USING SMALL GROUP DISCUSSIONS TO GATHER EVIDENCE OF MATHEMATICAL POWER

by

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ABSTRACT

The purpose of this study was to investigate, with or without prompts, students' small group discussions of their solutions to mathematical problems and to determine the extent to which the students demonstrate mathematical power. Mathematical power was defined in terms of student assessment standards (SAS) and their integration. SAS, each of which has associated with it categories of mathematical activities, comprise communication, problem solving, mathematical concepts, mathematical procedures, and mathematical disposition. Other insights perceived to be important from the discussions were also documented.

Grade 9 students of the regular school program were used for the study. There were 18 students in the class but only one group of students comprising 2 females and 2 males was the focus of the study. They responded to mathematical problems individually for 20 minutes and then used 40 minutes to discuss, in groups, their solutions to the problems. Also, they responded to questionnaire items. The group discussions were video recorded and analyzed. Data were collected on 7 different occasions using 7 different problems over a period of 3 months.

Results of the study indicate that students demonstrated mathematical power to the extent that at least one category of the mathematical activities associated with each SAS was reflected by the small group discussions of students' solutions to mathematical problems. Other results indicate that combining students written scripts with students' talk provides a better insight into the things about which students are talking. Also, monitoring students and providing them with prompts while they work in groups is useful in helping them accomplish tasks in which they are engaged. Finally, when students work in groups, they can shift their viewpoints consensually or conceptually to align their viewpoints with majority viewpoints.
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CHAPTER 1

INTRODUCTION

Background

Recently, the National Council of Teachers of Mathematics (NCTM) initiated a major reform in mathematics education throughout North America. This major reform involves the provision of standards for curriculum and evaluation in K-12 mathematics (NCTM, 1989) and standards for teaching K-12 mathematics (NCTM, 1991). The standards for curriculum and evaluation, and those for teaching, are the ones perceived by the NCTM as important if students are to attain "mathematical literacy" (NCTM, 1989, p. 2).

The curriculum standards involve what mathematics students are to learn in order to become mathematically literate (NCTM, 1989). These curriculum standards include mathematics as problem solving, mathematics as communication, mathematics as reasoning, mathematical connections, algebra, functions, trigonometry, statistics, and probability, among others. There are suggested mathematical activities that are associated with these curriculum standards and how classroom instruction should be organized to help students make sense of these mathematical activities. For example, it is suggested that "Classroom activities should provide students the opportunity to work both individually and in small- and large-group arrangements." (NCTM, 1989, p. 67). Consequently, suggesting small-group arrangement as a format of classroom instruction within the on-going reform means that the NCTM recognizes that format as an important one through which students can make sense of mathematical activities.

The evaluation standards are categorized by focus, namely general assessment, student assessment, and program evaluation (NCTM, 1989). These evaluation standards "propose changes in the processes and methods by which information is collected" (NCTM, 1989, p. 190) regarding all aspects of the reform. In particular, student assessment involves students' performance in those mathematical
activities that are associated with the curriculum standards, with emphasis on what information to gather and how to gather the information. Standards suggested by the NCTM for student assessment involve students' mathematical power, problem solving, communication, reasoning, mathematical concepts, mathematical procedures, and mathematical disposition (see NCTM, 1989, p. 189). To determine whether by engaging in mathematical activities students are providing information that is reflective of the student-assessment standards (SAS) listed above, the NCTM has suggested "written, oral, or computer-oriented" modes of assessment (NCTM, 1989, p. 192). Recommending an oral mode of assessment presupposes that the NCTM recognizes the importance of interaction and the talk that can result when students interact (as can occur in small groups) to make sense of mathematical activities.

Also, professional standards for teaching mathematics that support the changes in curriculum and evaluation standards have been suggested by the NCTM (1991). The professional standards are categorized into standards for teaching mathematics, standards for the evaluation of the teaching of mathematics, and standards for the professional development of teachers of mathematics. Three of the teaching standards relate to classroom discourse while the remaining three relate to classroom environment, worthwhile mathematical tasks, and analysis of teaching and learning (see NCTM, 1991, p. 19). Specifically, classroom discourse addresses issues that involve the teacher's role in discourse, the student's role in discourse, and the tools for enhancing discourse. Classroom discourse, in terms of "the ways of representing, thinking, talking, agreeing and disagreeing," is perceived as "central to what students learn about mathematics as a domain of human inquiry with characteristic ways of knowing" (NCTM, 1991, p. 34). Furthermore, the NCTM recognizes that students' talk should not be only in response to the teacher, but that students must also talk with one another (see NCTM, 1989, p. 34), and one of the several ways students can talk with one another is through the small group format.
The NCTM (1991, p. 57) envisions "serious mathematical thinking" as the central focus of the classroom environment. They indicate it is important to create a classroom environment that will permit students to represent ideas, think, talk, agree and disagree. One means to foster this desired classroom discourse and help students make sense of mathematics is to encourage students to work collaboratively through the small group format (NCTM, 1991). Also, mathematical tasks (or problems) should "facilitate significant classroom discourse" (NCTM, 1991, p. 25), which seems viable through the small group format, among others. Furthermore, to analyze the learning and teaching of mathematics, the NCTM (1989, p. 64) encourages teachers to observe "students participating in small group discussions" so that they can gain insights related to students' understanding of mathematics. So, whether in relation to classroom environment, mathematical tasks (or problems) or the analysis of teaching and learning mathematics, the importance of the small group format for promoting serious mathematical thinking has been recognized.

Within the on-going reform in mathematics education, in addition to the NCTM (1989, 1991), other educators (Artzt & Newman, 1990; Bishop, 1988, 1985; Bishop & Nickson, 1983; British Columbia Ministry of Education, 1990; Cobb, 1989; Davidson, 1989; Dees, 1991; Kroll, 1988; Vygotsky, 1978; Webb, 1991; and Yackel et al., 1990), have recognized the importance of small group format in promoting mathematics learning. Based on the perceived importance of the small group format it was decided to formally examine information from students' small group discussions and then determine what evidence there is that such information is indicative of students' mathematical power as suggested by the NCTM.

**Rationale for the Study**

The importance the NCTM (1989, 1991) attaches to the small group format in promoting mathematics teaching and learning has prompted the investigation into students' small group discussions. In addition, students' use of the small group format
to make sense of mathematics forms one of the important hallmarks of the revised curriculum in the Province of British Columbia (Ministry of Education, B. C., 1990; Robitaille, Schroeder, & Nicol, 1992). Furthermore, whereas several efforts are being made to examine students' sense making of mathematics using the small group format (Kamii, 1984; Kroll, 1988; Smith & Confrey, 1991; Webb, 1991; Yackel et al, 1991), none has been directed specifically at providing evidence about students' demonstration of mathematical power as defined by the NCTM. An in-depth examination of how students make sense of mathematics in one group as a case (Merriam, 1991; Yin, 1989) should provide mathematics educators with useful insights which can be used to influence the current reform.

To examine students' sense making of mathematics using the small group format and determine if information from the small group discussions is indicative of students' mathematical power, I have decided to focus on the student-assessment standards which the NCTM has recommended as criteria for judging whether or not "we have reached the Standards" (NCTM, 1989, p. 189). Gathering and analyzing information that is based on these student-assessment standards should provide educators with insights about the contributions of the small group format to the ongoing reform in mathematics education.

The seven focus areas of student assessment in K-12 mathematics listed by the NCTM are mathematical reasoning, problem solving, communication, mathematical concepts, mathematical procedures, mathematical disposition, and students' mathematical power. Although the NCTM places equal emphasis on all seven focus areas of student assessment, it recognizes the need for productive changes in the curriculum and evaluation standards (see NCTM, 1989, p. 189). The NCTM definition of students' mathematical power encompasses the definitions of what constitutes the other standards of student assessment and their integration (see NCTM, 1989, p. 205). So, gathering information on the remaining six student-assessment standards should provide evidence about students' demonstration of mathematical power (see NCTM,
In addition, the NTCM argues that "The goal of teaching mathematics is to help all students develop mathematical power." (NCTM, 1991, p. 21). Therefore, the current research investigated students' demonstration of mathematical power.

Also, in the present research, it was assumed that mathematical reasoning is a major method by which students gain mathematical knowledge. (Notice that mathematical knowledge involves all categories of mathematical activities associated with SAS.) Kline (cited in Clements & Ellerton, 1991) shares a similar view on the role of reasoning in gaining knowledge when he argues that even though authority, revelation, experience, and experimentation are important sources of obtaining knowledge, the major method is reasoning. Furthermore, mathematics educators recognize that mathematics is reasoning (NCTM, 1989) and for students to become autonomous in doing mathematics, they need to "gain confidence in their ability to reason and justify their thinking" (NCTM, 1989, p. 29). It follows then that mathematical reasoning should involve "the kind of informal thinking, conjecturing, and validating that helps children to see that mathematics makes sense" (NCTM, 1989, p. 29).

Many educators believe that when students work in groups in the mathematics classrooms, the kinds of informal thinking, conjecturing, and validating that students go through could be captured when they verbalize those activities (Artzt & Newman, 1990; Kroll, 1988; NCTM, 1991). This is because talking, or oral discourse, or verbalizing forms an important medium of exchanging ideas as students work in groups. Teachers of mathematics are encouraged to listen to students' ideas, ask students to clarify and justify their ideas orally, monitor students' participation in discussions, and urge students to talk in small groups (NCTM, 1991). Also, mathematics educators (NCTM, 1991) observe that "when students make public conjectures and reason with others about mathematics, ideas and knowledge are developed collaboratively" (p. 34). Consequently, mathematical reasoning through talk, should provide one basic source from which educators make inferences about students' mathematical power. Thus, focusing on students' reasoning during small
group discussions of mathematical activities should provide evidence about how students are meeting the student-assessment standards within mathematics education. So, to make inferences about students' demonstration of mathematical power, the focus in this study is on students' reasoning and arguments as the main source for deciding how students are meeting the student-assessment standards (SAS) of communication, problem solving, mathematical concepts, mathematical procedures, and mathematical disposition.

**Purpose of the Study**

The purpose of this study is to examine information from the discussions of a group of students and determine if in the information generated by the students, one can find the existence of data reflective of NCTM's definition of students' mathematical power. The focus of the study is on the information the students generate while they discuss their solutions to mathematical problems during student-student interactions. However, to a lesser extent, data were also gathered with respect to students' discussions following prompts from the researcher.

**Research question**

Specifically, the question addressed in this study is:

*To what extent is information from students' small group discussions of their solutions to mathematical problems indicative of students' mathematical power?*

By information, I mean mathematical information that is related to the problems the students have solved. This mathematical information involves what a group of four students said or wrote down individually as they engaged in student-student interactions to discuss their solutions to mathematical problems. The information also involves students' individual responses to questionnaire items given to the students.
each time they discussed their solutions to a mathematical problem. The problems for
the study relate to the mathematical topics covered by the students' teacher over the
period of the study. Furthermore, the mathematical problems are the types with which
the students were familiar. Excerpts from the students' discussions that are used to
provide evidence of students' mathematical power are such that they reflect NCTM's
intended meaning of the categories of mathematical activities associated with each
SAS. Furthermore, the existence and integration of information indicative of SAS
provide evidence of students' demonstration of students' mathematical power. In this
study, mathematical power is that which individual students demonstrated within the
small group context.

Assumptions of the Study

In this study, it is believed that students in small groups will be able to reason
and engage in a meaningful discourse over mathematical activities. Also, I
hypothesize that while students discuss mathematical activities, there will be
observable events (Locke, Spirduso & Silverman, 1987) that will form the basis for
evidence of students' demonstration of mathematical power. That is, as students
discuss their solutions to the problems used for the study, they will provide evidence of
their ability to communicate mathematically, use mathematical concepts, and use
mathematical procedures. Finally, students' discussions will provide evidence of their
ability to use mathematics to solve problems and of their mathematical disposition.

Limitations of the Study

Since an overlap in categories is possible whenever there is conceptual
categorization, one limitation to this study is the possible overlap of the categories of
mathematical activities associated with SAS. For example, evidence of students'
ability to communicate mathematically can also constitute evidence of their use of
mathematical concepts. So, to minimize this limitation, it should be necessary,
sometimes, to use the same evidence as indicative of students' ability regarding two or more SAS.

For this study where the problems are limited to the mathematical topics that the teacher taught the students, not all information that constitutes evidence that students meet SAS will be provided by the students as they discuss their solutions to the problems. Thus, the extent to which students' information meets SAS depends on the nature of the mathematical problems. For example, students were not asked to formulate problems so the absence of evidence from the discussions indicative of students' ability to formulate problems should not be construed to mean that students cannot formulate problems. Rather, it is the limitation of the problems that make it impossible to make inferences about students' ability to formulate problems. However, no single mathematical problem may permit students to generate enough information to reflect all categories of mathematical activities associated with all of SAS. In regard to mathematical communication, for example, a single mathematical problem may not provide information on all of what constitutes evidence of students' ability to communicate mathematically. However, as observed by Lappan and Friel (1993), the important thing is that the problems can permit students to significantly talk about the solutions and how they obtained those solutions.

Another limitation to this study is the amount of time used for gathering data. The longer the data gathering period, the deeper the insights into the discussions of the members of the small group. Resources allowed seven data gathering sessions of one hour per session, spread over three months. Even if more resources had been available, an impending teachers' strike would have terminated the data gathering period. Nevertheless, as reported in chapter three, the amount of data collected proved sufficient for the research question.
Summary

The reform in mathematics education suggested by the NCTM provides the mathematics education community with standards for curriculum and evaluation in school mathematics. A major component of the evaluation standards involves student-assessment standards in mathematics as criteria for making decisions about students' demonstration of mathematical power. These assessment standards relate to problem solving, communication, mathematical concepts, mathematical procedures, and mathematical disposition. To foster the attainment of students' mathematical power, the NCTM has suggested some teaching standards for school mathematics. Three of the teaching standards relate to classroom discourse while the rest relate to worthwhile mathematical tasks, classroom environment, and analysis of teaching and learning. Within all these standards, the small group format has been recommended as important for fostering classroom discourse that will permit students to make sense of mathematics. However, no studies have been directed specifically at examining the extent to which information from the small group format is indicative of students' mathematical power. So, to provide such insight, I have decided to examine information from discussions of one group of students.
In this chapter, I present a brief review of the background leading to the current reform in mathematics education. Then, I review the current reform, which has as one of its recommendations, the use of small groups for promoting the teaching and learning of mathematics. In addition, I review the literature on how educators are using small groups to foster mathematics teaching and learning. Finally, I provide a theoretical framework for using small groups to gather information on students' demonstration of mathematical power.

Background to Current Reform in Mathematics Education

The apparent reforms in mathematics education over the last fifty years have been due mainly to changes in societal expectations of what graduates of the mathematics curriculum should be capable of doing (Davis & Maher, 1993; Howson, Keitel, & Kilpartrick, 1981; Kline, 1973; NCEE, 1983; NCTM, 1980, 1989; NRC, 1989; Willoughby, 1990). In trying to meet societal expectations, emphasis on what teachers should teach within the mathematics curriculum has shifted several times, with new materials added and old ones removed from the mathematics curriculum. What teachers should teach shifted from basic mathematics (Howson, Keitel, & Kilpartrick, 1981; National Commission on Excellence in Education [NCEE], 1983) through new mathematics (National Research Council [NRC], 1989), back-to-basics mathematics (Kline, 1973), problem solving (National Council of Teachers of Mathematics [NCTM], 1980), and mathematical literacy, achievable through students' mathematical power, as the main emphasis within the current reform in the discipline (NCTM, 1989, 1991).
Basic mathematics era

In the 1950s, the major goal of mathematics education was for "training the mind" (Kline, 1973, p. 9). The belief by educators at the time was that training of the mind through numerous repetitions and memorization of mathematical rules and procedures ensures the sharpness of the mind. Having attained a sharp mind, students could then solve problems in science, engineering, and other careers in life (Christiansen et al., 1986; Kline, 1973). Despite the emphasis on the sharpening of the mind, Kline (1973) observed that not only were students' grades in mathematics lower than in other subjects, their dislike and dread for mathematics were widespread. Furthermore, he observed that "Educated adults retained almost nothing of the mathematics they were taught and could not operate simple operations with fractions" (p. 15). In addition, not only did the military discover during World War II that they were deficient in mathematics, but the Russians, instead of the Americans, were first to go to space in 1957 (Howson, Keitel, & Kilpartrick, 1981; Kline, 1973, NRC, 1989). Thus, educators of the time became convinced that the mathematics curriculum during that era, in terms of what it contained and how it was taught, did not prepare students to meet societal expectations. Consequently, there were moves to reform the existing mathematics curriculum thereby giving birth to new or modern mathematics.

Modern mathematics era

The modern mathematics era (the '60s) was characterized by two main features, namely "a new approach to the traditional mathematics, and new contents" (Kline, 1973, p. 21). Familiar high school mathematics, including Euclidean geometry, was abandoned. The belief was that technology had made the learning of most of the traditional topics obsolete and that abstract mathematics had become more important as the basis for the development of modern science. Consequently, topics like theory of numbers (including set theory), abstract algebra, linear algebra, projective geometry, topology, and calculus were emphasized. A new language involving set
theory was used to emphasize the "logic, the structure, and the unity of mathematics as a whole" (Kline, 1973. p. 18). Students were encouraged to be precise and rigorous in their use of mathematical concepts. A deductive approach to learning these new topics was emphasized by educators of the time. However, the modern mathematics program was criticized for its excessive rigor and abstractness (Howson, Keitel, & Kilpartrick, 1981; Kline, 1973; Willoughby, 1990) and for the inability of its graduates to perform mathematical computations efficiently (Kline, 1973). Naturally, there were moves to go back to the basics of skills development in mathematics.

Back-to-basics era

The mathematics curriculum for this era (the '70s) was informed by the belief that mathematical concepts arose from "physical situations or phenomena" and that "their meanings were physical for those who created mathematics in the first place" (Kline, 1973, p. 153). Furthermore, education (including mathematics education) should be broad rather than deep for elementary and secondary students so as to provide opportunities for students to integrate their activities and interests and see mathematics as part of whole knowledge, but not as a different knowledge (Kline, 1973). Thus, mathematics education should develop in students basic mathematical skills that will enable them to function properly in several careers. Despite the emphasis on the development of basic mathematical skills, it was soon to be realized that students could still not utilize the accumulated mathematical skills to solve word and real-life problems. So, around late '70s, there were moves by mathematics educators to make problem solving the focus of mathematics education for the '80s.

Problem-solving era

As the previous reforms had tended to be slogan oriented, mathematics educators considered problem solving as more than just a phrase or a slogan (Krulik & Reys, 1980). They considered problem solving as "the reason for teaching
Emphasis was placed on the development and the teaching of several problem-solving strategies, with Polya's problem-solving strategies as one of the notable ones (Polya, 1957). The enthusiasm for problem solving was great. There were attempts to define problem solving as a goal, process, and basic skill; suggest ways to pose problems properly; use pictorial language in problem solving; suggest how to supplement and understand textbook problems, use calculators to solve problems; and provide problem-solving experiences through recreational mathematics (Krulik & Reys, 1980).

However, despite all these efforts at focusing on problem solving as the reason for teaching mathematics, the needs of the North American society, which are mainly social and economic, are not being met (NCEE, 1983; NCTM, 1989; NRC, 1989). Furthermore, Willoughby (1990) observes that the "real motivation for reform is a change in society itself" (p. 2). He argues further that:

Never before has a change in technology made knowledge and understanding of mathematics so important to so many people. Never before has a change in technology made the kind of mathematics people have been learning so obsolete. The technological revolution will not go away. We will not collect and destroy all calculators and computers on some day in the future. The reformers may die, but the reforms now taking place will continue to live. Those who fail to benefit from these reforms will live less full and less productive lives than those who benefit from the reforms. Those societies that prepare people well for a technological future will become better places to live. Those that don't will wither. (p. 2)

Such strong beliefs, among others, have compelled educators to initiate the current reform in mathematics education, which encompasses all the reforms undertaken for the last 50 years.

**Current Reform in Mathematics Education**

The current reform in mathematics education for students reflects society's expectations that schools produce a mathematically literate work force (NCTM, 1989). To meet society's expectations, the NCTM has suggested some curriculum and evaluation standards of mathematics and what student activities are associated with
mathematics in such a curriculum (NCTM, 1989). Also, the NCTM has suggested teaching standards through which teachers can facilitate the attainment of the curriculum and evaluation standards (NCTM, 1991).

Curriculum standards

There are 13 curriculum standards for each of K-4 and grades 5-8, and 14 for grades 9-12. The curriculum standards that are common to the grades include mathematics as problem solving, mathematics as communication, mathematics as reasoning, and mathematical connections. Whereas estimation is considered as one standard under K-4, it is combined with computation under grades 5-8 and does not appear explicitly under grades 9-12. Topics involving numbers are grouped into three standards under K-4 as number sense and numeration, concepts of whole number operations, and whole number computations. Under grades 5-8, topics involving numbers are grouped under two standards as number and number relationships and number systems and number theory; there are no number topics explicitly grouped under grades 9-12. While there is geometry and spatial sense as one standard under K-4, it is simply geometry under grades 5-8, but separated into geometry from a synthetic perspective and geometry from an algebraic perspective under grades 9-12. There is measurement as one standard under K-4 and grades 5-8, but not under grades 9-12.

Other curriculum standards include patterns and relationships under K-4, patterns and functions under grades 5-8, and functions under grades 9-12. Fractions and decimals form one standard under only K-4. Statistics and probability form one standard under K-4, but form two separate standards under grades 5-8 and grades 9-12. Algebra as a standard appears first under grades 5-8 and then under grades 9-12. The remaining four standards under grades 9-12 include trigonometry, discrete mathematics, conceptual underpinnings of calculus, and mathematical structure.
The mathematical activities associated with these curriculum standards indicate overlaps across the standards and across grade levels. For example, under K-4, the standard involving mathematics as problem solving requires that students "verify and interpret results with respect to the original problem" (NCTM, 1989, p. 23). Similarly, the standard involving mathematics as reasoning requires that students "justify their answers and solution processes" (NCTM, 1989, p. 29). In these examples, the same evidence can be used to indicate that students are either verifying and interpreting results or that they are justifying their answers. Also, similar evidence can be used to indicate that students at K-4, grades 5-8, or grades 9-12 are verifying and interpreting results with respect to the original problem. The presence of the overlaps suggests that in gathering information indicative of students' achievement of the curriculum standards, a holistic approach needs to be taken by mathematics educators.

**Teaching standards**

There are six teaching standards suggested by the NCTM for K-12. These are grouped into four categories that are labeled tasks, discourse, environment, and analysis. Under the tasks category, the teaching standard involves worthwhile mathematical tasks which are "the projects, questions, problems, constructions, applications, and exercises in which students engage" (NCTM, 1991, p. 20). The discourse category has three teaching standards. These are the teacher's role in discourse, students' role in discourse, and tools for enhancing discourse. Discourse as used here "refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use to engage those tasks" (NCTM, 1991, p. 20). The teaching standard under the environment category involves the learning environment which is "the context in which the tasks and discourse are embedded" (NCTM, 1991, p. 20). Finally, under the analysis category, the teaching standard involves analysis of teaching and learning which is "how well the tasks, discourse, and
environment foster the development of every student's mathematical literacy and power" (NCTM, 1991, p. 20).

Evaluation standards

The evaluation standards are grouped into three categories. These are general assessment, student assessment, and program evaluation (see NCTM, 1989, p. 189). The general assessment category comprises alignment, multiple sources of information, and appropriate assessment methods and uses. The program evaluation category comprises indicators for program evaluation, curriculum and instructional resources, instruction, and evaluation team. The student assessment category, which comprises students' mathematical power, problem solving, communication, reasoning, mathematical concepts, mathematical procedures, and mathematical disposition, is reviewed more extensively because of its importance for this study.

Problem solving. Problem solving refers to students' abilities to use mathematics to solve problems. Students having the ability to use mathematics to solve problems should provide evidence that they can:

i) formulate problems;
ii) apply a variety of strategies to solve problems;
iii) solve problems;
iv) verify and interpret results;

Communication. Students having the ability to communicate mathematically should provide evidence that they can:

i) express mathematical ideas by speaking, writing, demonstrating, and depicting them visually;
ii) understand, interpret, and evaluate mathematical ideas that are presented in written, oral, or visual forms;

iii) use mathematical vocabulary, notation, and structure to represent ideas, describe relationships, and model situations (NCTM, 1989, p. 214).

Reasoning. Students who reason mathematically should provide evidence that they can:

i) use inductive reasoning to recognize patterns and form conjectures;

ii) use reasoning to develop plausible arguments for mathematical statements;

iii) use proportional and spatial reasoning to solve problems;

iv) use deductive reasoning to verify conclusions, judge the validity of arguments, and construct valid arguments;

v) analyze situations to determine common properties and structures;

vi) appreciate the axiomatic nature of mathematics (NCTM, 1989, p. 219).

Mathematical concepts. Students having knowledge and understanding of mathematical concepts should provide evidence that they can:

i) label, verbalize, and define concepts;

ii) identify and generate examples and nonexamples;

iii) use models, diagrams and symbols to represent concepts;

iv) translate from one mode of representation to another;

v) recognize the various meanings and interpretations of concepts;

vi) identify properties of a given concept and recognize conditions that determine a particular concept;
vii) compare and contrast concepts (NCTM, 1989, p. 223).

**Mathematical procedures.** Mathematical procedures generally mean computational methods, even though they may include geometric constructions. Students having knowledge of mathematical procedures should:

i) recognize when a procedure is appropriate;

ii) give reasons for steps in a procedure;

iii) reliably and efficiently execute procedures;

iv) verify the results of procedures empirically (e.g., using models) or analytically;

v) recognize correct and incorrect procedures;

vi) generate new procedures and extend or modify familiar ones;

vii) appreciate the nature and role of procedures in mathematics (NCTM, 1989, p. 228).

**Mathematical disposition.** Disposition does not involve only attitudes, but includes the "tendency to act in positive ways." Aspects of students' disposition towards mathematics include:

i) confidence in using mathematics to solve problems, to communicate ideas, and to reason;

ii) flexibility in exploring mathematical ideas and trying alternative methods in solving problems;

iii) willingness to persevere in mathematical tasks;

iv) interest, curiosity, and inventiveness in doing mathematics;

v) inclination to monitor and reflect on their own thinking and performance;

vi) valuing of the application of mathematics to situations arising in other disciplines and everyday experiences;
vii) appreciation of the role of mathematics in our culture and its values as a tool and as a language (NCTM, 1989, p. 233).

**Mathematical power.** Students' mathematical power refers to "all aspects of mathematical knowledge and their integration" (NCTM, 1989, p. 205). Aspects of students' mathematical knowledge include:

i) ability to apply their knowledge to solve problems within mathematics and in other disciplines;

ii) ability to use mathematical language to communicate ideas;

iii) ability to reason and analyze;

iv) knowledge and understanding of concepts and procedures;

v) disposition towards mathematics;

vi) understanding of the nature of mathematics;

vii) integration of these aspects of mathematical knowledge (NCTM, 1989, p. 205).

**Underlying assumptions and recommendations**

There are many assumptions that underlie the current reform in mathematics education (NCTM, 1989, 1991). One of them is the belief that successful implementation of the reform should result in students acquiring mathematical power. In fact, the development of students' mathematical power is seen as the goal for teaching mathematics within the current reform (see NCTM, 1991, p. 21). Another is the belief that social interaction (Artzt & Newman, 1990; Bishop, 1988, 1985; Bishop & Nickson, 1983; British Columbia Ministry of Education, 1990; Cobb, 1989; Davidson, 1989; Dees, 1991; Kroll, 1988; Vygotsky, 1978; Webb, 1991; and Yackel et al., 1990) is important for students' construction of mathematical knowledge. So, to implement the K-4 curriculum standards for example, the NCTM believes that children should be actively involved in doing mathematics "by interacting with the physical world,
materials, and other children" (NCTM, 1989, p. 17). Similar beliefs have been expressed for the implementation of the curriculum standards for grades 5-8 (see NCTM, 1989, p. 69) and for grades 9-12 (see NCTM, 1989, p. 124). For implementation of the teaching standards, the NCTM believes in the importance of interaction among children (see NCTM, 1991, p. 21). Also, to implement the evaluation standards, the NCTM believes in the importance of interaction among children (see NCTM, 1989, p. 192). Thus, interaction among children is considered very important for the implementation of the curriculum, teaching, and evaluation standards.

Arising out of these beliefs or assumptions is the NCTM recommendation that small group formats form an important context within which educators implement the curriculum, the teaching, and the evaluation standards (NCTM, 1989, 1991). For example, students are to be encouraged to engage mathematical activities in groups (see NCTM, 1989, p. 8), to be taught in groups (see NCTM, 1991, p. 36), and to be assessed in groups (see NCTM, 1989, p. 192). Also, learning in small groups has been recognized as one of the important hallmarks of the revised curriculum in the Province of British Columbia (Robitaille, Schroeder, & Nicol, 1992). It follows then that the use of small groups to promote the construction of mathematical knowledge (and consequently the development of students' mathematical power) should be taken seriously by mathematics educators.

**Small Group Learning**

Small group learning involves students collaborating in an intellectual endeavor to make sense of complex situations (Artzt & Newman, 1991; Davidson, 1990; NCTM, 1989). This collaboration involves "talking, listening, explaining, and thinking with others, as well as by oneself" (Davidson, 1990, p. 4). To collaborate successfully, it is important for students to observe some social norms that could guide their interactions (Davidson, 1990; Eichinger et al, 1991; Yackel et al, 1991; Webb,
Some useful social norms that Eichinger et al. (1991) identify include (1) the responsibility of all students to contribute to the group and to seek to understand other students' ideas, (2) commitment to helping others contribute successfully to the group, and (3) tolerance for diverse cultural backgrounds and working styles. The use of such social norms to guide classroom interaction is very useful in providing students with the opportunity to cooperate and benefit from the group interaction.

Having students cooperate for the common good should not be construed to mean that individual accountability is abandoned. In fact, individual accountability is one of the seven major defining characteristics of small group learning that Davidson (1990) identifies. Recognizing and maintaining individual accountability, and consequently individual ability, should promote small group learning since individuals form the group and their individual abilities influence the nature and outcome of the group interaction. An analysis of several studies involving group interactions (Webb, 1991), indicates that individual students of high ability tend to give the most explanation to other group members. Findings from these studies also indicate that a student with a particular level of ability may interact more actively and learn more in some group compositions than in others. Also, in many groups, the high ability and low ability students tend to form teacher-learner relationships, while medium ability students tend to participate less in group interactions than the highs and lows. However, in high-medium and medium-low ability groups, all students tend to be active participants in group interactions. Thus, the individual's ability has a significant influence on the outcome of the group interaction, and therefore the design of any study involving group interactions.

Rau and Heyl (1990) indicate that "self-selection, random assignment, and criterion-based selection are all possible" (p. 145) when forming small groups for learning purposes. Also, the type of group dynamics taking place influences the type of benefits the group members derive from the group. Each group member's role, involving who is giving help, who is receiving help, whether help is given when
needed or not, are all important considerations. Also, there are relationships that exist between gender and mathematics achievement. For example, boys tend to ask more specific and direct questions than girls and this gender difference seems to be related to achievement (Webb, 1991). Webb also observes that there is higher male achievement than female achievement within groups having higher male-female or female-male ratios, but for equal number of males and females, the achievement does not differ significantly. So, using the same number of males and females to form groups can minimize gender differences in the achievement attained in the small group setting. Also, Jungwirth (1991) observes that everything that happens in a social interaction (student-student) is determined by the behavior of the individual participants and that mathematical competence on the part of students can be established during such interaction.

Many educators recognize that social processes like peer regulation, feedback, support, and encouragement (Johnson & Johnson, 1985); kinds of help given or received by students (Webb, 1991); and group composition (Davidson & Kroll, 1991; Jungwirth, 1991; Yackel et al, 1991) influence mathematics learning. In fact, several studies (e. g., see Kroll, 1988; Webb, 1991; Yackel et al, 1991) show that students demonstrate a better understanding of mathematics when they learn in small groups. Consequently, many educators use small groups to encourage mathematics learning. For example, Kamii (1984) observes that "when children confront the ideas of other children for as brief as 10 minutes, higher levels of logical reasoning are often the outcome" (p. 414). Also, Smith & Confrey (1991) found that peer interactions "enhance the development of logical reasoning through a process of cognitive reorganization induced by cognitive conflict" (p. 4).

Theoretical Framework of the Study

In this section, I discuss the constructivist's perspective as a source of knowledge generation. Then, I discuss Vygotsky's perspectives on knowledge
generation through social interaction. Finally, I argue that constructivism and Vygotsky's perspectives on knowledge generation provide a useful theoretical framework for investigating students' small group discussions.

Constructivism

There are cognitive and methodological perspectives on constructivism, which in simple terms, involve how the individual makes sense of things (Noddings, 1990). The methodological perspective assumes that human beings are knowing objects, they organize knowledge, and their behavior is purposive. The cognitive perspective assumes that individuals construct all knowledge. Of this cognitive perspective, one school of thought is that the instruments of construction are cognitive structures that are innate while the other school believes that these cognitive structures are developmental. Most constructivists in mathematics education hold the developmental view (Noddings, 1990). Whichever perspective one holds, constructivists assert that it is the individual who has ownership of the knowledge he or she uses (Golding, 1990; Kamii and Kamii, 1990; Lerman, 1989; Lythcott & Duschl, 1990; Nodding, 1990; von Glasersfeld, 1990). The individual's knowledge is used to build interpretive frameworks for making sense of the world (Schoenfeld, 1987).

For example, when Yackel et al. (1990) asked second graders to solve $9 + 11 = \_\_$ in ways that made sense to them, the children offered varying solution methods that indicate different interpretive frameworks as illustrated below:

- **Brenda:** 9 and 9 is 18, plus 2 is 20.
- **Adam:** 7 and 7 is 14, so 8 and 8 is 16. 9 and 9 would be 18 so 9 + 11 must equal 20
- **Chris:** 11 and 11 equals 22. 10 and 11 equals 21. 9 and 11 equals 20.
- **Jane:** 11 and 9 more—12, 13, ..., 18, 19, 20. (p. 13).
Although the students in this study got the same solution as that expected by the investigators, students' interpretive frameworks do not always lead them to the expected solution. For example, in a study by Ginsburg and Kaplan cited by Rowan et al. (1989), a first grader was presented with the numerical sentence $7 + 6 = 13$ and asked if it was true; he responded positively. When he was presented with $13 = 7 + 6$ and then asked the same question, he responded that it was not true, and for it to be true it must be changed to $7 + 6 = 13$. Further probing showed that his notion of a numerical sentence being true is that the sentence must have two numbers joined by an operation on the left side of the equal sign and a single number to the right of the equal sign. These examples illustrate that students make use of different interpretive frameworks in making sense of problems. Educators therefore need an assessment framework that can capture how students, as autonomous individuals, make sense of problems and their solutions (Dreyfus, Jungwirth & Eliovitch, 1990; Vosnianou & Brewer, 1987).

By autonomy, I mean the ability of the individual to make decisions on his or her own. The importance of autonomy as an aim of education has been recognized by many educators (Wesson, 1986; Wenden, 1988; Weinstein, 1987; Lane & Lane, 1986; Haydon, 1983; Kamii, 1984). Haydon (1983) refers to autonomy as "some set of qualities of mind and character which persons can in principle have despite external constraints" (p. 220). She posits further that autonomy is a matter of degree. Persons are more or less autonomous to different degrees in different aspects of their lives. Implied in this ability to govern and make decisions for oneself is the concept of understanding. To make informed decisions while solving a problem, one needs to understand the problem and its context. Webb (1991) observes that: "The best indication of students' understanding is their ability to solve the problem on their own" (p. 369). However, students may get the answer (a product) to a problem without providing any indication as to how (a process) they arrived at that answer. It will therefore be inappropriate, I believe, to pass judgment on their understanding of
mathematics by relying solely on the answer to the problem. What educators need is an idea of what constitutes understanding and Pirie (1988) shares a similar view by observing that although one cannot fully comprehend the term "understanding" itself, mathematics educators must first have a viable model of understanding on which to make sense of students' work.

There have been attempts to theoretically identify different kinds of understanding (Pirie & Kieren, 1989) to elucidate what it means to understand mathematics. For example, Skemp (1978) distinguishes between instrumental and relational understanding, Buxton (1978) talks about formal or logical understanding, while Backhouse (1978) elaborates on symbolic understanding. Herscovics and Bergeron (cited in Pirie, 1988), in their attempt to categorize understanding, define four levels of understanding- intuitive understanding, initial conceptualization (procedural), abstraction (logico-physical), and formalization.

It is important to note, however, that single categories do not describe understanding well, nor do such categories capture understanding as a process rather than as a single acquisition. Ohlsson (1988), Herscovics and Bergeron (1988), Von Glasersfeld (1987), and Pirie & Kieren (1989), have made attempts at examining understanding as a process. The view for considering understanding as a process is that for all knowledge, as educators move nearer their goal, the goal itself recedes and successive models that they create can be no more than approximations. So, in assessing students' understanding, as in the present study, educators must be aware that what they are assessing may only be approximations. Nevertheless, educators can make more prudent decisions if they consider all relevant factors that influence students' mathematical understanding and use means that will permit gathering of a variety of information on such understanding.

One such relevant factor which influences students' mathematical understanding is the social configuration of the mathematics classroom. Educators recognize that social factors and their interrelationships influence students'
mathematical understanding when they work in groups (Artzt & Newman, 1991; Davidson, 1990; NCTM, 1989). However, educators observe also that individual construction does not have to conflict with concepts of social cognition (Smith & Confrey, 1991). Therefore, in dealing with the individual's construction of mathematical knowledge, the social context within which the knowledge construction takes places should be considered as important. The issue of social influence on mathematical understanding is explained by Vygotsky's ideas on how social interaction between the student and the teacher (or more capable peers) can lead to higher levels of generalization for the student. I address this issue next.

**Knowledge Through Social Interaction**

Basically, this perspective assumes that in order for an outsider to understand the individual, one must first understand the social relations in which the individual exists (Vygotsky, 1978; Wertsch, 1985). Vygotsky argues that, internal processes (*intrapsychological*, e.g., understanding) that result when an individual constructs knowledge, are related to external or social processes (*interpsychological*; e.g., competence) that result from the dynamics of the classroom setting. Also, a student on his or her own, might be able to solve a problem only to a certain level without being able to continue. But with adult guidance or collaboration with more capable peers, this same student might be able to solve the problem beyond the initial level. The difference in these two levels is what Vygotsky calls the *zone of proximal development*.

The zone of proximal development defines cognitive functions that are yet to mature in the student and Vygotsky (1978) calls these functions "buds" or "flowers" of development. The relevance of this zone of proximal development to assessing students' understanding in mathematics is that it permits educators to "delineate the child's immediate future and his dynamic developmental state, allowing not only for what already has been achieved developmentally, but also for what is in the course of maturing" (Vygotsky, 1978, p. 87). Having knowledge of what is in the course of
maturing, I believe, should help in the selection of mathematical activities that will promote the maturation. And having students discuss their solutions to mathematical problems should provide an opportunity for educators to gain an insight into what is in the course of maturing.

**Complementarity of Perspectives**

To initiate and sustain verbal interactions among students, some form of discourse is necessary. This discourse includes the way ideas are exchanged and what those ideas entail (NCTM, 1991). Throughout the discourse, the individual's ways of making sense of things (Golding, 1990; Kamii and Kamii, 1990; Lerman, 1989; Nodding, 1990; von Glasersfeld, 1990), are influenced by the social interaction that helps the individual to make sense of those things (Cobb, 1989; Cobb et al, 1990; Smith & Confrey, 1991; Vygotsky, 1978; and Wertsch, 1985). Accordingly, where individual students interact to discuss their solutions to mathematical problems, I believe it is important to consider both the individual's ways of making sense of the problems and the social interaction among the students which contributes to the generation of knowledge. Thus, the knowledge of constructivism and Vygotsky's ideas on knowledge generation, can provide educators with a useful theoretical framework for assessing students' demonstration of mathematical power in small group contexts.

**Summary**

In this chapter, I presented a brief summary of events leading to the current reform in mathematics education. Then, while I discussed the current reform in relation to the curriculum, teaching, and evaluation standards suggested by the NCTM, I emphasized the importance of the NCTM recommendation that small groups be used to promote construction of mathematical knowledge. In addition, I reviewed small group work and identified the need to investigate small group work to provide information on students' mathematical power. Finally, I discussed a theoretical framework within which to conduct small group investigations.
CHAPTER 3

METHOD

In this chapter, I describe how I conducted the study. To contextualize the study, I discuss my background, the teacher's background, the school and the class that participated in the study. Then, I detail the procedures used for collecting and analyzing data.

Context of the Study

My Background

Throughout my school life in Ghana, I scored well on classroom tests and external examinations. Furthermore, as early as primary five (grade five in the Canadian school system), my good academic standing gave me the opportunity to lead small group discussions that the classroom teacher encouraged. After each assignment, the teacher would put students into groups led by peers who did well in those assignments. As leader, I gave explanations to students who could not do the assignments properly on their own. Then as a student, I realized from these group discussions that I understood the assignments better after I tried explaining to the group members how I did those assignments.

My participation in these group discussions influenced my learning style for the rest of my school life. I always formed study groups and I was eager to contribute meaningfully to the group discussions. I studied hard privately and searched for new ideas or ways of solving problems that I used in giving direction to the group discussions. I also adopted the small group format when I was a mathematics teacher in the secondary school for nine years, and I used the same format when I taught mathematics for three years at the Federal College of Education, Katsina, in Nigeria.

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When I came to Canada, I realized that educators here place a lot of emphasis on individual learning and instruction, at the expense of group learning and instruction. My interest in the use of small group format for learning and teaching mathematics was rekindled when I read the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional Standards for Teaching Mathematics* (NCTM, 1991), and *Enabling Learners: The Year 2000 Document* (British Columbia Ministry of Education, 1990) and realized these documents recommend the use of a small group format for teaching and learning mathematics. When I had the opportunity to teach mathematics at the University of British Columbia during the summer of 1993 and 1994, I once again used the small group format. Informed by my personal experiences and success with this format, I decided to formally investigate students' small group discussions of mathematical activities.

**Teacher's Background**

To investigate students' discussions of mathematical activities while they work in small groups, I looked for a teacher who believed in and used a small group format for teaching and learning of mathematics. I met Ms Joanne Stansfield (pseudonym) at a British Columbia Association of Mathematics Teachers (BCAMT) conference held at the University of British Columbia (UBC). Joanne had obtained a bachelor's and master's degrees in mathematics and started classroom teaching in 1978.

Before I met Joanne, she had participated in several BCAMT workshops. In particular, a workshop on alternative teaching styles with a focus on cooperative learning caught her interest and she has since then adopted cooperative teaching strategies (Artzt & Newman, 1990; Davidson, 1990; NCTM, 1991; Yackel et al, 1990; Yackel, Cobb, & Wood, 1991) in her mathematics classroom. Her experience with using cooperative teaching strategies suggested we would not disagree on organization.
Joanne had just moved to a new school when I approached her to undertake the study in her class. Joanne's experience (and mine also) was that if norms of cooperation are not firmly laid, students soon resorted to their individual learning styles even though they were grouped together and expected to be working cooperatively. So, she always tried to encourage students to observe the norms of working cooperatively (see Appendix C). Joanne transferred to another school while she was collaborating with me to lay a firm foundation for using cooperative learning strategies in the class, so the study was abandoned for a while.

There was a time lapse of about 6 months before Joanne started teaching in her new school. When I asked whether I could continue my work with her, she readily accepted. I was curious to find out why she was still willing to work with me and she said that "working with other people, especially those researching, helps me keep abreast with current issues." So, I followed Joanne to her latest school to continue with my study. Here also, we had to collaborate to lay a firm foundation for using cooperative learning strategies in the class. However, our experience working together in the first school helped us to develop some trust for each other. For example, the principal of her latest school was hesitant in giving approval for the study to take place because at the time, there were incidents of student molestation by intruders to the school. Joanne convinced the principal of the school that she had worked with me before and I proved to be very responsible dealing with students. The principal then gave approval for the study to take place in the school.

The School

The school structure is old and large and serves a population of about 1800 students. There are three programs for students, the summit program, the flex program, and the regular program. According to Joanne, students in the summit program are those that teachers consider to be high achievers, those in the flex program are considered by teachers as highly motivated, and those in the regular
program are those remaining after the other two programs are full. The high achievers are those students who are very successful at completing their assignments and who get high marks, while the highly motivated are those students that show eagerness to learn but do not necessarily obtain very high marks. All other students are put in the regular program, including those who failed the previous grade but had to go on to the next grade because space limitations do not permit students to repeat grades.

The Participating Class

The grade 9 students who participated in the study were in the regular program. There were 18 of them, 10 girls and 8 boys. According to Joanne, none of the students had failed grade 8 and some of them could have been in either the summit or flex programs except for the fact that the two programs were full. Nevertheless, Joanne likes teaching students in the regular program because of her belief that "these students seem neglected but they also have to experience mathematics as something meaningful and interesting." Because these students had not experienced small group work prior to this grade, Joanne and I used our previous experiences to formally introduce these students to cooperating and learning in small groups.

For about four months (that is from the start of the school year) prior to data collection for my study, Joanne used the small group (or the cooperative) format for her mathematics lessons. She followed the mathematics topics as laid down in the curriculum guide. She introduced the students to some guidelines (see Appendix C) for working together in groups and I assisted the students by responding to their individual and group difficulties as they adjusted to sharing their thoughts among themselves. (I visited Joanne's class at least once every week during the four months.) Some of the difficulties that arose include some of the students not wanting to share their ideas, some students dominating all the discussions, while at other times, some of the students would talk about other things not related to the topic under discussion. Sometimes when any one of the group members was found to be
disruptive or not contributing to the discussions, Joanne and I had to reallocate that
group member. Other times we had to encourage group members to be more tolerant
of each other and their views as they discussed their solutions to problems provided in
class by Joanne. The mathematical problems were related to the topics taught and
they were taken from the mathematics textbook recommended for that grade. The aim
at this stage was to get students used to discussing their solutions because, according
to Joanne, it has not been their experience in previous classes to discuss their
solutions to mathematical problems. No formal data were gathered at this stage.

Procedures

Small Group Formation

For this study, it was desirable to have group members who would
 communicate with each other and feel comfortable sharing their ideas together if I
were to gather the sorts of information the group members generated as they
discussed their solutions to the mathematical problems. Also, it was desirable to have
group members who would validate their conjectures while others in the group try to
meaningfully criticize those conjectures. If a student was found not talking, that student
would be replaced, since students' talk was very vital for gathering information for the
study. So, using Joanne's rating of the students in terms of their ability to
communicate together and with the familiarity I developed with the students by
interacting with them for about four months, I organized the 18 students in the class
into two groups of four and two groups of five.

In order not to leave out any of the students while data were being gathered, all
students in the class participated in the study. However, only one group of students
was selected as the focus group for the study. Members of the focus group had stayed
together during the four month period, prior to the data gathering, when the students
were learning how to work in groups. It was the focus group's discussions that were
analyzed to provide an answer to the research question. A combination of a student's mathematical ability as decided by performance in Joanne's classes and the student's ability to talk with colleagues was used to decide the membership of the focus group. There were two high achievers (George and Jane) and two medium achievers (Paulina and Daniel) in the focus group. Another important consideration was to maintain an equal number of males and females in the focus group so as to minimize differences in achievement that might be due to gender. The decision to balance males and females in the focus group was informed by Webb's (1991) observation that for equal number of males and females, achievement does not differ significantly when students work in groups.

Whenever there was the need to change the membership of the focus group, the following criteria were used: 1) mathematical ability, 2) ability to talk in a group, and 3) balancing of males and females. For example, when a female student had to leave the focus group to join the school's basketball team, she was replaced by another female of comparable mathematical ability and the ability to talk with the other members of the focus group. Again, when one male was found not to be talking during the discussions for the first three data gathering sessions, he was replaced by another male using the criteria listed above for deciding group membership.

The Problems

The problems used for the study (see Appendix A) were developed by me or modified from books or projects in which I participated. Although what constitutes a problem varies for each student (Van de Walle, 1990), and that not all the problems could provide information indicative of all the parts of SAS, what is important is that the problem or task must have the "potential for students to engage in sound and significant mathematics as a part of accomplishing the task" (Lappan & Friel, 1993, p. 525). Furthermore, the problem should provide the students the opportunity to have something to talk about. In that regard, I tried to use problem types with which the
students were familiar. While I tried to relate each problem to the topic Joanne was teaching so as to make the concepts current for the students, I also provided the students with the opportunity to talk by asking them to explain their solutions or give reasons for solving the problems the way they did. During the study, the students did talk most of the time.

Data Collection Techniques

There are several data gathering techniques that researchers use for qualitative studies. These techniques include interviewing, participant observation, field notes, use of questionnaires, and video and audio taping (Borg & Gall, 1989; Fetterman, 1989; Guba & Lincoln, 1991; Hammersley & Atkinson, 1991; Merriam, 1991; Patton, 1987; Van Maanen, 1988). For this study, to gather information from the participants' perspectives (Hammersley & Atkinson, 1991; Patton, 1987), I video-recorded the focus group's discussions of their solutions to the problems. The remaining groups' discussions were audio recorded. In addition, I collected all students' written responses to the problems. Furthermore, all students responded to questionnaire items (see Appendix B).

The study was conducted from December 10, 1992 - March 1, 1993. There were 7 data gathering sessions. (An impending teachers strike within the Vancouver School District shortened my data gathering period. Nevertheless, the amount of data collected proved sufficient for my research question.) For each data gathering session, the students attempted to solve the assigned problems individually within 20 minutes and then later discussed the solutions they obtained with their group members for 40 minutes. I urged the students to focus on explaining and giving justifications for the solutions they obtained while they discussed their solutions. Occasionally, I gave students prompts either when they asked for help, or when I found they were stuck in their discussions. Except for the classroom teacher checking the
roll, sitting at her table, or occasionally moving around to ensure that the students were on task, I was in complete control of the class during the data gathering stage.

To ensure that all students in all the groups were on task, I also moved from group to group (sometimes with Joanne) to listen to their discussions. However, I spent most of the time listening to the focus group. I would stand at a distance from where I could clearly hear the discussions of the focus group. When there was the need for me to intervene (when students asked for help or I overheard a discussion that required further explanation), I would quickly move to the focus group to do so. I recorded into a notebook any other activities that I could not capture by the video or the tape recorders. For example, I took note of students who had to leave during the discussion stage for a basketball game. With about 10 minutes left in the discussions, all the students were to respond individually to the questionnaire items (see Appendix B) that I provided them. Occasionally, however, some of the students in all the groups attempted discussing the questionnaire items collaboratively.

Prompts

As stated in chapter 1, the main focus of the study was to gather evidence of students' mathematical power from student-student interactions within a small group context. As such, prompting from the researcher was minimal. However, some of the realities of conducting classroom research involving students are that the students might ask the researcher questions or the researcher may feel obligated to respond to actions of students.

Thus, when students raised their hands or attracted my attention by banging on the table for example, I approached the group to offer help. At times, the help was in the form of prompts. That is, statements or questions that required them to explain what they did, how they did it, or why they did it the way they did. For example, when students arrived at different answers after evaluating $C = 1.80 + 0.75(m-1)$ for $m = 5$, my questioning ("So what do you do with the brackets?" and "So after the brackets
what did you do next?" led students to explain their actions and provide reasons for them.

At other times, the help was in the form of "hints." Hints were more prescriptive and directed students toward a solution rather than prompts which question students' actions to encourage reflection. For instance, unlike above, a hint given to students for part (b) of the same problem as above was "substitute $25.05 for 'C'." Students followed this advice and found a solution.

Data Analysis

All the focus group's discussions were transcribed from the video tapes. The transcripts were then analyzed to provide answers to the research question. That is, I organized information from students' small group discussions around SAS which served as key constructs (or events) (Fetterman, 1989; Guba & Lincoln, 1991; Hammersley & Atkinson, 1991; Merriam, 1991). For this study, the unit of analysis (Merriam, 1991; Yin, 1989) is the information students generated in 40 minutes, as they discussed in small groups, their solutions to each of the mathematical problems. Any inferences or generalizations are not statistical, but rather analytical (Yin, 1989), and they are to "guide but not predict one's actions" (Merriam, 1991, p. 176). To provide results that are "trustworthy", efforts were made to ensure the "credibility" and "auditability" of the data and the results.

Credibility and Auditability

Issues of "trustworthiness" in qualitative studies include ensuring what Guba and Lincoln (1991, p. 105) call credibility (validity) and auditability (reliability). While credibility involves using multiple sources to confirm the data collected, auditability involves an outsider concurring that, given the data collected, the results make sense. However, Merriam (1991) argues that credibility and auditability are inextricably linked
and that ensuring credibility ensures auditability. So, for this study, I took several steps
to ensure the credibility of the data collected.

For example, to ensure that all what the students said was properly transcribed,
each video tape and transcript were re-examined together to further determine the
"compatibility of the transcript" with the conversation it represented. Also, I listened to
the audio recordings of the other three groups of students in the class and found the
discussions similar to those of the focus group in terms of what the students were
saying. Furthermore, some of the responses to the questionnaire items corroborated
what the students in the focus group said and wrote down during the discussions. The
video recordings were given to my research advisor and Joanne (the classroom
teacher) to view and they read my interpretations of the data. Both agreed with most of
my interpretation of the data. In fact, Joanne even went through all the transcripts and
she agreed the transcriptions portrayed correctly her recollection, students' discussions
during the study.

Coding of Transcripts

I went through the full transcript of each problem and coded portions of the
discussions as C1, C2, ...; MP1, MP2, ...; MC1, MC2, ...; PS1, PS2, ...; and MD1,
MD2, ...; which represent categories of mathematical activities associated with SAS
(see p. 16 - 19). Portions of the discussions involve some or all members of the group.
For a particular problem, after coding portions of the discussions related to that
problem, I sorted together all C1, C2, ..., MP1, MP2, ..., MC1, MC2, ..., and so on.
(See Appendix D for the distribution of excerpts related to categories of mathematical
activities associated with SAS.) I selected excerpts from these coded portions of the
discussions as examples that represent each category of mathematical activity. For
example, in PS4, "PS" refers to "problem solving" while "4" refers to an excerpt that
reflects the "fourth" category of students' mathematical activities listed under problem
solving, that is verify and interpret results. Then, from all portions of the discussions

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coded PS4, I selected one excerpt that, in my judgment, best illustrates students' ability to verify and interpret results using the NCTM definition. Several examples provided by the NCTM (NCTM, 1989, 1991) of what constitutes students' ability to verify and interpret results guided me in the selection of the excerpts. Transcripts for the other problems were treated similarly.

**Interpretation of Data**

A combination of criteria was used to decide the extent to which information from students' group discussions are indicative of students' mathematical power. I described and interpreted the excerpts to provide insights which are related to SAS. For example, for an excerpt illustrating students' ability to verify and interpret results, I would indicate whether the students checked that the answers they got for solving a problem satisfied the initial conditions of that problem and whether they were able to make sense of the answer to the problem. Also, I described the union of excerpts related to SAS during the 40 minutes that the discussions took place for each problem so as to provide a holistic picture of how students were integrating SAS. That is, for each problem, I described the extent to which each SAS used for the study was reflected in the discussions involving that problem. The extent to which students demonstrate mathematical power is then provided in terms of the interpretations of the excerpts relating to SAS and the union of those excerpts. What is important here is to provide a holistic picture of students' demonstration of mathematical power within problems and across problems.

I also documented any other insights related to the study that I perceived to be important while I analyzed the transcripts. Whenever possible, the responses to the questionnaire items by the focus group were used to collaborate what members of the group said during the discussions, or to provide group members' opinions on their participation in the study.
Summary

In this chapter, I provided the context of the study by describing my background, the teacher's background, the school in which I conducted the study, and the participating class for the study. Before gathering data, the students were exposed to 4 months of teaching and learning using the small group format. There was a focus group of 4 students for the study, 2 males and 2 females. After trying to solve mathematical problems individually for about 20 minutes, they got together to discuss their solutions. The discussions were video recorded. The transcripts of the discussions were analyzed by providing evidence indicative of students' ability to communicate mathematically, use mathematical concepts and procedures, use mathematics to solve problems, their disposition towards mathematics, and their demonstration of mathematical power.
CHAPTER 4

RESULTS AND DISCUSSIONS

In this chapter, I present and discuss the results of the study in relation to the research question. The unit of analysis for the study is the information students generated in 40 minutes while they discussed, in a small group, their solutions to the mathematical problems given them. The analysis is not to provide a synopsis or a profile of any particular member of the group, but it is to provide an interpretation of the group discussions which consist of individual contributions. The main focus of the study was on the information students generated on their own during the discussions, but occasionally, when the students asked for help or were stuck in their discussions, I examined information they generated as responses to my prompts. From students' small group discussions, I provide examples of excerpts that reflect students' ability to communicate mathematically, use mathematical concepts and procedures, and use mathematics to solve problems. Also, I provide examples of excerpts that are reflective of students' mathematical disposition.

Excerpts of students' discussions that are used to provide evidence of their abilities are presented verbatim. Comments in brackets "[]" following students' discussions are my comments to highlight what was happening at the time. Students' written responses to the problems and their written responses to the questionnaire items are used, whenever possible, to provide additional information that illuminates the answers to the research question. Also, I document any other insights I perceive to be important while I analyzed the transcripts. It is important to note that students might not necessarily be aware that they demonstrated any particular abilities. Rather, it is my interpretation of their discussions and the criteria provided in chapter three that permit me to make inferences about students' mathematical power. The interpretations are guided by the definitions of SAS as provided by the NCTM (NCTM,
To maintain anonymity, pseudonyms are used for the classroom teacher and the students.

**Research question**

To what extent is information from students' small group discussions of their solutions to mathematical problems indicative of students' mathematical power?

By information, I mean mathematical information that is related to the problems the students have solved. This mathematical information from the discussions is represented by excerpts that are indicative of students' ability to communicate mathematically and use mathematical concepts and procedures. Also, this information is represented by excerpts that are indicative of students' use of mathematics to solve problems and their disposition towards mathematics. Furthermore, the existence and integration of information indicative of SAS provide evidence of students' demonstration of mathematical power.

**Communication**

Students' ability to communicate mathematically was reflected by all three categories of mathematical activities associated with mathematical communication. However, it was difficult to get any one excerpt to simultaneously reflect all the mathematical activities associated with mathematical communication. One category of mathematical activities where students demonstrated the ability to communicate mathematically involves expressing mathematical ideas by speaking, writing, demonstrating, and depicting them visually. There were 13 excerpts from the discussions of all the seven problems suggesting that students in the group expressed mathematical ideas by speaking, writing, demonstrating, and depicting them visually; two excerpts from discussions involving problem 1, three excerpts involving problem 2, four excerpts involving problem 4, and one excerpt each involving problems 3, 5, 6,
and 7. Even here, while some of the excerpts involve speaking, others involve writing, demonstrating, or depicting mathematical ideas visually.

Figure 4.01, for example, illustrates students' attempts at expressing mathematical ideas by speaking. Students were speaking about the cost in dollars for talking on the phone (problem 1, Appendix A). While Paulina was talking about part (i) of the problem involving how much to pay for speaking for five minutes, Daniel was talking about part (ii) of the problem involving how much time is involved in $25.05 for speaking. What is important here is that Paulina and Daniel were expressing the mathematical idea of rate, in this case the cost in dollars per minute (or how many minutes for a dollar). Even though Jane did not speak about rate, it is possible from her response "I think it is true..." to Paulina's question "Five dollars in ten minutes?" that she was also referring to five dollars in ten minutes. The solutions of Paulina or Daniel may be wrong (as it turned out in this case), and all three of them may not know they were talking about rate, but the important thing is that they were talking about a mathematical idea and that they provided an opportunity to whoever was listening to help them develop an understanding for rate as used in mathematics.

Paulina: Five dollars in ten minutes?
Jane: I think it is true...
Daniel: Look, look, it's ten minutes for twenty five bucks.

Figure 4.01: December 10th, 1993

The excerpt in figure 4.02 shows students speaking about perimeter and area as they try to differentiate between the two and indicate how to find them. They spoke of perimeter as involving adding of sides and area as involving the multiplication of sides. The way the students spoke about perimeter and area suggests a procedural understanding rather than a conceptual understanding of these concepts. The excerpt
was from discussions of students' solutions to problem 4 involving the lottery game (Appendix A).

---

Jane: Okay, perimeter is adding all these sides and area is multiplying, right?
Daniel: Go on!
Paulina: No, perimeter...
Jane: Perimeter is add, area is multiply.
Daniel: Don't get it? Go on!
Paulina: Oh!
Jane: This gives you one area, so you multiply these...

---

**Figure 4.02:** February 10th, 1993

The excerpt in figure 4.03 shows that while students discussed how they arrived at their solutions to problem 7 involving a hockey game (Appendix A), they spoke about the ideas of 1) guess and check, 2) using charts (systematic charting) to arrive at the solution, and 3) the boundary conditions of the problem. Quincy's affirmative response to Jane's question "Did you use the same charts?" indicates that he used charts to solve the problem. Paulina's comment "Well, 10 games, 2 goals...20 goals in 10 games, 50 goals and then 14 games left, right?" illustrates her awareness of the conditions to be satisfied in order to solve the problem. When Jane asked "So, in other words, you just kept on going?", apparently she was, within the context of the discussions, referring to Paulina continuously guessing and checking her solutions, while using the boundary conditions set up in the problem to guide her guesses.

Daniel said he did not use guess and check to solve the problem but then talked about "4 times 6, 4 times 7, ...", a guess and check approach, which his script shows as the way he solved the problem. So, just listening to what was said might not be enough
for deciding what was done. Even though guess and check was used to solve this problem, it was hard to tell if the students saw guess and check as an authentic procedure for solving the problem, from discourse alone.

Jane: How did you do yours?

Paulina: Well, 10 games, 2 goals...20 goals in 10 games, 50 goals and then 14 games left, right? Then 14 games 50 goals and then I went 4 times 12 equals 48, 3 doesn't go to 50...[48 plus 3 doesn't give 50]. So 4 times 11... 44... it adds up to 50 but its not 14 numbers. So I went on with 4 times 10 , 4 times 9...

Jane: So, in other words, you just kept on going?

Paulina: I didn't do systematic chart here...

Jane: Did you use the same charts? [directing the question to Quincy].

Quincy: Yes.

Jane: Did you use the same charts? [directing the question to Daniel].

Daniel: I just took...

Jane: Guessed, right?

Daniel: No. 4 times 6, 4 times 7, ...

Quincy: That's what we did...and got the answer.

Figure 4.03: March 1st, 1993

While responding to item 1 of the questionnaire where students were to indicate the ideas they used in solving the problems, the most common ideas students referred to are addition and multiplication. This might be due to the fact that most of the problems could be solved using addition and multiplication. However, there were instances when students identified some other ideas like factoring in solving problem 5 and substitution in solving problem 2. Students might have used the idea of
factoring while discussing their solutions to problem 5 since factorize was used while posing the problem, but the idea of substitution, at least for discussions involving problem 2, was first used by the students themselves. Also, the idea of systematic charting was first used by the students in solving problem 7. Thus, for this study, students used some of these mathematical ideas without a direct influence from the researcher.

Checking students’ scripts showed that while students as a group spoke of mathematical ideas, they sometimes wrote about them, demonstrated them, and depicted them visually as individuals. A visual demonstration and depiction of systematic charting used in the discussion of problem 7 is illustrated in figure 4.04. It can also be inferred that a “systematic” guess and check was taking place. This is because, while maintaining 14 games as the number of games left to be played, the number of 3-goal games was systematically increased from one to 13, while the number of 4-goal games was systematically reduced from 13 to one. The total number of goals for each 3-goal and 4-goal combinations was found and compared with the required total of 50 goals. Once again, listening to students’ ideas and observing their written scripts provided insights into how students were using those ideas.

\[
\begin{array}{|c|c|c|c|}
\hline
3 \text{ goal} & 4 \text{ goals} & 14 \times & \text{total goals} \\
\hline
1 & 13 & 14 & 55 \\
2 & 12 & 14 & 54 \\
3 & 11 & 14 & 53 \\
\vdots & \vdots & 14 & 10 \\
13 & 1 & 14 & 50 \text{ or } 15 \\
\hline
\end{array}
\]

*Figure 4.04: Quincy, March 1st, 1993.*
For students to demonstrate the ability to communicate mathematically, they should also provide evidence that they can understand, interpret, and evaluate mathematical ideas that are presented in written, oral, or visual forms. In this study, the mathematical ideas that students dealt with were presented by the researcher in written and visual forms. There were four excerpts from the discussions involving problem 1 and at least one excerpt involving each of the remaining problems, when students demonstrated understanding and interpreted mathematical ideas presented to them in written and visual forms. For example, figure 4.05 below is an illustration of students' understanding and interpretation of what it means to complete a multiplication table and find out the sum of the entries in the table (problem 2, Appendix A). Paulina's comment "You can't use those at the edge," and Jane's comment "Don't add these...Add only those in the middle" go to show their understanding and interpretation of what it means to find out the sum of entries obtained from a multiplication table.

Daniel was having some difficulty finding the sum of the entries, but with comments from Jane and Paulina, he was able to successfully find the sum of entries using the multiplication table. Daniel's success reflects what Vygotsky calls the "zone of proximal development" in that with the help of the "more knowledgeable peers' Jane and Paulina, Daniel was able to achieve beyond his own "level of development." Notice that even though it might seem that students were only making entries into a multiplication table, what was actually being tested was their knowledge and understanding of indices and how to manipulate them.
Paulina: So how much money did you get without the $50?
Daniel: How much?
Paulina: Yes.
Daniel: 100, I don't know.
Jane: You had 99, right?
Paulina: Let me see [collecting Daniel's script].
Daniel: You add up all the numbers, m and n.
Paulina: Yeah...How did you get...?
Daniel: You add up all the numbers, m and n.
Paulina: Yes, but it's supposed to be 99 if you add them up.
Daniel: All of it?
Paulina: Yeah, how did you get...? add again...try again. Oh Jane, he [Daniel].
doesn't know how to add yet!
Jane: Oh, he adds these ones at the edge [Apparently Daniel was also adding
the column and row numbers to be multiplied].
Paulina: You can't use those at the edge.
Jane: Don't add these...Add only those in the middle. You add these...

Figure 4.05: January 18th, 1993.

Additional evidence of students' understanding and interpretation of what it
means to complete a multiplication table and find out the sum of the entries was
provided by their written responses to problem 2. Figure 4.06 is an illustration of this
evidence from Jane's script. Jane (as well as other students in the group) correctly
substituted the values of m = 2 and n = 3 to find the amount of money to be won.
Figure 4.06: Jane, January 18th, 1993.

However, when the students were to simplify the sum of the entries first (for an additional $50.00) before finding out the amount to be won, all of them could not do the simplification properly, even though they recognized that the total amount to be won if the simplification was properly done would be $149.00 (99 + 50). In this case, even though the students showed a clear conception of what the solution to the problem would be, they had difficulty in manipulating the algebraic expression generated by the sum of the entries in a manner that will preserve the meaning of the addition of algebraic terms and the meaning of indices and then lead them to the solution to this part of the problem. Apparently, lack of procedural knowledge hindered the solution of this part of the problem.

Finally, for students to be communicating mathematically, they should provide evidence of the use of mathematical vocabulary, notation, and structure to represent ideas, describe relationships, and model situations. There were 15 excerpts from the discussions indicative of such ability; one excerpt for solving problem 2, three excerpts
each for solving problems 3 and 7, and four excerpts each for solving problems 4 and 6. Discussions of problems 1 and 5 did not provide any such evidence and there was no evidence of students' modeling of situations. An example of students' use of mathematical vocabulary to represent mathematical ideas and to describe mathematical relationships is illustrated by figure 4.07. While the students were discussing their solutions to problem 6 (Appendix A) where students were to decide the larger of two algebraic expressions, they used "invert and multiply" and "reciprocal" which are mathematical vocabularies that represent mathematical ideas and describe mathematical relationships. From students' written responses it was clear that they also used mathematical vocabulary such as "decimals", "combinations", "exhausting possibilities", and "variables", all of which represent mathematical ideas and describe mathematical relationships.

Quincy: Now you put inverted multiply, you are supposed to put "invert and multiply."

Shawna: Is that the reciprocal thing?

Paulina: What?

Quincy: Invert and multiply.

Daniel Yeah...

Figure 4.07: February 22nd, 1993.

An example of students' use of mathematical notations to represent mathematical ideas and to describe mathematical relationships is illustrated by the excerpt in figure 4.08. Students were discussing problem 4 (Appendix A) and talking about ratio and how to represent it. Jane orally described ratio correctly as "...This, two dots, and that." Her script (see figure 4.09) showed that she could represent ratio also with a "slash" instead of a "double dot" and when Daniel responded to Jane's
question "Does it matter if you write it this way or that way?" by saying that "They are all the right answer", apparently, he was assuring Jane that both ways of representing ratio are correct. So, in addition to seeing how students represent notations, we recognize that they also "debate" its appropriateness.

Jane: So then if it is ratio, it will be like....
Paulina: Is this ratio? [Asking Daniel].
Daniel: Yes...
Jane: ...This, two dots, and that? Does it matter if you write it this way or that way? What do you think?
Daniel: They are all the right answer.

Figure 4.08: February 10th, 1993

An examination of Jane's script (figure 4.09) shows that "this" was referring to one part of the ratio, "two dots" was referring to the symbol for ratio, and "that" was referring to the other part of the ratio.

\[
\begin{align*}
2y^2 + 4y + 10 & \quad \text{perimeter} \\
-6y^3 & \\
\frac{2y^2 + 4y - 30}{4y^4 + 10} & = \frac{6y^3 + 3}{4y^4} \\
2y^2 + 4y - 30 & : 4y^4 + 10
\end{align*}
\]

Figure 4.09: Jane, February 10th, 1993.
There was an indication that the students continued to communicate mathematically when I used prompts to help them clarify some of their thinking and refocus their discussions. For example, when students were having difficulty using the equation $C = 1.80 + 0.75(m - 1)$ to figure out how much to pay for speaking for 5 minutes (problem 1, Appendix A), the following discussion ensued (figure 4.10). Jane, Paulina, and Daniel added 1.80 and 0.75 first before multiplying the result by 4 \((m - 1)\). They knew they had to evaluate (do the brackets) first what was in the brackets, \(m - 1\), but did not know they had to multiply the result of \(m - 1\) by 0.75 because, as put by Paulina, there should have been a bracket around 0.75. When I asked them what 0.75\((m - 1)\) standing alone (without 1.80) would mean, they were quick to recognize that they would have to multiply them together. Eventually, they solved this part of the problem.

Discussions involving the prompts I gave the students showed that they continued to express mathematical ideas through speaking. For example, they spoke of "do the bracket" as meaning evaluating what is within the brackets. They also interpreted mathematical ideas when they recognized that having brackets around \(m - 1\) in the equation means they had to evaluate that first. Also, the students used mathematical vocabulary to express mathematical relationships when they referred to $C = 1.80 + 0.75(m - 1)$ as an equation.

So, students did not only continue to communicate mathematically when given prompts, they also provided information on how they made sense of mathematical ideas. "Because of the...equation", Jane, Daniel, and Paulina performed the operations in the equation in the order in which they were presented, even though they recognized that they had to evaluate \(m - 1\) before using it in the equation. Later, Daniel recognized that he had to first multiply four (from \(m - 1\) where \(m = 5\)) by 0.75 before adding 1.80. Also, Paulina's question "Why don't we have the brackets around the seventy five [0.75] like that...[showing it]...and then..." suggests that it is only when the equation is of the form $C = 1.80 + (0.75)(m - 1)$ would she have multiplied 0.75 with
m - 1 before adding on 1.80. Apparently, there is a lack of procedural knowledge of how to manipulate the equation so as to preserve the meaning of the equation. Furthermore, the use of prompts provided an opportunity for students to move beyond their own "level of development" to solve the problem.

Sitsofe: Let's get back to what you did...let's look at it again...You said you added this [1.80] to what?

Paulina: Seventy five cents.

Sitsofe: Why?...why did you do that?

Paulina: Because of the ...equation [whispered by Jane]

Sitsofe: Is that what it is?...What do you have here? [pointing to the bracket part of the equation].

Paulina: Brackets.

Sitsofe: So what do you do with the brackets?

Jane: You do [emphasis mine] the brackets...

Paulina: But I did the brackets first...

Sitsofe: Did you do the brackets first? [using their language].

Paulina: Yeah!

Sitsofe: So after the brackets, what did you do next?

Paulina: That might be...

Daniel: Four minutes times this [referring to 0.75]...you do that one.

Jane: Do you do that one first? Is that it?

Sitsofe: You have to tell me.

Daniel: Yes, yes!...

Sitsofe: You have a reason for adding this one to this one first? [to Paulina].

Paulina: Why don't we have the brackets around the seventy five [0.75] like that...[showing it]...and then...

Sitsofe: What difference will it make to have the bracket around the 0.75?
Paulina: You know that you can multiply these two...
Jane: By the way...then you add the one eighty [1.80]...[Jane attempting to complete Paulina's statement].
Sitsofe: So you mean without the brackets around these, you wouldn't know that you got to multiply this [0.75] by this [m-1]?
Paulina: Add these two [180+0.75] first, then you multiply by 4 [m-1].
Sitsofe: What about if you have something like this form? [2(5 -3)] If you want to take something out of the bracket, what do you do to the brackets? What do the brackets mean?
Paulina: You do this one first [referring to m-1] then you multiply by this one [referring to 0.75].
Paulina: Multiplication first.
Sitsofe: So what does this mean [0.75(m-1)]? If you take out the brackets?...the 0.75...?
Paulina: Oh, now I got it...you have multiple...so you go in order...you do multiplication.
Sitsofe: So what is the order now? Can you now go ahead?
Paulina: Yeah, you understand, George?
George: Yeah!

Figure 4.10: December 10th, 1992

Summary of results pertaining to communication

Students' ability to communicate mathematically within the group was reflected by all three of NCTM's categories of mathematical activities associated with such ability. Students' discussions provided insights into how the students made sense of mathematical ideas and debated the appropriateness of those ideas. Even though students were communicating mathematically, it was sometimes difficult to tell if the
students understood the ideas they were talking about. Also, combining students' written scripts with their discussions provided further insights into students' group discussions. Finally, prompts related to the discussions helped students to clarify their thinking and they continued to communicate mathematically while responding to the prompts.

**Mathematical Concepts**

An indication of students' ability to use mathematical concepts is that they can *label, verbalize, and define concepts*. Four excerpts were identified during the discussions, 1 each while solving problems 2 and 5, and 2 instances while solving problem 2, when the students attempted either labeling, verbalizing, or defining concepts. For example, in figure 4.11 below, Jane sought the reaction of other members in the group as she verbalized and defined area and perimeter in her own words. Daniel urged her to go on, apparently not disagreeing so far with what she was saying. However, when Paulina's comment of "No, perimeter..." suggested a disagreement, Jane became emphatic and said "Perimeter is add, area is multiply." Again, the group situation seemed to help Jane to clarify her thinking and become more assertive when she was challenged by Paulina regarding what perimeter and area mean to Jane. Also, Jane's verbalization of perimeter and area suggests a procedural rather than a conceptual understanding of these mathematical ideas.

Jane: Okay, perimeter is adding all these sides and area is multiplying, right?
Daniel: Go on! [He was turning over a page of his script].
Paulina: No, perimeter...
Jane: Perimeter is add, area is multiply.

*Figure 4.11: February 10th, 1993*
Another indicator of students' ability to use mathematical concepts is that they can **identify and generate examples and nonexamples**. However, there was hardly any evidence during the discussions that students identified and generated examples or nonexamples. The study was to investigate the group discussions and identify information related to SAS, given the problems whose solutions were to be discussed by the students. Notice that the students were not asked to specifically identify and generate examples and nonexamples. One might argue then that one should not expect responses to questions not posed. However, that would mean not permitting and listening to what students can say during group discussions, even if not specifically asked for. As shown in figure 4.12, it is possible (and desirable) to have responses to questions not directly posed as was the case when Paulina provided an example of a concept using $2 : 1$ to represent ratio.

Also, another indicator of students' ability to use mathematical concepts is that they **use models, diagrams and symbols to represent those concepts**. Seven excerpts were identified throughout the discussions suggesting that students demonstrated this ability; one excerpt each while solving problems 1 and 7, two excerpts while solving problem 5, and 3 excerpts while solving problem 4. For example, Paulina used "double dot" as a symbol for ratio. While Jane was wondering if ratio is a fraction, Paulina suggested "No, ratio is two double dot one" [$2 : 1$] as representing a ratio, apparently not recognizing her representation of ratio as also a possible representation for fraction. This part of the discussion is illustrated in figure 4.12 (see also figure 4.09).
Another indicator of students' ability to use mathematical concepts is that they recognize the various meanings and interpretations of concepts. Again, there were few instances indicative of students' ability in this regard throughout the discussions and throughout the written responses to the problems and the questionnaire items. A possible reason for the lack of students' recognition of the various meanings and interpretations of concepts, but which is hard to confirm from this study, is that once students know a meaning of a concept, they might not want to search for other meanings. One such instance is illustrated by the excerpt represented by figure 4.12. Here, one sees two different interpretations of "ratio" as a concept and the group discussions provided the group members the opportunity to benefit from the two different interpretations. However, in response to Jane, one realizes Paulina does not see ratio as a fraction, but sees it as a representation of two numbers $a$ and $b$ in the form $a : b$.

Students provided the same meaning of concepts using different words or expressions, and as shown above (figure 4.12), they would not accept other interpretations. Figure 4.13 below illustrates how students used different words or phrases to describe reciprocal. Daniel referring to reciprocal or inverse multiply as...
"Change around and multiply" and Quincy referring to the same concepts as "Three over two divided by one over four becomes three over two multiplied by four over one" (figure 4.13) indicate understanding of these concepts. Except for Daniel whose script did not indicate details of how he arrived at his solution, scripts of the other members of the group reflect the meaning of the concepts as presented by Daniel and Quincy. Figure 4.14 illustrates how Paulina represented *invert* and *multiply*. In this case, similar information was obtained from the scripts as well as from the oral discourse.

Shawna: Reciprocal........inverse multiply.
Daniel: Change around and multiply.
Quincy: Three over two divided by one over four becomes three over two multiplied by four over one...

**Figure 4.13:** February 22nd, 1993

\[
\frac{3}{2} x^3 \div \frac{1}{4} x^2 = \frac{3}{2} \times \frac{4}{1} = \frac{12}{2} = 6 \\
\text{None of them is larger. They're equal.}
\]

**Figure 4.14:** Paulina, February 22nd, 1993

Finally, students should be able to *translate any given concept from one mode of representation to another* as an indication of their ability to use mathematical concepts. Students' use of mathematical concepts should also be reflected by their
ability to identify the properties of a given concept and to recognize the conditions that determine that concept. It was difficult to identify excerpts from either the discussions, the written responses to the problems, or the written responses to the questionnaire items indicating that students demonstrated this aspect of the ability to use mathematical concepts, except when Jane identified ratio as a fraction and having two parts. Notice that even though students readily talked, wrote and drew, they were translating from one mode of presentation to another, rather than from one mode of representation to another.

Giving prompts to students did not provide instances where students demonstrated the ability to use mathematical concepts differently from the ability they had demonstrated on their own during the group discussions. The significant thing to note however, is that giving of prompts apparently helped the students to clarify their thinking and have a better conception of how to proceed in solving the problem. For example, in the equation $C = 1.80 + 0.75(m - 1)$ (problem 1, Appendix A), Paulina did not know that she had to multiply the 0.75 by $(m-1)$ before adding the 1.80 because according to her, there was no bracket around 0.75. The prompts that followed apparently helped her to eventually solve this part of the problem successfully. Also, one striking situation when giving prompts provided some useful insight was students' continual use of "m two n" for "m squared n" even after prompts were given to correct them (figure 4.15). This situation might suggest that compatibility (getting students to do things by convention), cannot be dictated. Students may adopt the conventional way of doing things only if they find it helping them solve their problems.
Sitsofe: You normally call it "m two n", right? That might confuse you at some stage. It is "m squared n."

Daniel: m four plus m two [when he was referring to $m^4 + m^2$].

Paulina: Isn't m two n and m two the same? [referring to $m^2n$ and $m^2$ respectively].

Figure 4.15: January 18th, 1993

Summary of results pertaining to mathematical concepts

Students' use of mathematical concepts was not generally wide spread. It was reflected mainly by two of NCTM's seven categories of mathematical activities that are associated with the ability to use mathematical concepts. The two categories relate to the ability to label, verbalize, and define concepts, and the ability to use models, diagrams and symbols to represent concepts. Use of students' scripts in conjunction with the discussions provided additional insights into how students used mathematical concepts. Once again, the major impact of prompts was that they helped students to clarify their thinking while engaged in the group discussions and enabled the students to successfully solve the problems they were to solve, using mathematical concepts.

Mathematical procedures

An indicator of students' ability to use mathematical procedures is for them to recognize when a procedure is appropriate. Seven excerpts illustrating students' recognition of the appropriateness of a procedure were identified during the discussions; one excerpt each involving problems 3, 4, 5 and 6, and three excerpts involving problem 7. Figure 4.16 provides evidence that while Paulina and Jane recognized "guessing" as an appropriate procedure for solving problem 7 (Appendix A), Quincy also recognized "systematic charting" as an appropriate procedure for solving the same problem. Jane's written script revealed that what she referred to as
"guessing" is actually "guess and check," an appropriate mathematical procedure. A portion of the script is illustrated in figure 4.17.

Paulina: So Jane, how did you do it?
Jane: Well, I just guessed!
Paulina: That's a good way of approaching it...
Jane: I know...let's go on...
Paulina: So, how did you do it? [asking Quincy].
Quincy: Systematic charting.

Figure 4.16: March 1st, 1993

Figure 4.17: Jane, March 1st, 1993

Another indicator of students' ability to use mathematical procedures is when they give reasons for steps in a procedure and they reliably and efficiently execute procedures. Only 2 excerpts were identified throughout the discussions, 1 each involving problems 2 and 4, when students gave reasons for steps in a procedure. Figure 4.18 is an illustration of Paulina's reason for selecting $m^3$ first when she simplified polynomials and arranged the results in descending order as she solved
problem 2 (Appendix A). All other students in the group agreed with her reason by nodding their heads. The other instance when a reason was given for steps taken in a procedure is provided in figure 4.19. While the students were discussing how to find the perimeter of the geometric figure in problem 4 (Appendix A), Paulina’s last statement suggests that the reason she had to "add all these" (referring to the given values of the sides of the geometric figure) was because she wanted to "get the perimeter."

Paulina: I chose m cubed first, because that has the highest degree.
Jane: That's right [others nodded their heads in agreement].

Figure 4.18: January 18th, 1993

Paulina: Four y plus eleven, isn't that the perimeter?
Jane: You got four y plus eleven, at least.
Paulina: But you didn't get that...[referring to Daniel].
Daniel: I know how I got it [Laughing and apparently holding back some information from the group].
George: Right?
Paulina: I think this is right; you have to add all these to get the perimeter.

Figure 4.19: February 10th, 1993

Seven excerpts identified throughout the discussions illustrate when students reliably and efficiently executed procedures. There was one excerpt each involving problems 2, 5, and 7, and two excerpts each in solving problems 1 and 3. Figure 4.20 illustrates how students reliably and efficiently used factorization and exhaustion of
possibilities to solve problem 5 (Appendix A). They found all possible sets of three factors of 72. Then, they tested to find out those sets whose members add up to 14. The solution to the problem from their scripts showed that each solved the problem correctly, so apparently, they reliably and efficiently executed the procedure they used.

Jane: I found every three number that can add up to 14...
Paulina: Same here!...[pointing to her script]...all is 14...and then I times.
Jane: ...and then I added them together and I got 14, right?...and then I multiplied them together...Agreed?
Daniel: Yeah! [Quincy also nodded his head].

Figure 4.20: February 15th, 1993

Also, another indicator of students' use of mathematical procedures is when they verify the results of procedures empirically or analytically and recognize correct and incorrect procedures. Eight excerpts were identified during the discussions illustrating that students verified their results analytically; one excerpt each involving problems 1, 5, and 6, two excerpts involving problem 3, and three excerpts involving problem 7. There was no indication that they verified results empirically. For example, Shawna had to convince herself that "480²" was the same as "480 x 480" by using the calculator to verify the procedure for solving parts b(i) and b(ii) of problem 3 (Appendix A). She had first used the square function on the calculator and then checked the result by keying 480, the multiplication sign, 480, and then the equal sign. Figure 4.21 below illustrates the discussion that took place.
Shawna: Wait, wait, wait... Four eighty times four eighty times two.
Daniel: Write that down and you take the first one, then you take nine eighty times four eighty and then subtract.
Shawna: Four eighty to second power is different from four eighty times four eighty... oh no, oh no... I was right [she used calculator to check].

Figure 4.21: January 28th, 1993

There were 12 excerpts identified throughout the discussions illustrating that students recognized correct and incorrect procedures; one excerpt each involving problems 3, 5, 6, and 7, two excerpts involving problem 4, and three excerpts each involving problems 1 and 2. One such instance is illustrated in figure 4.22. This was when students were discussing solutions to problem 2 part (b) (Appendix A). Students were to simplify the polynomial expression obtained from the entries of the multiplication table. (Notice that students were referring to $m^2n$ as $m2n$). When Paulina asked the group if it is a correct procedure to add "m2n" and "m2n" to get "m4n", George's emphatic "No" suggests that he recognized Paulina's procedure as incorrect. An instance suggesting a recognition of a correct procedure is provided in figure 4.10. Paulina's comment that "But I did the brackets first" suggests that she recognized it as correct to evaluate what was in the brackets first while solving the equation $C = 1.80 + 0.75(m - 1)$. 

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Paulina: Add these two.
Jane: Yeah.
Sitsofe: Add which two?
Daniel: Add those
Paulina: Is m two n plus m two n equal to m four n? \[m2n + m2n = m4n].
George: No.

Figure 4.22: January 18th, 1993

Finally, students' ability to use mathematical procedures should be reflected in their appreciation for the nature and role of procedures in mathematics. For such an appreciation to occur, students should be seen, over a long period of time, to be using well known procedures, generating new ones, and extending or modifying familiar ones (NCTM, 1989). Even though the study was over a three-month period, the discussions, as well as their written responses to the problems and the questionnaire items, do not indicate that these students generated any new procedures, extended or modified familiar ones. However, evidence that they used some well known procedures like factoring, guess and check, and exhausting all possibilities, has already been discussed above. The effect of prompts on students' ability to use mathematical procedures was similar to those already discussed under communication and mathematical concepts as these prompts provided opportunities to the researcher for explaining some of these procedures to the students (figure 4.10).

Summary of results pertaining to mathematical procedures

Students' use of mathematical procedures was relatively wide spread. It was reflected by four of NCTM's seven categories of mathematical activities that are associated with the ability to use mathematical procedures. These four categories
relate to the ability to recognize when a procedure is appropriate; reliably and efficiently execute procedures; verify the results of procedures empirically or analytically, and recognize correct and incorrect procedures. Use of students' scripts in conjunction with the discussions provided additional insights into how students used mathematical procedures. The use of prompts provided an opportunity for students to clarify their thinking and it also provided the researcher an opportunity to do some explaining.

Problem solving

Two of the areas in which students' ability to use mathematics to solve problems should be reflected are their ability to formulate problems and apply a variety of strategies to solve problems. For this study, students were not asked specifically to formulate problems and there was no evidence from the discussions and the written responses to the problems and the questionnaire items that they did. Also, students were not asked to individually apply a variety of strategies to solve any of the problems, and they did not. However, the small group context provided students the opportunity to become aware of the variety of strategies other members of the group used for solving the same problems. Three excerpts were identified throughout the discussions indicating that each student used a different strategy to solve a particular problem. Some of these strategies used by students are factoring, guess and check, systematic charting, exhausting all possibilities, and substitution. Whether or not each student in the group understood the other strategies he or she did not use for solving the problem was hard to decide from the discussions, except for statements like "We all got the same answer" which suggests that the students recognize the other strategies as appropriate for solving the same problem. Figure 4.23 below is an illustration of one such instance when a variety of strategies (limited to 2 in this case) has been identified as being used within the group to solve a particular problem. When Jane said "I just guessed mine," her script showed that she was actually
attempting guess and check (fig. 4.17) and when Paulina said "I used pure knowledge," she was trying to systematically "exhaust" all possibilities by combining 4-goal games with 3-goal games to get 50 goals in 14 games (fig. 4.24).

Daniel: How did you get your answer?
Jane: I just guessed mine...I didn't know the logical thing to write.
Paulina: What?
Jane: I guessed it!
Paulina: I didn't...I used pure knowledge.
Jane: Pure knowledge?
Quincy: What?
Paulina: What?
Quincy: Look...the...
Paulina: It isn't true...[she interrupted Quincy].
Jane: We all got the same answer [Paulina said the same thing].

Figure 4.23: March 1st, 1993

\[ \begin{array}{c}
\text{24 games} \\
-10 \text{ games} \\
14 \\
\end{array} \]

\[ \begin{array}{c}
\text{4 games} \\
\text{4 games} \\
\text{goals} \\
\text{3 games} \\
\text{3 games} \\
\text{3 games} \\
\text{3 games} \\
\end{array} \]

\[ = 50 \text{ goals} \]

They won 8 games.

Figure 4.24: Paulina, March 1st, 1993
Another indicator of students' ability to use mathematics to solve problems is that they are able to solve problems. Out of the seven problems for the study, the students were able to solve four on their own (problem 3 in a group, and problems 5, 6, and 7 individually), but needed prompts from me before they were able to solve the rest (problems 1, 2, and 4). Figure 4.03 shows students' discussions of how the solved problem 7 (Appendix A), for example.

It is important to note that even though all the students could not provide complete solutions to problems 1, 2, and 4, they were able to provide solutions to some parts of these problems. There were 4 excerpts that captured students' partial solutions to these problems, one excerpt each involving problems 1 and 4, and two excerpts involving problem 2. For example, when students were to find the amount of money they could win in the lottery game (problem 2, Appendix A), they all got $99.00. However, they could not get the bonus prize of $50 because they could not simplify the polynomial generated by completing the multiplication table, something they had to do to win the bonus prize. The students knew they had to simplify the polynomial expression, and they could even speculate on what the final solution should be, but not knowing how to simplify the expression prevented them from arriving at the solution. Thus, lack of procedural knowledge had once more prevented the students from solving a problem. Students' discussions reflected what they had written on their scripts. An example of students' discussions indicating that they had solved part of a problem, while they had difficulty continuing, is provided in figure 4.25. When they could not continue to solve the rest of the problem, Jane banged on the table to indicate her frustration. Jane's act suggested to me she needed some help so I went to the group only to find out that no one in the group could continue with solving the problem. I realized then that the difficulty the students were having was with the simplification of the algebraic expression with which they were dealing. Thus, monitoring the students helped me identify the difficulty they were having. However, it
was just about time to end that session for gathering data, so I could not provide prompts to students.

Daniel: Okay, okay...ninety nine, ninety nine dollars, okay?
Paulina: Okay [Jane also said the same thing].
Jane: What are we trying to find? We are trying to find...[she paused and read over the problem again].
Paulina: So, it's ninety nine [she was confirming the $99 they got after checking with the calculator the addition she had performed].
Jane: Yeah!
Paulina: So we all agree it's ninety nine dollars? [they all nodded their heads in the affirmative].
Jane: If you simplify, you get a bonus of fifty dollars...
Paulina: Why don't we try to figure that out, so that we get one forty nine dollars? [$149].
Jane: Why can't we just...you see, we don't know what simplify...or at least, I don't...[she banged on the table in frustration].

Figure 4.25: January 18th, 1993

Another indicator of students' ability to use mathematics to solve problems is their ability to verify and interpret results of problems they have solved. Throughout the discussions, 20 excerpts that suggest students verified and interpreted the results of the problems they had solved were identified. There were two excerpts involving problem 6, three excerpts each involving problems 4, 5, and 7, and four and five instances involving problems 2 and 3 respectively. Figure 4.26 illustrates how the students verified their solutions to parts b(i) and b(ii) of problem 3 (Appendix A).
Paulina used the calculator to verify that the results of $980x - 2x^2$ and $2x(490 - x)$ are the same if $x = 480$.

Similarly, figure 4.27 illustrates how students made sense of the solution they obtained individually to problem 6 involving algebraic expressions by exchanging their scripts and checking how each arrived at the solution. When Paulina wondered how Quincy came by a 3.5 while solving the problem and how that helped him to solve the problem, he explained he divided seven by two to get three point five. For the part (ii) of this problem, Quincy simplified the original expression of $21x^4 + 7/2x^3$ to get $(21x)/3.5$. Students' responses to the questionnaire item asking them to provide reasons for thinking they solved the problems showed that they said they either "checked" their solutions or "knew" they were right.

Daniel: The same answer, the same answer
Paulina: It's the same answer, look...[she was showing her script to Shawna]
Daniel: Same answer, as far as I know
Paulina: You have to use the calculator...
Shawna: Okay
Paulina: Two times four eighty times four ninety minus four eighty [using calculator]
Daniel: You get the same answer

Figure 4.26: January 28th, 1993

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Daniel: They are equal, equal [he was talking to Shawna].
Paulina: I am doing okay.
Quincy: Can I check? [talking to Paulina].
Daniel: Look at that one!
Quincy: Can we change scripts?
Others: Yeah. [They changed scripts in clockwise direction].
Paulina: Three point five?
Quincy: Yeah. Seven divided by two is three point five.

Figure 4.27: February 22nd, 1993

Finally, students' ability to use mathematics to solve problems should be reflected by their ability to generalize solutions to mathematical problems. However, there was no evidence from the discussions suggesting that students tried to generalize solutions to the problems for the study. For example, even though the students were provided with the opportunity to comment on their solutions to problem 3 where they could come up with general statements like "the value of an expression is the same as the value of its factorized form", they did not provide any such generalizations. May be, teachers need to be providing students with the opportunity to be engaged with the new categories of mathematical activities that are being emphasized within the current reform so that students become aware that these categories are valued by the mathematics education community.

For the effect of prompts on students' ability to use mathematics to solve problems, evidence from the discussions suggests that the prompts helped the students to solve some of the problems which they could not solve on their own. For example, when the students were not sure of how to solve part (ii) of problem 1 (Appendix A), the discussion illustrated by figure 4.28 ensued and the students finally
solved the problem. At first, I had wanted to encourage the students to go over their work and see if they could detect any mistakes they made or steps they missed. However, when I realized during the two-minute pause that they were not ready to continue the discussions unless I provided them with some vital clues (the students appeared to be saying we can't continue on our own), I decided to suggest to them to replace the C in the equation by the $25.05 to see if they could find m, the number of minutes spoken. That provided a breakthrough for Jane and she led the group to solve the problem. Once more, prompts from a "more knowledgeable" person created the opportunity for the students to solve the problem.

Jane: This is how we got a different value, we multiplied.
Sitsofe: Yeah, but how come you got different answers?
Jane: We used the equation.
Paulina: It might be wrong.
Sitsofe: That's what you need to find out among yourselves if you missed any steps along the way. You have to justify your [Daniel & George] solution to your colleagues. This is what I did, and this the way I did it. And you [Paulina & Jane] also justify yours to them.
George: How they got that answer? [looking up to me].
Sitsofe: You question them, then find out. [There was a long pause (2 minutes) and students seemed not sure of what to do]. If you have a problem of this sort and you want to find the value of your m, you know the cost, okay… what is the cost? [Paulina and Jane pointed to a value on their script].
It's that much…
If you replace this [C=25.05] by cost, can you find the value of the m?
Jane: Yeah, I can do this…
Sitsofe: What does that sign [=] mean?
Jane: Balance, it means this side is equal to this side, right?
Sitsofe: Yeah, so if that's what that sign [=] means, and you are given the amount you'll pay, can you find out how many numbers...[The students set out to figure out the solution]. Can you go ahead and find the value of m?
Jane: Yeah.
Sitsofe: Can you do that?
Jane: Yeah.
Sitsofe: Try and see, try and see, it might turn out.

Figure 4.28: December 10th, 1992

Summary of results pertaining to problem solving

Students' use of mathematics to solve problems was not wide spread. It was reflected by two of NCTM's five categories of mathematical activities that are associated with the ability to use mathematics to solve problems. These two categories relate to the ability to solve problems and to verify and interpret results. Use of students' scripts in conjunction with the discussions provided additional insights into how students used mathematics to solve problems. The use of prompts provided an opportunity for students to solve some of the problems they could not solve on their own (either individually or in the group) and it also provided an opportunity to do some explaining.

Mathematical disposition

Students' disposition towards mathematics should be reflected by their confidence in using mathematics to solve problems, to communicate ideas, and to reason. Seven excerpts were identified during the discussions, one excerpt involving each problem, when students demonstrated confidence in using mathematics to solve problems, to communicate ideas, and to reason. For example, when students
discussed their solutions to problem 5 (Appendix A), they initially agreed that the set of numbers (8, 3, 3) should correspond to the ages of the three sons of the host. However, one student challenged that solution and the discussions that followed are illustrated in figure 4.29. Paulina and Quincy thought at first that the ages of the three boys could not be (6, 6, 2) since, as Quincy put it, “It’s only one who goes fishing” and that one must be the eldest among the three. Apparently, Paulina was not satisfied with the rejection of (6, 6, 2) as a possible solution to the problem when she cautioned “six, six, two, though.” Apparently, Quincy was supporting Paulina when he said there was no problem [of having (6, 6, 2) as a possible solution]. His position was confirmed when he responded to Paulina’s question of whether there was no eldest son by saying that there was. Then, when Quincy followed his response with the question whether the two boys of age 6 could have been born at the same time, Paulina answered by saying it would depend on what time means. Daniel contributed to the discussions by suggesting that the time when the two boys of age 6 were born could differ by 1 minute. Apparently, Paulina became more convinced that the three boys could have ages (6, 6, 2), as well as (8, 3, 3). Thus, for example, knowledge of mathematics and what time means, were confidently used to reason, communicate ideas, and to provide a solution to the problem.

Paulina: It can’t be six, six, two because all the...
Quincy & Daniel: All the sons go fishing? [Both were talking simultaneously].
Jane: There is nothing to do.
Quincy: It’s only one who goes fishing.
Paulina: six, six, two, though...
Quincy: There is no problem.
Paulina: There is no eldest son?
Quincy:  There is! You think they both were born at the same time? Well, it's possible...

Paulina:  It depends on what time means...

Daniel:  What about 1 minute?

Paulina:  Yeah, it could be six,.six,.two also [apparently convinced that even for age 6,.6,.2 one of the sons could be the eldest].

Figure 4.29:  February 15, 1993

Also, students' disposition towards mathematics should be reflected by their flexibility in exploring mathematical ideas and trying alternative methods in solving problems. Throughout the discussions, seven excerpts were identified to illustrate students' flexibility in exploring mathematical ideas and trying alternative methods in solving problems. There was one excerpt each involving problems 1, 2, 4, 6, and 7, and 2 excerpts involving problem 3. For example, when the students discussed their solutions to the area and perimeter they were to find for problem 4 (Appendix A), they tried to differentiate between area and perimeter and then tried several ways of combining the dimensions of the geometric figure to get values for the area and the perimeter. After Jane wrote down an expression for the perimeter of the geometric figure, she counted 7 terms altogether. Meanwhile, Daniel added up the terms and got 6y + 10 and wanted to know if he could still simplify that [6y + 10]. Paulina also added up the terms in the expression she wrote for the perimeter and got 4y + 11, different from what Daniel and Jane got. Paulina was confident her expression for the perimeter was right and urged the group to move on to finding an expression for the area. Jane indicated the area of the geometric figure should be the product of 2y - 6 and 5 + y, after several trials. When she noticed that other members in the group were multiplying all the given dimensions together, Jane cautioned that they could not multiply everything together. For example, Paulina's script shows how she multiplied
all given dimensions together to get the area (see fig. 4.31). Their discussions are illustrated in figure 4.30 below.

Jane: How many terms did you get [for the perimeter]? One, two,...[she started counting the number of terms Paulina wrote down].
Daniel: How much is six y plus ten [6y + 10]? [what he got after adding all the terms of the expression for the perimeter].
Jane: I got seven.
Daniel: six y plus ten [6y + 10].
Paulina: Four y plus eleven [4y plus 11]. What are you doing? Clarifying things?
Jane: Don't worry...what is it? Four?
Paulina: Why? You check! [Paulina checked]. Is it not five y plus four y [5y + 4y]?
Daniel: I got...
Paulina: Four y plus eleven [4y + 11], isn't that the perimeter?
Jane: You got four y plus eleven [4y + 11] at least...Yeah, yeah...
Paulina: But you didn't get that...
Daniel: I know how I got it.
George: Right?
Paulina: I think this is right. You have to add all these to get the perimeter. Part (b) is right....Now let's do part (a).
Daniel: How could (b) part be right?
Jane: Cross that and that...and that multiplied by that equals area, right? [Looking at the written script shows Jane canceled - 4y and replaced it by 2y - 6 , canceled 5y and replaced it 5+y and referred to multiplying (2y-6) by (5+y)].
Jane: Cause you can't multiply everything together, right?
Paulina: Then this has to equal negative four y then...because of this part.
Jane: Look, this is equal to six. Yeah, this equal to six times five.
Daniel: Don't you know that?
Jane: Yes, this is 6 up here...
Daniel: Yeah...

Figure 4.30: February 10th, 1993

\[ 3y \times 6 \times 5y \times 3y \times 6 \times 5 = \]

\[ 6 	imes 5 = 30 \]

\[-3y \times 5y = -15y^2 \times 30 \]

Figure 4.31: Paulina, February 10th, 1993

Another aspect of students' disposition towards mathematics is reflected by their willingness to persevere in mathematical tasks, and by their inclination to monitor and reflect on their own thinking and performance. Eleven excerpts were identified throughout the discussions that suggest students were willing to persevere in mathematical tasks; at least one excerpt each involving problems 3 to 7, and three excerpts each involving problems 1 and 2. Also, there were 13 excerpts throughout the discussions suggesting that students were inclined to monitor and reflect on their thinking and performance. Of these 13 excerpts, there was one each involving problems 2, 5, 6, and 7; two involving problem 4, three involving problem 3, and four involving problem 1.
After they solved part (b) of problem 2, the students were having difficulty simplifying the algebraic expression they got for the sum of the entries which were obtained after completing the multiplication table. Students did not give up trying to simplify the expression. Paulina was persistently providing encouragement to the group (especially Jane) by urging them to "try to figure that out" and by reminding the group that "that's what we've been doing." They tried to relate to what was previously done in class, they tried several ways of combining the terms of the algebraic expression. They persevered in trying to solve this part of the problem. Jane alone would have given up very early trying to solve this part of the problem, but the group encouraged her to persevere. Apparently, trying to figure out what to do next, being emphatic about a method of arriving at a solution or the solution itself being wrong, and agreeing and disagreeing among themselves are indications of their inclination to monitor and reflect on their own thinking and performance. Thus, situations that make students argue among themselves intellectually provide the opportunity for them to persevere, monitor and reflect upon their thinking and performance. Figure 4.32 illustrates students' willingness to persevere in mathematical tasks, and their inclination to monitor and reflect on their own thinking and performance.

Paulina: So we all agree it's ninety nine dollars? [they all nodded their heads in the affirmative].

Jane: If you simplify, you get a bonus of fifty dollars...

Paulina: Why don't we try to figure that out, so that we get one forty nine dollars? [$149].

Jane: Why can't we just...you see, we don't know what simplify...or at least, I don't...[she banged on the table in frustration].

Paulina: I do! If we have one of those questions...that's what we've been doing.

Jane: Come on, show me then? [She banged on the table in front of Paulina].

Paulina: So I think we haven't got...so we have to put all these m's...Agreed?
[Quincy and Daniel nodded their heads].

Jane: I did that.

Paulina: And you have to add, like m three n [she was referring to m^3n].

Jane: I did that!

Paulina: So what's next?

Jane: I put m two n equals three mn

Paulina: How did you get that? How did you get 3mn?

Jane: There are three that look the same thing...then same amount, except that...

Paulina: It's wrong! It's wrong! wrong!

Jane: No, it's the same quantity, except that it is different.

Paulina: Yes, but you have to add this little thing, three mn.

Jane: It's the same thing!

Paulina: It's not the same thing!

---

**Figure 4.32:** January 18th, 1993

Finally, students' disposition towards mathematics should be reflected by students' activities that indicate they value the application of mathematics to situations arising in other disciplines and everyday experiences. Also, disposition should be reflected in an appreciation of the role of mathematics in our culture and its value as a tool and as a language and by their interest, curiosity, and inventiveness in doing mathematics. Throughout the discussions and the written responses to the problems and the questionnaire items, it was difficult to find evidence to suggest that students demonstrated these aspects of personal disposition towards mathematics. One situation which might suggest that students valued the application of mathematics in other disciplines and everyday experiences was when they discussed solutions to problem 7 which involved hockey. While Janet was busy trying to figure out the
solution through guess and check, Paulina, Quincy, and Daniel were talking about
baseball, golf, and the Super Bowl. Jane felt the other three were not talking about
mathematics so she banged on the table and said "We are talking math!" Quincy
replied that what they were talking about had to do with mathematics and Daniel and
Paulina agreed with him. Apparently, the three wanted to impress it upon Jane that
since they used mathematics to solve the problem involving hockey, their talk about
baseball, golf, and Super Bowl could also be mathematically valuable. Even if
Paulina, Quincy and Daniel were justifying an "off topic" conversation, justifying it the
way they did suggests that they realize mathematics can be applied to those sports
activities. Figure 4.33 is an illustration of the ensuing discussions.

Sitsofe: If you say it didn't work, what do you mean?
Jane. When you multiply it, right?...The games and goals...It's supposed to
equal to...not...well, it's supposed to work out to...
Sitsofe: Are you listening to the explanation? [Referring to other group members].
Jane: Explanation? I don't need one. [She thought I was referring to her].
Sitsofe: No, I mean they listening to your explanation.
Paulina: I like to watch Super Bowl.
Jane: We are talking math!!! [Janet banged on the table].
Paulina: Oh, sorry!
Quincy: But that has to do with math [His discussions with Paulina and Daniel].
Daniel: Yeah!
Paulina: Yes, it's hockey, hockey [She was pointing to her script].
Sitsofe: Oh, I see!
Jane: You are telling me you're a Canucks fan?
Paulina: Yes, of course!

Figure 4.33: March 1st, 1993
One instance which might suggest students' appreciation of the role of mathematics in our culture and its values as a tool and as a language was when they suggested using the newspaper as a source for finding the solution to problem 7 (Appendix A). Apparently, when Paulina, Jane and Daniel suggested a newspaper as "another way" of getting access to "the real scores" that can be used to solve the problem, they were appreciating the role the newspaper can play in generating mathematical activity. Apparently, they were also appreciating the role mathematics can play in solving such a problem and communicating its results. A portion of the discussions is illustrated in figure 4.34 below.

<table>
<thead>
<tr>
<th>Sitsofe:</th>
<th>Could you have done it differently?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quincy:</td>
<td>No!</td>
</tr>
<tr>
<td>Sitsofe:</td>
<td>You [Jane] said you guessed. You [Quincy] said you used some chart...</td>
</tr>
<tr>
<td>Paulina:</td>
<td>Systematic charting...</td>
</tr>
<tr>
<td>Sitsofe:</td>
<td>That's fine. Could you have done it differently?</td>
</tr>
<tr>
<td>Jane:</td>
<td>Yeah, I could have done a chart. It would have been easier for me.</td>
</tr>
<tr>
<td>Sitsofe:</td>
<td>So apart from using the chart or guessing?</td>
</tr>
<tr>
<td>Jane:</td>
<td>Another way?</td>
</tr>
<tr>
<td>Paulina:</td>
<td>Go get newspaper and find out.</td>
</tr>
<tr>
<td>Daniel:</td>
<td>The real scores.</td>
</tr>
<tr>
<td>Jane:</td>
<td>Yeah...</td>
</tr>
<tr>
<td>Sitsofe:</td>
<td>How could you do that from a newspaper?</td>
</tr>
<tr>
<td>Jane:</td>
<td>It has to be the truth but...</td>
</tr>
</tbody>
</table>

Figure 4.34: March 1st, 1993

Students' interest or curiosity in doing mathematics was better captured by their response to the questionnaire item asking them to indicate if they would like to see
group discussions of solutions to mathematical problems as part of their normal mathematics classes, except for Quincy who said "No" on one occasion, a typical response from the students was "Yes, because it makes us understand mathematics better." Thus, group work provides a context that can make students interested or curious in mathematics and lead them to understand mathematics better. The overwhelming interest or curiosity in group work may be due to the support students get from the group as they share their ideas. Despite the interest or curiosity, there was no evidence to suggest their inventiveness in doing mathematics.

Categories of students' mathematical activities associated with disposition towards mathematics that students demonstrated when they discussed their solutions to the problems on their own were also reflected in the discussions that followed the prompts I gave them. Students frequently showed tendencies "to think and to act in positive ways" (NCTM, 1989, p. 233). There was evidence indicative of students' confidence to use mathematics to solve the problem, willingness to persevere in solving the problem, and monitoring and reflecting on their own thinking and performance. I decided to find out if the students could solve part "c" of problem 3 using the difference of squares, because this was presented in class by Joanne (the classroom teacher) as a useful and fast strategy sometimes used to solve some problems in mathematics. The students remembered the expression of the form $x^2 - y^2$ as representing the difference of two squares but could not factorize it properly. They got $x^2 - y^2 = (x - y)(x - y)$. Even though they recognized that the two negatives before $y$ should multiply to give a positive (which is different from the negative before $y^2$, they did not seem to resolve the impasse. Apparently, students memorized the factorized form of $x^2 - y^2$ without reproducing it correctly. So, they could not solve the problem. When I provided them with the acceptable factorized form of the expression, they were able to solve the problem. Notice that students seemed to have applied only procedural knowledge in this case, without any indication of conceptual understanding. However, they did not give up in the face of the difficulties they faced
but rather acted positively to solve the problem and to use the calculator to check the results they got from factorization. Prompts once more helped the students to think and to act in positive ways to solve the problem. A portion of the discussions of problem 3 (Appendix A) reflecting students' disposition towards mathematics when prompts were given, is illustrated in figure 4.35.

---

Sitsofe: You were checking yourself to see if what you did was right. So if you want to simplify...

Daniel: You must change the sign.

Sitsofe: So what would you get if you want to factorize \( x^2 - y^2 \)?

Paulina: We don't know [She was apparently speaking for the group].

Sitsofe: You tried and did something. You only checked and found out that it didn't turn out to be this. You want to check and see if it comes back to this...

Paulina: Yes, I checked.

Daniel: So if we multiply, do we change the signs?

Shawna: Yes.

Daniel: Negative, negative...

Shawna: Yes, two negatives make a positive.

Sitsofe: Why did you put the negative here?

Paulina: I don't know. Because people weren't looking and they got this...that's what we did, though!

Daniel: Yeah, why not?

Sitsofe: So without the negative...?

Paulina: Because...

Sitsofe: If you are given that \( x^2 - y^2 = (x+y)(x-y) \), can that help you do it?

Paulina: Yeah.

Sitsofe: How?
Paulina: I don't know what you mean? Getting the answer?
Sitsofe: Uhum!...Do you find any similarity between this and that? [comparing $x^2 - y^2$ with $64^2 - 36^2$].
Paulina: Yes, it's the same, but $x$ equals sixty four and $y$ equals thirty six.
Sitsofe: So how can you use that? If you want to factorize this and use this to solve it, how can you do that?
Paulina: Putting sixty four plus thirty six and sixty four take away thirty six?
Sitsofe: Try it and see! [It was successfully done and the results checked using the calculator].

Figure 4.35: January 18th, 1993

Summary of results pertaining to mathematical disposition

Students' disposition towards mathematics was evident from the discussions. It was reflected by four of NCTM's seven categories of mathematical activities that are associated with mathematical disposition. Two of the four categories relate to students' confidence in using mathematics to solve problems, to communicate ideas, and to reason and flexibility in exploring mathematical ideas and trying alternative methods in solving problems. The other two relate to students' willingness to persevere in mathematical tasks and inclination to monitor and reflect on their thinking and performance. Use of students' scripts in conjunction with the discussions provided additional insights into students' disposition towards mathematics. The use of prompts provided an opportunity for students to further demonstrate their disposition towards mathematics.
Mathematical power

Students' demonstration of mathematical power should be reflected by their:
1) ability to communicate mathematically, 2) ability to use mathematical concepts, 3) ability to use mathematical procedures, 4) ability to use mathematics to solve problems, and 5) disposition towards mathematics. In addition, students' mathematical power should be reflected by the extent to which students integrate all these aspects of what should constitute mathematical knowledge. Appendix D provides, from the discussions of each problem, a distribution of excerpts that are reflective of SAS. Table 4.01 is a summary of Appendix D. It provides the number of categories, out of the total for each SAS, that has been reflected by discussions of each problem. For example, 2/3 in the row of "C" (communication) and in the column of "Prob 1" (problem 1) means that there were excerpts from the discussions of problem 1 that reflect two out of the three categories of mathematical activities associated with communication. Notice that 2/3 is not used to mean two out the three equal categories; it is only used to mean that two of the three categories have been reflected. It is important to remember that the group discussions of each problem took a maximum of 40 minutes and that it is a holistic picture of students' demonstration of mathematical power that is being presented. Therefore, evidence that the discussions reflect mathematical activities associated with any two or more SAS (union of excerpts related to SAS) should constitute evidence for integration.
Table 4.01

Categories of Student Assessment Standards (SAS)
Reflected by Discussions of Problems

<table>
<thead>
<tr>
<th>SAS</th>
<th>Prob 1</th>
<th>Prob 2</th>
<th>Prob 3</th>
<th>Prob 4</th>
<th>Prob 5</th>
<th>Prob 6</th>
<th>Prob 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>2/3</td>
<td>3/3</td>
<td>3/3</td>
<td>3/3</td>
<td>2/3</td>
<td>3/3</td>
<td>3/3</td>
</tr>
<tr>
<td>MC</td>
<td>2/7</td>
<td>4/7</td>
<td>3/7</td>
<td>4/7</td>
<td>3/7</td>
<td>3/7</td>
<td>2/7</td>
</tr>
<tr>
<td>MP</td>
<td>4/7</td>
<td>3/7</td>
<td>4/7</td>
<td>2/7</td>
<td>2/7</td>
<td>3/7</td>
<td>5/7</td>
</tr>
<tr>
<td>PS</td>
<td>2/5</td>
<td>2/5</td>
<td>2/5</td>
<td>2/5</td>
<td>2/5</td>
<td>4/5</td>
<td>2/5</td>
</tr>
<tr>
<td>MD</td>
<td>4/7</td>
<td>3/7</td>
<td>4/7</td>
<td>4/7</td>
<td>3/7</td>
<td>3/7</td>
<td>6/7</td>
</tr>
</tbody>
</table>

**Note:**

C = Communication, MC = Mathematical concepts, MP = Mathematical procedures, PS = Problem solving, MD = Mathematical disposition

For the seven problems used for this study, evidence from the discussions suggest that students demonstrated abilities reflective of all of SAS. There is a union of excerpts reflecting categories of mathematical activities associated with communication, problem solving, mathematical concepts, mathematical procedures, and mathematical dispositions. Also, there is evidence that each SAS was reflected throughout the study. Communication was reflected by all three categories of mathematical activities associated with it, namely expressing mathematical ideas by speaking, writing, demonstrating, and depicting them visually; understanding, interpreting, and evaluating mathematical ideas that are presented in written, oral, or visual forms; and using mathematical vocabulary, notation, and structure to represent ideas and describe relationships. Mathematical concepts were reflected mainly by categories of mathematical activities that relate to labeling, verbalizing, and defining concepts, and using models, diagrams and symbols to represent concepts. Mathematical procedures were reflected mainly by four categories of mathematical
activities. These categories involve recognizing when a procedure is appropriate; reliably and efficiently executing procedures; verifying the results of procedures empirically or analytically; and recognizing correct and incorrect procedures.

For problem solving, it was reflected mainly by two categories of mathematical activities associated with it. These categories involve solving problems and verifying and interpreting results. Finally, mathematical disposition was reflected mainly by four categories of mathematical activities associated with it. These categories involve confidence in using mathematics to solve problems, to communicate ideas, and to reason; flexibility in exploring mathematical ideas and trying alternative methods in solving problems; willingness to persevere in mathematical tasks; and an inclination to monitor and reflect on their own thinking and performance.

Now because categories of mathematical activities associated with all of SAS were reflected during the discussions involving each problem and because categories of mathematical activities associated with all of SAS were reflected throughout the study, results of the study indicate that students demonstrated mathematical power. The extent of students' mathematical power demonstrated is provided by the distribution of the categories of mathematical activities that are associated with all of SAS (see Appendix D and Table 4.01).

Other insights perceived from the study

While I was analyzing the data for the study, some insights became apparent to me. These insights involve students shifting their viewpoints as a result of the discussions with the group, with or without prompts. Only shifts considered to be major are discussed. A shift is major if it seemed to prevail at least during the discussions of any of the problems. For example, a student who argued for the appropriateness of a particular mathematical procedure and later accepted the inappropriateness of the same procedure in light of evidence from the discussions is considered to have gone through a major shift in his or her mathematical reasoning and arguments. I discuss
below two types of major shifts that I observed during students' discussions of their solutions to the mathematical problems used for the study. I call these types of major shifts *consensual* and *conceptual*, respectively.

**Consensual shift**

By consensual shift, I mean a change in viewpoint to align with the viewpoint of the majority, seemingly because it is a majority viewpoint. Shifts in viewpoint due to consensus could benefit the person whose viewpoint is shifting if the majority viewpoint meets acceptable mathematical standards. However, shifts could be detrimental to the person in the long run if the majority viewpoint does not meet acceptable mathematical standards. From the discussions throughout the study, three excerpts were identified that suggest a consensual shift occurred without it apparently benefiting the person undergoing the shift. Figure 4.36 below illustrates an apparently non-beneficial consensual shift in George's initial viewpoint.

While students were discussing their solutions to *How many minutes will you speak if you have to pay $25.05?* (problem 1, Appendix A), Jane and Paulina agreed on 26 minutes (with Daniel nodding his head in support) as the time to speak if one had to pay $25.05 for speaking on the phone. Earlier, George had obtained 31 minutes as the time to speak for $25.05. Even though the procedures he used led him to get a better solution (he subtracted 1 from 32, using the expression \(m-1\) to get 31 minutes, without realizing that it is the \(m = 32\) minutes that one should speak), he abandoned his own solution and went with the consensus of the group to accept 26 minutes. George has undergone what I call a consensual shift and so long as the majority solution is not acceptable by mathematical standards, this shift in George's viewpoint could be detrimental to him. Notice here that the students got the 2 different solutions on their own without prompts from me. So, George changing his mind to go with the group suggests he was believing in the group more than in his own abilities.
Paulina: There's problem with the second part. We are still on the second part. Four eighty for five minutes. We divided four eighty by five to get how much for a minute and then twenty five-o-five dollars divided by ninety six cents...

Sitsofe: Go ahead and see what you get and explain it to us.

Paulina: Four eighty divided by five equals ninety six...

Jane: Cents per minute.

Sitsofe: Why did you divide by 5?

Jane: To find out how much it is per minute.

Paulina: Four eighty divided by five to find out how much it costs per minute...equals ninety six cents and twenty five-o-five divided by ninety six equals twenty six minutes...

Sitsofe: Do you think you could do it some other way?

Jane & Paulina: I don't know [In unison].

Sitsofe: You got 31 minutes first, how did you get 31 minutes? Yours is different from the 26 minutes. [Directed at George].

George: I added seventy five cents every minute until I got twenty five dollars five cents.

Sitsofe: How different is yours from what they [Jane and Paulina] got? Do you agree with what they both got?

George: Yes.

Sitsofe: And you disagree with what you did first?

George: Yes [nodding his head].

Figure 4.36: December 10th, 1992
Notice that there were several instances of consensual agreements, both from
the discussions and the written responses to the questionnaire, that were based on
students' acceptance of a solution as correct because they all got the same solution
after working individually. Apparently, no shifts are involved in such cases but it is
important to note that students use consensus to accept the validity of a solution to a
problem. Even though seeking consensus is desirable, the danger could be that if all
the students use the same or similar faulty procedures to come to the same solution,
there will not be any motivation to recheck how they arrived at the solution, since they
all got the same solution.

Throughout the discussions of the study, four excerpts that suggest a
consensual shift that could benefit the person undergoing the shift were identified.
Even though the majority viewpoint warranting the shifts in these four cases were
based on acceptable mathematical standards, it was hard to tell whether the shifts
actually benefited the person undergoing the shift. An example illustrating a
consensual shift that should benefit the person undergoing the shift is provided by
figure 4.37. After George changed his mind to accept the faulty solution of the group, I
gave them several prompts that led them to solve the problem correctly. I asked them
what the $C$ and $=$ in the equation stood for. When they identified $C$ as standing for
$25.05$ and that they had to maintain a balance of the two sides of the equation while
finding the value for $m$, they eventually solved the problem. Jane echoed the result
when she said "So, we spoke for thirty two minutes." George did not come up with any
solution at this stage but he still agreed with the group, apparently maintaining his
position of going with the group decision.
Paulina: We got to use thirty two to get thirty one [realizing that the time to speak is m but m-1 is in the equation, so if m-1 =31 then the time spoken should be 32 minutes].

Jane: So, we spoke for thirty two minutes.

Paulina: Yes.

Jane: You all agree? [She was asking for consensus from the group and they all (including George) nodded their heads].

Figure 4.37: December 10th, 1993

However, there is an indication that when students have a viewpoint which they can justify, they stick to it, even if a more knowledgeable person challenges that viewpoint. Apparently, if students can self-validate (Anderson, 1993) their solutions using the rules and standards of mathematics, they become confident justifying those solutions through several means that are acceptable within mathematics. For example, after students had by consensus agreed to the solution to problem 6 (involving students deciding between the larger of two algebraic expressions), they confidently justified the original solution despite the prompts I provided to challenge their viewpoint of the original solution as being right. They substituted different positive and negative numbers and concluded that 6x = 6x would always be true for any given value of x. This is illustrated in figure 4.38 below.

Sitsofe: You came up with 6x and 6x?
Daniel: Yeah!
Sitsofe: Why do you say they are the same?
Quincy: Because they came out to be the same.
Paulina: Because they are the same!...
Sitsofe: Yes, my question is $6x$...and you don't know the value of $x$. This is also $6x$ and you don't know the value of $x$, how do you claim they are the same?

Daniel: Because they are the same $x$.

Quincy: Because it didn't say that it was otherwise...

Sitsofe: I see.

Paulina: Yeah!...That's right.

Sitsofe: Do you think of any number for which this can be different? If you have a value for $x$...can you think of a value for $x$ which can make the two solutions different?

Paulina: No...

Sitsofe: Think about it, I'll be back...[students continued with the discussions].

Quincy: Make it different by finding a value for $x$?

Shawna: No, too bad!

Paulina: We don't know the value of $x$.

Quincy: Value? Give them a value of $x$ so that one of them is greater!

Paulina: I thought we were finished...Okay, three over two...give me a number.

Quincy: Has to do with a negative.

Paulina: Why is it going to be negative?

Daniel: Because we didn't factorize.

Quincy: Looks like...No, like what?

Daniel: Constructing composite factors.

Quincy: What?

Daniel: Factorize...How do you factorize? [There was a pause while students worked individually].

Paulina: Have we found the answer?

Quincy: Yeah.

Daniel: We are done!
Paulina: No, we aren't!
Sitsofe: Have you come up with something different [I was speaking to the group].
Paulina: Yes...
Sitsofe: What?
Quincy: No matter what...the same!
Sitsofe: No matter what? What did you try?
Quincy: Negative.
Paulina: Why not try positive number?
Quincy: Cause they won't change.
Paulina: How do you know that?
Quincy: I say it won't change, if you want to prove me wrong, go ahead!
Paulina: No...Okay.
Sitsofe: You think they will change.
Daniel: No.
Paulina: No, I said before it wouldn't ...
Sitsofe: I see...So are you satisfied you solved the problems?
Yeah [they all responded in unison].
Sitsofe: So that you can explain it to anybody, right?
Quincy: Yes.
Sitsofe: I'll get one of you to explain it to the whole class.
Quincy: I will do it because I will get to be the teacher [after some of the group members hesitated to offer themselves].

Figure 4.38: February 22nd, 1993

**Conceptual shift**

By conceptual shift, I mean holding a different conception from an earlier conception as a result of the group discussions. To be a major shift it must be seen to
prevail during the discussion of a problem or prevail over discussions of other problems. Any perceived conceptual shift only indicates that at least a different competing conception is prevailing. If this "new" conception is compatible with acceptable conceptions within mathematics, then it is likely to promote mathematics learning. On the other hand, if this new conception is not compatible with acceptable conceptions within mathematics, then the shift it brings about might be detrimental to the person experiencing the shift, unless the "cognitive conflict" it creates in the person is resolved.

The conceptual shifts observed for this study resulted in new conceptions that are compatible with conceptions within mathematics. From the group discussions, five excerpts suggesting this type of major shift were identified. An example of this shift occurred when Jane first thought that guess and check was not an appropriate method for solving problems: she said she guessed because she "didn't know the logical thing to write". Apparently, guess and check was not the "logical thing to write," more so when Paulina's script showed that "pure knowledge" meant exhausting all possibilities before getting the solution (see figure 4.24). Later, Jane changed her mind during the discussions and accepted guessing as an appropriate method when she reacted to Paulina's remark that guessing was a good way of approaching the problem by saying "I know...let's go on". This was when the group was discussing their solutions to problem 7 involving a hockey game. The discussions are illustrated in figure 4.39 below.

Jane: Every one got eight...
Paulina: But there are other possibilities, probably?
Quincy: No...[shaking his head emphatically].
Jane: You are supposed to knock this one out!
Paulina: I know.
Jane: Did you guess? [She was asking Quincy].
Quincy: No.

Jane: Did you guess? [She was asking Daniel].

Daniel: No.

Jane: I guessed...[It looked like Jane was trying to find out if other group members would accept guessing as authentic].

Paulina: You copied! [She was referring to Daniel].

Daniel: Yeah!

Paulina: I got it for all...you can ask him [She was pointing to me].

Daniel: You are right! How do you know? How do you know?

Paulina: Because these...like the first one...

Quincy: These vary...those too vary...and doesn't...no one else...

Daniel: How did you get your answer?

Jane: I just guessed mine...I didn't know the logical thing to write.

Paulina: What?

Jane: I guessed it!

Paulina: I didn't...I used pure knowledge [Apparently, she was implying that guessing does not constitute pure knowledge].

Jane: Pure knowledge?

Quincy: What?

Paulina: What?

Quincy: Look...the...

Paulina: It isn't true...[I came to the group at this time].

Jane & Daniel: We all got the same answer.

Sitsofe: Same answer?

Daniel: Yeah...

Jane: It's amazing we do...the same answer!

Sitsofe: And did you come by the answer the same way?
Quincy: Yes.
Daniel: Yeah.
Jane: No...not sure...Sorry, I guessed.
Paulina: We have to find out...
Sitsofe: So, may be...you can find out how it was done differently.
Paulina: So Jane, how did you do it?
Jane: Well, I just guessed!
Paulina: That's a good way of approaching it...
Jane: I know...let's go on...

Figure 4.39: March 1st, 1993

Summary of results pertaining to other perceived insights

Two major types of shifts were perceived to have taken place as students discussed their solutions to the problems given them. These major types of shifts are labeled *consensual* when the shift is to align an initial viewpoint with that of the majority, and *conceptual* when an initial conception is abandoned and changed to a different conception during the discussions. Four of the seven consensual shifts perceived to have taken place from this study involved majority viewpoints that are compatible with acceptable viewpoints within mathematics. The other three viewpoints are not compatible with acceptable viewpoints within mathematics. Apparently, students do not change consensually if they have a solid grasp of an initial viewpoint. Finally, conceptual shifts observed from the study resulted in conceptions that are compatible with standard conceptions within mathematics.
 CHAPTER 5

CONCLUSION

In this chapter, I provide a summary of the study, general discussions of the results of the study, and the significance of the results of the study. Also, I discuss the implications of the results of the study for practice, and finally, I make some suggestions for future research in light of the results of this study.

Summary of the study

The purpose of this study was to appraise information from the discussions of a group of students and to determine if the information indicates students’ mathematical power. Also, from the discussions, I documented any other insights that I perceived to be important. The focus of the study was on the information the students generated on their own while they discussed their solutions to mathematical problems. To a lesser extent, data were also gathered with respect to students’ discussions following prompts from the researcher. Grade 9 students in the regular school program participated in the study.

A small group format was used for the study. There were four groups of students, two groups of four students per group and two groups of five students per group. All the students were given problems based on the concepts their classroom teacher had taught them the previous week or two. They responded to the problems individually for about 20 minutes and then used about 40 minutes to discuss, in their respective groups, their solutions to the problems. In order not to leave out any of the students while data were being gathered, all students in the class participated in the study. However, only one group of students comprising two males and two females was chosen as the focus group for the study and the group’s discussions were video recorded and analyzed to provide answers to the research question. Students for the
study also responded to questionnaire items at the end of each problem session. Data were collected on seven different occasions using seven different problems over a period of three months.

Students' discussions were analyzed around SAS as the key constructs. SAS comprises communication, problem solving, mathematical concepts, mathematical procedures, and mathematical disposition. Associated with each SAS are categories of mathematical activities. From the discussions of the solutions of each of the problems used for the study, excerpts reflecting any of the categories of mathematical activities associated with any of SAS were identified. These excerpts were interpreted to provide evidence for students' demonstration of mathematical power. Students' written responses to the problems and their written responses to the questionnaire items were also used to illuminate the interpretation of the excerpts. Other insights from the discussions that were perceived to be important were also documented.

Results of the study indicate that through the small group discussions of their solutions to mathematical problems, students demonstrated mathematical power to the extent that excerpts of the discussions reflected categories of mathematical activities associated with SAS. Specifically, communication was reflected by all three categories of mathematical activities associated with it, namely expressing mathematical ideas by speaking, writing, demonstrating, and depicting them visually; understanding, interpreting, and evaluating mathematical ideas that are presented in written, oral, or visual forms; and using mathematical vocabulary, notation, and structure to represent ideas and describe relationships.

Mathematical concepts were reflected mainly by categories of mathematical activities that relate to labeling, verbalizing, and defining concepts, and using models, diagrams and symbols to represent concepts. Also, mathematical procedures were reflected mainly by four categories of mathematical activities. These categories involve recognizing when a procedure is appropriate; reliably and efficiently executing procedures; verifying the results of procedures empirically or analytically; and
recognizing correct and incorrect procedures. For problem solving, it was reflected mainly by two categories of mathematical activities associated with it. These categories involve solving problems and verifying and interpreting results.

Finally, mathematical disposition was reflected mainly by four categories of mathematical activities associated with it. These categories involve confidence in using mathematics to solve problems, to communicate ideas, and to reason; flexibility in exploring mathematical ideas and trying alternative methods in solving problems; willingness to persevere in mathematical tasks; and an inclination to monitor and reflect on their own thinking and performance.

Other results indicate that combining students' written scripts with students' talk provides a better insight into what students are talking about. Also, monitoring students and providing them with prompts while they work in groups is useful in helping them accomplish tasks in which they are engaged. Finally, when students work in groups, they can shift their viewpoints consensually or conceptually to align their viewpoints with majority viewpoints.

**General discussion**

In this study, I set out to investigate students' small group discussions and to determine if there is information indicative of students' demonstration of mathematical power. The importance the NCTM attaches to the use of small groups to promote the construction of mathematical knowledge and my own interest in the use of small groups motivated me to undertake the study. The design of the study whereby students discussed solutions after they attempted solving the problems individually was based on my belief that whenever a group of students reflect over mathematical problems with which they are familiar, they gain better insights into the problems. This belief is consistent with evidence from research that "people who think about their problem solving after they have solved a problem are better problem solvers than those who don't" (Willoughby, 1990, p. 43). So, for me, the important thing in this study
was to provide the opportunity for a group of students to talk about mathematical problems they have attempted to solve on their own.

Also important is that the problems should contain something mathematically significant to talk about, and they did. The major concern here was not to compare students' discussions from one problem to the other, but to examine their discussions, given the problems. To that effect, what I did was to relate the problems of the study to the mathematics content that the teacher, Ms Stansfield, taught the students. For example, problem 1 relates to the solving of equations while problem 2 relates to simplifying polynomial expressions, the manipulation of indices, and estimation. Problem 3 relates to factorization and evaluation of binomial expressions and the use of the strategy of the difference of two squares to solve problems. Problem 4 connects algebra with geometry and relates to simplification of algebraic expressions. Problem 5 relates to factorization while problem 6 refers to simplification of algebraic expressions. Finally, problem 7 relates to the use of variables to solve problems.

Notice that none of the problems as presented, and for that matter no single problem, would permit the students to provide information that reflects all categories of mathematical activities associated with each of SAS. Consequently, only some categories of the mathematical activities associated with each of SAS were reflected throughout the study by the students' discussions. Nevertheless, students' discussions provided sufficient information to suggest that they demonstrated mathematical power. It was difficult to tell if students were aware of (or even understood) some of the mathematical ideas that they talked about. Sometimes, the mathematical idea was inferred from their talk as in the case of rate, while other times, they specifically mentioned the mathematical idea, as in the case of exhausting possibilities. Even though the problems were related to concepts like functions, indices, and polynomials, students did not identify these as some of the mathematical concepts they used. It would have been desirable to probe for further understanding,
but this was not possible since students were to be talking on their own most of the time.

Also, students' demonstration of a particular ability might not imply that they understand related abilities. For example, although the proper use of a mathematical procedure would suggest an understanding of the concept underlying the use of that procedure, a student might use a procedure properly with or without an understanding of the concept underlying the use of that procedure. It was hard to tell if the students only reproduced "invert multiply" and the "reciprocal thing" as taught them by their teacher or they understood the concepts behind them.

Students were more inclined towards providing evidence for procedural usage rather than conceptual understanding. For example, students' use of perimeter as "adding of sides", area as "multiplying of sides", and ratio as "this, two dots, and that" all suggest procedural usage. Also, there were instances, involving mainly algebraic manipulations, when students' demonstrated abilities suggest they did not have a good grasp of the concepts and the procedures involved. For example, anytime it came to simplifying algebraic expressions (like \( m^2n + m^2n \)), the simplified versions the students got (\( m^4n \)) were not acceptable within mathematics. Given that the students successfully solved most of the problems involving only numbers but could not simplify algebraic expressions, they apparently were having problems with algebraic manipulations per se.

It was insightful to find students' small group discussions providing information about categories of mathematical activities that are not traditionally emphasized. For example, students verified and interpreted the solutions they obtained to problems. Also, they shared information on strategies they used for solving problems, and they monitored, not only their thinking and performance, but those of other members of the group. Students engaging in such activities is a positive sign that small group discussions of mathematical activities can provide a context for demonstrating students' mathematical power.
Major shifts were those that prevailed over the period of 40 minutes' discussions of a problem. There were more consensual shifts observed than conceptual shifts. It was hard to tell if both types of shifts were taking place at the same time. What would have been desirable is a consensual shift taking place alongside a conceptual shift and having these two shifts aligned to "new" viewpoints that are based on acceptable standards of mathematics. Accepting majority viewpoint just because one is conforming to the "norm" is not enough, especially if the majority viewpoint is faulty. Rather, I believe an acceptable conceptual shift is one that will make any shift beneficial to whoever is undergoing that shift. This should be so even if the shift is a result of prompts from a more knowledgeable person like the teacher or someone from the peer group. Emphasizing conceptual shifts on the part of the person shifting is important if we are to avoid students shifting their reasoning because of the "authority" of the teacher or more capable peers. In any case, having the five conceptual shifts that occurred and four out of the seven consensual shifts that occurred meet acceptable mathematical standards suggests that these shifts could benefit those experiencing the shifts.

It was only through prompts that I had the opportunity to influence the sorts of information students generated and the quality of such information. It was interesting to find how giving prompts provided useful teaching-learning situations. Students' "barriers" to solving the problems were identified and repeated questioning helped students to clarify their thinking and refocus them to solve the problems. Apparently, giving of prompts made instruction an integral part of the data gathering process.

Not all of students' discussions were audible, and not all of the discussions made sense to an observer. Some information was lost because students were talking in very low tones. Some information was also lost because students were using indefinite pronouns such as this and that and it was difficult to tell to what they were referring. So, it was difficult to pass judgment on these aspects of the students' discussion. Sometimes, however, combining the discussions with what they wrote
down provided better insights into what they were discussing. Thus, the discussions make more sense if they are interpreted within the context in which the discussions took place.

Students used calculators to solve most of the problems. Even though calculators could enhance students' mathematical performance, information on how they used the calculator was sometimes missing. One example where the order of computation was important was the case for problem 1, involving speaking on the phone. When students did not follow acceptable order for carrying out the computations, they arrived at the wrong solutions and became frustrated for not solving the problem. Other times, they became falsely confident as having successfully solved the problem, despite punching the wrong keys on the calculator.

It was desirable for this study to have students talk. So, when before the fourth data gathering session, I found out that George was not contributing appreciably to the group discussions, he was replaced by Quincy who contributed meaningfully to the group discussions. This changed the locus of interaction within the group. When Jane and Paulina were in the group, they did most of the talking with Daniel talking occasionally. George seldom talked. When Jane had to go for basketball training and could not attend two of the data gathering sessions, Shawna replaced her and the locus of the interaction shifted to Daniel, Paulina, and Quincy, with Shawna talking less but more frequently than George had. So, sometimes changing the group membership may introduce more dynamism into group discussions. Notice however that the reverse could also be true, but for this case it turned out that the new group of students talked more. May be keeping to the criteria for changing group members was the contributing factor in this case.

It was difficult to represent the discussions as sometimes taking place simultaneously. However, for the purposes of this study, that did not matter because in whatever order the students made their contributions to the discussions, the important thing was that differences in reasoning resulted in discussions leading to the
resolution of the differences, with or without prompts. Likewise, similarities in reasoning, irrespective of who said what first, only helped the students come to the same conclusions.

Significance of the results

The results of this study are important in at least four respects. Firstly, the results show that when students talk in small groups, they can provide information indicative of students' mathematical power. Since the attainment of students' mathematical power is the goal of the current reform in mathematics education, the use of small groups as recommended by the NCTM, at least provides a context for gathering information on students' demonstration of mathematical power. Secondly, students' talk, when combined with their written scripts, provides better insights into what they are talking about. Thirdly, when students work in groups, they can shift their viewpoints consensually or conceptually to align them with majority viewpoints. However, if they have a viewpoint that they can justify, they do not change it even in the presence of an authority. Finally, monitoring students and providing them with prompts while they work in groups is useful in helping them to accomplish tasks in which they are engaged.

Implications for practice

Results of the Study

Even though this is a "best case scenario," the results of the study suggest several implications for classroom practice. Since the small group discussions provided information indicative of students' mathematical power, the result suggests that the small group context can be used to gather such information. As such, mathematics teachers are encouraged to use it as a context for gathering information indicative of students' mathematical power. Also, mathematics teachers are
encouraged to consciously provide for all categories of mathematical activities that are associated with SAS if students are to meet the expectations of the reform. Limiting the categories will limit the extent to which students develop mathematical power. Also, when teachers adopt the use of small groups to gather information indicative of students' mathematical power, they are encouraged not to focus only on students' talk, since sometimes, combining students' talk with their written scripts provides better insights into the subject of discussion.

A classroom instructional process, which involves discussions of mathematical activities, may help improve students' proficiency in mathematics because as students shift their reasoning consensually or conceptually as a result of group discussions, they tend to align themselves with viewpoints that are compatible with acceptable viewpoints within mathematics. For students to confidently align themselves with acceptable viewpoints, teachers need to encourage their students to self-validate (Anderson, 1993) their solution. This was evidenced in the study by students not changing their solution when they could self-validate it. Thus, the ability to self-validate should provide the control element shaping the direction of the shifts.

Finally, teachers need to monitor the group discussions so that prompts can be given to challenge shifts not aligned with acceptable viewpoints within mathematics. Giving the appropriate prompts at the appropriate time means that the teachers are knowledgeable enough to detect students' difficulties (and strengths) and know what prompts to give to help clarify students' thinking. Monitoring is also necessary if teachers are to identify the "buds" or "flowers" that are "in the course of maturing" (Vygotsky, 1978, p. 87) and provide appropriate mathematical activities that will enhance the growth of those buds or flowers.

Reflections

Having gone through this study, I have gained insights concerning difficulties associated with capturing students' mathematical power through the student
assessment standards (SAS). I would like to share some of these insights with the mathematics education community. The circular definition of students' mathematical power makes it problematic when deciding what constitutes students' mathematical power. For example, the NCTM considers students' mathematical power as one of the student assessment standards and considers mathematical reasoning also as one of the student assessment standards. However, a category of mathematical activity associated with students' mathematical power involves mathematical reasoning also. Thus, conceptually, mathematical reasoning is presented as a subset of students' mathematical power and at the same time presented as of equal importance to students' mathematical power which is a student assessment standard. What constitutes students' mathematical power is therefore difficult to determine and some conceptual clarification is needed.

Talking about conceptual clarification brings to mind the difficulty I had deciding whether the mathematical power demonstrated by the students in the small group was for the group or for the individuals in the group. During the discussions some particular students seemed to talk frequently, but as responses to what other students, who seemed talk less frequently, said in the group. In either case, the talk reflected a category of mathematical activity associated with one of the students assessment standards. So, was it the student who talked more frequently that demonstrated mathematical power or the one who talked less frequently? Or was it the whole group that demonstrated mathematical power? It was a difficult decision for me to take and I found myself "buying" into the idea that in the small group context, the individual demonstrated mathematical power which was "mediated" by the group interaction. By that I mean there was some "group effect" on the individual's demonstration of mathematical power, and I am still grappling with how to determine the extent of that group effect.

Sometimes, deciding on which categories of mathematical activities particular information reflects was difficult because of the overlap of some of the categories
associated with SAS. Evidence that is indicative of a student's ability "to apply a variety of strategies to solve problems", for example, might also be indicative of that student's "flexibility in exploring mathematical ideas and trying alternative methods in solving problems." However, these two categories of mathematical activities are associated with two different SAS. Rather, instead of creating separate categories for such mathematical activities, efforts should be made to unify such categories so as to provide a more holistic picture of students' mathematical power.

**Possibilities for future research**

Several issues raised by this study could become the focus of further investigations. Even though students were using some mathematical vocabularies, this study did not probe the extent of students' understanding associated with the use of those vocabularies. Further investigations into students' understanding associated with the mathematical vocabularies they use should be insightful.

The study did not compare students' discussions by problems. It should be of interest to investigate the type of problems that will permit students to provide information reflective of a wider range of categories of mathematical activities associated with SAS. Should problems be open-ended, non routine? Or can textbook problems provide similar information?

For this study, the small group format provided a context for gathering information on students' demonstration of mathematical power. The focus was not on how the small group format was contributing to the attainment of students' mathematical power. Information on the extent to which the small group format can facilitate the attainment of students' mathematical power deserves attention in further research.

Also, George was moved to another group because he was not talking in the group he was in initially. A follow up investigation of George (or any student whose
group membership changes) to determine his oral interaction with members of the new group can become the focus for further research.

Finally, there could be further investigations into the effect of consensual shift on students' understanding of mathematics and how students' understanding of mathematics influences how they align themselves with majority view points. Investigations into these issues could provide insights that would guide the use of students' discussions through the small group format for assessing students' mathematical power.

**Final note**

This study demonstrates in a small way that from small group discussions, there can be *observable events* that reflect the categories of mathematical activities associated with SAS. To continue with the current reform within mathematics education, teachers should be encouraged to take risks to identify classroom events that reflect the seemingly rhetoric parts of the SAS. Teachers will need a lot of guidance and encouragement, and I hope this study provides an additional source of encouragement that it can be done. As reported in the March 1994 issue of the *Journal of Research in Mathematics Education* (volume 25 number 2, page 115):

perhaps the most obvious research-related response to the *Standards* is the identification and clarification of the research base for the recommendations contained in the document. The *Standards* document contains many recommendations, but in general it does not provide a research context for the recommendations, even when such a context is available.

Research Advisory Committee, "NCTM *Curriculum and Evaluation Standards for School Mathematics*: Responses from the research community" JRME, 1988, 19, p. 339

In line with the aspirations of the Research Advisory Committee, all that this study sought to do was to provide a research context for using the small group format to gather information indicative of students' mathematical power.


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Appendix A
(Problems 1 - 7)

Problem 1
Name_____________________________________ Sex_______

Instruction: Solve the following problem on your own and discuss the solution with your group members. Respond to the questionnaire as honestly as you can. Remember, this is not an examination.

Problem: B. C. Telephone has a way of determining the cost in dollars of making telephone calls from cities in B. C. to other cities in Canada. The cost of a telephone call from Vancouver to Calgary is determined from

\[ C = 1.80 + 0.75(m - 1), \text{ where } m \text{ is the number of minutes you speak.} \]

i). How much will you pay if you speak for 5 minutes?

ii). How many minutes will you speak if you have to pay $25.05?
Problem 2
Name_____________________________________ Sex_______

Instruction: Solve the following problem on your own and discuss the solution with your group members. Respond to the questionnaire as honestly as you can. Remember, this is not an examination.

Problem:

a) Is $2m^3 + 3m^{-2}$ a polynomial? Explain your answer.

b) A new type of lottery was introduced in Vancouver recently. A ticket costs $1.00 and it looks like the diagram below.

\[
\begin{array}{ccc}
  X & m & 2 \\
  mn & & \\
  n & & \\
  m & & \\
\end{array}
\]

The entries in the 9 boxes are obtained by completing the multiplication table. To win, you must pick without watching, any two digits from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in a container. The first number you pick is 'm' and the second number you pick is 'n'. The amount of money you win is the sum of all the entries in the 9 boxes.

i) If you pick $m = 2$ and $n = 3$, how much will you win?

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Note: If you *simplify* the sum of all the entries *before* finding out how much you win, you get a bonus of $50.

ii) Guess the values of 'm' and 'n' that will give you the largest win. Give reasons for your guess.
Problem 3

Name_____________________________________ Sex_____ 

Instruction: Solve the following problem on your own and discuss the solution with your group members. Respond to the questionnaire as honestly as you can. Remember, this is not an examination.

Problem:

a) Without using a calculator:
   i) Find the value of $x^2 - 6x + 8$ if $x = 5$.
   ii) Factorize $x^2 - 6x + 8$ and then find the factored value if $x = 5$.
   iii) Comment on your answers to (i) and (ii).

b) Using a calculator or otherwise:
   i) Find the value of $980x - 2x^2$ if $x = 480$.
   ii) Given that the factored form of the binomial in (i) is $2x(490 - x)$, find its value if $x = 480$.
   iii) Comment on your answers to (i) and (ii).

c) Suppose your dad gives you and your sister a square plot of land of side 64 meters. If your sister's share is a square plot of side 36 meters, what area of the land is left for you?
Problem 4

Name______________________________  Sex______

Instruction: Solve the following problem on your own and discuss the solution with your group members. Respond to the questionnaire as honestly as you can. Remember, this is not an examination.

Problem:

You have a plot of garden shown below with its dimensions measured in meters.

\[\begin{array}{c}
\text{2y} \\
\text{y} & \text{6} \\
\text{5} \\
\end{array}\]

a) Find and simplify an expression for the area of the plot.

b) Find and simplify an expression for the perimeter of the plot.

C) What is the ratio area/perimeter? Simplify your result.
Problem 5

Name_____________________________________ Sex_______

Instruction: Solve the following problem on your own and discuss the solution with your group members. Respond to the questionnaire as honestly as you can. Remember, this is not an examination.

Problem:

At a recent Valentine Day party in Vancouver, the host came up with a puzzle for the guests to solve. She told them: "I have three sons. The product of their ages is 72 and the sum of their ages is my car number. If my car number is VTM 014 and my eldest son likes to go fishing with his father, what are the ages of my three boys?" Solve the puzzle.
Problem 6

Name_____________________________________ Sex______

Instruction: Solve the following problem on your own and discuss the solution with your group members. Respond to the questionnaire as honestly as you can. Remember, this is not an examination.

Problem:

Which is larger and why (i) $\frac{3}{2}x^3 + \frac{1}{4}x^2$ or (ii) $21x^4 + \frac{7}{2}x^3$?

Explain your solution.
Problem 7

Name__________________________________ Sex_______

Instruction: Solve the following problem on your own and discuss the solution with your group members. Respond to the questionnaire as honestly as you can. Remember, this is not an examination.

Problem:

For the past hockey season, Vancouver Canucks scored 70 goals in the last 24 games. In 10 of these games, the team scored 2 goals in each game. In the other games they scored 3 or 4 goals. The Canucks won only when they scored 4 goals. How many games did the team win?
Appendix B
(Student-Questionnaire Items)

1. Name______________________ Sex_______

2. Has the group discussion about the solution to the problem helped you to understand the problem and its solution better? In what way? Give specific examples.

3. As you worked in groups, what mathematical concept(s) did you find important for solving the problem?

4. How did you use this (these) concept(s) you identified in (3) to solve the problem? Explain your solution.

5. Were you able to identify the same concept(s) (as in 3 above) when you were solving the problem on your own? If not, how did you learn about these concepts?

6. Why do you think you have solved the problem? Explain.

7. Would you like to see this type of group discussion of the solution to a mathematical problem as part of your normal mathematics classes? Why?
Appendix C

SOME POINTS TO NOTE WHEN YOU WORK IN GROUPS

• Ask questions.
• Discuss ideas.
• Make mistakes.
• Learn to listen to other's ideas.
• Respect other members' ideas.
• Offer constructive criticism.
• You must perceive that you are part of a team and that you have a common goal.
• Success of the group is the success of the Individual.
• Be prepared to talk with all members of the group.
• All members of the group must contribute to the group activity.
### Appendix D

**Frequency of Excerpts Reflecting Categories of Mathematical Activities**

Associated with Student Assessment Standards (SAS)

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**Note.** The columns represent the problems used for the study while the rows represent categories of mathematical activities associated with SAS. An entry in a cell represents the number of excerpts from the discussions of a particular problem that reflect categories of mathematical activities associated with each of SAS.