The use of Absorbing Boundaries in the Analysis of Bankruptcy

by

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Abstract

An explicit solution is given for the value of a risk neutral firm with stochastic revenue facing the possibility of bankruptcy. The analysis is conducted in continuous time. Uncertainty is modeled using an Ito process and bankruptcy is modeled as an absorbing boundary. The analysis yields an ordinary differential equation with a closed form solution. The value function is used to calculate the firm’s demand for high interest rate loans, showing a positive demand at interest rates which appear intuitively to be excessive. A value function is also derived for a risk neutral lender advancing funds to the firm. The borrowing and lending value functions are then used to examine various aspects of lender-borrower transactions under different bargaining structures. In a competitive lending market, the model shows that credit rationing occurs inevitably. In a monopoly lending market, the lender sets interest rates and maximum loan levels which reduce the borrower to zero profit. When a second borrower is introduced, the lender must allocate limited funds between two borrowers. A lender is shown to squeeze the smaller “riskier” borrower out of the market when the lender’s overall credit constraint is tight. Under each bargaining structure, the model is also used to examine changes in the respective “salvage” recoveries of the lender and borrower on bankruptcy.
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CHAPTER I - INTRODUCTION

This thesis uses the methods of continuous time finance to analyse the impact of potential bankruptcy on corporate decision making, and on transactions between borrowers and lenders. Bankruptcy is analysed from the perspective of a risk neutral firm, and also from the perspective of a risk neutral lender advancing funds to the firm. Different bargaining structures are imposed on the borrower and lender to determine resulting debt levels and interest rates. Ito processes are used to model uncertainty in the firm's revenue. Ordinary differential equations are derived to calculate the value of the firm and of debt securities which it issues.

The assumptions of the analysis are deliberately kept as simple as possible. For example, all agents use the same exogenously imposed discount rate, which is equal to the rate of return on physical capital. These assumptions result in some loss of generality. However, the offsetting benefit is that explicit closed form solutions are available for most of the valuation problems posed. The closed form solutions permit both qualitative and quantitative analysis, and generate intuition about the decision making of borrowers and lenders.
The analysis carried out in this thesis is close in spirit to that of Leland (1994), who also examines the value of a risk neutral firm facing potential bankruptcy. Leland uses many similar simplifying assumptions, such as the exogenous discount rate, and considers similar consequential issues to some of those addressed here.

A point of departure from Leland is the stochastic process used to model the firm's wealth. Leland uses a stochastic process based on conventional geometric brownian motion, whose instantaneous standard deviation is proportional to wealth. This thesis uses a stochastic process whose standard deviation is independent of the wealth level. It is argued that this stochastic process models more realistically the revenue of a financially distressed firm - there is neither theoretical nor empirical support for the position that the uncertainty in firm revenue decreases as wealth decreases, or as the firm comes close to bankruptcy. As a result of the different stochastic process used, the solutions derived in this thesis differ significantly from Leland's.

Chapter II sets out the firms' valuation problem and derives its
closed form solution. Chapter II also examines properties of the solution, which are used in subsequent chapters, and compares the solution to those found in other related papers, such as Leland (1994) and Milne and Robertson (1996).

Chapter III analyses a distressed firm’s demand for high interest rate loans. The firm can borrow additional capital at an interest rate which exceeds the firm’s own rate of return. Borrowing thus imposes a net interest cost on the firm. The offsetting benefit is that borrowed capital provides an additional cushion from the firm’s bankruptcy threshold. The closed form solution derived in Chapter II is used to calculate the firm’s demand for loan funds at specific interest rates. The results are set out in tables which show a positive demand for loan funds at interest rates which appear intuitively to be excessive.

Chapter IV refines the valuation problem of Chapter II, and also its solution, to analyse a firm’s optimal capital structure. The model now requires the firm to choose a combination of debt and equity to finance a project of fixed size. Bankruptcy is assumed to occur when the firm’s wealth has dropped to the amount of its
debt, thus endogenizing the bankruptcy threshold. The interest rate on debt is now less than the firm earns on capital it invests. Under these assumptions, a higher debt-equity ratio has both advantages and disadvantages for the firm. On the one hand, more debt increases leverage. On the other hand, more debt increases the risk of bankruptcy. Chapter IV derives the value function which results from these assumptions. In addition, Chapter IV incorporates the perspective of the lender, who must value the discounted stream of interest payments which he expects to receive from the borrower. The lender’s valuation problem also has a closed form solution, whose properties are examined.

Chapter V uses the results of Chapter IV to analyse borrower-lender equilibrium in a competitive lending market. Competition among lenders is modelled by a zero profit condition. Changes in the loan amount and interest rate affect both the lender’s return and the risk of default, creating a schedule of loan amounts and interest rates satisfying the zero profit constraint. The solution to the maximization problem is the optimal loan for the borrower, subject to the lender’s constraint.

The analysis of Chapter V leads to a straightforward condition
for credit rationing (defined as a condition in which the borrower is unable to borrow all that he wishes at the prevailing interest rate), which is shown to be inevitable in a competitive lending market. Chapter V also attempts to analyse the effects of changes in the "salvage" value recovered by a lender on the borrower's bankruptcy. Changes in the salvage recovery are used to model changes in legislation governing secured lending transactions (eg. by enabling the borrower to grant more effective security) and in bankruptcy legislation (eg. by re-allocating assets remaining at the time of bankruptcy as between the borrower and lender). An increase in the ability of a borrower to grant security is shown to increase loan levels, but to have an ambiguous effect of interest rates. No unambiguous results are available when bankruptcy legislation re-allocates salvage recovery from the lender to the borrower, but both interest rates and loan levels increase if only the borrower's salvage recovery increases.

Chapter VI examines the actions of a monopolist lender. The lender sets both an interest rate and a maximum loan level for the borrower. The lender knows the borrower's value function, and can thus predict how much he will borrow at any given
interest rate. The borrower can borrow any amount, up to the lender's specified maximum, at the interest rate set by the lender. With the introduction of a monopolist lender, a new issue arises: can equilibrium be sustained with the borrower earning positive profit? The analysis of Chapter VI answers this question in the negative. Chapter VI also considers the issue of credit rationing, but unlike the competitive model, the monopoly model produces no unambiguous prediction. Finally, Chapter VI also analyses the effects of changing salvage conditions. In the monopoly lending market, the results turn out to be heavily dependent on whether credit rationing was present in the initial equilibrium.

Chapter VII examines the conduct of a lender facing two competing borrowers. The lender is given an exogenous ceiling on his total loans, and must allocate the available funds between the borrowers. The two borrowers have different starting wealth levels. The stochastic processes defining their revenue streams are modified so that the standard deviation that applies to the smaller borrower is proportionately greater than that of the larger borrower. Chapter VII thus simulates competition for loan funds between a smaller "riskier" borrower and a larger "less
risky" borrower. Some of the results of Chapter VII are analytical, but others (owing the complexity of the equations) are numerical only. Analytically, it is shown that the assumptions of Chapter VII lead to both credit rationing and zero profits for borrowers. A numerical analysis is used to investigate the impact of changes in the aggregate loan limit imposed on the lender. That analysis predicts (as is often commonly asserted) that the smaller borrower will be squeezed out of the market first under tightening credit conditions.
CHAPTER II - THE VALUATION PROBLEM

This chapter derives the solution to the fundamental valuation problem considered in this thesis. Section 1 states the problem to be solved and section 2 derives the solution. Section 3 examines the marginal value of wealth to a firm facing bankruptcy. Section 4 gives some basic mathematical results which are used throughout the thesis in examining properties of the value functions that are derived.

1. Statement of the Problem

Consider a firm whose revenue is stochastic, and whose wealth thus evolves stochastically over time.

The firm's starting wealth will be denoted as \( w \). Its wealth, as a function of time, will be denoted as \( x(t) \) and will evolve according to:

\[
\frac{dx(t)}{dt} = (r x(t) + Y) dt + \sigma dz; \quad x(0) = w.
\]  

(1)

\( r \) represents a rate of return to capital, and \( Y \) represents a
fixed rate revenue occurring in time $dt$.

The term $odz$ is the source of uncertainty. $dz$ is a wiener process, and is the conventional means of modelling uncertainty in continuous time. $dz$ represents a random variable with mean zero and with standard deviation $\sqrt{dt}$.

The firm faces a risk of bankruptcy, which is assumed to occur when the firm's wealth reaches the value $x(t) = A$. It is assumed that $A < w$, and also that $rA + Y > 0$ (i.e. even at the point of bankruptcy, the firm still has positive expected cash flow). The latter assumption ensures that bankruptcy imposes some positive cost on the owners of the firm, beyond losing a speculative hope of profitable operation in the future.

The value $A$ can be given a number of economic interpretations. It could represent a wealth level at which secured creditors become entitled to liquidate the firm's assets. Alternatively, it could represent a minimum wealth level imposed under bankruptcy legislation. For purposes of this chapter, it is not necessary to adopt a specific interpretation of $A$. Nor is it helpful to consider an endogenous level for $A$, as the analysis
does not yet include the perspective of a lender to the firm. In subsequent chapters, the bankruptcy threshold will be related to specific firm decisions, and will be endogenized.

The impact of bankruptcy can be analysed using theorems on absorbing boundaries of stochastic processes. The effect of an absorbing boundary is that once \( x(t) \) reaches the boundary \( x=A \), \( x \) remains at \( A \) thereafter.

Mathematically, a process subject to absorption, \( x(t) \), can be defined in the following way. Let \( x^*(t) \) represent a stochastic process without absorption, and define \( t' \) as:

\[
    t' = \inf \{ t : x^*(t) \leq A \}, \text{ if that quantity is finite} \]

\[
    = \infty \text{ otherwise.}
\]

Then:

\[
    x(t) = x^*(t) \text{ for } t < t';
\]

\[
    = A \text{ for } t \geq t'.
\]

\( t' \) represents the time at which the process \( x^*(t) \) reaches the
boundary $x^*(t) = A$ for the first time. For all points of time $t < t'$, $x(t)$ and $x^*(t)$ are equal. However, for $t > t'$, $x(t)$ and $x^*(t)$ diverge, with $x(t)$ remaining fixed at $A$.

The valuation problem is examined from the perspective of firm management, rather that from the perspective of an investor. Firm management acts in a risk neutral way, valuing the firm as the discounted value of future revenue. The discount rate is imposed exogenously as $r$.

On bankruptcy the firm is assumed to lose all its value. Accordingly, the firm's value can be expressed as the limit:

$$E[\lim_{t \to t'} f(x(t))]$$

where $f(x(t))$ is defined by:

$$f(x(t)) = e^{-rt}x(t) \text{ for } t < t'$$

$$= 0 \text{ for } t \geq t'$$

with $t'$ defined as above. This value is clearly a function of starting wealth, $w$, and will be represented by $v(w)$. 
2. Derivation of the Solution

An explicit solution to the valuation problem exists. The starting point is Theorem 3.22 of Gihman and Skorohod.

If \( x(t) \) satisfies \( dx(t) = a(x(t))dt + dz(t) \) in the region \( 0 < x < 1, \ t > 0 \), with absorption at \( x=0 \) and \( x=1 \), and if \( a(x) \) is twice continuously differentiable in \( (0,1) \), then:

1. The function:

\[
    u(w, t) = E[f(x(t)) \exp\left(\int_0^t \phi(x(u)) du\right)]
\]

where \( f(x) \) and \( \phi(x) \) are twice continuously differentiable with \( f(0)=f(1)=0 \) is a solution of:

\[
    \frac{\partial u(w,t)}{\partial t} = a(w) \frac{\partial u(w,t)}{\partial w} + \frac{1}{2} \frac{\partial^2 u(w,t)}{\partial w^2} + \phi(w) u(w,t)
\]

with the boundary conditions:

- \( u(w,t) \) approaches \( f(w) \) as \( t \) approaches 0
- \( u(w,t) \) approaches 0 as \( w \) approaches 0 or 1.

2. The functions \( p_1(w,t) = \text{Prob}\{x(t)=0\} \) and
$p_2(w,t) = \text{Prob}\{x(t)=1\}$ are solutions to:

$$\frac{\partial p_i(w,t)}{\partial t} = a(w) \frac{\partial p_i(w,t)}{\partial w} + \frac{1}{2} \frac{\partial^2 p_i(w,t)}{\partial w^2}$$

for $i = 1, 2$. The applicable boundary conditions are:

- $p_i(w,t)$ approaches 1 as $w$ approaches 0, and 0 as $w$ approaches 1.

- $p_2(w,t)$ approaches 0 as $w$ approaches 0, and 1 as $w$ approaches 1.

- $p_1(w,t), p_2(w,t)$ approach 0 as $t$ approaches 0.

This theorem must be extended in two ways to provide a solution to the valuation problem.

First, the theorem must be extended to apply to "one-sided" boundaries. This leads to Result 1:

If $x(t)$ satisfies $x(0)=w$, $dx(t) = (rx(t)+Y)dt + dz(t)$ in the region $0 < x, t > 0$, with absorption at $x=0$, then:

Part 1. The function
is a solution of:

\[
\frac{\partial u(w, t)}{\partial t} = (rw + Y) \frac{\partial u(w, t)}{\partial w} + \frac{1}{2} \frac{\partial^2 u(w, t)}{\partial w^2} - ru(w, t)
\]

with the boundary conditions:

- \( u(w, t) \) approaches \( w \) as \( t \) approaches 0; and
- \( u(w, t) \) approaches 0 as \( w \) approaches 0.

Part 2. The function \( Q(w, t) = \text{Prob}\{x(t) = 0\} \) is a solution of

\[
\frac{\partial Q(w, t)}{\partial t} = (rw + Y) \frac{\partial Q(w, t)}{\partial w} + \frac{1}{2} \frac{\partial^2 Q(w, t)}{\partial w^2}
\]

with boundary conditions:

- \( Q(w, t) \) approaches 1 as \( w \) approaches 0
- \( Q(w, t) \) approaches 0 as \( t \) approaches 0.

The proof of Result 1 is given in the Appendix to Chapter II.

(Note that most of the proof applies to any \( E[e^{-rt}f(x)] \) where \( f(x) \) is twice continuously differentiable, \( f(x) \) has suitable growth behaviour as \( x \) approaches infinity, and \( f(0) = 0 \). However, the boundary conditions are examined only for the particular case \( f(x) = x \).)
Second, it is necessary to examine the limiting behaviour of the solution as \( t \) approaches infinity. This leads to Result 2 (also proven in the Appendix):

If \( x(t) \) satisfies the same conditions as in Result 1, define \( v(w) \) as:

\[
v(w) = \lim_{t \to \infty} E[e^{-rt}x(t)] \quad \text{where} \quad x(0) = w
\]

\( v(w) \) must satisfy:

\[
\frac{1}{2} \frac{d^2 v}{dw^2} + (rw + Y) \frac{dv}{dw} - rv = 0
\]

subject to \( v(0) = 0 \), and \( v(w) \) approaches \( w + Y/r \) as \( w \) approaches infinity.

These results are trivially extended to the situation where absorption occurs at a non-zero value (A), and also to the situation where the coefficient of dispersion \( = \sigma \neq 1 \). If \( x(t) \) is a process following \( dx(t) = (rx(t)+Y)dt + \sigma dz \) with absorption at \( x=A \), then it is simply necessary to consider the process \( x1(t) = (x(t)-A)/\sigma \). \( x1(t) \) clearly satisfies the conditions of both Result 1 and Result 2. Also:
\[
\lim_{t \to \infty} e^{-rt} E\left[ \frac{x(t) - A}{\sigma} \right] = \lim_{t \to \infty} e^{-rt} E\left[ \frac{x(t)}{\sigma} \right]
\]

It follows that if \( x(t) \) is a process with absorption at \( x=A \) satisfying \( x(0)=w, \, dx(t) = (rx(t) + Y)dt + \sigma dz \), and if \( f(x) \) and \( v(w) \) are defined as above, \( v(w) \) satisfies:

\[
\frac{1}{\sigma^2} \frac{d^2 v}{dw^2} + (rw+Y) \frac{dv}{dw} - rv = 0
\]

subject to \( v(A)=0 \) and \( v(w) \) approaches \( w+Y/r \) as \( w \) approaches infinity.

Equation 2 has two general solutions. The first is trivial. By inspection, the function \( f_1(w) = rw + Y \) solves the equation.

The second general solution can then be found by the process of order reduction, which permits a second order differential equation to be reduced to a first order equation if one solution is known.\(^1\) The second solution is equal to:

\(^1\) The process of order reduction is explained in Boyce and di Prima (1977). Consider a second order differential equation: \( a(t)x''(t) + b(t)x'(t) + c(t)x(t) = 0 \) to which one solution, \( x_1(t) \), is known. Assume that the second solution takes the form \( x_2(t) = y(t)x_1(t) \). Differentiating twice, then substituting into the original equation and rearranging terms yields (with the argument \( t \) suppressed):
The general solution thus takes the form \( v(w) = c_1 f_1(w) + c_2 f_2(w) \), where \( c_1 \) and \( c_2 \) are constants of integration determined by boundary conditions.

With the boundary conditions set out above, \( v(A) = 0 \) and \( v(w) \) approaches \( w + Y/r \) for large \( w \), the solution becomes:

\[
 v(w) = w + \frac{Y}{r} - \frac{(Y + rw)}{r \int_{\gamma + r}^{\infty} \frac{ds}{s^2}} \exp \left[ - \frac{s^2}{ro^2} \right] ds
\]

By the definition of \( x_1 \), the first term is zero. Making the substitution \( u = y' \), the above equation thus reduces to:

\[
 [ax_1^2 + bx_1' + cx_1] + y''[ax_1] + y'[2bx_1' + bx_1] = 0
\]

Since \( x_1 \) is known, this is just a first order equation in \( u \), which can be found by ordinary integration methods. \( u \) can then be used to solve for \( y \), and hence \( x_2 \).
This expression can be simplified using the following integration by parts:

\[
\int_{x}^{\infty} \frac{\exp(-s^2)}{s^2} ds = \frac{\exp(-x^2)}{x} - 2 \int_{x}^{\infty} \exp(-s^2) ds
\]

Equation (3) then becomes:

\[
v(w) = \frac{Y}{r} - \frac{\exp\left(-\frac{(Y+rw)^2}{r\sigma^2}\right) - \frac{2(Y+rw)}{r\sigma^2} \int_{Y+rw}^{\infty} \exp\left[-\frac{s^2}{r\sigma^2}\right] ds}{r \int_{Y+rw}^{\infty} \exp\left[-\frac{s^2}{r\sigma^2}\right] ds}
\]

The value function can thus be reduced to expressions involving the standard error function.

Intuitive interpretation of the function \(v(w)\) is assisted by defining the quantity \(M\) as follows:
As will be shown in Chapter IV, section 3, M represents the discounted sum (i.e. integral) of the future probabilities of bankruptcy.

Using the quantity M, the value function (3) can then be re-written in the following way:

\[ v(w) = w + \frac{Y}{r} - (A + \frac{Y}{r})M = w - AM + \left(\frac{Y}{r}\right)(1-M) \]

Starting wealth is w. However, with discounted probability M, A of that will be lost. In addition, the firm enjoys the value of revenue Y, whose discounted present value is equal to \( \frac{Y}{r} \). However, this revenue stream may also be lost, with probability M, and is thus "discounted" by the factor \((1-M)\).

The following graph gives a pictorial representation of the value function.
The value function approaches $w + Y/r$ asymptotically as $w$ approaches infinity, reducing the likelihood of bankruptcy.

Both the differential equation (2) and its solution (3) are similar to other such equations and solutions found in the finance literature. However, there are significant differences as well.

Merton (1992, chapter 11) derives the following fundamental partial differential equation for the value ($F$) of a contingent
claim to an underlying asset with value $V$:

$$0 = \frac{1}{2} \sigma^2 V^2 F_{vv} + rVF_v - F_t + rF.$$ 

A more generalized equation, allowing for payouts from the firm, is presented in Black and Cox (1976).

Leland (1994) uses the infinite time version of the equation stated by Black and Cox (in which the time variable becomes irrelevant) to value firm debt paying $C$ per period:

$$0 = \frac{1}{2} \sigma^2 V^2 F_{vv} + rVF_v + C - rF$$

Leland goes on to investigate properties of the explicit solution to his equation, which takes the form $F(V) = A_0 + A_1V + A_2V^2$ (where $X$ is determined by parameters and boundary conditions).

Direct application of the equations used to value contingent claims (as in Leland) presents another means of deriving equation (2). That method has not been used here, as the method used here proceeds directly from fundamental theorems on stochastic differential equations, and thus makes explicit many of the
assumptions regarding convergence which would normally be implicit in the application of contingent claims analysis.

The differential equations of Merton, Black and Cox, and Leland are different from (2) in their second order term. Equation (2) omits the factor $V^2$ from that term. The reason is that Merton, Black and Cox, and Leland base their analyses on an underlying stochastic process which exhibits geometric brownian motion:

$$dx = rxdt + \sigma xdz.$$ 

Presence of the factor "x" in the stochastic term (which is not present in (1), above) explains both the different forms of equation, and the different form of solution derived by Leland.

For the purposes of this thesis, the stochastic process which has been chosen appears more realistic than one based on geometric brownian motion. There is no reason to believe that a firm's revenue becomes any more certain or reliable as it comes closer to bankruptcy. If anything, the opposite is true.

Milne and Robertson (1996) investigate the discounted value
(F(x), where x represents instantaneous cash balances) of the dividend payments from a firm which faces bankruptcy. The problem they solve is thus closely related to the problem under consideration here. In addition, Milne and Robertson consider the same stochastic process as set out in (1). However, their model requires that the return to capital (r) be less than the discount rate (ρ). Milne and Robertson thus arrive at the following differential equation which is highly similar to (2):

$$0 = 1/2 \sigma^2 F_{xx} + (rx+u)F_x - \rho F$$

Milne and Robertson then go on to consider the circumstances under which an explicit solution to their equation exists. They identify such a solution (essentially a ratio of exponential values) in the particular case r=0. However, the model of Milne and Robertson does not allow for the case r=ρ, and thus does not yield the closed form solution presented in (3). The closed form solution derived in this thesis can be considered a limiting case, as the discount rate approaches the rate of return on capital from above, of the numerical results set out in Milne and Robertson.
3. The Marginal Value of Capital

A quantity of interest is \( v'(w) \), the marginal value of wealth. Intuitively, the marginal value of wealth is the combination of three factors, two positive and one negative:

1. The basic revenue stream from the marginal unit of wealth - a positive factor.

2. The risk of losing the revenue stream from the marginal unit of wealth owing to bankruptcy - a negative factor.

3. Added protection from bankruptcy - a positive factor.

The derivative of the value function is:

\[
\frac{dv}{dw} = 1 - \frac{\int_{Y+rw}^{\infty} \exp[-\frac{s^2}{rO^2}] ds}{\int_{Y+ra}^{\infty} \frac{s^2}{rO^2} ds} + \frac{\exp[-\frac{(Y+rw)^2}{rO^2}]}{(Y+rw) \int_{Y+ra}^{\infty} \frac{s^2}{rO^2} ds}
\]
For w>A, dv/dw must be positive, as the second term in (6) is less than one.

Equation (6) can also be simplified using integration by parts to:

\[
\frac{dv}{dw} = 1 + \frac{2}{r_0^2} \int_{y + rA}^{\infty} \exp\left[-\frac{s^2}{r_0^2}\right] ds
\]

\[
= 1 + \frac{2}{r_0^2} \int_{y + rA}^{\infty} \exp\left[-\frac{s^2}{r_0^2}\right] ds
\]

The derivative of the value function, like the value function itself, can thus be reduced to expressions involving the standard error function. Expression (7) also discloses that v'(w) is no less than 1 - the "saving" factor identified as 3, above, outweighs the negative factor identified as 2.

Finally, differentiation of (7) with respect to w yields:
which is unambiguously negative, confirming that \( v(w) \) is concave in \( w \). The threat of bankruptcy induces local risk aversion into a value function which would otherwise be risk neutral.

4. Limiting Values

Both \( v(w) \) and \( v'(w) \) have interesting limiting values.

Consider first the limiting value of \( v(w) \) as the coefficient of dispersion, \( \sigma \), becomes arbitrarily large. Reparameterizing with \( z^2 = s^2/\sigma_0^2 \) in equation (4) leads to:

\[
v(w) = w + \frac{Y}{r} - \frac{2 \exp \left[ -\frac{(Y+rw)^2}{\sigma^2} \right]}{\sigma_0^2(Y+ra)} - \frac{2}{\sigma_0^2(Y+ra)} \int_{Y+ra}^{\infty} \exp \left[ -\frac{s^2}{\sigma_0^2} \right] ds
\]
For large $\sigma$, the exponential terms in the numerator and denominator both approach $1$, while the integrals in the numerator and denominator both approach zero. It is thus apparent that the limiting value of $v(w)$ becomes:

$$w + \frac{Y}{r} - (A + \frac{Y}{r}) = w - A.$$

With infinite variance, the value function depends only on the starting distance from the boundary. The result is intuitive. A large variance eliminates any influence from either the constant drift factor, $Y$, or the return on capital, $r$.

It follows that as $\sigma$ approaches infinity, $v'(w)$ approaches $1$, a result which can also be derived directly from Equation (7).

A second limiting value of interest is the marginal value of additional capital as when the firm is on the verge of bankruptcy. As $w$ approaches $A$, $v'(w)$ approaches:
\[
\frac{dv}{dw}(A) = \exp\left[-\frac{(Y+rA)^2}{\sigma^2}\right] - \frac{2(Y+rA)}{\sigma^2} \int_{Y+rA}^{\infty} \exp\left[-\frac{s^2}{\sigma^2}\right] ds
\]

\[
= 1 + \frac{2(Y+rA)}{\sigma^2} \int_{Y+rA}^{\infty} \exp\left[-\frac{s^2}{\sigma^2}\right] ds
\]

Reparameterizing again with \(z^2 = \frac{s^2}{\sigma^2}\) leads to:

\[
\frac{dv}{dw}(A) = 1 + \frac{2(Y+rA)}{\sigma \sqrt{\sigma}} \int_{Y+rA}^{\infty} \exp\left[-z^2\right] dz
\]

The marginal value of capital as \(w\) approaches \(A\) can thus be written as a function of the single quantity \(\frac{(Y+rA)}{\sigma \sqrt{\sigma}}\). As this quantity approaches 0, it is apparent that \(v'(A)\) approaches 1. As \(\frac{(Y+rA)}{\sigma \sqrt{\sigma}}\) becomes arbitrarily large, \(v'(A)\) becomes infinite as well. Further information concerning \(v'(A)\) is given below in section 5, Related Functions.
5. Related Functions

There are several functions related to \( v(w) \) which appear in the proofs and discussions below. Some of the functions appear as components of the value function, in one or more of its various expressions. Others are used in evaluating the signs of derivatives of the value function and other related functions. In some cases, it is the logarithmic derivative of an expression appearing a value function which is of interest.

This section of this thesis investigates the properties of these related functions. At the end of this section, some of the results are applied in a further investigation of \( v'(w) \), the marginal value of wealth.

\[(I) \text{ The Function } I(x) = \int_x^\infty \exp(-s^2) ds\]

Consider first the function \( I(x) = \int_x^\infty \exp(-s^2) ds \) and let \( g(x) \) represent the negative of its logarithmic derivative, \(-I'(x)/I(x)\). \( g(x) \) is thus equal to:
\[ g(x) = \frac{\exp[-x^2]}{\int_{-\infty}^{x} \exp[-s^2] ds} \]

Subject to constants, \( I(x) \) is the standard normal distribution function, and appears directly in expressions for \( v(w) \) and \( v'(w) \) (equations (3) and (5) of this chapter). The properties of \( I(x) \) are well known and need no elaboration.

\( g(x) \) is used in subsequent chapters in evaluating the derivatives of the value function and related functions. Its properties are not obvious, and require analysis.

At \( x = 0 \), \( g(x) \) is clearly equal to \( 2/\sqrt{n} \). For large \( x \), l'Hôpital's rule is necessary, as both the numerator and the denominator approach 0. Taking derivatives yields the ratio:

\[ \frac{2x\exp[-x^2]}{\exp[-x^2]} \]

From this expression is apparent that for large \( x \), \( g(x) \) approaches \( 2x \). The following graph shows the locus of \( g(x) = -I'(x)/I(x) \).
Consider next the function \( h(x) = g(x) - 2x = -\frac{I'(x)}{I(x)} - 2x \).

\( h(x) \) does not appear directly in any of the expressions considered thus far. However, it is important in analysing the function \( J(x) \), which is defined and analysed below.

It can be shown that \( h(x) \) is monotonically decreasing in \( x \).
(Note that the results already given show that h(x) converges to 0 for large x). To save on notation, let $E = \exp[-x^2]$, and let $I$ represent the integral $I = \int_{x}^{\infty} \exp[-s^2] ds$. The objective is thus to show that $h(x) = E/I - 2x$ is a monotonically decreasing function of $x$.

The derivative of $h(x)$ is $h'(x) = E'/I - EI'/I^2 - 2$, but since $E' = -2xE$, and since $I' = -E$, $h'(x)$ can be rewritten as

$$h'(x) = E^2/I^2 - 2xE/I - 2$$

Thus, $h(x)$ will be monotonically decreasing if:

$$E^2/I^2 - 2xE/I < 2, \text{ or}$$

$$E/I - 2x < 2I/E, \text{ or}$$

$$h(x) < 2I/E$$

It has already been shown that $E/I$ is monotonically increasing, from which it follows that the RHS of (9) is monotonically decreasing. The following graph now illustrates why $h(x)$ must be
monotonically decreasing.

Figure 3 - Graph of h(x)

If the graph of the function were to cross the graph of 2I/E (as in the case of the dash line), that would imply that h(x) > 2I/E, which would in turn imply that h(x) is increasing. But with h(x) increasing and 2I/E decreasing, the condition h(x) > 2I/E would continue to hold for all points to the right of the point of intersection (marked A in the graph). h(x) would thus have to be increasing for all points to the right of the intersection,
contradicting the result that $h(x)$ approaches 0 for large $x$.

From the foregoing discussion, it also follows that the function

$$h(x)/x + 2,$$

or:

$$\frac{\exp[-x^2]}{\int_{x}^{\infty} \exp[-s^2] ds}$$

is monotonically decreasing in $x$ as well.

(III) The Function $J(x) = \int_{x}^{\infty} \frac{\exp[-s^2]}{s^2} ds$

Consider finally the function $J(x) = \int_{x}^{\infty} \frac{\exp[-s^2]}{s^2} ds$, which is clearly positive. $J(x)$ appears directly in expressions for the value function and its derivatives (equations (2) and (4)). It is of fundamental importance in many of the results derived in the following chapters.

Using integration by parts, $J(x)$ can be rewritten as:

$$J(x) = \exp[-x^2] - 2x \int_{x}^{\infty} \exp[-s^2] ds$$

This expression shows that $J(0) = 1$, and is also easily
differentiated to obtain:

\[
\frac{dJ}{dx} = -2 \int_{-\infty}^{\infty} \exp[-s^2] ds = -2I(x)
\]

which in turn implies that \( J''(x) = 2\exp(-x^2) > 0 \).

\( J(x) \) is thus strictly decreasing and convex. Inspection of the above equations also shows that \( J(x) \) must approach 0 for large \( x \). \( J(x) \) is thus graphically shown as follows.

Figure 4 - Graph of \( J(x) \)

\[
J(x)
\]

1

Now let \( p(x) = -J'(x)/J(x) = 2I(x)/J(x) \) and let \( q(x) = 1/p(x) = J(x)/I(x) \). \( q(x) \) can be re-written in the following way:
q(x) = -J(x)/J'(x)

\[ \frac{\exp(-x^2) - 2x \int_x^\infty \exp(-s^2) \, ds}{\int_x^\infty 2 \exp(-s^2) \, ds} = \frac{1}{2} \left[ \frac{\exp(-x^2)}{\int_x^\infty \exp(-s^2) \, ds} - 2x \right] \]

The term in brackets in the final expression was defined as h(x) in the previous section, and was shown to be monotonically decreasing. It follows that p(x) = 1/q(x) = 2I(x)/J(x) is monotonically increasing, and thus that the (negative of) the logarithmic derivative of J(x) is a monotonically increasing function. It also follows that the function xI(x)/J(x), as the product of two monotonic functions, is monotonically increasing as well.

(IV) Application to the analysis of v'(w)

An example of the application of some of these functions comes in determining the limiting value of v'(A) as Y+rA becomes large. v'(A) measures, in effect, the value of an additional unit of capital to a firm on the verge of bankruptcy. A large value of Y+rA implies that the firm's ability to generate positive cash
flow is strong near bankruptcy. \( v'(A) \) thus measures what the firm would pay to preserve this ability to generate cash flow.

Evaluated at \( w=A \), equation (8) gives the marginal value of capital as:

\[
\frac{dv}{dw}(A) = \frac{\exp\left[-\frac{(Y+rA)^2}{r\sigma^2}\right]}{\exp\left[-\frac{(Y+rA)^2}{r\sigma^2}\right] - \frac{2(Y+rA)}{r\sigma^2} \int_{Y+rA}^{\infty} \exp\left[-\frac{s^2}{r\sigma^2}\right] ds}
\]  

(10)

As both the numerator and denominator vanish for large \( x \), it is necessary to use l'Hopital's rule. To economize on notation, let \( E \) represent the numerator in the above expression, and let \( I \) represent the integral in the denominator. Differentiation of both the numerator and denominator with respect to \( (rA+Y) \) then yields the ratio:

\[
\lim_{A \to w} \left[ \frac{dv}{dw}(A) \right] = \frac{-2(rA+Y)}{r\sigma^2} \frac{E}{r\sigma^2 - E} - \frac{2}{r\sigma^2} \frac{I}{r\sigma^2}
\]

which is equal to \( (rA+Y)E/I \). However, as \( E(x)/I(x) \) approaches
2x/\sigma_0^2 \text{ for large } x, \text{ this expression is in turn equal to } \\
2(rA+Y)^2/\sigma_0^2. \text{ Accordingly, the marginal value of capital at the } \\
bankruptcy boundary is proportional to the square of the marginal 

cash flow at the boundary, as the latter value becomes large. 

(V) Application to the analysis of a change in \sigma 

Another question arising from the value function derived above is 
whether a firm on the verge of bankruptcy will have an incentive 
to increase the inherent risk level of its business. 

Differentiation of the value function (3) with respect to \sigma, the 
coefficient of dispersion, yields: 

\[
\frac{\partial V(w)}{\partial \sigma} = -2 \left( \frac{w+Y}{r} \right) \int_{rA}^{rA+s} \frac{\exp \left( -\frac{s^2}{\sigma_0^2} \right)}{s^2} ds \int_{rA}^{rA+s} \exp \left( -\frac{s^2}{\sigma_0^2} \right) ds \int_{rA}^{rA+s} \exp \left( -\frac{s^2}{\sigma_0^2} \right) ds \\
 \left[ \exp \left( -\frac{s^2}{\sigma_0^2} \right) \right] \left[ \frac{s^2}{\sigma_0^2} ds \right]^2 
\]

This expression reduces to an expression in the form: 

\[-2 \left( (rw+Y)I((rw+Y)/J(rw+Y) - (rA+Y)I((rA+Y)/J(rA+Y) \right)\]
By the results of (III), above, $xI(X)/J(x)$ is monotonically decreasing. Since $rw+Y>rA+Y$, it follows that even near the bankruptcy threshold, $v$ is a decreasing function of $\sigma$. The firm thus has no incentive to "gamble for resurrection" by switching to project with higher $\sigma$. 
I. PROOF OF RESULT 1

A proof will be given for Part 1. No proof is given for Part 2 because the proof given in Gihman and Skorohod (theorem 22.3) applies directly to the case under consideration.

Proof of part 1.

The proof follows Gihman and Skorohod, theorem 22.3, and begins with a similar result applicable to processes without absorption. The proof then defines a function \( g(x(t)) \) which is positive for \( x<0 \) and 0 for \( x>0 \). The function \( g \) is used to create an multiplying factor of \( \exp[-ng(x(u))]du \). For large \( n \), the multiplying factor vanishes for all sample paths which cross the boundary. This feature of the multiplying factor allows the result for processes without absorption to be adapted to processes with absorption.

Notation:

1. \( x_{w,s}^*(t) \) identifies the stochastic process for wealth at time
t>s, without absorption, given that \( x_{w,s}*(s)=w \). Thus,

\[
dx_{w,s}*(t) = rx_{w,s}*(t)dt + Ydt + dz; \quad x_{w,s}*(s) = w.
\]

2. \( x_{w,s}(t) \) identifies the corresponding process with absorption.

\[
dx_{w,s}(t) = rx_{w,s}(t)dt + Ydt + dz; \quad x_{w,s}(s)=w, \text{ with absorption at } x = 0.
\]

3. \( Q(w,t) \) denotes the probability that \( x(t) \) has reached the boundary \( x(t)=0 \) at or before time \( t \), given that \( x(0)=w \).

Assume that:

1. \( g(x) \) has continuous, bounded derivatives up to and including second order; \( g(x)=0 \) for \( x\geq0 \); \( g(x)>0 \) for \( x<0 \); \( g(x) \) is bounded.

2. \( f(x) \) is nonnegative, has bounded continuous first and second derivatives, and has suitable growth restrictions for large \( x \). Also, \( f(0)=0 \).

3. \( h(x) \) is twice continuously differentiable with \( h(x)=0 \) if \( x \) lies outside \( [0,b] \) for some arbitrary \( b>0 \). Also,

\[
h(0)=h'(0)=h''(0)=0 \quad \text{and} \quad h(b)=h'(b)=h''(b)=0.
\]

Define. \( u_n(w,s) = E[f(x_{w,s}*(t)) \exp\{-n \int g(x_{w,s}+(u)) du\}] \). From Gihman and Skorohod (#11, remark 1), each \( u_n \) satisfies:
\[
\frac{\partial u_n(w,s)}{\partial s} - (rw+Y) \frac{\partial u_n(w,s)}{\partial w} + \frac{1}{2} \frac{\partial^2 u_n(w,s)}{\partial w^2} - ng(w) u_n(w,s) \tag{A}
\]

subject to \(u_n(w,t) = f(w)\).

Define \(u_\infty(w,s) = \lim_{n \to \infty} u_n(w,s)\), which limit exists since \(u_n(w,t)\) is non-increasing as \(n\) approaches infinity. From the definition of \(u_\infty\), it follows that \(u_\infty(w,t) = f(w)\).

Using \(g(w)h(w)=0\), integrating (A) with respect to \(s\) from \(s_1\) to \(t\), then multiplying by \(h(w)\), then integrating with respect to \(w\) from minus infinity to infinity, and then integrating by parts with respect to \(w\) yields:

\[
- \int_{-\infty}^{\infty} u_n(w,t) h(w) \, dw + \int_{-\infty}^{\infty} u_n(w,s_1) h(w) \, dw =
- \int_{s_1}^{\infty} \int_{-\infty}^{\infty} u_n(w,s) \frac{\partial}{\partial w} \left((rw+Y) h(w)\right) \, dw \, ds + \frac{1}{2} \int_{s_1}^{\infty} \int_{-\infty}^{\infty} u_n(w,s) \frac{d^2}{dw^2} h(w) \, dw \, ds
\]

All integrals in this expression will be finite, since \(h(w)=0\) outside of some arbitrary interval \([0,b]\). Letting \(n\) go to infinity:
\[- \int_{-\infty}^{\infty} f(w) h(w) \, dw + \int_{-\infty}^{\infty} u_0(w, s) h(w) \, dw = \]
\[\int_{s_1}^{s_2} \int_{-\infty}^{\infty} u_0(w, s) \left[ - \frac{\partial}{\partial w} ((rw+Y) h(w)) + \frac{1}{2} \frac{d^2}{dw^2} h(w) \right] \, dw \, ds \]

As \( h(w) \) is 0 outside of \([0, b]\), integration with respect to \( w \) in this equation can be limited to that interval. Differentiating with respect to \( s \) then yields:

\[- \frac{\partial}{\partial s} \int_{s_1}^{s_2} u_0(w, s) h(w) \, dw = \int_{s_1}^{s_2} u_0(w, s) \left[ - \frac{\partial}{\partial w} ((rw+Y) h(w)) + \frac{1}{2} h''(w) \right] \, dw \]

\[\int_{0}^{b} h(w) \left[ \frac{1}{2} \frac{\partial^2 u_0}{\partial w^2} + (rw+Y) \frac{\partial u_0}{\partial w} + \frac{\partial u_0}{\partial s} \right] \, dw = 0 \]

Integration of this expression by parts yields:

This implies that \( u_0(s, w) \) is a generalized solution of:

\[- \frac{\partial}{\partial s} u_0(w, s) = (rw+Y) \frac{\partial}{\partial w} u_0(w, s) + \frac{1}{2} \frac{\partial^2}{\partial w^2} u_0(w, s) \quad (B) \]

This equation has thus been proven for all \( w \) in an arbitrary interval \((0, b)\). As \( b \) is arbitrary, the equation must hold for
all \( \omega \in (0, \infty) \).

It can now be shown that \( u_0(\omega, t) = \mathbb{E}[f(x(t))] \).

Note that \( \exp\left\{-n \int_{s}^{t} g(x^*(u)) \, du \right\} \to 0 \) as \( n \to \infty \) if there is any \( u \) in \([s, t]\) such that \( x^*_{\omega, s}(u) < 0 \). Conversely, if \( x^*_{\omega, s}(u) \geq 0 \) for all \( u \) in \([s, t]\), then \( \int_{s}^{t} g(x^*_{\omega, s}(u)) \, du = 0 \). Also, \( \mathbb{P}\{\min_{s \leq u \leq t} x^*_{\omega, s}(u) = 0\} = 0 \). Thus, since \( f(0) = 0 \):

\[
\mathbb{P}\left\{ \lim_{n \to \infty} f(x^*_{\omega, s}(t)) \exp\left\{-n \int_{s}^{t} g(x^*_{\omega, s}(u)) \, du \right\} \neq f(x^*_{\omega, s}(t)) \right\} \\
\leq \mathbb{P}\{\min_{s \leq u \leq t} x^*_{\omega, s}(u) = 0\} = 0
\]

Thus, with probability 1:

\[
\lim_{n \to \infty} f(x^*_{\omega, s}(t)) \exp\left\{-n \int_{s}^{t} g(x^*_{\omega, s}(u)) \, du \right\} = f(x^*_{\omega, s}(t))
\]

so that:

\[
\lim_{n \to \infty} \mathbb{E}[f(x^*_{\omega, s}(t)) \exp\left\{-n \int_{s}^{t} g(x^*_{\omega, s}(u)) \, du \right\}] = \mathbb{E}[f(x^*_{\omega, s}(t))]
\]

But the LHS of this expression is \( u_0(\omega, t) \). This proves that equation (B) is solved by \( \mathbb{E}[f(x^*_{\omega, s}(t))] \) for \( \omega \) in the interval \((0, b)\). As the proof has been given for an arbitrary interval
(0, b), the equation must hold for the entire domain \( w > 0 \).

Two further steps are necessary to complete the first part of the result. It is necessary to incorporate the discount factor, \( e^{-rt} \), and to eliminate the "backwards" nature of the equation.

Since stochastic process for \( x \) does not depend explicitly on time:

\[
e^{-r(t-s)} u_0(w, s) = E[e^{-r(t-s)} f(x_{w, s}(t))] = E[e^{-r(t-s)} f(x_{w, 0}(t-s))] \quad (C)
\]

Defining \( u(w, t) = E[e^{-rt} f(x_{w, 0}(t))] \), equation (C) implies that:

\[
u(w, t-s) = e^{-r(t-s)} u_0(w, t-s).
\]

Differentiating both sides with respect to \( s \) yields:

\[
\frac{\partial u(w, t-s)}{\partial s} = re^{-r(t-s)} u_0(w, t-s) + e^{-r(t-s)} \frac{\partial u_0(w, t-s)}{\partial s}
\]

Since \( \frac{\partial u}{\partial t} = - \frac{\partial u}{\partial s} \), this can be rewritten as:
\[ -\frac{\partial u(w, t-s)}{\partial t} = re^{-r(t-s)}u_0(w, t-s) + e^{-r(t-s)} \frac{\partial u_0(w, t-s)}{\partial s} \]

Substituting expression (B) for \( \frac{\partial u}{\partial s} \) leads to:

\[ -\frac{\partial u(w, t-s)}{\partial t} = ru(w, t-s) + e^{-r(t-s)} \frac{\partial u_0(w, t-s)}{\partial s} \]

Reparameterizing with \( t \) instead of \( t-s \) leads to the partial differential equation:

\[ \frac{\partial}{\partial t} u(w, t) = (rw+Y) \frac{\partial}{\partial w} u(w, t) + \frac{1}{2} \frac{\partial^2}{\partial w^2} u(w, t) - ru(w, t) \]

which is the equation to be proved.

The final step in the first part of the proof is establishment of
the boundary conditions. It is now possible to restrict the analysis to the function in question, name \( f(x) = x \).

The first condition, \( \lim_{t \to 0} u(w,t) = f(w) = w \), is self-evident.

To establish the second boundary condition, namely that \( \lim_{t \to 0} u(w,t) = 0 \), consider the process \( X_w(t) = x_w e^{-rt} \) (a process without absorption). \( X_w(t) \) then satisfies:

\[
dX_w(t) = Y e^{-rt} dt + e^{-rt} dz.
\]

Define the process \( z_w(t) = w + \int_0^t e^{-rs} dz \), so that \( Z_w(t) \) follows \( dZ_w = e^{-rt} dz \). Now define \( Z_w(t) \) to be a process corresponding to \( Z_w(t) \), but with absorption at \( Z=0 \).

Theorem 3, #12 of Gihman and Skorohod establishes that \( X_w(t) \) is absolutely continuous with respect to the process \( Z_w(t) \), and that the density of the former with respect to the latter process is:

\[
p = \exp \left[ \int_0^t Y dZ_w(s) - \frac{1}{2} \int_0^t Y^2 ds \right]
\]
If $A$ is some Borel set in $(0,\infty)$, then:

$$\text{Prob}\left\{0 < X_w^*(u) \text{ for } 0 < u \leq t ; X_w^*(t) \in A\right\} = E[pL_A]$$

where $L_A = 1$ if $0 < Z_w^*(u)$ for $0 < u \leq t$, $Z_w^*(t) \in A$, and $L_A = 0$ otherwise. The LHS of this expression represents the probability $\text{Prob}\{X_w(t) \in A\}$, and the RHS can be expressed as $E[K_A(Z_w(t)p)$ where $K_A$ is the indicator function for set $A$. Using these probabilities, for any function $f$ such that $f(0) = 0$:

$$E[f(X_w(t))] = E[f(Z_w(t))] \exp\left[\int_0^t YdZ_w^*(s) - \frac{1}{2} \int_0^t Y^2 ds\right]$$

or, in the case $f(x) = x$:

$$E[X_w(t)] = E[(Z_w(t))] \exp\left[\int_0^t YdZ_w^*(s) - \frac{1}{2} \int_0^t Y^2 ds\right]$$

To prove the boundary condition $\lim_{w\to 0} u(w,t) = 0$, it is thus necessary to show that the RHS of this expression goes to 0 as $w$ goes to 0. As the second term in brackets on the RHS is a constant for any given value of $t$, it suffices to analyze the expression:
\[ E[X(t)] = E[(Z(t)) \exp \left( \int_0^t YdZ^*(s) \right)] \]

By Holder's inequality:

\[ E[(Z(t)) \exp \left( \int_0^t YdZ^*(s) \right)] \leq (E[(Z(t))^q])^{1/q} (E[\exp \left( p \int_0^t YdZ^*(s) \right)])^{1/p} \]

for \( 1/q + 1/p = 1 \) and \( p, q \) positive.

It thus suffices to prove that the first factor on the right hand side goes to 0 as \( w \) approaches 0, and that the second factor of the right hand side is bounded.

To analyze the second factor on the RHS, consider a stochastic process:

\[ dR = e^{-rt}YRdz; \ R(0) = w. \]

From Arnold, corollary 8.4.3, \( R(t) \) is equal to:
\[ R(t) = \exp \left[ \int_0^t \frac{Y^2}{2} e^{-2\tau t} ds + \int_0^t Ye^{-\tau t} dz \right] \]

From Arnold, Theorem 8.4.5, \( R(t) \) has a finite pth order moment, indicating that the expectation:

\[ \left( E \left[ \exp \left[ p \int_0^t YdZ^+(s) \right] \right] \right)^{1/p} \]

must be finite.

Turning next to the expectation \( \left( E[(Z_w(t))^q] \right)^{1/q} \), from the definition of \( Z_w(t) \):

\[ Z_w(t) = w + \int_0^{t\wedge \tau} e^{-rs} dz \]

where \( \tau \) represents the first exit time to \(-w\) for the process \( \int e^{-rs} dz \). Thus:
\[ E|z_w|^q \leq EK_q[w^q + \int_0^t e^{-rs}dz(s)|^q] \]
\[ \leq K_q[w^q + E[\int_0^t e^{-rs}dz(s)|^q]] \]

for a constant \( K_q \). \( w^q \) clearly approaches 0 as \( w \) approaches 0.

To analyze the second term in brackets, define \( Z^*_0(t) \) as the process:

\[ Z^*_0(t) = \int_0^t e^{-rs}dw(s) \]

with \( Z_0(t) \) as the corresponding process with absorption at \( Z^*_0(t) = -w \). Thus,

\[ \text{Prob}\{Z_0(t) > -w\} \leq \text{Prob}\{\inf_{0\leq u\leq t}[Z^*_0(u)] > -w\} \]

From Arnold, theorem 8.2.10, \( Z^*_0(t) \) is normally distributed with mean 0. The variance solves:

\[ K'(t) = e^{-2rt} \]

This implies that the variance at time \( t \) is \( 1/2r[1 - e^{-2rt}] \), so that:
\[ \begin{align*}
&= \text{Prob}\left( \min_{0 \leq u \leq t} [Z^*_0(u) > -w] \right) \\
&= \text{Prob}\left( \max_{0 \leq u \leq t} [Z^*_0(u) < w] \right) \\
&\leq \frac{2\sqrt{2\pi}}{\sqrt{2\pi[1-e^{-2\pi t}]}} \int_0^w \exp\left(-\frac{2ry^2}{[1-e^{-2\pi t}]}ight) dy
\end{align*} \]

The latter expression clearly approaches 0 as w approaches 0, proving the second boundary condition. It follows that as w approaches 0, \( E[Z_w(t)] \) approaches 0, which completes the proof of the second boundary condition. QED

Corollary 1. The partial derivative \( \frac{\partial^2}{\partial w^2} Q(w,t) \geq 0 \).

Proof: It is self evident that \( \frac{\partial}{\partial w} Q(w,t) \leq 0 \). Similarly, it is clear that \( \frac{\partial}{\partial t} Q(w,t) \geq 0 \), since for \( s < t \), the set \( \{ \min_{0 \leq u \leq s} x(u) < 0 \} \) is a subset of the set \( \{ \min_{0 \leq u \leq t} x(u) < 0 \} \).

But \( \frac{\partial^2}{\partial w^2} Q(w,t) = 2 \left[ \frac{\partial}{\partial t} Q(w,t) - (rx+y) \frac{\partial}{\partial w} Q(w,t) \right] \), which must clearly be non-negative. QED

Corollary 2. For any given \( w > 0 \), the partial derivative \( \frac{\partial}{\partial w} Q(w,t) \) remains bounded for large t.

This follows immediately from Corollary 1, as \( \frac{\partial}{\partial w} Q(w,t) \) is both increasing and bounded from above by 0. QED.
II. PROOF OF RESULT 2

Notation:

1. $x^*(t)$ identifies the stochastic process for wealth, without absorption, with starting value $w$:
   $$dx^*(t) = rx^*(t)dt + Ydt + dz; x^*(0) = w.$$  

2. $x(t)$ identifies the corresponding process with absorption. Thus, $dx(t) = rx(t)dt + Ydt + dz$ with absorption at $x = 0$; $x(0) = w$. 

3. $V(w,t) = u(w,t)$ from the proof of Result 1, so that $V(w,t) = E[e^{-rt}x(t)]$.

4. $v_n(w) = V(w,n)$.

5. $h_n(w) = V_t(w,n)$. Thus, since $V$ satisfies:
   $$\frac{1}{2} V_{ww} + (rw+Y)V_w - rV = V_t.$$  
   It follows that $v_n(w)$ and $h_n(w)$ satisfy:
   $$\frac{1}{2} v_n''(w) + (rw+Y)v_n'(w) - rv_n(w) = h_n(w).$$

6. $v(w)$ is defined as $v(w) = \lim_{t \to \infty} V(w,t)$, which limit is shown to exist in proposition 8.

The proof uses Result 1 and proceeds as a series of propositions. The propositions ultimately show uniform convergence of each of $v_n''(w), v_n'(w), v_n(w)$, and $h_n(w)$. Proof of the result follows.
easily, once uniform convergence is shown.

Proposition 1: \( V(w,t) \) is bounded as \( t \) approaches infinity.

Specifically, \( V(w,t) \leq w + \frac{Y}{r}(1-e^{-rt}) \)

Proof: \( V(w,t) \) is equal to:

\[
V(w,t) = \int_0^\infty y Po(w,t,dy)
\]

where \( Po(w,t,dy) \) is the transition probability:

\[
\text{Prob}\{e^{-rt}x(t) < dy / x(0) = w\}
\]

To evaluate this integral, it is necessary to consider the corresponding integral for the process without absorption, \( x^*(t) \).

From Arnold, theorem 8.2.10, \( x^*(t) \) is normally distributed.

The mean value of \( x^*(t) \) solves the differential equation

\[
m'(t) = rm(t) + Y ; m(0) = w
\]

which implies that \( m(t) = w e^{rt} + \frac{Y}{r} \left[ e^{rt} - 1 \right] \).

The variance of \( x^*(t) \) solves the differential equation
\[ K'(t) = 2rK(t) + 1, \quad K(0) = 0 \]

which implies that \( K(t) = \frac{1}{2r} \left[ e^{2rt} - 1 \right] \)

These results imply that \( x^* e^{-rt} \) is distributed as:

\[ N(w + Y/r[1 - e^{-rt}], (1/2r)[1 - e^{-2rt}]) \]

Let \( \text{Po}^*(w,t,dy) = \text{Prob} \{ e^{-rt}x^*(t) \leq dy \mid x^*(0) = w \} \), the transition probability for \( x^*(t) \), the process without absorption. \( V(w,t) \) can be rewritten as:

\[ V(w, t) = \int_0^\infty y \left[ \text{Po}^*(w, t, dy) - \text{Prob} \{ x^*(t) \leq dy \cap \min_{0 \leq s \leq t} x^*(s) \leq 0 \} \right] \]

which can also be written as:

\[ V(w, t) = \int_0^\infty y \text{Po}^*(w, t, dy) - \int_0^\infty y \text{Prob} \{ x^*(t) \leq dy \cap \min_{0 \leq s \leq t} x^*(s) \leq 0 \} \]

For \( y < 0 \), \( \text{Prob} \{ x^*(t) \leq dy \} = \text{Prob} \{ x^*(t) \leq dy \cap \min_{0 \leq s \leq t} x^*(s) \leq 0 \} \).

Thus, \( V(w,t) \) can be written as:

\[ V(w, t) = \int_0^\infty y \text{Po}^*(w, t, dy) - \int_0^\infty y \text{Prob} \{ x^*(t) \leq dy \cap \min_{0 \leq s \leq t} x^*(s) \leq 0 \} \]
This first integral in this expression is just the expected value of \( e^{-rt}x^*(t) \), which is equal to \( w+Y/r(1-e^{-rt}) \). As well, it is clear that the second integral is positive. To see this, consider any sample path of \( x^*(t) \) which intersects the boundary at \( t=s \), so that \( x^*(s)=0 \). At the point when \( x^*(s)=0 \), 
\[
dx^*(s) = rx^*(s) + Ydt + \sigma dz = Ydt + \sigma dz.
\]
Accordingly, the process has positive drift at this point \( s=t \), implying that for \( t>s \), \( E[x^*(t)/x^*(s)=0] \) must be positive. In the result, 
\[
V(w,t) \leq w+Y/r(1-e^{-rt}).
\]
QED

Proposition 2: \( V(w,t) \) is an increasing function of \( t \).

Proof: \( V(w,t+dt) - V(w,t) = E[e^{-r(t+dt)}x(t+dt) - e^{-rt}x(t)] \)

Neglecting terms \( O(dt^2) \) and higher,
\[
= (e^{-rt} - e^{-rt}rdt) E[x(t+dt)] - e^{-rt} E[x(t)]
\]
\[
= e^{-rt} \{ E[x(t+dt)] - E[x(t)] - rdtE[x(t+dt)] \}
\]

It is clearly sufficient to show that the function in braces is non-negative.

\[
E[x(t+dt)] - E[x(t)] - rdtE[x(t+dt)] = \int y P(w,t+dt,dy) - \int y P(w,t,dy) - rdt \int y P(w,t+dt,dy)
\] (A)
where $P(w,t,dy)$ is the transition probability for the process $x(t)$: $P(w,t,dy) = \operatorname{Prob}\{x(t) \in dy / x(0)=w\}$.

The expression $\int_y P(w,t+dt,dy)$ can be replaced by:
$\int z P_1(w, t, dy, dt, dz)$

where $P_1(w, t, dy, dt, dz)$ is the transition probability:
$\operatorname{Prob}\{x(t+dt) \in dz \cap x(t) \in dy / x(0)=w\}$

The last integral is in turn equal to:
$\int z P(y,dt,dz) P(w,t,dy)$
$= \int \mathbb{E}[x(t+dt) / x(t)=y] P(w,t,dy)$

Substituting this last expression into (A) leads to:
$\mathbb{E}[x(t+dt)] - \mathbb{E}(x(t)) - r dt \mathbb{E}[x(t+dt)] =
\int P(w,t,dy)\{\mathbb{E}[x(t+dt)] / x(t)=y\} - y - r dt \mathbb{E}[x(t+dt) / x(t)=y]\}$

It thus suffices to show that:
$(1 - r dt) \mathbb{E}[x(t+dt)/x(t)=y] - y > 0$ (B)

The expression $\mathbb{E}[x(t+dt)/x(t)=y]$ is equal to:
$\int z P(y,dt,dz)$
\[ \int z P^2(y, dt, dz) \]

where \( P^2(y, dt, dz) \) is the transition probability:
\[
\text{Prob}\{x^*(dt) \in dz \cap \min_{0 \leq s \leq dt} \{x^*(s) > 0 \} / x^*(0) = y}\]

\[ = \int z P^*(y, dt, dz) - \int z P^3(y, dt, dz) \]

(C)

where \( P^3(y, dt, dz) \) is the transition probability:
\[
\text{Prob}\{x^*(dt) \in dz \cap \min_{0 \leq s \leq dt} \{x^*(s) \leq 0 \} / x^*(0) = y}\]

Consider first the integral \( \int z P^*(y, dt, dz) \). It can be shown that this integral is equal to \( y + rydt + Ydt + o(dt) \).

Using the same method as in the proof of Proposition 1, the probability distribution of \( P^*(y, dt, dz) \) is normal:
\[
N( ye^{rdt} + Y/r(e^{rdt} - 1), 1/2r(e^{2rdt} - 1) )
\]

The integral in question is thus:
\[
\frac{\sqrt{2r}}{\sqrt{2\pi e^{2rdt} - 1}} \int_{0}^{\infty} z \exp \left[-\frac{1}{2} \left( \frac{2r(z - ye^{rdt} - Y/r e^{rdt} + Y/r)^2}{e^{2rdt} - 1} \right) \right] dz
\]

Using the substitution
\[
u = \frac{\sqrt{2r}}{\sqrt{e^{2rdt} - 1}} [z - ye^{rdt} - Y/r e^{rdt} + Y/r]
\]

the integral becomes:
Writing $t$ in place of $dt$, and using Leibniz' Rule, this expression can be differentiated with respect to $t$ to yield:

$$
\frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2t}{e^{2rt}-1}}}^{\infty} e^{-\frac{y^2}{2}} \left[ u \frac{\sqrt{e^{2rt} - 1}}{\sqrt{2t}} + ye^{rt} + Ye^{rt} - Y/r \right] du
$$

As $t$ approaches 0, this expression approaches $ry + Y$, proving that the integral under consideration is $y + rydt + Ydt + o(dt)$.

Now consider the integral:

$$\int z P^3*(y,dt,dz)$$

It can be shown that this integral is $o(dt)$. To show this, consider the probability (again writing $t$ instead of $dt$):

$$\operatorname{Prob} \{ \min_{0 \leq s \leq t} \{ x^*(s) \} \leq 0 \} / x(0) = Y \} \text{ for any fixed } y.$$

This probability is simply the probability that the process $x^*(t)$ reaches the boundary of 0 during the interval $(0, t]$.

To show the order of this probability, it is sufficient to
examine a process without drift $x^\ast(t)$, where $dx^\ast(t) = dz$. This process clearly has a greater probability of reaching the $0$ boundary than a process with positive drift.

An explicit solution exists for the probability of $x^\ast(t)$ reaching the boundary in $(0,t]$, given that $x^\ast(0) = y$. The solution, given in Ross at p. 190, is equal to:

$$\frac{2}{\sqrt{2\pi t}} \int_{y/y}^{\infty} \exp\left[-\frac{x^2}{2}\right] dx$$

The derivative of this integral with respect to $t$ is:

$$\frac{y}{\sqrt{2\pi t^3}} \exp\left[-\frac{y^2}{2t}\right]$$

The limit of this expression as $t$ approaches $0$ is $0$. This shows that the probability of $x^\ast(t)$ reaching the $0$ boundary in $(t,t+dt]$ is $o(dt)$. The same must hold true for $x^\ast(t)$ and $x(t)$.

Using these results, expression (C) now becomes:

$$y + ry dt + Y dt + o(dt)$$

and the LHS of equation (B) can thus be written:
\[(1-r dt)(y + ry dt + Y dt + o(dt)) - y = Y + o(dt)\]

which is clearly positive for small \(dt\). QED

Proposition 3. The partial derivative of \(V\) with respect to \(t\), \(V_t\) is bounded. Specifically, \(V_t \leq Ye^{-rt}\).

Proof: For a given \(w\),

\[dV = [E[e^{r(t+dt)x(t+dt)} - E[e^{-rt}x(t)]] dt\]

\[= e^{-rt}[(1-r dt)E[x(t+dt)] - E[x(t)]] \text{ neglecting terms } o(dt)\]

\[= e^{-rt} \int P(w,t,dy) [(1-r dt)\int zP^*(w,dt,dz) - \int zP^*(w,dt,dz) - y] \]

using equation (C) from Proposition 3. Using the results of Proposition 3, this can also be written as:

\[= e^{-rt} \int P(w,t,dy) [(1-r dt)[y + ry dt + Y dt + o(dt) - o(dt) - y]\]

\[= e^{-rt} \int P(w,t,dy) [Y dt + o(dt)]\]

It is clear, however, that \(\int P(w,t,dy) \leq 1\), as the integral represents the probability that \(x\) has not reached the boundary at
time t.

It follows that \( V_t(w,t) \leq Y e^{-rt} \). QED

Proposition 4. The sequence of functions \( v_n'(w) \) is non-decreasing in \( n \), i.e. \( v_{n+1}'(w) \geq v_n'(w) \).

(For ease of notation, \( t' \) shall be used to indicate \( t+dt \))

Proof: Let \( x_1(t) \) and \( x_2(t) \) denote, respectively, processes with starting values \( w \) and \( w+dw \), i.e. \( x_1(0) = w, x_2(0) = w+dw \).

To compare \( v_{n+1}'(x) \) and \( v_n'(x) \), it suffices to look at the difference:

\[
E \left[ e^{-rt} x_2(t') - e^{-rt} x_1(t') - \{ e^{-rt} x_2(t) - e^{-rt} x_1(t) \} \right]
\]

\[
= e^{-rt} E[x_2(t') - x_2(t) - rdtx_2(t') + rdtx_1(t') + x_1(t) - x_1(t')]
\]

after neglecting terms of \( O(dt^2) \) and higher order.

Eliminating the factor \( e^{-rt} \):

\[
= (1-rt) \int z P(w+dw,t',dz) - \int z P(w+dw,t,dz)
\]
\[- \left[ (1 - rdt) \int z P(w, t', dz) - \int z P(w, t, dz) \right]\]

\[= \int (1 - rdt) z P(y, dt, dz) P(w + dw, t, dy) - \int y P(w + dw, t, dy)\]
\[- \int (1 - rdt) z P(y, dt, dz) P(w, t, dy) + \int y P(w, t, dy)\]

\[= \int (1 - rdt) z P(w + dw, t, dy) [P^*(y, dt, dz) - P^3(y, dt, dz)]\]
\[- \int (1 - rdt) z P(w, t, dy) [P^*(y, dt, dz) - P^3(y, dt, dz)]\]
\[- \int y P(w + dw, t, dy) + \int y P(w, t, dy)\]

\[= \int P(w + dw, t, dy) \left[ \int (1 - rdt) z [P^*(y, dt, dz) - P^3(y, dt, dz) - y]\right] - \int P(w, t, dy) \left[ \int (1 - rdt) z [P^*(y, dt, dz) - P^3(y, dt, dz) - y]\right]\]

From the proof of Proposition 2:

\[\int z P^*(y, dt, dz) = y + ry dt + Y dt + o(dt)\]
\[\int z P^3(y, dt, dz) = o(dt)\]

Substituting these results and eliminating terms of $o(dt)$, the above expression becomes:

\[= \int P(w + dw, t, dy) \left[ (1 - rdt) (y + ry dt + Y dt + o(dt) - o(dt) - y) \right] - \int P(w, t, dy) \left[ (1 - rdt) (y + ry dt + Y dt + o(dt) - o(dt) - y) \right]\]
\[ = \text{Y} dt \left[ \int P(w+\text{dw},t,\text{dy}) - \int P(w,t,\text{dy}) \right] \]

The two integrals represent, respectively, the probabilities that \( x_2 \) and \( x_1 \) have not reached the 0 boundary by time \( t \). As \( x_2 \) has a higher starting value, the difference between the integrals is unambiguously positive. In the result, the difference:

\[ E[e^{-rt}x_2(t') - e^{-rt}x_1(t') - (e^{-rt}x_2(t) - e^{-rt}x_1(t))] \]

is unambiguously positive. QED

Proposition 5. Each of the functions \( h_n(w) = v_e(w,n) \) is non-decreasing in \( w \).

Proof: As in the proof of proposition 4, let \( x_1(t) \) and \( x_2(t) \) denote, respectively, processes with starting values \( w \) and \( w+\text{dw} \), i.e. \( x_1(0) = w, x_2(0) = w+\text{dw} \). Again, \( t+\text{dt} \) will be denoted \( t' \).

To compare \( h_n(w+\text{dw}) \) and \( h_n(w) \), it suffices to look at the difference:

\[ E[e^{-rt}x_2(t') - e^{-rt}x_2(t) - (e^{-rt}x_1(t') - e^{-rt}x_1(t))] \]
\[ = E[e^{-\tau}x2(t') - e^{-\tau}x1(t') - (e^{-\tau}x2(t) - e^{-\tau}x1(t))] \]

The latter expression was already shown to be non-negative in the proof of proposition 4. QED

Proposition 6. The functions \( h_n(w) \) converge uniformly to \( h(w) = 0 \) on any closed, bounded interval \([a,b]\) with \( b > a > 0 \).

Proof: Proposition 1 shows that for any \( w \), \( V(w,t) \) is bounded as \( t \) approaches infinity. Proposition 2 shows that for any \( w \), \( V(w,t) \) is increasing in \( t \). It follows that \( V_t(x,t) \) approaches 0 as \( t \) approaches infinity, showing pointwise convergence of \( h_n(w) \) to 0.

To show uniformity of convergence, proposition 6 shows that each \( h_n(w) \) is non-decreasing in \( w \), or \( h_n(w) \leq h_n(b) \) for \( w \) in \([a,b]\).

Since \( h_n(b) \) converges to zero, for every \( \varepsilon > 0 \), there exists \( N \) such that \( n > N \) implies \( h_n(w) \leq h_n(b) \leq \varepsilon \) for all \( w \) in \([a,b]\). QED

Proposition 7. The sequence \( v_n(w) \) is uniformly convergent on any closed interval \([a,b]\) with \( b > a > 0 \).
Proof: It follows from proposition 3 that

\[ v_{n+k}(w) - v_n(w) \leq \int_0^{n+k} Ye^{-r_s} ds = Y \left[ e^{-rn} - e^{-r(n+k)} \right] \leq \frac{Y}{r} e^{-rn} \]

The final value is independent of \( w \), and can be made arbitrarily small by choosing \( n \) large. It follows that the sequence \( v_n(w) \) satisfies the Cauchy criterion for uniform convergence. QED

Proposition 8. \( v'_n(w) \geq 1 \).

Proof: From the boundary conditions applicable to \( V(w,t) \), \( V(w,0) = w \), from which it follows that \( \partial V(w,0)/\partial w = 1 \).

Propositions 4 and 5 show that \( \partial^2 V(w,t)/\partial w \partial t \geq 0 \), from which it follows that \( \partial V(w,t)/\partial w \geq 1 \).

Proposition 9. \( v''_n(w) \leq 0 \)

Proof: From the definition of \( v_n(w) \), each \( v_n(w) \) satisfies:

\[ v_n''(w) = 2[h_n + rv_n - (rw+y)v'_n(w)] \]
From proposition 3, \( h_n \geq Ye^{-m} \). As well, from Proposition 1, 

\[ V(w, n) \leq (w + Y/r(1-e^{-m})) \]

Thus, using the result from Proposition 8 that \( v'_n (w) \geq 1 \):

\[
v''_n (w) \leq 2[Ye^{-m} + rw + Y - rwe^{-m} - Ye^{-m} - rw - Y] \leq 2[-rwe^{-m}] \leq 0
\]

Proposition 10. For each \( w \), the sequences \( v'_n (w) \) and \( v''_n (w) \) are bounded as \( n \) approaches infinity.

Proof: Consider first the sequence \( v'_n (w) \) and assume that there exists some \( w^* \) such that \( v'_n (w^*) \) is unbounded for large \( n \). For any \( M > 0 \), there must then exist some \( j \) for which \( v'_j (w^*) > M/w^* \).

From Proposition 9, \( v''_j (w) \leq 0 \) for \( w \) in \( (0, w^*) \), from which it follows that \( v'_j (w) > M/w^* \) for all \( w \) in \( (0, w^*) \). This in turn implies that \( v_j (w^*) > M \), which would imply that \( v_n (w) \) is unbounded. This contradicts the result of Proposition 1. This completes the proof that \( v'_n (w) \) is bounded.

Boundedness of \( v''_n (w) \) follows easily. Each \( v''_n (w) \) satisfies:
\[ v_n''(w) = 2[h_n + rv_n(w) - (rw+Y)v_n'(w)] \]

As each term on the RHS is bounded for large n, the same must hold for \( v_n''(w) \).

**Proposition 11.** The sequence \( v_n'(w) \) is uniformly convergent on any interval \([a,b]\) with \( b>a>0 \).

**Proof:** Proposition 10 establishes that \( v_n'(w) \) is bounded as \( n \) increases. Proposition 4 establishes that \( v_{n+1}'(w) \geq v_n'(w) \). These two results establish pointwise convergence to a function \( v^*(w) \).

Proposition 9 establishes that each \( v_n'(w) \) is monotonic, so the same must hold for \( v^*(w) \). As a monotonic function, \( v^*(w) \) can have only jump discontinuities. In the neighbourhood of any such discontinuity, the sequence \( v_n''(w) \) would have to diverge. This, however, contradicts the second result of Proposition 10, that the sequence \( v_n''(w) \) is bounded. It follows that \( v^*(w) \) is continuous.

Uniform convergence of \( v_n'(w) \) now follows from Dini's Theorem,
which holds that a monotonic sequence of continuous functions
converging to a continuous limit function must converge uniformly
on a closed interval.

PROOF OF RESULT 2.

Result 2 can now be proved shortly from the above propositions.

Each \( v_n(w) \) satisfies:

\[ v_n''(w) + (rw+Y)v_n'(w) - rv_n = h_n(w). \]  \((F)\)

By Proposition 6, \( h_n(w) \) converges uniformly to 0 on any interval
\([a,b]\). \( v_n(w) \) converges uniformly by Proposition 7, to limit
function \( v(w) \).

By proposition 11 \( v_n'(x) \) converges uniformly as well. Thus, by
Apostol, theorem 9.13, \( v_n'(w) \) converges uniformly to \( v'(w) \).
Similarly, \( (rw+Y)v_n'(w) \) converges uniformly to \( (rw+Y)v'(w) \), as
both factors are bounded on any interval \([a,b]\).

Each \( v_n''(w) \) satisfies:
\[ v_n''(w) = 2\{ h_n(w) + rv_n(w) - (rw+Y)v_n'(w) \}. \]

Since each term on the RHS converges uniformly, so must \( v_n''(w) \).

By Apostol, theorem 9.13, \( v_n''(w) \) converges to \( v''(w) \).

It is thus possible to take limits on both sides of equation (F), which yields the equation to be proved:

\[ \frac{1}{2} v''(w) + (rw+Y)v'(w) - rv(w) = 0. \]

This equation has been proved for any interval \([a,b]\), with \( b > a > 0 \). Since both \( a \) and \( b \) are arbitrary, the equation must hold for \( w \geq 0 \).

Finally, the applicable boundary conditions follow easily. Since \( v_n(0) = 0 \) for all \( n \), it is clear that \( v(0) = 0 \). As well, since each \( v_n(w) \) approaches \( w + Y/r[1-e^{-m}] \) for large \( w \), it follows that \( v(w) \) approaches \( w + Y/r \) for large \( w \). QED
CHAPTER III - THE DEMAND FOR HIGH INTEREST RATE LOANS

The results of Chapter II can be used to analyse the willingness of a nearly bankrupt firm to borrow at high interest rates. In this chapter, the value function derived in Chapter II is adapted to account for such borrowing. The resulting value function is then used to calculate numerical examples, which show a positive demand for loans at strikingly high interest rates.

The analysis is most apt for small to medium sized firms, which often have few options to diversify risk or raise new capital when facing financial distress. In practical terms, such firms must often accept high interest rates to raise operating capital required to stave off bankruptcy. (Note that while finance literature often refers to the era of the "junk bond", this chapter does not really illustrate that phenomenon. Junk bonds are seldom floated as a means of avoiding imminent bankruptcy. This chapter more accurately portrays the demand for services of loan sharks, or the high interest rate lenders operating in secondary lending markets.)

As in Chapter II, assume the firm has steady revenue $Y + oz$, and
faces bankruptcy when its wealth drops to a specified level $A$. In absence of borrowing, the firm's wealth evolves according to:

$$dx(t) = rx(t)dt + Ydt + odz; x(0)=w$$

Now assume in addition that:

1. The firm is able to borrow funds ($B$) at an interest rate $R>r$.

2. The borrowed funds can be invested at the same rate as the firm's own capital, producing a return at rate $r$.

3. The additional borrowing does not change the bankruptcy threshold, which still occurs at $x(t)=A$.

If $A$ represents a minimum wealth level imposed by secured creditors, the additional borrowing can be interpreted as unsecured debt. The new lender earns his high interest rate, $R$, by accepting the risk of an unsecured loan.

The new debt creates debt service charges equal to $RBdt$. By
assumption, this is greater than the additional revenue the firm can earn \((r_B dt)\) by investing the borrowed funds. The offsetting advantage to the firm is that the borrowed funds provide an extra cushion against bankruptcy. As bankruptcy still occurs at the point \(x(t) = A\), the firm's initial margin from bankruptcy is \(w + B - A\), instead of \(w - A\).

The borrower's wealth now evolves according to:

\[
dx(t) = rx(t) \, dt + Y dt - R_B dt + \sigma dz
\]

with absorption at \(x = A\). The discount rate is again \(r\).

Define \(q = w + B\) as the total initial assets of the corporation, including borrowed funds, so that \(x(0) = q\). The borrower's value function is then:

\[
v(q) = v(w; B, R) = \lim_{t \to 0} E[e^{-rt} x(t)]
\]

As a function of \(q\), \(v\) must now satisfy
\[ \frac{1}{2\sigma^2} \frac{d^2v}{dq^2} + (rq + Y - RB) \frac{dv}{dq} - rv = 0 \]

subject to \( v(A) = 0; \) \( v \) tending to \( q + Y/r - RB/r \) for large \( q. \)

Expressed as a function of \( q \), the solution is:

\[
v(q) = (q + \frac{Y}{r} - \frac{RB}{r}) \left( 1 - \int_{rA + Y - RB}^{\infty} \frac{\exp\left(-\frac{s^2}{r\sigma^2}\right)}{s^2} ds \right) \]

This expression can be rewritten as a function of \( w, B, \) and \( R \) as follows:

\[
v(w; R, B) = (w + B + \frac{Y}{r} - \frac{RB}{r}) \left( 1 - \int_{rA + Y - RB}^{\infty} \frac{\exp\left(-\frac{s^2}{r\sigma^2}\right)}{s^2} ds \right)
\]
The corporation's demand for loan funds can be calculated using (1). For given starting wealth, \( w \), and for a given borrowing rate, \( R \), the corporation chooses the value of \( B \) which maximizes \( v \).

Differentiation of (1) yields expressions too difficult to manipulate into a meaningful first order condition. However, the following tables give numerical examples which illustrate the magnitude of the demand for high interest rate loans. The tables give values of \( v \) for differing levels of \( w \), \( B \), and \( R \). Other parameters are as follows: \( A=0 \), \( Y=5 \), \( c=10 \), and \( r=.1 \). Quantities marked with asterisks represent maximizing values of \( B \) (from among those calculated) for the given borrowing interest rate, \( R \).

Table 1 - Borrower's values: \( w=1 \)

<table>
<thead>
<tr>
<th></th>
<th>( R = 11% )</th>
<th>13%</th>
<th>15%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B = 1</td>
<td>13</td>
<td>12.9</td>
<td>12.9</td>
<td>12.7</td>
<td>12.4</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>22.5</td>
<td>22.1</td>
<td>21.8</td>
<td>20.9</td>
<td>19.3</td>
<td>17.7</td>
</tr>
<tr>
<td>5</td>
<td>29.4</td>
<td>28.6</td>
<td>27.9</td>
<td>26.1</td>
<td>22.7</td>
<td>19.6*</td>
</tr>
<tr>
<td>10</td>
<td>39.5</td>
<td>37.7</td>
<td>35.8</td>
<td>31.5*</td>
<td>23.5*</td>
<td>16.6</td>
</tr>
<tr>
<td>20</td>
<td>46.1</td>
<td>42.3*</td>
<td>38.4*</td>
<td>29.2</td>
<td>14</td>
<td>5.3</td>
</tr>
<tr>
<td>30</td>
<td>47.0*</td>
<td>41.4</td>
<td>35.6</td>
<td>22.4</td>
<td>6.7</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 2 - Borrowers' values: \( w=3 \)

<table>
<thead>
<tr>
<th></th>
<th>R = 11%</th>
<th>13%</th>
<th>15%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
</tr>
</thead>
<tbody>
<tr>
<td>B = 1</td>
<td>23.4</td>
<td>23.7</td>
<td>23.2</td>
<td>23.2</td>
<td>22.7</td>
<td>22.1</td>
</tr>
<tr>
<td>3</td>
<td>31</td>
<td>30.5</td>
<td>30.1</td>
<td>29</td>
<td>26.8</td>
<td>24.8</td>
</tr>
<tr>
<td>5</td>
<td>36.2</td>
<td>35.4</td>
<td>34.5</td>
<td>32.5</td>
<td>28.5*</td>
<td>24.8*</td>
</tr>
<tr>
<td>10</td>
<td>43.9</td>
<td>41.9</td>
<td>40.1</td>
<td>35.5*</td>
<td>26.8</td>
<td>19.3</td>
</tr>
<tr>
<td>20</td>
<td>48.8</td>
<td>44.8*</td>
<td>41.0*</td>
<td>31.6</td>
<td>15.6</td>
<td>6</td>
</tr>
<tr>
<td>30</td>
<td>49.4*</td>
<td>43.5</td>
<td>37.7</td>
<td>24.2</td>
<td>7.3</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3 - Borrower's values: \( w=5 \)

<table>
<thead>
<tr>
<th></th>
<th>R = 11%</th>
<th>13%</th>
<th>15%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
</tr>
</thead>
<tbody>
<tr>
<td>B = 1</td>
<td>32.6</td>
<td>32.5</td>
<td>32.3</td>
<td>32</td>
<td>31.2</td>
<td>30.5</td>
</tr>
<tr>
<td>3</td>
<td>38.1</td>
<td>37.6</td>
<td>37.1</td>
<td>35.8</td>
<td>33.3</td>
<td>30.9*</td>
</tr>
<tr>
<td>5</td>
<td>42</td>
<td>41</td>
<td>40.2</td>
<td>37.9</td>
<td>33.6*</td>
<td>29.4</td>
</tr>
<tr>
<td>10</td>
<td>47.8</td>
<td>45.8</td>
<td>43.8*</td>
<td>39.1*</td>
<td>30</td>
<td>21.9</td>
</tr>
<tr>
<td>20</td>
<td>51.3</td>
<td>47.3*</td>
<td>43.4</td>
<td>33.8</td>
<td>17.3</td>
<td>6.8</td>
</tr>
<tr>
<td>30</td>
<td>51.5*</td>
<td>45.6</td>
<td>39.8</td>
<td>26</td>
<td>8</td>
<td>-</td>
</tr>
</tbody>
</table>
These tables demonstrate a strong demand for loan funds, even at what seem like very high interest rates. For example, a borrower with starting wealth equal to 5 would appear to have substantial safety from bankruptcy, without additional borrowing. His starting distance from the boundary threshold is equal to $a/2$. In addition, he has expected positive cash flow equal to $5 + 5(.1) = 5.5$. In spite of this, even at interest rates of 30% and 40% he will borrow, respectively, amounts equal to 100% and 60% of his starting wealth.

It is readily apparent that the ability to borrow will be most valuable to a corporation close to bankruptcy. The marginal value of additional borrowing can be calculated from (1), by fixing $w$ and $R$, and by considering $v$ as a function of $B$. 

<table>
<thead>
<tr>
<th>B</th>
<th>$R = 11%$</th>
<th>13%</th>
<th>15%</th>
<th>20%</th>
<th>30%</th>
<th>40%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>49</td>
<td>48.7</td>
<td>48.5</td>
<td>48</td>
<td>47.0*</td>
<td>46.1*</td>
</tr>
<tr>
<td>3</td>
<td>51.5</td>
<td>50.9</td>
<td>50.3</td>
<td>48.8</td>
<td>45.8</td>
<td>42.9</td>
</tr>
<tr>
<td>5</td>
<td>53.3</td>
<td>52.3</td>
<td>51.3</td>
<td>48.8*</td>
<td>43.8</td>
<td>39.1</td>
</tr>
<tr>
<td>10</td>
<td>55.9</td>
<td>53.9*</td>
<td>51.8*</td>
<td>46.8</td>
<td>37.1</td>
<td>28.1</td>
</tr>
<tr>
<td>20</td>
<td>57.1*</td>
<td>53.1</td>
<td>49.1</td>
<td>39.3</td>
<td>21.5</td>
<td>9.08</td>
</tr>
<tr>
<td>30</td>
<td>56.8</td>
<td>50.8</td>
<td>44.9</td>
<td>30.6</td>
<td>9.9</td>
<td>-</td>
</tr>
</tbody>
</table>
Differentiation of (1) at the point \( w=A, B=0 \) yields:

\[
\frac{dv}{dB} = 1 + \frac{2(rA+Y)}{rO^2} \int_{rA-Y}^{\infty} \exp\left[-\frac{s^2}{rO^2}\right] ds
\]

This is the same expression that was derived in Chapter II, equation (8) as the marginal value of wealth at the bankruptcy boundary. As bankruptcy becomes imminent, avoiding bankruptcy becomes the paramount consideration. It does not matter whether the wealth used to do that comes from borrowed funds or from the corporation's own endowed wealth. It is also worth noting that this expression is independent of the borrowing rate \( R \). There is no theoretical maximum rate that the corporation will pay. Facing imminent bankruptcy, infinitesimal additional wealth will be borrowed at any interest rate if bankruptcy can be forestalled.
This and following chapters will examine several additional aspects of lender-borrower transactions. These will include the choice of an optimal debt-equity ratio, the effects of a change in the ability of a borrower to grant security, and the effects of changes in bankruptcy legislation.

To complete the analysis, both borrower and lender value functions must be derived. The value functions of this chapter are derived for a project of fixed size, w, which must be financed by a combination of debt, B, and equity, w-B. As the total size of the project is fixed, the value functions can be used to investigate the firm's optimal capital structure, but not the optimal scale for the project.

Section 1 adapts the borrower's value function, calculated in Chapter II, to a project financed partly by debt and partly by equity. Section 2 derives a value function for the lender, based on the discounted value of his expected interest revenue. Under the maintained assumptions, this problem as well has a closed form solution. Finally, section 3 investigates mathematical
properties of the value functions which are used in subsequent chapters.

In addition to calculating value functions, this chapter also calculates rates of return for both the borrower and lender. It turns out that in several of the subsequent problems (particularly those involving a single lender and a single borrower), more interesting results are derived in maximizing the parties' respective rates of return, as opposed to their value functions simpliciter.

1. Value Function of the Borrower / Project Owner

The structure of the borrower's problem is now slightly different. The borrower is seeking to finance a project of fixed size \( w \). \( w \) is thus both the size of the project and the borrower's total starting wealth.

To finance the project, the borrower must chose a combination of debt, \( B \), and equity, \( w-B \). As before, the borrower's invested wealth earns a rate or return \( r \), and interest is payable on debt at rate \( R \). However, it is now assumed that \( R < r \) i.e. the interest
rate payable on debt is less than the physical rate of return to capital. (In the analyses of some problems, it will also be assumed that the borrower incurs transaction costs in arranging a loan. These will be incorporated into the value function later on).

Bankruptcy no longer occurs at an arbitrary level A. Instead, bankruptcy occurs when the borrower's equity is lost, or when wealth falls to the level of the borrower's debt, B. In the result, the bankruptcy threshold is now endogenous.

These assumptions confront the borrower with realistic tradeoffs in choosing a debt equity ratio. More debt increases the borrower's leverage, in that the borrower is able to invest borrowed funds at a return (r) higher than his borrowing rate (R). However, increased debt also increases the risk of bankruptcy, and hence the risk that starting equity will be lost.

As before, x(t) will represent the evolution of the borrower's wealth over time, given that x(0) = w. w will be the nominal argument of the value function. In deriving the value function the debt level, B, and the interest rate on debt, R, will be
taken as parametric. As a practical matter, however, \( R \) and \( B \) will be considered as the choice variables. Once the value function is derived, the investigations of interest will be in determining the effect of different values for the "parameters", \( B \) and \( R \).

The stochastic component of revenue is still equal to \( \sigma dz \). All agents are risk neutral, and value their assets as the discounted expected value of future revenue. The discount rate is again set exogenously at \( r \).

From the assumptions set out above, \( x(t) \) evolves according to:

\[
dx(t) = rx(t)dt - RBdt + \sigma dz
\]

Let \( v(w;B,R) \) represent the value function of the borrower:

\[
v(w;B,R) = \lim_{t \to \infty} E[e^{-rt} x(t)].
\]

\( v(w;B,R) \) must then satisfy:
The boundary conditions are:

- \( v(w;B,R) \) approaches 0 as \( w \) approaches \( B \)
- \( v(w;B,R) \) approaches \( w-RB/r \) as \( w \) becomes arbitrarily large.

Using the same general solutions as were derived in Chapter II, \( v(w;B,R) \) is equal to:

\[
v(w;B,R) = \left( w - \frac{RB}{r} \right) \left( 1 - \frac{1}{s^2} \int_{RB-RB}^{\infty} \exp\left[ - \frac{s^2}{rO^2} \right] ds \right) \]

(1)

To allow for transaction costs equal to \( cb \), it is simply required to subtract this quantity from the value given above. To calculate the borrower's rate of return, the value function is divided by \( (w-B) \). The borrower's rate of return, with transaction costs included, is thus:
\[ v(w; B, R) = \left[ \frac{1}{w-B} \right] \left( w - \frac{RB}{r} \right) \left( 1 - \frac{sv}{s^2} \right) \right] - cb \]

\[ = \exp \left[ - \frac{s^2}{rO} \right] \int_{w-B}^{w-B} ds \]

2. **Value Function of the Lender**

The value function of the lender is more involved, and requires the application of different mathematical results.

(a) **Assumptions**

The assumptions concerning the lender's revenue and cost structure are as follows.

1. The lender receives interest revenue equal to RB until bankruptcy intervenes. Interest revenue ceases on bankruptcy.

2. The lender also receives "salvage value" on bankruptcy. The salvage value is equal to SaB.
3. The lender's cost of capital is at rate \( \rho < R < r \). Bankruptcy does not relieve the lender of this cost.

4. The lender faces transaction costs equal to \( c_1 \) in arranging a loan.

(b) The Basic Value Function - The Valuation of Interest Revenue

The value function must be derived in steps. The first step is to derive the basic value of expected future interest revenue \( (I_1(w;B,R)) \), leaving salvage value and transaction costs for later.

Interest revenue at time \( t \) is equal to \( RB \) if bankruptcy has not occurred prior to time \( t \), but 0 if bankruptcy has occurred prior to time \( t \). The expected discounted value of interest revenue at time \( t \) is thus equal to:

\[
e^{-rt} RB (1 - Q(w,t))
\]

where \( Q(w,t;B,R) \) is the probability that bankruptcy occurs at or
before time t, given starting wealth equal to w.

The discounted value of all future interest revenue, \( l_1(w; B, R) \), is thus equal to:

\[
\begin{align*}
   l_1(w; B, R) &= \int_0^\infty e^{-rt} RB (1 - Q(w, t; B, R)) \, dt \\
   &= \int_0^\infty e^{-rt} \frac{RB}{r} \, dt
\end{align*}
\]  

(For the remainder of this section, to economize on notation, B and R will be suppressed as arguments of Q and \( l_1 \)). To evaluate this expression, note that:

\[
\begin{align*}
   \frac{dl_1}{dw} &= \int_0^\infty e^{-rt} RB \left( 1 - \frac{\partial Q}{\partial w} \right) \, dt \\
   &= \frac{RB}{r} - \int_0^\infty e^{-rt} RB \frac{\partial Q}{\partial w} \, dt
\end{align*}
\]

and:

\[
\begin{align*}
   \frac{d^2 l_1}{dw^2} &= \int_0^\infty e^{-rt} RB \left( 1 - \frac{\partial^2 Q}{\partial w^2} - \frac{\partial Q}{\partial w} \frac{\partial Q}{\partial w} \right) \, dt \\
   &= \frac{RB}{r} - \int_0^\infty e^{-rt} RB \frac{\partial^2 Q}{\partial w^2} \, dt
\end{align*}
\]

Using these expressions:
However, by Result 1 of Chapter II, the expression in square brackets is equal to $\frac{\partial Q}{\partial t}$. Using integration by parts yields:

$$
\int_{0}^{\infty} e^{-rt} \frac{\partial Q}{\partial t} dt = -Q(w,0) + \int_{0}^{\infty} e^{-rt} Q(w,t) dt
$$

By the boundary conditions for $Q$, $Q(w,0)=0$. As well, the final term in this expression is equal to $rl_1(w)$. Thus, $l_1(w)$ must satisfy:

$$
\frac{1}{2} \sigma^2 \frac{d^2 l_1}{dw^2} + (rw-B(R-r)) \frac{dl_1}{dw} = \frac{RB}{r} - \int_{0}^{\infty} e^{-rt} \left[ \frac{1}{2} \sigma^2 \frac{\partial^2 Q}{\partial w^2} + (rw-B(R-r)) \frac{\partial Q}{\partial w} \right] dt
$$

This equation is identical to equation (2) of Chapter II, and has the same general solutions. The boundary conditions applicable to $l_1$ are:

- $l_1(B) = 0$
- $l_1$ approaches $RB/r$ as $w$ becomes arbitrarily large.

Using the solutions derived in Chapter II, $l_1$ is thus equal to:
To incorporate salvage value into the lender's valuation problem, it is necessary to consider the probability that bankruptcy occurs at a specific time, \( t \). If \( Q(w,t;B,R) \) represents the probability that bankruptcy has occurred at or before time \( t \), then \( \partial Q/\partial t \) must represent the probability that bankruptcy occurs at time \( t \).

Let \( l_2(w;B,R) \) represent the expected discounted value of the salvage recovered by the lender. If the lender recovers \( SaB \) when bankruptcy occurs, \( l_2(w;B,R) \) must be equal to:

\[
l_2(w;B,R) = \frac{RB}{r} - \frac{RB}{r} \left[ \int_{rB - RB}^{\infty} \frac{\exp\left(-\frac{S^2}{R \sigma^2}\right)}{s^2} ds \right]
\]

\[
\quad + \left(\frac{rB - RB}{rB - RB}\right) \int_{rB - RB}^{\infty} \frac{\exp\left(-\frac{S^2}{R \sigma^2}\right)}{s^2} ds
\]

\( (c) \) Salvage Value
\[ l_2(w; B, R) = \int_0^\infty e^{-rs} S \alpha B \frac{\partial Q(w, s; B, R)}{\partial s} ds \]

\[ = S \alpha B \int_0^\infty e^{-rs} \frac{\partial Q(w, s; B, R)}{\partial s} ds \]

However, it is clear that both \( Q(w, 0; B, R) = 0 \) and
\[ \lim_{t \to \infty} e^{-rt}[Q(w, t; B, R)] = 0. \]
Integration of the above expression by parts thus yields:

\[ l_2(w; R, B) = \frac{S \alpha B}{r} \int_0^\infty e^{-rs} Q(w, s; B, R) ds \]

The integral in this expression is the same integral encountered
in equation (3), in determining the value of the stream of interest income. Following the same steps used to solve (3), the
value of the salvage is thus:

\[ l_2(w; B, R) = S \alpha B \left[ \int_{rw-RB}^\infty \frac{\exp \left[ -\frac{s^2}{\rho \omega^2} \right]}{s^2} ds \right] \quad (5) \]

\[ + \int_{rB-RB}^\infty \frac{\exp \left[ -\frac{s^2}{\rho \omega^2} \right]}{s^2} ds \]
(d) The Value Function

Two more components are required to complete the value function.

First, it is assumed that the lender has a cost of capital equal to $\rho$, with $\rho < R < r$. Bankruptcy does not affect the lender’s cost of capital - he must continue to bear that cost even if his own loan to the borrower goes bad. The discounted cost of the capital used to fund a loan of size $B$ is thus $\rho B / r$.

Second, it is assumed that the lender incurs transaction costs equal to $c_l$ in negotiating the loan. Again, transaction costs are independent of the size of the loan.

The lender’s value function, $l(w;B,R)$ is thus equal to:

$$l(w;B,R) = l_1(w;B,R) + l_2(w;B,R) - \frac{\rho B}{r} - c_l.$$ 

Substituting expressions given above for $l_1$ and $l_2$, $l(w;B,R)$ is equal to:
The lender's rate of return, $L(w;B,R)$ will be equal to:

$$w;B,R = \frac{RB}{r} - \frac{\rho B}{r} + (sB - \frac{RB}{r}) \left( \begin{array}{c} \exp[-\frac{s^2}{r^2}] \\ \int_{rB-RB}^{rw-RB} \frac{s^2}{ro^2} ds \\ \int_{rB-RB}^{rw-RB} \frac{s^2}{ro^2} ds \end{array} \right)$$

Henceforth, $w$ will be suppressed as an argument of both $l$ (the value function) and $L$ (the rate of return function), which will be written respectively as $l(B,R)$ and $L(B,R)$.

Having derived the value functions, it is worthwhile to make explicit some of the restrictions that have been incorporated into the underlying assumptions.

First, the bankruptcy threshold has been endogenized, but only
partly so. The firm can alter its bankruptcy threshold by selecting a different value for B. However, once B is chosen the bankruptcy threshold is fixed - neither the borrower nor lender can change it. Other assumptions, permitting either the lender or borrower to set the bankruptcy threshold, could be used. For example, Leland (1994) examines cases in which the borrower has complete autonomy in setting the bankruptcy threshold, and other cases when lenders have contractual rights to force liquidation. There is no reason in principal why the value functions of this chapter could not be adapted to follow Leland’s analysis, and that represents a possible extension of this thesis.

Second, the form of contract between the borrower and lender is imposed exogenously. There is nothing in the analysis that makes a debt contract optimal - it is simply assumed as the form of contract.

Third, the analysis does not permit either renegotiation of the terms of the contract or dynamic adjustment of the firm’s capital structure over time. Dynamic adjustment of a firm’s capital structure was investigated by Mauer and Triantis (1994), who note the mathematical complications introduced by a variable capital
structure, and who resort to numerical analysis in the absence of a closed form solution. Debt renegotiation has been examined both in the case of sovereign debt (eg. Krugman (1988) and Eaton and Gersovitz (1981)) and in the case of a firm (eg. Mella-Barral and Perraudin (1997)). The latter paper begins with an analysis that follows Leland (1994), and then considers what renegotiations might occur near bankruptcy, when bankruptcy costs give the lender an incentive to avoid liquidation. A similar analysis using the value functions of this thesis represents another possible extension of this thesis. It is likely that with bankruptcy costs of sufficient magnitude, renegotiation could increase the value functions of both borrower and lender. As in Mella-Barral and Perraudin, it is also likely that the distribution of any gains from trade would depend on the bargaining structure imposed on the parties.

While the assumptions of this thesis may seem restrictive, many other analyses of bankruptcy involve some form of limitation on a firm's actions. For example, Leland (1994) assumes that the firm raises new equity to pay debt service charges, but raises no additional equity, even when bankruptcy is near. Milne and Robertson (1996) assume that the firm can pay out dividends, but
that raising new equity (i.e. a negative dividend) is impossible. Some limitation on firm action may be inevitable in any analysis of bankruptcy - a firm with unrestricted access to complete capital markets would likely never go bankrupt.

3. The function $M(B,R)$

Following the notation of Chapter II (and treating $w$ as parametric), define the function $M(B,R)$ as follows:

$$M(B,R) = \frac{(rw - RB) \int_{rw - RB}^{\infty} \exp\left(-\frac{s^2}{r_0^2}\right) ds}{(rB - RB) \int_{rB - RB}^{\infty} \exp\left(-\frac{s^2}{r_0^2}\right) ds}$$

A slightly different version of the function $M$ (under slightly different model specifications) was previously given in Chapter II, in analysing the value function derived in that chapter.

The function $l_2(w;B,R)$, derived in section 2(c) as the salvage value received by the lender on bankruptcy, assists in interpreting the function $M$. Equation (5) defines $l_2$ as:
From (7), this expression \( l_2 \) can be rewritten as \( l_2 = SaBM \).

Since \( SaB \) is the amount received by the lender when bankruptcy occurs, \( M \) must represent the discounted sum (i.e. integral) of the probabilities of bankruptcy occurring at all future times.

Again using (7), the value function of the lender (6) can be written as:

\[
l(w;B,R) = \frac{RB}{r} - \frac{\rho B}{r} + B(Sa-R/r)M
\]

This expression can be further rewritten as

\[
\frac{RB}{r}(1-M) - \frac{\rho B}{r} + BsaM, \text{ an expression which provides further insight into the lender's value function.}
\]

The terms \( \frac{\rho B}{r} + BSaM \) have already been explained. However, the term \( \frac{RB}{r}(1-M) \) requires further explanation. Absent bankruptcy
considerations, the “fundamental value” of the lenders’ revenue stream would be RB/r. This fundamental value is discounted by the value (1-M), with M representing the discounted sum of the probabilities of bankruptcy occurring at different points in the future. At the point of bankruptcy, M=1 and the “fundamental value” of the revenue stream disappears. A similar pattern appears manner of the other functions which will be considered later, and in the decision making rules for borrowers and lenders. The equations in question reflect a “fundamental” value which becomes irrelevant near bankruptcy, leaving as relevant only considerations directly affecting the probability of bankruptcy.

The function M appears in sufficiently many places that its properties merit investigation.

First, it should be pointed out that M permits of a simple “normalization”. Making the substitution $z^2 = s^2/(\sigma o^2)$ in both integrands above, M can be written as:
where \( J(x) \) is the function defined in Chapter II, section 5:

\[
J(x) = x \int_x^\infty \frac{\exp[-s^2]}{s^2} \, ds
\]

To analyse the derivatives of \( M \), it is convenient to define the quantities \( E_1, F_1, I_1 \) and \( J_1 \) as follows:

\[
E_1 = \exp\left[-\frac{(r \omega - \omega B)^2}{\omega_0^2}\right] \quad F_1 = \frac{\exp[-(r \omega - \omega B)^2/\omega_0^2]}{(r \omega - \omega B)^2} \\
I_1 = \int_{r \omega - \omega B}^\infty \exp[-s^2/\omega_0^2] \, ds \quad J_1 = (r \omega - \omega B) \int_{r \omega - \omega B}^\infty \frac{\exp[-s^2/\omega_0^2]}{s^2} \, ds
\]

Recall that by integration by parts, \( J_1 = E_1 - 2(r \omega - \omega B)I_1/\omega_0^2 \).

This expression is equal to 1 if \((r \omega - \omega B)\) is equal to zero. Also:
(a) if the function expressed as $J_1$ is differentiated with respect to the lower limit of its integral, $(r w - R B)$, the result is $-2 I_1 / (r^2)$; and

(b) if the function expressed as $I_1$ is differentiated with respect to the lower limit of its integral $(r w - R B)$, the result is $-E_1$.

$E_2, F_2, I_2$ and $J_2$ will have definitions corresponding to $E_1, F_1, I_1$ and $J_1$, with $(r w - R B)$ replaced by $(r B - R B)$. The relations of the preceding paragraph apply equally to $E_2, F_2, I_2$, and $J_2$.

Using the definitions given above, it is easy to show that $M_R > 0$.

$$M_R = 2 (R J_2 I_1 + (r - R) J_1 I_2) / (r^2 J_2^2) > 0.$$  

Evaluation of $M_R$ is only slightly more complex:

$$M_R = 2 B (J_2 I_1 - J_1 I_2) / (r^2 J_2^2)$$

$$= 2 B J_1 (I_1 / J_1 - I_2 / J_2) / (r^2 J_2)$$

By the results of Chapter II, section 5, $I(x)/J(x)$ is an
increasing function of its argument. As \( rw-RB \geq r_B-R_B \), it follows that \( M_r \geq 0 \). The inequality is not strict, as \( J_1=J_2, I_1=I_2 \) at the boundary \( w=B \). Thus, \( M_r=0 \) at this boundary.

Evaluation of the second partial derivatives of \( M \) is more difficult.

Differentiation of \( M_B \) with respect to \( B \) yields:

\[
M_{bb} = \frac{[J_2 (2R^2J_2E_1 - 2(r-R)^2J_1E_2) + (4(r-R)I_2/(ra^2)) (2RJ_2I_1 + 2(r-R)I_1I_2)]}{(J_2a/2)} \tag{10}
\]

If it is assumed that \( r \leq 2R \) (i.e. the lender's lending rate is greater than one half the inherent return on capital), the first two terms of the numerator of (10) can be analysed as follows:

\[
2R^2J_2E_1 - 2(r-R)^2J_1E_2 \\
\geq 2R^2(J_2E_1 - J_1E_2) \\
= 2R^2J_1J_2 \left( \frac{E_1}{J_1} - \frac{E_2}{J_2} \right)
\]

However, by the results of Chapter II, section 5, \( E/J \) is an increasing function. It follows that \( M_{bb} \) is positive.
The second derivative $M_{RR}$ is ambiguous in sign. Numerical evaluation confirms that $M_{RR}$ tends to be positive at low values of $B$, far away from bankruptcy, but negative for high values of $B$, near bankruptcy.

The source of the ambiguity in sign can be identified. After eliminating the factor $2B/(r_0^2)$, differentiation of $M_R$ with respect to $R$ yields:

$$M_{RR} = [J_2^2E_1 - J_1J_2E_2 - 4J_2I_1I_2/(r_0^2) + 4J_1I_2^2/(r_0^2)] [B/J_2^3]$$ \hspace{1cm} (11)

Combining the second and fourth terms yields:

$$4J_1I_2^2 / (r_0^2) - J_1J_2E_2 = J_1(4I_2^2/(r_0^2) - J_2E_2)$$ \hspace{1cm} (12)

To analyse this expression it is necessary to consider the integral $I_2$:

$$I_2 = \int_{R_{er}}^{R_{eb}} \exp\left(-\frac{s^2}{r_0^2}\right) ds$$

Making the substitution $z^2 = s^2/(r_0^2)$, this integral becomes:
$$I_2 = \int_{\frac{rB-RB}{\sigma \sqrt{r}}}^{\infty} \exp[-z^2]dz = I\left(\frac{(rB-RB)}{\sigma \sqrt{r}}\right)$$

where the function $I(x)$ is as defined in Chapter II, section 5.

The difference in (12) thus becomes $4I(x)^2 - E(x)J(x)$, where $x$ represents the quantity $(rb-RB)/(\sigma \sqrt{r})$. From the results of Chapter II, section 5, this difference is positive.

However, the remaining difference in (11),
$$J_2[ E_1J_2 - 4I_1I_2 / (\sigma \rho^2) ],$$
cannot be signed unambiguously, and is clearly negative when $w=B$.

Finally, the mixed second partial derivative $M_{BR}$, is also ambiguous in sign. From above, $M_R = 2B(J_2I_1 - J_1I_2)/J_2^2(\rho \sigma^2)$.

Inspection of this expression shows that $M_R=0$ at the two extreme values $B=0$ and $B=w$. Since $M_R$ assumes positive values for intermediate values of $B$, $M_{BR}$ must assume both positive and negative values.

Again it is useful to investigate the reason for the ambiguity in
sign. Differentiating $M_B$ with to $R$ yields:

\[
M_{BR} = \left(\frac{1}{J^2} \right) \left[ 2J^2 \left(2J2I1 - 2J1I2 + 4RB12I1/(r_0^2) + 2RBJ2E1 \right) \\
+ 4(r-R)BI1I2/(r_0^2) + 2(r-R)BJ1E2 \right] \\
- 8BRJ2^2I1I2/(r_0^2) - 8(r-R)BJ1J2I2^2/(r_0^2) ]
\]

which, after grouping terms, yields:

\[
M_{BR} = \left(\frac{1}{J^2} \right) \left[ 2J^2 \left(2J2I1 - J1I2 \right) + 2RBJ2^2E1 + 4(r-R)BJ2I1I2/(r_0^2) \right] \\
+ 2(r-R)BJ2J1E2 - 4BRJ2I1I2/(r_0^2) - 8(r-R)BJ1I2^2/(r_0^2) ]
\]

\[
= \left(\frac{1}{J^2} \right) \left[ 2J^2 \left(2J2I1 - J1I2 \right) + 2(r-R)BJ1 \left( E2J2 - 2I2^2/(r_0^2) \right) + 4(r-R)BI2/(r_0^2) \left( J2I1 - J1I2 \right) + 2RBJ2 \left( E1J2 - 2I1I2/(r_0^2) \right) \right]
\]

The first and third terms are positive. The second term is negative, and the fourth term is ambiguous.

The following table gives sample values of $M_R$ and $M_{BR}$, at different levels of $B$, for the following parameter values: 

$w=100$, $r=.08$, $R=.06$, $\sigma=30$. 
Table 5 - Values of $M_R$, $M_{BR}$

<table>
<thead>
<tr>
<th></th>
<th>B=0</th>
<th>B=20</th>
<th>B=40</th>
<th>B=60</th>
<th>B=80</th>
<th>B=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>0.106</td>
<td>0.175</td>
<td>0.281</td>
<td>0.44</td>
<td>0.671</td>
<td>1.00</td>
</tr>
<tr>
<td>$M_R$</td>
<td>0</td>
<td>0.414</td>
<td>0.985</td>
<td>1.52</td>
<td>1.52</td>
<td>0.00</td>
</tr>
<tr>
<td>$M_{BR}$</td>
<td>0.016</td>
<td>0.025</td>
<td>0.031</td>
<td>0.019</td>
<td>-0.027</td>
<td>-0.139</td>
</tr>
</tbody>
</table>
To make use of the value functions (and rates of return) that have been derived, it is necessary to impose a bargaining structure on the borrower and lender. In this chapter, the lender will be assumed to operate in a competitive lending market. In this context, the value functions of Chapter IV lead to an explanation for credit rationing (as defined below). In addition, they can be used to predict changes in loan levels and interest rates resulting from changes in ability of a borrower to grant effective security, or in the rules governing the disposition of a bankrupt company.

Competition among lenders will be modelled by assuming that the lender's value function must equal zero. For this purpose, it does not matter whether it is the lender's value function or rate of return which is used - both are zero simultaneously. Mathematically, the function easiest to manipulate turns out to be the lender's rate of return multiplied by the scalar $r$. For the purposes of this chapter, the function $L(B,R)$ will represent this quantity. $w$ is suppressed as an argument of this function as the size of the project is taken as given.
The borrower also seeks to maximize his rate of return, and again it is the rate of return multiplied by \( r \) which turns out to be the easiest to manipulate. For purposes of this chapter, \( V(B,R) \) will represent this quantity. No transaction costs are included in the functions in this chapter, as they are an unnecessary complication.

Section 1 will state the borrower’s maximization problem and the value functions to be used. Sections 2 and 3 will examine preliminary results concerning the value functions and the implications of the lender’s constraint \( L=0 \). Section 4 will examine credit rationing. Sections 5 and 6 will examine the implications of changes to the legal rules governing, respectively, a lender’s salvage recovery and bankruptcy legislation.

1. **Statement of the Problem**

Under the assumptions stated above, the borrower’s maximization problem can be stated as:

\[
\max_{B,R} V(B,R) \quad \text{subject to } L(B,R) = 0
\]
From Chapter IV, the rates of return to be analysed are:

\[ L(B, R) = R - \rho + (rSa - R) \left( \frac{\exp\left[-\frac{s^2}{rO^2}\right]}{rW-\text{RB}} \right) \]

\[ V(B, R) = \frac{1}{(w-B)} \left( rW-\text{RB} \right) \left( 1 - \frac{\exp\left[-\frac{s^2}{rO^2}\right]}{rW-\text{RB}} \right) \]

Inspection of (1) identifies a restriction which must be placed on the parameters immediately. If bankruptcy occurs immediately (i.e. the starting wealth level is B), the lender’s rate of return will be:

\[ L = rSa - \rho \]
As it makes no sense for the lender to profit from immediate bankruptcy, this quantity will be assumed negative.

As in the previous chapter, the symbols $E_1$, $F_1$, $J_1$ and $I_1$ will be defined as follows:

$$E_1 = \exp\left[-\frac{(r_{W-RB})^2}{r_0^2}\right]; \quad F_1 = \frac{\exp\left[-\frac{(r_{W-RB})^2}{r_0^2}\right]}{(r_{W-RB})^2};$$

$$I_1 = \int_{r_{W-RB}}^{\infty} \exp[-\frac{s^2}{r_0^2}]ds; \quad J_1 = (r_{W-RB}) \int_{r_{W-RB}}^{\infty} \frac{\exp[-\frac{s^2}{r_0^2}]}{s^2}ds$$

$E_2$, $F_2$, $I_2$, and $J_2$ will have definitions corresponding to $E_1$, $F_1$, $I_1$, and $J_1$, with $(r_{W-RB})$ replaced by $(r_{B-RB})$. Finally, as in Chapter IV, $M$ will denote the ratio $J_1/J_2$.

2. The Lender's Constraint

Substantial information about the problem can be gained by examining the set of points satisfying $L(B,R)=0$. The first step in this examination is an analysis of the partial derivatives $\partial L/\partial B$ and $\partial L/\partial R$. 
(a) The Partial Derivative $\frac{\partial L}{\partial B}$

It can be shown that the partial derivative of $L(B,R)$ with respect to $B$ is unambiguously negative. The intuition is simple. A higher debt level brings the firm closer to bankruptcy, which reduces the expected return.

Mathematically, differentiation of (1) with respect to $B$ yields:

$$\frac{\partial L}{\partial B} = (rS_a - R) \frac{J_1}{J_2} \left[ \frac{2R}{r\sigma^2 I_1} \frac{J_2}{J_1} + \frac{2(r-R)}{r\sigma^2} \frac{I_2}{J_2} \right]$$

As this first factor on the RHS is negative and the others are positive, the partial derivative must be negative.

(b) The Partial Derivative $\frac{\partial L}{\partial R}$

The sign of the partial derivative of $L(B,R)$ with respect to $R$ is ambiguous. The intuitive reason is that an increase in the interest rate has both negative and positive influences on the lender's rate of return. The positive influence of greater basic
interest revenue is offset by the negative influence of the increased risk of the borrower's bankruptcy. The ambiguity of sign can be confirmed by numerical examples.

However, it possible to show that \( \partial L/\partial R \) must be positive at the intercept point \( B=0 \).

Differentiating (1) with respect to \( R \) yields:

\[
\frac{\partial L(B,R)}{\partial R} = (1 - \frac{J1}{J2}) + (r_Sa - R) \frac{\partial 2B11}{J1} - \frac{\partial 2B12}{J2}
\]

At \( B=0 \), this is equal to \( 1-J1 \). By the results of Chapter II, section 5, \( J1 \) is less than 1 for positive \( (r_w - R_B) \), showing that the partial derivative is positive.

(c) The Intercepts

With knowledge about the signs of \( \partial L/\partial B \) and \( \partial L/\partial R \), it is possible to investigate the points at which the \( L=0 \) locus intercepts the two boundaries \( B=0 \) and \( R=r \).
Consider first the boundary $B=0$ and let $R^*$ (assuming it exists) represent a value of $R$ such that $L(0,R^*)=0$. Setting $L(B,R)$ and $B$ both equal to zero in (1) yields:

$$
R^* - \rho + (rSa - R^*) \left( rW \right) \int_{rW}^{\infty} \exp \left[ -\frac{s^2}{rC^2} \right] ds = 0
$$

Letting $N$ denote $(rW)$ times the integral in final term of (3), $R^*$ can be isolated in (3) as follows:

$$
R^* = \frac{\left( \rho - rSaN \right)}{1-N}
$$

If $R^*$ exists, it is clear that it must be unique. This follows from the fact that along the boundary $B=0$, $\partial L/\partial R$ is positive.

It remains to show that $R^*$ exists, but a simple argument shows that this must be the case for any sensible set of parameters. It is clear that if $L$ is negative everywhere along the boundary $B=0$, it will be negative for all values of $B$ (as $\partial L/\partial B$ is negative). Since no lending will occur if $L$ is everywhere negative, it is necessary to assume that $L$ is positive for at least some points along the boundary $B=0$. However, it is also
clear that $L$ is always negative for the extremal point of the $B=0$ locus $(B=0,R=\rho)$. By continuity and the intermediate value theorem, it follows that $L$ must be equal to zero for at least one point on the locus $B=0$ - i.e. a solution for $R^*$ must exist if any lending can occur. It follows that if any lending can take place, there is a unique point at which the $L=0$ locus intercepts the boundary $B=0$.

The other material intercept for the $L=0$ locus is the point at which it meets the $R=r$ locus. Another simple argument shows that this intercept must exist, and that it must be unique.

Uniqueness follows directly from the result stated above, that $\partial L/\partial B$ is positive.

Existence follows directly from continuity, the intermediate value theorem, and the facts that:

(a) $L(B=w,R=r)$ is negative, from inspection of (1) and the assumption that $rSa - \rho < 0$; and

(b) $L(B=0,R=r)$ is positive, from the proceeding analysis of the
boundary $B=0$.

If $B^*$ represents the value of $B$ for which $L(B^*,r)=0$, it follows from (1) that:

$$r - \rho + (rS-a-r)(rw-rB^*) \int_{rw-rB^*}^{\infty} \frac{\exp[-\frac{s^2}{2\rho^2}]}{s^2} ds = 0$$

which can be rewritten as:

$$(rw-rB^*) \int_{rw-rB^*}^{\infty} \frac{\exp[-\frac{s^2}{2\rho^2}]}{s^2} ds = \frac{r-\rho}{r-rSa}$$

Equation (4) confirms conditions which were imposed on the parameters of the problem in section 1. The LHS of (4) is equal to 1 if $w=B$, and is not greater than 1 in any event (Chapter II, section 5). It must thus be true that $1-\rho/r \leq 1-Sa$, which in turn implies the same condition derived previously, $rSa \leq \rho$. 
(d) The locus $L(B,R)=0$

From the foregoing results, the $L(B,R)=0$ locus meets the $B=0$ and $R=r$ boundaries at unique points. As well, the slope of the $L=0$ locus, $(-\partial L/\partial R)/(\partial L/\partial B)$, is ambiguous in sign, but is positive at $B=0$. Thus, the $L(B,R)=0$ locus must have the shape of one of the examples shown below.

Figure 5 - Graph of $L(B,R)=0$

It is now possible to see that while the slope of the $L(B,R)=0$
locus is ambiguous, the borrower will never select at point on that locus where its slope is negative. For any such point, there is always another point on the locus with the same debt level, but with a lower interest rate on debt. It is thus clear that at the point which solves the borrower's constrained maximization problem, $\partial L/\partial R$ will be positive and $\partial L/\partial B$ will be negative.

3. The Borrower's Rate of Return

The required information about the partial derivatives of $V(B,R)$ is more easily obtained.

(a) $\partial V(B,R)/\partial R$

$\partial V(B,R)/\partial R$ is unambiguously negative. Intuitively, higher interest rates impose higher debt service costs on borrowers with no offsetting advantage.

To prove this result, it is helpful to rewrite the function $V$ (equation (2), above) in the following way:
\[ V(B,R) = \frac{1}{(w-B)} \left[ rw-RB - (rB-RB) \frac{J_1}{J_2} \right] \]

It is clear that the factor \( \frac{1}{(w-B)} \) can be neglected.

Differentiation then yields:

\[
(w-B) \frac{dV}{dR} = -B \left( 1 - \frac{J_1}{J_2} \right) - (rB-RB) \left[ \frac{r_2}{J_1} - \frac{r_2}{J_2} \right]
\]

The first term is clearly negative, and the second is negative as well by the results of Chapter II, section 5.

\[(b) \quad \frac{\partial V(B,R)}{\partial B}\]

Just like \( \frac{\partial L(B,R)}{\partial R}, \frac{\partial V(B,R)}{\partial B} \) is ambiguous in sign. The intuitive reason is similar. Increasing debt load increases the borrower's leverage on existing equity, but also increases the risk of bankruptcy.

However, it can be shown that \( \frac{\partial V(B,R)}{\partial B} \) must be negative as \( B \) approaches \( w \).

It is clear that \( V(B,R) \) is positive for all \( B<w \). However, application of l'Hopital's rule to (2), above shows that \( V(B,R) \)
approaches 0 as $B$ approaches $w$. It follows that $\frac{\partial V(B, R)}{\partial B}$ must be negative as $B$ approaches $w$.

4. Credit Rationing

Using the results of sections 1, 2, and 3, the first aspect of lender borrower relations to be examined is credit rationing.

Literature on credit rationing has not been consistent in its terminology. Traditionally, the term "Type 1 credit rationing" has referred to a condition in which a firm can not borrow as much as it wishes at the going interest rate. "Type 2 credit rationing" has referred to a condition where, among identical borrowers, some can borrow and some can not (Blanchard and Fisher (1989), p. 479). More recently, Freixas and Rochet (1997) have argued that circumstances in which a borrower would like to borrow more at the interest rate he is paying should not properly be called credit rationing at all. Freixas and Rochet use the term "apparent credit rationing" to describe this condition.

"Credit rationing", as used in this thesis, means a condition in which a borrower would like to borrow more at the interest rate
he has negotiated with his lender. Whether this condition constitutes "Type 1 credit rationing" or merely "apparent credit rationing" is an issue which awaits a uniform definition of terms.

Models of credit rationing often rely on some form of market imperfection, such as asymmetric information (eg. Stiglitz and Weiss, (1981)). The analyses presented below will concentrate on determining whether credit rationing can be explained with less severe assumptions. More particularly, the analyses will focus on whether risk of the borrower's bankruptcy can explain credit rationing.

Using the results of sections 2 and 3, the borrower's maximization problem can be set up as a lagrangean.

The problem is:

\[
\text{Max } V(B,R) \text{ subject to } L(B,R)=0, \ R \geq R^*, \ R \leq r
\]

where \( R^* \) is interest rate solving \( L(0,R^*)=0 \) (see section 2(c), above).
The lagrangean is:

\[ V(B,R) - \lambda L(B,R) - \lambda_2 (R^* - R) - \lambda_3 (R - r) \]

where \( \lambda_2 \) and \( \lambda_3 \) must be non-negative.

Some simplification is possible immediately. It is clear that the borrower will never borrow at an interest rate equal to \( r \), so that \( \lambda_3 \) must be equal to zero.

With this simplification, the first order conditions for maximization are:

\[
\frac{\partial V}{\partial R} = \lambda \frac{\partial L}{\partial R} - \lambda_2 \\
\frac{\partial V}{\partial B} = \lambda \frac{\partial L}{\partial B}
\]

\[
\lambda_2 (R^* - R) = 0
\]

Consider first the case of an interior maximum, where \( \lambda_2 = 0 \).

From section 2(d) it is known that at the equilibrium point, \( \frac{\partial V}{\partial R} \) is negative and \( \frac{\partial L}{\partial R} \) is positive. These two conditions imply that \( \lambda \) is negative. As well, it is known that \( \frac{\partial L}{\partial B} \) is negative. With \( \lambda \) negative as well, \( \frac{\partial V}{\partial B} \) must be positive.
This, however, means that credit must be rationed. At the equilibrium debt level and interest rate, the borrower would like to borrow more.

At a non-interior maximum, no unambiguous result can be stated. However, it is also true that at a non-interior maximum, there is no borrowing in equilibrium. The result is thus that if no borrowing is observed, one can not tell whether it is because the borrower chooses not to borrow at all, or because the lender will not lend.

The following figures illustrate three different results. In the first there is positive borrowing in equilibrium, and credit rationing as well. The amount of the credit rationing is measured by the distance between the equilibrium debt level \((B_e)\) and the borrower's maximizing debt at the equilibrium interest rate \((B_m)\). Tangency occurs at \(B_m\) between an isoquant of the borrower and the vertical line through \(R_e\), the equilibrium interest rate.

In the second figure, there is no borrowing in equilibrium, but there is credit rationing.
In the third figure, there is no borrowing in equilibrium, but that is not the result of credit rationing.

Figure 6 - Borrowing with Credit Rationing
Figure 7 - Credit Rationing Without Borrowing
In the result, there is a relatively straightforward explanation for credit rationing, without resorting to asymmetric information. The intuitive explanation is that the borrower has an incentive to raise debt levels, as he keeps the benefit of additional leverage if bankruptcy is avoided. He is thus willing to accept a greater risk of bankruptcy. The lender rations credit because he is less willing to risk bankruptcy. The borrower’s additional leverage from higher debt levels provides no corresponding advantage to the lender.
While the preceding analysis has been done in the context of specific value functions, it is equally applicable to any generalized value functions for a borrower \((V^*(B,R))\) and lender \((L^*(B,R))\). If a zero profit condition is imposed on the lender, and if an interior maximum is assumed, the following problem results:

\[
\max_{B,R} V^*(B,R) \text{ subject to } L^*(B,R) = 0
\]

with the resulting first order conditions:

\[
V^*_B(B,R) = \lambda L^*_B(B,R)
\]
\[
V^*_R(B,R) = \lambda L^*_R(B,R)
\]

For obvious reasons, \(V^*_R\) and \(L^*_R\) will generally be negative and positive, respectively, implying that \(\lambda < 0\). This in turn implies that \(V^*_B(B,R)\) and \(L^*_B(B,R)\) will be opposite in sign. If \(L^*_B(B,R)\) is negative \(V^*_B(B,R)\) will be positive, implying credit rationing. It is reasonable to expect \(L^*_B(B,R)\) to be negative when increasing debt levels increase the threat of the borrower’s bankruptcy, and thus the threat of loan default.
This analysis adds insight into the results obtained by de Meza and Webb (1992). De Meza and Webb consider the following expected profit functions for the entrepreneur (e) and a bondholder (B).

\[
\begin{align*}
\Pi_e &= p[f(k,\Theta_H) - (1+r)k] + (1-p)\max[f(k,\Theta_L) - (1+r)k, 0] \\
\Pi_B &= p(1+r)k + (1-p)\max[f(k,\Theta_L), (1+r)k] - (1+p)k
\end{align*}
\]

Debt level is represented by the variable \( k \), and the entrepreneur’s capital structure is assumed to be 100% debt. \( \rho \) is the cost of capital, \( f \) is a production function, and \( \Theta \) is a random variable which takes a high value with probability \( p \) and a low value with probability \( 1-p \). De Meza and Webb impose a zero profit condition on the lender, and go on to conclude that the existence of credit rationing depends on whether \( f(k,\Theta) \) exhibits decreasing returns to scale, i.e. \( f_k < f/k \).

However, with the profit functions used by de Meza and Webb, their “decreasing returns” condition is equivalent to condition that the derivative of the lender’s function with respect to \( k \) be negative. Assuming that \( f(k,\Theta_L) < (1+r)k \) (which is the case of interest) the derivative of \( \Pi_B \) with respect to \( k \) is equal to:
\[ \Pi_{b,k} = p(1+r) + (1-p)f_k(k, \Theta_L) - (1+\rho). \]

Multiplying this expression by \( k \) and rearranging yields:

\[ k\Pi_{b,k} = [p(1+r)k + (1-p)f(k, \Theta_L) - (1+\rho)k] - (1-p)f(k, \Theta_L) + (1-p)f_k(k, \Theta_L)k \]

However, the term in square brackets are equal to \( \Pi_\theta \) which is equal to zero by assumption. Thus:

\[ k\Pi_{b,k} = (1-p) [-f(k, \Theta_L) + f_k(k, \Theta_L)k] \]

This expression shows that the decreasing returns condition derived by de Meza and Webb is identical to a condition that the derivative of the lender's profit function with respect to \( k \) be negative.

The results of de Meza and Webb have been subject to conflicting interpretation. Accordingly, the following should be noted in comparing the results of this chapter to those of de Meza and Webb.
1. Freixas and Rochet (1997) have argued that the model of de Meza and Webb does not illustrate credit rationing at all, but rather "apparent credit rationing." Freixas and Rochet argue that the term "credit rationing" should be reserved for conditions in which borrowers are able to fulfill the terms of their contracts, but are still denied credit. This is not the case in de Meza and Webb, where the risk of default drives at least some of the results.

2. The analysis set out above omits an important feature of the model of de Meza and Webb. In the model of de Meza and Webb, the amount invested in the firm's production process is optimal in spite of the credit rationing (or apparent credit rationing). In this respect, de Meza and Webb argue, the presence of credit rationing is not evidence of market failure. No measure of efficiency is available in the analysis presented here.

While there are unresolved issues surrounding the definition of credit rationing, it is submitted that the analysis presented here makes two points. First, the lagrangean analysis shows how simple it is to explain credit rationing in a competitive lending
market. With the signs of some derivatives being obvious, the analysis leads to one straightforward condition: credit rationing will be present if $\partial L/\partial B$ is negative. The assumptions leading to this condition are so general that it can be regarded as both necessary and sufficient. Second, it is submitted that the lagrangean analysis demonstrates a more general condition underlying the results of de Meza and Webb. While the lagrangean conditions include no efficiency criterion, they must be satisfied at an interior maximum. They are, it is submitted, more general and less model specific than the equivalent decreasing returns condition stated in de Meza and Webb.

5. Security Limitations

The above analysis can be extended to consider limitations on the ability of a borrower to provide security.

All Canadian jurisdictions (as well as the Federal government) have within the last twenty years enacted legislation designed to streamline the granting of security interests by borrowers. Often, the legislation attempts to make as many forms of property as possible amenable to security interests. Examples include:
(a) Section 88 of the Bank Act of Canada, which is designed to allow banks, with relatively little documentation, to take broadly based security over the assets of an operating business;

(b) The Personal Property Security Act of British Columbia, enacted in the late 1980's. This act (and similar acts passed in most other Canadian provinces) was intended to simplify procedures for granting, recording and enforcing security interests in personal property.

A change in the range of assets over which a borrower can grant effective security can be modelled as a change in the parameter $S_a$. An increase in $S_a$ represents an increase in the amount recovered by the lender on liquidation, i.e. an increase in the scope of the borrower's assets over which security has been obtained.

The following analysis assumes that the lender and borrower are at an interior maximum to the borrower's maximization problem. $V$ and $L$ will now be considered as functions of $(B,R;S_a)$. 
At an interior maximum, the first order conditions are:

\[ V_R = \lambda L_R \]
\[ V_B = \lambda L_B \]
\[ L(B, R; S_a) = 0. \]  \hspace{1cm} (6)

Combining the first two conditions yields \( V_R L_B = V_B L_R \). Total differentiation of this expression and the expression \( L=0 \) yields:

\[ L_B dB + L_R dR + L_{S_a} dS_a = 0 \]  \hspace{1cm} (7)

\[ dR (L_B V_{RR} + V_R L_{BR} - L_R V_{RB} - V_B L_{RR}) + dB (L_B V_{RB} + V_R L_{BB} - L_R V_{BB} - V_B L_{BR}) \]
\[ + dS_a (V_R L_{BS_a} + L_B V_{BS_a} - V_B L_{BS_a} - L_R V_{BSa}) = 0 \]

The coefficients in the first expression are clear. \( L_B \) has been shown negative, \( L_R \) has been shown positive (at equilibrium) and \( L_{S_a} = rM > 0 \).

The coefficients in the second expression of (7) are more complex. Let \( X, Y, \) and \( Z \) represent, respectively, the coefficients of \( dR, dB, \) and \( dS_a \) in the second expression.
Consider first $Z$, the coefficient of $dS_a$:

$$Z = V_R L_{BS_a} + L_{BS_a} V_{RS_a} - V_B L_{RS_a} - L_R V_{BS_a}$$

As $V$ does not depend in any way on $S_a$, the second and fourth terms are clearly zero. Also:

$$L_{BS_a} = r M_B > 0$$
$$L_{RS_a} = r M_R > 0.$$  

Since $V_R < 0$ and $V_B > 0$ (in equilibrium), $Z$ must be negative.

$X$ and $Y$ are more difficult to sign. Some of the components can be signed unambiguously. Others, however, are sufficiently complicated that unambiguous expressions cannot be derived. In some cases, the best that can be done is to show that "fundamental" values (which will prevail far from bankruptcy) are of the correct sign, and that the desired sign is also achieved near the bankruptcy boundary. These conclusions must be relied upon to conclude that the term in question will have the desired sign in general. Sometimes it is also necessary to derive restrictions on parameters to ensure that the desired signs are
X and Y can be rewritten as:

\[
X = L_b V_{RR} + V_R L_{BR} - L_R V_{RB} - V_B L_{RR} = L_b (V_{RR} - \lambda L_{RR}) - L_R (V_{RB} - \lambda L_{RB}) \tag{8}
\]

\[
Y = L_b V_{RB} + V_B L_{BB} - L_R V_{BB} - V_B L_{BR} = -L_R (V_{BB} - \lambda L_{BB}) + L_b (V_{RB} - \lambda L_{RB})
\]

The quantities \((V_{RR} - \lambda L_{RR})\) and \((V_{BB} - \lambda L_{BB})\) must be non-positive (and in general negative), as they are the first and second diagonal elements of the bordered hessian matrix for the maximization problem, which must be negative semi-definite at an interior maximum. Thus, since \(L_R > 0\) and \(L_b < 0\), X and Y will both be non-negative (and positive in general) if it can be shown that the term \((V_{RB} - \lambda L_{RB})\) is negative. As \(\lambda < 0\), it thus suffices to show that \(V_{RB}, L_{RB} < 0\).

Consider first \(L_{RB}\). As \(L = R - \rho + (rSa-R)M\), it follows that \(L_{RB} = -M_b + (rSa-R)M_{BR}\). From the previous chapter, \(M_b > 0\), but \(M_{RB}\) is ambiguous in sign.

The expression \(L_{RB} = -M_b + (rSa-R)M_{BR}\) can be simplified to some degree by invoking the zero profit condition imposed on the
lender. Since \( L = R - \rho + (rSa-R)M = 0 \), the expression \( (rSa-R) \) can be replaced by \( (\rho-R)/M \). Expanding the expression

\[
L_{BR} = -M_b + M_{BR}(\rho-R)/M \text{ then yields:}
\]

\[
L_{BR} = -(1/J^2) \left[ 2RJ2I1/(r_0^2) + 2(r-R)J1I2/(r_0^2) \right] - (R-\rho)/M(J2^4) \left[ J2^2 \left\{ 2J2I1/(r_0^2) - 2J1I2/(r_0^2) + 4RBI2I1/(r_0^2)^2 \right\} + 2RBJ2E1/(r_0^2) + 4(r-R)BI1I2/(r_0^2)^2 + 2(r-R)BJ1E2/(r_0^2) \right] - 2J2I2/(r_0^2) \left\{ 2RJ2I1/(r_0^2) + 2(r-R)J1I2/(r_0^2) \right\}
\]

As shown in Chapter IV, section 3, \( M_{BR} \) achieves negative values only in the vicinity of bankruptcy, and achieves its most negative values at the boundary \( w=B \). Thus, this boundary can be examined as a worst case scenario. Defining \( J=J1=J2, I=I1=I2, \) and \( E=E1=E2 \) (which conditions all hold at the boundary \( w=B \)) the above expression simplifies to:

\[
L_{BR} = (1/J^2) \left\{ -2rJ1/(r_0^2) - (R-\rho)\{2rBEJ/(r_0^2) - 4rBI^2/(r_0^2)^2\} \right\}
\]

By the assumptions placed earlier on parameters, \( (R-\rho)<r/2 \). It thus suffices to examine the expression:

\[
-2rJ1/(r_0^2) - r\{rBEJ/(r_0^2) - 2rBI^2/(r_0^2)^2\}
\]
Making a substitution of variables in the integrations in this expression yields:

\[
\left(\frac{r}{\sigma \sqrt{r}}\right) \left[ J(x)I(x) - \left(\frac{rw}{\sigma \sqrt{r}}\right)E(x)J(x) + \left(2\frac{rw}{\sigma \sqrt{r}}\right)I(x)\right]^2
\]

where \(x=(rw-Rw)/(\sigma \sqrt{r})\) and \(I(x), J(x), \text{ and } E(x)\) are the functions defined in Chapter II, section 5.

This expression contains only two "normalized" variables, \(x\) and \(rB/(\sigma \sqrt{r})\) and can be evaluated numerically.

At the point \(w=B, r=R, x\) becomes equal to 0, and the above expression will be negative if \(rw/(\sigma \sqrt{r})<3.1\). Numerical analysis of the expression for other values of \(x\) and \(rw/(\sigma \sqrt{r})\) confirms that this restriction on parameters is sufficient to ensure that \(L_{BR}\) remains negative along the entire boundary \(B=w\). This restriction on parameters is easily satisifed by all numerical examples used in this paper, and it is difficult to generate meaningful results without such a restriction. As well, it arises from a "worst case" scenario, the point \(B=w\). At the actual equilibrium point, where \(B<w, L_{BR}\) is even less likely to be positive. It can thus be taken safely that \(L_{BR}\) is negative.
Turning finally to $V_{BR}$, recall that $V$ can be written as:

$$V = \frac{[(rw-RB) - (rB-RB)M]/(w-B)}{\text{Letting } D \text{ represent the quantity } (rw-RB) - (rB-RB)M, V \text{ can be written as:}}$$

$$V = D/(w-B).$$

$V_{BR}$ is then equal to:

$$V_{BR} = \frac{D_{R}}{(w-B)^2} + \frac{D_{BR}}{(w-B)}$$

$D_{R} = -B(1-M) - (rB-RB)M_{R}$ which is unambiguously negative.

However, $D_{BR} = -1(1-M) - (r-R)M_{R} + BM_{B} - (rB-RB)M_{BR}$, which is ambiguous in sign. The "fundamental" value for $D_{BR}$ is -1, which is of the desired sign. However, near bankruptcy this term disappears and the expression is dominated by other terms whose cumulative sign is ambiguous.

It is possible, however, to use l'Hopital's rule to show that even near bankruptcy, $V_{BR}$ will be negative.
To apply l'Hopital's rule, it is first necessary to rewrite the limit of \( V_{BR} \) as follows:

\[
\lim_{B \to w} V_{BR} = \lim_{B \to w} \left[ \frac{D_{BR}}{w-B} + \frac{D_R}{(w-B)^2} \right] = \lim_{B \to w} \left[ \frac{D_{BR} + \frac{D_R}{w-B}}{w-B} \right]
\]

It is clear that the denominator of (9), \( w-B \), approaches 0 as \( B \) approaches \( w \). The numerator also approaches 0, since:

(i) \( D_R = -(1-M) - (RB-RR)M_R \) approaches 0 at \( B=w \);

(ii) \( w-B = 0 \) at \( w=B \); and

(iii) by l'Hopital's rule, \( D_R/(w-B) \) approaches \(-D_{BR}\) at \( w=B \).

Equation (9) can thus itself be evaluated by l'Hopital's rule, so that:
From (10) it follows that:

\[
\lim_{B \to w} [V_{BR}] = -\lim_{B \to w} [D_{BRB}] - \lim_{B \to w} [V_{BR}]
\]

or

\[
\lim_{B \to w} [V_{BR}] = -\left(\frac{1}{2}\right) \lim_{B \to w} [D_{BRB}]
\]

It can thus be shown that \( V_{BR} \) approaches a negative value at \( B = w \) if it can be shown that \( D_{BRB} \) approaches a positive value.

Differentiation of \( D_{BR} \) with respect to \( B \) yields:

\[
D_{BRB} = BM_{BB} - 2(r-R)M_{BR} + 2M_a - (rB-RB)M_{BRB}
\]
The analysis of Chapter IV, section 3 shows that at the boundary $B=w$, $M_{BB}>0$, $M_{BR}<0$, $M_{BRB}>0$, and $M_{BBB}<0$. It follows that $D_{BBB}$ is positive near the boundary $B=w$, and thus that $V_{BBB}<0$.

With $V_{BBB}$ having a negative fundamental value away from bankruptcy, and also a negative limiting value near bankruptcy, it can be concluded that $V_{BBB}$ will be general in negative.

Returning to the system (7):

\[ L_B dB + L_R dR + L_{Sa} dSa = 0 \]
\[ XdR + YdB + ZdSa = 0 \]

it follows from the foregoing analysis that in the second equation, both $X$ and $Y$ are positive, while $Z$ is negative.

Eliminating $dR$ from the system yields:

\[ dB \left[ Y - XL_B/L_R \right] = dSa \left[ -Z + XL_{Sa}/L_R \right] \]

Both coefficients are positive, with the result that $B$ increases with $Sa$. 
Eliminating dB from the system yields:

\[ dR \left[ X - YL_B/L_B \right] = dSa \left[ -Z + YL_{sa}/L_B \right] \]

In this expression, the coefficient of dR is positive, but the coefficient of dSa is ambiguous.

In the result:

(i) the indebtedness level, B, will increase as Sa increases;

and

(ii) the effect on the interest rate, R, is ambiguous.

It has thus been shown that permitting borrowers to grant effective security will increase the level of borrowing, but will have an ambiguous effect on the interest rate. These results coincide with intuition. With less risk of loss resulting from bankruptcy, the lender is more willing to lend, which increases the loan amount. The interest rate must adjust to ensure that the condition \( L=0 \) continues to be met, and this may result in either an increase or decrease.
Significant differences exist between the bankruptcy legislation prevailing in Canada and the United States. The most notable difference is Chapter 11 of the United States Bankruptcy Code. Chapter 11 is designed to permit the reorganization and survival of corporations which would otherwise face liquidation. For corporations able to invoke its protection, Chapter 11 provides broad protection from creditors. The most obvious justification for Chapter 11 is that it permits corporations to preserve their value as a going concern, which will far exceed their break-up value.

No similar legislation exists in Canada. The closest analog is the Companies Creditors Arrangements Act. Under that legislation, qualifying corporations can obtain a stay of legal proceedings while a reorganization plan is prepared and presented to creditors. The CCAA is not, however, amenable to protracted operation of a financially distressed business under bankruptcy protection, as has occurred under Chapter 11. Under the CCAA, a plan of reorganization must be presented and accepted within a limited period, or traditional bankruptcy results.
Chapter 11 has been criticized. Adler (1992) and Bradley and Rosenzweig (1992) argue that it motivates corporate management to undertake projects with excessive risk. Bradley and Rosenzeig base their argument on the contention that corporate managers act in the knowledge that insolvency will not cost them their jobs, which in turn removes the incentive for prudence. Adler adopts a less cynical view of corporate managers, arguing that excessive risk taking is a result of the re-allocation in favour of shareholders which results in many Chapter 11 reorganizations. Warren (1992) has disputed the conclusions of Bradley and Rosenzweig, arguing that they have misinterpreted empirical evidence upon which they rely. More importantly, Warren argues that detractors of Chapter 11 do not acknowledge the important redistributive effects of Chapter 11, which are perceived to be part of its inherent justice.

The model developed in this thesis can be used to examine the question of how bankruptcy protection will affect the decisions of borrowers and lenders.

Assume now that the model analysed above is altered to provide that upon bankruptcy, part of the remaining value of the firm is
recovered by the lender, while part is recovered by the borrower as well. In the context of the previous discussion, both the borrower and lender recover a "salvage" value. The effect of bankruptcy legislation is to apportion the salvage value between the borrower and lender.

Assume that on bankruptcy, the lender will receive $S_aB$, while the borrower receives $(1-S_a)B$.

The lender's rate of return is unchanged:

$$L = R - \rho + (rS_a-R)M$$

However, the borrower's rate of return (multiplied by $r$, for convenience) is now:

$$V = [(rw-RB) - (rB-RB)M + (r-rS_a)MB]/(w-B)$$

As before, the first order conditions are:

$$V_B = \lambda L_B$$

$$V_R = \lambda L_R$$ (11)
As before, total differentiation of the first order conditions yields:

\[ L_B dB + L_R dR + L_{Sa} dSa = 0 \]
\[ XdR + YdB + ZdSa = 0 \]  

where:

\[ X = L_B V_{RR} + V_R L_{BR} - L_R V_{RB} - V_B L_{RR} \]
\[ Y = L_B V_{RB} + V_R L_{BB} - L_R V_{BB} - V_B L_{BR} \]
\[ Z = V_R L_{BSa} + L_B V_{RSA} - V_B L_{RSA} - L_R V_{BSa} \]

X and Y remain positive, as in the previous analysis. However, the sign of Z has now become ambiguous.

Previously, V was independent of Sa, with the result that the second and fourth terms of Z were zero. As the first and third terms were, respectively, negative and positive, the sign of Z was unambiguously negative.
With $Z$ now depending on $Sa$, the second term of $Z$, $L_B V_{R5a}$, is now equal to:

$$L_B V_{R5a} = L_B \left[-BM_B/(w-B)\right] > 0.$$  

The fourth term, $L_R V_{S5a}$, is equal to

$$L_R \left[-M/(w-B) - BM_B/(w-B) - MB/(w-B)^2\right] < 0$$

With the second term of $Z$ positive and the fourth negative, the sign of $Z$ is ambiguous.

Eliminating $dR$, and then $dB$, from system (12) yields:

$$dB \left[Y - XL_B/L_R\right] = dSa[-Z + XL_{sa}/L_R] \text{ and}$$  
$$dR \left[X - YL_R/L_B\right] = dSa[-Z + YL_{sa}/L_B]$$

The coefficients of $dB$ and $dR$ are positive, as in the analysis of section 5. However, with ambiguity in the sign of $Z$, the coefficients of $dSa$ cannot be signed. In the result, a change in $Sa$ has an ambiguous effect on both loan levels and interest rates.
The ambiguous effect on loan levels is intuitive. An increase in $\Sa$ increases the lender's recovery on bankruptcy making him more willing to lend. However, it also decreases the borrower's recovery on bankruptcy, making him less willing to borrow. The change in loan levels will depend on which effect is greater.

The ambiguous effect on interest rates is less intuitive, as it would appear that the lender will be willing to accept lower interest rates, and that the borrower will only be able to pay lower interest rates. However, the ambiguity in $dR$ can be explained. As explained above, the change in lending levels is unpredictable. As a result, the borrower and lender may be moved to a higher lending level, which corresponds to a higher interest rate on the locus $L(B,R)=0$.

The result can be made unambiguous if the change in bankruptcy legislation affects only the borrower's recovery. As only differentials are relevant, a change in the borrower's recovery alone can be modelled by assuming that the lender recovers nothing on bankruptcy, so that his rate of return is now:

$$L = R - \rho - RM$$
The borrower’s rate of return is still:

\[ V = \frac{[(r_w-R_B) - (r_B-R_B)M + (r-r_{Sa})MB]}{(w-B)} \]

The first order conditions have the same form as before. The total differentials are now:

\[ L_B dB + L_R dR = 0 \]
\[ X dR + Y dB + Z dSa = 0 \]

As \( L \) no longer depends on \( Sa \), \( Z \) is equal to:

\[ Z = L_B V_{RSA} - L_R V_{BSa} \]

which is positive.

Solving for \( dB \) and \( dR \) now yields:

\[ dB \left[ Y - \frac{XL_B}{L_R} \right] = dSa [-Z] \]
\[ dR \left[ X - \frac{YL_R}{L_B} \right] = dSa [-Z] \]

The coefficients of \( dB \) and \( dR \) are both positive. The coefficient
of \( dS_a \) is negative. Accordingly, both \( B \) and \( R \) rise as \( S_a \) falls. A decrease in \( S_a \) represents an increase in the borrower's recovery on liquidation. This makes him more willing to borrow. His increased borrowing demand is split between an increased quantity of borrowing and an increased cost of borrowing (i.e. a higher interest rate). As the lender's rate of return is now independent of \( S_a \), the \( L=0 \) locus, on which any equilibrium must lie, is unchanged. The equilibrium point moves upward along the positively sloping \( L=0 \) locus, to higher levels for both debt and interest rates.
This Chapter adopts the opposite extreme assumption about the lending market - that the borrower faces a single monopolist lender. The lender occupies a position akin to a Stackelberg leader. He quotes the borrower an interest rate and a maximum loan amount. The lender sets the interest rate and loan ceiling with full knowledge of the borrower's value function. The borrower can borrow any amount, up to the lender's imposed maximum, at the prescribed interest rate. Forced borrowing is not allowed, in that the borrower need not borrow the maximum if he prefers to borrow less. Transaction costs of negotiating a loan are now included for both the borrower and lender to rule out degenerate solutions.

The monopoly assumption raises an issue not present in analysing a competitive lending market: can an equilibrium exist in which the borrower earns positive profits? Section 1 answers this question in the negative. Section 1 also deals with credit rationing, which does not inevitably occur in the case of a monopolist lender.
For most of the following analysis, the lender and borrower will seek to maximize their respective rates of return, as the model contains only a single borrower and single lender. For purposes of comparison, some examples will be considered in which the lender maximizes the value of his loan, as opposed to his rate of return.

Analytically, the results of this chapter are more complicated than those of Chapter IV. Accordingly, section 2 provides a numerical example of credit rationing, with the lender maximizing his rate of return. Section 3 provides a comparison to section 2, in which the lender maximizes the value of his loan as opposed to his rate of return. Finally, sections 4 and 5 consider the effects of changes in security and bankruptcy legislation. It turns out that effects of changes in the salvage recovery of the borrower or lender depend largely on whether credit rationing was present in the original equilibrium.
1. Analytical Interpretation - Credit Rationing

The first step is to derive certain results relating to the Lender's rate of return function. With transaction costs of $c_l$ included, this function is:

$$L = \frac{R}{r} - \frac{p}{r} + (S_a - \frac{R}{r})M - \frac{cl}{rB}.$$  

Its first and second derivatives with respect to $B$ are:

$$L_B = (S_a - \frac{R}{r})M + \frac{cl}{rB^2}$$

$$L_{BB} = (S_a - \frac{R}{r})M_{BB} - \frac{2cl}{rB^3}$$

Since $M_{BB}$ is positive, $L_{BB}$ is unambiguously negative.

The borrower's rate of return function is equal to:

$$V = [(w - RB/r) - (B - RB/r)M - cb] / (w - B)$$

Any values less than 1 for $V$ represents a losing proposition, which the borrower has no incentive to undertake.
A critical figure in the analysis is the maximum rate at which the borrower can borrow, while still breaking even. Let $R^*$ represent this value. For the following reasons, it is clear that $R^* < r$:

1. The borrower receives only a return of $r$ on his invested capital, and thus could only break even on such a loan if there were no risk of bankruptcy. The prospect of bankruptcy ensures that a loan at interest rate $r$ must be a losing proposition.

2. Transactions costs also prevent the borrower from recovering a full effective rate of $r$ on borrowed capital which he invests.

For each interest rate $R$, let $B^*_v(R)$ represent the borrower's optimal level of borrowing, subject to the proviso that $B^*_v(R)$ is only defined for those values of $R$ for which $V(R) > 1$, i.e. the borrower does not lose money by taking the loan. By definition $V(B^*_v(R^*), R^*) = 1$.

Define $B^*_l(R)$ similarly, as the lender's optimal loan level, for
a given interest rate $R$.

Two cases must now be considered, namely the case $B^*_L(R^*) < B^*_V(R^*)$ and the case $B^*_L(R^*) > B^*_V(R^*)$.

Consider first the case $B^*_L(R^*) < B^*_V(R^*)$, i.e. at the maximum rate the borrower can pay, his preferred debt level still exceeds what the lender wishes to loan. This condition is sufficient to ensure that credit is rationed, and also that the borrower cannot earn positive profits in equilibrium.

The situation is most conveniently shown graphically.
Equilibrium involving positive borrowing must lie somewhere on the locus $V=1$, between points $P$ and $Q$. This is seen by the following reasoning:

1. No point to the left of the $V=1$ locus can be an equilibrium, as the lender always has an incentive to increase the interest rate. Until he reaches his point of indifference ($V=1$), the borrower will accept the higher interest rate.

2. Any point lying above the $V=1$ locus is less advantageous for the lender than any point on the $V=1$ locus. This is because the $V=1$ locus, in the region in question, lies above the
locus $B^*_{L}(R)$, along which $L_{R}=0$. As $L_{BG}<0$, $L_{R}<0$ throughout the region above the locus $V=1$.

Finally, it can also be shown that an equilibrium along the locus from $P$ to $Q$ cannot be $Q$ itself - i.e. equilibrium cannot be precisely at the maximum interest rate at which the borrower can survive. This follows from the fact that slope of the $V=1$ locus ($-V_{R}/V_{B}$) at this point is infinite, since $V_{B}=0$. Accordingly, a slight reduction in the lending level along the $V=1$ locus, results in the following change to $L$:

$$dL = L_{B}dB + L_{R}dR = dB \left[ L_{R} - V_{R}/V_{B} \right] = L_{B}dB > 0 \text{ for } dB<0$$

These results support the following conclusions, which apply to any equilibrium involving positive borrowing when $B^*_{L}(R*) < B^*_{V}(R*)$:

1. Credit will be rationed to the borrower.

2. The borrower will be reduced to zero profits. However, the equilibrium interest rate will be less than the maximum interest rate at which the borrower can survive.
3. The equilibrium can be described as the solution to the following maximization problem:

$$\max_{B, R} L(B, R) \text{ subject to } V(B, R) = 1.$$ 

Consider now the case $B^*_L(R^*) > B^*_v(R^*)$, i.e. that the lender's preferred debt level lies above that of the borrower. Graphically, the situation is now:

The central question is whether any equilibrium involving positive borrowing will necessarily occur at point $Q$, the maximum
interest rate which the borrower can pay.

No unambiguous analytical answer can be given to this question. To see this consider a small decrease in the interest rate, $dR < 0$. As the lender's optimizing curve lies above the borrower's optimizing curve, the new debt level will still be on the borrower's optimizing curve, but at a slightly higher debt level. The borrower's optimizing curve is characterized by $V_B = 0$, and thus has slope $-V_{BB}/V_{BB}$. The resulting change in the lender's rate of return will thus be:

$$dL = V_B dB + V_R dR = dR(V_R - V_B V_{BB}/V_{BB})$$

The components of the coefficient of $dR$ are sufficiently complex that no unambiguously signed expression can be derived.

There is, however, a fairly clear intuitive argument showing that any equilibrium involving positive borrowing must be at Q. The lender's rate of return is likely to be highly sensitive to interest rate changes, while the slope of the locus $V_B = 0$ is unlikely to be. In other words, the lender's isoquants are likely to be steeper than the $V_B = 0$ locus, as shown in Figure 10.
above. If this is the case, a decrease in the interest rate will reduce the lender's rate of return, so that the lender will prefer point Q in the figure to any lesser interest rate. Equilibrium will therefore be at point Q, at which the borrower is still reduced to a condition of zero profit, but does not experience credit rationing.

2. Credit Rationing - A Numerical Example.

Owing to the complexity of the analytical results, a numerical example is instructive.

The rates of return of the borrower and lender are:

\[
V = \frac{\left[ (w-RB/r) - (B-RB/r)M - cb \right]}{(w-B)}
\]
\[
L = \frac{R/r - \rho/r + (Sa-R/r)M - c1/B}{(w-B)}
\]

Consider the following set of parameter values. The magnitude of the project, w, equals 100. The return to capital invested in the project, r, equals 8%. The lender's cost of capital, \( \rho \), is 4%. The coefficient of the lender's salvage recover, Sa, equals .5. Finally, the variance, \( \sigma \), is 20 and the transactions costs
of the borrower and lender, cb and cl, are each equal to 5.

The following tables set out the returns for the borrower and lender respectively. For the borrower, any value less than 1 represents a losing project, which will not be undertaken at the interest rate and debt combinations given. For the lender, negative values have the same connotation. Finally, figures with an asterisk represent maximizing debt levels, for interest rates given.
Table 6 - Borrower's Returns

<table>
<thead>
<tr>
<th>B</th>
<th>5%</th>
<th>5.5%</th>
<th>6%</th>
<th>6.5%</th>
<th>6.8%</th>
<th>6.9%</th>
<th>7%</th>
<th>7.5%</th>
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<tbody>
<tr>
<td>30</td>
<td>1.07</td>
<td>1.05</td>
<td>1.02</td>
<td>1.00</td>
<td>0.98</td>
<td>0.97</td>
<td>0.95*</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>1.07</td>
<td>1.07</td>
<td>1.04</td>
<td>1.01</td>
<td>0.99</td>
<td>0.98</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>1.13</td>
<td>1.09</td>
<td>1.05</td>
<td>1.02</td>
<td>1.00</td>
<td>1.00</td>
<td>0.98</td>
<td>0.95</td>
</tr>
<tr>
<td>45</td>
<td>1.16</td>
<td>1.15</td>
<td>1.07</td>
<td>1.03</td>
<td>1.00</td>
<td>1.00</td>
<td>0.99</td>
<td>0.95</td>
</tr>
<tr>
<td>50</td>
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<td>1.14</td>
<td>1.09</td>
<td>1.04</td>
<td>1.01</td>
<td>1.00</td>
<td>0.99*</td>
<td>0.94</td>
</tr>
<tr>
<td>55</td>
<td>1.23</td>
<td>1.17</td>
<td>1.11</td>
<td>1.05</td>
<td>1.01</td>
<td>1.00</td>
<td>0.99</td>
<td>0.94</td>
</tr>
<tr>
<td>60</td>
<td>1.27</td>
<td>1.20</td>
<td>1.12</td>
<td>1.06</td>
<td>1.02*</td>
<td>1.00*</td>
<td>0.99</td>
<td>0.93</td>
</tr>
<tr>
<td>65</td>
<td>1.31</td>
<td>1.22</td>
<td>1.14</td>
<td>1.06</td>
<td>1.02</td>
<td>1.00</td>
<td>0.99</td>
<td>0.92</td>
</tr>
<tr>
<td>70</td>
<td>1.35</td>
<td>1.25</td>
<td>1.15</td>
<td>1.06*</td>
<td>1.01</td>
<td>0.99</td>
<td>0.98</td>
<td>0.90</td>
</tr>
<tr>
<td>75</td>
<td>1.39</td>
<td>1.27</td>
<td>1.16*</td>
<td>1.05</td>
<td>1.00</td>
<td>0.98</td>
<td>0.96</td>
<td>0.87</td>
</tr>
<tr>
<td>80</td>
<td>1.41</td>
<td>1.28*</td>
<td>1.15</td>
<td>1.03</td>
<td>0.97</td>
<td>0.94</td>
<td>0.92</td>
<td>0.83</td>
</tr>
<tr>
<td>85</td>
<td>1.42*</td>
<td>1.26</td>
<td>1.11</td>
<td>0.97</td>
<td>0.90</td>
<td>0.88</td>
<td>0.85</td>
<td>0.75</td>
</tr>
<tr>
<td>90</td>
<td>1.34</td>
<td>1.16</td>
<td>0.99</td>
<td>0.84</td>
<td>0.75</td>
<td>0.73</td>
<td>0.70</td>
<td>0.59</td>
</tr>
<tr>
<td>95</td>
<td>0.95</td>
<td>0.73</td>
<td>0.54</td>
<td>0.36</td>
<td>0.27</td>
<td>0.25</td>
<td>0.22</td>
<td>0.10</td>
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</tbody>
</table>
### Table 7 - Lenders' Returns

<table>
<thead>
<tr>
<th>B</th>
<th>R=5%</th>
<th>5.5%</th>
<th>6%</th>
<th>6.5%</th>
<th>6.8%</th>
<th>6.9%</th>
<th>7.0%</th>
<th>7.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>-0.055</td>
<td>0.000</td>
<td>0.053</td>
<td>0.120</td>
<td>0.137</td>
<td>0.148</td>
<td>0.158</td>
<td>0.209</td>
</tr>
<tr>
<td>35</td>
<td>-0.034</td>
<td>0.019</td>
<td>0.071</td>
<td>0.121</td>
<td>0.151</td>
<td>0.162</td>
<td>0.172</td>
<td>0.222</td>
</tr>
<tr>
<td>40</td>
<td>-0.019</td>
<td>0.032</td>
<td>0.082</td>
<td>0.130</td>
<td>0.159</td>
<td>0.168</td>
<td>0.177*</td>
<td>0.222*</td>
</tr>
<tr>
<td>45</td>
<td>-0.009</td>
<td>0.040</td>
<td>0.088</td>
<td>0.133*</td>
<td>0.160*</td>
<td>0.168*</td>
<td>0.177</td>
<td>0.219</td>
</tr>
<tr>
<td>50</td>
<td>-0.002</td>
<td>0.045</td>
<td>0.089*</td>
<td>0.131</td>
<td>0.156</td>
<td>0.164</td>
<td>0.172</td>
<td>0.210</td>
</tr>
<tr>
<td>55</td>
<td>0.002</td>
<td>0.046*</td>
<td>0.088</td>
<td>0.127</td>
<td>0.149</td>
<td>0.156</td>
<td>0.163</td>
<td>0.197</td>
</tr>
<tr>
<td>60</td>
<td>0.004</td>
<td>0.045</td>
<td>0.083</td>
<td>0.118</td>
<td>0.138</td>
<td>0.144</td>
<td>0.150</td>
<td>0.180</td>
</tr>
<tr>
<td>65</td>
<td>0.004*</td>
<td>0.041</td>
<td>0.074</td>
<td>0.106</td>
<td>0.124</td>
<td>0.129</td>
<td>0.134</td>
<td>0.159</td>
</tr>
<tr>
<td>70</td>
<td>0.002</td>
<td>0.035</td>
<td>0.065</td>
<td>0.092</td>
<td>0.106</td>
<td>0.111</td>
<td>0.115</td>
<td>0.135</td>
</tr>
<tr>
<td>75</td>
<td>-0.001</td>
<td>0.027</td>
<td>0.052</td>
<td>0.075</td>
<td>0.086</td>
<td>0.090</td>
<td>0.093</td>
<td>0.108</td>
</tr>
<tr>
<td>80</td>
<td>-0.007</td>
<td>0.016</td>
<td>0.037</td>
<td>0.055</td>
<td>0.063</td>
<td>0.066</td>
<td>0.069</td>
<td>0.079</td>
</tr>
<tr>
<td>85</td>
<td>-0.015</td>
<td>0.004</td>
<td>0.019</td>
<td>0.032</td>
<td>0.038</td>
<td>0.040</td>
<td>0.042</td>
<td>0.048</td>
</tr>
<tr>
<td>90</td>
<td>-0.024</td>
<td>-0.017</td>
<td>-0.001</td>
<td>0.007</td>
<td>0.011</td>
<td>0.012</td>
<td>-0.013</td>
<td>0.016</td>
</tr>
<tr>
<td>95</td>
<td>-0.036</td>
<td>-0.030</td>
<td>-0.024</td>
<td>-0.021</td>
<td>-0.019</td>
<td>-0.018</td>
<td>-0.018</td>
<td>-0.017</td>
</tr>
</tbody>
</table>

The following graph sets out the "maximization" debt levels for both borrower and lender, at different interest rates. (Dashed lines represent maximizing values which will not be undertaken, as they represent losing propositions.)
Two specific points should be noted from the lender’s "maximization" curve:

1. The lender’s "maximization" curve is downward sloping. As the interest rate increases, so does the risk of the loan. The lender reduces debt level to offset the increased risk resulting from the higher interest rate.
2. The lender’s “maximization” curve tends towards a positive debt value as the interest rate approaches 8%, the borrower’s return on invested capital. In other words, the lender is still willing to lend a positive amount as the maximum allowable interest rate is approached (although there is no guarantee that the borrower will borrow at this interest rate). From the lender’s perspective, there is still profit to be made from the loan, even though the borrower is not earning a margin on borrowed capital.

The figures given in the tables demonstrate a simple example of credit rationing. Under the bargaining structure set out above, the equilibrium interest rate will be 6.9% (the maximum rate at which the borrower will borrow). Credit rationing occurs. With an interest rate of 6.9%, the borrower wishes to borrow 60 units of capital. But the lender lends only 45.

The intuitive reason for the credit rationing is simple. The borrower enjoys the benefit of leveraging off of increased debt levels. He retains all of the upside benefit should bankruptcy be avoided.
There is no corresponding benefit to the lender. He cannot recover more than contract interest - any upside beyond that is for the account of the borrower only. Accordingly, he limits debt to give added safety to his return. Debt is not reduced to infinitesimal levels because of the fixed costs involved - fixed costs greatly reduce the return on very small loans.

3. Comparison - Maximization of the Lender's Value Function

It is useful to compare the results of the exercise when the lender seeks to maximize his value function, as opposed to his rate of return. The following table sets out the lender's value function at different loan levels and interest rates, using the same parameter values used in section 2.
The lender's "maximization" curve is now nearly horizontal, approximately equal to the B=60 locus.

There is now no credit rationing, when the lender maximizes his value function. The equilibrium interest rate is again 6.9%, the maximum rate at which the borrower will borrow. But now, the debt level is equal to 60, the preferred level for both agents.
It is easy to demonstrate, however, that this result is not general, and that credit rationing can appear when parameter values are changed. The simplest parameter to change is $S_a$, the lender's recovery on bankruptcy. This parameter does not enter the borrower's value function in any way, and thus affects the decision of the lender only. The following table sets out the debt levels that maximize the lender's value function using the same parameter values as above, but with varying levels of $S_a$.

<table>
<thead>
<tr>
<th>$S_a$</th>
<th>5%</th>
<th>5.5%</th>
<th>6%</th>
<th>6.5%</th>
<th>7%</th>
<th>7.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>*</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>55</td>
</tr>
<tr>
<td>0.45</td>
<td>*</td>
<td>50</td>
<td>50</td>
<td>55</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>0.4</td>
<td>*</td>
<td>*</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>0.35</td>
<td>*</td>
<td>*</td>
<td>45</td>
<td>45</td>
<td>45</td>
<td>50</td>
</tr>
<tr>
<td>0.3</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>45</td>
<td>45</td>
<td>45</td>
</tr>
<tr>
<td>0.25</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>40</td>
<td>40</td>
<td>45</td>
</tr>
<tr>
<td>0.2</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>35</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>0.15</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>35</td>
<td>35</td>
<td>40</td>
</tr>
<tr>
<td>0.1</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>35</td>
<td>35</td>
</tr>
</tbody>
</table>

The following graph compares the results of Tables 6 and 9, to
show the maximizing debt levels of the borrower and lender. The graph shows the lender's "maximization" curve for different levels of Sa, to show that the absence of credit rationing is not a general result.
Figure 12 - Maximizing Debt Levels
The graph makes it clear that credit rationing must result for relatively low levels of $S_a$. There is no interest rate at which the lender will lend the amount that the borrower would choose.

4. **Change in Security Legislation**

As in Chapter V, the model under consideration can be used to evaluate the consequences of a change in lending legislation.

Consider first the case in which credit rationing is present (and in which the borrower is also operating at zero profit).

From section 1, equilibrium is the solution to:

$$\max_{B,R} L(B,R) \text{ subject to } V(B,R) = 1$$

with lagrangean:

$$L - \lambda V$$

The first order conditions are:

$$L_B = \lambda V_B$$
Differentiation of the first order conditions yields

$$X \, dR + Y \, dB + Z \, dS_a = 0$$

$$V_a \, dR + V_b \, dB = 0$$

where:

$$X = -L_{RR} V_R + L_R V_{RR} + V_R L_{RR} - L_R V_{RR} - V_R I_{RR} + V_b I_{RR} - L_R V_{RR} > 0$$

$$Y = V_s I_{RS} - L_R V_{RR} + L_R V_{RR} - V_R I_{RR} = V_s [I_{RR} - \lambda V_{RS}] + L_R V_{RR} - V_b I_{RR} > 0$$

$$Z = V_s I_{RS} - V_b I_{RS} < 0$$

Eliminating first \(dR\), and then \(dB\), from the above system yields:

$$dB [Y - XV_R/V_R] = -ZdS_a$$

$$dR [X - YV_R/V_a] = -ZdS_a$$

Both \(dB\) and \(dR\) can be signed unambiguously. As \(S_a\) increases, both \(B\) and \(R\) increase as well. The following graph illustrates the reason:
When no credit rationing is present, as in Figure 10, above, it is clear that an incremental change in the lender's salvage recovery will not affect the equilibrium point, which will remain at $Q$.

5. Bankruptcy Legislation

Following the analysis of Chapter V, section 6, consider first the situation in which bankruptcy legislation is altered to shift recovery on bankruptcy from the lender to the borrower.
The borrower's and lender's rates of return are now:

\[ V = \left( \frac{(w-RB/r) - (B-RB/r)M + MB(1/r-Sa) - cb}{w-B} \right) \]

\[ L = \frac{R/r - \rho/r + (Sa-R/r)M - cl/rB}{w-B} \]

Differentiation of the first order conditions now yields:

\[ XdR + YdB + ZdSa = 0 \]

\[ V_RdR + V_\delta dB + V_Sa dSa = 0 \]

where:

\[ X = -LBRV_B + LBL_S + V_{\delta L} V_B - LBL_S = -V_{\delta L} V_B - LBL_S > 0 \]

\[ Y = V_BL_L - LBL_S + V_{\delta L} V_B = V_R (L_{\delta R} - \lambda V_{\delta R}) + LBL_S - V_{\delta L} > 0 \]

\[ Z = V_BL_L + V_{\delta L} V_B - V_{\delta L} L_L - V_{\delta L} L_R \]

As in Chapter V, section 6, the sign of Z is now ambiguous. It follows that neither dB nor dR can be signed ambiguously when a change is imposed on dSa.

The situation is different in the situation where credit rationing does not exist. Equilibrium under these conditions is
at the maximum interest rate at which the borrower can survive. This is point Q in Figure 10, and is characterized by:

\[ V = 0 \]
\[ V_B = 0 \]  \hspace{1cm} (1)

Since the borrower is at a maximization point with respect to debt level, it must also be the case that \( V_{bb} < 0 \).

Total differentiation of the system (1), and then eliminating respectively \( dR \) and \( dB \), yields:

\[ dR[V_{BR} - V_{BB} V_B/V_B] = -dSa[V_{BSa} - V_{BB} V_S/V_B] \]
\[ dB[V_{BB} - V_{BR} V_B/V_R] = -dSa[V_{BSa} - V_{BR} V_S/V_R] \]

In the first equation, the coefficients of \( dR \) and \( dSa \) are, respectively, negative and positive. An increase in \( Sa \) represents a decrease in the borrower's recovery on bankruptcy, reducing his willingness to borrow and the maximum interest rate he can pay. In the second equation, the coefficient of \( dSa \) is ambiguous, so that no effect on debt levels can be predicted.
The final issue is what happens when only the borrower’s recovery on bankruptcy is altered, so that the respective rate of return functions are now:

\[ V = \frac{((rw-RB) - (rB-RB)M + MB(1-rSa) - cb)}{(w-B)} \]

\[ L = R - \rho - RM - cl/B \]

When credit rationing is present, differentiation of the first order conditions yields:

\[ XdR + YdB + ZdSa = 0 \]

\[ V_R dR + V_B dB + V_Sa dSa = 0 \]

where:

\[ X = -L_{RR}V_R + L_{RB}V_B + V_RL_{BR} - L_{RB}V_R = -V_R[L_{RR} - \lambda V_{RR}] + V_RL_{BR} - L_{RB}V_R > 0 \]

\[ Y = V_RL_{RB} - L_{RB}V_R + L_{RR}V_B - V_RL_{RB} = V_R[L_{RR} - \lambda V_{RR}] + L_{RB}V_R - V_RL_{RB} > 0 \]

\[ Z = V_{SSa}L_R - V_{SSa}L_R > 0 \]

Eliminating first dB and then dR yields:

\[ dR[X - YV_R/V_B] = -dSa[Z - YV_{Sa}/V_B] \]
\[ dB (Y - XV_g/V_r) = -dSa(Z - XV_s/V_R) \]

If \( Sa \) increases, \( R \) decreases and the effect on \( B \) is ambiguous.

Finally, it is clear that analysis of the case without credit rationing is the same as above (i.e. a reallocation from the lender to the borrower) as the result of that analysis depended only the characteristics of the borrower's rate of return function.
This chapter considers the decision making of a single lender who must allocate loanable funds between two borrowers. The lender is given an exogenous maximum on the total funds he can lend. He is not forced to lend all of the available funds, if holding some back is consistent with profit maximization. The lender acts with knowledge of the borrowers' objective functions, and sets both an interest rate and a maximum loan level for each borrower. The borrowers act independently, and are unable to collude. To avoid degenerate solutions, the borrowers also incur transaction costs.

A principal purpose behind the analysis is to examine the allocation of loanable funds between borrowers of different sizes. It is commonly asserted that small businesses are riskier than larger ones. In Canada, it has also been asserted that bank loans are not sufficiently available to small business. The borrower's maximization problem, and the stochastic process which underlies it, are modified to investigate these issues.

Each borrower is endowed with a given wealth level, $w_i$. He can
borrow additional capital, $B_i$, at the interest rate specified by the lender, $R_i$. He earns a return equal to $r > R_i$ on both borrowed and equity capital, with $r$ being common to the borrowers. The borrower’s transaction costs are equal to $c$, regardless of the amount borrowed.

To accommodate different “sized” borrowers, a borrower’s uncertainty now varies with the amount borrowed. The standard deviation built into the stochastic process for the borrower’s wealth is equal to $A + dB_i$, with $A$ being common to the borrowers. Setting the standard deviation in this way ensures that a “small” borrower (i.e. one with lower starting wealth, $w$) has a proportionately higher standard deviation (i.e. in relation to total wealth) for a given debt-equity ratio. Bankruptcy still occurs when a borrower has lost his equity capital, i.e. when his total wealth has fallen to $B_i$. The lender and borrowers all seek to maximize their value functions, which are derived in section 1.

The assumptions of this Chapter lead to a number of unambiguous results. Section 2 shows that credit rationing for all borrowers occurs inevitably, and section 3 shows that all borrowers are
reduced to a condition of zero profit.

Section 4 examines the effect of changes in the aggregate funds available to the lender. Here, only numerical results are available, as analytical results are too complex. The implication of the numerical results is clear, however. When overall credit is sufficiently tight, it is the small borrower who is squeezed out of the market.

Section 5 investigates the effect of changes in the salvage recovery of the borrower and lender. Again, only numerical results are available.

1. **The Value Functions of the Borrower and Lender**

Consider a borrower with "equity" w. If he borrows B, his starting wealth will be w+B. Under the assumptions stated above, the stochastic process representing the borrower's wealth over time, x(t), will evolve according to:

\[
dx(t) = rx(t)dt - RBdt + (A+\sigma B)dz
\]  

(1)
with absorption at $x(t) = B$.

Define $k = w + B$ as the starting value for $x(t)$, i.e. $x(0) = k$. As a function of $k$, the borrower's value function must solve:

$$\frac{1}{2} (A + \sigma B)^2 \frac{d^2 v}{dk^2} + (rk - RB) \frac{dv}{dk} - rv = 0$$

with $v$ defined as:

$$v(k) = \lim_{t \to \infty} E[e^{-r t} (x(t) - B)] = \lim_{t \to \infty} E[e^{-r t} x(t)] ; x(0) = k$$

The boundary conditions for $v$ are:

$$\lim_{k \to B} v(k) = 0 ; \lim_{k \to \infty} v(k) = k - \frac{RB}{r}$$

As $B$ is parametric and $k = w + B$, it is clearly possible to express this value function as $v(w; B, R)$. Following the analysis of Chapter II, the solution to the differential equation is given by:

$$v(w) = w + B - \frac{RB}{r} - (B - \frac{RB}{r}) \frac{\exp\left[-\frac{s^2}{r(A + \sigma B)^2}\right]}{s^2} \int_{rB - RB}^{\infty} \frac{s^2}{r(A + \sigma B)^2} ds - c$$

(3)
which can be rewritten as: \( v(w) = w + (B-RB/r)(1-M) - c \), where \( M \) is now defined by:

\[
M = \frac{\left( \int_{(rB-RB)}^{\infty} \exp\left(-\frac{s^2}{r(A+OB)^2}\right) \frac{s^2}{s^2} ds \right)}{\left( \int_{(rW+rB-RB)}^{\infty} \exp\left(-\frac{s^2}{r(A+OB)^2}\right) \frac{s^2}{s^2} ds \right)}
\]

(4)

If the lender recovers \( SaB \) on liquidation, the lender's value function can be expressed in the same form as in Chapter II:

\[
l(B,R) = RB/r - \rho B/r + (Sa-R/r)BM
\]

The following tables set out lender and borrower values for various levels of \( R \) and \( B \). Two borrowers are considered, a "small" borrower with \( w=50 \) and a "large" borrower with \( w=100 \). In addition, the following parameters are used: \( \rho=0.04, r=0.08, A=30, o=0.3, Sa=0.4 \).
Table 10 - Borrower’s values for w = 50

<table>
<thead>
<tr>
<th></th>
<th>R=4.5</th>
<th>R=5.0</th>
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Table 13 - Lender's values for \( w = 100 \)

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Selected results (i.e. for interest rates \( R=5\% \), \( 6\% \), \( 7\% \), and \( 7.5\% \)) are presented in the following graphs.
Figure 14 - Borrower’s values
Figure 15 - Lender's values: \( w=50 \)

Figure 16 - Lender's values: \( w=100 \)
While tables 10 to 13 are not extensive enough to show it, calculation of the borrower's value function for larger debt levels confirms that eventually \( v_B \) becomes negative. This is the intuitive result, given the assumption of this chapter that the magnitude of the stochastic component of revenue, \( (A+\sigma B)dz \), increases with \( B \). For large enough debt levels, increasing variability of revenue increases the risk of bankruptcy, to the point where that risk outweighs the benefits of increased leverage.

The function \( M(B,R) \), defined above in equation (4), is more complex than in previous chapters. However, some unambiguous results can be stated. As in Chapter III, section 3, define the functions \( E_l, F_l, I_l, \) and \( J_l \) as:

\[
E_l = \exp\left(-\frac{(rW+rB-RB)^2}{r(A+\sigma B)^2}\right) \quad \text{and} \quad F_l = \frac{\exp\left(-\frac{(rW+rB-RB)}{r(A+\sigma B)^2}\right)}{(rW+rB-RB)}
\]

\[
I_l = \int_{rW+rB-RB}^{\infty} \exp\left(-\frac{s^2}{r(A+\sigma B)^2}\right)ds \quad \text{and} \quad J_l = (rW+rB-RB) \int_{rW+rB-RB}^{\infty} \frac{s^2 \exp\left(-\frac{s^2}{r(A+\sigma B)^2}\right)}{s^2}ds
\]

\( E_2, F_2, I_2, \) and \( J_2 \) have corresponding meanings with \( rW+rB-RB \) replaced by \( rB-RB \).
The expression for $M_R$ is similar to that obtained in Chapter II:

$$M_R = 2BJ1 \left( \frac{I1}{J1} - \frac{I2}{J2} \right) / (r(A+\sigma B)^2)$$ (J2)

which is positive as $I(x)/J(x)$ is an increasing function, (see Chapter II, section 5).

However, it is no longer possible to sign $M_B$ unambiguously for all parameter values. After considerable manipulation and simplification, $M_B$ is equal to:

$$M_B = 2M \left[ \frac{(r\sigma - A(r-R))I1/J1 + A(r-R)I2/J2}{r(A+\sigma B)^3} \right]$$

It is plain that this function cannot be signed a priori, as $I1/J1 > I2/J2$. Intuitively, an increase in $B$ now has two offsetting effects on the probability of future bankruptcy. One the one hand, increased leverage increases expected positive cash flow, decreasing the risk of bankruptcy. On the other hand, an increase in $B$ increases the magnitude of the stochastic component of revenue, which increases the risk of bankruptcy.

If the "variability" associated with "equity" capital is the same
as that associated with "debt" capital, so that \( A = \omega \), then the expression for \( M_B \) reduces to:

\[
M_B = 2M \left[ \frac{RAI_1/J_1 + (r-R)AI_2/J_2}{r(A+\sigma B)} \right] / r(A+\sigma B)^3
\]

which is unambiguously positive. It follows that \( M_B \) will only be negative if the "variability" associated with equity capital is much greater than that associated with debt capital, i.e. \( A \gg \omega \), which is an inherently unlikely proposition. In addition, numerical analysis shows that \( M_B \) is positive for all of the values calculated in tables 10 - 13. The possibility that \( M_B \) is negative can thus be safely ignored.

Using these results both:

\[
v_B = (1-R/r)(1-M) - M_B(B-RB/r)
\]

and

\[
l_B = R/r - \rho/r + (S_a-R/r)(M + BM_B)
\]

are ambiguous in value.

The partial derivative of \( v \) with respect to \( R \):
\[ v_R = \frac{(-B)}{r} (1-M) - M_R (B - RB/r) \]

is unambiguously negative.

Finally, the partial derivative of \( l \) with respect to \( R \):

\[ l_R = \frac{(B)}{r} (1-M) + \frac{(Sa-R)}{r} (BM_R) \]

cannot be signed unambiguously. However, its "fundamental" value, \( B/r \), is clearly positive, and a negative result is only possible in the vicinity of bankruptcy, a possibility which can be ignored in equilibrium solutions.

2. The Existence of Credit Rationing

Under the model set out above, it can be shown relatively easily that credit rationing must exist. It turns out that the existence of credit rationing follows readily from the value functions of a single borrower and single lender, and is not dependent on:

(a) the presence of a second borrower, and the resulting
allocation problem for the lender; or

(b) the existence of an exogenous limit on the lender's funds, or on such a constraint being binding.

Consider first the borrower's value function. (As in previous chapters, the borrower's equity, \(w\), will be taken as parametric and suppressed. The choice variables are now \(B\) and \(R\)).

\[
v(B,R) = w + (B-RB/r)(1-M) - c
\]

For any given rate of interest, the borrower's preferred debt level will be characterized by the condition \(v_B = 0\).

Differentiating \(v\) with respect to \(B\) and setting the result to 0 yields:

\[
(1-R/r)(1-M) - (B-RB/r)M_B = 0,
\]

which implies in turn that:

\[
M + BM_B = 1.
\]
Now consider the lender's value function:

\[ l = RB/r - \rho B/r + (Sa-R/r)BM. \]

Differentiation with respect to B yields:

\[ l_B = R/r - \rho/r + (Sa - R/r)(M + BM_B) \]

As shown above, at the point \( v_B = 0 \), \( M + BM_B = 1 \), so that \( l_B \) reduces to:

\[ l_B = R/r - \rho/r + (Sa - R/r) = Sa - \rho/r. \]

In the analysis of previous chapters, the quantity \( Sa - \rho/r \) was assumed to be negative, and for an obvious reason. If the salvage value obtained by a lender on bankruptcy exceeds the cost of his loan, a lender can never lose money, even if bankruptcy occurs immediately. So long as an immediate bankruptcy poses a risk of loss for the lender, the quantity \( Sa - \rho/r \) must be negative, and thus \( l_B \) must be negative at the point where \( v_B = 0 \). By implication, the lender is always motivated to reduce credit from the borrower's desired level - i.e. to ration credit.
3. The Lender's Maximization Problem.

Using the results of sections 1 and 2, the lender's maximization problem can now be formally set up as a lagrangean.

Assume that there are two borrowers with wealth endowments \( w_1 \) and \( w_2 \). The borrowers value functions \( v_1(B_1,R_1) \) and \( v_2(B_2,R_2) \). The corresponding value functions for the lender are \( l_1(B_1,R_1) \) and \( l_2(B_2,R_2) \). Assume also that \( B_1^*(R_1) \) and \( B_2^*(R_2) \) represent the respective preferred borrowing levels for the first and second borrowers, for given values of \( R_1 \) and \( R_2 \). As forced borrowing is not allowed, the conditions \( B_1 \leq B_1^* \) and \( B_2 \leq B_2^* \) must prevail. As well, since no borrower would borrow if the result would be to reduce his net wealth, the lender's problem is subject to the constraints \( v_1 \geq w_1, v_2 \geq w_2 \). Finally, the lender's funds are constrained, so that \( B_1 + B_2 \leq B_{\text{max}} \), where \( B_{\text{max}} \) is exogenous.

The lender must now choose \( B_1, R_1, B_2 \) and \( R_2 \) to:

\[
\text{maximize } l_1(B_1,R_1) + l_2(B_2,R_2) \text{ subject to: }
\]

\[
B_1(R_1) \leq B_1^*(R_1);
\]
B2(R2) ≤ B2*(R2);

v1(B1,R1) ≥ w1;  \hspace{1cm} (5)

v2(B2,R1) ≥ w2;

B1+B2 ≤ Bmax.

The Lagrangean is equal to

\[ 11 + 12 - \alpha_1(w1-v1) - \alpha_2(w2-v2) - \gamma_1(B1-B1*) - \gamma_2(B2-B2*) - \lambda(B1+B2-Bmax) \]

with first order conditions:

\[
\begin{align*}
11_{B1} + \alpha_1 v_{1_{B1}} - \gamma_1 - \lambda &= 0 \\
12_{B2} + \alpha_2 v_{2_{B2}} - \gamma_2 - \lambda &= 0 \\
11_{R1} + \alpha_1 v_{1_{R1}} + \gamma_1 B1*_{r1} &= 0 \\
12_{R2} + \alpha_2 v_{2_{R2}} + \gamma_2 B2*_{r2} &= 0 \\
\alpha_1(w1-v1) &= 0 \\
\alpha_2(w2-v2) &= 0 \\
\gamma_1(B1-B1*) &= 0 \\
\gamma_2(B2-B2*) &= 0 \\
\lambda(B1+B2-Bmax) &= 0
\end{align*}
\]  \hspace{1cm} (6)

As shown above, however, credit rationing must prevail, implying
that each of \( y_1 \) and \( y_2 \) must be zero. The first order conditions can thus be reduced to the following:

\[
\begin{align*}
\ell_{B_1} + \alpha_1 v_{1_{B_1}} - \lambda &= 0 \\
\ell_{B_2} + \alpha_2 v_{2_{B_2}} - \lambda &= 0 \\
\ell_{R_1} + \alpha_1 v_{1_{R_1}} &= 0 \\
\ell_{R_2} + \alpha_2 v_{2_{R_2}} &= 0 \\
\alpha_1 (w_1 - v_1) &= 0 \\
\alpha_2 (w_2 - v_2) &= 0 \\
\lambda (B_1 + B_2 - B_{\text{max}}) &= 0
\end{align*}
\]

Since the derivative of the lender’s value functions with respect to \( R_1 \) and \( R_2 \) will be non-zero in equilibrium, it follows that each of \( \alpha_1 \) and \( \alpha_2 \) must be non-zero. From this it follows that \( w_1 = v_1 \) and \( w_2 = v_2 \). In other words, each of the borrowers is driven to the zero profit condition, and this result is independent of whether the lender’s overall lending constraint is binding or not.

In the result, the lagrangean formulation of the lender’s problem shows that the conditions of credit rationing and “zero profit” for borrowers are automatic properties of the model.
4. A change in the lending constraint

A natural question to ask is what the result of a change in Bmax will be. The expressions for the borrowers' and lender's value functions are too complex to produce analytical results. The following numerical results illustrate both what trends can be expected, and why analytical results are unlikely to be available.

Using the data in tables 10 to 13 (and interpolating with additional calculations where necessary), the following table sets out the loan amounts that will be allocated to the "small" and "large" borrowers under consideration. The maximizing values are rounded to the nearest multiple of 10.
Table 14 - Debt levels for different Lender’s constraints

<table>
<thead>
<tr>
<th>Bmax</th>
<th>B1</th>
<th>B2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>80</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>110</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>120</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>130</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>140</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>150</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>160</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>170</td>
<td>120</td>
<td>50</td>
</tr>
<tr>
<td>180</td>
<td>130</td>
<td>50</td>
</tr>
<tr>
<td>190</td>
<td>130</td>
<td>60</td>
</tr>
<tr>
<td>200</td>
<td>140</td>
<td>60</td>
</tr>
<tr>
<td>210</td>
<td>140</td>
<td>70</td>
</tr>
<tr>
<td>220</td>
<td>150</td>
<td>70</td>
</tr>
<tr>
<td>230</td>
<td>150</td>
<td>80</td>
</tr>
<tr>
<td>240</td>
<td>160</td>
<td>90</td>
</tr>
<tr>
<td>250</td>
<td>170</td>
<td>90</td>
</tr>
<tr>
<td>260</td>
<td>180</td>
<td>90</td>
</tr>
<tr>
<td>270</td>
<td>180</td>
<td>90</td>
</tr>
<tr>
<td>280</td>
<td>180</td>
<td>100</td>
</tr>
<tr>
<td>290</td>
<td>180</td>
<td>110</td>
</tr>
<tr>
<td>300</td>
<td>190</td>
<td>110</td>
</tr>
<tr>
<td>310</td>
<td>190</td>
<td>120</td>
</tr>
<tr>
<td>320</td>
<td>190</td>
<td>120</td>
</tr>
</tbody>
</table>

While the above table corresponds to a specific set of parameters, there is no reason to expect that the trends it shows would not generalize. A loan to the smaller borrower is relatively more risky, with the result that the lender will allocate all credit to the wealthier borrower, when the amount of loanable funds is small. As the wealthier borrower’s debt levels grow, the marginal return he pays on additional loan funds goes
down. The reason is that higher debt levels bring higher risk of bankruptcy, with resulting loss of the lender's capital. Accordingly, funds start to become available to the smaller borrower.

There is a threshold level at which the small borrower enters the market. Owing to his transaction costs, he must enter with a discrete, rather than infinitesimal, debt level. In the result, the credit available to the large borrower takes a discrete drop when the small borrower enters the market.

Finally, for values of Bmax in excess of the small borrower's "threshold" level, it is clear that the benefits of increasing credit (or alternatively the burden of tighter credit constraint) is shared between the small and large borrower.

5. A change in the salvage ratio.

In previous chapters, several changes in the legal relations between borrowers and lenders were modelled by varying Sa, the parameter governing the lender's recovery on insolvency. It is not proposed to duplicate each analysis here, as the maximization
This condition states, in effect, that the total incremental value of an incremental loan must be the same for each borrower in equilibrium. The first term in each expression is the direct increment in value for an incremental loan, while the second term is the indirect increment in value, resulting from the consequential change in the interest rate. The change in the interest rate ensures that the borrower remains in a condition of "zero profit" at the new loan level. Equation (8) illustrates why it is not feasible to derive analytic results in this section - the expressions governing the respective changes in the left and right hand sides of (8), when the value of $S_a$ is changed, are highly complex.

It is clear that at or near a corner solution, a change in $S_a$ will have a dramatic change in equilibrium loan levels. If
equilibrium lies at the threshold point, where the small borrower has just entered the market, a reduction in $S_a$ will force him back out of the market, resulting in a discrete reallocation of credit to the large lender. Conversely, an increase in $S_a$ could allow the small borrower into the market, resulting in a discrete reallocation of credit away from the large borrower, if the existing equilibrium only marginally excludes the small borrower.

Numerical results suggest, however, that at equilibrium solutions that are not at or near a corner, a change in $S_a$ is likely to have little effect on equilibrium loan levels.

The following values have been computed to show how the equilibrium value for $B_{\text{max}}=180$, as shown in table 14, change when $S_a = .3, .4, \text{ or } .5$. 
Table 16 - Lender's values (and corresponding interest rates) for "large" lender (w=100) at different values of Sa

<table>
<thead>
<tr>
<th></th>
<th>Sa = .3</th>
<th>Sa = .4</th>
<th>Sa = .5</th>
</tr>
</thead>
<tbody>
<tr>
<td>B=120; R=.0743</td>
<td>20.3</td>
<td>25.25</td>
<td>30.21</td>
</tr>
<tr>
<td>B=121; R=.0743</td>
<td>20.32</td>
<td>25.38</td>
<td>30.36</td>
</tr>
<tr>
<td>B=122; R=.0744</td>
<td>20.42</td>
<td>25.5</td>
<td>30.59</td>
</tr>
<tr>
<td>B=123; R=.0744</td>
<td>20.43</td>
<td>25.58</td>
<td>30.74</td>
</tr>
<tr>
<td>B=124; R=.0744</td>
<td>20.45</td>
<td>25.66</td>
<td>30.88</td>
</tr>
<tr>
<td>B=125; R=.0745</td>
<td>20.54</td>
<td>25.83</td>
<td>31.11</td>
</tr>
<tr>
<td>B=126; R=.0745</td>
<td>20.56</td>
<td>25.91</td>
<td>31.26</td>
</tr>
<tr>
<td>B=127; R=.0745</td>
<td>20.57</td>
<td>25.98</td>
<td>31.4</td>
</tr>
<tr>
<td>B=128; R=.0745</td>
<td>20.58</td>
<td>26.06</td>
<td>31.55</td>
</tr>
<tr>
<td>B=129; R=.0746</td>
<td>20.66</td>
<td>26.22</td>
<td>31.77</td>
</tr>
<tr>
<td>B=130; R=.0746</td>
<td>20.67</td>
<td>26.29</td>
<td>31.91</td>
</tr>
</tbody>
</table>
Table 17 - Lender’s values (and corresponding interest rates) for “small” lender (w=50) at different values of Sa

<table>
<thead>
<tr>
<th></th>
<th>Sa = .3</th>
<th>Sa = .4</th>
<th>Sa = .5</th>
</tr>
</thead>
<tbody>
<tr>
<td>B=50; R=.0631</td>
<td>0.1</td>
<td>4.2</td>
<td>6.84</td>
</tr>
<tr>
<td>B=51; R=.0634</td>
<td>1</td>
<td>4.33</td>
<td>7.03</td>
</tr>
<tr>
<td>B=52; R=.0636</td>
<td>1.66</td>
<td>4.24</td>
<td>7.19</td>
</tr>
<tr>
<td>B=53; R=.0638</td>
<td>1.69</td>
<td>4.52</td>
<td>7.35</td>
</tr>
<tr>
<td>B=54; R=.0640</td>
<td>1.72</td>
<td>4.62</td>
<td>7.51</td>
</tr>
<tr>
<td>B=55; R=.0642</td>
<td>1.75</td>
<td>4.71</td>
<td>7.68</td>
</tr>
<tr>
<td>B=56; R=.0644</td>
<td>1.78</td>
<td>4.81</td>
<td>7.84</td>
</tr>
<tr>
<td>B=57; R=.0646</td>
<td>1.81</td>
<td>4.91</td>
<td>8</td>
</tr>
<tr>
<td>B=58; R=.0648</td>
<td>1.84</td>
<td>5</td>
<td>8.17</td>
</tr>
<tr>
<td>B=59; R=.0650</td>
<td>1.87</td>
<td>5.1</td>
<td>8.33</td>
</tr>
<tr>
<td>B=60; R=.0652</td>
<td>1.9</td>
<td>5.2</td>
<td>8.5</td>
</tr>
</tbody>
</table>

Analysis of these figures shows that for Bmax=180, the value of Sa is largely irrelevant to the optimal loan levels. In each case, values of B1=129 for the large lender and B2=51 is optimizing. The result in not changed whether Sa is set equal to .3, .4 or .5.
CHAPTER VIII - CONCLUSION

This thesis has examined the implications of corporate bankruptcy through the use of stochastic differential equations and absorbing boundaries. The stochastic process used was different from that most commonly used in finance literature (geometric brownian motion), in that the magnitude of the stochastic term was constant, rather than proportional to the state variable under consideration.

A differential equation and closed form solution were derived for the discounted value of expected future revenue for a firm facing possible bankruptcy. The same equation and solution were also applicable in valuing a lender's stream of expected revenue and the expected salvage value recovered by a lender on bankruptcy. The differential equation derived was not homogeneous, as is often the case in models based on geometric brownian motion.

With value functions available for both a borrower and lender, various aspects of debtor - creditor relations were examined.

In a competitive loan market, a straightforward explanation for
credit rationing was provided, without resorting to market imperfections or asymmetric information. As well, loan levels were found to rise as legal changes were simulated to increase, ceteris paribus, either a lender's or borrower's salvage value on bankruptcy. However, when salvage recovery was reallocated from a borrower to a lender, the result was ambiguous. The borrower's increased demand was offset by the lender's decreased supply.

In a monopoly lending market, the borrower was generally reduced to a zero profit condition. However, credit rationing could not be demonstrated to occur generally. Where credit rationing was present, a legal change to increase the lender's recovery on bankruptcy would relax the credit constraint, increasing lending levels. Absent credit rationing, a change in the lender's salvage recovery did not affect any equilibrium variable. Conversely, an increase in the borrower's recovery increased the equilibrium interest rate whether or not credit rationing was present. A reallocation from lender to borrower produced no unambiguous results when credit rationing was present, but unambiguously increased interest rates (by increasing the maximum interest rate the borrower could afford) when credit rationing was absent.
The final case analysed was that of a monopolist lender allocating limited funds between two borrowers, one of which was smaller and relatively more risky. In this part of the analysis, a different stochastic process and maximization problem were used. Under the revised structure, credit rationing was shown to exist for all borrowers independently of the lender's allocation problem, and even when the lender's constraint on total lending was not binding. For relatively low values of the lender's constraint, the riskier borrower was cut out of the market altogether. As more aggregate funds became available, they were allocated between the two borrowers. However, no borrower ever received as much as he would prefer. A change in the lender's salvage recovery could only be evaluated numerically, but seemed to have relatively little effect on loan levels except at corner solutions.

The analysis has been carried out in a partial equilibrium framework, like most other studies focusing directly on the issue of bankruptcy. It remains to be seen whether the closed form solutions derived in this thesis are sufficiently tractable to be used in a general equilibrium framework.


