SELF-EXCITED VIBRATIONS OF ROTATING DISCS AND SHAFTS
WITH APPLICATIONS TO SAWING AND MILLING

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Abstract

This thesis investigates a number of major self-excited vibrations and instabilities that are involved in the interactions between moving flexible components, such as rotating discs or rotating shafts, and stationary systems, such as work-pieces or constrained systems. These investigations are applicable to a broad range of problems involved in the rotor-stator interactions.

As an application, an undesirable lateral self-excited vibration in wood cutting called washboarding is studied theoretically and experimentally. A thorough investigation in self-excited instabilities due to multiple moving interactions between the saw blade and work-piece such as regenerative cutting force, non-conservative following cutting forces and in-plane asymmetric stress fields are simulated and presented based on a number of new developments in modelling and algorithms.

A new theoretical approach for the prediction of chatter in milling is also proposed based on a dynamic milling model including a rotating spindle. The stability characteristics of this milling system are investigated by using the generalized Fourier series Method presented in this thesis.

A comprehensive experimental study on the self-excited lateral vibration in the wood cutting and washboarding phenomenon was conducted to verify the simulations. A side-cut experiment using the pendulum cutting rig was also conducted in order to support the modelling of regenerative cutting force.

To detect the self-excited vibration modes during cutting experimentally, several efficient methods and algorithms of identifying travelling modes, such as the Artificial Damping Method, are also proposed and presented in this thesis.
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Nomenclature

\( a, b \)  
inner and outer radii of the disc

\( \{b_k\}, \{a_k\} \)  
coefficients of Fourier series defined in (6.22)

\( [a_0], [a_1], [b_0], [b_1] \)  
coefficient matrices in condensed state space, defined in (8.44)

\( A \)  
cross sectional area

\( A_i \)  
modal residual defined in (8.19)

\( [A(t)] \)  
time dependent matrix associated with regenerative cutting forces, defined in (5.3)

\( [A_0], [A_1], [B_0], [B_1] \)  
coefficient matrices in the state space, defined in (8.35) and (8.38)

\( [A_r(t)], [A_{cr}(t)] \)  
time dependent matrix associated with stress fields produced by radial and tangential cutting forces, defined in (5.69) and (5.90)

\( [B_k], [A_k] \)  
coefficients of Fourier series defined by (5.7)

\( c_i, d_i, p_i, q_i \)  
modal parameters defined in (8.21)

\( c_{yy}, c_{yz}, c_{zy}, c_{zz} \)  
damping coefficients of the bearing

\( C_{mn}, S_{mn} \)  
generalized coordinates of plate defined in (2.28)

\( D \)  
bending rigidity of plate

\( [D_r(t)] \)  
time dependent matrix associated with radial follower cutting forces, defined in (5.47)

\( [D_t(t)] \)  
time dependent matrix associated with tangential follower cutting forces, defined in (5.36)

\( E \)  
Young's modulus

\( E_{mn} \)  
coefficients in radial mode shape functions (2.29)

\( E_s \)  
power of total vibrational energy of string or plate
$e^{-TD}$
time delay operator defined in (5.2)

$F_{B_i}, F_{B_{zi}}$ bearing forces defined by (6.16)

$F_n$ lateral non-conservative interactive force applied on the string

$F_{ij}(\theta_j)$ radial cutting force applied on the $j$th tooth

$F_{ij}(\theta_j)$ tangential cutting force applied on the $j$th tooth

$F_{ik}, F_{bn}, F_{ib}$ and $F_n$ circumferential components of interactive forces defined in (3.9)

$F_\mu$ friction force applied on the string

$f_c$ transverse cutting force applied on the saw-blade

$f_g$ interactive force due to the guide (constraint)

$f_t, f_l$ resonant frequency or tooth passing frequency of saw-blade

$\{f\}$ generalized force vector defined in (8.2)

$F_v, F_u$ cutting forces in space-fixed coordinates defined by (6.3) and (6.4)

$F_z$ general lateral interactive force applied on the disc

$g_j(\theta_j)$ step (or switch) function defined in (5.1)

$G = E /[2(1 + \nu)]$

$h$ thickness of disc or saw-blade

$h(\theta_j)$ chip thickness at the $j$th tooth in milling, defined by (6.2)

$H_{de}(j\omega)$ frequency response function defined by (8.16)

$H^{(1)}_{mnql}, H^{(2)}_{mnql}, H^{(3)}_{mnql}, H^{(4)}_{mnql}$ coefficients associated with tangential follower cutting forces, defined in (5.34) and (5.35)

$H^{(1)}_{mnql}, H^{(2)}_{mnql}, H^{(3)}_{mnql}, H^{(4)}_{mnql}$ coefficients associated with radial follower cutting forces, defined in (5.34) and (5.35)

$H_R(\omega), H_I(\omega)$ real and imaginary parts of measured FRF defined in (8.23)

$[H(t)]$ impulse response matrix defined by (8.30)

$[H_0], [H_1], [H_2], [L]$ system coefficient matrices defined by (5.14)-(5.17)
in-band and out-band FRF matrices in condensed space defined by (8.42)

in-band and out-band FRF matrices in physical space

moments of inertia of the rigid disc around X or Z axis

stiffness, mass and viscous damping coefficients of constraint

stiffness coefficients of the bearing

regenerative cutting force coefficient defined in (5.1)

cutting coefficients in tangential and radial directions, defined by (6.1)

stiffness and damping matrices of the bearing

membrane operators defined by (2.16) and (2.17)

= \(-L_a(w)\), membrane operator

residual function defined by (2.15)

modal participation matrix defined in (8.30)

numbers of nodal circles and nodal diameters of the disc

mass of the disc defined in (6.14)

bending moments and shear force in plate defined by (2.14)

system mass, stiffness, gyroscopic and damping matrices

system mass, stiffness, and damping matrices of the spindle

number of out-band modes used in (8.22)

number of saw teeth

coefficients associated with constraints, defined in (2.32) and (2.33)

rotational mass matrix of the beam

transverse force applied on the disc shown in Figure 8.2

modal parameter vector defined in (8.22)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>{Q}</td>
<td>generalized force vector defined by (4.9)</td>
</tr>
<tr>
<td>(Q_{rk}(r), Q_{\theta k}(r), Q_{\theta \theta k}(r))</td>
<td>coefficients of stress fields caused by radial cutting force, defined in (5.48)-(5.50)</td>
</tr>
<tr>
<td>(P(t))</td>
<td>transverse force applied on the disc</td>
</tr>
<tr>
<td>(q(r, \theta, t))</td>
<td>general distributed lateral force applied on plate</td>
</tr>
<tr>
<td>((r, \theta, z))</td>
<td>space-fixed polar coordinates</td>
</tr>
<tr>
<td>(R_i, R_o)</td>
<td>inner and outer radii of a given region in saw-blade</td>
</tr>
<tr>
<td>(R_{mn}(r))</td>
<td>radial mode shape functions of the disc</td>
</tr>
<tr>
<td>([R_1], [R_2])</td>
<td>coefficient matrices to be identified, defined in (8.45)</td>
</tr>
<tr>
<td>(S)</td>
<td>wave speed in the string</td>
</tr>
<tr>
<td>(S_i, S_o)</td>
<td>residual tensioning stresses at a given region in saw-blade</td>
</tr>
<tr>
<td>(S_L, S_U)</td>
<td>residual tensioning stresses at the boundaries of rolled region</td>
</tr>
<tr>
<td>([\tilde{S}])</td>
<td>Jacobian matrix defined in (8.29)</td>
</tr>
<tr>
<td>([S_{ap}])</td>
<td>auto-power spectrum matrix defined by (8.40)</td>
</tr>
<tr>
<td>([S_{im}])</td>
<td>spectrum matrix of imaginary parts of FRFs, defined by (8.41)</td>
</tr>
<tr>
<td>(T)</td>
<td>Tooth passing period</td>
</tr>
<tr>
<td>(T_e, P_e, U_e)</td>
<td>kinetic, potential or strain energy defined by (2.5), (2.6) or (3.1)</td>
</tr>
<tr>
<td>(T_D)</td>
<td>driving torque of the string defined in (3.10)</td>
</tr>
<tr>
<td>(T_k, T_m, T_c, T_n)</td>
<td>torques produced by circumferential forces defined in (3.9)</td>
</tr>
<tr>
<td>(T_{rk}(r), T_{\theta k}(r), T_{\theta \theta k}(r))</td>
<td>coefficients of stress fields caused by tangential cutting force, defined in (5.70)-(5.72)</td>
</tr>
<tr>
<td>([T_0], [T])</td>
<td>singular vector matrices defined in (8.40)</td>
</tr>
<tr>
<td>(u, v, w)</td>
<td>displacements in polar coordinates</td>
</tr>
<tr>
<td>(u_r(0,t), u_r'(0,t))</td>
<td>transverse velocities observed in space-fixed coordinates or in rotating coordinates</td>
</tr>
<tr>
<td>(\overline{u}_\theta(\theta,t))</td>
<td>average slope of the string at point (\theta)</td>
</tr>
</tbody>
</table>
right and left eigenvectors defined in (8.4) and (8.5)

$\{u\}_i, \{v\}_i$

displacement vector of the spindle defined in (6.17)

$[U_s(t)]$

transverse responses in space-fixed or plate-fixed coordinates

$w, w_r$

transverse velocity of the disc in plate-fixed coordinates

$\{w_d(t)\}$

artificially damped response defined by (8.18)

$\{w_h(t)\}$

resonant response of forced or self-excited vibration

$\{x\}$

generalized coordinate vector defined in (2.34)

$\{y\}$

state-space variable vector defined in (5.38)

$\{X(s)\}$

Laplace form of $\{x(t)\}$

$\{y\}$

state-space variable vector defined in (5.38)

$\{Y(s)\}$

Laplace form of $\{y(t)\}$

$\alpha, \beta$

constants defined in (2.32) and (2.33)

$[\Gamma(T,0)]$

transition matrix defined by (5.39)

$\delta = (v, u, \psi, \varphi)^T$

DOFs at one node of finite rotating shaft element

$\delta()$

Dirac delta function

$\delta_{qmt}^{(1)}, \delta_{qmt}^{(2)}, \delta_{qmt}^{(3)}$

coefficients associated with plate parameters, defined in (2.32) and (2.33)

$\{a\}$

change of modal parameter vector defined in (8.26)

$\Delta E$

total in-plane strains of the disc defined by (2.1)

$\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{r\theta}$

linear in-plane strains of the disc defined by (2.2)

$\varepsilon_{rrl}, \varepsilon_{\theta\theta l}, \varepsilon_{r\theta l}$

nonlinear in-plane strains of the disc defined by (2.3)

$\varepsilon_{rrm}, \varepsilon_{\theta\theta m}, \varepsilon_{r\theta m}$

artificial damping factor defined in (8.18)

$\zeta_a$

range of cutting sector defined in Figure 5.1

$\theta_{st}, \theta_{ex}$
\( \theta_j \) angular position of the \( j \)th tooth in space-fixed coordinates
\( \lambda \) characteristic variable of system
\( \lambda_i \) complex eigenvalue of the \( i \)th mode
\( \lambda_x \) wavelength of washboarding
\([\Lambda],[\Phi]\) eigenvalue and eigenvector matrices
\( \mu_k \) small constant at the \( k \)th iteration, defined in (8.29)
\( \nu \) Possion’s ratio
\( \rho \) mass per unit length or volume for the string or for the plate
\( \sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta} \) in-plane stresses of the disc defined by (2.4)
\( \sigma_i, \omega_i \) damping factor and natural frequency defined in (5.40)
\([\Sigma]\) singular value matrix
\( \phi_{di}(r_d, \theta_d) \) complex right eigenvector at \( (r_d, \theta_d) \) defined in (8.16)
\( \Phi = 12EI / kAGL^2 \) transverse shear coefficient of beam
\([\varphi_p],[\varphi_c]\) mode shape matrices defined by (8.47)
\( \Psi \) circumferential velocity of load in space-fixed coordinates
\( \omega \) angular frequency
\( \omega_{ok} \) angular natural frequency of the \( k \)th out-band mode
\( \Omega \) rotation speed
\( \nabla^4 \) bi-harmonic operator of plate defined by (2.8)

**Subscripts**

, partial differentiation
g, l, m, n mode number used in (2.32) and (2.33)

**Superscripts**

* complex conjugate
Acknowledgments

I greatly appreciate the support, the trust and the advice which I have received from my supervisor, Professor Stanley Hutton, throughout my graduate studies. I would like to thank Professors Yusuf Altintas and Gary Schajer for many useful discussions which I have enjoyed in the past and during the development of this work. Professor Ian Yellowley also deserves special thanks for his kindness and advice on the pendulum cutting test. I would also like to thank Dr. John Kishimoto for his assistance in the experiments conducted during this work.

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Chapter 1

Introduction

1.1 Background

Self-excited vibration and instabilities in rotating machines, such as a rotating circular disc or a rotating shaft subjected to various interactive excitations, have been a very active topic for many decades. Industrial applications involve wood-cutting saw-blades, computer recording disks, silicon wafer slicing cutters, rotor systems and metal milling. The present work aims to investigate several important aspects of self-excited vibration and instabilities in saw-blade wood cutting and metal milling cutting. This subject is a fascinating blend of theoretical approaches and experimental investigations.

Saw blade vibration, which arise from interactions and disturbances in the production environment, will always exist and directly contributes to production problems such as poor cutting accuracy, poor surface quality, excessive raw material waste, short tool life and high noise level. Therefore, efficient wood sawing is being pursued with the objective to reduce saw blade vibration and reduce sawdust. Because of the existence of various cutting and interactive forces between the cutter and the work-piece or other constraining systems, under certain circumstances, the saw blade can be self-excited. The self-excited vibration of the blade can not only widen the cutting path but can also produce an undesired washboarding pattern on the wood surface when the saw-blade is fully excited.

The term washboarding is used to describe a sinusoidal-like pattern on the wood cutting surface, as shown in Figure 1.1. This is a common problem in the lumber industry. This dimensional variation on the wood surface is generally undesirable because the sawn
lumber must be made thicker than normal to allow the planer to produce smooth surface, which may lead to significant material loss and labour waste.

The lateral vibration of the saw blade can be affected by many factors such as the dynamics of the saw-blade, the geometry of the tooth and the wood, the property and position of the wood and the operational conditions. When these factors are badly set, the interactive cutting forces can be in phase with the absolute velocity of the blade, thus, unstable cutting with large lateral vibration can occur. In this case the work done by the cutting forces becomes positive, which implies that the driving energy is switched into the transverse vibration energy of the blade.

Several distinct fundamental instability mechanisms in saw-blade cutting may be involved in the self-excited vibration which may cause the washboarding phenomenon. Therefore, in order to find out the dominant mechanisms in washboarding, all the possible instabilities due to different cutting forces involved in the saw-blade cutting will be investigated in this thesis.

Similarly, in metal cutting the self-excited vibration called machine tool chatter can also be induced by the interaction between the cutting tool and the work-piece. In a closed-loop milling system, if the cutting forces are in phase with the velocities of the rotating spindle at the cutting points observed in the spindle-fixed coordinates, the driving energy of the system
can be switched into the vibration modes of the spindle, then the amplitude of the milling cutter's vibration with respect to the work-piece may reach an unacceptable level. Chatter not only limits the rates of material removal but also affects both the surface finish and the dimensional accuracy of the work-piece. Furthermore, it may promote tool wear and even cause tool and work-piece damage. This thesis also deals with analysis of the self-excited vibration in milling using a general dynamic milling model with the gyroscopic effect of the rotating spindle.

1.2 Physics of Self-Excited Vibration in Sawing and Milling

![Diagram](image)

Figure 1.2: A rotating disc (a) or a rotating shaft (b) which interacts with a stationary constraint or work-piece

Circular sawing systems, milling machines and other rotating machines contain rotating discs or shafts which interact with other media, structure or work-pieces. Under certain circumstances, in such systems, the driving energy for the steady rotation can be channeled into lateral vibration modes, which may lead to an unstable behavior of the
system. In order to prevent such energy transmission, an understanding of these instability mechanisms is essential.

Figure 1.2(a) shows schematically a circular plate rotating at a constant angular velocity \( \Omega \), in contact with a stationary spring-mass-dashpot constraint or a feeding work-piece. The governing equation for transverse vibration of the blade in terms of the lateral displacement \( w(r, \theta, t) \) with respect to the space-fixed polar coordinates \( (r, \theta, z) \) can be expressed as (Chen and Bogy, 1992 [13]):

\[
DV^4 w + \rho h \left( w_{,tt} + 2\Omega w_{,\theta} + \Omega^2 w_{,\theta\theta} \right) + L_s(w) = L_n(w) + f_g(t) + f_c(t)
\]

where, \( D \) and \( \rho \) are the flexural rigidity and the mass density of the plate, respectively. \( DV^4 w \) represents the bending stiffness of the plate. \( \rho h \left( w_{,tt} + 2\Omega w_{,\theta} + \Omega^2 w_{,\theta\theta} \right) \) represent the inertia, gyroscopic and centrifugal forces of the plate, respectively. \( L_s \) is the membrane operator associated with the axisymmetrical stress field due to the centrifugal force and/or the stress tensioning (Hutton, et al., 1987 [5]). \( L_n \) is the membrane operator associated with the asymmetric stress fields generated by in-plane edge forces, such as in-plane cutting forces (Chen, 1994 [21]). \( f_g(t) \) represents the transverse forces generated by the interaction between the plate and the stationary mass-spring-dashpot constraints including friction. \( f_c(t) \) represents the transverse cutting forces generated by the interaction between the disc and the work-piece. Three types of cutting forces are considered in this thesis. They are regenerative cutting forces, the lateral components of follower cutting forces and the asymmetric stress fields generated by in-plane cutting forces moving with the blade.

The change of the vibration energy \( \Delta E \) of the plate caused by the interactive forces is given by (see Chapter 4 for the details):

\[
\Delta E = \int_0^T \iint_A \left( F_c(t) \dot{w}_r \right) r dr d\theta dt
\]
where, \( F_z(t) = L_n(w) + f_{g}(t) + f_{c}(t) \) denotes a general transverse force. \( \dot{w}_r \) represents the transverse velocity at the interactive point, observed in the disc-fixed coordinates.

It is evident from the expression of \( \Delta E \) that the change of total vibration energy \( \Delta E \) of this system will increase when \( F_z(t) \) is in phase with \( \dot{w}_r \). In this case, the driving energy for the steady rotation will be transferred into transverse vibration. In the case that the energy transferred from the driving device is greater than the energy dissipated by the system, unstable behavior will occur. Thus this equation gives the basic relationship which provides physical insight into the stability characteristics of the system.

In the case that a circular saw blade cuts a work-piece, although the energy of excitation is originally provided by the driving motor, the saw-blade is directly excited by the interactive cutting forces which are also the function of blade response. Therefore, this type of vibration is called self-excited vibration.

Figure 1.2(b) illustrates schematically a milling cutter mounted on the rotating spindle, which interacts with a feeding work-piece. Similarly, large self-excited vibration of the spindle will result if the work done by the milling force becomes positive when the milling force \( F_m(t) \) is in phase with the velocity of the spindle \( V_r \) at the cutting point, observed in the spindle-fixed coordinates.

1.3 Previous Research

In the past three decades, numerous papers have been published on the dynamics of rotating discs. Most of them have focused on the mathematical modeling and the solution for a specific interactive force, such as an elastic force produced by a stationary spring or a damping force from a stationary viscous damper. Few of them have looked into the instability mechanisms.

Mote et al. (1970 [10] and 1977 [11]) and Iwan and Stahl (1973 [17]) studied the instability of a stationary disc with a moving load system. Iwan and Moeller (1976 [14]), Hutton et al. (1987 [5]), and many other researchers mathematically investigated the
onset of instability for a rotating disc subjected to a stationary constrained system. Ono.
et al. (1991 [12]) and Chen and Bogy (1992 [13]) also looked into the instabilities of a
rotating disc subjected to a mass-spring-dashpot constraint with a friction effect. Shen
and Mote’s work (1991 [15]) appears to be the first paper attempting to present an
explanation of the instability mechanisms for a stationary circular plate subjected to a
rotating damped spring-mass system. In general, the physical mechanisms involved in
such rotating systems have not been adequately investigated and explained by previous
researchers, especially for instability caused by an arbitrary interactive force.

Although much research has been done in circular saw dynamics in the past two
decades, most of the research has focused on saw dynamics during idling. The self-
excited vibrations and instabilities due to the moving interactive cutting forces under
cutting conditions remain unsolved and an understanding of the dynamics of
washboarding was not available.

Carlin, et al. (1975 [7]) first studied the effects of a concentrated radial edge force on
the natural frequencies of a spinning disc, and it was found that the asymmetric stress
fields caused by the edge force can change the natural frequencies of the disc, and
therefore the critical speed (where the natural frequency of a given mode of the disc
becomes zero, observed in the space-fixed coordinates). Radcliffe and Mote (1977 [18])
considered a more general concentrated edge force with both radial and tangential
components. Chen and Bogy (1993 [19]) formulated the membrane stress fields in a
spinning disc produced by a stationary circumferential friction force, and it was found in
their paper that these asymmetric membrane stress fields cannot cause instability in a
spinning disc. Chonan et al. (1993 [20]) studied the self-excited vibration of a pre-
tensioned inner-teeth saw blade subjected to in-plane slicing force, and their results also
showed no significant effect of in-plane force on the natural frequencies and stability of
the blade because the cutting forces used in their study are too small to exhibit the
coupling instability regions.
Chapter 1. Introduction

Chen (1994 [21]) reformulated the equations for a rotating disc subjected to a stationary concentrated in-plane radial force with the effects of gyroscopic terms included. It was reported in his paper that both divergence and flutter instabilities can be induced by the stationary in-plane radial force at critical or supercritical speeds.

In an independent study, Shen and Song (1996 [22]) treated a cutting saw blade as a rotating disc subjected to a general stationary follower edge forces with both radial and tangential components and predicted the instabilities of this system through the multiple scale method. It was concluded in their paper that the asymmetric membrane stresses resulting from the stationary in-plane edge forces will transversely excite the disc to single-mode resonances as well as coupling resonances at critical and supercritical speeds. It was also concluded from their simulations that the radial edge force determines the rotating speeds where the instability occurs and the tangential edge force determines the width of the instability zones without introducing new unstable regions.

Recently Chen (1997 [23]) modeled a saw system as a spinning disc under a space-fixed periodically varying edge force as shown in Figure 1.3 and investigated the dynamic stability of a rotating disc subject to a pulsating in-plane radial force by using an extended multiple scale method.

In this thesis a more general cutting model of multiple moving concentrated cutting forces over a given sector in the space-fixed coordinates with the effect of the wood grain direction, as shown in Figure 1.3, was considered. The equation of motion for a spinning disc under multiple moving in-plane forces at the outer rim was first derived and the basic Fourier series method was extended to solve the stability problem for the equation with time-varying and time delay terms, which is called Extended Fourier Series Method.

In previous research the self-excited vibrations and instabilities in saw-blade cutting were assumed to be induced by a in-plane edge force. However, a more important cutting force, called lateral regenerative cutting force which may dominate the dynamics of saw-blade cutting has not been modeled in the previous studies. In cutting tests, it has been
found that self-excited vibration in the saw-blade cutting may be primarily produced by the lateral cutting forces due to the flank cutting of the teeth.

![Diagram of saw blade cutting forces](image)

Figure 1.3: Various cutting forces: stationary constant ($F_r = F_{r0}$), pulsating ($F_r = F_{r0} \sin \omega t$) or multiple moving forces ($F_{r0}$ is a constant force)

It can be seen from the A-A plot in Figure 1.4 that there is an extra lateral cutting area between two successive teeth if the blade oscillates laterally, which causes the lateral cutting force $f_c(t)$ associated with both the transverse response $w(t)$ of the current tooth and the transverse response $w(t - T)$ of the previous tooth at the same stationary location $\theta_A$ on the work-piece. Therefore it is called regenerative cutting force. This concept was originally introduced into the dynamics of saw-blade cutting from metal cutting dynamics, which may also induce unstable behavior in saw-blade cutting when the lateral cutting force is in phase with the velocity observed in rotating coordinates.
Figure 1.4: A rotating circular saw blade with lateral regenerative cutting forces produced by flank cut

Machine tool chatter is also caused by the interaction between the machine tool cutter and the work-piece. The fundamental chatter theory has been developed by Tobias and Fishwick (1958) [43] and Tlusty and Polacek (1963) [44]. Merritt (1965, [45]) presented a systematic theory for the stability analysis of the regenerative chatter based on Nyquist stability criterion in the well-established feedback control theory. Sridhar et al. [46, 49, 50] presented a comprehensive insight about milling dynamics and derived the detailed mathematical model with the time varying cutting force coefficients in milling.

Minis and Yanushevsky (1990 [51], 1993 [27]) reduced the stability analysis for the equation of motion with periodic coefficients and time delay to a analysis of a finite order characteristic equation with constant coefficients based on Floquet’s theorem [52]. In an independent study, Budak (1994, [53]) and Altintas and Budak (1995, [28]) also derived the finite-order characteristic equation for the stability analysis in milling through a practical
approach with clear physical meaning. From the literature review it was found that almost all the researchers voluntarily or involuntarily have employed Nyquist stability criterion and the concept of feedback control based on Tobias, Tlusty and Merritt' work to conduct the stability analysis in the frequency domain. A more detailed literature review on this topic can be found in Chapter 6.

In this thesis, a new dynamic milling model including a rotating spindle is developed and the generalized Fourier series method for eigenvalue analysis is proposed to conduct the stability analysis in milling. This approach reduces the stability analysis for the equation of motion containing periodic coefficients and time delay terms to an eigenvalue analysis of a finite order characteristic equation expressed by the system matrices. From this approach the chatter instability can be effectively predicted for the milling systems with both the time delay inherent and the gyroscopic effect of the rotating spindle. The gyroscopic effect of the spindle will make the natural frequencies of the system become rotating speed dependent and make one bending vibration mode of the spindle split into two modes (i.e., the forward-wave and the backward-wave modes). In this case, the chatter frequencies and the real parts of eigenvalues (i.e., the stability characteristics of the milling system) must be solved simultaneously at a given rotating speed of the spindle.

1.4 Objectives and Scope

This thesis investigates a number of self-excited vibrations and instabilities that are involved in the interactions between moving flexible components, such as rotating discs or rotating shafts, and stationary systems, such as work-pieces or constrained systems. The aim is predict how the saw blade and the milling cutter, the various cutting forces and the operating parameters affect the self-excited vibrations and instabilities in such systems.

To meet the above stated general objectives, the approach taken in this thesis consists of the following sub-objectives:
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1) To first use a relatively simple mathematical model involving an idealized rotating string constrained at one point to develop a clear understanding of the basic physical mechanisms of instabilities involved in such a system. Such an analysis can be expected to have relevance in the understanding of the behaviour of more complex systems.

2) To establish a generalized approach to study the physical mechanisms of instabilities caused by the interaction between a rotating flexible disc and a constraint system or a stationary work-piece.

3) To find the dominant instability mechanism involved in saw-blade cutting based on simulations of dynamic models of a rotating disc subjected to various interactive forces.

4) To develop the solution for the stability analysis of a rotating disc subjected to multiple moving concentrated regenerative cutting forces; the lateral components of follower cutting forces; and the asymmetric stress fields produced by in-plane cutting forces over a given sector of the work-piece in the space-fixed coordinates.

5) To extend the knowledge and the methods developed in saw-blade cutting into milling cutting, and to investigate the chatter instability mechanism of a rotating spindle with a milling cutter which interacts with a stationary work-piece based on the generalized Fourier series method.

6) To experimentally investigate the washboarding phenomenon under various operating conditions and to find out which type of cutting force is mainly responsible for the self-excited vibrations in saw-blade cutting. A supporting cutting test needs to be conducted to prove the existence of lateral cutting force due to flank cut.

7) To identify the travelling-wave modes excited by the interactive cutting forces which are not available. An effective method for the modal analysis of high-speed rotating systems has to be developed for supporting the whole experimental work in this research.
This thesis is presented in nine chapters. Chapter 2 is devoted to introducing the theoretical background of dynamics of a rotating disc subjected to a general interactive force.

Chapters 3 and 4 are concerned with the instability mechanisms of an idealized rotating circular string and a rotating disc subjected to a general interactive force. Through this effort, the possible occurrence of instability due to a general interactive force can be identified based on the energy flux analysis proposed in this research. According to the equations derived for energy flux analysis, the unified conditions for predicting instability occurrence are presented.

Chapters 5 and 6 are the central part of this thesis. In Chapter 5, stability analyses of a rotating disc subjected to various interactive forces including:

- interactive elastic, inertial, damping and friction forces due to a stationary constrained system;
- multiple moving lateral regenerative lateral forces caused by flank cuts;
- multiple moving lateral components of follower radial and tangential cutting forces; and
- asymmetric membrane stress fields resulting from multiple moving in-plane cutting edge forces

are presented. In this chapter the Fourier series method is generalized to solve the time-varying equations with time lag or without time lag. The point mapping scheme is also employed to conduct the stability analysis for the time-varying system without time delay.

Chapter 6 deals with the instability analysis of a rotating spindle with a milling cutter which interacts with a stationary work-piece. This topic enables the author to treat the self-excited vibrations in cutting processes as a whole. Through this study, a number of important and interesting results are presented in this chapter. For example, it was found from this analysis that the backward-wave modes of the rotating spindle are destabilized
by the interactive cutting forces in most of rotating speed regions but the forward-wave modes are primarily stabilized in the entire range of the speed.

A comprehensive experimental investigation on washboarding instability using a real circular saw rig and a supporting test for verifying the existence of transverse cutting force due to flank cutting of the blade using a pendulum cutting rig were conducted and the results are presented in Chapter 7. The kinematics of washboarding are also studied in this chapter. Based on the theoretical and experimental results, the major instability mechanism that dominates washboarding phenomenon in the saw-blade cutting can be discovered.

An effective algorithm for identifying travelling-wave modes for high-speed rotating systems, called Artificial Damping Method, is proposed in Chapter 8. This method can deal with the modal analysis for the self-excited systems where the excitation information is not available. A multiple reference modal identification method is also presented to identify two modes with identical natural frequencies but different mode shapes for a circular saw-blade.

Finally Chapter 9 presents conclusions and suggestions for further investigation.
Chapter 2

Dynamic Model of a Rotating Disc Subjected to Interactive Forces

2.1 Introduction

In this chapter, the equations of flexural motion of a rotating disc with transverse interactive forces and in-plane stress fields are derived in discrete form in space-fixed coordinates. In the formulation, the strain energy due to bending as well as the middle plane stretching are taken into account, which is considered to be realistic in the case of large amplitude vibrations of the disc. The energy approach (Hamilton’s principle) is applied to obtain the governing partial differential equation of motion from the expressions of potential and kinetic energies of the system.

Finally, Galerkin’s method is employed to determine a set of linear ordinary differential equations in the case of small deflections.

2.2 Equations of Motion

A rotating disc subjected to both transverse and in-plane interactive forces is illustrated in Figure 2.1. The space-fixed coordinate system \((r, \theta, z)\) is employed to describe this system. The model is restricted by the following assumptions:

1) The disc is homogeneous and isotropic.

2) The rotation speed of the disc is constant.

3) The plate thickness is small compared to the radius of the disc.

4) Rotational inertia and shearing deformation terms are small and can be neglected.
5) The linear strains and the squares of angles of rotation (whose definitions can be found in Equations (1-15) and (1-18) in reference [1]) are small compared to unity.

6) The angle of rotation about the z axis is negligibly small.

7) The stress in the z direction is much smaller than those in the r and θ directions (i.e., plane stresses).

Figure 2.1 A rotating disc subjected to transverse interactive forces (q or $F_z$)

These assumptions lead to the following strain-displacement relationship:

$$
\varepsilon_{rr} = \varepsilon_{rrl} + \varepsilon_{rrn}
$$

$$
\varepsilon_{\theta\theta} = \varepsilon_{\theta\theta l} + \varepsilon_{\theta\theta n}
$$

$$
\varepsilon_{r\theta} = \varepsilon_{r\theta l} + \varepsilon_{r\theta n}
$$

where the linear membrane and bending strains are [2]:

$$
\varepsilon_{rrl} = u_r - zw_{rr}
$$

$$
\varepsilon_{\theta\theta l} = \frac{1}{r}(u_\theta + v_\theta) - \frac{z}{r^2}(rw_{,r} + w_{,\theta})
$$

$$
\varepsilon_{r\theta l} = \frac{1}{r}(u_\theta - v) + v_r - \frac{2z}{r^2}(rw_{,r\theta} - w_{,\theta})
$$

where $u$, $v$, and $w$ are the radial, tangential and transverse displacements in the middle plane of the disc, respectively.
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The non-linear membrane strains can be expressed as [1]:

\[ \varepsilon_{rrn} = \frac{1}{2} w_{,r}^2 \]
\[ \varepsilon_{\theta\theta n} = \frac{1}{2r^2} w_{,\theta}^2 \]
\[ \varepsilon_{r\theta n} = \frac{1}{r} w_{,r} w_{,\theta} \] (2.3)

For plane stresses:

\[ \sigma_{rr} = \frac{E}{1-\nu^2} (\varepsilon_{rr} + \nu \varepsilon_{\theta\theta}) \]
\[ \sigma_{\theta\theta} = \frac{E}{1-\nu^2} (\varepsilon_{\theta\theta} + \nu \varepsilon_{rr}) \] (2.4)
\[ \tau_{r\theta} = G \varepsilon_{r\theta} \]

where \( E \) and \( \nu \) are Young’s modulus and Poisson’s ratio, respectively. \( G = \frac{E}{2(1+\nu)} \).

It is now necessary to obtain expressions for the kinetic energy and the potential energy of the rotating disc. The potential energy of the disc can be expressed as [2]:

\[ P = \int_0^{2\pi} \int_a^b \left[ \frac{1}{2} (\sigma_{rr} \varepsilon_{rr} + \sigma_{\theta\theta} \varepsilon_{\theta\theta} + \tau_{r\theta} \varepsilon_{r\theta}) - \rho r \Omega^2 u \right] dz - q w \] (2.5)

where \( \rho \) is the mass density of the disc. \( \Omega \) is the rotating speed of the disc. \( q \) is a general distributed transverse load applied on the disc. It is also possible to include any external concentrated interactive force, such as a cutting force \( F_z \), by means of \( q = F_z \delta \) (\( \delta \) is a singularity function). \( h \) represents the thickness of the disc and \( a \) and \( b \) are the inner and outer radii of the disc, respectively.

The kinetic energy of the disc is given by:
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\[ T = \int_0^{2\pi} \int_a^b \frac{1}{2} \rho \left[ (w_{,t} + \Omega w_{,\theta})^2 + (u_{,t} + \Omega u_{,\theta})^2 + (v_{,t} + \Omega v_{,\theta} + r\Omega)^2 \right] r dz dr d\theta \]  

(2.6)

Hamilton's principle:

\[ \delta \int_{t_1}^{t_2} (T - P) dt = 0 \]  

(2.7)

is then applied to develop the governing equations and the boundary conditions.

Substitution of Equations (2.5) and (2.6) into Equation (2.7) yields the following governing equations and boundary conditions in a space-fixed coordinates after a lengthy simplification procedure [3, 4]:

**Governing Equations:**

\[ D \nabla^4 w + \rho (w_{,tt} + 2\Omega w_{,\theta t} + \Omega^2 w_{,\theta \theta}) = \frac{h}{r} \left[ (r \sigma_{rr}, w_{,r} + \tau_{r\theta} w_{,\theta}), r + (\tau_{r\theta}, w_{,r} + \frac{1}{r} \sigma_{\theta \theta} w_{,\theta}), \theta \right] \]

\[ = q \]  

(2.8)

\[ \rho (u_{,tt} + 2\Omega u_{,\theta t} + \Omega^2 u_{,\theta \theta}) = \sigma_{rr}, r + \frac{1}{r} \tau_{r\theta}, \theta + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta \theta}) + \rho \Omega^2 r \]  

(2.9)

\[ \rho (v_{,tt} + 2\Omega v_{,\theta t} + \Omega^2 v_{,\theta \theta}) = \tau_{r\theta}, r + \frac{1}{r} \sigma_{\theta \theta}, \theta + \frac{2}{r} \tau_{r\theta} \]  

(2.10)

where

\[ \nabla^4 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \]  

is the bi-harmonic operator.

\[ D = \frac{Eh^3}{12(1-\nu^2)} \]  

is the bending rigidity.

**Boundary Conditions:**

\[ (u, v, w, w_{,r}) |_{r=a \text{ or } b} = 0 \]  

(2.11)
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\[(\sigma_{rr}, \tau_{r\theta}, M_{rr}, V_r)|_{r=a \text{ or } b} = 0\]  

(2.12)

where \( V_r \) is the transverse force at the boundary \((r = a \text{ or } b)\):

\[V_r|_{r=a \text{ or } b} = Q_{rr} + \frac{1}{r} M_{r\theta,\theta}|_{r=a \text{ or } b}\]  

(2.13)

The bending moments \( M_{rr} \) and \( M_{r\theta} \) and the shear force \( Q_{rr} \) are given by:

\[M_{rr} = -D\left[w_{rr} + \frac{V}{r^2}(w_{\theta\theta} + r w_r)\right]\]

\[M_{r\theta} = -D(1-v)\frac{1}{r^2}\left[rw_{r\theta} - w_{\theta}\right]\]  

(2.14)

\[Q_{rr} = -D(\nabla^2 w)_r\]

Equations (2.8) - (2.12) describe the nonlinear response of the rotating disc in terms of the coupled displacements \( u, v, w \) in the three coordinate directions.

For a thin disc with small deflections (i.e., the deflections of the plate are small in comparison with its thickness), only the equation of motion governing the transverse deflection \( w \) of the disc is needed to describe its dynamics. In this case the stress terms and displacements in Equation (2.8) become decoupled so that these stress-related terms only represent the stress fields induced by in-plane interactive forces such as in-plane cutting forces, or denote rotational centrifugal stresses and tensioning membrane stresses. For the linear case Equation (2.8) can be rewritten in the form:

\[L(w) = D\nabla^4 w + \rho h (w_{tt} + 2\Omega w_{r\theta} + \Omega^2 w_{\theta\theta}) + L_s(w) + L_a(w) - q(r, \theta, t) = 0\]  

(2.15)

where, \( L_s(w) = -\frac{h}{r}\left[\frac{(r \sigma_r w_r)_r}{r} + \frac{1}{r}(\sigma_{\theta\theta} w_{\theta})_\theta\right] \)

(2.16)

\[L_a(w) = -\frac{h}{r}\left[\frac{(r \sigma_r w_r + \tau_{r\theta} w_{\theta})_r}{r} + (\tau_{\theta\theta} w_r + \frac{1}{r}\sigma_{\theta\theta} w_{\theta})_\theta\right]\]  

(2.17)
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$L_s$ is the membrane operator associated with the symmetrical stress fields due to the centrifugal force and/or the residual stress tensioning. $L_a$ is the membrane operator associated with the asymmetric stress fields produced by in-plane edge forces, such as in-plane cutting forces.

$q(r, \theta, t)$ represents a general transverse force, which can be a single force or multiple transverse interactive forces. For a stationary mass-spring-dashpot system attached to the rotating disc, the interactive transverse force can be conveniently handled in the equation of motion in the following form [5]:

$$q(r, \theta, t) = -(m\omega^2 + c\omega + k\omega)(1/r)\delta(r - r_0)\delta(\theta - \theta_0)$$

(2.18)

where, $\delta(\cdot)$ is the Dirac delta function and $(r_0, \theta_0)$ defines the location of a stationary constraint in space-fixed coordinates. $k$, $m$ and $c$ are the stiffness, mass and viscous damping coefficients, respectively.

A general moving lateral force can be expressed as:

$$q(r, \theta, t) = (1/r)\delta[r - r_0(t)]\delta[\theta - \theta_0(t)]p(t)$$

(2.19)

where, $p(t) = P$ for a constant force and $p(t) = P\sin(\omega t + \psi)$ for a harmonic force. In the case of self-excited vibration, $p(t)$ can be a function of the displacement, velocity and/or acceleration of the disc.

2.3 Membrane Stress Fields due to Rotation and Tensioning

2.3.1 Centrifugal Stress Fields due to Rotation

The rotation of the disc about its central axis causes in-plane radial and circumferential stresses, $\sigma_{rr}$ and $\sigma_{\theta\theta}$. They are determined by solving Equation (2.9) where the in-plane inertia terms are neglected and in the case of axial symmetry $\tau_{r\theta} = 0$. In this case Equation (2.9) becomes [5]:

$$...$$
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\[ \sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho\Omega^2 r = 0 \]  \hspace{1cm} (2.20)

For this case the symmetric in-plane stresses can be written from Equation (2.4) as:

\[ \sigma_{rr} = \frac{E}{1 - v^2} \left( u_r + v \frac{\dot{u}}{r} \right) \]
\[ \sigma_{\theta\theta} = \frac{E}{1 - v^2} \left( -\frac{\dot{u}}{r} + v u_r \right) \]  \hspace{1cm} (2.21)

where \( u \) is the in-plane radial displacement.

For a disc clamped at the inner radius and free at the outside edge the boundary conditions are:

\[ u(a) = 0 \hspace{0.5cm} \text{and} \hspace{0.5cm} \sigma_{rr}(b) = 0 \]  \hspace{1cm} (2.22)

Solving Equations (2.20) and (2.21) with the boundary conditions (2.22) gives the centrifugal stress fields:

\[ \sigma_{rr} = (1/8)\rho b^2 \Omega^2 (C_1 + C_2 - C_3) \]
\[ \sigma_{\theta\theta} = (1/8)\rho b^2 \Omega^2 (C_1 - C_2 - C_4) \]  \hspace{1cm} (2.23)

where,

\[ C_1 = [(1 + v)(3 + v) + (1 - v^2)(a/b)^4]/[(1 + v) + (1 - v)(a/b)^2] \]
\[ C_2 = [a^2/r^2][(1 - v)(3 + v) - (1 - v^2)(a/b)^2]/[(1 + v) + (1 - v)(a/b)^2] \]
\[ C_3 = (3 + v)(r/b)^2 \hspace{0.5cm} \text{and} \hspace{0.5cm} C_3 = (1 + 3v)(r/b)^2 \]

2.3.2 Residual Stress Fields due to Tensioning

A rotating clamped disc with in-plane residual stresses is shown in Figure 2.2(a). The disc has been cold rolled so that plastic strains in the region \( R_L - R_U \) are induced. This process is used to optimize the lateral stiffness of circular saw blades. Research (Schajer, 1981, [6]; Carlin et al., 1975, [7]) shows that significant increases in blade stiffness can be achieved...
through the proper application of residual tensioning stresses. Outside this region the stresses can be determined in terms of radial stresses $S_L$ and $S_U$ on the boundaries of the cold rolled region which is narrow compared with its mean radius [6]. Stresses $S_L$ and $S_U$ can be related to the rolling process and the stress fields in the blade can be easily calculated once $S_L$ and $S_U$ are determined.

The solutions for the stress fields in the elastic regions I and III (a general case is shown in Figure 2.2(b)) in terms of the boundary radial stresses $S_i$ and $S_o$ follow from the Lamé equation as [6]:

$$
\sigma_{rr} = A_1 - \frac{A_2}{r^2} \quad \text{and} \quad \sigma_{\theta\theta} = A_1 + \frac{A_2}{r^2}
$$

(2.24)

with the boundary conditions:
\[ \sigma_{rr} = S_i \quad \text{at} \quad r = R_i \quad \text{and} \quad \sigma_{rr} = S_o \quad \text{at} \quad r = R_o \] (2.25)

Substitution of the boundary conditions (2.25) into Equation (2.24) yields the resultant tensioning stress fields in the elastic regions I and III:

\[
\sigma_{rr} = \frac{S_o [R_o^2 r^2 - R_i^2 R_o^2] - S_i [R_i^2 r^2 - R_i^2 R_o^2]}{(R_o^2 - R_i^2) r^2}
\]

\[
\sigma_{\theta\theta} = \frac{S_o [R_o^2 r^2 + R_i^2 R_o^2] - S_i [R_i^2 r^2 + R_i^2 R_o^2]}{(R_o^2 - R_i^2) r^2}
\] (2.26)

According to the equilibrium condition and the assumption that the width of plastic region II is very small compared with its radius, the tensioning stress fields in the region II can be simplified and expressed in the following form [6]:

\[
\sigma_{rr} = \frac{S_U + S_L}{2}
\]

\[
\sigma_{\theta\theta} = \frac{(S_U - S_L)(R_U + R_L)}{2(R_U - R_L)}
\] (2.27)

The axisymmetric stress fields due to centrifugal forces and initial stress tensioning can be added through Equation (2.16) to determine the magnitude of their effects on the dynamic buckling and dynamic characteristics of the disc. The asymmetric stress fields produced by in-plane cutting forces and their effects on the dynamics of the blade will be investigated and discussed in later chapters.

### 2.4 Solution Method

The governing equation (2.15) is a fourth order partial differential equation with extremely complicated boundary conditions. It is very difficult to obtain an analytical solution. In this
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study Equation (2.15) is solved by application of Galerkin's method. To this end, the displacement field of the rotating disc can be assumed in the modal expansion form [5]:

\[ w(r, \theta, t) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r) \left[ C_{mn}(t) \cos(n\theta) + S_{mn}(t) \sin(n\theta) \right] \right\} \]  \hspace{1cm} (2.28)

where \( C_{mn}(t) \) and \( S_{mn}(t) \) are unknown functions to be determined. \( m \) and \( n \) represent the numbers of nodal circles and nodal diameters, respectively. \( R_{mn}(r) \cos(n\theta) \) and \( R_{mn}(r) \sin(n\theta) \) are the shape functions of an unconstrained disc, in which \( R_{mn}(r) \) are the radial shape functions chosen to satisfy the inner and outer bending boundary conditions of the blade. \( R_{mn}(r) \) could be approximated by Bessel functions or polynomial functions. In this study it is assumed that \( R_{mn}(r) \) is a polynomial in \( r \) of the form:

\[ R_{mn}(r) = \sum_{i=1}^{5} E_{mni} r^{(m+i-1)} \]  \hspace{1cm} (2.29)

where \( E_{mni} \) are unknown coefficients determined from out-of-plane boundary conditions and the normalized condition \( R_{mn}(b) = 1 \).

In this chapter, for the sake of simplicity, only one stationary lateral force \( P \) at \( (r_0, \theta_0) \), one stationary spring at \( (r_c, \theta_c) \) and the axisymmetric stress fields due to centrifugal force or tensioning (i.e., Equation (2.16)) are included in Equation (2.15). Substituting Equation (2.28) into Equation (2.15) and applying Galerkin procedure:

\[ \int_{0}^{2\pi} \int_{a}^{b} L(w)(R_{q1} \cos l\theta) r dr d\theta = 0 \]  \hspace{1cm} (2.30)

\[ \int_{0}^{2\pi} \int_{a}^{b} L(w)(R_{q1} \sin l\theta) r dr d\theta = 0 \]  \hspace{1cm} (2.31)

\( (q = 0,1,2,\cdots,M; \ l = 0,1,2,\cdots,N) \)

lead to the following two equations:
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\(\sum_{m=0}^{M} \alpha_m \left[ \delta^{(1)}_{qlm} \dot{C}_m + \delta^{(2)}_{qlm} \dot{S}_m + \delta^{(3)}_{qlm} C_m \right] + \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ N^{(1)}_{mnql} C_{mn} + N^{(2)}_{mnql} S_{mn} \right] = PR_{ql}(r_0) \cos(l\theta_0) \)  

(2.32)

\(\sum_{m=0}^{M} \beta_m \left[ \delta^{(1)}_{qlm} \dot{S}_m - \delta^{(2)}_{qlm} \dot{C}_m + \delta^{(3)}_{qlm} S_m \right] + \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ N^{(3)}_{mnql} C_{mn} + N^{(4)}_{mnql} S_{mn} \right] = PR_{ql}(r_0) \sin(l\theta_0) \)  

(q = 0, 1, 2, ..., M; l = 0, 1, 2, ..., N)  

(2.33)

where

\[ \delta^{(1)}_{qlm} = \rho \pi \int_{a}^{b} r R_{ml}(r) R_{ql}(r) dr \]

\[ \delta^{(2)}_{qlm} = \rho \pi (2\Omega)^2 \int_{a}^{b} r R_{ml}(r) R_{ql}(r) dr \]

\[ \delta^{(3)}_{qlm} = D \pi \int_{a}^{b} \left[ (d^2 / dr^2 + (1 / r) d/dr - l^2) R_{ml}(r) \right] R_{ql}(r) dr \]

\[ \dot{\psi} = \frac{1}{\psi} \int_{a}^{b} \left[ (d / dr) \left[ r \sigma_r dR_{ml}(r) / dr \right] - (l^2 / r) \sigma_{\theta \theta} R_{ml}(r) \right] R_{ql}(r) dr \]

\[ \delta^{(1)}_{qlm} = kR_{mn}(r_c) R_{ql}(r_c) \cos(n\theta_c) \cos(l\theta_c) \]

\[ N^{(2)}_{mnql} = kR_{mn}(r_c) R_{ql}(r_c) \sin(n\theta_c) \cos(l\theta_c) \]

\[ N^{(3)}_{mnql} = kR_{mn}(r_c) R_{ql}(r_c) \cos(n\theta_c) \sin(l\theta_c) \]

\[ N^{(4)}_{mnql} = kR_{mn}(r_c) R_{ql}(r_c) \sin(n\theta_c) \sin(l\theta_c) \]

and \( \alpha_l = \begin{cases} 1 & (l \neq 1) \\ 2 & (l = 0) \end{cases} \) \( \beta_l = \begin{cases} 1 & (l \neq 1) \\ 0 & (l = 0) \end{cases} \)

Equations (2.32) and (2.33) can be rewritten in matrix form:

\[ [M]\dot{x} + [G][x] + [K][x] = f \]  

(2.34)

where
\[ \{x\} = (C_{00}, C_{10}, \ldots, C_{M0}, C_{01}, C_{11}, \ldots, C_{M1}, S_{01}, S_{11}, \ldots, S_{M1}, \ldots, C_{0N}, C_{1N}, \ldots, C_{MN}, S_{0N}, S_{1N}, \ldots, S_{MN})^T \]

Note that the element \( S_{mn}(t) \) where \( n=0 \) has been removed from \( \{x\} \). In general, \([M]\) is a real, symmetric and positive definite matrix, \([G]\) is a real and skew-symmetric matrix, and \([K]\) is real and symmetric but not necessarily positive definite. It is also important to note that the matrices \([G]\) and \([K]\) are functions of \( \Omega \) or \( \Omega^2 \) so that the dynamic characteristics of the system are also dependent on the rotating speed.

The general linear equation of motion (2.34) can be decoupled in the mode domain of the state space. Thus the response \( \{x(t)\} \) in the Galerkin space can be calculated so that the response in the physical space is obtained by transforming \( \{x(t)\} \) into \( w(r, \theta, t) \) through Equation (2.28).
Chapter 3

On the Mechanisms of Instability in a Constrained Rotating String

3.1 Introduction

In this chapter a relatively simple mathematical model involving an idealized rotating string constrained at one point is used to develop an understanding of the basic physical mechanisms that govern the development of unstable lateral vibration response. The purpose of this study is to investigate the instability mechanisms that are involved in the interaction between a rotating string and an interactive stationary system with elastic, inertial, and damping characteristics together with a general non-conservative force.

Rotating machinery often contains components that are high energy systems with latent instabilities. Under certain circumstances the driving energy in such systems can be channeled into lateral vibration modes leading to unstable behavior. In order to prevent such energy transmission, an understanding of these instability mechanisms is required.

The essential physical mechanisms involved in rotor shaft system instabilities have been studied by S. H. Crandall [8] through the use of simplified models. However, the physical mechanisms involved in rotating disc like systems with transverse vibration are not well understood especially for instability mechanisms caused by the interaction with stationary components. These mechanisms are involved in a broad range of problems involving rotor/stator interaction of greater complexity than the model considered in this chapter. Such problems are encountered in turbine discs, computer floppy discs and guided saw blades for example. Schajer (1984) [9] presented a theoretical stability analysis for a rotating circular string subjected to a stationary spring, and established the foundation in dynamics of rotating
string. Mote [10, 11], Chen, Ono and Bogy [12, 13], Iwan and Moeller [14] and many other researchers have presented mathematical analyses that predict the onset of instability for a rotating disc-stationary load system or for a stationary disc with a moving load system. However these studies do not make clear the physical mechanisms responsible for the onset of these instabilities. Shen and Mote [15] presented a physical explanation of the instability mechanisms of a stationary circular plate subjected to a rotating damped spring-mass system and Yang and Hutton [16] presented a physical explanation of the instability of a rotating circular string due to interaction with a stationary frictional force. However there is lacking clear physical explanations for the development of such instabilities in general.

In this chapter exact equations defining the relationship between the inertial, damping, elastic and other interactive forces are presented. Analysis of these equations leads to a clear understanding of the physical mechanisms involved in the development of divergence and flutter instability mechanisms. New developments involve the identification of the energy flux into the rotating system and an explanation of the role of circumferential forces caused by interaction with the non-rotating constraint. The equations developed suggest methods for minimizing the instability regions that are encountered in such rotating systems when operating above their lowest critical speed. Numerical examples are presented for the case of a rotating string to illustrate its instability characteristics when constrained by a single degree of freedom viscously damped system.

3.2 Equation Defining Energy Flux in a Constrained Rotating String

Figure 3.1 shows a circular string of radius $r$, rotating at a constant angular velocity $\Omega$, attached to a stationary spring-mass-dashpot system at $(r, \theta = 0)$. $k$, $c$, and $m$ are the stiffness, damping and mass parameters, respectively, of the constraint system. $F_n(0,t)$ represents an arbitrary lateral non-conservative interactive force such as the lateral component of a friction force between the string and a constraint. $T_d(t)$ is the applied torque required to maintain the angular velocity $\Omega$. 
For small displacements the strain energy $U$ and the kinetic energy $T$ of the deformed string system in the space-fixed coordinates can be written as:

\begin{align}
U &= \int_0^{2\pi} \frac{1}{2} P \left(\frac{u_\theta}{r}\right)^2 rd\theta + \frac{1}{2} ku^2(0,t) \\
T &= \int_0^{2\pi} \frac{1}{2} \rho (\Omega^2 r^2 + (u_\theta + \Omega u_\theta)^2) rd\theta + \frac{1}{2} mu^2(0,t) 
\end{align}

(3.1) (3.2)

where, $P$ and $\rho$ are the circumferential tension (assumed constant) and density per unit length of the string, respectively.

According to Hamilton’s principle and including the generalized non-conservative forces $F_i$ ($q_i$ is the $i$th generalized coordinate):

\begin{equation}
\delta \int_{t_1}^{t_2} (T - U) dt + \sum_i \int_{t_1}^{t_2} F_i \delta q_i dt = 0
\end{equation}

(3.3)

\begin{equation}
(\sum_i F_i \delta q_i = -[cu_\theta(0,t) + F_n(0,t)\delta u(0,t)] \quad \text{in this case})
\end{equation}

the governing equation of transverse motion $u(\theta,t)$, with respect to stationary coordinates, and the boundary conditions can be derived in the following forms:
Chapter 3. On the Mechanisms of Instability in a Constrained Rotating String

\[ u_{tt} + 2\Omega u_{t\theta} - \left(\frac{P}{\rho r^2} - \Omega^2\right)u_{\theta\theta} = 0 \]  
(3.4)

\[ u(0,t) - u(2\pi,t) = 0 \]  
(3.5)

\[ (S^2 - \Omega^2)[u_{\theta}(2\pi,t) - u_{\theta}(0,t)] + \frac{k}{\rho r} u(0,t) + \frac{m}{\rho r} u_{t\theta}(0,t) + \frac{c}{\rho r} u_{t\theta}(0,t) - \frac{1}{\rho r} F_n(0,t) = 0 \]  
(3.6)

where, \( S^2 = P / (\rho r^2) \) (\( S \) is the flexural wave speed).

The rate of total energy change (power) \( E_t \) is given by:

\[ E_t = U_t + T_t \]  
(3.7)

Substituting Equations (3.1) and (3.2) into Equation (3.7) and simplifying the result using the governing equation and the boundary conditions yields the following equation (the detailed derivation is given in the Appendix A):

\[ E_t = \Omega(F_k + F_m + F_c + F_n)\bar{u}_{\theta}(0,t) + F_c u_{t\theta}(0,t) + F_n u_{t\theta}(0,t) \]  
(3.8)

where, \( F_k = -ku(0,t) \), \( F_m = -mu_{t\theta}(0,t) \) and \( F_c = -cu_{t\theta}(0,t) \) are the lateral elastic, inertial and viscous damping forces acting on the string, as a result of the constraint at the location \((r, \theta=0)\). \( \bar{u}_{\theta}(0,t) / r = [u_{\theta}(2\pi,t) + u_{\theta}(0,t)] / (2r) \), is the average slope of the string at the constraint location. Note that Equation (3.8) is an exact expression derived from the exact solution of the string motion, which can also be rewritten in the following forms:

\[ E_t = -\Omega r(F_{\theta k} + F_{\theta m} + F_{\theta c} + F_{\theta n}) + F_c u_{t\theta}(0,t) + F_n u_{t\theta}(0,t) \]  
(3.9)

or

\[ E_t = \Omega T_D(t) + F_c u_{t\theta}(0,t) + F_n u_{t\theta}(0,t) \]  
(3.10)
where, $T_D(t)$ is the external applied driving torque. $F_{\theta k} = -F_k \tan \alpha = -F_k (\bar{u}_\theta / r)$, $F_{\theta m} = -F_m (\bar{u}_\theta / r)$, $F_{\theta c} = -F_c (\bar{u}_\theta / r)$ and $F_{\theta n} = -F_n (\bar{u}_\theta / r)$ are the circumferential components of the interactive forces acting on the string. The torques produced by these circumferential components can be expressed as: $T_k = F_{\theta k} r$, $T_m = F_{\theta m} r$, $T_c = F_{\theta c} r$ and $T_n = F_{\theta n} r$.

Figure 3.2 Interactive forces applied to the rotating string due to a spring constraint

Figure 3.2 illustrates the forces involved for the case where the constraint consists only of a stiffness component. As the string-spring interface is assumed frictionless then the spring applies a normal force $R$ to the deflected rotating string. The vertical component of this force $F_k$ must balance the spring force $k u(0, t)$ whereas the horizontal component of this force $F_{\theta k}$ will result in a torque $T_k = F_{\theta k} r$ applied to the string. This torque must be provided by the driving torque $T_D$ in order for the system to rotate at constant speed. Thus in the general case where inertial, damping, stiffness and non-conservative forces exist at the constraint interface the driving torque $T_D$ must supply a torque that exactly balances the sum of the
individual torques in order to maintain motion at constant speed. One implication of this result is that the applied torque required to produce constant speed is not constant. For the case of a pure spring constraint, it can be seen from Equation (3.10) that the work done by the lateral interactive force $F_k$ is equal to the work done by the driving torque $T_D$. If the work done by $T_D$ is greater than zero over a complete cycle of motion and the speed is maintained at a constant value, the driving energy will be channeled into vibration energy and unstable behavior will result.

To analyze the energy flux in this system, Equation (3.10) can be also expressed in the following form:

$$\Delta E = \int_0^\tau [N_{lk} + N_{lm} + (N_{lc} - N_{oc}) + (N_{ln} - N_{on})] dt$$

(3.11)

where, $\Delta E$ represents the change of total energy in the system during the time interval $0-\tau$. $N_{lk} = -T_k \Omega$, $N_{lm} = -T_m \Omega$, $N_{lc} = -T_c \Omega$ and $N_{ln} = -T_n \Omega$, are the components of the input power provided by the driving torque. $N_{oc} = cu_s^2(0,t)$ and $N_{on} = -F_n u_s(0,t)$, represent the power dissipated by a viscous damper and a non-conservative force. Thus, the resultant power into the system through these non-conservative elements are $N_{lc} - N_{oc}$ and $N_{ln} - N_{on}$, respectively.

From Equation (3.11) it should be noted that if the total energy of this system increases, as the rotational speed is constrained to be constant, the driving energy for steady rotation will be transferred into lateral vibration, i.e., unstable behavior will occur. Thus, Equation (3.11) is the basic relationship that provides physical insight into the stability characteristics of the system. Under stable conditions $\Delta E$ will be a non-zero function of time, but over a sufficiently long period its mean value will be zero.

If no interactive system is present, i.e., $k = m = c = F_n(0,t) = 0$, the system is always stable.
3.3 Divergence Instability in a Constrained Rotating String

Divergence is a standing wave instability and occurs at a critical speed of a rotating system. This type of instability can be mathematically defined as a state in which the real part of one of the system eigenvalues \( \sigma_n > 0 \) and the imaginary part \( \omega_n = 0 \). Such instabilities are to be found in rotating disc interaction problems [10-14].

Divergence instability cannot occur in the present model. When the rotating speed \( \Omega \) equals the wave speed \( S \), the backward-wave mode shape appears as a stationary wave for an unconstrained string system when observed from fixed coordinates. It can be noted from Equation (3.6) that divergence instability cannot occur at the wave speed in the constrained rotating string system because the total interactive forces between the string and constraint become zero when \( \Omega = S \), which means \( \Delta E(t) = 0 \).

3.4 Flutter Instability in a Constrained Rotating String

Flutter is a type of dynamic instability characterized by oscillations with increasing amplitude, and for a rotating string subjected to stiffness, mass and damping constraints it occurs at supercritical speeds i.e. in the present study at speeds above the wave speed. Mathematically, flutter instability can be defined as a state in which the real part of a system eigenvalue \( \sigma_n > 0 \) and the imaginary part \( \omega_n > 0 \). This type of instability always occurs as a result of the coupling of two vibratory modes in a given speed region. The onset of the flutter is characterized by the coincidences of mode frequencies and mode shapes. The coupled modes subsequently separate after some increase in rotating speed as shown in Figure 3.3(a).

Consider first the case of a stiffness constraint. The total energy change of two modes \( i \) and \( j \) can be expressed as:

\[
\Delta E = \Omega \int_0^r (F_{ki} + F_{kj})(\ddot{u}_i + \ddot{u}_j) \, dt
\] (3.12)
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Figure 3.3 Response characteristics due to stiffness constraint $(k' = kr/P = 1.0)$

(a): Nondimensional natural frequency (b): Phase difference $\Delta \phi_{ij}$ between $F_k$ and $\bar{u}_{ij}(0,t)$

or

$$\Delta E = -\Omega \int_0^T (F_{\theta ki} + F_{\theta kij} + F_{\theta kij} + F_{\theta kij}) dt = \Delta E_{ii} + \Delta E_{ij} + \Delta E_{ji} + \Delta E_{jj}$$ (3.13)

where, $\Delta E_{ii} = -\Omega \int_0^T F_{\theta kij} r dt = \Omega r \int_0^T F_{\theta kij} (\bar{u}_{ij}/r) dt , \quad \Delta E_{ij} = -\Omega \int_0^T F_{\theta kij} r dt = \Omega r \int_0^T F_{\theta kij} (\bar{u}_{ij}/r) dt$

$\Delta E_{ij} = -\Omega \int_0^T F_{\theta kij} r dt = \Omega r \int_0^T F_{\theta kij} (\bar{u}_{ij}/r) dt , \quad \Delta E_{ji} = -\Omega \int_0^T F_{\theta kij} r dt = \Omega r \int_0^T F_{\theta kij} (\bar{u}_{ij}/r) dt$

$\Delta E_{ii}$ and $\Delta E_{jj}$ represent the energy changes caused by the circumferential force components $F_{\theta kij}$ and $F_{\theta kij}$ of the $i(j)$th modes, respectively. $\Delta E_{ij}$ and $\Delta E_{ji}$ represent the energy changes due to the action of the coupled circumferential force components $F_{\theta kij}$ and $F_{\theta kij}$, respectively.
It can be easily proven that in the absence of damping the phase difference between the displacement $u_i(0,t)$ and the average slope $\bar{u}_{i\theta}(0,t)$ for a single mode is always $90^\circ$ in an uncoupled region. $\Delta E_{ii}$ is thus given by:

$$\Delta E_{ii} = \Omega \int_0^\tau F_{ki} \bar{u}_{i\theta}(0,t) dt = -k \Omega \int_0^\tau u_i(0,t) \bar{u}_{i\theta}(0,t) dt$$

$$= A_i(\Omega) \int_0^\tau \sin[\omega_i(\Omega)t + \phi_i(\Omega) + \pi] \sin[\omega_i(\Omega)t + \phi_i(\Omega) + \frac{\pi}{2}] dt$$

(3.14)

where, the magnitude $A_i(\Omega)$, angular frequency $\omega_i(\Omega)$ and phase angle $\phi_i(\Omega)$ are all functions of the rotating speed $\Omega$.

$\Delta E_{ii}$ is a periodic function of time with zero mean, which implies that there is only a periodically oscillating energy flow into and out of the system. This means that flutter instability due to a stationary stiffness cannot occur in a single mode by itself. In this case, the circumferential force component $F_{skii}$ is also a periodic function of time, so a periodic driving torque that does zero-mean work is required to balance $F_{skii}$ in order to keep the system rotating at constant speed.

Consider now two modes with different frequencies $\omega_{ia}$ and $\omega_{jb}$, that are approaching each other as shown in Figure 3.3(a), the coupling terms $\Delta E_{ij}$ and $\Delta E_{ji}$ that are responsible for the flutter instability of the system can be expressed as:

$$\Delta E_{ij} = \Omega \int_0^\tau F_{ki} \bar{u}_{i\theta}(0,t) dt = A_{ij}(\Omega) \int_0^\tau \sin[\omega_{ia}(\Omega)t + \phi_{ia}(\Omega)] \sin[\omega_{jb}(\Omega)t + \phi_{jb}(\Omega)] dt$$

(3.15)

where, $\phi_{ia}(\Omega)$ and $\phi_{jb}(\Omega)$ are the phase angles associated with the stiffness force of the $ith$ mode and with the mean slope of the $jth$ mode respectively. It can be verified that $\Delta E_{ij}$ is a periodic function with zero mean when $\omega_{ia} \neq \omega_{jb}$, which implies that flutter instability due to a stationary stiffness cannot occur in two modes with different frequencies. The
circumferential force component $F_{\theta ij}$ is also a periodic function with zero mean when the $i$th and $j$th modes have different frequencies.

As shown in Figure 3.3(a), when the two modes become coupled at point $c$, the frequencies of the coupled modes become identical, and the phase difference $\Delta \varphi_{ij}$ between the elastic force $F_{ki}$ and the slope $\bar{u}_j(0,t)$ becomes $90^\circ$. At this point the total energy change is still periodic with zero mean. In the flutter region, the coupled modes have identical frequencies and mode shapes. In this case the phase difference $\Delta \varphi_{ii}$ between $F_{ki}$ and $\bar{u}_i(0,t)$ varies from $90^\circ$ at point $c$ to $0^\circ$ at some rotational speed in the flutter area, it then returns to $90^\circ$ at point $d$ as shown in Figure 3(b). These results were obtained from the exact solution of the free response analysis of the string system.

It can be proven mathematically that $\Delta E_{ij}$ is an increasing positive oscillatory function of time when $\omega_{ia} = \omega_{jb}$ and $0 \leq \Delta \varphi_{ii} < 90^\circ$. In this case, the circumferential component $F_{\theta kii}$ caused by the two coupled modes is a periodically oscillating increasing function of time with a positive mean, so a corresponding driving torque that does positive work is required to balance it. Without any energy dissipation, the system becomes unstable.

For the case of a mass constraint, the circumferential component $F_{\theta mii}$ that is caused by two coupled modes is also periodic with positive mean when $0 \leq \Delta \varphi_{ii} < 90^\circ$, so a corresponding driving torque is required to balance the circumferential force, which leads to a positive energy flux into the system.

The preceding arguments also apply to the case of simultaneous mass and stiffness constraints, in which case the circumferential component is $F_{\theta k} + F_{\theta m}$. Figure 3.4 shows the flutter region caused by the circumferential force $F_{\theta k} + F_{\theta m}$ and the phase difference between the total interactive force $F_k + F_m$ and the slope $\bar{u}_\theta(0,t)$. It is noted that the phase difference is such that $0 \leq \Delta \varphi_{ii} < 90^\circ$ in the flutter region.
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From Equations (3.8) and (3.9) it can be noted that for the case of simultaneous mass and stiffness constraints flutter instability will disappear when

\[ F_{ek} + F_{em} = -(F_{k} + F_{m})\bar{\phi}(0,t) / r = 0. \]

Within the flutter region, the total circumferential force generated by the mass and stiffness constraints is given by:

\[ F_{ek} + F_{em} = m(\omega_{0}^{2} - \omega_{i}^{2})u(0,t)\bar{\phi}_{0}(0,t) / r \]

where, \( \omega_{0}^{2} = k / m \). \( \omega_{i} \) is some frequency in the flutter region. Therefore, the flutter instability can be minimized by setting \( \omega_{0}^{2} = \sigma_{i}^{2} \), where \( \sigma_{i} \) is an average frequency for the flutter region. Figure 3.5 illustrates four flutter instability regions before (Case 1) and after (Case 2) such a modification is made. The effect on minimizing the flutter regions can clearly be seen in Figure 3.5(case 2). The flutter regions reduce or even disappear and the real parts of the
eigenvalues are notably diminished due to a significant reduction of the resultant circumferential force.

Note also that, for the case of stiffness constraints a particular flutter region may be eliminated or minimized by putting two springs at appropriate locations on the string in order that:

\[
F_{ek} = [k_1 u(\theta_1, t) \bar{u}_\theta(\theta_1, t) + k_2 u(\theta_2, t) \bar{u}_\theta(\theta_2, t)] = 0
\]  

(3.17)

where, \( \theta_1 \) and \( \theta_2 \) represent two specific positions on the string. The system may then become more stable because the interactive circumferential force that causes positive energy flow into the system has been minimized.

![Figure 3.5 Effect of system parameters on stability characteristics](image)

Case 1: \( m' = m/(\rho r) = 0.6 \) and \( k' = 0.0 \); Case 2: \( m' = 0.6 \) and \( k' = 3.31 \) (\( \bar{\omega}_i = 2.35 \))
3.5 Terminal Flutter Instability

A special type of flutter instability which occurs at all speeds above a particular rotating speed is termed "terminal instability". The occurrence of terminal flutter usually involves a single mode and a non-conservative force. A typical example of this type of instability is the flutter instability caused by a constraint consisting of a stationary viscous damper. Figure 3.6 illustrates the effect of viscous damping on the eigenvalues of a rotating string, where modes $nF$ and $nR$ represent forward and reflected traveling wave modes with $n$ nodal diameters.

![Figure 3.6 Response characteristics due to viscous damper constraint](image)

Figure 3.6 Response characteristics due to viscous damper constraint

\[ (c' = c / \sqrt{\rho P} = 0.3) \]

From Figure 3.6(a), it is noted that the frequencies of the system are similar to those in a system without a constraint, with all the frequency curves going through the crossing points.
without coupling. From Figure 3.6(b), however, it can be noted that all the backward reflected wave modes become unstable at supercritical speeds and the instability involves a single mode. At supercritical speeds the real parts of the eigenvalues for reflected waves increase almost linearly with rotating speed but they reduce to zero when the reflected wave modes meet the forward wave modes. It can also be noted that the real parts of the eigenvalues for the forward wave modes have negative values and decrease linearly except near crossing points where they all take a common value. Thus, the forward wave modes associated with a stationary viscous damper constraint are always stable.

The resultant energy change in the system for a stationary damping constraint can be expressed (using Equations (3.8) and (3.10)) as:

$$\Delta E = \int_0^1 [\Omega F_c \bar{u}_\beta(0,t) - cu^2_j(0,t)] dt = \int_0^1 [\Omega \tau_{Dc} - cu^2_j(0,t)] dt$$

(3.18)

or

$$\Delta E = \int_0^1 [F_c u'_j(0,t)] dt$$

(3.19)

where, \(u'_j(0,t) = u_j(0,t) + \Omega \bar{u}_\beta(0,t)\), which is the relative transverse velocity at the constraint observed from rotating coordinates, while \(u_j(0,t)\) represents the absolute transverse velocity at the constraint observed from fixed coordinates. From Equation (3.18) it is noted that the energy into the system equals the difference between the total input energy required to overcome the torque induced by the damper force and the energy dissipated by the same damper.

The total input energy \(\Delta E_i\) introduced by the \(i\)th reflected wave mode is given by:

$$\Delta E_i = \frac{\Omega}{F_c} \int_0^\tau F_c \bar{u}_\beta(0,t) dt = A_{ii}(\Omega) \int_0^\tau \sin(\omega_it + \phi_{Fi}) \sin(\omega_{ii}t + \phi_{ii}) dt$$

(3.20)

where \(\phi_{Fi}\) and \(\phi_{ii}\) are the phase angles of \(F_c\) and \(\bar{u}_\beta(0,t)\), respectively.
Figure 3.7: Nondimensional energy changes of the 4th backward mode due to a stationary spring \((k' = 1.8)\) and a viscous damper \((c' = c / \sqrt{\rho P} = 0.75)\) at \(\Omega' = 1.72\)

Figure 3.7 shows the net energy changes for the 4th backward wave mode caused individually by a spring and by a viscous damper that provide approximately equal transverse forces to the string. It can been seen from this figure that the net energy into the system through the damper is relatively small compared to that of the spring because the damper dissipates a portion of the input energy.

The phase differences \(\Delta \phi_{F \alpha}\) between \(F_c\) and \(\bar{u}_\phi(0, t)\) for modes 1R(1B) and 4R(4B) are calculated and the results are shown in Figure 3.8. From this figure it can be seen that \(F_c\) is always out of phase with \(\bar{u}_\phi(0, t)\) at subcritical speeds and in phase with \(\bar{u}_\phi(0, t)\) at supercritical speeds. As a result, energy is removed from the system at subcritical speeds \((P_1)\) but is input into the system at supercritical speeds except at the speed corresponding to the crossing point \((P_2)\). It can be noted that \(\Delta \phi_{F \alpha}\) at the crossing point \((P_2)\) becomes \(90^\circ\). Therefore, the periodic energy flux into the system has zero mean and this mode becomes stable at the crossing points.
Figure 3.9 schematically illustrates the effects of a viscous damper on the backward wave mode instability. In Figure 3.9 the solid curve is the backward wave observed in space-fixed coordinates at supercritical speeds and the dashed curve represents the backward wave observed in the coordinates rotating with the string. From this figure it can be seen that at supercritical speeds the transverse damping force $F_c$ acting on the string is always in phase with the average slope $\bar{u}_\theta(0,t)/r$ at the constraint on the solid curve (i.e., $F_c < 0$ and $\bar{u}_\theta(0,t) < 0$ at the position shown in Figure 3.9). Therefore, the energy introduced by the damper $\Delta E_f$ is always greater than zero, which implies that the driving energy is switched into the vibratory energy of the reflected wave mode through the damper.

Figure 3.8: Response characteristics due to viscous damper constraint ($c' = 0.3$)

(a) Real parts of eigenvalues for modes $1B$ and $4B$  
(b) Phase difference $\Delta \varphi_{Fa}$
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Figure 3.9: Backward-wave mode instability due to viscous damper constraint or friction (Solid line: Observed in space-fixed coordinates; Dashed line: Observed in rotating coordinates)

Figure 3.10: Forward-wave mode instability due to viscous damper constraint or friction (Solid line: Observed in space-fixed coordinates; Dashed line: Observed in rotating coordinates)
According to Equation (3.19), the net energy change of the system equals the work done by the damping force $F_c$ with the relative transverse velocity $u_t'(0,t)$ indicated by the dash curve. It should be noted that $F_c$ only depends on $u_t'(0,t)$ which is dependent on the rotating speed of the string, while the direction of $u_t'(0,t)$ is independent of rotating speed. It can be seen that $F_c$ is in phase with $u_t'(0,t)$ at supercritical speed. Thus, $F_c$ does positive work on the reflected wave and destabilizes its motion. It can also be shown that $F_c$ is always out of phase with $\bar{u}_g'(0,t)$ or $u_t'(0,t)$. Therefore, the damper always stabilizes the backward wave at subcritical speeds. For the forward wave shown in Figure 3.10, it is can be seen that $F_c$ is always out of phase with $u_t'(0,t)$. As a result, the damper stabilizes the forward wave mode at all speeds.

From Figures 3.9 and 3.10 it can also be seen that in the case of friction the lateral component of a constant friction force $F_\mu$ (i.e., $F_n = F_\mu u_g / r$) is always out of phase with $u_t'(0,t)$ for the backward wave mode and is always in phase with $u_t'(0,t)$ for the forward wave mode, regardless of rotating speed. Therefore, stationary friction stabilizes the backward wave mode and destabilizes the forward wave mode at all speeds. This result agrees with the theoretical eigenvalue analysis in reference [16].

The above illustration also applies to the case where the constraint is an arbitrary non-conservative force $F_n$. The net energy into the system can be expressed from Equation (3.8) as:

$$\Delta E = \int_0^\tau F_n[u_t'(0,t) + \Omega \bar{u}_g(0,t)]dt = \int_0^\tau F_n u_t'(0,t)dt$$

(3.21)

In this case the stability of the system relies on the characteristics of $F_n$.

3.6 Summary

In this study the relationship between energy flux and the interactive forces for an idealized constrained string rotating at constant speed has been derived. This equation leads to
a clear understanding of the physical mechanisms involved in the development of vibrational
instabilities in this system. The instability mechanisms identified in this work can be expected
to have relevance in the understanding of the behaviour of more complex systems.

Some important conclusions pertaining to the model considered are summarized as:

1) The circumferential component of the interactive force between the stationary constraint
and the rotating string is a primary factor in all instability mechanisms.

2) When the interactive force applied to the rotating string is in phase with the average slope
$\bar{u}_\phi(0,t) / r$ at the constraint, driving energy will be switched into vibratory energy
leading to unstable behaviour. In this case, the circumferential component of the
interactive force has a positive mean value. Vibratory energy will be switched back into
driving energy when the interactive force is no longer in phase with $\bar{u}_\phi(0,t)$. At this time
the circumferential component of the interactive force has a negative mean value.

3) When the phase difference between the interactive force and $u'_\phi(0,t)$ is 90° there is a
periodically oscillating energy flow into and out of the system, in which case, the total
energy of the system is a periodic function of time with zero mean.

4) In the constrained rotating string divergence instability cannot occur at the critical speed
because the constraint forces must sum to zero as the string tension and inertia effects
cancel. Under such conditions no energy can be input into the system.

5) Flutter instability only occurs with stationary stiffness and mass constraints at supercritical
speeds. Terminal flutter instability occurs at supercritical speeds in conjunction with non-
conservative interactive forces. A stationary viscous damper, for example, can not only
dissipate vibratory energy but can also introduce net energy into the system at supercritical
speeds leading to the possibility of terminal flutter instability.

6) The severity of the instability in a given flutter region can be minimized by choosing
appropriate stationary stiffness and mass characteristics or by positioning constraints at the
locations where the circumferential force can be minimized.
Chapter 4

Instability Mechanism of a Rotating Disc Subjected to Various Transverse Interactive Forces — A General Approach

4.1 Introduction

This chapter deals with the instability mechanisms that are involved in the interaction between a rotating flexible disc and a constraining system or an arbitrary interactive force. This chapter is an extension of the work presented in Chapter 3.

In the past three decades, numerous papers have been published on disc dynamics. Most of them have focused on the mathematical modeling and the solution to problems involving a specific interactive force. The physical mechanisms involved in such rotating systems have not been satisfactorily explained by previous researchers, especially for the instability mechanisms caused by an arbitrary interactive force, such as a non-conservative cutting force of the type that arises in saw-blade cutting.

Chapter 4. Instability Mechanisms of a Rotating Disc — A General Approach

This chapter is designed to establish a generalized approach to investigate the instability mechanisms that are involved in the interaction between a rotating disc and an interactive stationary system such as a work-piece. Rotating disc models with different types of interactive forces are considered and used to develop an unified understanding of the physical mechanisms of the instabilities involved.

Equations defining the relationship between the total vibration energy and the interactive forces are presented for a rotating disc system, leading to a full understanding of the physics involved in the interactions. Based on the derived equations, unified conditions for the occurrence of instability are presented. Through this effort possible instabilities due to any types of interactive forces, including nonlinear interactive forces, can be identified based on an energy flux analysis, even without solving the equations of motion.

4.2 Energy Flux Equation of A Rotating Disc with Interactive Forces

Figure 4.1 shows a circular plate of inner radius $a$, outer radius $b$, and thickness $h$, rotating at a constant angular velocity $\Omega$, in contact with a stationary spring-mass-dashpot constraint or a work-piece that gives rise to some known interactive forces. The governing equation for transverse vibration in terms of the lateral displacement $w(r,\theta,t)$, with respect to the space-fixed coordinates, can be rewritten from Equation (2.15) as:

$$DV^4w + \rho w_{rr} + 2\Omega w_{r\theta} + \Omega^2 w_{\theta\theta} + L_s(w) = L_n(w) + f_g(t) + f_c(t)$$  \hspace{1cm} (4.1)

where $D$ and $\rho$ are the flexural rigidity and mass density of the plate, respectively.

$L_s$ is the membrane operator associated with the axisymmetrical stress fields due to the centrifugal force and/or the stress tensioning. $L_n$ is the membrane operator associated with the asymmetric stress fields generated by in-plane edge loads, such as in-plane cutting forces, which, under certain circumstances, may cause instability in such systems.
Chapter 4. Instability Mechanisms of a Rotating Disc — A General Approach

In Equation (4.1), \( f_g(t) \) represents the transverse forces generated by the interaction between the plate and the stationary mass-spring-dashpot constraint which includes the effect of friction,

\[
f_g(t) = -\sum_j \left[ (1/r) \delta(r - r_j) \delta(\theta - \theta_j) [m_j \dot{w}_r + c_j \dot{w}_\theta + k_j w + (1/r)\mu_j w_\theta ] \right]
\]

where, \( m_j \), \( k_j \), \( c_j \) and \( \mu_j \) are coefficients representing the mass, stiffness, damping and friction characteristics of the \( j \)th constraint, respectively. \(( r_j, \theta_j )\) defines the location of the \( j \)th constraint observed from the space-fixed coordinates.

\( f_c(t) \) represents the transverse cutting forces generated by the interaction between the saw-blade and the work-piece. Two types of multiple moving cutting forces, regenerative cutting forces and in-plane follower cutting forces, will be considered in later chapters.

For linear elastic behavior, the kinetic energy \( T \) and the strain energy \( U \) of a rotating plate with respect to the plate-fixed coordinates \(( w_r, r, \theta )\) are given by:
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\[ T = \frac{\rho h}{2} \iint_A [(w_{r,t})^2 + V^2(x,y)]dxdy \]  
(4.3)

and

\[ U = \frac{D}{2} \iint_A \left\{ (V^2 w_r)^2 - 2(1-v)[w_{r,xx} w_{r,yy} - (w_{r,xy})^2] \right\}dxdy \]  
(4.4)

where, \( V^2(x,y) = \Omega^2(x^2 + y^2) \). Without losing generality, only bending strain energy is considered here for the sake of simplicity.

The rate of the energy variation \( E_t \) of the plate, for a constant rotating speed, is given by (for a detailed derivation, see Appendix B):

\[ E_t = U_t + T_t = \iint_A [D V^4 w + \rho h(w_{r,t} + 2\Omega w_{r,\theta} + \Omega^2 w_{\theta,t})(w_{r,t} + \Omega w_{r,\theta})]rdrd\theta \]  
(4.5)

Substituting Equation (4.1) into Equation (4.5) and integrating it over a period \([0, \tau]\) yields:

\[ \Delta E = \int_0^\tau (\iint_A [q(w,r,\theta,t)w_r]rdrd\theta)dt \]  
(4.6)

where, \( q(w,r,\theta,t) = L_n(w) + f_k(t) + f_c(t) \), represents a generalized transverse force. \( \dot{w}_r = w_{r,t} + \Omega w_{r,\theta} \) represents the transverse velocity at the location of interaction, observed in rotating coordinates.

Note that \( L_n(w) \) is treated as a force term in this study instead of being put into the total strain energy of the plate. It should also be noted that if the strict form of the equation of motion along with the force boundary conditions is developed and employed instead of using Equation (4.1), the first term in Equation (B17) will disappear but the second and third terms (i.e., the boundary condition terms) will contain the effects of the interactive forces on the energy flux in the system.

From Equation (4.6) it is apparent that the change of total energy \( \Delta E \) of this system will increase when \( q(w,r,\theta,t) \) is in phase with \( \dot{w}_r \). In this case the driving energy for constant speed will be transferred into transverse vibration, and instability may occur. Thus, Equation (4.6) is the basic relationship that provides physical insight into the stability characteristics of
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the system. Under stable conditions $E_f$ will be a non-zero periodic function of time, but over a complete cycle the total energy change will be zero.

4.3 Instabilities Due to Moving Transverse Forces and Stationary Constraints

In the following sections, the possibility of instability occurring for a rotating disc subjected to different types of interactive forces will be studied based upon Equation (4.6).

4.3.1 Stationary or Moving Constant and Harmonic Point Forces

A moving force can be expressed as:

$$q(r, \theta, t) = \frac{1}{r} \delta[r - r_0(t)]\delta[\theta - \theta_0(t)]p(t)$$

where, $p(t) = P$ for a constant force. $p(t) = P \sin(\omega t + \psi)$ for a harmonic force.

Substituting Equation (4.7) into Equation (4.6) yields:

$$\Delta E = \int_0^T \dot{\omega}_r[r_0(t), \theta_0(t), t]p(t)dt$$

In the case of a stationary force, both $r_0(t)$ and $\theta_0(t)$ are constant. Therefore, $\Delta E$ increases with time only if $p(t)$ is time-varying with the same frequency as $\dot{\omega}_r(r_0, \theta_0, t)$ and is in phase with $\dot{\omega}_r(r_0, \theta_0, t)$. These conditions are satisfied only when $\omega$ is equal to the natural frequency of the disc measured from space-fixed coordinates. A constant stationary force can only cause an oscillating energy flux with zero mean.

For the case of a moving lateral force, both the magnitude and the phase angle of $\dot{\omega}_r[r_0(\tau), \theta_0(\tau), \tau]$ vary with the location of the moving force. Therefore, for a certain moving pattern of the force, $\Delta E$ may increase with time even when $p(t)$ is constant. This implies that instability may occur when moving forces (even constant moving forces) are applied.
From modal expansion theory, the response of the disc can be assumed to be a linear summation of mode shapes. The circumferential shape functions are assumed to be $\cos(n\theta)$ and $\sin(n\theta)$, and the radial shape function $R_{mn}(r)$ is chosen to satisfy the boundary conditions of the disc. By application of the Galerkin method, for a single mode case, the generalized force vector is:

$$\{\mathcal{Q}\} = \frac{\hbar}{\pi} \left\{ \begin{array}{c} \sin(n\theta) \\ \cos(n\theta) \end{array} \right\} R_{mn}(r) r dr d\theta = R_{mn}(r_0(t)) \left\{ \begin{array}{c} \sin(n\theta_0(t)) \\ \cos(n\theta_0(t)) \end{array} \right\} p(t)$$  \hspace{1cm} (4.9)

In the case when a harmonically varying force of frequency $\omega$ travels circumferentially at a constant rotating speed $\Psi$ measured from the space-fixed coordinates, Equation (4.9) can be written as:

$$\{\mathcal{Q}\} = R_{mn}(r_0(t)) \left\{ \begin{array}{c} \sin(n(\theta_0(t) + \Psi t)) \\ \cos(n(\theta_0(t) + \Psi t)) \end{array} \right\} P \sin(\omega t + \psi)$$  \hspace{1cm} (4.10)

Thus, resonance instability occurs when:

$$|\omega \pm n\Psi| = \omega_{mn} \quad (n \neq 0)$$  \hspace{1cm} (4.11)

where, $\omega_{mn}$ represents the natural frequency of the mode with $n$ nodal diameters and $m$ nodal circles.

For the case of a harmonically varying force of frequency $\omega$ travelling in the radial direction at a constant speed $v$, Equation (4.9) may be written as:

$$\{\mathcal{Q}\} = R_{mn}(r_0 + vt) \left\{ \begin{array}{c} \sin(n\theta_0) \\ \cos(n\theta_0) \end{array} \right\} P \sin(\omega t + \psi)$$  \hspace{1cm} (4.12)

For most boundary conditions and for modes with nodal circles, $R_{mn}(r_0 + vt)$ can be expressed in terms of periodic functions, such as $\sin(\alpha r)$. Similarly, the resonance occurs when:
where, $\alpha$ is a coefficient determined by the radial boundary conditions. For a simply-supported disc, $\alpha = m\pi / (b - a)$.

### 4.3.2 Stationary Mass-Spring Constraint With Conservative Forces

For a single constraint, substituting Equation (4.2) into Equation (4.6) yields:

$$\Delta E = \int_0^T F_z(t)[w_{\theta}(r_0, \theta_0, t) + \Omega w_{\phi}(r_0, \theta_0, t)] dt$$

(4.14)

where $F_z(t) = -[kw(r_0, \theta_0, t) + mw_{\phi}(r_0, \theta_0, t) + cw_{\theta}(r_0, \theta_0, t) + (1/r_0)\mu w_{\phi}(r_0, \theta_0, t)]$ (4.15)

$F_z(t)$ is the total lateral force caused by the interaction between the disc and the constraint at $(r_0, \theta_0)$. When $F_z(t)$ is in phase with the velocity $w_{\theta}(r_0, \theta_0, t) + \Omega w_{\phi}(r_0, \theta_0, t)$, driving energy is switched into vibration energy of the disc through the constraint.

For those cases when the interactive force is a conservative elastic or inertial force, $F_z(t)$ is always perpendicular to the velocity $w_{\theta}(r_0, \theta_0, t)$ measured in the space-fixed coordinates.

Thus, Equation (4.14) can be written as:

$$
\Delta E = \int_0^T \left[ F_k(t) + F_m(t) \right][\Omega w_{\theta}(r_0, \theta_0, t)] dt = \int_0^T \Omega r_0 [F_k(t) + F_m(t)] dt = \int_0^T \Omega (T_k + T_m) dt
$$

(4.16)

where $F_k(t) = -kw(r_0, \theta_0, t)$ and $F_m(t) = -mw(r_0, \theta_0, t)$. $F_{\theta k} = F_k[w_{\theta}(r_0, \theta_0, t) / r_0]$ and $F_{\theta m} = F_m[w_{\theta}(r_0, \theta_0, t) / r_0]$ are the circumferential components of the interactive elastic and inertial forces, respectively. The resistant torques generated by these circumferential forces are: $T_k = F_{\theta k} r_0$ and $T_m = F_{\theta m} r_0$. The externally applied driving torque $T_D$ must exactly balance the sum of the individual resistant torques $T_i(t)$ in order to maintain motion at constant speed. If the work done by the driving torque $T_D$ is greater than zero unstable behavior will result.
Chapter 4. Instability Mechanisms of a Rotating Disc — A General Approach

Figure 4.2: Instability regions of a rotating disc subjected to a stationary spring
\( k=2000\text{N/m} \) \( (a=0.28\text{m}, b=0.076\text{m}, h=1.12e^{-3}\text{m}, E=2.08e11\text{N/m}^2, \rho=7850\text{kg/m}^3) \)

Figure 4.3: Energy flux \( \text{(J)} \) at different rotating speeds due to a stationary stiffness
Figure 4.4: Instability regions of a rotating disc subjected to a stationary mass 

\( m = 0.5 \text{kg} \)

Figure 4.2 shows the Campbell diagram (i.e., the eigenvalue plots with respect to rotating speed) of a rotating disc constrained by a stationary spring. As shown in Figure 4.2, the divergence and the coupling flutter instabilities occur in several supercritical-speed regions where the interactive elastic force is in phase with \( w_i(r_0, \theta_0, t) + \Omega w_d(r_0, \theta_0, t) \). The coupling flutter instability will occur when the interactive elastic force caused by one of the two coupled modes is in phase with the velocity \( w_i(r_0, \theta_0, t) + \Omega w_d(r_0, \theta_0, t) \) of the other mode.

Figure 4.3 shows the energy flux \( \Delta E \) at different locations A, B, C and D illustrated in Figure 4.2. \( \Delta E \) is computed by using the Galerkin discretization and using an integration procedure in which the interactive forces are put into the generalized force vector.
Figures 4.4 and 4.5 show the Campbell diagram of a rotating disc which interacts with a stationary mass and the energy flux at Points A and B, respectively. From these figures it can be seen that the energy flux analyses agree very well with the Campbell diagrams.

4.3.3 Stationary Constraint With Damping and Friction Forces

For the cases when the interactive force is a non-conservative force, such as a viscous damping force or a friction force, Equation (4.14) can be written in the following forms:

For the case of a stationary viscous damping:

$$\Delta E = \int_0^T [\Omega r_0 F_\theta (t) - cw^2, (r_0, \theta_0, t)]dt = \int_0^T F_c (t)[w_\psi (r_0, \theta_0, t) + \Omega w_\phi (r_0, \theta_0, t)]dt$$  (4.17)

For the case of a stationary frictional force:
\[ \Delta E = \int_0^\infty F_\mu(t)[w_r(r_0, \theta_0, t) + \Omega w_\theta(r_0, \theta_0, t)]dt \]  \hspace{1cm} (4.18)

where, \( F_c(t) = -cw_r(r_0, \theta_0, t) \) and \( F_\mu(t) = -\mu w_\theta(r_0, \theta_0, t)/r_0 \).

From Equations (4.17) and (4.18) it is noted that there are two energy flows associated with these non-conservative interactive forces: one is the energy introduced by the damping or friction force and the other is that dissipated by the same element.

Figure 4.6 schematically illustrates the effects of viscous damping and friction on the instability of the travelling-wave modes. The solid curve represents the forward wave or the backward wave at supercritical speeds (where \( \Omega \) is greater than the wave speed \( S_n \) of a given mode) observed in the space-fixed coordinates, and the dashed curves represent the backward and forward waves observed in the coordinates rotating with the disc. From the interaction shown in this figure it can be seen that the lateral damping force \( F_c \) caused by the backward wave at supercritical speeds is always in phase with the backward wave velocity \( w_r(r_0, \theta_0, t) \) but the damping force \( F_c \) due to the backward wave at subcritical speeds is always out of phase with the velocity \( w_r(r_0, \theta_0, t) \) whose direction is independent of the rotating speed. It may also be noted that the damping force induced by the forward wave mode is out of phase with the forward wave velocity \( w_r(r_0, \theta_0, t) \) measured in the disc-fixed coordinates. Therefore, instability can only occur when the backward wave mode is subjected to the stationary damping force at supercritical speeds, which has been proven by the theoretical analysis shown in Figure 4.7. Figure 4.8 illustrates the energy flux \( \Delta E \) at three different locations \( A, B \) and \( C \) shown in Figure 4.7, which also match the analytical results.

From Figure 4.6 it may also be noted that, in the case of friction, \( F_\mu \) is always out of phase with \( w_r(r_0, \theta_0, t) \) for the backward wave mode and is always in phase with \( w_r(r_0, \theta_0, t) \) for the forward wave mode, regardless of rotating speed. This result agrees with the theoretical eigenvalue analysis shown in Figure 4.9.
Figure 4.6: Physical explanations for instabilities due to a viscous damper or a frictional force

Figure 4.7: Instability regions of a rotating disc due to a stationary viscous damper

\(c = 10 \text{ Ns/m}\)
Figure 4.8 Energy flux ($J$) due to stationary viscous damper ($c=10$ Ns/m) at different rotating speeds

Figure 4.9 Instability of a rotating disc subjected to a stationary friction (Mode (0,3), $\mu=10$ N, without centrifugal stiffening effect)
4.4 Summary

This chapter has established a generalized approach to investigate the instability mechanisms that are involved in the interaction between a rotating disc and an arbitrary interactive force. An energy flux equation has been developed, which leads to an understanding of the physical mechanisms of instability.

Specific conclusions are:

1) The possibility of the occurrence of instability due to any interactive forces may be identified based on the energy flux analysis, even without solving equations.

2) Instabilities will occur if the interactive forces are in phase with the velocity measured at the interactive point from rotating coordinates.

3) Instability cannot occur when a rotating disc subjected to a stationary constant lateral force, but a stationary harmonic lateral force, a moving constant lateral force or a moving harmonic lateral force may cause instability.

4) Conservative forces may only cause coupling instability associated with two modes, and non-conservative forces usually cause terminal instability where only one mode is involved.
Chapter 5

Self-Excited Vibrations in Saw-Blade Cutting — Theoretical Approaches

5.1 Introduction

This chapter deals with the self-excited vibrations that are involved in the interaction between a rotating flexible disc and a work-piece. The aim is to understand the stability characteristics of the blade subjected to various cutting forces in order to find the dominant mechanism of instability which matches the characteristics of self-excited vibration identified from saw-blade cutting. Although much research has been conducted in circular saw dynamics in the past two decades, most of this research has focused on the dynamics of an idling saw-blade. An analysis of the self-excited vibrations due to multiple moving interactive cutting forces had not been solved and an understanding of the dynamics of washboarding was not available.

Carlin, et al. (1975) [7] first studied the effects of a concentrated radial edge force on the natural frequencies of a spinning disc and analyzed the manner in which the asymmetric stress fields caused by the edge forces affect the natural frequencies of the disc. Radcliffe and Mote (1977) [18] considered a more general concentrated edge force with both radial and tangential components. Chen and Bogy (1993) [19] determined the membrane stress fields caused by a stationary circumferential friction force in a spinning disc, and found that these asymmetric membrane stress fields cannot cause instability in a spinning disc. Chonan et al. (1993) [20] studied the self-excited vibration of a pre-tensioned saw blade subjected to a small in-plane slicing force, and their results also showed no significant effect of in-plane force on the natural frequencies and stability of the blade.
Chen (1994) [21] reformulated the problem of a rotating disc subjected to a stationary concentrated in-plane radial force and included the effects of gyroscopic terms. It was reported in his paper that both divergence and flutter instabilities can be induced by stationary in-plane radial forces at critical or supercritical speeds. In an independent study, Shen and Song (1996) [22] treated a cutting saw blade as a rotating disc subjected to stationary follower edge forces with both radial and tangential components and predicted the instability of this system through the multiple scale method. It was reported in their paper that the radial edge force determines the rotating speed regions where the instability occurs and the tangential edge force only affects the width of the instability zones without introducing new unstable regions.

Recently Chen (1997) [23] modeled a saw-blade system as a spinning disc under a space-fixed periodically varying edge force and investigated the dynamic stability of the rotating disc subject to a pulsating in-plane radial force by using an extended multiple scale method.

In present study, a more realistic cutting model (i.e., multiple moving concentrated cutting forces over a given space-fixed sector, which include the effect of the wood grain direction) is suggested to investigate the stability characteristics of the saw-blade and work-piece interaction. Furthermore, in previous research, the self-excited instability in saw-blade cutting was assumed to be produced by a in-plane edge force. A more important cutting force which might dominate the dynamics of the blade during cutting has not been investigated in the previous studies, and this is the lateral regenerative cutting force. Machine tool chatter induced by regenerative cutting forces in turning and milling has been an active research topic for decades [24, 25, 26, 27, 28]. In this study the stability analysis associated with regenerative metal cutting forces is extended to saw-blade cutting.

In this chapter, the stability analysis for a rotating disc subjected to the following interactive cutting forces are considered:

1) multiple moving regenerative lateral cutting forces caused by flank cuts;
2) transverse components of multiple moving follower radial and tangential cutting forces;  
3) asymmetric membrane stress fields resulting from multiple moving in-plane cutting edge forces.

The equations of motion for a spinning disc under multiple moving lateral and in-plane forces applied at the outer rim of the blade are derived and solved for the first time.

New developments involve solution methods for the stability analysis of a rotating disc subjected to different types of multiple moving concentrated cutting forces over a given space-fixed sector. The basic Fourier series method is extended to solve the stability problem for time-varying equations with or without time lag terms, and the point mapping algorithm is employed to deal with the instability analysis of the time-varying system without time delay.

5.2 Instabilities Due to Multiple Moving Regenerative Cutting Forces

In the dynamics of saw-blade cutting the cutting forces have traditionally been modeled as constant or pulsating in-plane edge forces (Shen, et al., 1996 and Chen, 1997) [22, 23]. A further cutting force, produced by flank cutting which may dominate saw-blade cutting dynamics, has been neglected in previous research. A detailed theoretical study on the instability induced by multiple moving regenerative cutting forces over a given space-fixed sector is presented in this section.

Figure 5.1 shows a rotating circular blade which cuts a work piece over a space-fixed sector. It can be seen from the sectional plot A-A that, if the blade oscillates laterally, there is an extra lateral cutting area between two successive teeth associated with both the transverse response \( w(r_0, \theta_j, t) \) of the current tooth (the \( j \)th tooth) and the transverse response \( w(r_0, \theta_j, t-T) \) of the previous tooth (the \((j-1)\)th tooth) at a given location \((r_0, \theta_j)\) on the work-piece. \( T \) is the tooth passing period (i.e., the period between successive tooth engagements). Lateral compression between the work-piece and the teeth of the
blade in this extra cutting area causes the lateral cutting force $f_{cj}(t)$. This type of cutting force is called regenerative cutting force.

The following assumptions are made in this study:

- The effect of slope angles due to deflection of the blade on the cutting force is neglected because of very small deflections of the blade.

- The cutting force is a linear function of chip thickness because of small chip thickness.

Thus, the lateral regenerative cutting forces are assumed in the following linear form:

$$f_c(t) = \sum_{j=1}^{N_c} f_{cj}(t) = -\sum_{j=1}^{N_c} \left(1/r\right)K_r[w(r, \theta, t) - w(r, \theta, t - T)]\delta(r - r_0)\delta(\theta - \theta_j)g(\theta_j) \quad (5.1)$$
where, $N_t$ is the total number of saw teeth. $K_r$ is a cutting force coefficient which is determined by the geometry, properties, and speeds of the blade and by the characteristics of the work piece. For an anisotropic material, $K_r$ is a function of the position $\theta_j$ in the space-fixed coordinates.

$$w(r, \theta, t) \delta(r - r_0) \delta(\theta - \theta_j) = w(r_0, \theta_j, t)$$

and

$$w(r, \theta, t - T) \delta(r - r_0) \delta(\theta - \theta_j) = w(r_0, \theta_j, t - T)$$

represent the responses of present and previous teeth at the location $(r_0, \theta_j)$, respectively. $T = 2\pi \Omega (N_t)$. and $\theta_j = \theta_{st} + \Omega t + (j - 1) \theta_p$ ($\theta_1 = \theta_{st}$ when $t = 0$; $\theta_p$ is the angular tooth pitch)

$g_j(\theta_j) = 1$ when $\theta_{st} < \theta_j < \theta_{ex}$ and $g_j(\theta_j) = 0$ otherwise

where $\theta_{st}, \theta_{ex}$ are the start and exit immersion angles of the cutting range, respectively.

Under such assumptions, the major stability characteristics of the blade due to this type of cutting force still remains unchanged. These assumptions result in a linear equation of motion and an elegant theoretical solution such that numerical integration for solving the non-linear equations can be avoided.

Substituting Equation (5.1) into Equation (4.1) and applying Galerkin procedure lead to an equation of motion in the form:

$$[M][\ddot{x}(t)] + [G][\dot{x}(t)] + [K][x(t)] + (1 - e^{-TD})[A(t)][x(t)] = [0]$$

(5.2)

where, $e^{-TD}$ is a time delay operator (i.e., $e^{-TD}[x(t)] = [x(t - T)]$). $[M]$, $[G]$ and $[K]$ represent the mass, gyroscopic and stiffness matrices, respectively, which may contain centrifugal stiffening and/or stress tensioning effects (see Equation (2.34) for the details of $[M]$, $[G]$, $[K]$ and $\{x\}$). $[A(t)]$ is a time-varying matrix associated with the cutting forces, whose elements are defined in Equations (5.31) and (5.32).
It can be easily proven that each term in \([A(t)] \) is periodic with period \(T\) (i.e., \([A(t+T)] = [A(t)]\)). The proof is given as follows:

The general form of each term \(a_{kl}(t)\) in \([A(t)]\) can be expressed from Equations (5.31) and (5.32) as:

\[
a_{kl}(t) = \sum_{j=1}^{N} g(\theta_j) P_{kl}(\theta_j) = \sum_{j=1}^{N} Q(\theta_j) = \sum_{j=1}^{N} Q(t + (j-1)T)
\]  

(5.3)

where, \(\theta_j = \theta_{st} + \Omega t + (j-1)\theta_p = \theta_{st} + \Omega(t + (j-1)T)\). \(P_{kl}(\theta_j)\) is a function of \(\theta_j\) with finite bound. Therefore, based on the property of \(g(\theta_j)\), the terms \(Q(\theta_j) = g(\theta_j)P_{kl}(\theta_j)\) are always periodic with period \(N_tT\) that is the period of one revolution of the saw-blade,

\[
a_{kl}(t+T) = \sum_{j=1}^{N_t} g(\theta_j + \Omega T) P_{kl}(\theta_j + \Omega T) = \sum_{j=1}^{N_t} Q(t + jT) = \sum_{j=1}^{N_t} Q(t + jT) + Q(t + N_tT)
\] 

\[
= \sum_{j=1}^{N_t-1} Q(t + jT) + Q(t) = \sum_{i=1}^{N_t} Q(t + (i-1)T) = a_{kl}(t)
\]  

(5.4)

Hence, \([A(t)]\) is periodic with the period \(T\).

The solution of Equation (5.2) with periodic coefficients may be assumed in the form of a Fourier series with frequency components: \(\lambda \pm ik\omega\) \((k = 0,1,2, \cdots)\) \((\omega = 2\pi / T; i = \sqrt{-1})\) (Bolotin, V.V., 1964 [29] and Lengoc, 1995 [30]):

\[
\{x(t)\} = \left\{b_0\right\} / 2 + \sum_{k=1}^{\infty} \left\{a_k\right\} \sin k\omega t + \left\{b_k\right\} \cos k\omega t e^{i\lambda t}
\]  

(5.5)

where, \(\{b_0\}, \{b_k\}\) and \(\{a_k\}\) are time-invariant coefficient vectors. \(\lambda\) is the characteristic variable of the system.

Since \([A(t)]\) is periodic, it can also be expressed in a Fourier series form:
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\[ [A(t)] = \frac{1}{2} [B_0] + \sum_{k=1}^{\infty} ([A_k] \sin k\omega t + [B_k] \cos k\omega t) \quad (5.6) \]

where, \[ [B_0] = \frac{2}{T} \int_0^T [A(t)] dt \]
\[ [B_k] = \frac{2}{T} \int_0^T [A(t)] \cos(k\omega t) dt \]
\[ [A_k] = \frac{2}{T} \int_0^T [A(t)] \sin(k\omega t) dt \quad (5.7) \]

Substituting Equations (5.5) and (5.6) into Equation (5.2) and equating the coefficients of \( e^{\lambda_1} \), \( e^{\lambda_2} \sin k\omega t \) and \( e^{\lambda_2} \cos k\omega t \) to zero lead to the following form of the characteristic equation of the system:

\[ (\lambda^2 [H_2] + \lambda [H_1] + [H_0] + (1 - e^{-TA})[L])\{s\} = \{0\} \quad (5.8) \]

where, \( \{s\} = (\{b_0\} \{a_1\} \{b_1\} \cdots \{a_{N_s}\} \{b_{N_s}\})^T \). \( N_s \) is the number of components included in the Fourier series. The coefficient matrices \( [H_2] \), \( [H_1] \), \( [H_0] \) and \( [L] \) consist of \([K] \), \([G] \), \([M] \), \([B_0] \), \([B_k] \), \([A_k] \) and \( \omega \).

For the zero order approximation (i.e., \( N_s = 0 \)), Equation (5.8) can be written as:

\[ (\lambda^2 [M] + \lambda [G] + [K] + (1 - e^{-TA})[B_0] / 2)\{b_0\} = \{0\} \quad (5.9) \]

For the first order approximation (i.e., \( N_s = 1 \)), \( [H_2] \), \( [H_1] \), \( [H_0] \) and \( [L] \) are (for the sake of simplicity of expression, let \( M = [M] \), \( G = [G] \), \( K = [K] \), etc.):

\[ [H_2] = \begin{bmatrix} \frac{1}{2}M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix} \quad (5.10) \]
\[
\begin{bmatrix}
\frac{1}{2} G & 0 & 0 \\
0 & G & -2\omega M \\
0 & 2\omega M & G
\end{bmatrix}
\] (5.11)

\[
\begin{bmatrix}
\frac{1}{2} K & 0 & 0 \\
0 & K - \omega^2 M & -\omega G \\
0 & \omega G & K - \omega^2 M
\end{bmatrix}
\] (5.12)

\[
\begin{bmatrix}
\frac{1}{2} B_0 & \frac{1}{2} A_1 & \frac{1}{2} B_1 \\
\frac{1}{2} A_1 & \frac{1}{2} (B_0 - B_2) & \frac{1}{2} A_2 \\
\frac{1}{2} B_1 & \frac{1}{2} A_2 & \frac{1}{2} (B_0 + B_2)
\end{bmatrix}
\] (5.13)

\[
\{s\} = (\{b_0\} \ [a_1] \ [b_1])^T
\]

For a general case (i.e., the \( N_s \) th order approximation), the coefficient matrices in the characteristic equation (5.8) can be written as:

\[
\begin{bmatrix}
\frac{1}{2} M \\
M & 0 \\
& M \\
& & \ddots \\
0 & M & \cdots \\
& & & M
\end{bmatrix}
\] (5.14)

\[
\begin{bmatrix}
\frac{1}{2} G & 0 & 0 \\
G & -2\omega M & 0 \\
& 2\omega M & G \\
0 & 0 & G \\
& & -2N_s \omega M \\
& & 2N_s \omega M & G
\end{bmatrix}
\] (5.15)
\[ [H_0] = \begin{bmatrix}
\frac{1}{2} K & K - \omega^2 M & -\omega G & 0 \\
\omega G & K - \omega^2 M & 0 & 0 \\
0 & \omega G & K - \omega^2 M & -N_s \omega G \\
0 & N_s \omega G & K - N_s^2 \omega^2 M & 0
\end{bmatrix} \] (5.16)

\[ [L] = \\
\begin{bmatrix}
\frac{1}{2} B_0 & \frac{1}{2} A_1 & \frac{1}{2} B_1 & \cdots & \cdots & \frac{1}{2} A_{N_s} & \frac{1}{2} B_{N_s} \\
\frac{1}{2} A_1 & \frac{1}{2} (B_0 - B_2) & \frac{1}{2} A_2 & \cdots & \cdots & \frac{1}{2} (B_l - B_{2N_s-l}) & \frac{1}{2} (A_l + A_{2N_s-l}) \\
\frac{1}{2} B_1 & \frac{1}{2} A_2 & \frac{1}{2} (B_0 + B_2) & \cdots & \cdots & \frac{1}{2} (A_l + A_{2N_s-l}) & \frac{1}{2} (B_l + B_{2N_s-l}) \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\frac{1}{2} A_{N_s} & \cdots & \frac{1}{2} (B_l - B_{2N_s-l}) & \frac{1}{2} (A_l + A_{2N_s-l}) & \cdots & \frac{1}{2} (B_0 - B_{2N_s}) & \frac{1}{2} A_{2N_s} \\
\frac{1}{2} B_{N_s} & \cdots & \frac{1}{2} (-A_l + A_{2N_s-l}) & \frac{1}{2} (B_l + B_{2N_s-l}) & \cdots & \frac{1}{2} A_{2N_s} & \frac{1}{2} (B_0 + B_{2N_s})
\end{bmatrix} \] (5.17)

\[ \{s\} = (\{b_0\} \quad \{a_1\} \quad \{b_1\} \quad \cdots \quad \{a_{N_s}\} \quad \{b_{N_s}\})^T \]

where, the general expression of diagonal sub-matrices in \([L]\) can be written as:

\[ \begin{bmatrix}
\frac{1}{2} (B_0 - B_{2N_s}) & \frac{1}{2} A_{2N_s} \\
\frac{1}{2} A_{2N_s} & \frac{1}{2} (B_0 + B_{2N_s})
\end{bmatrix} \] (5.18)

and the non-diagonal sub-matrices above the diagonal sub-matrix shown in Equation (5.17) can be expressed in the following form:

\[ \begin{bmatrix}
\frac{1}{2} (B_{0+l} - B_{2N_s-l}) & \frac{1}{2} (-A_{0+l} + A_{2N_s-l}) \\
\frac{1}{2} (A_{0+l} + A_{2N_s-l}) & \frac{1}{2} (B_{0+l} + B_{2N_s-l})
\end{bmatrix} \quad (l = 1, 2, \ldots, N_s - 1) \] (5.19)
where, \( l \) is the number of sub-matrix rows counting from the diagonal sub-matrix up to the second row sub-matrix.

As an example of \([L]\) when \( N_s = 3\), the following matrix will be obtained:

\[
\begin{bmatrix}
\frac{1}{4} B_0 & \frac{1}{2} A_1 & \frac{1}{2} B_1 & \frac{1}{2} A_2 & \frac{1}{2} B_2 & \frac{1}{2} A_3 & \frac{1}{2} B_3 \\
\frac{1}{2} A_1 & \frac{1}{2} (B_0 - B_2) & \frac{1}{2} A_2 & \frac{1}{2} (B_1 - B_3) & \frac{1}{2} (-A_1 + A_3) & \frac{1}{2} (B_2 - B_4) & \frac{1}{2} (-A_2 + A_4) \\
\frac{1}{2} B_1 & \frac{1}{2} A_2 & \frac{1}{2} (B_0 + B_2) & \frac{1}{2} (A_1 + A_3) & \frac{1}{2} (B_1 + B_3) & \frac{1}{2} (A_2 + A_4) & \frac{1}{2} (B_2 + B_4) \\
\frac{1}{2} A_2 & \frac{1}{2} (B_1 - B_3) & \frac{1}{2} (A_1 + A_3) & \frac{1}{2} (B_0 - B_4) & \frac{1}{2} A_4 & \frac{1}{2} (B_1 - B_5) & \frac{1}{2} (-A_1 + A_5) \\
\frac{1}{2} B_2 & \frac{1}{2} (-A_1 + A_3) & \frac{1}{2} (B_1 + B_3) & \frac{1}{2} A_4 & \frac{1}{2} (B_0 + B_4) & \frac{1}{2} (A_1 + A_5) & \frac{1}{2} (B_1 + B_5) \\
\frac{1}{2} A_3 & \frac{1}{2} (B_2 - B_4) & \frac{1}{2} (A_2 + A_4) & \frac{1}{2} (B_1 - B_5) & \frac{1}{2} (A_1 + A_5) & \frac{1}{2} (B_0 - B_6) & \frac{1}{2} A_6 \\
\frac{1}{2} B_3 & \frac{1}{2} (-A_2 + A_4) & \frac{1}{2} (B_2 + B_4) & \frac{1}{2} (-A_1 + A_5) & \frac{1}{2} (B_1 + B_5) & \frac{1}{2} A_6 & \frac{1}{2} (B_0 + B_6)
\end{bmatrix}
\]

(5.20)

Müller's algorithm with deflation for solving nonlinear equations (Mathews, 1956 [31]) is employed to find the roots of Equation (5.8).

### 5.2.1 Single-Mode Solution

The single mode solution is assumed by:

\[
w(r, \theta, t) = R_{mn}(r)[C_{mn}(t)\cos(n\theta) + S_{mn}(t)\sin(n\theta)]
\]

(5.21)

Substituting Equation (5.21) into Equation (4.1) without considering centrifugal stiffening or tensioning and applying the Galerkin procedure lead to the following equation:

\[
\begin{bmatrix}
M_n & 0 \\
0 & M_n
\end{bmatrix}\begin{bmatrix}
\dot{C}_{mn}(t) \\
\dot{S}_{mn}(t)
\end{bmatrix} + \begin{bmatrix}
0 & 2\Omega n M_n \\
-2\Omega n M_n & 0
\end{bmatrix}\begin{bmatrix}
\dot{C}_{mn}(t) \\
\dot{S}_{mn}(t)
\end{bmatrix} + \begin{bmatrix}
K_n - \Omega^2 n^2 M_n & 0 \\
0 & K_n - \Omega^2 n^2 M_n
\end{bmatrix}\begin{bmatrix}
C_{mn}(t) \\
S_{mn}(t)
\end{bmatrix}
\]

+ \begin{bmatrix}
C_{mn}(t) \\
S_{mn}(t)
\end{bmatrix}(1 - e^{-TD})[A(t)] = \{0\}
\]

(5.22)
where, \( M_n = \rho h \pi \int_{a}^{b} R_{mn}^2(r) r dr \)

\[
K_n = D \pi \left[ \int_{a}^{b} \left( \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right) R_{mn}(r) \right) R_{mn}(r) r dr \right]
\]

and

\[
[A(t)] = \sum_{j=1}^{N} g(\theta_j) [P(\theta_j)]
\]

where, \([P(\theta_j)] = R_{mn}^2(r_0) \begin{bmatrix} K_r(\theta_j) \cos^2(n \theta_j) & K_r(\theta_j) \sin(2n \theta_j) / 2 \\ K_r(\theta_j) \sin(2n \theta_j) / 2 & K_r(\theta_j) \sin^2(n \theta_j) \end{bmatrix} \). \( K_r(\theta_j) \) is a function of position \( \theta_j \) for the case where the material of the work-piece is anisotropic.

The coefficient matrices \([B_0], [B_k], \text{ and } [A_k] \ (k = 1, 2, \cdots)\) in Equation (5.6) can be efficiently calculated by using the following scheme:

\[
[B_0] = \frac{2}{T} \int_{0}^{T} [A(t)] dt = \frac{2}{T} \int_{0}^{T} \sum_{j=1}^{N} g(\theta_j) [P(\theta_j)] dt = \frac{2}{T} \int_{0}^{T} \sum_{j=1}^{N} g(t + (j - 1)T) [P(t + (j - 1)T)] dt
\]

\[
= \frac{2}{T} \sum_{j=1}^{N} \int_{(j-1)T}^{jT} g(\tau) [P(\tau)] d\tau = \frac{2}{\Omega T} \int_{0}^{N \Omega} g(\theta) [P(\theta)] d\theta
\]

Since \( T = \frac{2\pi}{N \Omega} \), \([B_0]\) can be rewritten as:

\[
[B_0] = \frac{N \pi}{\Omega} \int_{0}^{\theta} g(\theta) [P(\theta)] d\theta = \frac{N \pi}{\Omega} \int_{0}^{\theta} [P(\theta)] d\theta
\]

Similarly,

\[
[B_k] = \frac{N \pi}{\Omega} \int_{0}^{\theta} [P(\theta)] \cos(kN, \theta) d\theta
\]

\[
[A_k] = \frac{N \pi}{\Omega} \int_{0}^{\theta} [P(\theta)] \sin(kN, \theta) d\theta
\]

Substituting Equations (5.24)-(5.26) into the characteristic equation (5.8) and employing Müller’s optimization algorithm to solve this complex nonlinear equations yield the eigenvalues of the system.
Figure 5.2 shows the Campbell diagram for a rotating clamped circular saw-blade \(a=0.08m, b=0.28m, h=0.0015m, E=2.08e+11\, N/m^2, \rho = 7850kg/m^3, \nu = 0.3, \theta_{st} = 20^\circ, \theta_{ex} = 50^\circ, N_r = 40\). From this figure it was found that the primary instability (i.e., the widest unstable region \(A\)) occurs when the tooth passing frequency \(f_t (= (\Omega / 2\pi)N_r\)) is greater than the natural frequency \(f_n(\Omega)\) and less than \(2f_n(\Omega)\), namely in the region \([f_n(\Omega), 2f_n(\Omega)]\) (Note that the natural frequency \(f_n\) itself is also a function of rotation speed or tooth passing frequency \(f_t\)).

It was also found, from this Campbell diagram, that the rotating blade, subjected to regenerative cutting forces, also becomes unstable when:
The general expression for unstable regions is given by:

\[ f_t = \left[ \frac{1}{k} f_n(\Omega), \frac{1}{k - 0.5} f_n(\Omega) \right] (k = 1, 2, \ldots) \]

(5.27)

For example, when \( k = 1 \) the instability region is \( f_t = [f_n(\Omega), 2 f_n(\Omega)] \), which is the primary instability region. From the simulations it was found that this expression is true for all modes.

Figures 5.3 illustrates the real parts of the eigenvalues for different \( K_r \) values. As may be noted, although the magnitudes of the real parts of the eigenvalues grow with the increase of \( K_r \), the instability regions remains almost unchanged for this set of \( K_r \). A similar situation occurs when the number of active cutting teeth increases based on the change of the cutting depth of the work-piece, as shown in Figure 5.4.

The results from different order approximations of Equation (5.8) are presented in Figure 5.5. From this figure it can be seen that the zero-order approximation can give very accurate predictions for the eigenvalues of the system even when \( K_r \) and time-varying components in \([A(t)]\) are very large compared to \([K]\) and the high-order approximations almost reach the same result as the one from the zero-order approximation. The reason for this fact is that the high-order response in Equation (5.6) is much smaller than that of the zero-order approximation. This fact was verified by numerous simulations.

Figures 5.6 illustrates the real parts of the eigenvalues of the saw-blade when \( K_r = 100(N/m) \) and the Figure 5.7 shows the energy flux calculated at the points (A-G) shown in Figure 5.6. It is clear that the energy flux analysis matches the eigenvalue analysis very well.
Figure 5.3 Effect of cutting coefficient $K_r$ on instability regions (backward-wave mode (0, 2), the zero-order approximation)

Figure 5.4 Effect of different numbers of active cutting teeth on instability regions (backward-wave mode (0, 2), zero order approximation, $K_r = 10 \text{ N/m}$)
Figure 5.5 A comparison of the real parts of eigenvalues with respect to different order approximations (Mode = (0, 2), $K_r = 1.0E+4$ N/m)
(The small peak is caused by one of $\lambda \pm ik\omega$ terms ($k \neq 0$))

Figure 5.6 Instability regions caused by moving regenerative cutting forces
($K_r = 100$ N/m, the zero-order approximation)
It should be noted, from the above single-mode analysis, that an important feature of the instability regions produced by the regenerative cutting force is that the self-excited vibration of the blade can be built up when the tooth passing frequency $f_t$ (or its multiples) is slightly greater than the corresponding natural frequency $f_n$. In these cases, the equation for energy flux analysis can be expressed from Equation (4.6) in the following form:

$$
\Delta E = \sum_i \int_0^T \dot{w}_i(t) f_{ci}(t) \, dt = \sum_i \int_0^T (\dot{w}_i + \Omega w_{,\theta_i}) f_{ci}(t) \, dt
$$

(5.28)

where, $f_{ci}(t)$ is the regenerative cutting force produced by the $i$th active cutting tooth. $\dot{w}_i(t)$ represents the velocity of a point on the disc at the location of the $i$th cutting tooth, observed in rotating coordinates. If the regenerative cutting forces are in phase with the velocities at

Figure 5.7 Energy flux at the different tooth passing frequencies defined in Figure 5.6 ($K_r = 100$ N/m)
the cutting points measured in rotating coordinates, the driving energy of this system will be channeled into the vibration energy of the blade, which implies instability for the system.

5.2.2 Multi-Mode Solution

In the case of multiple modes, the response at the location \((r_0, \theta_j)\) for a rotating circular saw subjected to regenerative cutting forces can be assumed to have the same form as the one in Equation (2.28), that is:

\[
 w(r_0, \theta_j, t) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r_0) \left[ C_{mn}(t) \cos(n\theta_j) + S_{mn}(t) \sin(n\theta_j) \right] \right\} 
\]  
(5.29)

and the response of the blade at the previous cut \((t-T)\) at the same location can be expressed as follows:

\[
 w(r_0, \theta_j, t-T) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r_0) \left[ C_{mn}(t-T) \cos(n\theta_j) + S_{mn}(t-T) \sin(n\theta_j) \right] \right\} 
\]  
(5.30)

where, \(S_{mn}(t)\) and \(C_{mn}(t)\) can be assumed to have the same form as the one in Equation (5.5) with frequency components: \( \lambda \pm ik\omega \) (\( k = 0,1,2,\cdots; \omega = 2\pi / T \)).

Substituting Equation (5.1) along with Equations (5.29) and (5.30) into Equation (4.1) (without the term \(L_n(w)\)) and applying the Galerkin procedure result in the following equations:

\[
 \alpha_i \left[ \delta_{qml} \ddot{C}_{ml} + \delta_{qml} \dot{S}_{ml} + \delta_{qml} C_{ml} \right] + K_r (1 - e^{-DT}) \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ H_{mnq}^{(1)} C_{mn} + H_{mnq}^{(2)} S_{mn} \right] = 0
\]  
(5.31)

\[
 \beta_i \left[ \delta_{qml} \ddot{S}_{ml} - \delta_{qml} \dot{C}_{ml} + \delta_{qml} S_{ml} \right] + K_r (1 - e^{-DT}) \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ H_{mnq}^{(3)} C_{mn} + H_{mnq}^{(4)} S_{mn} \right] = 0
\]  
(5.32)

\((q = 0,1,2,\cdots, M; \ l = 0,1,2,\cdots, N)\)
where,

\[ H_{mnq}^{(1)} = \sum_{j=1}^{N_t} g(\theta_j) R_{mn}(r_0) R_{ql}(r_0) \cos(n\theta_j) \cos(l\theta_j) \]

\[ H_{mnq}^{(2)} = \sum_{j=1}^{N_t} g(\theta_j) R_{mn}(r_0) R_{ql}(r_0) \sin(n\theta_j) \cos(l\theta_j) \]

\[ H_{mnq}^{(3)} = \sum_{j=1}^{N_t} g(\theta_j) R_{mn}(r_0) R_{ql}(r_0) \cos(n\theta_j) \sin(l\theta_j) \]

\[ H_{mnq}^{(4)} = \sum_{j=1}^{N_t} g(\theta_j) R_{mn}(r_0) R_{ql}(r_0) \sin(n\theta_j) \sin(l\theta_j) \]

and \( \delta_{qml}^{(1)}, \delta_{qml}^{(2)}, \) and \( \delta_{qml}^{(3)} \) are the same as the ones in Equation (2.32).

Figure 5.8 Instability regions caused by moving regenerative cutting forces
(Two modes (0,5) and (0,6), \( K_r = 100 \) N/m, the zero-order approximation)
As presented previously, the zero-order approximation of Equation (5.8) can predict the stability regions very accurately in such a system. Therefore, in the multi-mode analysis, only the zero order truncation of the characteristic equation will be considered. Figure 5.8, for example, shows a two-mode solution for the rotating circular saw blade which has the same geometry as that used for the results shown in Figure (5.2). As may be noted, the results related to each mode in the two mode solution are very similar to the ones from the single mode solution and no coupling instability regions are found at subcritical speeds. This figure also reveals that the real parts of the eigenvalues in mode \((0,5)\) are greater than the ones in mode \((0,6)\). From numerous stability simulations in saw-blade cutting and metal cutting it was found that the levels of the real parts of the eigenvalues for different modes are associated with the corresponding displacements at the cutting points. This implies, in this idealized model, that the displacements contributed by mode \((0,5)\) were greater than those from mode \((0,6)\).

Figure 5.9 shows a two-mode Campbell diagram including supercritical speeds. From this figure it can be seen that, in such a system, both the backward-wave modes become unstable at supercritical speeds. This simulation result may explain why a clamped saw usually has high levels of vibration and poor cutting accuracy at supercritical speeds.

Figure 5.10 illustrates the four-mode Campbell diagram of the same blade. From the plot of the real parts of the eigenvalues it can be seen that, for a saw-blade having many vibration modes, several modes of the blade can be excited simultaneously and different number of vibration modes may be excited at different tooth passing frequencies (or rotating speeds). It should be mentioned that in saw-blade cutting the levels of the self-excited vibration of the blade during cutting is not only related to its stability characteristics (i.e., the real parts of the eigenvalues of the system) but also associated with its initial vibration conditions.
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Figure 5.9 Instabilities regions at subcritical and supercritical speeds (Two modes (0,5) and (0,6), $K_r = 100$ N/m, the zero-order approximation)

Figure 5.10 Instabilities regions at subcritical speeds (Four modes (0,4) - (0,7), $K_r = 100$ N/m, the zero-order approximation)
5.3 Instabilities Due to Multiple Moving Follower Cutting Forces

This section deals with the self-excited vibration of a rotating circular saw-blade subjected to the lateral components of follower tangential and radial cutting forces which are schematically shown in Figure 5.11.

Most previous research has focused on the stability analysis for a rotating disc subjected to a stationary circumferential friction traction with applications to braking systems in automobiles, guided saw systems and computer floppy disks. Celep (1979) [32] examined a free circular plate subjected to a non-conservative radial force, for which case only a divergence instability was found. Ono et al. (1991) [12] studied the stability characteristics of a computer disk which interacts with the recording head. It was observed that the circumferential friction tends to destabilize certain vibration modes over the entire rotating speed range. Chan et al. (1995) [33] and Mottershead and Chan (1995) [34] employed the finite element method and a multiple scale algorithm to study the parametric instability of an annular braking disc loaded by rotating concentrated and distributed follower forces. Results similar to the ones found in reference [12] were reached in their research.
In the present study, the follower radial and tangential cutting forces are modeled as multiple moving concentrated forces which are more realistic than the models used in previous research. The self-excited vibrations due to these multiple moving forces are investigated and the point mapping algorithm for stability analysis is employed to deal with the resulting time-varying system without time delay.

5.3.1 Follower Tangential Cutting Forces

The moving circumferential cutting forces are assumed tangential to the blade. Thus, they are called follower tangential cutting forces and may play an important role in the self-excited vibration in saw-blade cutting. The transverse components of such forces can be modeled as follows:

\[
f_c(t) = -\sum_{j=1}^{N_c} \left[ \frac{1}{r} \delta(r - r_0) \delta(\theta - \theta_j) g(\theta_j) [F_{ij}(t) w_{ij}(t) / r] \right]
\]

where, \( F_{ij}(t) \) may be assumed to have the form: \( F_{ij}(t) = K_t H_j(t) \) \[28\]. \( H_j(t) = F \sin \theta_j \) is the cutting depth which can be easily proven to be a function of the position \( \theta_j \) for a circular saw blade. \( F \) is the feed per tooth of the work-piece. \( h \) is the thickness of the tooth. \( K_t \) is the cutting force coefficient determined by the cutting tool geometry and the work-piece property. For the case that the work-piece is anisotropic, \( K_t \) is also a function of the angular position \( \theta_j \).

In the case of multiple modes, the response of a rotating circular saw subjected to follower cutting forces can also be assumed to have the same form as the one in Equation (5.29). Substituting Equations (5.33) and (5.29) into Equation (4.1) (without \( L_n(w) \)) and applying the Galerkin procedure result in the following two equations:
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(5.34)

\[
\sum_{m=0}^{M} \alpha_i \left[\delta_{qml}^{(1)} \ddot{C}_{ml} + \delta_{qml}^{(2)} \dot{C}_{ml} + \delta_{qml}^{(3)} C_{ml}\right] + \sum_{n=0}^{N} \sum_{m=0}^{M} \left[\overline{H}_{mnq}^{(1)} C_{mn} + \overline{H}_{mnq}^{(2)} S_{mn}\right] = 0
\]

(5.35)

\[
\sum_{m=0}^{M} \beta_i \left[\delta_{qml}^{(1)} \ddot{C}_{ml} - \delta_{qml}^{(2)} \dot{C}_{ml} + \delta_{qml}^{(3)} S_{ml}\right] + \sum_{n=0}^{N} \sum_{m=0}^{M} \left[\overline{H}_{mnq}^{(3)} C_{mn} + \overline{H}_{mnq}^{(4)} S_{mn}\right] = 0
\]

\[(q = 0,1,2,\ldots,M; \ l = 0,1,2,\ldots,N)\]

where, \(\overline{H}_{mnq}^{(1)} = \sum_{j=1}^{N_t} C^t \sin(\theta_j)g(\theta_j)R_{mn}(r_0)R_{ql}(r_0)[-n \sin(n\theta_j)]\cos(l\theta_j) / r_0\)

\(\overline{H}_{mnq}^{(2)} = \sum_{j=1}^{N_t} C^t \sin(\theta_j)g(\theta_j)R_{mn}(r_0)R_{ql}(r_0)[n \cos(n\theta_j)]\cos(l\theta_j) / r_0\)

\(\overline{H}_{mnq}^{(3)} = \sum_{j=1}^{N_t} C^t \sin(\theta_j)g(\theta_j)R_{mn}(r_0)R_{ql}(r_0)[-n \sin(n\theta_j)]\sin(l\theta_j) / r_0\)

\(\overline{H}_{mnq}^{(4)} = \sum_{j=1}^{N_t} C^t \sin(\theta_j)g(\theta_j)R_{mn}(r_0)R_{ql}(r_0)[n \cos(n\theta_j)]\sin(l\theta_j) / r_0\)

\(C^t = K^t(\theta_j)hF\)

and \(\delta_{qml}^{(1)}, \delta_{qml}^{(2)}\) and \(\delta_{qml}^{(3)}\) are the same as the ones in Equation (2.32).

Equations (5.34) and (5.35) can be written in the following matrix form:

\[
[M][\ddot{x}] + [G][\dot{x}] + [K][x] + [D_1(t)][x] = \{0\}
\]

(5.36)

As proven in Section 5.2, \([D_1(t)]\) is periodic with the tooth passing period \(T\).

For a single mode case, \([D_1(t)]\) is as follows:
\[ [D_i(t)] = \frac{R_{mn}^2(r_0)}{r_0} \left[ \sum_{j=1}^{N_i} C_{\phi g}(\theta_j)(-n/2)\sin\theta_j \sin 2n\theta_j \sum_{j=1}^{N_i} C_{\phi g}(\theta_j)n\sin\theta_j \cos^2 n\theta_j \right. \\
+ \left. \sum_{j=1}^{N_i} C_{\phi g}(\theta_j)(-n)\sin\theta_j \sin^2 n\theta_j \sum_{j=1}^{N_i} C_{\phi g}(\theta_j)(n/2)\sin\theta_j \sin 2n\theta_j \right] \]
\[ (5.37) \]

The stability characteristics of Equation (5.36) can be efficiently determined by the generalized Fourier series method proposed in Section 5.2 or by the point mapping method for stability analysis. This latter method is most straightforward for dealing with a periodic system without time lag and is briefly described in this section.

According to the Floquet-Liapunov theorem [35, 36], a knowledge of the state transition matrix over one period is sufficient to determine the characteristics of the periodic system in the first order form (i.e., the state-space form) (Friedmann, et al., 1977, [37]):

\[ \{y\} = [A(t)]\{y\} \]
\[ (5.38) \]

where, \( \{y\} = (x \quad \dot{x})^T \), \([A(t)] = \begin{bmatrix} 0 & I \\ -M^{-1}[K + D_i(t)] & -M^{-1}G \end{bmatrix} \) and \([A(t + T)] = [A(t)].\)

Based upon the Floquet-Liapunov theorem, the stability of a periodic system can be predicted by the transition matrix \([\Gamma(T,0)]\) over one period \([0, T]\), which is defined by:

\[ \{y(T)\} = [\Gamma(T,0)]\{y(0)\} = e^{[\bar{A}]T}\{y(0)\} \]
\[ (5.39) \]

where, \([\bar{A}]\) is a time invariant matrix which determines the stability of the system.

If the eigenvalues of \([\bar{A}]\) are distinct, the \(i\)th damping factor \(\sigma_i\) and natural frequency \(\omega_i\) of the system are determined by:
\[ \sigma_i = \ln(A_{Ri}^2 + A_{Ai}^2) / (2T) \quad \text{and} \quad \omega_i = \tan(A_{Ri} / A_{Ai}) / T \] (5.40)

where \( \Lambda_{Ri} \) and \( \Lambda_{Ai} \) are the real and imaginary parts of the \( i \)th eigenvalue of \( \Gamma(T,0) \).

\( \Gamma(T,0) \) can be evaluated by several approximate methods (Hsu, 1972 [38], 1973 [39], 1974 [40]) where the most efficient one employs the fourth order Runge-Kutta scheme with Gill coefficients [37]. Based upon a Runge-Kutta integration scheme, the current response \( \{y_{j+1}\} \) can be expressed by using the previous one \( \{y_j\} \) with the step-size \( \Delta t \):

\[ \{y_{j+1}\} = [E(t_j)] \{y_j\} \] (5.41)

where \( [E(t_j)] \) can be found in reference [37].

Thus, \( \Gamma(T,0) \) can be obtained by one pass over one period with \( N_L \) intervals:

\[ \Gamma(T,0) = \prod_{j=1}^{N_L} [E(T - j\Delta t)] \] (5.42)

For this time-varying equation of motion (5.36) without time lag, the characteristic equation of the system is given by using the generalized Fourier series method:

\[ (\lambda^2[H_2] + \lambda[H_1] + [H_0] + [L])\{s\} = \{0\} \] (5.43)

and the stability characteristics can be easily analyzed by a classical eigenvalue solver.

Figure 5.12 shows the Campbell diagram of a rotating clamped circular saw-blade (\( a=0.076m, \ b=0.279m, \ h=1.117e-3m, \ E=2.08e+11 N/m^2, \ \rho = 7850 kg/m^3, \ \nu = 0.3, \ \theta_{st} = 20^\circ, \ \theta_{ex} = 50^\circ, \ N_t = 40 \) with respect to the tooth passing frequencies, which has been computed by the point mapping method. From this figure it can be seen that the forward-wave mode becomes unstable over the entire rotating speed range and the backward-wave mode is always stable, which is very similar to the result from the stationary friction case. An energy flux analysis was also conducted for this case, and found to give results that match those of the Campbell diagram very well.
Figure 5.12 Instability induced by multiple moving follower tangential cutting forces (without centrifugal stiffening effect) \( (C_f = 100 \text{ N}) \)

Figure 5.13 A comparison of the real parts of eigenvalues using different order approximations and algorithms \( (C_f = 1000 \text{ N}) \)
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Figure 5.14 Instability characteristics (follower tangential cutting force)
(Two mode solution, $C_f = 500 \text{ N}$, the zero-order approximation)

Figure 5.15 Instability characteristics (follower tangential cutting force)
(Four mode solution, $C_f = 1000 \text{ N}$, the zero-order approximation)
From numerous simulations it was found that the results from the point mapping method match the ones from the zero-order Fourier series method very well when $C_{fi}$ is relatively small compared to the diagonal elements of the stiffness matrix. However, a difference appears when $C_{fi}$ has the same level as the maximum value in the stiffness matrix. A comparison of the results from different methods and/or different orders is presented in Figure 5.13. It can be seen from this figure that the result from the point method is more accurate than the one from the zero-order approximation of the generalized Fourier series method (the GFS method), but it is not as accurate as the first and third order approximations of the GFS method.

Figures 5.14 and 5.15 show the two-mode and four-mode Campbell diagrams of the same saw blade with centrifugal stiffening. It may be noted that in the case shown in Figure 5.14 the modal interaction of two reflected-wave modes is always of the coupling type (such as region A). So is the modal interaction between one non-reflected wave mode and one reflected wave mode (such as region B). From Figure 5.15 it can also be seen that the real parts of the eigenvalues in the low-order modes are greater than the ones in the high-order modes due to relatively high stiffness in the high order modes compared to the time varying matrix $[D_i(t)]$ in Equation (5.36).

### 5.3.2 Follower Radial Cutting Forces

The radial components of cutting forces also exist in saw-blade cutting, and can be assumed to be tangential to the blade in the radial direction. These follower radial cutting forces may also play a role in the self-excited vibration of the blade. Their transverse components may be expressed as:

$$f_c(t) = -\sum_{j=1}^{N_c} (1/r) \delta(r-r_0) \delta(\theta - \theta_j) g(\theta_j) [F_{rj}(t)w_r(t)]$$  \hspace{1cm} (5.44)
where, the radial cutting force \( F_r(t) \) is assumed to be proportional to the tangential cutting force \( F_t(t) \) [28], namely, \( F_r(t) = K_r F_t = K_r h H(t) \). \( K_r \) is a ratio that is determined by the tooth geometry of the blade, the feed speed of the work-piece and so on.

Substituting Equations (5.44) and (5.29) into Equation (4.1) (without considering term \( L_n(w) \)) and applying the Galerkin procedure give the following equations:

\[
\sum_{m=0}^{M} \alpha_j \left[ \delta_{qml}^1 \ddot{C}_{ml} + \delta_{qml}^2 \dot{S}_{ml} + \delta_{qml}^3 C_{ml} \right] + \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ \tilde{H}_{mnql}^{(1)} C_{mn} + \tilde{H}_{mnql}^{(2)} S_{mn} \right] = 0 \quad (5.45)
\]

\[
\sum_{m=0}^{M} \beta_j \left[ \delta_{qml}^1 \ddot{S}_{ml} - \delta_{qml}^2 \dot{C}_{ml} + \delta_{qml}^3 S_{ml} \right] + \sum_{n=0}^{N} \sum_{m=0}^{M} \left[ \tilde{H}_{mnql}^{(3)} C_{mn} + \tilde{H}_{mnql}^{(4)} S_{mn} \right] = 0 \quad (5.46)
\]

where, \( \tilde{H}_{mnql}^{(1)} = \sum_{j=1}^{N} \frac{C_f r}{\sin(\theta_j)} g(\theta_j) \frac{\partial R_{mn}(r_0)}{\partial r} R_{mn}(r_0) R_{ql}(r_0) \cos(n \theta_j) \cos(l \theta_j) \)

\( \tilde{H}_{mnql}^{(2)} = \sum_{j=1}^{N} \frac{C_f r}{\sin(\theta_j)} g(\theta_j) \frac{\partial R_{mn}(r_0)}{\partial r} R_{mn}(r_0) R_{ql}(r_0) \sin(n \theta_j) \cos(l \theta_j) \)

\( \tilde{H}_{mnql}^{(3)} = \sum_{j=1}^{N} \frac{C_f r}{\sin(\theta_j)} g(\theta_j) \frac{\partial R_{mn}(r_0)}{\partial r} R_{mn}(r_0) R_{ql}(r_0) \cos(n \theta_j) \sin(l \theta_j) \)

\( \tilde{H}_{mnql}^{(4)} = \sum_{j=1}^{N} \frac{C_f r}{\sin(\theta_j)} g(\theta_j) \frac{\partial R_{mn}(r_0)}{\partial r} R_{mn}(r_0) R_{ql}(r_0) \sin(n \theta_j) \sin(l \theta_j) \)

\( C_f r = K_r K_l(\theta_j) h F \)

Equations (5.45) and (5.46) can be written in the matrix form:

\[
[M]\{\ddot{x}\} + [G]\{\dot{x}\} + [K]\{x\} + [D_r(t)]\{x\} = \{0\} \quad (5.47)
\]

where, \( [D_r(t)] \) is also periodic with period \( T \).
Figure 5.16 Instability characteristics (follower radial cutting force) (Three modes (0,3)-(0,5), $C_f=1000$ N, the zero-order approximation)

Figure 5.17 Instability characteristics (follower radial cutting force) (Four modes (0,1)-(0,4), $C_f=1000$ N, the zero-order approximation)
Similarly, the stability characteristics of Equation (5.47) can be easily analyzed by using the generalized Fourier series method and a classical eigenvalue solver.

Figure 5.16 shows a three-mode Campbell diagram of a rotating saw-blade with the same parameters as the one used in Figure (5.12), calculated using the zero-order approximation of the generalized Fourier series method. This figure draws attention to the fact that only the coupling type of instability occurs in this case and the modal interaction of two non-reflected wave modes is always of the veering type. On the other hand, the modal interaction between one reflected-wave mode and one non-reflected wave mode (the backward-wave mode in this case) is of the coupling type. Interestingly, it was also found in this example that the modal interaction two reflected-wave modes is also of the veering type. Figure 5.17 shows a four-mode Campbell diagram, from which the conclusions drawn from Figure 5.16 can be verified. It was also found that the coupling unstable regions among low-order modes are wider than those of high-order modes.

5.4 Instabilities Due to Asymmetric In-Plane Stress Fields Caused by Multiple Moving Cutting Forces

In previous studies (Shen and Song, 1996 [22]; Chen, 1997 [23]), the interaction between the rotating saw blade and the work-piece was modeled as a spinning disc subjected to asymmetric stress fields produced by space-fixed stationary or periodically varying edge forces. In the present study, a more realistic cutting model: the asymmetric in-plane stress fields induced by multiple moving concentrated cutting forces over a stationary sector is proposed to investigate these instabilities.

5.4.1 Instability Due to In-Plane Stress Fields Produced by Radial Cutting Forces

It is assumed that the stress fields produced by a constant stationary radial in-plane cutting force \( F_r \) can be expanded into a Fourier series [22, 23, 41]:
\[
\sigma_{rr} = F_r \sum_{k=0}^{\infty} Q_{rk}(r) \cos k\theta \\
(5.48)
\]
\[
\sigma_{\theta\theta} = F_r \sum_{k=0}^{\infty} Q_{\theta k}(r) \cos k\theta \\
(5.49)
\]
\[
\tau_{r\theta} = F_r \sum_{k=0}^{\infty} Q_{r\theta k}(r) \sin k\theta \\
(5.50)
\]

where, \( Q_{rk}(r) \), \( Q_{\theta k}(r) \) and \( Q_{r\theta k}(r) \) can be found in Appendix C.

![Diagram showing multiple moving radial and tangential cutting forces over a sector of work-piece](image)

**Figure 5.18** Multiple moving (a) radial and (b) tangential cutting forces over a sector of work-piece

However, the stress fields generated by multiple cutting forces \( F_{rj}(\theta_j) \) moving with the teeth, as shown in Figure 5.18(a), can be expressed as:

\[
\sigma_{rr} = \sum_{j=1}^{N_t} g(\theta_j) F_{rj}(\theta_j) \sum_{k=0}^{\infty} Q_{rk}(r) \cos k(\theta - \theta_j(t)) \\
(5.51)
\]
\[
\sigma_{\theta\theta} = \sum_{j=1}^{N_t} g(\theta_j) F_{rj}(\theta_j) \sum_{k=0}^{\infty} Q_{\theta k}(r) \cos (\theta - \theta_j(t)) \\
(5.52)
\]
\[
\tau_{r\theta} = \sum_{j=1}^{N_i} g(\theta_j) F_{rj}(\theta_j) \sum_{k=0}^{\infty} Q_{r0k}(r) \sin k(\theta - \theta_j(t))
\] (5.53)

where, \(g(\theta_j) = 1\) when \(\theta_{st} \leq \theta_j \leq \theta_{ex}\) (i.e., the \(j\)th tooth is in the cutting range), \(g(\theta_j) = 0\) for other cases. \(\theta_j = \theta_{st} + \Omega t + (j-1)\theta_p\) is the angular position of the \(j\)th tooth (\(\theta_p\) is the pitch angle). As described in Section 5.3, \(F_{rj}(t) = C_r \sin(\theta_j)\) for an isotropic work-piece, and \(F_{rj}(t) = C_r(\theta_j) \sin(\theta_j)\) for an anisotropic work-piece.

The force terms contributed by the in-plane stress fields can be expressed as [42]:

\[
L(w) = -\frac{h}{r} \left[ \frac{\partial}{\partial r} \left( r \sigma_{rr} \frac{\partial w}{\partial r} + \tau_{r\theta} \frac{\partial w}{\partial \theta} \right) + \frac{\partial}{\partial \theta} \left( \tau_{r\theta} \frac{\partial w}{\partial \theta} + \frac{1}{r} \sigma_{\theta\theta} \frac{\partial w}{\partial \theta} \right) \right]
\] (5.54)

Applying the Galerkin procedure to Equation (5.54), yields,

\[
\int_{b}^{a} \frac{2\pi}{\int_{b}^{a} L(w)[R_{ql}(r)\cos l\theta]d\theta d\phi} = 11 + 12 + 13 + 14 \quad (q,l = 0,1,2,...) \tag{5.55}
\]

\[
\int_{b}^{a} \frac{2\pi}{\int_{b}^{a} L(w)[R_{ql}(r)\sin l\theta]d\theta d\phi} = 15 + 16 + 17 + 18 \quad (q,l = 0,1,2,...) \tag{5.56}
\]

where,

\[
I_1 = \int_{b}^{a} \frac{2\pi}{\int_{b}^{a} \frac{\partial}{\partial r} \left( r \sigma_{rr} \frac{\partial w}{\partial r} \right) R_{ql}(r) \cos l\theta]d\theta d\phi
\] (5.57)

\[
I_2 = \int_{b}^{a} \frac{2\pi}{\int_{b}^{a} \frac{\partial}{\partial \theta} \left( \tau_{r\theta} \frac{\partial w}{\partial \theta} \right) R_{ql}(r) \cos l\theta]d\theta d\phi
\] (5.58)

\[
I_3 = \int_{b}^{a} \frac{2\pi}{\int_{b}^{a} \frac{\partial}{\partial \theta} \left( \tau_{r\theta} \frac{\partial w}{\partial \theta} \right) R_{ql}(r) \cos l\theta]d\theta d\phi
\] (5.59)

\[
I_4 = \int_{b}^{a} \frac{2\pi}{\int_{b}^{a} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \sigma_{\theta\theta} \frac{\partial w}{\partial \theta} \right) R_{ql}(r) \cos l\theta]d\theta d\phi
\] (5.60)

and

\[
I_5 = \int_{b}^{a} \frac{2\pi}{\int_{b}^{a} \frac{\partial}{\partial r} \left( r \sigma_{rr} \frac{\partial w}{\partial r} \right) R_{ql}(r) \sin l\theta]d\theta d\phi
\] (5.61)

\[
I_6 = \int_{b}^{a} \frac{2\pi}{\int_{b}^{a} \frac{\partial}{\partial \theta} \left( \tau_{r\theta} \frac{\partial w}{\partial \theta} \right) R_{ql}(r) \sin l\theta]d\theta d\phi
\] (5.62)
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\[ I7 = \int_b^a \left[ \frac{\partial}{\partial \theta} (\tau r \theta) \frac{\partial}{\partial r} R_{ql}(r) \sin l \theta \right] dr d\theta \]

(5.63)

\[ I8 = \int_b^a \left[ \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} R_{ql}(r) \sin l \theta \right] dr d\theta \]

(5.64)

Substituting Equations (5.51-5.53) and \( w(r, \theta, t) = \sum_{m=0}^{M} \sum_{n=0}^{N} R_{mn}(r)\left[ C_{mn}(t) \cos n \theta + S_{mn}(t) \sin n \theta \right] \)

into the Galerkin procedure yields,

\[ I1 = \sum_{j=1}^{N_1} g(\theta_j) P_{rj}(\theta_j) \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=0}^{N_k} A_{ik} \left[ (C_{ik} \cos k \theta_j + D_{ik} \sin k \theta_j) C_{mn}(t) + (E_{ik} \cos k \theta_j + F_{ik} \sin k \theta_j) S_{mn}(t) \right] \]

\[ = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=1}^{N_k} \left\{ \sum_{j=1}^{N_1} A_{ik} \left[ (C_{ik} B_{ck}(t) + D_{ik} B_{sk}(t)) C_{mn}(t) + (E_{ik} B_{ck}(t) + F_{ik} B_{sk}(t)) S_{mn}(t) \right] \right\} \]

(5.65)

where, \( B_{ck}(t) = \sum_{j=1}^{N_1} g(\theta_j) F_{rj}(\theta_j) \cos k \theta_j \) and \( B_{sk}(t) = \sum_{j=1}^{N_1} g(\theta_j) F_{rj}(\theta_j) \sin k \theta_j \)

\[ A_{ik} = \int_a^b \frac{\partial}{\partial r} (\tau r q_{ik}(r)) \frac{\partial}{\partial r} R_{ql}(r) dr \]

\[ C_{ik} = \int_0^{2\pi} \cos k \theta \cos n \theta \cos l \theta d\theta \]

\[ D_{ik} = \int_0^{2\pi} \sin k \theta \cos n \theta \cos l \theta d\theta \]

\[ E_{ik} = \int_0^{2\pi} \cos k \theta \sin n \theta \cos l \theta d\theta \]

\[ F_{ik} = \int_0^{2\pi} \sin k \theta \sin n \theta \cos l \theta d\theta \]

and \( N_k \) is the maximum number of Fourier components included in the calculation.

\[ I2 = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=1}^{N_k} \left\{ \sum_{j=1}^{N_1} A_{2ik} \left[ C_{ik} B_{ck}(t) + D_{ik} B_{sk}(t) \right] C_{mn}(t) + \sum_{k=1}^{N_k} \left[ E_{ik} B_{ck}(t) + F_{ik} B_{sk}(t) \right] S_{mn}(t) \right\} \]

(5.66)
where, \( A_{2k} = \int_{a}^{b} \frac{\partial}{\partial r} (Q_{r\theta k}(r)R_{mn}(r))R_{ql}(r)dr \)

\[
C_{2k} = \int_{0}^{2\pi} \sin k\theta(-n\sin n\theta)\cos l\theta d\theta \\
D_{2k} = \int_{0}^{2\pi} (-\cos k\theta)(-n\sin n\theta)\cos l\theta d\theta \\
E_{2k} = \int_{0}^{2\pi} \sin k\theta(n\cos n\theta)\cos l\theta d\theta \\
F_{2k} = \int_{0}^{2\pi} (-\cos k\theta)(n\cos n\theta)\cos l\theta d\theta
\]

\[ I3 = \sum_{m=0}^{M} \sum_{n=0}^{N} \left( \sum_{k=1}^{N_k} A_{3k} [C_{3k} B_{ck}(t) + D_{3k} B_{sk}(t)]C_{mn}(t) + \sum_{k=1}^{N_k} A_{3k} [E_{3k} B_{ck}(t) + F_{3k} B_{sk}(t)]S_{mn}(t) \right) \] (5.67)

where, \( A_{3k} = \int_{a}^{b} Q_{r\theta k}(r) \frac{\partial R_{mn}(r)}{\partial r} R_{ql}(r)dr \)

\[
C_{3k} = \int_{0}^{2\pi} \left( \sin k\theta \cos n\theta \right) \cos l\theta d\theta \\
D_{3k} = \int_{0}^{2\pi} \left( \sin k\theta \sin n\theta \right) \cos l\theta d\theta \\
E_{3k} = \int_{0}^{2\pi} \left( -\cos k\theta \cos n\theta \right) \cos l\theta d\theta \\
F_{3k} = \int_{0}^{2\pi} \left( -\cos k\theta \sin n\theta \right) \cos l\theta d\theta
\]

\[ I4 = \sum_{m=0}^{M} \sum_{n=0}^{N} \left( \sum_{k=1}^{N_k} A_{4k} [C_{4k} B_{ck}(t) + D_{4k} B_{sk}(t)]C_{mn}(t) + \sum_{k=1}^{N_k} A_{4k} [E_{4k} B_{ck}(t) + F_{4k} B_{sk}(t)]S_{mn}(t) \right) \] (5.68)

where, \( A_{4k} = \int_{a}^{b} \frac{1}{r} Q_{\theta k}(r)R_{mn}(r)R_{ql}(r)dr \)

\[
C_{4k} = \int_{0}^{2\pi} \left[ \cos k\theta(-n\sin n\theta) \right] \cos l\theta d\theta \\
D_{4k} = \int_{0}^{2\pi} \left[ \sin k\theta(-n\sin n\theta) \right] \cos l\theta d\theta
\]
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\[ E_{4k} = \int_{0}^{2\pi} \frac{\partial}{\partial \theta} [\cos k\theta (n \cos n\theta)] \cos l\theta d\theta \]

\[ F_{4k} = \int_{0}^{2\pi} \frac{\partial}{\partial \theta} [\sin k\theta (n \cos n\theta)] \cos l\theta d\theta \]

Similarly, 15, 16, 17 and 18 can be obtained by simply changing \( \cos l\theta \) to \( \sin l\theta \) in Equations (5.65)-(5.68).

Based on Equations (5.55) and (5.56), the equation of motion, which includes the contribution of the rotating stress fields produced by the multiple moving radial cutting forces, can be expressed in the following form:

\[
[M][\ddot{x}] + [G][\dot{x}] + [K][x] + [A^{(t)}][x] = \{0\} \tag{5.69}
\]

where, \( [A^{(t)}] \) is the matrix caused by the multiple radial cutting forces which are displacement dependent.

It should be noted from Equations (5.65)-(5.68) that the time-varying terms in \( [A^{(t)}] \) are \( B_{ck}(t) = \sum_{j=1}^{N_i} g(\theta_j) F_{\eta_j}(\theta_j) \cos k\theta_j \) and \( B_{sk}(t) = \sum_{j=1}^{N_i} g(\theta_j) F_{\eta_j}(\theta_j) \sin k\theta_j \). It can be easily shown that \( B_{ck}(t) \) and \( B_{sk}(t) \) are periodic with period \( T = 1/F_t \) (\( F_t \) is the tooth passing frequency). This feature provides the key to solve this stability problem.

For the time-varying equation of motion (5.69) without time lag, the characteristic equation of the system can be also expressed as the same form as Equation (5.43), which can be easily solved.

Figure 5.19 shows the Campbell diagram of a rotating circular saw-blade ( \( a=0.076m, b=0.279m, h=1.5e^{-3}m, \theta_{st} = 20^\circ, \theta_{ex} = 50^\circ, N_i = 40 \) ) subjected to the asymmetric in-plane stress fields produced by the multiple moving radial cutting forces, which is calculated by using the zero-order approximation of the generalized Fourier series method. This figure draws attention to the fact that both the coupling type of instability and divergence instability
can take place in this case. It is also noted that the modal interaction between one forward-wave mode and one backward-wave mode is of the veering type at subcritical speeds. On the other hand, the modal interaction between two backward-wave modes is of the coupling type even at subcritical speeds, which is different from the case of instability due to the transverse components of follower radial cutting forces, as shown in Figure 5.17. It is also noted that at supercritical speeds the modal interaction between one backward-wave mode and one reflected-wave mode is also of the coupling type.

Figure 5.19 Instabilities due to the asymmetric in-plane stress fields produced by radial cutting forces (Three modes (0,1)-(0,3), $C_r = -3000N$, $N_k = 10$)
Figure 5.20 Instabilities due to the asymmetric in-plane stress fields produced by radial cutting forces (Four modes (0,1)-(0,4), $C_r = -1000N$, $N_k = 10$)

Figure 5.21 Instabilities due to the asymmetric in-plane stress fields produced by radial cutting forces (Four modes (0,1)-(0,4), $C_r = -1000N$, $N_k = 20$)
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Figure 5.22 Instabilities due to the asymmetric in-plane stress fields produced by radial cutting forces (Two mode solution, \( C_r = -3000 N \), \( N_k = 10 \))

Figure 5.23 Instabilities due to the asymmetric in-plane stress fields produced by radial cutting forces for anisotropic work-piece (Two mode solution, \( C_{r0} = -3000 N \), \( N_k = 10 \))
In order to assess the effect of the number \( (N_k) \) of Fourier components taken in the stress field expressions, a comparison between the results from \( N_k = 10 \) and \( N_k = 20 \) is made in this study. The results are shown in Figures 5.20 and 5.21. From these figures it can be seen that the results are changed very little, which implies that it is sufficient for stability prediction to take the first ten Fourier components in the calculation in the case that only low-order modes are involved.

It was verified by numerous simulations that the coupling instability between two modes with \( n_1 \) and \( n_2 \) nodal diameters can occur only when a specific Fourier component of the stress fields: \( \cos(k\theta) \) or \( \sin(k\theta) \) with \( k = n_1 + n_2 \) is included in the calculation. For example, when \( n_1 = 3 \) and \( n_2 = 4 \), \( N_k \) should be at least 7 to make the model show the coupling type of instability. The physics behind this observation is that the vibration modes can be excited only when the distributed stress field of a specific Fourier component is in phase with the absolute velocity field which matches the stress field in the space. Therefore, this observation is also the rule to choose the minimum number of Fourier components included in the calculation.

Figures 5.22 and 5.23 illustrate the effect of an anisotropic work-piece upon the stability characteristics of the saw-blade cutting. For the case of an anisotropic work-piece, \( C_r \) can be assumed to have the form: \( C_r = C_{r0} f(\theta_j) \), where \( f(\theta_j) = 3\theta_j^3 \) in this example \( (\theta_{st} = 20^\circ \) and \( \theta_{ex} = 50^\circ \) so that \( F_{rj}(\theta_2)/F_{rj}(\theta_1) = 15.6 \), which means that the cutting force at \( \theta_{ex} = 50^\circ \) is about 16 times as large as that at \( \theta_{st} = 20^\circ \). It is evident from these figures that the introduction of an anisotropic property of the work-piece does not change the stability characteristics of the blade because the frequency of cutting force variation caused by the anisotropic property is much less than tooth passing frequency. This implies that the grain direction of the lumber cannot affect the stability characteristics in saw-blade cutting.

From this study it was also found that the stability of the blade due to the stress fields caused by multiple moving radial cutting forces over a sector of the work-piece is very similar to that produced by a single space-fixed pulsating radial force (Chen, 1997, [23]). This is not
surprising because, for the case of multiple moving cutting forces, the engagement and release of saw-teeth from the work-piece also cause pulsating edge forces with the tooth passing frequency.

### 5.4.2 Instability Due to In-Plane Stress Fields Produced by Tangential Cutting Forces

The stress field produced by a constant stationary tangential cutting force $F_t$, as shown in Figure 5.18(b), can be expressed as:

$$\sigma_{rr} = F_t \sum_{k=0}^{\infty} T_{rk}(r) \sin k\theta$$  \hspace{1cm} (5.70)

$$\sigma_{\theta\theta} = F_t \sum_{k=0}^{\infty} T_{\theta k}(r) \sin k\theta$$  \hspace{1cm} (5.71)

$$\tau_{r\theta} = F_t \sum_{k=0}^{\infty} T_{r\theta k}(r) \cos k\theta$$  \hspace{1cm} (5.72)

where, $T_{rk}(r)$, $T_{\theta k}(r)$ and $T_{r\theta k}(r)$ can be found in Appendix C.

However, the stress field generated by multiple tangential cutting forces $P_{ij}(\theta_j)$ moving with the teeth can be expressed as:

$$\sigma_{rr} = \sum_{j=1}^{N_i} g(\theta_j) F_{ij}(\theta_j) \sum_{k=0}^{\infty} T_{rk}(r) \sin k(\theta - \theta_j(t))$$  \hspace{1cm} (5.73)

$$\sigma_{\theta\theta} = \sum_{j=1}^{N_i} g(\theta_j) F_{ij}(\theta_j) \sum_{k=0}^{\infty} T_{\theta k}(r) \sin k(\theta - \theta_j(t))$$  \hspace{1cm} (5.74)

$$\tau_{r\theta} = \sum_{j=1}^{N_i} g(\theta_j) F_{ij}(\theta_j) \sum_{k=0}^{\infty} T_{r\theta k}(r) \cos k(\theta - \theta_j(t))$$  \hspace{1cm} (5.75)

where, $F_{ij}(t) = C_i \sin(\theta_j)$ for an isotropic work-piece, and $F_{ij}(t) = C_i(\theta_j)\sin(\theta_j)$ for an anisotropic work-piece.

Combining Equation (5.54) into the Galerkin procedure, yields,
\[
\int_{b}^{a} \int_{0}^{2\pi} L(w)[R_{ql}(r) \cos \theta] yrdrd\theta = J1 + J2 + J3 + J4 \quad (q, l = 0, 1, 2, \ldots) \quad (5.76)
\]

\[
\int_{b}^{a} \int_{0}^{2\pi} L(w)[R_{ql}(r) \cos \theta] yrdrd\theta = J5 + J6 + J7 + J8 \quad (q, l = 0, 1, 2, \ldots) \quad (5.77)
\]

where,

\[
J1 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \cos \theta | y | drd\theta
\]

\[
J2 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \cos \theta | y | drd\theta
\]

\[
J3 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \cos \theta | y | drd\theta
\]

\[
J4 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \cos \theta | y | drd\theta
\]

\[
J5 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \sin \theta | y | drd\theta
\]

\[
J6 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \sin \theta | y | drd\theta
\]

\[
J7 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \sin \theta | y | drd\theta
\]

\[
J8 = \int_{b}^{a} \int_{0}^{2\pi} \left( \frac{1}{r} \right) \left( \frac{\partial}{\partial r} \right) \left( \frac{\partial}{\partial \theta} \right) R_{ql}(r) \sin \theta | y | drd\theta
\]

Substituting Equations (5.73-5.75) into the Galerkin procedure yields,

\[
J1 = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left[ \overline{B}_{ck}(t) \overline{E}_{lk}(t) + \overline{D}_{lk}(t) \overline{S}_{mn}(t) \right] C_{mn}(t) + \sum_{k=1}^{\infty} \overline{B}_{sk}(t) \overline{S}_{mn}(t) \right\}
\]

\[
J1 = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=1}^{\infty} \left\{ \sum_{l=1}^{\infty} \left[ \overline{B}_{ck}(t) \overline{E}_{lk}(t) + \overline{D}_{lk}(t) \overline{S}_{mn}(t) \right] C_{mn}(t) + \sum_{k=1}^{\infty} \overline{B}_{sk}(t) \overline{S}_{mn}(t) \right\}
\]

where, \( \overline{B}_{ck}(t) = \sum_{j=1}^{N} g(\theta_j) F_j(\theta_j) \cos k \theta_j \) and \( \overline{B}_{sk}(t) = \sum_{j=1}^{N} g(\theta_j) F_j(\theta_j) \sin k \theta_j \)
\[ \bar{A}_{1k} = \int_{a}^{b} \frac{\partial}{\partial r} (r T_{rk}(r) \frac{\partial R_{mn}(r)}{\partial r}) R_{ql}(r) dr \]

\[ \bar{C}_{1k} = \int_{0}^{2\pi} \sin k\theta \cos n\theta \cos l\theta d\theta \]

\[ \bar{D}_{1k} = \int_{0}^{2\pi} \sin k\theta \sin n\theta \cos l\theta d\theta \]

\[ \bar{E}_{1k} = \int_{0}^{2\pi} (-\cos k\theta) \cos n\theta \cos l\theta d\theta \]

\[ \bar{F}_{1k} = \int_{0}^{2\pi} (-\cos k\theta) \sin n\theta \cos l\theta d\theta \]

\[ J2 = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=1}^{\infty} \bar{A}_{2k} [\bar{C}_{2k} B_{ck}(t) + \bar{D}_{2k} B_{sk}(t)]C_{mn}(t) + \sum_{k=1}^{\infty} \bar{A}_{2k} [\bar{E}_{2k} B_{ck}(t) + \bar{F}_{2k} B_{sk}(t)]S_{mn}(t) \]

(5.87)

where, \( \bar{A}_{2k} = \int_{a}^{b} \frac{\partial}{\partial r} (T_{rk}(r) R_{mn}(r)) R_{ql}(r) dr \)

\[ \bar{C}_{2k} = \int_{0}^{2\pi} \cos k\theta (-n \sin n\theta) \cos l\theta d\theta \]

\[ \bar{D}_{2k} = \int_{0}^{2\pi} \sin k\theta (-n \sin n\theta) \cos l\theta d\theta \]

\[ \bar{E}_{2k} = \int_{0}^{2\pi} \cos k\theta (n \cos n\theta) \cos l\theta d\theta \]

\[ \bar{F}_{2k} = \int_{0}^{2\pi} \sin k\theta (n \cos n\theta) \cos l\theta d\theta \]

\[ J3 = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=1}^{\infty} \bar{A}_{3k} [\bar{C}_{3k} B_{ck}(t) + \bar{D}_{3k} B_{sk}(t)]C_{mn}(t) + \sum_{k=1}^{\infty} \bar{A}_{3k} [\bar{E}_{3k} B_{ck}(t) + \bar{F}_{3k} B_{sk}(t)]S_{mn}(t) \]

(5.88)

where, \( \bar{A}_{3k} = \int_{a}^{b} \frac{\partial}{\partial r} (T_{rk}(r) \frac{\partial R_{mn}(r)}{\partial r}) R_{ql}(r) dr \)
$\bar{C}_{3k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} (\cos k\theta \cos n\theta) \cos \theta d\theta$

$\bar{D}_{3k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} (\sin k\theta \cos n\theta) \cos \theta d\theta$

$\bar{E}_{3k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} (\sin k\theta \cos n\theta) \cos \theta d\theta$

$\bar{F}_{3k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} (\sin k\theta \sin n\theta) \cos \theta d\theta$

\[ J_4 = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=1}^{\infty} \bar{A}_{4k} [\bar{C}_{4k} \bar{B}_{ck}(t) + \bar{D}_{4k} \bar{B}_{sk}(t)] \bar{C}_{mn}(t) + \sum_{k=1}^{\infty} \bar{A}_{4k} [\bar{E}_{4k} \bar{B}_{ck}(t) + \bar{F}_{4k} \bar{B}_{sk}(t)] S_{mn}(t) \]  

(5.89)

where, \( \bar{A}_{4k} = \frac{1}{\pi} \int_{0}^{1} T_{4k}(r) R_{mn}(r) R_{q1}(r) dr \)

\[ \bar{C}_{4k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} [\sin k\theta (-n \sin n\theta)] \cos \theta d\theta \]

\[ \bar{D}_{4k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} [(- \cos k\theta)(-n \sin n\theta)] \cos \theta d\theta \]

\[ \bar{E}_{4k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} [\sin k\theta (n \cos n\theta)] \cos \theta d\theta \]

\[ \bar{F}_{4k} = \frac{2\pi}{0} \frac{\partial}{\partial \theta} [(- \cos k\theta)(n \cos n\theta)] \cos \theta d\theta \]

Similarly, \( J_5, J_6, J_7 \) and \( J_8 \) can be obtained by replacing \( \cos \theta \) by \( \sin \theta \) in Equations (5.86)-(5.89).

The equation of motion, including the contribution of the rotating stress fields produced by the multiple moving tangential cutting forces, can be written in the following form:

\[ [M]\ddot{x} + [G][\dot{x}] + [K][x] + [A_{\alpha}(t)][x] = 0 \]  

(5.90)

where \([A_{\alpha}(t)]\) is periodic with period of \((1/F_t)\).
Figure 5.24 shows the Campbell diagram of a rotating circular saw-blade (the same blade as the previous one) subjected to the asymmetric in-plane stress fields produced by multiple moving tangential cutting forces. The stability characteristics of the blade are predicted by using the zero-order approximation of the generalized Fourier series method. From this figure it can be seen that all the forward-wave modes are destabilized and all the backward-wave modes are stabilized by these stress fields over the entire rotating speed range. This result is very similar to the instability due to the transverse components of multiple tangential cutting forces described in Section 5.4.1. From this figure it should also be noted that the modal interaction of the coupling type occurs mainly between two backward-wave modes or between one backward-wave mode and one reflected-wave mode.

Figure 5.25 presents a comparison of the real parts of the eigenvalues for different numbers of Fourier components \( N_k = 3 \) and \( N_k = 10 \), from which it is clear that the introduction of more terms in the Fourier series only results in very little change in predicting the stability of the blade. As may be noted the small peaks in the result from \( N_k = 10 \) result from the modal interactions.

Figures 5.26 and 5.27 present a comparison of instability in a saw blade cutting over different sectors of the work-piece \( (\theta_{st} = 20^\circ, \theta_{ex} = 50^\circ) \) and \( (\theta_{st} = 20^\circ, \theta_{ex} = 30^\circ) \), corresponding to the cases of three active teeth and one active tooth, respectively. This figure reveals that the instability regions of the system remain unchanged.

Figure 5.28 shows the effect of an anisotropic work-piece upon the stability characteristics of the blade subjected to stress fields due to tangential cutting forces. In the case of an anisotropic work-piece, \( C_r \) can be expressed as: \( C_r = C_{r0} f(\theta_j) \), where \( f(\theta_j) = 3\theta_j^3 \) for the case of \( (\theta_{st} = 20^\circ \) and \( \theta_{ex} = 50^\circ) \). Therefore, \( F_{rj}(\theta_2)/F_{rj}(\theta_1) = 15.6 \). The result is compared with the analysis in Figure 5.26. Clearly, the introduction of anisotropic properties of the work-piece does not change the stability characteristics of the blade.
Figure 5.24 Instabilities due to asymmetric in-plane stress fields produced by multiple moving tangential cutting forces (Four mode solution, \( C_t = -500N \), \( N_k = 10 \))

Figure 5.25 A comparison of real parts of eigenvalues for different numbers of Fourier components (Two mode solution, \( C_t = -1000N \), \( N_k = 10 \))
Figure 5.26 Instabilities due to the in-plane stress fields caused by multiple moving tangential cutting forces $\left(C_t = -1000\text{N}, N_k = 10, \theta_{st} = 20^\circ, \theta_{ex} = 50^\circ\right)$

Figure 5.27 Instabilities due to the in-plane stress fields caused by multiple moving tangential cutting forces $\left(C_t = -1000\text{N}, N_k = 10, \theta_{st} = 20^\circ, \theta_{ex} = 30^\circ\right)$
Figure 5.28 Instabilities due to the in-plane stress fields caused by tangential cutting forces for anisotropic work-piece (Two mode solution, $C_{t0} = -1000N$, $N_k = 10$)

5.5 Summary

In this chapter, the stability characteristics of a rotating disc subjected to various interactive cutting forces including (1) multiple moving lateral regenerative cutting forces caused by flank cuts, (2) the transverse components of multiple moving follower radial and tangential cutting forces and (3) the asymmetric membrane stress fields resulting from the multiple moving in-plane cutting edge forces are investigated in order to find the dominant mechanism of instability which matches the characteristics identified from the experimental results. New developments also involve the solution methods for stability analysis of the rotating disc subjected to different types of cutting forces. The basic Fourier series method is generalized to solve the stability problem for the time-varying systems with or without time lag.
Specific conclusions can be drawn from this study as follows:

1) Self-excited vibration due to the regenerative cutting forces can exist when the tooth passing frequency $f_t$ or its multiples lie in specific speed ranges related to the natural frequencies of the constrained disc system.

2) Both the coupling type of flutter instability and divergence instability can be induced by the transverse components of multiple moving radial cutting forces. In this case the modal interaction of two non-reflected wave modes is always of the veering type, but the modal interaction between one reflected-wave mode and one non-reflected wave mode is of the coupling type of instability.

3) When the saw blade is subjected to the transverse components of multiple moving tangential cutting forces, all the forward-wave modes of the blade are always unstable and all the backward-wave modes are always stable over the entire rotating speed range.

4) The asymmetric in-plane stress fields caused by the multiple moving cutting radial forces can only produce coupling type of instability and divergence instability. The modal interaction between one forward-wave mode and one backward-wave mode is of the veering type at subcritical speeds, but the modal interaction between one backward-wave mode and one reflected-wave mode is of the coupling type at supercritical speeds. The modal interaction between two backward-wave modes is of the coupling type even at subcritical speeds.

5) The asymmetric in-plane stress fields caused by the multiple moving cutting tangential forces can cause terminal instability. All the forward-wave modes are destabilized and all the backward-wave modes are stabilized by these stress fields over the entire rotating speed range, which is similar to the stability characteristics due to the transverse components of multiple tangential cutting forces.
6.1 Introduction

The use of modern tool materials has opened the possibility of dramatically reducing the machining time in metal removal operations such as milling by using higher cutting speeds and higher feed rates. However, there is an important phenomenon called chatter that limits the rates of material removal and, consequently, the rate of production. This chapter deals with self-excited vibration in milling and the mechanisms that are involved in the interaction between the milling cutter attached to a rotating spindle and the work-piece.

Figure 6.1: The regenerative instability mechanism in metal cutting
Machine tool chatter is the self-excited vibration produced by the interaction between the machine tool cutter and the work-piece. An initial disturbance generates relative motion between the tool and the work-piece which produces a waviness $z(t - T)$ ($T$ is the tooth passing period) on the cutting surface, as shown in Figure 6.1. This wavy surface will be removed by the succeeding tooth which is also oscillating due to the disturbance. The resulting chip thickness $(z(t) - z(t - T))$ is also wavy, which results in a fluctuating cutting force. Consequently, this force will regenerate a new wavy surface. The regeneration of waviness may induce self-excited vibration when the oscillating regenerative cutting force $F_z(t)$ is in phase with the absolute lateral vibration velocity $\dot{z}(t)$ of the spindle at the cutting point. In general, a typical self-excited vibration usually consists of the following components:

- An interaction between two objects, such as the cutter system and the work-piece.
- An interactive force such as the cutting force that is dependent of the displacement and/or velocity and/or acceleration of the system, namely $F_z(t) = f(h, \dot{z}, \ddot{z}, ...)$, where $h = \ddot{f}(z(t) - z(t - T))$ for the case of regenerative chatter.
- An energy flux (i.e., the change of vibration energy of the system): $\Delta E(t) = \int_0^t \dot{z}(\tau) \cdot F_z(\tau) d\tau$. $\Delta E(t)$ will increase when $F_z(t)$ is in phase with $\dot{z}(t)$, which implies that chatter instability may occur.
- An energy source (behind the scene) to drive the tool or work-piece. Under certain circumstances, the driving energy can be channeled into the vibration energy of the system.

The fundamental chatter theory has been developed by Tobias and Fishwick (1958) [43] and Tlusty and Polacek (1963) [44], who identified the "regeneration of waviness" phenomenon and mode coupling effects as the major chatter mechanisms. Tlusty and Polacek [44] investigated the instability mechanisms of regenerative chatter and mode coupling and derived a well-known equation for predicting the maximum depth of cut in the regenerative chatter analysis in terms of the cutting force coefficient of the work-piece and the frequency
response function of the system. Merritt (1965, [45]) presented a systematic theory for the stability analysis of regenerative chatter based on the Nyquist stability criterion using well-established feedback control theory. In fact, in their early chatter analysis, Tlusty and Polacek involuntarily used the Nyquist stability criterion as well [44]. Thereafter, methods and algorithms for the stability analysis in turning and milling based on the Nyquist criterion and the transfer function of the system dominated the frequency-domain stability research for many decades.

In the early research, most studies focused on turning where the cutting forces are time invariant. In milling, due to the rotating cutter which has multiple teeth, the cutting forces and their directions are time dependent functions of the cutter immersion angle. This leads to time varying characteristic matrices in the dynamic model of milling. This fact forced many researchers to look at stability problems in milling in the time domain. Tlusty and Ismail (1981 [47]) and Smith and Tlusty (1993 [48]) analyzed the stability problem in milling and generated the stability diagram based on time domain simulations.

Sridhar et al. [46, 49, 50] presented comprehensive insights about milling dynamics and derived a detailed mathematical model with time varying cutting force coefficients. Then they employed an averaging technique [46] to predict the stability characteristics of the milling system without time delay by examining the real parts of the eigenvalues of the state transition matrix calculated through a numerical procedure over one period of revolution of the cutter.

Minis and Yanushevsky’s work (1990 [51], 1993 [27]) can be considered a milestone in chatter analysis of milling. In their study the stability analysis of the equation of motion with periodic coefficients and time delay was reduced to the analysis of a finite order characteristic equation in terms of the FRF matrix of the system based on Floquet’s theorem [52] and the Nyquist criterion. In the present thesis this method is called the FLN method for short. In an independent study, Budak (1994, [53]) and Altintas and Budak (1995, [28]) also derived the finite-order characteristic equation for the stability analysis in milling through a practical
approach which had a clear physical meaning. They solved the eigenvalue problem by selecting a chatter frequency "around dominant structural modes" prior to the calculation instead of simultaneously solving the chatter frequency and the critical axial depth of cut by using the techniques in control theory [53]. The method proposed is very efficient for the milling model without considering the gyroscopic effect of the spindle.

From a detailed literature review, it was found that almost all the researchers voluntarily or involuntarily employed Nyquist stability criterion and the concept of feedback control based on Tobias, Tlusty and Merritt's work to conduct a stability analysis in the frequency domain. According to the author's knowledge the gyroscopic effects of the rotating spindle on the chatter frequencies, the mode shapes and the stability characteristics of the milling system have not been considered and incorporated in the milling model in the previous literature. The stability analysis of the milling cutter-rotating spindle system becomes more complicated than the case of stationary milling cutter. The gyroscopic effects of the spindle will make the natural frequencies become rotating speed dependent and make one bending vibration mode split into two modes (i.e., the forward-wave and the backward-wave modes). In this case, the frequency response functions (FRFs) or the system matrices of the milling system become speed dependent. Therefore, the chatter frequencies and the real parts of the eigenvalues (i.e., stability features) must be solved simultaneously at a specific rotating speed. Although the various methods [27, 28] of solving the characteristic equation of the milling system have been well established for the stationary milling models, they cannot be used in the case of the milling model with gyroscopic effects involved.

In this thesis, a dynamic milling model including a rotating spindle is developed for the first time. The generalized Fourier series method (the GFS method) is proposed to change the stability analysis for the time-varying milling system with time delay to the eigenvalue analysis of a finite-order characteristic equation in terms of the system matrices, instead of the transfer function matrix used in the traditional FLN method. The eigenvalue problem of the resulting
equation can be efficiently solved by using Müller's method with deflation (Müller, 1956 [54]; Gerald, et al., 1994 [55]) which is an optimization algorithm for solving the roots of complex non-linear equations. This procedure is called the GFS method for short. The stability lobes (i.e., the critical axial depths of cut at the different rotating speeds of the spindle), under which instability cannot occur, can also be predicted at different rotating speeds by using a nonlinear optimization procedure based on the Nyquist stability criterion, which is called the FSN method in this thesis.

6.2 Dynamic Milling Forces

The milling force model for the cutter with a zero helix angle has been well investigated for many decades [56, 57, 58, 59]. The milling force model used in this study follows the one in reference [28], which is simple but capable of describing the primary characteristics of dynamic milling forces.

![Dynamic milling force model](image)

Figure 6.2: Dynamic milling force model
Figure 6.2 shows a cross section of an end mill cutter which interacts with a work-piece. The tangential and radial cutting forces applied on the jth tooth in the coordinates R-T rotating with the spindle are given by [28]:

\[ F_{tj}(t) = K_{t}(\theta_j)ah(\theta_j), \quad F_{rj}(t) = K_{r}(\theta_j)F_{tj}(t) \]  \hspace{1cm} (6.1)

where, \( a \) is the axial depth of cut. \( K_{t}(\theta_j) \) and \( K_{r}(\theta_j) \) are the cutting coefficients determined by experiment, and are assumed to be constant for an isotropic material work-piece. \( h(\theta_j) \) is the chip thickness and can be expressed in the following form:

\[ h(\theta_j) = g(\theta_j)[V_f \sin \theta_j + (R_{j0}(\theta_j) - R_j(\theta_j))] \]  \hspace{1cm} (6.2)

where, \( V_f \) is the feed per tooth. \( R_{j0} \) and \( R_j \) represent the radial displacements of the cutter at the previous and present cuts at the position \( \theta_j \), respectively. \( g(\theta_j) \) is a switch function which determines whether the jth tooth is in or out of cut, namely \( g(\theta_j) = 1 \) when \( \theta_{st} \leq \theta_j \leq \theta_{ex} \) (where, \( \theta_{st} \) and \( \theta_{ex} \) are the start and exit angular immersions of the cutter, respectively). Otherwise, \( g(\theta_j) = 0 \).

In Equation (6.2), the term \( g(\theta_j)V_f \sin(\theta_j) \) represents the static component of the chip thickness. It can be easily shown that the summation of the cutting forces due to these static components is periodic with period \( 1/f_t \) where \( f_t \) is the tooth passing frequency. Therefore, the response to forced vibration may be amplified when \( f_t \) is near one of the natural frequencies of the system, which also plays an important role in the initialization of self-excited vibration. However, the periodic cutting force produced by these static components is independent of the response of the system, which does not give any contribution to the regenerative instability mechanism. Thus, it can be removed from Equation (6.2).
Chapter 6. Self-Excited Vibration in Milling Cutter with Flexible Rotating Spindle

The total dynamic cutting force $F_c(t)$ applied on the cutter teeth can be resolved into two components in the $Y$ and $Z$ directions in the stationary coordinates $(XYZ)$:

\begin{align*}
F_v(t) &= \sum_{j=0}^{N_t-1} F_{vj} = \sum_{j=0}^{N_t-1} (F_j \sin \theta_j - F_j \cos \theta_j) \\
F_u(t) &= \sum_{j=0}^{N_t-1} F_{uj} = \sum_{j=0}^{N_t-1} (-F_j \cos \theta_j - F_j \sin \theta_j)
\end{align*}

where, $N_t$ is the total number of the teeth. Substituting Equations (6.1) and (6.2) into Equations (6.3) and (6.4) with $R_j(\theta_j) = -u \sin \theta_j - v \cos \theta_j$ and regrouping, give,

\begin{align*}
\begin{bmatrix} F_v(t) \\ F_u(t) \end{bmatrix} &= \begin{bmatrix} a_{vv} & a_{vu} \\ a_{uv} & a_{uu} \end{bmatrix} \begin{bmatrix} v - v_0 \\ u - u_0 \end{bmatrix} = (1 - e^{-TD}) [a(t)] \begin{bmatrix} v \\ u \end{bmatrix}
\end{align*}

where,

\begin{align*}
a_{vv} &= a \sum_{j=0}^{N_t} g(\theta_j) [K_r(\theta_j) \sin 2\theta_j - K_r(\theta_j) K_u(\theta_j)(1 + \cos 2\theta_j)] / 2 \\
a_{vu} &= a \sum_{j=0}^{N_t} g(\theta_j) [K_r(\theta_j)(1 - \cos 2\theta_j) - K_r(\theta_j) K_u(\theta_j) \sin 2\theta_j] / 2 \\
a_{uv} &= a \sum_{j=0}^{N_t} -g(\theta_j) [K_r(\theta_j)(1 + \cos 2\theta_j) + K_r(\theta_j) K_u(\theta_j) \sin 2\theta_j] / 2 \\
a_{uu} &= a \sum_{j=0}^{N_t} -g(\theta_j) [K_r(\theta_j) \sin 2\theta_j + K_r(\theta_j) K_u(\theta_j)(1 - \cos 2\theta_j)] / 2
\end{align*}

$e^{-TD}$ is a time delay operator (i.e., $e^{-TD}v(t) = v(t-T) = v_0$). $v$ and $u$ represent the displacements of the spindle in the $Y$ and $Z$ directions, which are observed from the stationary
coordinates. \([a(t)]\) is referred to as the directional dynamic milling coefficient matrix which is periodic with the tooth passing frequency \(f_t\).

### 6.3 Dynamic Finite Element Model of Rotating Spindle

![Figure 6.3: A typical rotating shaft element](image)

A finite rotating shaft element using Timoshenko beam theory with gyroscopic terms is shown in Figure 6.3. There are two coordinate reference systems used to describe the motion of the element: a fixed coordinates \((XYZ)\) and a coordinates \((xyz)\) rotating with the shaft. The element is considered to be initially straight and is modeled as an eight DOF element. At a given node the element has four degrees of freedom: two displacements \(v\) and \(u\), and two slopes about the \(Z\) and \(Y\) axes which are \(\psi\) and \(\varphi\) respectively. Shear, rotational inertia and gyroscopic effects are included in this element. The elemental kinetic energy consists of both translational and rotational terms, and the potential energy of the element includes elastic bending, shear and axial contributions. The total elemental potential and kinetic energies are
developed, then the elemental translational and rotational mass matrices, elemental stiffness matrix and gyroscopic matrix are obtained by applying the extended Hamilton’s Principle, which are expressed in the following forms [60, 61] (the details of $[M_0]$, $[M_1]$, $[M_2]$, $[N_0]$, $[N_1]$, $[N_2]$, $[G_0]$, $[G_1]$, $[G_2]$, $[K_0]$ and $[K_1]$ matrices are given in Appendix D).

(1) Translational mass matrix:

$$[M] = [M_0] + \Phi [M_1] + \Phi^2 [M_2]$$

(6.10) where, \( \Phi \) is the transverse shear effect (\( \Phi = 12EI / kAGL^2 \)). \( EI \) is the bending stiffness per unit of curvature. \( G \) is the shear modulus. \( A \) is the cross sectional area of the shaft. \( k \) is a constant determined by the shape of the cross section of the shaft.

(2) Rotational mass matrix:

$$[N] = [N_0] + \Phi [N_1] + \Phi^2 [N_2]$$

(6.11) Therefore, the total elemental mass matrix is: $[M] + [N]$.

(3) Gyroscopic matrix:

$$[G] = [G_0] + \Phi [G_1] + \Phi^2 [G_2]$$

(6.12)

(4) Stiffness matrix:

$$[K] = [K_0] + \Phi [K_1]$$

(6.13)

(5) Disc mass element:

In the case of small displacements, the expression for the kinetic energy of a symmetric disc is given by [62]:

$$T_D = \frac{1}{2} m_D (u^2 + v^2) + \frac{1}{2} I_{Dx} (\psi^2 + \varphi^2) + \frac{1}{2} I_{Dz} (\Omega^2 + 2\Omega \psi \varphi)$$

(6.14) where, \( m_D \) is the mass of the disc. \( I_{Dx} \) and \( I_{Dz} \) are the moments of inertia of the disc around the \( X \) and \( Z \) (or \( Y \)) axes, respectively. The application of Lagrange’s equations to Equation (6.14) gives the mass and gyroscopic matrices in the space-fixed coordinates [62]:
where, \( \delta = (v, u, \psi, \phi) \).

(6) **Bearing stiffness and damping model:**

The elastic and viscous damping forces provided by the bearings supporting the spindle can be expressed in the form of

\[
\begin{pmatrix}
F_{BYi} \\
F_{BZi}
\end{pmatrix} = \begin{pmatrix}
k_{yy} & k_{yz} & v_i - v_b \\
k_{zy} & k_{zz} & u_i - u_b
\end{pmatrix} \begin{pmatrix}
\dot{v}_i - \dot{v}_b \\
\dot{u}_i - \dot{u}_b
\end{pmatrix}
\]

where, \( F_{BYi} \) and \( F_{BZi} \) are the elastic and damping forces of the bearing support at the \( i \)th node of the spindle in the \( Y \) and \( Z \) directions, respectively. \( v_i \) and \( u_i \) are the displacements at the \( i \)th node of the spindle in the \( Y \) and \( Z \) directions, respectively. \( v_b \) and \( u_b \) are the corresponding displacements of the bearing. \( k_{yy}, k_{yz}, k_{zy} \) and \( k_{zz} \) (or \( c_{yy}, c_{yz}, c_{zy} \) and \( c_{zz} \)) represent the stiffness (or viscous damping) coefficients of the bearing.

Figure 6.4: A dynamic finite element model with a rotating spindle in milling
The dynamic model shown in Figure 6.4 is based on the acceptance of the assumption that the milling machine structure holding the spindle and the work-piece clamping system are sufficiently rigid (or they will move together if they move) so that only the relative motion between the tool and the work-piece needs to be considered. Although a continuous spindle is modeled here, it is assumed that the whole axial depth of cut zone deflects the same amount because the stability characteristics of the cutting system are not changed by the very small slope of the spindle tip.

Assembling the elemental rotating shaft matrices into the global matrices of the spindle and combining the spindle, the bearing and the milling cutting force models generates the whole dynamic milling model which can be expressed in the following form:

\[
[M]\{\ddot{U}_s\} + [G]\{\dot{U}_s\} + [K]\{U_s\} + (1 - e^{-TD})[A(t)]\{U_s\} = \{0\}
\]  

where, \(\{U_s\} = \begin{bmatrix} \{\dot{U}_s\} \\ \{\ddot{U}_s\} \end{bmatrix}\) represents the whole displacement vector of the spindle including the vector \(\{\dot{U}_s\}\) associated with the degrees of freedom of the spindle tip which interacts with the work-piece and the vector \(\{\ddot{U}_s\}\) associated with the remaining degrees of freedom of the spindle.

\[
[M] = [M_s]
\]  

where, \([M_s]\) is the global mass matrix of the spindle.

\[
[K] = [K_s] + [K_b]
\]  

where, \([K_s]\) is the global stiffness matrix of the spindle. \([K_b]\) is the global bearing stiffness matrix associated with the stiffness matrix in Equation (6.16).
where, $[G_s]$ is the global gyroscopic matrix of the spindle. $[C_s]$ is the global structural damping matrix of the spindle, which will be assumed in the form of $[C_s] = \alpha [M_s] + \beta [K_s]$ ($\alpha$ and $\beta$ are damping ratios). $[C_b]$ is the global bearing damping matrix associated with the damping matrix in Equation (6.16).

$[A(t)]$ is a time-varying matrix associated with the cutting forces, which can be written in the form of

$$[A(t)] = \begin{bmatrix} [a(t)] & 0 \\ 0 & 0 \end{bmatrix}$$

where, $[a(t)]$ is given by Equations (6.6)-(6.9). It can be easily verified that $[A(t)]$ is periodic with the tooth passing period $T$. $e^{-TD}$ is a time delay operator.

### 6.4 Generalized Fourier Series Method for the Stability Analysis in Milling

As described in Chapter 5, the solution of Equation (6.17) with periodic coefficient terms can be assumed in the form of Fourier series with the frequency components: $\lambda \pm ik\omega$ ($k = 0,1,2,\ldots$, $i = \sqrt{-1}$ and $\omega = 2\pi / T$) [30]:

$$\{U_s(t)\} = \left\{ b_0 \right\} / 2 + \sum_{k=1}^{\infty} ([a_k] \sin k\omega t + [b_k] \cos k\omega t) e^{\lambda t}$$

(6.22)

where, $\{b_0\}$, $\{b_k\}$ and $\{a_k\}$ are time-invariant coefficient vectors. $\lambda$ is the characteristic variable of the system. Since $[A(t)]$ is periodic, it can also be written in a Fourier series form:

$$[A(t)] = \frac{1}{2} [B_0] + \sum_{k=1}^{\infty} ([A_k] \sin k\omega t + [B_k] \cos k\omega t) \quad (k = 0,1,2,\ldots)$$

(6.23)
where,

\[ [B_0] = \frac{2}{T} \int_0^T [A(t)] dt \]

\[ [B_k] = \frac{2}{T} \int_0^T [A(t)] \cos(k \omega t) dt \]

\[ [A_k] = \frac{2}{T} \int_0^T [A(t)] \sin(k \omega t) dt \]

Similarly, substituting Equations (6.22) and (6.23) into Equation (6.17) and equating the coefficients of \( e^{\lambda t} \), \( e^{\lambda t} \sin k \omega t \) and \( e^{\lambda t} \cos k \omega t \) to zero lead to the following form of the characteristic equation of the cutting system:

\[
(X^2 [H_2] + X [H_1] + [H_0] + (1 - e^{-T \lambda}) [L]) \{c\} = \{0\} \quad (6.24)
\]

where, \( \{c\} = (b_0 \, a_1 \, b_1 \, \cdots \, a_{N_s} \, b_{N_s})^T \). \( N_s \) is the maximum number of Fourier components. The coefficient matrices \([H_2], [H_1], [H_0] \) and \([L] \) consist of \([K], [G], [M], [B_0], [B_k], [A_k] \) \( (k = 0, 1, 2, \cdots) \) and \( \omega \). The details of these matrices can be found in Equations (5.9)-(5.20) in Chapter 5.

Müller’s optimization algorithm [31] with deflation for solving nonlinear complex equations can be used to find the complex eigenvalues \( \lambda \) of the characteristic equation (6.24) by specifying an axial depth of cut and a rotating speed. This procedure is called the GFS method. The critical axial depths of cut (i.e., stability lobes) can be predicted by letting \( \lambda = j \omega_c \) (\( \omega_c \) is the chatter frequency) in Equation (6.24) based on the Nyquist stability criterion. In this case, the chatter frequency and the critical axial depth of cut have to be solved simultaneously at a given rotating speed due to the gyroscopic effect of the rotating spindle. This procedure for predicting the stability lobes in milling is called the FSN method for short in this thesis.

### 6.5 Applications of the Stability Analysis in Milling

The stability analysis method developed in this chapter is applied to an end milling operation using the two spindle models (Model A and Model B) shown in Figure 6.5.
6.5.1 The Stability Analysis of a Relatively Simple Milling Model (Model A)

Consider a relatively simple dynamic milling model with six finite elements as shown in Figure 6.5(a), whose parameters are listed in Table 6.1. It is assumed that the motion of this model involves primarily the relative displacement between the cutter and the work-piece. A milling cutting with \( N_r = 8 \) straight teeth is used, and the start and exit immersion angles are \( \theta_{st} = 0^\circ \) and \( \theta_{ex} = 90^\circ \), respectively. Equation (6.24) and Müller’s optimization algorithm were used to predict the stability characteristics of this model. A complete investigation was conducted, and the results are presented in Figures 6.6-6.17.

![Model A](image)

Figure 6.5: The finite element model of the rotating spindle with the milling cutter (Model A and Model B)
Table 6.1: The parameters of the finite element model of the spindle (Model A)

<table>
<thead>
<tr>
<th>Element No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length(m)</td>
<td>0.0450</td>
<td>0.0400</td>
<td>0.0500</td>
<td>0.0200</td>
<td>0.2000</td>
<td>0.2000</td>
</tr>
<tr>
<td>Outer diameter(m)</td>
<td>0.0190</td>
<td>0.0745</td>
<td>0.0400</td>
<td>0.0600</td>
<td>0.0600</td>
<td>0.0600</td>
</tr>
<tr>
<td>Inner diameter(m)</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0400</td>
<td>0.0400</td>
</tr>
<tr>
<td>$E$(Pa)</td>
<td>4.5e+11</td>
<td>2.1e+11</td>
<td>2.1e+11</td>
<td>2.1e+11</td>
<td>2.1e+11</td>
<td>2.1e+11</td>
</tr>
</tbody>
</table>

(All the stiffness coefficients of the bearings located at Nodes 5 and 7 are: $k_{yy} = k_{zz} = 5.0E+8$ (N/m) and $k_{yz} = k_{zy} = 0$ (N/m))

Table 6.2: The parameters of the finite element model of the spindle (Model B)

<table>
<thead>
<tr>
<th>Element No.</th>
<th>Length(m)</th>
<th>Outer diameter (m)</th>
<th>Inner diameter (m)</th>
<th>$E$(Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.040</td>
<td>0.019</td>
<td>0.000</td>
<td>4.5e+11</td>
</tr>
<tr>
<td>2</td>
<td>0.040</td>
<td>0.019</td>
<td>0.000</td>
<td>4.5e+11</td>
</tr>
<tr>
<td>3</td>
<td>0.050</td>
<td>0.085</td>
<td>0.000</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>4</td>
<td>0.015</td>
<td>0.050</td>
<td>0.000</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>5</td>
<td>0.020</td>
<td>0.078</td>
<td>0.000</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>6</td>
<td>0.030</td>
<td>0.070</td>
<td>0.000</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>7</td>
<td>0.040</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>8</td>
<td>0.035</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>9</td>
<td>0.035</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>10</td>
<td>0.035</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>11</td>
<td>0.035</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>12</td>
<td>0.035</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>13</td>
<td>0.030</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>14</td>
<td>0.045</td>
<td>0.070</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
<tr>
<td>15</td>
<td>0.050</td>
<td>0.078</td>
<td>0.045</td>
<td>2.1e+11</td>
</tr>
</tbody>
</table>

(All the stiffness coefficients of the bearings located at Nodes 6, 7, 13 and 14 are: $k_{yy} = k_{zz} = 2.0E+8$ (N/m) and $k_{yz} = k_{zy} = 0$ (N/m))
Figures 6.6, 6.7 and 6.8 illustrate the regenerative instability characteristics (including the chatter frequencies) of the model A predicted by using the GFS method as the milling force level \((K_t a)\) increases. These results reveal that the backward-wave modes of the spindle are destabilized and the forward-wave modes of the spindle are stabilized by the cutting forces over the entire speed range except in a few small regions. As may be noted, the level of the milling force does not change the instability regions but it will effect the chatter frequencies of the milling system as the cutting force increases, especially at low rotating speeds of the spindle. From these figures it was also found that the lowest points, such as A and B, of the real parts of the eigenvalues in Figure 6.6 are located near the points where the tooth passing frequency \(f_t\) is equal to the chatter frequency \(f_{B1}\) of the first backward-wave mode (i.e., \(f_t = f_{B1}\)), or the multiples of \(f_t\) is equal to \(f_{B1}\) (i.e., \(Nf_t = f_{B1}\)). More detailed features of the regenerative instability will be discussed in relation to Figure 6.14.

The regenerative instability characteristics of the second mode is presented in Figure 6.9, which also draws attention to the fact that the backward-wave modes are the primary vibration modes excited by the milling forces and the chatter frequency varies notably at different rotating speeds, especially in the low speed region. As may be noted the lowest point A of the real parts of eigenvalues in Figure 6.9 is also located near the point where \(f_t = f_{B2}\) (\(f_{B2}\) is the chatter frequency of the second backward-wave mode).

In order to verify the effectiveness and correctness of the proposed method for the stability analysis of the milling system, a time-domain analysis of solving Equation (6.17) using direct integration was conducted at different rotating speeds. Figure 6.10 shows a spectral waterfall diagram of the time domain simulation at \(\Omega = 5090\) (RPM) (i.e., the point A shown in Figure 6.8, in which the first mode is stable but the second mode is unstable), from which it can be seen that the stability analysis from the GFS method agrees very well with the time domain simulation which can be considered as the exact solution, even when the zero-order approximation in the GFS method is used in the calculation.
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Figure 6.6: Regenerative instability characteristics of the first mode
($K_r a = 2.0E+3(N/m)$, $K_r = 0.3$, $\theta_{si} = 0^\circ$, $\theta_{ex} = 90^\circ$; the zero-order GFS)

Figure 6.7: Regenerative instability characteristics of the first mode
($K_r a = 2.0E+4(N/m)$, $K_r = 0.3$, $\theta_{si} = 0^\circ$, $\theta_{ex} = 90^\circ$; the zero-order GFS)
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Figure 6.8: Regenerative instability characteristics of the first mode \((K_r a = 2.0E+5(N/m), K_r = 0.3, \theta_{st} = 0^\circ, \theta_{ex} = 90^\circ; \text{ the zero-order GFS })\)

Figure 6.9: Regenerative instability characteristics of the second mode \((K_r a = 2.0E+5(N/m), K_r = 0.3, \theta_{st} = 0^\circ, \theta_{ex} = 90^\circ; \text{ the zero-order GFS })\)
Figure 6.10: Waterfall spectrum diagram at $\Omega = 5090$ (RPM) ($K_a = 2.0E+5$ (N/m), $K_r = 0.3$)

Figure 6.11 shows the response spectrum predicted by the direct time domain integration at the tool tip when the spindle rotates at $\Omega = 6000$ RPM. As may be noted the response of the system contains both vibration mode components and tooth passing frequency components (TPFCs). This observation is consistent with the assumptions in Equation (6.22). From various time domain simulations it was also found that the tooth passing frequency components of the response are always much smaller than the vibration mode components and the vibration mode components usually increase rapidly and will dominate the response of the system. This explains why the zero-order approximation of the GFS method usually gives excellent results.

Figure 6.12 presents a comparison of the stability characteristics predicted by different order approximations of the GFS method, in which a relatively large $K_a$ value is used. From this figure it can be clearly seen that the zero-order approximation of the GFS method can predict the stability regions very accurately.
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Figure 6.11: Dynamic response of vibration modes and tooth passing frequency components (TPFCs) at the tool tip ($K_Ia = 2.0E+5$ N/m, $K_r = 0.3$, $\Omega = 6000$ RPM)

Figure 6.12: A comparison of the stability characteristics predicted by different order solutions of the GFS method ($K_Ia = 1.0E+7$ N/m, $K_r = 0.3$)
A comparison of the instability regions resulting from different radial cutting constants $K_r$ is given in Figure 6.13, from which it was found that in the case of $K_r = 0$ (i.e., the radial milling forces $F_{rj}$ are zero) the real parts of the eigenvalues of the backward-wave modes are always greater than zero or equal to zero (see the curve (a) of the backward-wave mode), however, the real parts of the eigenvalues of the forward-wave modes are always less than zero or equal to zero (see the curve (a) of the forward-wave mode). Therefore, there is no stable region existing in this case. This figure also reveals that, for the backward-wave mode, the maximum value of the real parts of the eigenvalues grows but its minimum value decreases and becomes negative when the radial milling force increases. For the forward-wave mode, the maximum value of the real parts of the eigenvalues increases and has positive regions as the radial milling force increases (see the curves (b) and (c) in Figure 6.13). Such observations imply that the radial milling force can introduce several small stable regions for the backward-wave mode but it can also induce several small unstable regions for the forward-wave mode. As may be noted the introduction of the radial milling forces also changes the instability regions of the milling system. Specifically, the instability regions of the backward-wave mode move towards the lower speed direction, in contrast, the instability regions of the forward-wave mode move towards the higher speed direction as shown in Figure 6.13.

Figure 6.14 presents the stability lobes (i.e., the critical axial depths of cut at different rotating speeds) for the first mode and the corresponding chatter frequencies predicted from FSN method where a direct search optimization algorithm called the Complex method [63] and Nyquist criterion are utilized to solve the characteristic equation (6.24) by letting $\lambda = j\omega_c$ ($\omega_c$ is the chatter frequency). In this case, the chatter frequency and the critical axial depth of cut should be solved simultaneously at different speeds. From this figure it can be clearly seen that chatter frequency is highly dependent on the rotating speed and the axial depth of cut.
Figure 6.13: A comparison of the instability regions from different radial cutting constants $K_r$ ($K_r a = 2.0E+4 \text{ N/m}$)

Figure 6.14: The stability lobes predicted from FSN method

($K_x = 1.0e+7 \text{ N/m}^2$, $c_{yy} = c_{zz} = 2e+3 \text{ Ns/m}$)
Figure 6.15: A comparison of the stability results from the FSN (a) and the GFS (b) methods ($L_1$: $a=2\text{mm}$; $L_2$: $a=0.776\text{mm}$; $K_i = 1.0e+7 \text{ N/}m^2$, $c_{xy} = c_{zz} = 2e+3 \text{ Ns/m}$)

Figure 6.16: A comparison of the stability results from the different models (NGE: without gyroscopic effect; WGE: with gyroscopic effect)
Figure 6.17: A comparison of the stability results from the FSN method (with or without gyroscopic effect (WGE or NGE)) and the FLN method proposed by Altintas & Budak [28]

Figure 6.15 presents a comparison between the results from the different methods. The chatter stability lobes obtained from the FSN procedure are presented in Figure 6.15(a). This result is compared with the real parts of eigenvalues predicted from the GFS method as shown in Figure 6.15(b). As may be noted due to the introduction of the bearing damping \( c_{yy} = c_{zz} = 2e+3 \) Ns/m and \( c_{yz} = c_{zy} = 0 \) the real parts of eigenvalues \( (L_2) \) move down compared to the one shown in Figure 6.7. From this figure it is evident that the lowest points in the chatter stability lobes, such as \( A, B, \) and \( C \), are exactly corresponding to the maximum points \( A, B \) and \( C \) in the plot of the real parts of eigenvalues. If a critical axial depth of cut, such as the point \( A \) shown in Figure 6.15(a), is used in the GFS method, the real parts of the eigenvalues \( (L_2) \) of the system will move below the zero line. This implies that these two methods agree very well with each other.

In order to assess the gyroscopic effects of rotating spindle on the stability characteristics of the milling system, a comparison of the eigenvalues between the different milling models...
with or without gyroscopic effect is made and the results are shown in Figure 6.16. From this figure it should be noted that although the gyroscopic effect of the rotating spindle does not change the instability regions of the milling system, it does change the real parts of the eigenvalues up to about 30% for this model. It should also be noted from this figure that the gyroscopic effect of the rotating spindle will change the resonant frequencies of the system dramatically.

Figure 6.17 presents a comparison of the critical axial depths predicted from different procedures for the different milling models with or without gyroscopic effect. From this figure it can be seen that the FLN method proposed by Altintas and Budak [28], in which the characteristic equation in terms of the FRF matrix of the system is solved by specifying chatter frequencies around a given natural frequency of the spindle, gives the exactly same prediction as that from the FSN method when the gyroscopic effect of the spindle is neglected. In this case, the FLN method is numerically more efficient. From this figure it was found that the critical axial depth of cut predicted from the FSN method (WGE) is about 28% less than that from the FSN method (NGE) because the gyroscopic effect will reduce the modal stiffness of the backward-wave mode. This implies that the critical axial depths of cut predicted from the FSN method become less conservative when the gyroscopic effect of the spindle is neglected.

6.5.2 The Stability Analysis of a More Complicated Milling Model (Model B)

Consider a more complicated dynamic milling model with 15 rotating finite elements as shown in Figure 6.5(b), which is designed to investigate some detailed stability characteristics in milling. The parameters of model B are given in Table 6.2. The work-piece is machined by using 19 mm diameter, eight fluted end milling cutter. The start and exit immersion angles are $\theta_{si} = 0^\circ$ and $\theta_{se} = 90^\circ$, respectively. The zero-order approximation of the GFS method and
Müller’s optimization algorithm were used to predict the stability characteristics of the system.

Is there any relationship between the mode shapes of the spindle and the stability characteristics in milling? Figure 6.18 illustrates the first and second backward-wave mode shapes of the rotating spindle. From this figure, it can be seen that the modal displacement of the first bending mode at the tool tip is much larger than that of the second bending mode. Interestingly, it was found that the larger the modal displacement of the tool tip, the greater the corresponding real part of the system eigenvalue. This observation is verified by the results shown in Figures 6.19 and 6.20, from which it can be seen that the real parts of the eigenvalues of the second mode is much smaller than that of the first mode.

![Figure 6.18: The mode shapes of the rotating spindle with a milling cutter](image.png)
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Figure 6.19: The stability characteristics of the first mode \( (K_t a = 1.0\times10^5 \text{N/m}) \)

Figure 6.20: The stability characteristics of the second mode \( (K_t a = 1.0\times10^5 \text{N/m}) \)
Chapter 6. Self-Excited Vibration in Milling Cutter with Flexible Rotating Spindle

Figure 6.21: A comparison of the stability characteristics for different cutting immersion ranges \((K_t a = 1.0E+5(N/m))\)

Figure 6.22: A comparison of the stability characteristics for different \(N_i\) \((N_i = 4 \text{ and } 8)\) (the first mode, \(K_t a = 1.0E+5(N/m))\)
Figure 6.21 presents a comparison of the stability characteristics for different cutting immersion ranges: \((\theta_{st} = 0^\circ, \theta_{ex} = 90^\circ)\) and \((\theta_{st} = 0^\circ, \theta_{ex} = 120^\circ)\). It may be noted from this figure that despite the changes of the real parts of eigenvalues the instability regions of the milling system remain the same. This result suggests that the alteration of immersion range cannot modify the stability characteristics in milling.

Figure 6.22 presents a comparison of the stability characteristics of the milling system with four and eight fluted (teeth) end milling cutters. An interesting feature of these results is that the instability regions for the case of eight-teeth milling cutter can be obtained by shrinking the instability diagram from four-teeth milling cutter by half. The reason behind this observation is that the number of teeth \(N_t\) decides the tooth passing frequency \(f_t (= \Omega N_t / (2\pi))\) which determines the major frequency component in the milling forces, thus, plays a very important role in determining the instability regions where the cutting forces are in phase with the lateral velocities of the spindle measured from the coordinates rotating with the spindle.

On the other hand, as may be noted from this figure, the real parts of the eigenvalues for the case of the eight-teeth milling cutter are twice as much as that from the four-teeth milling cutter because greater milling forces are involved in the case that an eight-teeth milling cutter is used.

### 6.6 Summary

In this chapter, a new dynamic milling model of a rotating spindle is developed and the gyroscopic effect of the spindle on the stability characteristics of the milling system is investigated for the first time. The basic Fourier series method for solving the harmonically time-varying system without time lag is extended to handle a periodically time-varying system with time delay. This procedure will result in a finite-order characteristic equation for the milling system in terms of the system matrices. Two procedures: the GFS and the FSN
methods are proposed to solve the resulting characteristic equation. In the GFS procedure, the complex eigenvalues of the resulting equation can be efficiently predicted by using Müller's optimization algorithm with deflation. In the FSN procedure, the stability lobes can be predicted by using a nonlinear optimization procedure at different rotating speeds based on the Nyquist stability criterion.

Some important conclusions are summarized as follows:

1) The complex eigenvalues of the backward-wave and the forward wave modes of the milling system can be accurately predicted by the GFS method, and this information provides more detailed insights into the instability mechanisms in milling. However, the stability lobes contain more stability information at different axial depths of cut.

2) The FSN procedure proposed in this thesis gives the exactly same results as those from the FLN procedure proposed by Altintas and Budak when the gyroscopic effect of the spindle is neglected. However, the FLN method is numerically more efficient in this case.

3) The backward-wave modes of the spindle are primarily excited by the milling forces whereas the forward-wave modes of the spindle are stabilized by the milling forces over almost the entire speed range.

4) Although the gyroscopic effect of the rotating spindle does not change the instability regions in milling, it increases the real parts of the eigenvalues of the system or reduces the critical axial depth of cut. In other words, it makes the stability prediction less conservative.

5) The gyroscopic effect of the rotating spindle usually reduces the chatter frequency of the system, especially as the rotating speed increases.

6) The radial milling force can introduce several small stable regions for the backward-wave modes but it can also induce several small unstable regions for the forward-wave modes.
7) A relatively large modal deflection at the tool tip will lead to a relatively large real part of the eigenvalue of the closed-loop milling system.

8) For different materials or cutting immersion ranges, the instability regions of the milling system remain unchanged despite the changes in the real parts of the eigenvalues.
Chapter 7

Experimental Investigation on Self-Excited Vibration in Circular Saw Cutting
— Washboarding Phenomenon

7.1 Introduction

Washboarding is a pattern produced on the surface of the cut wood by the lateral motion of the saw blade during cutting, and occurs as a result of self-excited vibration. It is characterized by a sinusoidal-like pattern on the cut wood surface, as shown in Figure 7.1. This dimensional variation is undesirable because the lumber must be made thicker than normal to allow the planer to produce a smooth surface. This may lead to significant material and labour waste.

As described in the previous chapters, self-excited vibration of a saw-blade is induced by the interactions between the blade and the wood, and is affected by factors such as the characteristics of the saw-blade, the wood properties, the tooth and wood geometry and the operational conditions.
Several possible fundamental instability mechanisms involved in the self-excited vibrations of saw-blades during cutting were investigated theoretically in previous chapters. To complement this work, a comprehensive experimental investigation was conducted in order to fully understand the cause of washboarding and to relate the dynamics of the saw-blade to the washboarding pattern.

Zhan (1989 [64]) experimentally investigated the washboarding phenomenon in handsaws. The tests conducted by Zhan showed that the bandmill strain and the blade speed had little effect on washboarding. However, the causes of washboarding were not clearly explained in his study. Yokochi, Nakashima and Kimura (1990 [65]) observed that two types of travelling-wave modes (i.e., forward-wave and backward-wave modes) were self-excited at tooth passing frequencies slightly greater than the natural frequencies of the modes excited during cutting. A further study on the kinematics of washboarding in bandsaws was conducted by the same group (Kimura, et al., 1995 [66], 1996 [67]). In these studies, the relationships between the wavelength of the washboarding pattern, the feed speed, the tooth passing frequency and the natural frequency of the saw-blade were presented without consideration of the effect of travelling-wave modes.

In an independent study, Lehmann and Hutton (1997, [68]) presented a more detailed analysis of the kinematics of washboarding in bandsaws. This study included, in an approximate way, the effect of the travelling-wave modes of the saw-blade upon the washboarding pattern.

However, in all previous studies, the dynamics of washboarding instability has not been modeled and the link between the dynamics and kinematics of washboarding was missing. In this study, the problems mentioned above have been solved and the missing link has been established based on the simulations presented in Chapter 5 and a comprehensive experimental investigation, the results of which are presented in this chapter.
In this chapter a side-cut test using a pendulum cutting rig is also described, which verifies the existence of a lateral cutting force due to flank cutting. This force is required in order to support the regenerative cutting force model presented in Chapter 5.

7.2 Washboarding Caused by Self-Excited Vibration of the Saw-Blade

![Schematic diagram of experimental set-up](image)

**Figure 7.2** Schematic of the experimental set-up of saw-blade cutting ($H_p = 0.14m$)

Figure 7.2 shows a schematic diagram of the experimental set-up. The system consists of a commercial circular saw rig, a wood feed carriage and measurement devices including displacement probes and a data acquisition system. The rotating speed of the saw-blade and the feed speed of the wood can be controlled and measured in the system.
Figure 7.3 A comparison of the responses of the clamped 21” saw-blade without ($\Omega = 33$ Hz) and with ($\Omega = 39$ Hz) washboarding

Figure 7.4 Waterfall spectral diagram of the response with washboarding (Clamped 21” saw-blade, $f_i = 1572$ Hz)
The cutting tests were conducted at different rotating speeds using a clamped forty tooth saw-blade of inner radius $a=0.0952\text{m}$, outer radius $b=0.265\text{m}$ and thickness $h=0.00229\text{m}$, cutting two $36\text{mm}$ wood boards with $H_p=0.14\text{m}$.

It was found that the interaction between the saw-blade and the work-piece may reduce the lateral vibration of the blade at certain speeds (i.e., the idling vibration of the blade is greater than that during cutting ($AB$)), shown in Figure 7.3(a), and at different speeds it may induce self-excited vibration, as shown in Figure 7.3(b). From Figure 7.3(b) it can be clearly seen that when washboarding occurs the dynamic response of the blade is much larger than that of the idling blade.

Figure 7.4 shows the waterfall spectral diagram of the saw-blade response when washboarding occurs. As may be noted, in the case shown in Figure 7.4, three modes are self-excited during cutting and their natural frequencies are very high. Note that the time durations ($AB$) and ($BC$) in Figure 7.4 represent the idling and cutting periods, respectively. Also, it should be noted that only the modes with frequencies below the tooth passing frequency $f_t=1572\text{ Hz}$ are excited and the mode closest to $f_t$ has the largest response magnitude.

Figures 7.5 and 7.6 illustrate the response spectra of the saw-blade, at a tooth passing frequency $f_t=1598\text{ Hz}$, during idling and cutting, respectively. From these figures it may be noted that travelling-wave modes $(0, 11F)-(0, 8F)$ below the tooth passing frequency $f_t$ are self-excited during cutting. The component of response at $f_t$ may be aerodynamically induced by the interaction between the teeth of blade and the surrounding air [69, 70]. It should be mentioned that a new modal identification method for identifying the self-excited modes, called the Artificial Damping Method, is proposed and described in Chapter 8.

In order to understand the primary features of the self-excited vibration in saw-blade cutting, numerous cuts at different rotating speeds were conducted and the modes excited are illustrated in Figure 7.7. From this figure it was found that primarily forward-wave modes with frequencies below the tooth passing frequency $f_t$ are excited during cutting.
Figure 7.5 Idling response of the clamped 21" saw-blade ($f_r = 1598$ Hz)
(some modes are excited aerodynamically)

Figure 7.6 Self-excited response of the clamped 21" saw-blade during cutting
($f_r = 1598$ Hz)
Figure 7.7 Experimental Campbell diagram: the modes excited at different speeds

Figure 7.8 The Campbell diagram predicted from the natural frequencies at $\Omega = 0$
Figure 7.8 shows the Campbell diagram predicted from the modal parameters of the same stationary saw-blade, in which the stiffening of the blade due to rotation was neglected based on the fact that the stiffness of a high-frequency mode is much larger than the rotational stiffening of the blade. The number of nodal diameters is identified by using the method proposed in Chapter 8, which only involves the reading of the imaginary parts of the FRFs. By comparing this figure with Figure 7.7 it can be seen that the forward-wave modes predicted from the measured stationary data match the forward-wave modes excited during cutting. This result also confirms the identification results from the Artificial Damping Method.

Figure 7.9 illustrates the displacement response spectra of the blade, measured at the point shown in Figure 7.2, at different tooth passing frequencies with the same bite (0.020") per tooth. When the circular saw cuts wood at \( f_t = 1272 \) Hz, no washboarding is found on the cutting surface and no large response of the blade is observed right below the tooth passing frequency in the corresponding spectrum shown in Figure 7.9(a). When the saw cuts at \( f_t = 1400 \) Hz, washboarding appears on the cut wood surface. Accordingly, the mode \( A \) right below the tooth passing frequency is excited, as shown in Figure 7.9(b). Figure 7.9(c) shows the situation when the tooth passing frequency increases to \( f_t = 1473 \) Hz, in which no new modes are excited. When the tooth passing frequency \( f_t \) reaches 1590 Hz, a new mode near \( f_t \) (i.e., the mode \( B \)) is self-excited as shown Figure 7.9(d). Figure 7.9(e) shows the modes excited at \( f_t = 1680 \)Hz, from which it can be seen that several modes below \( f_t = 1680 \) Hz are excited during cutting. Another new mode (i.e., the mode \( C \)) is self-excited as the tooth passing frequency approaches \( f_t = 1820 \) Hz.

The waterfall spectral diagrams shown in Figure 7.10 clearly present the transition from saw idling to saw cutting and also show the transition from an old mode \( A \) to a new mode \( B \) excited during cutting as the rotating speed increases.
Figure 7.9 The response spectra of the blade during cutting at different tooth passing frequencies: (a) 1272Hz, (b) 1400Hz, (c) 1473Hz, (d) 1590Hz, (e) 1680Hz, (f) 1820Hz.
Figure 7.10 Waterfall spectral diagrams of the saw-blade at different rotating speeds 
(a) $f_i = 1587\text{Hz}$, (b) $f_i = 1624\text{Hz}$, (c) $f_i = 1783\text{Hz}$, (d) $f_i = 1810\text{Hz}$

Figures 7.10(a) and 7.10(b) present the modes excited at $f_i = 1587\text{Hz}$ and $f_i = 1624\text{Hz}$, respectively. Mode A with the largest response is identified as the mode (0, 1F). From these figures it can be seen that the magnitude of the excited mode decreases rapidly as the tooth
passing frequency increases above the natural frequency of this mode. As the tooth passing frequency increases to \( f_r = 1783 \text{Hz} \), a new mode with small magnitude, identified as the mode (0,12F) (i.e., the mode B), appears as shown in Figure 7.10(c). Its magnitude grows at \( f_r = 1810 \text{Hz} \) compared to that at \( f_r = 1783 \text{Hz} \), as shown in Figure 7.10(d).

From numerous cutting tests it was found that the response magnitude of the saw-blade is very sensitive to the difference between the tooth passing frequency \( f_r \) and the natural frequency \( f_n \) of the excited mode. The magnitude of response first increases very rapidly to a maximum value and then drops at a relatively slow rate to zero as the tooth passing frequency moves above the point \( f_r = f_n \). In other words, the rotating speed of the blade is a primary factor affecting the self-excited vibration in saw-blade cutting.

### 7.3 The Effect of Feed Speed on Self-Excited Vibration

In order to determine the dominant instability mechanism in washboarding, the detailed features of washboarding were studied. In this section the effect of feed speeds upon the magnitude of self-excited vibration of the saw-blade is assessed. This is necessary in order to identify the dominant instability mechanism from several possible instabilities in saw-blade cutting.

Figures 7.11(a)-(f) show the magnitudes of the saw-blade response with the same tooth passing frequency \( f_r = 1592 \text{Hz} \) (i.e., the same rotating speed), but at different feed speeds of the wood. Interestingly, from these figures, it was found that the maximum response magnitude decreases as the feed speed of the wood increases. The results from another run at different tooth passing frequency \( f_r = 1585 \text{Hz} \) are shown in Figures 7.12(a)-(f). This observation is verified by many other cuts with different saw-blades. A question arises, that is, which cutting force model can explain this important observation?
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Figure 7.11 Waterfall spectra at different feed speeds of the work-piece ($f_1 = 1592$ Hz)
Figure 7.12 Waterfall spectra at different feed speeds of the wood ($f_i = 1585$ Hz)
As the feed speed of the wood is increased, the bite per tooth also increases, which leads to increasing tangential and radial cutting forces. Therefore, as presented in Chapter 5, vibration of the saw-blade due to the follower cutting forces or the asymmetric in-plane stress fields could be excited more easily because of increased cutting forces. Thus, it can be concluded that the dominant instability during washboarding is not likely to be induced by follower cutting forces or the asymmetric in-plane stress fields.

Figure 7.13 The effects of the feed speed of the wood and the side clearance angle $\alpha_s$ of the tooth upon the regenerative cutting force

However, this is not the case if the instability of the blade is caused by the regenerative cutting forces, which are assumed to be proportional to the extra side cutting area as shown in Figure 7.13(a). From this figure it should be noted that the additional side cutting area which is a function of $(L-B)$ decreases as the feed speed of the wood (i.e., the bite per tooth $B$)
increases. Therefore, the regenerative cutting force decreases as the bite per tooth increases, which explains why the response of the saw-blade decreases as the feed speed of the wood increases.

From the case shown in Figure 7.13(b), it can been seen that the gap $g$ between the previous and current cuts is affected by the side clearance angle $a_s$ of the tooth and the bite per tooth $B$. If the side clearance angle $a_s$ and the bite $B$ are large enough, the extra side cutting area can be very small, or even eliminated. In this case, the regenerative instability in saw-blade cutting can be reduced or even eliminated.

### 7.4 The Effect of Tooth Profile on Washboarding

![Tooth profiles of the saw blade #1 (a) and the saw blade #2 (b)](image)

Figure 7.14 Tooth profiles of the saw blade #1 (a) and the saw blade #2 (b) with the same size: $a=0.0952m$, $b=0.265m$, $h=0.00229m$
In order to assess the effect of the tooth profile upon the self-excited vibration of the blade, cutting tests using the same size blade but different tooth profiles as shown in Figure 7.14 were conducted and some interesting results are presented in Figure 7.15.
Figures 7.15(a) and 7.15(b) show the modes excited during cutting with saw blade #1. Figures 7.15(c) and 7.15(d) show the development of the self-excited vibration with saw blade #2. As may be noted in the case of saw blade #1, the magnitude of the self-excited vibration reaches a maximum value in a very short time. However, in the case of saw blade #2, it takes longer for the growth of the excited mode A as shown in Figures 7.15(c) and (d). A third saw-blade was also used to assess the effect of the tooth profile upon the self-excited vibration of the blade. From numerous cuts it was found that the blade with the deep gullets (i.e., with more flexible teeth) usually gave a greater magnitude of vibration with a fast response growth. This observation is reasonable, because the blade with deep gullets usually has a large mode shape at the outer rim, which is especially true for the high-frequency modes. As discussed in the previous chapters, for a given mode, the larger the mode shape at the cutting point, the larger the real part of its eigenvalue. In other words, a saw-blade with deep gullets can be excited more easily by the cutting.

7.5 The Effect of Wood Geometry and Property on Washboarding

In this section, the effects of the geometry and the properties of the work-piece on the self-excited vibration of the blade are investigated and some of these results are presented.

Figures 7.16(a) and 7.16(b) illustrate the waterfall spectra of saw blade #1 at the same tooth passing frequency \( f_t = 1573.4 \text{Hz} \) but with different depths of cut (46mm and 92mm wood boards). From these figures it may be noted that the deflection magnitude of the blade in the case of cutting 92mm wood is greater than the one in the case of cutting 46mm wood, and it was also found that the resonant frequency \( f_n \) in 92mm wood cutting is about 4Hz higher than that in 46mm wood cutting.

Figures 7.16(c) and 7.16(d) show the waterfall spectra of the blade at the same tooth passing frequency \( f_t = 1812.3 \text{Hz} \) with different wood densities ((a): one 36mm low density wood board; (b): one 36mm high density wood board). From these figures it is clear that
cutting the denser wood results in a relatively large response magnitude at a little higher resonant frequency $f_n$ of the blade when interacting with the wood. This observation agrees with the simulation results presented in Chapter 5.

Figure 7.16 Waterfall spectra of the blade with different work-pieces

- (a) $f_n = 1549.9$ Hz
- (b) $f_n = 1553.8$ Hz
- (c) $f_n = 1782.9$ Hz
- (d) $f_n = 1788.8$ Hz

(a): 46mm wood, $f_i = 1573.4$ Hz; (b) 92mm wood, $f_i = 1573.4$ Hz; (c): 36mm low density wood, $f_i = 1812.3$ Hz; (d) 46mm high density wood, $f_i = 1812.3$ Hz
In this study it was found that, in the case of a clamped saw-blade, the resonant frequencies of the cutting system excited during washboarding are insensitive to the depth of cut, the property and the position of the work-piece because the modes excited usually have very high frequencies which involve the modal stiffnesses much higher than the stiffness added from the wood.

### 7.6 The Mode Shapes of the Blade and Washboarding

From the simulations it was found that a large displacement at the cutting point results in a large interactive force between the tooth and the work-piece, which usually corresponds to a large positive real part of the eigenvalue in the unstable region of the eigenvalue analysis. This knowledge is of significant assistance to those involved in the design of machine tools. A successful application of this principle is to use a guide to increase the lateral stiffness of the saw-blade at the cutting point in order to minimize the displacement of the blade in the cutting area. From this point of view blades with deep gullets are not a good design for resisting the self-excited vibration because the deep gullet design usually results in modes with large displacements at the outer rim of the blade, called “teeth modes”, especially for the high-frequency modes.

Figure 7.17 shows the radial mode shapes of saw blade #1 identified from the measured FRFs. As shown in this figure, modes (0, 9), (0,10) and (0,11) have large modal deflections at the rim of the blade, thus, they can be easily excited during cutting. However, it is very hard for mode (2,7) with small displacement at the outer rim to be excited. A complete mode shape for the mode (0,10) identified by using the method proposed in Chapter 8 is shown in Figure 7.18, which draws attention to the fact that this mode has fairly large displacements at the outer rim of the blade.

The simulations presented in Chapter 5 show that both the forward-wave modes and the backward-wave modes can be self-excited for clamped saw-blades. However, in the cutting
tests, only the forward-wave modes were found to be excited, as shown in Figure 7.7. Figures 7.19 and 7.20 show the frequency responses at the outer rim and in the middle of the stationary clamped blade #1, respectively. From these figures it is clear that, at the outer rim of this clamped blade assembly, the modes in the frequency range (1800Hz-3200Hz) have much smaller displacements than those at low frequencies. Once the blade is rotating the natural frequencies of the backward-wave modes will come down to reach the frequency region close to the tooth passing frequency of the blade. Because of their small displacement at the cutting point, these high-order modes do not show up in the time domain during cutting.

Figure 7.17 Radial mode shapes of the blade #1
Figure 7.19 The acceleration frequency response at the rim \( (r = b) \) of the blade #1 excited at an outer-rim point (A: the mode \((0, 11)\), B: the mode \((0, 10)\), C: the mode \((0, 9)\))
7.7 Washboarding Pattern on the Surface Finish

In the experimental studies on washboarding, it was found that a sparse washboarding pattern with a relatively large \( \lambda_x \) as shown in Figure 7.21(a) is always associated with the situation where the natural frequency \( f_n \) of the primary mode excited during cutting is very close to the tooth passing frequency \( f_t \). In this case the pattern on the wood surface is also relatively deep. On the other hand, a shallow but intense pattern with small \( \lambda_x \) as shown in Figure 7.21(b) always appears on the cutting surface when \( f_n \) is relatively far away from the tooth passing frequency \( f_t \).
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Figure 7.21 The washboarding pattern with different wavelengths $\lambda_x$

(a) $\lambda_x = 37.3\text{mm} - 37.9\text{ mm}$

(b) $\lambda_x = 19.6\text{mm} - 20.3\text{mm}$

Figure 7.22 The side view (a) and the top view (b) of the circular saw teeth during cutting
Figure 7.22 shows the positions of the teeth at the previous and current cuts. It can be assumed that Tooth \( A \) reaches its maximum lateral displacement \( A_0 \) at \( t = 0 \) when this tooth enters the wood, namely the lateral response of the blade at a point stationary in space can be expressed in the form \( w = A_0 \cos(2\pi f_n t) \) (where \( f_n \) is the resonant frequency of the blade), based on the assumption that only one primarily excited mode is involved and the response of the blade is harmonic. The next tooth (i.e., Tooth \( B \)) will take \( 1/f_r \) of time to reach the same vertical position. At this point of time, the blade vibrates laterally to a location where \( w_1 = A_0 \cos(2\pi f_n \Delta T) \). From Figure 7.22 it is clear that:

\[
\Delta T = 1/f_n - 1/f_r \tag{7.1}
\]

for the case that \( 2f_n > f_r > f_n \).

It is assumed that there are \( N \) teeth passing through the same horizontal line \( CD \) in the time period \( dt \) (\( dt \) must be such that \( N \) is an integer). Thus, the lateral displacement of the \( N \)th tooth can be expressed in the time domain as:

\[
w_N = A_0 \cos(2\pi f_n N\Delta T) \quad (N = 1, 2, \ldots) \tag{7.2}
\]

where, \( N = f_r dt \) and \( dt = x/V \) (where \( V \) is the feed speed of the wood).

Substituting Equation (7.1) and the above expressions for \( N \) and \( dt \) into Equation (7.2) results in the lateral displacement \( w(x) \) in the space domain \( x \):

\[
w(x) = A_0 \cos(2\pi f_r - f_n) = A_0 \cos\left(2\pi \frac{1}{\lambda_x} x\right) \tag{7.3}
\]

which is also the pattern produced by the teeth of the blade along an arbitrary horizontal line on the cut surface. Thus, the wavelength of the washboarding pattern can be written as:

\[
\lambda_x = \frac{V}{f_r - f_n} \tag{7.4}
\]
Note that Equation (7.4) is derived based on the assumption that the response of the blade is purely harmonic. From this equation, it can be seen that the wavelength $\lambda_x$ is proportional to the feed speed of the work-piece $V$ and is inversely proportional to the difference between the tooth passing frequency $f_t$ and the resonant frequency of the blade $f_n$.

Figures 7.23(a) and 7.23(b) show two washboarding patterns produced by the saw-blade with approximately equal feed speed of the work-piece ($V = 165$ ft/min) but different $\Delta f = f_t - f_n$. The differences between $f_t$ and $f_n$ can be evaluated from the responses of the blade shown in Figures 7.24(a) and 7.24(b) which correspond to the washboarding patterns shown in Figures 7.23(a) and 7.23(b), respectively. The result is that $\Delta f = 36.7$ Hz for (a) and $\Delta f = 59.8$ Hz for (b). Thus, the wavelengths of washboarding can be predicted from Equation (7.4), that is: $\lambda_x = 22.8$ mm for the case (a) and $\lambda_x = 14.0$ mm for the case (b), which agree with the measured wavelengths: $\lambda_x = 21.8 mm \sim 22.3 mm$ for the case (a) and $\lambda_x = 13.0 mm \sim 13.4 mm$.

Figure 7.23 The washboarding patterns at different tooth passing frequencies
(a): $\lambda_x = 21.8 mm \sim 22.3 mm$; (b): $\lambda_x = 13.0 mm \sim 13.4 mm$
Figure 7.24 The self-excited responses of the blade at different tooth passing frequencies ( (a): $f_t = 1590.3\text{Hz}, f_n = 1553.6\text{Hz}$; (b) $f_t = 1619.7\text{Hz}, f_n = 1559.9\text{Hz}$)

Figures 7.21(a) and 7.21(b) illustrate two washboarding patterns produced with different feed speeds ($V = 200\text{ ft/min}$ for (a) and $V = 100\text{ ft/min}$ for (b)) and similar differences between $f_t$ and $f_n$ ($\Delta f = 26.3\text{ Hz}$ ($f_t=1575.0\text{ Hz}$) for (a) and $\Delta f = 28.0\text{ Hz}$ ($f_t=1577.5\text{ Hz}$) for (b)). Then the wavelengths can be predicted as: (a) $\lambda_x = 38.6\text{ mm}$ and (b) $\lambda_x = 18.1\text{mm}$, which, in general, matches the measured wavelengths: (a) $\lambda_x = 37.3\text{~}37.9\text{ mm}$ and (b) $\lambda_x = 19.6\text{~}20.3\text{ mm}$. For the test rig used in this study the primary error of predicting the wavelength of washboarding pattern comes from the estimated feed speed of the wood.
Figure 7.25 The washboarding zones where the washboarding patterns appear. (The ranges of washboarding zones are about 1.8-2.5 Hz (or 108-150 RPM) of the rotating speed of the blade or about 70-100 Hz of the tooth passing frequency)

Figure 7.26 The washboarding pattern in the case of relatively large difference between $f_t$ and $f_n$ ($f_t - f_n = 88.6$ Hz)

Figure 7.25 illustrates the washboarding zones for the clamped saw-blade #1, in which the blade is fully excited and the washboarding patterns can be detected. From this figure it was found that the onset of the washboarding pattern always starts at the point where $f_t$ is slightly greater than $f_n$ and the washboarding range is normally about 1.8-2.5 Hz (or 108-150 RPM).
of the rotating speed or about 70~100 Hz of the tooth passing frequency for the clamped saw-blade. The washboarding pattern usually disappears when the tooth passing frequency is about 100Hz apart from the resonant frequency of the blade, in which case, the washboarding pattern is getting shallow and the wavelength $\lambda_x$ is getting small, as shown in Figure 7.26.

In this study, three different clamped saw-blades were used to investigate the relations between the self-excited vibration of the blade and the washboarding patterns on the cut surfaces, and similar results and conclusions were reached. However, the washboarding zones for different blades are different because different saw blades have different resonant frequencies.

7.8 A Supporting Test to Prove the Existence of Regenerative Cutting Force

A supporting test was conducted by using the pendulum cutting rig shown in Figure 7.27 to illustrate the existence of a lateral cutting force due to the side cut of the saw tooth, so that the existence of a regenerative cutting force can also be verified.

As shown in Figure 7.27, the pendulum cutting rig consists of a pendulum arm with a cutter attached, the wood work-piece of thickness $H = 4$ mm, the 3D piezo-electric force dynamometer and other measurement hardware. The pendulum arm is first lifted and then released under the influence of its own potential energy.

Figure 7.28(a) shows the top cutting of the wood, which is used to simulate the flank cuts of the teeth with zero or negative velocity towards the work-piece. In the case that the cutter cuts the wood along an arc which moves away from the work-piece, as shown in Figure 7.28(a), it was found that the cutting force applied on the wood in the Z direction is always downwards, in other words, this cutting force always pushes the wood and the cutter away from each other as long as there is an interactive region between them, even when the tooth is moving away from the work-piece. Figure 7.28(b) illustrates the side cut of the wood, which simulates the side cuts of the teeth with zero velocity towards the work-piece.
Figure 7.27 Schematic of the pendulum cutting rig and the experimental set-up

Figure 7.28 Schematic of the wood cutting using the pendulum cutting rig

(a) Top cutting  (b) Side cutting (Side view of (a))
Figure 7.29 The cutting forces from the top cuts along the wood grain with 30° rake angle cutter at different nominal chip thicknesses

(a) $h_c = 0.0625\text{mm}$

(b) $h_c = 0.125\text{mm}$

(c) $h_c = 0.167\text{mm}$

(d) $h_c = 0.250\text{mm}$
Figures 7.29(a)-(d) show the cutting forces applied on the wood in the (-Z) direction at different nominal chip thicknesses $h_c$ in the cases of the top cuts along the wood grain with the 30° rake angle cutter. From this figure, it can be seen that the cutting force increases approximately linearly as the nominal chip thickness increases. It should be mentioned that due to the compressibility of the work-piece and the flexibility of the cutting rig, the actual chip thickness may be very much different from the nominal chip thickness.

Figure 7.30 shows the cutting forces in the top cuts along the wood grain with the 90° rake angle cutter, which is similar to the side cut of real saw-blade cutting, from which it is clear that the cutting force also increases as the bite of the cutter increases. As may be noted for the same nominal chip thicknesses the amplitude of the cutting force of this case is about 20%~40% larger than those from the 30° rake angle cutter.

Figure 7.31 illustrates the cutting forces in the top cuts vertical to the wood grain with the 30° rake angle cutter. From these figures it is clear that for the same nominal chip thicknesses the amplitude of cutting force in the cuts vertical to the wood grain is much (2~3 times) larger than those in the cuts along the wood grain.

Figure 7.32 shows the cutting forces in the case of the side cuts along the wood grain using the cutter with 90° rake angle and 10° clearance angle ($H=4$mm). The set-up of this case is shown in Figure 7.28(b). In this case, the work-piece is moved towards the cutter in the $Y$ direction after each cut. As shown in Figure 7.32, the lateral cutting force increases with the increase of lateral chip thickness. Note that the oscillation in the cutting forces shown in this figure is caused by the structural vibration of the pendulum rig.

Figure 7.33 shows the cutting forces in the case of the side cuts vertical to the wood grain. Interestingly, it was found that the cutting force produced in the side cuts vertical to the wood grain was not as large as expected. This is because, in this case, the wood fiber is not severed but ripped off by the side edges of the saw teeth unlike the cases of top cuts shown in Figure 7.31.
Figure 7.30 The cutting forces from the top cuts along the wood grain with 90° rake angle cutter

Figure 7.31 The cutting forces from the top cuts vertical to the wood grain with 30° rake angle cutter
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Figure 7.32 The cutting forces from the side cuts along the wood grain

(a) \( h_c = 0.127\text{mm} \)  
(b) \( h_c = 0.254\text{mm} \)

Figure 7.33 The cutting forces from the side cuts vertical to the wood grain

(a) \( h_c = 0.127\text{mm} \)  
(b) \( h_c = 0.254\text{mm} \)
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7.9 Summary

An understanding of the dynamics of washboarding and the dominant instability mechanism in washboarding can be achieved through the experimental investigation presented in this chapter. In addition, in order to support the regenerative cutting force model, a cutting test using a pendulum cutting rig was conducted to verify the existence of the lateral cutting force due to the flank cut in the case that the velocity of the blade towards the work-piece is zero or the blade is moving away from the work-piece.

It may be concluded, from the information presented in this thesis, that the dominant self-excited vibration and the washboarding phenomenon in saw-blade cutting is most likely to be induced by regenerative lateral cutting forces based on the following facts and observations:

- Washboard happens at high frequency and subcritical speeds. In simulations only three types of cutting forces can cause such instabilities. They are (a) lateral regenerative cutting forces, (b) the lateral components of tangential cutting forces and (c) the stress fields caused by tangential cutting forces.

- In simulations it was found that the saw-blade subjected to regenerative cutting forces can be excited when the tooth passing frequency \( f_t \) (or its multiples) is greater than the natural frequencies \( f_n \) of the excited modes, but less than \( 2f_n \) in primary instability region. This is also observed in the experimental Campbell diagram shown in Figure 7.7.

- In cutting tests it was found that the magnitude of the self-excited vibration of the blade decreases as the feed speed of the work-piece increases. This phenomenon can only be explained in the lateral regenerative cutting force model.

- On the contrary, the lateral components of follower cutting forces or the stress fields caused by in-plane cutting forces increase as the feed speed of the work-piece increases, which will aggravate the instability in saw-blade cutting. This is contrary to the experimental results.
• According to numerous cutting tests and experiences, the resonant frequency of the dominant mode excited during cutting is always less than the tooth passing frequency of the blade. However, it was found from simulations that all the forward-wave vibration modes of the saw-blade can be self-excited by the lateral components or the stress fields caused by tangential cutting forces over the entire speed range. This implies that the dominant instability in saw-blade cutting is not caused by the lateral components or the stress fields due to tangential cutting forces.

Other conclusions are summarized as follows:

1) The closest mode to the tooth passing frequency usually has the largest magnitude of the self-excited vibration of the saw-blade.

2) For the clamped saw-blades, the modes excited during cutting are usually the forward-wave ones.

3) The severity of washboarding is associated with the mode shapes at the cutting points. Blades with the deepest gullets usually have the largest displacements at the outer rim, leading to large interactive cutting forces.

4) The resonant frequencies and the magnitude of the self-excited vibration in saw-blade cutting increases as the depth of cut and the density of the work-piece increases. However, for the clamped saw-blade, the resonant frequencies of the blade during cutting are insensitive to the changes of the depth of cut and the density of the wood.
8.1 Introduction

Experimental modal analysis is a technique used to obtain a mathematical model of the vibration characteristics of structures by means of curve fitting algorithms in either the frequency or the time domain. The theoretical basis for this technique involves the development of a general expression known as the frequency response function (FRF) relating the response to excitation. A general mathematical model for FRFs contains a set of unknowns that need to be identified in order to fit the model to the measured FRFs.
Experimental modal analysis usually involves measuring the frequency response function between the response at one location and the excitation at a driving point as shown in Figure 8.1. Excitation is measured by a force transducer at the excitation point, and the response, at the chosen degree of freedom, is usually measured by an accelerometer for a stationary structures as shown in Figure 8.1(a) or by an eddy-current or a laser displacement probe for rotating systems as shown in Figure 8.1(b).

Experimental modal analysis has been a widely accepted technique for achieving a greater understanding of the dynamic characteristics of stationary systems since the 1980's. Numerous sophisticated identification algorithms are available to estimate modal parameters from the frequency response data or time-domain response data [71, 72, 73]. Applying this technique to rotating machines gives considerable insight into their complicated dynamic characteristics.

The effects of rotation in rotating systems create a number of theoretical and practical barriers against the direct applications of conventional modal testing techniques [74]. The rotating components generate so-called gyroscopic forces which modify the dynamic properties and result in a skew-symmetric matrix in the equation of motion as described in the previous chapters. Also, a number of practical difficulties arise in the acquisition of valid measured data. There are some special issues that arise in the application of modal analysis to rotating machines:

- The rotating system must be sufficiently excited so that sufficient information for modal parameter identification can be obtained.
- Due to the fluctuation of rotating speed the responses of the rotating system must be sampled simultaneously in order to keep accurate phase information. This is especially important at high rotating speeds.
- In the case of self-excited vibration, such as a circular saw-blade interacting with the work-piece, the excitation forces (i.e., the multiple moving cutting forces) are not
measurable, thus, an effective identification algorithm dealing with this situation has to be
developed in this study.

- Noisy response signals due to operational excitation, electrical induction, misalignment,
aerodynamic excitation, friction, non-linearity in bearings and so on have to be dealt with.

A successful procedure of identifying modal parameters for rotating machines or self-
excited structures, such as a rotating disc interacting with the work-piece, is described in this chapter. A practical and effective identification method called the Artificial Damping Method is proposed to identify the traveling-wave modes for rotating systems with extremely noisy response signals. A multiple reference method of modal analysis, based on physical parameter identification, is also presented here to handle the situation of two modes with identical natural frequencies but different mode shapes. This situation is often encountered in rotating systems. In addition, a practical and efficient procedure is proposed in this chapter to identify the mode shapes of a circular disc by only reading the imaginary parts of the FRFs.

### 8.2 Special Requirements in Modal Testing of Rotating Discs

#### 8.2.1 Frequency Response Function for a Rotating Disc

The significant difference between a conventional structure and a rotating machine is that the effects of the rotation result in an asymmetric gyroscopic system matrix in the equation of motion, which requires a more general theoretical approach to solve this equation.

As described in Chapter 2, the displacement field of a rotating disc can be assumed in the modal expansion form:

\[
w(r, \theta, t) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r) \left[ C_{mn}(t) \cos(n\theta) + S_{mn}(t) \sin(n\theta) \right] \right\} \tag{8.1}
\]
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Figure 8.2 A rotating disc with a displacement transducer at \((r_d, \theta_d)\) and a shaker at \((r_e, \theta_e)\)

where \(C_{mn}(t)\) and \(S_{mn}(t)\) are unknown functions to be determined, \(m\) and \(n\) are the numbers of nodal circles and nodal diameters, respectively. Substituting Equation (8.1) into the governing equation of the rotating disc and applying the Galerkin procedure lead to the equations of motion in matrix form:

\[
[M]\{\ddot{x}\} + ([G] + [C])\{\dot{x}\} + [K]\{x\} = \{f\}
\]

(8.2)

where, \([M]\) and \([K]\) are the mass and stiffness matrices, respectively. In general, \([M]\) is a real and symmetric matrix, and \([K]\) is real and symmetric but not necessarily positive definite. \([G]\) is the skew-symmetric gyroscopic matrix. \([C]\) is the damping matrix of the system.

\[
\{x\} = (C_{00}, \ldots, C_{M0}, C_{01}, \ldots, C_{M1}, \ldots, C_{0N}, \ldots, C_{MN}, S_{01}, \ldots, S_{M1}, \ldots, S_{0N}, \ldots, S_{MN})^T
\]

\[
= (\cdots C_{mn}, \ldots, S_{mn}, \ldots)^T
\]

\[
\{f\} = P_e(t)\{Q\} \quad \text{and} \quad \{Q\} = (\cdots R_{ql}(r_e)\cos(l\theta_e)\cdots, \cdots R_{ql}(r_e)\sin(l\theta_e)\cdots) \quad (q, l = 0, 1, 2, \ldots).
\]

\(P_e(t)\) represents a lateral excitation force at \((r_e, \theta_e)\).
The equation of motion (8.2) can be written in the state-space form as:

$$[A]\{\dot{y}\} + [B]\{y\} = \{F\}$$

(8.3)

where, $[A] = \begin{bmatrix} G + C & M^T \\ M & 0 \end{bmatrix}$, $[B] = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}$, $\{y\} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ and $\{F\} = P_e(t)\begin{bmatrix} 0 \\ 0 \end{bmatrix} = P_e(t)\{\overline{Q}\}$.

The eigenvalue problem and the orthogonality properties of a general gyroscopic system can be expressed by the following four equations [75]:

$$\begin{align*}
(\lambda_i [A] + [B])\{u\}_i &= \{0\}, \\
(\lambda_i [A]^T + [B]^T)\{v\}_i &= \{0\} \\
\{v\}_j^T [A]\{u\}_i &= \delta_{ji}, \\
\{v\}_j^T [B]\{u\}_i &= -\delta_{ji}\lambda_i
\end{align*}$$

(8.4) (8.5)

where $\{u\}_i$ and $\{v\}_i$ are the right-hand and left-hand eigenvectors, respectively. $\delta_{ji}$ is the Dirac delta function.

It is assumed that the response $\{y(t)\}$ can be expressed as a linear combination of the right-hand mode shapes of the system. That is,

$$\{y(t)\} = \sum_{i=1}^{2N} \{u\}_i q_i(t)$$

(8.6)

where, $N$ is the number of modes included in the linear combination. $q_i(t)$ are the unknown coefficients which are also the variables in modal coordinates. Substituting Equation (8.6) into Equation (8.3) and simplifying the equations using the orthogonality properties lead to $2N$ decoupled equations in modal coordinates and their corresponding Laplace domain forms:

$$\begin{align*}
\{v\}_i^T [A]\{u\}_i \dot{q}_i(t) + \{v\}_i^T [B]\{u\}_i q_i(t) &= \{v\}_i^T P_e(t)\{\overline{Q}(r_e, \theta_e)\} \\
q_i(s) - q_i(0) - \lambda_i q_i(s) &= \{v\}_i^T P_e(s)\{\overline{Q}(r_e, \theta_e)\}
\end{align*}$$

(8.7) (8.8)
By letting \( q_i(0) = 0 \), Equation (8.8) can be written in the form:

\[
q_i(s) = \frac{P_e(s)\{v_i\}^T \{\tilde{Q}(r_e, \theta_e)\}}{s - \lambda_i} \tag{8.9}
\]

It therefore follows from Equation (8.6) that

\[
\{Y(s)\} = \left\{ \frac{X(s)}{sX(s)} \right\} = P_e(s) \sum_{i=1}^{2N} \{u_i\} \frac{\{v_i\}^T \{\tilde{Q}(r_e, \theta_e)\}}{s - \lambda_i}
= P_e(s)(\cdots \tilde{C}_{mn}(s)\cdots; \cdots \tilde{S}_{mn}(s)\cdots)^T \tag{8.10}
\]

where, \( \{X(s)\} \) and \( \{Y(s)\} \) are the Laplace expressions of \( \{x(t)\} \) and \( \{y(t)\} \), respectively. Extracting \( X(s) \) from Equation (8.10), yields:

\[
\begin{align*}
\{X(s)\} &= P_e(s) \sum_{i=1}^{2N} \{\varphi_u\}_i \left\{ \frac{\{\varphi_v\}_i^T \{\tilde{Q}(r_e, \theta_e)\}}{s - \lambda_i} \right\} \\
&= P_e(s)(\cdots \tilde{C}_{mn}(s)\cdots; \cdots \tilde{S}_{mn}(s)\cdots)^T \tag{8.11}
\end{align*}
\]

where, \( \{u\}_i = \left\{ \frac{\varphi_u}{\lambda_i} \right\} \) and \( \{v\}_i = \left\{ \frac{\varphi_v}{\lambda_i} \right\} \).

\( \{\tilde{C}_{mn}(s)\} \) and \( \{\tilde{S}_{mn}(s)\} \) can be expressed as follows:

\[
\begin{align*}
\{\tilde{C}_{mn}(s)\} &= \sum_{i=1}^{2N} \frac{\varphi_{ui,mm}^c \psi_{ei}(r_e, \theta_e)}{s - \lambda_i} \quad \text{and} \quad \{\tilde{S}_{mn}(s)\} = \sum_{i=1}^{2N} \frac{\varphi_{ui,mm}^s \psi_{ei}(r_e, \theta_e)}{s - \lambda_i} \tag{8.12}
\end{align*}
\]

where, \( \{\varphi_u\}_i = (\cdots \varphi_{ui,mm}^c, \cdots \varphi_{ui,mm}^s, \cdots) \). \( \varphi_{ui,mm}^c \) and \( \varphi_{ui,mm}^s \) are the mode shapes in \( \{\varphi_u\}_i \) corresponding to the terms \( C_{mn} \) and \( S_{mn} \) in \( \{x\} \), respectively.

and

\[
\psi_{ei}(r_e, \theta_e) = \{\varphi_v\}_i^T \{Q(r_e, \theta_e)\} \quad \begin{align*}
&= \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r_e) \left[ \varphi_{vi,mm}^c \cos(n \theta_e) + \varphi_{vi,mm}^s \sin(n \theta_e) \right] \right\} \tag{8.13}
\end{align*}
\]
where, \( \{ \varphi_i \} = \{ \cdots \varphi_{vi,mn}^c, \cdots \varphi_{vi,mn}^s \} \).

The displacement at \((r_d, \theta_d)\) in the Laplace domain can be expressed from Equations (8.1) and (8.11) as follows:

\[
\hat{w}(r_d, \theta_d, s) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r_d) \left[ \tilde{C}_{mn}(s) \cos(n\theta_d) + \tilde{S}_{mn}(s) \sin(n\theta_d) \right] \right\} P_e(s) \quad (8.14)
\]

Thus, the transfer function between the response at \((r_d, \theta_d)\) and the excitation at \((r_e, \theta_e)\) can be written as:

\[
H_{de}(s) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r_d) \left[ \tilde{C}_{mn}(s) \cos(n\theta_d) + \tilde{S}_{mn}(s) \sin(n\theta_d) \right] \right\} \quad (8.15)
\]

Letting \( s = j\omega \) and substituting Equation (8.12) into Equation (8.15) yield the frequency response function:

\[
H_{de}(j\omega) = \sum_{i=0}^{2N} \frac{\phi_{di}(r_d, \theta_d) \psi_{ei}(r_e, \theta_e)}{j\omega - \lambda_i} \quad (8.16)
\]

where, \( \phi_{di}(r_d, \theta_d) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r_d) \left[ \varphi_{vi,mn}^c \cos(n\theta_d) + \varphi_{vi,mn}^s \sin(n\theta_d) \right] \right\} \)

\[
\psi_{ei}(r_e, \theta_e) = \sum_{n=0}^{N} \sum_{m=0}^{M} \left\{ R_{mn}(r_e) \left[ \varphi_{vi,mn}^c \cos(n\theta_e) + \varphi_{vi,mn}^s \sin(n\theta_e) \right] \right\}
\]

As may be noted \( \phi_{di}(r_d, \theta_d) \) and \( \psi_{ei}(r_e, \theta_e) \) represent the complex right-hand and left-hand mode shapes in physical coordinates at the points \((r_d, \theta_d)\) and \((r_e, \theta_e)\), respectively.

In the case of stationary structures, such as a stationary disc, only the right-hand eigenvectors are involved in the FRF which can be expressed as:
8.2.2 Application of Excitation and Measurement of Response

Figure 8.3 A typical configuration of modal testing for a rotating disc

Figure 8.3 shows a schematic diagram of the experimental setup for the modal testing of a rotating disc. Due to the rotation of the test-piece, non-contact displacement probes and a non-contact electromagnetic exciter are utilized to acquire the input and output data of the system. It is possible to propose several ways to excite a rotating disc. The following two excitation methods are practical and easily implemented:

- use of a non-contact electromagnetic exciter with a force transducer built between the magnetic head and the base as shown in Figure 8.3.
- application of an impact hammer or rig with a force transducer attached.
The non-contact shaker excitation has the advantage that an input signal and its energy spectrum can be easily chosen and controlled, which enables the maximum amount of energy to be put into the frequency regions of interest or even put into one frequency component. This feature is very important for dealing with an extremely noisy system. The difficulty of applying shaker excitation to the rotating disc arises in the case that the measurement of excitation force is needed because it is no longer possible to attach the force transducer directly between the test-piece and the shaker rod. Instead the force transducer must be put between the electromagnetic head and its support (i.e., its base), thus, it is necessary to introduce some sort of supporting elements which can sustain the bending moment and the gravity of the heavy magnetic head and its rod. In this study, as shown in Figure 8.4, a strong fiber material is chosen to support the magnetic head instead of using bearing which may introduce friction. The main disadvantages of electromagnetic shaker excitation are that the excitation force is not concentrated but distributed over the cross-sectional area of the magnetic head and also that the vibration of the magnetic shaker may contaminate the force signal measured from the force transducer. Thus, the natural frequencies of the shaker should be avoided within the frequency range of interest.
The alternative to shaker excitation is to use impact excitation. The impact excitation rig is relatively simple and easy to make. It is also easy to change the excitation point from one location to another. However, the impact excitation has a relatively broad force spectrum which may not provide sufficient input energy in the frequency regions of interest. Furthermore, the repeatability of the impact excitation is relatively poor compared to shaker excitation.

In conventional modal testing for a stationary structure, the accelerometer is the most common device for response measurement. The main advantages of using accelerometers are that this type of transducer is reliable and it measures absolute motion. However, it is difficult to use an accelerometer to measure the response of a rotating component. Hence, in rotating machinery applications, it is more common to use non-contact probes such as eddy-current displacement probes or laser probes to measure the response of rotating components. The most important point is that non-contact probes usually measure the relative displacement between two interactive components such as a work-piece and a milling cutter, which is of direct interest in the study of self-excited vibration in cutting process.

In the modal testing for a rotating saw-blade, the use of laser displacement probe may eliminate many practical problems and provide clean response measurement for the saw-blade during idling [76], but it was found that the signals from laser probes may be contaminated by the saw dust during cutting. However, in this case, the eddy-current displacement probes arranged around the rim of the blade as shown in Figure 8.3 were found to give reliable response signals.

8.3 Modal Identification from Forced and Self-Excited Vibrations of a Rotating Disc — An Artificial Damping Method (ADM)

In the modal testing of a rotating disc, the extremely noisy response signals due to misalignment, aerodynamic excitation, electrical induction, friction and bearing nonlinearity
have to be dealt with. The primary difficulty in performing such a modal analysis is to choose a satisfactory excitation method and a proper type of excitation signal. The electromagnetic shaker shown in Figure 8.4 is used to excite the rotating blade and several types of input signals were tested. The input signals from the source module are amplified by a power amplifier and then input into the magnetic coil of the exciter to produce a magnetic flux which passes through the gap between the magnetic head of the exciter and the steel blade to produce an attraction force applied on the blade.

In the modal testing it was found that a white noise random waveform with broad frequency components could provide sufficient excitation energy to the system and due to its broad frequency range it also causes undesirable responses with extremely rich frequency components. It was also found that the best input signal for this application is to use sine waveform or sine sweeping signal in order to provide concentrated energy into the rotating system. For high-speed rotating systems, the responses at different locations should be acquired simultaneously because a small fluctuation of rotating speed may ruin the phase information in the responses, which is essential to the identification of traveling-wave modes.

In the case of a rotating system with stable speed, the sine sweeping procedure may give satisfactory FRFs, but this is not the case when the rotating component interacts with another object, such as a work-piece. In this case, the circumferential components of the interactive forces may cause oscillation at the rotating speed. In practice, it was found that a sinusoidal signal was the most appropriate for high-speed rotating systems with rotating speed fluctuations. Then, a critical problem arises, which is how to identify the travelling-wave modes by only using harmonic resonant response of the system? Since it is difficult to model and remove noises from the time domain responses, various time-domain identification algorithms were ruled out in this application. The problem still remains as how to identify the travelling-wave modes by using harmonic resonant response in the frequency domain?
A simple but effective scheme is presented in this study, in which artificial damping is applied to the harmonic resonant responses due to self-excited vibration (or forced vibration) in order to transform the harmonic responses \( \{w_h(t)\} \) into artificially damped free responses \( \{w_d(t)\} \):

\[
\{w_d(t)\} = \{w_h(t)\} e^{-\xi_d \omega_i t}
\]  

(8.18)

where, \( \xi_d \) is an artificial damping factor chosen by the user, \( \omega_i \) is the estimated angular natural frequency. The artificial frequency response functions (the AFRFs) can then be calculated from the damped free responses \( \{w_d(t)\} \). Therefore, the travelling-wave mode shapes which are essential for the system diagnosis can be identified from the AFRFs. It should be mentioned that the use of artificial damping was found in the frequency domain response analysis using discrete Fourier transforms for reducing aliasing errors [77].

The FRF of a linear rotating system can be rewritten from Equation (8.16) as:

\[
H_{de}(j\omega) = \sum_{i=1}^{N} \left( \frac{A_i}{j\omega - \lambda_i} + \frac{A_i^*}{j\omega - \lambda_i^*} \right)
\]  

(8.19)

where, \( A_i = \phi_{di}(r_d, \theta_d) \psi_{ei}(r_e, \theta_e) \), is called the \( i \)th modal residual. \( A_i^* \) is the complex conjugate of \( A_i \).

Equation (8.19) can also be written as:

\[
H_{de}(j\omega) = \sum_{i=1}^{N} \frac{-(j\omega p_i + q_i)}{\omega^2 + j\omega c_i - d_i}
\]  

(8.20)

where, \( \lambda_i = \frac{1}{2} \left( c_i + j\sqrt{4d_i - c_i^2} \right) \), \( A_i = \frac{1}{2} \left( p_i + j\frac{-(2q_i + p_i c_i)}{\sqrt{4d_i - c_i^2}} \right) \)

(8.21)

From Equation (8.21), it is clear that all the modal parameters can be determined if \( p_i, q_i, c_i \) and \( d_i \) are identified.
8.3.1 Modal Identification for a Single Mode

In the single-mode case, the FRF of the \( i \)th mode, together with the out-band modes (i.e., the modes not in the frequency range used in the identification), can be written as follows:

\[
H_{de}(j\omega) = \frac{-j\omega p_i + q_i}{\omega^2 + j\omega c_i - d_i} + \sum_{k=1}^{N_{\text{ob}}} \frac{j\omega E_k + F_k}{\omega^2 - \omega^2_{ok}}
\]  
(8.22)

where, \( N_{\text{ob}} \) is the number of out-band modes involved. \( \omega_{ok} \) is the \( k \)th out-band angular natural frequency. \( E_k \) and \( F_k \) are the noise model coefficients to be identified. Note that the damping terms are neglected in the out-band FRF model. In the cases of multiple modes, the noise from the out-band modes should be modeled and removed from the measured FRFs. It has been found that the noise model proposed in the study is not only effective for handling the out-band modes but also capable of modeling the general noise.

Multiplying \( \omega^2 + j\omega c_i - d_i \) on both sides of Equation (8.22), neglecting the high-order terms in the noise FRF model and regrouping the resulting equation into real and imaginary parts lead to the following curve fitting equation in matrix form:

\[
[h(\omega)]{p} = \{g(\omega)\}
\]  
(8.23)

where

\[
[h(\omega)] = \begin{bmatrix}
-\omega H_I(\omega) & -H_R(\omega) & 0 & 1 & \frac{1}{\omega^2 - \omega^2_{o1}} & 0 & \frac{1}{\omega^2 - \omega^2_{o2}} & 0 \\
\omega H_R(\omega) & -H_I(\omega) & \omega & 0 & 0 & \omega & \frac{\omega}{\omega^2 - \omega^2_{o1}} & 0 & \frac{\omega}{\omega^2 - \omega^2_{o2}}
\end{bmatrix}
\]

\[
{p}^T = (c_k \ d_k \ p_k \ q_k \ E_{k1} \ E_{k2} \ F_{k1} \ F_{k2})
\]
\[
\{g(\omega)\}^T = \begin{pmatrix} -H_R(\omega)\omega^2 & -H_I(\omega)\omega^2 \end{pmatrix}
\]

and \(H_{de}(j\omega) = H_R(\omega) + jH_I(\omega)\) is the measured FRF.

Thus, the least-square solution of the curve fitting equation is given by:

\[
\{\hat{p}\} = \left( [\tilde{H}]^T [\tilde{H}] \right)^{-1} [\tilde{H}]^T \{\tilde{G}\}
\]  

where,

\[
[\tilde{H}] = \begin{bmatrix} h(\omega_1) & h(\omega_2) & \cdots & h(\omega_I) \end{bmatrix}^T,
\]

\[
\{\tilde{G}\} = \begin{bmatrix} g(\omega_1) & g(\omega_2) & \cdots & g(\omega_I) \end{bmatrix}^T
\]

and \(\omega_1, \omega_2, \cdots, \omega_I\) represent the in-band frequency points selected in the calculation.

The method proposed in this section is a practical and effective algorithm which can give satisfactory results in modal identification for rotating systems, even when the modes are close to each other because of its special noise model. It should be mentioned that for the multiple mode identification orthogonal polynomials should be used to represent the FRFs in order to improve the numerical condition of the matrix \([\tilde{H}]^T [\tilde{H}]\).

### 8.3.2 Modal Identification For Multiple Modes

A typical method for multiple mode identification is based on iteration and Newton’s optimization algorithms. A basic identification algorithm and its improved version based on the sensitivity analysis of the FRFs is described in this section. The modified FRF \(H_{de}(\omega)\) due to changes of parameters \((\Delta a_k)\) can be approximately expressed in the first-order Taylor series form:

\[
H_{de}(\omega) = \tilde{H}_{de}(\omega) + \sum_k \frac{\partial H_{de}}{\partial a_k} \Delta a_k
\]  

where, \(\tilde{H}_{de}(\omega)\) is the original FRF (i.e., the measured FRF).
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For the case of \( N \) modes, the curve fitting equation can be written from Equation (8.25) in the matrix form:

\[
[S(\omega)]\{\Delta a\} = \{\Delta H(\omega)\} \tag{8.26}
\]

where,

\[
\{\Delta a\} = (\Delta c_1, \Delta d_1, \Delta p_1, \Delta q_1, \ldots, \Delta c_N, \Delta d_N, \Delta p_N, \Delta q_N)^T
\]

\[
[S(\omega)] = 
\begin{bmatrix}
\frac{\partial H_R(\omega)}{\partial c_1} & \frac{\partial H_R(\omega)}{\partial d_1} & \cdots & \frac{\partial H_R(\omega)}{\partial c_N} & \frac{\partial H_R(\omega)}{\partial d_N} \\
\frac{\partial H_I(\omega)}{\partial c_1} & \frac{\partial H_I(\omega)}{\partial d_1} & \cdots & \frac{\partial H_I(\omega)}{\partial c_N} & \frac{\partial H_I(\omega)}{\partial d_N}
\end{bmatrix}
\]

\[
\{\Delta H(\omega)\} = \begin{bmatrix} H_R(\omega) - \tilde{H}_R(\omega) \\ H_I(\omega) - \tilde{H}_I(\omega) \end{bmatrix}
\]

and \( H_{de}(j\omega) = H_R(\omega) + jH_I(\omega) \).

Thus, the least-square solution of Equation (8.26) is given by:

\[
\left( [\tilde{S}]^T [\tilde{S}] \right) \{\Delta a\} = [\tilde{S}]^T \{\Delta \tilde{H}\} \tag{8.27}
\]

where,

\[
[\tilde{S}] = 
\begin{bmatrix}
S(\omega_1) \\
\vdots \\
S(\omega_n)
\end{bmatrix}
\]

and \( \{\Delta \tilde{H}\} = \begin{bmatrix} \Delta H(\omega_1) \\ \vdots \\ \Delta H(\omega_n) \end{bmatrix} \)

and \( \omega_1, \omega_2, \ldots, \omega_i \) are the in-band frequency points selected in the identification.

Then, the updated parameters can be calculated by:

\[
\{a_j\} = \{a_{j-1}\} + \{\Delta a\} \tag{8.28}
\]
where, the subscript \( j \) represents the \( j \)th iteration.

However, the performance of this method depends highly on the initial guesses of the modal parameters. In order to improve the convergence of this method, the Levenberg-Marquardt optimization method (Fletcher, 1987 [78]; Gill, 1981 [79]) can be employed to solve Equation (8.27) so that the requirement for good initial guesses may be released to some extent. It can be shown that the Levenberg-Marquardt search direction is defined as the solution of the following equation:

\[
\left( [\tilde{S}]^T [\tilde{S}] + \mu_k [I] \right) \{ \Delta a \} = [\tilde{S}]^T \{ \Delta \tilde{H} \} \tag{8.29}
\]

where, \([\tilde{S}]\) is also called the Jacobian matrix with respect to the modal parameters. The parameter \( \mu_k \) is found such that \( [\tilde{S}]^T [\tilde{S}] + \mu_k [I] \) is positive definite, then Equation (8.29) is solved for \( \{ \Delta a \} \). If \( \mu_k \) is zero, \( \{ \Delta a \} \) becomes the Gauss-Newton direction, however, as \( \mu_k \) approaches infinity, it can be shown that \( \{ \Delta a \} \) becomes parallel to the steepest-descent direction [79]. Therefore, \( \mu_k \) generates an updated direction that varies between the Gauss-Newton direction and the steepest-descent direction, which also controls step size. This algorithm continuously varies search direction and step size to maintain an optimum descent rate, which retains the rapid rate of convergence of Newton’s method, but is also globally convergent and generally robust [78].

**8.3.3 Multiple Reference Modal Analysis Based on Physical Parameter Identification**

In the case of a circular disc, the problem of the identification of two modes with identical natural frequencies but different mode shapes is often encountered, but this problem cannot be identified correctly by using the algorithms described in sections 8.3.1 and 8.3.2. Therefore, in
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this section, it is necessary to present a multiple reference method of modal analysis based on physical parameter identification [80, 81], which can handle this situation.

For the case of multiple references, the impulse response matrix \([H(t)](m \times r)\) is a rectangular matrix, which can be written as [80]:

\[
[H(t)] = [\Phi]e^{[\Lambda]t}[L]
\]  \quad (8.30)

where, \([\Phi] (m \times 2N)\) is the mode shape matrix. \([\Lambda] (2N \times 2N)\) is the eigenvalue matrix. \([L] (2N \times r)\) is the matrix of modal participation factors. \(m\) is the number of measurement points. \(N\) is the number of modes of interest, which is usually much less than \(m\). \(r\) is the number of reference points.

Equation (8.30) and its first and second order derivatives can be transformed into the Laplace domain as follows:

\[
[H(s)] = [\Phi][R(s)]
\]  \quad (8.31)

\[
s[H(s)] - [H(t = 0)] = [\Phi][\Lambda][R(s)]
\]  \quad (8.32)

\[
s^2[H(s)] - s[H(t = 0)] - \dot{H}(t = 0) = [\Phi][\Lambda]^2[R(s)]
\]  \quad (8.33)

where, \([R(s)] = (s[I] - [\Lambda])^{-1}[L]\).

Equations (8.31)-(8.33) can be expressed in matrix form using \([H(t = 0)] = [\Phi][L]\) and \([\dot{H}(t = 0)] = [\Phi][\Lambda][L]\):

\[
\begin{bmatrix}
H(s) \\
sh(s) \\
s^2H(s)
\end{bmatrix} =
\begin{bmatrix}
0 & [\Phi] \\
[\Phi] & [L] \\
[\Phi \Lambda + s\Phi] & [\Phi \Lambda^2]
\end{bmatrix} =
\begin{bmatrix}
\Phi \\
\Phi \Lambda \\
\Phi \Lambda^2
\end{bmatrix} [R(s)]
\]  \quad (8.34)
Consider the characteristic equation of a general linear equation of motion (such as Equation (8.2)):

\[
[\Phi][\Lambda]^2 + [A_1][\Phi][\Lambda] + [A_0][\Phi] = \{0\} \tag{8.35}
\]

where, \([A_1] = [M]^{-1}([G] + [C]), \quad [A_0] = [M]^{-1}[K].\) Equation (8.35) can be also written in matrix form as

\[
\begin{bmatrix}
[A_0] & [A_1] & I \\
\Phi & \Phi \Lambda & \Phi \Lambda^2 \\
\end{bmatrix} \begin{bmatrix}
\Phi \\
\end{bmatrix} = \{0\} \tag{8.36}
\]

Multiplying \([A_0 \quad A_1 \quad I]\) on the both sides of Equation (8.34) and substituting Equation (8.37) into the resulting equation, yield:

\[
\begin{bmatrix}
[A_0] & [A_1] & I \\
\end{bmatrix} \begin{bmatrix}
H(s) \\
\Phi \Lambda \\
\Phi \Lambda^2 \\
\end{bmatrix} = \{0\} \tag{8.37}
\]

Equation (8.37) can be rewritten in the following form:

\[
[A_0][H(s)] + s[A_1][H(s)] + s[B_1] + [B_0] = -s^2[H(s)] \tag{8.38}
\]

where, \([B_1] = -[\Phi][L]\) and \([B_0] = -([A_1][\Phi] + [\Phi][A_1])[L]. \quad [A_0], [A_1], [B_0] \) and \([B_1]\) are unknown matrices to be identified. Equation (8.38) is the basic multiple reference curve fitting equation of modal analysis, which is set up by using the measured FRFs and solved for \([A_0], [A_1], [B_0] \) and \([B_1]\) in the least-square manner. Once the system matrices \([A_0]\) and
[A_i] are identified, the modal parameters can be calculated by solving the following general eigenvalue problem:

\[
\begin{bmatrix}
0 & I \\
-M^{-1}K & -M^{-1}(G+C)
\end{bmatrix} = \begin{bmatrix}
0 & I \\
-A_0 & -A_1
\end{bmatrix}
\]  

(8.39)

Unfortunately, Equation (8.38) must be solved in the space of \(2m\) DOFs, where \(m\) may be much larger than the number of modes \(N\) contained in the selected frequency band, which is not only time consuming but also it may result in some "calculation modes" (i.e., false modes). Therefore, a space-reduction technique should be employed to reduce the size of the curve fitting equation. The FRF matrix \([H(s)](s = j\omega)\) can be transformed into a much smaller matrix \([h(j\omega)]\) by using a linear transformation matrix \([T]\) which can be obtained from the singular value decomposition (SVD) of the auto-power spectrum matrix \([S_{ap}]\):

\[
[S_{ap}] = \text{Real Parts of } \sum_i [H(j\omega_i)][H(j\omega_i)]^T = [T_0][\Sigma][T_0]^T
\]

where, \([T_0]\) is called the singular vector matrix. \([\Sigma]\) is the singular value matrix. \([H(j\omega_i)]^T\) represents the complex conjugate matrix of \([H(j\omega_i)]\). Selecting the singular vectors corresponding to the \(n_r\) largest singular values forms the transformation matrix \([T]\) \((m \times n_r)\). The number of the relatively large singular values is an indicator of the rank of \([S_{ap}]\), which also shows the number of modes in the selected frequency band.

It was found in this study that the linear transformation matrix \([T]\) can be also obtained from the SVD of the matrix \([S_{im}]\) which can give a better indication of the number of in-band modes:

\[
[S_{im}] = \sum_i \text{Im}([H(j\omega_i)])(\text{Im}([H(j\omega_i)]))^T = [T_0][\Sigma][T_0]^T
\]

where, \(\text{Im}([H(j\omega_i)])\) represents the imaginary part of \([H(j\omega_i)]\).
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The condensed FRF matrix \( h(j\omega) \) can be written as:

\[
[h(j\omega)] = [T]^T [H(j\omega)] = [T]^T ([H_1] + [H_2]) = [h_1] + [h_2]
\] (8.42)

or

\[
[H(j\omega)] = [T]([h_1] + [h_2])
\] (8.43)

where, \([H_1]\) and \([H_2]\) represent the in-band and out-band FRF matrices, respectively. \([h_1]\) and \([h_2]\) are the corresponding in-band and out-band FRF matrices in the condensed space.

Substituting Equation (8.43) into Equation (8.38) and multiplying the resulting equation by \([T]\) yield,

\[
[a_0][h_1] + s[a_1][h_1] + s[b_0] + s^2[h_1] = -(T)^T [A_0][H_2] + s[T]^T [A_1][H_2] + s^2[T]^T [H_2])
\] (8.44)

where, \([a_0] = [T]^T [A_0][T] (n_r \times n_r)\)
\([a_1] = [T]^T [A_1][T] (n_r \times n_r)\)
\([b_0] = [T]^T [B_0] (n_r \times r)\)
\([b_1] = [T]^T [B_1] (n_r \times r)\)

By neglecting the damping in the out-band FRFs, \([H_2]\) can be written in the form:

\[
[H_2] = \sum_{k=1}^{N_{\text{eff}}} \left( \frac{[R_1]_i}{\omega^2 - \omega_i^2} + \frac{j\omega[R_2]_i}{\omega^2 - \omega_i^2} \right)
\] (8.45)

where, \([R_1]_i\) and \([R_2]_i\) are the unknown coefficient matrices.

Substituting Equation (8.45) into Equation (8.44) and simplifying the resulting equation lead to the following curve fitting equation in the condensed space:

\[
[a_0][h_1] + j\omega[a_1][h_1] + [\tilde{b}_0] + j\omega[\tilde{b}_1] + \sum_{k=1}^{N_{\text{eff}}} \left( \frac{[\tilde{R}_1]_i}{\omega^2 - \omega_i^2} + \frac{j\omega[\tilde{R}_2]_i}{\omega^2 - \omega_i^2} \right) = \omega^2[h_1]
\] (8.46)
where, \([a_0]\), \([a_1]\), \([\hat{b}_0]\), \([\hat{b}_1]\), \([\tilde{R}_1]\) and \([\tilde{R}_2]\) are the unknown matrices to be identified, in which only the system matrices \([a_0]\) and \([a_1]\) are used to identify modal parameters by solving the eigenvalue problem of the matrix:
\[
\begin{bmatrix}
0 & I \\
-a_0 & -a_1
\end{bmatrix}
\]

The mode shapes in physical coordinates \([\varphi_p]\) can be recovered from the mode shapes in the condensed coordinates \([\varphi_c]\) by:
\[
[\varphi_p] = [T][\varphi_c]
\] (8.47)

### 8.4 Applications of Modal Testing in Rotating and Stationary Discs

The artificial damping method (ADM) proposed in this chapter is applied to the forced and the self-excited vibrations of a rotating circular saw-blade to identify the mode shapes of the travelling-wave modes of interest. The ADM procedure can be described as follows:

1) to simultaneously measure the resonant responses of the test-piece (i.e., a disc in this application), excited in forced vibration or self-excited vibration;

2) to impose the artificial damping to the resonant responses of the test-piece by using Equation (8.18);

3) to select an appropriate identification method described in this chapter to identify the travelling-wave mode shapes of the system.

Trouble shooting in such rotating systems may then be conducted based on the information provided by the mode shape animation.

It is important to note that only response information is needed to identify the mode shapes in the artificial damping method. Therefore, the responses at different locations must be sampled simultaneously.
Another issue involved in implementing ADM is the determination of the best artificial damping for the purpose of identification. Fortunately, it was found that the results of modal identification are insensitive to the damping value chosen in the procedure. For example, for the case of a rotating disc, the mode shape of the travelling-wave mode \((0,1)\) can be described by \(w_1(\theta) = \sin(\omega t + n\theta)\) \((n = 1)\) (where \(\theta\) is the angular position in the space-fixed coordinates, and \(\omega\) is the angular natural frequency). The resonant responses at the locations \(\theta = 0°\), and \(30°\) are calculated, then they are transformed into the damped free responses by imposing the artificial damping. The identified phase differences of mode shapes between the two locations are \(30.0001°\), \(30.0013°\) and \(30.0099°\) which correspond to the artificial damping factors \(\zeta_a = 0.1\%\), \(2\%\) and \(5\%\), respectively. Thus, the differences caused by different damping factors are very small. For the case of \(\zeta_a = 2\%\), the phase differences of the identified mode shapes at the locations \(\theta = 0°\), \(30°\), \(60°\) and \(90°\) are \(\Delta\phi = 0°\), \(30.0013°\), \(60.0046°\) and \(90.0042°\), respectively. It is important to note that for the travelling-wave modes it is possible to identify the number of nodal diameters from the identified phase difference by using the following expression:

\[
n = \frac{\Delta\phi}{\Delta\theta}
\]

where, \(\Delta\phi\) is the phase difference of the identified mode shapes between two measurement points, and \(\Delta\theta\) is the angular difference between the two locations. In this example, the number of nodal diameters is \(n = 30° / 30° = 1\).

### 8.4.1 Modal Testing of a Rotating Disc Using Forced Resonant Responses

Figure 8.5 shows the experimental set-up for a rotating circular saw-blade with inner radius \(a=0.1334m\), outer radius \(b=0.4064m\) and thickness \(h=0.00299m\). Measurements were taken at eight locations by using displacement transducers (EMDT, Model PA12D03) around
the rim of the disc as shown in Figure 8.5. A self-designed electromagnetic shaker was employed to resonate the blade with periodic signals. Imposing artificial damping to the resonant responses of the blade leads to the damped free responses which can be used to calculate the artificial FRFs. Therefore, the mode shapes of travelling-wave modes can be identified based on the calculated FRFs.

Figure 8.6 shows the artificial FRFs at Points A and B resulting from the resonant responses with artificial damping $\zeta_a = 3\%$, excited at the frequency $f = 18.858$ Hz. The identified mode shapes are $\phi_{Ai} = 0.07829 - j0.006558$ and $\phi_{Bi} = 0.04333 - j0.06392$. The phase difference between $\phi_{Ai}$ and $\phi_{Bi}$ is $51.08^\circ$. The phase differences of $47.78^\circ$, $49.18^\circ$, $52.15^\circ$ and $52.30^\circ$ were also obtained by using different damping factors and frequency bands. From the identified phase difference, the number of nodal diameters can be calculated.
from Equation (8.48), namely \( n = \Delta \phi / \Delta \theta = 51.08 / 50 = 1 \), which corresponds to the backward-wave mode (0,1B).

Figure 8.7 shows the artificial FRFs with artificial damping \( \zeta_a = 1.8\% \) at Points A and B, excited at the frequency \( f = 50.257 \) Hz. The identified mode shapes at Points A and B are \( \varphi_A = 0.06565 - j0.05113 \) and \( \varphi_B = 0.07010 + j0.01483 \), thus, the phase difference between \( \varphi_A \) and \( \varphi_B \) is -49.86 which varies from -47.82° to -50.26° for different damping factors and frequency bands. The mode excited is corresponding to the forward-wave mode (0,1F) because \( n = \Delta \phi / \Delta \theta = -49.86 / 50 \approx -1 \).

Figure 8.8 shows the artificial FRFs (artificial damping \( \zeta_a = 2\% \)) of the saw-blade rotating at \( \Omega = 1200 \) RPM, subjected to a sinusoidal excitation at the frequency \( f = 36.116 \) Hz. The identified mode shapes at Points A and B are \( \varphi_A = -0.04084 - j0.03530 \) and \( \varphi_B = 0.01957 + j0.05444 \) which result in a phase difference of 151.05°. Thus, this mode is identified as (0,3B) because \( n = \Delta \phi / \Delta \theta = 151.05 / 50 = 3 \).

Figure 8.9 illustrates the artificial FRFs with artificial damping \( \zeta_a = 2\% \) at Points A and B (\( \Omega = 3000 \) RPM, \( f = 18.750 \) Hz). In this case, the relative angular position \( \Delta \theta \) is reduced from 50° to 45°. The identified phase difference \( \Delta \phi \) between \( \varphi_A \) and \( \varphi_B \) varies from -131.12° to -134.46°. Thus, this mode is identified as (0,3R) (where R represents the reflected-wave mode) because \( n = \Delta \phi / \Delta \theta = -134.46 / 45 \approx -3 \).

The mode shapes of the modes (0,1F) and (0,3B) at \( \Omega = 1200 \) RPM and the mode (0,3R) at \( \Omega = 3000 \) RPM are shown in Figure 8.10. They are linearly interpolated by using the mode shapes identified from 8 measured FRFs. From the mode shape animation the travelling-wave feature can be clearly observed.

From the results described above, it can be seen that for a rotating disc only two displacement probes are required to identify the mode shapes of travelling-wave modes with zero nodal circles. It was also found that the results of mode identifications are very consistent in the case that the rotating disc does not interact with other objects.
Figure 8.6 Artificial FRFs at Points A and B (Mode (0,1B), $\Omega = 1200$ RPM)

Figure 8.7 Artificial FRFs at Points A and B (Mode (0,1F), $\Omega = 1200$ RPM)
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Figure 8.8 Artificial FRFs at Points A and B (Mode (0,3B), $\Omega = 1200$ RPM)

Figure 8.9 Artificial FRFs at Points A and B (Mode (0,3R), $\Omega = 3000$ RPM)
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- **Figure 8.10** The mode shapes of the travelling-wave modes of a rotating disc

(a): Mode(0,1F): Forward-wave mode ($\Omega = 1000$ RPM)

(b): Mode(0,3B): Backward-wave mode ($\Omega = 1000$ RPM)

(c): Mode(0,3R): Reflected-wave mode ($\Omega = 3000$ RPM)

*Figure 8.10 The mode shapes of the travelling-wave modes of a rotating disc*
8.4.2 Modal Testing of a Self-Excited Circular Saw-Blade During Cutting

In this section, the artificial damping method is applied to the self-excited vibration of a rotating circular saw-blade during cutting in the case that the blade is fully self-excited (i.e., the washboarding phenomenon occurs) in order to identify the mode shapes of travelling-wave modes. Figure 8.11 shows a clamped circular saw-blade of inner radius \( a = 0.0952 \text{m} \), outer radius \( b = 0.265 \text{m} \) and thickness \( h = 0.00229 \text{m} \). The work-piece is a low density wood board of thickness \( H = 1.5 \text{ inch} \), which is fed at a constant speed \( V \). Only two displacement probes (EMDT, Model: PA12D03) are used to acquire the response data of the blade at the locations \( A \) and \( B \). Based on the identified phase difference \( \Delta \varphi \) between the mode shapes at Points \( A \) and \( B \), the number of nodal diameters of the mode excited during cutting can be identified.

Figure 8.11 Experimental modal analysis for a circular saw-blade which is self-excited during cutting
Because the interaction between the blade and the work-piece may modify the phase difference of mode shapes, it is important to know whether Equation (8.48) can still be used to predict the modes excited during cutting. From the simulations and the experiments, it was found that the interaction between the blade and the wood may severely affect the modes with low natural frequencies but it has very little effect on the modes with high natural frequencies where the blade is normally self-excited during cutting. For example, for the blade shown in Figure 8.11, subjected to an elastic force due to a stationary spring \( k = 10000 \) (N/m), the phase difference of the mode \((0, 1B)\) between two points with \( \Delta \theta = 10^\circ \) varies from \( \Delta \varphi = 4.39^\circ \) to \( \Delta \varphi = 16.14^\circ \) compared to \( \Delta \varphi = 10^\circ \) for the blade with no interaction. However, the phase difference of the mode \((0, 6F)\) is about \( \Delta \varphi = -60.021^\circ \sim -59.966^\circ \) compared to \( \Delta \varphi = -60^\circ \) for the idling blade. For the mode \((0, 10F)\) the phase difference is \( \Delta \varphi = -100.002^\circ \sim -100.000^\circ \) compared to \( \Delta \varphi = -100^\circ \) for the free blade. Therefore, it can be concluded that Equation (8.48) is still suitable for identifying the number of nodal diameters of the modes with high natural frequencies, which are the ones usually excited during cutting due to the high tooth passing frequency of the blade.

Figure 8.12 shows the artificial FRFs (the artificial damping \( \zeta_a = 0.2\% \)) at Points A and B of the saw-blade cutting at \( \Omega = 2400 \) RPM. The identified mode shapes at Points A and B are \( \varphi_{Ai} = 0.1738 - j0.1480 \) and \( \varphi_{Bi} = -0.3192 + j0.3205 \) for the mode with the identified natural frequency \( f_i = 1392.77 \) Hz, then, the phase difference \( \Delta \varphi \) between \( \varphi_{Ai} \) and \( \varphi_{Bi} \) is \( -175.30^\circ \). Therefore, the excited mode is the forward-wave mode \((0,10F)\) because \( n = \Delta \varphi / \Delta \theta = -175.30 / 18 = -10 \). For the mode with the identified natural frequency \( f_j = 1561.38 \) Hz, the identified mode shapes at Points A and B are \( \varphi_{Aj} = 0.2040 - j0.07735 \) and \( \varphi_{Bj} = -0.3607 + j0.03902 \). The corresponding phase difference \( \Delta \varphi \) is \( -194.59^\circ \). In this case the excited mode is the forward-wave mode \((0,11F)\) because \( n = \Delta \varphi / \Delta \theta = -194.59 / 18 = -11 \).
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Figure 8.12 Artificial FRFs of the saw-blade at Points A and B during cutting (Modes (0,10F) and (0,11F), $\Omega = 2400$ RPM)

Figure 8.13 Artificial FRFs of the saw-blade at Points A and B during cutting ($\Omega = 2394$ RPM) (The peak P is induced by the teeth of the saw-blade)
Figure 8.13 shows the artificial FRFs (the artificial damping $\zeta_a = 0.2\%$) at Points A and B of the saw-blade rotating at $\Omega = 2394$ RPM where the mode with a natural frequency $f_i = 1552.28$ Hz is self-excited. In this case, the angular distance between the two probes is $\Delta \theta = 27^\circ$. The identified mode shapes are $\varphi_{Ai} = -0.166578 - j0.052226$ and $\varphi_{Bi} = -0.148534 + j0.174264$. The corresponding phase difference $\Delta \varphi$ is $-293.04^\circ$ so that $n = \Delta \varphi / \Delta \theta = -293.04 / 27 = -11$. Therefore, this mode is the forward-wave mode $(0,1F)$.

It is important to note that a stable rotating speed of the blade and a stable feed speed of the wood are essential for the success of identification. This can be ensured by selecting a relatively thin, soft and uniform wood board as the work-piece which can still induce the self-excited vibration of the blade. Otherwise, in the case of variable operational conditions caused by the variations of wood properties, the identified $\Delta \varphi$ may vary in a wide range, such as $\pm 19^\circ$, which may lead to a false identification. For relatively constant cutting conditions, the variation of $\Delta \varphi$ can be usually controlled in $\pm 7^\circ$.

**8.4.3 Modal Testing of a Saw-Blade with Repeated Natural Frequencies**

The algorithm for multiple reference modal analysis based on the system matrix identification is applied to a clamped stationary circular saw-blade of inner radius $a = 0.0952m$, outer radius $b = 0.265m$ and thickness $h = 0.00229m$. The measurement hardware consists of a PCB impact hammer with a steel head (Model: 208A03) and a small B&K accelerometer (Type: 4393). Figure 8.14 shows two FRFs at the rim of the blade, from which only one mode appears to exist at each peak. However, the singular value sequence in a frequency band around the $(0,2)$ peak is $[1000.0, 721.26, 69.970, 21.927, 4.987... ]$, which implies that there are at least two major independent eigenvectors (i.e., two modes) included in this frequency band.
The mode shapes of the dual modes corresponding to each peak are shown in Figures 8.15 and 8.16. As may be noted the natural frequencies of the dual modes are very close, their damping factors are also similar, but their mode shapes have different phases although they have the same number of nodal diameters.

It is important to know that only the multiple reference algorithms of modal analysis can be used to identify the modes with identical natural frequencies but with different mode shapes. Only one mode with contaminated mode shape phases can be found if a conventional identification algorithm, such as the method described in sections 8.3.1 or 8.3.2, is used in the modal identification.
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Figure 8.15 The mode shapes of the modes (0, 2s) and (0, 2c)
(a) $f_i = 48.1822$ Hz, $\zeta_i = 0.2705\%$; (b) $f_i = 48.2836$ Hz, $\zeta_i = 0.2125\%$

Figure 8.16 The mode shapes of the modes (0, 3s) and (0, 3c)
(a) $f_i = 81.3176$ Hz, $\zeta_i = 0.1027\%$; (b) $f_i = 81.3696$ Hz, $\zeta_i = 0.1059\%$
8.4.4 An Effective Way of Identifying the Modes of a Circular Disc with Large Number of Nodal Diameters or Nodal Circles

It was found, from the modal analysis of circular saw-blades, that it is very time consuming and inefficient to identify the mode shapes of a circular disc by using a conventional procedure, namely accumulating FRF data from point to point then selecting an identification algorithm to identify the modal parameters including the mode shapes.

If the purpose of the modal analysis is to identify the mode shape associated with an undesirable resonant frequency, it is not necessary to follow the standard way to complete the identification procedure. A practical and efficient procedure is described in this section to identify the mode shapes of a rotating disc by only reading the imaginary parts of the FRFs, which carry all the information for drawing the mode shapes. Only an impact hammer, a small accelerometer (2.2g) and a FFT analyzer are needed in this procedure. The impact hammer is moved from point to point along a selected circle (c), and the sign of the imaginary part of FRF at each point are recorded as shown in Figure 8.17. From the information provided in Figure 8.17, the number of complete cycles can be immediately identified, which is exactly the number of nodal diameters. In the case shown in Figure 8.17, the number of nodal diameters is $n = 8$.

To show the mode shape of the blade, it is necessary to document the signs and the magnitudes of the imaginary parts of FRFs along a selected radial line (ab) and a circle (c). However, in this case, only a portion of the circle (c) which contains one complete cycle needs to be sampled. The mode shape of the mode (0,10) generated from the imaginary parts of the FRFs along a radial line and a circumferencial arc is shown in Figure 8.18.

In the applications, it was found that the procedure described for the mode shape identification using the signs of the imaginary parts of FRFs is much more reliable and efficient than the standard methods, especially for the modes with a large number of nodal diameters and/or nodal circles.
Figure 8.17 An effective scheme of identifying the mode shapes of a circular disc directly based on the imaginary parts of FRFs (Mode (0,8))

Figure 8.18 The mode shape of the mode (0, 10) identified by using the imaginary parts of the FRFs
8.5 Summary

An effective identification procedure called the Artificial Damping Method (ADM) is proposed to identify the mode shapes from the forced or the self-excited responses in rotating systems. This procedure can handle extremely noisy signals and it does not require the excitation information. Therefore, the method proposed can be used to deal with the self-excited vibration due to multiple excitations.

Specific conclusions are summarized as follows:

1) For a rotating disc, only two displacement probes are required to identify the mode shapes of travelling-wave modes. In the case that the saw-blade does not interact with other objects, the results of modal identifications are relatively consistent compared to those from the saw-blade in the cutting.

2) The expression \( n = \Delta \phi / \Delta \theta \) is still suitable for identifying the number of nodal diameters of the modes with high natural frequencies because the stiffness associated with the high frequency mode is relatively high such that the interaction between the saw-blade and the work-piece may only result in a very little change of mode shape phase.

3) Keeping a stable rotating speed of the blade and a steady feed speed of the wood is essential for the success of the modal shape identification. The consistency of the results can be improved by selecting a relatively thin, soft and uniform wood board as the work-piece.

In addition, a multiple reference modal identification method is presented in this chapter, which can handle the situation of two modes with identical natural frequencies but with different mode shapes. This situation is often encountered in the vibration of circular saw-blades.
The main contributions in this thesis can be summarized as follows:

1) **A generalized approach has been developed to study the instability mechanisms involved in the interaction between a rotating flexible disk and a constraint system or a stationary work-piece.**

A generalized approach to study the instabilities due to various lateral interactive forces has been established. The self-excited vibrations of an idealized constraint rotating string and those of a rotating flexible circular disc have been considered. The relationship defining the energy flux in such systems has been developed, leading to a clear understanding of the physical mechanisms involved in the development of vibrational instabilities. Based on the equations derived for energy flux analysis, unified conditions for the occurrence of instability for any lateral interactive forces have been presented. On the basis of these results, active or passive schemes for minimizing instability regions may be formulated.

It was found in this study that: (i) when a conservative space-fixed interactive force applied to a rotating disc is in phase with the tangential slope at the interactive point, driving energy will be switched into vibration energy leading to unstable behaviour, usually causing a coupling type of instability; (ii) a terminal type of instability often occurs when a non-conservative force applied to a rotating disc is in phase with the disc transverse velocity at the interactive point observed in rotating coordinates; (iii) the
possibility for the occurrence of the instability due to an arbitrary interactive force may be identified based on energy flux analysis, even without solving equations.

2) The lateral regenerative cutting forces in saw-blade cutting have been successfully identified and modeled for the first time. Dynamic models of a rotating disc subjected to various interactive cutting forces have been developed and solved.

Dynamic models of a rotating disc subjected to multiple moving concentrated cutting forces have been developed, which are more realistic than those used in previous studies. The stability characteristics of a rotating disc subjected to (a) multiple moving lateral regenerative cutting forces caused by flank cuts, (b) the transverse components of multiple moving follower radial and tangential cutting forces and (c) the asymmetric membrane stress fields resulting from multiple moving in-plane cutting edge cutting forces have been investigated in this thesis. New developments also involve the solution methods for the stability analysis of a rotating disk subjected to different types of cutting forces. The basic Fourier series method is generalized to solve the stability problem for time-varying systems with or without time lag terms.

It was revealed from the present study that (i) self-excited vibration due to regenerative cutting forces may develop when the tooth passing frequency $f_t$, or its multiples are greater than the natural frequency $f_n$ of the excited mode; (ii) both coupling type of flutter instability and divergence instability can be induced by the transverse components of multiple moving radial cutting forces or the asymmetric in-plane stress fields caused by multiple moving cutting radial forces; (iii) when the saw blade is subjected to the transverse components of multiple moving tangential cutting forces, all the forward-wave modes of the blade are destabilized and all the backward-wave modes are stabilized over the entire rotating speed range, and a similar situation
happens when the saw-blade is subjected to the asymmetric in-plane stress fields produced by multiple moving cutting tangential forces.

3) **A new dynamic milling model including the gyroscopic effect of a rotating spindle has been developed and the gyroscopic effect of the spindle on the stability characteristics of the milling system was investigated.**

Two procedures are proposed to solve the characteristic equation derived from the generalized Fourier series method for the dynamic milling models including the gyroscopic effect of the rotating spindle. In the first procedure (i.e., the GFS procedure), the complex eigenvalues of the resulting equation can be efficiently predicted by using Müller's optimization algorithm with deflation. In the second procedure (i.e., the FSN procedure), the stability lobes can be predicted by using a nonlinear optimization procedure at different rotating speeds based on the Nyquist stability criterion.

New findings are as follows: (i) the backward-wave modes of the spindle are primarily excited by the milling forces but the forward-wave modes of the spindle are stabilized by the milling forces in almost the entire speed range; (ii) although the gyroscopic effect of the rotating spindle does not change the instability regions in milling, it increases the real parts of the eigenvalues of the system or reduces the critical axial depth of cut; (iii) the gyroscopic effect of the rotating spindle usually reduces the chatter frequency of the system notably as the rotating speed increases; (iv) the radial milling force can introduce several small stable regions for the backward-wave modes but it can also induce several small unstable regions for the forward-wave modes; (v) a relatively large modal deflection at the tool tip will lead to a relatively large real part of the eigenvalue in milling system.

4) **An understanding of the dynamics of washboarding and the dominant instability mechanism in washboarding has been achieved through an comprehensive experimental investigation.**
Chapter 9. Conclusions

A comprehensive experimental investigation of washboarding has been conducted under various operating conditions. Based on the theoretical and experimental results presented in this thesis, the lateral regenerative cutting force is identified to be responsible for the self-excited vibration and the washboarding phenomenon in saw-blade cutting. A missing link between washboarding patterns and the dynamics of the saw-blade have been successfully established in this thesis. A supporting cutting test involving the use of a pendulum rig has been conducted to prove the existence of lateral cutting forces due to flank cutting.

In this experimental study it was found that (i) the saw blade can be self-excited only when the tooth passing frequency is greater than the natural frequencies of the excited modes in certain rotating speed ranges; (ii) the mode closest to the tooth passing frequency usually has the largest response magnitude; (iii) for clamped saw-blades, the forward-wave modes are usually excited during cutting; (iv) the magnitude of the self-excited response decreases as the feed speed of the work-piece increases in the operational feed speed range; (v) the occurrence of washboarding is associated with the displacements of the blade in the cutting area, and blades with the deepest gullets usually have the largest displacements at the outer rim, leading to large interactive cutting forces; (vi) the resonant frequencies and the magnitude of the excited response increases as the depth of cut and the density of the work-piece increases.

5) The Artificial Damping Method of identifying the travelling-wave mode shapes of a rotating disc from the self-excited resonant responses has been developed in this thesis.

The Artificial Damping Method (ADM) has been developed for identifying the travelling-wave mode shapes of rotating systems from the forced or the self-excited resonant responses by only using two displacement probes. ADM can handle extremely noisy systems and does not require excitation information. A multiple reference modal
identification method is also presented in this thesis to identify two modes with identical natural frequencies but with different mode shapes of a circular saw-blade.

Further research work in this area should concentrate on the following aspects:

- to study the effects of geometric nonlinearity of the disc upon the stability characteristics of a saw-blade subjected to multiple moving lateral regenerative cutting forces or tangential cutting forces, and to examine how the limit cycle in nonlinear vibration restrains the self-excited response in saw-blade cutting;
- to examine the effects of initial vibration conditions of the blade, such as the frequency of initial vibration, upon the self-excited response of the saw-blade.
- to conduct experimental work to study the effect of side clearance angle of the saw teeth upon the self-excited vibration of the saw-blade and to develop a nonlinear cutting force model including the effect of the side clearance angle.
- to experimentally and theoretically investigate self-excited vibrations in guided circular saws.
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Appendix A

Equation of Energy Flux in a Constrained Rotating String

The total power flow $E_t$ in a constrained rotating string system is given by Equation (3.7). Substituting Equations (3.1) and (3.2) into Equation (3.7) and assuming $\Omega = \text{constant}$ yields:

$$E_t = \int_0^{2\pi} \left( \frac{P}{r} u_t + \frac{\Omega}{r} u_\theta \right) d\theta + \int_0^{2\pi} r \left( u_t + \Omega u_\theta \right) d\theta + ku(0,t)u_t(0,t) + mu_t(0,t)u_t(0,t)$$

(A.1)

Rearranging Equation (A.1) gives:

$$E_t = \frac{P}{r_0} \left( \frac{P}{r^2} - \Omega^2 \right) u_\theta u_t d\theta + \frac{P}{r_0} \left( u_t + \Omega u_\theta \right) d\theta + \frac{P}{r_0} \left( u_t + 2\Omega u_\theta \right) d\theta$$

$$+ [ku(0,t) + mu_t(0,t)]u_t(0,t) = I_1 + I_2 + I_3 + I_4$$

(A.2)

where the first term $I_1$ is:

$$I_1 = \frac{P}{r_0} \left( \frac{P}{r^2} - \Omega^2 \right) u_\theta u_t d\theta = \frac{P}{r_0} \left( \frac{P}{r^2} - \Omega^2 \right) \left( u_\theta u_t \right)$$

Considering Equation (3.5), the above equation can be rewritten as

$$I_1 = \frac{P}{r_0} \left( \frac{P}{r^2} - \Omega^2 \right) \left[ u_\theta(2\pi,t) - u_\theta(0,t) \right] u_t(0,t) - \frac{P}{r_0} \left( \frac{P}{r^2} - \Omega^2 \right) \int_0^{2\pi} u_\theta u_t d\theta$$

Substitution of the force boundary condition (Equation (3.6)) yields:
Appendix A. Equation of Energy Flux in a Constrained Rotating String

\[ I_1 = -[ku(0,t) + mu_n(0,t) + cu_j(0,t) - F_n(0,t)]u_j(0,t) - \rho r \left( \frac{P}{pr^2} - \Omega^2 \right) \int_0^\theta \int_0^{2\pi} u,_{\theta\theta}^2 d\theta \]  

(A.3)

The second term in Equation (A.2) can be rewritten as:

\[ I_2 = \rho r \int_0^{2\pi} u_j(u,_{\theta\theta} + 2\Omega u_{\theta\theta})d\theta - \rho r \int_0^{2\pi} u_j u,_{\theta\theta} d\theta \]

where \[ \int_0^{2\pi} u_j u,_{\theta\theta} d\theta = \int_0^{2\pi} u_j u,_{\theta\theta} d\theta \]

Thus, \[ \int_0^{2\pi} u_j u,_{\theta\theta} d\theta = \frac{1}{2} [u_j(2\pi,t) - u_j(0,t)][u_j(2\pi,t) + u_j(0,t)] = 0 \]

Substituting the above equation and Equation (3.4) into \[ I_2 \] leads to:

\[ I_2 = \rho r \int_0^{2\pi} \frac{P}{pr^2} - \Omega^2 \int_0^{2\pi} u,_{\theta\theta} d\theta \]  

(A.4)

Substituting Equations (A.3) and (A.4) into Equation (A.2) and simplifying the result, \[ E_j \] can be expressed as:

\[ E_j = \rho r \Omega \int_0^{2\pi} \left( \frac{P}{pr^2} - \Omega^2 \right) u,_{\theta\theta} d\theta - [cu_j(0,t) - F_n(0,t)]u_j(0,t) \]  

(A.5)

where \[ \int_0^{2\pi} u,_{\theta\theta} d\theta = \int_0^{2\pi} u,_{\theta\theta} d\theta . \] Thus, \[ \int_0^{2\pi} u,_{\theta\theta} d\theta = \frac{1}{2} [u,_{\theta}(2\pi,t) - u,_{\theta}(0,t)] . \]

Substituting into Equation (A.5), gives:

\[ E_j = \rho r \Omega \left( \frac{P}{pr^2} - \Omega^2 \right) [u,_{\theta}(2\pi,t) - u,_{\theta}(0,t)][u,_{\theta}(2\pi,t) + u,_{\theta}(0,t)] / 2 \]

\[ -[cu_j(0,t) - F_n(0,t)]u_j(0,t) \]  

(A.6)
By substituting the force boundary condition (Equation (3.3)) into the above equation, the power $E_t$ can be expressed as follows:

$$E_t = \Omega[-ku(0,t) - mu_{u_t}(0,t) - cu_x(0,t) + F_n(0,t)\bar{u}_\theta(0,t)]$$

$$- [cu_x(0,t) - F_n(0,t)]u_x(0,t)$$

(A.7)

where $\bar{u}_\theta(0,t) = [u_\theta(2\pi,t) + u_\theta(0,t)]/2$.

Equation (A.7) enables a physical explanation of the instabilities that occur in a constrained rotating string system with transverse vibration to be presented.
Appendix B

Equation for Energy Flux Analysis in a Rotating Disc
Subjected to Interactive Forces

The total power $E_t$ (energy variation rate) of a thin plate with small deflections may be written as follows:

$$E_t = (U + T)_t = I_1 + DI_2 - (1 - v)DI_3 \quad (B.1)$$

where,

$$I_1 = \rho h \iint_A w_{r,t} w_{r,tt} \, dx \, dy \quad (B.2)$$

$$I_2 = \iint_A \left\{ (\nabla^2 w_r) \frac{\partial}{\partial t} (\nabla^2 w_r) \right\} \, dx \, dy \quad (B.3)$$

$$I_3 = \iint_A \frac{\partial}{\partial t} \left[ w_{r,xx} w_{r,yy} - (w_{r,xy})^2 \right] \, dx \, dy \quad (B.4)$$

and $w_r$ denotes the transverse displacement with respect to the plate-fixed coordinates.

Equation (B.3) can be rewritten in the following form:

$$I_2 = \iiint_A \nabla^4 w_r \, dx \, dy + \iiint_A \left\{ [w_{r,tt} \nabla^2 w_r]_x + [w_{r,tt} \nabla^2 w_r]_y \right\} \, dx \, dy \quad (B.5)$$

$$- \iiint_A \left\{ [w_{r,t} (\nabla^2 w_r)]_x + [w_{r,t} (\nabla^2 w_r)]_y \right\} \, dx \, dy$$

By employing Green's theorem,
Appendix B. Equation for Energy Flux Analysis in a Rotating Disc

\[
\int_A (\Phi_x + \Psi_y) dx dy = \oint_L (\Phi \alpha_{nx} + \Psi \alpha_{ny}) dl
\]  \hspace{1cm} (B.6)

( where, \( \alpha_{nx} \) and \( \alpha_{ny} \) are the angles between axis \( n \) and axis \( x \) and \( y \), respectively, as shown in Figure 1. \( \Phi \) and \( \Psi \) are assumed to be continuous functions in the domain \( A \), Equation (B.5) can be expressed as follows:

\[
I_2 = \int_A \nabla^4 w_r \cdot dx dy + \oint_L \nabla^2 w_r [w_{r,xt} \alpha_{nx} + w_{r,yt} \alpha_{ny}] dl
\]  \hspace{1cm} (B.7)

Equation (B.4) can now be rewritten in the form:

\[
I_3 = \int_A \left[ [w_{r,yy} w_{r,xt} - w_{r,xy} w_{r,yt}]_x + [w_{r,xx} w_{r,yt} - w_{r,xy} w_{r,xt}]_y \right] dx dy
\]  \hspace{1cm} (B.8)

Similarly, applying Green’s theorem to Equation (B.8) yields:

\[
I_3 = \oint_L \left[ [w_{r,yy} w_{r,xt} - w_{r,xy} w_{r,yt}] \alpha_{nx} + [w_{r,xx} w_{r,yt} - w_{r,xy} w_{r,xt}] \alpha_{ny} \right] dl
\]  \hspace{1cm} (B.9)

Substituting Equations (B.2), (B.7) and (B.9) into Equation (B.1) yields:

\[
E_I = \int_A (D \nabla^4 w_r + \rho hw_r,tt) w_r,tt dx dy - \oint_L (\nabla^2 w_r)_x \alpha_{nx} + (\nabla^2 w_r)_y \alpha_{ny} w_r,tt dl + I_4
\]  \hspace{1cm} (B.10)

where,
Appendix B. Equation for Energy Flux Analysis in a Rotating Disc

\[
I_4 = \oint_D [(w_{xx} - v w_{xy}) w_{xx} + (1 - v) w_{xy} w_{yy}] \alpha_{nx} + [(w_{yy} - w_{xx}) w_{xy} + (1 - v) w_{xy} w_{yy}] \alpha_{ny} \, dl
\]  

(B.11)

Noting that: \( w_{x,x} = w_{r,n} \alpha_{nx} - w_{r,s} \alpha_{ny} \) and \( w_{x,y} = w_{r,n} \alpha_{ny} + w_{r,s} \alpha_{nx} \) on the boundary shown in Figure 4.1, Equation (B.11) may be rewritten as:

\[
I_4 = \oint_D [(1 - v)(w_{xx} \alpha_{nx}^2 + 2 w_{xy} \alpha_{nx} \alpha_{ny} + w_{yy} \alpha_{ny}^2) + v \nabla^2 w_r] w_{r,n} \, dl
\]

- \( \oint_D [(1 - v)(w_{xx} - w_{yy}) \alpha_{nx} \alpha_{ny} - w_{xy} (\alpha_{nx}^2 - \alpha_{ny}^2)] w_{r,n} \, dl \)  

(B.12)

The bending moment intensity \( M_n \), twisting moment intensity \( M_{ns} \) and shear stress \( Q_n \) for the plate are given by:

\[
M_n = -D(1 - v)(w_{xx} \alpha_{nx}^2 + 2 w_{xy} \alpha_{nx} \alpha_{ny} + w_{yy} \alpha_{ny}^2) + v \nabla^2 w_r 
\]  

(B.13)

\[
M_{ns} = D(1 - v)[(w_{xx} - w_{yy}) \alpha_{nx} \alpha_{ny} - w_{xy} (\alpha_{nx}^2 - \alpha_{ny}^2)] 
\]  

(B.14)

\[
Q_n = -D[(\nabla^2 w_r)_{,x} \alpha_{nx} + (\nabla^2 w_r)_{,y} \alpha_{ny}] 
\]  

(B.15)

Combining Equations (B.12), (B.13), (B.14) and (B.15) into Equation (B.10) yields:

\[
E_s = \iint_A (DV^4 w_r + \rho hw_{r,u}) w_{r,s} \, dxdy + \oint_D Q_n w_{r,s} \, dl - \oint_L M_n \dot{w}_{r,n} \, dl - \oint_L M_{ns} \dot{w}_{r,s} \, dl 
\]  

(B.16)

If \( M_{ns} \) is continuous and the curve is smooth, the fourth integral in Equation (B.16) becomes:

\[
-\oint_L M_{ns} \dot{w}_{r,s} \, dl = -\oint_L (M_{ns} \dot{w}_r)_{,s} \, dl + \oint_L (M_{ns})_{,s} \dot{w}_r \, dl = \oint_L (M_{ns})_{,s} \dot{w}_r \, dl
\]
Appendix B. Equation for Energy Flux Analysis in a Rotating Disc

Based on the above formula, Equation (B.16) can be rewritten as follows:

\[
E_t = \oint_A (\nabla^4 w_r + \rho w_{r,rr})w_{r},dx\,dy - \oint_L M_n \dot{w}_{n,rr} \, dl + \oint_L \left[ Q_n + (M_n)_{,s} \right] w_{r,s} \, dl
\]  
(B.17)

From the natural and kinematic boundary conditions, it is apparent that:

\[
\oint_L M_n \dot{w}_{n,rr} \, dl = \oint_L \left[ Q_n + (M_n)_{,s} \right] w_{r,s} \, dl = 0
\]  
(B.18)

Therefore, Equation (B.17) becomes:

\[
E_t = \oint_A (\nabla^4 w_r + \rho w_{r,rr})w_{r}, dx\,dy
\]  
(B.19)

Equation (B.19) can also be expressed in the space-fixed cylindrical coordinates \((w, r, \theta)\):

\[
E_t = \oint_A \left[ \nabla^4 w + \rho h(w_{,r} + 2\Omega w_{,\theta} + \Omega^2 w_{,\theta\theta}) \right] (w_{,r} + \Omega w_{,\theta}) \, r\, dr\, d\theta
\]  
(B.20)
Appendix C

Stress Fields in the Disc due to Concentrated In-Plane Loads

C1. Stress Fields in the Disc due to a Concentrated Radial Load

With the fixed inner and free outer boundary conditions, the stress fields due to the concentrated radial in-plane load $F_r$ was derived by St. Cyr [5.13] and DuBois [5.14] as the following Fourier series form:

$$\sigma_{rr} = F_r \sum_{k=0}^{\infty} Q_{rk}(r) \cos k\theta$$  \hspace{1cm} (C.1)

$$\sigma_{\theta\theta} = F_r \sum_{k=0}^{\infty} Q_{ek}(r) \cos k\theta$$  \hspace{1cm} (C.2)

$$\tau_{r\theta} = F_r \sum_{k=0}^{\infty} Q_{r\theta k}(r) \sin k\theta$$  \hspace{1cm} (C.3)

where,

$$Q_{rk}(r) = \frac{-1}{\pi bh} \left[ a_k^1 \left(\frac{r}{b}\right)^{k-2} + a_k^2 \left(\frac{r}{b}\right)^k + a_k^3 \left(\frac{r}{b}\right)^{-k-2} + a_k^4 \left(\frac{r}{b}\right)^{-k} \right]$$

$$Q_{ek}(r) = \frac{-1}{\pi bh} \left[ b_k^1 \left(\frac{r}{b}\right)^{k-2} + b_k^2 \left(\frac{r}{b}\right)^k + b_k^3 \left(\frac{r}{b}\right)^{-k-2} + b_k^4 \left(\frac{r}{b}\right)^{-k} \right]$$  \hspace{1cm} (C.4)

$$Q_{r\theta k}(r) = \frac{-1}{\pi bh} \left[ c_k^1 \left(\frac{r}{b}\right)^{k-2} + c_k^2 \left(\frac{r}{b}\right)^k + c_k^3 \left(\frac{r}{b}\right)^{-k-2} + c_k^4 \left(\frac{r}{b}\right)^{-k} \right]$$

and \hspace{1cm} $a_0^1 = -b_0^1 = \frac{1}{2} \alpha^2 (1 / \gamma + \alpha^2)$ \hspace{1cm} $\alpha = a / b$, \hspace{0.5cm} $\beta = \frac{3 - \nu}{1 + \nu}$, \hspace{0.5cm} $\gamma = \frac{1 + \nu}{1 - \nu}$

$\alpha^2 = b_0^2 = \frac{1}{2} \gamma (1 / \gamma + \alpha^2)$
Appendix C. Stress Fields in the Disc due to Concentrated In-Plane Loads

\[ a_0^3 = b_0^3 = a_0^4 = b_0^4 = 0 \]
\[ a_1^1 = 1, \quad b_1^1 = c_1^1 = 0 \]
\[ a_2^2 = b_2^2 / 3 = c_2^2 = \frac{1 - \nu}{4} \left( \frac{1 + \gamma \alpha^2}{1 + \beta \alpha^4} \right) \]
\[ a_3^3 = -b_3^3 = c_3^3 = -\alpha^2 \frac{1 - \nu}{4} \left( \frac{\gamma - \beta \alpha^2}{1 + \beta \alpha^4} \right) \]
\[ a_4^4 = b_4^4 = c_4^4 = -(1 - \nu) / 4 \]

and for all \( k \) greater than 1,
\[ a_k^1 = -b_k^1 = -c_k^1 = \left[ k \beta \alpha^{-2k} + (k^2 - 1 + \beta^2) \alpha^2 - k(k - 1) \right] / d_k \]
\[ a_k^2 = -(k - 2)e_k / d_k, \quad b_k^2 = (k + 2)e_k / d_k, \quad c_k^2 = ke_k / d_k \]
\[ a_k^3 = -b_k^3 = c_k^3 = \left[ -k \beta \alpha^{2k} + (k^2 - 1 + \beta^2) \alpha^2 - k(k + 1) \right] / d_k \]
\[ a_k^4 = (k + 2)f_k / d_k, \quad b_k^4 = -(k - 2)f_k / d_k, \quad c_k^4 = kf_k / d_k \]
in which,
\[ d_k = 2[\beta(\alpha^k + \alpha^{-k})^2 + (k^2 - 1)(\alpha + \alpha^{-1})^2 + (\beta \alpha + \alpha^{-1})^2] \]
\[ e_k = \beta \alpha^{-2k} + k + 1 - k \alpha^{-2} \]
\[ f_k = k \alpha^{-2} + \beta \alpha^{2k} - k + 1 \]

C2. Stress Fields in the Disc due to a Concentrated Tangential Load

With the fixed inner and free outer boundary conditions, the stress fields due to the concentrated tangential in-plane load \( F_t \) was derived by Shen and Song [5.6] as follows:

\[ \sigma_{rr} = F_t \sum_{k=0}^{\infty} T_{rk}(r) \sin k\theta \quad (C.5) \]
\[ \sigma_{\theta \theta} = F_t \sum_{k=0}^{\infty} T_{\theta k}(r) \sin k\theta \quad (C.6) \]
Appendix C. Stress Fields in the Disc due to Concentrated In-Plane Loads

\[ \tau_{r\theta} = F \sum_{k=0}^{\infty} T_{r\theta k}(r) \cos k\theta \]  

(C.7)

For \( k = 0 \), \( T_{r0}(r) = 0 \), \( T_{\theta 0}(r) = 0 \), \( T_{r\theta 0}(r) = b/(2\pi hr^2) \)

For \( k = 1 \),

\[
\begin{align*}
T_{r1}(r) &= -\frac{2b_1}{r^3} + 2c_1 r + \frac{d_1}{r} + \frac{2}{\pi hr} \\
T_{\theta 1}(r) &= \frac{2b_1}{r^3} + 6c_1 r + \frac{d_1}{r} \\
T_{r\theta 1}(r) &= \frac{2b_1}{r^3} - 2c_1 r - \frac{d_1}{r}
\end{align*}
\]

where, \( b_1 = \frac{1}{8\pi h} \left[ \frac{(1+v)a^2 - (3+v)b^2}{(1+v)b^4 + (3-v)a^4} (1+v)b^4 + (3+v)b^2 \right] \)

\[
c_1 = \frac{1}{8\pi h} \frac{(1+v)a^2 - (3+v)b^2}{(1+v)b^4 + (3-v)a^4} (1+v)
\]

\[
d_1 = -\frac{1-v}{4\pi h}
\]

and for all \( k \) greater than 1,

\[
\begin{align*}
T_{r k}(r) &= a_k (k-k^2)r^{-k-2} - b_k (k+k^2)r^{-k-2} + c_k (2+k-k^2)r^k + d_k (2-k-k^2)r^{-k} \\
T_{\theta k}(r) &= a_k (k^2-k)r^{-k-2} + b_k (k+k^2)r^{-k-2} + c_k (2+3k+k^2)r^k + d_k (2-3k+k^2)r^{-k} \\
T_{r\theta k}(r) &= a_k (k-k^2)r^{-k-2} + b_k (k+k^2)r^{-k-2} - c_k (k+k^2)r^k - d_k (k-k^2)r^{-k}
\end{align*}
\]

(C.8)

The coefficients \( a_k, b_k, c_k, d_k \) (\( k = 2, 3, \ldots \)) can be obtained by solving the following equation:

\[
\begin{bmatrix}
(k-k^2)b^{k-2} & -(2+k-k^2)b^{-k-2} & (2+k-k^2)b^k & (2-k-k^2)b^{-k} \\
(k^2-k)b^{k-2} & -(2+k-k^2)b^{-k-2} & (2+k-k^2)b^k & (k-k^2)b^{-k} \\
(1+v)ka^{k-1} & (1+v)ka^{-k-1} & [4+(1+v)k]a^{k+1} & -[4-(1+v)k]a^{-k+1} \\
-(1+v)ka^{k-1} & (1+v)ka^{-k-1} & [4-(1+v)(2+k)]a^{k+1} & [4-(1+v)(2-k)]a^{-k+1}
\end{bmatrix}
\begin{bmatrix}
a_k \\
b_k \\
c_k \\
d_k
\end{bmatrix}
= \frac{1}{\pi bh}
\begin{bmatrix}
0 \\
-1 \\
0 \\
0
\end{bmatrix}
\]
Appendix D

Elemental Matrices of a Finite Rotating Shaft Element

1. Translational mass matrix: \[ [M] = [M_0] + \Phi[M_1] + \Phi^2[M_2] \]

where, \( \Phi \) is transverse shear effect (\( \Phi = \frac{12EI}{kAGl^2} \)).

\[
[M_0] = \frac{\rho Al}{420(1 + \Phi)^2}
\begin{bmatrix}
156 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 156 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -22l & 4l^2 & 0 & 0 & 0 & 0 & 0 \\
22l & 0 & 0 & 4l^2 & 0 & 0 & 0 & 0 & 0 \\
54 & 0 & 0 & 13l & 156 & 0 & 0 & 0 & 0 \\
0 & 54 & -13l & 0 & 0 & 156 & 0 & 0 & 0 \\
0 & 13l & -3l^2 & 0 & 0 & 22l & 4l^2 & 0 & 0 \\
-13l & 0 & 0 & -3l^2 & -22l & 0 & 0 & 4l^2 & 0
\end{bmatrix}
\]

\[
[M_1] = \frac{\rho Al}{420(1 + \Phi)^2}
\begin{bmatrix}
294 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 294 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -38.5l & 7l^2 & 0 & 0 & 0 & 0 & 0 \\
38.5l & 0 & 0 & 7l^2 & 0 & 0 & 0 & 0 & 0 \\
126 & 0 & 0 & 31.5l & 294 & 0 & 0 & 0 & 0 \\
0 & 126 & -31.5l & 0 & 0 & 294 & 0 & 0 & 0 \\
0 & 31.5l & -7l^2 & 0 & 0 & 38.5l & 7l^2 & 0 & 0 \\
-31.5l & 0 & 0 & -7l^2 & -38.5l & 0 & 0 & 7l^2 & 0
\end{bmatrix}
\]
Appendix D. Elemental Matrices of a Finite Rotating Shaft Element

\[ M_2 = \frac{\rho A l}{420(1 + \Phi)^2} \begin{bmatrix}
    140 & 0 & \cdots & 0 \\
    0 & 140 & \cdots & 0 \\
    0 & -17.5l & 3.5l^2 & \cdots \\
    17.5l & 0 & 0 & 3.5l^2 \\
    0 & 70 & 0 & 17.5l & 140 \\
    0 & 17.5l & -3.5l^2 & 0 & 17.5l & 3.5l^2 \\
    -17.5l & 0 & 0 & -3.5l^2 & -17.5l & 0 & 0 & 3.5l^2 \\
\end{bmatrix} \]

2. Rotational mass matrix: \[ N = [N_0] + \Phi [N_1] + \Phi^2 [N_2] \]

\[ [N_0] = \frac{\rho l}{30l(1 + \Phi)^2} \begin{bmatrix}
    36 & 0 & \cdots & 0 \\
    0 & 36 & \cdots & 0 \\
    0 & -3l & 4l^2 & \cdots \\
    3l & 0 & 0 & 4l^2 \\
    -36 & 0 & 0 & -3l & 36 \\
    0 & -36 & 3l & 0 & 0 & 36 \\
    0 & -3l & -l^2 & 0 & 0 & 3l & 4l^2 \\
    3l & 0 & 0 & -l^2 & -3l & 0 & 0 & 4l^2 \\
\end{bmatrix} \]

\[ [N_1] = \frac{\rho l}{30l(1 + \Phi)^2} \begin{bmatrix}
    0 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & 0 \\
    0 & 15l & 5l^2 & \cdots \\
    -15l & 0 & 0 & 5l^2 \\
    0 & 0 & 0 & 15l & 0 \\
    0 & 0 & -15l & 0 & 0 \\
    0 & 15l & -5l^2 & 0 & 0 & -15l & 5l^2 \\
    -15l & 0 & 0 & -5l^2 & 15l & 0 & 0 & 5l^2 \\
\end{bmatrix} \]
Appendix D. Elemental Matrices of a Finite Rotating Shaft Element

\[ [N_2] = \frac{\rho l}{30l(1 + \Phi)^2} \begin{bmatrix}
0 & 0 & \text{SYM} \\
0 & 0 & 10l^2 \\
0 & 0 & 0 & 10l^2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5l^2 & 0 & 0 & 0 & 10l^2 \\
0 & 0 & 0 & 5l^2 & 0 & 0 & 0 & 10l^2 \\
\end{bmatrix} \]

3. Gyroscopic matrix: \[ [G] = [G_0] + \Phi[G_1] + \Phi^2[G_2] \]

\[ [G_0] = \frac{\rho l\Omega}{15l} \begin{bmatrix}
0 & 36 & \text{Skew} & \text{SYM} \\
-3l & 0 & 0 \\
0 & -3l & 4l^2 & 0 \\
0 & 36 & -3l & 0 & 0 \\
-36 & 0 & 0 & -3l & 36 & 0 \\
-3l & 0 & 0 & l^2 & 3l & 0 & 0 \\
0 & -3l & -l^2 & 0 & 0 & 3l & 4l^2 & 10l^2 \\
\end{bmatrix} \]

\[ [G_1] = \frac{\rho l\Omega}{15l} \begin{bmatrix}
0 & 15l & 0 & \text{Skew} & \text{SYM} \\
0 & 0 & 0 \\
15l & 0 & 0 \\
0 & 0 & 15l & 0 & 0 \\
0 & 0 & 0 & 15l & 0 & 0 \\
15l & 0 & 0 & 5l^2 & -15l & 0 & 0 \\
0 & 15l & -5l^2 & 0 & 0 & -15l & 5l^2 & 0 \\
\end{bmatrix} \]
Appendix D. Elemental Matrices of a Finite Rotating Shaft Element

\[ [G_2] = \frac{\rho I \Omega}{15l} \begin{bmatrix}
    0 & 0 & \text{Skew} & \text{SYM} \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 10l^2 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & -5l^2 & 0 \\
    0 & 0 & 5l^2 & 0 \\
\end{bmatrix} \]

4. Stiffness matrix: \[ [K] = [K_0] + \Phi[K_1] \]

\[ [K_0] = \frac{EI}{l^3(1 + \Phi)} \begin{bmatrix}
    12 & 0 & 0 & \text{SYM} \\
    0 & 12 & 0 & 0 \\
    0 & -6l & 4l^2 & 0 \\
    6l & 0 & 0 & 4l^2 \\
    -12 & 0 & 0 & -6l \\
    0 & -12 & 6l & 0 \\
    0 & -6l & 2l^2 & 0 \\
    6l & 0 & 0 & 2l^2 \\
\end{bmatrix} \]

\[ [K_1] = \frac{EI}{l^3(1 + \Phi)} \begin{bmatrix}
    0 & 0 & 0 & \text{SYM} \\
    0 & 0 & l^2 & 0 \\
    0 & 0 & 0 & l^2 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & -l^2 & 0 \\
    0 & 0 & 0 & -l^2 \\
    0 & 0 & 0 & 0 \\
\end{bmatrix} \]