PION-NUCLEUS TOTAL CROSS-SECTIONS

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Abstract

In this work we present a method for analytically calculating the total cross section of $\pi - nucleus$ scattering. The momentum space approach is used. An optical potential is obtained using the Kerman-McManus-Thaler (KMT) multiple scattering formalism. Lippmann-Schwinger equation in momentum space is solved by a numerical method. This formalism includes the spin flip term and fermi motion averaging with 3-body kinematics. The calculation results of our code are in excellent agreement with the experimental data of $\pi^{-12C}$ scattering in the energy region $80 \text{ MeV} \sim 280 \text{ MeV}$.

Compared to other similar methods, our results better reproduce the $\pi^{-12C}$ resonance peak. It predicts the correct energy of the resonance peak, and gives scattering amplitudes which agree with the experimental data within 1%.
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Chapter 1

Introduction

1.1 Pion-nucleus scattering

GEANT 4 is a large international project to upgrade the elementary particle simulation package GEANT 3:21 [1] Nucleonic interactions are simulated using the PITHA package [2] optimized to simulate particle detectors operating with incident energies about a few GeV. Part of the objective of the upgrade is to improve the algorithms for simulating pion interactions with nuclei down energies to a few keV where decay becomes the dominant mechanism for positive pions and absorption the dominant mechanism for negative pions. The objective is to simulate pion interactions in a large number of materials used in the construction of experimental apparatus as well as in particle detectors such as hydrocarbons, sodium iodide and bismuth germanate.

The experimental data for pion interactions in nuclei up to 3 GeV are sparse. Data in the range up to 400 MeV were reviewed by Thomas and Landau [3] in 1980. The available data above 1 GeV/c incident momenta has been tabulated by the Durham Group [4]. The energy spectrum of the cross-sections in the energy up to 400 MeV are shown in Fig 1.1. There is a resonance structure in the energy region up to 400 MeV due to the
3,3 resonance in $\pi$-nucleon scattering. Above 1 GeV the cross-sections are quite energy independent. [5]

Measurement of positive and negative pions scattering from carbon and lead in the energy range from 600 MeV to 900 MeV have been reported by T. Takahashi [6]. Fits to the elastic differential cross-section using the PIPIT code are presented. The best fits we obtained by modifying the elementary amplitudes phenomenologically and by replacing the elementary pion nucleon amplitudes used in the PIPIT calculations with Fermi averaged amplitudes. The averaged amplitude were calculated by averaging every partial wave in the Fermi gas model.

Because the experimental data are sparse, it is not feasible to obtain an empirical fit as a function of energy and atomic number for all nuclei that will be useful for the GEANT simulation code. The decision was made to use a modified PIPIT code to calculate the cross-sections for all nuclei in the energy range up to 500 MeV and then to parameterize these cross-sections as a function of angle, energy and atomic number. The modifications to the PIPIT code to include three body kinematics, the spin flip amplitude and Fermi averaging is the subject of this thesis. Calculation techniques used in the original PIPIT code are described in Chapter 2. Also described in the chapter are the implementation of three body kinematics fermi momentum averaging and spin flip.

The computer code with details of the subroutines are described in chapter 3.

Calculations of $\pi^{-12C}$ scattering using the PIPIT code supplied by Arima [7] are shown in Chapter 4. Also shown in Chapter 4 is a comparison of calculations made with
Figure 1.1: Total cross section of $\pi$ with different nuclei reproduced from the article by Landau and Thomas [3]
the LPOTT program obtained from Rubin Landau [13]. Results of calculations using
the modified PIPIT code are also shown.

The results obtained in this work are summarized in chapter 5.

1.1.1 The model

In general, microscopic multiple scattering theories are successful when applied to the
pion-nucleus problem. The reasons for this are:

1. The simpler spin structure of pion($0 \otimes \frac{1}{2}$) than that of nucleon($\frac{1}{2} \otimes \frac{1}{2}$);

2. The availability of both $\pi^+$ and $\pi^-$ beams leads to more complete knowledge of the
   $\pi N$ scattering amplitude;

3. The dominance of the large $P_{33}$ amplitude at intermediate energies means that
   much of the pion-nucleus interaction appears simply absorptive and diffractive.

The main objective in this thesis is to study the total cross-section of pion-nucleus
interaction. Mainly, we will focus our interest on the interaction $\pi^- {^{12}}C$ because there
are adequate data for this interaction.

Momentum space methods for solving scattering problems offer a useful alternative to
the standard coordinate space approach. Following this approach we will first construct
the first-order $\pi$-nucleus optical potential in momentum space with suitable approxi-
mations according to the Kerman-McManus-Thaler (KMT)[8] formalism; Then, using a
numerical method, we'll solve the Lippmann-Schwinger equation with relativistic kine-
matics in momentum space.
CHAPTER 1. INTRODUCTION

There are already some computer programs to calculate the cross-section of π-nucleus. One is PIPIT by R.A. Eisenstein and F. Tabakin [9], this code does not include the spin-flip and Fermi motion. Our code, based on PIPIT, includes both spin-flip and Fermi motion. In addition, we introduce a 3-body model in place of the 2-body model used in PIPIT. Another code considered is LPOTT by R.H. Landau [10]. In this code, spin-flip and Fermi motion are included. Our new results are compared to these two previous calculations in chapter 4.
Chapter 2

Description of the model

In what follows we quote the major theoretical results that are necessary for an oper­
atinal understanding of our code. No attempt has been made to present a full treatment
of scattering theory in general or momentum space methods in particular. This descrip­
tion follows very closely the description of formalism used by Eisenstein and Tabakin
in their development of the PIPIT code which did not include spin flip, fermi-motion
and three-body kinematics. Many of the equations used by Eisenstein and Tabakin are
reproduced and where appropriate the spin flip terms are added. The fermi-motion and
3-body model are discussed in subsections 2.2.4 and 2.2.5.

2.1 General expressions for the cross section

The equation describing the scattering of relativistic pions from nuclei is the Lippmann-
Schwinger equation with relativistic kinematics. In momentum space the equation for
the transition matrix \( T' \) takes the form

\[
T'(\vec{k}, \vec{k}'; \vec{k}_0) = U(\vec{k}, \vec{k}') + \int \frac{T'(\vec{k}, \vec{k}''; \vec{k}_0)U(\vec{k}'', \vec{k}')}{{E(k_0) - E_\pi(k'') - E_A(k'')} + i\epsilon} d^3k''
\]  

(2.1)
CHAPTER 2. DESCRIPTION OF THE MODEL

Here \( E(k_0) \) is the total on-shell energy in the pion-nucleus \((\pi A)\) center-of-mass frame and \( E_{\pi}(k_0) = (k_0^2 + m_{\pi}^2)^{\frac{1}{2}} \). The quantities \( k, k' \) and \( k_0 \) are the pion momenta in the pion-nucleus center-of-mass frame. The optical potential \( U(\vec{k}, \vec{k'}) \) is obtained using the KMT multiple scattering formalism and is described more fully in section 2.2. In order to solve eq.(2.1), explicit knowledge of \( U \) off the energy shell is required. This feature is in fact one of the principal attractions of the method because different off-shell extrapolations of the potential may easily be studied.

In the KMT formalism the full pion-nucleus \( T \)-matrix differs from the \( T' \) defined above:

\[
T = \frac{A}{A - 1} T'
\]  

(2.2)

where \( A \) is the nuclear mass number. The cross sections for elastic scattering can be written either in terms of the transition matrix \( T \) or in terms of the reaction matrix \( R' \) which is related to \( T' \) via the Heitler equation[11]

\[
T'(\omega) = R'(\omega) - \pi i R'(\omega) \delta(\omega - H_0) T'(\omega)
\]  

(2.3)

The optical potential \( U(\vec{k}, \vec{k'}) \) and \( T' \)-matrix in eq.(2.1) have a central (non-spin-flip) and a spin-dependent (spin-flip) term:

\[
U(\vec{k}, \vec{k'}) = U_{NF}(\vec{k}, \vec{k'}) + i \sigma \cdot (\vec{k} \times \vec{k'}) U_F(\vec{k}, \vec{k'})
\]  

(2.4)

\[
T'(\vec{k}, \vec{k'}; k_0) = T'_{NF}(\vec{k}, \vec{k'}; k_0) + i \sigma \cdot (\vec{k} \times \vec{k'}) T'_{F}(\vec{k}, \vec{k'}; k_0)
\]  

(2.5)

where the subscript \( NF \) and \( F \) refer to spin non-flip and spin flip terms.
CHAPTER 2. DESCRIPTION OF THE MODEL

Eq.(2.1) can be reduced to a one-dimensional form by introducing the following angular momentum decomposition of $U(k, k')$, and $T'(k, k'; k_0)$ (or $R'(k, k'; k_0)$):

\[
\begin{pmatrix}
U_{NF}(\vec{k}, \vec{k}') \\
R_{NF}(\vec{k}, \vec{k'}; k_0)
\end{pmatrix} = \sum_l \begin{pmatrix}
U^l_{NF}(k, k') \\
R^l_{NF}(k, k'; k_0)
\end{pmatrix} P_l(\hat{k} \cdot \hat{k}') \tag{2.6}
\]

\[
\begin{pmatrix}
U_F(\vec{k}, \vec{k}') \\
R_F(\vec{k}, \vec{k'}; k_0)
\end{pmatrix} = \sum_l \begin{pmatrix}
U^l_F(k, k') \\
R^l_F(k, k'; k_0)
\end{pmatrix} P'_l(\hat{k} \cdot \hat{k}') \tag{2.7}
\]

where $P_l(\hat{k} \cdot \hat{k}')$ are the Legendre polynomials and $P'_l(x) = \frac{dP_l(x)}{dx}$.

However, orbital angular momentum $l$ is not conserved in the reaction, and consequently the dynamics are described in terms of eigenstates of the total angular momentum $j = l \pm \frac{1}{2}$:

\[
\begin{pmatrix}
U_{NF}(\vec{k}, \vec{k}') \\
R'_{NF}(\vec{k}, \vec{k'}; k_0)
\end{pmatrix} = \frac{1}{2\pi^2} \sum_l \left[ (l + 1) \begin{pmatrix}
U^{l+\frac{1}{2}}(k, k') \\
R^{l+\frac{1}{2}}(k, k'; k_0)
\end{pmatrix} \\
+ l \begin{pmatrix}
U^{l-\frac{1}{2}}(k, k') \\
R^{l-\frac{1}{2}}(k, k'; k_0)
\end{pmatrix} \right] P_l(\hat{k} \cdot \hat{k}') \tag{2.8}
\]

\[
\begin{pmatrix}
U_F(\vec{k}, \vec{k}') \\
R'_F(\vec{k}, \vec{k'}; k_0)
\end{pmatrix} = \frac{1}{2\pi^2} \sum_l \left[ \left( \begin{pmatrix}
U^{l+\frac{1}{2}}(k, k') \\
R^{l+\frac{1}{2}}(k, k'; k_0)
\end{pmatrix} \\
- \begin{pmatrix}
U^{l-\frac{1}{2}}(k, k') \\
R^{l-\frac{1}{2}}(k, k'; k_0)
\end{pmatrix} \right) \right] P'_l(\hat{k} \cdot \hat{k'}) \tag{2.9}
\]

Comparing eq.(2.8) and (2.9) to eq.(2.6) and (2.7), we obtain

\[
\begin{pmatrix}
U^l_{NF}(k, k') \\
R^l_{NF}(k, k'; k_0)
\end{pmatrix} = \frac{1}{2\pi^2} \left[ (l + 1) \begin{pmatrix}
U^{l+\frac{1}{2}}(k, k') \\
R^{l+\frac{1}{2}}(k, k'; k_0)
\end{pmatrix} \\
+ l \begin{pmatrix}
U^{l-\frac{1}{2}}(k, k') \\
R^{l-\frac{1}{2}}(k, k'; k_0)
\end{pmatrix} \right] \tag{2.10}
\]
CHAPTER 2. DESCRIPTION OF THE MODEL

\[
\begin{pmatrix}
U^i_j(k, k') \\
R^i_F(k, k', k_0)
\end{pmatrix}
= \frac{1}{2\pi^2} \begin{pmatrix}
U^{i+\frac{1}{2}}_j(k, k') \\
R^{i+\frac{1}{2}}_F(k, k', k_0)
\end{pmatrix}
- \begin{pmatrix}
U^{i-\frac{1}{2}}_j(k, k') \\
R^{i-\frac{1}{2}}_F(k, k', k_0)
\end{pmatrix}
\] (2.11)

Solving these equations gives:

\[
U_{l\pm\frac{1}{2}}(k, k') = \frac{2\pi^2}{2l + 1} \left[ U_{N,F}^l(k, k') + \left( \begin{array}{c}
l \\-(l + 1)
\end{array} \right) U_F^l(k, k') \right]
\] (2.12)

\[
R_{l\pm\frac{1}{2}}(k, k', k_0) = \frac{2\pi^2}{2l + 1} \left[ R_{N,F}^l(k, k', k_0) + \left( \begin{array}{c}
l \\-(l + 1)
\end{array} \right) R_F^l(k, k', k_0) \right]
\] (2.13)

The resulting integral equation for \( R_{l\pm\frac{1}{2}} \) is

\[
R_{lj}(k, k', k_0) = U_j(k, k') + \frac{2}{\pi} P \int \frac{R_{lj}(k, k'', k_0)U_j(k'', k')k'' dke''}{E_\pi(k_0) + E_A(k_0) - E_\pi(k'') - E_A(k'')} \] (2.14)

Here \( j = l \pm \frac{1}{2} \), and \( P \) denotes a principal value integral. This equation for \( R_{lj} \) is solved for each partial wave of interest by a matrix inversion method described in section 2.3. This is done in the main program. Eisenstein and Tabakin used a standing wave (principal value) Green function because it is real. Only \( U_j \) and \( R_{lj} \) are complex quantities.

By relating \( R_{lj} \) and \( T_{lj} \) to each other and to the (complex) scattering phase shift \( \delta_j \), the scattering cross sections can be calculated. The on-shell \( T_{lj} \) is related to the on-shell \( R_{lj} \) via

\[
T_{lj} = R_{lj}[1 - i\rho(E)T_{lj}]
\] (2.15)

where \( \rho(E) \) is a phase space factor given by

\[
\rho(E) = \frac{1}{(\hbar c)^2} \frac{2k_0E_\pi E_A}{E_\pi + E_A}
\] (2.16)

Using eq.(2.15) to obtain \( T_{lj} \) from \( R_{lj} \), followed by eq.(2.2) to obtain \( T_j \) and eq.(2.15) again to obtain \( R_j \), Eisenstein and Tabakin extracted the phase shift \( \delta_j \) from the relations

\[
\rho(E)T_j = -e^{i\delta_j} \sin \delta_j
\] (2.17)
and
\[ \rho(E)R_j = - \tan \delta_j \quad (2.18) \]
Here \( \delta_j \) is the phase shift relative to an undistorted outgoing spherical wave \( j_j(k_0r) \).

Note that according to KMT, eq.(2.2), which provides a \( T \)-matrix correctly accounting for multiple scatterings from the \( A \) target particles, applies only to \( T \)-matrices. Eisenstein and Tabakin treated principal value integrals numerically, \( R \)-matrices are introduced and converted when needed by using eq.(2.15).

Because of the large effect that the Coulomb interaction can have on the scattering cross sections it is desirable to include the Coulomb distortion in the calculation. This can be done very precisely using the method of Vincent and Phatak [12]. This method requires simple logarithmic derivative matching at a distance \( R_c \) chosen outside the region of nuclear forces but not so large as to jeopardize momentum space treatment of the inner potential region.

Using the phase shifts given in eq.(2.18) the asymptotic radial wavefunction can be written
\[ u_j \sim j_j + [\rho(E)R_j]n_jkr \quad (2.19) \]
with resulting logarithmic derivative \( \xi_j \) (evaluated at the appropriate cutoff radius \( R_c \))
\[ \xi_j = \left. \frac{u'_j}{u_j} \right|_{R_c} \quad (2.20) \]
New phase shifts \( \delta_j \) can now be defined relative to the outgoing Coulomb wave-functions. The asymptotic wavefunction is
\[ \bar{u}_j \sim f_j - [\rho(E)\bar{R}_j]G_j \quad (2.21) \]
where $F$ and $G$ are the regular and irregular Coulomb wave functions, respectively. We find for $\bar{T}_j$

$$-\rho(E)\bar{T}_j = \frac{F_j' - \xi_j F_j}{\xi_j G_j - G_j'}|_{r=R_c} \tag{2.22}$$

from which $\bar{T}_j$ can be computed via eq.(2.15). Finally, the $S$-matrix is related to $\bar{T}$ via

$$S_j(k_0) = 1 - 2i\rho(E)\bar{T}_j(k_0) \tag{2.23}$$

The differential elastic cross section is computed using the following formula:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + \sin\theta^2|g(\theta)|^2 \tag{2.24}$$

where

$$f(\theta) = f_c(\theta) + f_A(\theta) \tag{2.25}$$

$$f_c(\theta) = -\frac{\eta_c}{2p\sin^2\frac{\theta}{2}} \exp\{2i[\sigma_0 - \eta_c ln(sin \frac{\theta}{2})]\} \tag{2.26}$$

$$\eta_c = Z_A Z_\pi \frac{1}{p_\pi} \frac{\alpha E_\pi E_A}{E_\pi + E_A} \tag{2.27}$$

$$f_A(\theta) = \frac{1}{k_0} \sum_{l=0}^{LPIMAX-1} [(l+1)T_{l+\frac{1}{2}} + lT_{l-\frac{1}{2}}]P_l(\cos \theta) \tag{2.28}$$

$$g(\theta) = \frac{1}{k_0} \sum_{l=0}^{LPIMAX-1} [T_{l+\frac{1}{2}} - T_{l-\frac{1}{2}}]P'_l(\cos \theta) \tag{2.29}$$

$$\sigma_{elas} = \frac{4\pi}{k_0^2} \sum_{l=0}^{LPIMAX-1} [(l+1)|\bar{T}_{l+\frac{1}{2}}\rho|^2 + l|\bar{T}_{l-\frac{1}{2}}\rho|^2] \tag{2.30}$$

$$\sigma_{total} = \frac{4\pi}{k_0^2} \sum_{l=0}^{LPIMAX-1} [(l+1)Im(\bar{T}_{l+\frac{1}{2}}\rho) + lIm(\bar{T}_{l-\frac{1}{2}}\rho)] \tag{2.31}$$

Here $\theta$, $E_\pi$, $p_\pi$ and $k_0$ are the $\pi$-nucleus center-of-mass scattering angle, total energy, $\pi$ momentum and wave number, while $\sigma_l$ is the point charge Coulomb phase shift and $\alpha = \frac{e^2}{\hbar c}$. 
CHAPTER 2. DESCRIPTION OF THE MODEL

It is often convenient to express the scattering amplitudes $S_i$ directly in terms of the real and imaginary parts of the phase shifts:

$$S_i = e^{2i\delta_i} = e^{-2\delta_i^\prime} e^{2i\delta_i^R} = \eta_i e^{2i\delta_i^R}$$

(2.32)

In terms of $\bar{T}_i$ Eisenstein and Tabakin found, using eq.(2.17),

$$\bar{\eta}_i = |1 - 2i\rho(E) \bar{T}_i|$$

(2.33)

$$\bar{\delta}_i = \frac{1}{2} \tan^{-1} \left( \frac{-2\rho \bar{T}_i^R}{1 + 2\rho \bar{T}_i^T} \right) \text{ (radians)}$$

(2.34)

Comparison of results for the full $T$-matrix from eq.(2.22) to the results of single-scattering or the Born approximation, is of interest. The Born amplitude can be obtained from the leading term of eq.(2.14). Processing this through eqs.(2.15) and (2.2) yields the Born $t$-matrix which provides a useful gauge of the role of the multiple scattering terms summed by solving eq.(2.1).

The main function of this program is to set up and solve eq.(2.14) for all partial waves of interest. For each partial wave the Coulomb-matched $T$-matrix $\bar{T}_i$ is obtained, from which the quantities $\bar{\eta}_i$ and $\bar{\delta}_i$ are found. The $\bar{T}_i$ values are then used to calculate the cross sections of eqs.(2.25), (2.30) and (2.31) over a selected range of angles. For selected partial waves the coordinate space wave functions are calculated in a way which will be described later.

2.2 The optical potential

According to the KMT formalism with “factorization” approximations (these approximations do not consider fermi-motion which will be discussed in section 2.2.5.)
first-order $\pi - A$ optical potential in momentum space takes the form

$$U(\vec{k}', \vec{k}) = \frac{A - 1}{A} \{ \rho_p(\vec{q}) T_{\pi p}^{NF}(\vec{k}', \vec{k}; k_0) + \rho_n(\vec{q}) T_{\pi n}^{NF}(\vec{k}', \vec{k}; k_0)$$

$$+ i \cdot (\vec{k} \times \vec{k}') [\rho_p(\vec{q}) T_{\pi p}^{F}(\vec{k}', \vec{k}; k_0) + \rho_n(\vec{q}) T_{\pi n}^{F}(\vec{k}', \vec{k}; k_0)] \}$$

$$= U^{NF}(\vec{k}', \vec{k}) + U^{F}(\vec{k}', \vec{k}) \quad (2.35)$$

Here $\vec{k}$ and $\vec{k}'$ are relative pion-nucleus momenta and $\vec{q} = \vec{k}' - \vec{k}$ is the momentum transferred to the nucleus; the $\rho(\vec{q})$ are the Fourier transforms of the ground state nuclear density for neutrons and protons normalized to $N$ and $Z$, respectively. The quantities $T_{\pi n}^{NF}$ and $T_{\pi p}^{NF}$ are the $\pi$-neutron and $\pi$-proton non-spin-flip scattering matrices and $T_{\pi n}^{F}$ and $T_{\pi p}^{F}$ are the $\pi$-neutron and $\pi$-proton spin-flip scattering matrices, respectively, which are constructed from the isospin $-\frac{1}{2}$ and $-\frac{3}{2}$ components of the free $\pi N$ $t$-matrices.

### 2.2.1 Decomposition of $U(\vec{k}', \vec{k})$

In order to make a partial wave decomposition of the potential $U(\vec{k}', \vec{k})$ we first decompose the densities $\rho(\vec{r})$ and the $t$-matrices $t_\pi$. We find

$$\rho_{p,n}(q) = \sum_{l_1} \rho_{p,n}^{l_1}(k', k) P_{l_1}(\hat{k} \cdot \hat{k}') \quad (2.36)$$

and

$$T_{\pi p,n}^{NF}(\vec{k}', \vec{k}; k_0) = \sum_{l_2} T_{\pi p,n}^{NF,l_2}(k', k; k_0) P_{l_2}(\hat{k} \cdot \hat{k}') \quad (2.37)$$

$$T_{\pi p,n}^{F}(\vec{k}', \vec{k}; k_0) = \sum_{l_2} T_{\pi p,n}^{F,l_2}(k', k; k_0) P_{l_2}(\hat{k} \cdot \hat{k}') \quad (2.38)$$

Note that eqs.(2.37) and (2.38) define the decomposition conventions for $\rho$ and $T$. Thus we have the new equations

$$U^{NF}(\vec{k}', \vec{k}) = \left( \frac{A - 1}{A} \right) \sum_{l_1,l_2} (\rho_{p}^{l_1} T_{\pi p}^{NF,l_2} + \rho_{n}^{l_1} T_{\pi n}^{NF,l_2}) P_{l_1}(\hat{k} \cdot \hat{k}') P_{l_2}(\hat{k} \cdot \hat{k}') \quad (2.39)$$
CHAPTER 2. DESCRIPTION OF THE MODEL

\[ U^F(k', k) = \left( \frac{A - 1}{A} \right) \sum_{l_1, l_2} (\rho_{l_1}^{l_1} T_{\pi p}^{F_{l_2}} + \rho_{l_1}^{l_1} T_{\pi n}^{F_{l_2}}) P_{l_1}(\hat{k} \cdot \hat{k'}) P_{l_2}(\hat{k} \cdot \hat{k'}) \]  \hspace{1cm} (2.40)

which can be written

\[ U^{NF}(k', k) = \left( \frac{A - 1}{A} \right) \sum_{l_1, l_2} (\rho_{l_1}^{l_1} T_{\pi p}^{NF_{l_2}} + \rho_{l_1}^{l_1} T_{\pi n}^{NF_{l_2}}) \langle l_1 0 l_2 0 | 00 \rangle ^2 P_{l}(\hat{k} \cdot \hat{k'}) \] \hspace{1cm} (2.41)

\[ U^F(k', k) = \left( \frac{A - 1}{A} \right) \sum_{l_1, l_2} (\rho_{l_1}^{l_1} T_{\pi p}^{F_{l_2}} + \rho_{l_1}^{l_1} T_{\pi n}^{F_{l_2}}) \langle l_1 0 l_2 0 | 00 \rangle ^2 \frac{l(l + 1) + l_2 (l_2 + 1) - l_1 (l_1 + 1)}{2l(l + 1)} P_{l}(\hat{k} \cdot \hat{k'}) \] \hspace{1cm} (2.42)

Comparing this to eq.(2.6) and (2.7) we find

\[ U^{NF}(k', k) = \left( \frac{A - 1}{A} \right) \sum_{l_1, l_2} (\rho_{l_1}^{l_1} T_{\pi p}^{NF_{l_2}} + \rho_{l_1}^{l_1} T_{\pi n}^{NF_{l_2}}) \langle l_1 0 l_2 0 | 00 \rangle ^2 \] \hspace{1cm} (2.43)

\[ U^F(k', k) = \left( \frac{A - 1}{A} \right) \sum_{l_1, l_2} (\rho_{l_1}^{l_1} T_{\pi p}^{F_{l_2}} + \rho_{l_1}^{l_1} T_{\pi n}^{F_{l_2}}) \langle l_1 0 l_2 0 | 00 \rangle ^2 \frac{l(l + 1) + l_2 (l_2 + 1) - l_1 (l_1 + 1)}{2l(l + 1)} \] \hspace{1cm} (2.44)

In the above, \( \langle l_1 0 l_2 0 | 00 \rangle \) is a Clebsch-Gordan coefficient which is constructed in the subroutine \texttt{cgc2} and \( P_{l}(\hat{k} \cdot \hat{k'}) \) is a Legendre polynomial. The optical potential \( U_{l} \) is constructed in the subroutine \texttt{poten} using \( \rho' \) values from the subroutine \texttt{rhol} and \( T^l \) values from the subroutine \texttt{tnuc}.

2.2.2 Form factors

The Fourier transforms \( \rho_p(q) \) and \( \rho_n(q) \) of the ground state neutron and proton densities are calculated by Eisenstein and Tabakin in the subroutine \texttt{denq} as functions of momentum transfer \( q \). The coordinate space densities available are the \( p \)-shell gaussian, e.g. for
neutrons

\[ \rho(r) = \frac{2}{(c\sqrt{\pi})^3} \left[ 1 + \frac{N-2}{3} \left( \frac{r}{c} \right)^2 \right] e^{-\frac{(r/c)^2}{2}} \quad (2.45) \]

whose Fourier transform is

\[ \rho(q) = 2 \left( 1 + \frac{N-2}{12} (6 - q^2 c^2) \right) e^{-\frac{q^2 c^2}{4}} \quad (2.46) \]

and the "wine-bottle"

\[ \rho(r) = \rho_0 \left[ 1 + w \left( \frac{r}{c} \right)^2 \right] \frac{1}{1 + e^{r/c}/t} \quad (2.47) \]

whose Fourier transform can be shown to be

\[ \rho(q) = \frac{4\pi}{q} \rho_0 \left( \phi(q) - \frac{w}{c^2} \phi''(q) \right) \quad (2.48) \]

where

\[ \phi(q) = \pi t c [g(q) \sin(qc) + f(q) \cos(qc)] \]

\[ = -2t^2 \sum_{n=1}^{\infty} (-e^{-e/t})^{n} \frac{nqt}{[(qt)^2 + n^2]^2} \]

\[ g(q) = \frac{\pi t \cosh(qt\pi)}{c \sinh^2(qt\pi)} \]

\[ f(q) = -\frac{1}{\sinh(qt\pi)} \]

and

\[ \rho_0 = \frac{3N}{4\pi(c^3 + \pi^2 t^2 c)} \left\{ 1 + \frac{3w}{5} \left[ 1 + \frac{7}{3} \left( \frac{\pi t}{c} \right)^2 \right] \right\}^{-1} \]

For a given pair of \( \vec{k} \) and \( \vec{k}' \) values, the chosen density \( \rho(q) \) is decomposed numerically in the subroutine \( \text{rhол} \) using a gaussian integration technique. Eisenstein and Tabakin found, from eq.(2.36), using \( \mu = \vec{k} \cdot \vec{k}' \) and \( q^2 = k^2 + k'^2 - 2\mu kk' \)

\[ \rho^l(k, k') = \frac{1}{2} (2l + 1) \int_{-1}^{1} \rho(q) P_l(\mu) d\mu \]

\[ \approx \frac{1}{2} (2l + 1) \sum_{i=1}^{NRHOL} \rho(q_i) P_l(\mu_i) w_i \quad (2.49) \]
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Here the index \( i \) runs over the NRHOL gaussian points \( \mu_i \) with weights \( w_i \). The values for the \( \rho(q) \) at the gaussian point of interest are obtained by calculating the corresponding \( q_i \) and using Lagrange interpolation on the \( \rho(q) \) vector constructed by \( \text{denq} \).

The Fourier transform of a uniform, spherically symmetric ground state charge distribution

\[
\rho_c(r) = \begin{cases} 
\frac{3Z_e}{4\pi R_e^2}, & r \leq R_e, \\
0, & r \geq R_e.
\end{cases}
\]

is calculated to be, with \( X_e = qR_e \) and \( X = qR_{\text{cut}} \),

\[
V_{\text{coul}}(q) = (Ze^2 2\pi^2 q^2) \left[ \frac{3}{X_e} \left( \frac{\sin X_e}{X_e} - \cos X_e \right) - \cos X \right] (2.50)
\]

This is also evaluated at the points \( q_i \) in the subroutine \( \text{rhol} \) and is decomposed via eq.(2.49). This is added to the potential \( U_i \) in \( \text{poten} \) and represents the contribution of the Coulomb interaction inside \( R_e \).

2.2.3 Construction of T-matrix

We turn now to a consideration of the \( t \)-matrix elements \( T_\pi \) appearing in eq.(2.35).

Explicitly, the \( t \)-matrix element \( T_\pi \) is written

\[
T_\pi(k',\vec{p};k,\vec{p}_0) = \langle \vec{k'},\vec{p}_0 - \vec{q}|t(\omega_0)|\vec{k},\vec{p}_0 \rangle (2.51)
\]

where \( \vec{k} \) and \( \vec{k}' \) are the initial and final momenta of the \( \pi \) in the \( \pi - \text{nucleus} \) (\( \pi A \)) c.m., \( \vec{q} = \vec{k}' - \vec{k} \) is the moment transfer, and \( \omega_0 \) is the total \( \pi - \text{nucleon} \) (\( \pi N \)) collision energy in the \( \pi - \text{nucleus} \) c.m. The quantity \( \vec{p}_0 \) is an appropriate choice for the average nucleon
momentum; Eisenstein and Tabakin used the simple form

\[ \vec{p}_0 = -\frac{\vec{k}}{A} \]  

(2.52)

although other reasonable choices are advocated in the literature and can be used in this program. Writing eq.(2.51) in terms of the elementary isospin 2/2 and 3/2 t-matrices, they find the elements of the t matrix \( T_\pi \) can be expressed as follows:

\[ T_\pi^+(k',k;\vec{k}_0) = \left< \vec{k}',\vec{p}_0 - \vec{q} \left| \frac{2N}{3A}t_{\frac{1}{2}} + \frac{1}{3} \left( \frac{N + 3Z}{A} \right) t_{\frac{3}{2}} \right| \vec{k},\vec{p}_0 \right>, \]

\[ T_\pi^-(k',k;\vec{k}_0) = N \text{ and } Z \text{ interchanged in above}, \]  

(2.53)

\[ T_\pi^0(k',k;\vec{k}_0) = \frac{1}{3} \left( \frac{N + Z}{A} \right) \left< \vec{k}',\vec{p}_0 - \vec{q} | t_{\frac{1}{2}} + 2t_{\frac{3}{2}} | \vec{k},\vec{p}_0 \right> \]

In order to solve eq.(2.14) \( U_t \) both on (\( |\vec{k}| = |\vec{k}_0| = |\vec{k}'| \)) and off (\( |\vec{k}| \neq |\vec{k}_0| \neq |\vec{k}'| \)) the energy shell are needed. The \textit{on-shell} behavior of eq.(2.53) can be obtained from free \( \pi N \) scattering information, while the \textit{off-shell} behavior requires the specification of a model for the \textit{off-shell} extrapolation.

In addition, the expression(2.35) for the \( \pi - \text{nucleus} \) optical potential requires the \( \pi - N \) collision matrix as seen in the \( \pi - \text{nucleus} \) system. This means the T-matrices must be transformed to the \( \pi - \text{nucleus} \) system. For \textit{on-shell} amplitudes this is a straightforward matter, while for the \textit{off-shell} case it is ambiguous. Several prescriptions exist in the literature for this procedure; Eisenstein and Tabakin adopt the method of ref.[13]

General lorentzian invariance of probability allows a relation between the \( \pi - A \) and \( \pi - N \) amplitudes to written (with \( I \) labelling isospin):

\[ \left< \vec{k}',\vec{p}_0 - \vec{q} | t_{I}(\omega_0) | \vec{k},\vec{p}_0 \right> = \Gamma \left< \vec{\kappa}',\vec{t}_I(\omega_0) | \vec{k},\vec{p}_0 \right> \]  

(2.54)
where

\[
\Gamma = \left( \frac{E_n(\kappa)E_n(\kappa')E_N(\kappa)E_N(\kappa')}{E_n(k)E_n(k')E_N(p_0)E_N(|p_0 - q|)} \right) \frac{1}{2}
\]

(2.55)

and \( \kappa, \kappa' \) are the initial and final \( \pi \) momenta in the \( \pi N \) c.m. and \( \tilde{\omega}_0 \) the energy. The scattering angles in the \( \pi - A \) and \( \pi - N \) systems can be related via

\[
\kappa \cdot \kappa' = A + B \cos \theta_{\pi A} = A + B \mu
\]

(2.56)

The definitions of \( A \) and \( B \) are given in the appendix of Eisenstein and Tabakin's paper. In general, these also depend on \( \cos \theta_{\pi A} \). As outlined there, other choices for the transformation can easily be incorporated by changing the definitions of \( \Gamma, A \) and \( B \). These quantities are calculated in the subroutine \texttt{abgen}.

The \( \pi - N \) amplitudes can now be written

\[
T_I(\kappa', \kappa) = \langle \kappa' | \hat{t}_I^{NF}(\tilde{\omega}_0) + i \partial \cdot (\kappa \times \kappa') \hat{t}_I^{PF}(\tilde{\omega}_0) | \kappa \rangle
\]

(2.57)

where \( \hat{t}_I^{NF}(\tilde{\omega}_0) \) and \( \hat{t}_I^{PF}(\tilde{\omega}_0) \) are non-spin-flip and spin-flip \( t \)-matrices, separately. We can write them clearly as

\[
\langle \kappa' | \hat{t}_I^{NF}(\tilde{\omega}_0) | \kappa \rangle = \sum_{l=0}^{LNMAX-1} \sum_{j=\pm \frac{1}{2}}^1 (j + \frac{1}{2}) \langle \kappa' | \hat{t}_I^{NF}(\omega_0) | \kappa \rangle P_l(\kappa \cdot \kappa')
\]

(2.58)

\[
\langle \kappa' | \hat{t}_I^{PF}(\tilde{\omega}_0) | \kappa \rangle = \sum_{l=0}^{LNMAX-1} \sum_{j=\pm \frac{1}{2}}^1 2(j - l) \langle \kappa' | \hat{t}_I^{PF}(\omega_0) | \kappa \rangle P_l(\kappa \cdot \kappa')
\]

(2.59)

The summation indices \((l, j)\) represent the orbital and total angular momentum in the \( \pi - N \) system \((l \leq LNMAX - 1)\). Returning now to eq.(2.37), (2.38) and combining eqs.(2.52)-(2.55), we find for \( T_I^I(k', k; k_0) \) the isospin decomposed values

\[
T_I^{NF,l_2}(k', k; k_0) = \frac{1}{2}(2l_2 + 1) \sum_{l,j} (j + \frac{1}{2}) \int_{-1}^{1} \Gamma \langle \kappa' | \hat{t}_I^{NF}(\omega_0) | \kappa \rangle
\]

(2.57)
\[ T_{l}^{F,l_2}(k', k; k_0) = \frac{1}{2} (2l_2 + 1) \sum_{l,j} 2(j - l) \int_{-1}^{1} \Gamma \langle \kappa' | \bar{t}_{l,j}^{I}(\omega_0) | \kappa \rangle \cdot P_{l}^{I}(\bar{\kappa} \cdot \kappa') P_{l_2}(\mu) \, d\mu \]
\[ \approx \frac{1}{2} (2l_2 + 1) \sum_{l,j} 2(j - l) \sum_{m=1}^{N_{\text{PROJ}}} \Gamma \langle \kappa' | \bar{t}_{l,j}^{I}(\omega_0) | \kappa \rangle \cdot P_{l}^{I}(\bar{\kappa} \cdot \kappa') P_{l_2}(\mu_m) w_m \] (2.60)

The subroutine \texttt{tnuc} carries out the construction of the \( T_{l}^{F,l_2}(k', k; k_0) \) and \( T_{l}^{F,l_2}(k', k; k_0) \) according to these eqs. After calculation the necessary \( \pi N \) kinematics, the \textit{on-shell} values of \( \langle \kappa' | \bar{t}_{l,j}^{I}(\omega_0) | \kappa \rangle \) are obtained from the subroutine \texttt{wvdat}. The sum on the gaussian points is the performed, obtaining for each point the necessary \( \Gamma, A \) and \( B \) from the subroutine \texttt{abgen}.

There are several options in the program for the \textit{off-shell} factors \( \sigma_{l,j}^{I} \) calculated in the subroutine \texttt{tnuc}. The factor \( \sigma(\kappa, \kappa') \) extrapolates the \( t \)-matrix smoothly to zero as either the incident or outgoing pion momentum variable goes further \textit{off-shell}. Eisenstein and Tabakin took
\[ \langle \kappa' | \bar{t}_{l,j}^{I}(\omega_0) | \kappa \rangle = \langle \kappa_0 | \bar{t}_{l,j}^{I}(\omega_0) | \kappa_0 \rangle \sigma_{l,j}^{I}(\kappa', \kappa) \] (2.62)
with \( \sigma \) constructed from separable potentials that exactly reproduce the \textit{on-shell} scattering phase shifts:
\[ \sigma_{l,j}^{I}(\kappa, \kappa') = \frac{g_{l,j}^{I}(\kappa') g_{l,j}^{I}(\kappa)}{g_{l,j}^{I}(\kappa_0)^2} \] (2.63)
The options include:
1. the model of Londergan et al.[15] for which a table of $g$ factors is read in and the needed values obtained by Lagrange interpolation;

2. a model in which the off-shell damping is achieved using gaussian functions

$$g_{ij}^l(k) = \kappa^l e^{-\alpha_l k^2}. \quad (2.64)$$

The on-shell matrix element in eq.(2.62) is constructed from the tabulated[16] phase shifts and absorption parameter $\eta$ for the $\pi N$ problem:

$$\langle \kappa_0 | H_{ij}^l(\omega_0) | \kappa_0 \rangle = -\frac{E_\pi(\kappa_0) + E_N(\kappa_0)}{4\pi^2E_\pi(\kappa_0)E_N(\kappa_0)} \frac{\eta_{ij}^l(\omega_0) e^{2i\delta_{ij}^l(\omega_0)} - 1}{2i\kappa_0} \quad (2.65)$$

In the case when the $\pi$ energy is low, the amplitude is calculated from scattering lengths.

### 2.2.4 2-body and 3-body models

The four-dimensional invariant total energy is $\omega_0^2 = (k + p_0)^2$. In the 2-body model, we can express it as a function of the momentum of the $\pi$ and the $p$ in $\pi - A$ system,

$$\omega_{2B}^2 = (k + p_0)^2 = [E_\pi(k) + E_N(p_0)]^2 - (\vec{k} - \vec{p}_0)^2 \quad (2.66)$$

where

$$\vec{p}_0 \approx -\frac{\vec{k}}{A} \quad (2.67)$$

and $\vec{k}$ is the momentum of $\pi$ in $\pi - A$ system.

In the 3-body model, we consider the nucleus as two parts. (See Fig. 2.1) One is an active nucleon which interacts with $\pi$ with the momentum $\vec{p} + \vec{p}_0$, the other is a non-internal motion (A-1) core with the momentum $\vec{P} = -\vec{k} - \vec{p} - \vec{p}_0$. Now, the $\omega_0$ can be
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Figure 2.1: 3-body model

expressed as

\[ \omega_{3B}^2 = (k_N + k_A - P)^2 = [E_N(k) + E_A(-k) - E_{A-1}(\vec{P}) - |E_B|^2 - |\vec{P}|^2 \]  

(2.68)

After substituting for \( \vec{p}_0 \) from eq.(59), we make the approximation

\[ |\vec{P}|^2 \approx \left( \frac{A - 1}{A} \right)^2 |\vec{k}|^2 + |\vec{p}|^2 \]  

(2.69)

with \( |\vec{p}|^2 \approx \langle p^2 \rangle = p_F^2, \ p_F \approx 174 \text{ MeV} \) being a “surface value” for the Fermi momentum \( p \), and \( E_B \), the core-nucleon binding energy, is an input parameter.

2.2.5 Fermi-motion

We now consider the inclusion of Fermi motion. This means we will consider the momentum of nucleon in the nucleus. In principle, we can express the optical potential as

\[ U(k, k') = \frac{A - 1}{A} \sum_n \int d^3p \phi_n^*(\vec{p} - \vec{q}) (f|t^{\pi N}(\omega)|i) \phi_n(\vec{p}) \]  

(2.70)

where \( \vec{q} = k' - k \). Using the “folding” approximation, we integrate over the nucleon momentum \( d^3p \) of \( \langle f|t^{\pi N}(\omega)|i \rangle \). The \( t \) matrices, \( \langle \kappa_0|\vec{t}_{ij}(\omega_0)|\kappa_0 \rangle \), are first calculated in the \( \pi N \) c.m. as follows.
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\begin{equation}
\langle \kappa_0 | \tilde{t}_{ij}(\omega_0) | \kappa_0 \rangle_{pol}
= \int d^3 \bar{p} \sum_n \frac{1}{A} |\phi_n(\bar{p})|^2 \langle \kappa(\bar{p}) | \tilde{t}_{ij}(\omega(\bar{p})) | \kappa(\bar{p}) \rangle
= 2\pi \int_{-1}^{1} d(\cos \theta) \int_0^{\infty} dp \sum_n \frac{1}{A} |\phi_n(\bar{p})|^2 \langle \kappa(\bar{p}) | \tilde{t}_{ij}(\omega(\bar{p})) | \kappa(\bar{p}) \rangle
\approx 2\pi \int_{-1}^{1} d(\cos \theta) \int_0^{p_{cut}} dp \sum_n \frac{1}{A} |\phi_n(\bar{p})|^2 \langle \kappa(\bar{p}) | \tilde{t}_{ij}(\omega(\bar{p})) | \kappa(\bar{p}) \rangle
\approx 2\pi \sum_{ij} w_i w'_j \sum_n \frac{1}{A} |\phi_n(\bar{p})|^2 \langle \kappa(\bar{p}) | \tilde{t}_{ij}'(\omega(\bar{p})) | \kappa(\bar{p}) \rangle_{(\cos \theta)=(\cos \theta)_j} \tag{2.71}
\end{equation}

where \(\bar{p}\) is the momentum of nucleons in the lab system, and

\begin{align}
\omega(\bar{p})^2 &= s(\bar{p}) = m_\pi^2 + m_N^2 + 2E_\pi(\bar{p}_{lab})E_N(\bar{p}) - 2\bar{p}_{lab} \cdot \bar{p} \tag{2.72} \\
\kappa(\bar{p})^2 &= \frac{1}{4s(\bar{p})} [s(\bar{p}) - (m_\pi + m_N)^2] [s(\bar{p}) - (m_\pi - m_N)^2] \tag{2.73} \\
|\bar{p}_{lab}| &= \frac{\kappa_0}{m_N} [E_\pi(\kappa_0) + E_N(\kappa_0)] \tag{2.74}
\end{align}

The formulas used for \(\sum_n |\phi_n(\bar{p})|^2\) are from the harmonic oscillator shell model:

\begin{align}
\begin{cases}
AP_{1s}(p) & A \leq 4, \\
4P_{1s}(p) + (A - 4)P_{1p}(p) & 4 \leq A \leq 16, \\
4P_{1s}(p) + 12P_{1p} + \frac{A-16}{2} [P_{1d}(p) + P_{2s}(p)] & 16 \leq A \leq 40,
\end{cases}
\end{align}

where

\begin{align}
\begin{cases}
P_{1s}(p) &= N \exp[-(pa)^2] \\
P_{1p}(p) &= \frac{2}{3} N (pa)^2 \exp[-(pa)^2] \\
P_{1d}(p) &= \frac{4}{15} N (pa)^4 \exp[-(pa)^2] \\
P_{2s}(p) &= \frac{3}{2} N \left[1 - \frac{2}{3} (pa)^2\right]^2 \exp[-(pa)^2]
\end{cases}
\end{align}

and \(N = \frac{a^3}{\pi^{3/2}}, a = 1.66\text{ fm}\).

2.3 Solution for the equation for \(R_i'(k', k; \kappa_0)\)

With the optical potential \(U_i\) constructed as above, we now turn to the numerical solution of eq.(2.14). The method used by Eisenstein and Tabakin is simply to express eq.(2.14)
as a matrix equation on a carefully chosen grid of \( N \) gaussian integration points. They expressed the formal equation \( R' = U + R'G_0U \) as a matrix and solved it using the steps \( R' = U(1 - G_0U)^{-1} = U\Omega' \). The order \( R'G_0U \) is preferred so the \( \text{Mol} \)ller operator \( \Omega' \) is precisely the one needed to construct the wave function. Using the rule that \( w_i(k_i|1|k_j) \rightarrow \delta_{ij} \) for handling the unit operator, Eisenstein and Tabakin expressed eq.(2.14) for any choice of grid points as a non-singular matrix provided no grid point \( k_i(i < N) \) coincides with the incident energy value \( k_0 \equiv k_{N1} = k_{N+1} \):

\[
\langle k_{N1}|R'_i|k_i \rangle = \langle k_{N1}|U_i|k_i \rangle + \sum_{j=1}^{N1} \langle k_{N1}|R'_i|k_i \rangle G_0(k_j)\langle k_j|U_i|k_i \rangle \tag{2.77}
\]

The standing wave Green function is identified as

\[
G_{0l}(k_i) = \frac{2}{\pi} \frac{w_jk_j^2}{E - E_{n}(k_j) - E_{A}(k_j)}, \quad j \leq N \tag{2.78}
\]

where \( E \) is the on-shell \( \pi - \text{nucleus} \) c.m. collision energy \( E_{n}(k_{N1}) + E_{A}(k_{N1}) \) with \( E_{A}^2 = k^2 + (Am)^2 \). The matrix in eq.(2.77) has been extended to \( N + 1 \) points so as to permit calculation of phase shifts from the on-shell \( R \)-matrix \( \langle k_{N1}|R'|k_{N1} \rangle \) without generating any \( k_j = k_0 \) difficulties.

In this discussion Eisenstein and Tabakin set \( G_{0l}(k_{N1}) = 0 \); for extra stability the subtraction described in ref.[7] is actually incorporated into the code. Now eq.(2.77) can be rewritten, after suppressing the index \( l \), as

\[
\sum_{j=1}^{N1} R'_{N1,j}F_{ji} = U_{N1,j} \tag{2.79}
\]

where \( F \) is the finite \((N1) \times (N1)\) matrix \( \delta_{ji} - G_0(k_j)\langle k_j|U_i|k_i \rangle \). The inverse of \( F \) is identified later as \( w_j(k_j|\Omega'|k_i) \).
Eisenstein and Tabakin allowed $U$ and $R'$ to be complex ($G_0(k_j)$ remains real), eq.(2.60) separated into two coupled equations:

$$
\sum_{j=1}^{N_1} R_{N1,j}^R \{ \delta_{ji} - G_0(k_j)U_{ji}^R \} = U_{N1,j}^R \sum_{j=1}^{N_1} R_{N1,j}^I G_0(k_j)U_{ji}^I 
$$

(2.80)

$$
\sum_{j=1}^{N_1} R_{N1,j}^I \{ \delta_{ji} - G_0(k_j)U_{ji}^R \} = U_{N1,j}^I \sum_{j=1}^{N_1} R_{N1,j}^R G_0(k_j)U_{ji}^I 
$$

(2.81)

These can be combined into one “supermatrix” equation of dimension $(N + 1)^2$:

$$
(R_{N1,j}^R P_{N1,j}^I) \begin{pmatrix}
\delta_{ji} - G_0(k_j)U_{ji}^R \\
-G_0(k_j)U_{ji}^I
\end{pmatrix} = (U_{N1,j}^R U_{N1,j}^I) 
$$

(2.82)

or, in condensed notation $R'F = U$. The matrix $U$ is set up for each $l$ in the subroutine poten and the matrix $F$ is constructed from $U$ and $G_0$ in the main program. $F$ is then inverted by the subroutine zinver and the $R$ matrix is calculated via $R' = UF^{-1}$. The complex nature of the potential has been treated by a two-channel approach (real and imaginary parts). That provides the key to extending this approach to additional channels; one needs to simply build a larger supermatrix.

The integral of eq.(2.14) is made discrete by a careful choice of $N$ gaussian points and weights. To ensure enough points, both in the tail and near the principal value singularity, these points were suitably mapped with two-thirds of the points below $2k_0$ and one-third above. The mappings are $k_i = k_0(1 + \bar{x}_i)$, $w_i = \bar{w}_ik_0$ for $0 \leq k \leq 2k_0$ and $k_i = 2k_0(3 + \bar{x}_i)/(1 - \bar{x}_i)$, $w_i = \bar{w}_i8k_0/(1 - \bar{x}_i)^2$ for $2k_0 \leq k \leq \infty$. Here the integral is split into $0 \sim 2k_0$ and $2k_0 \sim \infty$ parts and then mapped to the $-1 \leq \bar{x}_i \leq 1$ interval where $\bar{x}_i$ and $\bar{w}_i$ are the standard gaussian points provided by the files lgr10.dat, lgr30.dat, lgr60.dat and lgr90.dat containing the Gaussian points and weights of integration by Gaussian quadrature with 10, 30, 60 and 90 points. This choice provided good stability on the matrix inversion results for the $U_l(k', k)$ described here; certainly other choices can be
made to reduce dimensions. For other potentials alternative mappings might be needed. The present discussion differs from the code in that here $G_{0l}(k_{N1})$ is set equal to zero; the program on the other hand involves a subtraction procedure to cope with the principal value singularity. The above discussion illustrates that with $G_{0l}(k_{N1}) = 0$ we have a numerical realization of the Fredholm theory, and this provides an easy demonstration that the inversion of $F$ yields the wave function as discussed in section 2.4. Nevertheless we must be certain of giving proper treatment to the principal value integral which depends on having the integration points straddling the singularity (an equal number of points on each side would be one possibility). Therefore the scaled on-shell quantity is given by

$$4 \frac{k_{N1}^2}{\pi} \frac{E_c(k_{N1})E_A(k_{N1})}{E_c(k_{N1}) + E_A(k_{N1})} \sum_{j=1}^{N} w_j \frac{k_{N1}^2 - k_j^2}{k_{N1}^2}$$

compared with eq.(2.65) for the on-shell case is incorporated in a subtraction procedure outlined in ref. [14].

**2.4 Generation of wave functions in coordinate space**

Momentum space wave functions which can be used directly as distorted waves in $p$-space inelastic codes are constructed in this program. However, for ease in visualizing the waves and for use in standard $r$-space inelastic codes, it is useful to provide wave functions in coordinate space. The relation between $r$ and $p$-space wave functions is simply a Bessel transform expressed on the same grid used to construct $R'$. Eisenstein and Tabakin considered first the standing wave solution for $R'$; later the properly normalized and matched wave function corresponding to $T$ will be discussed.

Using the formal relation $|\psi'\rangle = \Omega |k_0\rangle$, where the Møller operator is defined by
\( \Omega' = 1 + G_0 R' = (1 - G_0 U)^{-1} \), we can express the wave function for the \( l \)th partial wave as

\[
\psi'_{k_0 l}(r) = \int_0^\infty j_i(k r) k^2 dk \psi'_{k_0 l}(k) = \sum_{i=1}^N j_i(k_i r) k_i^2 w_i(k_i |\Omega'_l|k_0) \tag{2.84}
\]

Here the (real) standing wave Green function is denoted by \( G_0 = (E - H_0)^{-1} \). Eq. (2.84) shows that \( (k|\Omega'_l|k_0) \) is the momentum space wave function \( \psi'_{k_0 l}(k) \). Consequently, in solving eq. (2.1) for \( R' = U \Omega' \), Eisenstein and Tabakin generated not only on-shell information, i.e. asymptotic wave functions and phase shifts, but also the full wave function, because the matrix inversion generates \( w_i(k_i |\Omega'_l|k_0) \) from which \( \psi' \) is determined using eq. (2.84). The process of expressing formal operator equations as explicit matrices on an integration point grid entails replacing the unit operator (or Dirac delta function) by a Kronecker delta according to the rule

\[
w_i(k_i |1|k_j) = w_i \delta(k_i - k_j) \rightarrow \delta_{ij}.
\]

With this rule one sees that \( w_i(k_i |\Omega'_l|k_j) \) is given by \( F(K,K_P) \) in the code.

Thus the matrix \( F \) yields the wave function \( \psi' \) and its associated phase shifts \( \delta' \) as they arise from the optical potential \( U \). Recall that \( U \) is cut-off at \( R_c \) and that for \( r > R_c \) one needs to match to the coulomb waves. However, before that step one must also incorporate the \( A/(A - 1) \) factor of eq. (2.2) which accounts for the multiple scattering sequence according to KMT. Therefore the following steps are needed before generating the full wave functions. First the on-shell \( R' \) is used to obtain \( T' \) and from that \( T \). Eisenstein and Tabakin preferred to program principal values and to match using standing waves. They also generated \( R \) from \( T \) before finally obtaining \( R \) and \( T \) by Coulomb matching. These on-shell steps are now paralleled in the construction of the physical wave function. The formal steps that follow will hopefully help to explain that
process. The full outgoing wave function is constructed from the $\Omega^{(+)}$ McIver operator

$$\Omega^{(+)} = 1 + G_0 T$$  \hspace{1cm} (2.85)$$

whereas they first had the $\Omega'^{(+)}$ operator or, using

$$T' = \frac{A - 1}{A} T$$  \hspace{1cm} (2.86)$$

$$\Omega'^{(+)} = 1 + G_0 T' = 1 + \frac{A - 1}{A} G_0 T$$  \hspace{1cm} (2.87)$$

It follows that $\Omega^{(+)}$ is given by

$$\Omega^{(+)} = 1 + \frac{A}{A - 1} (\Omega'^{(+)} - 1) = \frac{1}{A - 1} + \frac{A}{A - 1} \frac{T'_i}{R'_i} \Omega'$$  \hspace{1cm} (2.88)$$

Here we return to the standing wave operator $\Omega'$ by introducing the complex phase relation $\Omega'^{(+)} = (T'_i/R'_i) \Omega'$. In explicit matrix form eq.(2.88) is

$$\hat{\Omega}_i(k_i, k_j) \equiv (A - 1) w_i(k_i) |\Omega'^{(+)}| k_j) = -\delta_{ij} + A \frac{T'_i}{R'_i} w_i(k_i) |\Omega'| k_j)$$  \hspace{1cm} (2.89)$$

Thus, from knowledge of $F$, $T'$ and $R'$ Eisenstein and Tabakin used eq.(2.89) to obtain the final McIver operator which includes the proper Coulomb matching and multiple scattering counting steps in the code the outgoing wave $u^+_{k_0 t}$ is finally generated using

$$u^+_{k_0 t}(r) = (k_0 r) \psi_{k_0 t}(r) = \frac{r}{R_c} \sum_{i=1}^{N} \frac{j_i(k_i r) \hat{\Omega}_i(k_i, k_0)}{\sum_{i=1}^{N} j_i(k_i R_c) \hat{\Omega}_i(k_i, k_0)} u^+_{k_0 t}(R_c)$$  \hspace{1cm} (2.90)$$

where $u^+_{k_0 t}$ is the Coulomb outgoing wave. Eq.(2.90) completes the discussion of constructing $r$-space wave functions. The calculations are carried out in the subroutine rewfn.
3.1 Program structure

The structure of the PIPIT program including the modifications described in this thesis are shown in Fig. 3.1. The program consists of the three main parts:

1. read.f, pitin.dat;
2. pion.f, pion.out;
3. hoel.tmp;
lgr10.dat, lgr30.dat, lgr60.dat, lgr90.dat.

The first part is to input the parameters for the particular calculation. The input is done via the interactive program read which interrogates the user for the input parameters. These parameters are output to the file pitin.dat.

The second part is the main program pion.f which reads the file pitin.dat for initialization and calculates the cross-section. The results of the calculation are stored in
the file pion.out.

The third part are the data files hoel.tmp containing hoehler \( \pi \)-nucleon *on-shell* phase shifts and the three files lgr10.dat, lgr30.dat, lgr60.dat and lgr90.dat containing the Gaussian points and weights of integration by Gaussian quadrature with 10, 30, 60 and 90 points.

### 3.2 Description of the subroutines

#### 3.2.1 Main program

After reading the input parameters from the input file pitin.dat, the program calls subroutines *rewfn* and *coufaz* to obtain Coulomb phase shifts [eq.(2.20)\(\sim\)2.22)] and the \( l = 0 \) Coulomb phases shift \( \sigma_0 \) [eq.(2.26)]. The subroutine *poten* is used to obtain the optical potential \( U_j \) described in eq.(2.14). The Lippmann-Schwinger equation [eq.(2.14)] is solved using the numerical method described in section 2.3, and the cross-section are calculated as described in eq.(2.24)\(\sim\)2.31).

#### 3.2.2 Subroutines rewfn and coufaz

Subroutine *rewfn* is used to calculate the Coulomb wave functions \( F_l, F'_l \) and \( G_l, G'_l \) in eq.(2.15) and (2.16). Subroutine *coufaz* is used to calculate the Coulomb phase \( \sigma_0 \) in eq.(2.26).

#### 3.2.3 Subroutine poten

The purpose of this subroutine is to set up the matrix \( U_l(k, k') \) for a given \( l \). The matrices \( urr \) and \( uii \), the real and imaginary parts of the optical potential matrix described in
CHAPTER 3. DESCRIPTION OF THE COMPUTER PROGRAM

BEGIN

Input from pitin.dat

Set parameters

Call rewfn to get Coulomb wave

Call coufaz to get Coulomb phases

Call poten to get optical potential U

Yes

Spin flip

Call zinver to get $1/F$

Solve the equations to get $T$ matrices

Calculate $\sigma$

Output to pion.dat

No

Set $T_{1/2} = 0$

Call tmuc to get $\pi$-N $T$ matrices $T$

Call rhol to get form factors $\rho(k,k')$

Call denq to get form factor $\rho(q)$

Figure 3.1: The flow diagram of pion.f
eq.(2.43) and (2.44), are constructed of all $l$ values and $(k, k')$ momentum pairs by calling the subroutines \texttt{tnuc} and \texttt{rhol}. The results are returned to the \texttt{main} program as a set of complex numbers $zuu$. 

3.2.4 Subroutine \texttt{rhol} and \texttt{denq}

In the subroutine \texttt{rhol} the partial wave decomposition of $\rho'(k, k')$ in eq. (2.49) and $V_{\text{coul}}(q)$ in eq.(2.50) is accomplished numerically for all $l$. To calculate the integral of (2.49), the proper values of $\rho(q_i)$ at the Gaussian points $q_i$ are obtained by calling subroutine \texttt{denq}.

3.2.5 Subroutine \texttt{zinver}

The purpose of this subroutine is to calculate the inverse of the matrix $F$ in eq.(2.79).

3.2.6 Subroutine \texttt{cgc2}

In this subroutine, the Clebsch-Gordan coefficient $\langle l_10l_20|00 \rangle$ is calculated.
CHAPTER 3. DESCRIPTION OF THE COMPUTER PROGRAM

3.2.7 Subroutine tnuc

Subroutine tnuc is the most important part of this program. The flow chart for this subroutine is shown in Fig. 3.2. The purpose of tnuc is to calculate the optical potential $T_{l_2}^{k,k'}(k,k')$ in eq.(2.60) and (2.61) for all values $l_2$, iso-spin $I$ and momentum pairs $(k,k')$ in the $\pi$–nucleus center-of-mass frame. The on-shell $t$ matrices in the $\pi$-nucleon center-of-mass frame are obtained by calling the subroutine wvdat, the coordinate system transformation parameters used in eq.(2.55) and (2.56) are obtained by calling the subroutine abgen. It also calls the subroutine tfold to include the effect of Fermi-motion.[See section 2.2.5]

3.2.8 Subroutine wvdat, dsplin and dsubsp

For a given total $\pi$-nucleon collision energy $\omega_0$ in the $\pi$-nucleon c.m., the Hoehler phase shifts are read from the file hoel.tmp with the subroutine wvdat. A spline fit to phase shifts is done in the subroutine dsplin in order to calculate the on-shell $\pi$-nucleon scattering amplitudes matrices $(\kappa_0|\tilde{t}_{l_2}^{k,k'}(\omega_0)|\kappa_0)$ at momentum $k_0$ in eq.(2.65) by the subroutine dsubsp.

3.2.9 Subroutine abgen

This subroutine is used to obtain the $\Gamma$ in eq.(2.15) and $A, B$ in eq.(2.16), the coefficients of transferring the off-shell amplitudes in the $\pi$–nucleon system to the off-shell amplitudes in the $\pi$–nucleus system.

3.2.10 Subroutine tfold

The subroutine tfold is the main modification to the original code PIPIT.f. The flow diagram of this subroutine is shown in the Fig. 3.3.
CHAPTER 3. DESCRIPTION OF THE COMPUTER PROGRAM

Figure 3.3: The flow diagram of subroutine tfold
The purpose of this subroutine is to calculate Fermi-motion average using the formalism in section 2.2.5. The subroutine reads the $\pi N$ momentum in $\pi - A$ c.m., and transforms it to the $\pi N$ momentum in $\pi - N$ c.m. for either the 2-body or 3-body model. The integration over the angle $\theta$ and the momentum $p$ of the active nucleon in eq.(2.71) is then performed to obtain the “folded” on-shell $T$ matrices in $\pi - N$ c.m. which are returned to the subroutine `tnuc`. The integral over both $\theta$ and $p$ is done using Gaussian quadrature with 10 points.

### 3.3 Running the program

#### 3.3.1 Input initial parameters

The steps to calculate a cross section as a function of energy are first to run the program `read` to input the run parameters. The program `read` creates a new file `pitin.dat`. This file will be the input file for the main program `pion`. The structure of the input file `pitin.dat` is shown in table:

<table>
<thead>
<tr>
<th>J1</th>
<th>K1</th>
<th>E1</th>
</tr>
</thead>
<tbody>
<tr>
<td>J2</td>
<td>K2</td>
<td>E2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>X1</td>
<td>X2</td>
<td>X3</td>
</tr>
</tbody>
</table>

where

- $I_1 = \text{the charge of the pion.}$
- $I_2 = \text{the number of the protons in the nucleus.}$
- $I_3 = \text{the number of the nucleons in the nucleus.}$
I4 = the number of the partial waves for optical potential $U$ in eq.(2.6) and (2.28)~(2.31).

I5 = 1 : p-shell gaussian form factor;
2 : wine-bottle form factor;
3 : modified harmonic oscillator form factor;
4 : 3-parameter fermi form factor;
5 : form factor $\rho = Z(1 - q^2 c^2 / 6)$;
6 : special form factor for $^4He$.

I6 = switch for charge form factor, 1 : on; other : off.

I7 = the number of the partial waves for $T$ matrix in eq.(2.37) and (2.38).

I8 = 0 : with Coulomb potential;
1 : without Coulomb potential.

I9 = 1 : no off-shell transformation;
2 : Gaussian off-shell model;
3 : Yukawa off-shell model.

I10 = the number of the energies to be calculated.

I11 = 1 : with spin-flip;
2 : without spin-flip.

I12 = 1 : 2-body model;
2 : 3-body model.

I13 = 1 : with Fermi-motion average;
2 : without Fermi-motion average.

J\text{i} = the number of the partial waves for $T$ matrix in $\pi - N$ system
for the $i$th energy. [see eq. (2.58) and (2.59)]

$K_i = 0$ : input data is the momentum in lab. system;

$1$ : input data is the kinetic energy in lab. system.

$E_i$ = the $i$th energy to be calculated.

$A$ = the size $R$ of the charged sphere for the Coulomb potential in eq. (2.50),

$$R_e = \sqrt{5/3} \cdot A$$

$B$ = coordinate space cut off radius $R_{\text{cut}}$ in eq. (2.50)

$C$ = the charge form factor parameter.

$X_i$ = the form factor parameters for proton distributions.

$Y_i$ = the form factor parameters for neutron distributions.

The typical input file for $\pi^- + ^{12}\text{C}$ is shown following:

-1 6 12 25 3 1 16 0 1 14 1 2 1
4 1 89.000
4 1 115.000
4 1 128.000
4 1 156.000
4 1 187.000
4 1 229.000
4 1 259.000
4 1 289.000
4 1 337.000
4 1 409.000
4 1 482.000
4 1 567.000
3.3.2 Main program

After running the main program pion, the results will be output into the file new.out, a typical output is:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>89</td>
<td>566.16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>115</td>
<td>672.19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>697.50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>156</td>
<td>711.83</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>187</td>
<td>678.40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>229</td>
<td>582.44</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>259</td>
<td>511.03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>289</td>
<td>443.94</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>337</td>
<td>361.55</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>409</td>
<td>295.31</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>482</td>
<td>257.34</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>567</td>
<td>261.69</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>696</td>
<td>245.99</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>855</td>
<td>329.37</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The first column is the π kinetic energy (in MeV) in Lab. system, the second column is the total cross-section (in mb).
The typical running time is around 2 ~ 3 minutes for one energy. More than half of the time is used to calculate the integral in the subroutine \texttt{tfold}. 
Chapter 4
Calculation results

4.1 Calculations with PIPIT code

Figure 4.1: Total cross section of $\pi^- + ^{12}C$ with PIPIT
Simulation results for \( \pi^- \) scattering from \(^{12}\text{C}\) obtained using the original PIPIT code are shown in Fig. 4.1. The parameters used in this calculation were: [see section 3.3.1]

\[
\begin{array}{ccccccccccccccc}
-1 & 6 & 12 & 25 & 3 & 1 & 16 & 0 & 2 & 14 & 2 & 1 & 2 \\
4 & 1 & 89.000 \\
4 & 1 & 115.000 \\
4 & 1 & 128.000 \\
4 & 1 & 156.000 \\
4 & 1 & 187.000 \\
4 & 1 & 229.000 \\
4 & 1 & 259.000 \\
4 & 1 & 289.000 \\
4 & 1 & 337.000 \\
4 & 1 & 409.000 \\
4 & 1 & 482.000 \\
4 & 1 & 567.000 \\
6 & 1 & 696.000 \\
6 & 1 & 855.000 \\
2.460 & 6.000 & 4.2638 \\
1.672 & 1.150 & 0.000 & 1.672 & 1.150 & 0.000
\end{array}
\]

In the energy range from \( \sim 100 \text{ MeV} \) to \( \sim 1 \text{ GeV} \), there are three serious discrepancies between the theoretical result and the experiment data.

1. The resonance peak of the calculated result is approximately 30 MeV below the experimental resonance peak;
2. The theoretical result over predicts the experimental data by approximately 30% in the region of the resonance;

3. The theoretical result under predicts the experimental data by approximately 15% in the region near 400 MeV.

In the following sections, we will discuss the results of the 3-body model and the results of including Fermi-motion and show how the first two discrepancies are removed using these techniques. These techniques, however, also under estimate the cross-section in the region of 400 MeV.

4.2 Comparison of calculations with 2-body and 3-body models

The calculated cross section using the 3-body model are shown in Fig. 4.2. Compared to the 2-body calculation, we see that the 3-body results are an improvement in the region near the resonance. The over prediction of the cross-section has been reduced, but the resonance has shifted to too high an energy. In the energy region above 250 MeV, the difference between the two models is insignificant.

The \(\pi\)-nucleus total cross section data for several nuclei reproduced from the paper of Landau and Thomas [3] are shown in Fig 1.1. A characteristic feature is these data is that the \(P_{33}\) resonance shifts to lower energies with increasing \(A\). The shift in the cross-section maximum to an energy lower than the energy at which the phase shift is 90° occurs also in \(\pi - p\) scattering.
This shift can be demonstrated by considering the partial wave decomposition of the forward (θ ~ 0°) scattering amplitude and the total cross section:

\[
Re f(0°) = \frac{1}{k} \sum_l (2l + 1) \sin \delta_l \cos \delta_l
\]

\[
\sigma(\text{total}) = \frac{4\pi}{k} Im f(0°) = \frac{4\pi}{k} \frac{1}{k} \sum_l (2l + 1) \sin^2 \delta_l
\]

In \(\pi - p\) scattering below 300 MeV, the \(P_{33}\) resonance completely dominates. Thus when \(\delta_l\) reaches 90° at \(T_{\pi}^{lab} \sim 195\) MeV \((S = (p + p')^2 \sim 1238\) MeV\), the sum in the imaginary part of the forward scattering amplitude in the expression for \(\sigma_{\text{total}}\) above will be maximum. However, because of the \(1/k\) factor, the energy at which the total cross section reaches
its maximum is reduced. The energy of the maximum cross section can be calculated by computing the zero of the derivative

\[ \frac{d}{dk} \left( \frac{1}{k} \sin^2 \delta_l \right) = 0 \implies \tan \delta_l = 2k\delta'_l \]

as \( \delta'_l > 0 \), so \( 2k\delta'_l \) is a positive finite number, and as \( \delta_l < 2\pi \), the cross section maximum occurs at the energy lower than the energy at which \( \text{Re} f(0^\circ) = 0 \). In fact, \( \text{Im} f(0^\circ) \) has its maximum shifted to \( \sim 187\text{MeV} \), whereas \( \sigma^{\pi\nu}(\text{total}) \), which is multiplied by an addition \( 1/k \) factor, is shifted down to \( \sim 180\text{MeV} \).

The \( \pi \)-nucleus scattering phenomena are similar. The inclusion of multiple scattering tends to broaden the the \( P_{33} \) resonance. Furthermore, the pion nucleus resonance is also shifted due to the 2-body approximation. In this case,

\[ s_{\pi N} \approx m_{\pi}^2 + m_N^2 + 2E_{\pi}(\vec{k})E_N \left( -\frac{\vec{k}}{A} \right) - 2\vec{k} \cdot \left( -\frac{\vec{k}}{A} \right) \]

where

\[ |\vec{k}|^2 = \frac{[s_{\pi A} - (m_{\pi} + m_A)^2][s_{\pi A} - (m_{\pi} - m_A)^2]}{4s_{\pi A}} \]

\[ s_{\pi A} = (p + p')^2 = (m_{\pi} + m_A)^2 + 2m_A T_k \]

and \( T_k \) is the energy of \( \pi \) in lab. system. When \( m_A \) increases, \( s_{\pi A} \) also increases. Consequently \( |\vec{k}| \) increases, so that \( s_{\pi N} \) will increase as well. Therefore in order that \( s_{\pi N} \) match the resonance energy (\( \sim 1238\text{MeV} \)), for the heavier nucleus, \( T_k \) will be corresponding lower at the resonance.

The 2-body calculation results show that the energy of resonance is shifted to too low an energy. Thus the 2-body approximation alone appears to be inadequate. In this model, the nucleus is assumed to be a non-internal-motion particle. After calculating the
In the 3-body model [See 2.2.4], we treat the nucleus in two parts, the first is the active proton (or neutron) which interacts with the pion, the other is a (A-1) core which does not participate in the interaction. This core is considered to be a single particle with no internal motion. In this case,

\[ s_{\pi N} \approx (p_\pi + p_A - p_{A-1})^2 \]

\[ \approx [E_{\pi}(\vec{k}) + E_A(\vec{k}) - E_{A-1}(\vec{P})]^2 - |\vec{P}|^2 \]

where

\[ \vec{P} = -\vec{k} - \vec{p}_0 - \vec{p} \]

where \( \vec{k} \) is the momentum of the pion in the in \( \pi - A \) system, \( \vec{p}_0 \) is the momentum of the active proton (neutron) in \( \pi - A \) system, and \( \vec{p} \) is the moment of the active proton (neutron) in lab. system.

When \( p = 0 \), \( E_A(\vec{k}) - E_{A-1}(\vec{P}) \sim E_N(\vec{p}_0) \), the expression for the invariant total energy \( s_{\pi N} \) is equivalent the 2-body model. But when \( p \neq 0 \), \( |\vec{P}| \) increases, and \( s_{\pi N} \) correspondingly decreases. This gives a reduction to the resonance energy, as required by the experiment data.

### 4.3 Fermi-motion averaging

In order to include Fermi-motion averaging in the LOPTT program, Landau used the “surface value” \( p_F = \sqrt{\langle p^2 \rangle} \approx 174 MeV \) as the Fermi momentum \( p \) of the (A-1) core, and only integrated the Fermi momentum of the active proton (or neutron). In our modified PIPIT program, both are integrated together. The results of these calculation are shown
Figure 4.3: Comparison of calculations with the LPOTT program of Landau and the modified PIPIT code with and without Fermi-motion averaging in Fig. 4.3. In this case the parameters for the LPOTT calculation are given in the follow: (see [10])
Figure 4.4: Total cross section of $\pi^- + ^{12}C$ with Fermi-motion
The calculated cross sections including Fermi-motion of the nucleons in the target nucleus are shown in Fig. 4.4. We see that both 2-body and 3-body models give better agreement with the data than the previous models. In the region of 80 ~ 250 MeV [See Fig. 4.5], the calculation of the cross-section with Fermi-motion averaging included, particularly in the case of the 3-body result agrees with the data within 1%. These results
Figure 4.5: Total cross section of $\pi^- + ^{12}C$ calculated with Fermi-motion included are consistent with our expectation that the inclusion of Fermi-motion will soften and broaden the peak.

### 4.4 Coulomb potential sensitivity

The sensitivity of the code to the Coulomb potential was investigated. The results of calculations of the total cross section with and without the inclusion of the Coulomb term (2.50) are shown in Fig. 4.6. Inclusion of the Coulomb term increases the cross section at low energies where it is expected to be most important.
4.5 *Off-shell* model sensitivity

In Fig. 4.7 shows the 3-body results with different *off-shell* model is compared. Simply using *on-shell* $t$ matrices as *off-shell* $t$ matrices is a very poor approximation and is disagreement with the experimental data. Using the separable model, see eq.(2.62), yields a better agreement. Different dampings, see eq.(2.64), yields insignificant changes. In the region of the $P_{33}$ resonance, the experimental data is fitted extremely well with the Gaussian model even to the lowest energy.
Figure 4.7: off-shell sensitivity

4.6 Other nuclei – $^4$He and $^{40}$Ca

Fig. 4.8 show the different simulation results for $\pi^4$He with 2-body and 3-body models. The experimental data is from [17]. The input file for the calculation is:
Figure 4.8: Total cross section for $^4$He
We see that the position of the resonance is reproduced correctly, but the magnitude of cross section is approximately 10% low in all energy region.

Fig. 4.9 shows the calculation result for $\pi^{40}$Ca with the modified PIPIT. There is no experimental data for calcium to compare.
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Figure 4.9: Total cross section for $^{40}$Ca
Chapter 5

Conclusions

An optical model potential and transition matrix have been constructed in order to calculate the elastic scattering cross section of negative pions from carbon and helium. The potential is constructed from the basic pion nucleon scattering amplitudes. In this work we have included the spin flip term, three body kinematics and fermi averaging over the magnitude of the momentum and angle of the struck nucleon.

With this new formalism the agreement between the calculated and measured cross section for $\pi^-$ scattering from carbon 12 is within 1% in the range from 90 to 140 MeV with no free parameters. The position of the resonance is correctly predicted by this model and the fit has been improved in the low energy region. Above the resonance region the calculated cross section is approximately 15% less than the measured cross section similar to previous calculation using PIPIT.

Similar results were obtained for $\pi^-$ scattering from $^4$He. The position of the resonance is predicted correctly but the magnitude of the cross section is approximately 10% low in at all energies. No data exists for calcium for comparison. The calculated cross section however shows a broad resonance in the same energy range as resonances seen in
other nuclei.
 Bibliography

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