MULTILATERAL APPROACHES TO THE
THEORY OF INTERNATIONAL COMPARISONS

by

KEIR G. ARMSTRONG

B.Sc., The University of Toronto, 1987
M.A., The University of British Columbia, 1989

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Department of Economics
The University of British Columbia
Vancouver, Canada

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ABSTRACT

The present thesis provides a definite answer to the question of how comparisons of certain aggregate quantities and price levels should be made across two or more geographic regions. It does so from the viewpoint of both economic theory and the "test" (or "axiomatic") approach to index-number theory.

Chapter 1 gives an overview of the problem of multilateral interspatial comparisons and introduces the rest of the thesis.

Chapter 2 focuses on a particular domain of comparison involving consumer goods and services, countries and households in developing a theory of international comparisons in terms of the the (Kontüs-type) cost-of-living index. To this end, two new classes of purchasing power parity measures are set out and the relationship between them is explored. The first is the many-household analogue of the (single-household) cost-of-living index and, as such, is rooted in the theory of group cost-of-living indexes. The second consists of sets of (nominal) expenditure-share deflators, each corresponding to a system of (real) consumption shares for a group of countries. Using this framework, a rigorous exact index-number interpretation for Diewert's "own-share" system of multilateral quantity indexes is provided.

Chapter 3 develops a novel multilateral test approach to the problem at hand by generalizing Eichhorn and Voeller's bilateral counterpart in a sensible manner. The equivalence of this approach to an extended version of Diewert's multilateral test approach is exploited in an assessment of the relative merits of several alternative multilateral comparison formulae motivated outside the test-approach framework.
Chapter 4 undertakes an empirical comparison of the formulae examined on theoretical grounds in Chapter 3 using an appropriate cross-sectional data set constructed by the Eurostat–OECD Purchasing Power Parity Programme. The principal aim of this comparison is to ascertain the magnitude of the effect of choosing one formula over another. In aid of this, a new indicator is proposed which facilitates the measurement of the difference between two sets of purchasing power parities, each computed using a different multilateral index-number formula.
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CHAPTER 1
INTRODUCTION

In a country like Canada, which covers a large territory and has a diverse economy, it is often necessary to compare regional real incomes or consumption levels so that federal transfers to poorer areas can be made according to some definite formula. Similarly, many international organizations require measures of real per-capita GDP which are comparable across their member countries in order to determine desirable directions and magnitudes for aid flows. Measures of this sort are also essential in understanding economic growth, and in investigating the international distribution of world income and the extent of world poverty.

Until as recently as fifteen years ago, no useful worldwide system of consistent international comparisons covering a substantial number of countries was available for general use. The effort to come up with such a system has been led by various national statistical agencies collaborating under the auspices of the United Nations International Comparison Project (ICP). Established in the late 1960s, the ICP's mandate was to bridge a gap in the world's statistical system and thereby enable the conversion of GDP and other national-accounts aggregates of different countries to a common currency in such a way as to make them directly comparable. At that time, anyone requiring such data had to make do with inter-country comparisons calculated using exchange rates as conversion factors.

According to Balassa (1964, p. 586), if "...international differences in productivity are greater in the sector of traded goods than in the non-traded goods sector," the ratio of a currency's relative purchasing power to its exchange rate will be an increasing function of relative per-capita real income. Since relative purchasing power and relative per-capita real income are not independent variables — their product being constrained to equal relative per-

---

1 The first such system, described by Summers et al. (1980), included estimates of real GDP and its components for 119 countries in 1950 and in each of the years between 1960 and 1977.
capita nominal income – this means that, although relative purchasing powers will tend to deviate from exchange rates in a systematic manner, there is no way to provide a general characterization of the former in terms of the latter. Thus, the relative purchasing power of currencies cannot be approximated using exchange rates.

The logic of Balassa's argument proceeds under the following assumptions: (i) capital flows and invisibles do not enter the balance of payments; (ii) there are no barriers to trade; (iii) transport costs are zero for traded goods; (iv) prices equal marginal costs; and (v) labour is mobile within countries but not between them. By assumptions (ii) and (iii), exchange rates will equate the prices of traded goods across countries. By assumption (iv), inter-country wage differences in the traded-goods sector will be positively related to productivity differentials. Assumption (v) implies that the wages of similar types of labour will be equalized within each country. Therefore, if international productivity differences among non-traded goods are smaller than those among traded goods, the former will be relatively more expensive in countries with greater productivity differentials in the production of the latter. Since relative purchasing powers are directly influenced by the prices of non-traded goods but exchange rates are not, the purchasing power of the currency of a low-productivity country relative to that of a high-productivity country will be lower than the corresponding equilibrium rate of exchange. The wider the productivity gap, the larger the differences in wages and non-traded-goods prices and, consequently, the larger the difference between relative purchasing power and equilibrium exchange rate.

Presently, the United Nations, the World Bank, the International Monetary Fund (IMF), the Organization for Economic Co-operation and Development (OECD), and the Statistical Office of the European Communities (Eurostat) publish per-capita real income data for various countries which were calculated using one of two methods favoured by the ICP. Notwithstanding the fact that either of these methods is an enormous improvement over the exchange-rate approach, all three suffer from the same apparent lack of grounding in
economic theory. This is also true of every other multilateral comparison formula developed
to date. Nowhere in the relevant literature is there any strong justification for the use of one
formula over another.

The present thesis investigates the problem of multilateral international comparisons
from the viewpoint of both economic theory and the test (or axiomatic) approach to index—
number theory. It endeavours to give a definite answer to the question of how interspatial
comparisons of aggregate quantities and price levels should be made.

The usual objective of studies of this sort is to facilitate comparisons of real GDP
among all the world’s countries based on all purchases of final—use commodities by economic
agents during a single calendar year. Since there is often a need for quantity comparisons
made on a restricted or altogether different basis, it is desirable that the recommended
procedure be applicable in as many contexts as possible. A flexible basis for making
place—to—place comparisons is established by the following definition: A domain of
comparison is (i) a time period, (ii) a list of final—use commodities, (iii) a list of geographic
regions, (iv) a matrix of positive regional commodity prices, (v) a matrix of regional quantities
purchased, and (vi) a vector of regional purchaser populations.

In its broadest interpretation, this definition could be used as the basis for comparing
standards of living among groups of people. Leisure, family size, length of life, quality of life,
\textit{etc.} could be treated as final—use commodities with associated prices and quantities
"purchased." Moreover, any stratification of society could be treated as a list of "regions"
over which levels of well—being are to be compared. In this way, systematic differences in the
prices faced by different types of people in the same location could be recognized.\textsuperscript{2}

\textsuperscript{2} "Rich" versus "poor," for example.
A natural way to begin thinking about international comparisons is in terms of the cost-of-living index. As shown in Chapter 2, the interspatial interpretation of this concept can be generalized to facilitate comparisons of real private consumption among groups of countries. It should be understood that, by necessarily limiting the domain of comparison to "private consumer goods and services," "countries" and "households" instead of the more general "final-use commodities," "geographic regions" and "purchasers," this approach cannot offer a complete answer to the question posed above. It is, however, an important first step towards doing so.

A novel multilateral test approach to the problem at hand is developed in Chapter 3 by generalizing Eichhorn and Voeller's (1983) bilateral counterpart in a sensible manner. The equivalence of this approach to an extended version of Diewert's (1986) multilateral test approach is exploited in an analysis of the major multilateral comparison formulae discussed in the literature. Two alternatives supported by the results in the second chapter are shown to be superior to the others in the sense of satisfying the most tests.

The final chapter undertakes an empirical comparison of the methods compared on theoretical grounds in Chapter 3. In aid of this, a new indicator is proposed which facilitates the measurement of the difference between the results of two different formulae applied to the same data set. It is discovered that the formulae found to be best from the perspective of the test approach yield results which are substantially different from those generated by other methods.
Any price index between two groups of countries is, by definition, a measure of "purchasing power parity." The most commonly required indexes of this sort involve single-country groups in which commodities are valued using different currencies. As explained in Chapter 1, there is no definite relationship between such a measure and the corresponding exchange rate parity — the most readily observable and seemingly appropriate proxy. How, then, should suitable measures of purchasing power parity be constructed?

In its current form, economic theory offers very little support for any of the practical procedures which have been developed to facilitate international comparisons of purchasing power. This is especially true of multilateral methods — i.e., those applicable to comparisons among two or more countries. The present chapter explores a novel consumer theory approach to the problem at hand with the object of finding an economic basis for one or more of the existing methods or for some heretofore undiscovered alternative.

Three classes of purchasing power parity measures are set out below. The first, developed in Sections 1 and 2, is an interspatial interpretation of the single-household cost-of-living index. The second, developed in Sections 3 and 6, is the many-household analogue of the first and, as such, is rooted in the theory of group cost-of-living indexes. Members of the third class are sets of (nominal) expenditure-share deflators, each corresponding to a system of (real) consumption shares for a group of countries. Sections 4 and 5 establish reasonable generalizations of the cost-of-living index and the Allen consumption index as a consistent pair within this class.
2.1 The Cost–of–Living Index

Consider a bloc comprising $K \geq 2$ countries indexed by the set $K := \{1, \ldots, K\}$. Within this bloc there are $N \geq 2$ well–defined types of consumer goods and services. Let $\mathcal{N} := \{1, \ldots, N\}$ denote the “general list” of these commodities and let $\mathcal{N}_k \subseteq \mathcal{N}$ denote the subset which is available in country $k \in K$. Every country–specific commodity list contains at least two items.\footnote{Formally, $|\mathcal{N}_k| \geq 2$.}

In each country $k \in K$, a representative household is assumed to purchase $x^k_n \geq 0$ units of commodity $n \in \mathcal{N}_k$ at a price of $p^k_n > 0$ country–$k$ currency units ($k$\$, for short).\footnote{Representative households are assumed for the sake of expositional clarity. The need for their existence may be expunged by regarding the bloc as a collection of households rather than as a collection of countries. As such, $\mathcal{K}$ would index the constituent households, $\mathcal{N}_k$ would denote a household–specific commodity list, and $x^k_n$ would be household $k$’s consumption of commodity $n$.} For any good or service $n \in \mathcal{N} \setminus \mathcal{N}_k$ which is unavailable in country $k$, $x^k_n \equiv 0$ and there is sufficient information to estimate a reservation price $p^k_n > 0$.\footnote{Using a hedonic price index, for example. See Griliches (1961)(1967) and Kravis and Lipsey (1971).} Let $x^k := (x^k_1, \ldots, x^k_N)^\prime \in \mathbb{R}^+_N$ denote the representative country–$k$ consumption bundle and let $p^k := (p^k_1, \ldots, p^k_N)^\prime \in \mathbb{R}^+_N$ denote the vector of country–$k$ commodity prices.\footnote{Notation: The “prime” symbol denotes the transpose operator.} The matrix of all commodity prices in the bloc and the corresponding matrix of representative quantities are denoted by $P := (p^1, \ldots, p^K)$ and $X := (x^1, \ldots, x^K)$, respectively.

The tastes of the $k$th (representative) household are represented by a preference ordering $\succeq^k$ defined over the commodity space indexed by $\mathcal{N}$. Letting $x \in \mathbb{R}^N_+$ and $\bar{x} \in \mathbb{R}^N_+$ denote arbitrary consumption bundles, $x \succeq^k \bar{x}$ means that household $k$ considers $x$ to be at least as good as $\bar{x}$. Assuming that $\succeq^k$ is complete, reflexive, transitive, continuous and increasing, there
exists a continuous, increasing function \( U^k : \mathbb{R}^N_+ \to \mathbb{R} \) such that \( U^k(x) \geq U^k(\bar{x}) \) if and only if \( x \succeq^k \bar{x} \). Thus, \( U^k \) is a (direct) utility function corresponding to the preference ordering \( \succeq^k \).\(^5\)

The regularity conditions for \( U^k \) are summarized as

R1. continuity and strict positive monotonicity.

Let \( \mathcal{R}(U^k) \) denote the range of \( U^k \) with its infimum value excluded.\(^6\) Household \( k \)'s expenditure function \( C^k : \mathbb{R}^N_+ \times \mathcal{R}(U^k) \to \mathbb{R} \) shows the minimum expenditure required to attain a given level of utility \( u \) at commodity prices \( p \in \mathbb{R}^N_+ \); i.e.,

\[
C^k(p, u) := \min_{x} \{ p'x \mid U^k(x) \geq u \}.
\] (2.1)

Given that \( U^k \) satisfies R1, \( C^k \) is non-decreasing, positively linearly homogeneous (PLH) and concave in \( p \), increasing in \( u \), jointly continuous in \( (p, u) \) and positive.\(^7\) Whenever \( C^k \) has these properties, it is said to satisfy

R2. positivity, continuity in \( (p, u) \), strict positive monotonicity in \( u \), and positive monotonicity, positive linear homogeneity and concavity in \( p \).

The (Konüs-type) cost-of-living index for household \( k \) is the ratio of the minimum expenditure required to attain a particular utility level under two different price regimes:

\[
\rho^k(p^r, p^s, u) := \frac{C^k(p^s, u)}{C^k(p^r, u)}.
\] (2.2)

Thus, given household \( k \)'s indifference map (or preference ordering) \( \succeq^k \), the household-\( k \) index

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\(^5\) Of course, any increasing transformation of \( U^k \) is also a utility function representing \( \succeq^k \).

\(^6\) Since the commodity space excludes the origin, the expenditure minimization problem defined below does not have a solution when the utility level is at its infimum value.

\(^7\) Proofs of these properties can be found in Diewert (1982).
depends on a base indifference curve \( v \) chosen from that map, a vector of reference prices \( p^r \) and a vector of comparison prices \( p^c \).

There are five basic properties of the household-\( k \) cost-of-living index which follow directly from its definition and R2.

P1. \( \rho^k \) is a positive function; i.e., for every \((p^r, p^c, v) \in \mathbb{R}_{++}^N \times \mathcal{U}(U^k)\), \( \rho^k(p^r, p^c, v) > 0 \).

P2. \( \rho^k \) is non-decreasing in the comparison prices; i.e., for every \((p^r, p^c, \bar{p}^c, v) \in \mathbb{R}_{++}^N \times \mathcal{U}(U^k)\) such that \( p^c < \bar{p}^c \), \( \rho^k(p^r, p^c, v) \leq \rho^k(p^r, \bar{p}^c, v) \).

P3. \( \rho^k \) is PLH in the comparison prices; i.e., for every \((p^r, p^c, \lambda, v) \in \mathbb{R}_{++}^{2N+1} \times \mathcal{U}(U^k)\),
\[
\rho^k(p^r, \lambda p^c, v) = \lambda \rho^k(p^r, p^c, v).
\]

P4. \( \rho^k \) is transitive (or circular) with respect to the reference and comparison prices; i.e., for every \((p^r, \bar{p}, p^c, v) \in \mathbb{R}_{++}^N \times \mathcal{U}(U^k)\),
\[
\rho^k(p^r, \bar{p}, p^c, v) \rho^k(p^r, p^c, v) = \rho^k(p^r, p^c, v).
\]

P5. \( \rho^k \) is concave in the comparison prices; i.e., for every \((p^r, p^c, \bar{p}^c, \lambda, v) \in \mathbb{R}_{++}^N \times (0, 1) \times \mathcal{U}(U^k)\),
\[
\rho^k(p^r, (1-\lambda)p^c + \lambda \bar{p}^c, v) \geq (1-\lambda)\rho^k(p^r, p^c, v) + \lambda \rho^k(p^r, \bar{p}^c, v).
\]

Eight additional properties are implied by one or more of the preceding five.

P6. If the comparison prices are equal to the reference prices then \( \rho^k \) is equal to unity; i.e., for every \((p^r, v) \in \mathbb{R}_{++}^N \times \mathcal{U}(U^k)\), \( \rho^k(p^r, p^r, v) = 1 \).

P7. If the comparison prices are proportional to the reference prices then \( \rho^k \) is equal to the factor of proportionality; i.e., for every \((p^r, \lambda, v) \in \mathbb{R}_{++}^{N+1} \times \mathcal{U}(U^k)\), \( \rho^k(p^r, \lambda p^r, v) = \lambda \).

P8. If the reference prices and the comparison prices are switched, the new \( \rho^k \) is the reciprocal of the old; i.e., for every \((p^r, p^c, v) \in \mathbb{R}_{++}^N \times \mathcal{U}(U^k)\), \( \rho^k(p^c, p^r, v) = 1/\rho^k(p^r, p^c, v) \).
P9. \( \rho^k \) is non-increasing in the reference prices; i.e., for every \((p^r, p^r, p^c, v) \in \mathbb{R}^3_{++} \times \mathcal{U}(U^k)\) such that \( p^r < \bar{p}^r \), \( \rho^k(p^r, p^c, v) \geq \rho^k(\bar{p}^r, p^c, v) \).

P10. \( \rho^k \) is positively homogeneous of degree minus one in the reference prices; i.e., for every \((p^r, p^c, \lambda, v) \in \mathbb{R}^{2N+1}_{++} \times \mathcal{U}(U^k)\), \( \rho^k(\lambda p^r, p^c, v) = \lambda^{-1} \rho^k(p^r, p^c, v) \).

P11. If the reference prices and the comparison prices are multiplied by the same positive scalar, \( \rho^k \) is unaffected; i.e., for every \((p^r, p^c, \lambda, v) \in \mathbb{R}^{2N+1}_{++} \times \mathcal{U}(U^k)\),
\[
\rho^k(\lambda p^r, \lambda p^c, v) = \rho^k(p^r, p^c, v).
\]

P12. \( \rho^k \) is bounded from below and above by, respectively, the smallest and the largest price relative \( \frac{p^e_n}{p^r_n}, n \in \mathcal{N} \); i.e., for every \((p^r, p^c, v) \in \mathbb{R}^{2N+1}_{++} \times \mathcal{U}(U^k)\),
\[
\min_{n \in \mathcal{N}} \left\{ \frac{p^e_n}{p^r_n} \right\} \leq \rho^k(p^r, p^c, v) \leq \max_{n \in \mathcal{N}} \left\{ \frac{p^e_n}{p^r_n} \right\}.
\]

P13. \( \rho^k \) is convex in the reference prices; i.e., for every \((p^r, \bar{p}^r, p^c, \lambda, v) \in \mathbb{R}^{2N+1}_{++} \times (0, 1) \times \mathcal{U}(U^k)\),
\[
\rho^k((1-\lambda)p^r + \lambda \bar{p}^r, p^c, v) \leq (1-\lambda)\rho^k(p^r, p^c, v) + \lambda \rho^k(\bar{p}^r, p^c, v).
\]

The necessary and sufficient conditions for an arbitrary function \( \rho^k : \mathbb{R}^{2N+1}_{++} \to \mathbb{R} \) to be a cost-of-living index are provided by the following theorem.

**Theorem 2.1.** Let \( \rho^k : \mathbb{R}^{2N+1}_{++} \to \mathbb{R} \) satisfy P1-P5 and, for some \( p^r \in \mathbb{R}^N_{++}, \) let \( C^k(p^c, v; p^r) := v \rho^k(p^r, p^c, v) \). Then \( C^k \) is an expenditure function which satisfies R2 and the money metric utility scaling property
\[
C^k(p^r, v) = v \forall v \in \mathbb{R}^N_{++}, \tag{2.3}
\]
and \( \rho^k \) is the cost-of-living index corresponding to the preferences that are dual to \( C^k \).
Conversely, given an expenditure function $C^k : \mathbb{R}^{N+1}_{++} \to \mathbb{R}$ which satisfies $R2$ and (2.3), $\rho^k(p^r, p^e, v) := C^k(p^e, v)/C^k(p^r, v)$ satisfies $P1-P5$ and $C^k(p^e, v) = v\rho^k(p^r, p^e, v)$.\(^8\)

The next theorem asserts that the cost–of–living index is invariant to changes in the dimensionality and/or ordering of prices. It follows from the fact that the introduction of such a change imposes no restrictions on the functional form of $\rho^k$.

**THEOREM 2.2.** Let $\tilde{I}_N$ be a permutation of the columns of the $N \times N$ identity matrix and, for some $\lambda := (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_{++}$, let $\tilde{\lambda}$ be the diagonal matrix with $\tilde{\lambda}_{nn} = \lambda_n$ for all $n \in \mathcal{I}$. Then $\tilde{U}^k : \mathbb{R}^N_{++} \to \mathbb{R}$ defined by

$$\tilde{U}^k(z) := U^k(\tilde{\lambda}\tilde{I}_N z) \tag{2.4}$$

is a utility function representing $z^k$, and $\tilde{\rho}^k : \mathbb{R}^{2N}_{++} \times \mathcal{K}(U^k) \to \mathbb{R}$ defined by (2.2) with $C^k := \tilde{C}^k$, $\tilde{C}^k : \mathbb{R}^{N}_{++} \times \mathcal{K}(U^k) \to \mathbb{R}$ defined by (2.1) with $U^k := \tilde{U}^k$, satisfies

$$\tilde{\rho}^k(\tilde{\lambda}\tilde{I}_N p^r, \tilde{\lambda}\tilde{I}_N p^e, v) = \rho^k(p^r, p^e, v) \tag{2.5}$$

for every $(p^r, p^e, v) \in \mathbb{R}^{2N}_{++} \times \mathcal{K}(U^k)$.\(^9\)

A quantity counterpart to the cost–of–living index can be obtained by using $\rho^k$ as a deflator for household $k$'s expenditure ratio between two different price–utility situations, $(p^r, v_r)$ and $(p^e, v_e)$. More precisely, the implicit household–$k$ (Konüs–type real) consumption index is defined as

$$\tilde{\phi}^k(p^r, p^e, v_r, v_e, v) := \frac{C^k(p^e, v_e)}{C^k(p^r, v_r)} \rho^k(p^r, p^e, v). \tag{2.6}$$

The result of substituting for $\rho^k(p^r, p^e, v)$ using (2.2) and setting $v$ equal to either $v_r$ or $v_e$ is the

---

\(^{8}\) A less concise version of this theorem was established by Diewert (1983, Theorem 1).

\(^{9}\) A version of this theorem was established by Samuelson and Swamy (1974, pp. 571–572).
household—k Allen (1949, p. 199) consumption index at prices \( p := p^e \) or \( p := p^r \):

\[
\phi^k(p, v, v_r) := \frac{C^k(p, v)}{C^k(p, v_r)} .
\]  

(2.7)

Since \( C^k \) is increasing in \( v \), for \( p \) constant, \( C^k(p, v) \) is a (money—metric) utility function representing \( \varepsilon^k \). Thus, \( \phi^k \) is a measure of welfare change for household \( k \) in moving from the reference situation to the comparison situation.\(^{10}\)

2.2 Country—Specific Indexes of Relative Purchasing Power

Setting \( p^r := p^i, j \in K, \) and \( p^e := p^i, i \in K, \) in (2.2) yields the household—\( k \) cost—of—living index associated with the class of international comparisons defined in Chapter 1:

\[
\rho^k(p^i, p^i, v) := \frac{C^k(p^i, v)}{C^k(p^i, v)} .
\]  

(2.8)

The number \( \rho^k(p^i, p^i, v) \) is the factor by which household \( k \)'s purchasing power at country—\( i \) prices must be deflated in order to make it equal to the same household's purchasing power at country—\( j \) prices. Thus, \( \rho^k(p^i, p^i, v) \) is of the dimensionality \( i \$/j \) — the number of units of country \( i \)'s currency per unit of country \( j \)'s.

To distinguish interspatial interpretations of the cost—of—living index from intertemporal ones, it is common practice to refer to the former as purchasing power parities (PPPs). Since there are \( K \) countries in the bloc, (2.8) defines \( K^2 \) PPPs for each of the \( K \) representative households. The \( K \) of these \( (K^2) \) PPPs for which \( p^i = p^j \) can be ignored because, by P6, they are always unity. Half of the \( K(K—1) \) PPPs for which \( p^i \neq p^j \) can be ignored because, by P8, they are the reciprocals of the other half. Each of the remaining

\(^{10}\) \( \phi^k \) is “reference—free” (i.e., independent of \( p \)) if and only if the household's preferences are homothetic. See Blackorby and Donaldson (1983, pp. 379—381) for a proof of this statement.
\[ K(K - 1)/2 \] PPPs corresponds to a distinct bilateral intra-bloc price level comparison. By P4, the \( K - 1 \) country-\( k \) PPPs for which \( j = k \) and \( i \neq k \) constitute a basis for these comparisons.

Clearly, since \( N \geq 2 \), the functional form of the country-\( k \) PPP index \( \rho^k \) depends on the functional form of the country-\( k \) expenditure function \( C^k \) which, in turn, depends on the functional form of the representative country-\( k \) utility function \( U^k \).\(^{11}\) If the functional form of \( U^k \) is unknown then so is that of \( \rho^k \). Under the assumptions made so far, the upper and lower bounds on \( \rho^k \) stated in P12 (with \( p^r := p^i \) and \( p^c := p^j \)) represent the best that can be done.

One way to do better is to assume that the household-\( k \) consumption bundle \( x^k \) is expenditure-minimizing; i.e.,

\[ p^k x^k = C^k(p^k, U^k(x^k)). \tag{2.9} \]

Equation (2.9) suggests what would seem to be the "natural" (although by no means the only) choice of an indifference curve on which to base the country-\( k \) PPP index: The one corresponding to the utility level \( u_k := U^k(x^k) \) attained by the representative country-\( k \) household when facing prices \( p^k \).

The "natural" country-\( k \) PPP index \( \rho^k(p^i, p^j, u_k) \) is of special interest because it can be bounded more tightly than that of the general case. Prior to demonstrating this, it is necessary to introduce the concept of an axiomatic PPP index.

Unlike the economic PPP indexes discussed above, axiomatic PPP indexes ignore household preferences and treat prices and quantities as independent variables. Examples of such indexes include the country-\( k \) Laspeyres PPP index and the country-\( k \) Paasche PPP

\[^{11}\text{For } N = 1, \text{ P7 implies that } \rho^k(p^i, p^j, u) = p^i/p^j.\]
index defined, respectively, by
\[ \rho_L(p^k, p^i, x^k, x^i) := \frac{p^{i^T}x^k}{p^{i^T}x^i} \]  \hspace{1cm} (2.10)
and
\[ \rho_F(p^k, p^i, x^k, x^i) := \frac{p^{i^T}x^i}{p^{i^T}x^i} . \]  \hspace{1cm} (2.11)

**Theorem 2.3** [Pollak (1971, p. 11)]. For all \((k, i) \in K \times K\),
\[ \min_{n \in \mathcal{J}} \left\{ \frac{p^i_n}{p^n_k} \right\} \leq \rho^k(p^k, p^i, u_k) \leq \rho_L(p^k, p^i, x^k, x^i). \]  \hspace{1cm} (2.12)

Note that since
\[
\rho_L(p^k, p^i, x^k, x^i) = \sum_{n=1}^{N} \frac{x^k_n}{p^{i^T}x^k} p^n_i
\]
\[= \sum_{n=1}^{N} \frac{p^k_n x^i_n}{p^{i^T}x^k} \left[ \frac{p^i_n}{p^n_k} \right]\]
\[\leq \max_{n \in \mathcal{J}} \left\{ \frac{p^i_n}{p^n_k} \right\} \text{ since } p^k \in \mathbb{R}^{N}_+, x^k \in \mathbb{R}^{N}_+ \text{ and } \sum_{n=1}^{N} \frac{p^k_n x^k_n}{p^{i^T}x^k} = 1, \]  \hspace{1cm} (2.13)
the upper bound in (2.12) is an improvement upon the one stated in P12 (with \(p^r := p^i\), \(p^c := p^i\) and \(u := u_k\)). The lower bound is the same in both cases.

**Corollary 2.3.1** [Pollak (1971, p. 12)]. For all \((k, i) \in K \times K\),
\[ \rho_F(p^i, p^k, x^i, x^k) \leq \rho^k(p^i, p^k, u_k) \leq \max_{n \in \mathcal{J}} \left\{ \frac{p^k_n}{p^n_i} \right\}. \]  \hspace{1cm} (2.14)

The preceding corollary follows directly from (2.12), the fact that \(\rho_F(p^i, p^k, x^i, x^k) = 1/\rho_L(p^k, p^i, x^k, x^i)\), P8 and P12.
The bounds on $p^k$ can be tightened still further by assuming that the representative households have identical preferences.

**Theorem 2.4** [Konüs (1924, pp. 20–21)]. Let $U : \mathbb{R}_+^N \to \mathbb{R}$ satisfy $R1$ and let $C : \mathbb{R}_+^N \times \mathcal{R}(U) \to \mathbb{R}$ satisfy $R2$. Suppose that $U^k := U$ and $(p^k, x^k)$ satisfies the expenditure minimization property (2.9) with $C^k := C$. Then, for all $(j, i) \in \mathcal{K} \times \mathcal{K}$, there exists a base utility level $v_j^i \in \mathcal{R}(U)$ bounded by $u_j := U(x^j)$ and $u_i := U(x^i)$ such that $\rho(p^j, p^i, v_j^i) := C(p^j, v_j^i)/C(p^i, v_j^i)$ is bounded by $\rho^j_{ij} := \rho^j(p^j, p^i, x^j, x^i)$ and $\rho^i_{ij} := \rho^i(p^j, p^i, x^i, x^j)$; i.e., $\forall (j, i) \in \mathcal{K} \times \mathcal{K}$, $\exists v_j^i \in \min\{u_j, u_i\} \leq v_j^i \leq \max\{u_j, u_i\}$ and

$$
\min\{\rho^j_{ij}, \rho^i_{ij}\} \leq \rho(p^j, p^i, v_j^i) \leq \max\{\rho^j_{ij}, \rho^i_{ij}\} \tag{2.15}
$$

A stronger version of Theorem 2.4 would assert the existence of a $v_j^i$ bounded by $u_j$ and $u_i$ such that $\rho(p^j, p^i, v_j^i)$ is equal to a particular average of $\rho^j_{ij}$ and $\rho^i_{ij}$. Unfortunately, since $\rho(p^j, p^i, v)$ could be close to either one of $\rho^j_{ij}$ and $\rho^i_{ij}$ but not the other for all $v$ between $u_j$ and $u_i$, there is, in general, no such $v_j^i$. If $\rho^j_{ij}$ and $\rho^i_{ij}$ are sufficiently close, however, there exists a $v_j^i$ between $u_j$ and $u_i$ such that $\rho(p^j, p^i, v_j^i)$ is approximately equal to any average of the two.\(^{12}\)

Empirical studies have demonstrated that $\rho^j_{ij}$ and $\rho^i_{ij}$ are usually too far apart to be of much practical use in approximating the "true" PPP index $\rho^i$. Ruggles (1967), for example, compared country-$j$ Paasche and Laspeyres PPP indexes for a bloc consisting of nineteen Central and South American countries and found that the average difference between the two was between thirty-five and forty-eight percent. Similar large disparities between these indexes were found by Kravis et al. (1975) for price level comparisons among OECD countries. Even if this sort of outcome were not the norm, the use of an (unweighted) average of Paasche and Laspeyres indexes to approximate $\rho^i$ would be problematic because no such average is transitive with respect to its component price vectors.

\(^{12}\) E.g., Fisher's (1927) "ideal" PPP index $(\rho^j_{ij} \rho^i_{ij})^{1/2}$.\)
2.3 Bloc-Specific Indexes of Relative Purchasing Power

The primary application of a set of PPPs is in extending the usefulness of national accounts data by making economically meaningful cross-country comparisons or combinations of such data feasible. This is achieved by using the PPPs as nominal-value deflators as in equation (2.6). The results of such calculations are needed for policy purposes and for the purposes of economic analysis by international organizations which exist to further the collective interests of a bloc of countries.

For example, policy decisions regarding desirable levels of intra-bloc aid from "have" to "have-not" countries require a measure of each country's per-capita consumption level relative to some numéraire. None of the country-specific indexes discussed above provide an appropriate basis for such a measure. This is due to the fact that, in general, different country-specific indexes yield different sets of PPPs for the members of the same bloc, and there is no good reason to choose one country's representative household over another's to represent the bloc as a whole. What is required, then, is an index that somehow reflects the preferences of all representative households in the relevant bloc. Under such a requirement, the purchasing power of one national currency relative to another will depend on whether or not some third country is a member of the same bloc. In other words, it will be "bloc specific."

Another way in which the appropriate PPP index might be context dependent is with respect to the nature of the purpose to which it is applicable. Different international organizations involving identical groups of countries may require different sets of PPPs just because they have different objectives. To borrow an example from Pollak (1971, p. 7), "... suppose the U.S. government wants to compare prices in Paris with those in Tokyo to decide on appropriate salary differentials for its diplomats." Quite clearly, such a comparison should be made using a U.S.-specific PPP index. Suppose instead that a certain multinational
corporation wants to make the same comparison to decide on appropriate salary differentials for its sales agents. If these sales agents are drawn from the U.S. and France, say, then some kind of bloc-specific PPP index is called for.

One way to construct such an index is by aggregating over the $K$ instances of the country-specific variety. Indexes defined under this approach in the intertemporal context are called group cost-of-living indexes because they measure the impact on a group of households of moving from one price regime to another and are constructed as weighted averages of single-household cost-of-living indexes. The theory of such index-number formulae was developed by Pollak (1980)(1981) and Diewert (1984).

Since the country-specific index $\rho^k$ is a measure of the relative purchasing power of a single household considered to be representative of all $H_k \geq 1$ households living in country $k \in K$, it may be desirable that a given bloc-specific counterpart take account of size differences among countries by considering the population vector $H := (H_1, \ldots, H_K)'$. Formally, then, a bloc-specific PPP index for country $i$ relative to country $j$ is constructed by choosing a real number $r$ and a set of weights $\alpha \in S^{K-1} \cap \mathbb{R}^+_{++},$ where $S^{K-1} := \{z \in \mathbb{R}^+_+ | \sum_{k=1}^K z_k = 1\}$ denotes the unit simplex of dimension $K-1$, and then applying a mean of order $r$ $M_{r,\alpha} : \mathbb{R}^+_{++} \rightarrow \mathbb{R}$ defined by

$$M_{r,\alpha}(z) := \begin{cases} \left[ \sum_{k=1}^K \alpha_k z_k^r \right]^{1/r} & \text{if } r \in \mathbb{R}\setminus\{0\} \\ \prod_{k=1}^K z_k^{\alpha_k} & \text{if } r = 0 \end{cases}$$

(2.16)

to the $K$ country-specific PPP indexes:

$$\rho(p^i, p^j, \mu, H) := M_{r,\alpha}[\rho^j(p^i, p^j, \mu_i), \ldots, \rho^K(p^j, p^j, \mu_K)] ,$$

(2.17)

where $\mu := (\mu_1, \ldots, \mu_K)' \in \times_{k=1}^K \mathbb{R}(U^j)$ is the vector of representative base utility levels. As a

---

13 Note that $\mu_k$ is not necessarily equal to $u_k := U^k(x^j)$. 

weighted average of the \( \rho^k \)'s, \( \rho \) inherits the dimensionality \( i/j \$ and depends on the preference orderings of all the representative households in the bloc.\(^{14}\)

If, for each \( k \in K \), \( \alpha_k \) is chosen to be the fraction of households living in country \( k \), then (2.17) defines a bloc–specific PPP index which is “democratic” in the sense of assigning weights to the associated country–specific indexes which are increasing in the number of households they represent. Alternatively, if \( \alpha_k \) is chosen to be the bloc expenditure share of the country–\( k \) households — defined in terms of prices \( p^j \), base utility levels \( \mu \) and populations \( H \), say — then (2.17) is “plutocratic” in the sense of giving more weight to \( \rho^k \)'s which represent higher spending.

In general, \( \alpha \) is a function of \( (p^j, p^i, \mu, H) \). Since \( M_{r,\alpha} \) is positive, continuous, non–decreasing and PLH, and since, for every \( r \in \mathbb{R} \), there exists an \( \alpha : \mathbb{R}^2_+ \times \mathcal{K}(U^1) \times \cdots \times \mathcal{K}(U^K) \times \mathbb{R}_+^K \rightarrow S^{K-1} \cap \mathbb{R}_+^K \) such that the right–hand side of (2.17) is transitive with respect to \( p^j \) and \( p^i \), there is an uncountably–infinite number of \( \rho \)s satisfying the first four properties of \( \rho^k \) (P1–P4). These are summarized as

R3. positivity, continuity in \( (p^j, p^i, \mu, H) \), positive monotonicity and positive linear homogeneity in \( p^i \), and transitivity with respect to \( p^j \) and \( p^i \).\(^{15}\)

Further investigation of the \( \rho \)–class of bloc–specific indexes defined by (2.17) is deferred until Section 6. The intervening sections that follow pursue alternative multilateral approaches to the construction of bloc–specific PPP indexes.

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\(^{14}\) Consequently, a more accurate (and cumbersome) notation for the image of this function would be \( \rho(p^j, p^i, \mu, H; \geq^1, \ldots, \geq^K) \).

\(^{15}\) Since P1–P4 implies P9 and P10, R3 implies negative monotonicity and positive homogeneity of degree minus one in \( p^i \).
2.4 The Multilateral–Konüs PPP Index

In this section, a reasonable generalization of $p^k$ is used to specify a PPP index which relates the general or average price level of each country to that of the bloc as a whole. The same outcome is derived indirectly in the next section by generalizing a different type of country–specific index. Using an exact index–number argument, both approaches are then shown to enable the justification of a particular system of axiomatic quantity indexes.

To begin with, let $\epsilon_k \in \mathbb{R}_{++}$ denote the price of a unit of country $k$'s currency ($1 \; k$) in terms of some numéraire. Consequently, the ratio $\epsilon_k/\epsilon_i$ ($i$/$/k$) is country $i$'s exchange rate with respect to country $k$. In the rest of this chapter, $\vec{p}^k := \epsilon_k p^k$ denotes the vector of numéraire–denominated country–$k$ commodity prices and $\vec{p} := (\vec{p}^1, ..., \vec{p}^K)$ denotes the associated matrix. Use of such a normalization allows the summation of household expenditure functions as in the following definition of country $i$'s share of (possibly hypothetical) bloc expenditure at utility levels $\mu$:

$$s^i(\vec{P}, \mu, H) := \frac{H_i C^i(\vec{p}^i, \mu_i)}{\sum_{k=1}^{K} H_k C^k(\vec{p}^k, \mu_k)}.$$ (2.18)

Likewise, it enables the definition of the bloc expenditure function:

$$C(\vec{P}, u, H) := \min_{z^1, ..., z^K} \left\{ \sum_{k=1}^{K} H_k \vec{p}^k z_k \mid U^k(z^k) \geq u_k, k \in K \right\}.$$ (2.19)

$$= \sum_{k=1}^{K} H_k \min_{z_k} \left\{ \vec{p}^k z_k \mid U^k(z^k) \geq u_k \right\}$$

$$= \sum_{k=1}^{K} H_k C^k(\vec{p}^k, u_k).$$ (2.20)

The substitution of $C$ for $C^k$ in (2.8) gives rise to a logical bloc–specific counterpart to the (Konüs–type) country–$k$ PPP index. Specifically, the multilateral-Konüs (MK) PPP index

\[\text{Note that } C \text{ is not a Scitovsky expenditure function since prices are not, in general, equal across countries.}\]
for country $i$ relative to the bloc as a whole is defined as the ratio of the minimum expenditure required to attain utility levels $u := (u_i, ..., u_K)'$ when every representative household faces the prices of country $i$ to the minimum expenditure required to attain the same utility levels when each household faces the prices of its home country:

$$
\delta_{MK}^i(\bar{P}, u, H) := \frac{C(\bar{P}, u, H)}{C(\bar{P}, u, H)}
$$

(2.21)

$$
= \frac{\sum_{j=1}^K H_j C^j(\bar{P}_i, u_j)}{\sum_{k=1}^K H_k C^k(\bar{P}_k, u_k)}, \text{ by (2.20)}.
$$

(2.22)

The number $\delta_{MK}^i(\bar{P}, u, H)$ may be interpreted as the factor by which cost-minimizing bloc expenditure at country–$i$ prices and actual utility levels must be deflated in order to make it equal to nominal bloc expenditure. Thus, the numerator of (2.22) is the sum of (hypothetical) bloc expenditures when the $j$th household ($j \in K$) faces the prices of country $i$, $\bar{P}_i$, and its utility level is held constant at the actual value $u_j$.

Use of (2.18) and (2.8) with $v := u_k$ in conjunction with (2.22) reveals that the MK PPP index is an expenditure-share–weighted sum of country–specific PPP indexes; i.e.,

$$
\delta_{MK}(\bar{P}, u, H) = \sum_{k=1}^K s^k(\bar{P}, u, H) \rho^k(\bar{P}_k, \bar{P}_i, u_k).
$$

(2.23)

Following directly from this fact is a corollary to Theorem 2.3 establishing bounds on $\delta_{MK}^i$.

**Corollary 2.3.2.** For all $i \in K$,

$$
\sum_{j=1}^K s_j \min_{n \in J} \left[ \frac{\rho^i_n}{\bar{P}_i} \right] \leq \delta_{MK}^i(\bar{P}, u, H) \leq \sum_{j=1}^K s_j \rho_j(\bar{P}_j, \bar{P}_i, x^j, x^i),
$$

(2.24)

where $s_j := \bar{P}_j(H_j x^j) / \sum_{k=1}^K \bar{P}_k(H_k x^k)$ denotes the actual bloc expenditure share for country $j$.

Since $\delta_{MK}^i$ is not defined over an arbitrary vector of base utility levels, there is no corollary to
Theorem 2.4 establishing tighter bounds on this index when the representative households have identical tastes.

2.5 The Multilateral–Allen Consumption–Share System

Setting \( k := i, \ u := u_i \) and \( v := \mu_i, \ i \in \mathcal{K} \), in (2.7) yields the country–i Allen consumption index:

\[
\phi^i(p, \mu_i, u_i) := \frac{C^i(p, u_i)}{C^i(p, \mu_i)}.
\] (2.25)

The number \( \phi^i(p, \mu_i, u_i) \) is a measure of household i's consumption at utility level \( u_i \) relative to that at utility level \( \mu_i \) using reference prices \( p \). If \( \mu_i \) is chosen so that \( C^i(p, \mu_i) = C^i(p, u_j) \) for some \( j \in \mathcal{K} \), then \( \phi^i \) is a per–household consumption index for country \( i \) relative to country \( j \). A natural way to generalize this bilateral country–specific measure into a multilateral bloc–specific one is to use it as the basic building block of a system of consumption shares:

\[
\frac{H_i \phi^i(p, \mu_i, u_i)}{\sum_{k=1}^{K} H_k \phi^k(p, \mu_k, u_k)} = \frac{H_i C^i(p, u_i) / C^i(p, \mu_i)}{\sum_{k=1}^{K} H_k C^k(p, u_k) / C^k(p, \mu_k)}
\]

\[
= \frac{H_i C^i(p, u_i) / C^i(p, u_j)}{\sum_{k=1}^{K} H_k C^k(p, u_k) / C^k(p, u_j)}
\] (2.26)

since \( C^k(p, \mu_k) = C^j(p, u_j) \) for all \( k \in \mathcal{K} \).

Thus, the multilateral–Allen (MA) consumption share for country \( i \) is defined as the ratio of the minimum country–i expenditure required to attain per–household utility level \( u_i \) at reference prices \( p \) to the minimum bloc expenditure required to attain per–household utility levels \( u \) at the same prices:
\[ \sigma_{MA}^i(p, u, H) := \frac{H_i C^i(p, u_i)}{\sum_{k=1}^{K} H_k C^k(p, u_k)} . \] (2.27)

The number \( \sigma_{MA}^i(p, u, H) \) is the fraction of total bloc expenditure which would be attributable to country-\( i \) households at reference prices \( p \).

The data set \( P \) admits \( K \) possible choices for the reference-price vector \( p \) in (2.27). If the country-\( i \) price vector \( p^i \) is chosen, \( \sigma_{MA}^i(p^i, u, H) \) is called the MA own-price consumption share for country \( i \). In general, \( \sigma_{MA}(P, u, H) := [\sigma_{MA}^1(p^1, u, H), ..., \sigma_{MA}^K(p^K, u, H)]' \) is only a quasi-consumption-share system since its components do not necessarily sum to unity.

The MA own-price expenditure-share deflator for country \( i \) is a bloc-specific PPP index \( \delta_{MA}^i \) defined implicitly by

\[ \delta_{MA}^i(P, u, H) \sigma_{MA}^i(p^i, u, H) = s^i(P, u, H) . \] (2.28)

The number \( \delta_{MA}^i(P, u, H) \) is the amount by which country \( i \)'s actual expenditure share must be deflated in order to make it equal to the same country's MA own-price consumption share. Since \( s(P, u, H) := [s^1(P, u, H), ..., s^K(P, u, H)]' \in \mathbb{R}^K \),

\[ \sum_{i=1}^{K} \delta_{MA}^i(P, u, H) \sigma_{MA}^i(p^i, u, H) = 1 . \] (2.29)

Using the definitions of \( \delta_{MA}^i, \sigma_{MA}^i \) and \( s^i \), \( \delta_{MA}^i \) can be shown to be equal to the MK PPP index \( \delta_{MK}^i \): From (2.28),

\[
\delta_{MA}^i(P, u, H) = \frac{s^i(P, u, H)}{\sigma_{MA}^i(p^i, u, H)} = \frac{H_i C^i(\bar{p}^i, u_i) \sum_{j=1}^{K} H_j C^j(\bar{p}^i, u_j)}{\sum_{k=1}^{K} H_k C^k(\bar{p}^k, u_k) H_i C^i(\bar{p}^i, u_i)} , \text{ by (2.18) and (2.27)}
\]
Thus, the MA own-price consumption-share system and the MK PPP index are completely consistent with one another. This fact together with (2.23) implies a corollary to Theorem 2.3 establishing bounds on $\sigma_{MA}^i$.

**COROLLARY 2.3.3.** For all $i \in \mathcal{K}$,

$$\left[ \sum_{j=1}^{k} \phi_{L}(\bar{p}, \bar{p}, H_i x^i, H_j x^j) \right]^{-1} \leq \sigma_{MA}^i(p, u, H) \leq \left[ \sum_{j=1}^{k} \frac{S_j}{S_i} \min_{n \in \mathcal{N}} \left\{ \frac{\bar{p}_n^i}{\bar{p}_n^j} \right\} \right]^{-1},$$

(2.31)

where $\phi_{L}(\bar{p}, \bar{p}, H_i x^i, H_j x^j) := \bar{p}^{i''}(H_j x^j)/\bar{p}^{ii}(H_i x^i)$ denotes the country-$i$ Laspeyres consumption index.

In addition to satisfying R1, suppose that $U^k$ is independent of $k$ and (positively) homothetic. Consequently, there exists a PLH function $U : \mathbb{R}^N \to \mathbb{R}$ and a continuously increasing function $\psi : \mathbb{R}(U) \to \mathbb{R}$ such that

$$U^k(x) = \psi(U(x)).$$

(2.32)

Since $\psi$ is continuous and increasing,

$$U(x) = \psi^{-1}(U^k(x)).$$

(2.33)

Thus, $U$ is a utility function representing $\succeq^k$. The associated expenditure function can be written as

$$C(p, \nu) = \nu c(p),$$

(2.34)
where
\[ c(p) := \min_{x} \{ p'x \mid U(x) \geq 1 \} \] (2.35)
is the unit expenditure function for \( U \).

A bilateral axiomatic per-household (real) consumption index for country \( i \) relative to
country \( j \) is a real-valued function \( \phi \) of the observed price and (per-household) quantity data
for the two countries. Such an index is defined to be exact for a PLH utility function \( U \) if, for
every \((p^i, x^i)\) and \((p^j, x^j)\) satisfying (2.9) with \( C^k := C \) and \( U^k := U, \ k \in \{j, i\}, \)
\[ \phi(p^i, p^j, x^i, x^j) = \frac{U(x^i)}{U(x^j)} . \] (2.36)
Diewert (1981, p. 181) noted that the price and quantity vectors in this equation are not
completely independent variables since \((p^k, x^k)_{k \in \{j, i\}}\) is assumed to be consistent with
expenditure-minimizing behaviour.

A bilateral axiomatic PPP index for country \( i \) relative to country \( j \) is a real-valued
function \( \rho \) of \( p^i, p^j, x^i \) and \( x^j \). This type of index is defined to be exact for a PLH utility
function \( U \) (and its dual unit expenditure function \( c \)) if, for every \((p^i, x^i)\) and \((p^j, x^j)\) satisfying
(2.9) with \( C^k := C \) and \( U^k := U, \ k \in \{j, i\}, \)
\[ \rho(p^i, p^j, x^i, x^j) = \frac{c(p^i)}{c(p^j)} . \] (2.37)
Since
\[ \arg \min_{x} \{ p^k'x \mid U(x) \geq U(H_k x^k) \} = H_k \arg \min_{x/H_k} \left\{ p^k' \frac{x}{H_k} \mid U \left( \frac{x}{H_k} \right) \geq U(x^k) \right\} = H_k x^k , \]
equation (2.36) is equivalent to
\[ \phi(p^i, p^j, H_j x^i, H_i x^j) = \frac{U(H_j x^i)}{U(H_i x^j)} . \] (2.38)
Substituting for $C^i := C$ and $C^j := C$ in (2.27) using (2.34) with, respectively, $v := u_i$ and $v := u_j$, and setting $u_k := U(x^k)$ for all $k \in K$ yields

$$
\sigma^i_{MA}(p, U(x^i), \ldots, U(x^K), H) = \left\{ \frac{K}{\sum_{j=1}^{K} \frac{U(H_jx^j)}{U(H_ix^i)}} \right\}^{-1}
$$

(2.39)

$$
= \left\{ \frac{K}{\sum_{j=1}^{K} \left[ \phi(p^j, p^i, H_jx^j, H_ix^i) \right]^{-1}} \right\}^{-1}, \text{ by (2.38)}
$$

(2.40)

where $\hat{H}$ is the $K \times K$ diagonal matrix with $\hat{H}_{kk} = H_k$ for all $k \in K$. Thus, given a bilateral axiomatic per-household consumption index which is exact for a PLH utility function representing the (homothetic) preferences of the (identical) representative households, the un-normalized country-$i$ own share of bloc consumption, $\sigma^i(P, X\hat{H})$, is a direct approximation for the MA consumption index for country $i$. The significance of this result is that it provides a rigorous exact index-number interpretation for Diewert's (1986, p. 25) own-share system of axiomatic quantity indexes.

Substituting for $C^i := C$ and $C^k := C$ in (2.22) using (2.34) with, respectively, $(p, v) := (\bar{p}^i, u_j)$ and $(p, v) := (\bar{p}^k, u_k)$, and setting $u_l := U(x^l)$ for all $l \in K$ yields

$$
\delta^i_{MK}(\bar{P}, U(x^1), \ldots, U(x^K), H) = \left\{ \frac{K}{\sum_{k=1}^{K} \left[ \frac{U(H_kx^k)}{U(H_kx^k)} \right]^{-1}} \right\}^{-1}
$$

(2.41)

Since $\rho$ must be homogeneous of degree zero in each of its quantity arguments in order to satisfy (2.37),

$$
\delta^i_{MK}(\bar{P}, U(x^1), \ldots, U(x^K), H) = \left\{ \frac{K}{\sum_{k=1}^{K} \sigma^k(P, X\hat{H})(\rho(\bar{p}^k, \bar{p}^i, H_kx^k, H_ix^i))^{-1}} \right\}^{-1} \tag{2.42}
$$

follows from (2.41) by (2.40) and (2.37). Therefore, the own-share-weighted harmonic mean

---

17 This number is measured in the "metric" of country $i$. Since real-world data are seldom in accord with the maintained assumptions about household preferences, the metric for one country is not, in general, compatible with that of another. Consequently, further adjustments are necessary in order to ensure that the shares sum to unity.
of the bilateral axiomatic PPP indexes for country $i$ is a direct approximation for the MK PPP index for country $i$ when preferences are homothetic and identical across households.

2.6 Plutocratic and Democratic PPP Indexes

Prais (1959) was the first to note that official group cost–of–living indexes like the Consumer Price Index assign an implicit weight to each constituent household’s consumption pattern which is proportional to its total expenditure. He called such indexes “plutocratic” and suggested an alternative “democratic” variety which treats all households equally. Pollak (1980) formalized these concepts by extending the theory of the (single–household) cost–of–living index to groups. The present section applies this extended theory to the $\rho$–class of bloc–specific PPP indexes and compares the results with those obtained in the intertemporal context.

Under the maintained international–comparisons interpretation, Pollak’s Scitovsky group cost–of–living index becomes the (Prais–Pollak) plutocratic PPP index\(^{18}\) and is defined as the ratio of the minimum bloc expenditure required to attain per–household utility levels $\mu$ at country–$i$ prices to that required at country–$j$ prices:

$$\rho_{PP}(p^i, p^j, \mu, H) := \frac{\sum_{k=1}^{K} H_k C^k(p^i, \mu_k)}{\sum_{l=1}^{L} H_l C^l(p^j, \mu_l)} . \quad (2.43)$$

Using (2.18) with $\bar{P} := (p^i, \ldots, p^j)$ and (2.8) with $v := \mu_k$, (2.43) can be re–written as an expenditure–share–weighted average of the corresponding country–specific PPP indexes:

$$\rho_{PP}(p^i, p^j, \mu, H) = \sum_{k=1}^{K} s^k(p^i, \ldots, p^j, \mu, H) \rho_k(p^i, p^j, \mu_k) . \quad (2.44)$$

Pollak’s democratic group cost–of–living index becomes the additive democratic PPP index\(^{18}\)

---

\(^{18}\) This term is due to Diewert (1984).
and is defined as a population–share–weighted average of the corresponding country–specific
PPP indexes; i.e.,

\[ \rho_{AD}(p^j, p^i, \mu, H) := \sum_{k=1}^{K} \theta^k(H) \rho^k(p^j, p^i, \mu_k) , \]  

(2.45)

where \( \theta^k(H) := H_k/\sum_{i=1}^{K} H_i \) is the country–k bloc population share. As shown in
Diewert (1984), a second type of democratic index can be constructed by replacing the
arithmetic average in (2.45) by its geometric counterpart. The result of doing so is called the
multiplicative democratic PPP index\(^\text{18}\) for country \(i\) relative to country \(j\):

\[ \rho_{MD}(p^j, p^i, \mu, H) := \prod_{k=1}^{K} \left[ \rho^k(p^j, p^i, \mu_k) \right]^{\theta^k(H)} . \]  

(2.46)

In both sorts of democratic PPP index, the use of population–share weights has the effect of
counting every household equally. In contrast, by implicitly weighting each household–
specific PPP index by its total expenditure, the plutocratic variety counts every dollar of
consumption spending equally.

Of the three bloc–specific PPP indexes just defined, \( \rho_{AD} \) is the only one that does not
satisfy R3 as it is not, in general, transitive with respect to \( p^j \) and \( p^i \). Worse still, it does not
even satisfy the weaker property of "country reversal" — the multilateral analogue to P8:

\[ \rho_{AD}(p^j, p^i, \mu, H) \geq \rho_{MD}(p^j, p^i, \mu, H) \]

\[ = 1/\rho_{MD}(p^i, p^j, \mu, H) \]

\[ \geq 1/\rho_{AD}(p^i, p^j, \mu, H) , \]  

(2.47)

where each of the two inequalities follows by the Theorem of the Arithmetic and Geometric
Means and the equality follows by the transitivity and positivity properties of $\rho_{MD}$. The existence of an alternative to $\rho_{AD}$ which is both “democratic” and transitive eliminates the need to consider this index any further. As suggested by the following theorem, any $\rho$-type PPP index satisfying R3 can be provided with an axiomatic characterization similar to that provided for $\rho^k$ in Theorem 2.1.

**Theorem 2.5.** Let $\rho : \mathbb{R}_+^{2N} \times \mathbb{R}_+^{2K} \to \mathbb{R}$ satisfy R3 and, for some $p^r \in \mathbb{R}_+^N$, $\mu_{-k} := (\mu_1, \ldots, \mu_k, \ldots, \mu_K)$, $p^r, \mu_k \in \mathbb{R}_+^K$ and for all $(p^c, \mu_k) \in \mathbb{R}_+^N \times \mathbb{R}_+$, $k \in K$, let $C^k(p^c, \mu_k; p^r, \mu_{-k}) := \mu_k \rho(p^r, p^c, \mu, e^k)$, where $e^k$ is the K-dimensional unit column vector with $e^k_k = 1$. Further, for all $(p^j, p^i, \mu, H) \in \mathbb{R}_+^{2N} \times \mathbb{R}_+^{2K}$, let $\rho$ satisfy

(i) $\rho(p^j, p^i, \mu, H) = \sum_{k=1}^K H_k \mu_k \rho(p^j, p^i, \mu, e^k)$ and $\mu_k = 0 \to \mu_k \rho(p^j, p^i, \mu, e^k) \equiv 0 \forall k \in K$; or

(ii) $\rho(p^j, p^i, \mu, H) = \prod_{k=1}^K [\rho(p^j, p^i, \mu, e^k)]^{\theta_k(H)}$.

Then, for all $k \in K$, $C^k$ is an expenditure function which satisfies R2 and the money-metric utility scaling property (2.3) with $v := \mu_k$ and, depending upon whether it satisfies (i) or (ii), $\rho$ is the plutocratic or multiplicative democratic PPP index for country $i$ relative to country $j$ corresponding to the preferences that are dual to $\{C^k\}_{k \in K}$. Conversely, given expenditure functions $C^k : \mathbb{R}_+^{N+1} \to \mathbb{R}$ which satisfy R2 and (2.3) with $v := \mu_k$ ($k \in K$), (a) $\rho(p^j, p^i, \mu, H) := \sum_{k=1}^K H_k C^k(p^i, \mu_k)/\sum_{k=1}^K H_k C^k(p^i, \mu_k)$ satisfies (i), R3 and

$$C^k(p^i, \mu_k) = \mu_k \rho(p^i, p^i, \mu, e^k);$$

and (b) $\rho(p^j, p^i, \mu, H) := \prod_{k=1}^K [C^k(p^i, \mu_k)/C^k(p^i, \mu_k)]^{\theta_k(H)}$ satisfies (ii), R3 and (2.48).\(^{20}\)

---

\(^{19}\) See Hardy, Littlewood and Pólya (1952, pp. 16–21) for a general statement and proof of this result.

\(^{20}\) A less concise version of this theorem was established by Diewert (1984, Theorems 6 and 10).
By rearranging the terms of the definition of the MA own-price consumption shares multiplied and divided by their sum, the former can be re-expressed as the product of the latter and the harmonic mean of the associated national expenditure ratios, each deflated by the corresponding plutocratic PPP index: From (2.27),

$$
\sigma_{MA}^i(\bar{p}^i, u, H) = \frac{H_i C^i(\bar{p}^i, u_i)}{\sum_{m=1}^{K} H_m C^m(\bar{p}^i, u_m)} \frac{\sum_{k=1}^{K} H_k C^k(\bar{p}^k, u_k) / \sum_{i=1}^{K} H_i C^i(\bar{p}^k, u_i)}{\sum_{k=1}^{K} H_k C^k(\bar{p}^k, u_k)}
$$

$$
= \left[ \sum_{k=1}^{K} \frac{H_k C^k(\bar{p}^k, u_k)}{H_i C^i(\bar{p}^i, u_i)} \frac{\sum_{m=1}^{K} H_m C^m(\bar{p}^i, u_m)}{\sum_{i=1}^{K} H_i C^i(\bar{p}^k, u_i)} \right]^{-1} \frac{\sum_{k=1}^{K} H_k C^k(\bar{p}^k, u_k)}{\sum_{k=1}^{K} H_k C^k(\bar{p}^k, u_k)}
$$

$$
= \left[ \sum_{k=1}^{K} \left[ \frac{H_i C^i(\bar{p}^i, u_i)}{H_k C^k(\bar{p}^k, u_k)} \rho_{PP}(\bar{p}^k, \bar{p}^i, u, H) \right] \right]^{-1} \sum_{k=1}^{K} \sigma_{MA}^i(\bar{p}^k, u, H), \quad (2.49)
$$

Dividing both sides of equation (2.49) by $\sum_{k=1}^{K} \sigma_{MA}^k(\bar{p}^k, u, H)$ reveals that the system of consumption shares that is dual to the plutocratic PPP index is the normalized MA own-price consumption–share system:

$$
\sigma_{PP}^i(\bar{P}, u, H) := \frac{\sigma_{MA}^i(\bar{p}^i, u, H)}{\sum_{k=1}^{K} \sigma_{MA}^k(\bar{p}^k, u, H)} \quad (2.50)
$$

$$
= \left[ \sum_{k=1}^{K} \left[ \frac{H_i C^i(\bar{p}^i, u_i)}{H_k C^k(\bar{p}^k, u_k)} \rho_{PP}(\bar{p}^k, \bar{p}^i, u, H) \right] \right]^{-1} \sum_{k=1}^{K} \sigma_{MA}^k(\bar{p}^k, u, H), \quad (2.51)
$$

by (2.49).

**THEOREM 2.6.** For any $\rho$ satisfying $R3$, the right-hand side of (2.51) with $\rho_{PP} := \rho$ defines a system of consumption shares $\sigma(\bar{P}, u, H) := [\sigma^1(\bar{P}, u, H), \ldots, \sigma^K(\bar{P}, u, H)]' \in S^{K-1}$ which is continuous in $(\bar{P}, u, H)$ and homogeneous of degree zero in each of $\bar{p}^1, \ldots, \bar{p}^{K-1}$ and $\bar{p}^K$. 
COROLLARY 2.6.1. The ratio of per-household consumption shares for countries \( i \) and \( j \) is equal to the per-household consumption index for country \( i \) relative to country \( j \) obtained by using \( \rho \) to deflate the per-household expenditure ratio between the two countries:

\[
\frac{\sigma^i(P, u, H)/H_i}{\sigma^j(P, u, H)/H_j} = \frac{C^i(p^i, u_i)}{C^j(p^j, u_j)} / \rho(p^i, p^j, u, H) \\
=: \tilde{\phi}(p^i, p^j, u_j, u_i, u, H) .
\]

Thus, under R3, any PPP index \( \rho \) and its quantity counterpart \( \tilde{\phi} \) are dual to some system of consumption shares \( \sigma \) and its PPP counterpart \( \tilde{\sigma} \), the \( i \)-th element of which is defined as the amount by which the actual expenditure share \( s_i \) must be deflated in order to make it equal to \( \sigma^i \).

The final result of this section shows that the translog PPP index (defined below) is exact for the multiplicative democratic PPP index evaluated at a particular vector of base utility levels when preferences are dual to a translog expenditure function.

THEOREM 2.7 [Diewert (1976), p. 122]]. Let \( C : \mathbb{R}^{N+1}_+ \to \mathbb{R}_+ \) be a general translog expenditure function defined by

\[
\ln C(p, u) := \alpha + \sum_{n=1}^{N} \beta_n \ln p_n + \beta_0 \ln u + \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \gamma_{mn} \ln p_m \ln p_n \\
+ \sum_{n=1}^{N} \gamma_{0n} \ln p_n \ln u + \frac{1}{2} \gamma_{00} (\ln u)^2 , \tag{2.53}
\]

\( \alpha \in \mathbb{R}_+, \beta_n \in \mathbb{R}_+ \text{ for all } n \in \mathcal{M} \cup \{0\} =: \mathcal{M}, \sum_{m=1}^{N} \beta_n = 1, \gamma_{mn} = \gamma_{nm} \in \mathbb{R}_+ \text{ for all } (m, n) \in \mathcal{M} \times \mathcal{M}, \text{ and } \sum_{n=1}^{N} \gamma_{mn} = 0 \text{ for all } m \in \mathcal{M}. \) For all \( k \in \mathcal{K}, \) suppose that \( (p^k, u_k) \in \mathbb{R}^{N+1}_+ \text{ and } x^k := \nabla_p C(p^k, u_k). \) Then

\[
\rho_{MD}(p^i, p^j, \nu_{ji}, \ldots, \nu_{ji}, H) = \frac{C(p^i, \nu_{ji})}{C(p^j, \nu_{ji})} = \rho_T(p^i, p^i, x^j, x^i) , \tag{2.54}
\]

where
\[ \rho_T(p^i, p^j, x^i, x^j) := \frac{\prod_{n=1}^{N} \left( p_n^i \right) (\omega_n^i + \omega_n^j)/2}{\prod_{n=1}^{N} \left[ p_n^i \right]} \]  

(2.55)

is the country-\( j \) translog PPP index, \( \omega_n^k := p_n^k x_n^k / p^{k'} x^k \) is the \( n^{th} \) household-\( k \) expenditure share and \( \nu_{ji} := (u_j u_i)^{1/2} \).

An obvious weakness of this result is its dependence on preferences being identical across households — a weakness not shared by the intertemporal analogue as comparison with Diewert (1984, Theorem 9) reveals. A similar asymmetry of outcomes is evident in the relative applicability of certain bounds on the plutocratic and additive democratic indexes: The bounding theorems for these indexes in the intertemporal context\(^{21} \) have no direct counterparts in the international one.

2.7 Concluding Remarks

The present chapter has extended the theory of the cost-of-living index into the realm of multilateral international comparisons. Such comparisons can be made from the viewpoint of an individual or from that of a group. Those that reflect the perspective of a group which includes a representative from each country being compared are called "bloc specific." Different classes of bloc-specific indexes can be distinguished by the types of comparisons they facilitate. The dual relationships among four of these — two comprising indexes of relative purchasing power and two comprising indexes of real consumption — were established above. Within this framework, the plutocratic PPP index and multilateral analogues to the Kontüs PPP index and the Allen consumption index were shown to be mutually consistent, and Diewert's (1986) own-share system of axiomatic quantity indexes was shown to be justifiable. In the next chapter, the theory of bloc-specific indexes is used to provide a solid foundation for the development of an appropriate multilateral test approach.

\(^{21}\) See Diewert (1984, Theorems 5 and 7).
CHAPTER 3

A RESTRICTED-DOMAIN MULTILATERAL TEST APPROACH

The economic approach to index number theory pursued in Chapter 2 has a number of limitations. First, in deriving empirically useful results, it relies heavily on separability assumptions about the underlying aggregator functions which are unlikely to be correct. The most objectionable of these is the requirement that tastes or technologies be identical or, at the very least, closely related across countries. Second, in some contexts, the key assumption that agents behave optimally in allocating their available resources may be inappropriate. Finally, implementation of the economic approach may require unobservable \textit{ex ante} expectations about future prices to enable the calculation of rental prices of durable goods.

The test (or axiomatic) approach gets around these problems by focusing exclusively on axiomatic indexes; \textit{i.e.}, those based on \textit{ex post} accounting data which are observable and treated as independent variables. Its ultimate objective is to specify a set of "reasonable" tests (or axioms or requirements) which is sufficient to determine a unique functional form for the index in question. Failing this, the specified tests can provide a basis for assessing the relative merits of alternative formulae motivated outside the test approach framework.

For the most part, the existing literature in this field is concerned with bilateral comparisons.\footnote{See, for example, Fisher (1927), Voeller (1981), and Eichhorn and Voeller (1983).} Working under the auspices of the United Nations International Comparison Project (ICP), Kravis \textit{et al.} (1975, p. 54) were the first to develop a set of tests that is applicable in a multilateral context. The latest version of this set was described by Gerardi (1982). Diewert (1986) proposed a more comprehensive system of multilateral tests and then used it to evaluate a number of different methods for making real output comparisons.
within a bloc of countries. Balk (1989) used Diewert's system to evaluate an additional output–comparison formula.

In the sections that follow, a new framework for making multilateral international comparisons is developed. The various tests that define this framework are set out in Section 1. Many of these tests can be justified as "reasonable" using the fact that they are direct analogues to properties of the cost–of–living index. Further support for the new approach is provided in Section 2 by showing that it is equivalent to an extended version of Diewert's (1986) multilateral test approach. Section 3 analyzes a number of alternative multilateral comparison formulae and establishes the relative superiority of two of them. Section 4 offers some concluding remarks.

3.1 Definitions

The maintained domain of comparison can be characterized as a bloc of countries \( K := \{1, \ldots, K\} \) with household populations \( H := (H_1, \ldots, H_K) \in \mathbb{R}^K_{++} \) a set of consumer goods and services \( N := \{1, \ldots, N\} \) with country–specific prices \( P := (p^1, \ldots, p^K) \in \mathbb{R}^{NK}_{++} \) and a vector of per–household consumption bundles \( X := (x^1, \ldots, x^K) \in \mathbb{R}^{NK}_{++} \). In this chapter, unlike the preceding one, the underlying preferences which generate \( X \) are ignored. Further, the elements of \( P, X \) and \( H \) are treated as independent variables.

From the viewpoint of the typical country–\( k \) household, the vectors \( p^i, p^j \) and \( x^k \) (\( j, i, k \in K \)) constitute the only available information which is relevant to the calculation of the purchasing power parity (PPP) between countries \( i \) and \( j \). Prices outside \( i \) and \( j \) have no bearing on the cost of a commodity bundle in one of these countries relative to the cost of the same bundle in the other. Consumption bundles other than \( x^k \) are generated by preferences which may be very different from those of the typical country–\( k \) household. Thus, it would
appear that the best way to make use of the available data in calculating PPPs which are specific to country $k$ is by means of the formula $\rho_k$ defined by

$$\rho_k(p^j, p^i, x^k) := \frac{p^i x^k}{p^j x^k}.$$  \hfill (3.1)

If the typical country-$k$ household has preferences which admit very little substitution among the commodity types in $\mathcal{J}$, or if the various price vectors are not very different from one another, then this index will be approximately exact.

The most obvious way to think about PPPs which are relevant to the bloc as a whole is as an aggregate of the $K$ country-specific PPPs. To reflect the democratic principle of "one person, one vote," the available population data could be used to provide appropriate weights for the different countries in constructing such an aggregate. Following this logic, a bloc-specific PPP index for country $i$ relative to country $j$ is a function $\rho : \mathbb{R}_{+}^{2N} \times \mathbb{R}_+^{K(N+1)} \to \mathbb{R}$ defined over (i) the price vectors for the pair of countries being compared, (ii) all of the per-household consumption bundles and (iii) the vector of household populations. Since there are $K - 2$ price vectors which are not arguments of this index (but could be, in principle), $\rho$ is called a restricted-domain index. Examples of such indexes are presented in Section 3 below.

The first and second vector of prices over which $\rho$ is defined can be thought of as reference and comparison prices, respectively. Given the results in Chapter 2, it seems reasonable to require that $\rho(p^j, p^i, X, H)$ depend on $p^j$ and $p^i$ in the same way as the (Kontis-type) cost-of-living index $\rho^k(p^*, p^c, u)$ depends on $p^*$ and $p^c$. Accordingly, the first four tests for $\rho$ encompass the direct analogues to properties P1–P4.2

Corresponding to P1 is the requirement that the value of $\rho$ be a positive number. The motivation for this test, called positivity, comes from the fact that the PPP between any two

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2 See Section 1 of Chapter 2.
countries is the number of currency units of the first country needed to buy a commodity bundle equivalent to one that can be bought with a single currency unit of the second country.

P.  **Positivity:** For all \((j, i) \in K \times K\), \(\rho(p^j, p^i, X, H) > 0\).

The analogue to \(P2\), called **positive monotonicity**, requires that an increase in one or more of the comparison prices cause the value of \(\rho\) to increase or remain the same.

M.  **Positive Monotonicity:** For all \((j, i) \in K \times K\) and for all \(\bar{p}^i \in \mathbb{R}^N_{++}\), if \(p^i < \bar{p}^i\) then

\[
\rho(p^j, p^i, X, H) \leq \rho(p^j, \bar{p}^i, X, H).
\]

The \(P3\)-analogue, **linear homogeneity**, requires that a common proportional change in all comparison prices cause the same proportional change in the value of \(\rho\).

H.  **Linear Homogeneity:** For all \((j, i) \in K \times K\) and for all \(\lambda \in \mathbb{R}_{++}\), \(\rho(p^j, \lambda p^i, X, H) = \lambda \rho(p^j, p^i, X, H)\).

The test for \(\rho\) which corresponds to \(P4\) is called **transitivity** or, more traditionally, **circularity**. It requires that the PPP between two countries be equal to the product of the PPP between the first country and any third country and the PPP between the same third country and the second country.

T.  **Transitivity:** For all \((j, i) \in K \times K\) and for all \(l \in K\),

\[
\rho(p^j, p^l, X, H) \rho(p^l, p^i, X, H) = \rho(p^j, p^i, X, H).
\]

Seven additional tests for \(\rho\) follow from the preceding four in the same way that properties \(P6\)–\(P12\) of the cost-of-living index follow from \(P1\)–\(P4\). The first of these additional tests, called **identity**, requires the value of \(\rho\) to be unity if the reference and comparison countries are one and the same.
I. **Identity:** For all $j \in \mathcal{K}$, $\rho(p^j, p^j, X, H) = 1$.

The second implied test for $\rho$, called *proportionality*, asserts that if the result of applying a common proportional change to a country’s prices is compared with its original situation, the value of $\rho$ is the factor of proportionality. Note that this requirement contains I as a special case.

**PP. Proportionality:** For all $j \in \mathcal{K}$ and for all $\lambda \in \mathbb{R}_{++}$, $\rho(p^j, \lambda p^j, X, H) = \lambda$.

The third implied test, *country reversal*, asserts that if the reference and comparison countries are switched, the new value of $\rho$ is the reciprocal of the old.

**CR. Country Reversal:** For all $(j, i) \in \mathcal{K} \times \mathcal{K}$, $\rho(p^j, p^i, X, H) = 1/\rho(p^i, p^j, X, H)$.

The fourth implied test, *negative monotonicity*, is the reference-price counterpart to M. It requires that an increase in one or more of the reference prices cause the value of $\rho$ to decrease or remain the same.

**NM. Negative Monotonicity:** For all $(j, i) \in \mathcal{K} \times \mathcal{K}$ and for all $\bar{p}^j \in \mathbb{R}^N_{++}$, if $p^j < \bar{p}^j$ then $\rho(p^j, p^i, X, H) \geq \rho(\bar{p}^j, p^i, X, H)$.

The fifth implied test, *homogeneity of degree minus one*, is the reference-price counterpart to H. It requires that a common proportional change in all reference prices cause the value of $\rho$ to change by the reciprocal of the factor of proportionality.

**HDM. Homogeneity of Degree Minus One:** For all $(j, i) \in \mathcal{K} \times \mathcal{K}$ and for all $\lambda \in \mathbb{R}_{++}$, $\rho(\lambda p^j, p^i, X, H) = \lambda^{-1} \rho(p^j, p^i, X, H)$.

The sixth implied test, *price dimensionality*, requires that a common proportional change in all reference and comparison prices have no effect on the value of $\rho$. 
PD. Price Dimensionality: For all \((j, i) \in K \times K\) and for all \(\lambda \in \mathbb{R}_{++}\), \(\rho(\lambda p^j, \lambda p^i, X, H) = \rho(p^j, p^i, X, H)\).

The final implication of the four basic tests for \(\rho\), the mean value test, asserts that the value of \(\rho\) lies between the smallest and the largest price relative \(p^i_n/p^j_n, n \in \mathcal{V}\).

MV. Mean Value Test: For all \((j, i) \in K \times K\),

\[
\min_{n \in \mathcal{V}} \left\{ \frac{p^i_n}{p^j_n} \right\} \leq \rho(p^j, p^i, X, H) \leq \max_{n \in \mathcal{V}} \left\{ \frac{p^i_n}{p^j_n} \right\}.
\]

**Theorem 3.1.** Suppose there exists a function \(\rho: \mathbb{R}^N_+ \times \mathbb{R}^{K(N+1)}_+ \to \mathbb{R}\) satisfying \(P\) and \(T\). Then \(\rho\) also satisfies (i) \(I\); (ii) \(PP\) if \(H\) holds; (iii) \(CR\); (iv) \(NM\) if \(M\) holds; (v) \(HDM\) if \(H\) holds; (vi) \(PD\) if \(H\) holds; (vii) \(MV\) if both \(H\) and \(M\) hold.

The direct analogue to the invariance property of the cost–of–living index with respect to the dimensionality of each price and the position of each commodity in the “general list” is encompassed by a pair of tests.\(^3\) The first of these, called commensurability, requires that a change in the unit of measure of each commodity\(^4\) have no effect on the value of \(\rho\).

C. Commensurability: For all \((j, i) \in K \times K\) and for all \(\lambda := (\lambda_1, \ldots, \lambda_N)' \in \mathbb{R}^N_+\),

\[\rho(\lambda p^j, \lambda p^i, \lambda^{-1}X, H) = \rho(p^j, p^i, X, H),\]

where \(\lambda\) is the \(N \times N\) diagonal matrix with \(\lambda_{nn} = \lambda_n\) for all \(n \in \mathcal{V}\).

The second part of the invariance analogue is captured by commodity symmetry: a change in the ordering of the items in the general commodity list has no effect on the value of \(\rho\).

\(^3\) See Theorem 2.2.

\(^4\) Such a change could include measuring the quantity of beer, say, in litres instead of gallons and the associated prices in currency units per litre instead of currency units per gallon.
CS.  *Commodity Symmetry:* For all \((j, i) \in K \times K\) and for any permutation of the columns of the \(N \times N\) identity matrix, denoted by \(\tilde{I}_N\),
\[
\rho(\tilde{I}_Np^j, \tilde{I}_Np^i, \tilde{I}_N X, H) = \rho(p^j, p^i, X, H).
\]

The nature of the dependence of \(\rho\) on the matrix of per-household quantities and the vector of household populations cannot be established by analogy to properties of the cost-of-living index because neither set of variables is in the domain of this (latter) function. Consequently, from a theoretical economic standpoint, no pertinent test for \(\rho\) can be considered to be as desirable as those discussed above. From certain applied standpoints, however, this conclusion may not hold. Political or other non-economic considerations could lead to the prioritization of a particular requirement for \(\rho\) which is not grounded in the economic approach.

One such requirement, *weight symmetry*, precludes the possibility that any country’s total consumption bundle (or weight) plays a special role in the determination of \(\rho\).

WS.  *Weight Symmetry:* For all \((j, i) \in K \times K\) and for any permutation of the columns of the \(K \times K\) identity matrix, denoted by \(\tilde{I}_K\),
\[
\rho(p^j, p^i, X\tilde{I}_K, \tilde{I}_K' H) = \rho(p^j, p^i, X, H).
\]

Another "ungrounded" test lives up to the name *population irrelevance* by granting equal treatment to every country, regardless of size.

PI.  *Population Irrelevance:* For all \((j, i) \in K \times K\) and for all \(\bar{H} \in \mathbb{R}^K\),
\[
\rho(p^j, p^i, X, \bar{H}) = \rho(p^j, p^i, X, H).
\]

An obvious counterpart to the price dimensionality axiom discussed earlier, *quantity dimensionality* requires that a common proportional change in all per-household quantities
together with a possibly different proportional change in all household populations have no effect on the value of $\rho$.

**QD.** *Quantity Dimensionality:* For all $(j, i) \in K \times K$ and for all $(\beta, \gamma) \in \mathbb{R}_+^2$,

$$\rho(p^j, p^i, \beta X, \gamma H) = \rho(p^j, p^i, X, H).$$

A stronger version of this requirement, *strong quantity dimensionality*, states that a common proportional change in the per-household quantities of any country has no effect on the value of $\rho$.

**SQD.** *Strong Quantity Dimensionality:* For all $(j, i) \in K \times K$, for all $l \in K$ and for all $\lambda \in \mathbb{R}_+$,

$$\rho(p^j, p^i, x^1, ..., x^{l-1}, \lambda x^l, x^{l+1}, ..., x^K, H) = \rho(p^j, p^i, X, H).$$

The importance of the distinction between total and per-household quantities implicit in the definition of $\rho$ is assessed by the *total quantities test*. It demands that a change in per-household quantities and populations which is such that all total quantities remain the same have no effect on the value of $\rho$.

**TQ.** *Total Quantities Test:* For all $(j, i) \in K \times K$,

$$\rho(p^j, p^i, X \hat{H}, 1_K) = \rho(p^j, p^i, X, H),$$

where $\hat{H}$ is the $K \times K$ diagonal matrix with $\hat{H}_{kk} = H_k$ for all $k \in K$ and $1_K$ is the $K$-dimensional column vector of ones.

Bilateral versions of the following test have been proposed by several authors, beginning with Fisher (1911).

**D.** *Determinateness:* If any scalar argument in $\rho$ tends to zero then the value of $\rho$ tends to a unique positive real number.
Opinions on the desirability of this requirement are usually expressed in a categorically unequivocal manner. At one extreme is Frisch (1930, p. 405) who "feel[s] a great repugnance against any index which does not satisfy the determinateness test." He justifies his position on practical grounds by adding that "...the withdrawal or entry of any [new] commodity will often have to be performed as a limiting case when either the quantity... or the money value... decreases toward zero, respectively increases from zero." At the other extreme are Samuelson and Swamy (1974, p. 572) who consider the determinateness test to be "...odd... and not at all... desirable... [because] it rules out the non-satiation assumptions often made in standard economic theory" thereby making it impossible for households to derive infinite utility when one or more prices vanish.

Next, three tests are considered which require that the set of PPPs change in a consistent manner as the size of the bloc changes; i.e., they require consistency-in-aggregation. First up is the country partitioning test. It says that if some country \( l \in K \) is partitioned into two new countries, each with the same per-household consumption bundle \( x^l \), then none of the PPPs among the rest of the countries are affected. If, in addition, the two new countries have the same price vector \( p^l \), then each inherits the PPPs of the original country \( l \).

**CP. Country Partitioning Test:** For all \( l \in K \) and for all \( \lambda \in (0, 1) \),

\[
\bar{\rho}(p^j, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_1, H_{l+1}, ..., H_K, \lambda H_l)
= \begin{cases} 
\rho(p^j, p^l, X, H) & \text{if } (j, i) \in (K\setminus\{l\}) \times (K \setminus \{l\}) \\
\rho(p^j, p^l, X, H) & \text{if } (j, i) \in \{l, K+1\} \times (K \setminus \{l\}) \\
\rho(p^j, p^l, X, H) & \text{if } (j, i) \in (K\setminus\{l\}) \times \{l, K+1\} \\
\rho(p^j, p^l, X, H) & \text{if } (j, i) \in \{l, K+1\} \times \{l, K+1\}
\end{cases}
\]

under the additional assumption that \( p^{K+1} = p^l \).

A stronger version of this requirement is the strong country partitioning test. It says that if some country \( l \in K \) is partitioned into two new countries, each with a per-household
consumption bundle which is possibly different from that of the other, then none of the PPPs among the rest of the countries are affected. If, in addition, the two new countries have the same price vector $p^i$, then each inherits the PPPs of the original country $l$.

SCP. **Strong Country Partitioning Test:** For all $l \in \mathcal{K}$ and for all $(x^l, x^{K+1}, \lambda) \in \mathbb{R}_+^{2N} \times (0, 1)$ such that $(1-\lambda)x^l + \lambda x^{K+1} = x^i$,

$$
\bar{\rho}(p^i, p^i, x^l, x^i, x^l, x^{l+1}, ..., x^K, x^{K+1}, H_l, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_i)
$$

$$
= \begin{cases} 
\rho(p^i, p^i, X, H) & \text{if } (j, i) \in (K \setminus \{l\}) \times (K \setminus \{l\}) \\
\rho(p^i, p^i, X, H) & \text{if } (j, i) \in \{l, K+1\} \times (K \setminus \{l\}) \\
\rho(p^i, p^i, X, H) & \text{if } (j, i) \in (K \setminus \{l\}) \times \{l, K+1\} \\
\rho(p^i, p^i, X, H) & \text{if } (j, i) \in \{l, K+1\} \times \{l, K+1\}
\end{cases}
$$

under the additional assumption that $p^{K+1} = p^i$.

The third consistency-in-aggregation requirement, tiny country irrelevance, states that if the population of some country $l \in \mathcal{K}$ tends to zero, the PPPs among the remaining countries tend to those that would prevail if the bloc excluded country $l$ altogether.

TCI. **Tiny Country Irrelevance:** For all $l \in \mathcal{K}$, for all $(j, i) \in (K \setminus \{l\}) \times (K \setminus \{l\})$ and for all $\lambda \in \mathbb{R}_{++}$,

$$
\lim_{\lambda \to 0} \rho(p^j, p^i, X, H, ..., H_{l-1}, \lambda H_l, H_{l+1}, ..., H_K) = \bar{\rho}(p^j, p^i, X_{-l}, H_{-l})
$$

The next axiom is called the product test because it asks that the product of the values of $\rho$ and a bloc-specific per-household consumption index $\phi : \mathbb{R}_+^{2N} \times \mathbb{R}_+^{2N+K(N+1)} \to \mathbb{R}$ be equal to the corresponding per-household expenditure ratio.

PT. **Product Test:** For all $(j, i) \in \mathcal{K} \times \mathcal{K}$,

$$
\rho(p^j, p^i, X, H)\phi(p^j, p^i, x^j, x^i, X, H) = \frac{p^i x^i}{p^j x^j}.
$$

(3.2)
Note that once a functional form is established for \( \rho \), \( \phi \) can be defined implicitly by equation (3.2). In this case, PT is a tautology.

The final axiom considered in this section is a strengthened version of PT. *Factor reversal* says that for any bilateral intra-bloc price level comparison given by \( \bar{p} : \mathbb{R}^{2N} \times \mathbb{R}^{2(N+1)} \rightarrow \mathbb{R} \), if the roles of prices and per-household quantities are reversed, the result can be regarded as the corresponding per-household consumption index.

**FR. Factor Reversal:** For any sub-bloc \( \bar{k} \subseteq \mathcal{K}, |\bar{k}| = 2 \), and for all \( (j, i) \in \bar{k} \times \bar{k} \),

\[
\bar{p}(p^i, p^i, x^j, x^j, H_j, H_i) = \frac{p^j x^i}{p^j x^i}.
\]

Note that FR is not a truly multilateral test since \( \bar{p} \), unlike \( \rho \), is not defined over all per-household quantities and populations. In bilateral contexts, the validity of this requirement has occasionally come into question during the past seventy years because of its lack of intuitive appeal. This is unfortunate because, as the following theorem demonstrates, FR is of critical importance in establishing the axiomatic characterization of bilateral PPP indexes.

**Theorem 3.2 [Funke and Voeller (1978)].** *The bilateral PPP index* \( \bar{p} : \mathbb{R}^{2N} \times \mathbb{R}^{2(N+1)} \rightarrow \mathbb{R} \) *satisfies CR, FR, WS and PI if and only if* \( \bar{p} \) *is the country-\( j \) Fisher PPP index; i.e.,*

\[
\bar{p}(p^i, p^i, x^j, x^j, H_j, H_i) = \left[ \frac{p^i x^j}{p^j x^i} \frac{p^i x^i}{p^j x^j} \right]^{1/2} =: \rho_F(p^i, p^i, x^j, x^j).
\]

### 3.2 Consumption–Share Equivalence

The focus of this section is the translation of Diewert's (1986) multilateral test approach into the maintained domain of comparison. Following a detailed review and extension of the associated set of tests, a subset therefrom is shown to be equivalent to a subset
of the restricted-domain tests developed in the preceding section. This result serves to enhance the validity and usefulness of both approaches.

In order to make it compatible with the test framework established above, Diewert's multilateral system of output indexes is treated as a system of bloc-specific (real) consumption indexes. Any such system is characterized by a function \( \sigma : \mathbb{R}_{++}^{KN} \times \mathbb{R}_{+}^{(N+1)} \to \mathbb{R}^K \) defined over (i) all of the price vectors, (ii) all of the per-household consumption bundles and (iii) the vector of household populations. The \( i \)th element \( (i \in K) \) of the associated image vector \( \sigma(P, X, H) := [\sigma^1(P, X, H), ..., \sigma^K(P, X, H)]' \) is to be interpreted as country \( i \)'s share of total bloc consumption. Desirable properties for \( \sigma \), called share tests, are denoted by \( S1, S2, etc. \)

The first such property is the fundamental share test — so named because it is essential to the interpretation of \( \sigma \) as a system of consumption shares.

**S1. Fundamental Share Test:** \( \sigma^i(P, X, H) > 0 \) for all \( i \in K \) and \( \sum_{i=1}^{K} \sigma^i(P, X, H) = 1 \).

The next share test is called weak proportionality. It says that if all of the price vectors are proportional to one another, all of the per-household quantity vectors are proportional to one another and all of the household populations are equal to one another, then country \( i \)'s share of total bloc consumption is equal to its (common) share in consumption of every item in the general commodity list.

**S2. Weak Proportionality:** For all \( i \in K \), for all \( l \in K \), for all \( \gamma \in \mathbb{R}_{++} \), for all \( \alpha := (\alpha_1, ..., \alpha_K)' \in \mathbb{R}_{++}^K \) and for all \( \beta := (\beta_1, ..., \beta_K)' \in \mathbb{R}_{++}^K \) such that \( \sum_{k=1}^{K} \beta_k = 1 \),

\[
\sigma^i(\alpha_1 p^l, ..., \alpha_K p^l, \beta_1 x^l, ..., \beta_K x^l, \gamma, ..., \gamma) = \beta_i.
\]

A stronger version of this requirement is called proportionality. It says that if any country's per-household quantity vector is multiplied by a positive scalar, then the ratio of the same country's consumption share to that of any other country is equal to the original (pre-
multiplication) consumption–share ratio times the scalar; all other consumption–share ratios remain the same.

S3. **Proportionality:** For all \( l \in K \) and for all \( \lambda \in \mathbb{R}_{++} \),

\[
\sigma^i(P, x^1, \ldots, x^{l-1}, \lambda x^l, x^{l+1}, \ldots, x^K, H) = \begin{cases} 
\frac{\sigma^i(P, X, H)}{1 + (\lambda - 1) \sigma^i(P, X, H)} & \text{if } i \in K \setminus \{l\} \\
\frac{\lambda \sigma^i(P, X, H)}{1 + (\lambda - 1) \sigma^i(P, X, H)} & \text{if } i = l 
\end{cases}
\]

The fourth property, called the *monetary unit test*, states that multiplying each price vector, the matrix of per–household quantities and the population vector by (possibly different) positive scalars has no effect on the consumption share of any country.

S4. **Monetary Unit Test:** For all \( i \in K \), for all \( \alpha := (\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^K_+ \) and for all \( (\beta, \gamma) \in \mathbb{R}^2_+ \),

\[
\sigma^i(\alpha_1 P^1, \ldots, \alpha_K P^K, \beta X, \gamma H) = \sigma^i(P, X, H).
\]

The fifth share test, *commensurability*, requires the consumption shares to be invariant to changes in the units of measure of commodities.

S5. **Commensurability:** For all \( i \in K \) and for all \( \lambda := (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N_+ \),

\[
\sigma^i(\lambda P, \lambda^{-1} X, H) = \sigma^i(P, X, H),
\]

where \( \hat{\lambda} \) is the \( N \times N \) diagonal matrix with \( \hat{\lambda}_{nn} = \lambda_n \) for all \( n \in \mathcal{N} \).

The sixth test is called *country symmetry* because it requires that \( \sigma \) treat the prices and quantities of every country in the same manner.
S6. **Country Symmetry:** For any permutation of the columns of the $K \times K$ identity matrix, denoted by $I_K$, 

$$\sigma(P \tilde{I}_K, X \tilde{I}_K, \tilde{I}_K'H) = \tilde{I}_K'\sigma(P, X, H).$$

The preceding axiom makes the names of countries irrelevant to the determination of consumption shares. **Commodity symmetry** does the same for commodity names.

S7. **Commodity Symmetry:** For all $i \in K$ and for any permutation of the columns of the $N \times N$ identity matrix, denoted by $I_N$, 

$$\sigma^i(I_N P, I_N X, H) = \sigma^i(P, X, H).$$

All three of the following tests for $\sigma$ are consistency—in—aggregation requirements. The **country partitioning test** says that if some country $l \in K$ is partitioned into two new countries, each with the same per—household consumption bundle $x^l$ and the same price vector $p^l$, then none of the consumption shares among the rest of the countries are affected and the consumption—share ratio between the two new countries is equal to the corresponding population ratio.

S8. **Country Partitioning Test:** For all $l \in K$ and for all $\lambda \in (0, 1)$,

$$\tilde{\sigma}^i(P, p^l, X, x^l, H_l, ..., H_{l-1}, (1-\lambda)H_1, H_{l+1}, ..., H_K, \lambda H_l)$$

$$= \begin{cases} 
\sigma^i(P, X, H) & \text{if } i \in K \setminus \{l\} \\
(1-\lambda)\sigma^i(P, X, H) & \text{if } i = l \\
\lambda\sigma^1(P, X, H) & \text{if } i = K + 1 
\end{cases}$$

The second consistency—in—aggregation requirement for $\sigma$, **tiny country irrelevance**, states that if the population of some country $l \in K$ tends to zero, the consumption shares among
the remaining countries tend to those that would prevail if the bloc excluded country \( l \) altogether.

S9. **Tiny Country Irrelevance**: For all \( l \in \mathcal{K} \), for all \( i \in \mathcal{K}\{l\} \) and for all \( \lambda \in \mathbb{R}_{++} \),

\[
\lim_{\lambda \to 0} \sigma^i(P, X, H_{1}, \ldots, H_{l-1}, \lambda H_{l}, H_{l+1}, \ldots, H_{K}) = \sigma^i(P-l, X-l, H-l).
\]

The last of the multilateral tests devised by Diewert (1986) is called *strong dependence on a bilateral formula*. Arguably the least compelling of the consistency-in-aggregation requirements, it asks that the consumption-share ratio between any two countries tend to the value given by some bilateral total-consumption index-number formula as the number of households in the rest of the bloc shrinks to zero.

S10. **Strong Dependence on a Bilateral Formula**: For all \( j \in \mathcal{K} \), for all \( i \in \mathcal{K}\{j\} \) and for all \( \lambda \in \mathbb{R}_{++} \), there exists a function \( \phi : \mathbb{R}^{2N}_{++} \times \mathbb{R}^{2N}_{+} \to \mathbb{R} \) such that

\[
\lim_{\lambda \to 0} \frac{\sigma^i(P, X, \lambda H_{1}, \ldots, \lambda H_{j-1}, H_{j}, \lambda H_{j+1}, \ldots, \lambda H_{l-1}, H_{l}, \lambda H_{l+1}, \ldots, \lambda H_{K})}{\sigma^j(P, X, \lambda H_{1}, \ldots, \lambda H_{j-1}, H_{j}, \lambda H_{j+1}, \ldots, \lambda H_{l-1}, H_{l}, \lambda H_{l+1}, \ldots, \lambda H_{K})}
= \phi(p^j, p^i, H_{j} x^j, H_{i} x^i).
\]

Each of the next five tests is original. The first, *monotonicity*, says that if one or more of the prices in some country are increased, *ceteris paribus*, then the percentage changes in the consumption shares of the other countries are at least as large as the difference between the percentage change in the inflated country's consumption share and the percentage change in its per-household expenditure all divided by the latter percentage change incremented by one.

S11. **Monotonicity**: For all \((j, i) \in \mathcal{K} \times \mathcal{K}\) and for all \(\bar{p}^j \in \mathbb{R}^N_{++}\), if \(p^j < \bar{p}^j\) then

\[
\hat{\sigma}_j \geq \frac{\hat{\sigma}_i - \hat{s}_i}{1 + \hat{s}_i},
\]

where
\[
\hat{\sigma}_j := \frac{\sigma^j(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)}{\sigma(P, X, H)} - 1,
\]
\[
\hat{\sigma}_i := \frac{\sigma^i(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)}{\sigma(P, X, H)} - 1,
\]
and
\[
\hat{s}_i := \frac{\bar{p}^{i'} x^i}{p^{i'} x^i} - 1.
\]

**Theorem 3.3.** \(S1\) and \(S11\) implies \(\hat{\sigma}_i \leq \hat{s}_i\).

By rearranging the terms which result from substituting for \(\hat{\sigma}_i\) and \(\hat{s}_i\) using their respective definitions, the preceding inequality can be interpreted as meaning that an increase in one or more of the prices of country \(i\) causes its expenditure deflator to increase or remain the same:

\[
\frac{H_i p^{i'} x^i}{\sigma^i(P, X, H)} \leq \frac{H_i \bar{p}^{i'} x^i}{\sigma^i(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)}.
\]  \(3.4\)

Since expenditure deflators are implicit PPP indexes, this requirement is clearly analogous to the (positive) monotonicity test for the explicit PPP index of Section 1.

The second new share test, *implicit identity*, asserts that if the prices of any two countries are equal to one another, the consumption–share ratio between them is equal to the corresponding total–expenditure ratio.

\(S12.\) *Implicit Identity:* For all \((j, i) \in K \times K,\)

\[
\frac{\sigma^j(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)}{\sigma^j(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)} = \frac{H_j p^{i'} x^i}{H_j p^{i'} x^i}.
\]

As above, this requirement can be re-stated in terms of expenditure deflators: If one country's prices are the same as another's, then so are their expenditure deflators.
The third new requirement for $\sigma$ is the total quantities test. It asks that a change in per–household quantities and populations which is such that all total quantities remain the same have no effect on the consumption share of any country.

S13. **Total Quantities Test**: For all $i \in K$,

$$\sigma^i(P, X\hat{H}, 1_K) = \sigma^i(P, X, H),$$

where $\hat{H}$ is the $K \times K$ diagonal matrix with $\hat{H}_{kk} = H_k$ for all $k \in K$ and $1_K$ is the $K$–dimensional column vector of ones.

A strengthened version of S8, the strong country partitioning test says that if some country $l \in K$ is partitioned into two new countries, each with the same price vector $p^l$ but possibly different per–household consumption bundles, then none of the consumption shares among the rest of the countries are affected and the consumption–share ratio between the two new countries is equal to the corresponding total–expenditure ratio.

S14. **Strong Country Partitioning Test**: For all $l \in K$ and for all $(\bar{x}^l, x^{K+1}, \lambda) \in \mathbb{R}_+^{2N} \times (0, 1)$ such that $(1-\lambda)\bar{x}^l + \lambda x^{K+1} = x^l$,

$$\sigma^i(P, p^l, x^l, ... , x^{l-1}, \bar{x}^l, x^{l+1}, ..., x^K, x^{K+1}, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l)$$

$$= \begin{cases} 
\sigma^i(P, X, H) & \text{if } i \in K \setminus \{l\} \\
(1-\lambda)[p^l, \bar{x}^l/p^l, x^l]\sigma^i(P, X, H) & \text{if } i = l \\
\lambda[p^l, x^{K+1}/p^l, x^l]\sigma^i(P, X, H) & \text{if } i = K + 1
\end{cases}$$

The last axiom considered in this chapter is the ratio test. It provides a link between the two multilateral test approaches defined above by requiring that the ratio of any two
countries' restricted-domain total consumption indexes\(^5\) be equal to the corresponding consumption-share ratio.

**RT. Ratio Test:** For all \((j, i) \in \mathcal{K} \times \mathcal{K}\) and for all \(k \in \mathcal{K}\),

\[
\frac{\phi_j(p^k, p^i, x^k, x^i, X, H)}{\phi_j(p^k, p^i, x^k, x^i, X, H)} = \frac{\sigma^i(P, X, H)}{\sigma^i(P, X, H)}.
\] (3.5)

Using this axiom together with three others, it is possible to derive the precise mathematical relationship between the consumption-share system \(\sigma\) and the restricted-domain PPP index \(\rho\).

**Lemma 3.1.** Suppose there exists a function \(\rho : \mathbb{R}^N_+ \times \mathbb{R}^{K(N+1)}_+ \to \mathbb{R}\) satisfying \(P\), and a function \(\sigma : \mathbb{R}^{KN}_+ \times \mathbb{R}^{K(N+1)}_+ \to \mathbb{R}\) satisfying \(S1\). Define the function \(\tilde{\phi} : \mathbb{R}^N_+ \times \mathbb{R}^{2N+K(N+1)}_+ \to \mathbb{R}\) implicitly by equation (3.2) and suppose that \((\tilde{\phi}, \sigma)\) satisfies RT. Then

\[
\sigma^i(P, X, H) = \left\{ \sum_{j=1}^{K} \frac{H_j p^j x_j}{H_j p^j x_j} \rho(p^j, p^i, X, H) \right\}^{-1}.
\] (3.6)

If, in addition, \(\rho\) satisfies \(T\) then

\[
\rho(p^i, p^i, X, H) = \frac{H_j p^j x_j}{H_j p^j x_j} \sigma^i(P, X, H).
\] (3.7)

Equation (3.6) enables the derivation of each of the "non-fundamental" share tests \((S2-S14)\) from one or more of the tests for \(\rho\).

**Lemma 3.2.** Suppose there exists a function \(\rho : \mathbb{R}^N_+ \times \mathbb{R}^{K(N+1)}_+ \to \mathbb{R}\) satisfying \(P\), and a function \(\sigma : \mathbb{R}^{KN}_+ \times \mathbb{R}^{K(N+1)}_+ \to \mathbb{R}\) satisfying \(S1\). Define the function \(\tilde{\phi} : \mathbb{R}^N_+ \times \mathbb{R}^{2N+K(N+1)}_+ \to \mathbb{R}\) implicitly by equation (3.2) and suppose that \((\tilde{\phi}, \sigma)\) satisfies RT. Then \(\sigma\) satisfies (i) \(S2\) if \(\rho\) satisfies \(H\) and \(HDM\); (ii) \(S3\) if \(\rho\) satisfies \(SQD\) and \(T\); (iii) \(S4\) if \(\rho\) satisfies \(H\), \(HDM\) and \(QD\); (iv) \(S5\) if \(\rho\) satisfies \(C\); (v) \(S6\) if \(\rho\) satisfies \(WS\); (vi) \(S7\) if \(\rho\) satisfies \(CS\); (vii) \(S8\) if \(\rho\) satisfies

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\(^5\) Recall that indexes of this sort can be defined implicitly in terms of a restricted-domain PPP index by equation (3.2).
CP; (viii) S9 if \( \rho \) satisfies TCI; (ix) S10 if \( \rho \) satisfies TCI, T and TQ; (x) S11 if \( \rho \) satisfies M; (xi) S12 if \( \rho \) satisfies T; (xii) S13 if \( \rho \) satisfies TQ; (xiii) S14 if \( \rho \) satisfies SCP.

The derivation of each of the tests for \( \rho \) – except P, T, PI, D, PT and FR – from one or more of the share tests is enabled by equation (3.7).

**Lemma 3.3.** Suppose there exists a function \( \rho : \mathbb{R}^{2N}_{++} \times \mathbb{R}^{K(N+1)}_{+} \rightarrow \mathbb{R} \) satisfying P and T, and a function \( \sigma : \mathbb{R}^{KN}_{++} \times \mathbb{R}^{K(N+1)}_{+} \rightarrow \mathbb{R}^K \) satisfying S1. Define the function \( \tilde{\phi} : \mathbb{R}^{2N}_{++} \times \mathbb{R}^{2N+K(N+1)}_{++} \rightarrow \mathbb{R} \) implicitly by equation (3.2) and suppose that \((\tilde{\phi}, \sigma)\) satisfies RT. Then \( \rho \) satisfies (i) M if \( \sigma \) satisfies S11; (ii) H if \( \sigma \) satisfies S4; (iii) C if \( \sigma \) satisfies S5; (iv) CS if \( \sigma \) satisfies S7; (v) WS if \( \sigma \) satisfies S6; (vi) QD if \( \sigma \) satisfies S4; (vii) SQD if \( \sigma \) satisfies S3; (viii) TQ if \( \sigma \) satisfies S13; (ix) CP if \( \sigma \) satisfies S8; (x) SCP if \( \sigma \) satisfies S14; (xi) TCI if \( \sigma \) satisfies S9.

Under the hypothesis that \( \sigma \) together with \( \tilde{\phi} \) defined implicitly in terms of \( \rho \) satisfies the ratio test, the next theorem establishes the equivalence of Diewert's (1986) multilateral test approach and that of Section 1 by combining the results presented in Lemmas 3.2 and 3.3.

**Theorem 3.4.** Suppose there exists a function \( \rho : \mathbb{R}^{2N}_{++} \times \mathbb{R}^{K(N+1)}_{+} \rightarrow \mathbb{R} \) satisfying P and T, and a function \( \sigma : \mathbb{R}^{KN}_{++} \times \mathbb{R}^{K(N+1)}_{+} \rightarrow \mathbb{R}^K \) satisfying S1. Define the function \( \tilde{\phi} : \mathbb{R}^{2N}_{++} \times \mathbb{R}^{2N+K(N+1)}_{++} \rightarrow \mathbb{R} \) implicitly by equation (3.2) and suppose that \((\tilde{\phi}, \sigma)\) satisfies RT. Then

(i) \( \sigma \) satisfies S3 if and only if \( \rho \) satisfies SQD;
(ii) \( \sigma \) satisfies S4 if and only if \( \rho \) satisfies H and QD;
(iii) \( \sigma \) satisfies S5 if and only if \( \rho \) satisfies C;
(iv) \( \sigma \) satisfies S6 if and only if \( \rho \) satisfies WS;
(v) \( \sigma \) satisfies S7 if and only if \( \rho \) satisfies CS;
(vi) \( \sigma \) satisfies S8 if and only if \( \rho \) satisfies CP;
(vii) \( \sigma \) satisfies S9 if and only if \( \rho \) satisfies TCI;
(viii) \( \sigma \) satisfies S11 if and only if \( \rho \) satisfies M;

(ix) \( \sigma \) satisfies S12;

(x) \( \sigma \) satisfies S13 if and only if \( \rho \) satisfies TQ;

(xi) \( \sigma \) satisfies S14 if and only if \( \rho \) satisfies SCP.

By stating that two independently developed test approaches imply one another, this theorem reinforces the "reasonableness" of both. It should be understood, however, that such equivalence holds only for a particular class of PPP indexes and a particular class of consumption-share systems. The next lemma shows that the transitivity axiom restricts the admissible \( \rho \) indexes to ratios of national price levels which are independent of foreign prices. The theorem that follows shows that national expenditures deflated by these price levels and then normalized to sum to unity comprise the class of admissible consumption shares. This restriction on \( \sigma \) is a direct consequence of the ratio test.

**Lemma 3.4** [Eichhorn (1978, pp. 156–157)]. The function \( \rho : \mathbb{R}^2_+ \times \mathbb{R}^{(N+1)}_+ \rightarrow \mathbb{R}_+ \) satisfies T if and only if, for some \( \delta : \mathbb{R}^N_+ \times \mathbb{R}^{(N+1)}_+ \rightarrow \mathbb{R}_+ \),

\[
\rho(p^i, p^i, X, H) = \frac{\delta(p^i, X, H)}{\delta(p^j, X, H)} .
\] (3.8)

**Theorem 3.5.** Suppose there exists a function \( \rho : \mathbb{R}^2_+ \times \mathbb{R}^{(N+1)}_+ \rightarrow \mathbb{R} \) satisfying P and T, and a function \( \sigma : \mathbb{R}^K_+ \times \mathbb{R}^{(N+1)}_+ \rightarrow \mathbb{R}^K \) satisfying S1. Define the function \( \bar{\phi} : \mathbb{R}^2_+ \times \mathbb{R}^{2N+(N+1)}_+ \rightarrow \mathbb{R} \) implicitly by equation (3.2) and suppose that \( (\bar{\phi}, \sigma) \) satisfies RT. Then, for some \( \delta : \mathbb{R}^N_+ \times \mathbb{R}^{(N+1)}_+ \rightarrow \mathbb{R}_+ \),

\[
\sigma(P, X, H) = \frac{H_i p^i x^i}{\delta(p^i, X, H)} \left\{ \sum_{j=1}^{K} \frac{H_j p^j x^j}{\delta(p^i, X, H)} \right\}^{-1} .
\] (3.9)

The practical value of consumption-share equivalence is that it enables the evaluation of indexes of the form (3.8) either directly via the axioms of Section 1 or indirectly via those of the present section. Consequently, any admissible restricted-domain PPP index can be
compared with any consumption-share system under the share-test approach. Such comparisons are undertaken at the end of the next section.

3.3 Some Examples

There are many different ways in which the available price and quantity data can be aggregated into a bloc-specific index of relative purchasing power. In this section, twelve such alternatives are introduced and evaluated in light of the foregoing pair of test approaches.

Patterned after the multiplicative democratic PPP index, the household democratic PPP index for country $i$ relative to country $j$ is defined as the population-share-weighted geometric mean of the $K$ country-specific PPP indexes given by (3.1):

$$
\rho_{HD}(p_i, p^j, X, H) := \prod_{k=1}^{K} \left[ \frac{p_i^k X^k}{p^j X^k} \right]^{\theta^k(H)},
$$

(3.10)

where $\theta^k(H) := H_k/\sum_{i=1}^{K} H_i$ denotes the country-$k$ bloc population share. By assigning each country-$k$ PPP index a weight which is proportional to the number of households that it represents, $\rho_{HD}$ affords equal treatment to all households in the bloc.

**Theorem 3.6.** The household democratic PPP index $\rho_{HD}$ satisfies all of the restricted-domain tests except $PI, TQ, SCP$ and $FR$.

**Corollary 3.6.1.** The associated system of consumption shares, $\sigma_{HD}$, defined by (3.6) with $\rho := \rho_{HD}$, satisfies all of the share tests except $S13$ and $S14$.

A weaker democratic aggregation rule would treat countries as equals rather than households. Accordingly, define the country democratic PPP index for country $i$ relative to

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6 See Section 6 of Chapter 2.
country $j$ as the unweighted geometric mean of the $p_k$'s:

$$
\rho_{CD}(p^i, p^j, X, H) := \prod_{k=1}^{K} \left[ \frac{p^i_k x^k}{p^j_k x^k} \right]^{1/K}.
$$

(3.11)

**Theorem 3.7.** The country democratic PPP index $\rho_{CD}$ satisfies all of the restricted-domain tests except CP, SCP and TCI.

**Corollary 3.7.1.** The associated system of consumption shares, $\sigma_{CD}$, defined by (3.6) with $\rho := \rho_{CD}$, satisfies all of the share tests except S8-S10 and S14.

Although $\rho_{CD}$ fails one fewer test than $\rho_{BD}$, the former's shortcomings can easily be seen to be much worse than the latter's. If, for example, the size of the bloc is likely to change over time, the benefit of satisfying PI, TQ and FR will be more than offset by the cost of satisfying none of the consistency-in-aggregation requirements.

The preceding PPP indexes can be regarded as examples of "external average" formulae. In each case, the per-household country-$k$ basket $x^k$ is priced at both $p^i$ and $p^j$ for all $k \in K$, and then an average over the resulting $K$ relative costs is calculated. An alternative methodology along similar lines would be to compute an average over the country-$k$ baskets before doing the costing at $p^i$ and $p^j$. Such an "internal average" formula was once used by the United Nations Economic Commission for Latin America (ECLA) for measuring relative purchasing powers among the countries of Central and South America. Specifically, the ECLA or average basket PPP index for country $i$ relative to country $j$ is defined as the ratio of the cost of the bloc per-household consumption bundle in the two countries being compared:

$$
\rho_{AB}(p^i, p^j, X, H) := \frac{p^{i'} \chi(X, H)}{p^{j'} \chi(X, H)},
$$

(3.12)

where $\chi(X, H) := \sum_{k=1}^{K} \theta_k(H) x^k$ denotes the average household consumption bundle purchased in the bloc. By substituting for $\chi$ and then $\theta_k$ using their respective definitions, (3.12) can be
re-written as the axiomatic analogue to the (Prais–Pollak) plutocratic PPP index:

\[ \rho_{AB}(p^i, p^j, X, H) = \frac{\sum_{k=1}^{K} H_k p^i x^k}{\sum_{l=1}^{L} H_l p^j x^l}. \] (3.13)

**Theorem 3.8.** The average basket PPP index \( \rho_{AB} \) satisfies all of the restricted-domain tests except \( P1, SQD \) and \( FR \).

**Corollary 3.8.1.** The associated system of consumption shares, \( \sigma_{AB} \), defined by (3.6) with \( \rho := \rho_{AB} \), satisfies all of the share tests except \( S3 \).

Since TQ is arguably neither "desirable" nor "undesirable" as a requirement for \( \rho \), comparison of Theorems 3.6 and 3.8 reveals that the relative merit of \( \rho_{AB} \) and \( \rho_{RD} \) depends on the relative desirability of SCP and SQD. However, due to the fact that weaker versions of these tests hold for both indexes, any preference for one over the other is unlikely to be very intense.

Most of the multilateral PPP indexes considered in the literature are not independent of prices outside the countries being compared. In order to accommodate this fact, it is necessary to introduce a class of bloc-specific PPP indexes which is more general than that of Section 1. Accordingly, an unrestricted-domain bloc-specific PPP index for country \( i \) relative to country \( j \) is a function \( \rho^{ji} : \mathbb{R}_{++}^{KN} \times \mathbb{R}_{++}^{K(N+1)} \rightarrow \mathbb{R} \) with image \( \rho^{ji}(P, X, H) \). For a given system of consumption shares \( \sigma \), \( \rho^{ji} \) is defined by the right-hand side of equation (3.7). Clearly, such an index has a restricted domain if and only if there exists a function \( \phi \) such that \( (\phi, \sigma) \) satisfies RT.

Kravis (1984, p. 10) pointed out that early multilateral comparison methods were based on bilateral index-number formulae. The simplest and most popular of these methods
involved the use of the Laspeyres formula in making binary comparisons between a pre-
selected base country and each of the other countries in the bloc. The first use of this sort of
"star system"\(^7\) was by the British Board of Trade (1908–1911) in a series of inquiries into the
costs of living of workers in the major industrial centres of the United Kingdom, Germany,
France, Belgium and the United States. In general, bilateral–formula–based multilateral
comparison methods can depend on any index–number formula of the form \(\phi(p^j, p^i, x^j, x^i)\).
Thus, for a given base country \(k \in K\), the country–k star system of consumption shares is
defined by
\[
\sigma_{k*}(P, X, H) := \frac{H_k \phi(p^k, p^i, x^k, x^i)}{\sum_{i=1}^{K} H_i \phi(p^k, p^i, x^k, x^i)} .
\] (3.14)

Recall that a bilateral PPP index for country \(i\) relative to country \(j\) is a function
\(\bar{\rho} : \mathbb{R}_+^{2N} \times \mathbb{R}_+^{2(N+1)} \to \mathbb{R}\) with image \(\bar{\rho}(p^j, p^i, x^j, x^i, H_j, H_i)\). If \(\bar{\rho}\) satisfies PI then, using
equation (3.2) with \(\rho := \bar{\rho}\) and \(H_j = H_i = 1\), the associated consumption index is defined as
\[
\phi(p^j, p^i, x^j, x^i) := \frac{p^{i} x^{i}}{p^{j} x^{j}} / \bar{\rho}(p^j, p^i, x^j, x^i, 1, 1) .
\] (3.15)

**Theorem 3.9.** Suppose \(\bar{\rho}\) satisfies \(P, M, PP, CR, SQD, C, CS, WS\) and PI. Then the country-k
star system \(\sigma_{k*}\) with \(\phi\) defined by (3.15) satisfies all of the share tests except \(S6, S9, S10, S12\)
and \(S14\). Moreover, \(\rho_{k*}^{ij}\) defined by the right-hand side of (3.7) with \(\sigma := \sigma_{k*}\) is not a
restricted-domain PPP index.

A second multilateral comparison method based on a bilateral formula is known by the
initials of its three independent re–discoverers, Eltető and Köves (1964) and Szulc (1964).

\(^7\) Named for the fact that its graph, constructed by associating nodes with countries and edges
with admissible binary comparisons, looks like a star.
The (generalized) EKS system of consumption shares is defined by
\[
\sigma^i_{EKS}(P, X, H) := \frac{H_i \prod_{k=1}^{K} [H_k^{-1} \phi(p^k, p^i, x^k)]^{1/K}}{\sum_{i=1}^{I} H_i \prod_{j=1}^{K} [H_j^{-1} \phi(p^j, p^i, x^j)]^{1/K}}. \tag{3.16}
\]

**Theorem 3.10.** Suppose \( \tilde{\rho} \) satisfies \( P, M, PP, CR, SQD, C, CS, WS \) and \( PI \). Then the EKS system \( \sigma_{EKS} \) with \( \phi \) defined by (3.15) satisfies all of the share tests except S8-S10, S12 and S14. Moreover, \( \rho^i_{EKS} \) defined by the right-hand side of (3.7) with \( \sigma := \sigma_{EKS} \) is not a restricted-domain PPP index.

A third bilateral–formula–based multilateral comparison method is due to Diewert (1986, p. 25). His own–share system of consumption indexes is defined by
\[
\sigma^i_{OS}(P, X, H) := \frac{H_i \{ \sum_{k=1}^{K} H_k [\phi(p^k, p^i, x^k)]^{-1} \}^{-1}}{\sum_{i=1}^{I} H_i \{ \sum_{j=1}^{K} H_j [\phi(p^j, p^i, x^j)]^{-1} \}^{-1}}. \tag{3.17}
\]

**Theorem 3.11.** Suppose \( \tilde{\rho} \) satisfies \( P, M, PP, CR, SQD, C, CS, WS \) and \( PI \). Then the own-share system \( \sigma_{OS} \) with \( \phi \) defined by (3.15) satisfies all of the share tests except S3, S12 and S14. Moreover, \( \rho^i_{OS} \) defined by the right-hand side of (3.7) with \( \sigma := \sigma_{OS} \) is not a restricted-domain PPP index.

The next three multilateral methods are based on weighted averages of the country–\( k \) star systems. Respectively, the democratic weights, plutocratic weights and quantity weights consumption-share systems are defined by
\[
\sigma^i_{BW}(P, X, H) := \frac{1}{K} \sum_{k=1}^{K} \sigma^i_k(P, X, H), \tag{3.18}
\]

---

8 In the version of this index advanced by Eltető and Köves (1964) and Szulc (1964), the Fisher formula was used in place of \( \phi \). The more general version stated here is due to Gini (1931, p. 12).
\[
\sigma_{kW}(P\hat{\epsilon}, X, H) := \sum_{k=1}^{K} s^k(P\hat{\epsilon}, X, H) \sigma_{k*}^i(P\hat{\epsilon}, X, H)
\]  
(3.19)

and

\[
\sigma_{QW}(P, X, H) := \sum_{k=1}^{K} \sigma_{k*}^i(P, X, H) \sigma_{k*}^i(P, X, H),
\]  
(3.20)

where

\[
s^k(P\hat{\epsilon}, X, H) := \frac{H_k (\epsilon_k p^k)'x^k}{\sum_{l=1}^{K} H_l (\epsilon_l p^l)'x^l}
\]  
(3.21)

is country \(k\)'s share of (nominal) bloc expenditure, \(\epsilon := (\epsilon_1, \ldots, \epsilon_K)'\) is a vector of exchange rates and \(\hat{\epsilon}\) is the \(K \times K\) diagonal matrix with \(\hat{\epsilon}_{kk} = \epsilon_k\) for all \(k \in K\).

**Theorem 3.12.** Suppose \(\bar{p}\) satisfies \(P, M, PP, CR, SQD, C, CS, WS\) and \(PI\). Then (i) the democratic weights system \(\sigma_{DW}\) with \(\sigma_{k*}^i\) defined by (3.14) and \(\phi\) defined by (3.15) satisfies all of the share tests except \(S3, S8-S10, S12\) and \(S14\); (ii) the plutocratic weights system \(\sigma_{PW}\) with \(\sigma_{k*}^i\) defined by (3.14) and \(\phi\) defined by (3.15) satisfies all of the share tests except \(S3, S4, S12\) and \(S14\); and (iii) the quantity weights system \(\sigma_{QW}\) with \(\sigma_{k*}^i\) defined by (3.17), \(\sigma_{k*}^i\) defined by (3.14) and \(\phi\) defined by (3.15) satisfies all of the share tests except \(S3, S12\) and \(S14\). Moreover, \(\rho_{DW}^i\) defined by the right-hand side of (3.7) with \(\sigma := \sigma_{DW}\), \(\rho_{PW}^i\) defined by the right-hand side of (3.7) with \(\sigma := \sigma_{PW}\) and \(\rho_{QW}^i\) defined by the right-hand side of (3.7) with \(\sigma := \sigma_{QW}\) are not restricted-domain PPP indexes.

Returning now to multilateral methods that are not based on a bilateral formula, two additional procedures deserve consideration. The first is a proposal by Geary (1958) which was later amplified by Khamis (1970)(1972); the second is van Ijzeren's (1956) weighted balanced method.
The Geary-Khamis or GK consumption shares are found by solving the following system of equations:

\[ \sigma_i = \sum_{n=1}^{N} \pi_n [H_i x_n^i], \quad i = 1, \ldots, K, \]  
(3.22a)

\[ \pi_n = \frac{\sum_{i=1}^{K} \omega_n^i \sigma_i}{\sum_{k=1}^{K} H_k x_n^k}, \quad n = 1, \ldots, N, \]  
(3.22b)

where \( \omega_n^i := \frac{p_n^i x_n^i}{p^{i'} x^i} \) is the \( n \)-th country-\( i \) per-household expenditure share. Equations (3.22b) define the "international price" of each commodity as the ratio of the per-household expenditure-share-weighted sum of the \( K \) consumption shares to the total quantity consumed. Equations (3.22a) define the share of bloc consumption for each country as the cost of its national basket at international prices.

The \( N + K \) equations (3.22) are not independent since each constituent set implies

\[ \sum_{n=1}^{N} \pi_n \sum_{i=1}^{K} H_i x_n^i = \sum_{i=1}^{K} \sigma_i \]  
(3.23)

and, consequently, at least one non-trivial solution exists. Khamis (1970, Section 3) showed that, subject to any normalization on the \( \sigma_i \)'s, the system consisting of any \( N + K - 1 \) of the equations (3.22) has a unique positive solution. Under the normalization \( \sum_{i=1}^{K} \sigma_i = 1 \), this solution is denoted by \( \sigma^{\text{GK}}(P, X, H) := [\sigma^{1 \text{GK}}(P, X, H), \ldots, \sigma^{K \text{GK}}(P, X, H)]' \).

**Theorem 3.13.** The Geary-Khamis system \( \sigma^{\text{GK}} \) satisfies all of the share tests except \( S3, S12 \) and \( S14 \). Moreover, \( \rho^{i \text{GK}} \) defined by the right-hand side of (3.7) with \( \sigma := \sigma^{\text{GK}} \) is not a restricted-domain PPP index.

The consumption shares associated with van Ijzeren's weighted balanced method are found by solving the following system of equations:
\[ \sum_{k \neq i} \alpha_k \frac{p_i^k x^k}{p_i^{k', x^{k'}}} \frac{H_k}{\sigma_k} \frac{\sigma_i}{\sigma_k} = \sum_{k \neq i} \alpha_k \frac{p_i^k x^k}{p_i^{k', x^{k'}}} \frac{H_i}{\sigma_k} \frac{\sigma_k}{\sigma_i}, \quad i = 1, \ldots, K, \]  

(3.24)

where \( \alpha_k \) is the country-\( k \) "weighting coefficient." If \( \xi_i \equiv p_i^1 (H_i x^i) / \sigma_i, \ldots, \xi_K \equiv p_K^1 (H_K x^K) / \sigma_K \) are called "equivalents," the left-hand side of (3.24) is the number of equivalents that would be required to buy, in country \( i \), the quantities in the weighted national baskets that can be bought for one equivalent in countries \( 1, \ldots, i-1, i+1, \ldots, K \). The right-hand side is the number of equivalents that would be required to buy, in each of countries \( 1, \ldots, i-1, i+1, \ldots, K \), the weighted quantities purchased in country \( i \) for one equivalent. The balanced method asserts that, for \( i = 1, \ldots, K \), these two quantities of money are equal.

Van Ijzeren (1956, pp. 25–27) showed that, subject to any normalization on the \( \sigma_i \)s, the system consisting of any \( K-1 \) of equations (3.24) has a unique positive solution. Under the normalization \( \sum_{i=1}^{K} \sigma_i = 1 \), this solution is denoted by \( \sigma_{VH}(P, X, H) := [\sigma_{VH}^1(P, X, H), \ldots, \sigma_{VH}^K(P, X, H)]' \) if \( \alpha_k := H_k \) and \( \sigma_{VQ}(P, X, H) := [\sigma_{VQ}^1(P, X, H), \ldots, \sigma_{VQ}^K(P, X, H)]' \) if \( \alpha_k := \sigma_k \). The former weighting scheme originates with van Ijzeren (1956, p. 4); the latter with van Ijzeren (1983, p. 45).

**Theorem 3.14.** The population-weighted van Ijzeren system \( \sigma_{VH} \) satisfies all of the share tests except S12-S14; the quantity-weighted van Ijzeren system \( \sigma_{VQ} \) satisfies all of the share tests except S3, S8-S10 and S12-S14. Moreover, \( \rho_{VH}^{ij} \) defined by the right-hand side of (3.7) with \( \sigma := \sigma_{VH} \) and \( \rho_{VQ}^{ij} \) defined by the right-hand side of (3.7) with \( \sigma := \sigma_{VQ} \) are not restricted-domain PPP indexes.

One multilateral comparison method is said to "dominate" another if, in addition to satisfying every potentially desirable share test satisfied by the second method, the first method satisfies at least one other such test. Using this criterion in conjunction with the three corollaries and the final six theorems of this section, a merit–based hierarchy can be
established among the associated methods. Since the EKS system satisfies S3 in addition to satisfying every share test satisfied by the democratic weights system, the former method dominates the latter. Due to the fact that the total quantities test S13 is value-neutral, the democratic weights method neither dominates nor is dominated by van Ijzeren's quantity-weighted balanced method. Consequently, since neither of these methods satisfies S8–S10, both are dominated by the GK, own-share and quantity weights methods in addition to being dominated by the EKS method. By virtue of satisfying S4, the GK, own-share and quantity weights methods dominate the plutocratic weights method as well. In turn, these methods are dominated by the average basket method which satisfies two further tests (S12 and S14).

By virtue of satisfying S12, the country democratic method dominates the EKS method. Since S13 is value-neutral, van Ijzeren’s population-weighted balanced method dominates the EKS method (by S8–S10), the k-star method (by S6, S9 and S10), and the GK, own-share and quantity weights methods (by S3). Similarly, the household democratic method dominates the country democratic method (by S8–S10) and the population-weighted balanced method (by S12). Thus, only the average basket and household democratic methods are undominated.

The hierarchy of multilateral comparison formulae is illustrated by Figure 3.1. Therein, the twelve methods under consideration are grouped in boxes according to the tests they satisfy: Methods satisfying the same tests are contained in the same box; methods satisfying different tests are contained in different boxes. These boxes are arranged so that the vertical distance between any pair of them is proportional to the difference in the number of tests satisfied by the methods inside. The higher up a given method is in the diagram, the more tests it satisfies. The dominance of one method over another is represented by a straight line connecting the boxes that hold them. Each of these lines is labelled with the names of the tests that are satisfied by the methods in the higher box but not by the methods in the lower one.
It is significant that the best multilateral methods from the test-theoretic perspective are those which have interpretations that are firmly rooted in the theory of group cost-of-living indexes. Even methods which are justifiable via an exact index-number argument are dominated by these restricted-domain formulae. The two methods which have such a justification are the own-share system — shown in Chapter 2 to be a direct approximation for the system of multilateral—Allen consumption indexes when based on a bilateral axiomatic per-household consumption index which is exact for a positively linearly homogeneous utility function — and the quantity-weighted balanced method — shown by Diewert (1995) to be exact for homogeneous quadratic utility functions.

3.4 Concluding Remarks

The novel feature of the test approach developed in the early part of this chapter is the imposition of an economically sensible restriction on the price domain of admissible PPP indexes. Consequently, most of the multilateral comparison methods proposed in the literature are summarily ruled out. That this should be the case is reinforced by the fact that, under an extended version of Diewert's (1986) test approach, the best methods are those associated with a restricted-domain PPP index.

Kravis et al. (1975, p. 66) stated that "[e]conomic theory gives no explicit procedure for...[determining PPPs] in the the sense of providing a specific computing algorithm." The present chapter in conjunction with the preceding one have demonstrated that this is not so. What remains to be established is whether or not the choice of one method over another is important from an empirical standpoint. Do different methods yield substantially different PPPs for a given bloc of countries? The next chapter endeavours to answer this question.
CHAPTER 4

THE IMPACT OF ALTERNATIVE FORMULAE

In Chapter 3, twelve different methods for aggregating microeconomic price and quantity data into a bloc–specific index of relative household purchasing power were evaluated in light of a novel test approach. It was determined that ten of these methods are “dominated” by the other two since the tests satisfied by any one of the former comprise a proper subset of the tests satisfied by one or both of the latter. What remains to be shown is that such dominance matters: that the choice of one formula over another can have a substantial impact on the resulting international comparisons.

The question of how to compare multilateral purchasing–power–parity formulae from an empirical standpoint has received scant attention in the literature. If the formulae under consideration satisfy a certain minimal set of requirements, then the application of any one of them to a bloc consisting of \( K \) countries yields a vector of \( K - 1 \) numbers which can serve as a basis for all possible binary comparisons within the bloc. The universal means by which two such vectors have been compared in the past has been an assessment of the component–wise percentage differences between them.\(^1\) This approach is unsatisfactory for a couple of reasons. First, the percentage difference between two numbers is an asymmetric indicator of the relative difference between them because it depends on which number is used as the point of comparison. To paraphrase an example from Törnqvist et al. (1985), 250 is twenty–five percent more than 200, or 200 is twenty percent less than 250. Second, component–wise comparisons between two vectors are unlikely to give rise to a very accurate assessment of the overall difference between them unless the components are few in number or the calculated differences exhibit little variation in size.

\(^1\) See, for example, Kravis et al. (1975, ch. 1 and 5) and Ruggles (1967).
Section 1 proposes a new index of the difference between the results of two multilateral comparison methods applied to the same data set. Based on the symmetric and additive log(arithmetic) difference indicator, this index overcomes the problems mentioned above to provide an appropriate summary measure of the differences between the purchasing power parities (PPPs) or consumption shares associated with the two methods. Section 2 describes the data used in Section 3 to undertake an empirical comparison of the methods compared on theoretical grounds in the preceding chapter. Section 4 concludes by explaining why different sources disagree on the values of the same PPPs.

4.1 A Summary Measure of the Differences between Alternative Formulae

Recall that the maintained domain of comparison consists of a bloc of countries \( K := \{1, \ldots, K\} \) with household populations \( H := (H_1, \ldots, H_K)' \in \mathbb{R}_+^K \), a set of consumer goods and services \( \mathcal{N} := \{1, \ldots, N\} \) with country–specific prices \( P := (p_1, \ldots, p^K) \in \mathbb{R}_+^{NK} \) and a vector of per–household consumption bundles \( X := (x^1, \ldots, x^K) \in \mathbb{R}_+^{NK} \). As in Chapter 3, the elements of \( P, X \) and \( H \) are treated as independent variables.

An unrestricted–domain (axiomatic) bloc–specific PPP index for country \( i \) relative to country \( j \) is a function \( \rho^i_j : \mathbb{R}_+^{KN} \times \mathbb{R}_+^{K(N+1)} \to \mathbb{R} \) with image \( \rho^i_j(P, X, H) \). It is assumed that, at the very least, this index is positive and transitive with respect to \( j \) and \( i \). The positivity requirement enables the usual interpretation of \( \rho^i_j(P, X, H) \) as the number of country–\( i \) currency units needed to buy a commodity bundle equivalent to one that can be bought with a single country–\( j \) currency unit.

P. \( \text{Positivity: For all } (j, i) \in K \times K, \rho^i_j(P, X, H) > 0. \)

The transitivity requirement guarantees that the results of applying \( \rho^i_j \) to a bloc comprising three or more countries are self–consistent.
T. Transitivity: For all \((j, i) \in K \times K\) and for all \(l \in K\),

\[\rho^j(P, X, H)\rho^i(P, X, H) = \rho^{ji}(P, X, H).\]

In addition to satisfying T, a self-consistent set of PPPs has two further properties. The first, called weak identity, requires the value of \(\rho^{ji}\) to be unity when \(i = j\).

WI. Weak Identity: For all \(j \in K\), \(\rho^{ji}(P, X, H) = 1\).

The second, country reversal, asserts that the value of \(\rho^{ij}\) is the reciprocal of the value of \(\rho^{ji}\).

CR. Country Reversal: For all \((j, i) \in K \times K\), \(\rho^{ij}(P, X, H) = 1/\rho^{ji}(P, X, H)\).

**Theorem 4.1.** If \(\rho^{ji}\) satisfies P and T then it also satisfies WI and CR.

Consider two sets of bloc-specific PPPs, \(A\) and \(B\), each computed using a different multilateral index-number formula satisfying P and T. For ease of exposition, let the \(K\)-dimensional square matrices \((\rho_A^{ji})\) and \((\rho_B^{ji})\) represent the elements of \(A\) and \(B\), respectively. Since P together with T implies WI so that \(\rho_A^{ij} = \rho_B^{ij} = 1\) for all \(j \in K\), there are up to \(K^2 - K\) possible differences between these matrices. Define the mean absolute log difference between \((\rho_A^{ji})\) and \((\rho_B^{ji})\) as the sum of the absolute log differences between corresponding off-diagonal elements divided by their number:

\[
\Delta_{A,B} = \frac{\sum_{j=1}^{K} \sum_{i \neq j} |\ln(\rho_B^{ji}/\rho_A^{ij})|}{K(K-1)}. \tag{4.1}
\]

Since both \(A\) and \(B\) are transitive, for any \(h \in K\), (4.1) can be re-written as

\[
\Delta_{A,B} = \frac{\sum_{j=1}^{K} \sum_{i \neq j} |\ln[(\rho_B^{jh}\rho_A^{hi})(\rho_A^{jk}\rho_B^{kj})]/(\rho_A^{jh}\rho_B^{ki})]|}{K(K-1)}
\]
\[
\frac{2}{K(K-1)} \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \left| \ln \left( \frac{\rho_{A}^{j}}{\rho_{A}^{i}} \right) - \ln \left( \frac{\rho_{B}^{j}}{\rho_{B}^{i}} \right) \right|, \text{ by CR.}
\] (4.2)

For a particular bloc of countries, let \( \mathcal{P} := \{A, B, \ldots\} \) denote the set of all sets of PPPs which satisfy \( P \) and \( T \).

**THEOREM 4.2.** \( \Delta \) defined by the right-hand side of (4.2) is a metric on \( \mathcal{P} \); i.e., \( \Delta \ldots \) is a real-valued function on \( \mathcal{P} \times \mathcal{P} \) which satisfies (i) \( \Delta_{A,B} \geq 0 \); (ii) \( \Delta_{A,B} = 0 \) if and only if \( A = B \); (iii) \( \Delta_{A,B} = \Delta_{B,A} \); and (iv) \( \Delta_{A,B} \leq \Delta_{A,C} + \Delta_{C,B} \) (triangle inequality).

Thus, \( \Delta \) possesses the most important properties of ordinary distance in \( \mathbb{R}^3 \) making it a reasonable and intuitive measure of the difference between alternative sets of PPPs.

Table 4.1 contains PPPs calculated by the Organization for Economic Cooperation and Development (OECD) using both the Eltető–Kőves–Szulc (EKS) method and the Geary–Khamis (GK) method.\(^2\) For comparison, the corresponding exchange rates are also included. The differences among these three sets of numbers can be summarized by computing the associated \( \Delta \) values using equation (4.2): \( \Delta_{EKS,GK} = 0.04773 \), \( \Delta_{EKS,ER} = 0.28832 \) and \( \Delta_{GK,ER} = 0.30469 \). Thus, the OECD PPPs differ from one another by about 4.8 percent and from the corresponding exchange rates by roughly thirty percent.

A system of bloc–specific (real) consumption indexes for countries \( 1, \ldots, K \) is a function \( \sigma: \mathbb{R}_{++}^{K+} \times \mathbb{R}_{++}^{K(N+1)} \to \mathbb{R} \) with image \( \sigma(P, X, H) := [\sigma^{1}(P, X, H), \ldots, \sigma^{K}(P, X, H)]' \). To enable the \( i \)th element (\( i \in K \)) of this system to be interpreted as country \( i \)'s share of total bloc consumption, \( \sigma \) is required to satisfy

S1. **Fundamental Share Test:** \( \sigma^{i}(P, X, H) > 0 \) for all \( i \in K \) and \( \sum_{i=1}^{K} \sigma^{i}(P, X, H) = 1 \).

\(^2\) See Section 3 of Chapter 3.
THEOREM 4.3. If \( \sigma \) satisfies S1 then \( \rho^\pi \) defined implicitly by

\[
\rho^\pi(P, X, H) = \frac{\sigma^i(P, X, H)}{\sigma^j(P, X, H)} = \frac{H_i p^i x^i}{H_j p^j x^j}
\]

(4.3)

satisfies \( P \) and \( T \), and

\[
\sigma^i(P, X, H) = \left\{ \sum_{j=1}^{K} \frac{H_i p^i x^i}{H_j p^j x^j} \rho^\pi(P, X, H) \right\}^{-1}.
\]

(4.4)

Under the assumptions of this theorem, the number \( \rho^\pi(P, X, H) \) is the amount by which the total bloc expenditure of country–i households relative to those of country \( j \) must be deflated in order to make it equal to the corresponding total consumption ratio.

Substituting for \( \rho^\pi \) in (4.2) using (4.3) yields an equivalent expression for the mean absolute log difference between multilateral comparison methods \( A \) and \( B \):

\[
\Delta_{A,B} = \frac{2}{K(K-1)} \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \left| \ln \left( \frac{\sigma^j_B}{\sigma^j_A} \right) - \ln \left( \frac{\sigma^j_A}{\sigma^j_B} \right) \right|.
\]

(4.5)

Thus, \( \Delta_{A,B} \) can be calculated from associated basis sets of PPPs using (4.2) or from the associated consumption–share systems using (4.5).

4.2 The Data

The raw price and expenditure data used in the empirical work of the next section are those of the Eurostat–OECD PPP Programme. These data cover the bloc comprising the twenty–four OECD countries of 1990 and the general commodity list made up of the 158 basic headings\(^3\) of the major aggregate called “Final Consumption of Resident Households.”

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\(^3\) In principle, a basic heading consists of a small group of similar well–defined goods or services. In practice, it is the lowest level of classification for which expenditures can be estimated. Consequently, an actual basic heading can cover a broader range of commodities than is theoretically desirable.
Let \( V := (v^k_n) \) denote the \((158 \times 24)\) matrix of national expenditures (in national currency units) at the basic heading level, and let \( \bar{P} := (\bar{p}^k_n) \) denote the corresponding matrix of basic-heading PPPs in national currency units per U.S. dollar. Hence, for all \( n \in \mathcal{N} \) and for all \( k \in \mathcal{K} \),

\[
v^k_n \equiv H_k p^k_n x^k_n \tag{4.6}
\]

and

\[
\bar{p}^k_n \equiv p^k_n / p^U_n. \tag{4.7}
\]

Several different sources were employed in the determination of the 1990 household population data \((H)\) presented in Table 4.2. For the United States and Japan, Turkey, and each of the Nordic countries excluding Iceland,\(^4\) the corresponding datum was furnished by, respectively, the United Nations (1993, Table 3), the State Institute of Statistics (1993, p. 92), and the Nordic Statistical Secretariat (1994, p. 168). The raw data used to form estimates of \( H_k \) for Austria, Switzerland and Iceland were provided by the United Nations (1993, Table 3) and the OECD (1993a, pp. 156–157) (1994, p. 210) whereas those used in the cases of Portugal and the Netherlands were provided by Eurostat (1993, pp. 121, 137) and the OECD (1994, p. 210). Eurostat (1993, pp. 121, 137), OECD (1993a, p. 157) (1994, p. 210) and United Nations (1993, Table 3) afforded a basis for the interpolation of \( H_k \) in each of the remaining countries of the European Community.\(^5\) Similar estimations for Canada, Australia and New Zealand were carried out using data from Statistics Canada (1992, p. 125), the Australian Bureau of Statistics (1993, p. 120), the United Nations (1993, Table 3) and the OECD (1993a, p. 157) (1994, p. 210).

\(^4\) Viz., Denmark, Finland, Norway and Sweden.

\(^5\) Viz., Belgium, France, Germany, Greece, Ireland, Italy, Luxembourg, Spain and the United Kingdom.
For any scale factor $\beta \in \mathbb{R}_{++}$, a matrix $\bar{X} := (\bar{x}_n^k)$ of scaled per-household quantities consistent with $(V, \bar{P}, H)$ is defined by

$$\bar{x}_n^k := \frac{\beta v_n^k}{\bar{p}_n^k h_k}.$$  \hfill (4.8)

$$\equiv \beta p_n^{DUS} x_n^k, \text{ by (4.6) and (4.7)}.$$  \hfill (4.9)

In the present chapter, a $\beta$-value of 100,000 was used in the construction of $\bar{X}$.

4.3 Empirical Results

The calculation of PPPs based on the data set $(\bar{P}, \bar{X}, H)$ can only be accomplished by means of formulae which, in addition to satisfying $P$ and $T$, satisfy two other axioms. The first of these, called quantity dimensionality with respect to $X$, requires that a common proportional change in all per-household quantities have no effect on the value of $\rho^\ddagger$.

QDX. Quantity Dimensionality with Respect to $X$: For all $(j, i) \in \mathcal{K} \times \mathcal{K}$ and for all $\beta \in \mathbb{R}_{++}$,

$$\rho^\ddagger(P, \beta X, H) = \rho^\ddagger(P, X, H).$$

The second, commensurability, requires that a change in the unit of measure of each commodity have no effect on the value of $\rho^\ddagger$.

C. Commensurability: For all $(j, i) \in \mathcal{K} \times \mathcal{K}$ and for all $\lambda := (\lambda_1, ..., \lambda_\mathcal{K})' \in \mathbb{R}_{++}^{\mathcal{K}}$,

$$\rho^\ddagger(\lambda P, \lambda^{-1} X, H) = \rho^\ddagger(P, X, H),$$

where $\lambda$ is the $N \times N$ diagonal matrix with $\lambda_n = \lambda$ for all $n \in \mathcal{K}$.

Theorem 4.4. $\rho^\ddagger(\bar{P}, \bar{X}, H) = \rho^\ddagger(P, X, H)$ if and only if $\rho^\ddagger$ satisfies QDX and C.
Similarly, the calculation of consumption shares based on the data set \((\bar{P}, \bar{X}, H)\) can only be facilitated by formulae which satisfy S1 and two other axioms. The first of these, called the *monetary unit test with respect to X*, states that multiplying the matrix of per-household quantities by a positive scalar has no effect on the consumption share of any country.

**S4X. Monetary Unit Test with Respect to X:** For all \(i \in \mathcal{K}\) and for all \(\beta \in \mathbb{R}_{++}\),

\[
\sigma^i(P, \beta X, H) = \sigma^i(P, X, H). 
\]

The second, *share commensurability*, requires the consumption shares to be invariant to changes in the units of measure of commodities.

**S5. Share Commensurability:** For all \(i \in \mathcal{K}\) and for all \(\lambda := (\lambda_1, \ldots, \lambda_R)' \in \mathbb{R}_+^N\),

\[
\sigma^i(\lambda P, \lambda^{-1}X, H) = \sigma^i(P, X, H). 
\]

**THEOREM 4.5.** \(\sigma(\bar{P}, \bar{X}, H) = \sigma(P, X, H)\) if and only if \(\sigma\) satisfies S4X and S5.

All twelve of the multilateral comparison methods examined in Chapter 3 satisfy S1, S4X and S5 or, equivalently, by Theorem 4.3, P, T, QDX and C. Consequently, the computation of consumption shares was a straightforward exercise involving simple substitutions into the various formulae of Section 3.3. Table 4.3 contains a selection of the results of this exercise. Included are the three restricted-domain methods — the household democratic (HD), the average basket (AB) and the country democratic (CD) — two of the three unrestricted-domain methods not based on a bilateral formula — the GK and the population-weighted van IJzeren (VH) — and three of the six bilateral-formula-based methods — the EKS, the own-share (OS) and the \(k\)-star with \(k := US\) (US*). The Fisher "ideal" consumption index \(\phi_F\) defined by
\[
\phi_F(p^j, p^i, x^j, x^i) := \left[ \frac{p^j x^j p^i x^i}{p^j x^i p^i x^j} \right]^{1/2}
\] (4.10)

was used as the basis for each of the bilateral–formula–based methods. Each of the other unrestricted–domain methods was calculated iteratively using the household–democratic consumption shares as initial values.

The mean absolute log differences among the eight methods of Table 4.3 and the exchange–rate approach (ER) are expressed as percentages in Table 4.4. If the cutoff between “substantial” and “insubstantial” is set at two percent, this table partitions the considered methods into five groups based on whether or not they are substantially different from one another. HD, AB and CD are grouped together since all of the differences among them lie below the cutoff while all of the differences involving just one of them lie above. Similarly, VH, EKS and OS form a group as do each of GK, US* and ER. Thus, the choice of one method over another can have a substantial impact on international comparisons of consumption. More importantly, this is true of the theoretically justifiable restricted–domain methods (HD and AB) relative to those of the unrestricted–domain variety.

Table 4.5 presents eight per–household consumption indexes derived from the results in Table 4.3 using the household population data in Table 4.2. Each of these indexes measures the consumption of the average household in each OECD country as a percentage of that in the

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6 The exchange–rate–based consumption–share system was calculated by substituting the country–i exchange rate with respect to country j for \( \rho^i(P, X, H) \) in equation (4.4).

7 To get a feel for what this means, consider a hypothetical international project to be financed by reference to 1990 OECD consumption shares. Using the own–share system instead of the US* system (100 \( \Delta_{US*,OS} \approx 2 \)) would change the average national contribution by 1.58 percent. For some countries, however, this switch in methods would change their contribution by as much as 3.99 percent. In an era of government fiscal restraint, four dollars per hundred can easily be viewed as a substantial difference.

8 Since it is so close to two, 100 \( \Delta_{US*,OS} = 1.992 \) is treated as a substantial difference.

9 An extended version of Table 4.4 would show that the quantity–weighted van Ijzeren, democratic weights, plutocratic weights and quantity weights methods also belong to this group.
United States. Figure 4.1 is a graphical representation of selected results in Table 4.5 along with those of the exchange-rate approach. Therein, the relevant countries\(^\text{10}\) are arranged from left to right along the horizontal axis in order of decreasing per-household consumption calculated via the household democratic method.

Constructed in the same manner as Figure 4.1, the next three figures serve to illustrate the "difference partition" established above. The close proximity of the per-household consumption lines in each of Figures 4.2 and 4.3 conveys the similarity of outcomes generated by methods belonging to the same group. By contrast, the relative separation of the corresponding lines in Figure 4.4 conveys the dissimilarity between methods belonging to different groups.

In Table 4.6, each entry is a PPP associated with the corresponding consumption share in Table 4.3 by means of equation (4.3). Comparison of Tables 4.6 and 4.1 reveals small differences between the two sets of GK and EKS PPPs.\(^\text{11}\) For both methods, these differences have arisen because the definition of private final consumption expenditure employed herein excludes expenditures by private non-profit institutions serving households whereas that of the OECD does not. For the EKS method, the differences are also due to the imposition of "fixity" by the OECD. Under this requirement, the "official" PPPs for the European Community (EC) must remain unchanged in any comparison involving a larger group of countries. The achievement of fixity is a two-step process.\(^\text{12}\) First, each OECD-specific PPP comparing two EC countries is replaced with the corresponding EC-specific PPP. Second,

\(^{10}\) The United States (US), Luxembourg (LUX), Switzerland (CHE), Canada (CAN), Italy (ITA), Japan (JAP), Australia (AUS), Iceland (ICE), France (FRA), the United Kingdom (UK), Germany (GER), Belgium (BEL), New Zealand (NZ), Austria (AUT), Spain (SPA), the Netherlands (NLD), Ireland (IRE), Finland (FIN), Denmark (DNK), Sweden (SWE), Norway (NOR), Greece (GRE), Portugal (PRT) and Turkey (TUR).

\(^{11}\) The mean absolute log-percentage differences are 0.313 and 0.472, respectively.

\(^{12}\) Such a process is necessary since \(\rho^{ij}_{EKS}\) is not invariant to changes in the size of the bloc.
each OECD–specific PPP comparing an EC country and a non-EC country is adjusted to restore transitivity. Thus, there are three distinct PPP concepts embedded within the OECD–EKS results.

4.4 Concluding Remarks

Gordon (1995, p. 7) notes that the use of PPPs from alternative sources can lead to very different assessments of the relative standards of living of countries. In an expression of the widespread confusion that exists among users of PPP data about this seemingly "fragile state of international... comparisons," he then goes on to ask the obvious question: "[W]hy [do] the sources differ so much?" There are three essential reasons.

First, different sources calculate the same PPPs in the context of different blocs of countries. An example of this was given above when the OECD–calculated EKS PPPs comparing two EC countries were contrasted with the corresponding EKS PPPs calculated by the author. The differences between them are due (in part) to the fact that the EKS index, like all other multilateral indexes, is bloc–specific: The comparison of two EC countries in the context of the BC is conceptually different from the comparison of the same two countries in the broader context of the OECD.

Second, different sources build the same aggregates from different baskets of goods and services. For example, Final Consumption of Resident Households consists of 159 basic headings under the OECD's classification and 215 under Eurostat’s. This fact reveals an additional dimension of conceptual disparity among the OECD–EKS PPPs since those which compare two EC countries were calculated on the basis of the latter classification while all the others were calculated on the basis of the former.

Third, different sources calculate the same PPPs using different methods of aggregation. Using a new a new type of difference indicator, the preceding section showed
that the choice of one method over another can have a substantial impact on the results obtained. More importantly, it was demonstrated that the restricted-domain methods motivated in Chapter 2 and shown to be theoretically superior in Chapter 3 are sufficiently different that they cannot be approximated by any other method.

Table 4.1. OECD-calculated PPPs for private final consumption expenditure in 1990 — national currency per U.S. dollar.

<table>
<thead>
<tr>
<th>Country</th>
<th>OECD-EKS</th>
<th>OECD-GK</th>
<th>Exchange Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belgium</td>
<td>40.4</td>
<td>39.1</td>
<td>33.3</td>
</tr>
<tr>
<td>Denmark</td>
<td>9.92</td>
<td>9.08</td>
<td>6.17</td>
</tr>
<tr>
<td>France</td>
<td>6.69</td>
<td>6.48</td>
<td>5.43</td>
</tr>
<tr>
<td>Germany</td>
<td>2.06</td>
<td>2.00</td>
<td>1.61</td>
</tr>
<tr>
<td>Greece</td>
<td>140.</td>
<td>132.</td>
<td>158.</td>
</tr>
<tr>
<td>Ireland</td>
<td>0.688</td>
<td>0.679</td>
<td>0.603</td>
</tr>
<tr>
<td>Italy</td>
<td>1,380.</td>
<td>1,332.</td>
<td>1,195.</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>36.6</td>
<td>36.0</td>
<td>33.3</td>
</tr>
<tr>
<td>Netherlands</td>
<td>2.15</td>
<td>2.02</td>
<td>1.82</td>
</tr>
<tr>
<td>Portugal</td>
<td>105.5</td>
<td>93.7</td>
<td>142.2</td>
</tr>
<tr>
<td>Spain</td>
<td>113.1</td>
<td>109.5</td>
<td>101.6</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>0.597</td>
<td>0.586</td>
<td>0.561</td>
</tr>
<tr>
<td>Austria</td>
<td>14.2</td>
<td>14.0</td>
<td>11.3</td>
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<td>58.3</td>
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<td>10.13</td>
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<tr>
<td>Sweden</td>
<td>9.50</td>
<td>9.02</td>
<td>5.92</td>
</tr>
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<td>1,232.</td>
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</tr>
<tr>
<td>Japan</td>
<td>207.</td>
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<td>145.</td>
</tr>
<tr>
<td>Canada</td>
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<td>1.31</td>
<td>1.17</td>
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<tr>
<td>United States</td>
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<td>1.00</td>
<td>1.00</td>
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Sources: OECD (1992, Table 2.5) (1993b, Table 2.8).
<table>
<thead>
<tr>
<th>Country</th>
<th>Persons (1,000)</th>
<th>Households (1,000)</th>
<th>Ratio</th>
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<tr>
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<td>Denmark</td>
<td>5,141</td>
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</tr>
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<td>France</td>
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</tr>
<tr>
<td>Italy</td>
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</tr>
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<td>Luxembourg</td>
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<tr>
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<td>2.5</td>
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TABLE 4.4. Mean absolute log—percentage differences.

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<td>1.40</td>
<td>1.39</td>
</tr>
<tr>
<td>New Zealand</td>
<td>1.63</td>
<td>1.64</td>
<td>1.61</td>
<td>1.58</td>
</tr>
<tr>
<td>Japan</td>
<td>211.0</td>
<td>213.0</td>
<td>211.0</td>
<td>187.0</td>
</tr>
<tr>
<td>Canada</td>
<td>1.31</td>
<td>1.31</td>
<td>1.31</td>
<td>1.31</td>
</tr>
<tr>
<td>United States</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
TABLE 4.6b. PPPs for private final consumption expenditure in 1990 — national currency per U.S. dollar.

<table>
<thead>
<tr>
<th>Country</th>
<th>VH</th>
<th>EKS</th>
<th>OS</th>
<th>US*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belgium</td>
<td>40.2</td>
<td>40.6</td>
<td>40.1</td>
<td>39.4</td>
</tr>
<tr>
<td>Denmark</td>
<td>9.68</td>
<td>9.80</td>
<td>9.67</td>
<td>9.77</td>
</tr>
<tr>
<td>France</td>
<td>6.64</td>
<td>6.71</td>
<td>6.63</td>
<td>6.62</td>
</tr>
<tr>
<td>Germany</td>
<td>2.04</td>
<td>2.07</td>
<td>2.03</td>
<td>1.98</td>
</tr>
<tr>
<td>Greece</td>
<td>140.</td>
<td>140.</td>
<td>141.</td>
<td>144.</td>
</tr>
<tr>
<td>Ireland</td>
<td>0.690</td>
<td>0.684</td>
<td>0.696</td>
<td>0.725</td>
</tr>
<tr>
<td>Italy</td>
<td>1,387</td>
<td>1,386</td>
<td>1,392</td>
<td>1,413</td>
</tr>
<tr>
<td>Luxembourg</td>
<td>37.0</td>
<td>37.1</td>
<td>37.1</td>
<td>37.1</td>
</tr>
<tr>
<td>Netherlands</td>
<td>2.11</td>
<td>2.15</td>
<td>2.10</td>
<td>2.07</td>
</tr>
<tr>
<td>Portugal</td>
<td>104.2</td>
<td>104.9</td>
<td>104.6</td>
<td>107.2</td>
</tr>
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<td>Spain</td>
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<td>113.1</td>
<td>113.8</td>
<td>117.4</td>
</tr>
<tr>
<td>Austria</td>
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<td>14.3</td>
<td>14.2</td>
<td>14.5</td>
</tr>
<tr>
<td>Switzerland</td>
<td>2.20</td>
<td>2.22</td>
<td>2.19</td>
<td>2.17</td>
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<tr>
<td>Finland</td>
<td>6.86</td>
<td>6.87</td>
<td>6.88</td>
<td>7.01</td>
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<tr>
<td>Iceland</td>
<td>90.3</td>
<td>90.9</td>
<td>90.4</td>
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<tr>
<td>Norway</td>
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<td>10.69</td>
<td>10.71</td>
<td>10.92</td>
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<tr>
<td>Sweden</td>
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<td>9.50</td>
<td>9.50</td>
<td>9.64</td>
</tr>
<tr>
<td>Turkey</td>
<td>1,559</td>
<td>1,598</td>
<td>1,559</td>
<td>1,565</td>
</tr>
<tr>
<td>Australia</td>
<td>1.43</td>
<td>1.45</td>
<td>1.43</td>
<td>1.43</td>
</tr>
<tr>
<td>New Zealand</td>
<td>1.64</td>
<td>1.65</td>
<td>1.63</td>
<td>1.62</td>
</tr>
<tr>
<td>Canada</td>
<td>1.33</td>
<td>1.35</td>
<td>1.33</td>
<td>1.31</td>
</tr>
<tr>
<td>United States</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Figure 4.1
Figure 4.2

Per-Household Consumption

Countries: US, CHE, ITA, AUS, FRA, GER, NZ, SPA, IRE, DNK, NOR, PRT, LUX, CAN, JAP, ICE, UK, BEL, AUT, NLD, FIN, SWE, GRE, TUR

Legend: HD, AB, CD
Figure 4.3
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APPENDIX A

PROOFS OF THEOREMS IN CHAPTER 2

PROOF OF P1. Since $C^k$ is positive,

$$\rho^k(p^r, p^c, v) := \frac{C^k(p^c, v)}{C^k(p^r, v)} > 0 .$$

PROOF OF P2. Since $C^k$ is non-decreasing in $p$,

$$\rho^k(p^r, p^c, v) := \frac{C^k(p^c, v)}{C^k(p^r, v)} \leq \frac{C^k(p^c, v)}{C^k(p^r, v)} =: \rho^k(p^r, p^c, v) .$$

PROOF OF P3. Since $C^k$ is PLH in $p$,

$$\rho^k(p^r, \lambda p^c, v) := \frac{C^k(\lambda p^c, v)}{C^k(p^r, v)} = \lambda \frac{C^k(p^c, v)}{C^k(p^r, v)} =: \lambda \rho^k(p^r, p^c, v) .$$

PROOF OF P4.

$$\rho^k(p^r, \bar{p}, v) \rho^k(\bar{p}, p^c, v) := \frac{C^k(\bar{p}, v)}{C^k(p^c, v)} \frac{C^k(p^c, v)}{C^k(\bar{p}, v)} = \frac{C^k(p^c, v)}{C^k(p^r, v)} =: \rho^k(p^r, p^c, v) .$$

PROOF OF P5. Since $C^k$ is concave in $p$, $\rho^k$ is concave in $p^c$.

PROOF OF P6. By P4, for any $p^c \in \mathbb{R}_+^N$,

$$\rho^k(p^c, p^r, v) \rho^k(p^r, p^r, v) = \rho^k(p^c, p^r, v) .$$

Thus, by P1,

$$\rho^k(p^r, p^r, v) = 1 .$$
Proof of P7.

\[ \rho_k(p_r, \lambda p_r, u) = \lambda \rho_k(p_r, p_r, u) , \quad \text{by P3} \]

\[ = \lambda , \quad \text{by P6.} \]

Proof of P8.

\[ \rho_k(p^e, p_r, u) = \frac{\rho_k(p^e, p^e, u) \rho_k(p_r, p_r, u)}{\rho_k(p^e, p_r, u)} , \quad \text{by P1} \]

\[ = \frac{\rho_k(p_r, p_r, u)}{\rho_k(p^e, p_r, u)} , \quad \text{by P4} \]

\[ = \frac{1}{\rho_k(p^e, p_r, u)} , \quad \text{by P6.} \]

Proof of P9.

\[ \rho_k(p_r, p^e, u) = \frac{1}{\rho_k(p^e, p_r, u)} , \quad \text{by P8} \]

\[ \geq \frac{1}{\rho_k(p^e, p^e, u)} , \quad \text{by P2} \]

\[ = \rho_k(p^e, p^e, u) , \quad \text{by P8.} \]

Proof of P10.

\[ \rho_k(\lambda p_r, p^e, u) = \frac{1}{p^e(p^e, \lambda p_r, u)} , \quad \text{by P8} \]

\[ = \frac{1}{\lambda \rho_k(p^e, p^e, u)} , \quad \text{by P3} \]

\[ = \lambda^{-1} \rho_k(p_r, p^e, u) , \quad \text{by P8.} \]

Proof of P11.

\[ \rho_k(\lambda p_r, \lambda p^e, u) = \lambda^{-1} \lambda \rho_k(p_r, p^e, u) , \quad \text{by P10 and P3} \]

\[ = \rho_k(p_r, p^e, u) . \]
PROOF OF P12.

\[
p_n^f := \min_{p_n^f} \left\{ \frac{p_n^e}{p_n^f} \right\} = \rho^k \left[ p^r, \frac{p_n^e}{p_n^f}, v \right], \text{ by P7}
\]

\[
\leq \rho^k(p^r, p^e, v), \text{ by P2 since } p^e \geq \frac{p_n^f}{p_n^f};
\]

\[
p_m^e := \max_{p_m^e} \left\{ \frac{p_m^e}{p_m^r} \right\} = \rho^k \left[ p^r, \frac{p_m^e}{p_m^r}, v \right], \text{ by P7}
\]

\[
\geq \rho^k(p^r, p^e, v), \text{ by P2 since } p^e \leq \frac{p_m^e}{p_m^r}. \Box
\]

PROOF OF P13. Define \( p^\lambda := (1-\lambda)p^r + \lambda \bar{p}^r \). By P5,

\[
\rho^k(p^e, p^\lambda, v) \geq (1-\lambda)\rho^k(p^e, p^r, v) + \lambda \rho^k(p^e, \bar{p}^r, v)
\]

\[
\geq [\rho^k(p^e, p^r, v)]^{1-\lambda}[\rho^k(p^e, \bar{p}^r, v)]^\lambda,
\]

by the Theorem of the Arithmetic and Geometric Means.

If P1 holds, this inequality is equivalent to

\[
\frac{1}{\rho^k(p^e, p^\lambda, v)} \leq \left[ \frac{1}{\rho^k(p^e, p^r, v)} \right]^{1-\lambda} \left[ \frac{1}{\rho^k(p^e, \bar{p}^r, v)} \right]^{\lambda}
\]

\[
\leq (1-\lambda) \left[ \frac{1}{\rho^k(p^e, p^r, v)} \right] + \lambda \left[ \frac{1}{\rho^k(p^e, \bar{p}^r, v)} \right],
\]

by the Theorem of the Arithmetic and Geometric Means.

The required inequality follows by P8. \( \Box \)
Proof of Theorem 2.1. To prove necessity, recall that \( P4 \) implies \( P6 \). Hence, \( C^k(p^r, v) := v \rho^k(p^r, p^r, v) = v \) and

\[ \rho^k(p^r, p^r, v) = \frac{v \rho^k(p^r, p^r, v)}{v} = \frac{C^k(p^r, v)}{C^k(p^r, v)}. \]

To prove sufficiency, note that \( \rho^k \) satisfies (2.2). Since \( C^k \) satisfies R2, \( \rho^k \) must also satisfy P1–P5. Finally, by (2.3), \( C^k(p^r, v) := C^k(p^r, v) \rho^k(p^r, p^r, v) = v \rho^k(p^r, p^r, v). \Box \)

Proof of Theorem 2.2. Since \( \lambda \in \mathbb{R}_{++}^N \), \( \bar{U}^k \) is an increasing transformation of \( U^k \) and, consequently, a utility function representing \( \succeq^k \). Equation (2.5) follows directly from

\[ \bar{C}^k(\bar{I}_N p, v) := \min_z \{ (\bar{I}_N p)^T z \mid \bar{U}^k(z) \geq v \} \]

\[ = \min_z \{ p^T x \mid U^k(x) \geq v \} =: C^k(p, v), \]

where \( x := \bar{I}_N z. \Box \)

Proof of Theorem 2.3.

\[ \rho^k(p^k, p^i, u_k) := \frac{C^k(p^i, u_k)}{C^k(p^k, u_k)} \]

\[ = \frac{C^k(p^i, u_k)}{p^{i^T}x^k}, \quad \text{by (2.9)} \]

\[ \leq \frac{p^{i^T}x^k}{p^{i^T}x^k}, \quad \text{since the minimum expenditure required to attain } u_k \text{ at prices } p^i \text{ cannot exceed } p^{i^T}x^k. \Box \]

Proof of Theorem 2.4. Fix \( j \) and \( i \), and define \( x^\lambda := (1 - \lambda)x^j + \lambda x^i \), \( \varphi(\lambda) := \rho(p^j, p^i, u_\lambda) \) for all \( \lambda \in [0, 1] \). Since both \( C \) and \( U \) are continuous over their respective domains, \( \varphi \) is continuous over \([0, 1]\). There are twenty-four \((= 4!)\) possible inequalities between the four numbers \( \varphi(0) = \rho(p^j, p^i, u_j), \varphi(1) = \rho(p^j, p^i, u_i), \rho^{ji} \) and \( \rho^{ij} \). However, by Theorem 2.3 and Corollary 2.3.1, \( \varphi(0) \leq \rho^{ji} \) and \( \rho^{ij} \leq \varphi(1) \). Subject to these restrictions, there are just six possible inequalities between the four numbers: (i) \( \varphi(0) \leq \rho^{ji} \leq \rho^{ij} \leq \varphi(1) \).
\( \rho^{ij} \leq \varphi(1) \); (ii) \( \varphi(0) \leq \rho^{ij} \leq \rho^{ij} \leq \varphi(1) \); (iii) \( \varphi(0) \leq \rho^{ij} \leq \varphi(1) \leq \rho^{ij} \); (iv) \( \rho^{ij} \leq \varphi(1) \leq \varphi(0) \leq \rho^{ij} \); (v) \( \rho^{ij} \leq \varphi(0) \leq \varphi(1) \leq \rho^{ij} \); (vi) \( \rho^{ij} \leq \varphi(0) \leq \rho^{ij} \leq \varphi(1) \). Since \( \varphi \) continuous on \([0, 1]\) implies that for all \( \rho \in \mathbb{R} \) between \( \varphi(0) \) and \( \varphi(1) \) there exists a \( \lambda \in [0, 1] \) such that \( \varphi(\lambda) = \rho \), it is clear that there exists a \( \lambda_i \in [0, 1] \) such that \( \rho^{ij} \leq \varphi(\lambda_i) \leq \rho^{ij} \) for case (i) or \( \rho^{ij} \leq \varphi(\lambda_i) \leq \rho^{ij} \) for cases (ii)–(vi). Thus, there exists a \( \nu_i \) between \( u_j \) and \( u_i \) which satisfies (2.15).

**Proof of Theorem 2.5.** Necessity:

\[
C^k(p^r, \mu_k) := \mu_k \rho(p^r, p^c, \mu, e^k)
\]

\[
= \frac{\mu_k \rho(p^r, p^r, \mu, e^k) \rho(p^r, p^r, \mu, e^k)}{\rho(p^r, p^r, \mu, e^k)},
\]

by the positivity property of \( \rho \)

\[
= \frac{\mu_k \rho(p^r, p^c, \mu, e^k)}{\rho(p^r, p^c, \mu, e^k)}, \quad \text{by the transitivity property of } \rho
\]

\[
= \mu_k;
\]

\[
\rho(p^i, p^i, \mu, H) = \frac{\rho(p^r, p^r, \mu, H)}{\rho(p^r, p^r, \mu, H)}, \quad \text{by the transitivity property of } \rho
\]

\[
= \frac{\sum_{k=1}^{K} H_k \mu_k \rho(p^r, p^r, \mu, e^k)}{\sum_{i=1}^{K} H_1 \mu_1 \rho(p^r, p^r, \mu, e^k)}, \quad \text{by (i)}
\]

\[
= \frac{\sum_{k=1}^{K} H_k C^k(p^i, \mu_k)}{\sum_{i=1}^{K} H_1 C^1(p^i, \mu_i)};
\]

\[
\rho(p^i, p^i, \mu, H) = \prod_{k=1}^{K} [\rho(p^i, p^i, \mu, e^k)]^{\theta^k(H)}, \quad \text{by (ii)}
\]
\[ \prod_{k=1}^{K} \left[ \frac{p(p^*, p^i, \mu, e^k) \rho(p^j, p^i, \mu, e^k)}{\rho(p^*, p^j, \mu, e^k)} \right]^{-1} \]

by the positivity property of \( \rho \)

\[ \prod_{k=1}^{K} \left[ \frac{\mu_k p(p^*, p^i, \mu, e^k)}{\mu_k p(p^*, p^j, \mu, e^k)} \right]^{-1} \]

by the transitivity property of \( \rho \) and since \( \mu \in \mathbb{R}_{++}^K \)

\[ = \prod_{k=1}^{K} \left[ \frac{C^k(p^i, \mu_k)}{C^k(p^j, \mu_k)} \right]^{-1} \]

Sufficiency: Straightforward.

**Proof of Theorem 2.6.** Since both \( \rho \) and \( C^k \ (k \in K) \) are positive and continuous in their respective arguments, so is \( \sigma \). Now,

\[ \sum_{i=1}^{K} \sigma^i(\bar{P}, u, H) := \sum_{i=1}^{K} \left\{ \frac{H_i C^i(\bar{p}^i, u_i)}{\rho(\bar{p}^k, \bar{p}^i, u, H)} \right\}^{-1} \]

\[ = \sum_{i=1}^{K} H_i C^i(\bar{p}^i, u_i) \left\{ \frac{\sum_{k=1}^{K} H_k C^k(\bar{p}^k, u_k) \rho(\bar{p}^j, \bar{p}^i, u, H)}{\rho(\bar{p}^j, \bar{p}^k, u, H)} \right\}^{-1} \]

by the positivity and transitivity properties of \( \rho \)

\[ = \sum_{i=1}^{K} \frac{H_i C^i(\bar{p}^j, u_i)}{\rho(\bar{p}^j, \bar{p}^j, u, H)} \left\{ \frac{\sum_{k=1}^{K} H_k C^k(\bar{p}^k, u_k)}{\rho(\bar{p}^k, \bar{p}^k, u, H)} \right\}^{-1} \]

\[ = 1. \]

Finally,

\[ \sigma^i(\bar{P}, u, H) := \left\{ \frac{H_i C^i(\bar{p}^i, u_i)}{\rho(\bar{p}^k, \bar{p}^i, u, H)} \right\}^{-1}^{-1} \]

\[ \sum_{k=1}^{K} \left[ \frac{H_i C^i(\bar{p}^i, u_i)}{H_k C^k(\bar{p}^k, u_k)} \right]^{-1} \]
\[
\left( \sum_{k=1}^{K} \frac{H_i C^i(e_{i} p^j, u_i)}{H_k C^k(e_{k} p^k, u_k)} \rho(e_k p^k, e_i p^i, u, H) \right)^{-1}
\]

since \( p^k := e_k p^k \)

\[
\left( \sum_{k=1}^{K} \frac{H_i C^i(p^i, u_i)}{H_k C^k(p^k, u_k)} \rho(p^k, p^i, u, H) \right)^{-1}
\]

since \( \rho \) is homogeneous of degree minus one in \( p^k \) and both \( \rho \) and \( C^i \) are PLH in \( p^i \)

\( =: \sigma^i(P, u, H). \Box \)

**Proof of Corollary 2.6.1.**

\[
\frac{\sigma^j(P, u, H)/H_i}{\sigma^j(P, u, H)/H_j} := \frac{C^i(p^j, u_i)}{C^j(p^i, u_j)} \frac{\sum_{k=1}^{K} H_k C^k(p^k, u_k) \rho(p^k, p^i, u, H)}{\sum_{k=1}^{K} H_k C^k(p^k, u_k) \rho(p^k, p^j, u, H)}
\]

by the positivity and transitivity properties of \( \rho \)

\( = \frac{C^i(p^j, u_i)}{C^j(p^i, u_j)} / \rho(p^j, p^j, u, H). \Box \)

**Proof of Theorem 2.7.**

\[
\ln \rho_{MD}(p^j, p^i, \nu_{ji}, \ldots, \nu_{ji}, H) := \ln \prod_{k=1}^{K} \left[ \frac{C(p^i, \nu_{ih})}{C(p^j, \nu_{ih})} \right]^{\theta_k(H)}
\]

\( = \ln \left[ \frac{C(p^j, \nu_{ih})}{C(p^i, \nu_{ih})} \right] \) since \( \sum_{k=1}^{K} \theta_k(H) = 1 \)

\( = \ln C(p^i, \nu_{ji}) - \ln C(p^j, \nu_{ji}) \)
\[ 
\sum_{n=1}^{N} \beta_n \ln p_n^i - \ln p_n^j \] + \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \gamma_{mn} \ln p_m^i - \ln p_m^j \ln \nu_{ji}, \quad \text{by (2.53)} 
\]

\[ 
= \sum_{n=1}^{N} \beta_n \ln p_n^i - \ln p_n^j + \sum_{n=1}^{N} \gamma_{0n} \ln p_n^i - \ln p_n^j \ln \nu_{ji} 
\]

\[ 
+ \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \gamma_{mn} \ln p_m^i + \ln p_m^j \ln p_n^i - \ln p_n^j 
\]

\[ 
= \frac{1}{2} \sum_{n=1}^{N} \left[ \beta_n + \sum_{m=1}^{N} \gamma_{mn} \ln p_m^i + \gamma_{0n} \ln u_j + \beta_n + \sum_{m=1}^{N} \gamma_{mn} \ln p_m^j \right] \ln p_n^i - \ln p_n^j 
\]

\[ 
\quad + \gamma_{0n} \ln u_j \ln p_n^i - \ln p_n^j \quad \text{since} \quad \ln \nu_{ji} = \frac{1}{2} \ln u_j + \frac{1}{2} \ln u_i 
\]

\[ 
= \frac{1}{2} \sum_{n=1}^{N} \left[ p_n^i \frac{\partial \ln C(p^i, u_i)}{\partial p_n^i} + p_n^j \frac{\partial \ln C(p^j, u_j)}{\partial p_n^j} \right] \left[ \ln p_n^i - \ln p_n^j \right] 
\]

from (2.53)

\[ 
= \frac{1}{2} \sum_{n=1}^{N} \left[ \frac{p_n^i x_n^i}{p_n^i x_n^i} + \frac{p_n^j x_n^j}{p_n^j x_n^j} \right] \left[ \ln p_n^i - \ln p_n^j \right] 
\]

\[ 
= \sum_{n=1}^{N} \ln \left[ \frac{p_n^i}{p_n^j} \right]^\gamma \left( \omega_n^i + \omega_n^j \right) / 2 
\]

\[ 
= \ln \prod_{n=1}^{N} \left[ \frac{p_n^i}{p_n^j} \right]^\gamma \left( \omega_n^i + \omega_n^j \right) / 2 
\]

\[ 
= \ln \rho_T(p^i, p^j, x^i, x^j) \]

\[ \square \]
APPENDIX B
PROOFS OF THEOREMS IN CHAPTER 3

Proof of Theorem 3.1.

(i) By T, for any \( l \in K \),

\[
\rho(p', p^j, X, H) \rho(p^j, p^j, X, H) = \rho(p', p^j, X, H).
\]

Thus, by P,

\[
\rho(p^j, p^j, X, H) = 1.
\]

(ii) \( \rho(p^j, \lambda p^j, X, H) = \lambda \rho(p^j, p^j, X, H) \), by H

\[
= \lambda, \text{ by I.}
\]

(iii) \( \rho(p^i, p^j, X, H) = \frac{\rho(p^i, p^i, X, H) \rho(p^j, p^j, X, H)}{\rho(p^i, p^j, X, H)} \), by P

\[
= \frac{\rho(p^j, p^j, X, H)}{\rho(p^j, p^j, X, H)}, \text{ by T}
\]

\[
= \frac{1}{\rho(p^j, p^j, X, H)}, \text{ by I.}
\]

(iv) For any \( \bar{p}^j > p^j \),

\[
\rho(p^j, p^i, X, H) = \frac{1}{\rho(p^i, p^j, X, H)}, \text{ by CR}
\]

\[
\geq \frac{1}{\rho(p^i, \bar{p}^j, X, H)}, \text{ by M}
\]

\[
= \rho(p^j, p^i, X, H), \text{ by CR.}
\]
(v) \[ \rho(\lambda p^i, p^i, X, H) = \frac{1}{\rho(p^i, \lambda p^i, X, H)}, \text{ by CR} \]
\[ = \frac{1}{\lambda \rho(p^i, p^i, X, H)}, \text{ by H} \]
\[ = \lambda^{-1} \rho(p^i, p^i, X, H), \text{ by CR}. \]

(vi) \[ \rho(\lambda p^i, \lambda p^i, X, H) = \lambda^{-1} \lambda \rho(p^i, p^i, X, H), \text{ by H and HDM} \]
\[ = \rho(p^i, p^i, X, H). \]

(vii) \[ p^i_1 := \min_{n \in \mathbb{N}} \left( \frac{p^i_n}{p^i} \right) = \rho \left( p^i, \frac{p^i_1}{p^i}, X, H \right), \text{ by PP} \]
\[ \leq \rho(p^i, p^i, X, H), \text{ by M since } p^i \geq \frac{p^i_1}{p^i}; \]
\[ p^i_m := \max_{n \in \mathbb{N}} \left( \frac{p^i_n}{p^i_m} \right) = \rho \left( p^i, \frac{p^i_m}{p^i}, X, H \right), \text{ by PP} \]
\[ \geq \rho(p^i, p^i, X, H), \text{ by M since } p^i \leq \frac{p^i_m}{p^i}. \]

**Proof of Theorem 3.2.** Necessity:

\[ [\bar{\rho}(p^i, p^i, x^i, x^i, H_j, H_i)]^2 \]
\[ = \frac{p^{i, i} x^i \bar{\rho}(p^i, p^i, x^i, x^i, H_j, H_i)}{p^{i, i} x^i \bar{\rho}(x^i, x^i, p^i, p^i, H_j, H_i)}, \text{ by FR} \]
\[ = \frac{p^{i, i} x^i}{p^{i, i} x^i} \bar{\rho}(p^i, p^i, x^i, x^i, H_j, H_i) \bar{\rho}(x^i, x^i, p^i, p^i, H_j, H_i), \text{ by CR} \]
\[ = \frac{p^{i, i} x^i}{p^{i, i} x^i} \bar{\rho}(p^i, p^i, x^i, x^i, H_j, H_i) \bar{\rho}(x^i, x^i, p^i, p^i, H_j, H_i), \text{ by WS} \]
\[ = \frac{p^{i, i} x^i}{p^{i, i} x^i}, \text{ by FR and PI.} \]
Sufficiency: Straightforward.

**Proof of Theorem 3.3.** Suppose $\hat{\sigma}_i > \hat{s}_i$. Since $p^i < \bar{p}^i \rightarrow \hat{s}_i \geq 0$, $\hat{\sigma}_i > 0$. By S1, $\hat{\sigma}_j \geq (\hat{\sigma}_i - \hat{s}_i)/(1 + \hat{s}_i) > 0 \ \forall j \in K \setminus \{i\}$. By S1, $\Sigma_{k=1}^K \sigma^k(p^1, \ldots, p^{i-1}, \bar{p}^i, p^{i+1}, \ldots, p^K, X, H) = \Sigma_{k=1}^K \sigma^k(P, X, H) = 1$. But $\hat{\sigma}_k > 0 \ \forall k \in K$ implies that $\Sigma_{k=1}^K \sigma^k(p^1, \ldots, p^{i-1}, \bar{p}^i, p^{i+1}, \ldots, p^K, X, H) > \Sigma_{k=1}^K \sigma^k(P, X, H)$. □

**Proof of Lemma 3.1.** By RT,

$$\frac{\sigma_i(P, X, H)}{\sigma^i(P', X, H)} = \frac{H_i \phi(p^i, p^i, x^i, x^i, X, H)}{H_i \phi(p^i, p^i, x^i, x^i, X, H)}$$

$$= \frac{H_i p^i x^i \rho(p^i, p^i, X, H) \rho(p^i, p^i, X, H)}{H_j p^j x^j \rho(p^i, p^i, X, H)} \rho(p^i, p^i, X, H) , \text{ by (3.2)}$$

$$= \left\{ \frac{H_i p^i x^i \rho(p^i, p^i, X, H) \rho(p^i, p^i, X, H)}{H_j p^j x^j \rho(p^i, p^i, X, H)} \right\}^{-1} , \text{ by T}$$

$$\Rightarrow \rho(p^i, p^i, X, H) = \frac{H_i p^i x^i \sigma_i(P, X, H)}{H_j p^j x^j \sigma^i(P, X, H)} .$$

From (B.1),

$$\sum_{j=1}^K \frac{H_i p^i x^i \rho(p^i, p^i, X, H)}{H_j p^j x^j \rho(p^i, p^i, X, H)} \rho(p^i, p^i, X, H) \sigma_i(P, X, H) = \sum_{j=1}^K \sigma^j(P, X, H)$$

$$\Rightarrow \sigma_i(P, X, H) = \left\{ \sum_{j=1}^K \frac{H_i p^i x^i \rho(p^i, p^i, X, H)}{H_j p^j x^j \rho(p^i, p^i, X, H)} \rho(p^i, p^i, X, H) \right\}^{-1} , \text{ by S1. □}$$

**Proof of Lemma 3.2.**

(i) $\sigma^i(\alpha_1 p^i, \ldots, \alpha_k p^i, \beta_1 x^i, \ldots, \beta_k x^i, \gamma, \ldots, \gamma)$

$$= \left\{ \sum_{j=1}^K \frac{\gamma(\alpha_j p^i)(\beta_j x^i) \rho(\alpha_j p^i, \alpha_i p^i, \beta_j x^i, \ldots, \beta_k x^i, \gamma, \ldots, \gamma)}{\gamma(\alpha_i p^i)(\beta_i x^i) \rho(\alpha_j p^i, \alpha_i p^i, \beta_j x^i, \ldots, \beta_k x^i, \gamma, \ldots, \gamma)} \right\}^{-1} , \text{ by (3.6)}$$
\[
\begin{align*}
\sigma^i(P, x^1, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H) &= \left\{ \sum_{j=1}^{K} \frac{H_i p_i^j x^j}{H_i p_i^j x^j \rho(p^i, p^j, x^j, X, H)} \right\}^{-1}, \\
&= \frac{\beta_i}{\sum_{j=1}^{K} \beta_j}, \\
&= \beta_i, \text{ since } \sum_{j=1}^{K} \beta_j = 1.
\end{align*}
\]

(ii) For any \(i \in K \setminus \{1\}, \)

\[
\begin{align*}
\sigma^i(P, x^1, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H) &= \left\{ \sum_{j \neq i}^{K} \frac{H_i p_i^j x^j}{H_i p_i^j x^j \rho(p^i, p^j, x^j, X, H)} \right\}^{-1} + \frac{H_i p_i^i x^i}{H_i p_i^i x^i \rho(p^i, p^i, x^i, X, H)}
\end{align*}
\]

by (3.6)

\[
\begin{align*}
&= \left\{ \frac{H_i p_i^i x^i}{H_i p_i^i x^i \rho(p^i, p^i, X, H)} \right\}^{-1}, \text{ by SQD}
\end{align*}
\]

\[
\begin{align*}
&= \left[ \sigma^i(P, X, H) \right]^{-1} + (\lambda - 1) \frac{H_i p_i^i x^i}{H_i p_i^i x^i \rho(p^i, p^i, X, H)}
\end{align*}
\]

by (3.6)

\[
\begin{align*}
&= \sigma^i(P, X, H) \left\{ 1 + (\lambda - 1) \frac{\sum_{j=1}^{K} H_i p_i^j x^j \rho(p^i, p^j, X, H)}{H_i p_i^j x^j \rho(p^i, p^j, X, H)} \right\}^{-1}, \text{ by T}
\end{align*}
\]

\[
\begin{align*}
&= \sigma^i(P, X, H) \left\{ 1 + (\lambda - 1) \frac{\sum_{j=1}^{K} H_i p_i^j x^j \rho(p^i, p^j, X, H)}{H_i p_i^j x^j \rho(p^i, p^j, X, H)} \right\}^{-1}, \text{ by T}
\end{align*}
\]

\[
\begin{align*}
&= \frac{\sigma^i(P, X, H)}{1 + (\lambda - 1) \sigma^i(P, X, H)}, \text{ by (3.6)}
\end{align*}
\]
\[ \sigma'(P, x^1, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H) \]

\[ = \left\{ \sum_{j=1}^{i} \frac{H_j p^{i'} x^j}{p^{i'} (\lambda x^j)} \rho(p^j, p^{i'}, x^1, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H) \right\}^{-1} + \frac{H_i p^{i'} (\lambda x^i)}{H_i p^{i'} (\lambda x^i)} \rho(p^i, p^{i'}, x^1, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H) \right\}^{-1}, \]

by (3.6)

\[ = \lambda \left\{ \sum_{j=1}^{K} \frac{H_j p^{i'} x^j}{p^{i'} (\lambda x^j)} \rho(p^j, p^{i'}, x^1, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H) \right\}^{-1} + (\lambda - 1) \right\}^{-1}, \] by SQD

\[ = \lambda \left\{ (\sigma'(P, X, H))^{-1} + (\lambda - 1) \right\}^{-1}, \] by (3.6)

\[ = \frac{\lambda \sigma^i(P, X, H)}{1 + (\lambda - 1) \sigma^i(P, X, H)} . \]

(iii) \[ \sigma^i(\alpha_1 p^1, \ldots, \alpha_K p^K, \beta X, \gamma H) \]

\[ = \left\{ \sum_{j=1}^{K} \frac{\gamma H_j (\alpha_1 p^j)'}{(\beta x^j) \rho(\alpha_1 p^j, \alpha_1 p^i, X, H)} \right\}^{-1}, \] by (3.6)

\[ = \left\{ \sum_{j=1}^{K} \frac{\gamma H_j p^{j'} x^j}{(\beta x^j) \rho(\alpha_1 p^j, \alpha_1 p^i, \beta X, \gamma H)} \right\}^{-1}\]

\[ = \sigma^i(P, X, H), \text{ by H, HDM, QD and (3.6).} \]

(iv) \[ \sigma^i(\lambda P, \lambda^{-1} X, H) \]

\[ = \left\{ \sum_{j=1}^{K} \frac{H_j (\lambda p^j)'}{(\lambda x^j) \rho(\lambda p^j, \lambda p^i, \lambda^{-1} X, H)} \right\}^{-1}, \] by (3.6)

\[ = \left\{ \sum_{j=1}^{K} \frac{H_j p^{j'} (\lambda x^j)}{p^{j'} (\lambda x^j) \rho(p^j, p^i, X, H)} \right\}^{-1}, \] by C

\[ = \left\{ \sum_{j=1}^{K} \frac{H_j p^{j'} \lambda x^j}{H_j p^{j'} \lambda x^j \rho(p^j, p^i, X, H)} \right\}^{-1}, \]

\[ = \sigma^i(P, X, H), \text{ by (3.6).} \]
Consider the bijective mapping $\varphi: \mathcal{K} \to \mathcal{K}$ which satisfies

$$\col_{\varphi(l)} \tilde{I}_K = \col_l I_K$$  \hspace{1cm} (B.2)

for any $l \in \mathcal{K}$. For all $k \in \mathcal{K}$, let

$$[\tilde{p}^{\varphi(k)}, \tilde{x}^{\varphi(k)}, \tilde{H}_{\varphi(k)}] := [P, X, H']\col_{\varphi(k)} \tilde{I}_K$$  \hspace{1cm} (B.3)

$$= [P, X, H']\col_k I_K, \text{ by (B.2)}$$

$$=: (p^k, x^k, H_k).$$  \hspace{1cm} (B.4)

Now,

$$\sigma^{\varphi(i)}(P \tilde{I}_K, X \tilde{I}_K, \tilde{I}_K'H)$$

$$= \left\{ \sum_{j=1}^{K} \frac{\tilde{H}_{\varphi(j)} \tilde{p}^{\varphi(j)}' \tilde{x}^{\varphi(j)} \rho(\tilde{p}^{\varphi(j)}, \tilde{p}^{\varphi(i)}, X \tilde{I}_K, \tilde{I}_K'H)}{\rho(\tilde{p}^{\varphi(j)}, \tilde{p}^{\varphi(i)}, X \tilde{I}_K, \tilde{I}_K'H)} \right\}^{-1}, \text{ by (3.6)}$$

$$= \left\{ \sum_{j=1}^{K} \frac{H_j p^j x^j \rho(p^i, p^j, X, H)}{H_i p^i x^i \rho(p^i, p^j, X, H)} \right\}^{-1}, \text{ by (B.4) and WS}$$

$$= \sigma^i(P, X, H), \text{ by (3.6)}.$$

(v)  \hspace{1cm} (vi)  \hspace{1cm}

$$\sigma^i(\tilde{I}_N P, \tilde{I}_N X, H) = \left\{ \sum_{j=1}^{K} \frac{H_j (\tilde{I}_N p^j)'(\tilde{I}_N x^j) \rho(\tilde{I}_N p^i, \tilde{I}_N p^j, \tilde{I}_N X, H)}{H_i (\tilde{I}_N p^i)'(\tilde{I}_N x^i) \rho(\tilde{I}_N p^i, \tilde{I}_N p^j, \tilde{I}_N X, H)} \right\}^{-1}, \text{ by (3.6)}$$

$$= \left\{ \sum_{j=1}^{K} \frac{H_j p^j (\tilde{I}_N' \tilde{I}_N) x^j \rho(p^i, p^j, X, H)}{H_i p^i (\tilde{I}_N' \tilde{I}_N) x^i \rho(p^i, p^j, X, H)} \right\}^{-1}, \text{ by CS}$$

$$= \left\{ \sum_{j=1}^{K} \frac{H_j p^j I_N x^j \rho(p^i, p^j, X, H)}{H_i p^i I_N x^i \rho(p^i, p^j, X, H)} \right\}^{-1}.$$
= σ^i(P, X, H), by (3.6).

(vii) For any i ∈ K\{l},

\[ \tilde{\sigma}^i(P, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l) \]

\[ = \left\{ \sum_{j \neq i} \frac{H_j p^{j'} x^j}{H_i p^{i'} x^i} \right\} \times \frac{\tilde{\rho}(p^i, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l)}{\tilde{\rho}(p^i, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l)} \]

\[ \times \left[ (1-\lambda) H_l p^{i'} x^i \right]^{-1} \times \left[ \frac{\tilde{\rho}(p^i, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l)}{\tilde{\rho}(p^i, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l)} \right]^{-1} \]

by (3.6)

\[ = \left\{ \sum_{j=1}^{K} \frac{H_j p^{j'} x^j}{H_i p^{i'} x^i} \rho(p^i, p^l, X, H) \right\}^{-1}, \text{ by CP} \]

\[ = \sigma^i(P, X, H), \text{ by (3.6)}; \]

\[ \tilde{\sigma}^i(P, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l) \]

\[ = \left\{ \sum_{j \neq i} \frac{H_j p^{j'} x^j}{(1-\lambda)H_l p^{i'} x^i} \right\} \times \frac{\tilde{\rho}(p^i, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l)}{\tilde{\rho}(p^i, p^l, X, x^l, H_1, ..., H_{l-1}, (1-\lambda)H_l, H_{l+1}, ..., H_K, \lambda H_l)} \]
\[
\begin{align*}
&\frac{[(1 - \lambda) + \lambda]H_1 p^{l'}x^l}{(1 - \lambda) H_1 p^{l'}x^l} \\
&\times \frac{\rho(p^l, p^l, X, x^l, H_1, ..., H_{l-1}, (1 - \lambda)H_1, H_{l+1}, ..., H_K, \lambda H_0)}{\rho(p^l, p^l, X, x^l, H_1, ..., H_{l-1}, (1 - \lambda)H_1, H_{l+1}, ..., H_K, \lambda H_0)}^{-1}, \quad \text{by (3.6)}
\end{align*}
\]

\[
\begin{align*}
&= \left\{ (1 - \lambda)^{-1} \sum_{j=1}^{K} \frac{H_j p^{j'}x^j \rho(p^j, p^l, X, H)}{H_1 p^{l'}x^l \rho(p^l, p^l, X, H)} \right\}^{-1}, \quad \text{by CP}
\end{align*}
\]

\[
\begin{align*}
&= (1 - \lambda) \sigma^i(P, X, H), \quad \text{by (3.6)};
\end{align*}
\]

\[
\begin{align*}
\bar{\sigma}_{K+1}(P, p^l, X, x^l, H_1, ..., H_{l-1}, (1 - \lambda)H_1, H_{l+1}, ..., H_K, \lambda H_0)
\end{align*}
\]

\[
\begin{align*}
&= \left\{ \sum_{j \neq l} \frac{H_j p^{j'}x^j}{\lambda H_1 p^{l'}x^l} \right\}^{-1} \\
&\times \frac{\rho(p^l, p^l, X, x^l, H_1, ..., H_{l-1}, (1 - \lambda)H_1, H_{l+1}, ..., H_K, \lambda H_0)}{\rho(p^l, p^l, X, x^l, H_1, ..., H_{l-1}, (1 - \lambda)H_1, H_{l+1}, ..., H_K, \lambda H_0)}^{-1} \\
&\times \frac{[(1 - \lambda) + \lambda]H_1 p^{l'}x^l}{\lambda H_1 p^{l'}x^l} \\
&\times \frac{\rho(p^l, p^l, X, x^l, H_1, ..., H_{l-1}, (1 - \lambda)H_1, H_{l+1}, ..., H_K, \lambda H_0)}{\rho(p^l, p^l, X, x^l, H_1, ..., H_{l-1}, (1 - \lambda)H_1, H_{l+1}, ..., H_K, \lambda H_0)}^{-1}, \quad \text{by (3.6)}
\end{align*}
\]

\[
\begin{align*}
&= \left\{ \lambda^{-1} \sum_{j=1}^{K} \frac{H_j p^{j'}x^j \rho(p^j, p^l, X, H)}{H_1 p^{l'}x^l \rho(p^l, p^l, X, H)} \right\}^{-1}, \quad \text{by CP}
\end{align*}
\]

\[
\begin{align*}
&= \lambda \sigma^i(P, X, H), \quad \text{by (3.6)}.
\end{align*}
\]
(viii) \[ \lim_{\lambda \to 0} \sigma^i(P, X, H_1, ..., H_{i-1}, \lambda H_i, H_{i+1}, ..., H_K) \]

\[ = \lim_{\lambda \to 0} \left\{ \sum_{j \neq i} \frac{H_i p^{i'} x^i \rho(p^{i'}, p^i, X, H_1, ..., H_{i-1}, \lambda H_i, H_{i+1}, ..., H_K)}{H_i p^{i'} x^i} \rho(p^i, p^i, X, H_1, ..., H_{i-1}, \lambda H_i, H_{i+1}, ..., H_K) \right\}^{-1}, \text{ by (3.6)} \]

\[ = \left\{ \sum_{j \neq i} \frac{H_j p^{i'} x^j \rho(p^{i'}, p^i, X, H_1, ..., H_{i-1}, \lambda H_i, H_{i+1}, ..., H_K)}{H_j p^{i'} x^j} \lim_{\lambda \to 0} \rho(p^i, p^i, X, H_1, ..., H_{i-1}, \lambda H_i, H_{i+1}, ..., H_K) \right\}^{-1} \]

\[ = \left\{ \sum_{j \neq i} \frac{H_j p^{i'} x^j \rho(p^i, p^i, X, H_1, ..., H_{i-1}, \lambda H_i, H_{i+1}, ..., H_K)}{H_j p^{i'} x^j} \lim_{\lambda \to 0} \rho(p^i, p^i, X, H_1, ..., H_{i-1}, \lambda H_i, H_{i+1}, ..., H_K) \right\}^{-1}, \text{ by TCI} \]

\[ = \tilde{\sigma}^i(P, X, H_1, ..., H_{i-1}), \text{ by (3.6)} \]

(ix) \[ \lim_{\lambda \to 0} \frac{\sigma^i(P, X, \lambda H_1, ..., \lambda H_{j-1}, H_j, \lambda H_{j+1}, ..., \lambda H_{i-1}, H_i, \lambda H_{i+1}, ..., \lambda H_K)}{\sigma^i(P, X, \lambda H_1, ..., \lambda H_{j-1}, H_j, \lambda H_{j+1}, ..., \lambda H_{i-1}, H_i, \lambda H_{i+1}, ..., \lambda H_K)} \]

\[ = \lim_{\lambda \to 0} \frac{\sum_{i \neq j, i} \frac{\lambda H_i p^{i'} x^i \rho(p^{i'}, p^i, X, \lambda H_i, ..., \lambda H_{i-1}, H_i, \lambda H_{i+1}, ..., \lambda H_K)}{H_j p^{i'} x^j} + 1 + \frac{H_i p^{i'} x^i \rho(p^i, p^i, X, \lambda H_i, ..., \lambda H_{i-1}, H_i, \lambda H_{i+1}, ..., \lambda H_K)}{H_j p^{i'} x^j}}{\sum_{k \neq j, i} \frac{\lambda H_k p^{i'} x^k \rho(p^{i'}, p^k, X, \lambda H_i, ..., \lambda H_{i-1}, H_i, \lambda H_{i+1}, ..., \lambda H_K)}{H_j p^{i'} x^j} + 1 + \frac{H_j p^{i'} x^j \rho(p^i, p^i, X, \lambda H_i, ..., \lambda H_{i-1}, H_i, \lambda H_{i+1}, ..., \lambda H_K)}} \]

by (3.6)

\[ = \frac{H_j p^{i'} x^j \rho(p^i, p^i, x^i, H_j, H_i)}{H_j p^{i'} x^j \rho(p^i, p^i, x^i, H_j, H_i) + H_j p^{i'} x^j \rho(p^i, p^i, x^i, H_j, H_i)} + \frac{H_i p^{i'} x^i \rho(p^i, p^i, x^i, H_j, H_i)}{H_i p^{i'} x^i \rho(p^i, p^i, x^i, H_j, H_i)} \]

\[ = \frac{H_j p^{i'} x^j \rho(p^i, p^i, x^i, H_j, H_i)}{H_i p^{i'} x^i \rho(p^i, p^i, x^i, H_j, H_i) + H_j p^{i'} x^j \rho(p^i, p^i, x^i, H_j, H_i)} + \frac{H_i p^{i'} x^i \rho(p^i, p^i, x^i, H_j, H_i)}{H_i p^{i'} x^i \rho(p^i, p^i, x^i, H_j, H_i) + H_j p^{i'} x^j \rho(p^i, p^i, x^i, H_j, H_i)} \]

by TCI
\begin{align*}
&= \frac{H_i p^i \pi^i}{H_j p_j \pi_j} \left/ \frac{\sigma(p^i, p^i, x^i, x^i, H_j, H_i)}{H_j p_j \pi_j} \right. \text{, by T} \\
&= \frac{p^i \pi^i (H_i x^i)}{p_j^i \pi^i (H_j x^i)} \left/ \frac{\sigma(p^i, p^i, H_j x^i, H_j x^i, 1, 1)}{p_j \pi_j (H_j x^i)} \right. \text{, by TQ} \\
&= \phi(p^i, p^i, H_j x^i, H_j x^i, H_j x^i, H_j x^i, 1, 1) \text{, by (3.2).}
\end{align*}

(x) For any \( \bar{p}^i > p^i \),

\[ \frac{\rho(p^i, p^i, X, H)}{\rho(p^i, p^i, X, H)} \leq \frac{\rho(p^i, p^i, X, H)}{\rho(p^i, p^i, X, H)} \text{, by P and M} \]

\[ \Rightarrow \frac{H_i \pi^i \pi^i \sigma^i(P, X, H)}{H_j \pi^i \pi^i \sigma^i(P, X, H)} \leq \frac{H_i \pi^i \pi^i \sigma^i(p^1, \ldots, p^{i-1}, \bar{p}^i, p^{i+1}, \ldots, p^K, X, H)}{H_j \pi^i \pi^i \sigma^i(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)} \]

by (B.1)

\[ \Rightarrow \frac{\sigma^i(p^1, \ldots, p^{i-1}, \bar{p}^i, p^{i+1}, \ldots, p^K, X, H)}{\sigma^i(P, X, H)} \]

\[ \geq \left[ \frac{p^i}{\pi^i} \right]^{-1} \frac{\sigma^i(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)}{\sigma^i(P, X, H)} . \]

(xi) If \( p^i = p^i \) then

\[ \rho(p^i, p^i, X, H) = 1 \text{, by I (=} P \land T) \]

\[ \Rightarrow \frac{H_i \pi^i \pi^i \sigma^i(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)}{H_j \pi^i \pi^i \sigma^i(p^1, \ldots, p^{i-1}, p^i, p^{i+1}, \ldots, p^K, X, H)} = 1 \text{, by (3.7).} \]

(xii) \[ \sigma^i(P, X^i, 1_K) = \left\{ \sum_{j=1}^{K} H_j \pi^j \pi^j \sigma^j(p^j, p^j, X^j, 1_K) \right\}^{-1} \]

\[ \text{by (3.6),} \]

\[ = \sigma^i(P, X, H) \text{, by (3.6).} \]
(xiii) For any \( i \in \mathcal{X}\setminus\{l\}, \)

\[
\bar{\sigma}^i(P, p^l, x^1, \ldots, x^{i-1}, \bar{x}^l, x^{i+1}, \ldots, x^K, x^{K+1},
\]

\[
\begin{align*}
&H_{1, \ldots, H_{i-1}, (1-\lambda)H_{i}, H_{i+1}, \ldots, H_K, \lambda H_i) \\
&= \left\{ \sum_{j \neq i} H_j p^{j^l} x^j \frac{\bar{\rho}(p^j, p^l, x^j, x^{j-1}, \bar{x}^j, x^{j+1}, \ldots, x^K, x^{K+1},}{H_i p^{i^l} x^i \bar{\rho}(p^i, p^l, x^i, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}}, \\
&\frac{H_{1, \ldots, H_{i-1}, (1-\lambda)H_{i}, H_{i+1}, \ldots, H_K, \lambda H_i)}{H_{1, \ldots, H_{i-1}, (1-\lambda)H_{i}, H_{i+1}, \ldots, H_K, \lambda H_i)} \right\}^{-1}
\end{align*}
\]

by (3.6)

\[
= \left\{ \frac{\sum_{j=1}^{K} H_j p^{j^l} x^j \bar{\rho}(p^j, p^l, X, H)}{H_i p^{i^l} x^i \bar{\rho}(p^i, p^l, X, H)} \right\}^{-1}
\]

by SCP and since \((1-\lambda)\bar{x}^i + \lambda x^{K+1} = x^l\)

\[
= \sigma^i(P, X, H), \text{ by (3.6)};
\]

\[
\bar{\sigma}^l(P, p^l, x^1, \ldots, x^{i-1}, \bar{x}^l, x^{i+1}, \ldots, x^K, x^{K+1},
\]

\[
\begin{align*}
&H_{1, \ldots, H_{i-1}, (1-\lambda)H_{i}, H_{i+1}, \ldots, H_K, \lambda H_i) \\
&= \left\{ \sum_{j \neq i} H_j p^{j^l} x^j \frac{\bar{\rho}(p^j, p^l, x^j, x^{j-1}, \bar{x}^j, x^{j+1}, \ldots, x^K, x^{K+1},}{(1-\lambda)H_i p^{i^l} x^i \bar{\rho}(p^i, p^l, x^i, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}}, \\
&\frac{H_{1, \ldots, H_{i-1}, (1-\lambda)H_{i}, H_{i+1}, \ldots, H_K, \lambda H_i)}{H_{1, \ldots, H_{i-1}, (1-\lambda)H_{i}, H_{i+1}, \ldots, H_K, \lambda H_i)} \right\}
\end{align*}
\]
\[ + H_i p^{j'} [(1-\lambda) x^l + \lambda x^{K+1}] \frac{\bar{\rho}(p', p^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)}{(1-\lambda) H_i p^{j'} x^l} \frac{\bar{\rho}(p', p^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)}{\bar{\rho}(p^l, p'^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)}, \]

\[ x^K, x^{K+1}, H_i, \ldots, H_i-1, (1-\lambda)H_i, H_{i+1}, \ldots, H_K, \lambda H_i) \]

\[ \frac{1}{x^K, x^{K+1}, H_i, \ldots, H_i-1, (1-\lambda)H_i, H_{i+1}, \ldots, H_K, \lambda H_i) \]^{-1},

by (3.6)

\[ = \left\{ (1-\lambda) \frac{p^{j'} x^l}{p'^l x^l} \bar{\sigma}(P, X, H) \right\}^{-1}, \]

by SCP and since \((1-\lambda)x^l + \lambda x^{K+1} = x^l \)

\[ (1-\lambda) p^{j'} x^l \sigma^i(P, X, H), \text{ by (3.6);} \]

\[ \sigma^{K+1}(P, p^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots, x^K, x^{K+1}, \]

\[ H_i, \ldots, H_i-1, (1-\lambda)H_i, H_{i+1}, \ldots, H_K, \lambda H_i) \]

\[ \sum_{j \neq i} \frac{H_j p^{j'} x^j}{\lambda H_i p^{j'} x^{K+1}} \frac{\bar{\rho}(p^j, p^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)}{\bar{\rho}(p^l, p'^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)} \]

\[ x^K, x^{K+1}, H_i, \ldots, H_i-1, (1-\lambda)H_i, H_{i+1}, \ldots, H_K, \lambda H_i) \]

\[ (1-\lambda)H_i, H_{i+1}, \ldots, H_K, \lambda H_i) \]

\[ + \frac{H_i p^{j'} [(1-\lambda) x^l + \lambda x^{K+1}] \frac{\bar{\rho}(p', p^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)}{\lambda H_i p^{j'} x^{K+1}} \frac{\bar{\rho}(p', p^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)}{\bar{\rho}(p^l, p'^l, x^l, \ldots, x^{l-1}, x^i, x^{i+1}, \ldots)}, \]

\[ x^K, x^{K+1}, H_i, \ldots, H_i-1, (1-\lambda)H_i, H_{i+1}, \ldots, H_K, \lambda H_i) \]

\[ \frac{1}{x^K, x^{K+1}, H_i, \ldots, H_i-1, (1-\lambda)H_i, H_{i+1}, \ldots, H_K, \lambda H_i) \]^{-1},

by (3.6)
= \left\{ \left[ \frac{\pi^{i'} x^{i+1}}{\pi^{i'} x^i} \right]^{-1} \left( \sum_{j=1}^{K} \frac{H_j \pi^{i'} x^i \rho(p^j, \pi^i, X, H)}{H_j \pi^{i'} x^i \rho(p^j, \pi^i, X, H)} \right) \right\}^{-1},

\text{by SCP and since } (1-\lambda)x^i + \lambda x^{i+1} = x^i

= \frac{\lambda \pi^{i'} x^{i+1}}{\pi^{i'} x^i} \sigma^i(P, X, H), \text{ by (3.6).} \Box

\text{Proof of Lemma 3.3.}

(i) For any } \bar{p}^i > p^i,

\rho(p^i, p^i, X, H) = \frac{H_i \pi^{i'} x^i \sigma^i(P, X, H)}{H_j \pi^{i'} x^i \sigma^i(P, X, H)}, \text{ by (3.7)}

\leq \frac{H_i \pi^{i'} x^i \bar{p}^i x^i \sigma^j(p^1, ..., p^{i-1}, \bar{p}^i, p^{i+1}, ..., p^K, X, H)}{H_j \pi^{i'} x^i \pi^{i'} x^i \sigma^i(p^1, ..., p^{i-1}, \bar{p}^i, p^{i+1}, ..., p^K, X, H)},

\text{by S11}

= \frac{H_i \pi^{i'} x^i \sigma^i(p^1, ..., p^{i-1}, \bar{p}^i, p^{i+1}, ..., p^K, X, H)}{H_j \pi^{i'} x^i \sigma^i(p^1, ..., p^{i-1}, \bar{p}^i, p^{i+1}, ..., p^K, X, H)}

= \rho(p^i, \bar{p}^i, X, H), \text{ by (3.7).}

(ii) \rho(p^i, \lambda p^i, X, H) = \frac{H_i (\lambda \pi^{i'}) x^i \sigma^i(p^1, ..., p^{i-1}, \lambda p^i, p^{i+1}, ..., p^K, X, H)}{H_j \pi^{i'} x^i \sigma^i(p^1, ..., p^{i-1}, \lambda p^i, p^{i+1}, ..., p^K, X, H)},

\text{by (3.7)}

= \frac{\lambda H_i \lambda \pi^{i'} x^i \sigma^i(P, X, H)}{H_j \pi^{i'} x^i \sigma^i(P, X, H)}, \text{ by S4}

= \lambda \rho(p^i, p^i, X, H), \text{ by (3.7).}
(iii) \[ \rho(\lambda p^i, \lambda p^i, \lambda^{-1}X, H) = \frac{H_1(\lambda p^i)(\lambda^{-1}x^i) \sigma^i(\lambda p, \lambda^{-1}X, H)}{H_1(\lambda p^i)(\lambda^{-1}x^i) \sigma^i(\lambda p, \lambda^{-1}X, H)} \], by (3.7)

\[ = \frac{H_1 p^i(\lambda \lambda^{-1}) x^i \sigma^i(p, X, H)}{H_1 p^i(\lambda \lambda^{-1}) x^i \sigma^i(p, X, H)}, \text{ by S5} \]

\[ = \rho(p^i, p^i, X, H), \text{ by (3.7) and since } \lambda \lambda^{-1} = I_N. \]

(iv) \[ \rho(\lambda p^i, \lambda p^i, \lambda^{-1}X, H) = \frac{H_1(\lambda p^i)(\lambda^{-1}x^i) \sigma^i(\lambda p, \lambda^{-1}X, H)}{H_1(\lambda p^i)(\lambda^{-1}x^i) \sigma^i(\lambda p, \lambda^{-1}X, H)} \], by (3.7)

\[ = \frac{H_1 p^i(\lambda \lambda^{-1}) x^i \sigma^i(p, X, H)}{H_1 p^i(\lambda \lambda^{-1}) x^i \sigma^i(p, X, H)}, \text{ by S7} \]

\[ = \rho(p^i, p^i, X, H), \text{ by (3.7) and since } \lambda \lambda^{-1} = I_N. \]

(v) Consider the bijective mapping \( \varphi : \mathcal{K} \to \mathcal{K} \) which satisfies (B.2) for any \( l \in \mathcal{K} \) and let (B.3) hold for all \( k \in \mathcal{K} \). Now,

\[ \rho(p^i, p^i, X \tilde{K}, \tilde{K}'H) = \rho(p^{\varphi(i)}, p^{\varphi(i)}, X \tilde{K}, \tilde{K}'H), \text{ by (B.4)} \]

\[ = \frac{H_{\varphi(i)} p^{\varphi(i)} x^{\varphi(i)} \sigma^{\varphi(i)}(P \tilde{K}, X \tilde{K}, \tilde{K}'H)}{H_{\varphi(j)} p^{\varphi(j)} x^{\varphi(j)} \sigma^{\varphi(j)}(P \tilde{K}, X \tilde{K}, \tilde{K}'H)} \], by (3.7)

\[ = \frac{H_1 p^i x^i \sigma^i(P, X, H)}{H_1 p^i x^i \sigma^i(P, X, H)}, \text{ by (B.4) and S6} \]

\[ = \rho(p^i, p^i, X, H), \text{ by (3.7).} \]

(vi) \[ \rho(p^i, p^i, \beta X, \gamma H) = \frac{(\gamma H_2)p^i(\beta x^i) \sigma^i(P, \beta X, \gamma H)}{(\gamma H_2)p^i(\beta x^i) \sigma^i(P, \beta X, \gamma H)} \], by (3.7)

\[ = \frac{H_1 p^i x^i \sigma^i(P, X, H)}{H_1 p^i x^i \sigma^i(P, X, H)}, \text{ by S4} \]

\[ = \rho(p^i, p^i, X, H), \text{ by (3.7).} \]
(vii) For any \((j, i) \in (\mathcal{K}\setminus\{l\}) \times (\mathcal{K}\setminus\{l\})\),

\[
\rho(p^j, p^i, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)
\]

\[
= \frac{H_i p^i x^i}{H_j p^j x^j} \frac{\sigma^j(P, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)}{\sigma^i(P, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)} , \text{ by (3.7)}
\]

\[
= \frac{H_i p^i x^i}{H_j p^j x^j} \frac{\sigma^j(P, X, H)}{\sigma^i(P, X, H)} \frac{1 + (\lambda - 1) \sigma^i(P, X, H)}{\lambda \sigma^i(P, X, H)} , \text{ by } S3
\]

\[
= \rho(p^j, p^i, X, H) , \text{ by (3.7)};
\]

\[
\rho(p^j, p^i, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)
\]

\[
= \frac{H_i p^i x^i}{H_j p^j x^j} \frac{\sigma^j(P, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)}{\sigma^i(P, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)} , \text{ by (3.7)}
\]

\[
= \frac{H_i p^i x^i}{H_j p^j x^j} \frac{\lambda \sigma^j(P, X, H)}{\lambda + (\lambda - 1) \sigma^i(P, X, H)} \frac{1 + (\lambda - 1) \sigma^i(P, X, H)}{\lambda \sigma^i(P, X, H)} , \text{ by } S3
\]

\[
= \rho(p^j, p^i, X, H) , \text{ by (3.7)};
\]

\[
\rho(p^j, p^i, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)
\]

\[
= \frac{H_i p^i x^i}{H_j p^j x^j} \frac{\sigma^j(P, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)}{\sigma^i(P, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)} , \text{ by (3.7)}
\]

\[
= \frac{H_i p^i x^i}{H_j p^j x^j} \frac{\lambda \sigma^j(P, X, H)}{\lambda + (\lambda - 1) \sigma^i(P, X, H)} \frac{1 + (\lambda - 1) \sigma^i(P, X, H)}{\lambda \sigma^i(P, X, H)} , \text{ by } S3
\]

\[
= \rho(p^j, p^i, X, H) , \text{ by (3.7)};
\]

\[
\rho(p^j, p^i, x^i, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H) = 1 , \text{ by } I (\Leftrightarrow P \land T)
\]

\[
= \rho(p^j, p^i, X, H) , \text{ by } I.
\]

(viii) \[
\rho(p^j, p^i, X \hat{H}, 1_K) = \frac{p^j i(H_i x^i) \sigma^j(P, X \hat{H}, 1_K)}{p^j i(H_j x^j) \sigma^i(P, X \hat{H}, 1_K)} , \text{ by (3.7)}
\]

\[
= \frac{H_i p^i x^i}{H_j p^j x^j} \frac{\sigma^j(P, X, H)}{\sigma^i(P, X, H)} , \text{ by } S13
\]
\[ \rho(p^i, p^j, X, H) = \rho(p^i, p^j, X, H), \text{ by (3.7).} \]

(ix) For any \((j, i) \in (\mathcal{K}\setminus \{i\}) \times (\mathcal{K}\setminus \{i\}), \)

\[ \bar{\rho}(p^i, p^j, X, x^i, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i) \]

\[ = \frac{H_i p^i x^i \bar{\sigma}^j(P, p^j, x^i, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i)}{H_j p^j x^j \bar{\sigma}^i(P, p^i, x^j, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i)}, \]

by (3.7)

\[ = \frac{H_i p^i x^i \bar{\sigma}^j(P, x^i, H)}{H_j p^j x^j \bar{\sigma}^i(P, x^j, H)}, \text{ by S8} \]

\[ = \rho(p^i, p^j, X, H), \text{ by (3.7).} \]

\[ \bar{\rho}(p^i, p^j, X, x^i, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i) \]

\[ = \frac{H_i p^i x^i \bar{\sigma}^j(P, p^j, x^i, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i)}{(1-\lambda)H_i p^i x^j \bar{\sigma}^i(P, p^i, x^j, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i)}, \text{ by (3.7)} \]

\[ = \frac{H_i p^i x^i (1-\lambda)\bar{\sigma}^j(P, x^i, H)}{(1-\lambda)H_i p^i x^j \bar{\sigma}^i(P, x^j, H)}, \text{ by S8} \]

\[ = \rho(p^i, p^j, X, H), \text{ by (3.7).} \]

if \(p^{K+1} = p^i\) then

\[ \bar{\rho}(p^i, p^{K+1}, X, x^i, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i) \]

\[ = \frac{\lambda H_i p^i x^i \bar{\sigma}^j(P, p^j, x^i, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i)}{H_j p^j x^j \bar{\sigma}^{K+1}(P, p^i, x^i, H_1, \ldots, H_l-1, (1-\lambda)H_i, H_{l+1}, \ldots, H_{\mathcal{K}}, \lambda H_i)}, \text{ by (3.7)} \]
\[
\frac{\lambda H_1 p_i^i x^i \sigma^i(P, X, H)}{H_j p^i x^i \lambda \sigma^j(P, X, H)} \quad \text{by S8}
\]

\[= \rho(p^j, p^i, X, H), \quad \text{by (3.7).} \]

(x) For any \((j, i) \in (\mathcal{K}\backslash \{l\}) \times (\mathcal{K}\backslash \{l\})\),

\[
\tilde{\rho}(p^j, p^i, x^i, \ldots, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l)
\]

\[= \frac{H_i p^i x^i \bar{\sigma}^i(P, p^i, x^i, \ldots, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l)}{H_j p^i x^i \bar{\sigma}^i(P, p^i, x^i, \ldots, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l)}, \quad \text{by (3.7)}
\]

\[= \frac{H_i p^i x^i \sigma^i(P, X, H)}{H_j p^i x^i \sigma^i(P, X, H)} \quad \text{by S14}
\]

\[= \rho(p^j, p^i, X, H), \quad \text{by (3.7);} \]

\[
\tilde{\rho}(p^j, p^i, x^i, \ldots, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l)
\]

\[= \frac{H_i p^i x^i \bar{\sigma}^i(P, p^i, x^i, \ldots, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l)}{(1-\lambda)H_i p^i x^i \bar{\sigma}^i(P, p^i, x^i, \ldots, x^{i-1}, \bar{x}^i, x^{i+1}, \ldots, x^K, x^{K+1}, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l)}, \quad \text{by (3.7)}
\]

\[= \frac{H_i p^i x^i \sigma^i(P, X, H)}{(1-\lambda)H_i p^i x^i \sigma^i(P, X, H)} \quad \text{by S14}
\]

\[= \rho(p^j, p^i, X, H), \quad \text{by (3.7);} \]

if \(p^{K+1} = p^i\) then
\[ \bar{\rho}(p^j, p^{K+1}, x^1, ..., x^{l-1}, \bar{x}^l, \bar{x}^{l+1}, ..., x^K, x^{K+1}, H_1, ..., H_{l-1}, (1-\lambda)H_i, H_{l+1}, ..., H_K, \lambda H_i) \]

\[ = \frac{\lambda H_1 p^{l_1} x^{K+1}}{H_1 p^{l_1} x^j} \bar{\sigma}^j(P, P^1, x^1, ..., x^{l-1}, \bar{x}^l, \bar{x}^{l+1}, ..., x^K, x^{K+1}, H_1, ..., H_{l-1}, (1-\lambda)H_i, H_{l+1}, ..., H_K, \lambda H_i), \text{ by (3.7)} \]

\[ = \frac{\lambda H_1 p^{l_1} x^{K+1}}{H_1 p^{l_1} x^j} \frac{p^{l_1} x^l}{\lambda p^{l_1} x^{K+1}} \sigma^l(P, X, H), \text{ by S14} \]

\[ = \rho(p^j, p^i, X, H), \text{ by (3.7).} \]

\[ \lim_{\lambda \to 0} \rho(p^j, p^i, X, H_1, ..., H_{l-1}, \lambda H_i, H_{l+1}, ..., H_K) \]

\[ = \lim_{\lambda \to 0} \frac{H_1 p^{l_1} x^i \sigma(P, p^j, X, x^j, H_1, ..., H_{l-1}, (1-\lambda)H_i, H_{l+1}, ..., H_K, \lambda H_i)}{H_j p^{l_1} x^j \sigma(P, p^j, X, x^j, H_1, ..., H_{l-1}, (1-\lambda)H_i, H_{l+1}, ..., H_K, \lambda H_i)}, \text{ by (3.7)} \]

\[ = \frac{H_1 p^{l_1} x^i \sigma(P, p^j, X, H_i)}{H_j p^{l_1} x^j \sigma(P, p^j, X, H_i)}, \text{ by S9} \]

\[ = \bar{\rho}(p^j, p^i, X, H_i). \]
Proof of Lemma 3.4. Necessity: By T,

\[ ρ(p^i, p^i, X, H) = \frac{ρ(p^i, p^i, X, H)}{ρ(p^i, p^i, X, H)}. \]

Since the left-hand side and, consequently, the right-hand side of this equation is independent of \( p^i \), it can be rewritten as

\[ ρ(p^i, p^i, X, H) = \frac{ρ(1_N, p^i, X, H)}{ρ(1_N, p^i, X, H)} = \frac{δ(p^i, X, H)}{δ(p^i, X, H)}, \]

where \( 1_N \) is the \( N \)-dimensional column vector of ones.

Sufficiency: Straightforward.

Proof of Theorem 3.5. By (3.6) and I (≡ P ∧ T),

\[ σ^i(P, X, H) = \left\{ \frac{\sum_{j=1}^{K} H_j p^{j,i} x^j}{\sum_{j=1}^{K} H_j p^{j,i} x^j} \right\}^{-1} \rho(p^i, p^i, X, H) \]

\[ = \left\{ \frac{\sum_{j=1}^{K} H_j p^{j,i} x^j}{\sum_{j=1}^{K} H_j p^{j,i} x^j} \right\}^{-1} \delta(p^i, X, H), \] by Lemma 3.4

\[ = \frac{H_i p^{i,i} x^i}{δ(p^i, X, H)} \left\{ \frac{\sum_{j=1}^{K} H_j p^{j,i} x^j}{\sum_{j=1}^{K} H_j p^{j,i} x^j} \right\}^{-1}, \]

\[ \square \]

Proof of Theorem 3.6. Positivity:

\[ ρ_{HD}(p^i, p^i, X, H) := \prod_{k=1}^{K} \left[ \frac{p^{i,k}}{p^{j,k}} \right]^{θ^k(H)} > 0. \]

Positive Monotonicity: For any \( p^i < p^j \),

\[ ρ_{HD}(p^i, p^i, X, H) := \prod_{k=1}^{K} \left[ \frac{p^{i,k}}{p^{j,k}} \right]^{θ^k(H)} \leq \prod_{k=1}^{K} \left[ \frac{p^{i,k}}{p^{j,k}} \right]^{θ^k(H)} =: ρ_{HD}(p^i, p^i, X, H). \]

Linear Homogeneity: For any \( λ \in \mathbb{R}_{++}, \)

\[ ρ_{HD}(p^j, λp^i, X, H) := \prod_{k=1}^{K} \left[ \frac{(λp^i)^{i,k}}{p^{j,k}} \right]^{θ^k(H)} = \lambda \prod_{k=1}^{K} \left[ \frac{p^{i,k}}{p^{j,k}} \right]^{θ^k(H)} =: λρ_{HD}(p^i, p^i, X, H). \]
Transitivity: For any \( l \in \mathcal{L} \),

\[
\rho_{HD}(p^i, p^l, X, H) \rho_{HD}(p^l, p^i, X, H) := \prod_{k=1}^{K} \left[ \frac{p^{i'}x^k}{p^{j'}x^k} \right] \theta^k(H)
\]

\[
= \prod_{k=1}^{K} \left[ \frac{p^{i'}x^k}{p^{j'}x^k} \right] \theta^k(H)
\]

\[
= \rho_{HD}(p^i, p^i, X, H).
\]

Commensurability: For any \( \lambda \in \mathbb{R}^{N}_{++} \),

\[
\rho_{HD}(\lambda p^i, \lambda p^i, \lambda^{-1}X, H) := \prod_{k=1}^{K} \left[ \frac{(\lambda p^i)'(\lambda^{-1}x^k)}{(\lambda p^i)'(\lambda^{-1}x^k)} \right] \theta^k(H)
\]

\[
= \prod_{k=1}^{K} \left[ \frac{p^{i'}(\lambda' \lambda^{-1})x^k}{p^{j'}(\lambda' \lambda^{-1})x^k} \right] \theta^k(H)
\]

\[
= \prod_{k=1}^{K} \left[ \frac{p^{i'}x^k}{p^{j'}x^k} \right] \theta^k(H) \quad \text{since } \lambda' \lambda^{-1} = I_N
\]

\[
= \rho_{HD}(p^i, p^i, X, H).
\]

Commodity Symmetry: For any permutation matrix \( \tilde{I}_N \),

\[
\rho_{HD}(\tilde{I}_N p^i, \tilde{I}_N p^i, \tilde{I}_N X, H) := \prod_{k=1}^{K} \left[ \frac{(\tilde{I}_N p^i)'(\tilde{I}_N x^k)}{(\tilde{I}_N p^i)'(\tilde{I}_N x^k)} \right] \theta^k(H)
\]

\[
= \prod_{k=1}^{K} \left[ \frac{p^{i'}(\tilde{I}_N' \tilde{I}_N) x^k}{p^{j'}(\tilde{I}_N' \tilde{I}_N) x^k} \right] \theta^k(H)
\]

\[
= \prod_{k=1}^{K} \left[ \frac{p^{i'}x^k}{p^{j'}x^k} \right] \theta^k(H) \quad \text{since } \tilde{I}_N' \tilde{I}_N = I_N
\]

\[
= \rho_{HD}(p^i, p^i, X, H).
\]
Weight Symmetry: Consider the bijective mapping $\varphi : \mathcal{K} \times \mathcal{K}$ which satisfies (B.2) for any $l \in \mathcal{K}$ and let (B.3) hold for all $k \in \mathcal{K}$. Now,

$$
\rho_{HD}(p^i, p^i, X^K, H) := \prod_{k=1}^{K} \left[ \frac{p_i^k x_{\varphi(k)}}{p_i^k x^k} \right] \theta^\varphi(k)(H) = \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x^k} \right] \theta^k(H) =: \rho_{HD}(p^i, p^i, X, H).
$$

Quantity Dimensionality: For any $(\beta, \gamma) \in \mathbb{R}^2_+$,

$$
\rho_{HD}(p^i, p^i, X, H) := \prod_{k=1}^{K} \left[ \frac{p_i^k (\beta x^k)}{p_i^k (\beta x^k)} \right] \theta^k(\gamma H) = \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x_k} \right] \theta^k(H) =: \rho_{HD}(p^i, p^i, X, H).
$$

Strong Quantity Dimensionality: For any $l \in \mathcal{K}$ and for any $\lambda \in \mathbb{R}^1_+$,

$$
\rho_{HD}(p^i, p^i, x^l, \ldots, x^{l-1}, \lambda x^l, x^{l+1}, \ldots, x^K, H) := \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x_k} \right] \theta^k(H) \left[ \frac{p_i^k (\lambda x^k)}{p_i^k (\lambda x^k)} \right] \theta^k(H) = \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x_k} \right] \theta^k(H) =: \rho_{HD}(p^i, p^i, X, H).
$$

Determinateness: For any $n \in \mathcal{N}$ and for any $l \in \mathcal{K},$

$$
\lim_{p_n^k \to 0} \rho_{HD}(p^i, p^i, X, H) := \lim_{p_n^k \to 0} \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x_k} \right] \theta^k(H) = \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x_k} \right] \theta^k(H) > 0;
$$

$$
\lim_{p_n^k \to 0} \rho_{HD}(p^i, p^i, X, H) := \lim_{p_n^k \to 0} \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x_k} \right] \theta^k(H) = \prod_{k=1}^{K} \left[ \frac{p_i^k x_k}{p_i^k x_k} \right] \theta^k(H) > 0;
$$
\[ \lim_{x_1^n \to 0} \rho_{BD}(p^i, p^j, X, H) := \lim_{x_1^n \to 0} \prod_{k=1}^K \left[ \frac{p_{i,x^k}}{p_{j,x^k}} \right]^{\theta_k(H)} \]

\[ := \prod_{k \neq i} \left[ \frac{p_{i,x^k}}{p_{j,x^k}} \right]^{\theta_k^j(H)} \left[ \frac{p_{j,x^i}}{p_{j,x^i}} \right]^{(1-\lambda)\theta_k^i(H)} \]

\[ > 0 ; \]

\[ \lim_{H_1 \to 0} \rho_{BD}(p^i, p^j, X, H) := \lim_{H_1 \to 0} \prod_{k=1}^K \left[ \frac{p_{i,x^k}}{p_{j,x^k}} \right]^{\theta_k(H)} \]

\[ := \prod_{k \neq i} \left[ \frac{p_{i,x^k}}{p_{j,x^k}} \right]^{\theta_k(H)} \left[ \frac{p_{j,x^i}}{p_{j,x^i}} \right]^{\theta^i(H)} \]

\[ > 0 . \]

Country Partitioning Test: For any \( l \in \mathcal{L} \) and for any \( \lambda \in (0, 1) \),

\[ \rho_{BD}(p^i, p^j, X, x^l, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l) \]

\[ := \prod_{k \neq i} \left[ \frac{p_{i,x^k}}{p_{j,x^k}} \right]^{\theta_k^j(H)} \left[ \frac{p_{j,x^i}}{p_{j,x^i}} \right]^{(1-\lambda)\theta^i(H)} \lambda \theta^i(H) \]

\[ =: \rho_{BD}(p^j, p^i, X, H) . \]

Tiny Country Irrelevance: For any \( l \in \mathcal{L} \),

\[ \lim_{\lambda \to 0} \rho_{BD}(p^i, p^j, X, H_1, \ldots, H_{l-1}, \lambda H_l, H_{l+1}, \ldots, H_K) \]

\[ := \lim_{\lambda \to 0} \prod_{k \neq i} \left[ \frac{p_{i,x^k}}{p_{j,x^k}} \right]^{H_k/(\Sigma_m H_m + \lambda H_l)} \left[ \frac{p_{j,x^i}}{p_{j,x^i}} \right]^{\lambda H_l/(\Sigma_m H_m + \lambda H_l)} \]
PROOF OF COROLLARY 3.6.1. Since \( \rho_{BD} \) satisfies \( P \),

\[
\sigma_{BD}(P, X, H) := \left\{ \sum_{j=1}^{K} H_i p^{i'} x^{j} \rho_{BD}(p^{i'}, p^{i}, X, H) \right\}^{-1} > 0 .
\]

Now,

\[
\sum_{i=1}^{K} \sigma_{BD}(P, X, H) = \sum_{i=1}^{K} \left\{ \sum_{j=1}^{K} H_i p^{i'} x^{j} \rho_{BD}(p^{i'}, p^{i}, X, H) \right\}^{-1}
= \frac{\sum_{j=1}^{K} H_j p^{i'} x^{j} \prod_{k=1}^{K} [p^{i'} x^{j}]^{-\theta_k(H)}}{\sum_{j=1}^{K} H_j p^{i'} x^{j} \prod_{k=1}^{K} [p^{i'} x^{j}]^{-\theta_k(H)}}
= 1 .
\]

Thus, \( \sigma_{BD} \) satisfies \( S_1 \). By Lemma 3.2, \( \sigma_{BD} \) satisfies \( S_2 \sim S_{12} \). □

PROOF OF THEOREM 3.7. Straightforward.

PROOF OF COROLLARY 3.7.1. Straightforward.

PROOF OF THEOREM 3.8. Positivity:

\[
\rho_{AB}(p^{i}, p^{i}, X, H) := \frac{p^{i'} \sum_{k=1}^{K} H_k x^k}{p^{i'} \sum_{k=1}^{K} H_k x^k} > 0 .
\]

Positive Monotonicity: For any \( p^{i} < \bar{p}^{i} \),

\[
\rho_{AB}(p^{i}, p^{i}, X, H) := \frac{p^{i'} \sum_{k=1}^{K} H_k x^k}{p^{i'} \sum_{k=1}^{K} H_k x^k} \leq \frac{\bar{p}^{i'} \sum_{k=1}^{K} H_k x^k}{\bar{p}^{i'} \sum_{k=1}^{K} H_k x^k} =: \rho_{AB}(p^{i}, \bar{p}^{i}, X, H) .
\]
Linear Homogeneity: For any $\lambda \in \mathbb{R}_{++}$,
\[
\rho_{AB}(p^i, \lambda p^i, X, H) := \frac{(\lambda p^i)' \sum_{k=1}^{K} H_k x^k}{p^i' \sum_{k=1}^{K} H_k x^k} = \lambda \frac{p^i' \sum_{k=1}^{K} H_k x^k}{p^i' \sum_{k=1}^{K} H_k x^k} =: \lambda \rho_{AB}(p^i, p^i, X, H).
\]

Transitivity: For any $j \in \mathcal{K}$,
\[
\rho_{AB}(p^j, p^j, X, H)\rho_{AB}(p^j, p^j, X, H) := \frac{p^j' \sum_{k=1}^{K} H_k x^k}{p^j' \sum_{k=1}^{K} H_k x^k} = \frac{p^j' \sum_{k=1}^{K} H_k x^k}{p^j' \sum_{k=1}^{K} H_k x^k} =: \rho_{AB}(p^j, p^j, X, H).
\]

Commensurability: For any $\lambda \in \mathbb{R}^N_{++}$,
\[
\rho_{AB}(\lambda p^j, \lambda p^j, \lambda^{-1}X, H) := \frac{(\lambda p^j)' \sum_{k=1}^{K} H_k(\lambda^{-1}x^k)}{(\lambda p^j)' \sum_{k=1}^{K} H_k(\lambda^{-1}x^k)} = \frac{p^j(\lambda' \lambda^{-1}) \sum_{k=1}^{K} H_k x^k}{p^j(\lambda' \lambda^{-1}) \sum_{k=1}^{K} H_k x^k} = \frac{p^j' \sum_{k=1}^{K} H_k x^k}{p^j' \sum_{k=1}^{K} H_k x^k} \quad \text{since } \lambda' \lambda^{-1} = I_N
\]
\[
=: \rho_{AB}(p^j, p^j, X, H).
\]

Commodity Symmetry: For any permutation matrix $\tilde{I}_N$,
\[
\rho_{AB}(\tilde{I}_NP^j, \tilde{I}_NP^j, \tilde{I}_NX, H) := \frac{(\tilde{I}_NP^j)' \sum_{k=1}^{K} H_k(\tilde{I}_N x^k)}{(\tilde{I}_NP^j)' \sum_{k=1}^{K} H_k(\tilde{I}_N x^k)}
\]

Weight Symmetry: Consider the bijective mapping $\varphi: \mathcal{X} \times \mathcal{X}$ which satisfies (B.2) for any $l \in \mathcal{X}$ and let (B.3) hold for all $k \in \mathcal{X}$. Now,

$$\rho_{AB}(p^i, p^j, X, H) := \frac{p^{ij}(\tilde{I}_N, \tilde{I}_N) \sum_{k=1}^{K} H_k x^k}{p^{ij}(\tilde{I}_N, \tilde{I}_N) \sum_{k=1}^{K} H_k x^k} = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} =: \rho_{AB}(p^i, p^j, X, H).$$

Quantity Dimensionality: For any $(\beta, \gamma) \in \mathbb{R}_+^2$,

$$\rho_{AB}(p^i, p^j, \beta X, \gamma H) := \frac{p^{ij} \sum_{k=1}^{K} (\gamma H_k)(\beta x^k)}{p^{ij} \sum_{k=1}^{K} (\gamma H_k)(\beta x^k)} = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} =: \rho_{AB}(p^i, p^j, X, H).$$

Total Quantities Test:

$$\rho_{AB}(p^i, p^j, X, H, 1_K) := \frac{p^{ij} \sum_{k=1}^{K} (1)(H_k x^k)}{p^{ij} \sum_{k=1}^{K} (1)(H_k x^k)} = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} =: \rho_{AB}(p^i, p^j, X, H).$$

Determinateness: For any $n \in \mathcal{N}$ and for any $l \in \mathcal{X},$

$$\lim_{p^k \to 0} \rho_{AB}(p^i, p^j, X, H) := \lim_{p^k \to 0} \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} > 0;$$

$$\lim_{p^i \to 0} \rho_{AB}(p^i, p^j, X, H) := \lim_{p^i \to 0} \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} > 0;$$
\[ \lim_{x_n \to 0} \rho_{AB}(p^i, p^j, X, H) := \lim_{x_n \to 0} \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} > 0 ; \]

\[ \lim_{H_l \to 0} \rho_{AB}(p^i, p^j, X, H) := \lim_{H_l \to 0} \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} > 0 . \]

**Strong Country Partitioning Test:** For any \( l \in K \) and for any \( \lambda \in (0, 1) \),

\[ \tilde{\rho}_{AB}(p^j, p^i, x^1, \ldots, x^{l-1}, x^l, x^{l+1}, \ldots, x^K, x^{K+1}, H_1, \ldots, H_{l-1}, (1-\lambda)H_l, H_{l+1}, \ldots, H_K, \lambda H_l) \]

\[ := \frac{p^{ij} \left[ \sum_{k=l+1}^{K} H_k x^k + (1-\lambda)H_l x^l + \lambda H_l x^{K+1} \right]}{p^{ij} \left[ \sum_{k=l+1}^{K} H_k x^k + (1-\lambda)H_l x^l + \lambda H_l x^{K+1} \right]} \]

\[ = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} =: \rho_{AB}(p^j, p^i, X, H) . \]

**Tiny Country Irrelevance:** For any \( l \in K \),

\[ \lim_{\lambda \to 0} \rho_{AB}(p^j, p^i, X, H_1, \ldots, H_{l-1}, \lambda H_l, H_{l+1}, \ldots, H_K) \]

\[ := \lim_{\lambda \to 0} \frac{p^{ij} \left[ \sum_{k=1}^{K} H_k x^k + \lambda H_l x^l \right]}{p^{ij} \left[ \sum_{k=1}^{K} H_k x^k + \lambda H_l x^l \right]} \]

\[ = \frac{p^{ij} \sum_{k=1}^{K} H_k x^k}{p^{ij} \sum_{k=1}^{K} H_k x^k} =: \tilde{\rho}_{AB}(p^j, p^i, X_{-l}, H_{-l}) . \]
PROOF OF COROLLARY 3.8.1. Since $\rho_{AB}$ satisfies $P$,

$$\sigma^i_{AB}(p, X, H) := \left\{ \sum_{j=1}^K \frac{H_j p^{ij} x^j \rho_{AB}(p^i, p^i, X, H)}{H_i p^{ij} x^j \rho_{AB}(p^i, p^i, X, H)} \right\}^{-1} > 0.$$ 

Now,

$$\sum_{i=1}^K \sigma^i_{AB}(p, X, H) = \sum_{i=1}^K \left\{ \sum_{j=1}^K \frac{H_j p^{ij} x^j p^{ij} \sum_{k=1}^K H_k x^k}{H_i p^{ij} x^j p^{ij} \sum_{k=1}^K H_k x^k} \right\}^{-1}$$

$$= \frac{\sum_{i=1}^K H_i p^{ij} x^j p^{ij} \sum_{k=1}^K H_k x^k}{\sum_{j=1}^K H_j p^{ij} x^j p^{ij} \sum_{k=1}^K H_k x^k}$$

$$= 1.$$ 

Thus, $\sigma_{AB}$ satisfies S1. By Lemma 3.2, $\sigma_{AB}$ satisfies S2 and S4–S14. □

PROOF OF THEOREM 3.9. The bilateral consumption index $\phi$ has the following seven properties.

Q1. Positivity:

$$\phi(p^i, p^i, x^i, x^i) := \frac{p^{ij} x^i}{p^{ij} x^j} / \overline{p}(p^i, p^i, x^i, x^i, 1, 1) > 0,$$ by $P$.

Q2. Identity: If $p^i = \lambda p^j$ and $x^i = x^j$ then

$$\phi(p^i, p^i, x^i, x^i) := \frac{(\lambda p^j)^i x^i}{p^{ij} x^j} / \overline{p}(p^i, p^i, x^i, x^i, 1, 1)$$

$$= \frac{\lambda}{\lambda}, \text{ by PP}$$

$$= 1.$$ 

Q3. Proportionality:

$$\phi(p^i, p^i, H_j x^j, H_i x^i) := \frac{p^{ij}(H_j x^j)}{p^{ij}(H_i x^i)} / \overline{p}(p^i, p^i, H_j x^j, H_i x^i, 1, 1)$$
\[ \frac{H_i}{H_j} p_i^{i'} x_i^{i'} / \tilde{p}(p^i, p^i, x^i, x^i, 1, 1), \text{ by SQD} \]

\[ =: \frac{H_i}{H_j} \phi(p^i, p^i, x^i, x^i). \]

**Q4. Strong Monetary Unit Test:**

\[ \phi(\alpha_j p^j, \alpha_i p^i, \beta x^i, \beta x^i) := \frac{(\alpha_j p^j)'(\beta x^i)}{(\alpha_i p^i)'(\beta x^i)} / \tilde{p}(\alpha_j p^j, \alpha_i p^i, \beta x^i, \beta x^i, 1, 1) \]

\[ = \frac{\alpha_i p_i^{i'} x_i^{i'}}{\alpha_j p_j^{i'} x_j^{i'}} \left\{ \frac{\alpha_i}{\alpha_j} \tilde{p}(p^i, p^i, x^i, x^i, 1, 1) \right\}^{-1}, \]

by H, HDM (\( \equiv H \land CR \)) and QD (\( \equiv SQD \))

\[ =: \phi(p^i, p^i, x^i, x^i). \]

**Q5. Commensurability:**

\[ \phi(\hat{\lambda} p^j, \hat{\lambda} p^i, \hat{\lambda}^{-1} x^i, \hat{\lambda}^{-1} x^i) := \frac{(\hat{\lambda} p^j)'(\hat{\lambda}^{-1} x^i)}{(\hat{\lambda} p^i)'(\hat{\lambda}^{-1} x^i)} / \tilde{p}(\hat{\lambda} p^j, \hat{\lambda} p^i, \hat{\lambda}^{-1} x^i, \hat{\lambda}^{-1} x^i, 1, 1) \]

\[ = \frac{p_i^{i'}(\hat{\lambda} \hat{\lambda}^{-1}) x_i}{p_j^{i'}(\hat{\lambda} \hat{\lambda}^{-1}) x_j} / \tilde{p}(p^i, p^i, x^i, x^i, 1, 1), \text{ by C} \]

\[ =: \phi(p^i, p^i, x^i, x^i) \text{ since } \hat{\lambda} \hat{\lambda}^{-1} = I_N. \]

**Q6. Country Reversal:**

\[ \phi(p^i, p^i, x^i, x^i) := \frac{p_i^{i'} x_i^{i'}}{p_j^{i'} x_j^{i'}} / \tilde{p}(p^i, p^i, x^i, x^i, 1, 1) \]

\[ = \left\{ \frac{p_i^{i'} x_i^{i'}}{p_j^{i'} x_j^{i'}} / \tilde{p}(p^i, p^i, x^i, x^i, 1, 1) \right\}^{-1}, \text{ by CR and WS} \]

\[ =: 1 / \phi(p^i, p^i, x^i, x^i). \]
Q7. Commodity Symmetry:

\[ \phi(\bar{I}_N p^i, \bar{I}_N p^i, \bar{I}_N x^j, \bar{I}_N x^j) := \frac{(\bar{I}_N p^i)'(\bar{I}_N x^j)}{(\bar{I}_N p^i)'(\bar{I}_N x^j)} / \bar{p}(\bar{I}_N p^i, \bar{I}_N p^i, \bar{I}_N x^j, \bar{I}_N x^j, 1, 1) \]

\[ = \frac{p^{i'}(\bar{I}_N' \bar{I}_N)x^j}{p^{i'}(\bar{I}_N' \bar{I}_N)x^j} / \bar{p}(p^i, p^i, x^j, x^j, 1, 1) , \text{ by CS} \]

\[ =: \phi(p^i, p^i, x^j, x^j) \text{ since } \bar{I}_N' \bar{I}_N = I_N. \]

Since \( \phi \) satisfies Q1–Q7, \( \sigma_{k*} \) satisfies S1–S5, S7 and S8 by Diewert (1986, Prop. 8).

For \( j \neq i \neq k \),

\[
\frac{\sigma_{k*}(p^1, \ldots, p^{i-1}, \bar{p}^i, p^{i+1}, \ldots, p^K, X, H)}{\sigma_{k*}(P, X, H)}
= \frac{H_i \phi(p^k, p^i, x^k, x^i) + \Sigma_{i=1}^K H_i \phi(p^k, p^i, x^k, x^i)}{\Sigma_{m \neq i} H_i \phi(p^k, p^m, x^k, x^m) + H_i \phi(p^k, p^i, x^k, x^i)}
\]

\[ = \frac{p^{i'} x^j H_i \{ \bar{p}^{i'} x^j / p^{i'} x^j \bar{p}(p^k, p^i, x^k, x^j, 1, 1) / \sigma_{k*}(P, X, H) \}}{\bar{p}^{i'} x^j \Sigma_{m \neq i} H_i \phi(p^k, p^m, x^k, x^m) + H_i \phi(p^k, p^i, x^k, x^i)} , \text{ by (3.15)} \]

\[ \geq \left[ \frac{\bar{p}^{i'} x^j}{p^{i'} x^j} \right]^{-1} \frac{H_i \{ \bar{p}^{i'} x^j / p^{i'} x^j \bar{p}(p^k, p^i, x^k, x^j, 1, 1) / \sigma_{k*}(P, X, H) \}}{\Sigma_{m \neq i} H_i \phi(p^k, p^m, x^k, x^m) + H_i \phi(p^k, p^i, x^k, x^i)} , \text{ by M} \]

\[ = \left[ \frac{\bar{p}^{i'} x^j}{p^{i'} x^j} \right]^{-1} \frac{\sigma_{k*}(p^1, \ldots, p^{i-1}, \bar{p}^i, p^{i+1}, \ldots, p^K, X, H)}{\sigma_{k*}(P, X, H)} , \text{ by (3.15) and (3.14)}. \]

Thus, \( \sigma_{k*} \) satisfies S11.
Now,

\[ \sigma^i_k(P, X, H, 1_k) := \frac{\phi(p^k, p^i, H_k x^k, H_i x^i)}{\sum^K_{i=1} \phi(p^k, p^i, H_k x^k, H_i x^i)} \]

\[ = \frac{[p^{i'}(H_i x^i)/p^{k'}(H_k x^k)]/\bar{p}(p^k, p^i, H_k x^k, H_i x^i, 1, 1)}{\sum^K_{i=1} [p^{i'}(H_i x^i)/p^{k'}(H_k x^k)]/\bar{p}(p^k, p^i, H_k x^k, H_i x^i, 1, 1)}, \text{ by SQD} \]

\[ = \frac{H_i \phi(p^k, p^i, x^k, x^i, 1, 1)}{\sum^K_{i=1} H_i \phi(p^k, p^i, x^k, x^i, 1, 1)}, \text{ by (3.15)} \]

\[ =: \sigma^i_k(P, X, H). \]

Thus, \( \sigma^i_k \) satisfies S13.

Since

\[ \sigma^i_k(P, X, H) := \frac{H_i \phi(p^k, p^i, x^k, x^i)}{H_j \phi(p^k, p^j, x^k, x^j)} \]

depends on prices other than \( p^j \) and \( p^i \), there does not exist a restricted-domain consumption index satisfying RT with \( \sigma := \sigma^i_k \). Therefore, \( \rho^i_k \) is not a restricted-domain PPP index. \( \square \)

**Proof of Theorem 3.10.** Since \( \phi \) satisfies Q1–Q7, \( \sigma_{EKS} \) satisfies S1–S7 by Diewert (1986, Prop. 8). The remaining parts are straightforward.

**Proof of Theorem 3.11.** Since \( \phi \) satisfies Q1–Q7, \( \sigma_{QS} \) satisfies S1, S2 and S4–S10 by Diewert (1986, Prop. 8). The remaining parts are straightforward.
PROOF OF THEOREM 3.12. Since $\phi$ satisfies Q1–Q7, $\sigma_{DW}$ satisfies S1, S2 and S4–S7, $\sigma_{PW}$ satisfies S1, S2 and S5–S10, and $\sigma_{QW}$ satisfies S1, S2 and S4–S10 by Diewert (1986, Prop. 8). The remaining parts are straightforward.

PROOF OF THEOREM 3.13. Since $X \in \mathbb{R}_+^{NK}$, $\sigma_{GK}$ satisfies S1, S2 and S4–S9 by Diewert (1986, Prop. 13). For all $k \in \mathcal{K}$, let $\sigma_k^i := \sigma_{GK}^k(P, X, \lambda H_1, \ldots, \lambda H_{j-1}, H_j, \lambda H_{j+1}, \ldots, \lambda H_{i-1}, H_i, \lambda H_{i+1}, \ldots, \lambda H_K)$. To show that $\sigma_{GK}$ satisfies S10, substitute for $\pi_n$ in (3.22a) using (3.22b), replace $H_k$ by $\lambda H_k$ for all $k \in \mathcal{K}\setminus\{j, i\}$ and take the limit as $\lambda \to 0$:

\[
\sum_{n=1}^{N} \left[ \sum_{k \neq j, i}^{K} \lim_{\lambda \to 0} \frac{H_k x_n^k}{H_i x_n^i} + H_j x_n^j \right] \frac{p_n^i x_n^i}{H_i x_n^i} = \sum_{n=1}^{N} \frac{H_i x_n^i}{H_i x_n^i} \frac{p_n^i x_n^i}{p^i x^i}
\]

\[
\sum_{n=1}^{N} \frac{H_i x_n^i}{H_i x_n^i} \frac{p_n^i x_n^i}{H_i x_n^i} = \frac{\left( \frac{H_j x_n^j}{H_i x_n^i} \right)}{\left( \frac{H_i x_n^i}{H_i x_n^i} \right)} = \frac{\left( \frac{H_j x_n^j}{H_i x_n^i} \right)}{\left( \frac{H_i x_n^i}{H_i x_n^i} \right)}
\]

The remaining parts are straightforward.

PROOF OF THEOREM 3.14. Proportionality: For all $k \in \mathcal{K}$, let $\sigma_k := \sigma_{PH}^k(P, x^1, \ldots, x^{i-1}, \lambda x^i, x^{i+1}, \ldots, x^K, H)$ and let $\sigma_k := \sigma_{PH}(P, X, H)$. For $i \neq l$,

\[
\sum_{k \neq l}^{K} H_k^2 \frac{p_k^i x_k^i}{p_l^i x_l^i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k} + H_l^2 \frac{p_l^i x_l^i}{p_l^i x_l^i} \frac{\bar{\sigma}_i}{\bar{\sigma}_l} = \sum_{k \neq l}^{K} H_k^2 \frac{p_k^i x_k^i}{p_k^i x_k^i} \frac{\bar{\sigma}_k}{\bar{\sigma}_i} + H_l^2 \frac{p_l^i x_l^i}{p_l^i x_l^i} \frac{\bar{\sigma}_l}{\bar{\sigma}_i}
\]

\[
\sum_{k=1}^{K} H_k^2 \frac{p_k^i x_k^i}{p_l^i x_l^i} \frac{\sigma_i}{\sigma_k} = \sum_{k=1}^{K} H_k^2 \frac{p_k^i x_k^i}{p_l^i x_l^i} \frac{\sigma_k}{\sigma_i}
\]
\[ \forall i \neq l, \quad \frac{\bar{\sigma}_i}{\sigma_i} = \lambda \frac{\sigma_i}{\sigma_i} \quad \text{and} \quad \frac{\bar{\sigma}_k}{\sigma_i} = \frac{\sigma_k}{\sigma_i} \quad \text{for all } k \in \mathcal{K}\setminus\{l\}. \]

Country Partitioning: For all \( k \in \mathcal{X} \), let \( \bar{\sigma}_k := \sigma^k_{\mathcal{X}H}(P, p^l, X, x^l, H_1, \ldots, H_{i-1}, (1 - \lambda)H_i, H_{i+1}, \ldots) \), \( H_K, \lambda H_l \) and let \( \sigma_k := \sigma^k_{\mathcal{X}H}(P, X, H) \). For \( i \neq l \),

\[
\sum_{k \neq l} H_k^2 \frac{p^k_i x^k_i}{p^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k} + [(1 - \lambda)H] H_i^2 \frac{p_{i+l}^j x^j_i}{p_{i+l}^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_l} + (\lambda H_i) H_i^2 \frac{p_{i+l}^j x^j_i}{p_{i+l}^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k}
\]

\[
= \sum_{k \neq l} H_k^2 \frac{p^k_i x^k_i}{p^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k} + H_i^2 \frac{p_{i+l}^j x^j_i}{p_{i+l}^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_l} + H_i^2 \frac{p_{i+l}^j x^j_i}{p_{i+l}^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k}
\]

\[ \Rightarrow \sum_{k=1}^{K} H_k^2 \frac{p^k_i x^k_i}{p^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k} = \sum_{k=1}^{K} H_i^2 \frac{p_{i+l}^j x^j_i}{p_{i+l}^l x^l_i} \frac{\bar{\sigma}_i}{\bar{\sigma}_k} \]

\[ \Rightarrow (1 - \lambda) \frac{\bar{\sigma}_i}{\bar{\sigma}_l} = \lambda \frac{\sigma_i}{\sigma_l} = \frac{\sigma_i}{\sigma_l} \quad \text{and} \quad \frac{\bar{\sigma}_k}{\sigma_i} = \frac{\sigma_k}{\sigma_l} \quad \text{for all } k \in \mathcal{K}\setminus\{l\}. \]

Thus,

\[ \sum_{k=1}^{K+1} \frac{\bar{\sigma}_k}{\bar{\sigma}_l} = \sum_{k \neq l} \frac{\sigma_k}{\sigma_l} + (1 - \lambda) \frac{\sigma_l}{\sigma_l} + \lambda \frac{\sigma_l}{\sigma_l} \]

\[ \Rightarrow \bar{\sigma}_i = \sigma_i \quad \text{since} \quad \sum_{k=1}^{K+1} \bar{\sigma}_k = \sum_{k=1}^{K} \sigma_k = 1 \]

\[ \Rightarrow \bar{\sigma}_i = (1 - \lambda)\sigma_l \quad \text{and} \quad \bar{\sigma}_{K+1} = \lambda \sigma_l. \]

Strong Dependence on a Bilateral Formula: For all \( k \in \mathcal{X} \), let \( \sigma^k_{\mathcal{X}H}(P, X, \lambda H_1, \ldots, \lambda H_{j-1}, H_i, \lambda H_{i+1}, \ldots, \lambda H_K) \). Now,
\[
\sum_{k \neq i,j} \lim_{\lambda \to 0} (\lambda H_k)^2 \frac{p^{i'}x^k}{p^{i'}x^i} \lim_{\lambda \to 0} \left( \frac{\sigma_i}{\sigma_{k,j}} \right) + H_i^2 \frac{p^{i'}x^j}{p^{i'}x^i} \lim_{\lambda \to 0} \left( \frac{\sigma_i}{\sigma_{k,j}} \right)
\]

\[
= \sum_{k \neq i,j} H_i^2 \frac{p^{i'}x^i}{p^{i'}x^j} \lim_{\lambda \to 0} \left( \frac{\sigma_k}{\sigma_{i,j}} \right) + H_i^2 \frac{p^{i'}x^j}{p^{i'}x^j} \lim_{\lambda \to 0} \left( \frac{\sigma_j}{\sigma_{i,j}} \right)
\]

\[
\Rightarrow \lim_{\lambda \to 0} \frac{\sigma_i}{\sigma_j} = \left[ \frac{p^{i'}(H_i x^i) p^{i'}(H_i x^i)}{p^{i'}(H_j x^j) p^{i'}(H_j x^j)} \right]^{1/2} =: \phi_F(p^i, p^i, H_j x^i, H_i x^i).
\]

The remaining parts are straightforward.
APPENDIX C

PROOFS OF THEOREMS IN CHAPTER 4

Proof of Theorem 4.1. By T, for any \( l \in \mathcal{K} \),

\[
\rho_{ij}^l(P, X, H) \rho_{ij}^l(P, X, H) = \rho_{ij}^l(P, X, H) .
\]

Thus, by P,

\[
\rho_{ij}^l(P, X, H) = 1 .
\]

Now,

\[
\rho_{ij}^l(P, X, H) = \frac{\rho_{ij}^l(P, X, H) \rho_{ij}^l(P, X, H)}{\rho_{ij}^l(P, X, H)} , \text{ by P}
\]

\[
= \frac{\rho_{ij}^l(P, X, H)}{\rho_{ij}^l(P, X, H)} , \text{ by T}
\]

\[
= \frac{1}{\rho_{ij}^l(P, X, H)} , \text{ by W.L.O.}
\]

Proof of Theorem 4.2. First, by (4.2), there is a real number \( \Delta_{A,B} \in \text{range } \Delta \) which is associated with any two elements \( A \) and \( B \) of \( \mathcal{P} \). Clearly, \( \Delta_{A,B} > 0 \) if \( B \neq A \). If \( B = A \) then

\[
\Delta_{A,A} = \frac{2}{K(K-1)} \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \ln \left( \frac{\rho_{ij}^A}{\rho_{ij}^A} \right) - \ln \left( \frac{\rho_{ij}^A}{\rho_{ij}^A} \right) = 0 .
\]

Next,

\[
\Delta_{A,B} = \frac{2}{K(K-1)} \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \ln \left( \frac{\rho_{ij}^B}{\rho_{ij}^A} \right) - \ln \left( \frac{\rho_{ij}^B}{\rho_{ij}^A} \right)
\]

\[
= \frac{2}{K(K-1)} \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \ln \left( \frac{\rho_{ij}^A}{\rho_{ij}^B} \right) - \ln \left( \frac{\rho_{ij}^A}{\rho_{ij}^B} \right)
\]

\[
= \Delta_{B,A} .
\]
Finally, for any \( C \in \mathcal{F} \),

\[
\Delta_{A,B} = \frac{2}{K(K-1)} \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \left| \ln \left( \frac{\rho_{h_{ij}}^b}{\rho_{h_{ij}}^a} \right) - \ln \left( \frac{\rho_{h_{ij}}^b}{\rho_{h_{ij}}^a} \right) \right|
\]

\[
\leq \frac{2}{K(K-1)} \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \left( \left| \ln \left( \frac{\rho_{h_{ij}}^c}{\rho_{h_{ij}}^a} \right) \right| + \left| \ln \left( \frac{\rho_{h_{ij}}^b}{\rho_{h_{ij}}^a} \right) \right| + \left| \ln \left( \frac{\rho_{h_{ij}}^c}{\rho_{h_{ij}}^a} \right) \right| \right)
\]

\[
= \Delta_{A,C} + \Delta_{C,B} \quad \Box
\]

**Proof of Theorem 4.3.** By S1,

\[
\rho^i(P, X, H) := \frac{H_i p^{ix_i}}{H_j p^{ix_j}} \frac{\sigma_i(P, X, H)}{\sigma_i(P, X, H)} > 0.
\]

Next,

\[
\rho^i(P, X, H)\rho^i(P, X, H) := \frac{H_i p^{ix_i}}{H_j p^{ix_j}} \frac{\sigma_i(P, X, H)}{\sigma_i(P, X, H)} \frac{\sigma_i(P, X, H)}{\sigma_i(P, X, H)} = \rho^i(P, X, H).
\]

Finally, from (4.3),

\[
\frac{H_i p^{ix_i}}{H_i p^{ix_i}} \rho^i(P, X, H) \sigma^i(P, X, H) = \sigma^i(P, X, H)
\]

\[
\Rightarrow \sum_{j=1}^{K} \frac{H_i p^{ix_i}}{H_i p^{ix_i}} \rho^i(P, X, H) \sigma^i(P, X, H) = \sum_{j=1}^{K} \sigma^i(P, X, H)
\]

\[
\Rightarrow \sigma^i(P, X, H) = \left\{ \sum_{j=1}^{K} \frac{H_i p^{ix_i}}{H_i p^{ix_i}} \rho^i(P, X, H) \right\}^{-1} \text{, by S1.} \quad \Box
\]
**Proof of Theorem 4.4.** By (4.7) and (4.9),

\[ \rho^\| (\bar{P}, \bar{X}, H) = \rho^\| ((\hat{\rho}^{\text{US}})^{-1} P, \beta \hat{\rho}^{\text{US}} X, H), \]

where \( \hat{\rho}^{\text{US}} \) is the \( N \times N \) diagonal matrix with \( \hat{\rho}^{\text{US}}_{nn} = \rho^{\text{US}}_n \) for all \( n \in \mathcal{N} \). The required equivalence follows from setting \( \lambda := (\hat{\rho}^{\text{US}})^{-1} \). \( \Box \)

**Proof of Theorem 4.5.** By (4.7) and (4.9),

\[ \sigma(\bar{P}, \bar{X}, H) = \sigma((\hat{\rho}^{\text{US}})^{-1} P, \beta \hat{\rho}^{\text{US}} X, H). \]

The required equivalence follows from setting \( \lambda := (\hat{\rho}^{\text{US}})^{-1} \). \( \Box \)