MODELS OF ENTREPRENEURIAL DECISIONS:
A DYNAMIC PROGRAMMING APPROACH

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in
THE FACULTY OF GRADUATE STUDIES
(School of Commerce and Business Administration)

We accept this thesis as conforming to the requested standard

THE UNIVERSITY OF BRITISH COLUMBIA

May 1998
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Abstract

Entrepreneurs make decisions that influence subsequent decisions and future performance. The dissertation studies such sequences of decisions by using dynamic programming. This approach allows one to describe the decision process over time and, in some cases, it prescribes how business performance can be improved. An analytical approach helps to contribute a new dimension to entrepreneurship research and it encourages multidisciplinary work by allowing existing methodologies from various (analytical) disciplines to be applied to entrepreneurial problems.

The dissertation focuses on research questions that invoke effort allocation in sequential decision-making at early development stages of a new venture creation. The dissertation is composed of three separate research studies.

What dominates the entrepreneur's decision process initially is the effort allocation problem in sharing time between an existing job and committing to the new venture. The first study describes how this time-sharing is done and characterizes when is the best time to leave the wage job and become a full-time entrepreneur. I also show that the optimal time-allocation policy is driven by the entrepreneur's tolerance for work and by how returns behave with respect to time allocation in the venture.

It is important to understand resource allocations to internal activities such as product development and customer recruitment. The second study focuses on new product development and it investigates how the flow of a new venture's funding affects the development of a new product. I prescribe the optimal release time for the new product and describe how this strategy is affected by the expected amount of funding and its uncertainty. I also identify industrial and entrepreneurial characteristics that generate various behaviors for the rate of change in the return on product quality as investment in the product is increased.

The newly developed product must be bought to make the business start-up successful. The third study investigates how an entrepreneur makes decisions over time in allocating effort to building and exploiting a customer base so as to maximize profit. I study what a rational entrepreneur will do when faced with the allocation of effort to different customer categories. I also provide guidelines for improving the performance of an entrepreneur who may not be acting optimally.

In these three investigations a dynamic programming approach is utilized to study various sequential decision processes of an entrepreneur during the development process of new venture creation.
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A mes parents
Richard et Pâquerette
qui m’ont toujours encouragée
Acknowledgement

I am indebted to my co-supervisor Kenneth MacCrimmon for believing in me and for always asking the questions I was omitting. I am also indebted to my co-supervisor Shelby Brumelle for brain storming with me on issues that have greatly benefited this dissertation. Ken and Shelby, I would not have done it without you. I also wish to thank my thesis committee members Raphael Amit and Hong Chen for constructive comments throughout my doctoral work. This dissertation would have never been completed without the constant support of my husband Lawrence Yuter. I wish to thank him too. Finally, I thank the journal Entrepreneurship Theory and Practice for publishing Chapter 2 of this dissertation. Chapter 2 represents a joint paper with Kenneth MacCrimmon titled “On the Interaction of Time and Money Invested in New Ventures” (forthcoming in 1998).
Chapter 1

Introduction

1.1 Entrepreneurial Decision-Making

Entrepreneurs identify business opportunities to create new wealth. These business opportunities trigger the commitment of resources to form new ventures. The success of these ventures is associated with the entrepreneur's talent, time commitment, and ability to assemble other human, physical, and financial resources as inputs to making appropriate business decisions. A primary responsibility for the entrepreneur is thus to make resource-allocation decisions that will allow a new venture to start, survive, and grow.

Many questions regarding practical decisions must be considered by entrepreneurs. When and how should one leave a salaried job to start a venture? Should one concentrate on opportunities in a familiar industry? What makes a venture innovative: the product, the process, or the positioning? How vigorously should one grow a venture? Which customer categories should be targeted? Who should one approach for more funding? How should one's time and money be allocated over time? Should one operate simultaneous businesses? Other issues involve how entrepreneurial opportunities are recognized and how entrepreneurial start-ups become successful. The answers to those questions will provide a substantial step in one's understanding of new venture success and greatly help practitioners in their decisions.

It is a serious challenge to researchers to successfully model the decision process of an entrepreneur due to various sources of uncertainty in any entrepreneurial setting. Sources of uncertainty reside in the venture itself. Product development, product feasibility, costs, and funding corresponds to dynamic processes that are particular to the venture. These processes vary in an unpredictable manner and, unlike large established corporations, past experiences with
those processes are usually not available to the entrepreneur, although they may be available to potential investors. Sources of uncertainty also reside in the responsiveness of the market and the effect of competition, thus making it hard to distinguish which decisions must be made to successfully run the new business. Additionally, sources of uncertainty for potential investors reside in the entrepreneur. The investor does not know with certitude if the entrepreneur has the appropriate skills to recognize a viable business opportunity, get that business off the ground, manage a team successfully, or allocate resources optimally.

The scientific literature on entrepreneurship has already looked at issues related to understanding why some ventures fail while others succeed. However, this literature has been dominated by empirical studies with no underlying model. This approach has been most appropriate for research on characteristics of entrepreneurs and other static topics (such as Hornaday and Aboud 1971, McClelland 1987, Kalker-Konijn and Plantenga 1988, Siegel, Siegel, and MacMillan 1993). Authors including Shapiro (1983), Bull and Willard (1993), and Cooper (1993) have suggested the introduction of a different set of methodologies that better address many fundamental questions associated with entrepreneurial decision-making. Some of these questions (including the ones addressed in this dissertation) involve an explicit consideration of the evolution of the new venture over time (Woo, Daellenbach, and Nicholls-Nixon 1994). I propose a methodology that will clearly integrate evolution patterns and capture the uncertain aspects of entrepreneurial phenomena.

1.2 An Analytical Approach to Entrepreneurship

My approach to decision-making is analytical. Analytical methods have not been common in books and journals on entrepreneurship (Bygrave 1989a, 1989b, 1993, Ireland 1997, West 1997). However, they have been quite helpful to decision-makers in large organizations. Typically, analytical modelers have studied the decisions regarding various functional areas
(including production, marketing, and financing) of established corporations separately. There has been progress though on simultaneous coordination of decisions with respect to these functional areas, particularly on cost reduction (see, e.g., Abad 1987). Decisions at new venture start-ups must be taken concurrently (since functional areas are not yet well defined). Hence, analytical modelers should be encouraged to develop further their findings with respect to coordinated decisions as they conceptually apply to entrepreneurship. The dissertation will focus on coordinated decisions in entrepreneurial studies where the time dimension takes center stage in the decision process.

An analytical approach allows one to model the behavior of the new venture as it changes over time. Hence, not only cause-effect relationships between the variables can be studied (as done by empirical studies), but rationale for those cause-effect relationships can be found. Moreover, an analytical approach allows one to prescribe the "optimal" course of action rather than limiting the search for the good decision strategies to identifying the best strategy from among the few that are actually observed in practice. Finally, this approach can provide clearer hypotheses to test and serve as a guide to the study of various entrepreneurial processes. An analytical approach can thus be a useful complement to empirical work.

Mathematicians have provided us with powerful techniques, namely stochastic processes, to model unpredictable changes over time in any system. Stochastic processes are well suited to describe the dynamics of entrepreneurial phenomena. Examples of processes that can be modeled stochastically include demand for a new product, development stages of a new product, entrepreneur's cashflow, status of a new venture (start-up, growth, IPO), size of a customer base, and entrepreneurial activity of a given industry (number of new start-ups).

Once a stochastic, or even a deterministic, process has been identified to describe the behavior of a given entrepreneurial phenomenon, a suitable mathematical methodology can be utilized to prescribe decision strategies. When results, analysis, and implications are driven by
time, dynamic programming can prescribe the strategy that maximizes business performance. This method is appropriate for decisions that influence the current state of the new venture as well as its subsequent states.

There are, of course, limitations to using an analytical approach. The measurements of some variables, parameters, and functional forms can be hard to obtain and thus testing the model empirically can be difficult. For mathematical tractability reasons, I will assume that the decision-maker is rational. However, entrepreneurs may violate the requirements of rationality in various ways such as taking decisions in a "non-algorithmic" fashion (Bygrave 1993) or being overly optimistic (Olson 1986). There are various bounded rationality models that could capture some of these concepts and my approach could be adapted accordingly.

An analytical approach helps to contribute to a new dimension in entrepreneurship research (Low and MacMillan 1988, Cooper and Gimeno-Gascon 1992, Woo, Daellenbach and Nicholls-Nixon 1994). It encourages multidisciplinary research by allowing existing methodologies from various analytical disciplines to be applied to entrepreneurial problems. In addition, the optimization perspective can be useful to practitioners who wish to improve their business decisions.

1.3 The Focus of the Dissertation

My goal is to understand what drives an entrepreneur's start-up decisions. I describe decision processes and characterize the effects of those processes on business performance so that the best business strategies can be proposed. My focus is on the entrepreneur's decision process in allocating resources at new venture creation.1 Entrepreneurs will be optimizing the performance of their new ventures where performance is a function that evolves over time according to previous resource allocations (Woo, Daellenbach, and Nicholls-Nixon 1994). A

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1Prior research on resource allocation is reviewed in Chapter 2.
dynamic programming approach will be utilized to capture the endogenous dynamics affecting the decision process of resource allocations and, therefore, the new venture's performance.

The research investigations included in this dissertation can be put in perspective with respect to the entrepreneurship literature by presenting an overview of the development process of new venture creation and locating each investigation in that process. The literature already offers various development process models that partially capture my focus (Quinn and Cameron 1983, Moore 1986, Bhave 1994). I have borrowed pieces of those models to create a development process of new venture creation that better captures the decision process of resource allocations.

The development process begins with opportunity recognition. Then a commitment (including start-up costs) to the new venture is made. Next comes the operations (which includes on-going and future needs), where the business takes shape and the opportunity is transformed into a released product. The development process ends once the product is on the market, and it is hoped that the venture will achieve success. The bottom part of Figure 1.1 presents an overview of the development process of new venture creation where the four stages are ordered with respect to time.

Business performance can be measured at any of these stages. One can define business performance by the entrepreneur's wealth - that is, the market value of the discounted net present value of future earnings. This definition is most appropriate for later stages as business equities become more significant. Business performance can also be defined by the expected total discounted profit. It is this concept that will be used throughout the dissertation.

In each of the four stages, resources must be allocated in order to move on to the subsequent stage and grow the business. The entrepreneur's time, talent, and money (capital raised) are the most important resources at new venture creation and must be allocated throughout the development process of the start-up. It is assumed in this dissertation that the
entrepreneur will not have access to equity financing, which is the most important financial source for growth-stage ventures. Therefore, the dissertation focuses on resource allocation at the commitment and operations stages as shown by the upper part of Figure 1.1. Resources will be allocated to three major business activities, namely other job demands, product development, and customer recruitment. These business activities can require the same (limited) resources simultaneously as suggested by the figure. Each of those three activities is associated with a study that contributes to the core of this dissertation.

![Figure 1.1 Development Process of New Venture Creation](image)

I have chosen other job demand, product development, and customer recruitment because these business activities cover critical financial, operational, and marketing decisions that govern the survival of the new business. Understanding the decisions regarding the other job demands is important as these decisions directly affect resource allocations to the internal operations of the business (such as product development and customer recruitment) and thus affect business performance. Each study focuses on individuals who are sole proprietors and is more thoroughly described in the next paragraphs.

Successful start-ups require a tremendous amount of the entrepreneur's time. However, the owners of America's fastest growing private companies listed in *Inc. 500* magazine in 1995 kept their wage jobs for an average of four months after the birth of their new ventures. Why did
they still need to keep their current jobs? The answer is their need for money. The earned wages cover the individual’s living expenses and serve as a partial source of capital for the business. The first study, presented in Chapter 2, investigates when the best time is for an entrepreneur to leave a wage job and become a full-time entrepreneur. The decision is based on trying to maximize the entrepreneur's total earnings.

I show in Chapter 2 that the optimal time-allocation policy is driven by the entrepreneur’s tolerance for work and by how the returns behave with respect to time allocation in the venture. High work tolerant individuals have opportunities for starting a new venture that lower work tolerant individuals may not have. When the nature of the business is one in which it takes a considerable time investment up front before any results are achieved, I demonstrate that low work tolerant individuals stay in the wage job while high work tolerant individuals commit themselves fully to the new venture. However, when the initial time investment in a venture pays immediate benefits that quickly diminish with additional time spent, the low work tolerant individuals will spend all their time on the venture but the high work tolerant individuals will split their time. I also investigate over multiple periods an entrepreneur’s time sharing between a wage job and starting a new venture.

Money raised as salary from the wage job or as profit from the new venture could be spent on developing new products. In fact, the average amount of starting capital raised by Inc. 500 entrepreneurs in 1996 was only $25,000 from which more than 50% was from personal savings. Entrepreneurs involved in ventures based on the development of new products, however, must rely on additional financing such as debt and equity financing. These two types of financing are usually uncertain in magnitude and timing, thus creating a trade-off between the advantage of waiting (to generate more money) and improving the product versus releasing the product before competition increases. The second study, covered in Chapter 3, examines how the flow of a new venture’s funding affects new product development.
More specifically, Chapter 3 focuses on business start-ups that are fully self-financed or those that rely on debt financing for additional funding. I prescribe the optimal time to release the new product and describe how this strategy is affected by the expected amount of funding and its uncertainty. Individuals who keep their salaried job as they work on their R&D projects can generate a constant flow of funding over time that allows them to release a product whose quality increases as wages increase. However, individuals who give up a salaried job to become fully committed to their R&D project will release their product at a quality that depends on their ability to raise funding. I demonstrate that individuals who are able to raise considerable amounts of funding will market their product at a quality that decreases as more funding can be raised, but those who are unable to generate large funding will market their product at a quality that increases as funding increases.

Entrepreneurs, particularly those at their first start-ups, may know a great deal about their products and their attributes (especially when they develop the product); nevertheless, they may have little knowledge or intuition regarding prospecting for customers. Entrepreneurs must thus allocate their effort between building a customer base and maintaining and exploiting that base. The third study, in Chapter 4, presents a decision model in which this trade-off can be examined. In the basic model, the market is assumed to consist of two types of customers, namely potential customers who did not buy the product yet and loyal customers who did. Each period the entrepreneur must decide on the number of contacts of each type to make so as to maximize profit. The analysis associated with the basic model is partially extended to a multiclass model with multiple types of customers.

I construct in Chapter 4 a framework that allows one to associate an expected benefit with each contact - that is, a framework that evaluates the lifetime value of each potential buyer. This approach allows me to utilize a simple policy of the form "among the contacts available, choose the one which has the largest expected benefit" as an optimal contact policy. On a short-term
prospecting horizon, I demonstrate that short-run losses from contacting potential customers with negative immediate expected profit (e.g., from high contacting costs) are not covered by the long-run expected benefits when contacts are made at periods relatively close to the end of the prospecting horizon. Therefore, critical time points after which no more potential customers must be contacted are derived.

The above studies can be extended in various ways. Chapter 5 presents some extensions and concludes the dissertation.
Chapter 2

Between Partial and Total Commitment in New Venturing

2.1 Introduction

In this chapter I try to deepen my understanding of nascent entrepreneurs. Starting a new business takes a variety of resources. Two of the most important inputs are the time and the money contributed by the entrepreneur. I focus on the issues of when and how entrepreneurs make the commitment to allocate time and money to start a venture. As a contrast to the usual qualitative or empirical studies, I believe that an analytical approach can provide a new focus and that is the approach I present here.

Most entrepreneurs take the initial steps in starting a business while working for someone else in a salaried job. It is often difficult to leave the certainty of a salaried job for the uncertainties of a new venture. In particular, the nascent entrepreneur by staying in the existing position can, for a given commitment of time, be assured of a given wage. To spend time in building a new business, one has to take time away from the salaried job. By doing so, one earns less which means not only is there less money to support a regular standard of living but also less money to put into the new business. Hence some difficult time and money trade-offs have to be made.

Different individuals have different levels of work tolerance. For some people 40 hours is more than a full work week while for others 80 hours might be deemed light. It is often thought that among their other requirements, entrepreneurs have to have a high capacity for work. I am interested in the extent to which individuals who have a high work tolerance have better prospects for success. Might individuals with a low work tolerance be deterred from even starting

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2 While they could take the time from leisure, I will assume that leisure has already been cut back as far as possible.
a new venture in the first place? Might there be a critical time investment that only high work tolerant individuals can attain?

I trace the pattern by which people allocate a greater proportion of time to the new venture and away from the wage job, and perhaps depending on how the venture develops, how they may reallocate time back to the wage job. Once an individual begins investing more and more time in the business, does it imply that a point will be reached where the entrepreneur will quit the wage job completely? Under what conditions might the proportion of time at the wage job be increased?

The money generated from both the wage job and from new venture profits can support various activities. It can be spent on advertising or marketing initiatives and hence build up the propensity from potential customers to buy. It can be spent on R&D and hence build up product quality. It can be spent on networking and relationship building to increase contractual possibilities and reputation. In this chapter, I will use the general terminology of making “business expenditures” to cover any of these cases, namely the marketing, the R&D, and the networking scenarios.

A theoretical framework describing the optimal time to leave a wage job and become a full-time entrepreneur is developed. Using a deterministic setting, the framework describes, over time, the optimal time allocation based on maximizing the entrepreneur's earnings. The framework also reveals how the optimal time-allocation strategy is affected by the entrepreneur's work tolerance and by the nature of the relationship between financial returns and time invested. The application of the model is restricted to a class of businesses where outside financing is difficult to get. To grow the venture, the entrepreneur is then forced to rely on wages and revenues from the new venture. Alternatively, the model is restricted to businesses at early stages of development as businesses in growth stage often rely on equity financing.

The major results can be summarised as follows. When it takes a considerable time
investment up-front before any results are achieved in the business, the low work tolerant individuals stay in the wage job and the high work tolerant individuals spend all their time on the venture. However, when the initial time investment in a venture pays immediate benefits but they diminish with additional time spent, I get intermediate strategies. The low work tolerant individuals will spend all their time on the venture but the high work tolerant individuals will split their time. Over multiple periods, when the return on demand from reputation decreases, high or medium work tolerant individuals never give up the wage job. Given constant business expenditure each period and increasing returns to reputation, time allocated to the venture will increase over time and low work tolerant individuals will eventually give up the wage job.

The rest of the chapter is organized as follows. The second section presents a background of related literature while the third section describes the decision problem. In the fourth section, a preliminary analysis is done for a one-period decision problem to provide insights for the multi-period case. The multi-period decision problem is investigated in the fifth section and optimal time-allocation strategies of a specific example are derived. The sixth section concludes the chapter.

2.2 Related Studies

In this section I will briefly mention two very diverse literatures that are related to this chapter. First, there are interesting empirical studies involving the allocation of time in new ventures. Second, there is fundamental analytical research to draw upon in developing my models.

McCarthy, Krueger, and Schoenecker (1990) study the relationship between a firm's stage of development and its founder's time allocation pattern. They focus on changes that occur during the firm's crucial early stages. Their sample from the retail industry shows that entrepreneurs spent more time dealing with employees at later points in these stages. However, this extra effort
may simply be due to the larger number of employees as a result of expansion. There were no significant differences in time allocations for dealing with record keeping, customers, production, maintenance, suppliers, financing, and planning. They note that the changes of more than the single dimension studied (i.e., time allocation) should be investigated due to the multidimensional concept of the development stages. A logical extension is the consideration of money allocation and, especially, the interaction of time and money.

Cooper, Ramachandran, and Schoorman (1996) construct a conceptual framework based on the occupational choice and entrepreneurial typology literatures in order to investigate how entrepreneurs allocate time. A key finding from a three-year longitudinal study was that craftsmen-entrepreneurs devote less time to administrative activities while entrepreneurs with managerial experience are less likely to follow the objectives of craftsmen-entrepreneurs and devote more time to administrative activities.

Ronstadt (1984) conducted a survey of startups and exits comparing ex-entrepreneurs with practicing entrepreneurs and serious non-starters. Ex-entrepreneurs were practicing entrepreneurs who decided to give up their ventures and go to work for someone else. A central question was: "Did ex-entrepreneurs tend to start their entrepreneurial careers on a part-time basis?" He found that 36% of ex-entrepreneurs started on a part-time basis. In a subsequent paper, Ronstadt (1985) describes why ex-entrepreneurs gave up their ventures. Of all the exits, 43% are due to a combination of reasons, 31% are due exclusively to financial reasons, 11% are due exclusively to personal/family reasons, and 15% are due exclusively to venture/environmental reasons. The large percentage associated with financial reasons motivates the use of money allocations as an important element of any conceptual model that attempts to describe an entrepreneur's time allocation pattern. Ronstadt's focus on ex-entrepreneurs suggests the usefulness of studying conditions under which an entrepreneur may be better off by going back to a wage job.
Campbell (1992) uses an economic decision model to study the decision to become an entrepreneur as an alternative to wage labor. The decision is based on comparing the expected net gain from an entrepreneurial venture (which depends on the probability of success) to the expected income from wage labor (which depends on the probability of employment). His model implies that an increase in the probability of success induces an increase in the incentive to start an entrepreneurial venture. Also, risk-takers should be less inclined than risk-avers to increase entrepreneurial activity when the probability of success increases.

The economic literature is quite rich in modeling time allocation. My analytical model was stimulated by Becker's (1965) time allocation framework where time and market goods are inputs consumed by an individual through leisure to produce commodities (seeing a play, for instance). To afford these market goods, the individual invests time in a wage job. By analogy, an entrepreneur consumes time and money through venturing to produce business value. Unlike Becker, in an entrepreneurial context, money is not only generated through a wage job but also through producing the commodity, that is, through revenue from the new venture.

Greenwood & Hercowitz (1991) utilize Becker's approach to construct a descriptive model of allocation of capital and time between business and household at the level of the whole U.S. economy. Their model captures the high level of household investment relative to non-residential business investment that has been observed from 1954 to 1988. It also captures the cyclical behavior of household investment that led the movement in business investment. Although their approach has some commonalities with ours, they address a different problem at a much more macro level.

The effort allocation model of Radner and Rothschild (1975) studied how to allocate limited effort among various activities which tend to deteriorate if unattended. They utilize an analytical approach based on bounded rationality, thus limiting their search to three attractive and simple types of behavior. They use survival probability and long-run average rate of growth to
measure performance. They show that under certain conditions survival is possible with positive probability if and only if survival is possible when effort is allocated to the worst performing activity each period. While they consider a variety of activities, they do not model any endogenous state variables that can determine the optimal policy.

The approach utilized to formulate my multi-period decision problem has been influenced by the advertising literature on dynamic optimal control models. Feichtinger, Hartl, and Sethi (1994) give a partial survey of some of the few optimal control models that allow interaction among marketing, production, and financing. Because of the complicated nature of the resulting optimal control problem (in particular, too many state variables), it is usually not possible to derive analytical results. However, my model is complete enough to capture the important elements of new venture start-ups, but simple enough to provide results. Also, I contribute to the literature on coordinated planning models by considering production capacity constraints.

2.3 Problem Description

Monetary requirements force entrepreneurs to allocate one of their most valuable resources, time, to multiple activities. In this paper, time is shared between a wage job and a new venture. While time allocated to the new venture helps to get it profitable, the funds generated through the venture may not be enough to sustain the venture. Thus, keeping the wage job allows the entrepreneur to invest more money into the new business.

There are tradeoffs between investing time and investing money in the new venture. Figure 2.1 expresses these tradeoffs by presenting the relationships between types of investments (time versus money) and their effects on business performance. The more time invested in the wage job, the less time available for the new venture, but the more money available for the new venture. The more time or money invested in the new business, the better the performance.

One should recognize, however, that new venture start-ups involve uncertainty in their
business performance. Consequently, investing more time and/or money could result in worse business performance. To generate insights regarding the interaction of time and money invested in new ventures, I choose to put uncertainty aside. I thus utilize expected business performance, which increases as more time and/or money is allocated to the venture.

The relationships (or effects) in Figure 2.1 hold from one period to another throughout the multi-period decision process. What may be changing from period to period is the magnitude of those effects. It is by comparing those effects, which depend on the evolution of the new venture, that the entrepreneur will select the optimal time to leave the wage job. Those effects are influenced by the demand function which also influences the amount of money that can be reinvested in the new venture in the following period. Consequently, the evolution of the demand function will contribute to the evolution of the new venture and will thus define the dynamics of this time allocation problem.

![Figure 2.1 The Effects of Time and Money on Business Success](image)

(+), representing a positive relationship, and (-), a negative relationship.
Over multiple periods, a decision problem of time allocation between a wage job and starting a new business is formulated. Each period, the entrepreneur has a limited amount of working time. The wage job’s output is such that for each hour worked at the wage job up to the limit of the entrepreneur’s work tolerance level, a fixed amount of money can be invested into the new venture. It is assumed that the product is non-durable (so there is no gain from keeping inventories). To keep the model simple without compromising insightful qualitative results, I also assume that the entrepreneur has no uninvested capital, has no unutilized credit, and is not considering equity financing.

In the next two sections, a decision problem is formulated to find when an entrepreneur should commit to a new venture so as to maximize earnings. The new venture has a demand function which depends on business expenditures and grows as represented by a Nerlove-Arrow (1962) advertising capital model. Each period, a production capacity constraint will force sales to be smaller than or equal to the production level. The production function specifies the number of units produced for each combination of time and money allocated to the new venture.

It may help here to give a concrete application that can be represented by this theoretical model. Consider a computer programmer who wishes to start her own food catering business. She can vary the amount of time she spends as a programmer and receives a fixed wage rate for this work. The catering business provides a product (meals) that is non-durable and cannot be inventoried. Hence, the entrepreneur takes orders from clients and produces what is ordered. To generate a clientele, the entrepreneur can make marketing expenditures (such as hiring students to distribute food menus to the community), can make R&D expenditures (such as trying out different recipes), and can make networking expenditures (such as setting up a web site).

2.4 Static Analysis: Levels of Work Tolerance and Returns to Time Invested

I will begin with a simple model in which the entrepreneur makes a one-time decision
about how much time to allocate to the wage job versus the new venture. Thus, the problem is first formulated as a one-period decision problem and optimal time-allocation policies are derived. This section provides results that will be used in the multi-period setting of next section.

2.4.1 The Entrepreneur's Decision Variables and Level of Work Tolerance

The key focus in this section will be the work ethic of the entrepreneur. The entrepreneur has a maximum of \( \tau \) working hours that can be devoted either to the current wage job or to developing the new venture. The key decision is how many hours, \( h \), to allocate to the new venture; then \( \tau - h \) hours will be devoted to the wage job. I will consider three types of entrepreneurs: (1) those who are low work tolerant and work less than \( \tau_0 \) hours per week, (2) those who are high work tolerant and work more than \( \tau_1 \) hours per week, and (3) those whose are medium work tolerant and work between \( \tau_0 \) hours and \( \tau_1 \) hours per week. The parameters \( \tau_0 \) and \( \tau_1 \) are not exogenously given as they will be a result of the optimization problem.

There are two sources of money for the new venture. The wage job generates income, at a rate of $w per hour. This money can be used to invest in the new venture as well as to pay regular living expenses. The other source of funding for the new venture is the reinvestment of profits from the venture.

There are two basic money requirements for a new venture: regular periodic business expenditures (for marketing, product development, etc.) and capital investment. Let \( e \) be the business expenditure and \( i \) the capital investment in the new venture. The variables \( h, e, \) and \( i \) are controlled by the entrepreneur and they define the decision variables of the optimization problem.

The income from the wage job added to the profit from the new venture minus the business expenditures and capital investment defines the entrepreneur's earnings, denoted by \( P \), that is,
\[ P(h,e,i) = \rho D(e) + w[ \tau - h] - e - i , \]

where \( \rho \) represents the known unit price for the product, \( D(e) \) is the demand function (also depending on the parameter \( \rho \)) which increases with business expenditure \( e \).\(^3\) The entrepreneur chooses time, business expenditures, and investment allocations that maximize earnings.

The production function, denoted by \( K(h,i) \), is a function of time and money allocated to the new venture, and must equal or exceed the demand function. The entrepreneur thus encounters a production capacity constraint expressed by \( D(e) \leq K(h,i) \). The one-period time-allocation problem is thus formulated as

\[
p^* = \max_{h,e,i} \{ \rho D(e) + w[ \tau - h] - e - i \}
\]

subject to:

\[
D(e) \leq K(h,i) \\
0 \leq h \leq \tau \\
e \geq 0, i \geq 0.
\]

It is straightforward to verify that when the decision variables \( h, e \) and \( i \) take the value zero, the total earnings will correspond to the maximal wages, that is, \( w_\tau \). An optimal objective that does not exceed the maximal wages \( (w_\tau) \) may thus be interpreted as a non-attractive business opportunity.

The various elements of the model are presented in Table 2.1.

\(^3\) For instance, the demand for the product increases as more business expenditures are spent on marketing initiatives such as promotions, or on improving the quality of the product, or on increasing reputation through better customer services.


Because the product is non-durable and cannot be inventoried, there is a loss from producing more than the demand. Consequently, at optimality the production capacity constraint will be binding leading to an optimal capital investment \( i^* = i^*(h,e) \) from solving \( D(e) = K(h,i^*) \).

Maximizing the total earnings \( P(h,e) \) (\( = P(h,e,i^*(h,e)) \)) over the business expenditure \( e \) gives \( e^* = e^*(h) \) so that

\[
P(h) = P_v(h) + w[r-h]
\]

where

\[
P_v(h) = \rho D(e^*(h)) - e^*(h) - i^*(h,e^*(h))
\]

which represents the profit from the new venture while \( w[r-h] \) represents the wages.

I should mention here that coordinate-wise optimization to this multi-variable problem is only guaranteed by special forms of function \( P(h,e,i) \). I should also mention that properties of the function \( P_v(h) \) will depend on the properties of the production function and the demand function, respectively \( K(h,i) \) and \( D(e) \). This will be explored further in the multi-period analysis through specific examples.

### 2.4.2 Different Levels of Return to Time Invested in New Ventures

Ventures differ in the amount of financial payoff that results from different levels of time
allocated to them. Some new ventures consume a lot of effort before they begin to pay off but when they do the financial payoff may become very large. This characterizes the case of increasing returns to entrepreneurial effort or when viewed in terms of the functional relationship between the financial payoff and time, I say that the function is convex. Other new ventures give significant immediate payoffs when effort is first invested but then give decreasing marginal payoffs. This is a case of diminishing returns to entrepreneurial effort and is represented by a concave function. In between is the case where every hour spent on the venture yields the same payoff, that is, the functional relation is linear. I will consider each of these three cases separately. Other possibilities can be formed from these basic cases such as having increasing returns and then decreasing returns (as represented by an s-shaped function).

Consider first the relationships shown in Figure 2.2 in which the returns from investing time in the new venture are increasing (i.e., the curve $P_v$ is convex in $h$). The income from the wage job is represented by $wh$ which is a linear function of $h$. The entrepreneur has a maximum of $\tau$ working hours to allocate between a wage job and a new venture. The points $\tau_0$ and $\tau_1$ mark the minimum and maximum levels characterizing an intermediate work tolerant individual where low work tolerant individuals have $\tau < \tau_0$ while high work tolerant individuals have $\tau > \tau_1$. Note that the straight line representing payoffs from the wage job lies above the curve representing payoffs from the venture for all values of $\tau$ which are less than $\tau_1$. For these cases, if time were only allocated to the wage job, then the entrepreneur would get $wt$ which is larger than $P_v(\tau)$ corresponding to what would be received if time were only allocated to the new venture. For example, if $\tau_0$ hours were allocated to the new venture while $\tau - \tau_0$ hours were allocated to the wage job, then the entrepreneur's total earnings would be $P_v(\tau_0) + w[\tau - \tau_0]$ which is smaller than $wt$. In fact, one can easily verify that if any proportion of working time $qt$, $0 < q < 1$, is allocated to the new venture, the corresponding total profit is smaller than $wt$ as long as $\tau \leq \tau_1$ (i.e. the entrepreneur has a low or a medium work tolerance). When $\tau > \tau_1$ so that the entrepreneur has a
high work tolerance, one can easily derive from Figure 2.2 that the entrepreneur's total profit is maximized by allocating all the working time to the new venture. At $\tau_0$, note that the return from the venture ($\frac{\partial P_v}{\partial h}$) equals the wage rate $w$ (i.e., the slope of the curve at $\tau_0$ is the same as the slope of the straight line). These relationships will be established more formally in the mathematical results to be given below.

![Figure 2.2 Increasing Returns from the New Venture](image)

**THEOREM 2.1** If the returns from investing time in the new venture are increasing (i.e. $P_v$ is convex in $h$), then for $0 < \tau \leq \tau_1$ choose $h^* = 0$ and for $\tau > \tau_1$ choose $h^* = \tau$, where $h^*$ is the optimal time allocated to the new venture.

The next case to consider is when $P_v$ is linear, thus the returns from investing time in the new venture are constant—each hour yields the same payoff. The corresponding figure in this case would simply be two straight lines with differing slopes. Hence, the optimal time allocation strategy is derived by simply comparing the return from the venture to the wage rate or, equivalently, by comparing the slope of $P_v$ to the slope of $wh$. If the slope of the wage line is higher, then one devotes all the time to the wage job while if the slope of the new venture return line is higher then one commits all one's time to the new venture.
THEOREM 2.2 If the profit from the new venture, $P_v$, is linear in $h$, then $h^* = 0$ whenever \( \frac{\partial P_v}{\partial h} \leq w \) and $h^* = \tau$ whenever \( \frac{\partial P_v}{\partial h} > w \).

This case is particularly interesting in that it does not depend on the work tolerance of the entrepreneur. Low, intermediate, or high work tolerant individuals will all adopt the same strategy depending only on the relative payoffs between the constant rate of return on the new venture and the wage rate. Note then that with increasing or constant returns from investing time in the new venture, optimal time-allocation policies are of the "all-or-nothing" type so that the entrepreneur should be fully committed to a wage job or fully committed to the new venture.

These results raise the question about whether it is ever optimal to be partially committed to the new venture. The answer is "yes"—in the case where the returns from investing time in the new venture are decreasing ($P_v$ is concave) as shown in the curve in Figure 2.3. At $\tau_0$, the rate of return from the venture equals the wage rate. At $\tau_1$, the profit from the new venture when $\tau_1$ hours are allocated to the venture coincides with the income from the wage job when the entrepreneur works at it for $\tau_1$ hours.

![Figure 2.3 Decreasing Returns from the New Venture](image)

When $\tau < \tau_0$ so that the entrepreneur has a low work tolerance, one can derive from Figure 2.3 that the entrepreneur's total earnings are maximized by allocating all the working time to the new venture. Figure 2.3 also shows the total earnings when the entrepreneur has a medium
work tolerance, that is, $\tau_0 \leq \tau \leq \tau_f$. If time were only allocated to the wage job, then the entrepreneur would get $w\tau$ which is smaller than $P_v(\tau)$ corresponding to what would be received if time were only allocated to the new venture. If $\tau_0$ hours were allocated to the new venture while $\tau-\tau_0$ hours were allocated to the wage job, then the entrepreneur's total earnings would be $P_v(\tau_0)+w[\tau-\tau_0]$ which is larger than $w\tau$. It is straightforward to verify that this latter time allocation provides the maximum total earnings as long as the entrepreneur has a medium or a high work tolerance ($\tau \geq \tau_0$, including $\tau > \tau_f$). Theorem 2.3 summarizes those results.

**Theorem 2.3** If the returns from investing time in the new venture are strictly decreasing ($P_v$ is strictly concave in $h$), then for $0 < \tau < \tau_0$ choose $h^* = \tau$ and for $\tau \geq \tau_0$ choose $h^* = \tau_0$, where $h^*$ is the optimal time allocated to the new venture.

The rationale for an optimal time allocation being split between the wage job and the new venture goes as follows. Allocating one more hour to the new venture increases the production level, and so the demand (because the production capacity constraint binds at optimality). To increase demand, more business expenditure (e.g., R&D, marketing) is required. As business expenditure increases, the marginal effects on demand decrease so that it becomes very expensive to even slightly increase demand. Consequently, it becomes more profitable to allocate the hour to the wage job.

To summarize, three kinds of businesses have been considered, namely ones with increasing returns, constant returns, and decreasing returns with respect to time allocated. Furthermore three types of entrepreneurs have been defined, namely low, medium, or high work tolerant individuals. Within the time they had available, entrepreneurs apportion their working hours either to their current wage job or to developing the new venture. By generating money as salary from the wage job or from profits from the new venture, they could make business
expenditures and hence build up demand. In one case (increasing returns), all of the time should be devoted to the wage job unless the entrepreneur was willing to work enough hours to make the venture successful and if the critical commitment level was exceeded then all the time should be spent on the new venture. In a second case (constant returns), the entrepreneur’s work tolerance was irrelevant and one simply evaluated the relative rate of return expected from the new venture versus the wage job and committed fully to either the venture or the wage job depending on which offered the higher returns. In a third case (decreasing returns), one finds a strategy that allows for splitting time. A low work tolerant individual will commit fully to the venture while individuals having a work tolerance higher than a critical level (where the rates of return are equal) will allocate time to both the wage job and the venture with the highest tolerant types spending a higher proportion of time on the wage job. Table 2.2 outlines my findings.

As an example of my findings consider a taxi driver who can earn a fixed wage (and tip) rate per hour by driving a taxi and is able to devote any number of hours per week to this activity. He is considering a car repair business and must consider how much time per week to devote to the business. Suppose the taxi driver has accumulated necessary repair equipment over the years and, by spending more time on the business, the rate of return will increase. Then the energetic taxi driver who has a high work tolerance has enough hours in the week to reach the critical level of commitment to the business and should thus be fully committed to it while a less active individual will not be able to reach this critical level and will be best advised to stick to driving the cab full time. However, suppose the nature of the business is such that, although it is relatively profitable (compared to cab driving) at lower levels of activity, there are a fixed number of customers in the neighborhood and more hours will not generate proportionately more business. In this case the less active cabbie will be advised to spend all of his time on the business and no time driving his taxi while the higher work tolerant individual will split his time. The very energetic individual will find himself spending a high proportion of time driving the
cab because the extra hours that could be committed to the business do not pay off.

### TABLE 2.2
*A Summary of Findings for the Static Analysis of the Time-Allocation Model*

<table>
<thead>
<tr>
<th>Optimal strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Increasing returns</strong></td>
</tr>
<tr>
<td>Low and medium work tolerance</td>
</tr>
<tr>
<td>The entrepreneur should allocate all available working time to the wage job because there are not enough available hours that can be invested to make the venture profitable.</td>
</tr>
<tr>
<td>High work tolerance</td>
</tr>
<tr>
<td>The entrepreneur commit fully to the venture because the time that will be invested will bring the venture to the stage where it is very profitable and will more than compensate for lost wages.</td>
</tr>
<tr>
<td><strong>Constant returns</strong></td>
</tr>
<tr>
<td>Low work tolerance</td>
</tr>
<tr>
<td>Time allocated is independent of work tolerance. Low, intermediate and high work tolerant individuals will all adopt the same strategy.</td>
</tr>
<tr>
<td><strong>Decreasing returns</strong></td>
</tr>
<tr>
<td>Low work tolerance</td>
</tr>
<tr>
<td>The entrepreneur should allocate all available time to the venture, but the entrepreneur would not be able to invest enough time to make the venture as profitable as possible.</td>
</tr>
<tr>
<td>Medium and high work tolerance</td>
</tr>
<tr>
<td>The entrepreneur should split time between the wage job and the new venture spending an increasing percent of time beyond some critical level (when the two rates are equal) on the wage job because the venture has peaked out.</td>
</tr>
</tbody>
</table>

Note, then, that in general the best time allocation depends on assessing both the prospective entrepreneur’s tolerance for work and the nature of the business (in terms of whether extra time yields a proportionately high or low return). It can be observed that high work tolerant individuals will spend at least as much time absolutely on a new venture as low work tolerant individuals and in some cases (increasing returns) will commit themselves fully to a business
when it is not feasible for individuals with a lower work tolerance to even get involved. On the other hand, if the business is such that even a small effort yields a good payoff, the low work tolerant individual will engage in the startup and commit fully to the venture. Even here, though, note that the high work tolerant individual will also start the venture and spend at least as much time on it on an absolute basis but when it reaches the point of decreasing returns (based on extra hours per week), the higher work tolerant individual will devote the extra hours to the wage job.

2.5 Dynamic Analysis: Patterns of Time Allocation

In the preceding section I considered the basic decision about allocating time to a new venture but the goal of this chapter is also to characterize the pattern of shifting time allocations from the wage job to the new venture. In particular, I want to identify the best time for an entrepreneur to leave a wage job and become fully involved in a new venture. To address this issue, one must study the time-allocation problem over time. In this section I will present a multi-period framework from which optimal time-allocation policies are derived.

2.5.1 Formulation and Optimal Time-Allocation Policies

It is difficult for many new businesses to get started because of the need to develop sufficient visibility or reputation. Expenditures on the business provide this visibility and build the reputation. The value of the business can be thought of as directly proportional to its reputation.

From one period to the next, the entrepreneur builds up a reputation which summarizes the effects of current and past business expenditure in building up demand. By "reputation" I mean goodwill, product quality, personal relationships, etc., that is, some continuing and cumulative index of the status of the venture. Demand is a function of reputation, as well as price, where the reputation level at period $t$ is a function of business expenditure prior to $t$. I utilize the Nerlove-Arrow (1962) advertisement capital model to express the dynamics from
Let $A_t$ be the reputation level for period $t$ (starting with period 0), $\delta$ the depreciation rate of reputation from one period to another, and $\gamma$ the effect on reputation of one unit of business expenditure. Hence,

$$A_{t+1} = (1-\delta)A_t + \gamma e_t, \quad A_0 = 0,$$

where $e_t$ is the business expenditure for period $t$ and, for period 0, the entrepreneur has no reputation accumulated yet.

If expenditures are viewed in marketing terms, then $e$ can be thought of as advertising which for each dollar expended contributes $\gamma$ but then has a forget rate of $\delta$ each period. Alternatively, $e$ can be considered as product development expenditures each dollar of which contributes $\gamma$ to product quality but then obsolescence occurs at a rate of $\delta$ each period. Still another interpretation is for $e$ to represent expenditures on networking and relationship building which contribute at a unit rate of $\gamma$ to reputation but then fall off at a rate of $\delta$ each period.

Let $\beta$ be the value of a unit of reputation so that on a short term planning horizon, say of length $T$ periods, the accumulated value of reputation is

$$\beta A_{T+1} = \beta \left\{ (1-\delta)^T e_0 + (1-\delta)^{T-1} e_1 + (1-\delta)^{T-2} e_2 + \ldots + (1-\delta) e_{T-1} + e_T \right\}. $$

I will utilize $\beta A_{T+1}$ as the value of the business. This implies that I focus on new ventures for which the value of non-liquid assets (such as facilities, equipment) is small compared to the value of accumulated reputation. This property is increasingly relevant in a post-industrial context where knowledge-based industries are predominant.

For the time interval considered, $[0,T]$, the entrepreneur’s total earnings are

$$P(h_0, \ldots, h_T, e_0, \ldots, e_T, i_0, \ldots, i_T) = \sum_{t=0}^{T} \lambda \left\{ \rho D(A_t) + w[\tau - h_t] - e_t - i_t \right\} + \beta A_{T+1}, \quad (2.1)$$

where $h_t, e_t, i_t, \lambda$ ($0 \leq \lambda \leq 1$), and $D(A_t)$ are the time allocated to the new venture at period $t$, the business expenditure at $t$, the capital investment at $t$, the discount factor on future earnings, and
the demand at $t$, respectively. For any given period, total earnings will correspond to the maximal wages, that is, $w_{\tau}$ whenever the decision variables take the value zero on that period. An optimal objective that does not exceed the total maximal wages ($\sum_{i=0}^{T+1} \lambda_i [\tau - h_i]$) may be interpreted as a non-attractive business opportunity.

The entrepreneur chooses a path of time allocation, a path of business expenditures, and a path of capital investment that maximizes total earnings. The multi-period optimization problem is

$$\text{Max} \left\{ \sum_{i=0}^{T} \lambda_i \left[ \rho D(A_i) + w[\tau - h_i] - e_i - i_i \right] + \beta A_{T+1} \right\}$$

subject to:

$$A_{t+1} = (1-\delta)A_t + \gamma e_t, \quad A_0 = 0$$

$$D(A_t) \leq K(h_t, i_t)$$

$$0 \leq h_t \leq \tau$$

$$e_t \geq 0, \quad i_t \geq 0$$

$$t = 0, 1, 2, \ldots, T,$$

where $K(h_t, i_t)$ is the production function which, at period $t$, only depends on time allocated and the investment level at period $t$.

It is still assumed that the product is non-durable and cannot be inventoried. Hence, each period the production capacity constraint will be binding, that is, $D(A_t) = K(h_t, i_t)$. From this equality, one derives $i_t^* = i_t^*(h_t, A_t) = i_t^*(h_t, e_0, e_1, \ldots, e_T)$. By substituting $i_t^*(h_t, e_0, e_1, \ldots, e_T)$ into $P(h_0, h_1, \ldots, h_T, e_0, e_1, \ldots, e_T, i_0, i_1, \ldots, i_T)$ of equation (2.1), one gets total earnings as a function of $h_0, h_1, \ldots, h_T, e_0, e_1, \ldots, e_T$ only, and it will be denoted by $P(h_0, h_1, \ldots, h_T, e_0, e_1, \ldots, e_T)$. Maximizing the total earnings $P(h_0, h_1, \ldots, h_T, e_0, e_1, \ldots, e_T)$ over the business expenditure $e_t$ gives $e_{t}^* = e_{t}^*(h_0, h_1, \ldots, h_T)$ so that

$$P(h_0, h_1, \ldots, h_T) = P_v(h_0, h_1, \ldots, h_T) + w \sum_{i=0}^{T} \lambda_i [\tau - h_i].$$

29
where the first term on the right-hand side represents the overall profit and the second term represents the accumulated wages.

The profit from the venture, denoted by \( P_v(h_0, h_1, ..., h_T) \), is said to be separable in \( h_0, h_1, ..., h_T \) when the time allocation for any given period does not affect the time allocation for any other period. This condition is mathematically expressed by

\[
P_v(h_0, h_1, ..., h_T) = \sum_{t=0}^{T} X^t P_v'(h_t).
\]

I will show in the next sub-section that this condition holds for an abstract example when returns on demand are constant with respect to level of reputation.

The following result follows from the symmetry, from one period to another, of the problem formulation. It will be utilized to construct optimal time-allocation policies for the multi-period problem which will be stated as Corollary 2.1.

**Theorem 2.4** Assume that \( P_v(h_0, h_1, ..., h_T) \) is separable in \( h_0, h_1, ..., h_T \) so that

\[
P_v(h_0, h_1, ..., h_T) = \sum_{t=0}^{T} X^t P_v'(h_t).
\]

If there exists a period \( t \) where the function \( P_v'(h_t) \) is linear in \( h_t \), then \( P_v'(h_t) \) is linear in \( h_t \) for any \( t \). Similarly, if there exists a period \( t \) where the returns from investing time in the venture are strictly increasing (respectively strictly decreasing), then the returns from investing time in the venture are strictly increasing (respectively strictly decreasing) for any \( t \).

It is not obvious to verify that \( P_v(h_0, h_1, ..., h_T) \) is separable in its arguments as one must go through a number of steps to express the profit from the venture as a function of time allocated only. In the next sub-section, I will present a particular case where a Cobb-Douglas production function and a linear demand function provide a \( P_v(h_0, h_1, ..., h_T) \) that is linear (and thus separable) in its arguments.

When the returns from investing time in the venture are strictly increasing (i.e.,
Theorem 2.1 provides the optimal time-allocation policies. In period \( t \), let \( \tau_i \) be the time allocation for which the entrepreneur is indifferent between spending that amount of time at a wage job versus spending that amount of time building up a new venture (see the intersection point \( \tau_i \) in Figure 2.2). If the entrepreneur has a low or medium work tolerance at period \( t \) (i.e., \( 0 < \tau \leq \tau_i \)), then Theorem 2.1 suggests that the entrepreneur allocates time in period \( t \) exclusively to the wage job. When the entrepreneur's tolerance for work is high (i.e., \( \tau > \tau_i \)), Theorem 2.1 suggests that the entrepreneur becomes completely involved in period \( t \) with the new venture. These results are stated as Corollary 2.1 (a).

When \( P_v'(h_t) \) is linear in \( h_t \), the return (or marginal revenue) from allocating time in the new venture, \( MR_v' = P_v' = \partial P_v' / \partial h_t \), is constant over \( h_t \). Therefore, the optimal time allocation for every period is found by comparing \( MR_v' \) to the wage rate \( w \), as presented in Theorem 2.2. At period \( t \), the entrepreneur would devote all the time to the new venture whenever the new venture return exceeds the wage rate. The entrepreneur would devote all the time to the wage job at period \( t \) whenever the new venture return does not exceed the wage rate. This result is stated as Corollary 2.1 (b). A natural question to ask is how those returns from the new venture behave over the periods. An example where periodical marginal revenues are ordered over time will be presented in the next sub-section.

When the returns from investing time in the venture are strictly decreasing (i.e., \( \partial P_v' / \partial h_t < 0 \)), Theorem 2.3 provides the optimal time-allocation policies. In period \( t \), let \( \tau_0 \) be the time allocation where the rate of return from the venture equals the wage rate, that is, \( \partial P_v' / \partial h_t = w \). If the entrepreneur has a low work tolerance at period \( t \) (i.e., \( 0 < \tau \leq \tau_0 \)), then Theorem 2.3 suggests that the entrepreneur allocates time in period \( t \) exclusively to the venture. When the entrepreneur's tolerance for work is medium or high (i.e., \( \tau > \tau_0 \)), Theorem 2.3 suggests that at period \( t \) the entrepreneur allocates \( \tau_0 \) hours to the venture and the rest of the time to the wage job.
These results are stated as Corollary 2.1 (c).

**COROLLARY 2.1** Assume that

$$P(h_0, h_1, \ldots, h_T) = \sum_{t=0}^{T} \lambda^t P^t(h_t) + w \sum_{t=0}^{T} \lambda^t [\tau - h_t]$$

and let $h_t^*$ be the optimal time allocated to the new venture at period $t$.

(a) Assume the returns from investing time in the venture are strictly increasing, that is, $\partial P^t / \partial h_t > 0$. Hence, $h_t^* = 0$ whenever $0 < \tau < \tau_1^*$ and $h_t^* = \tau$ whenever $\tau > \tau_t^*$, $t=1, 2, \ldots, T$.

(b) Assume the returns from investing time in the venture are constant, that is, $\partial P^t / \partial h_t = 0$. Hence, $h_t^* = \tau$ whenever $MR_v^t = \partial P_v^t / \partial h_t > w$ and $h_t^* = 0$ whenever $MR_v^t \leq w$, $t=1, 2, \ldots, T$.

(c) Assume the returns from investing time in the venture are strictly decreasing, that is, $\partial P^t / \partial h_t < 0$. Hence, $h_t^* = \tau$ whenever $0 < \tau < \tau_0^*$ and $h_t^* = \tau_0^*$ whenever $\tau > \tau_0^*$, $t=1, 2, \ldots, T$.

My findings for the multi-period case are based on the condition that the optimal time allocated to the venture for any given period does not affect the optimal time allocated to any other periods. Under these conditions, the rates of return from the venture (with respect to time allocated) would behave similarly over the periods, that is, these rates would all increase, would all be constant, or would all decrease. Depending on how these rates behave, I used my findings for the one-period model to derive optimal time-allocation policies for each period. I now present an abstract example where the condition holds and Corollary 2.1 can be applied.

**2.5.2 An Abstract Example**

A specific example is utilized to demonstrate some of the above results. I choose a Cobb-Douglas production function, that is, $K(h_t, i_t) = \kappa h_t^\alpha i_t^{1-\alpha}$ where $\kappa$ is the production level associated with investing a unit of time and a unit of money and $\alpha \in (0, 1)$. This production function is linearly homogeneous so that constant returns to scale at all input combinations are obtained ($K(\delta h, \delta i) = \delta K(h, i)$). Moreover, $0 < \alpha < 1$ provides a declining marginal product. This makes sense in situations where the productivity decreases as $h_t$ or $i_t$ increases which would be the case.
with my concrete application involving a computer programmer in a food catering business when the menu offered is customized. For any given period, the more time invested in the catering business, the more food made, but the individualized menus result in fatigue rather than efficiency, and so the lower the productivity. An analogous argument can apply to the taxi driver investing time in a car repair business. With respect to capital investment, the more the money invested in the business, the higher the cost base and so the lower the rate of return on capital.

A) Constant returns on demand from accumulating reputation

The structure of the example makes it more parsimonious to express the objective function in terms of the business expenditures, e_t's, instead of the time allocations, h_t's. I show in the appendix that when the demand function is linear in level of reputation, the total earnings are linear in business expenditures for any t, that is,

$$P(e_0, e_1, ... , e_T) = \sum_{t=0}^{T-1} C_t e_t + \sum_{t=0}^{T} \lambda' w_T,$$

where $C_t$ depends on the various parameters of the model and on t. Since there are no sales after period $T$, the business expenditures at the last period, $e_T$, will not affect the production. I have thus chosen $e_T=0$.

I also show in appendix A that the optimal time allocation at any period $t$, $h_t^*$, is proportional to the demand function $D$ (see equation (A.1)). Hence, when the demand function is linear in level of reputation, the optimal time allocations are linear in business expenditures. It thus follows that the total earnings $P$ are linear in time allocations (and so total earnings is a separable function of times allocated to the venture and Corollary 2.1 will be applicable). Consequently, during any period $t$ the marginal revenue (or return) from investing time in the new venture, $MR_v^t$, is constant with respect to the amount of time invested. However, these constant returns may vary from one period to another.
I next show in appendix A that, when future earnings are not discounted (i.e., the discount rate \( \lambda \) equals 1), these marginal returns are ordered over time, that is,

\[
MR^0 < MR^1 < ... < MR^{T-i} \quad \text{whenever} \quad \beta > \frac{1}{\gamma}
\]

and

\[
MR^0 > MR^1 > ... > MR^{T-i} \quad \text{whenever} \quad \beta < \frac{1}{\gamma}
\]

It follows that the marginal returns from the venture increase over time when the value of a unit of reputation (\( \beta \)) is larger than the cost of acquiring that unit of reputation (\( 1/\gamma \)) for any given period; those returns decrease otherwise. This phenomenon can be explained as follows. For any given period, the gain from each dollar of business expenditure is constant with respect to the total amount of dollars invested (because of the linearity assumption). However, when the market value of a unit of reputation exceeds its cost, this gain increases over time because the entrepreneur benefits more from selling the venture than from operating it. This implies that the returns from the venture will increase over time. When the market value of a unit of reputation is below its cost, the gain from each dollar of business expenditure decreases over time because the entrepreneur benefits more from operating the venture than from selling it (but operations may stop before the planned time due to low marginal gain from making business expenditures when one gets too close to stopping operations). It will imply that the returns from the venture will decrease over time.

As suggested by Corollary 2.1 (b), the optimal time-allocation policies are derived by simply comparing (each period) the rates of return to the wage rate independently of the entrepreneur's tolerance for work (the reader can also refer to the third row of Table 2.2). When the returns increase over time, the entrepreneur should delay the start-up of the new venture until

\[4\] Assuming no discounting is equivalent to assuming that the entrepreneur can borrow or save money at a rate equal to the inflation rate.
some critical time point \( r (MR_v^r < w < MR_v^{r+1}) \) after which the wage job should be discontinued and a full commitment made to the venture. This corresponds to a "all-or-nothing" optimal strategy.

Such an optimal time-allocation policy makes sense for startups where success highly depends on identifying a window of opportunity. The computer programmer may delay the startup of her catering business until the Christmas holidays when many people have large parties or until the summer months when weddings are popular. The cabbie might wait to open a car repair shop until after the winter as the salt used on the roads might cause damage to cars or as people repair their cars for summer vacation trips.

The behavior of the optimal time-allocation path is quite surprising when the returns decrease over time. The entrepreneur stops any business expenditure after some critical time point \( s (MR_v^s > w > MR_v^{s+1}) \) and only produces (allocates time) at the demand level induced by the business expenditure from earlier periods.

This last result seems to support Ronstadt's (1985) observation of 31% of ex-entrepreneurs who gave up their venture and went back to their wage job due to financial reasons alone. Ronstadt's study, which covers various types of ventures in diverse industries, does not provide sufficient detail for the reader to categorize financial reasons. Consequently, I cannot identify if leaving the entrepreneurial career (and thus moving back to a wage job) was due to decreasing returns from the venture. However, he observed that earlier exits from an entrepreneurial career encounter greater difficulties with financial concerns while later exits encounter them with personal and family problems. He rationalizes that "later exits are less encumbered by cash flow constraints, having solved these difficulties earlier in their careers."

This argument suggests that new venture startups are commonly ended by the entrepreneur due to cash flow problems pushing those ex-entrepreneurs to go back to a wage job. A new set of data, though, would need to be gathered to further support my findings.
B) Decreasing returns on demand from accumulating reputation

When the return on demand from accumulating reputation (through business expenditure, such as marketing effort) decreases as the level of reputation increases, the optimal time-allocation policies depend on the entrepreneur's tolerance for work. The objective (total earnings) cannot be explicitly expressed as a function of time allocated as done in Corollary 2.1, however, the following results are algebraically derived in appendix A assuming no discounting (i.e., \( \lambda = 1 \)).

If the entrepreneur's work tolerance is medium or high (i.e. \( \tau > \tau_0 \), where \( \tau_0 \) depends on the various parameters of the model and is defined in equation (A.3), appendix A), the entrepreneur's total earnings are maximized by never giving up the wage job. The optimal time-allocation path will be constant over time, except for the first and last periods where the time allocation must adjust to the initial and final conditions. One can verify that the capital investment in the new venture, the production level, and the demand are constant over time (except for the first and last periods). Moreover, the business expenditure will be constant over time, except for the first period and the last two periods where it must adjust to the assumptions for those periods. If the entrepreneur's work tolerance is low (\( \tau \leq \tau_0 \)), the total earnings are maximized by devoting all the time to the venture right after the first period and remaining a full-time entrepreneur until the second-to-last period (again here, the optimal time allocation must adjust to the initial and final conditions on the problem).

Although I cannot algebraically show for this abstract example that returns from the entrepreneurial activity decrease as time invested in it is increased, my results suggest that it is the case. For medium and high work tolerant individuals, the optimal time allocation found for any period corresponds to the strategy described in the last row of Table 2.2. For low work tolerant individuals, the time allocation corresponds to the strategy given by the fourth row of Table 2.2. This observation suggests that the results above apply to situations where decreasing returns on demand from accumulating reputation leads to decreasing returns from the venture...
profit as time invested is increased.

As an example of my discoveries think about a nurse who specializes in home-care for elderly people. The nurse earns a fixed wage per hour and can allocate any number of hours per week to this salaried job. She is interested in opening a beauty-salon business and must thus decide how to allocate time between these two activities. The business is relatively profitable compared to the nursing activity, but more marketing expenditure into the beauty salon will not generate proportionately more buying customers. The effective nurse who has a medium or a high tolerance for work should always be involved in both activities. The less effective nurse should be involved in only one activity by giving up the nursing job and being fully committed to the beauty-salon business.

2.5.3 Constant Business Expenditures

It is often observed that entrepreneurs gradually commit themselves to their new venture. Such behavior can be shown to be optimal under various conditions. Below I consider the case where business expenditures are required to be constant over time.

Assume that the entrepreneur chooses to spend a fixed amount $e$ in each period. For instance, to increase reputation the entrepreneur may contract with an advertising firm that only advertises at a fixed rate on $[0,T]$. The time-allocation problem becomes

$$
\text{Max} \sum_{t=0}^{T} \lambda_t \left[ \rho D(A_t) + w[t - h_t] - e - i_t \right] + \beta A_{T+1}
$$

subject to:

$$A_{t+1} = (1-\delta)A_t + \gamma e, \quad A_0=0$$

$$D(A_t) \leq K(h_t,i_t)$$

$$0 \leq h_t \leq \tau$$

$$e \geq 0, \quad i_t \geq 0$$

$$t = 0,1,2, \ldots, T,$$
where

\[
\beta A_{t+1} = \frac{\beta \gamma}{\delta} [1 - (1 - \delta)^{t+1}] e \quad \text{and} \quad A_t = \gamma \left( \frac{1 - (1 - \delta)^t}{\delta} \right) e, \quad t = 1, \ldots, T.
\]

When business expenditures are constant over the periods, I show in appendix A that the time allocated to the venture will increase over time whenever the returns on demand increase as more reputation is acquired. A low work tolerant entrepreneur will eventually give up the wage job. The rationale goes as follows. By keeping the business expenditure constant, the accumulated level of reputation will increase over time and so will the demand and the time that must be allocated to the new venture. Eventually, the demand will reach the level associated with the optimal time allocation (which will correspond to the total amount of working time for low work tolerant individuals). Once this level has been reached, the entrepreneur will fully commit to the business, inducing a constant optimal time allocation until the end of the planning horizon.

2.6 Conclusions

I explored in this chapter the decision that an entrepreneur must make between allocating time to a wage job and time for starting a new venture. An analytical framework was utilized to derive the optimal time-allocation policy between the two activities both in a single-period setting and a multiple-period setting. The application of the model was restricted to ventures at early stages of development or, alternatively, to firms where outside financing is difficult to obtain. The analytical framework allowed one to characterize two key elements that affect the entrepreneur's decision, namely the entrepreneur's tolerance for work and the rate of change in the return from the new venture as time invested in it is increased.

High work tolerant individuals have opportunities for starting a business that lower work tolerant individuals may not have. If the nature of the business is one in which it takes a
considerable time investment up-front before any results are achieved, the low work tolerant individuals may not have sufficient work time available to reach the critical point. In such situations, I see all-or-nothing strategies. The low work tolerant individuals stay in the wage job and the high work tolerant individuals commit themselves fully to the venture. There is no middle ground in which part of the time is spent in the wage job and part on the venture.

In cases in which the initial time investment in a venture pays immediate benefits but they diminish with additional time spent, I do get intermediate optimal strategies. The low work tolerant individuals will spend all their time on the venture but the high work tolerant individuals will split their time. Indeed, those with the very highest work tolerance will spend a greater proportion of their time on the wage job. These behaviors reflect the fact that, after a certain point, it does not pay to spend more time on the venture. If such time splitting is not observed in practice, it may imply that high work tolerant individuals get immersed in the venture and do not recognize the diminishing returns. Alternatively, an entrepreneur's utility may not only depend on earnings, but also depend on a number of other possible attributes such as need for control and independence. The utilization of such an objective function could better describe an entrepreneur who chooses to be fully committed to the venture although business returns diminish.

Over multiple periods, when the return on demand from reputation decreases, high or medium work tolerant individuals never give up the wage job. When the return on demand from reputation increases and business expenditures are constant over the periods, time allocated to the venture will increase over time and low work tolerant individuals will eventually give up the wage job.
Chapter 3

Effects of New Venture Funding on Product Development

3.1 Introduction

For the Inc. 500 1996 fastest growing companies, starting capital came mostly from personal savings (54%) and the rest came from external sources such as family members (10%), partners (7%), personal charge cards (6%), bank loans (4%), venture capital (4%), friends (3%), mortgaged property (3%), or other (8%). The average amount raised for start-up was $25,000.

Many business start-ups based on the development of a new product require substantial capital. Entrepreneurs must often rely on external funding for the first few months, or sometimes the first few years, as revenues will only appear once the new product is sold. If an entrepreneur chooses to keep a salaried job then fixed wages can be used as investment on the new start-up, however these wages are usually humble. Therefore, the entrepreneur may prefer to quit a salaried job to allocate effort in gathering funding from investors.

Investors offer two types of funding, namely debt financing and equity financing, where both the amount and the timing of capital may be uncertain to the entrepreneur. In this chapter I focus on sole proprietorship. Consequently, I consider business start-ups that are fully self-financed or those that rely on debt financing. I will try to understand how uncertainty in debt financing affects business start-ups, especially those that are heavily based on new product development.

Uncertainties in magnitude and timing of capital investment raise questions regarding the new product development process. The following questions will be addressed in this chapter. How do changes in funding affect the "best" time to release a product? Do business opportunities with large financing potential provide new products with higher quality? Or, do entrepreneurs with the ability to raise large amount of capital provide new products with higher quality? Is it possible that
the product quality at market release decreases as the expected amount of funding increases?

It is assumed that an entrepreneur undertakes new product development aimed at starting a new business. It is also assumed that products of higher quality are preferable to those of lower quality on the long term (everything else being equal). Business revenues thus depend on how good the product is and how fast on can get the product in market. To increase the quality of the product, the entrepreneur must invest financial capital and time, however the entrepreneur’s time will not be an explicit dimension in my framework as it can be bought out. There is thus a tradeoff between the advantage of waiting and improving the product quality versus introducing the product to the market before competition arrives. The entrepreneur must decide when to terminate the development of the new product with the objective of maximizing the success of the business.

Some industries have a high potential for capital investments while others do not. For those industries where capital investment is a scarce resource, the entrepreneur’s ability to raise funding may become an important characteristic. Entrepreneurs with persuasive skills and good networks will raise considerable amounts of money compared to those who do not have these capabilities.

I will identify industrial and entrepreneurial characteristics that generate various functional relationships between product quality and investment. I will also characterize when the new product should be put on the market and how the expected amount of funding affects the time to market for the product.

A theoretical framework prescribing how new venture funding affects new product development will also be developed. Using deterministic and stochastic settings, the framework will describe the optimal release time for the new product. The framework will also study how the release-time strategy is affected by the expected amount of funding and by its variability. In addition, the stochastic framework will allow one to characterize classes of revenue functions under which a simple decision rule prescribes when the entrepreneur should market the product.
The major results are summarized as follows. I utilize a revenue function that is expressed as a (concave) function of product quality added to a (linear) function of competition; the product has diminishing returns to quality improvement and competition has a constant marginal cut-off on revenue. The time to market for the product should decrease as funding increases whenever the marginal gain from investing a constant flow of funding (e.g., wages) in developing the new product is initially large compared to the marginal loss from competition. However, when this marginal gain is initially small, the time to market for the product should increase as funding increases. With respect to product quality, I show that at market release it should always increase as the constant flow of funding increases. When funding becomes uncertain (funds coming from debt financing), the return of investment in terms of product quality depends on the level of investment. If the entrepreneur has the ability to raise a large amount of capital each period, the product quality at market release should decrease as per-period expected amount of funding increases. On the other hand, the product quality at market release should increase as per-period expected amount of funding increases whenever the entrepreneur cannot raise enough capital.

The rest of the chapter is organized as follows. The second section presents a background of related studies while the third section describes the decision problem. In the fourth section, a simple analysis is done in a setting where inputs of funding are known to the entrepreneur to provide a point of comparison with the case of uncertain funding. The decision problem with uncertain funding is investigated in the fifth section and classes of revenue functions are defined for which stopping decision rules that maximize expected profit can be derived. The sixth section concludes the chapter.

3.2 Related Studies

Three major elements of studies in the economic-R&D literature are time to market, new product quality, and resources assigned to develop the product. These elements are interconnected
in that the limitation in development resources creates a relationship of tradeoff between time to market and product quality. This literature categorizes the existing research into decision-theoretic models and game-theoretic models. In the decision-theoretic approach, a dynamic R&D investment problem is treated as a one-firm optimization problem. The optimal R&D expenditures are then characterized over time and, if unknown, the optimal stopping time of the project is determined. Kamien and Schwartz (1982) and Granot and Zuckerman (1991) provide a review of this literature. In the game-theoretic approach, firms compete in the development of a new product. Reinganum (1989) gives a review of the game-theoretic literature.

The approach used in the present chapter is decision-theoretic. The main contribution is on the explicit consideration of a random input of capital and its effect on product quality. I next mention research papers from the market-entry literature and from the optimal-control literature that relate to my focus.

The empirical work of Lilien and Yoon (1990) indicates that "if a pioneer's market entry creates a new product class, entry too early may push an underdeveloped product into the marketplace; however, if entry is delayed too long, the firm may sacrifice the benefits of being first with a new product or technology." This empirical result provides insights on the choice of a functional form for the return function of the new product.

Cohen, Eliashberg, and Ho (1996) study the tradeoff between maximizing new product quality and minimizing time to market. They show that "an improvement in the speed of product improvement does not necessarily lead to an earlier time to market, but always leads to enhanced products." This has been empirically supported by Lilien and Yoon (1990) who rationalize that "if the quality of a follower's new product can be readily improved relative to that of the existing product, then delaying the market entry timing may lead to better market performance."

Deshmukh and Chikte (1977) utilize a semi-Markov decision process to derive dynamic
investment strategies for a risky R&D project where the terminal reward of a new product is a function of its relative quality which can be affected by competition. They show that "the project should be terminated if the current status is too low or too high to make further expenditures worthwhile. Otherwise, for an intermediate (promising) status, an aggressive investment strategy is shown to be optimal;" this result has also been supported by Lilien and Yoon (1990). Deshmukh and Chikte's model allows the decision-maker to choose each period how much investment is allocated to the project. This feature is not necessary for new venture start-ups since all the money will be invested in developing the new product. However, the notion of random inputs of funding (that characterizes entrepreneurial start-ups based on new product development) is not captured by Deshmukh and Chikte's model.

Chi, Liu, and Chen (1996) present an optimal stopping formulation of an investment project that takes an uncertain length of time to develop and can give partial return if not completed. They propose functional forms to characterize the uncertainty and learning potential about the project's completion time, but they do not investigate the effect of changes in funding on product quality at the project's completion.

Granot and Zuckerman (1991) study R&D projects that involve choices of activities at various stages of the development process. They derive the selection procedure of activities and the stopping time of the project that maximize expected discounted net return. They also incorporate learning aspects and resource allocation (e.g., capital investment) into their basic model. They propose a greedy-type procedure to derive the optimal resource allocation of a single-stage R&D project. Here again, randomness in capital investment has not been captured explicitly. However, their research provides some insights in extending my model to multiple business activities at new venture start-ups.
3.3 Problem Description

Consider an entrepreneur who undertakes a new product development activity aimed at starting a new venture. The total stream of revenues at termination of the activity depends on the product quality and the market development (measured by the level of competition) once the product is released. Figure 3.1 shows the relationships between the elements of the new product development project.

The product quality in Figure 3.1 changes as a result of R&D funding allocated to the project. It is assumed that the more the funding, the better the product quality (everything else being equal), and so the larger the expected total stream of revenues. On the other hand, the higher the level of competition, the smaller the expected total stream of revenues. The business performance is measured by the profit from the project, which is obtained by subtracting the total stream of R&D expenditures from the total stream of revenues.

A decision problem of time release for a new product is formulated. Each period, the entrepreneur must compare the gain from further developing the product at a cost (and so achieve a better product quality) to the loss from an increase in competition. It is assumed that the product
does not deteriorate over time. It is also assumed that competition increases over time at a constant rate. To keep the model simple without compromising insightful results, I also assume that the behavior of the R&D funding is identical from one period to another.

In the next two sections, a decision problem is formulated to evaluate when the entrepreneur should release a new product so as to maximize the total expected stream of profit from that product. From this formulation, comparative statics will provide insights on how the optimal time to release the product and the corresponding quality are affected by changes in new venture funding. Special functional forms for the revenue function will be used to make this optimization problem mathematically tractable.

It may be useful here to give a practical example that can be represented by this theoretical framework. Consider the computer programmer of Chapter 2 who now wishes to quit a wage job to develop an optical character recognition (OCR) software. Such software can reach a good level of quality as more capital (and time) is invested in its development. The developer invests her personal savings as initial capital, but it is not enough to reach a marketable quality. Since she insists on staying the sole proprietor of this software business, she must rely on debt financing for more funding. The quality of the OCR software can be measured by the percentage of characters/words in a page that can be correctly identified by the software per unit of time (precision and speed must be considered). The software developer must tradeoff putting on the market a product that has a reasonable precision-speed combination versus waiting to improve the product quality at the cost of having more of these software products on the market (because competition increases over time). The developer must choose when to stop developing the OCR software so as to maximize profits.

3.4 Deterministic Analysis: Time to Product Release and Returns to Investment

I will begin with a model in which the flow of R&D funding is constant and known over time. This setting is better suited to entrepreneurs who keep their salaried job thus generating a
constant flow of capital, or else to large established corporations that internally finance new product
development by assigning a fixed amount of capital investment each period. A simple optimization
formulation will be used to derive the optimal time to release the new product. This section
provides results that will be compared to the results of the stochastic setting of the next section,
which is more representative of business start-ups from entrepreneurs fully involved in new product
development activities.

3.4.1 Formulation and Assumptions

Each period during the conduct of the project, the status (or state) of the project is measured
in terms of the product quality, denoted $q_t$, achieved so far, the level of competition, denoted $c_t$, and
the total R&D funding so far. The level of competition at period $t$ will be interpreted as the
percentage of market shares taken by competition at $t$. This percentage can be estimated by the total
sales of competitors up to period $t$ divided by the maximal potential total sales for the product life.

The product quality changes as a result of R&D expenditures, which is assumed constant
over time and is denoted by $\mu$.\(^5\) It is assumed that each dollar invested will increase the quality of
the product of $\alpha$ units, where $\alpha$ is a positive constant; hence, the product quality will not
deteriorate over time. For the OCR software example, each dollar invested in the development of
the software will increase of $\alpha$ the percentage of characters/words in a page that can be correctly
identified by the software per second. The evolution of the quality of the product is thus
mathematically expressed by

$$q_{t+1} = q_t + \alpha \mu \quad \text{or, equivalently,} \quad q_{t+1} = q_0 + \alpha (t + 1) \mu,$$

where the initial product quality, denoted by $q_0$, is known and corresponds to the quality obtained as
the entrepreneur has invested all of her personal savings in developing the product.

\(^5\) A portion of $\mu$ will be used to cover fixed costs such as equipment, rents, and salaries.
This linear relationship follows from the assumption of a constant marginal effect on product quality from investment. For products that can reach a wide range of qualities over one period, this marginal effect may be increasing as investment increases \((q_t)\) is a convex function of \(\mu_t\), or it may be increasing until the total investment in the product has reached some threshold, and then start decreasing \((q_t)\) is s-shaped). The model will better represent products that do not change drastically in quality so that the evolution of quality can be approximated by a linear function.

The project status also depends on the level of competition which is assumed to increase at a constant rate over time. Its dynamic is expressed by

\[
c_{t+1} = c_t + \theta \quad \text{or, equivalently, } c_{t+1} = c_0 + (t + 1)\theta,
\]

where the initial competition level \(c_0\) is known and will often correspond to zero as entrepreneurs try to introduce products that are not yet available on the market. The parameter \(\theta\) represents the (constant) rate of competition from one period to the next. Products that require relatively low investments for their development and have a huge potential market may see their rates of competition increase over time. On the other hand, products that require huge investments but have a small potential market may see their rates of competition decrease over time. A constant rate of competition is most appropriate for products that require a reasonable amount of funding and have a reasonable potential market.

For each period \(t\), the entrepreneur must choose to stop or to continue the project; if the entrepreneur chooses to continue the project, a fixed amount of dollars, \(\mu_t\) will be allocated to the new product development activity. The entrepreneur will receive revenue \(R(q_t,c_t)\) if she chooses to stop at \(t\), where \(R(q_t,c_t)\) is the expected total stream of revenues associated with quality \(q_t\) and competition \(c_t\). The marginal revenues will increase with respect to quality \((dR/dq_t>0)\) while they will decrease with respect to competition \((dR/dc_t<0)\). One obtains the terminal net expected (non-discounted) profit, denoted by \(P_t\), by subtracting from the terminal revenue the R&D expenditure

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stream, that is,\(^6\)

\[ P_t = R(q_t, c_t) - \sum_{i=1}^{t} \mu = R(q_0 + \alpha t \mu, c_0 + t \theta) - t \mu. \]

The entrepreneur should thus determine time \( t^* \) that maximizes the net expected profit, that is,

\[ P^* = R(q_0 + \alpha t^* \mu, c_0 + t^* \theta) - t^* \mu = Max \{ R(q_0 + \alpha t \mu, c_0 + t \theta) - t \mu \}. \]

To compare the results from the present section to the results of the stochastic setting of section 3.5, I will be using the same revenue function. A special structure for the revenue function is thus utilized to minimize mathematical complexity for the stochastic formulation.

The revenue function \( R \) is assumed to be separable (additively) in product quality and competition level, that is,

\[ R(q_t, c_t) = f(q_t) + g(c_t) \]

where \( df/dq_t > 0 \) (marginal revenues increase with respect to quality) \( dg/dc_t < 0 \) (marginal revenues decrease with respect to competition). This functional form implies that for different competition levels, say \( c^1 \) and \( c^2 \), the corresponding revenue functions will have the same shape with respect to product quality, but the revenue function associated with the lower level of competition will be located above the revenue function associated with the higher level of competition. Similarly, for different qualities, say \( q^1 \) and \( q^2 \), the corresponding revenue functions will have the same shape with respect to competition, but the revenue function associated with the lower level of product quality will be located below the revenue function associated with the higher level of product quality.

The various elements of the model are presented in Table 3.1.

For the OCR software developer, the above assumptions can be translated as follows. The

\(^6\)By not discounting one assumes that the entrepreneur can borrow or save money at a rate corresponding to the inflation rate.
better the software's precision-speed combination, the larger the revenues, and the larger the market share already taken by competitors, the lower the revenues. The software developer must tradeoff between waiting and improving the product quality, and then increase \( f(q_t) \), versus stopping before competition increases, thus forcing \( g(c_t) \) to decrease. This is supported by the empirical findings of Lilien and Yoon (1990), as stated in section 3.2.

### TABLE 3.1

**A Summary of the Various Elements for the Deterministic Model**

| State variables              | \( q_t \) = product quality at period \( t \)  
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td></td>
<td>( c_t ) = level of competition at period ( t )</td>
</tr>
</tbody>
</table>
| Model parameters             | \( \mu \) = fixed R&D expenditures per period  
|                              | \( \alpha \) = quality improvement per dollar invested  
|                              | \( \theta \) = rate of competition            |
| Functions                    | \( R \) = revenue as a function of quality and competition  
|                              | \( P_t \) = net expected profit at period \( t \)  
|                              | \( f \) = effect of product quality on revenue  
|                              | \( g \) = effect of competition on revenue      |

New product development activities differ in the amount of financial payoff that results from different combinations of product quality and competition. Some products require a high quality before they begin to pay off, but when they start paying off it may be very lucrative. This characterizes the case of increasing returns to quality or, when viewed in terms of the functional relationship between the financial payoff and quality, one says that the function \( f \) is convex. Other products give significant immediate payoffs when quality is improved, but then give decreasing marginal payoffs. This is the case of diminishing returns to quality improvement and is represented by a concave function \( f \). In between is the case where every unit of quality improvement yields the same payoff, that is, \( f \) is linear.
Similarly, some markets require a high level of competition before it begins to be detrimental to the introduction of an additional product, but when they do the financial cut associated with the additional product may be very large. This corresponds to the case of negatively decreasing returns (increasing returns in absolute value) to competition or, equivalently, the function $g$ is concave. Other markets give significant immediate financial cuts as competition increases, but then give negatively increasing marginal cuts (decreasing marginal cuts in absolute value). This case corresponds to a convex function $g$. There is also the case where every unit of competition increase yield the same financial cut, that is, $g$ is linear. Other possibilities can be formed from the basic cases such as having increasing returns and then decreasing returns, as represented by a s-shaped function $f$ or $g$.

I focus on situations where the product has diminishing returns to quality improvement ($f$ concave, or, equivalently, $d^2f/dq_i^2 < 0$) and where the financial marginal cut-off from competition is constant ($g$ linear). Consequently, $g(c_t) = -bc_t$ or, since $c_t$ is a linear function of $t$, $g(t) = -b(c_0 + \theta t)$, where $b > 0$ is a scaling parameter.\footnote{Without loss of generality one can choose the intercept in the linear functional relationship between revenues and competition to be 0.} The decision problem is thus formulated by

$$\max \left\{ f(q_i) - bc_0 - t\mu \right\}$$

subject to:

$$q_i = q_0 + \alpha t\mu$$

$$d^2f/dq_i^2 < 0$$

$q_0$ given.

### 3.4.2 Optimal Release Time and Sensitivity Analysis

I derive the optimal release time for the product and investigate what happens to this release time when the expected amount of R&D expenditures, $\mu$, increases. Furthermore, the effects on product quality from an increase in R&D expenditures are characterized.
Since the product quality, \( q_t \), is a linear function of \( t \) and \( f \) is chosen concave in \( q_t \), \( f \) is also a concave function of \( t \). Figure 3.2 presents the function \( f \) with respect to time. It also shows the linear loss on revenue which corresponds to a loss from competition \((b[c_0 + \kappa t])\) added to R&D expenditures \((t\mu)\). The optimal time to release the new product thus occurs when the return from developing the product equals the return from competition and investment costs. Simple geometric considerations do not allow one to draw conclusions on the effects on the release time from an increase in funding because, as \( \mu \) increases, the slope of the total cost function will increase and, also, the function \( f \) will be pushed above the existing one. Hence, it is not clear if the new release time will be located before or after the existing \( t^* \). Similarly, simple derivations (from comparative statics) on a generic function \( f \) do not allow one to draw conclusions on the returns on product quality from additional funding.\(^8\) Consequently, more structures must be given to the decision problem, and this is done by selecting the class of functions from which \( f \) will be selected.

![Figure 3.2 The Effect of Product Quality on Revenue: The Function \( f \)](image)

The developer of an OCR software may expect that returns on revenues from improving the

---

\(^8\)With \( f \) concave and \( g \) linear, one can easily derive that

\[
\frac{d q_t}{d \mu} = \frac{1}{\alpha f'(q_0 + \alpha t^* \mu)} \cdot \frac{\alpha f''(q_0 + \alpha t^* \mu)}{\alpha \mu f''(q_0 + \alpha t^* \mu)},
\]

where the denominator is positive, but the sign of the numerator depends on the explicit form of the function \( f \).
software quality are high at the beginning and decrease as the software offers better precision and does it faster. After some threshold, better software will stop increasing the revenues because potential buyers will not appreciate higher quality. Mathematically, it is translated by a concave function $f$ that levels out after some threshold $Q_0$, that is,\footnote{One could choose $f$ pseudo-concave by requiring that the product reach a minimal quality before generating any revenues.}

$$f(q_i) = \begin{cases}  
  f_{\max} \left[ 1 - \frac{(q_i - Q_0)^2}{Q_0^2} \right], & 0 \leq q_i \leq Q_0 \\
  f_{\max}, & q_i > Q_0
\end{cases}$$

The entrepreneur's objective function becomes

$$\text{Max} \left\{ f_{\max} \left[ 1 - \frac{(q_o + \alpha \mu - Q_0)^2}{Q_0^2} \right] - b(c_o + \theta) - t \mu \right\}.$$

The first and second order optimality conditions provide

$$t^* = \left[ \frac{1}{\alpha \mu} \left( Q_o - q_o \cdot \frac{Q_o^2(b \theta + \mu)}{2 \alpha \mu f_{\max}} \right) \right]^+,$$

with $[.]^+$ representing the positive integer part of its argument; it is zero whenever the argument is smaller than 1. This optimal release time is positive as long as

$$2 \mu \left[ \frac{(Q_o - q_0) \alpha f_{\max}}{Q_0^2} \cdot \frac{1}{2} \right] \geq b \theta,$$

or, equivalently,

$$\mu \left[ \frac{df}{dq} \right]_{q=q_0} \geq \frac{dg}{dt}.$$  

The left-hand side is the marginal gain from investing $\mu$ dollars in the new product given that the quality level is $q_0$. The right-hand side is the (constant) marginal loss from competition.

An OCR-software developer will thus find attractive to move on to the software business ($t^* > 0$) whenever the marginal return from investing all of her personal savings into developing the
software (corresponding to quality $q_0$) is at least as large as the marginal financial loss from competition. Geometrically, this corresponds to a concave function $f$ that is not too flat with respect to low levels of quality.

How does one compare the optimal release time of an OCR-software developer who, by keeping a salaried job, collects high wages from being experienced to an unexperienced developer (collecting low wages) who work at a similar salaried job for the same number of hours? Said differently, how is the optimal release time affected by a variation in R&D expenditures? Since

$$\frac{dt}{d\mu} = \frac{1}{\alpha_0^2} \left[ \frac{Q_o^2}{\alpha f_{\max}} \left( \frac{b\theta + \mu}{\mu} - \frac{1}{2} \right) + q_o - Q_o \right],$$

the optimal release time increases with respect to an increase in R&D expenditures whenever

$$\mu \left[ \frac{(Q_o - q_o)\alpha f_{\max}}{Q_o^2} \cdot \frac{1}{2} \right] < b\theta$$

and decreases otherwise. The left-hand side represents half the marginal gain from investing in the new product given the initial quality level ($q_0$) while the right-hand side represents the marginal loss from competition. These results are summarized in Proposition 3.1.

**PROPOSITION 3.1** The time of the product to market should decrease as funding increases whenever the marginal gain from investing in the new product is initially large compared to the (constant) marginal loss from competition. When the marginal gain from the investment is initially small compared to the (constant) marginal loss from competition, the time of the product to market should increase as funding increases.

Figure 3.3 shows how the optimal time to release the product can be affected by an increase in funding, with $\mu_1 < \mu_2$. In this specific realization, an increase in funding provides a shorter time to market the product ($t_2^* < t_1^*$), suggesting that the marginal gain from investing in the new product is initially large compared to the marginal loss from competition. The intuition goes as follows. The
new stream of marginal gains produced by an increase in funding will be decreasing relatively fast compared to the decreasing rate of the stream of marginal gains produced before the increase in funding. Consequently, the marginal losses from competition and capital reimbursement will take over the marginal gain faster than before the increase in funding. Hence, the entrepreneur should introduce the product to the market earlier.

![Figure 3.3 The Effect of an Increase in Funding on Time to Release the Product: A Realization where $t_1^* > t_2^*$](image)

Assume now that the marginal gain from investing is initially small compared to the loss from competition (but high enough to make the R&D project worth it). The new stream of marginal gains produced by an increase in funding will thus be slightly under the stream of marginal gains produced before the increase (because $f$ will be relatively flat). Therefore, the gain from developing further the product will still overcome the loss inducing a delay in the optimal time to release the new product as presented in Figure 3.4. Said differently, one needs a product of better quality to overcome the negative effect from increasing competition. This behavior has been empirically validated by Lilien and Yoon (1990) when a follower's new product quality can be significantly improved compared to the current quality available on the market.
As an example of my findings consider a robotics engineer who wishes to build a vacuum-cleaner robot. By keeping a salaried job, the engineer generates over time a constant flow of funding that is allocated to the development of the robot. The quality of the robot can be measured by a combination of percentage of dirt removed and percentage of surface covered. If, for instance, the engineer focuses on the household market, then the competition level can be measured by the number of households that already have a similar robots (it may be zero at $t=0$) divided by the total number of households that have a vacuum cleaner (robot or not). In a market where competition is growing slowly, the more experienced engineer generates higher wages that will enable her to introduce to the market the robot quicker than a less experienced engineer with low wages would. However, in a highly competitive market, an experienced engineer would be advised to introduce to the market her robot later than a less experienced engineer would, as the experienced engineer has a significantly higher rate of improvement in the product quality.

Cohen, Eliashberg, and Ho (1996) show that increasing the speed of product improvement does not always induce an earlier time to market, however, it always provides a product of higher quality. In my model, an increase in the speed of product improvement is expected when more
funding is available. Proposition 3.1 also claims that such an increase will not always lead to an earlier time to market. Proposition 3.2 below will suggest that such an increase will also lead to an enhanced product, thus supporting the findings of Cohen, Eliashberg, and Ho.

The product quality at the suggested optimal release time is

\[ q^* = q_0 + \alpha t^* \mu = Q_0 \cdot \frac{Q_0^2 (b \theta + \mu)}{2 \alpha \mu f_{\text{max}}} \]

The effect of funding on product quality is thus

\[ \frac{dq^*}{d\mu} = Q_0^2 \left[ \frac{b \theta + \mu}{\mu} - 1 \right], \]

which is always positive. Hence, the level of product quality that maximizes the expected net profit increases as the amount of funding invested in the new product development increases. This result is stated as Proposition 3.2.

**Proposition 3.2** As the (constant) amount of funding invested in new product development increases, the product quality at market release should also increase.

The rationale goes as follows. If more money is invested in new product development, the rate of capital reimbursement per unit of quality stays fixed, but the rate of competition per unit of quality decreases (because it is quicker to reach any given level of quality). Therefore, the product quality at market release cannot be worse.

A computer programmer with high wages would generate more funding each period, thus releasing an OCR software of better quality than a computer programmer with low wages would. Equivalently, an experienced robotics engineer would accumulate (through a salaried job) more funding, thus introducing to the market a more reliable vacuum-cleaner robot than a less experienced engineer would.
3.5 Stochastic Analysis: Funding Thresholds in New Product Development

The deterministic analysis of the preceding section provided grounded results for situations where the flow of R&D funding (expenditures) was constant over time. This was most appropriate for entrepreneurs who kept their salaried job. In this section, the same conceptual model is used, but the behavior of the flow of R&D funding will be modeled by a stochastic process. This framework will better represent business start-ups where the entrepreneur must rely on investors’ willingness to provide (debt) financing. I will study the effects of variations in funding on the quality of the product at time to market.

3.5.1 Formulation and Stopping Decision Rules

The deterministic formulation of sub-section 3.4.1 is adapted to a stochastic setting. The product quality is expressed by

\[ q_{t+i} = q_i + \alpha Z_{t+i} \quad \text{or, equivalently,} \quad q_{t+i} = q_0 + \alpha \sum_{i=1}^{t+i} Z_i, \]

where the initial product quality \( q_0 \) is known and may still correspond to the quality reached once the entrepreneur has invested all of her personal savings. The parameter \( \alpha \) is still the positive effect on product quality from each dollar invested in the new product development. The variable \( Z_i \) represents the random input of funding at period \( t=i \). The \( Z_i \)s are assumed to be independent identically distributed random variables on \( [0, \infty) \) of mean \( \mu_Z \) and variance \( \sigma_Z^2 \).

In the previous deterministic setting the entrepreneur could evaluate over time the stream of product qualities. Here, the stream becomes a stochastic process, and at any given time the entrepreneur cannot anticipate with certitude the product quality. An even more realistic model would additionally involve randomness in quality not attributed to funding, but to the general

\[ \text{\footnotesize{\textsuperscript{10}} One can argue that the assumptions on the } Z_i \text{s are too strong because venture capitalists have more incentive to invest in ventures that have already got prior funding from other investors. However, for mathematical tractability reasons, I keep those assumptions on the } Z_i \text{s.} \]
industrial environment.

The total (non-discounted) expected net profit if the R&D project is terminated at time \( t \) is

\[ E \left[ R(q_t, c_t) - \sum_{i=1}^{t} Z_i \right] \]

which is a function of \( q_0, c_0, \{Z_t, Z_{t-1}, \ldots, Z_1\} \) and where \( c_t \) is the level of competition as defined by the deterministic analysis. The expectation is taken according to the distribution of the random variables \( \{Z_t, Z_{t-1}, \ldots, Z_1\} \).

I will derive classes of revenue functions where an optimal stopping lemma due to Derman and Sacks (1960) can be applied to characterize the optimal stopping rule (see e.g., Chow, Robbins, and Siegmund (1971) for further reference on the theory of optimal stopping). In the next subsection, I will derive such a rule analytically for the revenue function of sub-section 3.4.2 and compare my results. Later, an alternative approach to characterize generic classes of revenue functions for which the lemma can be applied will be proposed.

Derman and Sacks’s result is presented in Lemma 3.1. A myopic property about future rewards makes the intuition behind this result quite clear: the decision-maker should stop her activity (such as developing a new product) whenever there is no expected gain from continuing that activity one more period.

**LEMMA 3.1 (Derman & Sacks, 1960)** Let \( \{\mathcal{F}_t, t=1,2,\ldots\} \) be a sequence of \( \sigma \)-fields of a sample space \( \Omega \) with \( \mathcal{F}_t \subseteq \mathcal{F}_{t+1}, t=1,2,\ldots \). Let \( \{P_t\} \) be a sequence of random variables with \( P_t \) measurable with respect to \( \mathcal{F}_t \) and such that \( EP_t \) exists and is finite for all \( t \). Let \( \mathcal{Y} \) be the class of all stopping rules such that \( EN < \infty \). If there exists a stopping rule \( N^* \) with

(i) \( EN^* < \infty \)

(ii) \( E[P_t | \mathcal{F}_{t-1}] \geq P_{t-1} \) when \( t \leq N^*(x) \)

\( \leq P_{t-1} \) when \( t > N^*(x) \)

for almost all \( x \in \Omega \) and if there is some \( \zeta < \infty \) such that

(iii) for all \( t \), \( E[|P_{t+1} - P_t| | \mathcal{F}_t] \leq \zeta \).
then

\[ EP^*_N = \text{Max}_{N \in \Psi} EP_N \cdot \]

In the context of the present chapter, the \( \sigma \)-field \( \mathcal{F} \) is generated by the random funding \( Z_t, \ldots, Z_t \). The random variable \( P_t \) is the terminal net (non-discounted) profit, that is, the total stream of profit from introducing a product of quality \( q_t \) to a market that has reached a competition level of \( c_t \). The total profit is mathematically expressed by

\[ P_t = R(q_t, c_t) - \sum_{i=1}^{t} Z_i. \]

Condition (iii) of the lemma holds as long as the expected increment in profit from one period to the following one is bounded. This condition is satisfied when the identically distributed non-negative random variables \( Z_s \) are bounded and the revenue function, \( R \), is also bounded.

Condition (i) will be satisfied when the incremental benefit from improving the product quality becomes eventually smaller than the incremental loss from competition and R&D expenditures. Since competition increases at a constant rate, condition (i) holds when the marginal effect on revenue from improving the product quality decreases as quality increases (i.e., \( R \) is concave in \( q_t \)) and when this marginal effect becomes small enough so that the loss from competition and R&D expenditures takes over. I will make the appropriate assumptions on the revenue function so that the expected optimal development length of the new product will be finite.

Condition (ii) of Lemma 3.1 is used to determine the optimal stopping rule and is quite intuitive. The condition suggests that the entrepreneur stops developing the product once the profit from continuing investing in the product one more period and then stop is smaller then stopping the development now. When \( g(c_t) = -bc_t \) with \( b > 0 \) and \( c_t = c_0 + \theta t \) (as in sub-section 3.4.1), condition (ii) becomes
\begin{align*}
(ii)' \quad & E[f(q_t + aZ_{t+1}) - f(q_t) \mid q_t] 
\geq & \quad b\theta + \mu_Z \text{ when } t \leq N^* \\
< & \quad b\theta + \mu_Z \text{ when } t > N^*,
\end{align*}

since \(q_t\) is a sufficient statistic for \(\{Z_1, Z_2, \ldots, Z_t\}\) and \(\mu_Z\) is the mean of the variable \(Z_t\). The expectation is taken with respect to \(Z_{t+1}\).

Let \(F_Z\) be the distribution function of \(Z_t\) (which is the same for any \(t\)) and consider the equation

\[ E[f(q_t + \alpha Z_{t+1}) - f(q_t) \mid q_t] = \int_{-\infty}^{\infty} f(q_t + \alpha z) dF_Z(z) - f(q_t) = b\theta + \mu_Z. \]  \hfill (3.1)

I show next that for a specific function \(f\) a simple optimal stopping rule that satisfies (ii)' is defined by

\[ N^* = \sup \{ t : q_t = q_0 + \alpha \sum_{i=1}^{t} Z_i \leq QT \} \]

where the "quality threshold" denoted by \(QT = QT(a, b, \theta, \mu)\) corresponds to the quality \(q_t\) that satisfies equation (3.1).

### 3.5.2 Effects of Funding Levels on Optimal Product Quality

As for the deterministic analysis, I choose

\[ f(q_t) = \begin{cases} 
  f_{\max} \left[ 1 - \frac{(q_t - Q_0)^2}{Q_0^2} \right], & 0 \leq q_t \leq Q_0 \\
  f_{\max}, & q_t > Q_0
\end{cases} \]

It follows that

\[ E[f(q_t + \alpha Z_{t+1}) \mid q_t] = \int_{0}^{Q_0} f_{\max} \left[ 1 - \frac{(q_t + \alpha Z_{t+1} - Q_0)^2}{Q_0^2} \right] dF_Z + \int_{Q_0}^{\infty} f_{\max} dF_Z \]

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Whenever the distribution function $F_Z$ has a small right tail (that is, for any $t$, the variable $Z_t$ has a negligible probability of being larger than some threshold), the integral term

$$
\int_0^\infty f_{\max} \left( I - \frac{(q, + \alpha Z_{t+1} - Q_0)^2}{Q_0^2} \right) dF_Z + \int_{\frac{Q_0}{\alpha(Q_t Q_0)}}^\infty f_{\max} \frac{(q, + \alpha Z_{t+1} - Q_0)^2}{Q_0^2} dF_Z
$$

go to zero as $Q_0$ goes to infinity. Hence, for $Q_0$ large enough,

$$
E[f(q, + \alpha Z_{t+1}) - f(q,)] \approx E[f_{\max} \left( I - \frac{(q, + \alpha Z_{t+1} - Q_0)^2}{Q_0^2} \right) I] - f_{\max} \left( I - \frac{(Q_t - Q_0)^2}{Q_0^2} \right)
$$

$$
= \frac{f_{\max}}{Q_0^2} \left( 2\alpha q, Q_0 \right) \mu_Z + \alpha^2 (\sigma_Z^2 + \mu_Z^2)
$$

where, for any $t$, $\mu_Z$ is the mean and $\sigma_Z^2$ is the variance of variable $Z_t$.

Condition (ii) of Lemma 3.1 (or, equivalently, (ii)') thus becomes

$$
\frac{f_{\max}}{Q_0^2} \left[ 2\alpha (q, - Q_0) \mu + \alpha^2 (\sigma^2 + \mu^2) \right] \geq b\theta + \mu_Z \text{ when } t \leq N^*
$$

$$
< b\theta + \mu_Z \text{ when } t > N^*,
$$

which is satisfied if

$$
N^* = \sup \{ t : q, \leq QT = Q_0 \cdot \frac{1}{2\alpha\mu_Z} \left[ \frac{(b\theta + \mu_Z) Q_0^2}{f_{\max}} + \alpha^2 (\sigma_Z^2 + \mu_Z^2) \right] \}
$$

Therefore, a robotics engineer would continue investing the random capital into developing a vacuum-cleaner robot until the efficiency of the robot (a combination of percentage of dirt removed and percentage of surface covered) reaches some threshold denoted by $QT (QT \leq Q_0)$. 
How does one compare the quality (or efficiency) at market release of a robot developed by an engineer who is skilled with respect to raising capital to the quality of a robot developed by an engineer who is unable to raise considerable funds? Said differently, what does happen to the quality threshold as the expected amount of funding, $\mu_Z$, increases? Since

$$\frac{\partial QT}{\partial \mu_Z} = \frac{1}{2\alpha \mu_Z^2} \left[ \frac{b\theta Q_0^2}{f_{\text{max}}} + \alpha^2 (\sigma_z^2 \cdot \mu_Z^2) \right],$$

it follows that $\partial QT/\partial \mu_Z > 0$ if and only if

$$\mu_Z < \left[ \frac{b\theta Q_0^2}{\alpha^2 f_{\text{max}}} + \sigma_z^2 \right].$$

Consequently, the quality threshold increases as the expected funding increases whenever the ability for the entrepreneur to generate funding is low. When the entrepreneur can raise a considerable amount of funding, the quality threshold decreases as expected funding increases. These results are outlined in Proposition 3.3.

**Proposition 3.3** When the flow of funding is uncertain, the returns on product quality with respect to expected funding depend on the level of funding. If the entrepreneur's ability to generate funding is low, then the product quality at market release should increase as the expected per period amount of funding increases. If the entrepreneur's ability to generate funding is high, then the product quality at market release should decrease as the expected per period amount of funding increases.

These findings can be rationalized as follows. When the entrepreneur's ability to raise capital is low, the entrepreneur makes considerable gains from the "big hits" (i.e., large inputs of funding on any given periods). Consequently, the entrepreneur has a high incentive to develop the product more as funding increases in hope of getting a big hit. This result is supported by Cohen, Eliashberg, and Ho (1996) who show that an increase in the speed of product improvement
(corresponding here to an increase in funding) always leads to enhanced products, regardless of the expected amount of funding. Figure 3.5 represents that result and it corresponds to Figure 3.4 where the horizontal axis is interpreted as the product quality. This equivalence holds because product quality is linear in time. A low expected amount of funding is characterized by a small slope for the "loss line".

Figure 3.5 The Effect of an Increase in Funding on Product Quality:
A Realization where $q_i < q_2$

If the entrepreneur is expected to raise a large amount of funding, then the stream of marginal gains produced by an increase in funding will be decreasing relatively fast compared to the stream of marginal gains before the increase in funding (the realization of $f$ will be fairly steep for $q$ small). Consequently, the marginal losses from competition and capital reimbursement will take over the marginal gain faster than before the increase in funding. Hence, to succeed the entrepreneur should introduce the product to the market sooner, which translates into a diminished product. Figure 3.6 geometrically represents this result.
Proposition 3.3 also applies when the product's industry is characterized with respect to financing potential rather than the entrepreneur's ability to raise money. One would then conclude from the model that for a robotics engineer located in North America, where the robotics industry has a fair amount of wealthy investors, the quality of the robot at market release should decrease as the engineer gathers more money. However, an OCR-software developer in a third world country may encounter less funding opportunities and would thus be advised to introduce the software to the market at a quality that increases as funding increases.

My framework thus suggests that an increase in funding will not necessarily lead to an enhanced product in industries where funding is uncertain and expected to be high. This observation challenges Cohen, Eliashberg, and Ho (1996) findings.

One should finally notice that the quality threshold, $QT$, decreases as the uncertainty in funding, $\sigma^2$, increases. Therefore, the more the uncertainty in funding, the worse the quality of the product at time to market. As uncertainty increases, the chance of having a big hit decreases (although the magnitude of the hits may increase). Hence, the loss encountered by an increase in competition would be larger than the expected gain from continuing the development of the product, forcing the entrepreneur to market the product at a lower quality level. Proposition 3.4
summarizes the result.

**PROPOSITION 3.4** The product quality at market release should decrease as the uncertainty in funding increases.

As an additional example of my results consider a chemist who tries to develop a paint that can be applied under freezing temperatures. The quality of the paint can be measured by how long it takes to peel off. A chemist who keeps a salaried job thus generating a constant flow of funding (sufficient to get the product marketable) would, regardless of her ability to raise money from outside investors, market the paint at a quality that increases as wages increase. On the other hand, a chemist who gives up a salaried job to solely rely on debt financing as funding would market the paint at a quality that increases as expected funding increases only if she is unable to raise a considerable amount of money from the investors.

**3.5.3 Technical Extensions**

In the preceding sub-section I derived the optimal stopping rule of a small group of revenue functions thus limiting my analysis. I wish to extend, at least technically, my analysis to larger classes of revenue functions. An alternative approach is proposed, various assumptions on the revenue function are made and Lemma 3.1 is applied.

Let \( \Delta P(q_t,c_t) \) be the expected incremental profit from continuing investing in the R&D project one more period and then stop compared to stopping now. Formally, the expected incremental profit at period \( t \) is

\[
\Delta P(q_t,c_t) = E[R(q_{t+1},c_{t+1}) - \sum_{i=t+1}^{t+i} Z_i | Z_1, Z_2, \ldots, Z_t] - \{R(q_t,c_t) - \sum_{i=t}^{t+i} Z_i \}
\]

\[
= E[R(q_{t+1},c_{t+1}) | q_t] - R(q_t,c_t) - \mu_Z,
\]

since \( E[Z_{t+1}] = \mu_Z \) and \( q_t \) is a sufficient statistic for \( \{Z_1, Z_2, \ldots, Z_t\} \). The expectation is taken with
respect to the random variable \( Z_{t+1} \).

Condition (ii) of Lemma 3.1 will be satisfied when the expected incremental profit is such that \( \Delta P(q_0, c_0) \) is positive, there exists \( \tau \geq 0 \) such that \( \Delta P(q_0, c_t) \) is negative, and \( \Delta P(q_t, c_t) \) is negative for any \( t \geq \tau \). One can easily induce from these conditions on \( \Delta P(q_t, c_t) \) that once the expected incremental profit becomes negative there is no hope for increasing expected profit, and so the entrepreneur should stop developing the product.

What assumptions should one make on the revenue function \( R \) so that condition (ii) holds? Let me start with a revenue function that is additively separable in its arguments, as used in the preceding sub-sections, that is, \( R(q_t, c_t) = f(q_t) + g(c_t) \). Hence, the expected incremental profit function can be expressed by

\[
\Delta P(q_t, c_t) = \Delta f(q_t) + \Delta g(c_t) - \mu Z_t
\]

with \( \Delta f(q_t) = E[f(q_{t+1}) - f(q_t) \mid q_t] \) and \( \Delta g(c_t) = E[g(c_{t+1}) - g(c_t) \mid c_t] \).

It is natural to choose a revenue function that increases as product quality increases (i.e., \( df/dq_t > 0 \)) and decreases as competition increases (i.e., \( dg/dc_t < 0 \)), as assumed earlier. Condition (ii) of Lemma 3.1 will be satisfied when \( \Delta f(q_t) \) (which is positive since the state component \( q_t \) increases as \( t \) increases) decreases with respect to \( q_t \), while \( \Delta g(c_t) \) (which is negative since the state component \( c_t \) increases as \( t \) increases) stays constant with respect to \( c_t \). This requirement is satisfied when choosing \( f \) to be an increasing concave function of \( q_t \) and \( g \) to be a decreasing linear function of \( c_t \), say \( g(c_t) = a - bc_t \).\(^{11}\) However, for condition (i) to be also satisfied (i.e., a finite development length is required), these requirements are not sufficient. One also needs that \( \Delta f(q_t) > -\Delta g(c_t) + \mu = b \theta + \mu \) for \( t \) smaller than some finite time \( \tau \) and \( \Delta f(q_t) \) must eventually become smaller than \( b \theta + \mu \) (i.e., \( f \) must

\(^{11}\)\( f \) increasing and concave is sufficient because \( q_{t+1} = q_t + aZ_{t+1} \geq q_t \). Since the level of competition is assumed to increase at a constant rate over time, \( g \) is a decreasing linear function of \( t \).
becomes sufficiently flat).

The above description includes the revenue function used in the preceding sub-section as a particular case. A class of more generic revenue functions has just been constructed where Derman and Sacks' lemma applies and shows the existence of an optimal stopping rule. This class of revenue functions is defined by

$$E = f(q_t, C_t) = f(q_t) + g(c_t)$$ where \( g(c_t) = a - bc_t, \quad b > 0, \quad f > 0, \quad \frac{df}{dq_t} \geq 0, $$

$$\frac{df}{dq_t} > b \theta + \mu_z \text{ for small } t, \quad \frac{df}{dq_t} < b \theta + \mu_z \text{ for large } t, \quad \frac{d^2 f}{dq_t^2} \leq 0. $$

If I choose a revenue function that is multiplicatively separable in its arguments, that is, \( R(q_t, c_t) = f(q_t) \cdot g(c_t) \), then it is straightforward verifying that condition (ii) is satisfied (and thus an optimal stopping rule exists) by making the assumptions presented in \( \Xi \) on the functions \( \log f \) and \( \log g \). A second class of revenue functions where Derman and Sacks' lemma applies thus corresponds to the following description: the return from the product quality, \( f \), must be an increasing log-concave function of product quality and the return from competition, \( g \), must be a decreasing log-linear function of competition. Moreover, the incremental gain from improving quality must be larger than the incremental loss from competition and R&D expenditure at early development, but it must become smaller later on (and thus condition (i) is satisfied).

In some industries, the returns from product quality may be s-shaped with respect to product quality. This means that returns may be constant and small for low levels of product quality and then, for intermediate levels, these returns increase until the product quality reaches some threshold after which the returns start to decrease. A third class of revenue functions where Derman and Sacks' lemma could be applied would restrict those s-shaped functions as follows: revenue is additively separable in product quality and competition \( (R = f + g) \), for small levels of product quality the marginal returns from product quality are constant and larger than the marginal loss from
competition \( (df/dq_i = constant > -dg/dc_i) \), for intermediate levels of product quality the marginal returns from product quality increase as product quality increase \( (df/dq_i \ increases) \), and for high levels of product quality the marginal returns from product quality decrease as product quality increase \( (df/dq_i \ decreases) \).

### 3.6 Conclusions

I explored in this chapter the effects of funding on operational decisions in new product development, especially for entrepreneurs who choose to start a venture based on the development of these new products. The effects on time to market the product and its quality at market release were particularly investigated. An analytical framework was utilized to derive the optimal release time for the new product and to characterize how the release-time strategy was affected by the expected level of funding and its randomness. I utilized a revenue function where the product had diminishing returns to quality improvement and, independently, competition had a constant marginal cut-off on revenue. The entrepreneur's ability to raise R&D funding was shown to be the key element that affects the entrepreneur's decision regarding the best time to market the product.

Entrepreneurs who keep a salaried job, thus generating over time a constant flow of funding for their R&D project, should see the quality of their products at market release increase as their wages increase, independently of their ability to raise debt financing (equity financing was not an option). However, entrepreneurs who give up their salaried job, thus relying on uncertain funding from debt financing exclusively, should see the quality of their product at market release be affected by their ability to raise money.

Entrepreneurs who are unable to raise a considerable amount of capital should have a lot to gain from continuing developing the product in anticipation of a considerable amount of extra funding. I argued that the quality of the product at market release should increase when extra investment in developing the new product is expected. Nevertheless, entrepreneurs who are able to
raise a considerable amount of capital should see the quality of their products at market release
decrease as the expected amount of funding increases because the effect of competition becomes
more destructive than the gain from developing further the product. The deterministic effect of
competition basically forced the entrepreneur to eventually stop developing the product as not much
gain was accumulated after reaching some level of product quality.

I have discussed a simple model in the context of starting a new venture based on the
development of a new product both in a deterministic setting and in a stochastic setting. A
deterministic framework supported results from existing literature in that increasing the speed of
product improvement (which occurs when increasing the expected amount of R&D funding) does
not always induce an earlier time to market, however, it always provides a product of higher
quality. A stochastic framework suggested that an increase in funding would not necessarily lead to
an enhanced product in contexts where funding is uncertain and expected to be high, thus providing
new results to investigate empirically.
Chapter 4

Prospecting for Customers in New Venturing

4.1 Introduction

The first test of success as an entrepreneur is to have a paying customer. To reach the stage where someone is interested in buying the product or service is a major task. However, as exciting as the first customer might be, a business will not be successful unless it can build up a customer base that will generate future revenues. This chapter studies the decisions that an entrepreneur must make in allocating time to building and exploiting a customer base so as to maximize profit. Such allocation decisions can be very difficult and the success of the enterprise depends on making them correctly.

Before the first sale, the market consists of only "potential" customers. As purchases are made, customers are added to the "purchased" category. Individuals from this category will be called "loyal" customers. The product or service considered is a non-durable good and so each individual may repeat their purchase. The entrepreneur has to decide how to allocate time in the face of different costs and/or benefits accruable from contacting potential customers versus contacting loyal customers who have made purchases.

I assume that at any point in time each customer is either potential or loyal and that the entrepreneur can make only a limited number of contacts due to time restrictions. Typically, the immediate expected revenue from contacting a potential customer is lower than the immediate expected revenue from contacting a loyal customer (mainly because the probability of making a purchase is lower for a potential customer than for a loyal customer). Why then spend time with the less profitable category? Obviously, because the only way to have loyal customers is to convert them from the potential customer category through making a purchase. By incurring
some losses in the short run one hopes to make higher profits in the long run. However, if an entrepreneur incurs large short run losses then there may not be any long run. One thus needs to look at the funds available to cover short term losses, and to estimate the long run benefits of spending those funds on contacting customers.

I use various formal models to derive results that make theoretical and practical sense. Taking a descriptive view, I study what a rational entrepreneur will do when faced with the allocation of time to different customer categories. Taking a prescriptive view, I provide guidelines for improving the performance of a "real" entrepreneur who may not be acting optimally. Although this investigation is addressed to entrepreneurs, the results can also be applied to any firm wishing to implement contact strategies.

The key results can be summarized as follows. On an infinite planning horizon, an expected gain is evaluated for each contact, taking into account that contacted loyal customers can only generate immediate revenue while potential customers are expected to generate revenue at every period following the contact. The optimal policy thus takes a simple form: to decide on the next customer to contact, choose from those available who have maximum expected gain. When a constraint on cash balance is imposed each period, I utilize a numerical example to show how this optimal contact policy must be adjusted. On a finite planning horizon, the short run losses from contacting potential customers with negative immediate expected profits are not covered by the long run expected benefits when contacts are made at periods relatively close to the end of the planning horizon. Therefore, critical time points after which no more potential customers should be contacted are derived. I also obtain critical time points after which loyal customers are more profitable to contact than potential customers.

The chapter is organized as follows. The second section presents a background of related studies while the third section describes the problem of prospecting for customers. In the fourth section, a simple discrete-time Markov decision model is utilized to derive optimal time
allocations on an infinite planning horizon. Optimal contact policies are drawn in two different settings created by short run considerations. On a finite planning horizon, optimal contact policies can be derived conditional on the growth of the loyal customer base. To identify those optimal policies, the fifth section formulates the decision problem as a continuous-time deterministic optimal control model. The sixth section provides a few extensions of the Markov decision model while the seventh section concludes the chapter.

4.2 Related Studies

In this section I describe how the development of Markov decision process theory was inspired by the development of a customer base. I then present studies from the marketing literature that relates to my focus.

Howard (1978) describes his work in the 1950s on decision-making involving the development of a customer base. A study was undertaken for the catalog division of Sears, Roebuck and Company to review the operation by which they send catalogues to present and prospective customers. He modeled the problem in a stochastic environment by creating the notion of Markov decision processes, and derived optimal catalog mailing decision rules based on purchasing behavior of Sears' customers.

Markov processes have long been of interest to marketers, particularly in the context of frequently purchased goods. Massy, Montgomery, and Morrison (1970) is a classic reference where stochastic modelling is utilized to describe buying behavior from one period to another. In one respect their models are more general than mine in that each individual may have his own probability of purchasing; however, they do not address any decisional issues allowing the firm to optimize payoffs. Schmittlein, Morrison, and Colombo (1987) and, more recently, Schmittlein and Peterson (1994) use data on customer's past purchases to infer future purchasing behavior. Although these models allow the size and growth rate of a customer base to be monitored, the
pattern in repeated purchases of a new product to be evaluated, and a subgroup of customers for advertising and promotions to be targeted, it does not directly address optimal decision-making issues. In the concluding chapter, I will suggest how their approach could help to generalize my model.

The sales force literature has a history of looking at loyal and potential customers, starting with Lodish's (1971) Callplan model. This model addresses the problem of allocating a salesperson's time between customers and prospects in his territory by allowing customers to require different amount of time. My model will also allow potential and loyal customers to require different amount of time, but at the expense of having fractional buyers. Callplan does not incorporate time dynamics into its allocation system. I thus provide a basis to understand time dynamics better, as well as some insight into the nature of carryover effects as they relate to sales calls. Allocation models in the sales force operations literature are thoroughly reviewed in Vandenbosch and Weinberg (1993).

Closely related to my formulation is the work of Bitran and Mondschein (1996) who studied mailing decisions in the catalog sales industry using a sophisticated mathematical model with stochastic responses from potential buyers and with dynamic evolution of a customer base. Their customer base is divided according to a recency, frequency, and monetary (RFM) classification, which is also used to define the dynamics of their system. Their model includes a mailing-budget constraint, periodical cashflow and inventory. A heuristic based on a simplified version of the model is developed and shown to give satisfactory results. I suggest a formulation that is more restricted than Bitran and Mondschein's. Nonetheless, my approach is more intuitive in that the lifetime value of each potential buyer can be evaluated so that each possible contact can be ordered and the exact optimal policy derived.

The advertising literature proposes normative guidelines to prospecting for customers in new venturing. Advertising diffusion models use deterministic dynamic optimal control
formulations to study effort allocation to customer categories. Muller (1983)'s dynamic model of a new product introduction is closely related to my model. His model is based on a diffusion process which makes the distinction between increasing awareness and changing predisposition to buy. It is more general than my model in that three categories of customers are considered and two types of (marketing) effort are allocated. However, its complex structure does not allow one to derive the specific results I propose. My model also offers something new in that potential buyers must be contacted individually, rather than as a mass-market.

4.3 Problem Description

Time is allocated to contacting individuals from the entrepreneur's loyal and potential customer bases. A contact may take many forms including a phone call, a letter, an electronic mail, or even an on-site visit. It is assumed that a contact takes the same amount of time for each type of customers; however, I will argue in section 4.6 that this assumption can be relaxed at the expense of having fractional customers. The total number of contacts an entrepreneur can make each period will thus be fixed and, because of time restrictions, it will also be limited to a positive integer, denoted by $N$. The product or service sold is non-durable so that each individual may repurchase at any given period.

The immediate expected profit from contacting a loyal customer, denoted by $\pi^L$, is typically larger than the immediate expected profit from contacting a potential customer, denoted by $\pi^P$. The immediate expected profit from not contacting a loyal customer, denoted by $\pi_L$, is positive since a proportion of loyal customers is expected to buy the product again without being contacted. Table 4.1 summarizes the four possible immediate expected profits.
TABLE 4.1
Immediate Expected Profits

<table>
<thead>
<tr>
<th></th>
<th>Contacted</th>
<th>Non-Contacted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loyal Customer</td>
<td>( \pi_c^L )</td>
<td>( \pi_c^L )</td>
</tr>
<tr>
<td>Potential Customer</td>
<td>( \pi_c^P )</td>
<td>0</td>
</tr>
</tbody>
</table>

It is assumed that knowledge about the entrepreneur's product or service is obtained only from contact with the entrepreneur. I thus focus on new ventures where advertising or word-of-mouth is negligible. For instance, it is the case with new software companies introducing their products through demos that are presented to potential clients competing with each other.

It is also assumed that potential customers who do not buy in the period during which they are contacted forget about the contact. For a software company it could mean that a new demo is required for whoever has not acquired the product yet, as the product is significantly upgraded from period to period. However, loyal customers who have already bought the product are constantly informed about upgrading and do not need a demo to repeat purchases.

Each contact of a potential customer has the same probability of success, denoted by \( \beta \), regardless of past contacts.\(^{12}\) Hence, a potential client who has seen a demo of an earlier version of the product offers by the software company would have the same propensity to buy the product as a potential client who has never seen the company's products would. A stationary response probability is relevant in a context where the offered product or service evolves from period to period. Figure 4.1 shows flows of current buyers. Unless they are contacted again, potential customers never purchase the product. Any current buyer who was not part of the loyal customer base is added to that base.

\(^{12}\) This is a conservative property that can be relaxed by using a multi-class model. Section 4.6 will discuss a particular case of multi-class models.
Direct Contact        No Contact

\[
\begin{array}{c}
\text{loyal customer base} \\
<1 \uparrow 1 \downarrow \beta \downarrow <1 \\
\text{current buyers} \\
\beta \\
\text{potential customer base}
\end{array}
\]

Figure 4.1 Customer Flow Diagram

In the next few sections, various formulations are used to find the decisions that an entrepreneur must make in allocating time to building and exploiting a customer base so as to maximize profit. The analysis will be first made on a long-term planning horizon without accounting for short-term losses. Second, the analysis will consider these short-term losses to insure survival on the long run. Finally, the time-allocation decisions will be made on a short-term planning horizon.

Before I go further, I present a concrete illustration using a company providing carpet cleaning. I chose the carpet cleaning industry because it captures the fact that there are not significant word-of-mouth effects. Each quarter, the carpet cleaning company's owner contacts private homes to offer carpet cleaning services. A contacted home can belong to a loyal customer (someone who has already had his carpet cleaned by this company), or it can belong to a potential new customer (someone who has never had his carpet cleaned by this company). The company selects a limited number of homes to contact each quarter. Those who have already experienced the company's cleaning work are aware of the quality of service offered by the company. Consequently, the immediate expected profit from contacting loyal customers is higher than the immediate expected profit from contacting potential customers. Some loyal customers purchase
the service from the cleaning company when not contacted, but non-contacted potential customers will never purchase the service. Each quarter, the cleaning company must decide who should be contacted so as to maximize its profit. Although loyal customers are more profitable during the period of contact, the company has an incentive to contact potential customers because that is the only way of building a base of loyal customers.

4.4 Prospecting for Customers on an Infinite Horizon: A Markov Decision Model

In this section the pool of potential buyers will be large enough so that at least \( N \) potential customers are available each period. This setting is well represented by entrepreneurs who cover large and populated territories, but cannot service that many customers each period. Mail orders of large item hand made by the entrepreneurs are such examples. I show that an expected benefit can be associated with each contact so that a simple policy of the form "among the contacts available, choose the one which has the largest expected benefit" is optimal.

4.4.1 Formulation and Optimal Contact Policies for the Basic Model

The entrepreneur is assumed to have an unlimited line of credit (without interest) to contact potential buyers so that she can always financially afford an immediate expected loss from a contact. For instance, the entrepreneur collects fixed wages each period allowing her to cover the cost of contacting \( N \) of the most costly potential buyers. The entrepreneur's goal is to allocate time to the class of potential buyers that is most profitable in the long run. I derive conditions that allow the immediate expected losses encountered from contacting potential customers (when the immediate expected profit is negative due to high contacting cost relative to immediate expected revenue) to be eventually covered by future expected gains.

I formulate the time allocation problem as a discrete-time, discrete-state Markov decision process. For more details on Markov decision process the reader is referred to Puterman (1994).
Let $S=\{0,1,2,...\}$ be the state space where an element $s \in S$ represents the number of loyal customers. The set of allowable actions given state $s$ is

$$A(s) = \{ (a_L,a_P) : a_L+a_P \leq N, \ 0 \leq a_L \leq s, \ a_P \geq 0 \},$$

with $a_L$ and $a_P$ representing respectively the number of loyal and potential customers contacted, and $N$ the maximum number of contacts each period. A randomized Markovian decision rule, denoted $d_t=(d_{t,L},d_{t,P})$, is a mapping of the state space $S$ into the set of probability distributions on the action space $A(s)$. The first component $d_{t,L}$ is associated with the contact of loyal customers while $d_{t,P}$ is associated with the contact of potential customers. A contact policy $\varphi = \{d_t : t=1,2,3,...\}$ prescribes at each period $t$ a decision rule $d_t$. A randomized decision rule in which the probability distribution on the set of actions is degenerated (i.e., a non-randomized decision rule) will specify for each state how many loyal and potential customers are contacted so that, for any state $s \in S$, $d_t(s) = (a_L(s),a_P(s)) \in A(s)$.

The contacted potential customers who purchase during period $t$ add to the loyal customer base at the beginning of period $t+1$. Given the state $s_t$ at period $t$ and a non-randomized decision rule $d_t(s_t)=(a_L(s_t),a_P(s_t))$, $s_t \in S$, one obtains the dynamic equation

$$s_{t+1} = s_t + \sum_{i=1}^{a_P(s_t)} \xi_i,$$

where $\xi_i=1$ if the $i$-th contacted potential customer at period $t$ buys during period $t$, and 0 if he does not. In the preceding section I have defined the probability of buying for a contacted potential customer to be the same for any contacted potential customer and I have denoted it by $\beta$. Hence, the $\xi_i$s are independent identically distributed random variables with $E(\xi_i)=\beta$, for any $i$ and any $t$. It follows that $s_{t+1} - s_t$ has a binomial distribution with parameters $\beta$ as the probability of success and $a_P(s_t)$ as the number of trials. Under decision rule $d_t(s_t)=(a_L(s_t),a_P(s_t))$, the expected number of loyal customers at the beginning of period $t+1$ given the size of the loyal customer base at the beginning of period $t$, $s_t$, is thus
\[
E_{d_i}(s_{t+1}|s_t) = s_t + \beta a_p(s_t)
\]

The entrepreneur's objective is to maximize the expected total discounted profit. The immediate expected reward or profit if \(s\) customers are in the loyal customer base, \(a_L\) loyal customers are contacted, and \(a_P\) potential customers are contacted is

\[
r'(s,a_L,a_P) = s\pi_L + a_L(\pi_L^C - \pi_L) + a_P\pi_P^C.
\]

The three terms in the reward function represent the expected profit from the loyal customer (obtained regardless of whether they are contacted or not), the incremental expected profit obtained from the loyal customers who are contacted, and the expected profit from the potential customers who are contacted.

In this formulation, the reward (or immediate expected profit) \(r'(s,a_L,a_P)\) is unbounded on the state space which causes some technical problems. Although this definition of reward function is natural and techniques are available for handling such problems (e.g., Lippman 1975), I present an alternative formulation with a bounded reward function and for which the optimal policy is easily determined and is of simple form. The alternative formulation is equivalent in the sense that any policy that is optimal for one formulation is also optimal for the other.

I define three gain quantities, \(G_P\), \(G_P^C\), and \(G_L\). The first two are the expected net present value from contacting a potential customer with no followup in subsequent periods and contacting a potential customer and following up in each subsequent period if the potential customer buys, respectively. Direct marketing practitioners refer to these quantities as the lifetime value of the customer. The third gain, \(G_L = \pi_L^C - \pi_L\), is the expected incremental profit obtained by contacting a loyal customer over and above the expected profit \(\pi_L\) which is obtained regardless of contact.

Contacting a potential customer gives an immediate expected profit of \(\pi_P^C\). In addition, there is an expected revenue of \(\beta\pi_L\) in each subsequent period if there are no followup contacts. I
assume that future expected gains are discounted with a discount rate of $\lambda$ so that the expected net present value of contacting a potential customer with no followup contacts is

$$G_p = \pi_p^c + \sum_{i=1}^{\infty} \frac{\lambda^i \beta \pi_L^c}{1 - \lambda}.$$

With followup contacts each subsequent period, the contact of a potential customer generates an immediate expected profit of $\pi_p^c$ and an expected profit of $\beta \pi_L^c$ in each subsequent period. So the expected net present value of contacting a potential customer with followup contacts is

$$G_p^c = \pi_p^c + \sum_{i=1}^{\infty} \frac{\lambda^i \beta \pi_L^c}{1 - \lambda} = G_p + \sum_{i=1}^{\infty} \frac{\lambda^i \beta G_L}{1 - \lambda}.$$

In my new formulation of the Markov decision process, the reward function is redefined to be

$$r(s, a_t, a_{t+1}) = a_L G_L + a_R G_R,$$

which is bounded by $N \times \max\{G_L, G_R\}$. Note that, in the new formulation, the entire expected net present value of contacting a potential customer, $G_p$, is accrued during the period of contact, whereas in the original formulation, a contact results in a revenue stream with the same expected net present value, $G_p$, but which is accrued over time.

The expected net present value of using policy $\varphi$ in the new formulation when there are initially $s$ loyal customers is

$$V_\varphi(s) = E \left[ \sum_{t=0}^{\infty} r(s_t, a_t, s_{t+1}) | s_0 = s \right].$$

I must verify that the two formulations are equivalent. It is sufficient to show that the difference between the value of a policy under the original formulation and the value of the same policy under the new formulation does not depend on the policy implemented. Since the revenue derived from the first $s_t$ loyal customers are taken away in the new formulation, I argue that the difference must be equal to
The objective is to find an optimal policy with value $V^*(s)$ where

$$V^*(s) = \sup_{\varphi} V_\varphi(s).$$  \hspace{1cm} (4.1)

From a well-known result on Markov decision process with bounded rewards (e.g., Puterman 1994), the optimal value function is associated with a policy that belongs to the class of non-randomized stationary policies. A non-randomized stationary policy $\varphi$ takes the following form:

$$\varphi = (d, d, d, \ldots) = (d)'^\infty$$

where $d$ is a decision rule such that $d(s) = (a_L, a_P)$, $s \in S$.

I have assumed that each contact takes the same amount of time. A natural policy to implement is now described. Divide the period into $N$ time slots. If there are $s \leq N$ loyal customers available, then assume that they will be contacted during the first $s$ time slots, if at all. Then starting at time slot 1, ask which alternative (contact a loyal customer, contact a potential customer, or make no contact) maximizes the immediate expected gain, ignoring any possible interactions with other time slots. Then do the same for time slot 2, and so forth. Such a policy essentially manages each time slot independently of the others, with the allocation of loyal customers to the first slots, and will be referred to as a time slot management policy. To analyze time slot management policies, it is convenient to explicitly introduce the parameter $N$ into the expected value function, so that $V_\varphi(N, s)$ is the expected net present value of starting with $s$ loyal customers and $N$ time slots. If $\varphi$ is of the time slot management type, then one has the decomposition

$$V_\varphi(N, s) = sV_\varphi(1, 1) + (N-s)V_\varphi(1, 0) \text{ if } s \leq N$$

$$NV_\varphi(1, 1) \text{ if } s > N.$$

The optimal contact policy, denoted $\varphi^*$, will take one of the following forms, depending
on the relative values of $G_P$, $G_P^c$, $G_L$, and 0. In each case, the policy is stationary with the choice of $a_L^*$ and $a_P^*$ a function of $s$ only. These forms are indicated in Figure 4.2 and stated explicitly in Theorem 4.1 which follows this discussion. The cartesian plane is divided in four regions corresponding to the four policies. The vertical and horizontal axes represent $G_P$ and $G_L$, respectively. The diagonal line in the positive quadrant is the graph of $G_L = G_P$. The diagonal line in the lower right quadrant is the graph of $G_P^c = 0$.

![Figure 4.2 Summarization of the Optimal Policies on an Infinite Horizon](image)

I now argue that the proposed contact policies are optimal. There are two trivial cases. If $G_P$ is positive and larger than $G_L$, then the entrepreneur can obtain an expected value of $G_P$ per contact by only contacting potential customers. Any other action will decrease the expected value. So clearly only potential customers will be contacted. This is case (a) of Theorem 4.1.

Another trivial case is when neither $G_P$ nor $G_L$ is positive. In this case contacting customers provides no benefit and the optimal policy is to make no contacts. This is case (b) of Theorem 4.1.

Optimality for the other two cases will be shown as follows. Because of the bounded reward function and the finite number of actions available, one knows from Denardo (1967) that
The optimal value function (4.1) is the unique solution to the optimality equation

\[ V^*(N, s) = \max_{(a_L, a_P) \in \mathcal{A}(s)} \left\{ a_L G_L + a_P G_P + \lambda EV^*(N, s + \sum_{i=1}^{q_i} \xi_i) \right\} . \tag{4.2} \]

One way to verify optimality is to compute the expected value function for the given policy and check that (4.2) holds.

The interesting case is part (c) of Theorem 4.1 where \( G_L \) is positive and larger than \( G_P \). In this situation, there is some potential tradeoff between the larger return of contacting a loyal customer versus the benefit of contacting a potential customer who builds the customer base. Because a loyal customer is profitable to contact and preferred to a potential customer, I argue that a maximum number of loyal customers should be contacted each period. Let \( \varphi^* \) be the policy where this is accomplished, that is, \( a_L^* = \min\{s, N\} \). In this case, when there is a loyal customer and one time slot, the loyal customer will be contacted each period so that

\[ V_{\varphi^*}(1, 1) = \frac{G_L}{1 - \lambda} . \tag{4.3} \]

Whether or not a potential customer should be contacted depends on the sign of \( G_P \). Part (c.i) of Theorem 4.1 considers the case in which \( G_P^c > 0 \) and so potential customers should be contacted during any time slot for which loyal customers are not available, that is, \( a_P^* = \max\{N-s, 0\} \). The value function for \( \varphi^* \) with one time slot can be computed by noting that

\[ V_{\varphi^*}(1, 0) = G_P + \lambda[\beta V_{\varphi^*}(1, 1) + (1 - \beta) V_{\varphi^*}(1, 0)] , \]

and then substitute using (4.3) and solving the resulting equation to obtain

\[ V_{\varphi^*}(1, 0) = \frac{G_P + \frac{\lambda}{1 - \lambda} \beta G_L}{1 - \lambda + \lambda \beta} = \frac{G_P^c}{1 - \lambda + \lambda \beta} . \tag{4.4} \]

Since \( G_P^c > 0 \), any value of \((a_L, a_P)\) which achieves the maximum in (4.2) must have \( a_L + a_P = N \). Therefore, (4.2) can be rewritten as
\[ V^*(N, s) = \max_{0 \leq k \leq s} \left\{ (s-k)G_L + (N-s+k)G_P + \lambda E V^*(N, s + \sum_{i=0}^{N-s+k} \xi_i) \right\}. \] (4.5)

To verify that policy \( \varphi^* \) satisfies (4.5) is equivalent to showing that the maximum is satisfied with \( k=0 \). Let me pick a positive \( k \leq s \). Then the right-hand side of (4.5) corresponds to a policy \( \varphi \) which contacts \( k \) more potential customers in the first period than \( \varphi^* \) would, and then use policy \( \varphi^* \) from period 2 onward.

The incremental expected loss in the first period from using \( \varphi \) instead of \( \varphi^* \) is \( k(G_L-G_P) \). Policy \( \varphi \) thus gives up in the first period \( k(G_L-G_P) \) in hope that this loss will be compensated by the additional loyal customers gained by the extra \( k \) contacts of potential customers. What is this extra expected gain?

Let \( X(n) \) be the number of (new) loyal customers which result from \( n \) contacts of potential customers; then \( X(n) \) has a binomial distribution with parameters \( \beta \) as the probability of success and \( n \) as the number of trials. The number of additional loyal customers gained in the first period with policy \( \varphi \) is \( X(N-s+k) \) which is equivalent to \( X(N-s)+X(k) \) with \( X(N-s) \) and \( X(k) \) being independent binomial variables of parameters \( (\beta,N-s) \) and \( (\beta,k) \), respectively. The number of additional loyal customers gained in the first period with policy \( \varphi^* \) is also binomial, say \( X'(N-s) \), and has the same distribution as \( X(N-s) \). Without loss of generality, I can assume that \( X(N-s)=X'(N-s) \). Then there will be

\[ Z = \min \{ X(N-s)+X(k), N-s \} - X(N-s), \]

additional loyal customers in period 2 if \( \varphi \) is used instead of \( \varphi^* \). The incremental expected gain from the second period on due to the extra \( k \) contacts of potential customers is thus

\[ \beta E(Z)[V_{\varphi^*}(1,1)-V_{\varphi^*}(1,0)]. \]

Taking the difference of (4.3) and (4.4) gives

\[ V_{\varphi^*}(1,1)-V_{\varphi^*}(1,0) = \frac{G_L-G_P}{1-\lambda + \lambda \beta}. \] (4.6)
The expected value of $Z$ can be easily bounded by noting that $Z \leq X(k)$ and, if $k > 0$, then with positive probability $Z < X(k)$; so that,

$$EZ < k\beta.$$  \hfill (4.7)

The optimality equations (4.5) are satisfied by $\varphi^*$ if the expected loss from contacting $k$ fewer loyal customers during the first period is not covered by the expected gain from the extra loyal customers available in period 2 onward generated by these contacts. It is thus sufficient to check that

$$\beta E(Z)[V_{\varphi^*}(1,1) - V_{\varphi^*}(1,0)] \leq k[G_L - G_P].$$  \hfill (4.8)

Using (4.6) and (4.7), the left-hand side is less than

$$\lambda k\beta \left[ \frac{G_L - G_P}{1 - \lambda + \lambda\beta} \right].$$

Since $\lambda\beta/(1 - \lambda + \lambda\beta) < 1$, (4.8) is verified.

The remaining case is when $G_P^C \leq 0$. In this case, there is no benefit derived from contacting a potential customer. So in checking (4.2), one only needs to consider $(a_L, a_P)$ with $a_P = 0$. However, I already argued that in case (c), $a_L = \min\{s, N\}$ dominates other choices. Hence, policy $\varphi^*$ is optimal.

**Theorem 4.1** Let $G_P$, $G_P^C$, and $G_L$ be defined as above.

(a) If $G_P > \max\{G_L, 0\}$, then potential customers are profitable to contact and are preferred to loyal ones so that the policy which chooses $a_P^* = N$ and $a_L^* = 0$ for all states is optimal.

(b) If $G_P \leq 0$ and $G_L \leq 0$, then it is not profitable to contact either loyal or potential customers so that $a_L^* = a_P^* = 0$ for all states.

(c) If $G_L > 0$ and $G_P \leq G_L$, then loyal customers are profitable to contact and preferred to potential ones so that $a_L^*(s) = \min\{s, N\}$.

(i) If, in addition, $G_P^C > 0$, then potential customers are also profitable to contact and thus any remaining resources should be used to contact them so that $a_P^*(s) = \max\{N - a_L^*(s), 0\}$.

(ii) If, on the other hand, $G_P^C \leq 0$ or $s \geq N$, then no potential customers are contacted.
I illustrate the application of Theorem 4.1 by returning to the example of the carpet cleaning company. I assume that the company plans to contact 200 homes each quarter. The matrix of immediate expected profits is given by Table 4.2.\footnote{The numbers in Table 4.2 can be generated as follows: Choose an expected revenue of $100 per sale, a cost of $7 per contact (for both loyal and potential customers), a probability of buying of 30\%, 15\%, and 5\% for, respectively, a contacted loyal customer, a non-contacted loyal customer, and a contacted potential customer.} On one hand, a contacted potential customer incurs a negative immediate expected profit because the probability of buying is very low. On the other hand, a loyal customer already knows about the great customer service offers by the company and by being contacted again has a large incentive to buy the service again, thus incurring a large immediate expected profit per contact.

**TABLE 4.2**

<table>
<thead>
<tr>
<th></th>
<th>Contacted</th>
<th>Non-Contacted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loyal Customer</td>
<td>$\pi_\ell=23$</td>
<td>$\pi_\ell=15$</td>
</tr>
<tr>
<td>Potential Customer</td>
<td>$\pi_p=-2$</td>
<td>0</td>
</tr>
</tbody>
</table>

In spite of the immediate expected losses incurred by contacting potential customers, the company realizes that it may be in its best interest to contact them because, say, 5\% of them will buy and be converted into loyal customers. Since, for $\lambda=0.95$, one can easily verify that $G_p > \max\{G_L,0\}$, the carpet cleaning company is advised to contact 200 potential customers and no loyal customers each quarter.

What would happen to the optimal policies of Theorem 4.1 if the entrepreneur were prospecting for customers on a finite number of periods? The expected net present values of contacting a potential customer, $G_P$ and $G_P^c$, would become time dependent. Let $T$ be the total finite number of periods. It is natural to define $G_P(t) = \pi_p^c + (T-t)\beta \pi_L$ and $G_P^c(t) = G_P + (T-t)\beta G_L$ as the expected net present value of contacting a potential customer with no followup contacts and with followup contacts, respectively. In the next paragraph I present a theoretical example...
showing that it is more profitable to contact a potential customer even though, at period \( t \), the incremental gain from contacting a loyal customer is larger than the expected net present value from contacting a potential customer \((G_L > G_P(t))\). The rational goes as follows. On a finite horizon, it is harder to reach a total of \( N \) loyal customers. Consequently, more value should be given to the contact of a potential customer than \( G_P(t) \) accounts for.

Let \( V_\varphi(t,N,s) \) be the expected net present value of policy \( \varphi \) from starting at period \( t \) with \( s \) loyal customers and \( N \) contacts. Let \( \varphi^* \) be the policy where as many loyal customers are contacted each period as possible and let \( \varphi^I \) be the policy where 1 more potential customer is contacted in the first period than \( \varphi^* \) would, and then using \( \varphi^* \) from the next period onward. If \( G_L > G_P(1) \), one would expect that a maximum number of loyal customers should be contacted on the first period. Consequently, one should have

\[
V_{\varphi^*}(t,N,s) \geq V_{\varphi^I}(t,N,s)
\]

for any combination of \( t, N, \) and \( s \). I show that this is not always true. Choose \( N=2, T=2, \) and \( \lambda=1 \). Then, for \( t=1 \) and \( s=1 \),

\[
V_{\varphi^*}(1,2,1) = G_L + G_P(1) + \beta 2G_L + (1 - \beta) [G_L + G_P(2)]
\]

and

\[
V_{\varphi^I}(1,2,1) = 2G_P(1) + \beta^2 2G_L + 2\beta(1 - \beta) 2G_L + (1 - \beta)^2 [G_L + G_P(2)].
\]

With \( \beta = 0.5 \) and \( \pi_P = 0 \) (so that \( G_P(2) = 0 \)),

\[
V_{\varphi^*}(1,2,1) = 2.5G_L + G_P(1) \quad \text{and} \quad V_{\varphi^I}(1,2,1) = 1.75G_L + 2G_P(1),
\]

and so

\[
V_{\varphi^*}(1,2,1) < V_{\varphi^I}(1,2,1) \quad \text{whenever} \quad 0.75G_L < G_P(1).
\]

Therefore, by choosing \( G_L \) and \( G_P(1) \) so that \( 0.75G_L < G_P(1) < G_L \), it is possible to prefer contacting potential customers even though, at period \( t=1 \), the incremental gain from contacting a loyal customer
customer is larger than the expected net present value from contacting a potential customer.

It is not clear how the expected net present values from contacting a potential customer, \( G_P \) and \( G_P^c \), must be defined so that the methodology developed in the present sub-section can be applied to a finite horizon setting. In section 4.5 a new methodology will be proposed where the decision problem will be formulated on a finite planning horizon with deterministic numbers of buyers from each category. Once a contact policy has been chosen, the state of the system (i.e., number of loyal customers) will be known. It will then be possible to eliminate the difficulties presented by the previous theoretical example by restricting the number of loyal customers available after some critical time point to be larger than the contact limit.

4.4.2 A Partial Investigation of a Cashflow Model

The optimal policies of Theorem 4.1 work well when survival is not affected by short run considerations such as cashflow. In this sub-section, the basic model is expanded to examine the balancing of short run survivability with long run optimality.

It is assumed that the entrepreneur has no uninvested capital and has no unutilized credit. Cash reserves are generated by buyers (contacted or not), and they are used the following periods to contact more potential buyers. To make the problem mathematically tractable, the total discounted future stream of revenues from contacting a potential customer will be accrued in a deterministic way during the period of contact. That is, the entrepreneur is assured of getting a fixed amount of revenue \( (\beta x_L) \) at each period following the contact. In practice this happens when the entrepreneur sells a product or a service that requires followups. Examples include software/hardware that are sold with a maintenance contract and natural water distributors.

I show that it is still optimal to contact as many potential customers as possible whenever loyal customers are not profitable to contact \( (G_L \leq 0) \). However, when loyal customers become profitable to contact, I give a numerical example where a loyal customer is more profitable to
contact than a potential customer \((G_L > G_P)\), but potential customers only should be contacted. Such a result makes sense here because the contacting costs \(a_L\) and \(a_P\) (for a loyal and a potential customer, respectively) limit the numbers of contacts that can be made from each category of potential buyers.

I formulate this time allocation problem as a discrete-time, continuous-state Markov decision process. Let \(S = \{(c,s): c \in R, s \in I\}\) be the state space where \(c\) represents the cash balance, \(s\) is the number of loyal customers, \(R\) is the set of non-negative real numbers, and \(I\) the set of non-negative integers. To insure that the cash balance of the period following the contacts does not become negative (and thus the entrepreneur does not go out of business), I require that the total contacting cost does not exceed the cash on hand. The set of allowable actions given state \((c,s)\) is thus

\[
A(c,s) = \{ (a_L,a_P): a_L + a_P \leq N, \quad a_L a_L + a_P a_P \leq c, \quad 0 \leq a_L \leq s, \quad a_P \geq 0 \},
\]

with \(a_L\) and \(a_P\) representing respectively the number of loyal and potential customers contacted, \(a_L\) and \(a_P\) representing respectively the cost of contacting a loyal and a potential customer, and \(N\) the maximum number of contacts each period.\(^{14}\) A contact policy, \(\varphi = \{d_t : t=1,2,...\}\), (possibly randomized) specifies an action \(d(c,s)=(a_L,a_P) \in A(c,s)\) for each state \((c,s)\) and time period \(t\), where \((a_L,a_P)=(0,0)\) whenever \(c\) reaches 0. Because the future stream of revenues from contacting a potential customer will be accrued in the period of contact, state \((0,s)\) is absorbent for any \(s \in I\).

The contacted potential customers who purchase during period \(t\) add to the loyal customer base at the beginning of period \(t+1\). Given the state \((c_t,s_t)\) at period \(t\) and a decision rule \(d(c_t,s_t)=(a_L(c_t,s_t),a_P(c_t,s_t))\), \((c_t,s_t) \in S\), one obtains the dynamic equation

\[^{14}\text{This time allocation problem thus has constraints on three resources which are number of contacts, cash on hand, and number of loyal customers. The basic model of section 4.4 only had two constraints, namely number of contacts and number of loyal customers.}\]
\[ s_{t+1} = s_t + \sum_{i=1}^{a_L(c_t,s_t)} \xi_i, \]

where \( \xi_i = 1 \) if the \( i \)-th contacted potential customer at period \( t \) buys during period \( t \), and 0 if he does not. In addition, contacted loyal and potential customers will generate profit that will add to the current cash balance. Let \( \rho \) be the (fixed) revenue from a sale, \( \eta_i = 1 \) if the \( i \)-th contacted loyal customer buys, and 0 otherwise, \( \xi_i = 1 \) if the \( i \)-th contacted potential customer buys, and 0 if he does not buy, and \( \lambda \) be the discount rate. One obtains a second dynamic equation

\[ c_{t+1} = c_t + \rho \sum_{i=1}^{a_L(c_t,s_t)} \eta_i + \left[ \rho + \frac{\lambda}{1-\lambda} \pi_L \right] \sum_{i=1}^{a_P(c_t,s_t)} \xi_i - [a_L a_L(c_t,s_t) + a_P a_P(c_t,s_t)]. \]

The four terms in the cash balance equation represent the current cash balance, the revenue from contacted loyal customers, the revenue from contacted potential customers (where the total discounted future stream of revenues is assumed to be accrued in a deterministic way during the period of contact), and the contacting cost.

As defined earlier, the \( \xi_i \)'s are independent identically distributed random variables with \( E(\xi_i) = \beta \). It thus follows that, under decision rule \( d_t(c_t,s_t) = (a_L(c_t,s_t), a_P(c_t,s_t)) \), the expected number of loyal customers at the beginning of period \( t+1 \) given state \((c_t,s_t)\) at the beginning of period \( t \), is

\[ E_{d_t}(s_{t+1} \mid c_t,s_t) = s_t + \beta a_P(c_t,s_t). \]

Moreover, the \( \eta_i \)'s are independent identically distributed random variables with \( E(\eta_i) = \gamma \) where \( \gamma \) represents the incremental proportion of loyal customers who would not purchase if not contacted (remember that the benefit from the non-contacted loyal customers who purchase is deterministically accounted at the period at which they move from the potential customer base to the loyal customer base). With
\[ G_L = \pi_L^c - \pi_L = \gamma p - \alpha_L \]  

and

\[ G_p = \beta p - \alpha_p + \beta \frac{\lambda}{1 - \lambda} \pi_L = \beta \pi_p + \frac{\lambda}{1 - \lambda} \beta \pi_L \]

it follows that the expected cash balance at the beginning of period \( t+1 \), given state \((c_t, s_t)\) at the beginning of period \( t \), is

\[ E_{d_t} \left( c_{t+1} \mid c_t, s_t \right) = c_t + a_L(c_t, s_t)G_L + a_P(c_t, s_t)G_P. \]

The increment on the expected cash balance is thus a linear combination of the gain from contacting a loyal customer and the gain from contacting a potential customer.

The entrepreneur's objective is to maximize the total expected cash on hand at any period \( t \), that is,

\[ \text{Max } E_{d_t} \left( c_t \mid c_{t-1} \right), \text{ for any } t. \]

Notice that each period the cash generated from contacting is used the following period to contact more potential buyers. Consequently, the accumulated expected discounted profit may stay at zero, and so it is not appropriate for this cashflow formulation to maximize the expected total discounted profit.

The immediate expected profit if action \((a_L, a_P) \in A(c, s)\) is implemented while in state \((c, s)\) is

\[ r(c, s, a_L, a_P) = a_L G_L + a_P G_P, \]

which adds up to the current cash balance to provide the expected cash balance of the following period. When loyal customers are not profitable to contact \((G_L \leq 0)\), the entrepreneur's choice is reduced to contact nobody or contact potential customers. If nobody is contacted during the current period, then the expected cash balance of the following period does not change. A contacted potential customer generates an expected profit of \( G_P \). Whenever \( G_P > 0 \), it is profitable (in average) to contact a potential customer so that a total of,

\[ ^{15} \text{It is equivalent, but more intuitive, to write } \gamma = \beta_c \gamma L \text{ where } \beta_c \text{ and } \beta_L \text{ are, respectively, the proportion of contacted and non-contacted loyal customers who buy. Hence, } \pi_c = \beta_c \rho + \alpha_L \text{ and } \pi_L = \beta \rho \text{ so that } G_L = (\beta_c - \beta_L)(\rho - \gamma L). \]
say, \( P \) potential customer should be contacted where \( P \) is the integer part of \( c/\alpha_p \). When \( G_P \leq 0 \), nobody should be contacted.

As an illustration of my findings consider a software engineer who is developing a software that remotely rebuild desktop computers. The potential buyers are companies with enough machines to make it worth it to pay the fixed basic cost. In an industry where these companies compete with each other for technology, knowledge of the product from word-of-mouth will be negligible. The software engineer contacts potential buyers by visiting them for a demo. Loyal customers who have already bought the product are contacted to buy an upgrade of the product; the engineer incurs the same per unit profit from the sells of an upgrade as it costs less to the loyal customer, but it also costs less to the entrepreneur for the demo (the loyal customer is already familiar with the product). A potential customer who buys the software incurs constant revenues each period following the contact as the software must be serviced.

After a few calculations, the software engineer realizes that upgrading is not a profitable business but servicing is \((G_L \leq 0 \text{ and } G_P > 0)\). Therefore, Theorem 4.1 (a) applies and the engineer should use all of her cash reserves and time on contacting companies that have not yet acquired the software.

When loyal customers become profitable to contact, there is a tradeoff between contacting potential versus loyal customers (upgrading and servicing are both profitable businesses, but the engineer has a limited budget and limited time for demos). I give a numerical example showing that Theorem 4.1(c) does not hold. Assume that \( \pi_L^c = 8, \pi_L = 3, \pi_P = 2, \beta = .1, \alpha_L = 10, \alpha_P = 1, N \geq 10, \lambda = .8, \) and, at the current period, \( c_i = 10 \) and \( s_i \geq 1 \). Hence, \( G_L = 5 \) and \( G_P = 3.2 \) so that \( G_L > 0 \) and \( G_P \leq G_L \) corresponding to case (c) of Theorem 4.1. The entrepreneur can only afford to contact one loyal customer who will generate a total expected benefit of 5. By contacting potential customers, she can afford 10 contacts generating a total expected benefit of 32. Although a loyal customer is more profitable to contact than a potential customer, it
maximizes the expected cash on hand for the next period to contact as many potential customers as it can be afforded and to contact no loyal customers.

This example showed that the optimal contact policies derived for the basic model must be adjusted for the cashflow model because the number of contacts is not only limited by $N$, but also by the cost of each contact. Under the assumptions of unlimited contacting time and fractional buyers, section 4.6 will present an extension of the basic model where the time constraint is replaced by a cashflow constraint.

4.5 Prospecting for Customers on a Finite Horizon: An Optimal Control Model

I showed that possible transfers of potential customers into the loyal customer base made it worthwhile for the entrepreneur to incur short term losses from contacting in anticipation of offsetting future gains. The analysis was done on an infinite planning horizon with stochastic numbers of buyers from the various categories of Table 4.1. Will this result stay true on a finite planning horizon? I have shown that the methodology used cannot be easily extended to a finite horizon setting. By modeling the numbers of buyers from each category with a deterministic process, I will demonstrate that the result stays true as long as there is sufficient time left after the contacts. The analysis is made possible because the deterministic approach provides an explicit expression for the size of the loyal customer base as a function of time, allowing one to make the appropriate assumptions on the problem's parameters so that optimal contact policies can be determined.

Throughout this section, the entrepreneur has unlimited credit to contact potential buyers (e.g., the entrepreneur collects fixed wages each period that cover the cost of contacts). The total number of potential buyers on the overall finite planning horizon, denoted by $M$, is assumed to be finite and does not change over time. This works well for products with a short life and predetermined, small markets. Optimal contact policies will be evaluated with and without
discounting.

4.5.1 Formulation of the Optimal Control

I formulate the time allocation problem as a continuous-time deterministic linear optimal control model. Let $s_t$ be the number of loyal customers at $t$ so that $M-s_t$ denotes the number of potential customers at $t$; $s_t$ represents the state variable of the linear control problem. Let $a_L(t)$ be the number of individuals from the loyal customer base contacted at $t$ and $a_P(t)$ be the number of individuals from the potential customer base contacted at $t$; $a_L(t)$ and $a_P(t)$ are the decision (or control) variables. Since there is a limit $N$ on the maximum number of contacts, $a_L(t)+a_P(t)$ must be smaller than or equal to $N$ at all times.

The potential customers contacted at time $t$ who decide to purchase at time $t$ add to the loyal customer base. One thus obtains the dynamic equation on the state variable $s_t$

$$\frac{ds_t}{dt} = \beta a_P(t).$$

The entrepreneur maximizes the present value of the profit stream up to the end of the planning horizon $T$. The profit at $t$ is

$$s_t \pi_L + a_L(t) G_L + a_P(t) \pi_P^c,$$

where $G_L = \pi_L^c - \pi_L$ is still the incremental gain from contacting a loyal customer and $\pi_L, \pi_L^c, \text{ and } \pi_P^c$ are the immediate expected profits of Table 4.1. With a discount factor $\lambda$ and no loyal customers at start-up, one obtains a linear optimal control problem (LOCP) defined by

$$\operatorname{Max}_{a_L(t), a_P(t)} \int_0^T e^{-\lambda t} \left\{ s_t \pi_t + a_L(t) G_L + a_P(t) \pi_P^c \right\} dt$$

subject to:

$$\frac{ds_t}{dt} = \beta a_P(t) \quad (4.9)$$

$$0 \leq a_L(t) \leq s_t \quad (4.10)$$

$$0 \leq a_P(t) \leq M - s_t \quad (4.11)$$
\[ a_L(t) + a_P(t) \leq N \tag{4.12} \]
\[ s_0 = 0. \tag{4.13} \]

The three terms in the objective function represent the expected profit from the loyal customers (obtained regardless of whether they are contacted or not), the incremental expected profit from the loyal customers who are contacted, and the expected profit from the potential customers who are contacted. Equation (4.9) expresses how the size of the loyal customer base evolves over time. Inequality (4.10) (respectively (4.11)) insures that the number of loyal customers contacted (respectively potential customers contacted) does not exceed the size of the loyal customer base (respectively the size of the potential customer base). Inequality (4.12) guarantees that the total number of contacts does not exceed the maximum limit and equation (4.13) states the initial condition on the state variable.\(^{16}\)

Pontryagin's Maximum Principle for mixed constraints is used on (LOCP). The constraints are mixed because (4.10) and (4.11) depend on both the control variables \(a_L(t), a_P(t)\) and the state variable \(s_t\). For more details on Pontryagin's Maximum Principle, the reader is referred to Seierstad and Sydsaeter (1987). For more references on optimal control theory for use in management problems see Sethi and Thompson (1981) and Kamien and Schwartz (1991).

The generalized Hamiltonian is

\[
H(s,a_L,a_P,p,q_1,q_2,q_3,q_4,q_5,t) = e^{\int t \left\{ s_t \pi_L + a_L(t) G_L + \pi_P a_P(t) \right\} + p(t) \beta a_P(t) + q_1(t) [s_t - a_L(t)] + q_2(t) [M - s_t - a_P(t)] + q_3(t) a_L(t) + q_4(t) a_P(t) + q_5(t) [N - a_L(t) - a_P(t)]},
\]

where \(p(t)\) is the adjoint variable associated with the differential equation (4.9), and \(q_1(t), q_2(t), q_3(t), q_4(t), q_5(t)\) are the multipliers associated with constraints (4.10), (4.11), and (4.12). The adjoint variable \(p(t)\) represents the shadow price associated with the potential customers who

\(^{16}\)The results derived would still hold if \(s_0\) was a positive integer.
purchase at \( t \), and thus become "new" loyal customers. The multiplier \( q_i(t) \) measures the marginal change in the optimal value of the objective function with respect to changes in the corresponding constraint. \( H \) can be interpreted as the instantaneous profit rate which includes the value of an increase in the size of the loyal customer base created by contacting

\[
p(t)ds/dt = p(t)\beta a_r(t)
\]

and the marginal values of each constraint on the decision variables. Concavity of the generalized Hamiltonian and quasi-concavity of the mixed constraints in \((s_i,a_L(t),a_P(t))\) allow one to find sufficient and necessary conditions for an optimal solution (e.g., Seierstad and Sydsaeter (1987) pp. 271 and 287).

One must find \( s_i^*, a_L^*(t), a_P^*(t) \), a corresponding adjoint function \( p(t) \), and multipliers \( q_i(t), q_2(t), q_3(t), q_4(t), q_5(t) \) satisfying the following system \((\Sigma)\):

\[
\begin{align*}
\partial H^*/\partial a_c &= G_i e^{-\lambda_1 t} - q_i(t) + q_2(t) - q_3(t) = 0 \quad (4.14) \\
\partial H^*/\partial a_P &= \pi_P e^{\lambda_2 t} + \beta p(t) - q_2(t) + q_4(t) - q_5(t) = 0 \quad (4.15) \\
-\partial H^*/\partial s &= dp(t)/dt = q_2(t) - q_5(t) - \pi L e^{\lambda_3 t}, p(T) = 0 \quad (4.16) \\
q_i(t) \geq 0 \quad (=0 \text{ if } a_c^*(t) < s_i^*) \quad (4.17) \\
q_2(t) \geq 0 \quad (=0 \text{ if } a_P^*(t) < M - s_i^*) \quad (4.18) \\
q_3(t) \geq 0 \quad (=0 \text{ if } a_L^*(t) > 0) \quad (4.19) \\
q_4(t) \geq 0 \quad (=0 \text{ if } a_P^*(t) > 0) \quad (4.20) \\
q_5(t) \geq 0 \quad (=0 \text{ if } a_L^*(t) + a_P^*(t) < N), \quad (4.21)
\end{align*}
\]

where \( H^* \) is the generalized Hamiltonian evaluated at \((s_i^*, a_L^*(t), a_P^*(t))\).

Equation (4.14) states that at optimality the marginal profit from contacting a loyal customer at \( t \) (i.e., \( G_i e^{-\lambda_1 t} \)) equals the total marginal change in profit associated with the constraints on the number of loyal customers that can be contacted (i.e., \( q_i(t) - q_3(t) + q_5(t) \)). Similarly, equation (4.15) states that at optimality the marginal profit from contacting a potential customer at \( t \) (i.e., \( \pi_P e^{\lambda_2 t} + \beta p(t) \)) equals the total marginal change in profit with respect to the
constraints on the number of potential customers that can be contacted (i.e., $q_2(t) - q_4(t) + q_5(t)$). The additional component in the marginal profit, $\beta p(t)$, comes from the fact that a potential customer does not only generate an immediate profit (or loss), but he also generates future gains by building up the loyal customer base. Equation (4.16) corresponds to the equilibrium relation for investments in contacting. It states that the marginal opportunity cost of investments in contacting potential customers ($q_2$) should equal the sum of the marginal profit from contacting loyal customers ($q_1$), the increment from not contacting a loyal customer ($\pi_i e^{\mu t}$), and the capital gain ($dp/dt$). Equation (4.17), (4.18), (4.19), (4.20), and (4.21) insures that there is no marginal effect on profit from making one more or one less contact when the corresponding constraint is not binding.

4.5.2 Optimal Contact Policies for a Small Number of Buyers

I assume first that the total number of potential buyers is smaller than the available number of contacts, that is, $M < N$. Hence, constraint (4.12) can be eliminated from the (LOCP). Consequently, multiplier $q_3(t)$ is not required and the problem is reduced to finding $s_i^*, a_L^*(t)$, $a_P^*(t)$, a corresponding adjoint function $m(t)$, and multipliers $u_1(t), u_2(t), u_3(t), u_4(t)$ that satisfy

\begin{align*}
G_L - u_1(t) + u_2(t) &= 0 \\
\pi_i e^{\mu t} + \beta m(t) - u_3(t) + u_4(t) &= 0 \\
dm(t)/dt &= \lambda m(t) + u_3(t) - u_4(t) - \pi_{L_0} \\
u_i(t) &\geq 0 \quad ( = 0 \text{ if } a_L^*(t) < s^*(t) ) \\
u_2(t) &\geq 0 \quad ( = 0 \text{ if } a_P^*(t) < M - s^*(t) ) \\
u_3(t) &\geq 0 \quad ( = 0 \text{ if } a_L^*(t) > 0 ) \\
u_4(t) &\geq 0 \quad ( = 0 \text{ if } a_P^*(t) > 0 )
\end{align*}

where, for clarity reasons, each lines of system $(\Sigma)$ has been multiply by $e^{\lambda t}$, $e^{\mu t} p(t)$ has been replaced by $m(t)$, and $e^{\mu t} q_i(t)$ has been replaced by $u_i(t), i = 1, 2, 3, 4$. 

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It is straightforward (but tedious) to verify that the contact policies, adjoint function, and multipliers given in Table 4.3 satisfy the above system. The optimal contact policies are presented in an alternative manner in Theorem 4.2 which follows this discussion. There are no tradeoffs between contacting a potential customer versus contacting a loyal customer due to the large number of contacts \( M < N \). Consequently, the optimal policies only depend on the sign of the immediate profits and, for contacted potential customers incurring an immediate loss, they also depend on the time left until the end of the planning horizon.

**TABLE 4.3**

An Optimal Solution to (LOCP): \( M < N \)

<table>
<thead>
<tr>
<th>( \pi_i^* \geq 0 )</th>
<th>( \pi_i^* &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_i \leq 0 )</td>
<td>( G_i &gt; 0 )</td>
</tr>
<tr>
<td>( a_i^* = 0, t \in [0,T] )</td>
<td>( a_i^* = 0, t \in [0,T] )</td>
</tr>
<tr>
<td>( a_p^* = M \cdot s'(t), t \in [0,T] )</td>
<td>( a_p^* = M \cdot s'(t), t \in [0,T] )</td>
</tr>
<tr>
<td>( m(t) = [\lambda + \beta] \cdot (\pi_i^* - \pi_i^*) \cdot e^{[\lambda + \beta]T_0 - 1}], t \in [0,T] )</td>
<td>( m(t) = [\lambda + \beta] \cdot (\pi_i^* - \pi_i^*) \cdot e^{[\lambda + \beta]T_0 - 1}], t \in [0,T] )</td>
</tr>
<tr>
<td>( u_i(t) = 0, t \in [0,T] )</td>
<td>( u_i(t) = 0, t \in [0,T] )</td>
</tr>
<tr>
<td>( u_d(t) = \pi_i^* \cdot \beta \cdot m(t), t \in [0,T] )</td>
<td>( u_d(t) = \pi_i^* \cdot \beta \cdot m(t), t \in [0,T] )</td>
</tr>
<tr>
<td>( u_d(t) = 0, t \in [0,T] )</td>
<td>( u_d(t) = 0, t \in [0,T] )</td>
</tr>
<tr>
<td>( s_i^* = M \cdot [1 - e^{-\beta}], t \in [0,T] )</td>
<td>( s_i^* = M \cdot [1 - e^{-\beta}], t \in [0,T] )</td>
</tr>
</tbody>
</table>

where \( \ln(a) = -\infty \) when \( a \leq 0 \)
\[
\tau(\lambda) = \max \left\{ 0, T - \left| \frac{1}{\lambda} \ln \left( 1 - \frac{\lambda \max\{-\pi^c_P, 0\}}{\beta \pi_L} \right) \right| \right\},
\]

\[
\eta(\lambda) = \max \left\{ 0, T - \left| \frac{1}{\lambda} \ln \left( 1 - \frac{\lambda \max\{-\pi^c_P, 0\}}{\beta \pi_L} \right) \right| \right\},
\]

and

\[
\tau(0) = \max \left\{ 0, T - \frac{\max\{-\pi^c_P, 0\}}{\beta \pi_L} \right\}, \quad \eta(0) = \max \left\{ 0, T - \frac{\max\{-\pi^c_P, 0\}}{\beta \pi_L} \right\}.
\]

The critical time point \(\tau(0)\) and \(\eta(0)\) can be derived from \(\tau(\lambda)\) and \(\eta(\lambda)\) by using l'Hospital's rule. The state variable \(s_t^*\) gives how many loyal customers the entrepreneur will have at time \(t\) if she uses those optimal policies. The number of individuals in the potential customer base at time \(t\) is obtained by subtracting \(s_t^*\) from \(M\).

Whenever \(\pi^c_P < 0\), every contact of a potential customer results in an immediate loss. Therefore, the only hope for profit is from turning those potential customers into loyal customers who will make future purchases that result in profits. Nobody from the potential customer base should be contacted after \(\tau(\lambda)\) or \(\eta(\lambda)\) because not enough time is left after the contact of a potential customer to make it worthwhile for the entrepreneur to absorb the immediate loss from the contact.

The critical time points \((\tau(\lambda)\) and \(\eta(\lambda)\) are smaller than \(T\) only when \(\pi^c_P < 0\). They are next interpreted for the case \(\lambda = 0\). In the expression for \(\tau(0)\) (which is only used when \(G_L \leq 0\)), the numerator \(-\pi^c_P\) represents the immediate loss from contacting a potential customer and the denominator \(\beta \pi_L\) represents the continuous future gain from the contact. It thus takes \(-\pi^c_P/\beta \pi_L\) units of time to cover the loss from contacting. Notice that if a transfer occurs, that is, a sale from the contact is incurred, the new customer will never be contacted (when \(G_L \leq 0\) nobody in the loyal customer base is contacted), and thus a gain is generated only by not contacting this individual. Similarly, in the expression for \(\eta(0)\) (only used when \(G_L \geq 0\)), the denominator \(\beta \pi_L^c\)
represents the continuous future gain from followup contacts. Therefore, it takes $-\pi_r^L/\beta\pi_L^L$ units of time to cover the loss of contacting. In that case, the transfer of a potential customer will generate a future gain by contacting that new customer (since everybody in the loyal customer base will be contacted). Similar interpretations can be described for the discounting case.

**Theorem 4.2** Let $G_L, \tau(\lambda), \eta(\lambda)$ be defined as above and $M<N$. The optimal contact policy depends on the sign of the immediate profits and they are

<table>
<thead>
<tr>
<th>$\pi_r^L \geq 0$</th>
<th>$\pi_r^L &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_L \leq 0$</td>
<td>contact nobody from the loyal customer base at any time; contact everybody from the potential customer base at all times</td>
</tr>
<tr>
<td>$G_L &gt; 0$</td>
<td>contact everybody from the loyal customer base at all times; contact everybody from the potential customer base at all times</td>
</tr>
</tbody>
</table>

By using the Taylor expansion of the function $\ln(1+x)$, it can be verified that $\tau(\lambda) < \tau(0)$ and that $\eta(\lambda) < \eta(0)$ for any $\lambda > 0$. Consequently, when discounting is required, the entrepreneur has a smaller time interval during which it is worthwhile contacting an individual from the potential customer base.

**4.5.3 Optimal Contact Policies for a Small Number of Contacts**

When contact resources or, to use the terminology of section 4.4, time slots are limited so that the maximum number of contacts each period is smaller than the overall market size (i.e., $N \leq M$), there is a tradeoff between contacting a loyal customer versus a potential customer whenever loyal customers are profitable to contact and preferred to potential ones. It is then possible that, after having contacted all the loyal customers, there still are time slots remaining. Those time slots should be used on potential customers whenever the time-dependent expected net present value from contacting a potential customer is positive. This value is not well defined as argued in section 4.4. Technical difficulties are overcome by imposing conditions on the speed of growth of the loyal customer base so that, once loyal customers become more profitable to
contact than potential ones, there will not be any time slots remaining for potential customers.

The contact policies, adjoint function, and multipliers given in Table 4.4 (a) and (b) satisfy system (Σ) of sub-section 4.5.1 for a special case where the discount rate \( \lambda \) is zero and, when loyal customers become profitable to contact, \( M \geq 2N \), and \( G_L < T \beta \pi_L + \pi_F^c \). The conditions on \( M \) and \( G_L \) will insure that the loyal customer base grows quickly enough so that when loyal customers become more profitable to contact than potential customers, the \( N \) time slots will be allocated to loyal customers only. The optimal contact policies are presented in an alternative manner in Theorem 4.3 which follows this discussion.

**TABLE 4.4**
(a) An Optimal Solution to (LOCP): \( \lambda=0 \) and \( G_L \leq 0 \)

<table>
<thead>
<tr>
<th>( \pi_F^c \geq 0 ) (( \tau=T ))</th>
<th>( \pi_F^c &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu &lt; \tau )</td>
<td>( \mu \geq \tau )</td>
</tr>
<tr>
<td>( a_1^*(t) = 0, t \in [0, T] )</td>
<td>( a_1^*(t) = 0, t \in [0, T] )</td>
</tr>
<tr>
<td>( a_2^*(t) = N, t \in [0, \mu] )</td>
<td>( a_2^*(t) = N, t \in [0, \mu] )</td>
</tr>
<tr>
<td>( M^2 (t), t \in [\mu, T] )</td>
<td>( M^2 (t), t \in [\mu, T] )</td>
</tr>
<tr>
<td>( p(t) = \pi_L (\mu t)^{-2} - \beta \pi_L \pi_F^c \frac{e^{-\beta t}}{1-\beta^2 \pi_L^2}, t \in [0, \mu] )</td>
<td>( p(t) = \pi_L (\mu t)^{-2} - \beta \pi_L \pi_F^c \frac{e^{-\beta t}}{1-\beta^2 \pi_L^2}, t \in [0, \mu] )</td>
</tr>
<tr>
<td>( q(t) = 0, t \in [0, T] )</td>
<td>( q(t) = 0, t \in [0, T] )</td>
</tr>
<tr>
<td>( q(t) = 0, t \in [0, \mu] )</td>
<td>( q(t) = 0, t \in [0, \mu] )</td>
</tr>
<tr>
<td>( s^*_L = \beta N t, t \in [0, \mu] )</td>
<td>( s^*_L = \beta N t, t \in [0, \mu] )</td>
</tr>
<tr>
<td>( M - N e^{-\beta \pi F^c}, t \in [\mu, T] )</td>
<td>( M - N e^{-\beta \pi F^c}, t \in [\mu, T] )</td>
</tr>
</tbody>
</table>

| \( \mu \geq \tau \) | | |
|\( a_1^*(t) = 0, t \in [0, T] \) | \( a_1^*(t) = 0, t \in [0, T] \) |
| \( a_2^*(t) = N, t \in [0, T] \) | \( a_2^*(t) = N, t \in [0, T] \) |
| \( p(t) = \pi_L (T-t)^{-2} - \beta \pi_L \pi_F^c \frac{e^{-\beta (T-t)}}{1-\beta^2 \pi_L^2}, t \in [0, T] \) | \( p(t) = \pi_L (T-t)^{-2} - \beta \pi_L \pi_F^c \frac{e^{-\beta (T-t)}}{1-\beta^2 \pi_L^2}, t \in [0, T] \) |
| \( q(t) = 0, t \in [0, T] \) | \( q(t) = 0, t \in [0, T] \) |
| \( q(t) = 0, t \in [0, \mu] \) | \( q(t) = 0, t \in [0, \mu] \) |
| \( q(t) = -G_L + q(t), t \in [0, T] \) | \( q(t) = -G_L + q(t), t \in [0, T] \) |
| \( q(t) = 0, t \in [0, \mu] \) | \( q(t) = 0, t \in [0, \mu] \) |
| \( q(t) = \pi_L < \beta \pi_F^c (T-t), t \in [0, T] \) | \( q(t) = \pi_L < \beta \pi_F^c (T-t), t \in [0, T] \) |
| \( s^*_L = \beta N t, t \in [0, T] \) | \( s^*_L = \beta N t, t \in [0, T] \) |
| \( M - N e^{-\beta \pi F^c}, t \in [\mu, T] \) | \( M - N e^{-\beta \pi F^c}, t \in [\mu, T] \) |
| \( M - N e^{-\beta \pi F^c}, t \in [\mu, T] \) | \( M - N e^{-\beta \pi F^c}, t \in [\mu, T] \) |
(b) An Optimal Solution to (LOCP): \( \lambda = 0, M \geq 2N, G_L > 0 \), and \( 0 < G_L \pi^* \leq T \beta \pi_L \)

<table>
<thead>
<tr>
<th>( G_L \pi^* \leq 0 ) (( \tau = T ))</th>
<th>( 0 &lt; G_L \pi^* &lt; T \beta \pi_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu &lt; \tau )</td>
<td>( \mu \geq \tau )</td>
</tr>
<tr>
<td>( a^*_L(t) = 0, t \in [0, \mu) )</td>
<td>( a^*_L(t) = 0, \mu \leq t \leq T )</td>
</tr>
<tr>
<td>( a^*_L(t) = N, t \in [\mu, T] )</td>
<td>( a^*_L(t) = N, t \in \tau, T )</td>
</tr>
<tr>
<td>( M^*_T(t), t \in [\mu, T] )</td>
<td>( N, t \in [\tau, T] )</td>
</tr>
<tr>
<td>( p(t) = x_1(\mu-t) + \beta \left[ \pi^<em>_L - \sigma \right] \left[ e^{\pi T - t} \right] + \beta^</em> \left[ \pi^*_L - \sigma \right] \left[ e^{T - \pi T - t} \right] ), t \in [0, \mu) )</td>
<td>( p(t) = x_1(\mu-t) + \beta \left[ \pi^<em>_L - \sigma \right] \left[ e^{T - \pi T - t} \right] + \beta^</em> \left[ \pi^*_L - \sigma \right] \left[ e^{T - \pi T - t} \right] ), t \in [\mu, T] )</td>
</tr>
<tr>
<td>( q_L(t) = 0, t \in [0, T] )</td>
<td>( q_L(t) = 0, \mu \leq t \leq T )</td>
</tr>
<tr>
<td>( q_L(t) = 0, \mu \leq t \leq T )</td>
<td>( q_L(t) = 0, \mu \leq t \leq T )</td>
</tr>
<tr>
<td>( q_L(t) = \pi^*_L - \beta \theta L(t), \mu \leq t \leq T )</td>
<td>( q_L(t) = \pi^*_L - \beta \theta L(t), \mu \leq t \leq T )</td>
</tr>
<tr>
<td>( s^*_L = \beta N \mu, t \in [0, \mu) )</td>
<td>( s^*_L = \beta N \mu, t \in [\mu, T] )</td>
</tr>
<tr>
<td>( M-N e^{\theta L(o)} ), t \in [\mu, T] )</td>
<td>( M-N e^{\theta L(o)} ), t \in [\mu, T] )</td>
</tr>
</tbody>
</table>

where

\[
\mu = \frac{M - N}{\beta N} \quad \text{and} \quad \tau = \max \left\{ 0, T - \frac{\max \left\{ -\pi^*_L + \max \{G_L, 0\}, 0 \right\}}{\beta \pi_L} \right\}.
\]

The critical time point \( \mu = (M-N)/\beta N \) represents the time necessary for the size of the potential customer base to become smaller than \( N \).\(^{17}\) When \( G_L \leq 0 \), nobody from the loyal

\(^{17}\)If \( M = \infty \), then \( \mu \geq \tau \).
customer base should be contacted and \( N \) potential customers should be contacted until the smallest critical time point between \( \mu \) and \( \tau \). The critical time point \( \tau \) is different from \( T \) only when the immediate profit from contacting a potential customer \( (\pi_{p,c}) \) is negative. When this happens, it is not worthwhile after \( \tau \) to contact any potential customer. Notice that the critical time point \( \tau \) of Theorem 4.3 is the same as \( \tau(0) \) of Theorem 4.2 so that the same arguments hold for its interpretation.

When \( G_L > 0 \), loyal customers become profitable to contact. One thus needs to evaluate the tradeoffs between contacting someone from the potential customer base versus someone from the loyal customer base. These tradeoffs generate a critical time point, \( \tau \), after which loyal rather than potential customers should be contacted. This critical time, however, needs to be considered only when the immediate profit from contacting a potential customer is smaller than the surplus from contacting a loyal customer, that is, when \( \pi_{p,c} < G_L \). The total immediate loss from contacting a potential customer becomes \( \pi_{p,c} - G_L \). The continuous future expected gain from that contact (which comes only from not contacting the customer once transferred) is \( \beta \pi_L \). With \( \lambda = 0 \), it thus takes \(- (\pi_{p,c} - G_L)/\beta \pi_L\) units of time to cover the loss from contacting.

**Theorem 4.3** Let \( G_L \), \( \mu \), \( \tau \) be defined as above and \( \lambda = 0 \).

a) If \( G_L \leq 0 \), then nobody from the loyal customer base should be contacted. For the potential customer base, the optimal contact policies are:

<table>
<thead>
<tr>
<th>( \pi_{p,c} \geq 0 ) (( \tau = T ))</th>
<th>( \pi_{p,c} &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu &lt; \tau )</td>
<td>Contact ( N ) individuals up to ( \mu ), contact everybody after ( \mu )</td>
</tr>
<tr>
<td>( \mu \geq \tau )</td>
<td>Contact ( N ) individuals at all times</td>
</tr>
</tbody>
</table>
b) If \( G_L > 0, M \geq 2N \) and \( G_L \pi^c_P \leq \bar{T} \beta \pi_L \), the optimal contact policies are

<table>
<thead>
<tr>
<th>( \mu &lt; \tau )</th>
<th>( G_L \pi^c_P \leq 0 ) (( \tau = T ))</th>
<th>( 0 &lt; G_L \pi^c_P \leq \bar{T} \beta \pi_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>From the loyal customer base, contact nobody up to ( \mu ), contact ( N ) minus the number of potential customers contacted after ( \mu ).</td>
<td>From the loyal customer base, contact nobody up to ( \mu ), contact ( N ) minus the number of potential customers contacted between ( \mu ) and ( \tau ), and contact ( N ) loyal customers after ( \tau ).</td>
<td></td>
</tr>
<tr>
<td>From the potential customer base, contact ( N ) individuals up to ( \mu ), contact everybody after ( \mu ).</td>
<td>From the potential customer base, contact ( N ) individuals up to ( \mu ), contact everybody between ( \mu ) and ( \tau ), and contact nobody after ( \tau ).</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \mu \geq \tau )</th>
<th>From the loyal customer base, contact nobody at any time.</th>
<th>From the potential customer base, contact ( N ) individuals at all times.</th>
</tr>
</thead>
<tbody>
<tr>
<td>From the potential customer base, contact ( N ) individuals up to ( \mu ), contact everybody between ( \mu ) and ( \tau ), and contact nobody after ( \tau ).</td>
<td>From the potential customer base, contact nobody after ( \tau ).</td>
<td></td>
</tr>
</tbody>
</table>

It is tedious but straightforward to verify that when the discount factor \( \lambda \) is positive, Theorem 4.3 still holds. However, adjoint functions, multipliers, and size functions for the loyal customer base \( (s^*_L) \) of Table 4.4 (a) and (b) are different. Moreover, the critical time point \( \tau \) becomes (with \( \ln(a) = -\infty \) when \( a \leq 0 \))

\[
\tau_\lambda = \max \left\{ 0, T - \frac{1}{\lambda} \ln \left( 1 - \lambda \left[ \min \{ \pi^c_P - \max \{ G_L, 0 \}, 0 \} \right] \right) \right\}.
\]

By using the Taylor expansion of the function \( \ln(1+x) \), it can be easily verify that \( \tau_\lambda < \tau \) for any \( \lambda > 0 \). Consequently, when discounting is required, the entrepreneur has a smaller time interval during which it is worthwhile contacting an individual from the potential customer base.

As an illustration of my findings consider a secretary starting a printing company that focuses on business supplies such as business cards, name tags, glossys, etc. for French Canadian businesses. The secretary has estimated a potential market of \( M = 1000 \). She plans to visit around \( N = 100 \) French businesses every month to display her merchandise. The immediate expected profits in hundreds of dollars are respectively \( \pi^c_L = 3 \), \( \pi^c_P = 2 \), and \( \pi^c_P = -0.5 \) for contacted loyal customers, non-contacted loyal customers, and contacted potential customers. The finite planning
horizon is $T=60$ months.\(^{18}\)

Although an immediate loss is incurred by contacting potential customers ($\pi_p^c = -0.5$), the secretary realizes that it may be in her best interest to contact them because 25% of them will buy and be converted into loyal customers. Since one can easily verify that $M \geq 2N$, $G_L > 0$, $0 < G_L - \pi_p^c < T\beta \pi_L$, $(M-N)/\beta N = 36$, and $T = 57$, it follows from Theorem 4.3 (assuming no discounting) that for the first 35 months (i.e., for any $t \in [0, 36]$) 100 potential customers and no loyal customers should be visited. For any month $t$ where $t \in [36, 57)$, $100e^{-t/36}M$ potential customers and $100(1-e^{-t/36})N$ loyal customers should be visited. The reader should notice that $a_L^*(t) = 100(1-e^{-t/36})$ is smaller than the size of the loyal customer base which is $1000 - 100e^{-t/36}$. Finally, for the last three months (i.e., for $t \in [57, 60]$) 100 loyal customers and no potential customers should be visited. Figure 4.3 shows the complete optimal contact policy in continuous time.

![Figure 4.3 An Optimal Contact Policy: Number of Contacts versus Time](image)

\section*{4.5.4 Valuing the Size of the Customer Base}

The objective function has not accounted for a value of the entrepreneur's venture at the end of the finite planning horizon. I choose to value the size of the loyal customer base. The

\footnote{These profits can be generated as follows: Choose an expected revenue of 10 per sale, a cost of 3 per potential customer contacted and a cost of 2 per loyal customer contacted, a probability of buying of 50%, 20%, and 25% for, respectively, a contacted loyal customer, a non-contacted loyal customer, and a contacted potential customer.}
previous analysis (with no discounting) can be repeated for the objective function

\[
\max_{a_L(t),a_P(t)} \int_0^\tau \left\{ s_L \pi_L + a_L(t)G_L + a_P(t)\pi_P^c \right\} dt + \delta s_T,
\]

where \( \delta \) represents the value of a loyal customer at the end of the planning horizon. One can verify that the optimal contact policies are only affected through their critical time points \( \tau(0) \), \( \eta(0) \), and \( \tau \) which become, respectively,

\[
\tau_\delta(0) = \max \left\{ 0, T - \frac{\max\{-\pi_P^c - \beta \delta, 0\}}{\beta \pi_L} \right\},
\]

\[
\eta_\delta(0) = \max \left\{ 0, T - \frac{\max\{-\pi_P^c - \beta \delta, 0\}}{\beta \pi_L^c} \right\},
\]

\[
\tau_\delta = \max \left\{ 0, T - \frac{\max\{-\pi_P^c - \beta \delta + \max\{G_L,0\}, 0\}}{\beta \pi_L} \right\}.
\]

Therefore, when valuing the size of the loyal customer base the entrepreneur should contact potential customers a little longer (e.g., \( \tau_\delta(0) > \tau(0) \), \( \delta > 0 \)) because she gains from growing the loyal customer base.

4.6 Extensions

This section presents a few extensions from the Markov decision model of section 4.4.

I assumed that each type of potential buyer takes the same amount of time to contact. This assumption can be relaxed at the expense of having fractional buyers. Let \( m_L \) and \( m_P \) be the number of minutes to contact a loyal and a potential customer, respectively. Theorem 4.1 can be applied by using \( G_L' \) and \( G_P' \) where \( G_L' \) is the return from \( l/m_L \) loyal customers \( (G_L' = G_l/m_l) \) and \( G_P' \) is the return from \( l/m_P \) potential customers \( (G_P' = G_P/m_P) \). For instance, if \( G_P' > \max\{G_L',0\} \), then it is optimal to contact \( l/m_P \) potential customers, where \( l \) represents the total amount of time allocated each period to prospecting for customers.

In the previous paragraph, the time constraint \( a_L + a_P \leq N \) has been replaced by \( m_L a_L + m_P a_P \).
Now, a constraint on time can be replaced by a constraint on cashflow where \( a_L a_L + a_p a_p \leq c_t \), with \( c_t \) being the current cash balance and \( a_L \) and \( a_p \) being the contacting cost of a loyal and a potential customer, respectively. The only difference is that unlike \( \Gamma \), \( c_t \) varies over the periods.

Assuming that the contact of individuals is fractional and there is no limits on the number of contacts, I can propose simple optimal contact policies by comparing the marginal return from contacting a loyal customer \( \frac{G_L}{a_L} \) to the marginal return from contacting a potential customer \( \frac{G_P}{a_p} \). For instance, if \( \frac{G_L}{a_L} > \frac{G_P}{a_p} \) and \( G_L > 0 \), then it would be optimal to contact \( \min\{s_0, c_t / a_L\} \) loyal customers. If there is any cash balance left, then it should be used to contact potential customers whenever \( G_P \) is large enough. If one restricts the number of contacts so that a maximum of \( \Gamma \) units of time can be allocated to contacting potential buyers, then the optimal number of loyal customers to contact becomes \( \min\{s_0, c_t / a_L, \Gamma / m_t\} \). Potential customers would be contacted only if there are time and cash balance left (and, of course, \( G_P \) is large enough). This analysis does not hold if the contact of an individual cannot be fractional. It becomes a combinatorial problem which appear to be hard to solve.

A loyal customer might decide not to buy any more, say with probability \( \theta \) each period, perhaps switching to a competitor. If we make the appropriate independence assumptions, then a customer will be loyal for a geometrically distributed number of periods. Then Theorem 4.1 can be modified by redefining \( G_P \) and \( G_P^c \). The expected net present value of contacting a potential customer with no followup contacts is

\[
G_p = \pi_p + \frac{\lambda(1-\theta) \beta \pi_L}{1 - \lambda(1-\theta)}
\]

and with followup contact is

\[
G_p^c = G_p + \frac{\lambda(1-\theta) \beta G_L}{1 - \lambda(1-\theta)}.
\]

The rest of the proof is almost the same as in section 4.4 and technical details are given in
Another extension allows degree of loyalty. As before, suppose that contacting a potential customer results in that customer's becoming loyal with probability $\beta$. But now suppose, in addition, that when a potential customer makes his first purchase, he moves into category 1 loyalty; and when a customer in category 1 is contacted, there is a probability $\beta_1$ that the customer will move into category 2 loyalty. As in the case of a potential customer, if there is no contact, then the customer will never move into the next category. In this case, a theorem similar to Theorem 4.1 can be proved using the same type of argument, although the state space becomes two-dimensional and there are many more cases. One case is similar to (c.i). Let $G_p$ be defined as in section 4.4 as the expected net present value of a contact of a potential customer with no followup contacts. Similarly, let $G_1$ be the expected net present incremental value of a contact of a category 1 customer with no followup (compared to the expected return from a category 1 customer with no contact) and let $G_2$ be the expected net present incremental value of a contact of a category 2 customer. If $G_2 > G_1 > G_p > 0$ it is optimal to contact as many of the category 2 customers as possible. And if there are time slots left, contact as many of the category 1 customers as possible. And then, if there are still time slots left, contact as many potential customers as possible. Other cases involve considering $G_p^c$ which would assume followup contacts when the customer becomes category 1 but not category 2, as well as $G_p^{cc}$ which assumes followup contacts at both loyalty levels and $G_1^c$ which assumes followup contacts at loyalty level 2.

Schmittlein and Peterson (1994) do a customer base analysis of an industrial market to predict individual purchase patterns based on past purchase behavior. Their work provides powerful techniques that could be adapted to estimate the many parameters of a multiclass model including the transition probabilities governing the dynamics of such a model.
4.7 Conclusions

Entrepreneurs involve in their first start-ups may know a lot about their products and their characteristics, but have often little understanding regarding prospecting for customers. I discussed several models in the context of allocating time to different types of customers. Infinite and finite planning horizons have been investigated. An incremental lifetime value has been evaluated for each contact and optimal allocations have been determined for two situations involving long-run optimization with and without constraints on short-run cashflow.

For each infinite-horizon model with no constraints on short-run cashflow, a benefit was allocated to each type of contact. The result was that each time slot could essentially be managed independently of the others based on a comparison of the possible benefits. The optimal policies were of the form "among the contacts available during a time slot, choose the one which has the largest expected benefit."

When the planning horizon was infinite, the loyal customer base grew as long as the short run losses from contacting potential customers with negative immediate expected profits was eventually covered by the long run expected gains from converting a proportion of those potential customers into loyal customers. This happened when the expected proportion of buyers among contacted potential customers ($\beta$), or the discount rate ($\lambda$), or the immediate expected profit from not contacting a loyal customer ($\pi_L$) was large enough. In the basic model, the entrepreneur had an unrestricted amount of credit to contact customers. I then showed that it is optimal every period to contact only potential customers. If loyal customers were profitable to contact and preferred to potential ones, then a mixed contact policy where both loyal and potential customers are contacted was optimal whenever $\beta$, $\lambda$, or $\pi_L$ is large enough and the number of loyal customers is smaller than the number of time slots.

In the cashflow model, the entrepreneur had no access to a line of credit and the cashflow
of the period following the contacts had been restricted to be non-negative. When loyal customers were not profitable to contact, I demonstrated that it is optimal each period to contact as many potential customers as permissible by the cash reserves and the contact limit. When loyal customers were profitable to contact, a numerical example showed that one cannot always find the optimal contact policy by comparing the gain from contacting a loyal customer to the gain from contacting a potential customer.

When the planning horizon was finite, I observed that short run losses from contacting potential customers with negative immediate expected profits were not covered by the long run expected benefits when contacts were made at periods relatively close to the end of the planning horizon. Therefore, critical time points after which no more potential customers must be contacted were derived. Moreover, when loyal customers were profitable to contact and preferred to potential ones, I obtained a critical time point after which only loyal customers should be contacted. Since the number of buyers from each category were deterministic, the framework allowed one to predict for every period the number of buyers from each category (and so the number of loyal customers) once a contacting policy had been specified.
Chapter 5

General Discussion and Conclusions

Entrepreneurs make decisions that influence subsequent decisions and future performance. The dissertation studied such sequences of decisions by using dynamic programming. This approach provided a description of the decision process over time and, in some cases, it prescribed how business performance could be improved.

Fundamental questions addressed by entrepreneurs at new venture creations were investigated. Those questions involved allocation of resources at various decisional stages of business start-ups. The dissertation is composed of three such questions that are separately investigated in Chapter 2 to Chapter 4.

Chapter 2 focused on the demands for time between another job and starting a new venture. More specifically, it determined how and when an entrepreneur makes the commitment to allocate time and money to start a new venture so as to maximize total earnings. It was found that when a venture requires a large investment in time at early stages, entrepreneurs with low work tolerance stay in the wage job and entrepreneurs with high work tolerance spend all their time on the venture. When the initial time investment in a venture pays immediate benefits that diminish with additional time spent, intermediate strategies where entrepreneurs with low work tolerance spend all their time on the venture but entrepreneurs with high work tolerance split their time were found.

The demands for time between another job and starting a new venture were also investigated over multiple periods. From one period to the next, the entrepreneur accumulates reputation which summarized the effects of current and past business expenditures in building up demand. By reputation I meant some continuing and cumulative index of the status of the venture.
such as goodwill and product quality. It was found that when the return on demand from reputation decreases, entrepreneurs with high or medium work tolerance should never give up the wage job. Given constant business expenditure each period and increasing returns to reputation, time allocated to the venture should increase over time and entrepreneurs with low work tolerance should eventually give up the wage job.

In Chapter 2 I focused on the core aspect of the time-allocation decisions. There are a number of useful directions for extending the results to capture more of the complexity in real entrepreneurial decisions. In the following paragraphs, I suggest extensions on how complexity in both the entrepreneur (e.g., other objectives) and the environment (e.g., uncertainty) can be dealt with.

To set up the new venture, the entrepreneur may choose giving up time not only from a wage job, but also from leisure. I assumed in the dissertation that leisure had been cut back as far as possible. By relaxing this assumption, one can address a new set of questions. Would the proportion of time in leisure decrease as the new venture grows? Would the patterns in time allocated to leisure depend on the entrepreneur's work tolerance? Might there be entrepreneurs with high work tolerance who have any time to allocate to leisure? This new problem requires an additional decision variable for the time allocated to leisure and also requires adding to the objective function (total earnings) a penalty or a bonus function of time allocated to leisure (it is a penalty if money is spent during leisure time, but a bonus if the entrepreneur values leisure).

I also assumed that the time spent on the wage job was flexible. For some salaried jobs it would be more realistic to introduce constraints on the allowable time allocations to the wage job. For instance, a lecturer in a university must commit to teaching zero, one, two, or three courses in any given semester, and is thus restricted to choosing among four allowable discrete numbers of hours to spend on a wage job. I anticipate that my qualitative results would still hold. More specifically, when marginal returns from investing time in the venture are constant on a
given period but ordered over the periods (this was the case with a Cobb-Douglas production function, linear demand and no discounting), the entrepreneur would be committed to only one activity at a time – wage job or venture – and there would be a critical period after which the entrepreneur would move from one activity to the other. When returns on demand from accumulated reputation are decreasing, some high work tolerant entrepreneurs could be deterred from ever starting a new venture as the required minimum time-investment into the wage job would not allow them to reach the critical level for their new venture.

For many entrepreneurs, a key objective when starting a business is to keep the business alive (Inc. 500 1994). By focusing on sole proprietorships, Holtz-Eakin, Joulfaian, and Rosen (1994) argue that "if entrepreneurs cannot borrow to attain their profit-maximising levels of capital, then those entrepreneurs who have substantial personal financial resources are more successful than those who do not." By successful they mean more likely to survive and perform better if they survive. A stochastic framework could be utilized to describe the best time to become fully involved in a new venture based on maximizing business survival. A tradeoff between the probability of business survival and expected profit would be introduced by defining an objective function that maximizes expected profit subject to maintaining a high survival probability, or an objective function maximizing the survival probability subject to making a minimum specified expected profit. A complementary study to Holtz-Eakin, Joulfaian, and Rosen’s work would also include those entrepreneurs who have benefited from equity financing.

My results were derived in a deterministic environment where one could predict from one period to the next what the reputation level and production level would be given a time-allocation policy. When there is considerable uncertainty in the demand function or the production function, would the proposed time-allocation patterns still hold? Would tolerance for work and rate of change in returns from allocating time to the venture still be the key variables? One could address these questions by formulating the time-allocation problem as a continuous-
time stochastic optimal control model as it exists in the resource economics literature.

By generating money as salary from the wage job or as profit from the new venture, money could be spent on the development of the product to increase its quality. Chapter 3 focused on the allocation of resources within the new venture, particularly the allocation of money to R&D activities. It investigated how entrepreneurs who undertake a new product development activity aimed at starting a new business were affected by the uncertainty in their funding. The tradeoff between the advantage of waiting (to generate more money) and improving the quality of the product versus releasing the product before competition increases was studied. The framework prescribed an optimal time to release the new product and described how this release-time strategy was affected by the expected amount of funding and its uncertainty. Chapter 3 focused on business start-ups that are self-financed or those that rely on debt financing for additional funding.

For entrepreneurs who generated fixed financing (e.g., wages) over time, an optimal time to market the product was derived. This optimal time was shown to decrease as financing increases whenever the marginal gain from investing in developing the new product was initially large compared to the marginal loss from competition. The optimal time to market the product was shown to increase as financing increases whenever the marginal gain from investing was initially small compared to the marginal loss from competition.

Entrepreneurs who generated fixed wages over time saw the quality of their products at market release increase as wages increased. However, for entrepreneurs who did not collect wages over time (thus relying on an uncertain flow of funding from debt financing) the effect on product quality of an increase in capital investment depended on the magnitude of the investment. Consequently, entrepreneurs who could not raise considerable amounts of money saw the quality of their products at market release increase as the expected amount of funding increased. Entrepreneurs with the ability to raise large amounts of money saw the quality of their
products decrease as the expected amount of funding increased.

Chapter 3 concentrated on the effects of capital investment on product quality. There are useful directions for extending this line of research that capture the further uncertainties in entrepreneurial decisions. These include allowing the product to randomly deteriorate over time and modeling the arrival of competitors by a stochastic process (e.g., a Poisson process). These additional features would increase the mathematical complexity of the problem, but it would better describe the many levels of uncertainty faced by real entrepreneurs.

A classical approach to project investment was utilized where the entrepreneur’s objective was to maximize the net present value (NPV) of the R&D project. Dixit and Pindyck (1994) propose a real options approach to investment that accounts for irreversibility (the investment can somehow be undone) and option value. The opportunity to invest in a project thus becomes like holding an option - “the decision maker has the right but not the obligation to buy an asset at some future time of its choosing.” Unlike the classical NPV approach, their approach suggests that one should invest when the value of a unit of capital overcomes its purchase and installation cost by at least the value of keeping the investment option alive. Dixit and Pindyck’s theory of irreversible investment under uncertainty thus propose a model of uncertainty about future rewards from an R&D investment that allows the inclusion of an opportunity cost for postponing action to get more information about the future.

Equity financing has been overlooked throughout the dissertation as I focused on sole proprietors. However, especially in the context of Chapter 3 where start-ups are based on new product development, equity financing is commonly utilized. Would the results regarding the effect of an increase in funding on product quality still hold in a framework where funding could come from equity financing? Would the entrepreneur’s ability to raise money still be a key variable in determining how the quality of the product behaves with respect to changes in funding?
As uncertainty in funding directed some of the entrepreneur's operational decisions in Chapter 3, it did not capture any aspects associated with attracting customers. Chapter 4 introduced various models in the context of allocating time (and money) to different types of customers. These models studied what a rational entrepreneur will do when faced with the allocation of time to different customer categories and they also provided guidelines for improving the performance of an entrepreneur who may not be acting optimally.

The framework constructed allowed one to evaluate the incremental lifetime value of each potential buyer. When no constraints were assigned to the short run cashflow, the optimal policies were of the form "among the contacts available, choose the one which has the largest expected benefit." If the entrepreneur had an unrestricted budget to contact, then it was optimal for the entrepreneur to contact every period a maximum number of potential customers because the expected future stream of revenues covered the cost of contact. If loyal customers were profitable to contact and preferred to potential ones, then mixed contact policies where both loyal and potential customers were contacted were optimal (under certain conditions based on the model parameters). When the entrepreneur had no up-front savings to contact and, in addition, the cashflow of the period following the contacts was restricted to be non-negative, then the optimal contact policy was not only limited by the entrepreneur's time, but also it was limited by the cash reserves accumulated through sales. I demonstrated that, if loyal customers are not profitable to contact, then it is optimal each period to contact as many potential customers as permissible by the cash reserves and the contact limit. When loyal customers became profitable to contact, I showed that the simple optimal policy "among the contacts available, choose the one which has the largest expected benefit" did not necessarily hold.

When the entrepreneur had limited time to prospect for customers, I observed that short run losses from contacting potential customers with negative immediate expected profits were not covered by long run expected benefits when contacts were made at periods relatively close to
the end of the planning horizon. Critical time points after which no more potential customers should be contacted were derived.

Chapter 4 already presents various extensions of the stochastic model including a model where different types of contacts require different levels of effort and a multiclass model. In the two following paragraphs, I propose a few more extensions dealing with complexity in both the resource available (e.g., credit) and customer behavior.

On one hand, the entrepreneur had access to enough capital to contact as many potential buyers as profitable. On the other hand (only for the stochastic formulation), the entrepreneur had no unutilized credit and could not contact unless expected gains from buyers were accumulated. What if the entrepreneur had access to a line of credit? Would potential customers still be contacted first? One would need that, when a potential customer is contacted with borrowed money, the interest paid each subsequent period be smaller than the expected gain received each subsequent period. Might there exist additional critical time points after which borrowed money should not be used to contact? The formal derivations of the optimal policies will involve more decision variables since, to express the dynamic of debt (or credit left), one will need to know how many contacts are made with cash reserves and how many are made with credit.

More insights on building and exploiting a customer base would be provided if some of the assumptions were relaxed. Specifically, repeated contacts of a potential customer might not have the same probability of success. Some loyal customers might be lost over time and the probability of doing so in any period would vary from customer to customer. Repeated contact of loyal customers might affect (most probably negatively) their propensity to buy.

Besides the additional investigations proposed for each chapter, one can explore the effect of economic conditions on entrepreneurial venturing. For instance, a recession would lower wage rates, make it harder to get financing, and provide a lower discount rate. In the context of Chapter 2 the entrepreneur would venture longer. More specifically, when demand is linear in reputation
and marginal returns from the business increase, the entrepreneur should start to be fully involved with the new venture faster during a recession that before the recession. When the marginal returns from the business decrease, she should leave the venture later than she would before the recession. In the context of Chapter 3, a recession would decrease funding thus generating products of lower quality, unless the entrepreneur’s ability to raise money is high in which case a decrease in funding would still deliver products of better quality. A recession in the context of Chapter 4 would translate into allocating more contacting effort to loyal customers on a long-term planning horizon. On the short-term, entrepreneurs would have fewer time periods available to make the contacts of potential customers profitable.

The dissertation presented an analytical approach to various models of entrepreneurial decisions. For these models to be of real value, one must be able to validate them. Consequently, a few more questions must be answered. How would one test the various models? How would one apply these models in a practical setting? What information would one need and is the information out there?

In the context of Chapter 2, data from federal individual tax returns would provide part of what is needed to empirically test the time-allocation model. Longitudinal data from tax returns would identify proprietorship and entrepreneurs who get revenues from a wage job. It would also provide business expenditures and sales; this information could be utilized to estimate the demand function. Given the industry (employer) of the wage job, time allocated to a wage job could be estimated by taking the average salary from the industry and extracting from the total wages the estimated number of hours spent at the wage job. If one knows the functional form of the production function, one can estimate the time allocated to the new venture as follows. Since the production function only depends on time allocated to the business and capital investments and since capital investments can be extracted from the tax returns, time allocated to the venture can be derived. If one can identify the entrepreneur’s tolerance for work (e.g., low work tolerant
individuals work for 40 hours a week and high work tolerant individuals work for 60), one can also estimate the time allocated to the venture and derive the production function. The database so formed would allow one to verify if the model’s predictions are representative of what real entrepreneurs actually do.

Once the framework of Chapter 2 has been validated, it can be applied to help practitioners to improve their business decisions regarding time-allocations in new ventures. If one knows the entrepreneur’s tolerance for work and the behavior of the business returns from investing time into the business, then one could suggest whether the entrepreneur should ever leave the wage job. Over multiple periods, one would need to estimate the demand as a function of reputation and verify that the production function is well represented by a power function of time allocated to the venture and capital investment.

In the context of Chapter 3, the validation of the model involves identifying sole proprietors whose business start-ups are based on the development of a new product. The software industry should provide a good sample of such entrepreneurs. However, data on new product development projects are not common on new venture start-ups. Hence, one would need to collect data by following up on start-ups until the product is put on the market and the revenue function can be estimated from first sales. One could then identify the evolution of the product quality (and thus the parameter associated with quality improvement). The evolution of competition would depend on the industry and could be estimated. The flow of funding would be observed for the selected entrepreneurs. Since the revenue function has a special structure in the mathematical model, one would need to focus on R&D projects where the estimated revenue function would have the assumed behavior. Going back to the data, one could observe how the product quality has been influenced by the value of financing and its uncertainty.

The framework of Chapter 3 has descriptive merit. It is used to understand practitioners’ decisions with respect to financing and new product development. The prescriptive nature of this
model is hard to use in practice because the model parameters and the function underlying the model can hardly be estimated up-front. Moreover, the entrepreneur's ability to raise money is a key variable in the model and it is very hard to measure.

The main contribution in Chapter 4 resides in proposing an approach where the lifetime value of each potential contact can be evaluated and a simple optimal contact policy can be implemented by ordering these lifetime values. In a practical setting, the performance of the basic model can be validated with real purchase data. Such data are already available in many large companies and have been commonly used by marketing researchers and practitioners. However, my model is more representative of small start-up companies, which must build a customer base that focuses on products that are publicized by direct contacts from the entrepreneur who cannot rely on, or afford, mass advertising. In these small start-up companies, real purchase data are not as commonly available because it is too costly (the entrepreneur has no time and no money to allocate to this task).

One way to validate the basic model of Chapter 4 would consist of asking owners of business start-ups their estimations of the various parameters of the model. More specifically, one would ask the following questions: How many contacts can be handled during a period? What is the propensity for a potential customer to purchase? What are the expected profits for each type of potential buyer? One could then compare their contact strategy to what is shown to be optimal by the model. A longer (but more accurate) way to validate the model would consist of building up a database of real purchase data of new venture start-ups, as it has been done for larger established corporations. It would involve keeping track of each contact and each purchase over a certain period of time. This database could be used to estimate the number of contacts the entrepreneur can make over time, the probability of purchasing for a potential customer and the expected profit from each type of potential buyers. Then, contact policies utilized by practicing entrepreneurs could be compared to the optimal contact policies proposed by the model. If the
more successful entrepreneurs choose contact policies similar to the ones suggested by the model, then start-up companies that may not be acting optimally would be encouraged to implement the optimal contact policies suggested by the validated model.

This dissertation is only the beginning of a long journey into the conceptualization and understanding of entrepreneurial decision-making. It focused on resource allocations at the commitment phase and at the operation phase of a new venture creation. A dynamic programming approach has been most appropriate for investigating fundamental questions that involved an explicit consideration of the evolution of the new venture.

There are various methods for gaining insight into decision-making processes of entrepreneurs. As a contrast to the usual qualitative or empirical studies, I presented an analytical approach that provided a new focus to entrepreneurship research. Analytical methods are useful in that they provide sharp answers to focused questions and thus extend one's knowledge of the issue being studied. When the time dimension takes center stage in the decision process, analytical methods permit one to explicitly model the evolution of the business process as it varies over time. Thus, cause-effect relationships between the variables can be identified and rational for those relationships can be found. An analytical approach to entrepreneurship can also provide clearer hypothesis to test, thus guiding empirical studies of various entrepreneurial processes. Moreover, the optimization perspective allows decision-makers to find the optimal strategy rather than limiting the search for the best action to a small set of possible actions.

Finally, I reiterate the three major areas of impact I anticipate from this research. First, an analytical approach contributes to a reorientation of the scientific literature on entrepreneurship which has been dominated by empirical work. Second, this dissertation encourages researchers to fulfil the need for a multidisciplinary emphasis in any research work on entrepreneurship. Third, the techniques developed will be useful to practitioners who wish to improve their business decisions.
Bibliography


Inc. 500 (1994), "How to Start an Inc. 500 Company," page 64.


Appendix A

Technical Detail for the Abstract Example of Sub-Section 2.5.2

The production capacity constraint will be binding at optimality, that is, \( D(A_t) = K(h_t, i_t) \). Hence, the optimal capital investment can be expressed as a function of \( h_t, e_{0,1}, ..., e_{t-1} \) and replaced in (2.1) of sub-section 2.5.1, for every period \( t \), to give

\[
P(h_0, h_1, ..., h_T, e_0, e_1, ..., e_T) = \sum_{t=0}^{T} \beta^t \left[ \rho D(A_t) + w(\tau - h_t) - e_t - \left( \frac{D(A_t)}{K} \right)^{1 - \alpha} h_t^{1 - \alpha} \right] + \beta A_{T+1},
\]

where \( A_t = \gamma \left[ (1 - \delta)^t e_0 + (1 - \delta)^{t-1} e_1 + ... + e_{t-1} \right], t = 1, ..., T, A_0 = 0. \)

In this specific example, the optimal time allocation policies are found by characterizing the earnings function \( P \) with respect to the \( e_i \)'s. It is easier to express \( P \) as a function of the \( e_i \)'s because those decision variables define the dynamic of the problem and because, unlike the \( h_i \)'s, they have no constraints on their upper bound. \( P \) is thus maximized over \( h_i \) (\( P \) is concave in \( h_i \)) to give

\[
h_i^*(e_0, e_1, ..., e_{t-1}) = \left( \frac{\alpha}{w(1 - \alpha)} \right)^{1 - \alpha} \frac{D((1 - \delta)^t A_0 + \gamma (1 - \delta)^{t-1} e_0 + \gamma (1 - \delta)^{t-2} e_1 + ... + \gamma e_{t-1})}{\kappa}.
\]  

(A.1)

For every period, the optimal time allocation \( h_t^* \) is proportional to the demand function \( D(A_t) \). The level of reputation at period \( t, A_t \), is a linear function of business expenditures prior to \( t \), except for \( t=0 \) where \( h_0^* \) is simply determined by \( A_0 \).

The objective function can now be expressed by

\[
P(e_0, e_1, ..., e_T) = \left[ \rho - \frac{1}{\kappa(1 - \alpha)} \right] \sum_{t=0}^{T} \beta^t D((1 - \delta)^t A_0 + \gamma (1 - \delta)^{t-1} e_0 + \gamma (1 - \delta)^{t-2} e_1 + ... + \gamma e_{t-1})
\]

\[
+ \beta \gamma (1 - \delta)^T e_0 + (1 - \delta)^{T-1} e_1 + ... + e_T \right].
\]  

(A.2)

Notice that the business expenditure at the last period does not affect the production capacity constraints since there are no sales after period \( T \). This translates into an objective function that is always linear in \( e_T \). If the coefficient of \( e_T \) in the objective function is negative, then the entrepreneur maximizes total earnings by choosing \( e_T = 0 \). If this coefficient is positive, then \( e_T \) should be infinity. It is assumed that the entrepreneur does not encounter any business expenditures (for marketing, product development, etc.) at the last period and so \( e_T = 0 \).
I first assume that \( D(A_t) = A_t \) (or \( D(A_t) = \theta A_t \) for some positive real number \( \theta \)). From (A.2), \( P \) is linear in \( e_t \) for any \( t \) and is expressed by

\[
P(e_0, e_1, \ldots, e_T) = \sum_{t=0}^{T-1} C_t e_t + \sum_{t=0}^{T} \lambda^t w_t,
\]

where \( C_t = \lambda^t (\rho - C)^{\gamma} \delta |1 - (1 - \delta)^{\gamma}| - \lambda^t + \alpha (1 - \delta)^{\gamma} t \) and \( C = \frac{1}{\kappa(1 - \alpha)} \left( \frac{w(1 - \alpha)}{\alpha} \right)^{\gamma} \),

for \( t = 0, 1, 2, \ldots, T-1 \) (remember that \( e_T = 0 \)).

When future earnings are not discounted (and so the discount rate \( \lambda \) is equals to 1), it is straightforward verifying that the \( C_t \)'s are ordered so that they decrease over the periods whenever \( (\rho - C) < \beta \delta \) or, equivalently, whenever

\[
w < \frac{\alpha}{1 - \alpha} \left( \frac{1}{\kappa(1 - \alpha)} \right)^{\gamma},
\]

and they increase otherwise. When \( C_0 < C_1 < \ldots < C_T \), there exist \( n \) such that \( C_0 < C_1 < \ldots < C_n < 0 < C_{n+1} < \ldots < C_T \). Consequently, \( e_0^* = e_1^* = \ldots = e_n^* = 0 \) and, from (A.1), \( h_1^* = h_2^* = \ldots = h_{n+1}^* = 0 \) (with \( A_0 = 0 \) and \( D(A_t) = A_t \)). Furthermore, \( e_{n+1}^* \), \( e_{n+2}^* \), \ldots, \( e_{T-1}^* \) are positive and as large as possible so that \( h_{n+2}^* = h_{n+3}^* = \ldots = h_T^* = \tau \). It is thus optimal to allocate no time to the new venture until period \( n+1 \), and allocate all the working time to the new venture from period \( n+2 \) to the end of the planning horizon. The optimal business expenditure only starts at period \( n \).

When \( C_0 > C_1 > \ldots > C_T \), there exist \( k \) such that \( C_0 > C_1 > \ldots > C_k > 0 > C_{k+1} > \ldots > C_T \). Consequently, \( e_0^* = e_1^* = \ldots = e_k^* = 0 \) and so \( h_{k+2}^* = h_{k+3}^* = \ldots = h_T^* = \tau \). In addition, \( e_{k+1}^* = e_{k+2}^* = \ldots = e_{T-1}^* = 0 \) and \( h_{k+2}^* = h_{k+3}^* = \ldots = h_T^* = \tau \).

One can better interpret the above results by noticing that the coefficient \( C_t \) is positive if and only if \( w < MR_t^\gamma \) where, for \( t = 0, 1, \ldots, T-1 \),

\[
MR_t^\gamma = \frac{\alpha}{1 - \alpha} \left( \frac{1}{\kappa(1 - \alpha)} \right)^{\gamma} \frac{\rho \gamma (1 - (1 - \delta)^{\gamma}) - \beta \gamma (1 - \delta)^{\gamma} + \alpha (1 - \delta)^{\gamma}}{\gamma (1 - (1 - \delta)^{\gamma})}
\]

is the marginal revenue (or return) from the venture at period \( t \). One can verify that these periodical marginal revenues are ordered, that is,

\[
MR_0^\gamma < MR_1^\gamma < \ldots < MR^\gamma_T \quad \text{whenever} \quad \beta > \frac{1}{\gamma}
\]

and \( MR_0^\gamma > MR_1^\gamma > \ldots > MR^\gamma_T \) whenever \( \beta \leq \frac{1}{\gamma} \).

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19 Since, from (A.1), \( h_t^* (e_0, e_1, \ldots, e_T) \) is linear in \( e_t \)'s, \( P \) is also linear in \( h_t \)'s.
I now show that when earnings are not discounted ($\lambda=1$) and the return on demand from reputation is decreasing ($D$ concave in $A_t$), the optimal time-allocation policy is driven by optimizing $D$ with respect to $e_t$ rather than binding the constraints $e_t \geq 0$, $h_t \geq 0$, and/or $h_t \leq \tau$. It implies that the returns from allocating time in the new venture are independent of time, and so the optimal time-allocation policy behaves quite differently than it did in the above discussion.

It is tedious but straightforward to show that the objective function given by (A.2) is concave in $(e_0, e_1, \ldots, e_T)$ whenever the demand, $D(A_t)$, is concave in $A_t$ and the per unit price for the product is high enough, that is,

$$\rho - \frac{1}{\kappa(1-\alpha)} (\frac{w(1-\alpha)}{\alpha})^a > 0.$$ 

First-order optimality conditions can be utilized to derive the optimal time-allocation path. With $D(A_t) = A_t^{\gamma/2}$,

$$e_0^* = \frac{1}{\gamma} \left\{ \frac{\gamma (\rho - C)}{2 \delta} \right\}^2 \cdot (1-\delta) A_0$$

$$e_{t-1}^* = \frac{1}{\gamma} \left\{ \frac{\gamma (\rho - C)}{2[1-\beta \gamma (1-\delta) \cdot \frac{\gamma (\rho - C)}{2 \delta}]^2 \cdot (1-\delta) \cdot \frac{\gamma (\rho - C)}{2 \delta} \right\},$$

where $C = \frac{1}{\kappa(1-\alpha)} (\frac{w(1-\alpha)}{\alpha})^a$.

The business expenditures are thus constant over time, except for the first and the last two periods where those expenditures must adjust to the problem requirements (i.e., $A_0=0$ and $e_T=0$).

From (A.1),

$$h_t = \ldots = h_{T-1}^* = \text{Min} \left\{ \tau, \tau_0 = \frac{\alpha}{w(1-\alpha)} \left\{ \frac{\gamma}{2 \delta \kappa} \left[ \rho - \frac{1}{\kappa(1-\alpha)} \left( \frac{w(1-\alpha)}{\alpha} \right)^a \right] \right\}, \right\} \quad \text{(A.3)}$$

$$h_T^* = \text{Min} \left\{ \tau, \left( \frac{\alpha}{w(1-\alpha)} \right)^{1-a} \frac{\gamma}{2[1-\beta \gamma (1-\delta) \cdot \kappa] \left[ \rho - \frac{1}{\kappa(1-\alpha)} \left( \frac{w(1-\alpha)}{\alpha} \right)^a \right] \right\}.$$

The time-allocation path is constant over time, except for the first period where the time allocation ($h_0^*$) is predetermined by the initial level of reputation ($A_0$) and for the last period where it is affected by the value of a unit of reputation ($\beta$).\(^{20}\)

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\(^{20}\) One can verify that $h_T^* = h_{T-1}^*$ when $\delta=1$, that is, when reputation at $t+1$ does not depend on reputation at $t$, but only on business expenditure at $t$. One can also verify that $h_T^* = h_{T-1}^*$ when $\beta=1/\gamma$. 

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I finally derive the optimal time allocation for the case where the business expenditure is constant over time. With a Cobb-Douglas production function, one finds after some algebraic manipulations that, for $t=1,2,...,T$,

$$h_t^*(e) = \left( \frac{\alpha}{w(1-\alpha)} \right)^{1-\alpha} \frac{D(t)^{\gamma[1-(1-\delta)]}}{\delta \kappa}.$$

It follows that

$$\frac{\partial h_t^*}{\partial t} = \frac{1}{\kappa} \left( \frac{\alpha}{w(1-\alpha)} \right)^{1-\alpha} \frac{dD}{dt} \frac{\partial A_t}{\partial t}.$$

Since $\frac{\partial A_t}{\partial t} > 0$,

$$\frac{\partial h_t^*}{\partial t} > 0 \text{ if and only if } \frac{dD}{dt} > 0.$$

Therefore, the optimal time allocation increases with respect to time whenever the demand increases with respect to reputation.

**Appendix B**

**Technical Detail for the Equivalence of the Two Formulations of Section 4.4.1**

Let $s_1,s_2,s_3,...$ be a realization of the states, and $V_{\phi}^O(s_1 \mid s_1,s_2,s_3,...)$ and $V_{\phi}(s_1 \mid s_1,s_2,s_3,...)$ be, respectively, the value of policy $\phi$ under the original (unbounded-reward) formulation and the new (bounded-reward) formulation, given the state realization. I show that

$$V_{\phi}^O(s_1 \mid s_1,s_2,s_3,...) - V_{\phi}(s_1 \mid s_1,s_2,s_3,...)$$

is independent of policy $\phi$.

One has

$$V_{\phi}^O(s_1 \mid s_1,s_2,s_3,...) = E_d, d_1, L(s_1) G_L + E_d, d_1, P(s_1) \pi_P + s_1 \pi_L$$

$$+ \lambda \left[ E_d, d_2, L(s_2) G_L + E_d, d_2, P(s_2) \pi_P + (s_2 - s_1) \pi_L + s_1 \pi_L \right]$$

$$+ \lambda^2 \left[ E_d, d_3, L(s_3) G_L + E_d, d_3, P(s_3) \pi_P + (s_3 - s_2) \pi_L + (s_2 - s_1) \pi_L + s_1 \pi_L \right]$$
+ \cdots + \lambda^k l \left[ E_{d_k} d_{kL} (s_k) G_L + E_{d_k} d_{kP} (s_k) \pi_p^c + (s_k - s_{k-1}) \pi_L + \cdots + (s_2 - s_1) \pi_L + s_1 \pi_L \right]

\text{and}

V_\varphi (s_1|s_1, s_2, s_3, \ldots) = E_{d_1} d_{1L} (s_1) G_L + E_{d_1} d_{1P} (s_1) \pi_p^c + \frac{\lambda (s_2 - s_1)}{1 - \lambda} \pi_L

+ \lambda E_{d_2} d_{2L} (s_2) G_L + \lambda E_{d_2} d_{2P} (s_2) \pi_p^c + \frac{\lambda^2 (s_3 - s_2)}{1 - \lambda} \pi_L

+ \lambda^2 E_{d_3} d_{3L} (s_3) G_L + \lambda^2 E_{d_3} d_{3P} (s_3) \pi_p^c + \frac{\lambda^3 (s_4 - s_3)}{1 - \lambda} \pi_L

+ \cdots

+ \lambda^k l E_{d_k} d_{kL} (s_k) G_L + \lambda^k l E_{d_k} d_{kP} (s_k) \pi_p^c + \frac{\lambda^k l (s_{k+1} - s_k)}{1 - \lambda} \pi_L

+ \cdots

so that everything cancels out, but the last term on each line in \(V_\varphi^0 (N, s_1|s_1, s_2, s_3, \ldots)\). Therefore,

\[V_\varphi^0 (s_1|s_1, s_2, s_3, \ldots) \cdot \varphi (s_1|s_1, s_2, s_3, \ldots) = \sum_{i=0}^{\infty} \lambda^i s_1 \pi_L = \frac{s_1 \pi_L}{1 - \lambda},\]

which does not depend on \(\varphi\).

**Appendix C**

**Technical Detail for the Attrition Extension of Section 4.6**

Let \(X\) be the number of periods a loyal customer stays loyal, then \(X+1\) is geometrically distributed with parameter \(\theta\). Let \(Y_t\) be the expected gain on period \(t\) from having contacted a potential customer with no followup contacts, then \(E[Y_t] = \beta \pi_L\) for any period \(t\) subsequent to the period of contact. The expected net present gain from contacting a potential customer with no followup contacts is

\[G_p = \pi_p + E \left[ \sum_{t=1}^{X} \lambda^t Y_t \right].\]

Assuming that \(X+1\) and the \(Y_t's\) are independent variables,
where 

\[ E[\lambda^X] = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \theta(1 - \theta)^{n-1} = \frac{\theta}{1 - \lambda(1 - \theta)} \]

and 

\[ E[\sum_{i=1}^{X} \lambda^i] = \beta \pi_L \sum_{n=2}^{\infty} \lambda(\frac{1 - \lambda^{n-1}}{1 - \lambda}) \theta(1 - \theta)^{n-1} = \beta \pi_L \frac{\lambda(1 - \theta)}{1 - \lambda(1 - \theta)} \]

Consequently,

\[ G_p = \pi_p^c + \frac{\lambda(1 - \theta)\beta \pi_L}{1 - \lambda(1 - \theta)} \]

In addition,

\[ V_{\phi^*}(1,1) = E[\sum_{i=1}^{X} \lambda^{i-1} G_L + \lambda^{(X+1)-1} V_{\phi^*}(1,0)] \]

where 

\[ E[\lambda^X] = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \theta(1 - \theta)^{n-1} = \frac{\theta}{1 - \lambda(1 - \theta)} \]

and 

\[ E[\sum_{i=1}^{X} \lambda^{i-1} G_L] = \frac{(1 - \theta)G_L}{1 - \lambda(1 - \theta)} \]

so that 

\[ V_{\phi^*}(1,1) = \frac{(1 - \theta)G_L}{1 - \lambda(1 - \theta)} + \frac{\theta}{1 - \lambda(1 - \theta)} V_{\phi^*}(1,0) \]

Moreover,

\[ V_{\phi^*}(1,0) = G_p + \lambda \beta V_{\phi^*}(1,1) + \lambda(1 - \beta) V_{\phi^*}(1,0) = \frac{G_p + \frac{\lambda \beta(1 - \theta)G_L}{1 - \lambda(1 - \theta)}}{1 - \lambda(1 - \beta)} \frac{\lambda \beta \theta}{1 - \lambda(1 - \theta)} \]

which is positive whenever the expected net present value from contacting a potential customer with followup contacts, that is,

\[ G_p^c = G_p + \frac{\lambda(1 - \theta)\beta G_L}{1 - \lambda(1 - \theta)} \]

is positive. It follows that

\[ V_{\phi^*}(1,1) - V_{\phi^*}(1,0) = \frac{(1 - \theta)[G_L - G_p]}{I - \lambda(1 - \theta)(1 - \beta)} \]

and so inequality (4.8) of sub-section 4.1.1 is satisfied.