$ from this information. Therefore the strategy X^* is well defined. Finally, recall that in our first model and in Wang's (1993) competitive model, the economy must reach (or begin from) a long–run steady state in order for the equilibrium to hold. This ensures that the uninformed investors' filtered estimates of the informed investors' private information have reached a steady state. Since no filtering is involved in the present model, this type of assumption is not required here.[4]

Like the equilibrium arising in our first model, this equilibrium is of the rational expectations type. All investors rationally expect the signal z_Θ to be revealed to the price takers in equilibrium.[5] By definition, we have $\mu_1 > 0$ and $\mu_Y < -a_\Theta$. Therefore, from the price taker's perspective, the dynamics (3.2) differ from those of $-\Theta$. Consequently, the "no–trade" theorem prevailing in the first model does not hold in this case. The residual supply $1 - Y$ has a higher mean reversion rate and a smaller mean than the total supply. The strategist reduces the variance of the residual supply relative to that of the total supply $1 + \Theta$ and takes, on average, a long position in the stock. To implement the policy (c^*, X^*) described in Proposition 3.1,

[4] We could conceivably include a filtered estimate of Θ as a state variable for the price taker's problem. However, given the dynamics (3.2) and (3.3), Y alone suffices for this purpose.

[5] Alternately, the device of an artificial market in which z_Θ is exogenously revealed to price takers can be used. Price takers can enforce the pricing rule Q^X in this artificial market, and since z_Θ is revealed by equilibrium excess returns, they can also enforce it in the actual market. The supply shock z_Θ plays the same role as the signal ζ in our first model.

the strategist need only observe her wealth, the supply noise Θ, and her current shareholdings. Since $\mu_Y < -a_\Theta < 0$, (3.4) implies that the strategist's order rate dX_t^*/dt is decreasing in her current shareholdings X_t^* and increasing in the total supply $1 + \Theta_t$. The strategist's decision to buy or sell stock is therefore determined by the size of her shareholdings relative to the total supply. She will further increase an existing long or short position in the stock only if her current shareholdings are sufficiently small relative to the total supply. Like our first model, this behavior reflects the fact that the strategist's value function J is decreasing in $|X|$. For a given amount of wealth W, the strategist is worse off the greater the proportion of wealth (long or short) she has invested in the stock. Due to her market power, it is costly for the strategist to unwind a large position in the stock should market conditions eventually turn against her. In anticipation of this possibility, she may choose to reduce her current position and forego short–term profits that might be obtained by increasing it further.

A change in the proportion ω of price takers in the population changes the balance between opposing forces. As ω falls, the relative increase in strategic trading causes the residual supply to be smoothed to a greater degree. This tends to reduce stock price variability. On the other hand, since the strategist's shareholdings have no Brownian component, her trades only smooth the residual supply over finite time horizons. On an infinitesimal time scale, her trades do not smooth the supply at all; the price takers must absorb the instantaneous supply shocks dz_Θ. Since fewer price takers are available to do this as ω decreases, this tends to increase stock price variability. Thus, like Wang's model, where ω is the proportion of uninformed investors in the economy, important aspects of the model change simultaneously as ω varies. Unlike Wang's model, our model currently requires ω to take the form $1 - N^{-1}$, where N is a positive integer. However, the following generalization has a meaningful economic interpretation for any rational $\omega \in]0, 1[$. Suppose that the strategist is a mutual fund manager acting on behalf of $M \geq 1$ unit holders. The fund manager seeks to maximize the utility of a representative unit holder having a *pro–rata* claim to a fraction $1/M$ of the fund's security holdings and consumption stream.[6] It is natural to consider equilibria in which the fund manager trades strategically, even if the number

[6]Here we ignore details related to the fund manager's compensation, assume that the number of unit holders M is constant over all states and time, and require all unit holders to refrain from holding securities outside of the fund. One justification for the last of these assumptions is that mutual fund investors face information costs or other frictions preventing them from investing outside the fund. A similar assumption appears in Basak and Cuoco (1998).

of mutual fund investors M is a relatively small fraction of the total population N. Let MX_t be the fund manager's shareholdings at time t. Assuming once again that the per capita supply of shares is $1+\Theta$, the market clearing condition takes the form $(N-M)X'_t + MX_t = N(1+\Theta_t)$, or

$$\omega X'_t + (1-\omega)X_t = 1 + \Theta_t,$$

where $\omega = (N-M)/N$. This condition coincides with that of Wang (1993, Equation (4.16)). In this context, Definition 3.1 has a meaningful interpretation for any rational ω between 0 and 1. Furthermore, Lemma 3.1 and Proposition 3.1 remain true for all rational $\omega \in]0,1[$. As ω decreases, these results tell us how the model behaves as the proportion of mutual fund investors increases.

3.4 A benchmark model

To better understand the implications of the model, it is helpful to use a conventional benchmark as a basis for comparison. For this, we employ a model in which identical price takers each hold $1+\Theta$ shares (*i.e.* $\omega = 1$). In this case, a standard argument establishes that the market clearing share price is $P_t^b = \Phi_t - \left(p_0^b + p_\Theta^b \Theta_t\right)$, where

$$p_0^b = \gamma' \frac{\sigma_D^2}{(r+k)^2}, \qquad p_\Theta^b > 0.$$

The share price in this case is simply the expected discounted value Φ_t of future dividends minus a risk premium $p_0^b + p_\Theta^b \Theta_t$.[7] The expected risk premium p_0^b is proportional to each agent's risk aversion coefficient γ'. The larger γ' is, the greater the expected risk premium and the expected excess stock return. As shown in Proposition 3.1, the equilibrium share price for our second model takes a similar simple form, where the risk premium is

$$\Phi_t - P_t = \left(\frac{b_{Q\Theta}}{\sigma_\Theta} + \frac{\lambda}{1-\omega}\right)\frac{\mu_1}{r} + \frac{b_{Q\Theta}}{\sigma_\Theta} Y_t.$$

[7] Compare to Wang (1993, Theorem 3.1).

Recalling the various definitions, we obtain the following simple result.

Lemma 3.2 *If $\gamma = \gamma'$, then*

$$\begin{aligned} \mathbb{E}\left[\Phi_t - P_t\right] &= p_0^b \left(\frac{1-\omega}{\omega}\right)\left(1 + \frac{r}{a_\Theta - \Xi}\right) \frac{a_\Theta(r + 2a_\Theta)}{\omega\Xi^2 + (1-\omega)(r + a_\Theta)(r + 2a_\Theta)} \\ &< \frac{p_0^b}{\omega}. \end{aligned} \tag{3.5}$$

The expected risk premium is a multiple of its benchmark counterpart p_0^b. The multiplier depends on the population parameter $\omega \in]0,\ 1[$, the term Ξ arising in the definition of equilibrium, and the supply noise mean reversion rate a_Θ. These quantities play no role in the benchmark model's expected risk premium. Unfortunately, we are unable to provide a comparably simple and interesting lower bound for the expected risk premium.

Since the expressions for both the share price variability $b_{Q\Theta}^2 + b_{QD}^2$ and the risk premium are somewhat unwieldy, the remainder of this chapter investigates the model through numerical examples. We find that there are two distinct regimes. There appears to be only one equilibrium when sufficiently many investors are price takers. However, below a certain threshold value of ω, multiple equilibria may exist. We examine the two regimes separately below.

3.5 Some comparative statics for a price taking majority

The proof of Lemma 3.1 addresses two separate cases; one where the denominator of $\Xi(b)$ has no negative roots, and the other where it does. Figure 3.1 illustrates the former case when $\omega = 0.9$.[8] The function $\Xi(b)$ (the dashed curve) is plotted together with the roots of the quadratic $\psi(\bullet; b)$ (the solid curves) over the negative b axis. We assume that both types of investors have the same preference parameters; *i.e.* $\rho = \rho'$ and $\gamma = \gamma'$. The equilibrium value $b_{Q\Theta}$ is the point b where $\Xi(b)$ coincides with one of the two roots. Figure 3.2 plots the ratio σ_p^2/σ_b^2 vs. ω,

[8] Here we use somewhat different parameter values than those appearing in Wang (1993). While the qualitative behavior of the model is unaffected by changing these values, this particular choice of parameters emphasizes the interesting features of the equilibria.

where $\sigma_p^2 = b_{Q\Theta}^2 + b_{QD}^2$ is the equilibrium price variability and σ_b^2 is the price variability for the benchmark model. Figure 3.3 plots the corresponding ratio $\mathbb{E}[\Phi_t - P_t]/p_0^b$ of expected risk premia. As one moves from left to right along the ω axes, mutual fund participation declines and price taking activity increases. As $\omega \uparrow 1$, both ratios converge to one. This is because the drift and diffusion coefficents of the two–dimensional process $[Q^X, -Y]^\top$ converge to those of $[Q^b, \Theta]^\top$, where Q^b is the cumulative excess stock return for the benchmark model.[9] This is interesting because no matter how large ω might be, as long as it is smaller than one, the price takers cannot observe the supply noise Θ. However, at $\omega = 1$, Θ is revealed to the price takers in equilibrium. As $\omega \uparrow 1$, the equilibria converge to the benchmark equilibrium despite this informational discontinuity. Wang (1993, p. 275) notes that similar phenomena arise in his competitive model. However, in his model, this may lead to instabilities where equilibria do not converge to the equilibrium prevailing at the discontinuity point.

As shown in Figures 3.2 and 3.3, the price variability and expected risk premium may be monotone increasing, monotone decreasing, or hump–shaped as ω decreases from 1. The shape of these curves reflects the trade–off between finite–horizon supply smoothing caused by the fund manager's trades and the availability of price takers to absorb instantaneous supply shocks. When risk aversion is low, the fund manager smooths the residual supply more aggressively and price takers are more willing to absorb supply shocks. As a result, the former effect dominates, causing price variability and risk premia to fall as ω falls. When risk aversion is high, the fund manager smooths the supply less aggressively and price takers are less willing to absorb supply shocks. In this case the latter effect dominates, causing price variability and risk premia to rise as ω falls. This behavior is consistent with anecdotal evidence, reported in the popular press, that with liquidity reduced, price volatility tends to rise while price itself tends to fall.[10]

The hump–shaped curves in Figures 3.2 and 3.3 demonstrate that the trade–off between these effects can be relatively complex at intermediate levels of risk aversion. A similar shape appears in Wang (1993, Figure 2), but it arises for somewhat different reasons related to informational asymmetry. The case where mutual fund participation and risk aversion are relatively high is particularly interesting in this numerical example. When $\gamma = \gamma' = 4$ and $\omega = 1/2$, the

[9] While we have not formally proved this assertion, a search over a wide range of parameter values failed to produce a counterexample.

[10] For example, see "The risk business," *The Economist*, Oct. 17–23, 1998, p. 21.

stock price variability for our model is thirty times its counterpart for the benchmark model and almost nine hundred times the variability of the supply noise and the dividend. The expected risk premium is roughly twice that of the benchmark. Spiegel (1998) notes that conventional representative agent models typically require a large amount of supply noise in order to produce the stock price variability observed in empirical studies. Our model provides a means of generating substantial price variability with a relatively small amount of supply noise. Since supply noise variations are generally thought to be much smaller than price variations, this is a desirable property.

Figure 3.4 plots the serial correlation $\rho_{Q^+Q^-}(\tau)$ in excess stock returns, where

$$\rho_{Q^+Q^-}(\tau) = \lim_{t\longrightarrow\infty} \left.\frac{\mathrm{Cov}(Q^X_{t+\tau} - Q^X_t, Q^X_t - Q^X_{t-\tau})}{\mathrm{Var}(Q^X_{t+\tau} - Q^X_t)}\right|_{X=X^*}.$$

As in our first model, the quantities appearing here can be computed in closed form using standard methods. $\rho_{Q^+Q^-}(\tau)$ is plotted for several values of ω at $\gamma = 4$. The notable feature of Figure 3.4 is that excess stock returns are negatively serially correlated. This is consistent with well known empirical studies [*e.g.* Fama and French (1987)]. Moreover, this negative serial correlation becomes more pronounced as ω falls; *i.e.* as the number of mutual fund investors grows and the number of price takers diminishes. This relationship is reversed for small values of γ; excess returns are still negatively serially correlated, but become less so as ω decreases. At intermediate values of γ, the serial correlation is no longer monotonic in ω; the curves in Figure 3.4 may cross at various points.[11] However, in each case, these curves approach their benchmark counterpart as $\omega \uparrow 1$.

The mechanism behind the serial correlation is the following. Suppose the fund manager sells shares over the time period $[t-\tau,\, t]$. Then the residual supply increases and the stock price tends to fall; *i.e.* the fund manager is a price chaser. Since $\mu_Y + a_\Theta < 0$, (3.3) implies that the expected excess stock return over the subsequent time period $[t,\, t+\tau]$ rises. Provided $|\mu_Y|$ is moderately large, this results in negative serial correlation in stock returns. The magnitude of the correlation depends on both the nature of the fund manager's trades and the extent to which the stock price moves in response to them; *i.e.* the sensitivity of price to residual

[11] Wang observes a similar nonmonotonicity in his model (1993, Figure 6), but it arises for different reasons.

supply variations. This sensitivity rises as price takers become fewer in number. Conversely, as mutual fund participation increases, the fund manager's trades smooth the residual supply to a greater extent. The balance between these opposing forces changes as the risk aversion coefficient γ varies. At high levels of risk aversion, serial correlation strengthens as ω falls; the increasing sensitivity of price to residual supply variations outweighs the impact of increased residual supply smoothing. At low levels of risk aversion, the opposite effect is observed.

3.6 Multiple equilibria for a strategic majority

If sufficiently many investors are price takers, the equilibrium described in Section 3.5 appears to be unique. However, for smaller values of ω, the situation is somewhat different. Figure 3.5 is the counterpart of Figure 3.1 when $\omega = 0.1$. In this case the function $\Xi(b)$ has two poles on the negative b axis. More importantly, multiple roots exist. The rightmost root below the b axis is the one guaranteed by Lemma 3.1. However, there are four additional roots, two lying above the b axis and two lying below. Each of the three roots lying below the b axis corresponds to a valid equilibrium. As long as investors agree on the pricing rule implied by one of these roots, the stock market clears when they optimize their respective utilities of consumption.[12] The leftmost of these roots lies at $b = -145.56$, well outside the range of Figure 3.5. The equilibrium corresponding to this root has a large expected risk premium; it is 9.96 times that of the benchmark model. In other words, the expected risk premium is within a half percent of attaining the upper bound (3.5).[13] The price variability in this equilibrium is enormous; it is three orders of magnitude larger than that of the benchmark model. This behavior is similar to that of the "negative root" equilibrium in Spiegel's (1998) overlapping generations model. However, the comparative statics of this equilibrium differ from those of Spiegel's. As σ_Θ decreases, the two leftmost roots of Figure 3.5 eventually converge to a common intermediate root when σ_Θ reaches some positive value σ^*_Θ. These roots do not exist if $\sigma_\Theta < \sigma^*_\Theta$. By contrast, Spiegel's negative root equilibrium exists at all levels of supply

[12]Like Spiegel (1998), we do not attempt to Pareto rank the various equilibria. Also note that the roots above the b axis correspond to a mean averting residual supply: *i.e.* $\mu_Y > 0$. Since this leads to implausible stock price behavior, we choose not to examine these roots.

[13]Recalling the expression for p^b_0, the investors' risk aversion coefficient must be increased by the factor 9.96 in order to obtain the same expected risk premium using the benchmark model.

variability. In fact, price variability in the negative root equilibrium approaches infinity as the supply variability approaches zero [Spiegel (1998, Theorem 1)]. By comparison, the price variability associated with the leftmost root of our model approaches infinity as ω approaches zero. In other words, pervasive strategic trading can greatly destabilize the share price relative to the benchmark model. As price takers become scarce, the share price variability consistent with their absorption of the instantaneous supply shocks grows without bound.

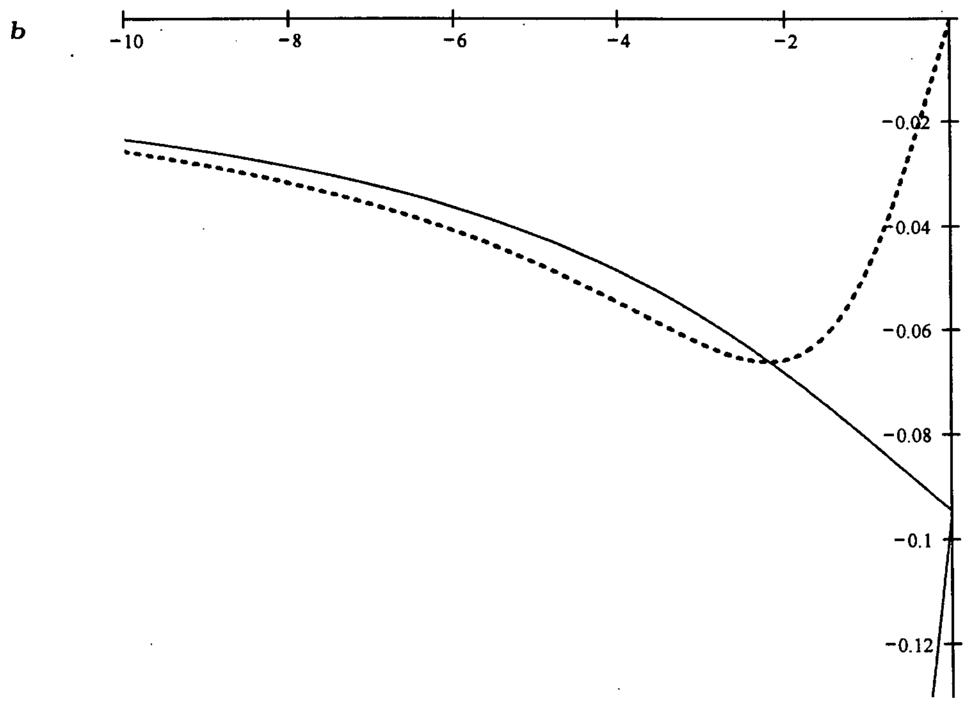

Figure 3.1 Model II - Equilibrium

$r = 0.05,\ \sigma_D = \sigma_\Theta = 1.0,\ k = a_\Theta = 0.4,\ \gamma' = \gamma = 3.0,\ \omega = 0.9$

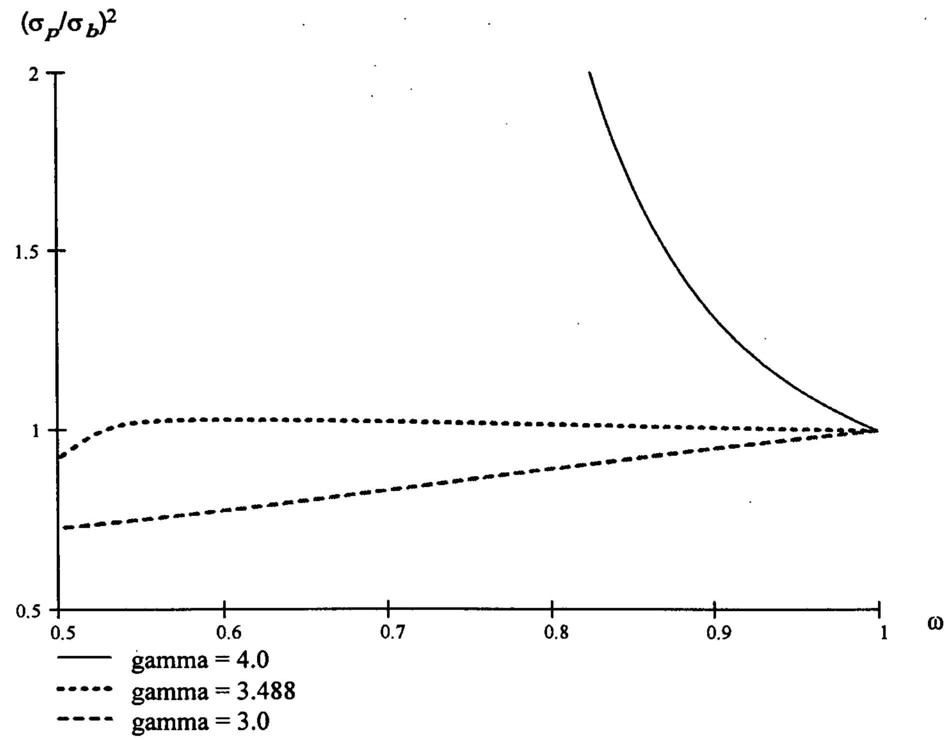

Figure 3.2 Model II - excess variability

$r = 0.05,\ \sigma_D = \sigma_\Theta = 1.0,\ k = a_\Theta = 0.4,\ \gamma' = \gamma$

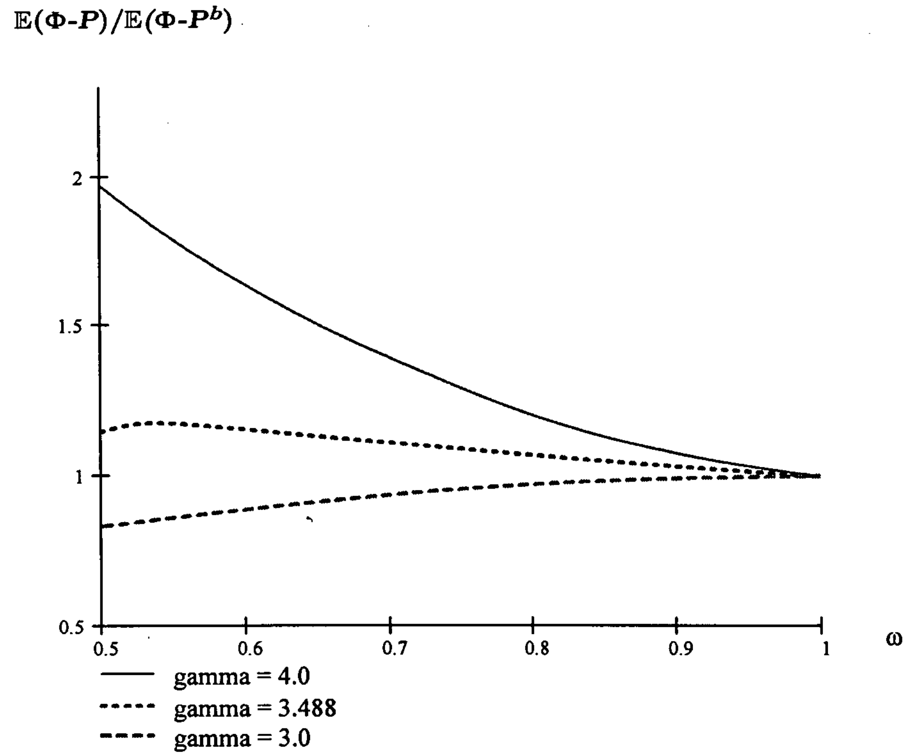

Figure 3.3 Model II - risk premium

$r = 0.05,\ \sigma_D = \sigma_\Theta = 1.0,\ k = a_\Theta = 0.4,\ \gamma' = \gamma$

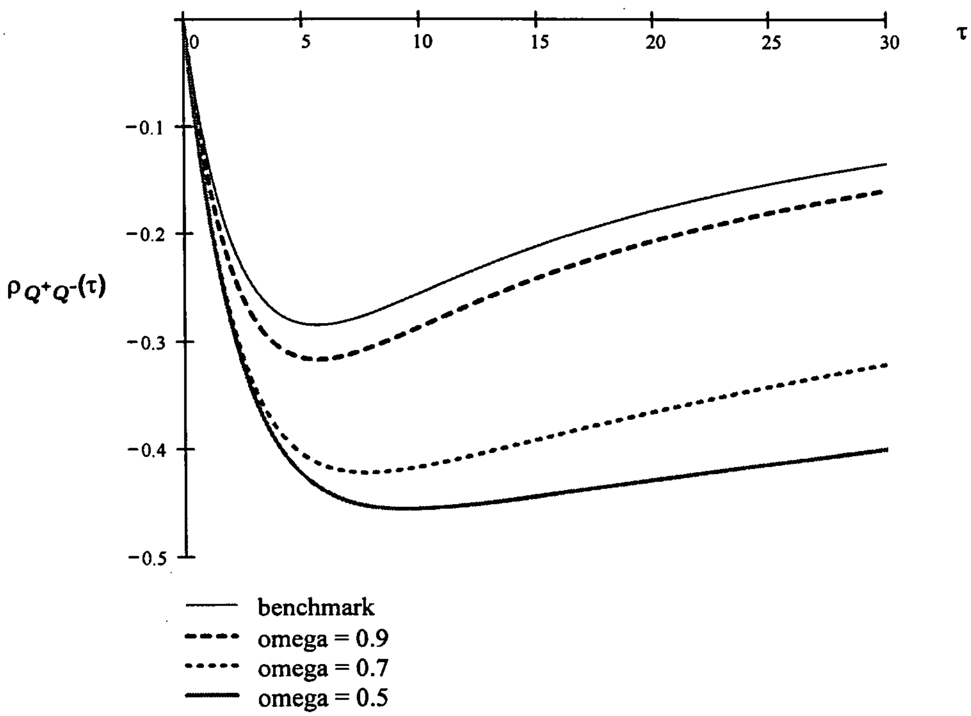

Figure 3.4 Model II - returns autocorrelation

$r = 0.05,\ \sigma_D = \sigma_\Theta = 1.0,\ k = a_\Theta = 0.4,\ \gamma' = \gamma = 4$

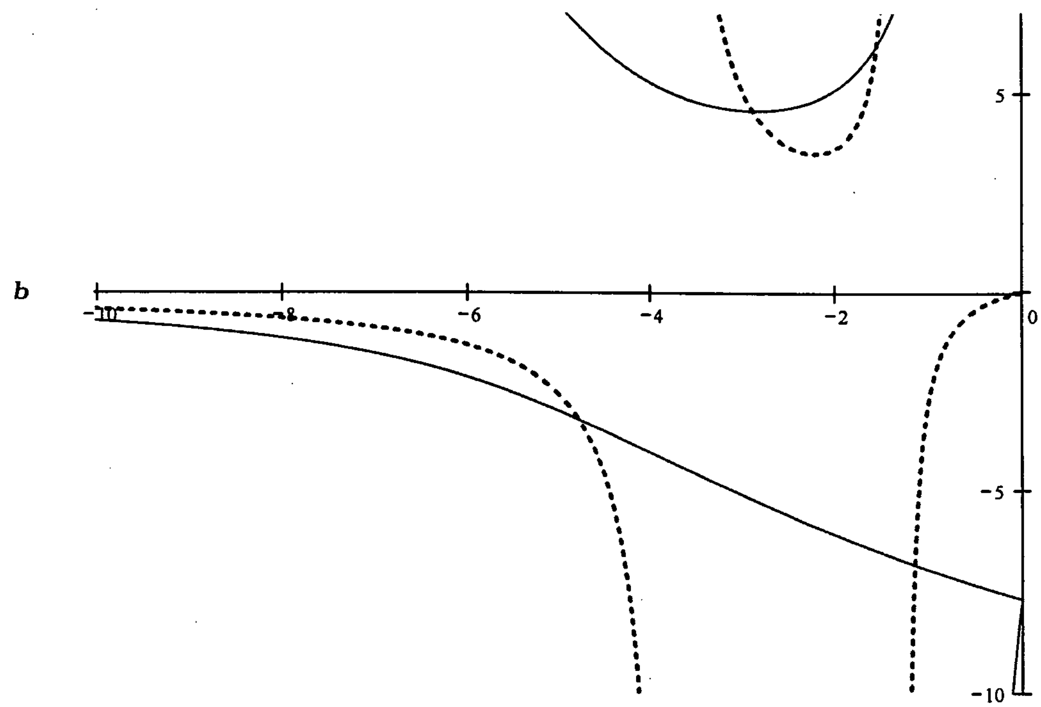

Figure 3.5 Model II - multiple equilibria

$r = 0.05,\ \sigma_D = \sigma_\Theta = 1.0,\ k = a_\Theta = 0.4,\ \gamma' = \gamma = 3.0,\ \omega = 0.1$

Chapter 4

Insider Trading with Incomplete Information

In one of his many seminal contributions to finance, Fischer Black (1986) conjectures that the price of a stock and its fundamental value must display several qualitative properties. Chief among these are the following. First, price tends to move toward value over time, and the farther price moves from value, the faster it will tend to move back. This is due to informed traders taking larger positions as the deviation between price and value increases. Second, both the price and value processes look like geometric Brownian motions with time–varying means. Changes in tastes, technology, and wealth cause the mean of the value process to change over time, while the mean of the price process changes in response to changes in the relationship between price and value. Third, Black conjectures that short term price movements can be decomposed into movements due to changes in value and movements due to noise. When the variance of these two components is the same, the variance of day–to–day price moves is roughly twice the variance of corresponding value moves. Over time, however, the variance of price movements tends toward the variance of value movements. The purpose of this chapter is to determine the share price in a greatly simplified version of Black's model, where a single trader has exclusive knowledge of a signal related to V_1, the terminal liquidation value of the share. The model is similar to the Kyle (1985) model of insider trading. The share is continuously traded throughout the time interval $[0, 1]$. Traders submit their orders to a market maker, who sets a market clearing share price in response to the total order. There are two types of traders; the insider and uninformed noise traders who submit random orders. Since the market maker only observes the combined order of the insider and noise traders, he is unable to distinguish noise trades from insider trades. Unlike the Kyle model, we assume that the insider cannot necessarily observe the liquidation value V_1 precisely at times $t < 1$. Instead, at each time t, the insider observes a signal V_t, where V is a stochastic process terminating at V_1 at time $t = 1$. V_t could simply be the insider's expectation of V_1, conditioned on her time t information set. Alternately, provided default can be ignored, V_t could be the leveraged value of the firm; *i.e.* the time t value, on a per share basis, of the firm's assets in place and future growth opportunities less outstanding debt. We show that, within a parametric class of diffusion models for V_t, an equilibrium analogous to Kyle's exists. We also obtain a closed form representation for the equilibrium price process. For example, if the signal V follows a geometric Brownian motion, then so does price.

Despite the increased generality of this model, the equilibrium is very similar to that of the Kyle model. In fact, the pricing rule and the distribution of trading volume, conditioned on the market maker's information set, are identical to their counterparts in the Kyle model. Thus, in contrast to Black's conjecture, the model displays constant price volatility when the underlying value process follows a geometric Brownian motion. Moreover, altering the arrival pattern of the insider's information by introducing systematic variations in the volatility of the signal V only serves to increase or decrease the price volatility. It has no intertemporal impact upon prices or upon the market maker's perception of trading volume. This property is much different from that of the Admati–Pfleiderer (1988) model, in which the insider's informational advantage is short–lived. There, the insider acts on private information immediately, imparting an informational component to price changes. By contrast, the insider's equilibrium trading strategy in this model conveys no information to the market maker about shocks to the fundamental value. As in the Kyle model, the insider's trading strategy has continuous sample paths of finite variation. Therefore her trades are locally correlated with neither the fundamental value nor the noise trades. The equilibrium trading strategy admits a relatively simple description. At each point in time, the insider adjusts her holdings in proportion to the size of the market maker's error in estimating the fundamental value. The proportionality constant grows without bound as $t \uparrow 1$. As a consequence, the share price is driven toward its fundamental value as

$t \uparrow 1$. An interesting feature of the model is the impact of mean reversion in value on price volatility. A uniform decrease in the rate with which value reverts to the mean increases the price volatility and reduces market efficiency.

Like the Kyle model, the insider's expected profits in equilibrium are identical to those of a perfectly discriminating monopsonist who refrains from trading until the last possible moment. This observation, together with those of the previous paragraph, might suggest that there are only minor differences between the Kyle model and the model under present consideration. However, the present model differs substantially in the way that information is impounded in the insider's investment strategy. If, as in the Kyle model, the insider knows V_1 throughout $[0, 1]$, she can simply "look ahead" to the terminal realization V_1 to construct an optimal portfolio at any time $t < 1$. By contrast, in the model presented here, the insider must form her portfolio based only on past realizations of the stock price and her private signal. The fact that the equilibrium pricing rules and expected profits should coincide for these models is interesting and not at all obvious. The model primitives are presented in preliminary form in Section 4.1 and developed in full detail in Section 4.2. Various qualitative properties of the equilibrium are also described in Section 4.2. Proofs appear in Appendices D, E, and F.

4.1 Model III

Apart from the insider's uncertainty about the terminal share value V_1, the model described below is identical to the Kyle model. Thus, the description is brief and closely parallels the discussion in Back (1992, §1). A non-dividend paying share is available for trade throughout the time interval $[0, 1]$. At each time $t \in [0, 1]$, a risk–neutral insider and noise traders simultaneously submit order quantities to a market maker, who sets a market clearing share price in response to the combined order. At time $t = 1$, the share liquidation value V_1 is announced. None of the market participants can observe V_1 until the announcement date. However, at each time $t \in [0, 1]$, the insider observes a private signal V_t, where $V = \{V_t\}$ is a diffusion process on $[0, 1]$ with terminal value V_1. We call V_t the *fundamental value* at time t.[1] For

[1] The fundamental value usually has the more specific interpretation as the discounted expected value of all future cash flows. [*e.g.* Summers (1986)]. However, we use this terminology in this more general context as a

concreteness, suppose that $V_t = g(\xi_t)$, where $g \in C^2(\mathbb{R})$ is strictly increasing and $\xi = \{\xi_t\}$ is the Gauss–Markov process satisfying

$$d\xi = [F(t)\xi_t + f(t)]\,dt + q(t)dB, \quad \xi_0 \sim N(\mu, \phi^2), \tag{4.1}$$

on the time interval $[0,1]$. Here $B = \{B_t\}$ is a standard Brownian motion independent of ξ_0 and F, f, and q are continuous, deterministic functions on $[0,1]$. This specification nests the geometric Brownian motion commonly appearing in continuous–time models of the firm. It also exhibits mean reversion if $F < 0$.[2]

Let X_t be the number of shares held by the insider at time t. The total order at time t, Y_t, is the sum of X_t and Z_t, where Z_t is the total number of shares held by noise traders at time t. $Z = \{Z_t\}$ is a Brownian motion independent of ξ_0 and B satisfying $\mathbb{E}\left[Z_t^2\right] = \sigma^2 t$ for all $t \in [0,1]$, where σ is a positive constant. Thus, noise trades are uncorrelated with shocks to the fundamental value. Since the market maker only observes the total order process $Y = \{Y_t\}$, the time t share price P_t can depend only upon realizations of Y up to time t. Following Back (1992), we restrict attention to the case where P_t depends only upon Y_t; *i.e.*

$$P_t = H(Y_t, t), \tag{4.2}$$

where $H \in C^{2,1}(\mathbb{R}\times]0,1[)$ is continuous on $\mathbb{R}\times[0,1]$ and strictly increasing in its first argument. H is called the market maker's *pricing rule.* The function H is assumed to belong to $\mathcal{H}$, a class of functions that satisfy technical conditions described in the following section.

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matter of convenience.

[2]For example, suppose the unleveraged value of the firm has constant proportional volatility σ_v, a constant proportion δ_v of the firm's assets are paid to security holders per unit time, and the instantaneous return $\mu_v(t)$ on the firm's assets is a deterministic function of time [Leland and Toft (1996, equation (1))]. Suppose that, at time 0, the firm issues a discount bond of face value D maturing at $t = 1$, and the unleveraged firm value is lognormally distributed. Then if the possibility of default can be ignored, we define $F(t) = 0$, $f(t) = \mu_v(t) - \delta_v - \sigma_v^2/2$, $q(t) = \sigma_v$, and $g(x) = e^x - D$. The time dependence of the drift coefficients is intended to capture the impact of (deterministic) changes in the economic factors mentioned in the introduction. In general, the time dependence of the dispersion coefficient $q(t)$ reflects changes in the firm's leverage and changes in the rate of arrival of information about the firm's prospects [Black (1986, p. 533)].

Since H is strictly increasing in its first argument, the insider can deduce the total order Y_s from the share price prevailing at time s. By subtracting X_s from Y_s at each time $s < t$, she can then infer the history of noise trades $\{Z_s, 0 \leq s < t\}$ before submitting her order X_t at time t. Since the insider also observes the fundamental value process, it is natural to require her trading strategy X to be adapted to the information filtration $\mathcal{F}^{V,Z}$ generated by V and Z.[3] Specifically, we assume that X belongs to $\mathcal{X}_H$, a set of $\mathcal{F}^{V,Z}$–adapted semimartingales satisfying technical conditions described in the following section.[4] The requirement that X be a semimartingale is somewhat stronger than the technical conditions usually imposed on trading strategies in competitive models [*i.e.* Harrison and Pliska (1981, p. 239)]. However, as Back notes, this assumption permits a useful integration by parts formula that is instrumental to the construction of an equilibrium.

If $t > 0$, let $X_{t-} = \lim_{s \uparrow t} X_s$ and define $X_{0-} = 0$. In view of the insider's risk–neutrality, her initial wealth may be assumed to be zero. Given the pricing rule $P_t = H(Y_t, t)$, the insider's terminal wealth W_1 is given by

$$W_1 = \int_0^1 X_{t-} dP_t + (V_1 - P_1) X_1.$$

W_1 is the sum of the cumulative gain from trade over the time interval $[0, 1]$ and the capital gain at the announcement date $t = 1$. The self–financing budget constraint implicit in the expression for the cumulative gain is similar to its counterpart in competitive models [Back (1992, pp. 391–392)]. However, as explained in Kyle (1985, p. 1327), it better reflects the impact of trades on the share price. In effect, trades are priced at the end of the instant in which they occur. Accordingly, the cumulative gain is expressed as $\int_0^1 X_{t-} dP_t$ instead of $\int_0^1 X_t dP_t$. Since X is a semimartingale, the generalized Itô formula [Jacod and Shiryaev (1987,

[3]More precisely, $\mathcal{F}^{V,Z}$ denotes the null–augmented filtration generated by V and Z. [Karatzas and Shreve (1991, §2.7)].

[4]Loosely speaking, the $\mathcal{F}^{V,Z}$–adapted semimartingales comprise a vector space containing all Itô processes; *i.e.* all processes of the form $x_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dV_s + \int_0^t \sigma'_s dZ_s$, where $x_0 \in \mathbb{R}$ and the coefficient processes μ_t, σ_t, σ'_t are adapted and satisfy standard integrability conditions. However, the class of semimartingales also contains jump processes, which is a natural requirement in the present context.

Theorem I.4.57)] implies

$$W_1 = \int_{[0,1]} (V_t - H(X_{t-} + Z_t, t))\, dX_t + \int_0^1 X_{t-} dV_t + [V - P, X]_1, \tag{4.3}$$

where $[V - P, X]_t$ is the quadratic (co)variation of the processes $V - P$ and X.[5] The first integral in (4.3) explicitly includes the left endpoint of the interval $[0, 1]$ because X may jump there.

Following Back, we now define an *equilibrium* to be a pair $(H, X) \in \mathcal{H} \times \mathcal{X}_H$ such that given the pricing rule H, the insider's trading strategy X is *optimal*; *i.e.* the insider's expected terminal wealth $\mathbb{E}\left[W_1 \mid V_0\right]$ is maximized over all strategies in $\mathcal{X}_H$; and given the insider's trading strategy X, the pricing rule H is *rational*; *i.e.*

$$H(Y_t, t) = \mathbb{E}\left[V_1 \,\middle|\, \mathcal{F}_t^Y\right] \quad \text{for every } 0 \leq t \leq 1, \tag{4.4}$$

where $\mathcal{F}^Y$ is the information filtration generated by the total order process Y. It is important to notice that in equilibrium, the insider takes the pricing rule H as given, rather than the price process P. This requires the insider to account for the impact of her own trades on the share price when selecting a trading strategy. In this sense, the insider rationally anticipates the effect of her orders on the equilibrium share price.

4.2 Main results

In this section, we define the class $\mathcal{H}$ from which the market maker chooses his pricing rule H and the class $\mathcal{X}_H$ from which the insider selects her trading strategy X. An equilibrium is then constructed. Closed form expressions for the equilibrium pricing rule and an optimal insider trading strategy are also obtained. We begin by noting that equation (4.1) can be solved explicitly [Karatzas and Shreve (1991, §5.6C)]. Given $t \in [0, 1]$ and $\xi_t = \xi \in \mathbb{R}$, we have

[5] The square bracket $[\cdot, \cdot]_t$ is a natural generalization of its more familiar counterpart $\langle \cdot, \cdot \rangle_t$ appearing in the Itô process literature [*i.e.* Back (1991)].

$\xi_1 = \tilde{\xi}(\xi, t)$ almost surely, where

$$\tilde{\xi}(\xi,t) = e(F)_{t,1}\left[\xi + \int_t^1 \frac{f(s)}{e(F)_{t,s}}ds + \int_t^1 \frac{q(s)}{e(F)_{t,s}}dB_s\right], \tag{4.5}$$

$$e(F)_{t,T} = \exp\left[\int_t^T F(s)ds\right]. \tag{4.6}$$

For every $t \in [0,1]$, let $C(t)$ be the covariance matrix of the random vector $\left(\tilde{\xi}(\xi,t), Z_1 - Z_t\right)$ and define

$$\|C(t)\| = \max\left\{x'C(t)x;\;\; x \in \mathbb{R}^2, |x| = 1\right\}, \tag{4.7}$$

where $|x| = \sqrt{x_1^2 + x_2^2}$. $C(t)$ evidently has no dependence upon ξ.

Assumption 4.1 *There exists $p > 1$ such that $|g(\xi_1)|^{2p}$ and $g'(\xi_1)^{2p}$ have finite expectations. Moreover, there exists $\delta > 0$ such that $\sup_{t\in[0,1]} \|C(t)\| < 1/2\delta$ and*

$$\int_{-\infty}^{\infty} e^{-\delta x^2}|g(x)|dx < \infty.$$

Assumption 4.1 holds if $\max\{|g(x)|, g'(x)\} \leq K_1 e^{K_2 x}$ for constants K_1, $K_2 < \infty$. This is true, for example, when the fundamental value follows a geometric Brownian motion ($g(x) = e^x$).[6]

Definition 4.1 *If $h : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing and $(\xi, y) \in \mathbb{R}^2$, define*

$$j(\xi, y; h) = \int_y^{h^{-1}\circ g(\xi)} [g(\xi) - h(x)]\, dx. \tag{4.8}$$

Since h is strictly increasing, it is easy to verify that $j(\xi, y; h) \geq 0$ with equality iff $g(\xi) = h(y)$. As in Back (1992), the function (4.8) plays a key role in the construction of an equilibrium.

[6]Back's counterpart to Assumption 4.1 is simply the requirement that $g(Z_1)$ be square integrable (1992, p. 390). The randomness of the fundamental value process assumed in the present model mandates the more complicated assumptions appearing here.

It has the following straightforward economic interpretation. If $h(\cdot) = H(\cdot, 1)$, $j(\xi_1, Y_{1-}; h)$ is the area in price–quantity space bounded by the "supply curve" $P = h(Q)$ and the lines $P = g(\xi_1)$, $Q = Y_{1-}$, and $Q = h^{-1} \circ g(\xi_1)$. Thus, it represents the profit available to a perfectly discriminating monopsonist at the last instant before the liquidation value $V_1 = g(\xi_1)$ is announced.

Definition 4.2 *With p as in Assumption 4.1, let $p' > 1$ be the conjugate of p:* i.e. $1/p + 1/p' = 1$. *$\mathfrak{h}_{p,C}$ denotes all strictly increasing functions $h \in C^2(\mathbb{R})$ such that $|h(Z_1)|^{2p}$ and $|h^{-1} \circ g(\xi_1)|^{2p'}$ have finite expectations, and such that*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\delta(\xi^2+y^2)} j(\xi, y; h) d\xi dy < \infty \tag{4.9}$$

for some $\delta > 0$ satisfying $\sup_{t\in[0,1]} \|C(t)\| < 1/2\delta$.

To illustrate, suppose $|h(Z_1)|^{2p}$ has a finite expectation, $|h^{-1} \circ g(x)| \leq |\text{poly}(x)|$ for some polynomial $\text{poly}(x)$, and $\int_{-\infty}^{\infty} e^{-\delta y^2} |h(y)| dy < \infty$ for some constant δ as described above. Then h satisfies the conditions of Definition 4.2. The integrability condition (4.9) follows from the bound

$$j(\xi, y; h) \leq |h^{-1} \circ g(\xi) - y| |g(\xi) - h(y)|.$$

In particular, the conditions of Definition 4.2 are satisfied when $h(x) = g(\alpha x + \beta)$, $\alpha > 0$, and $|g(x)| \leq K_1 e^{K_2 x}$ for constants K_1, K_2.[7] With these definitions in place, we now define the class $\mathcal{H}$ of admissable pricing rules.

Definition 4.3 *$\mathcal{H}$ denotes the class of functions H on $\mathbb{R} \times [0,1]$ of the form*

$$\begin{aligned} H(y,t) &= \mathbb{E}\left[h(Z_1)\,|Z_t = y\right] \\ &= \mathbb{E}h(y + Z_1 - Z_t), \end{aligned} \tag{4.10}$$

[7] Simpler technical conditions analogous to Definition 4.2 appear in Back (1992, p. 400). $|h(Z_1)|$ must have a finite expectation and the term $J(v, 0, 0)$ defined in equation (22) of Back must be finite.

where $h \in \mathfrak{h}_{p,C}$.

The definition of $\mathcal{H}$ is motivated by the equilibrium pricing rule found by Back (1992, Theorem 1). It is consistent with the rationality condition, which requires the equilibrium price to be the conditional expectation of a function of a normally distributed random variable.[8] We now turn to the definition of the class $\mathcal{X}_H$ of admissable insider trading strategies.

Definition 4.4 *Given* $H \in \mathcal{H}$, $\mathcal{X}_H$ *consists of the* $\mathcal{F}^{V,Z}$*-adapted semimartingales* X *on* $[0,1]$ *such that*

$$\mathbb{E}\int_0^1 H(X_{t-}+Z_t,t)^2 dt < \infty \quad \text{and} \tag{4.11}$$

$$\mathbb{E}\int_0^1 I_h(\xi_t, X_{t-}+Z_t, X_{t-},t)^2 dt < \infty, \tag{4.12}$$

where $h(\cdot) = H(\cdot,1)$ *and* $I_h(\xi,y,x,t) = \mathbb{E}\left[\partial j/\partial\xi\left(\tilde{\xi}(\xi,t), y+Z_1-Z_t; h\right) + xg'(\tilde{\xi}(\xi,t))\right]$.

The integrability conditions (4.11), (4.12) are analogous to those ruling out "doubling strategies" in competitive models [*i.e.* Dybvig and Huang (1988)]. Condition (4.11) also appears in Back (1992). It can be interpreted to mean that noise traders would not lose money on average if they could trade at the midpoint of the spread [Back (1992, pp. 394–395)].[9] We finally turn to the construction of an equilibrium. To this end, two additional assumptions are required.

Assumption 4.2 *The dispersion coefficient* $q(t)$ *in* (4.1) *satisfies a Lipschitz condition at* $t = 1$.

Assumption 4.3 *There exists* $\lambda > 0$ *such that*

$$\int_0^t \left[\sigma^2\lambda^2 - e(F)_{s,1}^2 q(s)^2\right] ds \le \phi^2 e(F)_{0,1}^2 \quad \textit{for every } t \in [0,1], \tag{4.13}$$

[8] In fact, *any* equilibrium pricing rule $H \in C^{2,1}(\mathbb{R}\times]0,1[)$ must take this form provided the insider's Bellman equation [Equation (D.2) below] admits a smooth solution that is linear in the state variable x and which has a Feynman–Kac representation [Compare to Back (1992, Theorem 2)]. We do not pursue the issue of uniqueness further.

[9] Unfortunately, condition (4.12) doesn't appear to have a clear intuitive interpretation.

with equality iff $t = 1$. *Moreover,*

$$\sigma^2\lambda^2 > q(1)^2. \tag{4.14}$$

Assumption 4.3 is illustrated in Figure 4.1. The shaded area below the line $y = \sigma^2\lambda^2$ less the shaded area above is equal to $\phi^2 e(F)^2_{0,1}$ at $t = 1$, and is strictly less than $\phi^2 e(F)^2_{0,1}$ for every $t < 1$. As indicated in the figure, it follows that $\sigma^2\lambda^2 \geq e(F)^2_{1,1}q(1)^2 = q(1)^2$. The strict inequality assumed in (4.14) is required in order to construct the equilibrium described below. It is evident from Figure 4.1 that (4.13) holds if $e(F)^2_{s,1}q(s)^2$ is nonincreasing in s. The latter is true, for example, when q is constant and $F \geq 0$. (4.13) also holds if the graph of the function $y = e(F)^2_{s,1}q(s)^2$ lies below the line $y = \sigma^2\lambda^2$ for all $0 \leq s \leq 1$. In Section 4.3, we will see that when $g(x) = e^x$, the volatility of the equilibrium price is $\sigma\lambda$. Condition (4.14) then states that, in equilibrium, the terminal volatility of the fundamental value process is strictly smaller than the price volatility.

The conditions under which (4.13) holds when q and F are constant are summarized in the following simple lemma, whose proof we omit.

Lemma 4.1 *Suppose the coefficients q and F in* (4.1) *are constant. If $F \geq 0$, then Assumption* 4.3 *holds. If $F < 0$, Assumption* 4.3 *holds iff*

$$\phi^2 > q^2 \left[\frac{(2F+1)e^{-2F} - 1}{2F}\right]. \tag{4.15}$$

Thus, in order to accommodate a constant coefficient mean reverting specification (4.1), inequality (4.15) states that the ratio ϕ^2/q^2 must be larger than a strictly positive threshold. In other words, the uncertainty about the initial realization ξ_0 must be sufficiently large relative to the rate at which information about ξ_t, $t > 0$, subsequently arrives.

Proposition 4.1 *Suppose that Assumptions* 4.1 *through* 4.3 *hold. Define, for every* $0 \leq t < 1$,

$$\Pi(t) = \frac{1}{e(F)^2_{t,1}} \int_t^1 \left[\sigma^2\lambda^2 - e(F)^2_{s,1}q(s)^2\right] ds, \tag{4.16}$$

$$\begin{aligned}
A(t) &= \frac{\lambda\sigma^2}{\Pi(t)e(F)_{t,1}}, && (4.17)\\
\hat{A}(t) &= -\frac{\lambda}{e(F)_{t,1}}A(t), && (4.18)\\
a(t) &= -\gamma(t)A(t), && (4.19)
\end{aligned}$$

where

$$\gamma(t) = \frac{1}{e(F)_{t,1}}\left[\mu e(F)_{0,1} + \int_0^t e(F)_{s,1}f(s)ds\right], \quad 0 \le t \le 1. \tag{4.20}$$

Define the function

$$h(x) = g(\lambda x + \gamma(1)). \tag{4.21}$$

Then if $h \in \mathfrak{h}_{p,C}$, *the pricing rule* $H(y,t) = \mathbb{E}h(y + Z_1 - Z_t)$ *and the insider trading strategy*

$$X_t = \begin{cases} \int_0^t \left[A(s)\xi_s + \hat{A}(s)Y_s + a(s)\right] ds & \text{if } 0 \le t < 1 \\ \lambda^{-1}\left[\xi_1 - \gamma(1)\right] - Z_1 & \text{if } t = 1 \end{cases} \tag{4.22}$$

constitute an equilibrium. With probability one, the sample paths of X *are continuous and of finite variation on* $[0,1]$. *Moreover, the insider's expected terminal wealth in equilibrium is*

$$\mathbb{E}\left[W_1 | V_0\right] = \mathbb{E}\left[j(\xi_1, Z_1; h) | \xi_0\right]. \tag{4.23}$$

Proof. See Appendix F.

Assumption 4.3 ensures that $\Pi(t) > 0$ for all $t \in [0,1[$, so the terms in (4.17) – (4.19) and (4.22) are well defined. Proposition 4.3 below shows that the equilibrium pricing rule H coincides with Back's pricing rule [Back (1992, Equation (12))]. Furthermore, by substituting $F = f = q = 0$ into (4.16) through (4.22), the resulting trading strategy X reduces to the strategy obtained by Back (1992, Equation (13)). In general, the entire history of fundamental value shocks is impounded in the equilibrium trading strategy through the term $A(s)\xi_s$ in the integrand of

(4.22). However, in this special case, the fundamental value is constant throughout the time interval $[0, 1]$ and there are no fundamental shocks to be impounded in the trading strategy. Recalling the discussion following Definition 4.1, the insider's expected terminal wealth is the expected profit of a perfectly discriminating monopsonist who does not trade until the last instant before the announcement date. While the insider cannot implement monopsonistic price discrimination instantaneously, she can approximate it by accumulating a large position in many small increments over a short time period [Kyle (1985, p. 1329)]. (4.23) implies that in the continuous time limit, the insider can expect to earn perfect monopsony profits in equilibrium. This is consistent with Back (1992, Lemma 2).

Proposition 4.2 *In equilibrium, $dP = [\partial H/\partial y(Y_t, t)]\, dY$, and the process $\partial H/\partial y(Y_t, t)$ is a martingale relative to the market maker's information filtration $\mathcal{F}^Y$. Moreover, conditioned on $\mathcal{F}^Y$, the total order process Y is a Brownian motion with the same distribution as Z.*

Proof. See Appendix F.

Proposition 4.2 is a direct counterpart to Back's Theorem 3. It says that the slope dP/dY of the residual supply curve at time t is a martingale relative to the market maker's information set. This property eliminates trading schemes in which the insider obtains arbitrarily large expected profits [Kyle (1985, p. 1329)]. Since the residual supply curve has no predictable bias, it follows that buy and sell orders are equally likely to arrive [Back (1992, pp. 389–390)]. Thus, the total order process is also a martingale relative to the market maker's information set. In fact, from the market maker's perspective, only the noise traders appear to trade.

4.3 Properties of the equilibrium

Proposition 4.1 is based in part on the Kalman filter [Liptser and Shiryaev (1977, Chapter 8)], which specifies the dynamics of the conditional expectation $\hat{\xi}_t = \mathbb{E}\left[\xi_t \middle| \mathcal{F}_t^Y\right]$ [See equation (E.3) in Appendix E]. Deviations of $\hat{\xi}_t$ away from ξ_t correspond to incorrect expectations by the market maker about the fundamental value. Accordingly, they represent profitable trading opportunities for the insider. This can be seen as follows. Equations (4.16) through (4.22), together with the Kalman filter equation, imply that $Y_t = [\hat{\xi}_t - \gamma(t)]/\hat{\sigma}(t)$, where $\hat{\sigma}(t) = \lambda/e(F)_{t,1}$ [Equation (E.8)]. Substituting this expression into (4.22) yields

$$X_t = \int_0^t A(s)\left[\xi_s - \hat{\xi}_s\right] ds$$

for all $t \in [0,1[$. At each time t, the insider takes an (incremental) position dX_t proportional to the error $\xi_t - \hat{\xi}_t$. This action counteracts continued movement of $\hat{\xi}_t$ away from ξ_t through its impact on the total order $Y_t = X_t + Z_t$, from which the market maker forms his expectations. The proportional multiplier $A(t)$ is a measure of the aggressiveness with which the insider pursues profitable trading opportunities.[10] Since $\lim_{t\uparrow 1} A(t) = \infty$, the insider becomes unboundedly aggressive in her trades as the announcement date approaches.

The insider's increasing aggressiveness is reflected in the dynamics of $\hat{\xi} - \xi$, which are derived in Lemma F.3 of Appendix F:

$$d\left(\hat{\xi} - \xi\right) = [F(t) - \hat{\sigma}(t)A(t)]\left(\hat{\xi}_t - \xi_t\right) dt + \hat{\sigma}(t)dZ - q(t)dB, \quad 0 \leq t < 1. \tag{4.24}$$

Since $\lim_{t\uparrow 1} \hat{\sigma}(t)A(t) = \infty$, the drift term in (4.24) opposes any deviation of $\hat{\xi}_t$ away from ξ_t with unboundedly increasing strength as $t \uparrow 1$. In this sense, $\hat{\xi} - \xi$ behaves much like a Brownian bridge process [Karatzas and Shreve (1991, §5.6B)]. In particular, $\hat{\xi}_t - \xi_t \longrightarrow 0$ almost surely as $t \uparrow 1$. Consequently, as $t \uparrow 1$,

$$\begin{aligned} P_t = H\left(\frac{\hat{\xi}_t - \gamma(t)}{\hat{\sigma}(t)},\ t\right) \quad &\longrightarrow \quad H\left(\frac{\xi_1 - \gamma(1)}{\lambda},\ 1\right) \\ &= \quad h\left(\frac{\xi_1 - \gamma(1)}{\lambda}\right) \\ &= \quad g(\xi_1) \end{aligned}$$

almost surely. In other words, the share price converges, with probability one, to the liquidation value $V_1 = g(\xi_1)$ as the announcement date approaches.

We now turn to the issue of price volatility. Consider the special case where $g(x) = e^x$. The fundamental value process is a generalization of geometric Brownian motion which exhibits

[10] $A(t)$ is comparable to Kyle's $\beta(t)$ [Kyle (1985, p. 1326)].

mean reversion (or aversion) and time–dependent volatility $|q(t)|$. According to Proposition 4.1, the equilibrium pricing rule is given by

$$\begin{aligned} H(y,t) &= \mathbb{E}\exp\left[\lambda(y+Z_1-Z_t)+\gamma(1)\right] \\ &= \exp\left[\lambda y+\gamma(1)+\lambda^2\sigma^2(1-t)/2\right]. \end{aligned}$$

It then follows from Proposition 4.2 that

$$\frac{dP}{P_t}=\lambda dY. \tag{4.25}$$

Thus, the share price has constant volatility $\lambda\sigma$. In contrast to Black's conjecture regarding the relative volatilities of price and value, the volatility of fundamental value may, at times, be larger than the price volatility. However, if the fundamental volatility is constant over time, (4.14) implies that the price volatility is strictly greater than the fundamental volatility. Although price volatility is constant over time, it is not independent of the volatility of the fundamental value. Holding the variance of noise trades σ constant, (4.13) implies that a uniform increase of $|q(t)|$ over the time interval $[0,1]$ causes λ, hence the price volatility, to increase. Price volatility also depends upon the mean reversion coefficient $F(t)$. A uniform increase of $F(t)$ over the time interval $[0,1]$ (*i.e.* a reduction of the tendency to revert to the mean) also causes λ, therefore the price volatility, to increase. As indicated in (4.25), the share price is more sensitive to changes in the total order for larger values of λ. In this sense, market efficiency declines as λ increases [Kyle (1985, pp. 1316–1317)].

Equation (4.25) implies that, relative to the insider's information filtration $\mathcal{F}^{V,Z}$, the share price process is a geometric Brownian motion with constant volatility and time varying drift rate $\lambda dX_t/dt$. The drift rate changes over time in response to the insider's increasingly aggressive pursuit of profitable trading opportunities. It also changes in response to movements in ξ_t. However, Proposition 4.2 implies that, relative to the market maker's filtration $\mathcal{F}^Y$, the share price follows a geometric Brownian motion with constant drift rate $-\lambda\sigma^2/2$. Thus, given the order flow history (equivalently, the history of prices), changes in mean returns due to insider trading cannot be detected. The distribution of the equilibrium price is identical to what it would have been in the absence of insider trading. In other words, using terminology similar to Summers (1986, p. 599), divergences of price from value leave no discernable trace in the historical record of returns alone.

The following proposition provides an alternative representation for the equilibrium pricing rule.

Proposition 4.3 *Let* $v_1(x) = \mathbb{P}(V_1 \le x)$ *denote the distribution function of* $V_1 = g(\xi_1)$*. Then with* h *as defined in* (4.21)*, we have*

$$h(x) = v_1^{-1} \circ N(x),$$

where $N(x)$ *is the* $N(0, \sigma^2)$ *distribution function.*

Proof. We have

$$\begin{aligned}\mathbb{P}(V_1 \le x) &= \mathbb{P}(g(\xi_1) \le x) \\ &= N_1\left(g^{-1}(x)\right),\end{aligned}$$

where N_1 is the distribution function for ξ_1. Hence, by definition, $v_1^{-1}(x) = g \circ N_1^{-1}(x)$. Let us conjecture that $N_1^{-1} \circ N(x) = \alpha x + \beta$ for some constants α, β, where $\alpha > 0$. Then we require

$$\mathbb{P}(\xi_1 \le \alpha x + \beta) = N(x),$$

which is true iff

$$\frac{\xi_1 - \beta}{\alpha} \sim N(0, \sigma^2).$$

The latter is true iff $\beta = \mathbb{E}\xi_1$ and $\mathrm{Var}\,(\xi_1)/\alpha^2 = \sigma^2$. Setting $t = 0$ and $\xi = \xi_0$ in (4.5), it follows that $\alpha = \lambda$ and $\beta = \gamma(1)$. But then

$$\begin{aligned}v_1^{-1} \circ N(x) &= g \circ N_1^{-1} \circ N(x) \\ &= g(\lambda x + \gamma(1)) \\ &= h(x).\end{aligned}$$

This completes the proof. □

The representation of the pricing rule appearing in Proposition 4.3 coincides with the equilibrium pricing rule derived by Back (1992, Equation (12))]. However, while the pricing rules are identical for the two models, the insider trading strategy described in Back (1992, Equation (13)) is not admissable under the present model. Innovations to this strategy depend explicitly upon the realization V_1 at each time $t \in [0, 1]$. Since the insider is unable to see V_1 until the announcement date in the present model, this trading strategy fails to be $\mathcal{F}^{V,Z}$–adapted.

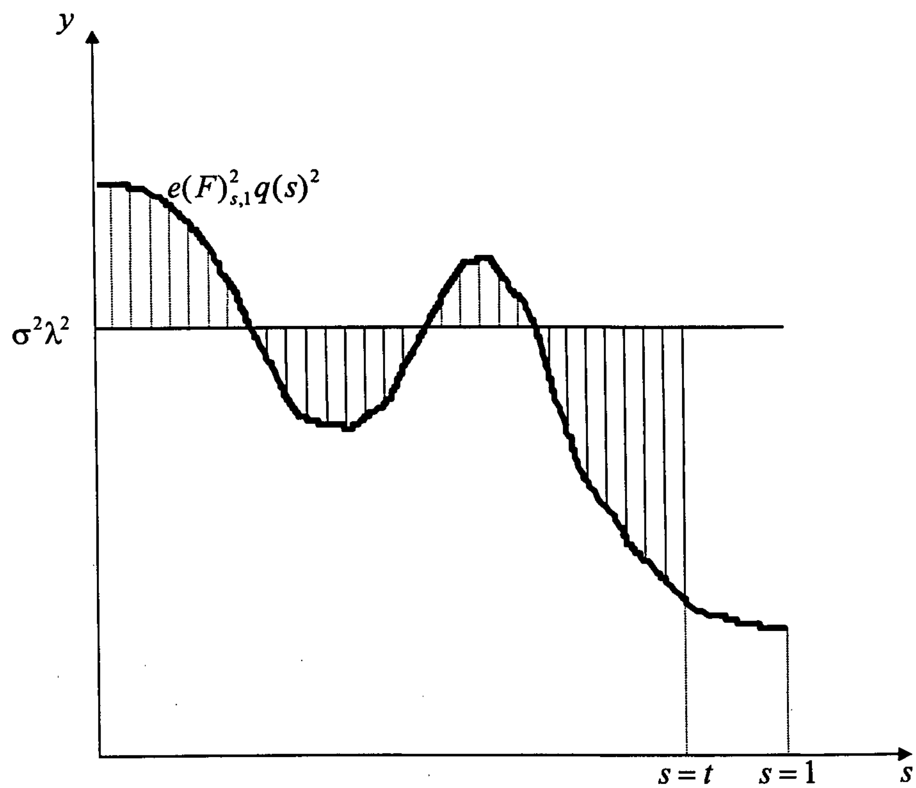

Figure 4.1

A regularity condition

Chapter 5

Conclusion and Suggestions for Future Research

5.1 Summary and concluding remarks

This thesis presents three noncompetitive equilibrium models of asset pricing under asymmetric information. The agents in the economy rationally extract information from all available signals. Those agents with superior information also account for the impact of their trades on the price. Conditions under which equilibria exist are obtained and equilibria are derived in closed form. Agents with superior information may rationally act as price chasers and those with less information may act as contrarians. In our first model, strategic trading by a single informed trader tends to stabilize the stock price. In our second model, strategic trading by, or on behalf of, a small proportion of the population may either stabilize or destabilize the stock price depending on the investors' level of risk aversion. Pervasive strategic trading in the second model can result in multiple equilibria, some of which involve extremely high price variability. The role of mutual funds in contributing to stock price volatility is a recent topic of interest in the popular press. The preceding analysis, although highly stylized, suggests that the presence of very large traders such as mutual funds may indeed contribute significantly to stock price volatility. Our third model demonstrates that the Kyle model can be further extended to a case where the insider observes a noisy signal of the terminal liquidation value prior to the announcement date. We find that the equilibrium price process looks much like the insider's private signal. In particular, when the signal follows a geometric Brownian motion, then so does the price. As time progresses, the insider's increasingly aggressive trades gradually drive the share price to its terminal liquidation value. However, the insider's trading activity cannot be detected by the market maker. Conditioned on public information, only the noise traders appear to submit orders.

5.2 Suggestions for future research

5.2.1 Alternative equilibria and generalizations of Models I and II

There is a simple one–period counterpart of Model II in which the stock price is adapted to the residual supply and the equilibrium is fully revealing. Given an initial amount of wealth W_0^i at time zero, investor i can consume c^i units of wealth, purchase X^i shares in a risky asset, and invest B^i in a riskless bond bearing no interest. The asset pays a liquidating dividend of $D \sim N(\mu_D, \sigma_D^2)$ at time one. Subject to her budget constraint $W_0^i = B^i + X^i P + c^i$, investor i solves the problem

$$\max_{(c^i, X^i, B^i)} -e^{-\gamma c^i} - \mathbb{E}\left[e^{-\gamma W_1^i}\right],$$

where $W_1^i = B^i + X^i D$. At time zero, the total supply of stock is $1 + \Theta$. Suppose there are two investors, a price taker ($i = p$) and a strategist ($i = s$) who assumes that her trades move the share price P. What is the equilibrium share price at time zero? Defining $1 - Y = 1 + \Theta - X^s$ to be the residual supply available to the price taker, it is easy to verify that the pricing rule

$$P = \mu_D - \gamma\sigma_D^2 + \gamma\sigma_D^2 Y \tag{5.1}$$

and the strategist policy $X^s = (1+\Theta)/3$ comprise an equilibrium; the stock market clears when both investors are at an optimum. In equilibrium, the price is an affine function of the supply noise Θ, so the price reveals Θ to both investors. This simple model can be extended to the case where the liquidating dividend takes the form $D = \Pi + D'$, where the strategist knows Π at time zero while the price taker does not. Provided Π and D' are normally distributed, the equilibrium share price reveals a linear combination of the strategist's private information and the supply noise, as in Wang (1993). The model can be further extended to a multiperiod discrete–time setting in which the stock pays a dividend at each date, Θ follows a mean–reverting random walk, and the strategist has private information about the next dividend. At each trading date t, an affine pricing rule $P = a_1(t) + a_Y(t)Y$ similar to (5.1) holds.

Following the intuitive argument in Kyle (1985), is it possible to obtain a continuous–time analogue of this equilibrium by allowing the time between trades to approach zero? We conjecture that it is; by holding the investment horizon fixed at T and allowing the time between trading dates to approach zero, we should obtain a limiting pricing rule in which the price increment dP takes the form $dP_t = \alpha_1(t,T)dt + \alpha_Y(t,T)dY_t$. As indicated, the coefficients α_i in this relationship should depend explicitly on calendar time t and the investment horizon T.[1] We further conjecture that as the investment horizon T approaches infinity, the slope coefficient α_Y either grows without bound or converges to zero. The resulting limit does not have a meaningful interpretation in either case. The latter conjecture is based on our inability to construct an infinite–horizon, continuous–time model such that $dP = \alpha_1 dt + \alpha_Y dY$ for constants α_i. Under a pricing rule of this form, we find that an equilibrium can hold only if an overdetermined system of equations is satisfied. In general, this system fails to have a solution.[2]

The equilibria described in Chapters 2 and 3 need not be the only ones involving strategic trade. By analogy to Wang (1993), we might propose a pricing rule of the form

$$P = \Lambda^\top \Psi, \quad \Psi = [1, X, D, \Theta, \Pi, \Pi^*]^\top,$$

where Λ is a constant 6–vector and Π^* is the uninformed agents' expectation of Π under the belief $X = X^*$. Since the informed investor's shareholdings X explictly enter the share price,

[1] This is consistent with the findings of Baruch (1997) for a similar model.

[2] Recalling footnote 11 of Chapter 2, a similar failure occurs under the more general conjecture

$$dP = \alpha_1(D,Y)dt + \alpha_Y dY + \alpha_D dD,$$

where $\alpha_1(D,Y)$ is an affine function of its arguments and α_Y, α_D are constant.

she will trade strategically in equilibrium. In fact, if we assume that $\partial P/\partial X = \Lambda_2 > 0$ and parametrize the informed investor's value function as

$$J(B, \Psi) = -\exp\left(-r\gamma B + \frac{1}{2}\Psi^\top v \Psi\right),$$

where B is her riskless bond holdings and v is a constant 6×6 symmetric matrix, then the equilibrium condition reduces to a system of simultaneous quadratic equations involving the same number of equations as unknowns. Unfortunately, this system of equations is formidably large, and we are, as yet, unable to provide conditions guaranteeing it to have a solution. However, an equilibrium of this form, should one exist, would be more easily compared to Wang's competitive counterpart than is an equilibrium of the type described in Chapters 2 and 3.

Another concern is the indifference of the informed investor over a large number of trading strategies (Lemma 2.3). It would be more appealing if the informed investor's equilibrium trading strategy was uniquely determined by her utility maximization incentives rather than by the market clearing condition. This might simply be a matter of including the uninformed investors' belief X^* as a state variable in her utility maximization problem. As in our existing models, the informed investor's Bellman equation would take the form

$$0 = \max_{c,X} \{(\text{slope})dX + (\text{intercept})\}$$

(Compare to (A.1) in Appendix A). This requires the slope term to vanish identically, as before. However, instead of forcing the intercept term to vanish, as we do in the existing models, we might now seek conditions under which the intercept term takes the form $-\alpha^2(X - X^*)^2$, where $\alpha \neq 0$. The informed investor's trading strategy would then be uniquely determined as $X = X^*$.[3]

[3] A similar mechanism arises in Back *et al* (1997).

5.2.2 Investor size and market power

In our first and third models, the strategic investor's market power arises through her informational advantage. By contrast, there is no external mechanism determining why the strategic investor is singled out in the second model. Like Cuoco and Cvitanić (1998), we simply postulate at the outset that she has market power. The fact that there exists an equilibrium in which this postulate is true says very little about how the market participants might arrive at a consensual belief in which one of them has market power and the others don't. Typically, market power is associated with large investors, but the notion of investor size does not explicitly enter the second model. Since order size is critical in determining real–world price dynamics,[4] it would be very desirable to explore the link between investor size and market power.

5.2.3 Incorporating a solvency constraint

In conventional portfolio selection problems, the solvency constraint $W \geq K$ is often imposed, where W is the investor's (nominal) wealth and K is an exogenously specified constant. This constraint rules out continuous–time versions of double–or–nothing strategies that result in arbitrage profits [Dybvig and Huang (1988)]. We have not imposed this constraint in any of the models presented here. If such a constraint exists, the uninformed investors must not only filter the informed investor's private information, but they must do so while conditioning on the constraint $W > K$. (Here we assume that as soon as the barrier $W = K$ is reached, the informed investor publicly announces her insolvency and ceases trading). A similar problem arises in Duffie and Lando (1998), where corporate outsiders must determine whether or not the value of a firm's assets has reached a bankruptcy trigger level, based only on periodic and noisy observations of firm value. Incorporating a solvency constraint into our models is a more difficult problem than that solved by Duffie and Lando. In their case the bankruptcy trigger process is exogenously specified, while in our case it is controlled by the informed investor. Moreover, instead of making periodic observations, our uninformed investors continuously observe the signals available to them. Both of these differences are likely to involve technical issues not

[4]Taleb (1997, p. 74) notes that even highly liquid markets can be influenced by orders representing only 0.2% of total volume.

addressed by Duffie and Lando.

5.2.4 Multiple strategists

Each of the models presented in this thesis involves only one strategic investor. However, in our second model, by assuming that the strategic investor trades on behalf of several individual investors, we obtain the most interesting price dynamics. Similar interesting behavior is likely to arise if each of these investors trades strategically on her own behalf. In particular, as the proportion of strategic investors approaches one, we should expect to find equilibria in which the price variability approaches infinity, since the number of price takers available to absorb instantaneous supply shocks approaches zero. In light of recent results of Back *et al* (1997), it would also be interesting to investigate the possibility of supporting diversely informed strategic traders in the infinite horizon context of our first two models. The implications of such a model would differ from those of Back *et al*, where the existence of a finite announcement date plays a crucial role. As Back *et al* demonstrate, these efforts will be complicated by the fact that each of the strategic investors competes with all the others.

5.2.5 Eliminating supply noise

In our first and third models, an information asymmetry arises because uninformed investors cannot disentangle the informed investor's private information from the supply noise. We might expect an information asymmetry to prevail in almost any model in which uninformed investors cannot distinguish between two or more sources of uncertainty, whether they arise from supply noise or not.[5] For example, suppose informed investors have private information about the dividend stream but there is no supply noise. Then the uninformed can observe the informed investor's trades. However, if they do not know the informed investor's risk aversion coefficient γ, they will be unable to disentangle the effects of the informed investor's risk aversion from those of her private signal. Unfortunately, the uninformed investors' filtering problem is unlikely to admit a closed form solution in this case. Likewise, suppose the informed investor

[5] Wang (1994), for example, involves an information asymmetry in which there are investors having private knowledge of the dividend stream and private investment opportunities.

has private information about the dividend stream and receives a private endowment stream. If she maximizes a (known) utility function of the CRRA type; *i.e.*

$$\mathbb{E}\left[\int_0^\infty c_s^p \, ds \,\middle|\, \mathfrak{I}_0\right], \quad 0 < p < 1,$$

then even in the absence of supply noise, the uninformed will be unable to disentangle the informed investor's endowment stream from her private information. Similar effects will arise if the informed investor has two separate pieces of private information about stock fundamentals. For example, our third model might be revised so that there are no noise traders, but the insider has exclusive knowledge of both the fundamental value V_t and its drift rate.[6] The market maker will be able to observe the insider's trades, but will be unable to isolate the extent to which they reflect the fundamental value or its drift rate.

[6]This precludes the conventional definition of V_t as the time t expectation of the terminal share payoff. (See footnote 1 of Chapter 4). More generally, we could assume that V is a two (or more) parameter process, and that the insider has exclusive knowledge of one of the parameters.

Bibliography

[1] A. Admati and P. Pfleiderer (1988), "A theory of intraday patterns: volume and price variability," *Review of Financial Studies* **1**: 3–40.

[2] K. Back (1991), "Asset pricing for general processes," *Journal of Mathematical Economics* **20**: 371–395.

[3] K. Back (1992), "Insider trading in continuous time," *Review of Financial Studies* **5**: 387–409.

[4] K. Back (1993), "Asymmetric information and options," *Review of Financial Studies* **6**: 435–472.

[5] K. Back, H. Cao, and G. Willard (1997), "Imperfect competition among informed traders," *Journal of Finance*, forthcoming.

[6] K. Back and H. Pedersen (1998), "Long–lived information and intraday patterns," *Journal of Financial Markets* **1**: 385–402.

[7] S. Baruch (1997), "Insider trading and risk aversion," working paper, Washington University in Saint Louis.

[8] S. Basak (1995), "Dynamic consumption and portfolio choice and asset pricing with non–price–taking agents," working paper, The Wharton School, University of Pennsylvania.

[9] S. Basak and D. Cuoco (1998), "An equilibrium model with restricted stock market participation," *Review of Financial Studies* **11**: 309–341.

[10] F. Black (1986), "Noise," *Journal of Finance* **41**: 529–543.

[11] H. Cao (1998), "On the possibility of informationally efficient markets," working paper, Haas School of Business.

[12] D. Cuoco and J. Cvitanić (1998), "Optimal consumption choices for a 'large' investor," *Journal of Economic Dynamics and Control*, **22**: 401–436.

[13] D. Duffie and D. Lando (1998), "Term structures of credit spreads with incomplete accounting information," working paper, Stanford University.

[14] P. Dybvig and C.–f. Huang (1988), "Nonnegative wealth, absence of arbitrage, and feasible consumption plans," *Review of Financial Studies* **1**: 377–401.

[15] E. Fama and K. French (1987), "Permanent and transitory components of stock prices," *Journal of Political Economy* **96**: 246–273.

[16] S. Grossman (1981), "An introduction to the theory of rational expectations under asymmetric information," *Review of Economic Studies* **48**: 541-559.

[17] J. Harrison and S. Pliska (1981), "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and their Applications* **11**: 215–260.

[18] U. Haussmann (1986), *A Stochastic Maximum Principle for Optimal Control of Diffusions.* Longman Scientific and Technical.

[19] E. Hewitt and K. Stromberg (1965), *Real and Abstract Analysis.* Springer–Verlag.

[20] J. Jacod and A. Shiryaev (1987), *Limit Theorems for Stochastic Processes.* Springer–Verlag.

[21] R. Jarrow (1992), "Market manipulation, bubbles, corners, and short squeezes," *Journal of Financial and Quantitative Analysis* **27**: 311–336.

[22] I. Karatzas and S. Shreve (1991), *Brownian Motion and Stochastic Calculus.* Springer–Verlag.

[23] R. Kihlstrom (1998), "Monopoly power in dynamic securities markets," working paper, The Wharton School, University of Pennsylvania.

[24] A. Kyle (1984), "Market structure, information, futures markets, and price formation," in G. Storey *et al* (eds.), *International Agricultural Trade: Advanced Readings in Price Formation, Market Structure, and Price Instability*, Westview Press, pp. 45–64.

[25] A. Kyle (1985), "Continuous auctions and insider trading," *Econometrica* **53**: 1315–1335.

[26] H. Leland and K. Toft (1996), "Optimal capital structure, endogeneous bankruptcy, and the term structure of credit spreads," *Journal of Finance* **51**: 987–1019.

[27] R. Liptser and A. Shiryaev (1977), *Statistics of Random Processes. Volumes I, II.* Springer–Verlag.

[28] J. Rauch (1991), *Partial Differential Equations.* Springer–Verlag.

[29] M. Spiegel (1998), "Stock price volatility in a multiple security overlapping generations model," *Review of Financial Studies* **11**: 419–447.

[30] A. Subrahmanyam (1991), "Risk aversion, market liquidity, and price efficiency," *Review of Financial Studies* **4**: 417–441.

[31] L. Summers (1986), "Does the stock market rationally reflect fundamental values?" *Journal of Finance* **41**: 591–601.

[32] N. Taleb (1997), *Dynamic Hedging: Managing Vanilla and Exotic Options.* Wiley.

[33] J. Wang (1993), "A model of intertemporal asset prices under asymmetric information," *Review of Economic Studies* **60**: 249–282.

[34] J. Wang (1994), "A model of competitive stock trading volume," *Journal of Political Economy* **102**: 127–168.

Appendix A

Infinite horizon pricing rules and trading strategies

Proof of Lemma 2.1. Given the pricing rule Q^X, the Brownian motion $b_Q z$ is $\mathfrak{I}$-adapted. Therefore, by Itô's formula, it suffices to show that the function $J(W, X)$ is the unique C^2 solution to the problem

$$0 = \max_{c,x} \left\{ -e^{-\gamma c} - \rho J + (rW - c + X\lambda[a_\Theta X + x]) J_W + x J_X + \frac{1}{2} X^2 \|b_Q\|^2 J_{WW} \right\}. \tag{A.1}$$

The first order conditions for this maximization problem are

$$0 = X\lambda J_W + J_X, \tag{A.2}$$

$$0 = \gamma e^{-\gamma c} - J_W. \tag{A.3}$$

We now show that these conditions uniquely determine J. Since the characteristic curves of the PDE (A.2) are of the form $W - \lambda X^2/2 =$ constant, J must take the form

$$J(W, X) = j\left(W - \frac{\lambda}{2} X^2\right)$$

for some C^2 function j on the real line [Rauch (1991, pp. 42, 53)]. Substituting this condition and the first order conditions into (A.1), we obtain the differential equation

$$0 = -\gamma^{-1}j' - \rho j + (rW + \gamma^{-1}\log(\gamma^{-1}j') + \lambda a_\Theta X^2)j' + \frac{1}{2}X^2\|b_Q\|^2 j'',$$

where j, j', and j'' are evaluated at the argument $\tilde{W} = W - \lambda X^2/2$. Rearranging terms, it follows that

$$\left(\gamma^{-1} - \left[r\tilde{W} + \gamma^{-1}\log(\gamma^{-1}j'(\tilde{W}))\right]\right)j'(\tilde{W}) + \rho j(\tilde{W}) = \lambda\left(a_\Theta + \frac{r}{2}\right)X^2 j'(\tilde{W}) + \frac{1}{2}X^2\|b_Q\|^2 j''(\tilde{W}).$$

Since X does not appear in isolation on the left hand side of this equation, both sides must vanish. Hence,

$$0 = \left(\gamma^{-1} - \left[r\tilde{W} + \gamma^{-1}\log(\gamma^{-1}j'(\tilde{W}))\right]\right)j'(\tilde{W}) + \rho j(\tilde{W}), \tag{A.4}$$

$$0 = \lambda\left(a_\Theta + \frac{r}{2}\right)j'(\tilde{W}) + \frac{1}{2}\|b_Q\|^2 j''(\tilde{W}). \tag{A.5}$$

Since $\|b_Q\|^2 = \lambda\gamma^{-1}(1 + 2a_\Theta/r)$, (A.5) implies that $j(\tilde{W})$ must be of the form $C_1\exp(-r\gamma\tilde{W}) + C_2$, where C_1 and C_2 are constants of integration. It then follows from (A.4) that

$$C_1 = -\frac{1}{r}\exp\left(1 - \frac{\rho}{r}\right), \quad C_2 = 0. \tag{A.6}$$

Recalling that $\tilde{W} = W - \lambda X^2/2$, we conclude that $J(W, X) = C_1\exp(-r\gamma\tilde{W})$ is the only possible C^2 solution to (A.2) through (A.3). This solution also satisfies the second order conditions with respect to c, while any choice of x is optimal given (A.2). □

Proof of Lemma 2.2. Let J be as defined in Lemma 2.1; *i.e.* $J(W_t, X_t) = C_1 e^{-r\gamma\tilde{W}_t}$, where C_1 is defined in (A.6) and $\tilde{W}_t = W_t - \lambda X_t^2/2$. By definition of $\tilde{W}$ and Q^X, we have

$$\begin{aligned} d\tilde{W}_t &= dW_t - \lambda X_t dX_t \\ &= (rW_t - c_t)dt + X_t dQ_t^X - \lambda X_t dX_t \\ &= (rW_t - c_t + \lambda a_\Theta X_t^2)dt + X_t b_Q dz_t. \end{aligned}$$

By hypothesis, the drift term of this equation is bounded below by

$$\frac{r\lambda}{2}\left(1+\frac{2a_{\Theta}}{r}\right)X_t^2-\epsilon=\frac{r\gamma}{2}\|b_Q\|^2X_t^2-\epsilon.$$

It then follows that $r\gamma\tilde{W}_t \geq r\gamma\tilde{W}_0 - r\gamma\epsilon t - \log\mathcal{E}_t(X)$, where $\mathcal{E}(X)$ denotes the exponential process

$$\mathcal{E}_t(X)=\exp\left[-\frac{1}{2}\int_0^t(r\gamma X_s\|b_Q\|)^2\,ds-\int_0^t r\gamma X_s b_Q dz_s\right].$$

Consequently,

$$e^{-\rho t}|J(W_t,X_t)|\leq|C_1|e^{-r\gamma\tilde{W}_0-\kappa t}\mathcal{E}_t(X), \tag{A.7}$$

where $\kappa=\rho-r\gamma\epsilon>0$. Given $t\in\mathbb{R}_+$ and $\delta>0$, choose $T\in[t,\infty[$ so that

$$\mathbb{P}\left\{|C_1|e^{-r\gamma\tilde{W}_0-\kappa T}\mathcal{E}_t(X)>\delta\right\}<\frac{\delta}{2}. \tag{A.8}$$

Then choose a constant $n<\infty$ so that the $\mathcal{I}_0$–measurable event $F=\{|K_T|\leq n\}$ has probability larger than $1-\delta$. Since $|K_T|\leq n$ on F, Corollaries 1.3 and 2.4 of Haussmann (1986) imply that $\mathcal{E}(1_F X)$ is a martingale on $[0,T]$. Corollary 2.3 of Haussmann also implies that $[\mathcal{E}_T(1_F X)]^p$ is integrable for some constant $p>1$. It then follows from Doob's maximal inequality that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[\mathcal{E}_t(1_F X)]^p\right)<\infty. \tag{A.9}$$

Let $\{\tau_k\}$ be a sequence of bounded $\mathcal{I}$–stopping times such that $\lim_{k\longrightarrow\infty}\tau_k=\infty$ a.s. Clearly

$$\mathbb{E}_t\left[e^{-\kappa\tau_k}\mathcal{E}_{\tau_k}(1_F X)\right]\leq\mathbb{E}_t\left[1_{\{\tau_k\leq T\}}\mathcal{E}_{\tau_k\wedge T}(1_F X)\right]+\mathbb{E}_t\left[e^{-\kappa T}\mathcal{E}_{\tau_k}(1_F X)\right],$$

for every k, where $\mathbb{E}_t[\bullet]=\mathbb{E}[\bullet|\mathcal{I}_t]$. By (A.9), the first term on the right–hand side approaches zero in L^1, hence in probability, as $k\longrightarrow\infty$. Furthermore, since τ_k is a bounded $\mathcal{I}$–stopping time, the second term is equal to $e^{-\kappa T}\mathcal{E}_{\tau_k \wedge t}(1_F X)$, which approaches $e^{-\kappa T}\mathcal{E}_t(1_F X)$ almost surely as $k \longrightarrow \infty$. Therefore, if k is sufficiently large,

$$\mathbb{P}\left\{\mathbb{E}_t\left[e^{-\kappa\tau_k}\mathcal{E}_{\tau_k}(1_F X)\right] > e^{-\kappa T}\mathcal{E}_t(1_F X) + \delta|C_1|^{-1}e^{r\gamma\tilde{W}_0}\right\} < \frac{\delta}{2}.$$

Since $\mathbb{P}F > 1-\delta$, this implies

$$\mathbb{P}\left\{\mathbb{E}_t\left[e^{-\kappa\tau_k}\mathcal{E}_{\tau_k}(X)\right] > e^{-\kappa T}\mathcal{E}_t(X) + \delta|C_1|^{-1}e^{r\gamma\tilde{W}_0}\right\} < \frac{3}{2}\delta.$$

Combining this with (A.7) and (A.8), we conclude that $\mathbb{P}\left\{\mathbb{E}_t\left[e^{-\rho\tau_k}|J(W_{\tau_k}, X_{\tau_k})|\right] > 2\delta\right\} < 2\delta$. Since $t \in \mathbb{R}_+$ and $\delta > 0$ may be chosen arbitrarily, it follows that (c, X) is Q^X–admissible. □

Proof of Lemma 2.3. The strategist's Bellman equation has the solution $J(W,X) = \jmath(\tilde{W}) \stackrel{def.}{=} C_1\exp(-r\gamma\tilde{W})$, where C_1 is the constant defined in (A.6) and $\tilde{W} = W - \lambda X^2/2$. Let (c, X) be a Q^X–admissible consumption–investment strategy, and let W be the corresponding wealth process. Given $t \in \mathbb{R}_+$ and a bounded $\mathcal{J}$–stopping time $\tau \geq t$, the generalized Itô formula implies

$$\begin{aligned} e^{-\rho\tau}J(W_\tau, X_\tau) - e^{-\rho t}J(W_t, X_t) &= \int_t^\tau e^{-\rho s}\,\tilde{\jmath}_{s-}\left\{-r\gamma d\tilde{W}_s - \rho ds + \frac{1}{2}(r\gamma)^2 d[\tilde{W}^c, \tilde{W}^c]_s\right\} \\ &\quad + \sum_{t<s\leq\tau} e^{-\rho s}\,\tilde{\jmath}_{s-}\left\{e^{-r\gamma\Delta\tilde{W}_s} - 1 + r\gamma\Delta\tilde{W}_s\right\}, \quad \text{(A.10)} \end{aligned}$$

where $\tilde{W}_s = W_s - \lambda X_s^2/2$ and $\tilde{\jmath}_s = \jmath(\tilde{W}_s)$. $\tilde{W}^c$ denotes the continuous local martingale component of the process $\tilde{W}$, $[\tilde{W}^c, \tilde{W}^c]$ is its quadratic variation, and $\Delta\tilde{W}_s = \tilde{W}_s - \tilde{W}_{s-}$.[1] Again by the generalized Itô formula,

$$\begin{aligned} \frac{\lambda}{2}d\left(X_s^2\right) &= \lambda X_{s-}dX_s + \frac{\lambda}{2}d[X^c, X^c]_s + \frac{\lambda}{2}\{\Delta(X^2)_s - 2X_{s-}\Delta X_s\} \\ &= \lambda X_{s-}dX_s + \frac{\lambda}{2}d[X^c, X^c]_s + \frac{\lambda}{2}(\Delta X_s)^2, \end{aligned}$$

[1]See Jacod and Shiryaev (1987, Theorem I.4.57) for the generalized Itô formula. Despite the conflict with the notation of Section 2.3, we use the symbol Δ as indicated above, in keeping with conventional usage. This terminology is confined to the present proof, and the context in which it is used should be clear.

where X^c is the continuous local martingale component of X, $\Delta(X^2)_s = X_s^2 - X_{s-}^2$, and $\Delta X_s = X_s - X_{s-}$. Combining this with (2.7), we have

$$d\tilde{W}_s = (rW_s - c_s)ds + \lambda a_\Theta X_s^2 ds + X_{s-} b_Q dz_s - \frac{\lambda}{2} d[X^c, X^c]_s - \frac{\lambda}{2} (\Delta X_s)^2 . \tag{A.11}$$

Here we have used the fact that, with probability one, $X_{s-}X_s = X_s^2$ for almost every $s \geq 0$, since X has right–continuous, left–limited sample paths. (A.11) implies that the right–hand side of (A.10) can be written as

$$\begin{aligned}\int_t^\tau e^{-\rho s}\, \tilde{\jmath}_s &\left\{-r\gamma(rW_s - c_s + \lambda a_\Theta X_s^2) + \frac{1}{2}(r\gamma)^2 X_s^2 \|b_Q\|^2 - \rho\right\} ds \\ &+ \frac{r\gamma\lambda}{2} \int_t^\tau e^{-\rho s}\, \tilde{\jmath}_{s-} d[X^c, X^c]_s + M_\tau - M_t \\ &\qquad + \sum_{t<s\leq\tau} e^{-\rho s}\, \tilde{\jmath}_{s-} \left\{e^{-r\gamma\Delta\tilde{W}_s} - 1 + r\gamma\Delta\tilde{W}_s + \frac{r\gamma\lambda}{2}(\Delta X_s)^2\right\},\end{aligned} \tag{A.12}$$

where

$$M_\bullet = -r\gamma \int_0^\bullet e^{-\rho s}\, \tilde{\jmath}_{s-} X_{s-} b_Q dz_s.$$

Using the fact that $\|b_Q\|^2 = \lambda\gamma^{-1}(1 + 2a_\Theta/r)$, the first integral in (A.12) simplifies to

$$\int_t^\tau e^{-\rho s}\, \tilde{\jmath}_s \left\{-r\gamma(rW_s - c_s) + \frac{1}{2}\lambda\gamma r^2 X_s^2 - \rho\right\} ds.$$

Combining these results, it follows that

$$\begin{aligned}-\int_t^\tau & e^{-\rho s - \gamma c_s} ds - [e^{-\rho t} J(W_t, X_t) - e^{-\rho\tau} J(W_\tau, X_\tau)] = \\ &\int_t^\tau e^{-\rho s} \left(-e^{-\gamma c_s} + \tilde{\jmath}_s \left\{-r\gamma(rW_s - c_s) + \frac{1}{2}\lambda\gamma r^2 X_s^2 - \rho\right\}\right) ds \\ &\quad + \frac{r\gamma\lambda}{2} \int_t^\tau e^{-\rho s}\, \tilde{\jmath}_{s-} d[X^c, X^c]_s + M_\tau - M_t \\ &\qquad + \sum_{t<s\leq\tau} e^{-\rho s}\, \tilde{\jmath}_{s-} \left\{e^{-r\gamma\Delta\tilde{W}_s} - 1 + r\gamma\Delta\tilde{W}_s + \frac{r\gamma\lambda}{2}(\Delta X_s)^2\right\}.\end{aligned} \tag{A.13}$$

Elementary calculus establishes that the first integrand on the right–hand side of (A.13) is ≤ 0 with equality iff

$$c_s = -\gamma^{-1}\log(-r\tilde{\jmath}_s) = rW_s - \frac{1}{2}r\lambda X_s^2 - \gamma^{-1}(1-\rho/r).$$

Since $\tilde{\jmath} < 0$, the second integral on the right–hand side of (A.13) is ≤ 0 a.s. with equality iff $X^c = 0$ a.s. Moreover, since $e^x - 1 \geq x$ for all real x,

$$\tilde{\jmath}_{s-}\left\{e^{-r\gamma\Delta\tilde{W}_s} - 1 + r\gamma\Delta\tilde{W}_s + \frac{r\gamma\lambda}{2}(\Delta X_s)^2\right\} \leq \tilde{\jmath}_{s-}\frac{r\gamma\lambda}{2}(\Delta X_s)^2 \leq 0,$$

with equality iff $\Delta X_s = 0$. Provided the conditional expectation $\mathbb{E}\left[M_\tau\,|\mathfrak{I}_t\right]$ exists, it follows that

$$\mathbb{E}\left[-\int_t^\tau e^{-\rho s-\gamma c_s}ds\,\middle|\,\mathfrak{I}_t\right] \leq e^{-\rho t}J(W_t,X_t) + \mathbb{E}\left[M_\tau - M_t\,|\mathfrak{I}_t\right] - \mathbb{E}\left[e^{-\rho\tau}J(W_\tau,X_\tau)\,|\mathfrak{I}_t\right], \quad \text{(A.14)}$$

with equality if (c, X) satisfies the conditions stated in the Lemma. By definition of M, there exists a nondecreasing sequence of bounded $\mathfrak{I}$–stopping times $\{\tau_k\}$ such that $\tau_0 = t$, $\tau_k \longrightarrow \infty$ a.s., and $\mathbb{E}\left[M_{\tau_k} - M_t\,|\mathfrak{I}_t\right] = 0$ for every $k \in \mathbb{N}$. Setting $\tau = \tau_k$ in (A.14) and letting $k \longrightarrow \infty$, it follows that

$$\mathbb{E}\left[-\int_t^\infty e^{-\rho s-\gamma c_s}ds\,\middle|\,\mathfrak{I}_t\right] \leq e^{-\rho t}J(W_t,X_t)$$

with equality if (c, X) satisfies the conditions stated in the Lemma. Convergence on the left–hand side of the inequality follows from the (conditional) monotone convergence theorem, while convergence on the right–hand side follows from the definition of Q^X–admissibility. The inequality in (A.14) is strict unless (c, X) satisfies the conditions stated in the Lemma. A straightforward refinement of the preceding limit argument establishes that the final inequality is also strict unless (c, X) satisfies these conditions. $\square$

Appendix B

Model I – equilibrium

In order to prove the claims in Section 2.3, define the process δ by

$$\begin{aligned} \delta_0 &= \mathbb{E}[\Pi_0|\mathfrak{U}_0] - \Pi_0, \\ d\delta_t &= -a_\Pi \delta_t dt \; + \; [\Gamma_{11}, \;\; \Gamma_{12}] \begin{bmatrix} -\delta_t dt + \sigma_D dz_D(t) \\ -\mu_\Delta \delta_t dt - \sigma_\Theta dz_\Theta(t) \end{bmatrix} \; - \; \sigma_\Pi dz_\Pi(t), \end{aligned} \tag{B.1}$$

where

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = \left(H\Omega + \alpha\sigma^\top \right)^\top (\beta\beta)^{-1} \tag{B.2}$$

and

$$H = \begin{bmatrix} 1 & 0 \\ \mu_\Delta & 0 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 0 & 0 \\ 0 & -\sigma_\Theta \end{bmatrix}, \quad \beta = \begin{bmatrix} \sigma_D & 0 \\ 0 & \sigma_\Theta \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_\Pi & 0 \\ 0 & \sigma_\Theta \end{bmatrix}.$$

μ_Δ and Ω are as defined in Section 2.3. Since $\mathbb{E}^*[\Pi_0|\mathfrak{U}_0] = \mathbb{E}[\Pi_0|\mathfrak{U}_0]$ for any belief $X = X^*$, we have $\delta_t = \Delta_t$ at $t = 0$. The following Lemma specifies a belief under which this equality holds for all $t \geq 0$.

Lemma B.1 *Under Assumptions* 2.1 *and* 2.2, *the process* $X_t^* = X_0 - \int_0^t (\mu_\Delta \delta_s + a_\Theta X_s^*)\, ds$ *satisfies*

$$\frac{dX^*}{dt} = -\mu_\Delta \Delta - a_\Theta X^* \quad a.s.$$

Proof. Define $\tilde{\Pi}_t = \Pi_t + \delta_t$. Then $\tilde{\Pi}_0 = \mathbb{E}[\Pi_0 | \mathfrak{U}_0]$ and

$$\begin{aligned} d\tilde{\Pi}_t &= d\Pi_t + d\delta_t \\ &= a_\Pi(\bar{\Pi} - \tilde{\Pi}_t)dt + [\Gamma_{11},\ \Gamma_{12}] \left\{ \begin{bmatrix} dD_t \\ dY_t \end{bmatrix} - \begin{pmatrix} \tilde{\Pi}_t - kD_t \\ -a_\Theta Y_t \end{pmatrix} dt \right\}. \end{aligned}$$

Therefore, $\tilde{\Pi}$ is $\mathfrak{U}$–adapted. The dynamics of the processes $\Pi_t^* = \mathbb{E}^*[\Pi_t | \mathfrak{U}_t]$ and $\Theta_t^* = \mathbb{E}^*[\Theta_t | \mathfrak{U}_t]$ can now be obtained by solving the following filtering problem. The state equation is

$$\begin{bmatrix} d\Pi_t \\ d\Theta_t \end{bmatrix} = \left(F \begin{bmatrix} \Pi_t \\ \Theta_t \end{bmatrix} + f \right) dt + \sigma \begin{bmatrix} dz_\Pi(t) \\ dz_\Theta(t) \end{bmatrix},$$

where $F = \mathrm{diag}[-a_\Pi,\ -a_\Theta]$, $f = [a_\Pi \bar{\Pi},\ 0]^\top$, and σ is as defined previously. Defining the $\mathfrak{U}$–adapted process $h_t = [-kD_t,\ -\mu_\Delta \tilde{\Pi}_t - a_\Theta Y_t]^\top$, the observation equation is

$$\begin{bmatrix} dD_t \\ dY_t \end{bmatrix} = \left(H \begin{bmatrix} \Pi_t \\ \Theta_t \end{bmatrix} + h_t \right) dt + \alpha \begin{bmatrix} dz_\Pi(t) \\ dz_\Theta(t) \end{bmatrix} + \begin{bmatrix} \sigma_D dz_D(t) \\ 0 \end{bmatrix},$$

where H and α are as defined above. By Liptser and Shiryaev (1977, Theorem 12.7), the filter dynamics are

$$\begin{aligned} \begin{bmatrix} d\Pi_t^* \\ d\Theta_t^* \end{bmatrix} = \left(F \begin{bmatrix} \Pi_t^* \\ \Theta_t^* \end{bmatrix} + f \right) dt + \\ \left(H\Sigma_t + \alpha\sigma^\top \right)^\top (\beta\beta)^{-1} \left\{ \begin{bmatrix} dD_t \\ dY_t \end{bmatrix} - \left(H \begin{bmatrix} \Pi_t^* \\ \Theta_t^* \end{bmatrix} + h_t \right) dt \right\}, \end{aligned} \tag{B.3}$$

where Σ_t is the (unique) solution of the Riccati equation

$$\Sigma_0 = \Omega, \qquad \dot{\Sigma}_t = -\left(H\Sigma_t + \alpha\sigma^\top\right)^\top (\beta\beta)^{-1}\left(H\Sigma_t + \alpha\sigma^\top\right) + \sigma\sigma^\top + F\Sigma_t + (F\Sigma_t)^\top.$$

Straightforward algebra establishes that $\Sigma_t = \Omega$ for all $t \geq 0$. Thus, recalling (B.2), we have $(H\Sigma_t + \alpha\sigma^\top)^\top(\beta\beta)^{-1} = \Gamma$. Substituting this into (B.3) and simplifying,

$$\begin{bmatrix} d\Pi_t^* \\ d\Theta_t^* \end{bmatrix} = \begin{bmatrix} a_\Pi(\bar{\Pi} - \Pi_t^*) \\ -a_\Theta\Theta_t^* \end{bmatrix} dt + \Gamma \begin{bmatrix} -(\Pi_t^* - \Pi_t)dt + \sigma_D dz_D(t) \\ -\mu_\Delta(\Pi_t^* - \Pi_t)dt - \sigma_\Theta dz_\Theta(t) \end{bmatrix}. \tag{B.4}$$

Subtracting $d\Pi_t$ from the first row of this equation and comparing to the definition of δ, it follows that $\delta_t = \Pi_t^* - \Pi_t = \Delta_t$ a.s. for every $t \geq 0$. □

Proof of Proposition 2.1. Suppose the uninformed agents hold the belief $X = X^*$, where X^* is the process described in Lemma B.1. The equality

$$P_t = \mathbb{E}^*\left[\int_t^\infty e^{-r(s-t)} D_s ds \,\middle|\, \mathcal{U}_t\right] = m_D D_t + m_\Pi \Pi_t^* + m_1 \tag{B.5}$$

follows immediately from (2.1), (2.2) and the law of iterated expectations. (B.4) then implies that the excess return $dQ = dP + Ddt - rPdt$ satisfies

$$dQ_t = -\left(m_D + m_\Pi\phi(\mu_\Delta)\left[\sigma_D^{-2} + \sigma_\Theta^{-2}\mu_\Delta^2\right]\right)\Delta_t dt + b_{QD}dz_D(t) + b_{Q\Theta}dz_\Theta(t).$$

It then follows from (2.11) that

$$dQ_t = -\lambda\mu_\Delta\Delta_t dt + b_{QD}dz_D(t) + b_{Q\Theta}dz_\Theta(t), \tag{B.6}$$

where $\lambda = (b_{QD}^2 + b_{Q\Theta}^2)\gamma(1 + 2a_\Theta/r)^{-1}$. Now under the belief $X = X^*$, the term $-\mu_\Delta\Delta_t dt$ can also be written as $a_\Theta X_t dt + dX_t$. Recalling the definition of Q^X in Lemma 2.1, it follows that

$$dQ_t^X = dQ_t\Big|_{X=X^*}.$$

This establishes the second of the equilibrium conditions in Definition 2.1.

If we can show that (c^*, X^*) is a Q^X–admissible consumption–investment policy, the remainder of the proof follows from Lemma 2.3. In view of (B.5), Δ is an affine combination of D, Π, and P and is therefore $\mathcal{I}$–adapted. Therefore X^* is also $\mathcal{I}$–adapted. The self–financing condition ensures that c^* is also $\mathcal{I}$–adapted. It suffices now to show that that X^* satisfies the conditions of Lemma 2.2. To this end, (B.1) implies $d\Delta_t = a_\Delta \Delta_t dt + b_\Delta dz_t$, where $a_\Delta \in \mathbb{R}$ and b_Δ is a constant 1×3 row vector. Therefore

$$\Delta_t = e^{a_\Delta t}\Delta_0 + a_\Delta \int_0^t e^{a_\Delta (t-s)} b_\Delta z_s ds + b_\Delta z_t. \tag{B.7}$$

Defining $z_t^* = \max_{0 \le s \le t} \|z_s\|$, it follows that there are continuous, deterministic functions K_i such that $|\Delta_t| \le K_1(t)|\Delta_0| + K_2(t) z_t^*$. Likewise, since $dX_t^* = -\mu_\Delta \Delta_t dt - a_\Theta X_t^* dt$, there are continuous, deterministic functions $K_j(t)$ such that

$$|X_t^*| \le K_3(t)\,|X_0| + K_4(t)\,|\Delta_0| + K_5(t) z_t^*.$$

We conclude that X^* meets the conditions of Lemma 2.2, so (c^*, X^*) is Q^X–admissible. □

Proof of Proposition 2.2. Let Σ_t be the solution to the Riccati equation defined in the proof of Lemma B.1. By Theorem 12.7 of Liptser and Shiryaev (1977), $[\Pi_t, \Theta_t]^\top$ is conditionally Gaussian given $\mathcal{U}_t$, with covariance matrix Σ_t. Since $\Sigma_t = \Omega$ for every $t \ge 0$, the first part of the proposition is immediate. Turning to the second part, (12.65) of Liptser and Shiryaev implies that

$$\begin{bmatrix} d\tilde{z}_D(t) \\ d\tilde{z}_Y(t) \end{bmatrix} = \beta^{-1} \left\{ \begin{bmatrix} dD_t \\ dY_t \end{bmatrix} - \left(H \begin{bmatrix} \Pi_t^* \\ \Theta_t^* \end{bmatrix} + h_t \right) dt \right\} = \begin{bmatrix} \sigma_D^{-1}(\Pi_t - \Pi_t^*)dt + dz_D(t) \\ \sigma_\Theta^{-1}(dY_t + a_\Theta Y_t dt) \end{bmatrix}$$

is a Brownian increment with respect to $\mathcal{U}$. In view of (2.1), we may write

$$\begin{aligned} dD_t &= (\Pi_t^* - kD_t)dt + \sigma_D d\tilde{z}_D(t), \\ dY_t &= -a_\Theta Y_t dt + \sigma_\Theta d\tilde{z}_Y(t). \end{aligned}$$

Moreover, (B.3) implies

$$\begin{bmatrix} d\Pi_t^* \\ d\Theta_t^* \end{bmatrix} = \begin{bmatrix} a_\Pi(\bar{\Pi} - \Pi_t^*) \\ -a_\Theta \Theta_t^* \end{bmatrix} dt + \Gamma\beta \begin{bmatrix} d\tilde{z}_D(t) \\ d\tilde{z}_Y(t) \end{bmatrix}.$$

By definition of Γ, we conclude that

$$d\Pi_t^* = a_\Pi(\bar{\Pi} - \Pi_t^*)dt + \sigma_D^{-1}\phi(\mu_\Delta)d\tilde{z}_D(t) + \sigma_\Theta^{-1}\mu_\Delta\phi(\mu_\Delta)d\tilde{z}_Y(t).$$

This completes the proof. □

Appendix C

Model II – equilibrium

Proof of Lemma 3.1. For any fixed ξ, we have

$$\lim_{b\uparrow 0} b^2\psi(\xi;b) = \frac{b_{QD}^4}{2}\left(\frac{\gamma}{1-\omega}\left[1+\frac{2a_\Theta}{r}\right]^{-1}\xi+\frac{r\gamma'}{\omega}\right)^2.$$

Since the mapping $(b,\xi)\mapsto b^2\psi(\xi;b)$ is continuous at the origin, it follows that

$$b^2\psi(\Xi(b);b) \longrightarrow \frac{b_{QD}^4}{2}\left(\frac{r\gamma'}{\omega}\right)^2 > 0$$

as $b\uparrow 0$. Therefore $\psi(\Xi(b);b) > 0$ if $b < 0$ is sufficiently close to zero. To complete the proof, two separate cases must be considered. First, suppose there exists $b < 0$ such that $(1-\omega)b+\sigma_\Theta\lambda(b) = 0$. Let b_1 be the largest such value (there are at most two such values). Then if $\epsilon > 0$, there exists $k\in\{1,2\}$ such that

$$\Xi(b_1+\epsilon) = C_1\epsilon^{-k}+o(\epsilon^{-k}),$$

where $C_1\neq 0$ and $o(\epsilon^{-k})/\epsilon^{-k}\longrightarrow 0$ as $\epsilon\longrightarrow 0$.[1] This implies

$$\psi\left(\Xi(b_1+\epsilon);b_1+\epsilon\right) = -C_2\epsilon^{-2k}+o(\epsilon^{-2k}),$$

[1] $k=1$ if the quadratic $(1-\omega)b+\sigma_\Theta\lambda(b)$ has two roots. Otherwise $k=2$.

where $C_2 > 0$ and $o(\epsilon^{-2k})/\epsilon^{-2k} \longrightarrow 0$ as $\epsilon \longrightarrow 0$. Since $\Xi(\bullet)$ is continuous on the open interval $]b_1, 0[$, the mean value theorem yields $b_{Q\Theta} \in\,]b_1, 0[$ such that $\psi(\Xi(b_{Q\Theta}); b_{Q\Theta}) = 0$. Moreover, by definition of b_1, $\Xi(b_{Q\Theta}) < 0$. On the other hand, suppose that $(1-\omega)b + \sigma_\Theta \lambda(b) > 0$ for all $b < 0$. Then $\Xi(\bullet)$ is continuous and strictly negative on $]-\infty, 0[$, so it suffices to find $b' < 0$ such that $\psi(\Xi(b'); b') < 0$. To this end, we first note that if b is sufficiently negative, then $\psi(\bullet; b)$ has two roots. The leftmost root $\xi_0(b)$ approaches a finite limit $\xi_0 < 0$ as $b \longrightarrow -\infty$ and the rightmost root $\xi_1(b)$ approaches zero. Moreover, the quadratic formula implies

$$\xi_1(b) = \frac{(1-\omega)(r+2a_\Theta)\left(\frac{r}{2}+a_\Theta\right)}{r\sigma_\Theta\gamma} b^{-1} + o(b^{-1}),$$

where $o(b^{-1})/b^{-1} \longrightarrow 0$ as $b \longrightarrow -\infty$. By comparison, it is easy to show that

$$\Xi(b) = \frac{(1-\omega)(r+2a_\Theta)(r+a_\Theta)}{r\sigma_\Theta\gamma} b^{-1} + o(b^{-1}).$$

It follows that $\Xi(b')$ lies between $\xi_0(b')$ and $\xi_1(b')$ for sufficiently negative b'. Since $\psi(\bullet; b')$ is convex for large negative b', it follows that $\psi(\Xi(b'); b') < 0$ for sufficiently negative b'. $\square$

Proof of Proposition 3.1. Under the pricing rule Q^X and the belief $X = X^*$ described in Proposition 3.1, the price taker observes the excess return dynamics (3.3), (3.2). Assume for the moment that his shareholdings do not jump. Then his Bellman equation is

$$\begin{aligned} 0 \;=\; & \max_{(c,X)} \left\{ -e^{-\gamma' c} - \rho' J' + \left(rW - c + X \frac{\lambda}{1-\omega} \left[\mu_1 + (a_\Theta + \mu_Y) Y \right] \right) J'_W \right. \\ & \left. + (\mu_1 + \mu_Y Y) J'_Y - X \sigma_\Theta b_{Q\Theta} J'_{WY} + \frac{1}{2} X^2 \|b_Q\|^2 J'_{WW} + \frac{1}{2} \sigma_\Theta^2 J'_{YY} \right\}. \end{aligned} \tag{C.1}$$

Wang (1993, Appendix B) proves that this equation has a solution J' of the form

$$J'(W,Y) = -\exp\left(-r\gamma' W - \frac{1}{2} \Psi^\top v \Psi \right),$$

where $\Psi = [1, Y]^\top$ and v is a constant 2×2 symmetric matrix

$$v = \begin{bmatrix} v_{11} & v_{1Y} \\ v_{1Y} & v_{YY} \end{bmatrix}.$$

As Wang points out, the first–order conditions are necessary and sufficient for a maximum. In fact, if

$$\begin{aligned} v_{1Y} &= (b_{Q\Theta}\sigma_\Theta)^{-1}\Big(\|b_Q\|^2 r\gamma'/\omega - \lambda\mu_1/(1-\omega)\Big), \\ v_{YY} &= -(b_{Q\Theta}\sigma_\Theta)^{-1}\Big(\|b_Q\|^2 r\gamma'/\omega + \lambda\Xi/(1-\omega)\Big), \end{aligned}$$

they imply that the price taker optimally holds X' shares, where $\omega X' = 1 - Y$. Thus, the stock market clears. Substituting the first–order conditions into the right–hand side of (C.1) and dividing through by J' yields a quadratic in Y, each of whose coefficients must vanish. Provided v_{1Y} and v_{YY} are as defined above, the coefficient of Y^2 is $\psi(\Xi; b_{Q\Theta})$, which vanishes by definition of $b_{Q\Theta}$. By our choice of μ_1, the coefficient of Y^1 also vanishes. Finally, the coefficient of Y^0 is of the form $rv_{11}/2 + \eta$, where η depends on the model parameters, the μ_is, and the b_{Qj}s. Accordingly, we simply choose $v_{11} = -2\eta/r$. Let W be the wealth process corresponding to the price taker's optimal shareholdings. By Itô's formula, $J'(W_t, Y_t) = e^{(\rho'-r)t}J'(W_0, Y_0)\mathcal{E}'_t$, where $\mathcal{E}'$ is an exponential martingale of the form

$$\mathcal{E}'_t = \exp\left[-\frac{1}{2}\int_0^t \|\nu_1 + \nu_Y Y_s\|^2 ds + \int_0^t (\nu_1 + \nu_Y Y_s)\, dz_s\right],$$

the ν_is being constant 1×2 row vectors. Since Y satisfies the growth condition required of X in Lemma 2.2, the argument used in the proof of Lemma 2.2 implies that $J'(W_T, Y_T)$ satisfies a suitable transversality condition. A verification theorem similar to Lemma 2.3 then establishes that it is indeed suboptimal for the price taker's shareholdings to jump.[2]

Having established that the price taker chooses to clear the stock market, it suffices to show that (c^*, X^*) is a Q^X–admissible consumption–investment policy. The remainder of the proof then follows from Lemma 2.3. Let us temporarily assume that the equilibrium stock price satisfies $P_t = P'_t$ for every $t \geq 0$, where $P'_t = \Phi_t + p_0 + p_Y Y_t$. Then Y is clearly $\mathcal{I}$–adapted. Since

$$dX^* = \left[\frac{\mu_1}{1-\omega} - a_\Theta X^* + \frac{\mu_Y + a_\Theta}{1-\omega} Y\right] dt,$$

[2]Implicit in this claim is the understanding that the price taker's consumption–investment policy belongs to a set of admissible policies satisfying a transversality condition relative to $J'(\bullet)$.

both X^* and $\Theta = (1-\omega)X^* - Y$ are $\mathfrak{I}$–adapted. The self–financing condition then establishes that c^* is also $\mathfrak{I}$–adapted. An argument similar to that used in the proof of Proposition 2.1 yields the inequality

$$|X_t^*| \leq K_1(t)|X_0| + K_2(t)|\Theta_0| + K_3(t) \max_{0 \leq s \leq t} \|z_s\|,$$

where the K_is are deterministic, continuous functions. Consequently the conditions of Lemma 2.2 are satisfied, so (c^*, X^*) is Q^X–admissible. Finally, $P_0 = P_0'$ by assumption and $dP_t' + D_t dt - rP_t' dt = dQ_t^X \Big|_{X=X^*}$ for all $t \geq 0$. Therefore $P = P'$, which confirms our standing assumption about the equilibrium stock price. □

Appendix D

Model III – the insider's problem

The proofs of Propositions 4.1 and 4.2 require several preliminary results. We begin as follows. Given $v \in \mathcal{S}_{++}$, the set of symmetric positive definite 2×2 matrices, let

$$p(v, x) = \frac{1}{2\pi\sqrt{\det v}} \exp -\frac{1}{2}\left(x'v^{-1}x\right) \tag{D.1}$$

be the density for a normally distributed random vector with covariance matrix v. Defining $\|v\| = \max\left\{x'vx;\ \ x \in \mathbb{R}^2, |x| = 1\right\}$, we then have the following.

Lemma D.1 *Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a Borel–measurable function satisfying*

$$\int_{\mathbb{R}^2} e^{-\delta|x|^2}|f(x)|dx < \infty$$

for some $\delta > 0$. Suppose $v : [0, 1[\longrightarrow \mathcal{S}_{++}$ such that the mappings $t \mapsto v_{ij}(t)$, $1 \leq i, j \leq 2$, are continuously differentiable on $]0, 1[$. Then the function

$$\psi(x, t) = \int_{\mathbb{R}^2} f(y)p(v(t), y - x)dy$$

is well defined on $\mathbb{R}^2 \times U_\delta$, where $U_\delta = \{t \in [0, 1[,\ \ \|v(t)\| < 1/2\delta\}$. Moreover, for every $i, j \geq 0$

and $(x,t) \in \mathbb{R}^2 \times U_\delta$,

$$\frac{\partial^{i+j}\psi}{\partial x_1^i \partial x_2^j}(x,t) = \int_{\mathbb{R}^2} f(y) \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} p(v(t), y-x)dy,$$

while

$$\frac{\partial \psi}{\partial t}(x,t) = \int_{\mathbb{R}^2} f(y) \frac{\partial}{\partial t} p(v(t), y-x)dy$$

for every $x \in \mathbb{R}^2$ *and* $t > 0$ *in* U_δ.

Proof. This is a straightforward two–dimensional extension of Karatzas and Shreve (1991, Problem 4.3.1). □

We now address the insider's optimization problem. The approach is much the same as in Back (1992, Lemmas 1 and 2). However, in this case we must account for the additional stochastic state variable ξ_t. Proceeding heuristically, suppose the insider's value function is of the form $J = J(\xi_t, Y_t, X_t, t)$ and the market maker adopts the pricing rule $P_t = H(Y_t, t)$. The Bellman principle states that

$$\max_{dX_t} \mathbb{E}_t \left[dW_t + dJ_t \right] = 0,$$

where $dW_t = [V_t - H(Y_t, t)]\, dX_t + X_t dV_t$ is the instantaneous profit realized by the insider between t and $t + dt$. Assuming the order rate $\theta_t = dX_t/dt$ exists and that J is sufficiently smooth in its arguments, it follows from Itô's formula that

$$\max_{\theta \in \mathbb{R}} \left\{ \left(\frac{\partial J}{\partial x} + \frac{\partial J}{\partial y} + g(\xi) - H(y,t) \right) \theta + \frac{\partial J}{\partial t} + \mathcal{D}_t J + x\nu(\xi, t) \right\} = 0 \tag{D.2}$$

for all $(\xi, y, x, t) \in \mathbb{R}^3 \times]0,1[$. $\mathcal{D}_t$ is the differential operator

$$\mathcal{D}_t = [F(t)\xi + f(t)] \frac{\partial}{\partial \xi} + \frac{1}{2} q(t)^2 \frac{\partial^2}{\partial \xi^2} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2},$$

and $\nu(\xi,t)$ is the drift coefficient in the differential representation $dV = \nu dt + \Sigma dB$ for the fundamental value process $V_t = g(\xi_t)$. The next lemma, inspired by Lemma 1 of Back (1992), provides conditions under which (D.2) has a solution.

Lemma D.2 *Suppose $h \in \mathfrak{h}_{p,C}$ and $H(y,t) = \mathbb{E}h(y + Z_1 - Z_t)$ for every $(y,t) \in R \times [0,1]$. Then there exists $J \in C^{2,1}\Big(\mathbb{R}^3 \times]0,1[\Big)$ solving* (D.2) *subject to the boundary condition*

$$J(\xi,y,x,1) \geq 0 \text{ with equality iff } g(\xi) = h(y). \tag{D.3}$$

The boundary condition (D.3) reflects the fact that the insider can extract a positive capital gain whenever the share price differs from its fundamental value at the announcement date.

Proof. Define, for every $(\xi,y,x,t) \in \mathbb{R}^3 \times [0,1]$,

$$J(\xi,y,x,t) = \Phi(\xi,y,t) + x\Psi(\xi,t), \tag{D.4}$$

where

$$\begin{aligned} \Phi(\xi,y,t) &= \mathbb{E}j\Big(\tilde{\xi}(\xi,t), y + Z_1 - Z_t; h\Big), & \text{(D.5)} \\ \Psi(\xi,t) &= \mathbb{E}g\Big(\tilde{\xi}(\xi,t)\Big) - g(\xi). & \text{(D.6)} \end{aligned}$$

Notice that $\Phi(\xi,y,1) = j(\xi,y;h)$ and $\Psi(\xi,1) = 0$. Provided the covariance matrix $C(t)$ is nonsingular, we can write

$$\Phi(\xi,y,t) = \int_{\mathbb{R}^2} j(x_1,x_2;h)\phi(x,\xi,y,t)dx,$$

where, recalling (D.1),

$$\phi(x,\xi,y,t) = p\left(C(t), \begin{bmatrix} x_1 - \int_t^1 e(F)_{s,1}f(s)ds - e(F)_{t,1}\xi \\ x_2 - y \end{bmatrix}\right).$$

In view of (4.9), Lemma D.1 implies that Φ is well defined on $\mathbb{R}^2 \times [0,1]$. Moreover, since $(\phi_t + \mathcal{D}_t\phi) = 0$, Lemma D.1 also implies that $\Phi \in C^{2,1}(R^2 \times]0,1[)$ and

$$\Phi_t + \mathcal{D}_t\Phi = 0 \tag{D.7}$$

on $\mathbb{R}^2 \times]0,1[$. Similarly, since

$$\Psi(\xi,t) + g(\xi) = \int_{\mathbb{R}^2} g(x_1)\phi(x,\xi,y,t)dx,$$

Assumption 4.1 and Lemma D.1 imply

$$\Psi_t + \mathcal{D}_t(\Psi + g) = 0 \tag{D.8}$$

on $\mathbb{R} \times]0,1[$. Since $\mathcal{D}_t g(\xi) = \nu(\xi,t)$, (D.7) and (D.8) yield

$$J_t + \mathcal{D}_t J + x\nu = 0 \tag{D.9}$$

on $\mathbb{R}^3 \times]0,1[$.

Now since $\partial j/\partial y = h(y) - g(\xi)$, Lemma D.1 implies

$$\begin{aligned} \Phi_y(\xi,y,t) &= \mathbb{E}h(y + Z_1 - Z_t) - \mathbb{E}g\left(\tilde{\xi}(\xi,t)\right) \\ &= H(y,t) - \Psi(\xi,t) - g(\xi) \end{aligned}$$

for every $(\xi,y,t) \in \mathbb{R}^2 \times [0,1]$. Since $J_y = \Phi_y$ and $J_x = \Psi$, it follows that

$$J_x + J_y + g(\xi) - H(y,t) = 0 \tag{D.10}$$

for every $(\xi,y,x,t) \in \mathbb{R}^3 \times [0,1]$. Since the objective in (D.2) is linear in θ, it follows from (D.9) and (D.10) that J satisfies (D.2). Finally, since $J(\xi,y,x,1) = j(\xi,y;h)$, the boundary condition (D.3) is satisfied. A similar argument establishes the desired result when $C(t)$ is singular. □

Definition (D.5) is easily recognized as the Feynman–Kac representation for the solution to (D.7) subject to the boundary condition $\Phi(\cdot, 1) = j(\cdot\,; h)$. However, condition (4.9) is weaker than the polynomial growth condition usually imposed in order to ensure such a representation [*i.e.* Karatzas and Shreve (1991, Remark 5.7.8 iv)].

The heuristic argument preceding Lemma D.2 is based on the presumption that the insider's optimal strategy X has differentiable sample paths. The following counterpart to Lemma 2 of Back (1992) establishes that this is indeed the case.

Lemma D.3 *Suppose the conditions of Lemma* D.2 *hold, and define* J *as in* (D.4). *Then for any trading strategy* $X \in \mathcal{X}_H$, *the insider's expected profit conditioned on* $V_0 = g(\xi_0)$ *is no larger than* $J(\xi_0, 0, 0, 0)$. *Any* $X \in \mathcal{X}_H$ *having continuous sample paths of bounded variation such that* $X_0 = 0$ *and for which* $H(X_1 + Z_1, 1) = V_1$ *a.s. is optimal. Any other* $X \in \mathcal{X}_H$ *yields an expected profit strictly less than* $J(\xi_0, 0, 0, 0)$.

Proof. Let $S_t = (\xi_t, Y_t, X_t)$. Itô's formula implies

$$\begin{aligned} J(S_1, 1) \;&=\; J(S_{0-}, 0) + \int_{[0,1]} (J_x + J_y)\, dX_t + \int_0^1 J_y dZ_t + \int_0^1 J_\xi q dB_t \\ &\quad + \int_0^1 (J_t + \mathcal{D}_t J)\, dt \\ &\quad + \frac{1}{2}\int_0^1 (J_{xx} + J_{yy} + 2J_{xy})\, d\,[X^c, X^c]_t \\ &\quad + \int_0^1 (J_{xy} + J_{yy})\, d\,[X^c, Z]_t \\ &\quad + \int_0^1 (J_{\xi x} + J_{\xi y})\, q\, d\,[X^c, B]_t \\ &\quad + \sum_{0 \le t \le 1} \left[\Delta J(S_t, t) - (J_x + J_y)\,(S_{t-}, t)\Delta X_t\right], \end{aligned} \tag{D.11}$$

where $S_{0-} = (\xi_0, 0, 0)$, X^c is the continuous martingale component of X, $\Delta J(S_t, t) = J(S_t, t) - J(S_{t-}, t)$, and $\Delta X_t = X_t - X_{t-}$. Each of the integrands is evaluated at (S_{t-}, t). Differentiating (D.10) through, we have

$$J_{xx} + J_{xy} \;=\; 0,$$

$$
\begin{aligned}
J_{xy} + J_{yy} &= H_y, \\
J_{\xi x} + J_{\xi y} &= -g'(\xi).
\end{aligned}
$$

Plugging these equalities and (D.9), (D.10) into (D.11), it follows that

$$
\begin{aligned}
J(S_1, 1) &= J(S_{0-}, 0) + \int_{[0,1]} (H - g)\, dX_t + \int_0^1 J_y dZ_t + \int_0^1 J_\xi q dB_t \\
&\quad - \int_0^1 x\nu\, dt \\
&\quad + \int_0^1 \frac{1}{2} H_y d\,[X^c, X^c]_t + H_y d\,[X^c, Z]_t - qg' d\,[X^c, B]_t \\
&\quad + \sum_{0 \le t \le 1} \left[\Delta J(S_t, t) - (H(Y_{t-}, t) - g(\xi_t))\, \Delta X_t\right]. \qquad \text{(D.12)}
\end{aligned}
$$

Using the previous notation, (4.3) can be written as

$$
\begin{aligned}
W_1 &= \int_{[0,1]} (g - H) dX_t + \int_0^1 x\, dV_t + [V - P, X]_1 \\
&= \int_{[0,1]} (g - H) dX_t + \int_0^1 x\nu\, dt + \int_0^1 xg' q\, dB_t \\
&\quad + \int_0^1 g' q\, d\,[X^c, B]_t \; - \; [P, X]_1.
\end{aligned}
$$

Combining this with (D.12), it follows that

$$
\begin{aligned}
W_1 - J(S_{0-}, 0) &= -J(S_1, 1) + \int_0^1 J_y dZ_t + \int_0^1 (J_\xi + xg')\, q dB_t \\
&\quad + \frac{1}{2} \int_0^1 H_y \left(d\,[X^c, X^c]_t + 2d\,[X^c, Z]_t\right) \; - \; [P, X]_1 \\
&\quad + \sum_{0 \le t \le 1} \left[\Delta J(S_t, t) - (H(Y_{t-}, t) - g(\xi_t))\, \Delta X_t\right].
\end{aligned}
$$

Now by definition of quadratic variation,

$$
\begin{aligned}
[P, X]_1 &= [P^c, X^c]_1 + \sum_{0 \le t \le 1} \Delta P_t \Delta X_t \\
&= \int_0^1 H_y d\,[Y^c, X^c]_t + \sum_{0 \le t \le 1} \Delta P_t \Delta X_t \\
&= \int_0^1 H_y \left(d\,[X^c, X^c]_t + d\,[X^c, Z]_t\right) + \sum_{0 \le t \le 1} \Delta P_t \Delta X_t.
\end{aligned}
$$

Here $\Delta P_t = H(Y_t, t) - H(Y_{t-}, t)$, and the second equality follows from the equality $dP^c = H_y dY^c$. Hence

$$\begin{aligned} W_1 - J(S_{0-}, 0) &= -J(S_1, 1) + \int_0^1 J_y dZ_t + \int_0^1 (J_\xi + xg')\, q dB_t \\ &\quad - \frac{1}{2}\int_0^1 H_y d[X^c, X^c]_t \\ &\quad + \sum_{0 \le t \le 1} \left[\Delta J(S_t, t) - (H(Y_{t-}, t) - g(\xi_t))\, \Delta X_t - \Delta P_t \Delta X_t\right]. \end{aligned} \tag{D.13}$$

Combining (D.10) with (D.6), we have $J_y(\xi, y, x, t) = H(y, t) - \mathbb{E}[g(\xi_1) | \xi_t = \xi]$. Assumption 4.1 and (4.11) then imply that $\mathbb{E}\int_0^1 J_y^2 dt < \infty$. Therefore

$$\mathbb{E}\left[\int_0^1 J_y dZ_t \middle| V_0\right] = \mathbb{E}\left[\int_0^1 J_y dZ_t\right] = 0.$$

Similarly, since $J_\xi + xg'(\xi) = e(F)_{t,1} I_h$, (4.12) implies that

$$\mathbb{E}\left[\int_0^1 (J_\xi + xg')\, q dB_t \middle| V_0\right] = 0.$$

Now plugging (D.4) into (D.13), the jump term can be written as

$$\sum_{0 \le t \le 1} \left[\Delta\Phi(\xi_t, Y_t, t) + \Delta X_t \Psi(\xi_t, t) - (H(Y_t, t) - g(\xi_t))\, \Delta X_t\right].$$

Since $\Psi - H + g = J_x - H + g = -J_y = -\Phi_y$, this simplifies to

$$\sum_{0 \le t \le 1} \left[\Delta\Phi(\xi_t, Y_t, t) - \Phi_y(\xi_t, Y_t, t)\Delta X_t\right]. \tag{D.14}$$

Now $\Phi_{yy} = J_{yy} = J_{xy} + J_{yy} = H_y > 0$. Hence Φ is strictly convex in y. Since $\Delta X_t = \Delta Y_t$, the term in (D.14) is ≤ 0 with equality iff $\Delta X_t = 0$ for all t with probability one. Finally, since $H_y > 0$, $\int_0^1 H_y d[X^c, X^c]_t \ge 0$ with equality iff $X_t^c = 0$ for all t with probability one. Now taking expectations through (D.13) and recalling the boundary condition (D.3), the lemma is proved. $\square$

Appendix E

Model III – the market maker's filtering problem

Having characterized optimal insider trading strategies corresponding to pricing rules in $\mathcal{H}$, we turn to the specific pricing rule and trading strategy described in Proposition 4.1. Henceforth, we refer to the quantities $\Pi, A, \hat{A}, a, \gamma, h$, and X defined in equations (4.16) through (4.22). The approach taken here is somewhat different from Back's in its use of linear filtering theory.

Given $t \in [0, 1[$, define

$$\zeta_t = \int_0^t A(s)\xi_s ds + Z_t. \tag{E.1}$$

Since $Z_t = Y_t - X_t$, (4.22) implies

$$\zeta_t = Y_t - \int_0^t \left[\hat{A}(s)Y_s + a(s)\right] ds.$$

Conversely, direct substitution establishes the equality

$$Y_t = e(\hat{A})_{0,t} \left[\int_0^t \frac{a(s)}{e(\hat{A})_{0,s}} ds + \int_0^t \frac{1}{e(\hat{A})_{0,s}} d\zeta_s\right],$$

recalling the notation of (4.6). Therefore

$$\mathcal{F}_t^\zeta = \mathcal{F}_t^Y \text{ for every } t \in [0,1[, \tag{E.2}$$

where $\mathcal{F}^\zeta$, $\mathcal{F}^Y$ denote the augmented filtrations generated by ζ and the total order process Y, respectively.

It is easy to verify that $\Pi(t)$ satisfies the differential equation

$$\begin{aligned} \frac{d\Pi}{dt} &= -\frac{(\Pi A)^2}{\sigma^2} + 2F\Pi + q^2, \ \ 0 \le t < 1, \\ \Pi(0) &= \phi^2, \end{aligned}$$

suppressing the "t" argument. This is the Riccati equation corresponding to the linear filter with state process ξ and observation process ζ. Accordingly, if $t < 1$ and if $\hat{x}_t \sim N(0, \Pi(t))$, the solution $\hat{\xi}$ to the Kalman filter equation

$$\begin{aligned} d\hat{\xi} &= (F\hat{\xi} + f)ds + \frac{\Pi A}{\sigma^2}\left(d\zeta - A\hat{\xi}ds\right), \ \ 0 \le s \le t, \\ \hat{\xi}_0 &= \mu, \end{aligned} \tag{E.3}$$

satisfies

$$\mathbb{E}\left[G(\xi_t) | \mathcal{F}_t^\zeta\right] = \hat{\mathbb{E}}G(y + \hat{x}_t)\Big|_{y=\hat{\xi}_t} \ \ a.s., \tag{E.4}$$

whenever both sides of (E.4) are well defined. Here the expectation $\hat{\mathbb{E}}$ is evaluated with respect to the distribution of $\hat{x}_t$. Equations (E.2) through (E.4) play key roles in establishing the following property.

Lemma E.1 *For every $t \in [0,1[$, $H(Y_t, t) = \mathbb{E}\left[g(\xi_1) \middle| \mathcal{F}_t^Y\right]$ a.s.*

Proof. For every $t \in [0,1]$, define

$$\hat{\sigma}(t) = \frac{\lambda}{e(F)_{t,1}}. \tag{E.5}$$

Now fix $t \in [0,1[$. On the interval $[0,t]$, we have

$$\begin{aligned} d\hat{\xi} &= \left(F\hat{\xi} + f\right) ds + \hat{\sigma}\left(d\zeta - A\hat{\xi} ds\right), && \text{(E.6)} \\ d(\hat{\sigma}Y) &= \hat{\sigma} dY - \hat{\sigma}' Y ds. && \text{(E.7)} \end{aligned}$$

The first of these equations follows by substituting (4.16) and (4.17) into (E.3). The second is a simple consequence of Itô's formula. Subtracting (E.7) from (E.6) and recalling the definitions of ζ and Y, we obtain

$$d\left(\hat{\xi} - \hat{\sigma}Y\right) = \left[(F - \hat{\sigma}A)\left(\hat{\xi} - \hat{\sigma}Y\right) + f - \hat{\sigma}a\right] ds \ \text{ on } [0,t].$$

Since $\gamma' = F\gamma + f = (F - \hat{\sigma}A)\gamma + f - \hat{\sigma}a$ and $\gamma(0) = \hat{\xi}_0 - \hat{\sigma}(0)Y_0 = \mu$, it follows that

$$\hat{\xi}_s - \hat{\sigma}(s)Y_s = \gamma(s) \ \text{ a.s.} \tag{E.8}$$

for all $s \in [0,t]$. Setting $s = t$, (E.8) and (4.21) imply

$$\begin{aligned} H(Y_t, t) &= \mathbb{E}h(y + Z_1 - Z_t)\big|_{y=Y_t} \\ &= \mathbb{E}g\left(\lambda\left[\frac{y - \gamma(t)}{\hat{\sigma}(t)} + Z_1 - Z_t\right] + \gamma(1)\right)\Bigg|_{y=\hat{\xi}_t}. \end{aligned} \tag{E.9}$$

On the other hand,

$$\begin{aligned} \mathbb{E}\left[g(\xi_1)\,\middle|\mathcal{F}_t^Y\right] &= \mathbb{E}\left[g(\xi_1)\,\middle|\mathcal{F}_t^\zeta\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[g(\xi_1)\,\middle|\mathcal{F}_t^{B,Z}\right]\middle|\mathcal{F}_t^\zeta\right] \\ &= \mathbb{E}\left[\mathbb{E}\, g\left(\tilde{\xi}(\xi,t)\right)\Big|_{\xi=\xi_t}\middle|\mathcal{F}_t^\zeta\right] \\ &= \hat{\mathbb{E}}\mathbb{E}\, g\left(\tilde{\xi}(y + \hat{x}_t, t)\right)\Big|_{y=\hat{\xi}_t}, \end{aligned} \tag{E.10}$$

where the first equality follows from (E.2), the second from the law of iterated expectations, the third from (4.5), and the fourth from (E.4). Comparing (E.9) to (E.10), it suffices to show that

$$\begin{aligned}
\frac{\lambda}{\hat{\sigma}(t)} &= e(F)_{t,1}, \\
-\frac{\lambda\gamma(t)}{\hat{\sigma}(t)} + \gamma(1) &= e(F)_{t,1} \int_t^1 \frac{f(s)}{e(F)_{t,s}} ds,
\end{aligned}$$

and

$$\lambda^2 \sigma^2 (1-t) = e(F)_{t,1}^2 \left[\int_t^1 \frac{q(s)^2}{e(F)_{t,s}^2} ds + \Pi(t) \right].$$

These equalities follow directly from the definitions of $\hat{\sigma}$, γ, and Π. $\square$

Appendix F

Model III – equilibrium

With the possible exception of the announcement date $t = 1$, Lemma E.1 establishes that the pricing rule defined in Proposition 4.1 is rational. The next three lemmas address the behavior of the pricing rule and the trading strategy (4.22) at $t = 1$.

Lemma F.1 *Let $\hat{\sigma}(t)$ be as defined in* (E.5)*. Then there exist constants $K < \infty$ and $1 < c < \infty$ such that*

$$\sup_{0 \le t < 1} \left| \hat{\sigma}(t) A(t) - \frac{c}{1-t} \right| \le K. \tag{F.1}$$

Proof. By Assumption 4.2, $t \mapsto e(F)^2_{t,1} q(t)^2$ is Lipschitz at $t = 1$. Thus, there exist constants $K_1 < \infty$ and $\epsilon > 0$ such that $\left| q(1)^2 - e(F)^2_{t,1} q(t)^2 \right| \le K_1(1-t)$ for every $t \in]1-\epsilon, 1[$. Integrating this inequality over the interval $[t, 1[$, it follows that

$$\left| q(1)^2(1-t) - \int_t^1 e(F)^2_{s,1} q(s)^2 ds \right| \le \frac{K_1}{2}(1-t)^2 \tag{F.2}$$

for every $t \in]1-\epsilon, 1[$. Recalling (4.14), we may choose ϵ so small that

$$1 - t \le 2 \left| \frac{1}{\lambda^2\sigma^2 - q(1)^2} \right| \left| \int_t^1 \left[\lambda^2\sigma^2 - e(F)^2_{s,1} q(s)^2 \right] ds \right| \tag{F.3}$$

for every $t \in]1-\epsilon, 1[$. Substituting (F.3) into the right–hand side of (F.2), we obtain

$$\begin{aligned}
&\left|\left[\lambda^2\sigma^2 - q(1)^2\right](1-t) - \int_t^1 \left[\lambda^2\sigma^2 - e(F)^2_{s,1}q(s)^2\right] ds\right| \\
&\qquad \leq K_2(1-t)\left|\int_t^1 \left[\lambda^2\sigma^2 - e(F)^2_{s,1}q(s)^2\right] ds\right|,
\end{aligned}$$

where $K_2 < \infty$ is a constant. Rearranging terms, it follows that

$$\left|\frac{1}{\int_t^1 \left[\lambda^2\sigma^2 - e(F)^2_{s,1}q(s)^2\right] ds} - \frac{1}{\left[\lambda^2\sigma^2 - q(1)^2\right][1-t]}\right| \leq K_3 \tag{F.4}$$

for all $t \in]1-\epsilon, 1[$, where $K_3 < \infty$ is constant. Since $\int_t^1 \left[\lambda^2\sigma^2 - e(F)^2_{s,1}q(s)^2\right] ds > 0$ on $[0, 1-\epsilon]$ by Assumption 4.3, (F.4) may be assumed to hold for all $t \in [0, 1[$.

Now from (4.17) and (E.5),

$$\hat{\sigma}(t)A(t) = \frac{\lambda^2\sigma^2}{\int_t^1 \left[\lambda^2\sigma^2 - e(F)^2_{s,1}q(s)^2\right] ds}$$

for every $t \in [0, 1[$. (F.4) then implies

$$\left|\hat{\sigma}(t)A(t) - \frac{\lambda^2\sigma^2}{\left[\lambda^2\sigma^2 - q(1)^2\right][1-t]}\right| \quad \leq \quad K_3\lambda^2\sigma^2.$$

It follows that (F.1) holds with

$$c = \frac{\lambda^2\sigma^2}{\lambda^2\sigma^2 - q(1)^2}. \tag{F.5}$$

This completes the proof. □

Lemma F.2 *Let $a, b : [0, 1[\longrightarrow \mathbb{R}$ be Borel–measurable functions such that*

$$k \leq \frac{|a(t)|}{(1-t)^c} \leq K, \quad k \leq |b(t)| \leq K$$

for all $t \in [0,1[$, *where* $0 < k \leq K < \infty$ *and* $c > 1/2$. *Given the standard Brownian motion* $W = \{W_t\}$ *on* $[0,1]$, *define*

$$M_t = \int_0^t \frac{b(s)}{a(s)} dW_s, \quad 0 \leq t < 1.$$

Then for every $\epsilon > 0$,

$$\limsup_{t \uparrow 1} \frac{|a(t) M_t|}{(1-t)^{1/2-\epsilon}} = 0 \quad a.s.$$

Proof. The quadratic variation process $\langle M \rangle_t$ is strictly increasing and satisfies

$$\frac{k^2}{K^2} \int_0^t \frac{ds}{(1-s)^{2c}} \leq \langle M \rangle_t \leq \frac{K^2}{k^2} \int_0^t \frac{ds}{(1-s)^{2c}} \tag{F.6}$$

for every $t \in [0,1[$. Since

$$\int_0^t \frac{ds}{(1-s)^{2c}} = \frac{(1-t)^{-2c+1} - 1}{2c-1}, \tag{F.7}$$

It follows that $\lim_{t \uparrow 1} \langle M \rangle_t = \infty$. Accordingly, for every $s \geq 0$ we may define $B_s = M_{T(s)}$, where $\langle M \rangle_{T(s)} = s$. $B = \{B_s\}$ is a standard Brownian motion on $\mathbb{R}_+$ by Lévy's Theorem [Karatzas and Shreve (1991, Theorem 3.3.16)]. The law of the iterated logarithm [*ibid*, Theorem 2.9.23] then implies

$$\limsup_{t \uparrow 1} \left| \frac{B_{\langle M \rangle_t}}{\omega(\langle M \rangle_t)} \right| = 1 \text{ a.s.},$$

where $\omega(x) = \sqrt{2x \log \log x}$. Since $B_{\langle M \rangle_t} = M_t$, it follows from (F.6) and (F.7) that

$$\limsup_{t \uparrow 1} \left| \frac{M_t}{\omega([1-t]^{-2c+1})} \right| \leq K_1 \text{ a.s.},$$

where $K_1 < \infty$ is a constant. Consequently, if $\epsilon > 0$,

$$\begin{aligned}
\limsup_{t\uparrow 1} \frac{|a(t)M_t|}{(1-t)^{1/2-\epsilon}} &\leq K_1 \limsup_{t\uparrow 1} \frac{\left|a(t)\omega\left([1-t]^{-2c+1}\right)\right|}{(1-t)^{1/2-\epsilon}} \\
&\leq KK_1 \limsup_{t\uparrow 1} \frac{\left|(1-t)^c\omega\left([1-t]^{-2c+1}\right)\right|}{(1-t)^{1/2-\epsilon}} \\
&= KK_1 \limsup_{t\uparrow 1} \left|(1-t)^\epsilon \sqrt{2\log\log[1-t]^{-2c+1}}\right| \\
&= 0.
\end{aligned}$$

This completes the proof. □

Lemma F.3 $\lim_{t\uparrow 1} X_t = X_1$ *a.s. Moreover,*

$$h(X_1 + Z_1) = g(\xi_1) \quad a.s. \tag{F.8}$$

Proof. First, note that (4.24) follows directly from (E.6), (4.1), and (E.1). The solution to (4.24) admits the following representation [Karatzas and Shreve (1991, §5.6C)]:

$$\hat{\xi}_t - \xi_t = e(F - \hat{\sigma}A)_{0,t}\left[\mu - \xi_0 + M_t\right] \quad \text{for all } t \in [0,1[, \tag{F.9}$$

where $e(\cdot)$ is defined in (4.6) and

$$M_t = \int_0^t \frac{\hat{\sigma}(s)dZ_s - q(s)dB_s}{e(F-\hat{\sigma}A)_{0,s}}.$$

Now with c as defined in (F.5), we have

$$\begin{aligned}
\frac{e(F-\hat{\sigma}A)_{0,t}}{(1-t)^c} &= \exp\left\{\int_0^t \left[F(s) - \hat{\sigma}(s)A(s)\right]ds - c\log(1-t)\right\} \\
&= \exp\left\{\int_0^t \left[F(s) - \hat{\sigma}(s)A(s) + \frac{c}{1-s}\right]ds\right\}.
\end{aligned}$$

Since F is bounded on $[0,1]$, Lemma F.1 then implies

$$k \leq \frac{e(F-\hat{\sigma}A)_{0,t}}{(1-t)^c} \leq K \quad \text{for every } t \in [0,1[, \tag{F.10}$$

for some constants $0 < k \leq K < \infty$. Consequently, $e(F - \hat{\sigma} A)_{0,t}(\mu - \xi_0) \longrightarrow 0$ almost surely as $t \uparrow 1$. Furthermore, applying Lemma F.2 with $a(t) = e(F - \hat{\sigma} A)_{0,t}$ and $b(t) = \sqrt{\sigma^2 \hat{\sigma}(t)^2 + q(t)^2}$, we have $e(F - \hat{\sigma} A)_{0,t} M_t \longrightarrow 0$ a.s. as $t \uparrow 1$. (F.9) then implies $\hat{\xi}_t - \xi_t \longrightarrow 0$ a.s. as $t \uparrow 1$. Consequently, by (E.8) and (E.5),

$$\lim_{t \uparrow 1} Y_t = \frac{\xi_1 - \gamma(1)}{\lambda} \quad \text{a.s.}$$

Recalling the definition of X_1 from (4.22), it follows that $\lim_{t \uparrow 1} X_t = X_1$ a.s. By definition of h, we also have

$$\begin{aligned} h(X_1 + Z_1) &= h\left(\frac{\xi_1 - \gamma(1)}{\lambda}\right) \quad \text{a.s.} \\ &= g(\xi_1). \end{aligned}$$

This completes the proof. □

Corollary

$$H(Y_1, 1) = \mathbb{E}\left[V_1 \,\middle|\, \mathcal{F}_1^Y\right] \quad \text{a.s.} \tag{F.11}$$

Proof. By (F.8), $V_1 = g(\xi_1)$ is $\sigma(Y_1)$–measurable, so both sides of (F.11) evaluate to $g(\xi_1)$. □

Lemma E.1 and the preceding Corollary establish the rationality of the pricing rule H defined in Proposition 4.1. It remains to show that the trading strategy (4.22) is optimal and belongs to the class of admissable strategies $\mathcal{X}_H$.

Lemma F.4 *With probability one, the sample paths of the trading strategy* (4.22) *have finite variation on the interval* $[0, 1]$.

Proof. Combining (E.8), (E.5), (4.18) and (4.19), we can write

$$X_t = \int_0^t A(s) \left(\xi_s - \hat{\xi}_s\right) ds$$

for every $t \in [0,1[$. Accordingly, we must show that $\int_0^1 \left|A(t)\left(\xi_t - \hat{\xi}_t\right)\right| dt < \infty$ with probability one. Since the integrand is continuous on $[0,1[$, it suffices to show that, with probability one, there exists $\epsilon > 0$ such that

$$\int_{1-\epsilon}^{1} \left|A(t)\left(\xi_t - \hat{\xi}_t\right)\right| dt < \infty. \tag{F.12}$$

By (F.9), (F.10) and Lemma F.2, we have

$$\limsup_{t \uparrow 1} \frac{\left|\xi_t - \hat{\xi}_t\right|}{(1-t)^{1/3}} = 0 \text{ a.s.}$$

Therefore, for almost every sample path, there exists $\epsilon > 0$ such that $\left|\xi_t - \hat{\xi}_t\right| \leq \hat{\sigma}(t)(1-t)^{1/3}$ on $[1-\epsilon, 1[$. Consequently, the left hand side of (F.12) is bounded above by

$$\begin{aligned} \int_{1-\epsilon}^{1} \left|\hat{\sigma}(t)A(t)\right|(1-t)^{1/3} dt &\leq \int_{1-\epsilon}^{1} \left[\frac{c}{(1-t)^{2/3}} + K(1-t)^{1/3}\right] dt \\ &< \infty, \end{aligned}$$

where the first inequality follows from Lemma F.1. □

Lemma F.4, together with Lemma D.3 and (F.8), implies that X is an optimal trading strategy, provided it belongs to $\mathcal{X}_H$. We establish this below, thereby completing the proof of Proposition 4.1. First, note that the definition $H(y,t) = \mathbb{E}h(y + Z_1 - Z_t) = \mathbb{E}h(y + Z_{1-t})$ implies

$$H_t + \frac{1}{2}\sigma^2 H_{yy} = 0 \tag{F.13}$$

[Karatzas and Shreve (1991, p. 254)]. It then follows from Itô's formula that

$$dP = H_y(Y_t, t)dY. \tag{F.14}$$

Back (1992, Lemma 5) and (F.14) then yield

Lemma F.5 *Relative to the filtration $\mathcal{F}^Y$, Y is a Brownian motion with the same distribution as Z.*

PROOF OF PROPOSITION 4.1. The formula (4.23) for expected terminal wealth follows from Lemma D.3 and the equalities

$$J(\xi,0,0,0)=\Phi(\xi,0,0)=\mathbb{E}\left[j(\xi_1,Z_1;h)\,|\,\xi_0=\xi\right].$$

It remains to prove the regularity conditions (4.11) and (4.12). Since $H(Z_t,t)=\mathbb{E}\left[h(Z_1)\,|\,Z_t\right]$, we have

$$\begin{aligned}\mathbb{E}\int_0^1 H(Z_t,t)^2dt &= \int_0^1 \mathbb{E}H(Z_t,t)^2dt\\ &\leq \int_0^1 \mathbb{E}h(Z_1)^2dt\\ &= \mathbb{E}h(Z_1)^2,\end{aligned}$$

where the inequality follows from Jensen's inequality. Since $h\in\mathfrak{h}_{p,C}$ and $\mathbb{E}\int_0^1 H(Y_t,t)^2dt=\mathbb{E}\int_0^1 H(Z_t,t)^2dt$ by Lemma F.5, condition (4.11) follows. We now turn to (4.12). Define

$$\begin{aligned}I_1(\xi,y,t) &= \mathbb{E}\left[\frac{\partial j}{\partial\xi}\left(\tilde{\xi}(\xi,t),y+Z_1-Z_t;h\right)\right],\\ I_2(\xi,x,t) &= x\mathbb{E}g'\left(\tilde{\xi}(\xi,t)\right).\end{aligned}$$

We first show that

$$\mathbb{E}\left[I_1(\xi_t,Y_t,t)^2\right]\leq M \ \text{ for every } t\in[0,1], \tag{F.15}$$

where $M<\infty$ is a constant. To this end, notice first that Jensen's inequality implies

$$I_1(\xi,y,t)^2\leq\mathbb{E}\left[\frac{\partial j}{\partial\xi}\left(\tilde{\xi}(\xi,t),y+Z_1-Z_t;h\right)^2\right]. \tag{F.16}$$

By definition of j,

$$\frac{\partial j}{\partial\xi}=\int_y^{h^{-1}\circ g(\xi)}\left[g'(\xi)-h(x)\right]dx.$$

Therefore,

$$\left|\frac{\partial j}{\partial \xi}\right| \le \max_{x \in I} |g'(\xi) - h(x)| \left|h^{-1} \circ g(\xi) - y\right|,$$

where I is the closed interval with endpoints y, $h^{-1} \circ g(\xi)$. Consequently

$$\left|\frac{\partial j}{\partial \xi}\right| \le (g'(\xi) + |g(\xi)| + |h(y)|) \left|h^{-1} \circ g(\xi) - y\right|.$$

Recalling the definition of p and p', Hewitt and Stromberg (1965, Corollary 13.3) then implies that

$$\begin{aligned}\left|\frac{\partial j}{\partial \xi}\right|^2 &\le \frac{(g'(\xi) + |g(\xi)| + |h(y)|)^{2p}}{p} + \frac{\left|h^{-1} \circ g(\xi) - y\right|^{2p'}}{p'} \\ &\le \frac{3^{2p}}{p}\left(g'(\xi)^{2p} + |g(\xi)|^{2p} + |h(y)|^{2p}\right) + \frac{2^{2p'}}{p'}\left(|h^{-1} \circ g(\xi)|^{2p'} + |y|^{2p'}\right).\end{aligned}$$

Hence, the expectation on the right hand side of (F.16) is bounded above by

$$\begin{aligned}&\mathbb{E}\left[\frac{3^{2p}}{p}\left(g'\left(\tilde{\xi}(\xi,t)\right)^{2p} + \left|g\left(\tilde{\xi}(\xi,t)\right)\right|^{2p}\right) + \frac{2^{2p'}}{p'}\left|h^{-1} \circ g\left(\tilde{\xi}(\xi,t)\right)\right|^{2p'}\right] \\ &\quad + \mathbb{E}\left[\frac{3^{2p}}{p}|h(y + Z_1 - Z_t)|^{2p} + \frac{2^{2p'}}{p'}|y + Z_1 - Z_t|^{2p'}\right].\end{aligned}$$

By definition of $\tilde{\xi}(\xi, t)$ and by Lemma F.5, this bound can also be written as

$$\begin{aligned}&\mathbb{E}\left[\frac{3^{2p}}{p}\left(g'(\xi_1)^{2p} + |g(\xi_1)|^{2p}\right) + \frac{2^{2p'}}{p'}\left|h^{-1} \circ g(\xi_1)\right|^{2p'} \middle| \xi_t = \xi\right] \\ &\quad + \mathbb{E}\left[\frac{3^{2p}}{p}|h(Y_1)|^{2p} + \frac{2^{2p'}}{p'}|Y_1|^{2p'} \middle| Y_t = y\right].\end{aligned}$$

Taking unconditional expectations through (F.16) then yields (F.15). Furthermore,

$$\mathbb{E} I_2(\xi_t, X_t, t)^2 \;=\; \mathbb{E}\left[X_t^2 \mathbb{E}\left[g'(\xi_1)|\xi_t\right]^2\right]$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\frac{|X_t|^{2p'}}{p'} + \frac{\mathbb{E}\left[g'(\xi_1)|\xi_t\right]^{2p}}{p}\right] \\
&\leq \frac{2^{2p'}}{p'}\mathbb{E}\left[|Y_t|^{2p'} + |Z_t|^{2p'}\right] + \frac{1}{p}\mathbb{E}\left[g'(\xi_1)^{2p}\right] \\
&= \frac{2^{2p'+1}}{p'}\mathbb{E}\left[|Z_t|^{2p'}\right] + \frac{1}{p}\mathbb{E}\left[g'(\xi_1)^{2p}\right], \qquad \text{(F.17)}
\end{aligned}
$$

where (F.17) follows from Lemma F.5. Combining (F.15) and (F.17), we conclude that (4.12) holds. This completes the proof of Proposition 4.1. □

PROOF OF PROPOSITION 4.2. In view of (F.14) and Lemma F.5, it only remains to prove that $\partial H/\partial y(Y_t, t)$ is a martingale relative to $\mathcal{F}^Y$. This follows directly from Proposition 4.3 and the proof of Theorem 3 in Back (1992). □