Urban Development Under Discrete Random Shocks

By

ZhenGuo Lin

MA in Economics, The University of Western Ontario, 1996

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

In

THE FACULTY OF GRADUATE STUDIES

Division of Urban Land Economics
The Faculty of Commerce and Business Administration

We accept this thesis as conforming to the required standard

The University of British Columbia

November 1998

© ZhenGuo Lin, 1998
In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Commerce

The University of British Columbia
Vancouver, Canada

Date Dec 2, 1998

DE-6 (2/88)
Abstract

This paper investigates the effects of discrete random shocks to the urban development decision and the price of land. The urban development has three important characteristics: investment irreversibility, uncertainty of future payoffs and an option to postpone the investment. In contrast to the existing literature that treats uncertainty as continuous time stochastic process, this paper considers uncertainty as a discrete time stochastic process (discrete random shocks) with two different sources of uncertainty. One is the uncertainty of the time when shock occurs, and the other is the uncertainty of the jump size shock incurs.

This paper contributes to the real option literature in the following ways. First, the approach developed here can be applied to other lump sum cost investment decision under discrete random shocks; second, this paper is the first to study how discrete random shocks affect the urban development and the price of urban and agricultural land. We show that (i). The price of urban land is uniquely determined by the current rent, the mean arrival rate of random shocks and the expected value of jump size; (ii). If jump sizes with shock occurrence are all positive, landowners will convert their agricultural land to urban use when the urban rent first hits agricultural rent plus capital opportunity cost; (iii). If jump size with shock occurrence is up and down with positive probability, then (a). The higher probability of random shocks may (a.1) deter urban development and decrease the price of agricultural land; (a.2) deter urban development and increase the price of agricultural land; (a.3) encourage urban development and increase the price of agricultural land; (b). The more spread of random jumps may (b.1) deter urban development and decrease the price of agricultural land; (b.2) deter urban development and increase the price of agricultural land; (b.3) encourage urban development and increase the price of agricultural land.

Keywords: Random shock jump process, Random shock, Random jump, Trigger rent
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td></td>
<td>ii</td>
</tr>
<tr>
<td>Section I</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Section II</td>
<td>The Model</td>
<td>5</td>
</tr>
<tr>
<td>Section III</td>
<td>The Trigger Rent</td>
<td>8</td>
</tr>
<tr>
<td>Section IV</td>
<td>The Results</td>
<td>11</td>
</tr>
<tr>
<td>Section V</td>
<td>Conclusion</td>
<td>22</td>
</tr>
<tr>
<td>Reference</td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>Appendix</td>
<td></td>
<td>26</td>
</tr>
</tbody>
</table>
Section I  Introduction

Many capital-investment decisions share three important characteristics. First, the investment is irreversible. Second, there is uncertainty over the future profits from the investment. Third, investors have an option to postpone the investment.

How should investors, facing uncertainty over future market conditions, decide whether to invest in a new project? The traditional rule suggests the following way. First, calculate the present value of the expected stream of profits this project will generate. Second, calculate the present value of the stream of expenditures required to complete this project. Finally, determine whether the difference between the two—the net present value (NPV) of the investment—is greater than zero. If it is, go ahead and invest.

The recent literature has shown that the NPV rule, to invest when the present value of expected cash flows exceeds the cost of the investment, is no longer valid when the future profits are uncertain. In other words, waiting to invest might have more value even if the net present value is positive. Intuitively, when investors make an irreversible investment, they exercise or "kill" their option to invest. They give up the possibility of waiting for new information to arrive that might affect the timing of the investment. They can't disinvest should market conditions change adversely. This lost option value is an opportunity cost that must be included as part of the cost of the investment. As a result, the NPV rule must be modified: to invest when the present value of expected cash flows exceeds the cost of the investment plus the value of keeping the investment option alive (real option value).

---

1 This paper is initiated from discussions with William Strange. The author is indebted to Robert Helsley and William Strange for valuable comments and encouragement. Of course, any error is mine.
In the 1980s, real option theory originated from perpetual American-option models of Samuelson (1965), Black and Scholes (1973) and Merton (1973). In the past decade, real option theory has been developed, and numerous papers have studied how real option value affects the investment decisions. For example, Brennan and Schwartz (1985) study how uncertainty of its output prices, which follows geometric Brownian motion, affects the decision of developing the natural resource projects. McDonald and Siegel (1986) solve for the optimal timing of investment in an irreversible project where the benefits from the project and the investment cost also follow continuous-time stochastic processes, specifically, geometric Brownian motion.

In the existing real option literature, uncertainty is most commonly assumed to follow Brownian motion. There are two reasons for this assumption. First, Brownian motion is a good approximation of high degree of uncertainty like stock prices; Second, Brownian motion has some good mathematical properties. For instance, at any time \( t \), the stochastic process is normally distributed with mean zero and variance \( \sigma^2 t \).

In the recent real option literature, Dixit (1989, 1991) investigates investment (entry) and abandonment (exit) decisions when the price follows geometric Brownian motion. He finds a pair of trigger prices for investment (entry) and abandonment (exit). The investment (entry) trigger exceeds the variable cost plus the option value of waiting, and the exit trigger is less than the variable cost minus option value of waiting. These gap produces "hysteresis". Bar-I and Strange (1998) study how real option value affects the sequential investment decision involving two stages to complete when the output price follows geometric Brownian motion. They find several interesting results. First, if allowed to suspend, the first-stage trigger is possibly below the second-stage trigger; without suspension, the first-stage trigger is always above the second-stage trigger. Second, exploratory investment is much likely with a low first-stage cost given constant investment lag and net present value of the total cost; Third, the investment with longer first-stage are more likely to be characterized by exploratory investment.
The option feature in urban development decisions was first identified by Titman (1985), who applied the binomial option pricing models of Cox, Ross and Rubinstein (1979) to land-use decisions under uncertainty. The more recent literature on land conversion under uncertainty, in which rent follows Brownian motion, includes Capozza and Helsley (1990), Capozza and Sick (1991), Capozza and Li (1994) and Grenadier (1996), Bar-II and Strange (1996).

Titman (1985) presents a two dates, two states of nature, model for determining the value of the vacant land when the future price of building is uncertain. He uses this framework to explain why the vacant land may be more valuable as a potential site than it is as an actual site for constructing any particular building at the present time. Capozza and Helsley (1990) assume developers are risk neutral and rent follows arithmetic Brownian motion. They show that rent uncertainty delays the conversion of land from agricultural to urban use, adds option value to agricultural land and reduces the expected city size. Bar-II and Strange (1996) consider urban development with conversion lag, when rent follows geometric Brownian motion. They find (i) it is possible that uncertainty may hasten development with conversion lags; (ii) development is less sensitive to uncertainty than in the absence of lags; (iii) Lags may lead to leapfrog development.

The price and rent are assumed to have two dates and two states, it might be too simple. However, the price and rent are assumed to follow Brownian motion\(^2\), it might be also not good approximation to the reality. Even stock prices, although, changes frequently, it can take large jumps or falls if a big event like Asian financial crisis happens. Therefore, the changes of price might be better modelled as discrete random shocks. There are two different kinds of uncertainty in discrete random shocks: the time of shock occurrence and the jump size of shock effect. We coin this process as random shocks jump process.

This paper attempts to study how the uncertainty under random shock jump process affects the urban development decision and the price of land. Using the concept of

\[^2\text{It} \text{implies that (i) they change continuously; (ii) they change in any interval time with probability one.}\]
stopping time, we solve the landowners' optimal urban development condition. The approach developed here can be adapted to analyze other lump sum cost investment decision under random shock jump process.

The random shock jump process has two different and "independent" sources of uncertainty: the time of shock occurrence and the jump size of shock effect. In the rest of the paper, we call the first uncertainty as random shock and the second uncertainty as random jump. Since these two uncertainties don't affect each other\(^5\), their effects on the urban development decision and price of land can be identified separately.

The main results of this paper are as follows: (i) The price of urban land is uniquely determined by the current rent, the mean arrival rate of random shocks and the expected value of jump size; (ii). If jump sizes with shock occurrence are all positive, landowners will convert their agricultural land to urban use when the urban rent first hits agricultural rent plus capital opportunity cost; (iii). If jump size with shock occurrence is up and down with positive probability, then (a). The higher probability of random shocks may (a.1) deter urban development and decrease the price of agricultural land; (a.2) deter urban development and increase the price of agricultural land; (a.3) encourage urban development and increase the price of agricultural land. (b). The more spread of random jumps may (b.1) deter urban development and decrease the price of agricultural land; (b.2) deter urban development and increase the price of agricultural land; (b.3) encourage urban development and increase the price of agricultural land.

This paper is organized as follows. Section II describes the model. Section III solves the urban development condition. Section IV carries out our analytical results. Section V concludes.

\(^5\) Random shocks don't affect the distribution of random jump and random jumps don't affect the Poisson process of random shocks.
In sprit, this paper is close to Capozza and Helsley (1990). In their paper, they assume urban rent follows arithmetic Brownian motion.

The urban rent is assumed to follow arithmetic Brownian motion. This implies that (i) urban rent changes continuously; (ii) urban rent changes in any interval time with probability one. In reality, urban rent changes much less "frequently" than arithmetic Brownian motion suggests. However, the change of urban rent is also unpredictable, specifically when change will occur and how much the change will incur. Therefore, urban rent might be better modelled as discrete random shocks with two different sources of uncertainty: (i) the uncertainty of the number of random shocks during any time interval and (ii) the uncertainty of the jump size with shock occurrence.

In this paper, we consider the rent adjustment by discrete random shocks. We assume the number of random shocks \( N(s) \) during an interval time, \( s \), follows Poisson process with rate \( \lambda \). The rate \( \lambda \) is the mean arrival rate of one random shock. During a time interval of infinitesimal length \( dt \), the probability that one shock will occur is given by \( \lambda dt \), and the probability that one shock will not occur is given by \( 1 - \lambda dt \). If one shock occurs, the jump size is also uncertain. We assume its probability density function is \( f(J) \).

Formally, we can express this urban rent adjustment as following random shock jump process,

\[
R(s + t, z) = R(t, z) + \sum_{i=1}^{N(t)} J_i
\]

(1)

where:

- \( R(t, z) \) is the rent at location \( z \) at time \( t \);
- \( \{N(s), s \geq 0\} \) is a Poisson process with rate \( \lambda \) (\( \lambda > 0 \)) time interval from \( t \);
- \( J_1, J_2, J_3, \ldots \) are a sequence of independent and identically distributed random jumps and their probability density function is \( f(J) \).
Mathematically, \( \{N(s), s \geq 0\} \) is a Poisson process having rate \( \lambda (\lambda > 0) \), which implies,

(i) \( N(0) = 0 \);
(ii) The process has independent increments;
(iii) The number of jumps in any interval of length \( t \) is Poisson distributed with mean \( \lambda t \).

That is, for all \( t, s \geq 0 \),
\[
\text{Prob}\left( N(t+s) - N(t) = n \right) = e^{-\lambda s} \left( \frac{\lambda t}{n!} \right)^n, n = 0, 1, 2, \ldots
\]

In this paper, we assume the landowners, if they convert their agricultural land to urban use, will receive urban rent forever, which follows the random shock jump process (1). Otherwise, they receive agricultural rent. They also have an option to decide when to make this conversion. When agricultural land is converted to urban use, it will incur lump sum cost \( C \), which is irreversible. The landowners choose the optimal conversion time to maximize their profit. As in Capozza and Helsley (1990), we assume the landowners are risk neutral and conversion from agricultural land to urban use doesn’t take any time.

**Urban Development**

Since agricultural land may be converted to urban land use in the future, the stream of its rent can be grouped into two parts:

- Agricultural rent until the land converted to urban land use;
- Urban rent forever after conversion.

Here we implicitly assume that land not in urban use is used in agriculture. All agricultural land earns agricultural rent \( A \) regardless of location.

The landowners choose the conversion time to maximize their profit. Suppose the landowner sets the conversion time at \( t + s \) (\( t \) is current time), then the expected price of the agricultural land is:
The landowners choose optimal conversion time to maximize its expected profit, which is equal to the price of agricultural land, formally,

\[
P^A(t, z) = \max_s \{E[P^A(t, s, z)]\}
\]

Actually, the optimal conversion time to (3) is not certain, because it depends on how the future rents actually evolve. However, this problem can be transformed into the stopping time problem.

Let \( R^* \) be the trigger rent at which it is optimal to convert land from rural to urban use. The optimal time of conversion, \( s^* \), known as the first hitting time, can be defined by

\[
s^* = \min \{s \geq 0 \mid R(t + s, z) \geq R^* \}
\]

Then, (3) is equivalent to the following problem:

\[
P^A(t, z) = \max_R \{E[P^A(t, s, z)] \mid R \}
\]

Therefore,

\[
R^* = \arg \max \{E[P^A(t, s, z)] \mid R \}
\]

In the next section, we will solve this urban development condition.
As in Henderson (1988, 1996), we consider landowners who have complete power to determine when agricultural land converts to urban land. Landowners choose urban development when the rent first hits the trigger rent. In doing so, they will receive the highest profit. Therefore, trigger rent is the key factor for urban development.

The landowners choose their optimal conversion time to maximize their profit. This optimal condition can be expressed as the following theorem.

**Theorem 1 (Trigger Rent):** If developers are risk-neutral, urban rent follows (1), and urban development has no investment lags, then (i). The urban development rule is to develop when first hitting the trigger rent \( R^* \); (ii). \( R^* \) must satisfy:

\[
R^* = \arg \max \left\{ \frac{\lambda \int J f(J) dJ}{(R + \frac{x}{r} - A - rC)} \left[ \sum_{n \in \Omega^R} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \right] \right\}
\]

where:

\[
\Omega^R = \left\{ n \mid n = \min \left\{ N(s) \mid \sum_{i=1}^{N(s)} J_i \geq R - R(t, z) \right\} \right\}
\]

\[
\Psi^R = \left\{ (J_1, J_2, \ldots, J_n) \mid \sum_{i=1}^{n} J_i \geq R - R(t, z) \text{ and } \sum_{i=1}^{m} J_i < R - R(t, z) \text{ for } m < n \right\}
\]

\[
F^R(n) = \int_{(J_1, J_2, \ldots, J_n) \in \Psi^R} \prod_{i=1}^{n} f(J_i) dJ_1 \ldots dJ_n
\]

Proof: From (4),

\[
P^A(t, z) = \max_R \left\{ E[P^A(t, s, z) \mid R] \right\}
\]

\[\text{If jump size is discrete then } \int \prod_{i=1}^{n} f(J_i) dJ_1 \ldots dJ_n \text{ should be replaced by } \sum_{(J_1, J_2, \ldots, J_n) \in \Psi^R} \prod_{i=1}^{n} f(J_i).\]
And from (2),

\[ E[P^A(t, s, z)|R] = \int_0^\infty Ae^{-r\tau} d\tau + E\left\{ \int_0^\infty [R(s + \tau, z) - A]e^{-r(s+\tau)} d\tau - Ce^{-rn} | R(t, z), R \right\} \]

\[ = \frac{A}{r} + E\left\{ \int_0^\infty [R(s_n + \tau, z) + \sum_{i=1}^{N(\tau)} J_i - A]e^{-r(s_n+\tau)} d\tau - Ce^{-rn} | R(t, z), R \right\} \]

(where \( n \in \Omega^r \))

\[ = \frac{A}{r} + E\left\{ \int_0^\infty [R + \sum_{i=1}^{N(\tau)} J_i - A]e^{-r(s_n+\tau)} d\tau - Ce^{-rn} | R(t, z), R \right\} \]

Next, we prove the following result,

\[ E\int_0^\infty \left( \sum_{i=1}^{N(\tau)} J_i e^{-r\tau} d\tau \right) = \frac{\lambda \int f(J)dJ}{r^2} \]

Since \( E\left\{ \sum_{i=1}^{N(\tau)} J_i e^{-r\tau} d\tau \right\} \) strictly increases with \( \tau \), therefore,

\[ E\left\{ \sum_{i=1}^{N(\tau)} J_i e^{-r\tau} d\tau \right\} = \lim_{\tau \to \infty} E\left\{ \sum_{i=1}^{N(\tau)} J_i e^{-r\tau} ds \right\} \]

By Ward's equation\(^5\),

\[ \lim_{\tau \to \infty} E\left\{ \sum_{i=1}^{N(\tau)} J_i e^{-r\tau} ds \right\} = \lim_{\tau \to \infty} E[N(\tau)]E[J_i]e^{-rn} ds \]

\[ = E[J_i] \lim_{\tau \to \infty} \int_0^\tau \lambda s e^{-rn} ds \]

(Here we use the result: If \( N(s) \) is a Possion process with rate \( \lambda \), then \( E[N(s)] = \lambda s \))

By parts yields,

\[ E[J_i] \lim_{\tau \to \infty} \int_0^\tau \lambda s e^{-rn} ds = E[J_i] \lambda \lim_{\tau \to \infty} \left\{ \frac{1}{r} \left( e^{-rn}s \right|_0^\tau - \int_0^\tau e^{-rn} ds \right\} \]

\[ = \frac{\lambda E[J_i]}{r^2} = \frac{\lambda \int f(J)dJ}{r^2} \]

\(^5\) Ward Equation: If \( E[N(s)] \) is limited, then \( E\left[ \sum_{i=1}^{N(\tau)} J_i \right] = E[N(s)]E(J_i) \). In details, see Ross (1983).
Therefore, 
\[ E[P^A(t, s, z)|R] = \frac{A}{r} + E\left( \frac{\lambda}{r} \int_{-\infty}^{\alpha} f(J) dJ - \frac{A}{r - C} e^{-r_s} | R(t, z), R \right) \]

\[ = \frac{A}{r} + \frac{1}{r} (R + \frac{\lambda}{r} \int_{-\infty}^{\alpha} f(J) dJ - A - rC) E\{ e^{-r_s} \} \quad (6) \]

Since \( s_n = \sum_{i=1}^{n} t_i \) (where \( n \in \Omega, t_i \) denotes the time between \((i - 1)\)st and the \(i\)th event), \( s_n \) has a Gamma distribution with parameters \( n \) and \( \lambda \) (see Ross (1983)). The moment generating function of Gamma distribution is,

\[ E\{ e^{\lambda t} | n \} = \left( \frac{\lambda}{\lambda - t} \right)^n, \quad t \in (-\infty, +\infty) \]

So,

\[ E\{ e^{-r_s} \} = E\{ E(e^{-r_s} | n) \} = E\{ \left( \frac{\lambda}{\lambda + r} \right)^n \} = \sum_{n=0}^{\infty} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \quad (7) \]

Therefore,

\[ R^* = \arg\max \left\{ (R + \frac{\lambda}{\lambda + r} \int_{-\infty}^{\alpha} f(J) dJ - A - rC) \left[ \sum_{n=0}^{\infty} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \right] \right\} \]

Q.E.D

Theorem 1 tells us how we can solve for the trigger rent \( R^* \). If we solve for the trigger rent, the optimal time of urban development is also determined. That is when the urban rent first hits the trigger rent \( R^* \). Formally,

\[ s^* = \min \left\{ s \geq 0 | R(t + s, z) \geq R^* \right\} \]
From the definition of \( s^* \), \( s^* \) depends on how the future rent actually evolves. When shocks actually happen more frequently and jump sizes are bigger, \( s^* \) would be shorter. Otherwise, it would take longer. Therefore, the optimal conversion time, \( s^* \), is uncertain.

The next question we concern would be whether the trigger rent exists. Theorem 2 has the result.

**Theorem 2.** The hurdle rent \( R^* \) always exists.

If the trigger rent \( R^* \) exists, then \( R^* \neq +\infty \). Otherwise, urban development will never occur. Therefore, we need to prove there exists \( R^* \) and \( R^* < +\infty \). As to the existence of \( R^* \), we consider two cases: (1) the price of agricultural is bounded, then there must exist the highest price. Suppose \( R^* \) is the solution to (5), we need to prove \( R^* < +\infty \); (2) the price of agricultural is unbounded, then the highest price will be \( +\infty \). We need to prove it happens when \( R^* < +\infty \). In details, see appendix 1.

**Section IV  The Results**

In this section, we study how the uncertainty under the random shock jump process affects the urban development decision and the price of urban and agricultural land. Here we don't consider urban redevelopment\(^6\). If agricultural land is converted to urban land, landowners will receive urban rent forever, which follows the random shock jump process (1). Landowners will convert agricultural land to urban use when it first hits the trigger rent. Different random shock jump processes will affect the trigger rent and consequently affect the price of agricultural land. How they affect the urban development decision and the price of agricultural land will be mainly discussed in this section.

\(^6\) As to urban redevelopment, see Brueckner (1980), Wheaton (1982) and Capozza and Sick (1991).
First, we investigate how the uncertainty under the random shock jump process (1) affects the prices of urban land. For urban land, although the rent may be below agricultural rent, the landowners will still receive the urban rent. We implicitly assume the conversion cost is very high if urban land converted to agricultural land.

(i). The Price of Urban Land

In the competitive market and under risk neutrality, the price of urban land is the expected present value of future urban rents. It can be expressed as follows,

\[ P^U(t, z) = \mathbb{E}\left[ \int_t^\infty R(t+s, z)e^{-rs}ds|R(t, z) \right] \quad (z \leq b_1) \quad (8) \]

where:
- \( t \) is the current time;
- \( r \) is the common discount rate;
- \( R(t, z) \) is the rent at location \( z \) and at time \( t \);
- \( b_1 \) is the boundary of the city at time \( t \).

Since urban rent follows (1), substituting it into (2) yields theorem 3.

**Theorem 3:** In the competitive market and under risk neutrality, if urban rent follows (1), then the price of urban land is uniquely given by,

\[ P^U(t, z) = \frac{1}{r} \left[ R(t, z) + \frac{\lambda}{r} \int J f(J)dJ \right] \quad \text{(or) } \frac{1}{r} \left[ R(t, z) + \frac{\lambda}{r} E(J) \right] \]

Proof:

From (1), \( R(t+s, z) = R(t, z) + \sum_{i=1}^{N(t)} J_i \)

\[ \Rightarrow \mathbb{E}\left[ \int_0^\infty R(t+s, z)e^{-rs}ds|R(t, z) \right] = \int_0^\infty R(t, z)e^{-rs}ds + \mathbb{E}\left[ \sum_{i=1}^{N(t)} J_i e^{-rs}ds \right] \]
Since 
\[ \int_0^\infty R(t, z) e^{-rs} ds = R(t, z) \int_0^\infty e^{-rs} ds = \frac{R(t, z)}{r} \]

In theorem 1, we already have the result,

\[ E[\sum_{i=1}^{N(z)} J_i e^{-r \tau_i}] = \frac{\lambda \int_0^\infty Jf(J) dJ}{r^2} \]

Therefore,

\[ E\left( \int_0^\infty R(t+s, z) e^{-rs} ds \mid R(t, z) \right) = \frac{1}{r} \left[ R(t, z) + \frac{\lambda \int_0^\infty Jf(J) dJ}{r^2} \right] \]

\[ \text{Q.E.D} \]

Theorem 3 tells us that two different sources of uncertainty affect the price of urban land in a "simple" way. The same percentage change of mean arrival rate (\( \lambda \)) of random shock affects the prices of urban land exactly in the same way as the same percentage change of the expected value (\( E(J) \)) of a random jump. In other words, 1% increase in \( \lambda \) adds the value of the urban land as the same as 1% increase in \( E(J) \). If either \( \lambda \) or \( E(J) \) is equal to zero, the uncertainty of the rent disappears and the rent will stay the same forever. In this case, the prices of urban land are easily to be calculated as current rent over discount rate. The same result can be obtained by using theorem 1.

Theorem 3 also tells us that the higher probability (bigger \( \lambda \)) of random shocks increases (decreases) the price of urban land when the expected value of jump size is positive (negative). In other words, its effect on the price of urban land depends on the expected value of random jump. Intuitively, with good news (\( E(J) > 0 \)), the higher probability of random shocks favors the price of urban land. Otherwise, fewer shocks are better.
Keeping the distribution of random jumps fixed, the higher probability of random shocks (bigger \( \lambda \)) will increase the uncertainty of urban rent and affect the price of urban land. Under the assumption of Brownian motion\(^7\), uncertainty will not affect the price of urban land. However, the growth rate does matter.

The variance\(^8\) doesn't matter in the price of urban land with either Brownian motion or discrete random shocks, since we assume risk neutral landowners. Without this assumption, the variance would also matter.

(ii). **Urban Development and the Prices of Agricultural Land**

In section III, urban development occurs when the urban rent first hits the trigger rent. Under discrete random shocks, it's almost impossible to solve the closed form of trigger rent. Therefore, the following study will focus on comparative analysis.

To begin, we consider a trivial case: Jumps \( J \) are all positive. Suppose developers have the following three strategies, which include all possible strategies they can choose,

(I). Urban development sometime before urban rent first hits \( A + rC \);

(II). Urban development right after urban rent first hits \( A + rC \);

(III). Urban development sometime after urban rent second hits \( A + rC \).

In the strategy (I), the best strategy is that urban development right before urban rent first hits \( A + rC \). Otherwise, developers will lose more money, because urban rent is lower than agricultural rent. In the strategy (III), the best strategy is that urban development

\(^7\) The price of urban land is: \( P^u(t, z) = \frac{1}{r} \left[ R(t, z) + \frac{g}{r} \right] \) under \( dR = gdt + \sigma dz \). The price of urban land becomes: \( P^u(t, z) = \frac{R(t, z)}{r - g} \), if we assume \( \frac{dR}{R} = gdt + \sigma dz \).

\(^8\) The variance refers to \( \sigma \) in Brownian motion case and it refers to the variance of random jump size in our paper.
right after urban rent second hits $A + rC$. In doing so, they will not miss the opportunity of gaining more money, because urban rent is higher than agricultural rent.

Now, we compare the following three strategies,
(1) Urban development right before urban rent first hits $A + rC$;
(2) Urban development right after urban rent first hits $A + rC$;
(3) Urban development right after urban rent second hits $A + rC$.

Compared with strategy (2), strategy (1) will lose money from the time of urban development to the time when urban rent first hits $A + rC$, because urban rent is lower than agricultural rent during that period. However, strategy (3) will miss the opportunity of gaining more money from the time when urban rent first hits $A + rC$ to the time of urban development, because urban rent is higher than agricultural rent in that period. Therefore, strategy (2) is the best. Consequently, strategy (II) is optimal among all the strategies and $R^* = A + rC$. In the following analysis, we assume urban rent jumps up and down with positive probability.

The random shock jump process (1) has two "components": random shocks and random jumps, neither of which "affects" each other. How they separately affect the urban development decision and the price of agricultural land will be mainly discussed here.

First, we investigate how the possibility of random shocks ($\lambda$) affects the urban development and the price of agricultural land.

Holding the distribution of random jump ($f(J)$) fixed, we study how increases in the mean arrival rate, $\lambda$, of Poisson process affect urban development decision and the price of agricultural land. An increase in the probability of random shocks ($\lambda$) will increase the expected number of shock occurrence, consequently will increases the uncertainty of urban rent. Since developers have an option to postpone urban development, if things change unfavorably for them, it will encourage them to wait for more time and deter urban development. Otherwise, it will encourage urban development. Intuitively, the
higher probability of random shocks will encourage urban development if good news (jump size, \( J \), is big) with shock occurrence more favorably happens \( (E(J) \gg 0)\)

9 because waiting will miss good opportunity to gain more money. However if good news with shock occurrence does not favorably happens \( (E(J) \approx 0)\)

10 the higher probability of random shocks will increase the possibility of bigger loss (bad news principle) and consequently deter urban development. If bad news with shock occurrence more favorably happens \( (E(J) \ll 0)\)

11 the higher probability of random shocks will encourage developers wait for more time until more good news comes. Accordingly, it will deter urban development.

In theorem 3, we already know that if the expected value of the random jump size is zero, an increase in \( \lambda \) (higher probability of random shocks) will not affect the price of urban land. As to the price of agricultural land, since developers have an option to postpone urban development, it will affect the price of agricultural land. Intuitively, when \( E(J) \approx 0 \), developer will wait to avoid future loss until more good news comes. Therefore, the higher probability of random shocks will deter urban development. However, it will impart uncertainty premium (the value of waiting) to agricultural land and increase the price of agricultural land. When \( E(J) \gg 0 \), the higher probability of random shocks will increase the expected number of good news that actually happens. Thus, it will increase the price of agricultural land. When \( E(J) \ll 0 \), the higher probability of random shocks will increase the expected number of bad news that actually occurs. Accordingly, it will decrease the price of agricultural land. The above intuitive analyses are confirmed by the following theorem.

**Theorem 4:** Considering two cities (city 1 and city 2) with urban rents following (1), they are identical except their mean arrival rates \( (\lambda_1 > \lambda_2)\), then their trigger rents and the prices of agricultural land have the following relations:

\[ E(J) \gg 0 \text{ means } E(J) \text{ is much bigger than zero.} \]
\[ E(J) \approx 0 \text{ means } E(J) \text{ is around zero.} \]
\[ E(J) \ll 0 \text{ means } E(J) \text{ is much less than zero.} \]
(i). There exists $\delta (\delta > 0)$, (1) $R_1^* > R_2^*$, when $E(J) < \delta$; (2) $R_1^* < R_2^*$, when $E(J) > \delta$; and (3) $R_1^* = R_2^*$, when $E(J) = \delta$.

(ii). There exists $\omega (\omega < 0)$, (1) $P_1^A(t,z) > P_2^A(t,z)$, when $E(J) > \omega$; (2) $P_1^A(t,z) < P_2^A(t,z)$, when $E(J) < \omega$; (3) $P_1^A(t,z) = P_2^A(t,z)$, when $E(J) = \omega$.

Proof: (i). Let's denote, $\Theta(R, \lambda) = \sum_{n \in \mathbb{N}} F^R(n) \times \left(\frac{\lambda}{\lambda + r}\right)^n$ then $\Theta(R, \lambda) > 0$

From (5), $R_1^* = \arg \max \left\{ (R + \frac{\lambda_1 E(J)}{r} - A - rC) \left[ \sum_{n \in \mathbb{N}} F^R(n) \times \left(\frac{\lambda_1}{\lambda_1 + r}\right)^n \right] \right\}$

$R_1^*$ must satisfy the following first-order condition $^{12}$,

$R_1^* + \frac{\lambda_1 E(J)}{r} - A - rC) \Theta_R(R_1^*, \lambda_1) + \Theta(R_1^*, \lambda_1) = 0 \quad (9)$

We can easily prove $\Theta_R(R, \lambda_1) > \Theta_R(R, \lambda_2)$. From appendix 1, $\Theta_R(R, \lambda) < 0$

Therefore,

$\Theta_R(R, \lambda_2) < \Theta_R(R, \lambda_1) < 0$; Moreover, $\Theta(R, \lambda_1) > \Theta(R, \lambda_2)$

$\Rightarrow \Delta = (R_1^* + \frac{\lambda_1 E(J)}{r} - A - rC) \Theta_R(R_1^*, \lambda_1) - \Theta_R(R_1^*, \lambda_2) + \Theta(R_1^*, \lambda_1) - \Theta(R_1^*, \lambda_2) > 0$

If $E(J) \leq (\geq)$ denote $\delta > 0$, then we can have,

$(R_1^* + \frac{\lambda_1 E(J)}{r} - A - rC) \Theta_R(R_1^*, \lambda_2) + \Theta(R_1^*, \lambda_2) \leq (\geq)0$

$R_2^*$ must satisfy the following first-order condition,

$(R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC) \Theta_R(R_2^*, \lambda_2) + \Theta(R_2^*, \lambda_2) = 0$

Second order condition yields, $R_1^* \geq (\leq) R_2^*$

$^{12}$ Here we implicitly assume the function $\Theta(\cdot, \cdot)$ is smooth. If it is not smooth, similarly we can prove the same results by using "max" definition that $R_1^*$ satisfies:
Therefore,
\[ E(J) \leq (\geq) \delta \Rightarrow (R_1^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_1^*, \lambda_2) + \Theta(R_1^*, \lambda_2) \leq (\geq) 0 \]
\[ \Rightarrow R_1^* \geq (\leq) R_2^* \]

(ii). \( \Theta(R, \lambda_2) < \Theta(R, \lambda_1) \) when \( \lambda_2 < \lambda_1 \)
\[ (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_2) < (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_1) \]
\[ \leq (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_1) \text{ if } E(J) \geq 0 \]
\[ \leq (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_1) \quad (\because R_1^* \text{ is optimal soln}) \]

When \( E(J) < 0 \) and becomes smaller, \( (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_2) \) decreases faster than \( (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_2) \), because \( \lambda_1 > \lambda_2 \).

Thus, there exists \( w = \frac{r((R_1^* - A - rC)\Theta(R_1^*, \lambda_1) - (R_2^* - A - rC)\Theta(R_2^*, \lambda_2)) - \lambda_2 \Theta(R_2^*, \lambda_2) + \lambda_1 \Theta(R_2^*, \lambda_1)}{\lambda_2 \Theta(R_2^*, \lambda_2) - \lambda_1 \Theta(R_2^*, \lambda_1)} < 0 \)

If \( E(J) \leq (\geq) \omega \), we can have,
\[ (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_2) \geq (\leq) (R_1^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_1^*, \lambda_1) \]

From (5), (6), (7), we know,
\[ P_1^A(t, z) = \frac{A}{r} + \frac{1}{r} (R_1^* + \frac{\lambda_1 E(J)}{r} - A - rC)\Theta(R_1^*, \lambda_1) \]
\[ P_2^A(t, z) = \frac{A}{r} + \frac{1}{r} (R_2^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_2^*, \lambda_2) \]

Therefore,
\[ E(J) \leq (\geq) \omega \Rightarrow P_1^A(t, z) \leq (\geq) P_2^A(t, z) \]
Q.E.D

\[ (R_1^* + \Delta R + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_1^* + \Delta R, \lambda_2) \leq (R_1^* + \frac{\lambda_2 E(J)}{r} - A - rC)\Theta(R_1^*, \lambda_2) \quad \forall \Delta R \geq 0 \]
Thus, there exists a threshold ($\delta > 0$), when the expected value of random jump size is higher than this threshold, the higher probability of random shocks encourages urban development. Otherwise, it deters urban development (when $E(j) < \delta$).

Similarly, there exists $\omega$ ($\omega < 0$). The higher probability of random shocks increases (decreases) the price of agricultural land when the expected value of random jump size is larger (less) than this threshold. Therefore, the higher probability of random shocks will increase the price of agricultural land if the expected value of random jump size is zero. In other words, the higher probability of random shocks imparts more option value to agricultural land when $E(j) = 0$. As to urban land, since there is no option value, it won't affect the price of urban land when $E(j) = 0$.

Second, we investigate how the spread of random jumps affects the urban development and the price of agricultural land.

Keeping other factors fixed, suppose the random jump size becomes more spread out, $J_1 = \zeta J_2$ ($\zeta > 1$) and its distribution doesn't change. We investigate how it will affect the urban development decision and the price of agricultural land.

Intuitively, if $E(J_2) \gg 0$, the more spread out the jump size is, the better news with shock occurrence will favorably happen. Therefore, it will encourage urban development and increase the price of agricultural land. If $E(J_2) \approx 0$, the more spread out, either better news or worse news with shock occurrence will favorably happen. The more uncertainty of bad news will deter urban development. However, the more uncertainty of good news will bring premium to agricultural land and increase the price of agricultural land. If $E(J_2) \ll 0$, the more spread out, the worse news with shock occurrence will favorably happens. As a result, it will deter urban development and decrease the price of agricultural land.

$^\dagger$ We call $\zeta$ the spread degree of random jump sizes. Obviously, $J_1$ is more spread out than $J_2$. 

19
Our findings support these intuitions. We show that there exists \( \kappa (\kappa > 0) \), when the expected value of random jump size is less than \( \kappa \), more spread of random jump will deter urban development. Otherwise, it will encourage urban development. Furthermore, there exists \( s (s < 0) \), when the expected value of random jump size is bigger (less) than \( s \), more spread of random jump size will increase (decrease) the prices of agricultural land. Therefore, more spread of random jump will increase the price of agricultural land when the expected value of random jump size is zero (option value). These findings can be summarized in the following theorem.

**Theorem 5:** Considering two cities (city 1 and city 2) with urban rents following (1), they are identical except their random jump sizes \( J_1 = \zeta J_2 \), where \( \zeta > 1 \), \( J_1 \) and \( J_2 \) are random jump sizes for city 1 and city 2, respectively; Then,

(i). There exists \( \kappa (\kappa > 0) \), \( E^2(J_2) \leq (\geq) \kappa \Rightarrow R_1^* \geq (\leq) R_2^* \).

(ii). There exists \( s (s < 0) \), \( E^2(J_2) \leq (\geq) s \Rightarrow P_1^A(t, z) \geq (\leq) P_2^A(t, z) \)

**Proof:** (i). Let's denote, \( \Theta(R, J_1) = \sum_{n=1}^{\infty} F_R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \)

\( R^*_2 \) must satisfy the following first order condition,

\[
(R_2^* + \frac{\lambda E^2(J_2)}{r} - A - rC)\Theta_R(R_2^*, J_2) + \Theta(R_2^*, J_2) = 0
\]

Similar to the proof of Theorem 4,

\[
\Delta = (R_2^* + \frac{\lambda E^2(J_2)}{r} - A - rC)\Theta_R(R_2^*, J_2) - \Theta_R(R_2^*, J_1) + \Theta(R_2^*, J_2) - \Theta(R_1^*, J_2) > 0
\]

Denoting \( \kappa = \frac{r\Delta}{(1-\zeta)A\Theta_R(R_2^*, J_1)} > 0 \), then we have

\[
E^2(J_2) \leq (\geq) \kappa \Rightarrow (R_2^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta_R(R_2^*, J_1) + \Theta(R_2^*, J_1) \leq (\geq) 0
\]

\( R_1^* \) must satisfy the following first order condition,

\[
(R_1^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta_R(R_1^*, J_1) + \Theta(R_1^*, J_1) = 0
\]
Second order condition yields,
\[
(R_2^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta(R_2^*, J_1) + \Theta(R_2^*, J_1) \leq (\geq) 0 \quad \Rightarrow \quad R_1^* \geq (\leq) R_2^*
\]

Therefore, \( E^2(J_2) \leq (\geq) \kappa \quad \Rightarrow \quad R_1^* \geq (\leq) R_2^*

(ii). \( \therefore \quad \Theta(R, J_2) < \Theta(R, J_1) \)
\[
\therefore \quad (R_2^* + \frac{\lambda E^2(J_2)}{r} - A - rC)\Theta(R_2^*, J_2) < (R_2^* + \frac{\lambda E^2(J_2)}{r} - A - rC)\Theta(R_2^*, J_1)
\]
\[< (R_2^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta(R_2^*, J_1) \quad \text{if} \quad E^2(J_2) \geq 0
\]
\[\leq (R_1^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta(R_1^*, J_1)
\]

When \( E(J) < 0 \) and becomes smaller, \( (R_1^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta(R_1^*, J_1) \) decreases faster than \( (R_2^* + \frac{\lambda E^2(J_2)}{r} - A - rC)\Theta(R_2^*, J_2) \), because \( E^1(J_1) = \zeta E^2(J_2), \zeta > 1 \).

Thus, there exists \( s = \frac{r((R_2^* - A - rC)\Theta(R_2^*, J_1) - (R_1^* - A - rC)\Theta(R_1^*, J_2))}{\lambda(1 - \zeta)(\Theta(R_2^*, J_2) - \Theta(R_1^*, J_1))} < 0
\)

And if \( E(J) \leq (\geq) \omega \), we have,
\[
(R_2^* + \frac{\lambda^2 E^2(J_2)}{r} - A - rC)\Theta(R_2^*, J_2) \leq (\leq) (R_1^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta(R_1^*, J_1)
\]

From (5), (6), (7), we know,
\[
P^A_1(t, z) = \frac{A}{r} + \frac{1}{r}(R_1^* + \frac{\lambda E^1(J_1)}{r} - A - rC)\Theta(R_1^*, J_1)
\]
\[
P^A_2(t, z) = \frac{A}{r} + \frac{1}{r}(R_2^* + \frac{\lambda E^2(J_2)}{r} - A - rC)\Theta(R_2^*, J_2)
\]

Therefore,
\[
E^2(J_2) \leq (\geq) s \quad \Rightarrow \quad P^A_1(t, z) \geq (\leq) P^A_2(t, z)
\]

Q.E.D
In Black and Scholes (1973), the option value of waiting always increases with the uncertainty. In our paper, the option value of waiting may decrease with uncertainty. For example, if the expected value of the jump size, \( E(J) < 0 \), the more spread of jump size will decrease the option value of waiting and consequently decrease the price of agricultural land.

Third, considering the random jump \( J_1 \) first degree stochastic dominates the random jump \( J_2 \), we study what's the difference between their urban development decisions and their prices of agricultural land. Intuitively, the urban development will be more active and the price of agricultural land will be higher in the random jump \( J_2 \) than the random jump \( J_1 \). The following theorem supports this intuition. Since the proof is very similar to the proof of Theorem 5, we just give the result.

**Theorem 6:** Considering two cities (city 1 and city 2) with urban rents following (1), they are identical except that \( f_1(J) \) in city 1 first-degree stochastic dominates \( f_2(J) \) in city 2; Then,

(i). \( R_1^* < R_2^* \)

(ii). \( P_1^A(t, z) > P_2^A(t, z) \).

**Section V Conclusion**

This paper investigates the effects of random shocks to urban development decision and the price of urban and agricultural land. In contrast to the existing literature that treats uncertainty as continuous time stochastic process, this paper considers uncertainty as discrete time stochastic process with two different sources of uncertainty. One is the uncertainty of time when shock occurs, and the other is the uncertainty of the size shock incurs.
Using the concept of stopping time, we solve the urban development condition and show its existence. Our main results are as follow: (1) the price of urban land is uniquely determined by the current urban rent, the mean arrival rate of Poisson process and the expected value of jump size. The variance of random jump doesn't affect the price of urban land. (2) If random jump is all positive, the urban development condition is: convert from agricultural land to urban use when the urban rent first hits agricultural rent plus capital opportunity cost \((A + rC)\). (3) If random jump is up and down with positive probability, we show that, (I). There exists a threshold \((\delta > 0)\), when the expected value of random jump size is bigger (smaller) than this threshold, the higher probability of random shocks encourages (deter) urban development. As to the price of agricultural land, there exists \(\omega \ (\omega < 0)\). The higher probability of random shocks increases (decreases) the prices of agricultural land, when the expected value of random jump size is larger (less) than this threshold. (II). There exist \(\kappa \ (\kappa > 0)\) and \(s \ (s < 0)\). When the expected value of random jump size is smaller (larger) than \(\kappa\), more spread of random jump size will deter (encourage) urban development. In addition, more spread of random jump will increase (decrease) the prices of agricultural land, when the expected value of random jump size is bigger (smaller) than \(s\). (III). If the bigger jump happens with the higher probability than used to be, it will encourage urban development and increases the price of agricultural land.
Reference:


Appendix 1

The Proof of Theorem 2,

From Theorem 2, we have,

\[
R^* = \arg \max \min \left\{ \frac{\lambda \int_{-\infty}^{\infty} f(j) \, dj}{(R + \frac{-\infty}{-\infty} - A - rC)[\sum_{n \in \Omega^R} F^R(n) \times \left(\frac{\lambda}{\lambda + r}\right)^n]} \right\}
\]

It's easy to show that \( \Omega^R_i \subseteq \Omega_{R_2} \) and \( F_{R_1}(n) < F_{R_2}(n) (\forall n \in \Omega^R_i) \), when \( R_i > R_2 \),

So,

\[
\sum_{n \in \Omega^R_i} F_{R_1}(n) \times \left(\frac{\lambda}{\lambda + r}\right)^n = \sum_{n \in \Omega^R_i \text{ and } n \in \Omega^R_i} F_{R_1}(n) \times \left(\frac{\lambda}{\lambda + r}\right)^n + \sum_{n \in \Omega^R} F_{R_1}(n) \times \left(\frac{\lambda}{\lambda + r}\right)^n
\]

Thus,

\[
\sum_{n \in \Omega^R} F^R(n) \times \left(\frac{\lambda}{\lambda + r}\right)^n \text{ decreases with } R \tag{10}
\]

Obviously,

\[
\frac{\lambda \int_{-\infty}^{\infty} f(j) \, dj}{(R + \frac{-\infty}{-\infty} - A - rC)} \text{ increases with } R
\]

Therefore, there may exist \( A \leq R^* < +\infty \), and \( R^* \) solves the following problem,

\[
\max_{R} \left\{ \frac{\lambda \int_{-\infty}^{\infty} f(j) \, dj}{(R + \frac{-\infty}{-\infty} - A - rC)[\sum_{n \in \Omega^R} F^R(n) \times \left(\frac{\lambda}{\lambda + r}\right)^n]} \right\}
\]

If \( R^* \) exists, urban development would occur after the rent first hits it. Here we should emphasize that \( R^* \neq +\infty \). If so, urban development will never occur, and implicitly there doesn't exist optimal timing of urban development. Next, we will prove that \( R^* \) always exists.

There are two possible cases,
\[
\begin{align*}
(1) & \quad \max_{R} \left\{ \lambda \int_{-\infty}^{\infty} f(j) dj \right\} (R + \frac{1}{r} - A - rC) \left[ \sum_{n=\Omega^R} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \right] = +\infty \quad \text{when } A \leq R^* \leq +\infty \\
(2) & \quad \max_{R} \left\{ \lambda \int_{-\infty}^{\infty} f(j) dj \right\} (R + \frac{1}{r} - A - rC) \left[ \sum_{n=\Omega^R} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \right] < +\infty \quad \text{when } A \leq R^* \leq +\infty
\end{align*}
\]

In the first case\(^{14}\), if we can prove

\[
\lim_{R \to +\infty} \left\{ \lambda \int_{-\infty}^{\infty} f(j) dj \right\} (R + \frac{1}{r} - A - rC) \left[ \sum_{n=\Omega^R} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \right] \neq +\infty, \text{ then } A \leq R^* < +\infty
\]

Actually,

\[
\lim_{R \to +\infty} \left\{ \lambda \int_{-\infty}^{\infty} f(j) dj \right\} (R + \frac{1}{r} - A - rC) \left[ \sum_{n=\Omega^R} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \right] = 0 \quad (11)
\]

In the second case,

\[
\therefore (R + \frac{1}{r} - A - rC) \left[ \sum_{n=\Omega^R} F^R(n) \times \left( \frac{\lambda}{\lambda + r} \right)^n \right] > 0 \quad \text{when } A \leq R < +\infty
\]

From (11), we can conclude that $A \leq R^* < +\infty$, which completes our proof.

Q.E.D

\(^{14}\) In reality, this case would never happen, because the prices of urban land wouldn't reach infinity. Here we consider this case just for mathematical completeness.