PUBLIC REPORT, PRIVATE INFORMATION,
MANAGERIAL COMPENSATION AND EFFORT ALLOCATION

By

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Abstract

The shareholder-management relationship of a publicly traded corporation is a classic agency relation. The conflict of interest arises because of the separation of ownership and control. The alignment of the goals of the firm’s owners and the goals of the managers has been the central focus of various theories. Many theories, including agency theory, are advanced to examine why firms choose the management compensation levels and structures they do and what effects these variables are likely to have in terms of providing proper managerial incentives. Empirical evidence indicates that accounting- and market-based information signals are frequently used as managerial performance measures. The premise is that tying management compensation to firm performance helps align the goals of managers with the goals of the firm’s owners.

However, prior agency research has ignored the impact of the investors’ endogenous private information acquisition on managerial incentives. Because endogenous information acquisition influences the informativeness of the market price, it likely affects the compensation contracts written on stock price, and hence influences managerial incentives. Since investors in financial markets endogenously trade off the cost and benefit of information to decide whether to acquire private information signals, careful examination of investors’ private information acquisition decision can enrich our understanding of management compensation arrangements.

Our study focuses on this issue. We start with a one-period, general asymmetric information structure, single risky asset (the firm) model similar to Grossman and Stiglitz (AER 1980), and establish the rational market price in the presence of a public report about the firm. It is shown that more investors decide endogenously to acquire the costly
private signal as the firm’s inherent uncertainty increases. The general formulation considered here permits identification of the key factors affecting various relations. Second, we examine the shareholder-management relation in a multi-task agency setting. The compensation contract is assumed to be based on two signals. It is shown that the Banker and Datar (JAR 1989) result that the relative weights on the signals equals the ratio of a measure of sensitivity times precision of those signals still holds with proper extension of the sensitivity measure. The extended sensitivity measure incorporates the signals’ congruity with the firm’s gross payoff (Feltham and Xie, AR 1994) and the first-best allocation of effort. A second performance measure can be used in the compensation contract to expand the implementable effort set. Third, we examine the relative use of the price and earnings in providing incentives. Our result suggests that the inclusion of accounting earnings facilitates the incentive alignment between the managers and shareholders. Fourth, we derive comparative statics and discuss empirical implications. The relative weights on the public report and a filtered price are independent (dependent) of the firm’s inherent uncertainty if the private information acquisition decision is endogenous (exogenous). The impact of an increase in the firm’s inherent uncertainty on the relative weights on the public report and the observed market price is reduced if information acquisition is endogenous instead of exogenous. Finally, we compare a stock ownership contract with an option contract. For any given level of implementable effort, an ownership contract always generates higher net surplus to the shareholders than does an option contract. Furthermore, the optimal ownership contract induces higher second-best effort level than does the optimal option contract.
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Chapter 1

Introduction

There are various theories about why firms choose the compensation levels and structures they do, and what effects these variables are likely to have in terms of providing proper managerial incentives. The evidence indicates that accounting- and market-related information signals are frequently used in constructing performance measures on which management compensation contracts are based. In particular, executive compensation contracts are commonly observed to be written on the combination of an accounting-based measure and the market price of the firm.\footnote{Empirical research includes Antle and Smith (1986), Jensen and Murphy (1990), Lambert and Larcker (1987), Healy (1985), and Sloan (1993).} For example, the annual bonus plans were used by only 35% of the U.S. major manufacturing firms in 1960 and by 99% currently. Restricted stocks were initially introduced in the mid-seventies and are currently popular as well (See Figure 1.1).

A variety of reasons have been advanced to explain the popularity of these two types of performance measures.\footnote{See Kaplan and Atkinson (1989, pp 724-725), Rappaport (1986), Watts and Zimmerman (1986, pp 201-202), and O'Byrne (1990).} One of the most common reasons is as follows. The stock price reflects the investors’ private information about the market value of the firm, including information about both short- and long-term factors influencing the firm’s value. In addition, the market-based performance measure “is not directly subject to the moral hazard which may underlie financial accounting or other performance measures reported by the executives being evaluated” (Antle and Smith 1986, p.6). On the other hand, accounting earnings reflect the cash flow of the firm during the accounting period, adjusted...
Chapter 1. Introduction

Figure 1.1: The prevalence of executive compensation plans in major U.S. manufacturing firms from 1960 to 1988 (Source: Sloan 1993)
for accruals, which represents the short-term factors influencing the firm's value (Rappaport, 1986). Accounting earnings are less affected by events beyond the control of the managers (Kaplan and Atkinson, 1989), but are myopic. Sloan (1993, p.59) reports that 99% of large U.S. manufacturing corporations tie their executive compensation plans to accounting earnings, and provides empirical evidence in support of the hypothesis that earnings-based incentives help shield executives from market-wide factors in stock price.

The above explanations are drawn from the agency literature (both empirical and analytical) which can be classified into two distinctive parts. One part ignores prices, or at least ignores the process by which the market prices are formed, and the other explicitly considers prices and models the process by which prices are formed.

1.1 Empirical Evidence

Broadly speaking, the empirical evidence can be classified into three categories: evidence of relative use of accounting earnings-based and stock price-based managerial incentives; the cost imposed by using the accounting earnings-based incentives; and the benefit generated by using the accounting earnings-based incentives.

Faithfully following the implications of analytical results, Lambert and Larcker (1987) investigate the relative use of the accounting earnings-based and the stock price-based incentives. Their evidence suggests that firms place relatively more weight on market performance (and less weight on accounting performance) in compensation contracts for situations in which

1. the variance of the accounting measure of performance is high relative to the market measure of performance,

2. the firm's experiencing high growth rates in assets and sales, or
3. the value of the manager's ownership of his firm's stock is low.

Focusing on firms' annual bonus plans which consist of a ceiling and a floor, Healy (1985) and Holthausen, Larcker and Sloan (1995) investigate the cost imposed by using the accounting earnings-based incentives. They find evidence in support of the hypothesis that the piece-wise incentive compensation plan provides undesirable managerial incentives. On the one hand, managers use *earnings increasing accounting methods* to report earnings if the earnings fall between the ceiling and floor, and on the other hand, managers use *earnings decreasing accounting methods* if earnings are higher than the ceiling of the bonus plan. If earnings are higher than the ceiling, management has incentive to manipulate earnings downwards because the management gets nothing for the excess amount of earnings reported. In sum, earnings-based annual bonus plans induce managers to manipulate *accounting methods* in ways harmful to the shareholders of the firm. Furthermore, Dechow and Sloan (1991) find that CEOs in their final years in office manage discretionary investment expenditures to improve short-term earnings performance (the horizon problem). As a result, some economists and compensation consultants (see Jensen 1989, Stewart 1989, 1991, and Rappaport 1990) recommend elimination of accounting earnings-based performance measure in management incentive compensation contracts.

To examine the benefit side of using the accounting earnings-based incentives, Sloan (1993) investigates the incremental information conveyed by accounting earnings in top executive compensation contracts. He provides evidence in support of the hypothesis that earnings-based incentives help shield executives from market-wide factors in stock prices.

This creates a major question for both practitioners and academic researchers: Are earnings-based incentives costly or beneficial? Although more empirical investigations are
warranted, analytical examinations of those issues are needed to provide better guidance for empirical studies.

1.2 Prior Analytical Research

Holmstrom (1979) is one of the pioneers in the area of contracting research. He characterizes an optimal contract with a single performance measure and provides an informativeness condition for identifying whether a second measure is valuable. He shows that if one signal is a sufficient statistic for a second signal with respect to the manager’s action, then the second signal receives zero weight in the optimal incentive contract. This result is useful in deciding whether to include an information signal in an incentive contract, but it is silent as to how much weight should be placed on the signals that are used.

Banker and Datar (1989) (hereafter BD), in a single-task agency setting, identify necessary and sufficient conditions under which linear aggregation of information signals is optimal for evaluating managerial performance. They show that the relative weight on the individual signals in the optimal linear aggregation is equal to the ratio of a measure of sensitivity times the precision of those measures.

Bushman and Indjejikian (1993) (hereafter BI) extend BD’s analysis in two dimensions. First, they combine the incentive issues in the agency literature with the information content of stock price in the capital market literature (also see Kim and Suh, 1993). They examine how the “information content” of accounting earnings affects the characteristics of the compensation contract, and explain that the inclusion of the accounting earnings as a performance measure helps to remove some of the “non-outcome-related” noise contained in price. The “information content” of the stock price, \textit{per se}, is not the focus of their study. Consequently, they adopt a noisy rational expectations model in which the allocation of pretrading information is exogenously specified. Second, they
model the manager’s action as multi-dimensional and examine how the noise in the stock price affects the agent’s allocation of his effort levels across tasks when both the price and earnings are used as performance measures. Feltham and Xie (1994) (hereafter FX) formally examine the congruity and diversity of performance measures in multi-task agency relations. They show that the incentive contract based on a noncongruent measure induces suboptimal effort allocation across tasks, whereas performance measure noise results in suboptimal effort intensity. They also show that while the market price is an efficient aggregation of investors’ information relative to assessing the value of the firm, the market price need not be an efficient aggregation of the contractible information.

1.3 Prior Studies’ Limitations

Prior agency research (both empirical and analytical) has either ignored private information acquisition by investors or treated that information as exogenously determined. However, the information acquired by investors depends on its anticipated costs and benefits, and the benefits are influenced by the existence of other sources of information, such as earnings reports. In financial markets, it is common to observe that traders expend valuable resources to acquire information from active markets in advisory services (e.g., full versus discount service brokers, consultant firms) and other forms of information (e.g., investment newsletters). This endogenous private information acquisition process may fundamentally change the information content of the market price in the competitive financial market.

Moreover, managers often have more than one task to perform. Their tasks include the management of current assets in place, strategic planning, market-share-building, identification of growth opportunities, and investments in the R&D of new products or technologies, etc. These tasks are different from each other in nature, and hence can only
be adequately measured by different performance criteria. This creates the need for multiple performance measures. Prior studies, like BI and FX, model the multi-task agency relation and recognize the diversity effect on the characterization of the optimal incentive linear contract. BI assume that all information signals are purely noisy representations of the underlying risk asset and consider a special case in which the accounting earnings numbers reflect only one of the managerial tasks. Even in the special case, their results are too complicated for them to identify the effect of the performance measures’ diversity on the optimal incentive contract. FX discuss the diversity of performance measures in a multi-task agency setting, but do not utilize this factor in characterizing the optimal incentive compensation contract.

Furthermore, the impact of the scale (i.e., linear transformation) of an information signal has not been adequately examined in the incentive environment. In particular, analytical research has not identified the effects of an information signal’s scale on managerial incentive compensation contracts. As a result, the following comment is commonly applicable to current research. “One difference between the specification of the theory and the empirics is that the theoretical results are derived in terms of the rescaled (emphasize added) accounting numbers, while the compensation regressions are conducted using the unscaled (emphasize added) accounting numbers,” Lambert (1993, p.120) comments. In particular, “If the scale factor is also removed from the noise terms, the variance of the noise in the now unscaled accounting numbers might very well be lower than that of the noise in stock return. This would reverse (emphasize added) the prediction that the relative weight on earnings declines as the correlation between the noise terms increases.” (See p.121).
1.4 Our Contributions to the Literature

Our study combines a multi-task agency model with a noisy rational expectations model similar to Grossman and Stiglitz (1980) (hereafter GS), in which there is a costly private information signal. Rational investors decide *endogenously* whether to acquire a private signal at a fixed cost, as opposed to BI’s assumption that the investors are *exogenously* endowed with free personal information signals. It is shown that the precision of the derived rational market price increases in the fraction of informed investors and decreases in the uncertainty of the risky asset dividend. In addition, the investors’ private information acquisition decisions are affected by the uncertainty of the risky asset dividend. Thus, there exists an indirect effect of the uncertainty of the risky asset dividend on the precision of the price if the investors *endogenously* decide whether to acquire the private information. Lambert (1993, pp.109-111) points out that the basic results obtained with a rational expectations model in which information acquisition decision is exogenous could be obtained with a simple model in which the investors receive non-contractible public information. However, the rational expectations model plays a crucial role when investors make endogenous information choices.

We make five contributions to the existing agency and capital market literature. First, we introduce a general information structure into the noisy rational expectations model to examine the impact of the investors’ endogenous private information acquisition on the equilibrium price. The existing papers often assume a particular representation of the information and, hence, the inferences drawn from their analyses only apply to the assumed structure. The more general formulation considered here permits identification of the key factors affecting various relations. Linear transformations (i.e., scaling) of information signals have no impact on their information content, even though such transformations affect the means and variances of the signals. The impact of investors’
private information and the public report on the market price depends entirely on the correlations among the private signal, the public report, and the terminal value of the firm. The scale of the market price depends on the unit of value employed in measuring the firm's terminal dividend. However, the scale used in the accounting report is arbitrary in this model. While the scale does not influence the informativeness of the information signal with respect to the manager's actions, it does influence the relative weights assigned to the accounting earnings and the stock price in the optimal linear incentive contract.

Second, we extend BD's single-task analysis to a multi-task setting with optimal linear incentive contracts. It is shown that BD's result that the relative weight on the individual signals equals the ratio of a measure of sensitivity times the precision of those performance measures still holds with proper extension of the measure of sensitivity. The extended measure of sensitivity incorporates FX's measure of congruity and the first-best allocation of effort. It is shown that the diversity of performance measures is a very important factor influencing our measure of sensitivity in a multi-task agency setting. In the single-task setting, since performance measures are either congruent with the firm's gross payoff or action-irrelevant, they are never diverse with each other. Contracts based on two diverse performance measures can induce the first-best allocation of effort, but contracts based on two non-diverse performance measures cannot induce the first-best allocation of effort unless at least one of them is congruent with the firm's gross payoff.

Third, we re-examine the relative use of the stock price and accounting earnings in providing managerial incentives. Since the price is traditionally assumed to reflect the total firm value (both current and future), while earnings are typically assumed to provide information about a firm's current operating performance, they are diverse information signals. If both are employed as performance measures, they can be used to induce the agent to exert any allocation of effort. In other words, the implementable second-best
effort set is the entire effort space. This gives the shareholders flexibility to tradeoff the agent's effort intensity with effort allocation across tasks. This flexibility reduces the risk premium paid to the agent without weakening the agent's incentives. Therefore, the inclusion of accounting earnings as a performance measure facilitates incentive alignment between the manager and the shareholders.

Fourth, we examine the impact of investors' endogenous information acquisition on the managerial incentive contracts. Our analysis implies that the endogenous private information acquisition reduces the impact of an increase in the risky asset's volatility on the use of accounting-based incentives relative to market-based incentives. If information of those firms which are heavily followed by financial analysts is endogenously acquired by investors, while the information about those firms which are not followed by financial analysts is exogenously made available to investors due to disclosure regulations, we hypothesize that the former firms place relatively more weight on market performance (and less weight on accounting performance) in their management compensation contracts than that of the latter firms.

Finally, we compare a stock ownership incentive compensation contract with a stock option incentive compensation contract. It is shown that, for any given level of implementable effort, an ownership contract always generates higher net surplus for the shareholders than does an option contract. This is because the use of options in the incentive contract increases the cost of risk imposed on the managers. The intuition is that the inducement of a given level of effort requires many more options than shares. The increase in the number of options is sufficient to result in higher risk with options than with shares. Hence, the shareholders have to pay a higher risk premium to the manager and obtain a lower net surplus with options contracts than with ownership contracts.
1.5 Organization of the Dissertation

The rest of the dissertation is organized as follows. Chapter 2 derives the equilibrium price of the firm in a noisy rational expectations framework. Chapter 3 develops optimal linear incentive compensation contract in a multi-task agency setting, and focuses on the impact of diversity of performance measures on the characterization of the optimal incentive contract. Chapter 4 links the analyses of Chapters 2 and 3 and examines how the investors' endogenous private information acquisition influences the optimal incentive contract. Chapter 5 compares a stock ownership incentive compensation contract and an option incentive compensation contract.
Chapter 2

Noisy Rational Expectations Equilibrium Price

In this chapter, we consider a one-period, general asymmetric information structure, single risky asset (the firm) model similar to Grossman and Stiglitz (1980) (hereafter GS), and establish the rational market price in the presence of a public report about the firm. The general formulation considered here permits identification of the key factors affecting various relations.

The public report is an important aspect of our general formulation of the information structure. In accounting literature, extensive research has been done to examine the impact of financial reports (e.g., annual reports, financial statements, etc.) on the prices of firms. The existence of such a source of publicly available information influences the investors’ decision whether to acquire private information. In particular, if the public report is a sufficient statistic for inferring the firm’s terminal value, no one would choose to purchase the costly private information. Like GS, the major thrust of our model is the investors’ endogenous acquisition process of private information. It is shown that more investors decide endogenously to acquire the costly private information as the firm’s inherent uncertainty increases.

The other important aspect of our general formulation of the information structure is that neither the public report nor the private information has to be a noisy representation of the underlying risky asset. The GS model is a special case of ours, in which the public report is uninformative and the noise term in the private signal equals the noise of the risky asset plus white noise (i.e., the private information is a noisy representation of the
risk asset). Because of this aspect, we can show that a linear transformation of any information signal (i.e., either the public report or the private information or both) does not affect the signal’s information content impounded in the equilibrium price. In other words, if we change the scale of an information signal by dividing by a positive constant, this scale affects neither the mean nor the variance of the equilibrium price although both the mean and standard deviation of the signal are changed. The equilibrium price is completely characterized by the correlation relations of the information structure.

2.1 Model Elements

2.1.1 Pure exchange economy

Consider a pure exchange economy which consists of three types of investors who own or trade a riskless bond and a single risky asset — the firm. The riskless bond is the numeraire, and its per-unit payoff is one unit of the consumption good at the termination date (see the time line in Figure 2.1). The risky asset’s total random dividend (the terminal value of the firm) is denoted as:

$$
\bar{X} = \bar{X} + \tilde{\epsilon}_x
$$

where $\bar{X}$ is the mean and $\tilde{\epsilon}_x$ is a normally distributed random variable with mean zero and variance $\sigma_x^2 > 0$, i.e., $\tilde{\epsilon}_x \sim N(0, \sigma_x^2)$.

The first type of trader consists of the initial shareholders of the firm (e.g., large institutional investors), who are assumed to take a "buy and hold" position. They are assumed to have a controlling interest in the firm. Hence, they make and approve all

2.1 For notational simplicity, we assume that this type of trader initially holds all the equity of the firm. Nothing of substance would change if this type of trader held some fraction of the firm’s equity and liquidity traders were endowed with the remainder. In later chapters, we refer to this type of investor as the principal of the agency relation, who is interested in the firm’s end-of-period value instead of the current market value. This is a special case of the objective function considered by Miller and Rock (1985), who consider a weighted average of the current and terminal values of the firm.
important investment and financial decisions for the firm, e.g., hire or fire managers, set managerial compensation contracts, etc.

The second type of trader consists of liquidity investors who randomly change their holdings for reasons beyond our model (e.g., to re-balance their portfolios, or to obtain cash from the market). As a result, these changes create a random supply of the risky asset. Let $\tilde{Z}$ represent the fraction of the risky asset not held by the first two types of traders.

The third type of trader consists of rational investors who engage in information acquisition and act on all available information. The equilibrium price is such that they absorb all shares not held by the first two types of traders. There are $N$ homogeneous rational investors.\footnote{Although the differences in belief, risk attitude, and endowment among this type of trader would create trades among themselves, their aggregate demand for the risky asset would remain unchanged. Since we are only interested in their collective trading with those outside their type, this homogeneous trader assumption is not restrictive for our modelling purposes.} Each is risk averse and has a negative exponential utility function with risk aversion parameter $\nu$. 

\begin{figure}[h]
\centering
\begin{tabular}{lll}
Endowment Date & Trading Date & Termination Date \\
all three types of traders endowed. & private information & risky dividend \\
& $\tilde{\psi}$ acquired; & $\tilde{X}$ realized; \\
& public report & claims paid; \\
& $\tilde{Y}$ released; & terminal wealth \\
& trading occurs & determined. \\
& at equilibrium price. & \\
\end{tabular}
\caption{The sequence of events in the pure exchange economy}
\end{figure}
We express the fraction of the firm \( Z \) not held by the first two types of investors in per capita terms (i.e., \( \bar{z} = \frac{Z}{N} \)) with respect to the third type of investors, and assume that \( \bar{z} \) is a normally distributed random variable with mean zero and variance \( \sigma_z^2 > 0 \), i.e., \( \bar{z} \sim N(0, \sigma_z^2) \). For simplicity, we assume it is independent of any other random variable in the economy.

### 2.1.2 Public report and private information

There is a public report \( Y \) available to everyone in the economy:

\[
\tilde{Y} = Y + \bar{\varepsilon}_y,
\]

where \( \bar{\varepsilon}_y \) is the mean and \( \bar{\varepsilon}_y \) is a normally distributed random variable with mean zero and variance \( \sigma_y^2 > 0 \), i.e., \( \bar{\varepsilon}_y \sim N(0, \sigma_y^2) \). The correlation between \( \bar{\varepsilon}_y \) and \( \bar{z} \) is denoted as \( \rho_{xy} \). An example of such a public report is the firm’s accounting earnings announcement.

It is assumed that there is a single source of private information, and access to that source involves a positive cost \( \kappa > 0 \), e.g., a financial analyst sells a financial forecast for the firm to anyone who pays the price \( \kappa \). Furthermore, this private signal is assumed to take the following form:

\[
\tilde{\psi} = \bar{\psi} + \bar{\varepsilon}_0,
\]

where \( \bar{\psi} \) is the mean and \( \bar{\varepsilon}_0 \) is a normally distributed random variable with mean zero and variance \( \sigma_0^2 > 0 \), i.e., \( \bar{\varepsilon}_0 \sim N(0, \sigma_0^2) \). This private information signal is correlated with the risky asset dividend and the public report; the correlations are denoted as \( \rho_{0x} \) and \( \rho_{0y} \), respectively.

### 2.1.3 Information structure

The three information signals, the risky asset dividend \( \tilde{X} \), the public report \( \tilde{Y} \), and the private information \( \tilde{\psi} \), are assumed to be jointly normally distributed. The relationship
among them establishes the information structure for the economy. In the following, we provide two ways to represent this information structure, and discuss some special structures.

Covariances and correlations

The covariance matrix describing the relation among $\tilde{X}$, $\tilde{Y}$, and $\tilde{\psi}$ is given by

$$\Omega = \begin{bmatrix}
\sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_z^2
\end{bmatrix} \begin{bmatrix}
\sigma_x^2 & \sigma_x \sigma_y \rho_{xy} & \sigma_x \sigma_0 \rho_{xz} \\
\sigma_y \sigma_{xy} & \sigma_y^2 & \sigma_y \sigma_0 \rho_{yz} \\
\sigma_z \sigma_{zx} & \sigma_0 \sigma_y \rho_{zy} & \sigma_0^2
\end{bmatrix} \cdot \quad (2.4a)$$

Since the means $\bar{X}$, $\bar{Y}$, and $\bar{\psi}$, are washed out in the information structure, they are immaterial in this chapter (see Proposition 2.9). As a result, in this chapter, we interchangeably use the payoff information ($\bar{X}$) and the payoff uncertainty ($\tilde{\varepsilon}_x$), the public report information ($\bar{Y}$) and the public report noise ($\tilde{\varepsilon}_y$), the private signal ($\bar{\psi}$) and the private signal noise ($\tilde{\varepsilon}_0$), respectively. Since the mean of the risky asset supply is assumed zero, we can interchangeably use the risky asset supply information $\tilde{z}$ and the risky asset supply shock $\tilde{\varepsilon}_z$, for the same reason mentioned above. However, when examining managerial incentive issues in later chapters like Chapters 3 and 4, we can no longer use these terminologies interchangeably, because the means of the performance measures directly reflect managerial effort.
Denote the determinants of matrix $\Omega$'s sub-matrices as follows:

\[
\begin{align*}
& s_1 \equiv \begin{vmatrix}
\sigma_{0x} & \sigma_{xy} \\
\sigma_{0y} & \sigma_y^2
\end{vmatrix} = \sigma_{0x} \sigma_y^2 - \sigma_{xy} \sigma_{0y} = \sigma_0^2 \sigma_y \rho_1, \\
& \quad \rho_1 \equiv \rho_{0x} - \rho_{xy} \rho_{0y}; \\
& s_2 \equiv \begin{vmatrix}
\sigma_y^2 & \sigma_{0y} \\
\sigma_{0y} & \sigma_0^2
\end{vmatrix} = \sigma_y^2 \rho_1^2 - \sigma_{0y}^2 = \sigma_0^2 \sigma_y^2 \rho_2, \\
& \quad \rho_2 \equiv 1 - \rho_{0y}^2 > 0; \\
& s_3 \equiv \begin{vmatrix}
\sigma_x^2 & \sigma_{xy} \\
\sigma_{xy} & \sigma_y^2
\end{vmatrix} = \sigma_x^2 \sigma_y^2 - \sigma_{xy}^2 = \sigma_x^2 \sigma_y^2 \rho_3, \\
& \quad \rho_3 \equiv 1 - \rho_{xy}^2 > 0; \\
& s_4 \equiv \begin{vmatrix}
\sigma_x^2 & \sigma_{0x} \\
\sigma_{xy} & \sigma_{0y}
\end{vmatrix} = \sigma_x^2 \sigma_{0y} - \sigma_{0x} \sigma_{xy} = \sigma_x^2 \sigma_{0y} \rho_4, \\
& \quad \rho_4 \equiv \rho_{0y} - \rho_{0x} \rho_{xy}; \\
& s_5 \equiv \begin{vmatrix}
\sigma_x^2 & \sigma_{0x} \\
\sigma_{0x} & \sigma_0^2
\end{vmatrix} = \sigma_x^2 \sigma_0^2 - \sigma_{0x}^2 = \sigma_x^2 \sigma_0^2 \rho_5, \\
& \quad \rho_5 \equiv 1 - \rho_{0x}^2 > 0; \\
& s_6 \equiv \begin{vmatrix}
\sigma_{xy} & \sigma_{0x} \\
\sigma_{0y} & \sigma_0^2
\end{vmatrix} = \sigma_{xy} \sigma_0^2 - \sigma_{0x} \sigma_{0y} = \sigma_{xy} \sigma_0^2 \rho_6, \\
& \quad \rho_6 \equiv \rho_{xy} - \rho_{0x} \rho_{0y}; \\
& s_0 \equiv |\Omega| = \sigma_0^2 \sigma_x^2 \sigma_y^2 \rho_0, \\
& \quad \rho_0 \equiv 1 - \rho_{0x}^2 - \rho_{0y}^2 - \rho_{xy}^2 + 2\rho_{0x} \rho_{0y} \rho_{xy} > 0.
\end{align*}
\]

Observe that the positiveness of $\rho_2$, $\rho_3$, $\rho_5$, and $\rho_6$ are ensured by the positive definiteness of the correlation matrix. However, the other three relations, $\rho_1$, $\rho_4$, and $\rho_6$, do not have definite sign.

Now, let us examine $\rho_1 = \rho_{0x} - \rho_{xy} \rho_{0y}$ as an example.\(^{24}\) It is clear that $\rho_1$ describes a three way relation among the three information signals $\tilde{X}$, $\tilde{Y}$ and $\tilde{Y}$. $\rho_{0x}$ describes the pair-wise correlation between the private signal and the risky asset dividend, while $\rho_{xy} \rho_{0y}$ is the indirect correlation between the private signal and the risky asset dividend conveyed through the public report. The pair-wise correlation can be interpreted as the total correlation between two information signals; the indirect correlation represents the component in the total correlation due to the presence of a third information signal (i.e., the

\(^{24}\) The discussion about $\rho_4$ and $\rho_6$ are analogous. That is, $\rho_6$ measures the direct/net correlation between the public report and the risky asset dividend; $\rho_4$ measures the direct/net correlation between the public report and the private signal.
public report). Consequently, $\rho_1$ can be interpreted as the direct/net correlation between the private signal and the risky asset dividend (i.e., the difference between the total and indirect correlations). Loosely speaking, $\rho_1$ is attributable to events that influence both the risky asset dividend and the private information, but do not influence the public report.

For example, if $\tilde{Y}$ is independent of either $\tilde{\psi}$ or $\tilde{X}$, it is clear that the indirect correlation between $\tilde{\psi}$ and $\tilde{X}$ conveyed through $\tilde{Y}$ is zero. Hence, the direct/net correlation is simply the total correlation between $\tilde{\psi}$ and $\tilde{X}$, i.e., $\rho_1 = \rho_{2z}$. In other words, the total correlation between two information signals is completely due to the direct correlation between the two if a third information signal is independent of either of them.

Thus, the following definition follows intuitively.

**DEFINITION 1** We refer to $\rho_l$ ($l = 1, 4, 6$) as the direct correlation between two information signals.

Observe that $\rho_1$ and $\rho_6$ represent the direct correlations of the private information with the risky asset and the public report with the risky asset, respectively, while $\rho_4$ represents the direct correlation of the private information with the public report. However, we focus our following discussion on correlations $\rho_1$ and $\rho_6$, because investors are interested in the correlation of an information signal with the firm's terminal value. Investors are not interested in the correlation between the public report and the private information signal unless the correlation provides useful information regarding the underlying risky asset.$^{25}$

In Parts I and IV, the indirect correlation $\rho_{ij}\rho_{jz}$ between the information signal $i$ and the risky asset dividend through the information signal $j$ does not reverse the sign of the total correlation $\rho_{iz}$, so that the direct/net correlation has the same sign as the total correlation. Parts I and IV occur if the indirect correlation $\rho_{ij}\rho_{jz}$ is either non-significant

---

$^{25}$In fact, our later analysis shows that $\rho_1$ and $\rho_6$ are more often used than $\rho_4$. 
Table 2.1: The information structure of the economy

<table>
<thead>
<tr>
<th>Direct Correlation (\forall j = 1, 6)</th>
<th>Total Correlation (\forall i = 0, y)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \rho_{ix} &gt; 0 )</td>
<td>I</td>
<td>III</td>
</tr>
<tr>
<td>( \rho_j &gt; 0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho_j &lt; 0 )</td>
<td>II</td>
<td></td>
<td>IV</td>
</tr>
</tbody>
</table>

Part I: Both direct and total correlations are positive

Part II: Direct correlation is positive and total correlation is negative

Part III: Direct correlation is negative and total correlation is positive

Part IV: Both direct and total correlations are negative

or of the opposite sign to the total correlation.

In parts II and III, on the other hand, the indirect correlation \( \rho_{ij} \rho_{jz} \) between the information signal \( i \) and the risky asset through the information signal \( j \) does reverse the sign of the total correlation \( \rho_{iz} \), so that the direct/net correlation has the opposite sign as the total correlation. Parts II and III occur if the indirect correlation \( \rho_{ij} \rho_{jz} \) is of the same sign as the total correlation and significant as well.
Sufficient statistics with respect to the firm value

In the preceding subsection, we introduced two representations of the information structure: covariance and correlation. In particular, we examined the direct/net correlation between two information signals. As a result, new concepts of direct, indirect, and total correlations were formally defined, and positive/negative direct/net correlations were discussed.

Now, let us examine two special information structures in which the private signal and the public report are not directly correlated with the risky asset’s payoff information, i.e., the total correlations are entirely due to the indirect correlations: $\rho_1 = 0$ and $\rho_6 = 0$. In such settings, the total correlation between an information signal and the risky asset can be equivalently conveyed through the other information signal, implying that there exists a redundant information signal in the economy.

**LEMMA 2.1** The public report (private information) is a sufficient statistic for both the public report and the private information with respect to the risky asset dividend if, and only if, $\rho_1 = 0$ ($\rho_6 = 0$), i.e.,

$$\rho_{0x} = \rho_{xy}\rho_{0y} \quad (\rho_{xy} = \rho_{0x}\rho_{0y}).$$

(2.5)

They cannot be sufficient statistics simultaneously.

**Proof:** See Appendix A.1. Q.E.D.

2.2 Informed and Uninformed Investors’ Demands for the Risky Asset

We say a rational investor is informed (uninformed) if he does (not) purchase the private information signal at cost $\kappa$. An informed investor conditions his beliefs about $\tilde{X}$ on both the public report and the private signal, and allocates his initial wealth $W_0$ at time $t = 0$
between the riskless asset and the risky asset so that his net holding of the risky asset is \( D_i \). An uninformed investor’s beliefs about \( \bar{X} \) is a function of the public report and the price from which he makes rational inferences about the private information acquired by the informed investors. He allocates his initial wealth \( W_0 \) at time \( t = 0 \) between the riskless asset and the risky asset so that his net holding of the risky asset is \( D_u \).

The market-clearing price of the risky asset is denoted as \( \tilde{P} \). At the termination date \( t = 2 \), the terminal wealth of the informed and the uninformed investors are, respectively,

\[
\begin{align*}
\tilde{W}_{2i} &= W_0 + D_i \left( \bar{X} - \tilde{P} \right) - \kappa; \\
\tilde{W}_{2u} &= W_0 + D_u \left( \bar{X} - \tilde{P} \right).
\end{align*}
\]

Thus, the informed and uninformed investors’ investment decisions are, respectively,

\[
\begin{align*}
\max_{D_i} E_i \left[ -e^{-\nu\tilde{W}_{2i}} \left| \tilde{Y}, \bar{X} + \varphi \right. \right] & \text{ Subject to (2.6a), and} \\
\max_{D_u} E_u \left[ -e^{-\nu\tilde{W}_{2u}} \left| \tilde{Y}, \bar{X} + \bar{P} \right. \right] & \text{ Subject to (2.6b)}.
\end{align*}
\]

Assume that the market value of the risky asset is a linear function of the private information signal \( \bar{y} \), the public report \( \bar{Y} \), and the non-observable random supply \( \bar{z} \):\(^{26}\)

\[
\tilde{P} = a_0 + a_1 \bar{y} + a_2 \bar{Y} - a_z \bar{z}, \quad \text{with } a_z \text{ nonzero.} \tag{2.7}
\]

Given the linearity of the equilibrium pricing relation (2.7), the investors make self-fulfilling conjectures about the fraction of informed investors (\( \lambda \)). Then, both the informed and uninformed investors decide their optimal demands for the risky asset:

\[
D_i = \frac{E_i \left[ \bar{X} \left| \bar{Y}, \bar{y}, \varphi \right. \right] - \tilde{P}}{\nu \text{Var} \left( \bar{X} \left| \bar{Y}, \varphi \right. \right)} \quad \text{and} \quad D_u = \frac{E_u \left[ \bar{X} \left| \bar{Y}, \bar{P} \right. \right] - \tilde{P}}{\nu \text{Var} \left( \bar{X} \left| \bar{Y}, \bar{P} \right. \right)}.
\]

\(^{26}\)Any solution with \( a_z = 0 \) will be regarded as an infeasible economic solution to our problem.
Based on the covariance matrix describing the relation between $\tilde{X}$, $\tilde{Y}$, $\tilde{\psi}$, and $\tilde{P}$,

$$
\begin{bmatrix}
\sigma^2_x & \sigma_{xy} & \sigma_{0x} & a_1\sigma_{0x} + a_2\sigma_{xy} \\
\sigma_{xy} & \sigma^2_y & \sigma_{0y} & a_1\sigma_{0y} + a_2\sigma^2_y \\
\sigma_{0x} & \sigma_{0y} & \sigma^2_0 & a_1\sigma^2_0 + a_2\sigma_{0y} \\
\sigma_{0z} + a_2\sigma_{xy} & a_1\sigma_{0y} + a_2\sigma^2_y & a_1\sigma^2_0 + a_2\sigma_{0y} & a_1^2\sigma^2_0 + a_2^2\sigma^2_y + 2a_1a_2\sigma_{0y} + a_2^2\sigma^2_z
\end{bmatrix},
$$

the informed investors form their beliefs based on the public report and the purchased private signal, and the uninformed investors form their beliefs based on the public report and inferences made from the equilibrium price.

The informed investors' conditional mean for the risky asset can be written as

$$
E_i \left[ \tilde{X} \mid \tilde{Y}, \tilde{\psi} \right] = B_0 + B_1\tilde{Y} + B_2\tilde{\psi},
$$

(2.9a)

where $B_0 \equiv \tilde{X} - B_1\tilde{Y} - B_2\tilde{\psi}$, and

$$
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix} \equiv \begin{pmatrix}
\sigma^2_y & \sigma_{0y} \\
\sigma_{0y} & \sigma^2_0
\end{pmatrix}^{-1} \begin{pmatrix}
\sigma_{xy} \\
\sigma_{0x}
\end{pmatrix} = \begin{pmatrix}
\frac{\sigma_{xy}}{\sigma_y\rho_2} & \frac{\sigma_{0y}}{\sigma_0\rho_2} \\
\frac{\sigma_{0x}}{\sigma_0\rho_2} & \frac{\sigma_{xy}}{\sigma_y\rho_2}
\end{pmatrix}.
$$

(2.9b)

The conditional variance can be written as

$$
V_i \equiv Var \left( \tilde{X} \mid \tilde{Y}, \tilde{\psi} \right) = \sigma^2_x - B_1\sigma_{xy} - B_2\sigma_{0x} = \sigma^2_\rho_0/\rho_2.
$$

(2.9c)

The uninformed investors' conditional mean for the risky asset can be written as

$$
E_u \left[ \tilde{X} \mid \tilde{Y}, \tilde{P} \right] = b_0 + b_1\tilde{Y} + b_2\tilde{P},
$$

(2.10a)

where $b_0 \equiv \tilde{X} - b_1\tilde{Y} - b_2\tilde{P}$, and

$$
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix} = \begin{pmatrix}
\sigma^2_y & a_1\sigma_{0y} + a_2\sigma^2_y \\
a_1\sigma_{0y} + a_2\sigma^2_y & a_1^2\sigma^2_0 + a_2^2\sigma^2_y + 2a_1a_2\sigma_{0y} + a_2^2\sigma^2_z
\end{pmatrix}^{-1} \begin{pmatrix}
\sigma_{xy} \\
a_1\sigma_{0x} + a_2\sigma_{xy}
\end{pmatrix}
= \begin{pmatrix}
\frac{a_1^2\sigma^2_{xy} - a_1^2\sigma_{0y}\sigma_{xy} + a_2^2\sigma^2_y + 2a_1a_2\sigma_{0y} + a_2^2\sigma^2_z}{\sigma_y(a_1^2\sigma^2_0 + a_2^2\sigma^2_y)} \\
\frac{a_1^2\sigma_{0x}\sigma_{xy} + a_2^2\sigma^2_z}{a_1^2\sigma^2_0 + a_2^2\sigma^2_z}
\end{pmatrix}.
$$

(2.10b)
Their conditional variance can be written as:

\[ V_u = \text{Var} \left( \bar{X} \mid \bar{Y}, \bar{P} \right) = \sigma_x^2 - b_1\sigma_{xy} - b_2 \left( a_1\sigma_{ux} + a_2\sigma_{uy} \right) = \frac{\sigma_x^2 \left( \frac{\rho_0}{\rho_2} + \frac{\rho_2}{\rho_3} \right)}{\rho_1^2 \sigma_x^2 + \sigma_y^2} \]  

(2.10c)

With these conditional means, variances, and demand functions (2.8) for the risky asset, we directly apply the Proposition 3.1 of Admati and Pfleiderer (1987) to this single risky asset case and obtain the following lemma.

**Lemma 2.2** The ex-ante expected utilities of the informed and the uninformed investors are, respectively,

\[
E \left[ U^i \right] = -\exp \left\{ -\nu W_0 - \frac{\nu^2 (\bar{X} - \bar{P})^2}{2 \text{Var}(\bar{X} - \bar{P})} \right\} \sqrt{\frac{\text{Var}(\bar{X} \mid \bar{Y}, \bar{P})}{\text{Var}(\bar{X} - \bar{P})}},
\]

\[
E \left[ U^u \right] = -\exp \left\{ -\nu W_0 - \frac{\nu^2 (\bar{X} - \bar{P})^2}{2 \text{Var}(\bar{X} - \bar{P})} \right\} \sqrt{\frac{\text{Var}(\bar{X} \mid \bar{Y}, \bar{P})}{\text{Var}(\bar{X} - \bar{P})}}.
\]

The rational investors maximize their ex-ante expected utilities when they decide whether to purchase the private information. The above expected utilities will be used later in Section 2.5 to establish the equilibrium fraction of informed investors. However, in the next two sections, we exogenously fix the fraction of informed investors, and derive the market price and its relationship with the fraction of informed investors.

### 2.3 Equilibrium Prices with Exogenous Information Acquisition

Assume that there are a fraction \( \lambda \) of the rational investors are informed. Then, the market clearing condition equilibrating the aggregate demand of the informed and the uninformed investors and the supply is:

\[ \lambda D_i + (1 - \lambda) D_u = \bar{z}. \]

After substituting demand function (2.8) into the above equation and using (2.9a)
and (2.10a), we obtain the following equation by multiplying $\nu V_i$ and collecting terms,

$$\begin{align*}
B \bar{P} &= \left[ \lambda B_0 + (1 - \lambda) (V_i/V_u) b_0 \right] + \lambda B_2 \bar{\psi} \\
&+ \left[ \lambda B_1 + (1 - \lambda) (V_i/V_u) b_1 \right] \bar{Y} - \left[ \nu V_i \right] \bar{z},
\end{align*}$$

(2.11)

where $B \equiv \lambda + (1 - \lambda) (V_i/V_u) (1 - b_2)$. Substituting (2.7) into the left-hand-side of the above equation, we collect the corresponding coefficients so that:

$$\begin{align*}
B a_0 &= \lambda B_0 + (1 - \lambda) (V_i/V_u) b_0; \\
B a_1 &= \lambda B_2; \\
B a_2 &= \lambda B_1 + (1 - \lambda) (V_i/V_u) b_1; \\
B a_z &= \nu V_i.
\end{align*}$$

(2.12a-d)

Since $a_z \neq 0$, dividing (2.12d) by $a_z$ yields that $B = \frac{\nu V_i}{a_z} = \frac{\nu \rho_2}{a_z \rho_2}$ [see (2.9c)], which is not equal to zero because neither $\nu$ nor $\rho_0$ is zero. Thus, by (2.12b) and (2.9b), it is clear that

$$\frac{a_1}{a_z} = \frac{\lambda \rho_1}{\nu \sigma_0 \sigma_x \rho_0}.$$  

(2.13)

Substituting the above equation into (2.10c), we have the following ratio

$$\frac{V_i}{V_u} = \frac{\lambda^2 \rho_1^2 \rho_2 + \nu^2 \sigma_x^2 \sigma_0^2 \rho_0^2}{\lambda^2 \rho_1^2 \rho_2 + \nu^2 \sigma_x^2 \sigma_0^2 \rho_0 \rho_2 \rho_3},$$

(2.14)

which implies the following lemma, since $\rho_2 \rho_3 = \rho_0 + \rho_1^2$.

**Lemma 2.3** For any exogenously given $\lambda \in [0, 1]$, an informed trader's conditional variance of the risky asset dividend is lower than that of an uninformed trader. These two conditional variances are equal if, and only if, $\rho_1 = 0$.

When the public report is a sufficient statistic for both the public report and the private signal, an uninformed investor ultimately knows exactly the same amount of
information as an informed investor does, in terms of inferring the firm's terminal value. Therefore, their conditional variances for the risky asset dividend are identical.

On the other hand, since identical conditional variances for the risky asset dividend imply that the conditional means are identical as well, the public report is a sufficient statistic for both the public report and the private signal (two parameter distribution).

In Lemma 2.1, we discussed two special information structures in which there exists a redundant information signal. Lemma 2.3 implies that the other redundant information structure ($\rho_6 = 0$) does not induce identical conditional variances of the risky asset dividend for the informed and uninformed. The reason is simple. If $\rho_6 = 0$, i.e., the private information is a sufficient statistic with respect to the firm's terminal value, then the informed investors no longer need the public report to form their beliefs about the risky asset dividend, but the uninformed investors still do. The market price impounds the private information, but does not fully reveal it because of the supply shock of the risky asset. The uninformed cannot completely infer the private information from the market price. Consequently, the uninformed and the informed have different beliefs about the risky asset dividend.

The above lemma describes a special case. In general, we have the following result.

**PROPOSITION 2.1** For any exogenously given $\lambda \in [0, 1]$, there exists a unique rational expectations equilibrium price within the class of functions of the form given by (2.7). This price has coefficients:

\[ a_0 = \bar{X} - a_1 \bar{Y} - a_2 \bar{Y}; \]  
\[ a_1 = \frac{\lambda \rho_1 \sigma_x (\lambda \rho_1^2 + \lambda^2 \sigma_1^2 \sigma_2^2 \rho_1 \rho_3)}{\sigma_0 (\lambda^2 \rho_1^2 \rho_2 + \lambda^2 \sigma_1^2 \sigma_2^2 \rho_0 \rho_1 \rho_2 + \nu^2 \sigma_2^2 \sigma_3^2 \rho_0^2)}, \]  
\[ a_2 = \frac{\sigma_x (\lambda^2 \rho_1^2 \rho_6 - \lambda^2 \rho_1^2 \rho_4 + \nu^2 \sigma_2^2 \sigma_3^2 \rho_0 \rho_3)}{\sigma_x (\lambda^2 \rho_1^2 \rho_4 + \nu^2 \sigma_2^2 \sigma_3^2 \rho_0 \rho_3)}, \]  
\[ a_2 = \frac{\nu \sigma_2^2 \sigma_0 (\lambda \rho_1^2 + \nu^2 \sigma_2^2 \sigma_0 \rho_0 \rho_3)}{\lambda^2 \rho_1^2 \rho_2 + \lambda^2 \sigma_2^2 \sigma_3^2 \rho_0 \rho_1 \rho_2 + \nu^2 \sigma_2^2 \sigma_3^2 \rho_0^2}. \]
To see the effect of the exogenous fraction of informed investors on equilibrium price, we give two special equilibrium prices. From Proposition 2.1, it is immediate:

**COROLLARY 2.2** If none are informed or all are informed, then

\[
\hat{P}(\lambda = 0) = \bar{X} + \frac{\sigma_{xY}}{\sigma_{Y}^2} \left( \bar{Y} - \bar{Y} \right) - \nu \sigma_{x}^2 \rho_3 \bar{z}, \\
\hat{P}(\lambda = 1) = \bar{X} + \frac{\sigma_{x\rho_1}}{\sigma_{\rho_2}} \left( \bar{\psi} - \bar{\psi} \right) + \frac{\sigma_{x\rho_6}}{\sigma_{Y}\rho_{2}} \left( \bar{Y} - \bar{Y} \right) - \frac{\nu \sigma_{x}^2 \rho_0}{\rho_{2}} \bar{z}.
\]

If no one is informed, the private information is unused in the economy and hence cannot affect the equilibrium price. It is dropped out of the equilibrium price completely. The public report is impounded in the equilibrium price and the regression coefficient between the two represents the impact of the public report on the equilibrium price.

On the other hand, if all investors are informed, the private information becomes public information. Now, both the "private" information and the public report are known by all. The coefficient of the private information (the public report) in the equilibrium price is zero if, and only if, \( \rho_1 = 0 \) (\( \rho_6 = 0 \)). Consequently, one of the two signals has no impact on the equilibrium price if, and only if, the other signal is a sufficient statistic for both of them.

### 2.4 Price Impact of Exogenous Information Acquisition

In the previous section, we characterized the equilibrium price of the firm under the assumption that the fraction of the informed investors is exogenously given. Corollary 2.2 gives the equilibrium prices under two extreme cases of information. When \( \lambda = 0 \), the private information signal is unused information, and hence the equilibrium price cannot impound it. As a result, there is no surprise that the equilibrium price depends only on
the public report $\tilde{Y}$, and $\sigma_{xy}/\sigma_y^2$ is the coefficient of regressing the risky asset dividend information onto the public report. When $\lambda = 1$, the private information signal is in fact public information. The equilibrium price now impounds both information signals, and the coefficients are adjusted for the correlation (i.e., $\rho_2$) between them. As a result, there is no surprise that the equilibrium price depends only on one of them if, and only if, the other is sufficient statistic with respect to inferring the risky asset dividend.

By Proposition 2.1 and (A.2) and (A.5), it follows that the signs of the price coefficients for the the private signal and the public report are unambiguously determined.

COROLLARY 2.3 For any given $\lambda > 0$, the price coefficient $a_1 > (=, <) 0$ if, and only if, $\rho_1 > (=, <) 0$. Furthermore, the price coefficient $a_2$ is positive (negative) if both $\rho_6$ and $\rho_{xy}$ are positive (negative). Assume that $\rho_6 = 0$, then $a_2 = 0$ if, and only if, either $\lambda = 1$ or $\rho_{xy} = 0$.

Before moving to the next section in which the equilibrium information acquisition is characterized, we examine how the investors' exogenous information acquisition decision affects the equilibrium price. This provides a benchmark for analyzing the equilibrium price effect of the investors' endogenous information acquisition.

PROPOSITION 2.4 For any exogenously given $\lambda \in [0, 1]$, the absolute value of the price coefficients $a_1$ and $a_2$ increases as the fraction of informed investors increases, i.e.,

$$
\frac{da_1}{d\lambda} = \begin{cases} 
> 0, & \text{if } \rho_1 > 0; \\
< 0, & \text{if } \rho_1 < 0.
\end{cases}
$$

and

$$
\frac{da_2}{d\lambda} = \begin{cases} 
> 0, & \text{if } \rho_6 > \rho_{xy}; \\
< 0, & \text{if } \rho_6 < \rho_{xy}.
\end{cases}
$$

Proof: See Appendix A.1.

Since Corollary 2.3 shows that $a_1 > (=) 0$ if, and only if, $\rho_1 > (=) 0$, this proposition suggests that the private signal has a more positive (negative) coefficient in the equilibrium price as the fraction of informed investors increases, if its direct/net correlation with
the risky asset is positive (negative). In other words, as more investors are informed, the private signal has a larger absolute coefficient in the equilibrium price. Its sign, however, depends on the sign of its direct correlation with the risky asset.

Notice that we refer to $\rho_6$ and $\rho_{xy}$ as the direct or net and total correlation between the public report and the risky asset, respectively. The condition $\rho_6 > \rho_{xy}$ implies that the "net" correlation between the public report and the risky asset is larger than the total correlation. This means the presence of the private signal strengthens the correlation between the pair of signals. In particular, we obtain from Corollary 2.3 that $a_2 > 0$ if, in addition, $\rho_{xy} > 0$; $a_2 < 0$ if, in addition, $\rho_6 < 0$. This proposition implies that the absolute value of the price coefficient $a_2$ increases as the fraction of informed investors increases.

Observe that $\rho_1 \rho_{0y} \equiv (1 - \rho_{0y}) (\rho_{xy} - \rho_6)$. We have an alternative way to express the above proposition. From (2.16), the following result is immediate by the above identity.

**COROLLARY 2.5** The fraction of informed investors has the same impact on both the price coefficients $a_1$ and $a_2$ if, and only if, the total correlation between the private signal and the public report is negative, i.e.,

$$\frac{da_1}{d\lambda} \frac{da_2}{d\lambda} > 0 \quad \text{if, and only if,} \quad \rho_{0y} < 0.$$  \hspace{1cm} (2.17)

An increase in the exogenously given fraction of informed investors increases or decreases both the price coefficients $a_1$ and $a_2$ if, and only if, the total correlation between the public report and the private information is negative. However, an increase in the exogenously given fraction of informed investors increases one coefficient and at the same time decreases the other if, and only if, the total correlation between the public report and the private information is positive.

To finish this section, we present some numerical examples (hereafter NE). As the fraction of the informed investors increases, other things equal, whether $a_1$ and $a_2$ increase
or decrease depends upon the information structure.

Table 2.2 lists eight numerical examples under three different fractions of the informed investors. In numerical examples $NE(+, -)$, $NE(+, 0)$ and $NE(+, +)$, the direct/net correlation between the private information and the risky asset is positive (i.e., $\rho_1 > 0$). The coefficient $a_1$ in these examples are zeros at $\lambda = 0$, which are smaller than 1.8571, .3590, .1935 at $\lambda = 1/2$, which in turn are smaller than 2, 2/3, 2/7 at $\lambda = 1$, respectively. That is, as the fraction of the informed investors increases, $a_1$ increases if $\rho_1 > 0$.

In numerical examples $NE(-, -)$, $NE(-, 0)$ and $NE(-, +)$, the direct/net correlation between the private information and the risky asset is negative (i.e., $\rho_1 < 0$). The coefficient $a_1$ in these examples are zeros at $\lambda = 0$, which are larger than $-0.6897$, $-0.6559$, $-0.2454$ at $\lambda = 1/2$, which in turn are larger than $-19/13$, $-2/3$, $-1/3$ at $\lambda = 1$, respectively. That is, as the fraction of the informed investors increases, $a_1$ decreases if $\rho_1 < 0$.

In the numerical examples $NE(+, -)$, $NE(+, 0)$, $NE(+, +)$, $NE(-, -)$, we have $\rho_1 \sigma_{0y} > 0$. The $a_2$’s are $1/6$, $1/2$, $1/2$, $-1$ at $\lambda = 0$, which are larger than $-0.4524$, $0.2308$, $0.3549$, $-1.4598$ at $\lambda = 0.5$, which in turn are larger than $-1/2$, 0, 2/7, $-19/13$ at $\lambda = 1$, respectively. That is, as the fraction of the informed investors increases, $a_2$ decreases, if $\rho_1 \sigma_{0y} > 0$.

In numerical examples $NE(-, 0)$ and $NE(-, +)$, we have $\rho_1 \sigma_{0y} < 0$. The $a_2$’s are $-2/5$, $3/4$ at $\lambda = 0$, which are smaller than $-0.0064$, $0.9341$ at $\lambda = 0.5$, which in turn are even smaller than 0, 1 at $\lambda = 1$, respectively. That is, as the fraction of the informed investors increases, $a_2$ increases, if $\rho_1 \sigma_{0y} < 0$. 
Table 2.2: Numerical examples of equilibrium prices with exogenous information acquisition (i.e., fixed fraction of informed investors $\lambda = 0, 0.5, 1$)

<table>
<thead>
<tr>
<th>Information Structure</th>
<th>Equilibrium Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x^2$</td>
<td>$\sigma_{xy}$</td>
</tr>
<tr>
<td>-----------------------</td>
<td>--------------------</td>
</tr>
<tr>
<td>$\lambda = 0$</td>
<td></td>
</tr>
<tr>
<td>$NE(+,-)$</td>
<td>2</td>
</tr>
<tr>
<td>$NE(+,0)$</td>
<td>2</td>
</tr>
<tr>
<td>$NE(+,+)$</td>
<td>2</td>
</tr>
<tr>
<td>$NE(0,-)$</td>
<td>20</td>
</tr>
<tr>
<td>$NE(0,+)$</td>
<td>9</td>
</tr>
<tr>
<td>$NE(-,-)$</td>
<td>20</td>
</tr>
<tr>
<td>$NE(-,0)$</td>
<td>4</td>
</tr>
<tr>
<td>$NE(-,+)$</td>
<td>5</td>
</tr>
</tbody>
</table>

| $\lambda = 0.5$ |
|-----------------------|--------------------|
| $NE(+,-)$ | 2 | 1 | 1 | 6 | 2 | 1 | 4 | 2 | -1 | 1.8571 | -0.4524 | 0.9286 |
| $NE(+,0)$ | 2 | 2 | 2 | 4 | 3 | 3 | 2 | 3 | 0 | 0.3590 | 0.2308 | 1.0769 |
| $NE(+,+)$ | 2 | 2 | 2 | 4 | 3 | 3 | 2 | 7 | 2 | 0 | 0.1935 | 0.3549 | 0.7741 |
| $NE(0,-)$ | 20 | -3 | 6 | 2 | -4 | 10 | 0 | 4 | -6 | 0 | -1.5 | 4.3056 |
| $NE(0,+)$ | 9 | 3 | 2 | 6 | 4 | 4 | 0 | 8 | 4 | 0 | 0.5 | 0.9375 |
| $NE(-,-)$ | 20 | -3 | -4 | 3 | -2 | 10 | -18 | 26 | -38 | 0 | -1.4598 | 0.7663 |
| $NE(-,0)$ | 4 | -2 | -4 | 5 | 3 | 6 | -14 | 21 | 0 | 0 | -0.0064 | 0.5622 |
| $NE(-,+)$ | 5 | 3 | 2 | 4 | 3 | 3 | -1 | 3 | 3 | 0 | 3/4 | 1.4725 |

| $\lambda = 1$ |
|-----------------------|--------------------|
| $NE(+,-)$ | 2 | 1 | 1 | 6 | 2 | 1 | 4 | 2 | -1 | 2 | -1/2 | 1/2 |
| $NE(+,0)$ | 2 | 2 | 2 | 4 | 3 | 3 | 2 | 3 | 0 | 2/3 | 0 | 1 |
| $NE(+,+)$ | 2 | 2 | 2 | 4 | 3 | 3 | 2 | 7 | 2 | 0 | 2/7 | 2/7 | 4/7 |
| $NE(0,-)$ | 20 | -3 | 6 | 2 | -4 | 10 | 0 | 4 | -6 | 0 | -3/2 | 5/2 |
| $NE(0,+)$ | 9 | 3 | 2 | 6 | 4 | 4 | 0 | 8 | 4 | 0 | 1/2 | 1/2 |
| $NE(-,-)$ | 20 | -3 | -4 | 3 | -2 | 10 | -18 | 26 | -38 | 0 | -19/13 | -19/13 | 5/13 |
| $NE(-,0)$ | 4 | -2 | -4 | 5 | 3 | 6 | -14 | 21 | 0 | 0 | -2/3 | 2/7 |
| $NE(-,+)$ | 5 | 3 | 2 | 4 | 3 | 3 | -1 | 3 | 3 | 0 | 1/3 | 1 |

Notes:

1. $\nu = \sigma_x^2 = 1$; $s_1 \equiv \sigma_{0x}\sigma_y^2 - \sigma_{xy}\sigma_{0y}$, $s_2 \equiv \sigma_y^2\sigma_0^2 - \sigma_{0y}^2$, $s_6 \equiv \sigma_0^2\sigma_{xy} - \sigma_{0x}\sigma_{0y}$.

2. The first (second) component in the bracket of $NE(\cdot, \cdot)$ represents the sign of $s_1$ ($s_6$) (see Definition 1, and Lemma 2.1).

3. $NE(0, 0)$ would be a degenerate example (i.e., $s_2$ has to be zero if $s_1 = s_6 = 0$) and is omitted from the table (see Lemma 2.1).
2.5 Equilibrium Information Acquisition

In the previous two sections, we assume that the fraction of informed investors is exogenously given, and characterize the equilibrium price, and the information’s impact on equilibrium prices. Furthermore, we show that the absolute value of the coefficient in the equilibrium price of the private signal is strictly increasing in \( \lambda \), i.e., the private signal has a larger absolute impact on the equilibrium price if more investors know it. We now assume the investors make their own information acquisition decision, i.e., \( \lambda \) is endogenously determined.

Since the investors make their information acquisition decision based on their ex-ante expected utilities (see Lemma 2.2), we now examine them formally.

**Lemma 2.4** The ratio of the informed and uninformed investors expected utilities is

\[
\Gamma(\lambda) \equiv \frac{E[U_i^\lambda]}{E[U_u^\lambda]} = e^{\nu}\left(1 + \frac{\nu^2\sigma_2^4\lambda^2\rho_0^2}{\lambda^2\rho_1^2\rho_2 + \nu^2\sigma_2^4\rho_0^2}\right)^{-\frac{1}{2}},
\]

(2.18)

a strictly (concave) increasing function of \( \lambda \).

If \( \kappa = 0 \), then it is clear that \( \Gamma(\lambda) \leq 1 \), for all \( \lambda \in [0, 1] \). Therefore, \( \lambda = 1 \) achieves the maximal \( \Gamma(\lambda) \), which is less than the unity.

From this lemma, it is clear that the equilibrium level of \( \lambda \) is determined as follows. In equilibrium, \( \lambda^* = 0 \) if \( \Gamma(0) > 1 \), \( \lambda^* = 1 \) if \( \Gamma(1) < 1 \), and \( \lambda \) is solved by \( \Gamma(\lambda^*) = 1 \), otherwise (see Figure 2.2). Consequently, the fact that \( \rho_1 = 0 \) implies \( \Gamma(\lambda) = e^{\nu\kappa} > 1 \) from (2.18) shows that \( \lambda = 0 \) is the equilibrium fraction of informed if \( \rho_1 = 0 \).

**Proposition 2.6** None of the investors chooses to be informed if the public report is a sufficient statistic with respect to the risky asset dividend for both the public report and the private signal, i.e., \( \lambda^* = 0 \) if \( \rho_1 = 0 \).
Figure 2.2: Endogenous information acquisition

This figure illustrates all three possible cases of function $\Gamma(\lambda)$. First, the curve entirely lies above 1, labelled by $\Gamma_A(\lambda)$, in which no one chooses to be informed. Second, the curve crosses 1, labelled by $\Gamma_C(\lambda)$, in which some investors choose to be informed and the rest choose not. Third, the curve entirely lies below 1, labelled by $\Gamma_B(\lambda)$, in which everyone chooses to be informed.
This result is intuitively obvious, since no one would acquire such a private signal at a positive cost if the public report is a sufficient statistic for both the public report and the private signal.

In general, the endogenous fraction of informed investors depends critically upon the cost of the private information signal. For the sake of tractability, we assume that such a cost is constant.²⁷

**DEFINITION 2** The private information is excessively expensive if \( \kappa \geq \kappa^0 \), moderately expensive if \( \kappa \in (\kappa_0, \kappa^0) \), and inexpensive if \( \kappa \leq \kappa_0 \), where

\[
\kappa_0 = \frac{1}{2\nu} \ln \left[ 1 + \frac{\nu^2 \sigma_2^2 \rho_1^2 \rho_0}{\rho_1^2 \rho_2 + \nu^2 \sigma_2^2 \sigma_2^2 \rho_0^2} \right], \quad \text{and} \quad \kappa^0 = \frac{1}{2\nu} \ln \left[ 1 + \frac{\rho_1^2}{\rho_0} \right], \quad (2.19)
\]

are the threshold points of the cost of the private information signal.

It is clear that \( \kappa_0 \leq \kappa^0 \) since \( \rho_1^4 \rho_2 \geq 0 \). The equality holds if, and only if, \( \rho_1 = 0 \). This is intuitively simple. If the public report is a sufficient statistic (i.e., \( \rho_1 = 0 \)) for both the private information and the public report, no one would purchase the private signal. The cost of the private signal is zero and the two threshold points are the same, and equal zero.

**PROPOSITION 2.7** The equilibrium fraction of informed investors is given by \( \lambda^* = \ldots \)

²⁷An alternative approach would be to assume the rational investors differ in their costs of acquiring information and are rank ordered by such costs. Then, there exists a rational investor who is indifferent between being informed and uninformed. Denote that cost as \( \kappa^* \). All the rational investors whose cost is lower than \( \kappa^* \) decide to be informed, and all the rational investors whose cost is higher than \( \kappa^* \) decide to be uninformed.

We can alternatively make this cost \( \kappa \) depend on some other factors of the economy. For example, let cost \( \kappa \) be an increasing function of the variance of the risky asset dividend. However, this would only be relevant in doing comparative statics. I would like to thank Professor Joshua Ronen for this point.
\[ \nu \sigma_x \sigma_q, \text{ where } \nu \equiv e^{2\nu} - 1, \text{ and } \nu = 0, \text{ if } \kappa \in [\kappa^0, +\infty); \]
\[ q \equiv \begin{cases} 0, & \text{if } \kappa \in [\kappa^0, +\infty); \\ \sqrt{\rho_0 (\rho_1^2 - \nu \rho_0)/ (\rho_1^2 \rho_2)}, & \text{if } \kappa \in (\kappa_0, \kappa^0); \\ 1/(\nu \sigma_x \sigma_q), & \text{if } \kappa \in [0, \kappa_0]. \end{cases} \] (2.20)

**Proof:** See Appendix A.1. \( Q.E.D. \)

This proposition identifies that investors are uniformly uninformed (i.e., \( \lambda^* = 0 \)) or uniformly informed (i.e., \( \lambda^* = 1 \)), in two of the cost regions, and a fraction of them are informed in the other cost region. Assume investors have different threshold points at which they are indifferent whether or not to purchase the private signal. Then, \( \kappa_0 \) can be interpreted as the threshold point below which every investor decides to purchase the private signal; and \( \kappa^0 \) is the threshold point above which every investor decides not to purchase the private signal.

Now, let us examine the equilibrium fraction of the informed investors.

**PROPOSITION 2.8** If the private information is excessively expensive, no one chooses to purchase it (i.e., \( \lambda^* = 0 \)). If the private signal is inexpensive, everyone chooses to acquire it (i.e., \( \lambda^* = 1 \)). If the private signal is moderately expensive, the equilibrium fraction of the informed investors is

1. independent of \( \sigma_0^2 \) and \( \sigma_2^2 \), i.e., \( \frac{d\lambda^*}{d\sigma_0^2} = 0, \frac{d\lambda^*}{d\sigma_2^2} = 0; \)

2. increasing in \( \sigma_x^2, \sigma_z^2 \), i.e., \( \frac{d\lambda^*}{d\sigma_x^2} > 0, \frac{d\lambda^*}{d\sigma_z^2} > 0; \)

3. decreasing in \( \kappa \), i.e., \( \frac{d\lambda^*}{d\kappa} < 0. \)

²8 We state this proposition as it stands without specifying the condition that \( \rho_1 \neq 0 \), because as \( \rho_1 \rightarrow 0 \), we have \( 1 - \frac{\kappa^0}{\rho_1^2} \rightarrow -\infty \), implying that \( 1 - \frac{\kappa^0}{\rho_1^2} < 0 \), i.e., \( q = 0 \). Hence, \( \lambda^* = 0 \). This is consistent with Proposition 2.6.
Table 2.3: Comparative statics of equilibrium fraction of informed investors

<table>
<thead>
<tr>
<th>Equilibrium Fraction</th>
<th>$\sigma_x^2$</th>
<th>$\sigma_y^2$</th>
<th>$\sigma_0^2$</th>
<th>$\sigma_y^2$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^* \in (0, 1)$</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
<td>0</td>
<td>0</td>
<td>$\downarrow$</td>
</tr>
</tbody>
</table>

$\uparrow$: Standing for positive partial derivative;

$\downarrow$: Standing for negative partial derivative;

0: Standing for zero partial derivative.

Proof: See Appendix A.1. Q.E.D.

If the private signal is excessively expensive, it does not matter how volatile the risky asset dividend or random supply shock are, no one chooses to purchase the private signal. The private signal is unused information in the economy. If the private signal is inexpensive, it does not matter what the other economic parameters are, every rational investor chooses to purchase the private signal. At the extreme $\kappa = 0$ (i.e., free information), every one gets the private information signal.

From now on, we assume that the cost of the private information is moderately expensive so that $\lambda^* \in (0, 1)$.

This proposition establishes that the variance of the private signal and the public report do not influence the investors’ information acquisition decision. The variance of a signal merely reflects its scale given that we hold the correlations fixed. For example, expressing earnings in thousands of dollars instead of dollars does not change the
information content of the earnings report, and hence does not influence the investors' information acquisition decision. Investors are interested in the correlations among the information signals since only the correlations are relevant.

The key insight of this proposition is that the higher the volatility of the risky asset dividend or the supply shock of the risky asset, the more investors choose to purchase the private information. It suggests that while the primary use of the acquired private information is to reduce the uncertainty about the risky asset, there are two sources of that uncertainty if one is uninformed: the inherent uncertainty of the risky asset and the noise created by the market price.

Although the acquisition of the private signal does not directly help reveal the supply shock of the risky asset since they are assumed to be independent of each other, more investors choose to purchase the private signal as the risky asset supply shock increases. The intuition is that an increase in the supply shock increases the noise in the equilibrium price, and the acquired private signal helps reduce the noise in the equilibrium price.

Finally, for obvious reasons, the higher the cost of the private information signal, the smaller the fraction of investors who choose to purchase the private signal.

2.6 Equilibrium Price With Endogenous Information Acquisition

In Section 2.3, we fixed the fraction of informed investors, and derived the equilibrium price. Given $\lambda$, investors make self-fulfilling conjectures about the equilibrium price, and decide how much of the risky asset to acquire. In Section 2.5, we allow the investors to make trade-offs between the cost and benefit of acquiring the private information.

Proposition 2.7 formally characterizes the equilibrium fraction of informed investors. Since investors simultaneously make their investment decisions and self-fulfilling conjectures (equilibrium price and fraction of the informed), we immediately obtain the
following proposition by simply combining Propositions 2.7 and 2.1.

**PROPOSITION 2.9** If the investors' endogenously decide whether to acquire the private information, there exists a unique rational expectations equilibrium price within the class of functions of the form given by (2.7). If \( \lambda^* = 0 \) or 1, this price is:

\[
\begin{align*}
\tilde{P}_0 &= \bar{X} + \frac{\sigma_x \rho_{y\hat{Y}}}{\sigma_{y}} (\hat{Y} - \bar{Y}) - \frac{\nu \sigma_x^2 \rho_0}{\rho_2} \hat{z}; \\
\tilde{P}_1 &= \bar{X} + \frac{\sigma_x \rho_1}{\sigma_0 \rho_2} (\hat{\psi} - \bar{\psi}) + \frac{\sigma_x \rho_6}{\sigma_y \rho_2} (\hat{Y} - \bar{Y}) - \frac{\nu \sigma_x^2 \rho_0}{\rho_2} \hat{z}.
\end{align*}
\]

If \( \lambda^* \in (0, 1) \), this price is:

\[
\tilde{P} = \bar{X} + a_1 (\hat{\psi} - \bar{\psi}) + a_2 (\hat{Y} - \bar{Y}) - a_2 \hat{z},
\]

where the coefficients are as follows:

\[
\begin{align*}
a_0 &= \bar{X} - a_1 \bar{\psi} - a_2 \bar{Y}; \\
a_1 &= \frac{\sigma_x (q \nu \sigma_x \sigma_z \rho_3 + \rho_1^2 - \rho_0)}{\sigma_z (1 + q \nu \sigma_x \sigma_z) \rho_1 \rho_2}; \\
a_2 &= \frac{\sigma_x (-q \nu \sigma_x \sigma_z \rho_4 + \rho_1 \rho_6 + \rho_0 \rho_0 \rho_0)}{\sigma_y (1 + q \nu \sigma_x \sigma_z) \rho_1 \rho_2}; \\
a_2 &= \frac{\nu \sigma_x^2 (q \rho_1^2 + \nu \sigma_x \sigma_z \rho_0 \rho_3)}{\sigma_x (1 + q \nu \sigma_x \sigma_z) \rho_1^2}.
\end{align*}
\]

2.7 Price Impact of Information Signals

2.7.1 Linear transformation of an information signal

Observe that Proposition 2.9 highlights two things. First, it is clear from (2.21c) that the means of the private signal and the public report have no impact on the equilibrium price. Their effect is directly removed.

Second, the coefficients \( a_1 \) and \( a_2 \) given by (2.22b) and (2.22c) are scaled by the standard deviation of their corresponding information signals. Hence, neither the mean
nor the noise of the information signals plays a role in determining the market equilibrium price of the risky asset dividend. The noise, as measured by the standard deviation of the signal, can be interpreted as merely reflecting the "scale" with which the information signal precision is measured. This proposition shows that the driving factors of the firm’s equilibrium price are the risky asset dividend’s expected value and its correlations with the information signals (the private signal and the public report).

Since the scale of each information signal (i.e., linear transformation) only affects its mean and standard deviation, and does not affect its correlations with other variables, we have the following proposition immediately. \[^{2.9}\]

**PROPOSITION 2.10** The scale of an information signal (the public report or the private information) does not influence the equilibrium price of the risky asset.

Here, we have to distinguish the covariance representation with the correlation representation of the information structure. The scale of an information signal changes the covariance structure because it affects the signal’s standard deviation, but it does not influence the correlation structure.

### 2.7.2 Private information signal

By (2.21c), it is clear that the noise term (i.e., \(\varepsilon_0\)) in the investors’ private information is explicitly impounded in the equilibrium price; its marginal impact is denoted as \(a_1\). The mean of the investors’ private information is irrelevant, and the standard deviation of the private signal is only an information scaling factor (see Proposition 2.10).

\[^{2.9}\]This is a very important point to notice here. In this pure exchange economy, the mean of an information signal is removed from the equilibrium price and hence plays no role. The scale effect is reflected only by the variance of the information signal. In later chapters, we assume that managerial effort affects information signal’s mean but not variance. Since a change in the scale of an information signal affects the signal’s mean and variance, the scale of a performance measure affects the managerial incentive compensation rate applied to that performance measure although the scale does not influence the optimally induced effort.
In the following, we identify some conditions under which the private signal has a positive (negative) coefficient on the equilibrium price [i.e., \( a_1 > (<) 0 \)], and examine how this price coefficient changes with respect to the noise contained in the information signals.

Observe that the direct correlation (see Definition 1) \( \rho_1 \equiv \rho_{xy} - \rho_{zy} \rho_{yz} \) represents the net correlation between the private signal and the risky asset dividend: the difference of the total and indirect correlations.

**PROPOSITION 2.11** The private signal is not impounded in the equilibrium price if, and only if, none of the investors choose to be informed, i.e., \( a_1 = 0 \) if, and only if, \( \lambda^* = 0 \). If \( \lambda^* > 0 \), then \( a_1 \) is positive (negative) if, and only if, \( \rho_1 \) is positive (negative). Furthermore, an increase of the noise in the private information reduces the absolute price coefficient for the private information (i.e., \( \frac{\partial \rho_1}{\partial \sigma_0^2} < 0 \)). The noise in the public report does not influence the price coefficient for the private information (i.e., \( \frac{\partial a_1}{\partial \sigma_0^2} = 0 \)).

**Proof:** See Appendix A.1. \( \quad \text{Q.E.D.} \)

If no one purchases the private signal, the private information clearly has no impact on the equilibrium price of the risky asset dividend. On the other hand, if the private information has no impact on the equilibrium price, one of the following conditions must have been met. First, the public report reveals all the information included in the private signal, i.e., the public report is a sufficient statistic with respect to the firm value. In such a case, when investors decide whether to acquire the private information, no one would purchase the private signal at a positive cost. Thus, the equilibrium fraction of informed investors is *endogenously* determined to be zero. Second, no one purchases the private signal if it is too costly, even if it would contribute information beyond what is included in the public report.
If positive \( \lambda \) were exogenously given, the necessary and sufficient condition for the investors' private information to have no impact on the equilibrium price would be that the public report is a sufficient statistic with respect to the firm value. That is, \( a_1 = 0 \) if, and only if, the direct/net correlation between the private information and the risky asset dividend is zero (i.e., \( \rho_1 = 0 \)). Since the equilibrium fraction of informed investors is endogenously determined, it is not surprising that the necessary and sufficient condition for the private signal to have no impact on the equilibrium price becomes \( \lambda^* = 0 \) instead of \( \rho_1 = 0 \). When \( \lambda^* \neq 0 \), the sign of \( \rho_1 \) alone determines the sign of \( a_1 \).

Furthermore, since the noise in the private signal is just a scale effect, it does not influence its relationship with other information signals, such as the risky asset dividend information and the public report. This scale effect is directly transferable to the impact of the private signal on the equilibrium price, i.e., the product of the price coefficient and standard deviation (i.e., \( a_1 \sigma_0 \)) is constant. Therefore, as the noise in the private signal increases, the price coefficient of the information decreases. Finally, it is intuitively clear that the price coefficient of the private signal is independent of the noise of the public report since the correlations of the information structure are fixed.

2.7.3 Public report

We now consider the conditions under which the public report has a positive (negative) coefficient on the equilibrium price [i.e., \( a_2 > (<) 0 \)], and examine how \( a_2 \) changes with respect to the noise contained in information signals. Clearly, no matter how many investors are informed, the public report always has an impact on the price as long as the private signal is not a sufficient statistic with respect to firm value. Therefore, that the private signal is a sufficient statistic is a necessary condition for the public report to have no impact on the market price.
PROPOSITION 2.12 If the private signal is a sufficient statistic for the public report with respect to the firm value, then the public report has no impact on the equilibrium price if, and only if, either all investors choose to be informed (i.e., $\lambda^* = 1$) or the public report is independent of the risky asset payoff (i.e., $\rho_{xy} = 0$).

Furthermore, the absolute value of the price coefficient for the public report decreases in the noise of the public report (i.e., $\partial |a_2|/\partial \sigma_y^2 < 0$) if $\rho_{xy}\rho_0 \geq 0$. The price coefficient for the public report is not affected by the noise of the private signal (i.e., $\partial a_2/\partial \sigma_0^2 = 0$).

**Proof:** See Appendix A.1. Q.E.D.

The informed investors ignore the public report if the private signal is a sufficient statistic with respect to the firm value. The uninformed investors, however, would still use the public report to form their beliefs about the firm value unless it is not correlated with the firm value (i.e., $\rho_{xy} = 0$), because they cannot perfectly infer the private signal from the price.

By Lemma 2.1, conditions $\rho_{xy} = 0$ and $\rho_0 = 0$ imply that $\rho_{0x}\rho_{0y} = 0$. If $\rho_{0x} = 0$ (i.e., the private signal is independent of the risky asset dividend), it is clear that the public report cannot be used to infer any information about the risky asset dividend either directly (due to $\rho_{xy} = 0$) or indirectly through the private signal (due to $\rho_{0x} = 0$). If $\rho_{0y} = 0$ (i.e., the public report is independent of the private signal), it then is clear that the public report reveals no information about either the risky asset dividend (due to $\rho_{xy} = 0$) or the private signal (due to $\rho_{0y} = 0$). Therefore, $\rho_{0x}\rho_{0y} = 0$ implies that the public report is uninformative about either the private signal or the risky asset dividend. Consequently, it has no impact on the equilibrium price.

Observe that Corollary 2.3 shows that $a_2$ has the same sign as $\rho_{xy}$ if $\rho_{xy}$ and $\rho_0$ have the same sign, i.e., $\rho_{xy}\rho_0 > 0$. The total and direct correlations between the public
report and the risky asset dividend jointly determine the sign of the price coefficient for the public report in the equilibrium price. That is, if the public report is positively correlated with the risky asset dividend and the direct correlation is also positive, then the price coefficient \( a_2 \) is positive; if the public report is negatively correlated with the risky asset dividend and the direct correlation is also negative, then the price coefficient \( a_2 \) is negative. This implies that the total correlation determines the sign of \( a_2 \) as long as the indirect correlation does not make the sign of the direct correlation different from that of the total correlation.

For example, if the private signal is independent of either the public report or the risky asset dividend, the indirect correlation between the public report and the risky asset dividend is zero (i.e., \( \rho_{0x}\rho_{0y} = 0 \)). Hence the "net" correlation equals the total correlation (i.e., \( \rho_y = \rho_{xy} \)). The price coefficient \( a_2 \) has the same as the sign of \( \rho_{xy} \). On the other hand, if the public report is independent of the risky asset dividend, the price coefficient \( a_2 \) has the same sign as that of the direct correlation: \( \rho_y \).

### 2.7.4 Risky asset supply

So far, we have examined the coefficients of the equilibrium price with respect to the investors' private signal and the public report. Now, we examine the coefficient of the equilibrium price with respect to the risky asset supply. By (A.1) and Proposition 2.11, it is immediate that \( a_x > 0 \).

**Proposition 2.13** The risky asset's random supply has a negative impact on the equilibrium price.

The intuition is clear. The requirement that demand equals supply implies that the equilibrium price decreases as the supply of the risky asset increases, and increases as the supply of the risky asset decreases.
2.7.5 Ratio of price coefficients

The ratio of the price coefficients for the private information and the risky asset's random supply (i.e., \( a_1/a_x \)) in the equilibrium price has been widely used in the literature. For example, Grossman and Stiglitz (1980) use it in deriving the equilibrium price. Grundy and McNichols (1990), Holthausen and Verricchia (1990), Bushman (1991), and Indjejikian (1991), etc., utilize it to characterize the equilibrium price function. One common feature in these models is the simple information structure: all information signals are noise representations of the underlying risky asset. To examine the characteristics of this ratio in our setting, we derive it as:

\[
\frac{\lambda^* \rho_1}{\nu \sigma_0 \sigma_x \rho_0} = \begin{cases} 
\frac{\rho_1}{\nu \sigma_0 \sigma_x \rho_0} & \text{if } \lambda^* = 1; \\
\frac{\rho_1 \sigma_x \sqrt{\rho_1^2 - 1} \rho_0}{[\rho_1 \sigma_x \sqrt{\rho_1^2 - 1}]^2} & \text{if } \lambda^* \in (0, 1),
\end{cases}
\]  

(2.23)

which can be interpreted as a measure of how well price transmits the privately acquired information, with respect to the supply uncertainty of the risky asset. Direct observations of (2.23) and simple derivatives give the following results immediately.

**PROPOSITION 2.14** The ratio of the price coefficients for the private information and the risky asset's random supply (i.e., \( b \equiv a_1/a_x \)) has the following characteristics.

1. \( b \) is zero if, and only if, \( \lambda^* = 0 \), and is positive (negative) if, and only if, \( \rho_1 > (<) 0 \).

2. \( b \) is independent of the public report noise (i.e., \( \partial b/\partial \sigma_y = 0 \)).

3. \( b \)'s absolute value decreases in the private signal's noise (i.e., \( \partial |b|/\partial \sigma_y < 0 \)).

4. \( b \)'s absolute value decreases in the investor's risk aversion (i.e., \( \partial |b|/\partial \nu < 0 \)).

5. If all rational investors choose to be informed (i.e., \( \lambda^* = 1 \)), then

   - \( b \)'s absolute value decreases in the risky asset's uncertainty (i.e., \( \partial |b|/\partial \sigma_x^2 < 0 \));

   - \( b \) is independent of the risky asset's supply uncertainty (i.e., \( \partial b/\partial \sigma_x^2 = 0 \));
• \( b \) is independent of the cost of the private signal (i.e., \( \partial b / \partial \kappa = 0 \)).

6. If a fraction of rational investors choose to be informed [i.e., \( \lambda^* \in (0, 1) \)], then

• \( b \) is independent of the risky asset's uncertainty (i.e., \( \partial b / \partial \sigma_x^2 < 0 \));

• \( b \)'s absolute value increases in the risky asset's supply shock (i.e., \( \partial |b| / \partial \sigma_x^2 > 0 \));

• \( b \)'s absolute value decreases in the cost of the private signal (i.e., \( \partial |b| / \partial \kappa < 0 \)).

2.8 Numerical Examples

We next introduce some numerical examples to illustrate the intuitive results when the investors' private information acquisition decision is endogenously determined in the model.

Table 2.4 lists the eight examples with three different exogenously given cost levels of the private information signal.

1. Numerical examples \( NE(0, -) \) and \( NE(0, +) \), in which \( \rho_1 = 0 \), show that the equilibrium fraction of informed investors is zero for all cost levels, as Proposition 2.6 suggested.

2. Numerical example \( NE(+, +) \) illustrates that \( \lambda^* \) decreases from 1 for \( \kappa = 0.1 \) to .5096 for \( \kappa = 0.5 \), and further down to .2168 for \( \kappa = 1 \). Numerical example \( NE(+, 0) \) illustrates that \( \lambda^* \) decreases from 1 for \( \kappa = 0.1 \) to .2338 for \( \kappa = 0.5 \), and further down to 0 for \( \kappa = 1 \). These reductions in \( \lambda^* \) due to an increase in cost are consistent with Proposition 2.8.

3. Comparing the columns of \( a_1 \) and \( \lambda^* \) yields that \( a_1 = 0 \) if, and only if, \( \lambda^* = 0 \). That is, the private signal does not influence the equilibrium price if, and only if, no one chooses to be informed. Furthermore, \( a_1 > (\) 0 if, and only if, \( \rho_1 > (\) 0, as Proposition 2.11 suggested.
Table 2.4: Numerical examples of endogenous information acquisition and equilibrium prices with fixed cost (κ) of private information

<table>
<thead>
<tr>
<th>Information Structure</th>
<th>Equilibrium Information Acquisition</th>
<th>Equilibrium Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s&lt;sub&gt;1&lt;/sub&gt;</td>
<td>s&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td>κ = 0.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NE(+, −)</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>NE(+, 0)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>NE(+, +)</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>NE(0, −)</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>NE(0, +)</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>NE(−, −)</td>
<td>−18</td>
<td>26</td>
</tr>
<tr>
<td>NE(−, 0)</td>
<td>−14</td>
<td>21</td>
</tr>
<tr>
<td>NE(−, +)</td>
<td>−1</td>
<td>3</td>
</tr>
<tr>
<td>κ = 0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NE(+, −)</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>NE(+, 0)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>NE(+, +)</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>NE(0, −)</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>NE(0, +)</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>NE(−, −)</td>
<td>18</td>
<td>26</td>
</tr>
<tr>
<td>NE(−, 0)</td>
<td>14</td>
<td>21</td>
</tr>
<tr>
<td>NE(−, +)</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>κ = 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NE(+, −)</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>NE(+, 0)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>NE(+, +)</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>NE(0, −)</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>NE(0, +)</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>NE(−, −)</td>
<td>−18</td>
<td>26</td>
</tr>
<tr>
<td>NE(−, 0)</td>
<td>−14</td>
<td>21</td>
</tr>
<tr>
<td>NE(−, +)</td>
<td>−1</td>
<td>3</td>
</tr>
</tbody>
</table>

Notes:

1. \( \nu = \sigma_z^2 = 1; s_1 \equiv \sigma_x \sigma_y^2 - \sigma_{xy} \sigma_{0y}, s_2 \equiv \sigma_y^2 \sigma_0^2 - \sigma_{0y}^2, s_6 \equiv \sigma_0^2 \sigma_{xy} - \sigma_x \sigma_{0y}. \)

2. The first (second) component in the bracket of \( NE(\cdot, \cdot) \) represents the sign of \( s_1 (s_6) \) (see Definition 1, and Lemma 2.1).

3. \( NE(0, 0) \) would be a degenerate example (i.e., \( s_2 \) has to be zero if \( s_1 = s_6 = 0 \) and is omitted from the table (see Lemma 2.1).
4. Finally, we notice that in numerical examples $NE(+, 0)$ and $NE(-, 0)$ when $\kappa = 0.1$, $a_2 = 0$ because $\rho_6 = 0$ and $\lambda^* = 1$ (see Proposition 2.11).
Chapter 3

Performance Measures and Incentive Contracts in a Multi-Task Agency

3.1 Introduction

Banker and Datar (1989) (hereafter BD) analyze managerial incentive issues in a setting in which two performance measures are used to induce the agent (manager) to expend single-dimensional effort. They show that the coefficient for one performance measure relative to the other measure in the optimal incentive contract equals the ratio of sensitivity times precision of the two performance measures. Sensitivity measures the extent to which the expected value of a performance measure changes with the agent’s action, and precision is the inverse of variance of a performance measure.

A key factor here is that the performance measures are always perfectly aligned with each other. If both performance measures are influenced by the agent’s action, the performance measures’ means change with the agent’s action. The gradients of these means with respect to the agent’s action are non-zeros and point in the same direction because the action has only one dimension. If one of these performance measures is not influenced by the agent’s action (i.e., action-irrelevant), its gradient is zero and hence does not affect the other performance measure’s gradient direction. We say that these two performance measures are still aligned with each other. In both cases, the diversity between performance measures is not a factor influencing the agent’s incentives.

In the multi-task agency setting, however, firm value is affected by different managerial tasks. Different performance measures are usually affected differently by the different
managerial tasks, that is, they may not be aligned with each other. In this setting, the
agent's incentive to allocate his effort levels across tasks is affected by both the perfor-
mance measures' alignment with the firm's terminal value and their alignment with one
another. Consequently, the diversity of the performance measures becomes an additional
important factor influencing the relative coefficient for the two performance measures in
an optimal incentive contract.

Focusing on this factor, we examine, in this chapter, optimal incentive linear com-
pensation contracts based on two performance measures. The discussion of incentive
contracts based on the rational expectations equilibrium price and accounting earnings
is postponed to Chapter 4.

The rest of this chapter is organized as follows. Section 3.2 describes the basic multi-
task agency relationship. Section 3.3 integrates BD's measure of sensitivity with Feltham
and Xie (1994) (hereafter FX) congruity and diversity concepts for performance measures,
and then discusses some new concepts in the multi-task agency setting. Section 3.4 char-
acterizes the optimal incentive linear compensation contract based on two managerial
performance measures. A general relationship between the managerial effort levels and
the coefficients for the two performance measures in the incentive contract is established.
In Subsection 3.4.1, we focus on perfectly aligned performance measures, and show that
the relative coefficient for the two performance measures in the optimal incentive contract
does not differ much from the relative coefficient in the single-task agency setting. In
Subsection 3.4.2, we focus on diverse performance measures. It is shown that the diver-
sity between performance measures is an additional key factor influencing the incentive
coefficient for the two performance measures. In Subsection 3.4.3, we examine the ef-
effect of action-irrelevant performance measure on the incentive coefficients. Although an
action-irrelevant information signal does not convey information about the managerial
action directly, it may be still useful in the incentive contract because it helps reduce
Chapter 3. Performance Measures and Incentive Contracts in a Multi-Task Agency 49

the risk imposed on the agent. In Subsection 3.4.4, we formally examine how a linear transformation of a performance measure affects its coefficient in the optimal incentive compensation contract.

3.2 The Multi-Task Agency Relationship

A classical agency relationship exists between the shareholders and management of a publicly owned corporation. Conflict of interests arise in several dimensions because of the separation of ownership and control. Moral hazard is a major issue if the shareholders do not have complete information regarding the management’s activities and investment opportunities. Agency theory assumes that a managerial incentive compensation contract can help align the goals of the management with the goals of the shareholders since such a compensation contract gives managers incentives to select and implement actions that increase shareholders’ wealth.

Assume there is a risk neutral Board of Directors (the principal) acting on the behalf of the first type of traders (e.g., large institutional investors) in the economy considered in Chapter 2. The board is delegated to hire a manager (the agent) to actually run the firm and to design a managerial compensation contract that will induce the manager to provide the optimal intensity and allocation of effort across two tasks.

3.2.1 The agent

The agent/manager has discretion to allocate his effort between two different tasks\(^3\) which influence the expected dividend of the risky asset. The multi-dimensional effort can include, for example, current management of assets in place, strategic planning, identification of growth opportunities, and investments in the research and development

\(^3\)Our analysis can be straightforwardly extended to settings in which there are more than two managerial tasks and two performance measures.
of new products or technologies. Let \( e \equiv (e_1, e_2) \), where \( e_i \in [0, +\infty), \forall i = 1,2 \), represent the managerial effort levels in the two tasks.

**DEFINITION 3** Any effort vector \( e \) consists of two components: intensity and allocation/direction, i.e., \( e \equiv I v \). We refer to \( I \equiv \sqrt{e_1^2 + e_2^2} \) and \( v \equiv (v_1, v_2) \equiv (\frac{e_1}{I}, \frac{e_2}{I}) \) as the effort intensity and allocation/direction, respectively.\(^{32}\)

We assume a constant-returns-to-scale production technology, and the expected dividend of the risky asset is additive with respect to the two tasks:

\[
\bar{X}(e) \equiv g_1 e_1 + g_2 e_2. \tag{3.1a}
\]

We interpret positive constant \( g_i (\forall i = 1, 2) \) as the rate of change in the expected dividend per unit of effort in task \( i \); it measures the incremental profitability of an increase in managerial effort. Similarly, the means of the two information signals are:

\[
\bar{Y}^1(e) \equiv f_1 e_1 + f_2 e_2, \quad \text{and} \quad \bar{Y}^2(e) \equiv h_1 e_1 + h_2 e_2. \tag{3.1b}
\]

We interpret constants \( f_i, h_i (\forall i = 1, 2) \) as the rates of change in the expected values of the two information signals per unit of effort in task \( i \), respectively.

The agent's utility function is negative exponential, \(-\exp\{-W/R\}\), where \( W \) is the agent's incremental end-of-period wealth net of his personal cost of effort, and \( R \) is the agent's risk tolerance parameter. He is effort averse and has a convex quadratic personal cost function of effort:

\[
C(e) \equiv \left( \frac{1}{2} \right) C_{11} e_1^2 + C_{12} e_1 e_2 + \left( \frac{1}{2} \right) C_{22} e_2^2, \tag{3.2}
\]

where \( C_{ij}'s (i,j = 1, 2) \) are constants, \( C_{11} > 0, C_{22} > 0, \) and \( C_{11} C_{22} > C_{12}^2 \).

In the first-best world, the Board of Directors would require the manager to provide the effort \( e \) that maximizes the net surplus: \( \bar{X}(e) - C(e) \). Given the agent's personal

\(^{32}\)Clearly, \( v \) is a unit vector. If \( I = 0 \), we define \( v \) to be the vector: \((1/\sqrt{2}, 1/\sqrt{2})\).
cost function (3.2) of effort, the first-order conditions $g_i = \frac{\partial C(e)}{\partial e_i} (\forall i = 1, 2)$ determine the optimally induced managerial effort levels to be:\(^3^3\)

$$e^* = (e_1^*, e_2^*) = \left( \frac{C_{22}g_1 - C_{12}g_2}{C_{11}C_{22} - C_{12}^2}, \frac{C_{11}g_2 - C_{12}g_1}{C_{11}C_{22} - C_{12}^2} \right).$$  \hspace{1cm} (3.3)

It is the first-best effort level that the shareholders desire the agent to undertake. By Definition 3, we obtain $e^* \equiv I^*v^*$, where $I^* \equiv \sqrt{(e_1^*)^2 + (e_2^*)^2}$ and $v^* \equiv \left( \frac{e_1^*}{I^*}, \frac{e_2^*}{I^*} \right)$ are the first-best effort intensity and allocation, respectively.

### 3.2.2 The incentive contract

There are two publicly observable information signals which provide information about effort $e$, but are contaminated by random events beyond the control of the agent. Denote them by

$$Y^1 = \bar{Y}^1(e) + \varepsilon_{y^1} \quad \text{and} \quad Y^2 = \bar{Y}^2(e) + \varepsilon_{y^2}.$$  \hspace{1cm} (3.4)

Examples of such information are accounting earnings, divisional profit, returns on investments, and stock prices. We assume that $\bar{Y}^1$ and $\bar{Y}^2$ are jointly normally distributed with the following covariance matrix:

$$\Sigma_{y^1y^2} = \begin{bmatrix} \sigma_{y_1}^2 & \sigma_{y_1y_2} \\ \sigma_{y_1y_2} & \sigma_{y_2}^2 \end{bmatrix}.$$

The managerial compensation contract is assumed to take the following linear form:\(^3^4\)

in the performance measures:

$$w(\bar{Y}^1, \bar{Y}^2) = \alpha + \gamma \bar{Y}^1 + \beta \bar{Y}^2.$$  \hspace{1cm} (3.5)

\(^3^3\)To guarantee the positiveness of the first-best effort levels, we impose a regularity condition that $C_{12}$ is sufficiently small compared to $C_{11}$ and $C_{22}$.

\(^3^4\)A linear contract is not necessarily optimal in this setting, but it is a common assumption in the multi-task agency literature. See Holmstrom and Milgrom (1987), Bushman and Indjejikian (1993) (hereafter BI), FX, and Dye (1985). Further, we assume the outcome $\bar{X}$ of the agent's effort is not observed at the contract settlement date $t = 2$, and hence it cannot be used as a contract variable.
Figure 3.1: The sequence of events in the agency relation

<table>
<thead>
<tr>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
<th>t = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contract Date</td>
<td>Action Date</td>
<td>Contract Settlement Date</td>
<td>Termination Date</td>
</tr>
<tr>
<td>contract specified and signed.</td>
<td>effort levels allocated; personal costs of effort incurred.</td>
<td>information signals $Y^1$ and $Y^2$ released; managerial compensation determined.</td>
<td>$X$ realized; other claims paid; terminal wealth determined.</td>
</tr>
</tbody>
</table>

After signing the contract at $t = 0$ (see Figure 3.1), the agent chooses his optimal effort allocation at $t = 1$. The agent is paid the amount of $w(Y^1, Y^2)$, which has been specified in the compensation contract. Consequently, the agent's net return is $\left[w(Y^1, Y^2) - C(e)\right]$ from this agency relationship. The firm liquidates at $t = 3$. The Board of Directors' (i.e., the risk neutral long-term shareholders) decision problem is:\(^{35}\)

\[
\max_{\alpha, \gamma, \beta, e_1, e_2} E_{\tilde{e}_s, \tilde{e}_{y1}, \tilde{e}_{y2}, \tilde{e}} \left[ X - \left( \alpha + \gamma Y^1 + \beta Y^2 \right) \right]
\]  
\[
\text{Subject to:}
\]

Participation Constraint:
\[
E_{\tilde{e}_s, \tilde{e}_{y1}, \tilde{e}_{y2}, \tilde{e}} \left[ - \exp \left\{ - \frac{\alpha + \gamma \tilde{Y}^1 + \beta \tilde{Y}^2 - C(e)}{R} \right\} \right] \geq -1
\]

Incentive Constraint:
\[
(e_1, e_2) \in \operatorname{argmax} E_{\tilde{e}_s, \tilde{e}_{y1}, \tilde{e}_{y2}, \tilde{e}} \left[ - \exp \left\{ - \left( \frac{1}{R} \right) \left( \alpha + \gamma \tilde{Y}^1 + \beta \tilde{Y}^2 - C(e) \right) \right\} \right].
\]

---

\(^{35}\)We state this as if the "board" owns all the shares, but the results are the same if they own a constant fraction.
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Since the risk neutral shareholders have a controlling interest in the firm, their objective function is to maximize their expected return, net of management compensation. Condition (3.6b) is the participation constraint of the agency relation. Since the agent can always do nothing and earn zero incremental end-of-period wealth, his reservation utility is $-1$. Constraint (3.6c) is the incentive constraint, i.e., the agent is motivated to exert the chosen effort levels.

We ignore risk sharing in this principal-agent relationship because the risk averse manager’s ability to bear risk is negligible compared to the risk neutral Board of Directors acting on the behalf of the initial shareholders. For this reason, if there were no moral hazard problem, it would be optimal for the principal to bear all the financial risk, and leave the agent fully insured against any source of fluctuation of his income. However, to induce the agent to provide intensive effort, the principal must impose some risk on the agent. Therefore, the compensation contract can be understood to balance the desire to provide incentives to the agent with the desire to impose as little risk on him as possible.

The agent’s effort choice

The incentive constraint (3.6c) characterizes the agent’s effort decision:

$$\max_{e_1, e_2} \mathbb{E}_{\tilde{e}_1, \tilde{e}_2} \left[ -\exp \left\{ -\left( \frac{1}{R} \right) \left( \alpha + \gamma \tilde{Y}^1 + \beta \tilde{Y}^2 - C(e_1, e_2) \right) \right\} \right]. \quad (3.7a)$$

That is, the agent maximizes his ex-ante expected utility by allocating his managerial effort levels across the two tasks.

Combining the normality assumption of $\tilde{Y}^1$ and $\tilde{Y}^2$ and the agent’s negative exponential utility function establishes that maximizing the agent’s objective function in (3.7a) is equivalent to maximizing the agent’s certainty equivalent:

$$\max_{e_1, e_2} \alpha + \gamma \tilde{Y}^1(e) + \beta \tilde{Y}^2(e) - C(e_1, e_2) - \left( \frac{1}{2R} \right) \text{Var} \left( \alpha + \gamma \tilde{Y}^1 + \beta \tilde{Y}^2 \right). \quad (3.7b)$$
Observe that the certainty equivalent is equal to the agent’s expected compensation minus his opportunity cost of providing effort \((e_1, e_2)\), and minus his cost of bearing risk imposed by the contract.

The risk associated with the wage contract (3.5) is

\[
\text{Var}\left( \alpha + \gamma \tilde{Y}^1 + \beta \tilde{Y}^2 \right) = \gamma^2 \text{Var}\left( \tilde{Y}^1 \right) + \beta^2 \text{Var}\left( \tilde{Y}^2 \right) + 2\gamma \beta \text{Cov}\left( \tilde{Y}^1, \tilde{Y}^2 \right),
\]

(3.8)

which is independent of the agent’s effort levels: \(e_1\) and \(e_2\).\(^{3,6}\)

Taking derivatives of (3.7b) with respect to \(e_1\) and \(e_2\) yields:

\[
\gamma \frac{\partial \bar{Y}^1(e)}{\partial e_i} + \beta \frac{\partial \bar{Y}^2(e)}{\partial e_i} = \frac{\partial C(e)}{\partial e_i}, \quad \forall i = 1, 2.
\]

(3.9a)

The above first-order conditions are both necessary and sufficient for the agent to maximize his expected utility. They imply that the agent selects his effort levels in such a way that his marginal gain from incremental effort equals his marginal personal cost of providing such additional effort. The marginal gain is the increased expected compensation,

\[
\frac{\partial E\left[ W\left( \bar{Y}^1, \bar{Y}^2 \right) \right]}{\partial e_i} = \gamma \frac{\partial \bar{Y}^1(e)}{\partial e_i} + \beta \frac{\partial \bar{Y}^2(e)}{\partial e_i};
\]

and the marginal cost is \(\frac{\partial C(e)}{\partial e_i}\).

**Lemma 3.1** The optimal managerial effort levels and the weights received by the two

\(^{3,6}\)This property is due to the special structure of our production function (3.1b), in which the managerial effort is assumed to affect the expected returns of the tasks but not to affect their uncertainties. In other words, the managerial effort only shifts the payoff distribution curve horizontally.
performance measures have the following relationship:\(^3\)

\[
\frac{\partial (e_1, e_2)}{\partial (\gamma, \beta)} = \begin{bmatrix}
\frac{\partial e_1}{\partial \gamma} & \frac{\partial e_1}{\partial \beta} \\
\frac{\partial e_2}{\partial \gamma} & \frac{\partial e_2}{\partial \beta}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
f_1 & h_1 \\
f_2 & h_2
\end{bmatrix}. \tag{3.9c}
\]

**Proof:** See Appendix A.2. \(Q.E.D.\)

The principal's decision problem

Note that the participation constraint (3.6b) is always binding in equilibrium, i.e.,

\[
E[\alpha + \gamma Y^1 + \beta Y^2] = C(e) + \left(\frac{1}{2\mathcal{R}}\right) \text{Var}\left(\alpha + \gamma Y^1 + \beta Y^2 - C(e)\right).
\]

The expected amount of managerial compensation exactly covers the agent's personal cost of managerial effort plus the risk premium. In other words, the agent always obtains zero certainty equivalent in equilibrium. Substituting it into the objective function (3.6a) gives the objective function of the principal:

\[
\max_{\gamma, \beta, e_1, e_2} \bar{X}(e) - C(e) - \left(\frac{1}{2\mathcal{R}}\right) \left(\begin{array}{c}
\gamma \\
\beta
\end{array}\right)^T \Sigma_{Y^1Y^2} \left(\begin{array}{c}
\gamma \\
\beta
\end{array}\right). \tag{3.10}
\]

This is also the total certainty equivalent of the principal and the agent.

**LEMMA 3.2** The optimal incentive linear compensation contract has coefficients for the two performance measures as follows:

\[
\left(\begin{array}{c}
\gamma \\
\beta
\end{array}\right) = \left[\frac{1}{\mathcal{R}} \Sigma_{Y^1Y^2} + \begin{bmatrix}
f_1 & f_2 \\
h_1 & h_2
\end{bmatrix} \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
f_1 & f_2 \\
h_1 & h_2
\end{bmatrix} \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{\partial \bar{X}}{\partial e_1} \\
\frac{\partial \bar{X}}{\partial e_2}
\end{bmatrix}\right]. \tag{3.11}
\]

\(^3\)In particular, the linear production function (3.1b) and the quadratic cost function of effort (3.2) imply that (3.9a) becomes:

\[
\begin{bmatrix}
f_1 & h_1 \\
f_2 & h_2
\end{bmatrix} \left(\begin{array}{c}
\gamma \\
\beta
\end{array}\right) = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix} \left(\begin{array}{c}
e_1 \\
e_2
\end{array}\right). \tag{3.9b}
\]

That is, there exists a linear closed form relation between the incentive weights on the two performance measures and the optimally induced effort levels and if the performance measures' means are linear in the agent's effort levels and the agent's personal cost function of effort is quadratic.
Proof: See Appendix A.2. Q.E.D.

Notice that the intercept ($a$) of the linear compensation contract (3.5) drops out as a decision variable, because it only represents the wealth transfer between the principal and the agent. With negative exponential utility functions, wealth has no effect.

3.3 Performance Measure Sensitivity, Congruity, and Diversity

Before characterizing the optimal compensation contract in the next section, we introduce some new concepts. When the managerial effort is single-dimensional, BD defines a measure of sensitivity of an information signal, e.g., $\tilde{Y}^1$, with respect to $e$ by:

$$\zeta_{y^i} = \frac{\partial \tilde{Y}^1(e)}{\partial e} - \frac{\sigma_{y^1y^2}}{\sigma_{y^2}^2} \frac{\partial \tilde{Y}^2(e)}{\partial e}. \quad (3.12)$$

That is, the sensitivity measures the extent to which the expected value of signal $\tilde{Y}^i$ changes with the manager's action, adjusted for the correlation with the other signal $\tilde{Y}^j$ whose expected value also changes with the manager's action.

When the managerial effort is multi-dimensional, this measure of sensitivity must be extended for the following reason. The sensitivity measure is a scaler, whereas $\frac{\partial \tilde{Y}^1(e)}{\partial e}$ and $\frac{\partial \tilde{Y}^2(e)}{\partial e}$ are inherently vectors in the multi-dimensional case, since there are as many first-order partial derivatives of a signal's mean as the dimensions of the manager's effort. The sensitivity measure must map the vectors into a scaler. This mapping would be trivial if each signal's mean were identically affected by the manager's multi-dimensional effort levels, i.e., $\frac{\partial \tilde{Y}^1(e)}{\partial e_1} = \frac{\partial \tilde{Y}^1(e)}{\partial e_2}$ ($i = 1, 2$). In such a case, $\frac{\partial \tilde{Y}^i(e)}{\partial e}$ can be expressed by $\frac{\partial \tilde{Y}^i(e)}{\partial e_1}$ (1, 1), the product of a scaler and the vector of ones.
3.3.1 Directional derivative of the first-best allocation of effort

Observe that the first-order partial derivatives, e.g., \( \frac{\partial Y^1}{\partial e_1} \) and \( \frac{\partial Y^1}{\partial e_2} \) of signal \( Y^1 \), give the rates of change of the signal's expected value \( Y^1(e) \) at \( e_0 \), measured in the directions of the positive \( e_1 \)- and \( e_2 \)-axes, respectively.

Since the principal maximizes the firm's expected value net of the agent's compensation, which is endogenously influenced by the agent's personal cost of effort, the principal would desire to induce the first-best allocation of effort if the risky asset dividend was contractible information and the noise could be completely filtered out. Given that the performance measures generally differ from the risky asset dividend and their noise can not be entirely eliminated, our analysis establishes that how fast \( Y^1(e) \) and \( Y^2(e) \) change value along the first-best effort allocation \( \nu^* \) is still important in characterizing the optimal linear contract. That is, the \( \nu^*-directional \ derivative, \) measuring the changes of the signals' means with respect to the manager's effort levels \( e_1 \) and \( e_2 \), is a key factor in the characterization of the optimal linear contract.

Denote the gradient vectors for \( X(e) \), \( Y^1(e) \), and \( Y^2(e) \), with respect to \( e \), as:

\[
\nabla \tilde{X}(e) \equiv \left( \frac{\partial \tilde{X}(e)}{\partial e_1}, \frac{\partial \tilde{X}(e)}{\partial e_2} \right) = (g_1, g_2), \quad (3.13a)
\]

\[
\nabla Y^1(e) \equiv \left( \frac{\partial Y^1(e)}{\partial e_1}, \frac{\partial Y^1(e)}{\partial e_2} \right) = (f_1, f_2), \quad (3.13b)
\]

\[
\nabla Y^2(e) \equiv \left( \frac{\partial Y^2(e)}{\partial e_1}, \frac{\partial Y^2(e)}{\partial e_2} \right) = (h_1, h_2). \quad (3.13c)
\]

Denote the \( \nu^*-directional \ derivatives of signals \( \tilde{Y}^1 \) and \( \tilde{Y}^2 \) as, respectively,\(^{3,8}\)

\[
D_{y^1} \equiv \nu^* \cdot \nabla \tilde{Y}^1(e) = v_{1}^* f_1 + v_{2}^* f_2, \quad (3.14a)
\]

\[
D_{y^2} \equiv \nu^* \cdot \nabla \tilde{Y}^2(e) = v_{1}^* h_1 + v_{2}^* h_2. \quad (3.14b)
\]

\(^{3,8}\)The risky asset's directional derivative along the first-best allocation of effort is:

\[
D_{x} \equiv \nu^* \cdot \nabla \tilde{X}(e) = v_{1}^* g_1 + v_{2}^* g_2.
\]
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They measure the extent to which the expected values of signals $\tilde{Y}^1$ and $\tilde{Y}^2$ change with the manager’s actions $e$ along the first-best effort allocation $v^e$. 3.9

3.3.2 Congruity and diversity

We now extend FX’s measure of congruity. Let

$$Q_{lm} \equiv \nabla \tilde{l}(e) \times \nabla \tilde{m}(e) = \tilde{l}_1 \tilde{m}_2 - \tilde{l}_2 \tilde{m}_1, \quad (3.15)$$

be the magnitude of the gradient cross product of the variables $\tilde{l}$ and $\tilde{m}$ with respect to the agent’s effort level $e$ ($l, m = X, Y^1, Y^2$).

FX use $Q_{el} \equiv g_1 \tilde{l}_2 - g_2 \tilde{l}_1$ to define a measure of the degree of congruence between the impact of the agent’s actions on the performance measure $\tilde{l}$ and on the firm’s gross payoff $\tilde{X}$. Since the cross product of two vectors represents the area of the parallelogram formed by the two vectors, the area is zero if, and only if, either the two vectors are parallel with each other, or one of them is the zero vector. We say that signal $\tilde{l}$ is congruent with the risky asset $\tilde{X}$ in the former case, is action-irrelevant in the latter, and is non-congruent otherwise (i.e., $Q_{el} \neq 0$).

Observe that $Q_{Y_1Y_2}$ measures the diversity of two performance measures, i.e., the relative alignment with each other. We say that $\tilde{Y}^1$ and $\tilde{Y}^2$ are perfectly aligned with each other if $Q_{Y_1Y_2} = 0$. Clearly, $Q_{Y_1Y_2} = 0$ if, and only if, either the gradients of $\tilde{Y}^1(e)$ and $\tilde{Y}^2(e)$ are parallel with each other or one of them is the zero vector. The latter case is degenerate, indicating that there is only one informative performance measure about the agent’s effort. We say $\tilde{Y}^1$ and $\tilde{Y}^2$ are diverse, 3.10 if $Q_{Y_1Y_2} \neq 0$. BI consider two cases.

In their first case, $h_1 = f_1 = g_1$, and $h_2 = f_2 = g_2$, i.e., both $\tilde{Y}^1$ and $\tilde{Y}^2$ reflect the firm’s

3.9 Alternatively, we can write $D_l = v^i \cdot \nabla \tilde{X}(e), l = Y^1, Y^2$, measuring the firm’s expected net value reflected by the information signal $\tilde{l}$ since $v^i$ is the agent’s induced effort allocation if signal $\tilde{l}$ is the sole performance measure. We abandon this interpretation because we wish to emphasize the shareholders’ desire for the first-best effort level.

3.10 Graphically, vectors $\nabla \tilde{Y}^1(e)$ and $\nabla \tilde{Y}^2(e)$ point in different directions.
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total value, and hence, $\tilde{Y}^1$ and $\tilde{Y}^2$ are not diverse and $Q_{y'Y^2} = 0$. In their second case, $h_1 = f_1 = g_1$, $h_2 = g_2$, and $f_2 = 0$, i.e., $\tilde{Y}^2$ reflects the total terminal payoff of the firm, and $\tilde{Y}^1$ only reflects a partial value of the firm, and hence, $\tilde{Y}^1$ and $\tilde{Y}^2$ are diverse and $Q_{y'Y^2} = g_1g_2 \neq 0$.

**DEFINITION 4** Information signals $\tilde{Y}^1$ and $\tilde{Y}^2$ are action-irrelevant if $\nabla\tilde{Y}^1(e) = 0$ and $\nabla\tilde{Y}^2(e) = 0$, respectively, and are action-relevant otherwise. Action-relevant signals $\tilde{Y}^1$ and $\tilde{Y}^2$ are congruent with the firm’s gross payoff if $Q_{xy^1} = 0$ and $Q_{xy^2} = 0$, respectively, and are non-congruent otherwise. $\tilde{Y}^1$ and $\tilde{Y}^2$ are non-diverse/perfectly aligned with each other if $Q_{y'Y^2} = 0$, and are diverse otherwise.\textsuperscript{3,11} Moreover, $Q_{xy^2}Q_{y'Y^1}$ is the adjusted congruity of $\tilde{Y}^1$ through signal $\tilde{Y}^2$, and $Q_{xy^1}Q_{y'Y^2}$ is the adjusted congruity of $\tilde{Y}^2$ through signal $\tilde{Y}^1$.

The adjusted congruity of a signal through another signal equals zero if, and only if, either the second signal is congruent with the firm’s payoff or the two signals are perfectly aligned with each other. The latter is necessary and sufficient for both adjusted congruities to be zero. Although an action-irrelevant signal is not influenced by the agent’s effort, it may still be useful in compensation contracts for reducing risk.

### 3.3.3 Sensitivity

Having discussed those new concepts in detail, we now use them to formally define the sensitivity of signals $\tilde{Y}^1$ and $\tilde{Y}^2$ with respect to $e$, respectively:

\[
\zeta_{y^1} \equiv D_{y^1} - \frac{\sigma_{y^2y^2}}{\sigma_{y^1}^2} D_y - \frac{\mathcal{R}}{\sigma_{y^2}^2(C_{11}C_{22} - C_{12}^2)I_x} Q_{xy^2}Q_{y'y^1},
\]

\[
\zeta_{y^2} \equiv D_{y^2} - \frac{\sigma_{y^2y^2}}{\sigma_{y^1}^2} D_y - \frac{\mathcal{R}}{\sigma_{y^1}^2(C_{11}C_{22} - C_{12}^2)I_x} Q_{xy^1}Q_{y'y^2}.
\]

\textsuperscript{3,11}By this definition, an action-irrelevant signal is always non-diverse/perfectly aligned with any other signal. We could exclude this special case from the category of non-diverse/perfectly aligned measures, but for simplicity, we leave the definition as it stands.
That is, the extended sensitivity measure reflects the extent to which the mean of the signal changes with the manager’s effort in the first-best direction/allocation, and takes into account both the correlation with and the adjusted congruity through the other signal, if its mean changes with the manager’s effort e.\(^{3.12}\)

### 3.4 Optimal Managerial Effort Levels and Linear Compensation Contract

We solve the Board of Directors decision problem (3.6) and explicitly characterize the optimal linear contract.

**PROPOSITION 3.1** The weights on the two information signals are\(^{3.13}\)

\[
\gamma = RI^c \sigma_{y^1}^2 \zeta_{y^1} / \mathcal{L} \quad \text{and} \quad \beta = RI^c \sigma_{y^2}^2 \zeta_{y^2} / \mathcal{L},
\]

where \(\mathcal{L}\) is positive.\(^{3.14}\) Hence, the weight on the performance measure \(\bar{Y}^1\) relative to the weight on the performance measure \(\bar{Y}^2\) is

\[
\frac{\gamma}{\beta} = \frac{\sigma_{y^2}^2 \zeta_{y^1}}{\sigma_{y^1}^2 \zeta_{y^2}}. \tag{3.18}
\]

\(^{3.12}\)We can alternatively express our sensitivities (3.16a) and (3.16b) as follows:

\[
\zeta_{y^1} = \nu^x \cdot \left( \nabla \bar{Y}^1(e) - \frac{\sigma_{y^2}^2}{\sigma_{y^1}^2} \nabla \bar{Y}^2(e) \right) - \frac{\mathcal{R}}{\sigma_{y^1}^2 (C_{11} C_{22} - C_{12}^2)^{1/2}} \sigma_{y^1}^2 \sigma_{y^2} \sigma_{y^1}^2 \left(1 - \rho_{y^1 y^2}^2\right),
\]

\[
\zeta_{y^2} = \nu^x \cdot \left( \nabla \bar{Y}^2(e) - \frac{\sigma_{y^2}^2}{\sigma_{y^1}^2} \nabla \bar{Y}^1(e) \right) - \frac{\mathcal{R}}{\sigma_{y^1}^2 (C_{11} C_{22} - C_{12}^2)^{1/2}} \sigma_{y^1}^2 \sigma_{y^2} \sigma_{y^1}^2 \left(1 - \rho_{y^1 y^2}^2\right).
\]

They include BD’s measure of sensitivity as the special case in which the adjusted congruities are zero and \(\nu^x\) is simply one.

\(^{3.13}\)Clearly, \(\lim_{\sigma_{y^2}^2 \rightarrow +\infty} \beta = 0, \lim_{\rho_{y^1 y^2} \rightarrow +\infty} \gamma = 0, \lim_{\sigma_{y^2}^2 \rightarrow 0} \beta = 0,\) and \(\lim_{\rho_{y^1 y^2} \rightarrow 0} \gamma = 0.\) That is, an infinitely noisy information signal is always excluded from the incentive contract, and the fixed payment contract is efficient for an extremely risk averse agent. Therefore, in the rest of the analysis, we assume that both performance measures contain finite noise and the agent is not extremely risk averse.

\(^{3.14}\)\(\mathcal{L} \equiv \mathcal{R} \left[ TV^1 \left( \nabla \bar{Y}^1(e) - \frac{\sigma_{y^2}^2}{\sigma_{y^1}^2} \nabla \bar{Y}^2(e) \right) \sigma_{y^2}^2 + TV^2 \left( \nabla \bar{Y}^2(e) - \frac{\sigma_{y^2}^2}{\sigma_{y^1}^2} \nabla \bar{Y}^1(e) \right) \sigma_{y^1}^2 \right] - \frac{\mathcal{R}^2 \sigma_{y^1}^2 \sigma_{y^2}^2}{C_{11} C_{22} - C_{12}^2} + \sigma_{y^1}^2 \sigma_{y^2}^2 \left(1 - \rho_{y^1 y^2}^2\right),\) where \(TV^1 \equiv \left( C_{11} C_{22} - C_{12}^2, C_{11} C_{22} - C_{12}^2 \right), TV^2 \equiv \left( C_{11} C_{22} - C_{12}^2, C_{11} C_{22} - C_{12}^2 \right).\)
The manager is induced to select the optimal effort levels:
\[
\begin{pmatrix}
e^*_1 \\
e^*_2
\end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} f_1 & h_1 \\ f_2 & h_2 \end{bmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}.
\] (3.19)

**Proof**: See Appendix A.2. Q.E.D.

Observe that (3.18) states that the relative weight on the two performance measures equals the ratio of their extended sensitivity measures times the inverse of variances of those performance measures. Our measure of sensitivity not only utilizes directional derivatives – the natural extension of a partial derivative, but also incorporates the performance measures' congruity and diversity. In the following, we discuss how congruent and/or diverse performance measures influence the sensitivity measure.

Table 3.1 lists sixteen special scenarios in which the two performance measures contain different information about the firm’s terminal value.

### 3.4.1 Perfectly aligned performance measures

When a single performance measure is used in the design of an incentive contract, the primary concern is its congruity and precision. Congruity determines how close the agent’s induced effort allocation is to the first-best allocation, while the precision influences the intensity of the induced effort.

When multiple performance measures are used in the design of an incentive contract, their congruity and precision are important, but an additional factor, the diversity (or the relative alignment) between these signals is also important. This additional factor is not an issue if the manager’s effort is single-dimensional, since any signal is either congruent or action-relevant. In either case, \( Q_{y_1y_2} = 0 \). Thus, Proposition 3.1 includes the following special case in which these performance measures are not diverse:

\[
\gamma = \mathcal{R} \left[ \sigma_{y y}^2 D_{y^1} - \sigma_{y y^2} D_{y^2} \right] / \mathcal{L},
\]  
(3.20a)
Table 3.1: Relative weight of optimal incentive contract

<table>
<thead>
<tr>
<th>$\psi^2$</th>
<th>$\psi^1$</th>
<th>$\psi_2^2$</th>
<th>$\psi_1^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_x$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
<td>$\beta = 0$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
</tr>
<tr>
<td>$[h_1 = 0; h_2 = 0]$</td>
<td>$\gamma = 0$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
</tr>
<tr>
<td>$[h_1 = 0; h_2 = 0]$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
</tr>
<tr>
<td>$[h_1 = 0; h_2 = 0]$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
<td>$Q_{y1}y_2 = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$</td>
</tr>
</tbody>
</table>

$\beta = \frac{\sigma_y^2 - \sigma_{y1}^2}{\sigma_{y1}^2}$

$D_x^e = f_1 \psi_1^2 + f_2 \psi_2^2$

$D_x^e = h_1 \psi_1^2 + h_2 \psi_2^2$

$D_x = g_1 \psi_1^2 + g_2 \psi_2^2$

$\nu_2 = (\nu_1^0, \nu_2^0) = \left( \frac{C_{22} - C_{12}^1}{(C_{11} C_{22} - C_{12}^1)^{1/2}}, \frac{C_{22} - C_{12}^1}{(C_{11} C_{22} - C_{12}^1)^{1/2}} \right)$

$Q_{y1}y_2 = (f_1 h_2 - f_2 h_1)$

$\text{Cons}_A = \frac{R_{g1}^2 \psi_2^2}{(C_{11} C_{22} - C_{12}^1)^{1/2}}$
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\[ \beta = \mathcal{R} \left[ \sigma^2_{v_1} D_{v_1} - \sigma_{v_1 v_2} D_{v_1} \right] / \mathcal{L}; \]  
(3.20b)

which imply that the relative weight is

\[ \frac{\gamma}{\beta} = \frac{\sigma^2_{v_2} \left[ D_{v_2} - \frac{\sigma_{v_1 v_2}}{\sigma^2_{v_1}} D_{v_1} \right]}{\sigma^2_{v_1} \left[ D_{v_1} - \frac{\sigma_{v_1 v_2}}{\sigma^2_{v_1}} D_{v_1} \right]}. \]  
(3.20c)

This is almost identical to BD's result in a single-task agency setting, except that the directional derivatives of the first-best allocation of effort replace the partial derivatives. The substantial modification of the relative weight on the public report compared to BD's is due to the diversity of the performance measures.

**PROPOSITION 3.2** In the multi-task agency setting, congruence does not influence the relative weight on the performance measures in the optimal contract if the performance measures are not diverse.

If performance measures are not diverse, they only provide information about the aggregate effect of the agent's effort in a single direction, and the multi-dimensionality of effort cannot be reflected. Thus, the multi-dimensional action aspect has no real impact on the relative weight on the performance measures [see (3.18)].

There is no problem if performance measures are not diverse, but at least one is congruent with the firm's gross payoff. The first-best allocation of effort is induced and multiple measures are used to reduce the risk imposed. If performance measures are not diverse and neither of them is congruent, then the first-best allocation of effort is never induced, but an additional measure is still useful for reducing risk.

In the following, we examine an important example (one of BI's special case) of perfectly aligned performance measures: both signals are congruent with the firm's gross payoff.
Let \( f_1/f_2 = h_1/h_2 = g_1/g_2 \). Then, the two gradients of the signals are parallel to the same vector \( \nabla \tilde{X}(e) \), and hence \( Q_{y_1y_2} = 0 \). Substituting it into (3.14) yields: \( D_{y_1} = D_{y_2} = D_z \).

**COROLLARY 3.3** If both information signals are congruent performance measures with respect to the firm value, the incentive weights are:

\[
\begin{align*}
\gamma &= \frac{\mathcal{R}D_x \left( \sigma_{y_2}^2 - \sigma_{y_2y_2} \right)}{\mathcal{R}D_x \left( \sigma_{y_1}^2 + \sigma_{y_2}^2 - 2\sigma_{y_1y_2} \right) + \sigma_{y_1}^2 \sigma_{y_2}^2 (1 - \rho_{y_1y_2}^2)}, \\
\beta &= \frac{\mathcal{R}D_x \left( \sigma_{y_1}^2 - \sigma_{y_2y_2} \right)}{\mathcal{R}D_x \left( \sigma_{y_1}^2 + \sigma_{y_2}^2 - 2\sigma_{y_1y_2} \right) + \sigma_{y_1}^2 \sigma_{y_2}^2 (1 - \rho_{y_1y_2}^2)},
\end{align*}
\]

which imply that the relative weight is

\[
\frac{\gamma}{\beta} = \frac{\sigma_{y_2}^2 - \sigma_{y_2y_2}}{\sigma_{y_1}^2 - \sigma_{y_2y_2}}.
\]

From (3.21a) and (3.21b), it is obvious that the signs of the individual weights depend on the covariance structure of the information:

\[ \text{Sign} \{\gamma\} = \text{Sign} \left\{ \sigma_{y_2}^2 - \sigma_{y_2y_2} \right\} \quad \text{and} \quad \text{Sign} \{\beta\} = \text{Sign} \left\{ \sigma_{y_1}^2 - \sigma_{y_2y_2} \right\}. \]

That is,

\[ \gamma > 0 \quad \text{if, and only if,} \quad \sigma_{y_2}^2 > \sigma_{y_1}^2 \rho_{y_1y_2}, \]
\[ \beta > 0 \quad \text{if, and only if,} \quad \sigma_{y_1}^2 > \sigma_{y_1}^2 \rho_{y_1y_2}. \]

Both incentive weights are positive if, and only if, \( \min \left\{ \sigma_{y_1}^2, \sigma_{y_2}^2 \right\} > \sigma_{y_1y_2} \). The noise contained in either of the two performance measures is greater than the covariance between them.

Since the covariance matrix is positive definite, i.e., \( \sigma_{y_1}^2 \sigma_{y_2}^2 > \sigma_{y_1y_2}^2 \), the larger of the two variances, \( \sigma_{y_1}^2 \) and \( \sigma_{y_2}^2 \), must be greater than \( \sigma_{y_1y_2} \). Consequently, the following proposition is immediate.
PROPOSITION 3.4 If both performance measures are congruent with respect to the firm value, the weight of performance measure $\hat{Y}^1$ ($\forall l,m = 1,2$ and $l \neq m$) is positive (zero, or negative) if, and only if,

$$ \frac{\sigma_{y^1}}{\sigma_{y^2}} > (\equiv, <) \rho_{y^1 y^2}. $$

Furthermore, the less noisy performance measure always receives a positive weight.

3.4.2 Diverse performance measures

In a multi-dimensional effort setting, performance measures are usually diverse. Their congruity with respect to the firm’s gross payoff is no longer binary (i.e., either congruent or action-irrelevant); they can be non-congruent. The following corollary illustrates three special cases in which the two signals are diverse.

COROLLARY 3.5 The relative weight of the two performance measures is as follows if they are diverse.

1. If $\hat{Y}^1$ is non-congruent (with $f_1 = g_1$ and $f_2 = 0$), and $\hat{Y}^2$ is congruent, then

$$ \frac{\gamma}{\beta} = \frac{v_1^x g_1 - \frac{\sigma_{y^1}}{\sigma_{y^2}} (v_1^x g_1 + v_2^x g_2)}{(v_1^x g_1 + v_2^x g_2) - \frac{\sigma_{y^1}}{\sigma_{y^2}} v_1^x g_1 + \frac{R}{\sigma_{y^1}(C_{11}C_{22} - C_{12}^2)} g_1^2 g_2^2}.$$

2. If $\hat{Y}^1$ is congruent and $\hat{Y}^2$ is non-congruent (with $h_1 = 0$ and $h_2 = g_2$), then

$$ \frac{\gamma}{\beta} = \frac{(v_1^x g_1 + v_2^x g_2) - \frac{\sigma_{y^1}}{\sigma_{y^2}} v_1^x g_2 + \frac{R}{\sigma_{y^2}(C_{11}C_{22} - C_{12}^2)} g_1^2 g_2^2}{v_2^x g_2 - \frac{\sigma_{y^1}}{\sigma_{y^2}} (v_1^x g_1 + v_2^x g_2)}.$$

3. If both $\hat{Y}^1$ and $\hat{Y}^2$ are non-congruent (with $f_1 = g_1$, $h_2 = g_2$, $f_2 = h_1 = 0$), i.e., each signal reflects the payoff information of only one managerial task, then,

$$ \frac{\gamma}{\beta} = \frac{v_1^x g_1 - \frac{\sigma_{y^1}}{\sigma_{y^2}} v_2^x g_2 + \frac{R}{\sigma_{y^2}(C_{11}C_{22} - C_{12}^2)} g_1^2 g_2^2}{v_2^x g_2 - \frac{\sigma_{y^1}}{\sigma_{y^2}} v_1^x g_1 + \frac{R}{\sigma_{y^1}(C_{11}C_{22} - C_{12}^2)} g_1^2 g_2^2}.$$
The first two parts of the corollary show that the congruent signal always has an incremental amount of incentive relative to the non-congruent signal. For example, in case 1, the incentive weight of congruent signal $\tilde{Y}^2$ is shifted upwards by an amount of $\frac{R}{\sigma^2_{\tilde{y}_1}(C_{11}C_{22}-C_{12}^2)1^2} g_1^2 g_2^2$. Consequently, this diversity term is referred to as the incremental incentive provided by congruent signal $\tilde{Y}^2$ over non-congruent signal $\tilde{Y}^1$. It is zero if $\tilde{Y}^1$ is infinitely noisy ($\sigma^2_{\tilde{y}_1} = +\infty$).

If both performance measures are non-congruent, i.e., each reflects the payoff information of one activity only, it is shown that the incremental incentive term is added to each of the information signals. That is, the incentive weights of the two signals $\tilde{Y}^1$ and $\tilde{Y}^2$ are shifted upwards by amounts of $\frac{R}{\sigma^2_{\tilde{y}_1}(C_{11}C_{22}-C_{12}^2)1^2} g_1^2 g_2^2$ and $\frac{R}{\sigma^2_{\tilde{y}_2}(C_{11}C_{22}-C_{12}^2)1^2} g_1^2 g_2^2$, respectively. The rationale is analogous. They represent the incremental incentives provided by one performance measure over the other.

Directly applying BD’s sensitivity measure to each action, BI are able to relate their results to BD’s. They explain that “$\gamma$ and $\beta$ are simply weighted sums of the ‘sensitivity times precision’ with respect to $c_1$ and $c_2$” (BI, p.13), and they recognize that these weights ($\mu_1$ and $\mu_2$), given by their eqs. (11) and (17), are the Lagrange multipliers associated with the constraints. Since those weights are complicated not only by the coefficients ($a_x$ and $a_y$) of the rational market price, but also by the incentive weights ($\beta$ and $\gamma$), no intuition is given. In their eq. (11), the complexity of those weights can be removed, since the first components of those weights are coincidentally the same, and hence can be dropped out. The second components which are the determining factor of those weights are, in fact, the first-best effort levels. In their eq. (17), however, those weights are radically different from each other. The complexity prevents them from explaining those weights any further.

BI recognize that there is diversity but they only consider the special cases in which at least one of the performance measures is congruent with the firm’s gross payoff. Although
they observe "that with two managerial actions and a limited information earnings variable, the use of relative weights (on the performance measures) does not eliminate the potentially confounding effects of manager- and firm-specific characteristics" (BI, p.5), they fail to recognize that adjusted congruity is a key determinant of sensitivity because of their unexplainably complex weights [their eq. (17), in particular]. They have to re-specify their determinants of sensitivities, $S_p$ and $S_y$ in their eq. (11), in order to explain the result of their eq. (16): "that $\beta$ and $\gamma$ are still weighted sums of sensitivities times precision". However, "there is a distinct breakdown in the symmetry of the solution" (BI, p.16). In short, BI's special information structure does not allow them to extend BD's measure of sensitivity to the multi-task agency settings, to restore that breakdown in the symmetry of their solution.

3.4.3 Action-irrelevant performance measure

Investors in real financial markets expend resources and undertake numerous activities to acquire information provided by, for example, active markets in advisory services, and/or other sources (e.g., investment newsletters). It is evident that financial markets possess diverse information. Although it may be reasonable to assume that private information obtained by rational investors is a noisy representation of the risky asset dividend (e.g., as in BI and Kim and Suh 1993), it is also noteworthy that some information pertains strictly to non-controllable events that influence the risky asset dividend. The latter type includes information about future competitive product prices, interest rates, and exchange rates. That is, the latter type of private information is action-irrelevant.

We now focus on the setting in which the second information signal $\bar{Y}_2$ is action-irrelevant, i.e., $h_1 = h_2 = 0$. Hence,

$$\bar{Y}^2 \equiv \bar{Y}_2(e) + \bar{e}_y^2 = \bar{e}_y^2.$$
In this setting, $D_{2} = 0$ and $Q_{2y_{2}} = Q_{2y_{1}} = 0$. Substituting them into Proposition 3.1 provides the following result.

**COROLLARY 3.6** If signal $\tilde{Y}^{2}$ is action-irrelevant, then the incentive weights on the two performance measures $\tilde{Y}^{1}$ and $\tilde{Y}^{2}$ are:

$$\gamma = \frac{RI D_{y_{1}}^{2}}{R \cdot T v_{1} \cdot \nabla \tilde{Y}^{1}(e) + \sigma_{y_{1}}^{2} \left(1 - \rho_{y_{1}y_{2}}^{2}\right)}, \quad (3.23a)$$

$$\beta = -\frac{RI D_{y_{2}}^{2}}{\sigma_{y_{2}}^{2} \left[R T v_{1} \cdot \nabla \tilde{Y}^{1}(e) + \sigma_{y_{1}}^{2} \left(1 - \rho_{y_{1}y_{2}}^{2}\right)\right]}, \quad (3.23b)$$

which imply that the relative weight is

$$\frac{\gamma}{\beta} = -\frac{\sigma_{y_{2}}^{2}}{\sigma_{y_{1}y_{2}}^{2}}. \quad (3.23c)$$

It is clear from (3.23b) that the incentive weight $\beta$ equals zero if performance measure $\tilde{Y}^{2}$ is either independent of performance measure $\tilde{Y}^{1}$, or infinitely noisy. It obviously is uninformative in the latter case, and hence valueless as a performance measure. In the former case, since knowing $\tilde{Y}^{2}$ informs us nothing about $\tilde{Y}^{1}$, $\tilde{Y}^{2}$ cannot be used to reduce the noise contained in $\tilde{Y}^{1}$ about managerial effort. Therefore, it is clearly intuitive to exclude such an action-irrelevant signal $\tilde{Y}^{2}$ from the incentive contract.

The incentive weight $\gamma$ is always positive and independent of the precision of signal $\tilde{Y}^{2}$. This exactly depicts the following intuition. First, the positiveness of weight $\gamma$ is due to the fact that $\tilde{Y}^{1}$ is the sole informative signal about the agent's actions. To induce the agent's effort levels, we have to put a positive weight on the sole informative signal. Second, it is the correlation between signals $\tilde{Y}^{1}$ and $\tilde{Y}^{2}$ that is effectively employed by the incentive contract. Observation of $\tilde{Y}^{2}$ is informative about $\tilde{Y}^{1}$ if, and only if, $\tilde{Y}^{1}$ and $\tilde{Y}^{2}$ are correlated with each other.
PROPOSITION 3.7 If signal $\hat{Y}^2$ is action-irrelevant and $\hat{Y}^1$ is action-relevant, the incentive weight received by $\hat{Y}^1$ is always positive and independent of $\hat{Y}^2$'s variance.\textsuperscript{3.15} The incentive weight received by $\hat{Y}^2$ is positive (zero or negative) if, and only if, the correlation between $\hat{Y}^1$ and $\hat{Y}^2$ is negative (zero or positive).

Having discussed the individual incentive weight of the two signals, we now examine the relative weight of the two information signals: $\gamma/\beta = -\frac{\sigma_{\hat{Y}^2}^2}{\sigma_{\hat{Y}^1}^2 \rho_{\hat{Y}^1 \hat{Y}^2}}$. This is incredibly simple in form, and is identical to BD's result (see Proposition 5 of BD, p.32) in the single-task setting.

PROPOSITION 3.8 If signal $\hat{Y}^2$ is action-irrelevant, the relative weight is independent of the sensitivity of action-relevant signal $\hat{Y}^1$, i.e., $\gamma/\beta = -\frac{\sigma_{\hat{Y}^2}^2}{\sigma_{\hat{Y}^1}^2 \rho_{\hat{Y}^1 \hat{Y}^2}}$.

Proof: See Appendix A.2. Q.E.D.

Of particular note is the fact that, even in a multi-task setting, the relative weight on the two signals $\gamma/\beta$ is completely independent of the impact of the manager's effort on signal $\hat{Y}^1$. This relative weight keeps this form as long as the signal $\hat{Y}^1$ is action-relevant and the signal $\hat{Y}^2$ is action-irrelevant. The degree of $\hat{Y}^1$'s congruity with respect to the firm's gross payoff does not affect this relative weight, except for the special case in which signal $\hat{Y}^1$ is action-irrelevant as well.

As we discussed in the above sub-section, a second performance measure can be employed to expand the implementable effort set. However, expansion does not occur when signal $\hat{Y}^2$ is action-irrelevant. The implementable effort set based on the two signals is the same as that based on signal $\hat{Y}^1$ alone. The first-best allocation of effort is implementable if, and only if, the signal $\hat{Y}^1$ is congruent with the firm's gross payoff.

\textsuperscript{3.15}Their correlation ($\rho_{\hat{Y}^1 \hat{Y}^2}$) is fixed to be a constant.
Despite of the loss of this facilitating role, the signal $Y^2$ may still be used in the optimal incentive contract because it helps reduce the risk imposed on the agent. It is trivial that the signal $Y^2$ is excluded (i.e., $\beta = 0$) if it is infinitely noisy. Thus, we focus on the case in which the signal $Y^2$ contains bounded noise and is correlated with the signal $Y^1$.

To see how an action-irrelevant signal (i.e., $Y^2$) helps to reduce the risk imposed on the manager, we observe that $\beta$ is positive (negative) if $\rho_{Y^1 Y^2} < (>) 0$; and $\gamma$ is always positive (note that $Y^1$ is action-relevant). If $Y^1$ and $Y^2$ are positively correlated (as they might be if $Y^1$ reflects the firm’s earnings and $Y^2$ is a measure of general market conditions), then $\gamma$ is positive and $\beta$ is negative. The firm’s earnings are influenced by controllable events and the market condition provides information about those uncontrollable events. Deducting a multiple of $Y^2$ partially eliminates the impact of the uncontrollable events in $Y^1$, and thereby reduces the risk imposed on the manager without reducing his incentive to expend effort.

To conclude this subsection, we consider the special case in which both information signals are action-irrelevant.

**Proposition 3.9** If both signals $Y^1$ and $Y^2$ contain finite amount of noise, then the fixed payment contract is optimal if, and only if, both $Y^1$ and $Y^2$ are action-irrelevant.

If both signals $Y^1$ and $Y^2$ are action-irrelevant, neither of them is then informative about the managerial effort levels. Intuitively, they should be optimally excluded from the incentive contract.

On the other hand, if the fixed payment is optimal, then the signals must have been uninformative about the agent’s actions, given that each contains finite amount of noise. That is, both signals are action-irrelevant.
3.4.4 Scale effect of performance measures

We now examine how the scale of an information signal influences the relative weight given to the two performance measures. The scale of a performance measure has no impact on its information content and therefore has no impact on the second-best effort level induced with the performance measure. However, the scale does influence the weight assigned to the information signal in the optimal linear incentive contract.

**PROPOSITION 3.10** Scaling information signals $\tilde{Y}^1$ and $\tilde{Y}^2$ by positive constants $K_1$ and $K_2$ results in their individual incentive weights being scaled by $\frac{1}{K_1}$ and $\frac{1}{K_2}$, respectively.

$$\gamma_{\text{new}} = \frac{\gamma}{K_1}, \text{ and } \beta_{\text{new}} = \frac{\beta}{K_2}. \quad (3.24)$$

The relative weight on the scaled signals equals that on the non-scaled signals (i.e., $\gamma_{\text{new}}/\beta_{\text{new}} = \gamma/\beta$) if, and only if, the scaling constants are equal (i.e., $K_1 = K_2$). In general, scaling the performance measures differently affects the relative weight given to the performance measures. For example, if $\tilde{Y}^1$ is initially expressed in dollars, and is then expressed in thousands dollars, then $K_1 = 1/1000$ and $\gamma_{\text{new}}/\beta_{\text{new}} = 1000 \gamma/\beta$. Since scaling is important in making empirical inferences, care is called for when interpreting some comparative static results with respect to the relative weight on the public report.

3.5 Performance Measure Diversity and Effort Allocation

"When there are multiple tasks, incentive pay serves not only to allocate risks and to motivate hard work, it also serves to direct the allocation of the agent’s attention among their various duties" (Holmstrom and Milgrom, 1991, p.25). In this section, we focus on the allocation of effort, and not on the effort intensity."
DEFINITION 5 Sub-optimal allocation of effort occurs if the induced second-best effort allocation differs from the first-best, i.e.,

$$\frac{v_1^*}{v_2^*} \neq \frac{v_1^F}{v_2^F}$$  \quad (3.25)

The optimal incentive contract characterized by Proposition 3.1 is based on two performance measures. To provide a benchmark for our analysis and discussion in this section, we characterize the optimal incentive compensation contracts based on individual performance measures separately, and solve for the optimal effort levels. That is, we solve the optimal compensation contracts $W(\alpha_1, \gamma_a) = \alpha_1 + \gamma_a \tilde{Y}^1$, and $W(\alpha_2, \beta_a) = \alpha_2 + \beta_a \tilde{Y}^2$.

LEMMA 3.3 The incentive weights of the optimal incentive contracts which are based on $\tilde{Y}^1$ and $\tilde{Y}^2$ separately are

$$\gamma_a = \frac{RI\sigma_{y}^{y_1}}{\sigma_v^{y_1} + RTv^1 \cdot \nabla \tilde{Y}^1(e)} \quad \text{and} \quad \beta_a = \frac{RI\sigma_{y}^{y_2}}{\sigma_v^{y_2} + RTv^2 \cdot \nabla \tilde{Y}^2(e)},$$  \quad (3.26)

and the induced second-best effort levels are

$$e^{v1} = \gamma_a T^{v1} \quad \text{and} \quad e^{v2} = \beta_a T^{v2},$$  \quad (3.27)

where $T^{v1} \equiv \left( \frac{C_{12}h_2 - C_{12}h_1}{C_{11} h_2 - C_{11} h_1}, \frac{C_{12}h_1 - C_{12}h_2}{C_{11} h_2 - C_{11} h_1} \right)$, and $T^{v2} \equiv \left( \frac{C_{22}h_1 - C_{22}h_2}{C_{11} h_2 - C_{11} h_1}, \frac{C_{22}h_2 - C_{22}h_1}{C_{11} h_2 - C_{11} h_1} \right)$.

Proof: See Appendix A.2. \quad Q.E.D.

In this benchmark case, it is clear that the allocation of the induced second-best effort levels $e^{v1}$ and $e^{v2}$ crucially depend on the congruity of the individual performance measure. For example, if $\tilde{Y}^2$ is congruent with the firm’s terminal value (i.e., $h_1 = g_1$, $h_2 = g_2$), we obtain that $e^{v2} = \beta_a e^*$, implying that the second-best effort level is a multiple of the first-best effort level. In other words, $\beta_a$ equals the relative second-best effort level with respect to the first-best effort level. It represents the incentive weight provided by the information signal $\tilde{Y}^2$. 
If both performance measures are simultaneously used in the incentive compensation contract, their diversity is an important factor affecting the managerial effort allocation. By (3.19) and (3.27), the following result is immediate.

**PROPOSITION 3.11:** The second-best effort level, induced by the optimal linear incentive contract based on both performance measures, is a weighted sum of the second-best effort levels, $e^\gamma_1$ and $e^\gamma_2$, induced by performance measures $\tilde{Y}^1$ and $\tilde{Y}^2$ separately:

$$e^* = \left( \frac{\gamma}{\gamma_a} \right) e^\gamma_1 + \left( \frac{\beta}{\beta_a} \right) e^\gamma_2. \quad (3.28a)$$

When incentive contracts are based on $\tilde{Y}^1$ and $\tilde{Y}^2$ separately, we have optimal compensation contracts: $\gamma_a\tilde{Y}^1$ and $\beta_a\tilde{Y}^2$. Scaling the performance measures by their individual incentive weights (i.e., $\beta_a$ and $\gamma_a$), we obtain that $\beta/\beta_a$ and $\gamma/\gamma_a$ are the incentive weights assigned to these two scaled performance measures (see Proposition 3.10).

Another interesting aspect of those coefficients (i.e., $\beta/\beta_a$ and $\gamma/\gamma_a$) is their magnitude. Take $\beta_a$ for example, it represents the relative incentive power provided by performance measure $\tilde{Y}^2$ in the presence of another measure $\tilde{Y}^1$ with respect to that provided by itself alone. Notice that

$$\lim_{\sigma^2_{y^2} \rightarrow +\infty} \left( \frac{\beta}{\beta_a} \right) = \frac{\mathcal{RT} \cdot \nabla \tilde{Y}^2(e) + \sigma^2_{y^2}}{\mathcal{RT} \cdot \nabla \tilde{Y}^2(e) + \sigma^2_{y^2} \left( 1 - \lim_{\sigma^2_{y^1} \rightarrow +\infty} \rho^2_{y^1 y^2} \right)} = 1$$

if, and only if, $\lim_{\sigma^2_{y^1} \rightarrow +\infty} \rho^2_{y^1 y^2} = 0$. If the second performance measure (e.g., $\tilde{Y}^1$ in this example) is infinitely noisy and is not correlated with the first measure, it is useless and hence excluded. The optimal incentive contract, in fact, becomes the contract based only on $\tilde{Y}^2$. For example, if the information signals are merely a noisy representation of the risky asset dividend, as in BI and Kim and Suh (1993), we have $\rho^2_{y^1 y^2} = \frac{\sigma^2_{y^2}}{\sigma^2_{y^2}}$. Hence, $\lim_{\sigma^2_{y^1} \rightarrow +\infty} \rho^2_{y^1 y^2} = 0$. Another example is that the two performance measures are not correlated with each other (i.e., $\rho^2_{y^1 y^2} = 0$).

The manager's base salary is unimportant since it depends on his reservation utility level.
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We now focus on the impact of performance measures' diversity on the optimal compensation contract in terms of providing incentives for the manager to allocate his multi-dimensional effort. Rewriting (3.28a) in terms of effort allocation, we have the second-best effort allocation:

\[ v^* = \left( \frac{\gamma I^y}{\gamma_a I^*} \right) v^y + \left( \frac{\beta I^z}{\beta_a I^*} \right) v^z, \quad (3.28b) \]

where \( I^y \), \( I^z \), \( I^* \) are the second-best effort intensities.

**Corollary 3.12** If the optimal linear incentive contract is based on two perfectly aligned performance measures, it never induces the first-best allocation of effort unless at least one performance measure is congruent with the firm's gross payoff.

**Proof:** See Appendix A.2. Q.E.D.

The second-best effort allocation coincides with the vector \( v^z \) (i.e., vectors \( v^x \), \( v^y \) and \( v^z \) all overlap one another, see Figure 3.2) if either the two performance measures are congruent with the firm's gross payoff (i.e., \( f_i = h_i = g_i \), \( i = 1, 2 \)), or one is congruent and the other is action-irrelevant. Whether the agent exerts the first-best effort intensity depends on the precision of the performance measures. The agent exerts zero effort intensity (i.e., the origin point 0) if the signals are infinitely noisy; and he exerts the first-best effort intensity if the signals are noiseless. In this setting, the agency cost is entirely due to the noise contained in the performance measures.

If the two performance measures are perfectly aligned with each other but are not congruent with the firm's gross payoff, then the first-best allocation of effort is never induced. Again, the effort intensity along the sub-optimal allocation of effort is determined by the precision of the performance measures.
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Figure 3.2: The effort space

Vector $e^* = I^x v^x$ is the first-best effort level. Vectors $e^{v1} = I^v v^{v1}$ and $e^{v2} = I^v v^{v2}$ are the second-best effort levels induced by performance measures $\tilde{Y}^1$ and $\tilde{Y}^2$ separately. Vector $e^* = \frac{\gamma}{\gamma_0} e^{v1} + \left(\frac{\beta}{\beta_0}\right) e^{v2}$ is the second-best effort level induced by both $\tilde{Y}^1$ and $\tilde{Y}^2$. It is a weighted sum of $e^{v1}$ and $e^{v2}$. 
COROLLARY 3.13 If the optimal linear incentive contract is based on two diverse performance measures, it can induce any allocation of effort. However, the first-best allocation of effort is implemented if, and only if, the relative weights on the two performance measures equals the ratio of the congruity for the two performance measures:

\[ \gamma / \beta = Q_y z / Q_y v. \]  

(3.29)

Proof: See Appendix A.2. Q.E.D.

When the performance measures are diverse with each other, individual performance measure congruity is unimportant. Even if neither of the performance measures is congruent, the first-best allocation of effort can still be implemented, but it will not be optimal unless the precision of the two performance measures are "proper". The first-best allocation of effort is implemented if, and only if, the vectors \( \left( \frac{\gamma v^1}{\gamma v^2} \right) v^{v_1} \) and \( \left( \frac{\beta v^2}{\beta v^1} \right) v^{v^2} \) form a parallelogram for which \( v^z \) forms the diagonal direction. See Figure 3.2. Equation (3.29) provides the necessary and sufficient condition for this result to hold. Its graphic interpretation is simply that the relative length of vector \( \left( \frac{\gamma v^1}{\gamma v^2} \right) v^{v_1} \) with respect to that of vector \( \left( \frac{\beta v^2}{\beta v^1} \right) v^{v^2} \) equals the relative area of the triangle formed by \( v^{v^2} \) and \( v^z \) with respect to that of the triangle formed by \( v^z \) and \( v^{v_1} \).

Take BI's second special case as an example. That is, suppose the information signal \( \tilde{Y}^1 \) (the accounting earnings) provides information only about the first activity's payoff (i.e., \( f_1 = g_1, f_2 = 0 \)), and the information signal \( \tilde{Y}^2 \) (the market price) reflects the firm's terminal value (i.e., \( h_i = g_i, i = 1,2 \)). From Figure 3.2, vectors \( v^z \) and \( v^{v^2} \) overlap with each other, and \( v^{v_1} \) points in the positive direction of the \( e_1 \)-axis. The manager under(over)-allocates his effort level on the second activity if the endogenously derived weight on the accounting earnings (\( \gamma \)) is positive (negative). Since \( Q_y z = 0 \), and \( Q_y v_1 = g_1 f_2 - g_2 f_1 = -g_1 g_2 \), condition (3.29) is satisfied if, and only if, \( \gamma = 0 \). The
first-best allocation of effort is implemented in this example if, and only if, the accounting earnings are excluded as a performance measure. The market price alone could be used to induce the agent to allocate his effort levels across tasks along the first-best allocation, but it is optimal to also include the accounting earnings since the amount of risk imposed on the agent is reduced. The inclusion of the accounting earnings results in sub-optimal allocation of effort, but greater effort intensity.
Incentive Contracts With Endogenous Information Acquisition

4.1 Introduction

A financial market provides a forum in which to raise funds and share risks. The process of trading ownership claims creates incentives for investors to spend valuable resources to acquire private information. Under the efficient market hypothesis, this information in turn becomes reflected in the stock price. That is, the stock price impounds both the public and private information about short-term as well as long-term factors influencing the firm's value. It is an efficient indicator of the economic well-being of the firm and hence, it is one of the indicators the firm's shareholders would like to use to evaluate the performance of their management. It is well documented that stock price is a widely used managerial performance measure. 4.1

However, the lack of a direct causal relationship between managerial performance and stock market performance limits the desirability of the stock price as a performance measure. Many noncontrollable events, such as general business conditions, unexpected material, energy, or labour shortage, decrease the precision of such a performance measure, and hence impose an additional component of noncontrollable risk on the manager. To mitigate this non-controllability problem, a performance measure based on the internal evaluation of the economic well-being of the firm is promoted. Examples of such a performance measure include earnings per share and return on shareholders' investment.

4.1 Empirical research includes Antle and Smith (1986), Jensen and Murphy (1990), Lambert and Larcker (1987), etc.
These variables are more under management's control than stock prices and, at least in the long run, should correlate with the economic welfare of the firm. Sloan (1993) provides empirical evidence in support of the hypothesis that the accounting-based performance measure helps shield the executives' incentive compensation from market-wide fluctuations in equity values reflected in the market-based performance measure.

Accounting-based performance measures are useful in helping to filter out the noise of the market price created by non-controllable events. However, they create additional moral hazard problems when used as managerial performance measures. Healy (1985) and Holthausen, Larcker and Sloan (1995) find empirical evidence in support of the hypothesis that earnings-based annual bonus plans induce managers to manipulate accounting methods in ways harmful to the shareholders of the firm. Dechow and Sloan (1991) find that CEOs in their final years of office manage discretionary investment expenditures to improve short-term earnings performance. As a result, some economists and compensation consultants\(^4^2\) recommend that accounting earnings-based performance measures should be eliminated.

In this chapter, we use the market price and the public report as two managerial performance measures. First, we explicitly model the investors' endogenous information acquisition and examine its impact on the managerial incentive compensation contract. It is shown that the investors' endogenous acquisition of private information reduces the impact of an increase in the risky dividend's volatility on managers' accounting earnings-based incentives relative to his market price-based incentives. Second, the market price, endogenously derived from the rational expectations model in Chapter 2, impounds both the public report and the investors' private information, and reflects short-term as well as long-term factors influencing the firm's value. The accounting earnings-based performance measure, however, is conventionally believed to reflect only short-term factors.

\(^{4^2}\)For example, see Jensen (1989), Stewart (1989, 1990), and Rappaport (1990).
influencing the firm's value. That is, those two signals are diverse. In multi-task agency setting, the diversity of performance measures is a very important factor influencing managerial incentives. It facilitates an efficient tradeoff between the manager's effort intensity and allocation, and hence the efficiency of the incentive compensation contract.

The rest of the chapter is organized as follows. Section 4.2 discusses the incremental information conveyed by the market price. It is shown that, in the presence of the public report, contracting on the market price is equivalent to contracting on the incremental information conveyed by the market price (i.e., the filtered price). This latter form of contract representation is analytically useful because it sharpens the economic insights. Section 4.3 examines how the investors' endogenous information acquisition influences the precision of the incremental information of the market price. This is central to our comparative static analysis in Section 4.5, where we not only highlight the economic intuition of the relative use of the accounting-based and the market-based information signals, but also re-examine the relative use of the public report and the "raw" market price. Empirically testable hypotheses are discussed. Section 4.4, a transition section, clarifies the notation of the optimal incentive linear contract based on the public report and the filtered price. A special information structure is examined in the last section of the chapter, Section 4.6.

4.2 Incremental Information Conveyed by the Market Price

Based on two performance measures, the rational expectations equilibrium price \( \hat{P} \) and the public report \( \hat{Y} \), we can express the linear managerial incentive compensation contract (3.5) as follows:

\[
    w(\hat{P}, \hat{Y}) = \hat{\alpha} + \hat{\beta}\hat{P} + \hat{\gamma}\hat{Y}.
\]  

(4.1)

Notice that the rational expectations price determined in Proposition 2.9 reflects the
gross return of the risky asset at time $t = 2$, before the managerial compensation. Thus, using the equilibrium price of the firm (reflecting the gross return of the risky asset minus managerial compensation), the Board of Directors’ decision problem (3.6) has an additional constraint:

\[ \text{Cum Compensation Market Price: } \bar{P} = a_0 + a_1 \bar{\psi} + a_2 \bar{Y} - a_2 \bar{z} - \left( \alpha + \beta \bar{P} + \gamma \bar{Y} \right). \] (4.2a)

This condition represents the market-based performance measure which is endogenously determined in a noisy rational expectations equilibrium, net of the management compensation.\(^{43}\) The compensation contract is determined at the contract settlement date $t = 2$. The cum-compensation price reflects the firm’s liquidating value. We can think of the firm borrowing against the terminal value at time $t = 3$ to pay the agent at time $t = 2$.

Re-writing the rational market price (4.2a) yields:

\[ \bar{P} = \left( \frac{1}{1 + \beta} \right) \left[ a_0 + a_1 \bar{\psi} + (a_2 - \gamma) \bar{Y} - a_2 \bar{z} - \alpha \right]. \] (4.2b)

Observe that the price $\bar{P}$ is a function of both the investors’ private signal ($\bar{\psi}$) and the public report ($\bar{Y}$). The public report has two effects on managerial compensation if both the stock price and the public report are simultaneously used as performance measures since the latter is impounded in the former. To make this more transparent, we substitute the rational market price (4.2b) into the wage contract (4.1) and collect terms:

\[ w(\bar{P}, \bar{Y}) = \gamma \bar{Y} + \frac{\beta (a_2 - \gamma)}{1 + \beta} \bar{Y} + \frac{\alpha + a_0 \beta}{1 + \beta} \bar{\psi} \bar{z} + \frac{a_1 \beta}{1 + \beta} \left( \bar{\psi} - \frac{a_2}{a_1} \bar{z} \right). \] (4.3)

\(^{43}\)The coefficients $a_0$, $a_1$, $a_2$, and $a_3$ are given by (2.22). The key reason that we can separate the agency problem from the equilibrium pricing process in this way is that the agent’s effort levels affect only the expected means of the information signals including $\bar{X}$, $\bar{Y}$, and $\bar{\psi}$, and do not affect the covariance/correlation relations of the information structure. That is, the agent’s effort levels only horizontally shift the payoff distributions of the information signals in the first-order stochastic dominance sense. Consequently, the cum-compensation price can be directly derived from the ex-compensation price by subtracting the compensation amount. I thank Professors Lane Daley, Alex Dontoh, and Bharat Sarath with whom I had interesting discussion about this point.
The compensation has two terms based on the public report, $\hat{\gamma}\hat{Y}$ and $\frac{\beta(a_2-\hat{\gamma})}{1+\beta}\hat{Y}$, which are referred to as the direct and the indirect wage effects of the public report, respectively.

Consequently, it is useful to transform the price to identify the component associated with the investors' private information, and thereby remove the public report's second wage effect. The filtered price is defined by:

$$\tilde{P} \equiv \left\{ (1+\hat{\beta})\tilde{P} - \left[ a_0 - \hat{\alpha} + (a_2 - \hat{\gamma})\hat{Y} \right] \right\} / a_1 = \tilde{\psi} - (a_2/a_1)\tilde{\zeta}. \quad (4.4)$$

Clearly, the filtered price, being a noisy representation of the investors' private information, exactly captures the incremental information provided by the rational market price. In the presence of the public report as a performance measure, it is the investors' private information impounded in the stock price that is incremental and effectively employed by the incentive contract.

Given this transformation, the managerial compensation contract (4.1), which is written in terms of the "raw" rational market price $\tilde{P}$ and the public report $\hat{Y}$, can be re-expressed in terms of the filtered price $\tilde{P}$ and the public report $\hat{Y}$:

$$w(\tilde{P}, \hat{Y}) = \alpha + \beta\tilde{P} + \gamma\hat{Y}. \quad (4.5)$$

The equivalence of those two wage contracts [i.e., $w(\tilde{P}, \hat{Y}) = w(\tilde{P}, \hat{Y})$] is ensured in the following lemma by straightforward algebraic manipulations.

---

4.4See also Kim and Suh (1993). From now on, we use the market-based performance measure and the filtered stock price exchangeably. Sometimes we use effective market-based performance measure to emphasize the incremental information of the stock price over the accounting-based measure. Furthermore, if $a_1 = 0$, then $\tilde{P}$ is infinitely affected by the supply uncertainty of the risky asset, i.e., $\sigma^2_{\tilde{P}} = +\infty$. Proposition 2.11 shows that $a_1 = 0$ if, and only if, $\lambda^* = 0$, i.e., the private signal is not utilized by any investor. In this case, the filtered price does not contain any relevant information, reflected by the fact that $\sigma^2_{\tilde{P}} = +\infty$. Therefore, we state the filtered price as it is without specifying that $a_1 \neq 0$. 

---
LEMMA 4.1 There exists an one-to-one correspondence between these two sets of parameters \( \{\hat{\alpha}, \hat{\beta}, \hat{\gamma}; \hat{P}\} \) and \( \{\alpha, \beta, \gamma; \tilde{P}\} \): 4.5

\[
\begin{align*}
\hat{\alpha} &= \frac{(a_1 \alpha - a_0 \beta)}{(a_1 - \beta)} \\
\hat{\beta} &= \frac{\beta}{(a_1 - \beta)} \\
\hat{\gamma} &= \frac{(a_1 \gamma - a_2 \beta)}{(a_1 - \beta)} \\
\hat{P} &= (a_0 - \alpha) + (a_1 - \beta) \hat{P} + (a_2 - \gamma) \hat{Y};
\end{align*}
\]

or equivalently

\[
\begin{align*}
\alpha &= \frac{(\hat{\alpha} + a_0 \hat{\beta})}{(1 + \hat{\beta})} \\
\beta &= \frac{a_1 \hat{\beta}}{(1 + \hat{\beta})} \\
\gamma &= \frac{(\hat{\gamma} + a_2 \hat{\beta})}{(1 + \hat{\beta})} \\
\tilde{P} &= \left[ (\hat{\alpha} - a_0) + (1 + \hat{\beta}) \tilde{P} + (\hat{\gamma} - a_2) \tilde{Y} \right] / a_1.
\end{align*}
\]

This lemma is important because it allows us to use either of the two managerial compensation contracts \( w(\hat{P}, \hat{Y}) \) and \( w(\tilde{P}, \tilde{Y}) \). Furthermore, the weight on the public report relative to the weight on the market price \( (\hat{\gamma} / \hat{\beta}) \) is related to the weight on the public report relative to the weight on the filtered price \( (\gamma / \beta) \):

LEMMA 4.2

\[
\frac{\hat{\gamma}}{\hat{\beta}} = a_1 \left( \frac{\gamma}{\beta} \right) - a_2 = a_1 \left( \frac{\gamma}{\beta} + \frac{\sigma_0 \rho_{0y}}{\sigma_y} \right) - \frac{\sigma_0 \rho_{0y}}{\sigma_y}. \tag{4.6}
\]

Proof: See Appendix A.3. Q.E.D.

Analytically, we use the filtered price and the public report as two performance measures, because it is the filtered price that captures the incremental information utilized by the incentive contract. Therefore, we use the incentive contract (4.5) to examine the

\footnote{4.5 We do not specify the relationship between the two sets of coefficients when \( a_1 = 0 \), because "\( a_1 = 0 \)" implies that the price is a noisy representation of the public report \( \hat{Y} \). Hence, \( \hat{\beta} = 0 \). As a result, \( \beta = \hat{\beta} = 0 \); and \( \alpha = \hat{\alpha} \); \( \gamma = \hat{\gamma} \).}
relative weight on the performance measure, and use Lemma 4.2 to derive the results for
the incentive contract (4.1).

4.3 Precision of Market Price’s Incremental Information

The inverse of an information signal’s variance is defined as the precision of the signal.\footnote{In Chapter 2, our general information structure shows that the correlations among information signals are the determinants of the equilibrium price but the variances are not. As a result, the inverse of an information signal’s variance does not capture the essence (i.e., the correlations) of the signal. To be consistent with literature, we still use this definition but great care has been exercised.} We now examine the precision of the filtered price.

For any exogenously given $\lambda$, by (4.4), we obtain that $\tilde{P} \sim N (\bar{P}, \sigma^2_{\bar{P}})$ and $\sigma_{\bar{P}} = \sigma_0$, where $\bar{P} = \bar{\psi}$, and $\sigma^2_{\bar{P}} = Var(\tilde{P}) = \sigma^2_0 + (a_2/a_1)^2 \sigma^2_2 = \sigma^2_0 N^2$, with

$$N \equiv \sqrt{1 + \frac{\nu^2 \sigma^2_2 \rho^2_{\psi}}{\lambda^2 \rho^2_1}}. \quad (4.7a)$$

That is, the standard deviation ($\sigma_{\bar{P}}$) of the endogenous filtered price equals the standard deviation ($\sigma_0$) of the private signal scaled by a factor $N$. The fact that this factor $N$ is greater than one reflects the essence of the equilibrium price: an information signal which is always noisier than is the private signal itself. In other words, the market, in which there are informed, uninformed and liquidity trader, adds noise to the private information, and hence reduces the precision of the private information. The magnitude of such a reduction of the precision is determined by the filtered price precision factor $N$ (hereafter precision factor).

We assume that the cost of the private information is within $(\kappa_0, \kappa^0)$ so that we can focus our analysis on the case in which only a fraction of the investors in equilibrium decide to acquire the private information.\footnote{If $\kappa \leq \kappa_0$ or $\kappa \geq \kappa^0$, we obtain $\lambda^* = 1$ or 0 respectively. Since the cost of the private information signal is exogenous in our model, condition that $\kappa \leq \kappa_0$ or $\kappa \geq \kappa^0$ would be equivalent to letting $\lambda$ exogenously be 1 or 0 respectively. As a result, we exclude them from our analysis as we deal with managerial incentives with investors’ endogenous private information acquisition.} Since $\lambda^* = \nu \sigma_{x} \sigma_{\bar{P}}$ (see Proposition 2.7, $\mathcal{N}$
Chapter 4. Incentive Contracts With Endogenous Information Acquisition

becomes

\[ N = \sqrt{1 + \frac{\rho_0 \rho_2}{\rho_1^2 - \rho_0}}. \]  

(4.7b)

Clearly, \( N \)'s properties are important for us to understand the endogenous precision of the effective market-based performance measure (i.e., the filtered price).

**Lemma 4.3** If the investors endogenously decide whether to acquire the private information signal, \( N \) given by (4.7b) is independent of the precision of the private signal, the precision of the public report, the risky asset dividend's volatility, the risky asset's supply shocks, but is increasing in the cost of the private signal and the risk aversion of the rational investors. See the summary table below.

<table>
<thead>
<tr>
<th>Endogenous information acquisition with ( \lambda^* \in (0, 1) )</th>
<th>Exogenous information acquisition with ( \lambda \in (0, 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \frac{dN}{d\sigma^2} )</td>
<td>= 0</td>
</tr>
<tr>
<td>2. ( \frac{dN}{d\rho_2} )</td>
<td>= 0</td>
</tr>
<tr>
<td>3. ( \frac{dN}{d(1/\sigma_0^2)} )</td>
<td>= 0</td>
</tr>
<tr>
<td>4. ( \frac{dN}{d(1/\sigma_1^2)} )</td>
<td>= 0</td>
</tr>
<tr>
<td>5. ( \frac{dN}{d\lambda} )</td>
<td>n/a</td>
</tr>
<tr>
<td>6. ( \frac{dN}{d\alpha} )</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>7. ( \frac{dN}{d\nu} )</td>
<td>&gt; 0</td>
</tr>
</tbody>
</table>

Because \( \frac{1}{\sigma_p^2} = \frac{1}{\sigma_2^2 N^2} \), the following proposition is immediate by Lemma 4.3.

**Proposition 4.1** If the investors endogenously decide whether to acquire the private information signal, the precision of the filtered price increases in the precision of the private signal and is independent of the precision of the public report, the risky asset
dividend's volatility, and the risky asset's supply shocks. See the summary table below.

<table>
<thead>
<tr>
<th>Endogenous information acquisition with $\lambda^* \in (0, 1)$</th>
<th>Ezogenous information acquisition with $\lambda \in (0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d(1/\sigma_p^2)}{d\sigma_i^2}$ = 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>$\frac{d(1/\sigma_p^2)}{d\sigma_i^2}$ = 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>$\frac{d(1/\sigma_p^2)}{d(1/\sigma_i^2)}$ &gt; 0</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>$\frac{d(1/\sigma_p^2)}{d(1/\sigma_i^2)}$ = 0</td>
<td>= 0</td>
</tr>
<tr>
<td>$\frac{d(1/\sigma_p^2)}{d\lambda}$ (n/a)</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>$\frac{d(1/\sigma_p^2)}{d\lambda}$ (n/a)</td>
<td>(n/a)</td>
</tr>
<tr>
<td>$\frac{d(1/\sigma_p^2)}{d\alpha}$ &lt; 0</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>

The most important results of the above proposition are the first two. Whether the investors endogenously decide to acquire the private information fundamentally changes the relations between the filtered price precision and the dividend volatility and supply shocks of the risky asset.

When the fraction of informed investors is exogenously given (i.e., investors use whatever information is exogenously given to them), rational traders have to absorb the risky asset uncertainty (due to the dividend volatility and the supply shocks) that has not been eliminated by their received information signals. That is, the informed traders absorb the risky asset uncertainty not eliminated by the public report and the private signal, while the uninformed investors absorb the risky asset uncertainty not eliminated by the public report and the inference from the equilibrium price. As a result, the precision of the equilibrium price decreases in the risky asset uncertainty.

However, when the fraction of informed investors is endogenously determined (i.e., investors decide endogenously whether to acquire the private information signal), rational traders can decide how much risky asset uncertainty they wish to absorb by choosing
whether or not to acquire the private information signal. Although the uninformed investors have to absorb the risky asset uncertainty not eliminated by the public report and the inference from the equilibrium price, that amount of uncertainty is exactly what they are willing to absorb, because, in equilibrium, rational investors are indifferent between being informed or uninformed. Since the endogenous fraction of informed investors increases in the risky asset's uncertainty (see Proposition 2.8), an increase in the risky asset's uncertainty has two offsetting effects on the filtered price: it directly decreases the precision of the filtered price (i.e., direct effect) and it indirectly increases the precision of the filtered price via the increased fraction of informed investors (i.e., indirect effect). In equilibrium, the direct and indirect effects exactly offset with each other. Therefore, the precision of the filtered price is not affected by the risky asset uncertainty.

To see this more clearly, let us perturb the equilibrium fraction (i.e., $A^*$) of informed investors. On the one hand, suppose that there are more than $A^*$ fraction of investors choosing to be informed. This would result in a higher price precision than the equilibrium precision otherwise, because price precision increases as more investors purchase the private information. Such an increase in the price precision benefits the uninformed investors because they can now use more precise price and the public report to infer the risky asset dividend. This ultimately induces investors to become uninformed, and hence the fraction of the informed investors decreases.

On the other hand, suppose that there are fewer investors choosing to be informed than the equilibrium fraction $A^*$. The price precision would be lower than the equilibrium precision. This decrease in the price precision reduces the uninformed investors' ability to infer the risky asset dividend because the uninformed investors form their beliefs on the price and the public report. As a result, investors are motivated to purchase the private information signal, and hence the fraction of the informed investors increases. In equilibrium, the tradeoff between the increased precision of information due to the private
signal and the cost incurred to acquire it is made so that the investors are indifferent between being informed and uninformed, that is, the equilibrium fraction of informed investors is achieved.

The remaining results are important but less interesting, since whether the investors endogenously decide to acquire the private signal or not does not qualitatively affect the relations between the filtered price precision and other economic factors. For example, items 3 and 4 show that the price precision remains increasing in the precision of the private signal and independent of the precision of the public report, no matter whether the private information signal is endogenously acquired or exogenously provided. The reason for the former result is that the filtered price ultimately reflects the incremental information (i.e., beyond the public report information) impounded in the market price. The intuition behind the latter result is simply that the public report’s information content is removed from the market price by the definition of the filtered price.

4.4 Optimal Incentive Contract

Since the optimal incentive linear compensation contract based on any two information signals was formally examined in Chapter 3, it is trivial to obtain the optimal incentive contract based on the filtered price and the public report by applying Proposition 3.1. For the sake of clarity, in this section, we reproduce the results using the filtered price and the public report as two performance measures.

The weights on the public report and the filtered price are

\[
\gamma = R \sigma^2 y \zeta y / \mathcal{L} \quad \text{and} \quad \beta = R \sigma^2 p \zeta p / \mathcal{L},
\]

where \( \mathcal{L} \) is positive. The measures of sensitivity of these two performance measures,
Chapter 4. Incentive Contracts With Endogenous Information Acquisition

the public report $\tilde{Y}$ and the filtered price $\tilde{P}$, are as follows:

$$\zeta_Y = D_Y - \frac{\sigma_{P}^2}{\sigma_p^2} D_P - \frac{\mathcal{R}}{\sigma_p^2 (C_{11}C_{22} - C_{12}^2) I_x} Q_{x_P} Q_{P_Y}, \quad (4.9a)$$

$$\zeta_P = D_P - \frac{\sigma_{P}^2}{\sigma_p^2} D_Y - \frac{\mathcal{R}}{\sigma_y^2 (C_{11}C_{22} - C_{12}^2) I_x} Q_{x_Y} Q_{Y_P}. \quad (4.9b)$$

The directional derivatives of performance measures $\tilde{Y}$ and $\tilde{P}$ are:

$$D_Y \equiv \nabla^x \cdot \nabla \tilde{Y}(\epsilon) = v_1^x f_1 + v_2^x f_2, \quad (4.10a)$$

$$D_P \equiv \nabla^x \cdot \nabla \tilde{P}(\epsilon) = v_1^x h_1 + v_2^x h_2. \quad (4.10b)$$

Therefore, the weight on the public report relative to the weight on the filtered price (hereafter, the relative weight on the public report) is:

$$\frac{\gamma}{\beta} = \frac{\sigma_Y^2 \zeta_Y}{\sigma_P^2 \zeta_P}. \quad (4.11)$$

Table 4.1 lists some special cases in which the private information and the public report contain different information about the firm's payoff.

4.5 Comparative Static Analysis

4.5.1 Relative weight on the public report versus the filtered price

Several researchers have done extensive analyses of the relative weight assigned to performance measures in incentive contracts,\textsuperscript{4,9} but none has considered endogenous information acquisition. Information acquisition decisions likely affect the market-based performance measure, and generally depend on the amount of public information available. Inspired by the pioneering effort in this direction by CS,\textsuperscript{4,10} we examine the impact of investors' endogenous information acquisition on the management incentive contract.


Table 4.1: Relative weight of the optimal incentive contract

<table>
<thead>
<tr>
<th>$D_y = \delta y$</th>
<th>$D_z$</th>
<th>$[f_1 = 0; f_2 = 0]$</th>
<th>$[f_1 = 0; f_2 = 0]$</th>
<th>$[f_1 = 0; f_2 = 0]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma^*$</td>
<td>$\frac{\partial D_y}{\partial \gamma^*}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^*}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^*}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^*}$</td>
</tr>
<tr>
<td>$D_y = 0$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
</tr>
<tr>
<td>$D_y = 0$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
</tr>
<tr>
<td>$D_y = 0$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
</tr>
<tr>
<td>$D_y = 0$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
<td>$\frac{\partial D_y}{\partial \gamma^<em>} = \frac{\partial D_y}{\partial \gamma^</em>}$</td>
</tr>
</tbody>
</table>

\[ \gamma^* = \left[ \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} \right] \]
\[ \left[ \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} \right] \]

\[ \gamma^* = \left[ \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} \right] \]
\[ \left[ \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} \right] \]

\[ \gamma^* = \left[ \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} \right] \]
\[ \left[ \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} - \frac{\partial D_y}{\partial \gamma^*} \right] \]
The comparative static analysis of the relative weight on the performance measures is of interest not only in terms of comparison with the results in the literature, but also in terms of providing guidance for empirical investigation of management compensation.

As in BI, if all investors are exogenously endowed with private information (i.e., $\lambda = 1$), the uncertainty of the risky asset directly reduces the precision of the filtered price. Thus, the uncertainty of the risky asset increases the relative weight on the public report.

However, if the investors endogenously decide whether to acquire private information, the precision of the market-based performance measure (i.e., the filtered price) is endogenously affected by this decision. Proposition 4.1 shows that the precision of the filtered price, on the one hand, increases in the fraction of the informed investors, and on the other hand, decreases in the risky asset’s uncertainty. Since the equilibrium fraction of informed investors increases in the risky asset’s uncertainty, the uncertainty of the risky asset has an offsetting impact on the precision of the price. Consequently, the relative weight on the public report is affected not only by the investors’ private information precision directly, but also by the equilibrium fraction of informed investors. More precisely, we have the following proposition.

**PROPOSITION 4.2** If the investors endogenously decide whether to acquire the private information signal, the relative weight on the public report ($\gamma/\beta$) is independent of the risky asset’s uncertainty. Furthermore, with some regularity conditions, the relative weight on the public report decreases in the precision of the private signal, increases in the precision of the public report, the cost of the private information signal, and the risk aversion of the investors. The effect of the agent’s risk tolerance on the relative

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4,11 Items 1 and 2 are completely general when information acquisition decision is endogenous. However, the rest of the results are contingently derived on some regularity conditions which guarantee that the signs of the individual weights on the performance measures (interested readers are referred to the proof). For example, Sloan (1993) assumes that $\sigma_p < \min\{\sigma_p^2, \sigma_y^2\}$ to guide his empirical hypothesis based on BD's results.
weight on the public report is ambiguous in general, but is zero if, and only if, the two performance measures are not diverse. See the summary table below.

<table>
<thead>
<tr>
<th></th>
<th>Endogenous information acquisition with $\lambda^* \in (0, 1)$</th>
<th>Exogenous information acquisition with $\lambda \in (0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\frac{d(\gamma/\beta)}{d\sigma^2}$</td>
<td>$= 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>2. $\frac{d(\gamma/\beta)}{d\sigma^2}$</td>
<td>$= 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>3. $\frac{d(\gamma/\beta)}{d(1/\sigma^2)}$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>4. $\frac{d(\gamma/\beta)}{d(1/\sigma^2)}$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>5. $\frac{d(\gamma/\beta)}{d\lambda}$</td>
<td>$n/a$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>6. $\frac{d(\gamma/\beta)}{d\kappa}$</td>
<td>$&gt; 0$</td>
<td>$n/a$</td>
</tr>
<tr>
<td>7. $\frac{d(\gamma/\beta)}{d\nu}$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>8. $\frac{d(\gamma/\beta)}{d\kappa}$</td>
<td>ambiguous</td>
<td>ambiguous</td>
</tr>
</tbody>
</table>

Proof: See Appendix A.3. Q.E.D.

The major results in this proposition are the first two items, because whether the investors endogenously decide to acquire the private information signal fundamentally changes the relation between the relative weight on the public report and the risky asset uncertainty. The intuition is as follows. When the investors decide endogenously to acquire the private information signal, the precision of the filtered price is affected by offsetting factors: the fraction of informed investors and the risky asset’s uncertainty. The precision of the filtered price increases in the fraction of informed investors, but decreases in the risky asset’s uncertainty. Since an increase in the risky asset’s uncertainty increases the fraction of informed investors, it decreases the precision of the filtered price directly on the one hand, but increases the precision of the filtered price indirectly through the fraction of informed investors on the other hand. As a result, the equilibrium fraction of the informed investors changes such that it exactly offsets the impact of the risky asset’s
uncertainty on the precision of the filtered price. That is, the precision of the filtered
price is independent of the risky asset’s uncertainty if the fraction of informed investors
is endogenously determined. Consequently, the relative weight on the public report is
independent of the risky asset’s uncertainty.

The agent’s risk attitude affects the relative weight on the public report if, and only
if, the performance measures are diverse. Since performance measures’ diversity (i.e., the
cross product is always zero) is not an issue if the managerial effort is single-dimensional,
it is not surprising that the agent’s risk aversion does not affect the relative weight on
the public report in BD’s single-task agency setting.

4.5.2 Relative weight on the public report versus the market price

The above analysis demonstrates that the relative weight on the public report with re-
spect to the filtered price is independent (dependent) of the risky asset dividend’s volatil-
ity if the information acquisition decision is endogenous (exogenous). If we believe that
investors’ private information acquisition decision is endogenous, the above result implies
that there is no change in the weight on the public report relative to the weight on the
filtered price as the risky dividend’s volatility increases, ceteris paribus. To empirically
test this hypothesis, we face a problem: the filtered price is not observable. Econometric
techniques may be useful in generating a proxy for our filtered price.

Alternatively, we can extend our analysis to restate the above results in terms of the
weight on the public report relative to the weight on the market price which is observed.
The following analysis adopts the latter approach and examines the impact of the risky
dividend’s volatility on the weight on the public report relative to the market price.

Given (4.6), we obtain:

$$
\frac{\partial (\gamma / \beta)}{\partial \sigma_x} = \frac{\partial a_1}{\partial \sigma_x} \left( \frac{\gamma}{\beta} + \frac{\sigma_0 \rho_{xy}}{\sigma_y} \right) + a_1 \frac{\partial (\gamma / \beta)}{\partial \sigma_x} - \frac{\rho_{xy}}{\sigma_y}.
$$ (4.12)
COROLLARY 4.3 Given appropriate regularity conditions (such as $a_1 > 0$, $\gamma/\beta > 0$, $\rho_{0y} > 0$, and $\rho_{xy}$ is sufficiently small relative to $\sigma_y$), the weight on the public report relative to the weight on the market price (i.e., $\gamma/\beta$) increases in the risky asset dividend's volatility, $\sigma_y^2$. In addition, $\gamma/\beta$ increases at a faster rate with respect to the risky asset dividend's volatility if the information acquisition decision is exogenous than that if the information acquisition decision is endogenous.

Proof: See Appendix A.3. Q.E.D.

Note that $\gamma/\beta$ is independent of $\sigma_y^2$ if investors' information acquisition decision is endogenous, and increases otherwise (items 1 and 2 of Proposition 4.2). The second term of (4.12), $a_1 \frac{\partial(\gamma/\beta)}{\partial \sigma_y}$, is zero if the investors spontaneously acquire the private signal, and is positive otherwise. This positive incremental change implies that an increase in the risky asset dividend's volatility has a larger impact on $\gamma/\beta$ under an exogenous information acquisition system than under an endogenous information acquisition system, ceteris paribus.

Implication: Investors' endogenous acquisition of private information reduces the impact of an increase in the risky dividend's volatility on managers' accounting earnings-based incentives relative to his market price-based incentives.$^{412}$

$^{412}$Developing an empirical measure for $\sigma_y^2$ is difficult. Ideally, such a measure would represent the investors' \textit{ex ante} uncertainty with respect to the price at the date at which they expect to sell their shares. In the single period model examined here, the latter price is equal to the terminal dividend. However, in a multi-period setting, one must consider the prices at the date at which the rational investors will reverse their speculative positions. Given an estimate of the length of time investors hold speculative positions, one might use the \textit{ex post} variability of prices over that interval as a proxy for $\sigma_y^2$, but that is not a direct measure of variance in the investors' \textit{ex ante} beliefs.

In developing estimates of $\sigma_y^2$, it is important to keep in mind that there is no distinction between firm-specific and systematic risk in the single risky asset model examined in this dissertation. Assuming the investors and the manager can eliminate the impact of systematic risk by holding an appropriate share of the market portfolio, firm-specific risk is the key factor impacting the investors' information acquisition decision and the risk premium that must be paid to the manager. This suggests that the theoretical analysis regarding the impact of changes in $\sigma_y^2$ is best interpreted as a theoretical analysis of
Figure 4.1: Relative incentive weight and the risky dividend's volatility

An increase in the risky dividend's volatility has a larger impact on the relative incentive weight on the public report if investors' private information is exogenously endowed rather than endogenously acquired.
Loosely speaking, the intuition is as follows. An increase (decrease) in the risky asset dividend’s volatility reduces (improves) the precision of the market price. We refer to this as the *direct effect*. When the information acquisition decision is exogenous, the risky asset dividend’s uncertainty has only this direct effect on the market price.

When the investors information acquisition decision is *endogenous*, however, there exists an additional effect which we refer to as the *indirect effect*. To see this indirect effect, observe that an increase (decrease) in the risky asset dividend’s volatility induces more (fewer) investors to acquire the private signal, and an increase (decrease) in the fraction of informed investors makes the market price more (less) informative. As a result, an increase (decrease) in the risky asset dividend’s volatility improves (reduces) the market price precision *indirectly* through the resulting higher (lower) fraction of informed investors. That is, the indirect effect offsets the direct effect.

Since an optimal contract assigns a larger (smaller) weight to a more (less) informative performance measure, the latter half of Corollary 4.3 implies that the impact of an increase in the risky asset dividend’s volatility on the relative weights assigned to the public report and the market price is smaller when the investors’ private information is endogenously acquired than when the investors’ private information is exogenously endowed.

A number of *empirical hypotheses* can be developed from this implication. For example, firms place relatively more weight on market performance (and less weight on accounting performance) in compensation contracts for situations in which

1. many financial analysts are actively seeking information about the firm,

2. the ownership concentration is low.

The impact of firm-specific risk. Consequently, empirical measures should focus on firm-specific risk, as in Sloan (1993).
4.6 A Special Information Structure

When we derived the equilibrium price in Chapter 2, we showed that it was the correlation relations of the information structure that determine the price. In Sections 4.3 and 4.5, we fixed the correlations of the information structure to examine the comparative static analyses.

In this section, we introduce a special information structure which has been widely used in the literature. Assume that all information signals are a noisy representation of the underlying risky asset. That is, both the public report and the investors' private information equal the risky asset plus white noise:

$$\tilde{Y} = \tilde{X} + \tilde{\varepsilon}_1 \quad \text{and} \quad \psi = \tilde{X} + \tilde{\varepsilon}_2.$$  

(4.13)

Their variances equal the variance of the risky asset dividend plus the variances of their white noise terms, i.e., $\sigma_y^2 = \sigma_x^2 + \sigma_1^2$ and $\sigma_0^2 = \sigma_x^2 + \sigma_2^2$. The covariance matrix describing the relation among $\tilde{X}$, $\tilde{Y}$, and $\psi$ is:

$$
\Omega = \begin{bmatrix}
\sigma_x^2 & \sigma_x^2 & \sigma_x^2 \\
\sigma_x^2 & \sigma_x^2 + \sigma_1^2 & \sigma_x^2 \\
\sigma_x^2 & \sigma_x^2 & \sigma_x^2 + \sigma_2^2
\end{bmatrix}.
$$

The covariances between any two of the three signals, $\tilde{X}$, $\tilde{Y}$, and $\psi$, are equal to the variance of the risky asset dividend. Their correlations are as follows.

$$
\rho_{xy} = \frac{\sigma_x}{\sigma_y}, \quad \rho_{0x} = \frac{\sigma_x}{\sigma_0}, \quad \text{and} \quad \rho_{0y} = \frac{\sigma_x^2}{\sigma_0 \sigma_y}.
$$

(4.14)

Obviously, as the risky asset dividend's variance increases, not only will the variances of the associated information signals (i.e., the public report and the investors' private information) increase (i.e., $\frac{d\sigma^2_x}{d\sigma_x^2} = 1 > 0, \ l = y, 0$), but the correlations will also increase. This
is exceedingly important when we do comparative static analysis and investigate empirical implications, because interpretation of these analyses require careful identification of what we mean by *ceteris paribus*.

Whether the correlation structure is fixed with respect to the variance of the risky asset dividend or increases in the variance of the risky asset dividend is an empirical question. It also depends on the issues to be addressed.

To illustrate the impact of this special information structure on comparative static analyses, we briefly examine the filtered price precision.

**Lemma 4.4** In this special information structure, the comparative statics of the filtered price factor $N$ are as follows.

<table>
<thead>
<tr>
<th></th>
<th>Endogenous information acquisition with $\lambda^\ast \in (0, 1)$</th>
<th>Exogenous information acquisition with $\lambda \in (0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\frac{dN}{d\sigma^2}$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>2. $\frac{dN}{d\sigma^2}$</td>
<td>$= 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>3. $\frac{dN}{d(1/\sigma^2)}$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>4. $\frac{dN}{d(1/\sigma^2)}$</td>
<td>$&gt; 0$</td>
<td>$= 0$</td>
</tr>
<tr>
<td>5. $\frac{dN}{d\lambda}$</td>
<td>$n/a$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>6. $\frac{dN}{d\xi}$</td>
<td>$&gt; 0$</td>
<td>$n/a$</td>
</tr>
<tr>
<td>7. $\frac{dN}{dv}$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
</tr>
</tbody>
</table>
The comparative statics of the filtered price’s precision are as follows.

<table>
<thead>
<tr>
<th>Endogenous information acquisition with $\lambda^* \in (0, 1)$</th>
<th>Exogenous information acquisition with $\lambda \in (0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\frac{d(1/\sigma^2_p)}{d\sigma^2_p}$</td>
<td>?</td>
</tr>
<tr>
<td>2. $\frac{d(1/\sigma^2_p)}{d\sigma^2_r}$</td>
<td>= 0</td>
</tr>
<tr>
<td>3. $\frac{d(1/\sigma^2_p)}{d(1/\sigma^2_r)}$</td>
<td>&gt; 0</td>
</tr>
<tr>
<td>4. $\frac{d(1/\sigma^2_p)}{d(1/\sigma^2_t)}$</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>5. $\frac{d(1/\sigma^2_p)}{d\lambda}$</td>
<td>n/a</td>
</tr>
<tr>
<td>6. $\frac{d(1/\sigma^2_p)}{dn}$</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>7. $\frac{d(1/\sigma^2_p)}{d\varphi}$</td>
<td>&lt; 0</td>
</tr>
</tbody>
</table>

Proof: See Appendix A.3 and Figure 4.2. Q.E.D.

Let us compare the comparative statics in Lemma 4.4 with Lemma 4.3 and Proposition 4.1. Items 2, 5, 6, and 7 are the same. The intuition is clear since changes in those parameters do not affect the correlation structure of the information signals.

Item 4 implies that an increase in $\sigma^2_r$ decreases $N$ and hence increases the precision of the filtered price if the private information is endogenously acquired; it does not affect $N$ and hence does not influence the precision of the filtered price if the private information is exogenously given. The former is different from that in Lemma 4.3 and Proposition 4.1.

Item 3 implies that an increase in $\sigma^2_r$ increases $N$ and decreases the precision of the filtered price. $N$ is no longer independent of $\sigma^2_r$ if the private information is endogenously acquired.

Item 1 implies that an increase in $\sigma^2_r$ decreases $N$ in this special information structure. This is very different from that in Lemma 4.3. The key is that an increase in $\sigma^2_r$ not only increases the variances of the public report and the private signal but also the correlations of the risky asset dividend with those two information signals. If the private information
Figure 4.2: The filtered price’s precision changes with respect to the variance of the risky asset dividend

The filtered price’s precision, plotted as a function of the risky asset dividend’s variance (from 5 to 10) and \( t \) ([.5, .95]), increases first and then decreases as the risky dividend’s variance increases. The range of the second variable \( t \) is set so that the cost of the private information is moderate expensive, where \( \sigma_z = \sigma_1 = \sigma_2 = \nu = 1 \).

is exogenously given, the direct effect of the risky asset’s variance on the filtered price’s precision is still negative. However, if the private information is endogenously acquired, there exists an additional (i.e., the indirect) effect of the risky asset’s variance on the filtered price’s precision. This indirect effect is now not clear because of the change of correlation relations due to the increase in \( \sigma_x^2 \). Consequently, the impact of an increase in the variance of the risky asset dividend on the precision of the filtered price is ambiguous (see Figure 4.2).
Chapter 5

Stock Ownership Versus Option Incentive Compensation Contracts

5.1 Introduction

It has been reported that stock option plans were used by 75% of the major U.S. manufacturing firms in 1960 and by 84% in 1988. Restricted stock plans were introduced in mid-seventies and were used by 34% of the major U.S. manufacturing firms in 1988 (see Figure 1.1). Both stock options and restricted stock are popular components in managerial incentive compensation contracts.

Stock option and ownership contracts provide market-based incentives and are long-term orientated. They are almost perfectly associated with the economic well-being of the firm and hence reflect the firm's long-run value. This is the desired perspective from the long-term shareholders' standpoint. Although the values of shares and options fluctuate with the market price of the stock, the payoffs of shares and options are very different. The former equals the \textit{ex post} market price and the latter equals either zero if the \textit{ex post} market price is lower than the pre-set exercise price or the amount of the market price exceeding the exercise price.

The elimination of downside risk seems to suggest that option contracts impose less risk on the managers than do the ownership contracts. This is true if one merely compares one share to one option. The payoff of one share is exactly the \textit{ex post} market price, but the payoff of one option is either zero or the excess of the \textit{ex post} market price over the exercise price. That is, the value fluctuation of one option may be considerably lower.
than that of one share, depending on the exercise price. For instance, if the exercise price is substantially larger than the current stock price, then the option is surely worthless and has zero volatility.

The risk imposed on a risk averse manager is costly to the firm’s owners because the manager demands a risk premium to take on such risk. Consequently, one option should be less costly than one share to the firm’s owners. This raises an important question: Does an option contract produce higher net surplus for the firm’s owners than does a stock ownership contract? The answer to this question is no. Ownership contracts not only insure a real mutuality of interest between the firm’s shareholders and management but also impose less risk on the risk averse managers than do options contracts. The Wall Street Journal (December 5, 1989) reported Peter Magowan’s (the chairman and chief executive of Safeway, a large grocery store chain) comment: “We have our money involved (ownership). Corporate management is given large salaries, stock options and that sort of thing but they haven’t got their own money at risk. We have. If this thing fails, we are going to lose a substantial portion of what we’ve invested. That’s a powerful incentive to get better performance. We’ve tried to carry that down with incentive plans to the level of store management.”

In this chapter, we compare two types of incentive compensation contracts: an ownership contract versus an option contract. We assume that the manager’s effort level affects the firm’s gross return, and the manager’s wage contract consists of two components: a fixed wage component and an incentive compensation component. The incentive component depends on the stock price of the firm and the exercise price is set _ex ante_ by the shareholders. In the ownership contract setting, the exercise price is set very low so that the incentive component equals a multiple of the stock price. In the option contract setting, the exercise price is usually set at or near the current stock price. The payoff of one option is the larger of zero and the stock price minus the exercise price.
Options are widely used and assumed to be desirable because they shelter the manager from the downside risk, thereby reducing the risk for which the manager must be compensated. However, for any given level of implementable effort, our analysis shows that an ownership contract generates higher total net surplus than does an option contract. This implies that an option contract in fact increases the cost of risk imposed on the manager. This arises because inducement of a given level of managerial effort requires many more options than shares. The increase in the number of options is sufficient to result in higher risk with options than with stock, and hence induces a higher risk premium and lower total net surplus.

Furthermore, it is shown that the entire effort space is implementable with a stock ownership contract, but only a strict subset of the effort space is implementable with an option contract. The intuition is as follows. In our model, the variance of the stock price is independent of and the expected value of the stock price is increasing in the managerial effort level. As a result, an increase in managerial effort does not induce higher risk imposed by the manager’s wage contract. That is, the risk associated with the wage contract is not a factor influencing the manager’s effort decision. He only needs to tradeoff his incremental gain from his wage contract with the increased personal cost of his effort. As long as the coefficient (i.e., incentive weight) for the performance measure (i.e., the stock price) is large enough, the incremental gain resulted from the manager’s wage contract is sufficient for the manager to cover his personal cost of effort. Therefore, the entire effort space is implementable.

On the other hand, it is shown that not only the mean but also the variance of an option increases as the manager increases his effort level. The increased variance induces higher risk on the manager. Now, the risk associated with the incentive contract becomes an important factor influencing the manager’s effort decision. He has to tradeoff his incremental gain from his wage contract with not only the increased personal cost of
effort but also the increased risk. We therefore cannot merely increase the coefficient (i.e., incentive weight) for the performance measure (i.e., the option's payoff) to induce the manager to expend higher effort level. An increase in the coefficient raises not only the expected gain received by the manager in the wage contract but also the risk associated with the wage contract. The increased risk due to higher managerial effort level ultimately becomes non-bearable for the risk averse manager so that he stops expending higher effort levels, i.e., the effort becomes non-implementable.

The rest of the chapter is organized as follows. Section 5.2 defines the manager's wage contract. It is shown that the exercise price and the managerial effort level are offsetting factors influencing the probability of the option being in the money, which in turn affects the mean and variance of the option payoff. Section 5.3 describes the manager's effort decision problem, which is characterized by a first-order approach. Section 5.4 formally introduces the agency relation between the shareholders and the manager. We first provide a bench-mark analysis: the ownership contract case. Then, we discuss the tradeoff between the effort level and the exercise price. Finally, we derive the optimal option contract. Section 5.5 examines the total net surplus of the agency relationship generated by these types of contracts. Implications of our analysis are discussed in Section 5.6.

5.2 Agent's Wage Contract

The Board of Directors, acting on the behalf of the shareholders of the firm, designs the manager's wage contract. Assume the wage contract consists of a fixed wage ($\alpha$) plus an incentive component of some number ($\beta$) of (call) options issued with exercise price $K$ per option. The ex post value of one option is $\tilde{v} = \max\{\tilde{P} - K, 0\}$, where $\tilde{P}$ is the ex post price of the underlying stock. The price is assumed to be a normally distributed
variable with mean $\bar{P}$ and variance $\sigma_p^2$. Hence the manager’s wage contract is:

$$\tilde{W} \equiv \alpha + \beta \tilde{v}, \quad (5.1a)$$

where

$$\tilde{v} \equiv \begin{cases} \tilde{v}^*, & \text{if } \tilde{v}^* > 0; \\ 0, & \text{if } \tilde{v}^* \leq 0, \end{cases} \quad (5.1b)$$

$$\tilde{v}^* \equiv \bar{P} - K \sim N((\bar{P} - K), \sigma_p^2). \quad (5.1c)$$

Observe that $\tilde{v}^*$ is a normally distributed variable and $\tilde{v}$ is a censored normal variable. The latter variable is truncated from below at zero, and its economic interpretation is the payoff of one option. For example, if a realization of the stock price $p$ is higher than the exercise price $K$ (i.e., $p > K$), the option will be exercised by the holder and will give the holder a value of $p - K$ (assuming frictionless market, e.g., no transaction costs). On the other hand, if a realization of the stock price $p$ is not higher than the exercise price $K$ (i.e., $p \leq K$), the option’s holder will let it lapse without exercise and hence it is worthless.

By Theorem 22.3 of Greene (1993, p.692), the mean and variance of the option payoff are as follows:

$$E[\tilde{v}] = \sigma_p [\phi - (1 - \Phi)\xi], \quad (5.2a)$$

$$\text{Var}(\tilde{v}) = \sigma_p^2 \left[ (1 - \Phi - \phi^2) - \phi(2\Phi - 1)\xi + (1 - \Phi)\xi^2 \right]. \quad (5.2b)$$

Where $\xi \equiv \frac{K - P}{\sigma_p}$ represents the number of standard deviations that the exercise price $K$ is away from the value of the stock price, $\phi \equiv \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\xi^2\right\}$ is the standard normal density function evaluated at $\xi$, and $\Phi \equiv \int_{-\infty}^{\xi} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} dt$ is the standard normal cumulative probability function evaluated at $\xi$.

Alternatively, if $\tilde{v}$ is censored from above at zero (i.e., $\tilde{v} = 0$ if $\tilde{v}^* \geq 0$, and $\tilde{v} = \tilde{v}^*$ otherwise), its economic interpretation is the payoff of one put option. It can also be interpreted as the budgeting process with $K$ being the targeted budget. In this setting, the mean and variance of the option payoff are: $E[\tilde{v}] = -\sigma_p (\phi + \Phi\xi)$ and $\text{Var}(\tilde{v}) = \sigma_p^2 \left[ (\Phi - \phi^2) - \phi(2\Phi - 1)\xi + (1 - \Phi)\xi^2 \right]$. The wage contract (5.1a) is a penalty contract and its analysis is analogous. I would like to thank my advisor Professor Jerry Feltham and Professor J. Lundessgaard for bringing this point to my attention.
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The mean and variance of the option payoff $\tilde{v}$ are influenced by the exercise price $K$, which is set by the principal, and the mean of the stock price $\bar{P}$, which is influenced by the manager's effort. In this chapter, we assume that the mean of the stock price $\bar{P}$ is linear in the manager's effort level $e$, i.e., $\bar{P} = ge$, where $g > 0$ presents the marginal productivity. The managerial effort is assumed to be single-dimensional.\textsuperscript{52} Hence, the mean of the stock price is an increasing function of managerial effort level $e$, i.e., $\frac{d\bar{P}}{de} = g > 0$.

Given these basic assumptions, we now consider the impact of an increase in the agent's effort $e$ and/or the exercise price $K$ on the mean and variance of the option payoff. To accomplish this, we first consider the impact of an increase in effort $e$ and exercise price $K$ on the probability of the option being out of the money. That probability is equal to $\Phi$.

PROPOSITION 5.1 The probability of the option being out of the money (i.e., $\Phi$)

1. decreases in the managerial effort level $e$, i.e., $\frac{\partial \Phi}{\partial e} < 0$;
2. increases in the exercise price $K$, i.e., $\frac{\partial \Phi}{\partial K} > 0$.

Proof: See Appendix A.4. Q.E.D.

See Figure 5.1. This probability $\Phi$ is of particular importance in evaluating the option's payoff. It represents the probability mass with which the option is worthless (i.e., out of the money). In this agency relationship, this probability is a function of the agent's effort level. For any fixed exercise price, increasing the agent's effort increases not only the expected value of the stock price ($\bar{P} = ge$); but also the probability of the option being in the money ($1 - \Phi$). Thus, an increase in the agent's effort level increases the mean of the option.

\textsuperscript{52}Multi-tasks could be analyzed in conjunction with an additional component of annual bonus plans in the wage contract. However, due to the complexity of the analysis, we do not pursue it here.
On the other hand, for any given effort level $e$, an increase in the exercise price $K$ decreases the probability of the option being in the money and it decreases the expected option value. Formally, the following proposition characterizes the relationship between the mean and variance of the option payoff with the managerial effort level $e$ and the exercise price $K$.

**PROPOSITION 5.2** The managerial effort level and the exercise price are two offsetting factors influencing the mean and variance of the option payoff.

1. The expected option value is non-negative, and increases in the managerial effort
level and decreases in the exercise price, i.e., \( \frac{\partial E[\phi]}{\partial e} > 0 \) and \( \frac{\partial E[\phi]}{\partial K} < 0 \).

2. The variance of the option payoff increases in the managerial effort level and decreases in the exercise price, i.e., \( \frac{\partial \text{Var}(\phi)}{\partial e} > 0 \) and \( \frac{\partial \text{Var}(\phi)}{\partial K} < 0 \).

Proof: See Appendix A.4. Q.E.D.

Observe that no one would exercise an option if the stock price is lower than the exercise price. The option is worthless. The holder of an option will always exercise it, and realize a positive value of \( p - K \), if the stock price is higher than the exercise price (i.e., \( p > K \)). Therefore, it is intuitively obvious that the expected value of the option cannot be negative.

If the exercise price is minus infinity,\(^{53}\) the option is always exercised. That is, an option is equivalent to holding a stock with expected value \( \bar{p} = ge \) and variance \( \sigma^2 \). The expected option value increases in direct proportion to the agent’s effort level, and the variance of the option payoff is independent of the agent’s effort. Therefore, a higher effort level is strictly preferred to a lower effort level in the first-order stochastic dominance sense.

However, if the exercise price is finite, then, an increase in the agent’s effort level increases both the mean and the variance of the option payoff. An increase in the variance induces higher risk. As a result, it now is not clear that a higher effort level is preferred to a lower effort level because of the tradeoff between the increased expected value and the induced higher risk of the option.

\(^{53}\)For technical simplicity, we have assumed that the stock price follows the normal distribution over the entire space: \((-\infty, +\infty)\). A minus infinity exercise price makes an option equivalent to a stock although this negative exercise price may not be a practically plausible assumption. What is important here is the idea that such an exercise price makes an option equivalent to holding a stock. We can alternatively assume that the stock price follows a truncated normal distribution over the positive space: \([0, +\infty)\). Then, the zero exercise price makes an option equivalent to holding a stock.
Figure 5.2: The mean and variance functions of the option's payoff

Let $\sigma_p = 100$ and $g = 2$. Both the mean function (top) and the variance function (bottom) increase in the managerial effort level ($e \in [0, +200]$) and decrease in the exercise price ($K \in [-200, +200]$).
On the other hand, if the effort level is fixed, both the mean and variance of the option payoff decrease in the exercise price $K$. The intuition is simple because an increase in the exercise price decreases the probability of the option being in the money. At one extreme case, for instance, as $K \to +\infty$, we obtain that $E[\tilde{v}] \to 0$ and $\text{Var}(\tilde{v}) \to 0$. The option is worthless for sure.

Consequently, the exercise price and the agent's effort level are two offsetting factors influencing the mean and the variance of the option payoff (see Figure 5.2). To see this more clearly, we observe that both the mean and the variance of the option payoff are decreasing functions [see (A.6)] of $\xi$, the number of standard deviations that $K$ is away from the expected value of the stock price $\bar{P} = ge$. In the special case where the exercise price $K$ is set equal to the expected value of the stock price, we obtain that $\xi = 0$. Hence, the probability of the option being in the money is constant: $\frac{1}{2}$.

5.3 Agent's Decision Problem

In the previous section, we assumed that the agent's wage contract consists of two components: a fixed wage and some options with exercise price $K$. We have shown that both the mean and the variance of the option payoff increase in the effort level and decrease in the exercise price.

Given the compensation contract in place, we now examine the agent's effort level decision. For the sake of tractability, we assume that the agent selects his effort level $e$
to maximize the following mean-variance objective function:  

$$
\max_e E[\bar{W} - C(e)] - \frac{1}{2R} \text{Var}(\bar{W} - C(e)),
$$

where $R$ is the agent’s risk tolerance parameter and $C(e)$ is the agent’s personal cost of expending effort level $e$. The personal cost is assumed to take the following simple form:

$$
C(e) = \frac{1}{2} e^2.
$$

The first-order condition of the agent’s decision problem is

$$
\beta \frac{\partial E[\bar{v}]}{\partial e} - e - \frac{\beta^2}{2R} \frac{\partial \text{Var}(\bar{v})}{\partial e} = 0,
$$

or equivalently,

$$
\beta (1 - \Phi) g - e - \beta^2 \sigma_p \Phi \frac{\phi - (1 - \Phi) \xi}{R} = 0.
$$

In general, we cannot provide a closed form solution for the optimal effort level $e$ since $\Phi$ and $\phi$ are functions of $\xi$ and $\xi = (K - ge)/\sigma_p$. However, in two special cases in which the exercise price is (plus or minus) infinity, i.e., $k \rightarrow \pm \infty$, we can. In the former case, we obtain that $e = 0$; and in the latter case, we obtain that the above equation becomes $\beta g = e$, which is identical to what we had in the stock ownership compensation contract case examined in the previous chapters. Before analyzing this condition further, we formally describe the principal’s decision problem.

5.4 Notice that the agent’s wage contract $\bar{W}$ is not a normally distributed random variable if the exercise price is finite. Hence, this mean-variance objective function (5.3) of the agent is no longer identical to the maximization of his ex ante expected (negative exponential) utility. Fortunately, it is a first-order approximation of the expected utility since

$$
E[-e^{-X/R}] = -\left[1 - \left(\frac{1}{R}\right) X + O(X)\right] \left[1 + \left(\frac{1}{2R^2}\right) \text{Var}(X) + O(X^2)\right]
$$

$$
= -1 + \left(\frac{1}{R}\right) \left[X - \left(\frac{1}{2R}\right) \text{Var}(X)\right] + O(X)
$$

$$
\propto X - \left(\frac{1}{2R}\right) \text{Var}(X).
$$

Readers are referred to Hanoch and Levy (1969), Pulley (1981), and Chamberlain (1983) for more discussion about this point. For application, see Grinblatt and Hwang (1989).
5.4 Principal’s Problem of Incentive Compensation Contract Design

In this section, we first exogenously specify the agent’s effort level \( e \) and characterize the optimal contract for inducing this effort level. Then, we briefly characterize the optimally induced second-best effort levels under both the ownership incentive contract and the option incentive contract.

Assume the agent’s expected wage exactly covers his personal cost of expending effort level \( e \) and the risk premium of taking on the risk associated with the wage contract,

\[
E[\tilde{W}] = C(e) + \frac{1}{2\mathcal{R}} \text{Var}(\tilde{W}).
\]

Substituting this constraint into the shareholders’ objective function (i.e., maximization of the net surplus) results in the following characterization of the principal’s decision problem:

\[
\max_{\beta} \bar{X} - E[\tilde{W}] = ge - \frac{1}{2}e^2 - \frac{\beta^2}{2\mathcal{R}} \text{Var}(\tilde{v}), \quad \text{subject to:} \quad (5.3). \quad (5.6)
\]

If the agent’s incentive constraint (5.3) is characterized by first-order condition (5.5), the Lagrangian function of the principal’s decision problem is:

\[
L \equiv ge - \frac{1}{2}e^2 - \frac{\beta^2}{2\mathcal{R}} \text{Var}(\tilde{v}) + \mu \left[ \beta \frac{\partial E[\tilde{v}]}{\partial e} - e - \frac{\beta^2}{2\mathcal{R}} \frac{\partial \text{Var}(\tilde{v})}{\partial e} \right], \quad (5.7)
\]

where \( \mu > 0 \) is the Lagrangian multiplier for the agent’s incentive constraint. The first-order conditions with respect to \( \beta \) and \( \mu \) are as follows:

\[
-\frac{\beta}{\mathcal{R}} \text{Var}(\tilde{v}) + \mu \left[ \frac{\partial E[\tilde{v}]}{\partial e} - \frac{\beta}{\mathcal{R}} \frac{\partial \text{Var}(\tilde{v})}{\partial e} \right] = 0; \quad (5.8a)
\]

\[
-\frac{\beta^2}{2\mathcal{R}} \frac{\partial \text{Var}(\tilde{v})}{\partial e} + \beta \frac{\partial E[\tilde{v}]}{\partial e} - e = 0. \quad (5.8b)
\]

Effort level \( e \) is only implementable if there exist \( \beta \) options such that (5.8b) is satisfied. The left-hand side of (5.8b) is a quadratic function of \( \beta \), which has real solutions for \( \beta \).
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if, and only if\(^5.5\)

\[
\left( \frac{\partial E[\tilde{v}]}{\partial e} \right)^2 \geq \frac{2e}{\mathcal{R}} \left( \frac{\partial \text{Var}(\tilde{v})}{\partial e} \right).
\]

We refer to (5.9) as the necessary and sufficient condition for implementable effort.\(^5.6\)

5.4.1 Stock ownership incentive contract

To provide a benchmark analysis for the option compensation contract, we first examine the stock ownership compensation contract. Since the stock price is assumed to be normally distributed, the exercise price has technically to be minus infinity (i.e., \(K = -\infty\). See footnote 5.3 for more discussion of this point). This ensures that any realization of the stock price is higher than the exercise price \(K\) and the option is always exercised; hence the option contract is equivalent to a stock ownership contract. The expected option value increases in proportion to the agent's effort level, and the variance of the option payoff is independent of the agent's effort level. The first-order conditions (5.8) become:

\[
-\frac{\beta}{\mathcal{R}} \sigma_e^2 + \mu g = 0;
\]

\[
\beta g - e = 0.
\]

The above equation system always has a solution, implying that the entire effort space \([0, +\infty)\) is implementable. Thus, for any effort level, we can solve the above equation

\(^5.5\)By Proposition 5.2, it is clear that (5.8b) has no solution for negative \(\beta\).

\(^5.6\)We can alternatively write this condition as follows:

\[
\mathcal{R} \geq 2e \left( \frac{\partial \text{Var}(\tilde{v})}{\partial e} \right) \left( \frac{\partial E[\tilde{v}]}{\partial e} \right)^{-2}.
\]

This implies that the manager has to be risk tolerant enough for the option compensation contract to be effective. In other words, the option compensation does not work for a very risk averse manager (i.e., \(\mathcal{R} = \text{very small}\)). For example, in the extreme case, the manager is extremely risk averse (i.e., \(\mathcal{R} = 0\)), only the zero effort level is implementable under the option compensation contract. On the other hand, however, if the manager is risk neutral (i.e., \(\mathcal{R} = \infty\), the entire effort space is implementable under the option contract.
system and obtain the optimal number of shares granted to the agent and the Lagrangian multiplier:

\[
\beta_s = \frac{e}{g}, \quad (5.10a)
\]
\[
\mu_s = \frac{e\sigma^2}{g^2\mathcal{R}}, \quad (5.10b)
\]

where the subscript \(s\) denotes stock ownership.

For any given effort level \(e\), the optimal number of shares \(\beta_s\) is independent of the agent's risk aversion and the performance measure's noise. This is not a surprising insight for the ownership contract, since the agent's action does not influence the stock price variance and he trades-off his gain from his ownership of the firm with his personal cost of effort.

The agent is motivated by this incentive contract to expend effort to maximize his objective function. In equilibrium, the agent's objective function is maximized (i.e., his incentive constraint is satisfied) and the total net surplus of the agency (i.e., the net surplus of the principal) is maximized.

**PROPOSITION 5.3** The net surplus induced by ownership contract is a concave quadratic function of effort level, and the optimally induced second best effort level is:

\[
e^*_s = \frac{g^3}{g^2 + \sigma^2_\epsilon / \mathcal{R}}. \quad (5.11a)
\]

**Proof:** See Appendix A.4. Q.E.D.

Observe that, for any given number of shares \(\beta\), the agent's response is characterized by (5.5b), which becomes \(e_s = \beta g\) in the ownership contract case. On the other hand, for any given effort level \(e\), the principal's response is characterized by (5.10a), which becomes \(\beta_s = e/g\). Both functions take exactly the same form and are independent of all other parameters except the productivity parameter \(g\).
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Given the response functions of both the agent and the principal, we combine equations (5.10a) and (5.11a) and obtain the equilibrium numbers of shares:

\[ \beta^* = \frac{g^2}{g^2 + \sigma^2_p/R}. \]  

(5.11b)

Both the equilibrium effort level \( e^*_s \) and the equilibrium number of shares \( \beta^*_s \) are increasing in the manager’s risk tolerance and the stock price precision (i.e., \( \sigma^2_p \)). If the manager is risk neutral (i.e., \( R = +\infty \)) or the stock price is noiseless (i.e., \( \sigma^2_p = 0 \)), it is optimal to sell the entire firm to the manager (i.e., \( \beta^*_s = 1 \)) and the manager expends the first-best effort level (i.e., \( e^*_s = g \)).

5.4.2 Implementable effort set

In the above benchmark (stock ownership compensation contract) case, we showed that the entire effort space is implementable. In general, when the exercise price is not equal to minus infinity, the implementable effort set is a strict subset of the effort space. For instance, if the exercise price is plus infinity, there is only a single implementable effort level: zero. The intuition is clear because an option with an exercise price of plus infinity always lapses without being exercised and hence is worthless. The agent’s wage contract equivalently becomes a fixed wage contract. There is no surprise that the agent only expends zero effort.

The agent’s risk aversion affects effort implementability as well. For example, if the agent is risk neutral, the variance of the option payoff is immaterial and the agent demands no risk premium for taking on risk. The expected option value is the sole factor that the agent uses to tradeoff his personal cost of effort. In that setting, the number of options required to induce effort level \( e \) is \( \beta_0 = \frac{e}{g(1-\Phi)} \) which is greater than \( \beta^*_s = \frac{e}{g} \). The number of options exceeds the number of shares required to induce a given action because the impact of effort on the expected option value [i.e., \( \frac{\partial E[S]}{\partial e} = (1 - \Phi)g \)] is less
than its impact on the expected stock price (i.e., $\frac{\partial X}{\partial e} = g$). The number of shares relative to the number of options required to implement a given action $e$ is:

$$\frac{\beta_{\epsilon}}{\beta_{o}} = 1 - \Phi,$$

(5.12)

That is, it is equal to the probability of the option being in the money. The higher is the probability of the option being in the money, the lower is the ratio of options to shares required to induce a given action $e$. If the exercise price is set equal to the expected value of the stock price (i.e., $K = ge$), there is a 50% of the chance that the option will be in the money (i.e., $1 - \Phi = \frac{1}{2}$) and hence two options are equivalent to one share in terms of providing the same incentive for the manager to expend a given effort level $e$.

The agent's effort implementibility is not an issue in the above special cases: the stock ownership contract and the risk neutral agent. In both cases, the first term of the left-hand-side of (5.8b) vanishes, i.e., the stock price variance is independent of effort level in the former case and $R = \infty$ in the latter. The key insight here is that the agent's effort does not affect the risk associated with the incentive contract. In the stock ownership case, it is obvious by assumption. In the risk neutral agent case, the risk is immaterial. Observe that the second term of the left-hand-side of (5.8b) is positive because the expected value is positively related to the agent's effort. As the agent increases his effort level $e$, condition (5.8b) can always be satisfied by increasing $\beta$ correspondingly. The intuition is clear, because the agent need not to worry about the risk associated with his compensation contract as he increases his effort level, he merely trades off his marginal gain from the wage contract with his personal cost of effort. As long as $\beta$ is large enough, his gain will be large enough to offset his increased cost of effort. Therefore, the entire effort space is implementable.

However, the risk associated with the agent's wage contract is a major factor influencing the agent's effort decision in general. In true option case (i.e., the exercise price
is finite), the variance of the option payoff is an increasing function of the agent’s effort level (see Proposition 5.2). This positive relation between the variance of the option payoff and the agent’s effort level influences a risk averse agent’s effort decision. That is, an increase in effort increases not only the expected option value but also the variance of the option payoff. Observe that increases in the mean and variance of the option payoff have opposite influences on the risk averse agent, which is captured by the opposite signs in the first two terms of the left-hand-side of (5.8b). Now, we cannot merely increase $\beta$ to compensate the agent’s personal cost of his incremental effort, because an increase in $\beta$ also imposes more risk on the agent. Consequently, there exists a cut-off effort level above which the effort level is not implementable.

Rewriting (5.9) yields a more explicit representation of the agent’s effort implementability:

$$e \leq \frac{\mathcal{R} \left( \frac{\partial E[I]}{\partial e} \right)^2}{2 \left( \frac{\partial \text{Var}(I)}{\partial e} \right)} = \frac{g(1 - \Phi)^2 \mathcal{R}}{4 \sigma_p^2 \Phi \left[ \phi - (1 - \Phi) \xi \right]} \equiv A(K).$$ \hspace{1cm} (5.13)

$A(K)$ defines the upper bound on the set of the implementable effort levels, i.e., the implementable effort set is $[0, A(K)]$. The upper bound is a function of the exercise price $K$, the variance of the stock price $\sigma_p^2$, the firm gross payoff productivity $g$, and the agent’s risk tolerance $\mathcal{R}$.

It is obvious that the implementable effort set increases as the agent is more risk tolerant (i.e., $\frac{\partial A(K)}{\partial \mathcal{R}} > 0$). The relationship between $A(K)$ and $K$, $\sigma_p^2$, and $g$, is not clear because $\phi$ and $\Phi$ are functions of all three.

The necessary and sufficient condition (5.9) captures the important tension between the exercise price and the effort level. The intuition behind this tension is that the exercise price $K$ and the effort level $e$ have offsetting effects on both the mean and variance of the option payoff (see Proposition 5.2). The following proposition exactly characterizes
Figure 5.3: The upper bound function of the implementable effort set

Let $\sigma_p = 500$, $e = 1$, $R = 1$, and $g = 100$. The implementable effort set’s upper bound decreases in the exercise price ($K \in [-400, 500]$).

This economic insight.

**Proposition 5.4** The implementable effort set $[0, A(K)]$ contracts as the exercise price $K$ increases, i.e., $\frac{dA(K)}{dK} < 0$.

**Proof**: See Appendix A.4. \(Q.E.D\).

This proposition establishes that there is a general relation between the exercise price and the managerial effort level: the higher is the exercise price, the smaller is the implementable effort set (see Figure 5.3). This relation stems from the offsetting impact of the exercise price and the effort level on the probability of the option being in the money (see Proposition 5.1). It is illustrated by the two limiting cases in which the exercise price is plus or minus infinity. In the former case, only the zero effort level is implementable, and in the latter (ownership incentive contract case), the entire effort
space is implementable.

In the stock ownership case, the mean (i.e., $ge$) increases in managerial effort, while the variance (i.e., $\sigma^2_p$) is independent of managerial effort. An increase in effort level increases the mean of the stock price and does not affect the riskiness of the stock. As a result, any effort level is implementable as long as the incentive weight (i.e., $\beta_*$) is large enough so that the gain in the wage contract due to the increased mean offsets the agent’s personal cost of effort. The constant risk of the wage contract results in a constant risk premium which does not influence the agent’s effort decision.

In the option contract setting (i.e., the exercise price is finite), an increase in effort level increases both the mean and variance of the option payoff (see Proposition 5.2). The increased variance affects the manager’s effort decision. The manager now has to tradeoff the incremental gain from his wage contract with not only the increased personal cost of his effort but also the increased risk. An increase in the coefficient (i.e., incentive weight) for the performance measure (i.e., the option’s payoff) may not be sufficient to induce the manager to expend higher effort level due to the increased risk associated with the wage contract. For some high effort levels, the increased risk associated with the wage contract may become non-bearable for the risk averse manager so that he simply does not want to expend those effort levels (i.e., non-implementable).

5.4.3 Stock option incentive contract

Given any implementable effort $e$ [i.e., (5.9) holds], we can solve for $\beta$, the optimal number of options granted to the agent to induce the given effort level $e$.

**PROPOSITION 5.5** For any implementable effort $e$, there exists a unique solution to the Board of Directors’ contract design problem (5.6)

\[
\beta_o = \frac{\mathcal{R} (1 - \Phi) - \sqrt{\Delta}}{2 \sigma_p \Phi (\phi - (1 - \Phi) \xi)}, \quad \text{and} \quad \mu_o = \frac{\beta_o \text{Var}(\tilde{v})}{\mathcal{R} \sqrt{\Delta}},
\]

(5.14a)
where $\Delta \equiv (1 - \Phi)^2 - \frac{4se}{\gamma K} \Phi [\phi - (1 - \Phi)\xi]$.

**Proof:** See Appendix A.4. Q.E.D.

For any given level of implementable effort, the expected option value decreases as the exercise price $K$ rises (see Proposition 5.2). To induce a given effort level from the agent, we should expect that the number of options (i.e., $\beta_o$) increases as the exercise price $K$ increases. In fact, numerical examples (see Figure 5.4) show that the number of options indeed increases in the exercise price $K$. Moreover, the number of options increases so fast that $\beta_o^2 \text{Var}(\tilde{v})$ increases in the exercise price as well, despite the fact that $\text{Var}(\tilde{v})$ decreases in the exercise price (see Proposition 5.2).

This implies that the risk premium associated with the agent's wage contract increases in the exercise price $K$. For any given level of effort, it is immediate that an stock ownership contract requires the lowest risk premium. For tractability, we focus our analysis on option incentive contracts in which the exercise price equals the expected value of the underlying stock price. Therefore, from now on, the exercise price of the option is fixed at the *ex ante* mean of the underlying stock price.

It is common practice that the exercise price of incentive options granted to managers is set equal to the *ex ante* mean of the stock price. If the *ex post* stock price is lower than the *ex ante* mean of the stock price, the options lapse without being exercised and hence their holders are penalized. If the *ex post* stock price is higher than the *ex ante* mean of the stock price, the options will be exercised and their holders realize a positive gain.

For any given implementable effort level $e$, fixing the exercise price at $K = ge$, we obtain $\xi = 0$, $\phi = \frac{1}{\sqrt{2\pi}}$, and $\Phi = \frac{1}{2}$. In this case, whether the option ends up in the money or out of the money is a fair coin flip. Furthermore, the Lagrangian multiplier,
Figure 5.4: Option's variance, optimal numbers of options, and risk premiums

Let \( \sigma_p = 10, e = 1, R = 1, g = 20, \) and \( K \in [-10, 15] \). The option's variance (top left) decreases in \( K \), the optimal numbers (bottom left) of options increases in \( K \), the risk premium of the option contract (top right) increases in \( K \), and the difference of the risk premiums between the option contract and the ownership contract is positive.
\( \mu_o = \left(1 - \frac{8e\sigma_p}{\sqrt{2\pi gR}}\right)^{-\frac{1}{2}} - 1 \), is positive, and the optimal number of options is given by:

\[
\beta_o = \left(1 - \sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi gR}}} \right) \frac{\sqrt{2\pi R}}{2\sigma_p}.
\] (5.14b)

The implementable effort set is \([0, \frac{\sqrt{2\pi gR}}{8\sigma_p}]\).

So far, we have focused on examining the optimal incentive weight \(\beta_o\) for any given level of effort. To conclude this section, let us now characterize the optimal effort level which maximizes the net surplus.

**PROPOSITION 5.6** The net surplus induced by the option compensation contract (with an exercise price equal to the ex ante mean of the stock price) is a concave function of managerial effort. The optimally induced second-best effort level is characterized by:

\[
e^*_o = g + \frac{(\pi - 1)\sigma_p}{\sqrt{2\pi g}} \left(1 - \frac{1}{\sqrt{1 - \frac{8e\sigma_p^{*}}{\sqrt{2\pi gR}}}}\right).
\] (5.15)

**Proof:** See Appendix A.4. Q.E.D.

Substituting \(e^*_o\) into (5.14b), we obtain the optimal number of options \(\beta^*_o\) to maximize the shareholders’ net surplus which is also the total net surplus of the agency relation.

**5.5 “Welfare” Implication of Ownership Versus Option Incentive Contracts**

In the previous section, we defined the principal’s problem of incentive compensation contract design, examined the ownership and option contracts, and derived the optimal incentive weights and the induced second best effort levels.

In this section, we compare these two types of incentive contracts in terms of the principal’s net surplus and the optimally induced second-best effort levels. The former is examined for every given implementable effort level, and the latter is examined in the equilibrium.
PROPOSITION 5.7 For any implementable effort level \( e \), the ownership incentive contract always generates a higher net surplus than does the option incentive contract.

Proof: See Appendix A.4. Q.E.D.

Since the implementable effort set for the option contract is a strict subset of the implementable effort set for the ownership contract (see Proposition 5.4), we can only compare the net surplus of the option contract with part of the net surpluses of the ownership contract. For any given effort level in the implementable effort set, it is clear that the firm's gross value is the same under these two types of contracts, and so is the agent's personal cost of effort. The difference between the net surpluses of the ownership and option contracts is only due to the differences in the risk premium.

It is shown that the risk premium difference between the option contract and the ownership contract is positive for any implementable effort level. As a result, the net surplus of the ownership contract is always higher than the net surplus of the option contract. Since the above result holds for any given effort level in the implementable effort set, it is clear that the optimal stock ownership contract [i.e., \((\beta^*, e^*_*)\)] is better than the optimal option contract [i.e., \((\beta^*_o, e^*_o)\)].

Now, let us determine which type of compensation contract induces the higher second best effort level.

PROPOSITION 5.8 The optimal ownership incentive contract always induces higher second-best effort level than does the optimal option incentive contract (with an exercise price equal to the ex ante mean of the stock price).

Proof: See Appendix A.4. Q.E.D.
5.6 Implications

Options are widely used and assumed to be desirable because they shelter the manager from downside risk ex post. However, it is shown that, to induce a given level of managerial effort, ex ante risk associated with an option contract is higher than that associated with a stock ownership contract. Hence, the risk premium with an option contract is higher than that with a stock ownership contract. Consequently, the firm's shareholders enjoy a higher net surplus with an ownership contract than that with an option contract. The intuition is that the number of options required to induce a given level of managerial effort is much larger than the number of shares. The increase in the number of options is sufficient to result in higher risk imposed on the agent \textit{ex ante} with options than with shares, although an option contract completely eliminates the downside risk \textit{ex post}.

In addition, our analysis shows that the entire effort space is implementable with an ownership contract, while only a strict subset of the effort space is implementable with an option contract. It is shown that, in equilibrium, the second-best effort level induced by an option contract is lower than that induced by an ownership contract. In sum, our results suggest that ownership contracts dominate options contracts.

However, our results were derived from a stylized model in which the payoff distribution is normal, the principal is risk neutral, and the agent is risk and effort averse. Moreover, for the sake of tractability, we assumed that the agent makes his effort decision to maximize a mean-variance criterion-based objective function. The standard agency result (see Holmstrom 1979) shows that the combination of the normal payoff distribution and the agent's negative exponential utility function implies that the optimal incentive compensation contract is concave. Consequently, there is no surprise that an ownership (i.e., a linear) contract beats an option (i.e., piece-wise convex linear) contract. That is, the former is a better approximation of the optimal incentive concave contract than is
the latter. Furthermore, it may be worthwhile to also examine penalty contracts (i.e., piece-wise concave linear, see footnote 5.1). A penalty contract may locally approximate the optimal incentive concave contract better than does an ownership contract. We cannot argue that this is a general result since it is based on a simple mean-variance objective function. Interesting extensions would examine combinations of different payoff distributions and utility functions. For instance, is an option contract preferred to an ownership contract if the optimal incentive contract is convex, such as when the agent has a square-root utility function and the probability function is exponential?
Bibliography


Appendix A

Proofs

A.1 Rational Expectations Equilibrium (Chapter 2)

Proof of Lemma 2.1

Proof: The differences of the conditional variances and means of the risky asset payoff \( \hat{X} \) under the two different information structures \( \{ \psi, \hat{Y} \} \) and \( \{ \hat{Y} \} \) are, respectively,

\[
\text{Var}\left( \hat{X} \mid \hat{Y} \right) - \text{Var}\left( \hat{X} \mid \psi, \hat{Y} \right) = \frac{\sigma_x^2 \rho_1^2}{\rho_2},
\]

\[
E\left( \hat{X} \mid \hat{Y} \right) - E\left( \hat{X} \mid \psi, \hat{Y} \right) = -\frac{\sigma_x \rho_1}{\sigma_0 \rho_2} \left[ \frac{\sigma_0 y}{\sigma_y^2} \left( \hat{Y} - \bar{Y} \right) - \left( \hat{\psi} - \bar{\psi} \right) \right].
\]

They are equal to zero if, and only if, \( \rho_1 = 0 \), i.e., \( \rho_{0x} = \rho_{0y} \rho_{xy} \).

The differences of the conditional variances and means of the risky asset payoff \( \hat{X} \) under the two different information structures \( \{ \psi, \hat{Y} \} \) and \( \{ \hat{\psi} \} \) are, respectively,

\[
\text{Var}\left( \hat{X} \mid \psi \right) - \text{Var}\left( \hat{X} \mid \hat{\psi}, \hat{Y} \right) = \frac{\sigma_x^2 \rho_6^2}{\rho_2},
\]

\[
E\left( \hat{X} \mid \psi \right) - E\left( \hat{X} \mid \hat{\psi}, \hat{Y} \right) = \frac{\sigma_x \rho_6}{\sigma_y \rho_2} \left[ \frac{\sigma_0 y}{\sigma_y^2} \left( \hat{\psi} - \bar{\psi} \right) - \left( \hat{Y} - \bar{Y} \right) \right].
\]

They are equal to zero if, and only if, \( \rho_6 = 0 \), i.e., \( \rho_{xy} = \rho_{0x} \rho_{0y} \).

Since a normal distribution is completely determined by its mean and variance, the conditional distribution of the risky asset payoff \( \hat{X} \) under different information structures \( \{ \hat{\psi}, \hat{Y} \} \) and \( \{ \hat{Y} \} \) are identical, i.e., the information structure \( \{ \hat{Y} \} \) is a sufficient statistic for the information structure \( \{ \hat{\psi}, \hat{Y} \} \) if, and only if, \( \rho_1 = 0 \). By analogy, the information
structure \( \{\hat{\psi}\} \) is a sufficient statistic for the information structure \( \{\tilde{\psi}, \tilde{Y}\} \) if, and only if, \( \rho_6 = 0 \).

Furthermore, since \( \rho_2 > 0 \) implies that \( \rho_{0y} \neq \pm 1 \), we have that \( \rho_{xy} \neq \rho_{0y} \rho_{zy} = \rho_{0y} \rho_{ox} \) (i.e., \( \rho_6 \neq 0 \)) if \( \rho_{0y} \rho_{zy} = \rho_{ox} \) (i.e., \( \rho_1 = 0 \)). This is sufficient to show that both \( \rho_1 \) and \( \rho_1 \) cannot be zero simultaneously. \( \text{Q.E.D.} \)

**Proof of Proposition 2.1**

**Proof:** By the definitions of \( B_0 \) and \( b_0 \), (2.12a) becomes

\[
Ba_0 = \lambda (\tilde{X} - B_1 \tilde{Y} - B_2 \hat{\psi}) + (1 - \lambda) (V_i/V_a) (\tilde{X} - b_1 \tilde{Y} - b_2 \tilde{P})
\]

\[
= [\lambda + (1 - \lambda) (V_i/V_a)] \tilde{X} - [\lambda B_1 + (1 - \lambda) (V_i/V_a) (b_1 + b_2 a_2)] \tilde{Y}
\]

\[
- [\lambda B_2 + (1 - \lambda) (V_i/V_a) b_2 a_1] \hat{\psi} - (1 - \lambda) (V_i/V_a) b_2 a_0
\]

\[
= [\lambda + (1 - \lambda) (V_i/V_a)] \tilde{X} - [Ba_2 + (1 - \lambda) (V_i/V_a) b_2 a_2] \tilde{Y}
\]

\[
- [Ba_1 + (1 - \lambda) (V_i/V_a) b_2 a_1] \hat{\psi} - (1 - \lambda) (V_i/V_a) b_2 a_0
\]

\[
= [B + (1 - \lambda) (V_i/V_a) b_2] (\tilde{X} - a_2 \tilde{Y} - a_1 \hat{\psi}) - (1 - \lambda) (V_i/V_a) b_2 a_0.
\]

The second equality above follows from the substitution of \( \tilde{P} = a_0 + a_1 \hat{\psi} + a_2 \tilde{Y} \). The third equality above follows from (2.12c) and (2.12b). The last equality follows from the definition of \( B \) that \( \lambda + (1 - \lambda) (V_i/V_a) = B + (1 - \lambda) (V_i/V_a) b_2 \). Therefore, (2.15a) is immediate by moving the last term of the right-hand-side (i.e., \( -(1 - \lambda) (V_i/V_a) b_2 a_0 \)) to the left-hand-side and then dividing the equation by non-zero factor: \( B + (1 - \lambda) (V_i/V_a) b_2 \).

This completes the proof of \( a_0 \).

On the one hand, if \( \lambda \rho_1 \neq 0 \), then, (2.13) can be re-written as

\[
a_2 = \frac{\nu \sigma_0 \sigma_x \rho_0}{\lambda \rho_1} a_1.
\]  \( \text{(A.1)} \)
Appendix A.1: Rational Expectations Equilibrium (Chapter 2)

Substituting (A.1) into (2.10b) yields

\[ b_2a_1 = \frac{\sigma_0 \sigma_x \rho_1}{\sigma_0^2 \rho_2 + (a_z^2/a_1^2) \sigma_z^2} = \frac{\lambda^2 \sigma_x \rho_1^2}{\sigma_0 (\lambda^2 \rho_1 \rho_2 + \nu^2 \sigma_z^2 \rho_0^2)} \]

Substituting the above equation into \( B \) and re-writing (2.12d) result in:

\[
[\lambda \pm (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u}] a_1 = \frac{\lambda \sigma_x \rho_1}{\sigma_0 \rho_2} + (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u} b_1 a_1
\]

\[
= \frac{\lambda \sigma_x \rho_1}{\sigma_0 \rho_2} + \frac{(1 - \lambda) (V_i/V_u) \lambda^2 \sigma_x \rho_1^3}{\sigma_0 (\lambda^2 \rho_1^2 \rho_2 + \nu^2 \sigma_z^2 \rho_0^2)}
\]

\[
= \frac{\lambda \sigma_x \rho_1 [\lambda \mp (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u}] \lambda \rho_1^2 \rho_2 + \nu^2 \rho_0^2 \sigma_z^2 \rho_2^2}{\lambda^2 \rho_1^2 \rho_2 + \nu^2 \rho_0^2 \sigma_z^2 \rho_2^2}
\]

Hence,

\[
a_1 = \frac{\lambda \sigma_x \rho_1 [\lambda \rho_1^2 \rho_2 + \nu^2 \rho_0^2 \sigma_z^2 \rho_2^2]}{\sigma_0 \rho_2 (\lambda^2 \rho_1^2 \rho_2 + \nu^2 \rho_0^2 \sigma_z^2 \rho_2^2)}.
\]

By (2.14), we straightforwardly obtain \( a_1 \) and then \( a_2 \) by (2.13).

Now, let us prove \( a_2 \). The left-hand-side of (2.12c), \( B a_2 \), after substituting \( B \equiv \lambda + (1 - \lambda) (V_i/V_u) (1 - b_2) \), becomes:

\[
[\lambda \pm (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u}] a_2 - (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u} b_2 a_2.
\]

Since it is clear from (2.10b) that

\[ b_1 = \frac{\sigma_0^2 \sigma_x \rho_6 - \left( \frac{a_z^2}{a_1^2} \right) \sigma_y \rho_0 \sigma_x \rho_1 + \left( \frac{a_z^2}{a_1^2} \right) \sigma_z^2 \sigma_x \rho_{xy}}{\sigma_y \left[ \sigma_0^2 \rho_2 + \left( \frac{a_z^2}{a_1^2} \right) \sigma_z^2 \right]} = \frac{\sigma_0^2 \sigma_x \rho_6 + \left( \frac{a_z^2}{a_1^2} \right) \sigma_z^2 \sigma_x \rho_{xy}}{\sigma_y \left[ \sigma_0^2 \rho_2 + \left( \frac{a_z^2}{a_1^2} \right) \sigma_z^2 \right]} - b_2 a_2,
\]

the right-hand-side of (2.12c), \( \lambda B_1 + (1 - \lambda) (V_i/V_u) b_1 \), becomes:

\[
\lambda B_1 + (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u} \frac{\sigma_0^2 \sigma_x \sigma_y \rho_6 + \left( \frac{a_z^2}{a_1^2} \right) \sigma_z^2 \sigma_x \sigma_y \rho_{xy}}{\sigma_0^2 \sigma_y \rho_2 + \left( \frac{a_z^2}{a_1^2} \right) \sigma_z^2 \rho_{y}} - (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u} b_2 a_2.
\]

Observe that \( (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u} b_2 a_2 \) is a common term in (A.3) and (A.4). Let us first remove it; second, divide the equation [(A.3) = (A.4)] by a non-zero \( \left[ \lambda \mp (1 - \lambda) \frac{(V_i/V_u)}{V_j/V_u} \right] \)
finally, substitute (2.14), (2.9b), and (A.1) into the equation. We immediately obtain

\[
\begin{align*}
\lambda B_1 + (1 - \lambda) \left( V_i / V_u \right) \frac{\sigma_x^2 \sigma_y \rho_6 + \left( \frac{\sigma_x^2}{\sigma_y^2} \right) \sigma_x^2 \sigma_y \rho_{xy}}{\sigma_y^2 \sigma_y \rho_6 + \left( \frac{\sigma_x^2}{\sigma_y^2} \right) \sigma_x^2 \sigma_y \rho_{xy}} \\
A + (1 - A) \left( V_i / V_u \right) \\
\lambda \sigma_x^2 \left[ \frac{\lambda (\rho_6 / \rho_2) + (1 - \lambda) \left( V_i / V_u \right) \frac{\lambda^2 \rho_6^2 \rho_{xy} + \nu^2 \sigma_x^2 \sigma_y \rho_{xy}}{\lambda^2 \rho_3^2 \rho_{xy} + \nu^2 \sigma_x^2 \sigma_y \rho_{xy}}}{\sigma_y \left[ \lambda + (1 - \lambda) \left( V_i / V_u \right) \right]} \right]
\end{align*}
\]

(A.5)

The rest of \( a_2 \) is straightforward.

On the other hand, if \( \lambda = 0 \), then \( a_1 = 0 \) from (2.12b) since \( B \neq 0 \). Hence, \( b_2 = 0 \) by (2.10b). Therefore, by (2.12c) and (2.10b), it is clear that \( a_2 = b_1 = \sigma_{xy} / \sigma_y^2 \); and by (2.12d), \( a_2 = \nu V_u = \nu \sigma_x^2 \rho_3 \). That is, they are (2.15b), (2.15c), and (2.15d) as \( \lambda = 0 \), respectively. If \( \rho_1 = 0 \), then (2.10b) implies that \( b_2 = 0 \) and Lemma 2.3 implies that \( V_i = V_u \), and hence \( B = 1 \neq 0 \). As a result, \( a_1 = 0 \) from (2.12b). Thus, by (2.12c),

\[
a_2 = \frac{\lambda \sigma_x \rho_6}{\sigma_y \rho_2} + \frac{(1 - \lambda) \sigma_{xy}}{\sigma_y^2} = \frac{\sigma_{xy}}{\sigma_y^2},
\]

and the second equality follows from the identity: \( \rho_6 = \rho_2 \rho_{xy} + \rho_1 \rho_{xy} \). Finally, (2.12d) implies that \( a_x = \frac{\nu \sigma_x^2 \rho_6}{\rho_2} \) in this setting. That is, they are (2.15b), (2.15c), and (2.15d) as \( \rho_1 = 0 \), respectively.

Q.E.D.

Proof of Proposition 2.4

Proof: The derivative of \( a_1 \) with respect to \( \lambda \) is

\[
\frac{da_1}{d\lambda} = \frac{\nu^2 \sigma_x^2 \sigma_y \rho_0 \left[ \sigma_x^2 \lambda (2 - \lambda) + \nu^2 \sigma_x^2 \sigma_y \rho_{xy} \right]}{\sigma_y \left( \lambda^2 \rho_1^2 \rho_2 + \lambda \nu^2 \sigma_x^2 \sigma_y \rho_0 \rho_{xy} + \nu^2 \sigma_x^2 \sigma_y \rho_{xy} \right)^2} \rho_1,
\]

which has the same sign as \( \rho_1 \). The derivative of \( a_2 \) with respect to \( \lambda \) is

\[
\frac{da_2}{d\lambda} = \frac{\nu^2 \sigma_x^2 \sigma_y \rho_0 \left[ \sigma_x^2 \lambda (2 - \lambda) + \nu^2 \sigma_x^2 \sigma_y \rho_0 \rho_{xy} \right] (1 - \rho_{xy})}{\sigma_y \left( \lambda^2 \rho_1^2 \rho_2 + \lambda \nu^2 \sigma_x^2 \sigma_y \rho_0 \rho_{xy} + \nu^2 \sigma_x^2 \sigma_y \rho_{xy} \right)^2} \left( \rho_6 - \rho_{xy} \right),
\]

which is positive (negative) if, and only if, \( \rho_6 > \rho_{xy} \).

Q.E.D.
Proof of Proposition 2.7

Proof: Since $\Gamma(\lambda)$ is a strictly increasing function of $\lambda$ by Lemma 2.4, $\Gamma(0) < \Gamma(\lambda) < \Gamma(1)$. Therefore, first, if $1 - \frac{\rho_0}{\rho_1} \leq 0$, then

$$\Gamma(\lambda) > \Gamma(0) = e^{\nu \kappa} \left(1 + \frac{\rho_1^2}{\rho_0}\right)^{-\frac{1}{2}} \geq 1,$$

implying that $\lambda^* = 0$. Second, if $1 - \frac{\rho_0}{\rho_1} \geq \frac{\rho_2}{2 \sigma_x^2}$, then

$$\Gamma(\lambda) < \Gamma(1) = e^{\nu \kappa} \left(1 + \frac{\nu^2 \sigma_x^2 \sigma_z^2 \rho_0 \rho_1^2}{\rho_1^2 \rho_2 + \nu^2 \sigma_z^2 \sigma_x^2 \rho_0} \right)^{-\frac{1}{2}} \leq 1,$$

implying that $\lambda^* = 1$. Third, if $1 - \frac{\rho_0}{\rho_1} \in (0, \frac{\rho_2}{2 \sigma_x^2})$, then

$$\Gamma(0) < 1 \text{ and } \Gamma(1) > 1.$$

The equilibrium fraction of informed investors is solved from $\Gamma(\lambda) = 1$, i.e.,

$$e^{\nu \kappa} \left(1 + \frac{\nu^2 \sigma_x^2 \sigma_z^2 \rho_0 \rho_1^2}{\rho_1^2 \rho_2 + \nu^2 \sigma_z^2 \sigma_x^2 \rho_0} \right)^{-\frac{1}{2}} = 1,$$

which gives the middle equation of (2.20) immediately. $Q.E.D.$

Proof of Proposition 2.8

Proof: Clearly, as $\kappa \geq \kappa^0$ (i.e., the cost of the private signal is excessively expensive), $q = 0$, hence $\lambda^* = 0$. As $\kappa \in [0, \kappa_0]$ (i.e., the cost of the private signal is inexpensive), $q = 1/ (\nu \sigma_x \sigma_z)$, hence $\lambda^* = 1$.

As $\kappa \in (\kappa_0, \kappa^0)$ (i.e., the cost of the private signal is moderately expensive), $q$ is independent of $\sigma_0^2, \sigma_y^2, \sigma_z^2$, and $\sigma_x^2$; thus, $\lambda^*$ is independent of $\sigma_0^2$ and $\sigma_y^2$, and increasing in $\sigma_x^2$ and $\sigma_z^2$. Since $\frac{\partial q}{\partial \kappa} > 0$ and $\frac{\partial q}{\partial \kappa} < 0$, we have immediately obtained that $\frac{d\lambda^*}{d\kappa} < 0$. $Q.E.D.$
Proof of Propositions 2.11 and 2.12

Proof: By (A.2), it is clear that $a_1 = 0$ if, and only if, $\lambda^* \rho_1 = 0$. Since Proposition 2.6 implies that $\lambda^* = 0$ if $\rho_1 = 0$, we have $\lambda^* \rho_1 = 0$ if, and only if, $\lambda^* = 0$. Therefore, when $\lambda^* > 0$, it is implied that $a_1 \neq 0$ and that $a_1$ and $\rho_1$ have the same sign.

Substituting $\rho_6 = 0$ and $\Gamma(\lambda) = 1$ (i.e., $V_i/V_u = e^{-2\nu \kappa}$) into (A.5) yields:

$$a_2 = \frac{e^{-2\nu \kappa} \nu^2 \sigma_x^2 \sigma_y^2 \rho_6^2}{\sigma_y \left( \lambda^* \rho_1^2 + \nu^2 \sigma_x^2 \sigma_y^2 \rho_6^2 \right) \left[ \lambda^* + (1 - \lambda^*) e^{-2\nu \kappa} \right]} \left[ (1 - \lambda^*) \rho_{xy} \right],$$

which is equal to zero if, and only if, $(1 - \lambda^*) \rho_{xy} = 0$. This completes the first part of the two propositions.

Since Proposition 2.8 shows that $\lambda^*$ is independent of $\sigma_y^2$ and $\sigma_0^2$, it is clear that $V_i/V_u$ is independent of $\sigma_y^2$ and $\sigma_0^2$ by (2.14). As a result, from (A.5) and (A.2), we immediately conclude that both $a_1 \sigma_0$ and $a_2 \sigma_y$ are independent of $\sigma_0^2$ and $\sigma_y$.

Observe that the sign of $a_1$ is the same as that of $\rho_1$. Therefore, $a_1$ becomes less positive as $\sigma_0^2$ increases if $\rho_1 > 0$, and less negative as $\sigma_0^2$ increases if $\rho_1 < 0$. That is, the absolute value of $a_1$ becomes smaller as $\sigma_0^2$ increases. In other words, an increase of the noise ($\sigma_0^2$) in the private signal reduces the absolute price coefficient for the private signal in the equilibrium price.

Observe that the sign of $a_2$ is definite [see (A.5)] if $\rho_{xy}$ and $\rho_6$ have the same sign, i.e., $\rho_{xy} \rho_6 > 0$. In other words, if the public report is positively correlated with the risky asset dividend and so is the direct/net correlation, the price coefficient $a_2$ is positive; if the public report is negatively correlated with the risky asset dividend and so is the direct/net correlation, the price coefficient $a_2$ is negative. An increase in the public report's noise ($\sigma_y^2$) decreases the absolute value of the price coefficient for the public report.

Q.E.D.
A.2 Multi-Task Agency Relationships (Chapter 3)

Proof of Lemma 3.1

Proof: Taking the first-order derivatives of (3.9a) with respect to $\gamma$ and $\beta$ yields

$$
\begin{pmatrix}
 f_1 \\
 f_2
\end{pmatrix} =
\begin{bmatrix}
 C_{11} & C_{12} \\
 C_{21} & C_{22}
\end{bmatrix}
\begin{pmatrix}
 \frac{\partial x(e)}{\partial \gamma} \\
 \frac{\partial x(e)}{\partial \beta}
\end{pmatrix}
and
\begin{pmatrix}
 h_1 \\
 h_2
\end{pmatrix} =
\begin{bmatrix}
 C_{11} & C_{12} \\
 C_{21} & C_{22}
\end{bmatrix}
\begin{pmatrix}
 \frac{\partial e_1}{\partial \beta} \\
 \frac{\partial e_2}{\partial \beta}
\end{pmatrix}
.$$ 

The Hessian matrix of the agent's certainty equivalent (3.7b) is negative definite because the agent's personal cost function is convex. Thus, the first-order conditions are sufficient, i.e., the relationship (3.9c) characterizes the global maximum. Q.E.D.

Proof of Lemma 3.2

Proof: The first-order conditions with respect to $\gamma$ and $\beta$ are:

$$
\begin{pmatrix}
 \frac{\partial x(e)}{\partial \gamma} \\
 \frac{\partial x(e)}{\partial \beta}
\end{pmatrix} - \begin{pmatrix}
 \frac{\partial c(e)}{\partial \gamma} \\
 \frac{\partial c(e)}{\partial \beta}
\end{pmatrix} - \left( \frac{1}{R} \right) \Sigma_{\gamma \delta} \begin{pmatrix}
 \gamma \\
 \beta
\end{pmatrix} = 0.
$$

By the chain rule, we have

$$
\begin{pmatrix}
 \frac{\partial x(e)}{\partial \gamma} \\
 \frac{\partial x(e)}{\partial \beta}
\end{pmatrix} =
\begin{bmatrix}
 \frac{\partial e_1}{\partial \gamma} & \frac{\partial e_1}{\partial \beta} \\
 \frac{\partial e_2}{\partial \gamma} & \frac{\partial e_2}{\partial \beta}
\end{bmatrix}
\begin{pmatrix}
 \frac{\partial x(e)}{\partial e_1} \\
 \frac{\partial x(e)}{\partial e_2}
\end{pmatrix}
= \begin{bmatrix}
 \frac{\partial}{\partial (\gamma, \beta)} \begin{pmatrix}
 e_1 \\
 e_2
\end{pmatrix}
\end{pmatrix}^t
\begin{pmatrix}
 \frac{\partial x(e)}{\partial e_1} \\
 \frac{\partial x(e)}{\partial e_2}
\end{pmatrix},
$$

$$
\begin{pmatrix}
 \frac{\partial c(e)}{\partial \gamma} \\
 \frac{\partial c(e)}{\partial \beta}
\end{pmatrix} =
\begin{bmatrix}
 \frac{\partial e_1}{\partial \gamma} & \frac{\partial e_1}{\partial \beta} \\
 \frac{\partial e_2}{\partial \gamma} & \frac{\partial e_2}{\partial \beta}
\end{bmatrix}
\begin{pmatrix}
 \frac{\partial c(e)}{\partial e_1} \\
 \frac{\partial c(e)}{\partial e_2}
\end{pmatrix}
= \begin{bmatrix}
 \frac{\partial}{\partial (\gamma, \beta)} \begin{pmatrix}
 e_1 \\
 e_2
\end{pmatrix}
\end{pmatrix}^t
\begin{pmatrix}
 f_1 \\
 h_1 \\
 f_2 \\
 h_2
\end{pmatrix} \begin{pmatrix}
 \gamma \\
 \beta
\end{pmatrix}.
$$

The last equality follows from (3.9b). Substituting them and (3.9c) into the above first order necessary conditions and collecting terms yield the result immediately.

The negative definiteness of the Hessian matrix of the objective function (3.10) is guaranteed by the fact that the agent's personal cost function is convex and the covariance matrix of the two performance measures is positive definite. Q.E.D.
Proof of Proposition 3.1

Proof: By Lemma 3.2, it is immediate that

\[
\begin{pmatrix}
\gamma \\
\beta
\end{pmatrix} = R \left[ \sum \gamma \nu - 1 + R \begin{bmatrix}
f_1 & f_2 \\
h_1 & h_2
\end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}^{-1} \begin{bmatrix} f_1 & f_2 \\
h_1 & h_2
\end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}^{-1} \begin{bmatrix} g_1 \\
g_2
\end{bmatrix} \right].
\]

Expanding the above matrices yields that

\[
\gamma \mathcal{L} / \mathcal{R} = \left[ \sigma^2 \nu (C_{22} f_1 g_1 - C_{12} f_2 g_1 - C_{12} f_1 g_2 + C_{11} f_2 g_2) \\
+ \sigma^2 \nu (C_{22} f_1 h_1 + C_{12} g_2 h_1 + C_{12} g_1 h_2 - C_{11} h_1 h_2)
\right] / (C_{11} C_{22} - C_{12}^2)
\]

\[
= \{ \sigma^2 \nu [f_1 (C_{22} g_1 - C_{12} g_2) + f_2 (C_{11} g_2 - C_{12} g_1)] \\
- \sigma^2 \nu [h_1 (C_{22} g_1 - C_{12} g_2) + h_2 (C_{11} g_2 - C_{12} g_1)] \\
+ R (g_1 h_2 - g_2 h_1) (f_1 h_2 - f_2 h_1) \} / (C_{11} C_{22} - C_{12}^2)
\]

\[
= \sigma^2 \nu (f_1 e_1 + f_2 e_2) - \sigma^2 \nu (h_1 e_1 + h_2 e_2)
\]

\[
\beta \mathcal{L} / \mathcal{R} = \left[ \sigma^2 \nu (C_{22} h_1 - C_{12} g_2 h_1 - C_{12} g_1 h_2 + C_{11} g_2 h_2) \\
+ \sigma^2 \nu (C_{22} f_1 g_1) + C_{12} f_2 g_1 + C_{12} f_1 g_2 - C_{11} f_2 g_2)
\right] / (C_{11} C_{22} - C_{12}^2)
\]

\[
= \{ \sigma^2 \nu [h_1 (C_{22} g_1 - C_{12} g_2) + h_2 (C_{11} g_2 - C_{12} g_1)] \\
- \sigma^2 \nu [f_1 (C_{22} g_1 - C_{12} g_2) + f_2 (C_{11} g_2 - C_{12} g_1)] \\
+ R (f_1 g_2 - f_2 g_1) (f_1 h_2 - f_2 h_1) \} / (C_{11} C_{22} - C_{12}^2)
\]

\[
= \sigma^2 \nu (h_1 e_1 + h_2 e_2) - \sigma^2 \nu (f_1 e_1 + f_2 e_2)
\]

\[
- \mathcal{R} (g_1 f_2 - g_2 f_1) (f_1 h_2 - f_2 h_1) / (C_{11} C_{22} - C_{12}^2)
\]
\[
\mathcal{L} = \left[ \sigma^2_y + \mathcal{R} \left( \frac{f_1 f_2}{h_1 h_2} + \frac{C_{11} C_{12}^{-1}}{C_{21} C_{22}} \right) \right] 
\]

where

\[
\mathcal{R} = \left[ \sigma^2_y \mathcal{R} c_{12} - \sigma^2_{y_1} c_{11} \right] \left[ \sigma^2_{y_2} + \mathcal{R} c_{12} \right] \left( \sigma^2_y + \mathcal{R} c_{11} \right) \left( \sigma^2_{y_2} + \mathcal{R} c_{12} \right)
\]

and \( T^v = \left( \frac{c_{11} c_{22} - c_{12}^2}{c_{11} c_{22} - c_{12}^2} , \frac{c_{11} c_{22} - c_{12}^2}{c_{11} c_{22} - c_{12}^2} \right) \), \( T^v = \left( \frac{c_{11} c_{22} - c_{12}^2}{c_{11} c_{22} - c_{12}^2} , \frac{c_{11} c_{22} - c_{12}^2}{c_{11} c_{22} - c_{12}^2} \right) \). The rest of the proof is straightforward. 

**Proof of Proposition 3.8**

**Proof:** If the information signal \( \tilde{Y}^2 \) is action-irrelevant, we have \( D_{y_2} = 0 \) and \( Q_{y_2 y_2} = Q_{y_1 y_1} = 0 \). Hence, by (3.16), it is clear that \( \zeta_{y_1} = D_{y_1} \) and \( \zeta_{y_2} = -\sigma_{y_1 y_2}^2 / \sigma_{y_1}^2 \). Substituting them into (3.18) yields our result immediately. 

**Proof of Lemma 3.3**

**Proof:** Let us take information signal \( \tilde{Y}^1 \) as an example, that is, the managerial incentive compensation contract is \( W(\alpha_1, \gamma_a) = \alpha_1 + \gamma_a \tilde{Y}^1 \). The Board of Directors' (i.e., the risk
neutral long-term shareholders) decision problem is:

\[
\max_{\alpha_1, \gamma_\alpha, e_1, e_2} E_{\tilde{e}_1, \tilde{e}_2, \tilde{z}} \left[ X \left( \alpha_1 + \gamma_\alpha \tilde{Y}^1 \right) \right]
\]

Subject to:

Participation Constraint: 
\[
E_{\tilde{e}_1, \tilde{e}_2, \tilde{z}} \left[ -\exp \left\{ -\frac{\alpha_1 + \gamma_\alpha \tilde{Y}^1 - C(e)}{\mathcal{R}} \right\} \right] \geq -1
\]

Incentive Constraint: 
\[
(e_1, e_2) \in \arg\max E_{\tilde{e}_1, \tilde{e}_2, \tilde{z}} \left[ -\exp \left\{ -\left( \frac{1}{\mathcal{R}} \right) \left( \alpha_1 + \gamma_\alpha \tilde{Y}^1 - C(e) \right) \right\} \right].
\]

Observe that the opportunity compensation is always binding in equilibrium, and the incentive constraint can be characterized by first-order condition. That is, the participation constraint becomes:

\[
E \left[ \alpha_1 + \gamma_\alpha \tilde{Y}^1 \right] = C(e) + \left( \frac{1}{2\mathcal{R}} \right) \text{Var} \left( \alpha_1 + \gamma_\alpha \tilde{Y}^1 - C(e) \right)
\]

\[
= C(e) + \left( \frac{1}{2\mathcal{R}} \right) \gamma_\alpha^2 \sigma_{\tilde{Y}^1}^2.
\]

The incentive constraint becomes:

\[
\gamma_\alpha \nabla \tilde{Y}^1(e) - \frac{dC(e)}{de} = 0.
\]

Thus, 
\[
e = \gamma_\alpha \nabla \tilde{Y}^1(e) \mathcal{C}^{-1} \equiv \gamma_\alpha T^\nu^1,
\]

where 
\[
T^\nu^1 = \left( f_1, f_2 \right) \begin{bmatrix}
\frac{C_{22}}{C_{11}C_{22} - C_{12}^2} & -\frac{C_{12}}{C_{11}C_{22} - C_{12}^2} \\
-\frac{C_{12}}{C_{11}C_{22} - C_{12}^2} & \frac{C_{22}}{C_{11}C_{22} - C_{12}^2}
\end{bmatrix} = \left( \frac{C_{22}f_1 - C_{12}f_2}{C_{11}C_{22} - C_{12}^2}, \frac{C_{11}f_2 - C_{12}f_1}{C_{11}C_{22} - C_{12}^2} \right).
\]

Substituting the above results into the principal's objective function yields the following unconstrained optimization problem:

\[
\max_{\gamma_\alpha} \tilde{X}(e) - C(e) - \left( \frac{1}{2\mathcal{R}} \right) \gamma_\alpha^2 \sigma_{\tilde{Y}^1}^2.
\]

The first-order condition with respect to \( \gamma_\alpha \) is the necessary and sufficient condition to achieve the optimal solution, that is

\[
g_1 T_{1}^\nu - g_2 T_{2}^\nu - \left( C_{11}e_1 T_{1}^\nu + C_{12}e_2 T_{1}^\nu + C_{12}e_2 T_{2}^\nu + C_{22}e_2 T_{2}^\nu \right) - \gamma_\alpha \sigma_{\tilde{Y}^1}^2 / \mathcal{R} = 0.
\]
Appendix A.2: Multi-Task Agency Relationships (Chapter 3)

Since $e = \gamma_a \mathbf{T}'$, collecting terms yields:

$$
\gamma_a \left[ \sigma_{v'}^2 + \mathcal{R} \begin{pmatrix} T_{1v'}^y & T_{2v'}^y \end{pmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{pmatrix} T_{1v'}^y \\ T_{2v'}^y \end{pmatrix} \right] = \mathcal{R} (g_1, g_2) \begin{pmatrix} T_{1v'}^y \\ T_{2v'}^y \end{pmatrix}.
$$

Note that $(T_{1v'}^y, T_{2v'}^y) = C^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{C_{11} f_1 - C_{12} f_2}{C_{11} C_{22} - C_{12} C_{21}} \\ \frac{C_{12} f_1 - C_{11} f_2}{C_{11} C_{22} - C_{12} C_{21}} \end{pmatrix}$. Therefore,

$$
(T_{1v'}^y, T_{2v'}^y) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} T_{1v'}^y \\ T_{2v'}^y \end{pmatrix} = \mathbf{T}' \mathbf{C}^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{T}' : \nabla Y'(e), \quad \text{and}
$$

$$(g_1, g_2) \begin{pmatrix} T_{1v'}^y \\ T_{2v'}^y \end{pmatrix} = (g_1, g_2) \begin{pmatrix} \frac{C_{11} f_1 - C_{12} f_2}{C_{11} C_{22} - C_{12} C_{21}} \\ \frac{C_{12} f_1 - C_{11} f_2}{C_{11} C_{22} - C_{12} C_{21}} \end{pmatrix} = \mathbf{e}_x \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = I^x \mathbf{v}^x \cdot \nabla Y'(e) = I^x \mathcal{D}_{y^2},
$$

where "·" denotes dot product: $(a, b) \cdot (c, d) = ac + bd$. The rest of the proof is clear. Q.E.D.

Proof of Corollary 3.12

Proof: Assume that the two performance measures are not diverse with each other, that is, $Q_{y_1 y_2} = 0$. On the one hand, if one of them is action-irrelevant, say $\nabla Y^1$, we have $e_{v_1}^y = e_{v_2}^y = 0$, and hence $I^y = 0$. By (3.28b), it is clear that $\frac{v^*_1}{v^*_2} = \frac{v^2_1}{v^2_2}$. Thus, in order to induce the first-best allocation of effort, information signal $\mathcal{D}$ must be congruent with the firm's terminal value.

On the other hand, suppose that neither signal is action-irrelevant. Then, $Q_{y_1 y_2} = 0$ implies that $\frac{f_2}{f_1} = \frac{h_2}{h_1}$, and $e_{v_1}^y = \left( \frac{h_1}{h_1} \right) e_{v_2}^2$. Hence, $\frac{v^*_1}{v^*_2} = \frac{\frac{e_{v_1}^2}{e_{v_2}^2} \frac{(h_2/h_1) e_{v_2}^2 (\gamma/\gamma_0) (\beta_+ / \beta)}{e_{v_2}^2 \gamma_0 / \beta}}{e_{v_2}^2}, \text{i.e.,} \frac{v^*_1}{v^*_2} = \frac{v^2_1}{v^2_2}$. Thus, $\nabla Y$ must be congruent with the firm's terminal value in order to induce the first-best allocation of effort. Q.E.D.
Proof of Corollary 3.13

Proof: Observe that \( e^{y^2} \times v^x = \left( e^{y^2} \times e^x \right) / (I^{y^2} I^x) \), and

\[
e^{y^2} \times e^x = e_1^{y^2} e_2^x - e_2^{y^2} e_1^x
\]

\[
= \beta_a \left[ C_{11} C_{22} (h_{1g_2} - h_{2g_1}) + C_{12}^2 (h_{2g_1} - h_{1g_2}) \right] / (C_{11} C_{22} - C_{12}^2)
\]

\[
= \beta_a \left[ \nabla Y^2(e) \times \nabla X(e) \right] / (C_{11} C_{22} - C_{12}^2)
\]

By analogy, \( e^x \times e^{v^l} = \gamma_a Q_{v^l x} / (C_{11} C_{22} - C_{12}^2) \). Putting them together yields:

\[
\frac{v^{y^2} \times v^x}{v^x \times v^{v^l}} = \frac{\left( e^{y^2} \times e^x \right) / (I^{y^2} I^x)}{(e^x \times e^{v^l}) / (I^y I^x)} = \frac{Q_{v^l x}}{Q_{v^l x}} \left( \frac{\beta_a}{\gamma_a} \right) \left( \frac{I^y I^x}{I^{v^l}} \right).
\]

Therefore, by (3.28b), \( \frac{v^x}{v^l} = \frac{v^y}{v^l} \) if, and only if,

\[
\frac{v_1^x}{v_2^x} = \left( \frac{\beta_a}{\beta_x I^x} \right) v_1^y + \left( \frac{\gamma_a}{\gamma_x I^x} \right) v_1^l = \frac{v_1^y + v_1^l}{v_2^y + v_2^l} \left( \frac{\gamma_a}{\gamma_x} \right) \left( \frac{I^y}{I^l} \right)
\]

which in turn holds if, and only if,

\[
\left( \frac{\gamma}{\beta} \right) \left( \frac{\beta_a}{\gamma_a} \right) \left( \frac{I^y}{I^l} \right) = \frac{v_1^y v_2^x - v_2^y v_1^x}{v_1^y v_2^l - v_2^y v_1^l} = \frac{v^{y^2} \times v^x}{v^x \times v^{v^l}} = \frac{Q_{v^l x}}{Q_{v^l x}} \left( \frac{\beta_a}{\gamma_a} \right) \left( \frac{I^y}{I^l} \right).
\]

Q.E.D.
Proof of Lemma 4.2

Proof: The first equality is trivial from Lemma 4.1. To prove the second equality of (4.6), we need to re-express \( a_2 \) in terms of \( a_1 \). Substituting the following two identities

\[
\rho_6 = \rho_2 \rho_{xy} - \rho_1 \rho_{0y}, \quad \text{and} \quad \rho_4 = -\rho_1 \rho_{xy} + \rho_3 \rho_{0y},
\]

into (2.22c) yields:

\[
a_2 = \frac{\sigma_x \left[-q \nu \sigma_z \sigma_x \rho_2 \left(-\rho_1 \rho_{xy} + \rho_3 \rho_{0y}\right) + \rho_1 \left(\rho_2 \rho_{xy} - \rho_1 \rho_{0y}\right) + \rho_0 \rho_{0y}\right]}{\sigma_y \left(1 + q \nu \sigma_x \sigma_z \right) \rho_1 \rho_2} \\
= \frac{\sigma_x \left[(q \nu \sigma_x \sigma_z \sigma_x + 1) \rho_1 \rho_2 \rho_{xy} - q \nu \sigma_x \sigma_z \rho_2 \rho_{0y} - (\rho_1^2 - \nu \rho_0) \rho_{0y}\right]}{\sigma_y \left(1 + q \nu \sigma_x \sigma_z \right) \rho_1 \rho_2} \\
= \frac{\sigma_x \rho_{xy}}{\sigma_y} - \frac{\sigma_x (q \nu \sigma_x \sigma_z \rho_2 + \rho_1^2 - \nu \rho_0) \sigma_0 \rho_{0y}}{\sigma_0 \left(1 + q \nu \sigma_x \sigma_z \right) \rho_1 \rho_2 \sigma_y} \\
= \frac{\sigma_x \rho_{xy}}{\sigma_y} - \frac{a_1 \sigma_0 \rho_{0y}}{\sigma_y}.
\]

This completes the result. \( Q.E.D. \)

Proof of Proposition 4.2

Proof: Since Proposition 4.1 implies that the endogenously determined precision of the filtered price is independent of \( \sigma_x^2 \) and \( \sigma_z^2 \), the measures of sensitivity (4.9) are independent of \( \sigma_x \) and \( \sigma_z \). This proves the first part of the proposition. To prove the rest, we consider the following special cases.
First, perfectly aligned performance measures

If the two performance measures are aligned with each other and neither is action-irrelevant, we re-write (4.11) as follows:

\[
\frac{\gamma}{\beta} = \frac{\sigma_y^2 \left[ D_x - \frac{\sigma_y}{\sigma_p^2} D_p \right]}{\sigma_y^2 \left[ D_p - \frac{\sigma_y}{\sigma_p^2} D_y \right]} = \frac{\sigma_0 N}{\sigma_y^2} D_y - \rho_{py} D_p
\]

Imposing some regularity conditions\(^{A.1}\) such that both the numerator and denominator are positive, we obtain:

\[
\frac{\partial (\gamma/\beta)}{\partial N} > 0, \quad \frac{\partial (\gamma/\beta)}{\partial \sigma_0} > 0, \quad \text{and} \quad \frac{\partial (\gamma/\beta)}{\partial \sigma_y} < 0.
\]

Since \(N\) is independent of \(\sigma_0^2\) and \(\sigma_y^2\), it is obvious that:

\[
\frac{d (\gamma/\beta)}{d \sigma_0} = \frac{\partial (\gamma/\beta)}{\partial \sigma_0} + \frac{\partial (\gamma/\beta)}{\partial N} \frac{\partial N}{\partial \sigma_0} = \frac{\partial (\gamma/\beta)}{\partial \sigma_0} > 0;
\]

\[
\frac{d (\gamma/\beta)}{d \sigma_y} = \frac{\partial (\gamma/\beta)}{\partial \sigma_y} + \frac{\partial (\gamma/\beta)}{\partial N} \frac{\partial N}{\partial \sigma_y} = \frac{\partial (\gamma/\beta)}{\partial \sigma_y} < 0.
\]

From Lemma 4.3, the rest of the results are immediate by the chain rule.

Second, diverse performance measures

We illustrate the case in which the performance measures are diverse by considering BI’s special case. That is, the private signal is congruent with the firm’s terminal value, and the public report (e.g., accounting earnings numbers) is only influenced by the manager’s first action. Now, it is clear that the relative weight on the public report becomes

\[
\frac{\gamma}{\beta} = \frac{\frac{\sigma_0 N}{\sigma_y^2} g_1 v_1^x - \rho_{py} D_x}{\frac{1}{\sigma_0 N} D_x - \rho_{py} g_1 v_1^x + \frac{\sigma_0^2}{\sigma_0^2 - \sigma_y^2} \frac{\sigma_0 N}{\sigma_y^2} D_y - \rho_{py} D_y}.
\]

\(^{A.1}\)In particular, if both the information signals are congruent with the firm’s terminal value, we have \(D_p = D_y\). The above regularity conditions are then identical to Sloan’s regularity condition: \(\min \{\sigma_0^2, \sigma_y^2\} > \sigma_p \sigma_y \rho_{py}\).
Clearly, it increases in $N$ and $\sigma_0$ under the assumption that $\frac{\gamma}{\beta} > 0$. Since $N$ is independent of $\sigma_0^2$ (Lemma 4.3), the relative weight increases in $\sigma_0^2$. The rest of the proof follows straightforwardly from Lemma 4.3.

Third, action-irrelevant performance measure

By Proposition 3.8, we only need to show that the results hold if the private information is action-irrelevant. In such a setting, we obtain that $D_P = 0$ and

$$\frac{\gamma}{\beta} = -\frac{\sigma_0}{\sigma_y \rho_{0y}} N = -\frac{\sigma_0}{\sigma_y \rho_{0y}} \sqrt{1 + \frac{i \rho_2 \rho_0}{\rho_1^2 - 4 \rho_0}}, \quad \text{if } \lambda^* \in (0, 1).$$

Clearly, $\frac{\gamma}{\beta} > 0$ if, and only if, $\rho_{0y} < 0$. Imposing this regularity condition\(^{A.2}\) $\rho_{0y} < 0$, we can obtain our results immediately by Lemma 4.3. \(Q.E.D.\)

Proof of Corollary 4.3

Proof: By the expression of $a_1$ in (2.22b), we have:

$$\frac{\partial a_1}{\partial \sigma_x} = \frac{(\rho_1^2 - \rho_0) + q \nu \sigma_x \sigma_z \rho_0 (2 + q \nu \sigma_x \sigma_z)}{\sigma_0 (1 + q \nu \sigma_x \sigma_z)^2 \rho_1 \rho_2} = sign\{\rho_1\} = sign\{a_1\}.$$  

Observe that the first item of Proposition 4.2 shows that $\frac{\partial (\gamma/\beta)}{\partial \sigma_x} = (> 0)$ if the information acquisition decision is endogenous (exogenous). Thus, given regularity conditions such as $a_1 > 0$, $\gamma/\beta > 0$, $\rho_{0y} > 0$, and $\rho_{xz}$ is sufficiently small relative to $\sigma_y$, the first half of the corollary is immediate by (4.12).

Since $\frac{\partial (\gamma/\beta)}{\partial \sigma_x} > 0$ if the information acquisition is exogenous, the second term of (4.12) is positive. Therefore, $\frac{\partial (\gamma/\beta)}{\partial \sigma_x}$ is larger if the information acquisition decision is exogenous than that if the information acquisition decision is endogenous. \(Q.E.D.\)

\(^{A.2}\)We make this condition to simplify our analysis. Empirically, Antle and Smith (1986) exclude those data with negative relative weights.
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Proof of Lemma 4.4

Proof: Substituting (4.14) into (2.4b) yields the following correlations immediately.

\[ \rho_1 = \frac{\sigma_x \sigma_1^2}{\sigma_0 \sigma_y^2}, \quad \rho_2 = \frac{\sigma_x^2(\sigma_1^2 + \sigma_2^2) + \sigma_2^2 \sigma_0^2}{\sigma_0 \sigma_y^2}, \quad \rho_3 = \frac{\sigma_1^2}{\sigma_y^2}, \]

\[ \rho_4 = 0, \quad \rho_5 = \frac{\sigma_2^2}{\sigma_0^2}, \quad \rho_6 = \frac{\sigma_x \sigma_2^2}{\sigma_0 \sigma_y^2}, \quad \rho_0 = \left(\frac{\sigma_1 \sigma_2}{\sigma_0 \sigma_y}\right)^2. \]

They all are positive except \( \rho_4 \). Hence, by (4.7b), we have

\[ \mathcal{N} = \sqrt{\frac{\sigma_2^2 (\sigma_1^2 \sigma_0^2 - \nu \sigma_2^2 \sigma_y^2)}{\sigma_0^2 (\sigma_1^2 \sigma_x^2 - \nu \sigma_2^2 \sigma_y^2)^2}}, \]

and the corresponding derivatives:

\[ \frac{d\mathcal{N}^2}{d\sigma_x^2} = -\frac{\nu \sigma_x^2 \sigma_2^2 (\sigma_1^2 \sigma_0^2 - \nu \sigma_2^2 \sigma_y^2) + \sigma_0^2 \sigma_1^2 (\sigma_1^2 \sigma_0^2 - \nu \sigma_2^2 \sigma_y^2)}{\sigma_0^2 \sigma_2^2 (\sigma_1^2 \sigma_x^2 - \nu \sigma_2^2 \sigma_y^2)^2} < 0, \]

\[ \frac{d\mathcal{N}^2}{d\sigma_y^2} = 0, \]

\[ \frac{d\mathcal{N}^2}{d\sigma_1^2} = \frac{\sigma_0^2 \sigma_x^2 (\sigma_1^2 \sigma_0^2 + \nu \sigma_2^2 \sigma_y^2)}{\sigma_0^2 \sigma_2^2 (\sigma_1^2 \sigma_x^2 - \nu \sigma_2^2 \sigma_y^2)^2} > 0, \]

\[ \frac{d\mathcal{N}^2}{d\nu} = -\frac{\nu (1 + \nu) \sigma_1^4 \sigma_2^4}{(\sigma_1^2 \sigma_x^2 - \nu \sigma_2^2 \sigma_y^2)^2} < 0, \]

\[ \frac{d\mathcal{N}^2}{d\sigma_0^2} = \frac{\sigma_x^2 \sigma_2^2 \sigma_0^2 \left[ \sigma_1^2 (\sigma_1 + \sigma_2^2) + \sigma_2^2 \sigma_0^2 \right]}{(\sigma_1^2 \sigma_x^2 - \nu \sigma_2^2 \sigma_y^2)^2} > 0. \]

Since \( \sigma_p^2 = \mathcal{N}^2 \sigma_0^2 \), the results regarding the precision of the filtered price are straightforward. Observe that

\[ \frac{d\sigma_p^2}{d\sigma_x^2} = -\nu \sigma_1^4 \sigma_2^4 + 2 \nu \sigma_1^2 \sigma_2^2 \left( -\sigma_1^2 + \nu \sigma_2^2 \right) \sigma_x^2 + \left( -\sigma_1^2 + \nu \sigma_2^2 \right)^2 \sigma_x^4, \]

whose sign is not determined (see Figure 4.2).

By (4.7a), we have:

\[ \mathcal{N} = \sqrt{1 + \frac{\nu^2 \sigma_2^2 \sigma_0^2}{\lambda^2 \sigma_0^2}}. \]

The rest of the proof is clear.
Appendix A.4: Ownership Versus Option Incentive Contracts (Chapter 5)

A.4 Ownership Versus Option Incentive Contracts (Chapter 5)

Proof of Proposition 5.1

Proof: Given that the stock price is normally distributed with mean $\bar{P}$ and variance $\sigma_p^2$, and the exercise price of the option is $K$, we have that

$$
\text{Prob. \{the option is out of money\}} = \int_{-\infty}^{K} \frac{1}{\sqrt{2\pi \sigma_p}} \exp \left\{ -\frac{1}{2\sigma_p^2} (t - \bar{P})^2 \right\} \, dt
$$

$$
= \int_{-\infty}^{(K - \bar{P})/\sigma_p} \frac{1}{\sqrt{2\pi \sigma_p}} \exp \left\{ -\frac{s^2}{2} \right\} \, d(s\sigma_p + \bar{P})
$$

$$
= \Phi.
$$

This proves that $\Phi$ is the probability of an option being out of the money. Moreover, the partial derivatives with respect to the effort level $e$ and the exercise price $K$ are:

$$
\frac{\partial \Phi}{\partial e} = \Phi \frac{\partial \xi}{\partial e} = \frac{\phi}{\sigma_p} < 0,
$$

$$
\frac{\partial \Phi}{\partial K} = \Phi \frac{\partial \xi}{\partial K} = \frac{\phi}{\sigma_p} > 0.
$$

This completes the proof. Q.E.D.

Proof of Proposition 5.2

Proof: It is clear that $E[\bar{v}]$ decreases in $\xi$, i.e.,

$$
\frac{dE[\bar{v}]}{d\xi} = \sigma_p \left[ -\phi \xi - (1 - \Phi) + \phi \xi \right] = -\sigma_p (1 - \Phi) < 0.
$$

(A.6a)

Observe that $\lim_{\xi \to +\infty} \Phi = 0$ and $\lim_{\xi \to +\infty} (1 - \Phi)\xi = 0$, we have $\lim_{\xi \to +\infty} E[\bar{v}] = 0$.

Combining the above facts yields that $E[\bar{v}]$ is positive.

It is also clear that $\text{Var}(\bar{v})$ decreases in $\xi$, i.e.,

$$
\frac{d\text{Var}(\bar{v})}{d\xi} = -2\sigma_p^2 \Phi \left[ \phi - (1 - \Phi)\xi \right] < 0,
$$

(A.6b)
the last inequality holds because \( E[\hat{\nu}] > 0 \). By the chain rule, it then is straightforward to derive the following partial derivatives

\[
\frac{\partial E[\hat{\nu}]}{\partial e} = \frac{dE[\hat{\nu}]}{d\xi} \frac{\partial \xi}{\partial e} = (1 - \Phi)g > 0, \\
\frac{\partial E[\hat{\nu}]}{\partial K} = \frac{dE[\hat{\nu}]}{d\xi} \frac{\partial \xi}{\partial K} = -(1 - \Phi) < 0, \\
\frac{\partial \text{Var}(\hat{\nu})}{\partial e} = \frac{d\text{Var}(\hat{\nu})}{d\xi} \frac{\partial \xi}{\partial e} = 2g\sigma_p \Phi [\phi - (1 - \Phi)\xi] > 0, \\
\frac{\partial \text{Var}(\hat{\nu})}{\partial K} = \frac{d\text{Var}(\hat{\nu})}{d\xi} \frac{\partial \xi}{\partial K} = -2\sigma_p \Phi [\phi - (1 - \Phi)\xi] < 0.
\]

Q.E.D.

**Proof of Proposition 5.3**

**Proof:** It is clear that the net surplus of the principal under the stock ownership contract is: \( TNS_e \equiv ge - \frac{1}{2} e^2 - \frac{\sigma_p^2}{2\sigma^2} e^2 \). This net surplus is a concave quadratic function of effort level \( e \). The maximal net surplus is achieved at the effort level \( e^*_e \).

Q.E.D.

**Proof of Proposition 5.4**

**Proof:** Let \( K = -\infty \), i.e., the option is equivalent to holding the underlying stock. It is clear that

\[
\frac{\partial E[\hat{\nu}]}{\partial e} \bigg|_{K=-\infty} = (1 - \Phi)g \bigg|_{K=-\infty} = g > 0, \\
\frac{\partial \text{Var}(\hat{\nu})}{\partial e} \bigg|_{K=-\infty} = 2g\sigma_p \Phi [\phi - (1 - \Phi)\xi] \bigg|_{K=-\infty} = 0.
\]

The necessary and sufficient condition (5.9) can be satisfied for all effort levels. Thus, the whole effort space is implementable.
Figure A.1: The function $F(\xi) \equiv \phi(1 + \Phi) [\phi - (1 - \Phi)\xi] - \Phi(1 - \Phi)^2$ is non-negative ($\xi \in [-5, 5]$).

Given (5.13), we can rewrite the necessary and sufficient condition (5.9) as $e \leq A(K)$, whose right-hand-side is a function of $K$ and left-hand-side is independent of $K$. To prove the proposition, we only need to show that $A(K)$ decreases in $K$. In fact,

$$
\frac{dA(K)}{dK} = -\frac{g\mathcal{R}(1 - \Phi) \{ \phi(1 + \Phi) [\phi - (1 - \Phi)\xi] - \Phi(1 - \Phi)^2 \}}{4\Phi^2 \sigma_p^2 [\phi - (1 - \Phi)\xi]^2}
$$

$$
= -\frac{g\mathcal{R}(1 - \Phi) F(\xi)}{4\Phi^2 \sigma_p^2 [\phi - (1 - \Phi)\xi]^2}
$$

$$
< 0,
$$

since the entire function of $F(\xi) \equiv \phi(1 + \Phi) [\phi - (1 - \Phi)\xi] - \Phi(1 - \Phi)^2$ is indeed positive (see Figure A.1). Q.E.D.
Proof of Proposition 5.5

Proof: Given any implementable effort \( e \) [i.e., (5.9) holds], we know that (5.8b) always has two positive real solutions to \( \beta \):

\[
\beta = \frac{-\frac{\partial E[\theta]}{\partial e} \pm \sqrt{\left(\frac{\partial E[\theta]}{\partial e}\right)^2 - \frac{2 \sigma}{\mathcal{R}} \frac{\partial \text{Var}(\theta)}{\partial e}}}{-\frac{\partial \text{Var}(\theta)}{\partial e}} \mathcal{R}.
\]

Substituting them into the other first-order condition (5.8a), we can solve for \( \mu \) as follows:

\[
\mu = \frac{\beta \text{Var}(\bar{\theta})}{\mathcal{R} \left[ \frac{\partial E[\theta]}{\partial e} - \frac{\beta}{\mathcal{R}} \frac{\partial \text{Var}(\theta)}{\partial e} \right]} = \pm \frac{\beta \text{Var}(\bar{\theta})}{\mathcal{R} \sqrt{\left(\frac{\partial E[\theta]}{\partial e}\right)^2 - \frac{2 \sigma}{\mathcal{R}} \frac{\partial \text{Var}(\theta)}{\partial e}}}
\]

We know that \( \mu \) is the Lagrangian multiplier and should be positive. Therefore, one of the two real solutions cannot be an economic solution to the principal's problem of incentive compensation contract design. Hence, there exists a unique solution: \( \beta_0 \). Q.E.D.

Proof of Proposition 5.6

Proof: It is clear that the principal's net surplus under the option incentive contract is:

\[
TNS_o \equiv ge - \frac{1}{2} e^2 - \frac{\text{Var}(\bar{\theta})}{2} \beta_0^2.
\]

If \( K = \bar{P} \), we have \( \beta_0 = \left( 1 - \sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi g\mathcal{R}}}} \right) \frac{\sqrt{2\pi}}{2\sigma_p} \),

\[
\text{Var}(\bar{\theta}) = \frac{\pi - 1}{2\pi} \sigma_p^2,
\]

and hence,

\[
TNS_o = ge - \frac{e^2}{2} - \frac{(\pi - 1)\mathcal{R}}{4} \left( 1 - \frac{4e\sigma_p}{\sqrt{2\pi g\mathcal{R}}} - \sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi g\mathcal{R}}}} \right).
\]

Since

\[
\frac{d}{de} \left( \sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi g\mathcal{R}}}} \right) = -\frac{4\sigma_p}{\sqrt{2\pi g\mathcal{R}}} \frac{1}{\sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi g\mathcal{R}}}}}, \quad \text{and}
\]

\[
\frac{d^2}{de^2} \left( \sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi g\mathcal{R}}}} \right) = -\frac{8\sigma_p^2}{\pi g^2\mathcal{R}^2} \left( 1 - \frac{8e\sigma_p}{\sqrt{2\pi g\mathcal{R}}} \right)^{-\frac{3}{2}} < 0,
\]
it is obvious that \( \frac{dTNS_a}{de^2} < 0 \). This has shown that the principal's net surplus is a concave function of the implementable effort level \( e \). The maximal net surplus is therefore achieved at the effort level \( e^*_0 \) characterized by the first-order condition: \( \frac{dTNS_a}{de} = 0 \), i.e.,

\[
g - e + \frac{(\pi - 1)\sigma_p}{\sqrt{2\pi}g} \left( 1 - \frac{1}{\sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi}gR}}} \right) = 0.
\]

The above equation has a unique solution, because the left-hand-side is a decreasing function of \( e \), and is positive as \( e = 0 \), and approaches minus infinity as the implementable effort level \( e \) approaches its upper bound: \( \frac{\sqrt{2\pi}gR}{8\sigma_p} \).

**Proof of Proposition 5.7**

**Proof:** For any given implementable effort level \( e \), we can calculate the difference between the net surpluses generated by the ownership and option compensation contracts:

\[
TNSD = \left[ \bar{X} - C(e) - \frac{1}{2\mathcal{R}} \beta_p^2 \text{Var}(\bar{v})|\kappa = -\infty \right] - \left[ \bar{X} - C(e) - \frac{1}{2\mathcal{R}} \beta_p^2 \text{Var}(\bar{v})|\kappa = p \right]
\]

\[
= \frac{1}{2\mathcal{R}} \left[ \frac{(\pi - 1)\mathcal{R}^2}{2} \left( 1 - \frac{4e\sigma_p}{\sqrt{2\pi}g\mathcal{R}} - \sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi}g\mathcal{R}}} \right) - \frac{e^2}{g^2} \sigma_p^2 \right].
\]

Differentiating with respect to \( e \) provides

\[
\frac{dTNSD}{de} = \frac{1}{2\mathcal{R}} \left[ \frac{2(\pi - 1)\mathcal{R}\sigma_p}{\sqrt{2\pi}g} \left( \frac{1}{\sqrt{1 - \frac{8e\sigma_p}{\sqrt{2\pi}g\mathcal{R}}} - 1 \right) - \frac{2e\sigma_p^2}{g^2} \right];
\]

\[
\frac{dTNSD}{de^2} = \frac{\sigma_p^2}{\mathcal{R}g^2} \left[ \frac{2(\pi - 1)}{\pi \left( 1 - \frac{8e\sigma_p}{\sqrt{2\pi}g\mathcal{R}} \right)^{\frac{3}{2}}} - 1 \right] > 0.
\]

The last inequality follows from the fact that \( \frac{2(\pi - 1)}{\pi} > 1 \geq \left( 1 - \frac{8e\sigma_p}{\sqrt{2\pi}g\mathcal{R}} \right)^{-\frac{3}{2}} \). Therefore, \( \frac{dTNSD}{de} \) is an increasing function of the implementable effort \( e \). Observe that at \( e = 0 \),
the value of \( \frac{dTNSD}{de} \) equals zero. This implies that \( \frac{dTNSD}{de} \) is positive, and hence \( TNSD \) is an increasing function of the implementable effort \( e \). Moreover, \( TNSD = 0 \) at \( e = 0 \), therefore, \( TNSD \) is always positive. That is, the net surplus generated by the ownership incentive contract is higher than that by the option incentive contract. \( Q.E.D. \)

**Proof of Proposition 5.8**

**Proof**: Suppose that \( e^*_o < e^*_s \). Observe that both \( TNS_s \) and \( TNS_o \) equal zero if the effort is zero and are concave functions of effort in the implementable effort set \([0, A(\hat{P})]\). It is clear that \( TNS_s \) decreases and \( TNS_o \) increases in the subset of implementable effort: \([e^*_o, e^*_s]\) because \( e^*_s \) and \( e^*_o \) are the optimal effort levels under the ownership and option incentive contracts respectively. Therefore, \( TNSD \equiv TNS_s - TNS_o \) decreases in effort set: \([e^*_o, e^*_s]\). This contradicts the fact that \( TNSD \) increases in the entire implementable effort set (see proof of Proposition 5.7). Therefore, \( e^*_s \geq e^*_o \). \( Q.E.D. \)