PRICING PERISHABLE INVENTORIES BY USING MARKETING RESTRICTIONS WITH APPLICATIONS TO AIRLINES

By

Michael Zhi-Feng Li

B. Sc. (Mathematics) Beijing Normal University

Ph. D. (Mathematics) University of Regina

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

FACULTY OF COMMERCE AND BUSINESS ADMINISTRATION

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April 1994

© Michael Zhi-Feng Li, 1994

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Faculty of Commerce and Business Administration
The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1Z1

Date: April 18, 1994

Abstract

This thesis addresses the problem of pricing perishable inventories such as airline seats and hotel rooms. It also analyzes the airline seat allocation problem when two airlines compete on a single-leg flight. Finally, several existing models for seat allocation with multiple fares on a single-leg flight are compared.

The pricing framework is consistent with modern yield management tools which utilize restrictions such as weekend stayover to segment the market. One model analyzed considers a restriction which is irrelevant to one set of consumers, but which the others find so onerous that they will not purchase a restricted ticket at any price. If the consumers who do not mind the restriction are less price sensitive than those who find the restriction onerous, then the thesis shows that there is an optimal policy for a monopolist which will sell fares at no more than three price levels.

When two restrictions are allowed in the model, if one is more onerous than the other in the sense that the set of consumers who would not buy a ticket with the first restriction is a subset of those who would not buy it with the second restriction, then the restrictions are said to be *nested.* If the sets of consumers who would not buy tickets with the first restriction is disjoint from those who would not buy with the second restriction, then the restrictions are said to be *mutually exclusive.* If two restrictions are either nested or mutually exclusive, then a monopolist needs at most four price levels with three types (i.e. combinations of restrictions) of product. With two general restrictions, the monopolist may need five price levels with four types of product.

The pricing model is applied to restrictions which are based on membership in a particular organization. For example, employees of an airline are frequently eligible

for special fares. Some airlines provide special fares for government employees or for employees of certain corporations. An analysis is given to help airlines understand the costs and benefits of such arrangements.

A model of two airlines competing on a single-leg flight is developed for the case where the airlines have fixed capacity and fixed price levels for two types of fares — full and discount. The airlines compete by controlling the number of discount fares which they sell. The split of the market between the airlines is modelled in two different ways. First, the airlines might share the market for a fare class proportionally to their allocation of seats to that fare class. In this case, under certain conditions, there exists an equilibrium pair of booking limits for the discount fare such that each airline will protect the same number of seats for the full fare customers, even when the demands are random and stochastically dependent. The second market sharing model assumes that the two airlines share the market demand equally. In this case, when the demands are deterministic, then there is an equilibrium solution where each airline will protect enough seats to split equally the market for the full fares.

Finally, three existing seat allocation models for multi-fare single-leg flights with stochastically independent demands are compared. It is shown that the optimality conditions for each of these models are analytically equivalent, thus providing a unified approach to this problem.

Table of Contents

List of Tables

Acknowledgement

With sincere appreciation I first acknowledge the advice and encouragement from the members of my supervisory committee, Professors Shelby L. Brumelle, Tae H. Oum and Michael W. Tretheway. In particular, I want to thank my supervisor, Professor Shelby L. Brumelle, for his tireless effort of helping me to finish this dissertation. I have also benefited a great deal from Professor Michael W. Tretheway for helping me to improve the presentation of this thesis.

I am deeply grateful to the Faculty of Commerce and Business Administration of the University of British Columbia for supporting my first two years of graduate studies through a MacPhee Memorial Fellowship. I feel very fortunate that I had been awarded a University Graduate Fellowship by the Faculty of Graduate Studies of the University of British Columbia during the early stage of this dissertation. I am also indebted to Professor Shelby L. Brumelle for supporting me in the last two years of my study.

Chapter 1

Introduction and Background

1.1 Introduction

Since the first important paper of Littlewood (1972,[131]) on the airline seat allocation problem, the subject of yield management has become a main topic for revenue enhancement for most airlines in the world. For example, American Airlines, a leader in airline yield management research and implementation, has experienced a tremendous success in the last decade or so due at least in part to its yield management system. It is estimated that during the period of 1990 to 1992, the annual benefit to American Airlines from its yield management programs was approximately 500 million dollars, or about a 2 percent improvement on total passenger revenues.[1]

Yield management is far from completely understood. For example, yield management is sometimes considered to be a *pricing mechanism.*[2] But actually, yield management is a tool to assist firms in determining the maximum number of reservations to accept at given price levels so as to maximize total expected revenues. This is not the same as setting prices — prices are taken as given. The research so far on yield management shows no indication that it is capable of providing the optimal set of prices for the firm. In fact, any seat inventory control system must have a set of pre-specified prices so that demand can be forecasted at these price levels. For airlines, the actual pricing

[1]Refer to Smith et al (1992, [221]) for a survey on yield management practice at American Airlines.

[2]This view is often held by professionals from hotel industry, for example, Relihan (1989,[193]) and Hanks et al (1992, [83]).

decisions are traditionally made separately, by a different department.[3] The *first goal of this thesis* is to enhance seat management techniques by developing a tactical pricing model for perishable inventories like airline seats and hotel rooms. In fact, the main part of this thesis is dedicated to this goal. The key innovation is to explicitly incorporate the use of artificial restrictions as marketing mechanisms. In particular, the questions I want to get answers include: (1) is it necessary to use the restrictions? (2) what are the optimal pricing structures? (3) do we have a tractable characterization for the optimal pricing structures? (4) how can the model be used in industries, such as airlines?

Another shortcoming of the existing yield management literature is that airlines do not operate in isolation of each other. The existing research in yield management still does not address the issue of competitive response to other airlines' seat allocation and pricing decisions. To analyze the strategic aspect of airline operations, we need to understand the nature of the strategic interaction among airlines. An early view was that the airline industry is a contestable industry.[4] But empirical studies show that the airline market is not perfectly contestable, which implies that the strategic interaction among airlines is oligopolistic. Typical oligopolistic behaviour has three forms: (1) *price-type competition*, also known as Bertrand competition, where price is the primary strategic variable; (2) *quantity-type competition*, also known as Cournot competition, where quantity is the primary strategic variable; and (3) *collusion*, where firms collude with each other to exploit their monopolistic power. A recent empirical study by Brander and Zhang (1990, [33]) shows that the competition among airlines is a quantity-type competition. This evidence suggests that it is important to develop a new generation of seat allocation

[3]There is a trend that some airlines now use a single department to handle pricing and yield management together. This suggests that airlines are starting to realize the importance of coordinating the pricing decisions and inventory control decisions.

[4]An industry is *contestable* when firms charge the same prices and supply the same quantities as a competitive industry would, even there are only a few firms in the industry. A key argument that the airline industry is contestable is that airplanes – the key assets for airlines – are *mobile*.

models which explicitly consider competitive quantity decision responses. The *second goal* of this thesis is to make *the first step* toward a full understanding of the seat allocation problem in the presence of competition.

It is well-known that modern yield management techniques critically depend on a proper understanding of the seat allocation problem for single-leg flights with multiple fares. The most recent development along this direction include three independently developed articles, Curry (1990, [52]), Wollmer (1992, [269]), and Brumelle and McGill (1993, [38]), which solve the seat allocation problem for single-leg flights with multiple fares when the random demands for different fare classes are independent. It is interesting to note that these three papers have utilized three different analytical tools to determine optimal booking policies; and they have three different optimality rules for booking policies. This raises the issue as to which approach is the best and which one has a computational advantage over the other two. A later part of this thesis clarifies this issue by showing that all three approaches are in fact analytically equivalent in the sense that the optimal policy rules in fact are identical. The thesis provides a unified approach that may shed useful light in dealing with the seat allocation problem for multi-fare single-leg flights with dependent random demands.

Before moving on to the main development of the thesis, brief overviews of airline seat inventory control and airline fare pricing are presented. These overviews provide the necessary background information needed to evaluate the contributions of this thesis and serve as motivation for the general class of problems to be addressed.

1.2 Airline Seat Inventory Control

1.2.1 The Seat Allocation Problem for Single-leg Flights

This is the most studied area in yield management. For a flight with two fare classes with independent demand, an optimal booking policy has been characterized by Littlewood (1972, [131]). He proves that the optimal *protection level* η^* *for the high fare*[5] is given by:

$$\eta^* = \text{Min}\{\eta \geq 0 : f_l \geq f_h P(Y > k - \eta)\}$$

where f_l and f_h are the fare price for a low fare seat and a high fare seat respectively, Y is the demand for the high fare class, k is the flight capacity, and $\text{P}(\cdot)$ denotes a probability distribution. The formula gives a *decision rule* that tells an airline reservations controller what to do when a booking request for the low fare arrives. Since the airline will not reject a high fare booking, the issue is when to *stop booking* the low fare tickets. The formula simply says that the airline should accept bookings for the low fare until the *expected* seat revenue from an uncertain high fare sale is greater than the certain low fare revenue f_l.[6] If the demands are continuous random variables, then we have the following closed-form formula:

$$f_l = f_h P(Y > k - \eta^*).$$

This result was generalized by Brumelle et al (1990, [37]) to the case of two-fare flights with *dependent* demands.

The seat allocation problem for single-leg flights with multiple fares with independent demands has been solved independently by Brumelle and McGill (1993, [38]), Curry

[5] A *protection level* for the high fare is the total number of seats that will be *protected* (i.e., *reserved*) for the high fare customers.

[6] It is worthwhile to mention that the main reason why the Littlewood's formula involves the sure value of a low fare ticket — f_l — is that each low fare booking, when it arrives, corresponds to a sure sale. On the other hand, if there is no booking request for the low fare, then the decision of rejecting or accepting a booking for the low fare never occurs. So by default, the formula is still valid.

(1990, [52]), and Wollmer (1992, [269]). Belobaba (1987,[13], and 1989, [14]) introduced an heuristic method — the so-called *Expected Marginal Seat Revenue* (EMSR) method — to handle the seat allocation problem for a single-leg flight with multiple fare classes. The major advantage of this method is computational since it is a technique based on Littlewood's result. Consequently, if the random demands are independent, then the EMSR method gives optimal solutions for any *two fare classes in isolation*, but unfortunately this will not be optimal for the whole problem.[7] The current challenge is to address the seat allocation problem for general multi-fare flights (that is, not limited to two fares) with dependent demands.

1.2.2 The Seat Allocation Problem for Multi-leg Flights

This problem turns out to be a major headache for all airlines. If we ignore the randomness of the demands, then the seat allocation problem for multi-leg flights can be formulated as a network flow optimization problem. This had been a main focus of research for one decade or so.[8] But there are two main drawbacks to this approach:

1. It cannot handle random demands;

2. It is of little use for modern computer reservation systems.[9]

To my knowledge, there is no major airline in the world that has actually implemented this approach. What the airlines need is a framework that gives either optimal or heuristic solutions and is compatible with their current reservations systems.

[7]Simulation studies show that (1) the difference between the revenue derived from the EMSR method and the optimal revenue is only 0.5 percent; and (2) the optimal booking policies may be drastically different from the policies derived from the EMSR heuristic. Refer to Brumelle et al (1990, [37]) for a reason why revenue functions are insensitive to booking limits.

[8]For example, refer to Ladany and Hersh (1977, [118]), Hersh and Ladany (1978, [89]), Glover et al (1982, [80]), and Dror, Trudeau and Ladany (1988, [63]).

[9]This will become more clear from the discussions that follow.

There are two other basic approaches in dealing with the seat allocation problem for multi-leg flights, which are the *segment-based method*[10] and the *revenue-based method.* Segment-based methods allocate seats on a flight in an effort to *maximize revenue on each individual flight segment*, independent of other flight segments. Currently, most airlines manage seat inventories by flight segment only. A typical segment-based seat allocation model involves two stages:

1. The airline first decides how many seats that will be made available for each segment; and

2. With the given number of seats allocated to each segment, the airline specifies booking policies for fare classes on each specific segment.

As claimed in Williamson and Belobaba (1988, [260]), several major airlines had implemented the EMSR method, which was originated by Littlewood for single-leg flights and was later extended by Belobaba (1987, [12]) for multi-leg flights. The main idea from the EMSR model is to protect seats for a higher fare class as long as the expected marginal revenue of the seats is greater than marginal revenues of the seats at a lower fare class. In the multi-leg EMSR approach, protection levels and booking limits are determined for each fare or each nest[11] on each flight segment. The *main drawback* for the segment-based approach is the lack of consideration for the *interaction of traffic across flight segments*

[10] A *segment* is an origin-destination (O-D) itinerary on the same flight number. For example, if legs A-H and H-B have the same flight number, then there are three separate segments: A-H, H-B and A-B. Refer to Curry (1990, [52]) for these definitions.

[11] A nest is a group of similar fares on the same O-D on the flight. For example, for a two-leg flight A-B via H, the airline may pool all fare classes on the segment H-B into a single nest. Clearly, on each segment, the maximum number of nests is the total number of fare classes on the segment. Traditionally, airlines offer as many as 15 fare classes on each flight segment, which implies that for a two-leg flight, there will be 45 different fare classes on the flight and the CRS must keep track of booking information for all these 45 fare classes regularly. Therefore, grouping fares on each segment into 4 or 5 different nests will substantially reduce the amount of information that needs to be displayed on the CRS. This is precisely why the deterministic network approach is not implementable in the current reservation systems.

within the network in the determination of seat allocations and booking limits. On the other hand, the major advantage of a segment-based approach is its *compatibility* with current airline reservations systems.[12]

Revenue-based methods have been developed and implemented by American Airlines. The main ideas are:

- *Indexing* all fare classes on a flight on the basis of the absolute ticket revenue of each fare class, where a lower index means a higher priority;[13] and
- Grouping all those fare classes with the same index, regardless of their O-D itineraries, into a *bucket.*

In practice, only the buckets — not the actual inventory of seats by fare classes — are displayed on the reservations systems. This is why this method is often known as the *virtual nesting method.* Consequently, a bucket is also known as *a virtual inventory class.* The optimization aspects of a revenue-based approach include:[14]

- Development of a systematic method of indexing fare classes and a simple procedure of updating the indices during the booking period; and
- Finding the protection levels, or equivalently the booking limits, for each bucket to maximize total revenues of the flight.

[12]There are several extensions for Belobaba's EMSR methods, for example, refer to Wong (1990, [270]), Belobaba (1991, [15]), and most recently Wong et al (1992, [271]).

[13]This is a critical component of any revenue-based method. Unfortunately, there is no public literature that addresses this issue. In the meantime, it is worthwhile to mention that this problem is also an interesting theoretical issue. Clearly, the most efficient and consistent way of indexing fare classes on the whole network is to have an index table that include all fare classes on all flights on the network, rather than indexing fare classes on a flight-by-flight basis. So the theoretical challenge here is to develop a model for such an *universal indexing table.* Of course, the first question is whether or not such a table is possible.

[14]Relevant papers on this problem include Buhr (1982, [41]), Wang (1983, [253]), Simpson (1985, [217]), Simpson (1985, [218]), Smith and Penn (1988, [222]), Curry (1989, [51]), Curry (1990, [52]), Curry et al (1990, [54]), and Smith et al (1992, [221]).

Fare Class	*Segment A–B*	*Segment A–H*	*Segment H–B*
Y	\$1000	\$750	\$700
M	\$800	\$500	\$470
Q	\$540	\$300	\$290
B	\$330	\$220	\$200

Table 1.1: An Example of Fare Classes for a Two-leg Flight

Buckets (Virtual Classes)	*O-D Classes*
V_0	Y_{AB}
V_1	Y_{AH}, M_{AB}, Y_{HB}
V_2	M_{AH}, B_{AB}, M_{HB}
V_3	B_{AH}, Q_{AB}, B_{HB}
V_4	Q_{AH}, Q_{HB}

Table 1.2: An Example of Virtual Nesting Classes for a Two-leg Flight

According to Smith et al (1992, [221]), American Airlines have also used the EMSR method in determining the booking limits for each bucket.

To see how this approach works, let us consider Flight 101 from A to B via the hub city H. This is a two-leg flight with three O-D itineraries, that is, A-B, A-H and H-B. Suppose that on each O-D itinerary, the airline offers four fare classes, Y, M, Q and B. Refer to Table 1.1 for the made-up revenues for each of the 16 fare classes for this flight. For the purpose of illustration, assume that the airline plans to group these fares into *five buckets*, labeled as V_0, V_1, V_2, V_3, and V_4. Refer to Table 1.2 for one specification of these five buckets. In the actual booking process, suppose there is a request for the fare class M on segment H-B of Flight 101. When a travel agent looks at the system and if there still are some seats available for bucket V_2, the agent can authorize this booking. But during the reservation process, the actual fare class M_{HB} on Flight 101 never shows up.

1.2.3 The Seat Allocation Problem for Multiple Flights

Equally important, airlines need to address the seat allocation problem in the presence of other flights. This becomes increasingly important because of the hub-and-spoke networks airlines have developed especially since U.S. airline deregulation in 1978. Most flights originating from a hub will pick up a substantial amount of connecting traffic from other flights destined to the hub. Without explicitly incorporating the demand from the connecting traffic, a yield management system cannot realize the full potential revenue for each flight. For example, consider Flight 201 from city H (the hub) to city B and Flight 202 from city A to city H. There are two key questions that are important to airlines:

- *How much should the airline charge a traveller who flies from city A to city B using flights 201 and 202 with a restricted ticket? And how much should be charged for an unrestricted ticket?*
- *How many seats should be allocated on flight 202 for travellers from city A to city B?*

These problems are closely related, but are not simple. Along this direction, except for the network flow approach, there is no other theoretical work done in this area.[15]

1.2.4 The Overbooking Problem

Since airline seats cannot be consumed before the time of the flight, they are sold in the form of reservations. This causes an operating problem for airlines, since some customers will either cancel their reservations at the last minute or simply not show up. Since the

[15]Refer to Ladany (1977, [116]), Hersh and Ladany (1978, [89]), Glover et al (1982, [80]) and Dror et al (1988, [63]).

airline has no time to resell these tickets, some seats will be unused. With overbooking,[16] the airline can generate additional revenue on these otherwise empty seats. The airline overbooking problem has two important components:

- Determining the optimal overbooking levels, which are associated with the booking limits for fare classes or buckets; and
- In the event that the flight is overbooked at the departure time, specifying a procedure for denying boardings to some passengers.

The first part is an optimization problem. Along this direction, pioneering research can be traced back to Beckmann (1958a, [9]), Beckmann (1958b, [10]), Thompson (1961, [235]) and Taylor (1962, [234]). Further analytical development has been studied in many papers in the seventies.[17] For most recent development, refer to Rothstein (1985, [203]), Brumelle et al (1990, [37]), and Bodily and Pfeifer (1992, [24]).[18]

The second issue is an implementation problem. The fact is that when no-shows and cancellations are low, the flight becomes truly oversold, which means that the number of passengers expecting to board the flight is larger than the flight seat capacity. Consequently, some passengers must be *bumped*, or denied boarding. This issue had been a focal point in the early stage of overbooking practice. The early practice of involuntary bumping created some serious consequences, including lawsuits against airlines. Eventually, the Simon-auction method, proposed by Simon (1968, [215]), was implemented by many airlines. The basic idea of a Simon-auction is: the airline asks passengers who do not mind being bumped to submit a sealed bid for compensation and the airline will

[16]Simply speaking, we say that an airline overbooks a flight if during the booking period, the total number of accepted bookings for the flight is larger than the flight-capacity.

[17]For example, refer to Rothstein (1971a, [199]), Rothstein (1971b, [200]), Simon (1972, [216]), Vickrey (1972, [250]), Rothstein (1975, [202]) and Shlifer and Vardi (1975, [214]).

[18]For the overbooking problem in the context of hotels, refer to Rothstein (1974, [201]), and Liberman and Yechiali (1977, [127]; 1978, [128]).

choose those passengers with the lowest bids to be bumped. Since the whole process is voluntary, it has no legal complications for the airline. Nowadays, most airlines use variations of the original Simon-auction in their practice.

1.2.5 The Seat Allocation Problem in the Presence of Competition

So far we know very little about how to manage the competition problem in the context of yield management. A recent study by Brander and Zhang (1990, [33]) shows that the Cournot model is more consistent for the data in their study than either the Bertrand or cartel models. This is a very interesting and important result for seat management research, since the quantity competition problem in the airline industry is closely related to the seat allocation problem under competition.

Consider that two airlines each have a scheduled flight from city A to city B, and that there are two fixed fares f_h and f_l, which both carriers charge. With random demand for both of the two fare classes, each airline's strategic decision variable is its *protection level for the high fare*, or equivalently, its *booking limit for the low fare*, since one airline's decision on its protection level for the high fare will have revenue impact on the other airline.[19] This *seat allocation game* can be analyzed in several ways. The most natural approach is to treat the game as a *bimatrix game*, since the protection level for the high fare is an integer bounded by the flight capacity.[20]

On the other hand, there are also many different theories of oligopoly in economics

[19]The key issue here is *how to split the demand* when both airlines allocate a positive number of seats for a particular fare class.

[20]A bimatrix game is a non-cooperative two-person non-zero sum game with a finite set of pure strategies for each player. There is a substantial amount of literature on bimatrix games. For example, refer to Bohnenblust et al (1950, [29], Nash (1950, [162]; 1951, [163]), Shapley and Snow (1950, [212]), Vorob'ev (1958, [252]), Kuhn (1961, [112]), Lemke and Howson (1964, [123]), Mangasarian (1964, [137]), Millham (1968, [147]; 1972, [148]; 1974, [149]), Raghavan (1970, [188]), Owen (1971, [175]), Parthasarathy and Raghavan (1971, [177]), Kreps (1974, [111]), Shapley (1974, [211]), Heuer (1975, [90]; 1979, [91]), Heuer and Millham (1976, [92]), Jansen (1981a, [93]; 1981b, [94]), and Okada (1984, [165]).

and marketing that are useful for airlines to gain a general understanding of the strategic behaviour of the firms. But they are not very useful at the operational level.[21] Therefore, I am not going to investigate the allocation game along this direction. For an excellent survey on theories of oligopoly, refer to Shapiro (1989, [210]).[22]

1.3 Airline Fare Pricing

1.3.1 Introduction

Since U.S. airline deregulation in 1978, airlines have enjoyed complete freedom in setting fare levels. In fact, airline fare pricing has evolved from the pre-deregulation simple and static practice to a more dynamic and complex structure. Marketing innovations have been developed and used to segment consumers into groups with different demand elasticities and to stimulate market demand and create consumers loyalty. Two classical examples are the use of artificial restrictions on discount seats and the establishment of frequent-fliers programs. On the other hand, even though the application of these marketing innovations is well-known in the airline industry, there is very little academic attention in economics and marketing literature to the theoretical implications of these marketing innovations.[23] In fact, in the vast literature of pricing research in economics

[21]In the context of homogeneous product with capacity constraints, refer to Levitan and Shubik (1972, [125]), Kreps and Scheinkman (1983, [110]), Allen and Hellwig (1986, [2]), Davidson and Deneckere (1986, [59]), and Klemperer and Meyer (1986, [107]). In the context of differentiated products, refer to Dixit and Stiglitz (1977, [61]), Hart (1979, [87]), Salop (1979, [205]), Shaked and Sutton (1982, [208]), and Singh and Vives (1984, [219]), Bulow et al (1985, [40]), Mas-Colell (1975, [138]), Moorthy (1985, [153]), Perloff and Salop (1985, [178]), Moorthy (1988, [154]), Anderson et al (1989, [5]), Economides (1989, [71]), and Cremer et al (1991, [47]).

[22]Also, general discussions on the competition issue in the airline industry can be found in many other papers. For example, refer to Graham et al (1983, [82]), Levine (1987, [124]), Reiss and Spiller (1989, [192]), Tretheway (1989, [244]), Brueckner (1990, [36]), Hansen (1990, [85]), Morrison and Winston (1990, [157]), Sorenson (1990, [223]), and Strassmann (1990, [233]). These studies are policy-oriented and they are useful for us to have a better understanding of the airline industry, in particular, airline deregulation.

[23]A recent article of Gale and Homles (1993, [74]) studies the economic impact of using *advance-purchasing* in the context of airlines. The primary focus of this paper is on the *mechanism design*.

and marketing, there is no analytical framework that is directly applicable to airline fare pricing.[24] In this section, we will briefly discuss several aspects of airline operations in connection with marketing restrictions.

1.3.2 Current Operating Environment

Before moving on to the discussion of the modelling practice in the economics literature that may be relevant to airline fare pricing, let us first summarize some recent corporate changes in the airline industry and some unique operating characteristics related to airline fare pricing. First of all, as it is well-known, an airline fare class is characterized by a price and a set of *restrictions.* The restrictions are intended to prevent business travellers from buying deep discount fares targeted to leisure travellers. The main fare pricing decisions for an airline include:

- Evaluating the impact of restrictions and choosing profit maximizing restrictions; and
- Choosing profit maximizing prices and deciding how many fare classes to offer.

One of the main consequences of U.S. airline deregulation is the emergence of complicated fare structures. Before deregulation, airlines had limited freedom in setting fare prices and selecting the type of fare classes on a particular route. After deregulation, the fare structure had become so complicated that on some flights, there are over 15 different fare classes with all kind of restrictions or fences. This in turn means that a two-leg flight will have 45 different fare classes. For carriers with a large operating network, the maintenance of such a complicated fare structure became a major operating challenge.

[24]There are many empirical studies of airline demand. For example, refer to Oum and Gillen (1983, [171]) and Oum et al (1986, [172]). These studies, together with traditional economic theories, can provide basic guidelines for airlines. But with the sheer number of flights and the diverse nature of different city pairs, their usefulness is very limited for daily operations and pricing decisions.

In a surprising move in April of 1992, American Airlines (AA) suggested a simplified fare structure and hoped that other major airlines would cooperate. Even though such an idea won a widespread praise from other airlines and the travel agency industry, this proposal was short-lived. By the end of April of 1992, none of U.S. major airlines including AA decided to fully implement it. Such a resistance might be in part due to the fact that airlines did not know how to price their products by explicitly incorporating the marketing restrictions they had been using for many years. In fact, there was no theoretical justification for such a simplified fare structure, and AA did not provide any satisfactory explanation.[25] There is another interesting development recently emerging from the airline industry. Again lead by American Airlines, many airlines in the world started to merge the traditional separate functional departments for pricing and yield management into one department.[26] This indicates that airlines now realize the importance of coordinating the pricing decisions with yield management techniques.

With these new developments in the marketplace, it is urgent for airlines to develop a tactical pricing model on a flight-by-flight basis that is consistent with their existing yield management system. The main task of this thesis is to take this challenge. In particular, I will develop a general pricing model for perishable inventories, such as airline seats and hotel rooms, which explicitly incorporates the use of artificial restrictions.[27]

Since the pricing model in this thesis is intended to be applicable for airlines, it is important to understand some basic, but unique, characteristics of the airline business that are related to fare pricing. There include:

[25]Since AA had never explicitly revealed *where the new simplified fare structure come from*: the rationale behind the simplified fare structure is a secret.

[26]Other examples include British Airways and Cathay Pacific. I have checked with additional major airlines through telephone inquiries and have found that even though they have two different functional departments, the interaction between the two is on a daily basis.

[27]As we will see later, the pricing model developed in this thesis will have optimal fare structures that at least in part *coincide* with the new simplified fare structure suggested by AA, that is, offering three coach fare classes with at least one unrestricted coach fare.

- The airline product is perishable and consumers can neither store nor enjoy the product before a given time;
- Capacity is fixed;
- Costs are either fixed or sunk;
- Airlines are operating in a network environment;
- Market demand is easily segmented, for example, leisure travellers and business travellers;
- Airlines practice price discrimination by use of artificial restrictions;
- Matching competitors' prices is the only way to survive because of the *commodity nature* of airline products; and
- Sophisticated computer reservation systems are powerful tactical tools to allow airlines to make better decisions on allocating a limited number of seats among many different fare classes, which is very difficult for competitors to mimic because these are an airline's *private information.*

The traditional view of pricing perishable products, such as produce and dairy products, is that the price will *decrease* as the product approaches the end of its life. This is mainly due to the fact that:

- such a product can be consumed at any time during its life span; and
- all consumers prefer to purchase and consume the product early.

For airline seats, travellers will enjoy the pleasure and feel the value *all in the same period*, that is, the period of the flight time. Because of this, consumers for airline

seats can *delay* their purchasing decisions. The striking implication of this behaviour is that airlines simply cannot *sell down.* If they do so, rational consumers may wait for the cheapest price.[28] Therefore, it suggests that the traditional pricing theories for perishable products are not useful for airlines at all and new approaches must be called for.

1.3.3 Modelling Practice

Apparently the main reason that an airline uses marketing restrictions, such as Saturday-night stayover, advance booking, and corporate membership, is to effectively segment market demand in order (1) to serve each market with a proper product and service, and (2) to improve the airline's revenue by extracting the maximum possible revenue from each market segment. Therefore, a precise understanding of the impact of marketing restrictions must be an integral part of airline fare pricing. This is also important to aircraft manufacturers since the seat configuration in an aircraft is the basis for airlines to offer different products and services.

On the other hand, modern yield management tools, in particular, the seat inventory control models, require the airline to pre-specify a set of prices so that yield management specialists can conduct demand forecasting on this set of prices and make proper decisions on booking requests for different fare classes.[29] Airline seats on a particular flight invariably are put on sale several months in advance of the actual departure time. Therefore airlines must make pricing decisions before the inventory control models are in effect. The traditional approach was to offer a large number of fare classes and let

[28]Of course, there are some consumers who just want to purchase the seats early in order to have their reservations confirmed. The point here is that consumers no longer have any incentive to purchase early.

[29]In the actual implementation of seat allocation models, *demand forecasting* is critical. Littlewood (1972, [131]) had some discussions on this issue. The latest development is the *censored regression model* of McGill (1989, [144]).

the yield management system take care of the rest. As a consequence, airline yield management specialists face a very complicated fare structure for each flight. But the lack of progress on the theoretical work for the optimal seat inventory control problem for multi-fare flights with random dependent demands becomes a major setback for such an approach since it is even difficult to evaluate the performance of heuristic methods. Furthermore, complicated fare structures require more manpower and other resources, which is again a cost burden to airlines.

One of the main features of the modern airline fare pricing is that airlines offer multiple fare classes on each flight. Then it is natural that we should try to find answers from economic pricing theories that suggest the use of multiple prices. Let us start with the argument, due to Lott and Roberts (1991, [132]), that airline fare pricing is a form of *product differentiation.* According to Lott and Roberts, the main reason why those customers travelling on short notice are paying a substantially higher price than those who can book early is *the cost of the service*: providing the consumer with an "ability to purchase a ticket at the last minute"(p. 21). They argue that

> In the case of airline seats, the opportunity cost of keeping some seats available until the last minute is that they may go unused. The limited number of seats that airlines make available for advance purchase discounts shows how easy it is for airline to sell advance tickets. If they could not sell many of these discount tickets, the restriction on the number sold would be superfluous. This also explains why airlines have penalties for cancellations of these discount fares and not for reservations at the higher price. ...
>
> For airlines to be willing to hold seats for last minute travellers, they must make the same expected revenue from these seats as they do from those seats

> purchased in advance. ...(p.22)

They then conclude that

> If this explanation is correct, we should observe similar pricing for the "perishable" goods with uncertain demand. ...
>
> ...Our explanation does not prove that pricing anomalies are due only to cost differences. ...Our explanation is able to explain, however, why advanced reservation discounts exist even on highly competitive routes after deregulation. (p.22)

But in their footnote 10, they feel that the use of restrictions is still a puzzle in airline fare pricing:

> Another puzzle is the requirement that consumers have to spend a Saturday night at their destination in order to receive the discount. This is typically explained as an example of price discrimination against business travellers, but it may only be a form of peak-load pricing if those who stay over Saturday night travel on Sunday, the quietest time of the week. The puzzle remains as to why there is not an explicit discount for returning on Sunday, but this is also a problem for the price discrimination explanation. (p.21)

These qualitative arguments are indeed interesting. But I do not think that Lott and Roberts provide a convincing argument as to why the traditional product differentiation model is appropriate for airline fare pricing with marketing restrictions.

Now, consider the argument that airlines engage in price discrimination. It is well-known that airlines offer many different price levels with the identical product. This suggests that airlines are practising *price discrimination.* There are three types of price

discrimination in the economics literature.[30] *First-degree* price discrimination, also called *perfect price discrimination*,[31] "involves the sellers charging a different price to each unit of product in such way that the price charged for each unit is equal to the maximum willingness to pay for the unit." (p. 600) An intuitive way of describing this type of price discrimination is that the seller makes a single *take-it-or-leave-it* offer to each consumer that extracts the maximum amount possible from the market. But the "leave-it" threat lacks *credibility*: it is not a rational way of bargaining since the firm will take the risk of losing the consumer and incurring a permanent loss of revenue. On the other hand, even if the seller had a way of making such a commitment, he typically lacks *full information about the buyers' preferences.* He cannot determine for certain whether his offer will be actually accepted. The requirement of full information about buyers' preference is the main drawback for first-degree price discrimination. For the sales of airline tickets, such an informational requirement is simply impossible.

Second-degree price discrimination, also called *nonlinear pricing*, occurs when prices differ depending on the number of units of the good brought. In other words, consumers face the same *price schedule*, but the schedule involves different prices for different amounts of the good purchased. This type of pricing critically relates to the existence of market segments and use of self-selection constraints. But with unit demand for airline tickets from most consumers, the quantity schedule is not meaningful in the context of airline fare pricing.[32]

Third-degree price discrimination means that different consumers are charged different prices, but each consumer pays a constant amount for each unit of the good brought.

[30]This classification of the forms price discrimination is due to Pigou (1920, [182]).

[31]Refer to Varian (1989, [249]).

[32]One exception here might be the pricing problem for *group booking*, where there are bulk sales of tickets to tour operators, sport teams, etc. The fact that bulk sales are also subject to the ordinary restrictions used on other tickets makes the direct application of second degree price discrimination questionable.

This pricing scheme requires that the market demand be segmented and different market segments be *perfectly sealed.* To seal different market segments, firms use certain mechanisms, such as discriminating by age. The determination of different groups of consumers is taken exogenously by the model. Even though most real life price discrimination falls into this category, the requirement of perfectly sealed market segments is too strong to allow us to use this type of price discrimination to explain the use of multiple prices in airline fare pricing.

The idea of using *a marketing mechanism* to help firms to segment the market is too good to throw away. Telephone companies effectively segment the market by offering discounts for residential usage; theatres successfully attract additional consumers by offering student discounts and senior discounts. All these cases are classic examples of effective third-degree price discrimination. One striking feature is that telephone companies and theatres use *artificial restrictions* that are *preventive* to some consumers. This is of great importance to the understanding of airline fare pricing. The difficulty is that third-degree price discrimination requires a *perfect marketing mechanism* in the sense that it will perfectly seal the different consumer groups. As demonstrated by Gerstner and Holthausen (1986, [76]), the action of perfectly sealing the market segments may not be in a monopolist's best interests. On the other hand, the pricing model in Gerstner and Holthausen (1986, [76]) only addresses the case of two segments, where each segment consists of identical consumers. The problem for airlines is that the restrictions, such as advance booking and Saturday-night stay, cannot effectively segment the market into several consumer groups so that each group consists of identical travellers. Therefore, a more delicate analytical framework is required to accurately model the fare pricing problem for modern airlines. This, in fact, is the main motivation for pricing model in this thesis. More precisely, the model proposed in this thesis is a third-degree price

discrimination model using restrictions which are *imperfect marketing mechanisms.*[33]

1.3.4 New Directions

Clearly, there are many important issues in airline fare *pricing* that need to be investigated. The most important task should be the development of a fare pricing model that *explicitly incorporates the use of marketing restrictions.* Such a model is a fundamental step toward a full understanding of the economic benefits of marketing restrictions. The second important issue should be the *design of marketing restrictions.* From the theoretical point of view, it would be interesting to know what kind of marketing restrictions are plausible. But in practice, an optimally designed restriction (in theory) may not be implementable since the actual implementation requires the physical specification of a marketing restriction (such as Saturday stay condition). In this regard, the second issue should be more empirically oriented. Another important issue is how to *incorporate the network structure*, in particular, the hub-and-spoke system, into fare pricing. Again this is a theoretically challenging question with direct impact on the market place. As long as airlines are operating on a network environment, different markets in the network will interact with each other, which will be reflected in the demand structure over the network. Therefore, such an interaction should be explicitly considered in the corresponding pricing problem.

In order to develop useful models for airlines, there are two basic criteria that should be followed:

[33]Recently, Gale and Holmes (1993, [74]) investigate the use of *advance-purchase* as a mechanism of discounting in the pricing problem for a monopoly airline. Their discussion focuses on how to find an optimal *direct-revelation mechanism* that will lead each consumer to make a rational choice between a *peak flight* and an *off-peak flight.* Their model is based on the assumption of the existence of different *time costs* over the two flights for a continuum of risk-neutral consumers. They concludes that advance-purchase discount is an optimal selling mechanism. Their model also predicts that no seats on the peak flight are sold at a discount, which is not consistent with airline fare pricing practice. In reality, airlines sometimes do make some discount seats available on peak flights.

- the model solution, either an optimal pricing structure or an optimal restriction, must be consistent with yield management practice;[34] and
- the model must consider the technological constraint of the computer reservations systems.[35]

With these two objectives in mind, I will dedicate most of this thesis to the development of a fare pricing model that implicitly incorporates the use of marketing restrictions. During the development, instead of limiting the discussion to airlines, I will cast the model as a general monopoly pricing model for perishable inventories that will be applicable to airline seats, hotel rooms, rental cars and other sectors, where the monopoly firm has to decide whether a marketing restriction should be used and how it should be used.

1.4 Objectives and an Overview of This Thesis

This section provides a summary of the objectives and main results of the remaining chapters of this thesis.

Summary of Chapter 2: Pricing Perishable Inventories Using A Restriction

This chapter addresses the monopoly pricing problem for perishable inventories, such as airline seats, using a single restriction to segment the market. The following assumptions are made

- the demand function is decreasing and is a step function defined on a finite set of prices;

[34] In particular, with analytical models and specialized yield management software.

[35] For example, most of existing computer reservations systems use the concept of *nesting* or *virtual nesting* to display the information on reservations.

- the impact of the restriction is such that as price increases, the percentage of customers who can accommodate the restriction in the market is decreasing;
- the monopolist offers the product for sale from the lowest price to the highest price; and
- the restricted units are sold first on some lower prices and unrestricted units are sold later at some higher prices.

The chapter proposes to answer the following questions:

- Is it necessary to use restrictions? What is the economic benefit?
- What is the optimal pricing structure? With how many prices?

The main contributions of this chapter are:

- It gives a detailed analysis of Wilson's pricing problem for a capacity-constrained monopolist in the context of unit demand. Additional results are derived so that they can be used to formulate the pricing problem when the monopolist uses restrictions.
- It presents a formal analytical model for the pricing problem of perishable inventories by using a single restriction.
- It further characterizes some optimal pricing polices that can be derived from a series of linear programming problems, which have *at most three active price levels.*
- It demonstrates that except for some trivial cases, the monopolist can do strictly better by offering some restricted units.

- It illustrates how the pricing model developed in this chapter can be used in airline fare pricing. It shows that the assumption of the impact of restriction is equivalent to saying, in the airline context, that leisure travellers are more price sensitive than the business travellers.

Summary of Chapter 3: General Optimality Results and Other Properties

This chapter addresses two important issues that are assumed away in Chapter 2. Although the current pricing practice by airlines and hotels is consistent with the assumption that restricted prices are lower than unrestricted prices, it is important, from a theoretical point of view, to understand the issue of *optimal pricing practice* when using an artificial restriction. The model in Chapter 2 does not explain why the monopolist should limit itself to the policies that restricted units are sold first at some lower prices and the unrestricted units are sold subsequently at some higher prices — a practice which will be called *primary policies.* Practically speaking, it might make sense to sell some unrestricted units at some very low price and then open sales of some restricted units later. The main goal of this chapter is to investigate whether such a pricing practice is necessary. Another issue is that the model in Chapter 2 assumes that the monopolist opens sales at one price at a time. This is not consistent with current practice. For example, airlines make all offered prices together with allocated quantities available simultaneously. Technically speaking, the model in Chapter 2 is not affected by this issue since it is a static model. But to airlines, this is very important simply because the optimal pricing structures, if implemented, must be consistent with yield management techniques.

The main contributions of this chapter are:

- Any general pricing policy, which allows the monopolist to sell restricted units at any price level, is always *weakly dominated* by a primary policy, where the

dominance is defined in terms of *realized revenue for the policy.* This result implies that any optimal pricing policy in the class of primary policies is also optimal in the class of general policies. Therefore, it resolves the issue of optimal pricing practice when a monopolist uses artificial restrictions; and

- Under a behavioral assumption that if restricted units and unrestricted units are offered at the same price level simultaneously every consumer in the market will try to buy an unrestricted unit first, it is shown that there always exists an optimal policy in the class of general policies with the property that even all offered prices together with the allocated quantities are made *available simultaneously*, there will be no negative impact on the realized revenue.

Summary of Chapter 4: Pricing Models with Two Types of Restrictions

By using one restriction, the monopolist can introduce two kinds of products to the market. In this chapter, I will use the same techniques developed in Chapters 2 and 3 to discuss the pricing problem when the monopolist can use two types of restrictions, which will allow the monopolist to introduce three to four different kinds of product to consumers. This will enrich the techniques developed in previous two chapters to handle more realistic pricing problems.

The main findings of this chapter are:

- For the cases of two nested restrictions and two mutually exclusive restrictions, the monopolist's pricing problem can be formulated as a mathematical programming problem with four constraints. It is further shown that this formulation can be relaxed into a linear programming problem. The main result is that the monopolist needs to offer at most four price levels with at most three types of product to maximize its revenue; and

- For the general case of two restrictions, it is shown that under certain plausible conditions, there exists optimal pricing polices that consist of at most four kinds of products with five price levels to maximize its revenue.

Summary of Chapter 5: Airline Pricing by Using Membership and Product Restrictions

The purpose of this chapter is to present some interesting applications of the pricing models developed in Chapter 4 to airline fare pricing when a product restriction, such as Saturday-night stay and advance booking, is incorporated in association with the use of membership fares. Note that besides the conventional product restrictions, there are many other forms of restrictions in airline industry, which further segment the market demand. For example, airlines usually have special agreements with some corporate clients and government clients for special fares; and airlines are under constant requests from tour operators for additional discounts. Also, airlines must deal with internal travellers (i.e., employees) and travellers from a partner airline. Each of these traveller groups constitutes a certain form of *membership*. The traditional view of membership is that it gives the members certain privileges, which are not available to the general public. Therefore, the presence of a membership can be considered as a form of restriction. This chapter discusses the issue of membership together with a product restriction. Studies in this chapter enable an airline to understand the operating environment for three commonly used membership privileges. It is further shown that the pricing models developed in previous chapters can be used to evaluate certain corporate commitments associated with membership deals.

The main findings of this chapter are:

- For the case that the membership privilege is limited to a restricted fare at a price level that is lower than the public restricted fare (for example, tour operators and

corporate retreat programs), the airline has an optimal fare structure with the following properties:

  - it consists of at most four fare classes; and
  - the demand for restricted membership fares will be exhausted right after the sales of these restricted fares targeted to members;

- For the case that the membership privilege allows the members to purchase cheaper unrestricted fares only (for example, interval travellers and major service corporations), the pricing model for two mutually exclusive restrictions can be used, which implies that the airline needs at most four different fare classes to maximize the flight revenue.

- It is well-known that most airlines have interval travel policies for employees, which indicates that if an employee pays a small fraction of the (public) full fare, then he/she will have a confirmed seat on the flight. Usually, these interval fares are cheaper than the public restricted fares, which is in fact a corporate commitment to employees. It is shown that unless an airline has a systematic method to monitor some high-demand flights, such a commitment may cost the airline a lot of money;

- If the membership privileges include cheaper restricted fares and cheaper unrestricted fares (for example, government employees and travellers from partner airlines), it is shown that the airline needs to use the pricing model for two general restrictions developed in Chapter 4. Sufficient conditions are given for airlines pursuing simple optimal fare structures. These models can be used to evaluate whether or not certain corporate commitment is indeed consistent with the operating environment; and

- The chapter concludes with the following insight: in order to fully exploit these market segments derived from the existence of memberships, an airline must have a systematic approach that can be computerized so that:
  - given a particular membership, it can *tag* those flights that will be available to these members; and
  - it is capable of identifying the most favorable membership group on the flight-by-flight basis.

Chapter 6: Seat Allocation Game on Flights with Two Fares

The main purpose of this chapter is to propose a major initiative in modern airline yield management research, that is, to investigate the seat allocation problem in the presence of competition. In particular, I will focus on single-leg two-fare flights with random demands (not necessarily independent) shared by two airlines, where each airline's strategic variable is the booking limit for the low fare, or equivalently, the protection level to the high fare. The discussion is based on how the two airlines split the market demand. If the splitting rule is proportional according to their respective allocations, two main results emerge:

- At equilibrium, each airline will protect the same number of seats for the high fare; and
- At equilibrium, the total number of seats that are available for low fare class from two airlines is strictly smaller than the total number of seats that would be available for the low fare if two airlines cooperate.

Under the equal splitting rule, the discussion of the seat allocation game is limited to the case of deterministic demands. It is shown that allocating enough seats to capture half

of the high fare demand is an equilibrium strategy for both airlines.

Chapter 7: A Note on Three Models for Multi-fare Seat Allocation Problem

The seat allocation problem for single-leg multi-fare flights with independent random demands has been solved independently by Curry (1988, [50]), McGill (1988, [143]) and Wollmer (1988, [268]). On the other hand, there are some differences in the field on which model is more computationally efficient. The purpose of the this chapter is two-fold. First, I want to clarify this issue by proving that the optimal conditions from three models are in fact analytically equivalent, which therefore implies that they are all computationally equivalent too. Second, by unifying the existing approaches, I provide further insight on the multi-fare seat allocation problem with dependent random demands.

Chapter 2

Pricing Perishable Inventories by Using a Restriction

2.1 Introduction

This chapter develops a model for monopoly pricing a perishable product with a capacity constraint.[1] Examples might include airline seats and hotel rooms. Firms, such as airlines and hotels, typically offer multiple prices and impose artificial restrictions to segment the market. Surprisingly, there is no useful framework in economics or marketing to deal with this kind of problem. Models in the economics literature are driven by simplicity and elegance, but these typically do not provide operational rules for management to follow. Pricing models in the marketing literature are aimed at the managerial level, and are intended to be realistic and therefore possibly to be implemented in practice. But it is very hard to achieve these two objectives at a satisfactory level without some degree of compromise. For example, Dobson and Kalish (1988, [62]) develop an operational, heuristic procedure to position and price a line of related, substitute products. Their mathematical programming formulation, however, is not computationally tractable. The goal of this chapter is to develop a model that is analytically tractable and practically interesting.

Recently, there has been a growing interest in the dynamic monopoly pricing problem of inventories in the operations research (OR) literature. In particular, Rajan, Rakesh and Steinberg (1992, [189]) present a model of simultaneous pricing and inventory control for

[1] While the model is developed for the case of a monopolist, it has some applicability for any firm with some degree of market power.

a monopolist retailer who orders, stocks and sells a single perishable, but *storable* product facing a known demand function. The solution to the model gives the optimal dynamic price[2] and the optimal cycle length.[3] Other related papers along this direction include Wernerfelt (1986, [258]), Stadie (1990, [226]), and Gallego and van Ryzin (1992, [75]). The main differences between these OR models and the model in this chapter become very clear after knowing the following aspects of the new model:

- The market consists of at least two types of consumers with different willingness to pay;
- The product is perishable, but *non-storable*;
- The firm imposes *an artificial restriction* on the product at lower prices;[4] and
- The availability of the product at lower prices is *controlled.*

It will be useful to have a pricing framework that explicitly incorporates these characteristics. The purpose of this chapter is to take up this challenge. The model in this chapter fills an important theoretical gap in the pricing research.

This chapter is organized as follows. Section 2.2 first motivates the notion of rationing, which is directly related to the specification of the residual demand when sales are rationed. I then discuss a pricing model developed by Wilson, which provided the inspiration for the development of this optimal pricing model. Section 2.3 formally presents an optimal pricing model that explicitly incorporates the impact of using one restriction on the product at some lower prices. Section 2.4 discusses optimal pricing strategies that

[2]A dynamic price is a *price functional* of time.

[3]A cycle length is the length of the time period when the retailer will not make another ordering decision. It is interesting to mention that the optimal dynamic price is independent of the choice of cycle length.

[4]The reason I call the restriction *artificial* is because the action of imposing it does not cost the firm anything. In contrast when a firm segments the market by *product differentiation*, the firm must expend resources in order to achieve it.

have tractable characterizations. Section 2.5 discusses an application of the model to the airline industry, where the market is segmented into business travellers and leisure travellers. This section also provides a simple example to illustrate the application. Finally, the last section is a summary.

2.2 Rationing Rules and Wilson's Pricing Model

2.2.1 Rationing Rules

The notion of rationing is commonly used in the duopoly model of price competition when two firms are capacity-constrained. Historically, the use of capacity constraints in a pricing game, first suggested by Edgeworth (1897, [68]), was motivated to resolve the well-known Bertrand Paradox.[5] The key question related to rationing is: if two firms offer different prices, then what is the residual demand for the high-price firm after the sales of the low-price firm? If the low-price firm's capacity is large enough to satisfy the whole market demand, then there is no residual demand for the high-price firm. Thus the main issue becomes specifying the residual demand when the low-price firm cannot exhaust total market demand at its price level. As a general rule, in models with capacity constraints, firms make positive profit and the market price is greater than the marginal cost.

The rationing issue also arises for a capacity-constrained monopolist. The innovation over the traditional model is that a monopolist does not need to allocate all its capacity at single price level. In fact, the monopolist can offer an allocation schedule at different price levels to maximize its total revenue.[6] This problem was first studied by Wilson

[5]Simply speaking, the Bertrand Paradox says that when two firms produce identical goods with identical and constant marginal cost, then there exists a unique equilibrium such that both firms price at marginal cost and make no profit. This result is considered to be a paradox by some because it is hard to believe that firms in industries with few firms never succeed in manipulating the market price to make profits.

[6]We should notice that from a consumer's point of view, it is irrelevant who is offering a higher price.

(1988, [262]). As in the duopoly case, the solution to this problem is critically related to a more specific assumption concerning the manner in which sales are *rationed.*

Two rationing rules have often been considered in the literature.[7] Assume that a monopolist produces a homogeneous product and each consumer only needs one unit of the product. If we let $D(p)$ be the demand function for the product defined in the interval $[p_0, \infty)$ with $N \equiv D(p_0) < \infty$ and $p_0 \geq 0$, then for $p \geq p_0$, $D(p)$ can be interpreted as the size of the market that consists of those consumers who have reservation prices of p or higher.[8] Now suppose that the monopolist first offers q_1 units at price $p_1 \geq p_0$. The *efficient-rationing rule* presupposes a residual-demand function given by:

$$d(p; p_1) = \begin{cases} D(p) - q_1 & \text{if } D(p) > q_1 \\ 0 & \text{otherwise.} \end{cases}$$

This residual demand function essentially assumes the most eager consumers buy the product at price p_1. This rationing is efficient because it maximizes consumer surplus. Therefore, it is the *most undesirable* rationing scheme for the monopolist. Also, the residual demand function defined by the efficient rationing rule is the one that would be obtained if the consumers were able to costlessly resell the good to each other, that is, to engage in arbitrage.[9]

In the duopoly case, the higher price is offered by another firm; and in the monopoly case, the higher price is offered by the same firm. The point here is that if the rationing makes economic sense in the duopoly case, it will also be rationed in the monopoly case.

[7]The following discussions follow from Chapter 5 of Tirole (1988, [236]), pp. 212–214.

[8]This is a typical way of handling unit demand when a firm does not have perfect information about consumers' willingness to pay. Such an approach has been extensively used in the economics literature in many different contexts, for example, refer to Harris and Raviv (1981, [86]), Wolinsky (1984, [264]), and Perloff and Salop (1985, [178]). On the other hand, if we define $F(p) = 1 - D(p)/N$, then we can interpret $F(p)$ as *the probability distribution function* of a consumer's reservation price. Therefore $1 - F(p)$ is the probability that a consumer has a reservation price of p or higher. Such an interpretation is also consistent with the notion of imperfect information in economics when consumers have inelastic demand.

[9]The use of the efficient-rationing rule can be found in many articles from economics literature. For example, refer to Beckmann (1965, [11]), Levitan and Shubik (1972, [125]), Kreps and Scheinkman (1983, [110]), and Perry (1984, [179]).

Another popular rationing rule is called the *proportional-rationing rule.* It assumes that all consumers have *the same probability of being rationed.* Given that the monopolist offers q_1 units at price $p_1 \geq p_0$, the probability of not being able to buy at price p_1 is

$$\frac{D(p_1) - q_1}{D(p_1)}.$$

Hence, the residual demand function after the sales of q_1 units at price p_1 is given by

$$d(p; p_1) = D(p)\frac{D(p_1) - q_1}{D(p_1)} \text{ for } p > p_1.$$

This rule is not efficient for consumers since some consumers with valuation $p' : p_1 < p' < p$ are able to buy the good at the bargaining price p_1, while some consumers with the higher reservation prices are unable to. However, the monopolist prefers this rule to the efficient rule since the residual demand is higher at each price. Typical conditions for the proportional-rationing rule are:

- consumers arrive in a random order and are served on a first-come-first-served basis; and
- there is no reselling opportunity.

Because of these conditions, the proportional-rationing rule is better than the efficient-rationing rule for some industries with perishable inventories because these conditions are easily met, for example, by airline seats and hotel rooms.[10]

2.2.2 Wilson's Model

Wilson (1988, [262]) is the only published paper that deals with rationing sales in the context of a monopolist. His primary goal is to explain the existence of price dispersion

[10]Papers that have used this rationing rule include Beckmann (1965, [11]), Allen and Hellwig (1986, [2]), Davidson and Deneckere (1986, [59]), and Wilson (1988, [262]).

by a monopolist, when consumers arrive in a random order and are served on the first-come-first-served basis. His paper uses the proportional-rationing rule.

Consider that a monopolist produces a homogeneous good with a fixed capacity. Assume that the market demand consists of *heterogeneous* consumers with unit demand. Let $D(p)$ be the market demand function, which is interpreted as the number of consumers whose valuation for the product is at least p. To simplify his analysis, Wilson makes a further assumption on the demand structure:

- $D(p)$ is a left-continuous, non-increasing step function on a finite set of prices $\{p_1, p_2, \cdots, p_n\}$ where $0 \leq p_1 < p_2 < \cdots < p_n < \infty$.

In other words, the demand function is specified by:

$$D(p) = \begin{cases} D_1 & \text{if } p \leq p_1 \\ D_i & \text{if } p \in (p_{i-1}, p_i] \text{ and } i > 1 \\ 0 & \text{if } p > p_n, \end{cases}$$

where $\infty > D_1 > D_2 > \cdots > D_n > 0$.

Remark: Since the demand function is a step function, the set of prices that a rational firm will consider is finite, which in fact is the set $\{p_1, p_2, \cdots, p_n\}$. For any price $p \in (p_{i-1}, p_i)$, the firm will be better off charging p_i rather than p, since increasing from p to p_i does not result in any loss of the demand.

Since the choice set of prices is finite, the firm only needs to decide how many units of the product to make available at the finite number of prices. This implies that the pricing decision becomes an allocation problem. In this regard, Wilson introduces the following definition of a pricing policy.

Definition 2.2.1 *A* **pricing policy** *for the monopolist is a vector* $\{q_1, q_2, \cdots, q_n\}$, *where* $q_i \geq 0$ *is the number of units for sale at price* p_i.

The monopolist's decision problem is to find a pricing policy to maximize its total revenue $\sum_{i=1}^{n} p_i q_i$. The key is to characterize the *set of feasible pricing policies.* The feasibility of a pricing policy is guided by two market forces: the demand and the supply. The supply is bounded by the capacity, which implies that we must have

$$\sum_{i=1}^{n} q_i \leq k,$$

where k is the firm's *capacity limit.* I call this *the supply constraint.*

I now want to derive the feasibility condition for the demand side.[11] First, since the product is assumed to be homogeneous and non-storable, a rational firm will only consider selling the product from the lowest price to the highest price, since otherwise those consumers with higher reservation prices would *prefer to wait* to get the product at a lower price.[12] More specifically, for any given pricing policy $\{q_1, q_2, \cdots, q_n\}$, the selling process is as follows: the firm first puts q_1 units of the products on sale at price p_1; after selling q_1 units at p_1, the firm allocates another q_2 units on sale at price p_2; and so on and so forth. The key aspect of this selling process is that if after the sales at prices $p_1, \cdots, p_i$ the demand at price p_i is not exhausted, the residual demands at higher prices are positive, since we assume that the consumers arrive in random order.[13]

Given a pricing policy $\{q_1, q_2, \cdots, q_n\}$, for $k \geq i$ and $i \geq 1$, let $d_{i,k}$ be the *residual demand* at price p_k *after* the sales of the product at prices $p_1, \cdots, p_i$, according to $q_1, \cdots, q_i$.

[11]The feasibility condition on the demand side is more involved since we allow the possibility of *multiple prices.*

[12]This issue is more subtle than it appears here. For many products, if the action of delaying the purchase decision creates a substantial loss of productivity or a substantial amount of disutility, then the firm may consider selling the product from the highest price to the lowest price. Wilson's model here and my model later do not address the pricing problem for this case.

[13]This assumption allows us to use the proportional-rationing rule.

Then according to the proportional-rationing rule,

$$d_{i+1,k} = d_{i,k} - \frac{d_{i,k}}{d_{i,i+1}} q_{i+1}, \text{ for } k \geq i+1, \tag{2.1}$$

where the second part in the right hand side is the *leakage* from the residual consumer group consisting of these consumers with the reservation price p_k or higher who actually purchase the product at the price p_{i+1}. Also, for ease of presentation, we will denote

$$d_{0,k} \equiv D_k \text{ for } k \geq 1.$$

The following lemma gives a simple updating formula for the residual demand, which is in fact the key in Wilson's development.

Lemma 2.2.1 *For every $i \geq 1$ and all $k \geq i$, we have*

$$d_{i,k} = D_k(1 - \sum_{t=1}^{i} \frac{q_t}{D_t}), \tag{2.2}$$

where $\sum_{t=1}^{i} \frac{q_t}{D_t}$ can be interpreted as the **total leakage ratio** *of the market demand due to the sales of the product according to the partial plan $\{q_1, \cdots, q_i\}$.*

Proof: I prove the result by using induction on argument i. For $i = 1$, by definition,

$$d_{1,k} = D_k - \frac{D_k}{D_1} q_1 = D_k(1 - \frac{q_1}{D_1}).$$

Therefore, the result is true for $i = 1$. Now assume that the result is true for $i \geq 1$. I want to prove that the result is also true for $i+1$. By (2.1) and (2.2), it follows that

$$\begin{aligned} d_{i+1,k} &= d_{i,k} - \frac{d_{i,k}}{d_{i,i+1}} q_{i+1} \\ &= D_k(1 - \sum_{t=1}^{i} \frac{q_t}{D_t}) - \frac{D_k}{D_{i+1}} q_{i+1} \\ &= D_k(1 - \sum_{t=1}^{i+1} \frac{q_t}{D_t}). \end{aligned}$$

This proves that the lemma holds for $i+1$. So by induction, the lemma is proved. □

The demand side feasibility condition is the following: *we must maintain a non-negative residual demand at the end of the sales at each possible price level.* This is equivalent to requiring that $d_{k,k} \geq 0$ for all k, that is,

$$\sum_{i=1}^{k} \frac{q_i}{D_i} \leq 1, \ \forall\, k \geq 1,$$

which is equivalent to

$$\sum_{i=1}^{n} \frac{q_i}{D_i} \leq 1,$$

since q_i's are all non-negative and D_i's are all positive. I call this the *demand constraint.*

Finally, putting the objective function, the supply constraint, and the demand constraint all together, Wilson concludes that the firm's pricing problem can be formulated as the following linear programming problem:

$$\begin{aligned} \text{Max} \quad & \textstyle\sum_{i=1}^{n} p_i q_i \\ \text{s.t.} \quad & \\ & \textstyle\sum_{i=1}^{n} \frac{q_i}{D_i} \leq 1 && (2.3) \\ & \textstyle\sum_{i=1}^{n} q_i \leq k && (2.4) \\ & q_i \geq 0 \ \forall i. \end{aligned}$$

From now on, we will call this formulation *Wilson's Pricing Model* or simply *the Wilson-model.* The following result is an immediate consequence of this linear programming formulation.

Theorem 2.2.2 [Wilson (1988, [262])] *To maximize revenue, the firm needs to charge no more than two prices.*

Proof: This follows immediately from the fact that the number of non-zero variables in any *basic optimal solution* for a linear programming is no more than the number of structural constraints. □

The following result, not proved in Wilson's paper, shows that any optimal solution in the Wilson-model will make the demand constraint binding.

Theorem 2.2.3 *If $k \geq D_n$, then for any optimal solution in the Wilson-model, the demand constraint (2.3) is* **binding.**

Proof: Note that if $k = D_n$, it is clear that the Wilson-model has an unique optimal solution $\{p_n, D_n\}$, which will automatically make the demand constraint (2.3) binding. Therefore, we only have to prove the theorem for the case that $k > D_n$.

I will first prove that the theorem is true for any *basic optimal solution.* Note that *any basic optimal solution* in the Wilson-model consists of a pair $\{q_i^*, q_j^*\}$ $(i < j)$. If both quantities are positive, (2.3) and (2.4) must be binding. So the theorem holds for this case.

Now suppose that an optimal solution consists of *only one* price, say p_i, with the optimal quantity q_i^*. I want to show that

$$\frac{q_i^*}{D_i} = 1. \tag{2.5}$$

Suppose (2.5) is not true, which must imply that:

$$q_i^* < D_i.$$

Therefore the constraint (2.3) is not binding. Since the given optimal solution is a basic solution, then the constraint (2.4) must be binding, that is

$$q_i^* = k.$$

Furthermore, it must be true that

$$k > D_{i+1},$$

since otherwise the firm can sell all k units at price level p_{i+1} that will realize a total revenues of $p_{i+1}k$, which contradicts to the optimality of $q_i^* = k$ at price level p_i. Also, since $k > D_n$, we must have $k < n$. In summary, we get:

- $q_i^* = k$;
- $0 < D_n \leq D_{i+1} < k < D_i$.

Define a new allocation plan only with $q_i > 0$ and $q_{i+1} > 0$ such that

$$\begin{aligned} \frac{q_i}{D_i} + \frac{q_{i+1}}{D_{i+1}} &= 1 \\ q_i + q_{i+1} &= k. \end{aligned}$$

Solving this system, we obtain

$$\begin{aligned} q_i &= \frac{k - D_{i+1}}{D_i - D_{i+1}} D_i \\ q_{i+1} &= \frac{D_i - k}{D_i - D_{i+1}} D_{i+1}. \end{aligned}$$

Now the revenue under $\{q_i, q_{i+1}\}$ is given by

$$R = p_i q_i + p_{i+1} q_{i+1} > p_i(q_i + q_{i+1}) = p_i k,$$

because $q_{i+1} > 0$. This contradicts the assumption that q_i^* is an optimal solution, which implies that (2.5) is true. This proves that the theorem is true for any *basic* optimal solution. For the general case, we note that any optimal solution in the Wilson-model is a *convex combination of basic optimal solutions.* Therefore, the theorem is true for any optimal solution since both (2.3) and (2.4) are *convex constraints.*[14] □

[14] A constraint is said to be a convex constraint if for any two feasible vectors to the constraint, any convex combination of these two vectors remains to be feasible to the constraint. In fact, any linear constraint is a convex constraint.

Remarks: We can make the following interesting observations from the above theorems.

- Theorem 2.2.3 tells us that if a firm's capacity is not so small that the firm can actually sell all units at the highest positive price level — p_n, the firm's pricing decision is primarily driven by the market demand in the sense that there will be no residual demand after the sales at the last positive allocation.
- Theorem 2.2.2 and Theorem 2.2.3 together imply that a firm needs to offer two prices only when the firm cannot reach its best possible revenue at the price that exhausts all its capacity. This occurs when the single-price revenue function is not concave.

It is worthwhile to notice that $k < D_n$ says that the firm can sell all its inventories at the highest possible price level p_n, which consequently implies that there is no need to ration the sales at all. Clearly, this is a trivial case. From now, unless explicitly stated, I will rule out this case. In other words, throughout this chapter I will assume that

$$k \geq D_n.$$

Now for a given demand structure, since the optimal revenue value in Wilson's formulation only depends upon the capacity level k, we can write this *derived revenue function* as $R^w(k)$. Wilson proves the following properties for $R^w(k)$:

Theorem 2.2.4 [Wilson (1988, [262])] It is true that

1. $R^w(k)$ *is a* **concave and non-decreasing function** *of* k*;*
2. $R^w(k)$ *is* **piece-wise linear** *in* k*;*
3. *if* $R^w(k)$ *is piece-wise linear in a neighborhood of* k*, then* $R^w(k) > R^s(k)$*,*

where $R^s(k)$ *is the* **best single-price revenue** *defined by*:

$$R^s(k) = \max_{0 \le x \le k} x D^{-1}(x).$$

The first property is true in general. The second property is a consequence of assumption that the demand function is a step function. On the other hand, if the best single-price revenue function is concave, the firm always charges one price. The significance of Theorem 2.2.4 is that if the best single-price revenue function is not concave, then the firm can achieve a concave revenue function that is strictly better than the best single-price revenue.

2.3 An Optimal Pricing Model by Use of Restrictions

2.3.1 The Model Settings

Consider that a firm plans to sell a fixed number of units of a certain product or service. For example, an airline wants to sell a fixed number of seats on a particular flight; a hotel needs to sell a fixed number of rooms on each particular day; and a car rental company wishes to rent out a fixed number of cars on each day. The main characteristic of the products we are concerned with here is *perishability*, that is, if the firm cannot sell the product before a certain time, the product has no further salvage value. Because of this, the firm hopes to sell as many units as possible before the product totally losses its value. Also, I will assume that the product is *not storable* for consumers in the sense that the product can not be stored and consumed before a given time.

Before moving on to the formal development of the model, it is necessary to highlight the model's assumptions:

- Each consumer will purchase at most one unit of the product;
- The firm has a fixed capacity k and its goal is to maximize revenue;

- The firm may choose to impose a restriction on the product to divide the consumer group into two subgroups — those who do not mind the restriction and those who do. For this, let $D_r(p)$ be the demand for the product at price p with the restriction; and let $D_u(p)$ be the demand for the product at price p *only when the restriction is not attached.* Then *the total demand for the product at the unrestricted price* p is given by:

$$D(p) = D_r(p) + D_u(p); \text{ and}$$

- The firm chooses a set of prices, each of which will have the restriction attached.

To highlight the impact of the restriction on market demand, let $\alpha(p)$ be the percentage of those consumers with reservation prices of p or higher *who will be unable to buy the product because of the restriction.* Then $\alpha(p)$ is given by:

$$\alpha(p) = \frac{D_u(p)}{D(p)} = 1 - \frac{D_r(p)}{D(p)}.$$

As in the Wilson-model, to further simplify the analysis, assume that both $D_r(p)$ and $D_u(p)$ are *step functions* defined on a set of prices $\{p_1, p_2, \cdots, p_n\}$, where $p_0 \leq p_1 < p_2 < \cdots < p_n < \infty$. More specifically speaking, for $l = r, u$,

$$D_l(p) = \begin{cases} D_{l,1} & \text{if } p \leq p_1 \\ D_{l,i} & \text{if } p \in (p_{i-1}, p_i] \text{ for } 2 \leq i \leq n \\ 0 & \text{if } p > p_n, \end{cases}$$

where $D_{l,i}$ is decreasing (not necessarily strictly decreasing) in i. Denoting

$$D_i = D_{r,i} + D_{u,i}, \text{ for } i = 1, \cdots, n,$$

then the total market demand function for unrestricted product is given by:

$$D(p) = \begin{cases} D_1 & \text{if } p \leq p_1 \\ D_i & \text{if } p \in (p_{i-1}, p_i] \text{ for } 2 \leq i \leq n \\ 0 & \text{if } p > p_n. \end{cases}$$

Consequently, $\alpha(p)$ is also a step function:

$$\alpha(p) = \begin{cases} \alpha_1 & \text{if } p \leq p_1 \\ \alpha_i & \text{if } p \in (p_{i-1}, p_i] \text{ for } 2 \leq i \leq n \\ 1 & \text{if } p > p_n, \end{cases}$$

where $\alpha_i \equiv \alpha(p_i)$ for $i = 1, \cdots, n$.

Before introducing the definition of a pricing policy, I need to highlight all the *assumptions* used in this chapter:

- D_i is *strictly* decreasing in i;
- $D_{l,i}$ is decreasing for $l = r, u$;
- α_i is *strictly increasing* in i, which implies that as the price for restricted product increases, the percentage of potential consumers who can accommodate the restriction is decreasing;
- The monopolist will sell the restricted units first and the unrestricted units later;[15]
- The monopolist sells its product from the lowest price level to the highest price level; and
- k is the capacity and is such that $k \geq D_n$, which rules out the trivial case of the pricing problem.[16]

Since the monopolist sells the restricted product first, it is natural to choose a value of m such that (1) $1 \leq m \leq n$, and (2) the prices $p_1, \cdots, p_m$ are attached with the restriction. Of course, the value of m is controlled by the monopolist. Then the firm's problem can be divided into two parts:

[15]In Chapter 3, I will call this type of policy *a primary policy*.
[16]This will rule out the trivial case as pointed out in Theorem 2.2.3.

- For *any given value* m, characterize the optimal pricing policies and the corresponding revenues, say $R(m)$; and
- Find the value of m that *maximizes* $R(m)$.

This section focuses on the formulation of the monopolist's pricing problem for a given m; and the next section will discuss the characterization of optimal choices on m.

Note that for a given value m, as argued in Wilson's model, the firm's pricing problem is in fact *an allocation problem.* More specifically, I give the following definition of a pricing policy, which is very similar to Definition 2.2.1 as in Wilson's model.

Definition 2.3.1 *For a given integer* m*:* $1 \leq m \leq n$*,* **a** m**-policy** *for the firm is a vector* $\{q_{r,1}, \cdots, q_{r,m}; q_{u,m}, \cdots, q_{u,n}\}$*, where* $q_{r,i}$ *is the number of units of the product allocated to sell at the price* p_i *with restriction for* $1 \leq i \leq m$*, and* $q_{u,j}$ *is the number of units of the product allocated to sell at the price* p_j *without restriction for* $m \leq j \leq n$*.*

Remark: For convenience, from now on, I call a price with the restriction *a restricted price* and a price without the restriction *an unrestricted price.* If a unit of the product is sold at a restricted price, I call it *a restricted product.* Similarly, if a unit of the product is sold at an unrestricted price, I call it *an unrestricted product.*

As we can see here, Definition 2.3.1 allows the possibility that the firm may offer restricted products and unrestricted products at the same price p_m. This is a technical trick that will enable us to give a simple characterization of an optimal pricing strategy, as we will see in the next section.

2.3.2 The Demand Constraints

Let me first discuss market demand at the restricted prices $p_1, p_2, \cdots, p_m$, which correspond to the market demands $D_{r,1}, D_{r,2}, \cdots, D_{r,m}$. Since $D_{r,i}$ is decreasing in i, then Wilson's technique of rationing demand works here. Thus we have the following *demand constraint* for the market of restricted products at prices $p_1, \cdots, p_m$:

$$\sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}} \leq 1.$$

To analyze the residual unrestricted market after the sales of restricted products according to the quantity schedule $\{q_{r,1}, \cdots, q_{r,m}\}$, for $j \geq m$, we introduce the following notation:

- $d^r_{m,j}$ be the residual demand for the restricted product at price p_j;
- $d_{m,j}$ be the total residual demand at the (unrestricted) price p_j;
- $d^u_{m,j} \equiv d_{m,j} - d^r_{m,j}$.

Evidently, the residual demand $d_{m,j}$ is *a function* depending upon $\{q_{r,1}, \cdots, q_{r,m}\}$. Fortunately, the following lemma shows that this functional relationship has a simple form.

Lemma 2.3.1 *After the sales of the product with restrictions at prices $\{p_1, \cdots, p_m\}$ according to $\{q_{r,1}, \cdots, q_{r,m}\}$, the residual demand for the unrestricted product at price p_j ($j \geq m$) is given by:*

$$d_{m,j} = D_j - D_{r,j} \sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}},$$

where the second term in the right hand side is the **fulfilled part** *of the restricted demand after the sales of the restricted products.*

Proof: Define $d^r_{m,j}$ to be the residual demand for the *restricted product* at price p_j after the sales of the restricted products according to $\{q_{r,1}, \cdots, q_{r,m}\}$. Since the demand market

for the restricted products is nested, then it follows from Lemma 2.2.1 (in Section 2.2) that

$$d^r_{m,j} = D_{r,j}(1 - \sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}}).$$

Therefore,

$$d_{m,j} = D_{u,j} + d^r_{m,j} = D_j - D_{r,j} + d^r_{m,j} = D_j - D_{r,j} \sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}}.$$

This proves the lemma. □

From the proof of the above lemma it is clear that $d^r_{m,j}$ is decreasing in j. Since $D_{u,j}$ is also decreasing by assumption, it implies that the residual demand function $d_{m,j}$ is decreasing too. So the residual market for unrestricted product defined on the price set of $\{p_m, \cdots, p_n\}$ is consistent with the settings in Wilson's model as discussed in the previous section. Therefore, the demand constraint for the unrestricted product in the residual market is given by:

$$\sum_{j=m}^{n} \frac{q_{u,j}}{d_{m,j}} \leq 1;$$

which, by Lemma 2.3.1, is equivalent to

$$\sum_{j=m}^{n} \frac{q_{u,j}}{D_j - D_{r,j} \sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}}} \leq 1.$$

As we can see here, this constraint is *no longer a linear constraint.*

2.3.3 The Formulation of the Optimal Pricing Model

Together with the objective function, the capacity constraint (or the supply constraint) and the two demand constraints, we have successfully formulated the pricing problem as a mathematical programming problem, which is summarized in the following theorem:

Theorem 2.3.2 *For a given m: $1 \leq m \leq n$, a monopolist's pricing problem can be formulated as choosing a vector of quantities $\{q_{r,1}, \cdots, q_{r,m}; q_{u,m}, \cdots, q_{u,n}\}$ to solve the following mathematical programming problem:*

$$\text{Max} \quad \sum_{i=1}^{m} p_i q_{r,i} + \sum_{j=m}^{n} p_j q_{u,j}$$

s.t.

$$\sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}} \leq 1 \tag{2.6}$$

$$\sum_{j=m}^{n} \frac{q_{u,j}}{D_j - D_{r,j} \sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}}} \leq 1 \tag{2.7}$$

$$\sum_{i=1}^{m} q_{r,i} + \sum_{j=m}^{n} q_{u,j} \leq k \tag{2.8}$$

$$q_{r,i}, q_{u,j} \geq 0, \forall i \text{ and } \forall j.$$

Remark: For ease of reference, the above pricing model is called *BL Pricing Model*, or simply, *the BL-model.*[17] The optimal value of the revenue derived from the BL-model will be denoted by $R(m)$. We will also call any optimal solution to the BL-model with given m as an *optimal m-policy.*

2.3.4 Basic Properties of Optimal m-Policies

Note that the formulation in Theorem 2.3.2 is computationally inconvenient because it involves a non-linear constraint. On the other hand, it is evident that if the constraint (2.6) in the BL-model is binding, then it will lead to a simple *linear programming problem*, which substantially simplifies the process of finding an optimal pricing strategy. This is the main motivation for the discussions in the next section. Before moving on to the next section, we first need to establish a comparison result between $R(m)$ and R^w – the revenue derived from the Wilson-model. Recall that the Wilson-model solves the pricing

[17]Here "BL" stands for Brumelle-Li.

problem *without using a restriction*, which is given by:

$$\begin{array}{ll} \text{Max} & \sum_{i=1}^{n} p_i q_i \\ \text{s.t.} & \\ & \sum_{i=1}^{n} \frac{q_i}{D_i} \leq 1 \\ & \sum_{i=1}^{n} q_i \leq k \\ & q_i \geq 0 \ \forall i. \end{array}$$

The following theorem establishes the following result: if the Wilson-model fails to generate an optimal pricing strategy consisting of only one price, then the firm has a strict incentive to use the restriction. As we will see later, this result plays a key role in my discussions in the next section.

Theorem 2.3.3 *Suppose an optimal solution in the Wilson-model consists of* **two** *prices, say p_i and p_j $(i < j)$. Then:*

$$R(i) - R^w \geq (p_j - p_i)\frac{(k - D_j)(\alpha_j - \alpha_i)D_i D_j}{(D_{r,i} - D_{r,j})(D_i - D_j)}.$$

So consequently, $R(i) - R^w > 0$ since α_i is strictly increasing in i.

Proof: Let $\{q_{w,i}, q_{w,j}\}$ be an optimal solution in the Wilson-model with $i < j$. Then we must have

$$\begin{aligned} q_{w,i} &= \frac{k - D_j}{D_i - D_j} D_i \\ q_{w,j} &= \frac{D_i - k}{D_i - D_j} D_j \end{aligned}$$

and

$$D_j < k < D_i.$$

The corresponding revenue is given by:

$$R^w = p_i q_{w,i} + p_j q_{w,j}.$$

Consider the following allocation plan:

- The firm offers $q_{r,i}$ units of the restricted product at p_i and $q_{u,j}$ units of the unrestricted product at p_j; and
- $q_{r,i}$ and $q_{u,j}$ satisfy the following system:

$$\begin{aligned} q_{u,j} &= D_j - D_{r,j}\frac{q_{r,i}}{D_{r,i}}; \\ q_{u,j} &= k - q_{r,i}. \end{aligned}$$

It is easy to check that the above policy is feasible, that is, the allocations satisfy all three constraints (2.6), (2.7) and (2.7) in the BL-model. Now solving the above system, we get

$$\begin{aligned} q_{r,i} &= \frac{k - D_j}{D_{r,i} - D_{r,j}} D_{r,i}; \\ q_{u,j} &= \frac{D_{r,i}D_j - D_{r,j}k}{D_{r,i} - D_{r,j}}. \end{aligned}$$

Denote the corresponding revenue to be $R = p_i q_{r,i} + p_j q_{u,j}$. Since $q_{r,i} + q_{u,j} = k$ and $q_{w,i} + q_{w,j} = k$, it is true that

$$R - R^w = (p_j - p_i)(q_{u,j} - q_{w,j}).$$

Now note that

$$\begin{aligned} q_{u,j} - q_{w,j} &= \frac{D_{r,i}D_j - D_{r,j}k}{D_{r,i} - D_{r,j}} - \frac{D_i D_j - D_j k}{D_i - D_j} \\ &= (k - D_j)\frac{D_{r,i}D_j - D_{r,j}D_i}{(D_{r,i} - D_{r,j})(D_i - D_j)} \\ &= (k - D_j)\frac{(1-\alpha_i)D_i D_j - (1-\alpha_j)D_j D_i}{(D_{r,i} - D_{r,j})(D_i - D_j)} \\ &= (k - D_j)\frac{(\alpha_j - \alpha_i)D_i D_j}{(D_{r,i} - D_{r,j})(D_i - D_j)} \end{aligned}$$

This proves the theorem. □

The above theorem presents a comparison result for the case that Wilson's model has an optimal policy that consists of two prices. At the end of the next section, I will establish a similar result for the case that Wilson's model has an optimal policy that consists of exactly *one price.*

Before concluding this section, I present two additional properties on optimal m-policies.

Theorem 2.3.4 *Any optimal m-policy does not need more than* **two** *unrestricted prices.*

Proof: Let m be given and $\{q^*_{r,1}, \cdots, q^*_{r,m}; q^*_{u,m}, \cdots, q^*_{u,n}\}$ be any optimal m-policy. Define

$$\beta \equiv \sum_{i=1}^{m} \frac{q^*_{r,i}}{D_{r,i}}, \text{ and } k_r \equiv \sum_{i=1}^{m} q^*_{r,i}.$$

To prove the theorem, it suffices to show that $\{q^*_{u,m}, \cdots, q^*_{u,n}\}$ is an optimal solution of the following linear programming problem:

$$\begin{array}{ll} \text{Max} & \sum_{j=m}^{n} p_j q_{u,j} \\ \text{s.t.} & \\ & \sum_{j=m}^{n} \frac{q_{u,j}}{D_j - \beta D_{r,j}} \leq 1 \\ & \sum_{j=m}^{n} q_{u,j} \leq k - k_r \\ & q_{u,j} \geq 0, \forall j. \end{array}$$

I name the above linear programming problem as *the Reduced Problem.*

First of all, it is clear that $\{q^*_{u,m}, \cdots, q^*_{u,n}\}$ is *feasible* to the reduced problem. Now suppose that it is not an optimal solution to the reduced problem. Then it implies that

there is another set of quantities $\{q'_{u,m}, \cdots, q'_{u,n}\}$ that constitutes a feasible solution to the reduced problem and is such that

$$\sum_{j=m}^{n} p_j q'_{u,j} > \sum_{j=m}^{n} p_j q^*_{u,j}.$$

On the other hand, it is obvious that $\{q^*_{r,i}, \cdots, q^*_{r,m}; q'_{u,m}, \cdots, q'_{u,n}\}$ is a feasible solution to the BL-model associated with m. The corresponding revenue is given by

$$\sum_{i=1}^{m} p_i q^*_{r,i} + \sum_{j=m}^{n} p_j q'_{u,j} > \sum_{i=1}^{m} p_i q^*_{r,i} + \sum_{j=m}^{n} p_j q^*_{u,j},$$

which contradicts the assumption that $\{q^*_{r,i}, \cdots, q^*_{r,m}; q^*_{u,m}, \cdots, q^*_{u,n}\}$ is an optimal m-policy. Therefore, the $\{q^*_{u,m}, \cdots, q^*_{u,n}\}$ must be an optimal solution to the reduced problem.

Finally, since the reduced problem is a Wilson-type formulation, by Theorem 2.2.2, the monopolist needs no more two of $q^*_{u,j}$'s to be positive. This implies that the monopolist needs at most two unrestricted prices. So the theorem is proved. □

As a consequence of Theorem 2.2.3 and Theorem 2.3.4, we get the following result.

Theorem 2.3.5 *Suppose that $k > D_n$. If $\{q^*_{r,i}, \cdots, q^*_{r,m}; q^*_{u,m}, \cdots, q^*_{u,n}\}$ is an optimal m-policy for the BL-model, then the demand constraint for unrestricted product (2.7) in the BL-model is always binding; that is,*

$$\sum_{j=m}^{n} \frac{q^*_{u,j}}{D_j - D_{r,j} \sum_{i=1}^{m} \frac{q^*_{r,i}}{D_{r,i}}} = 1. \tag{2.9}$$

Proof: Let $k_r = \sum_{i=1}^{m} q^*_{r,i}$ and $k_u = \sum_{j=m}^{n} q^*_{u,j}$. Suppose that (2.9) is not true. As an immediate consequence of this, we know that the supply constraint (2.8) must be binding, that is,

$$k_r + k_u = k.$$

As before, denote

$$\beta = \sum_{i=1}^{m} \frac{q^*_{r,i}}{D_{r,i}}.$$

From the proof of Theorem 2.3.4, we know that given the allocation plan for the restricted product, $\{q^*_{r,1}, \cdots, q^*_{r,m}\}$, the reduced problem in the residual market for the unrestricted product is a Wilson-type problem. Therefore, by Theorem 2.2.3, if (2.9) is not true, then it must be the case that

$$k_u < D_n - \beta D_{r,n}, \tag{2.10}$$

which obviously implies that the monopolist will sell all unrestricted units at price p_n, that is,

$$q^*_{u,j} = \begin{cases} 0 & \text{if } j < n \\ k_u & \text{if } j = n. \end{cases}$$

Define

$$i_0 = \max\{i \leq m : q^*_{r,i} > 0\}.$$

Choose $\epsilon > 0$ such that $k_u + \epsilon \leq D_n - \beta D_{r,n}$ and $q^*_{r,i_0} \geq \epsilon$.

Case 1: $i_0 < n$

For this case, define a new policy $\{q'_{r,1}, \cdots, q'_{r,m}; q'_{u,m}, \cdots, q'_{u,n}\}$ as follows:

$$q'_{r,i} = \begin{cases} q^*_{r,i} & \text{if } i \neq i_0 \\ q^*_{r,i_0} - \epsilon & \text{if } i = i_0; \end{cases}$$

and

$$q'_{u,j} = \begin{cases} 0 & \text{if } j < n \\ k_u + \epsilon & \text{if } j = n. \end{cases}$$

I claim that this new policy is a feasible policy to the BL-model associated with m. In fact, it is easy to see that

$$\sum_{i=1}^{m} \frac{q'_{r,i}}{D_{r,i}} < \sum_{i=1}^{m} \frac{q^*_{r,i}}{D_{r,i}} = \beta \leq 1,$$

which implies that the plan is feasible for the restricted market. For the unrestricted market, first note that the residual demand for unrestricted product at price p_n, after the sales according to $\{q'_{r,1}, \cdots, q'_{r,m}\}$, is given by

$$d'_{m,n} = D_n - (\beta - \frac{\epsilon}{D_{r,i_0}})D_{r,n} > D_n - \beta D_{r,n} \geq k_u + \epsilon,$$

which indicates that the allocation plan $\{q'_{u,m}, \cdots, q'_{u,n}\}$ is also feasible in the residual market for unrestricted product. Therefore, this new plan is indeed feasible for the BL-model associated with m. Now note that the total revenue generated by this new plan is

$$\begin{aligned} R' &= \sum_{i=1}^{m} p_i q'_{r,i} + \sum_{j=m}^{n} p_j q'_{u,j} \\ &= \sum_{i=1}^{m} p_i q^*_{r,i} + \sum_{j=m}^{n} p_j q^*_{u,j} + (p_n - p_{i_0})\epsilon > R(m), \end{aligned}$$

which contradicts to the fact that $\{q^*_{r,1}, \cdots, q^*_{r,m}; q^*_{u,m}, \cdots, q^*_{u,n}\}$ is an optimal m-policy. This contradiction leads to the conclusion that (2.9) must be true.

Case 2: $i_0 = n$

Clearly under this case, we must have that $m = n$. It is clear that

$$q^*_{r,n} + k_u \leq D_n.$$

Then since $k_r + k_u = k > D_n$, we must have some $i < n$ such that $q^*_{r,i} > 0$. Because of this, define

$$i'_0 = \max\{i < n : q^*_{r,i} > 0\}.$$

Now choose $\epsilon > 0$ such that $q^*_{r,i'_0} \geq \epsilon$ and $k_u + \epsilon \leq D_n - \beta D_{r,n}$. Define a new policy as follows:

$$q''_{r,i} = \begin{cases} q^*_{r,i} & \text{if } i = i'_0 \\ \min\{q^*_{r,n}, d^r_{n-1,n}\} & \text{if } i \neq i'_0; \end{cases}$$

and $q''_{u,n} = k_u + \epsilon$. Similarly, by the same argument used in case 1, we know that this new policy is feasible to the BL-model associated with m, which generates a total revenue of

$$R'' = \sum_{i=1}^{n} p_i q''_{r,i} + p_n q''_{u,n} = R(m) + (p_n - p_{i'_0})\epsilon > R(m),$$

since $i'_0 < n$. This again leads to a contradiction. Therefore (2.9) is also true for this case.

In summary, we show that (2.9) must be always true. □.

2.4 Optimal Pricing Strategies

The main purpose of this section is to demonstrate that the monopolist's pricing problem can be reduced to solving n linear programming problems. First, note that the previous section solves the monopolist's pricing problem for a given value of m. On the other hand, the choice of m is completely controlled by the monopolist. Therefore, it is the monopolist's best interest to choose a value of $\bar{m}$ such that

$$R(\bar{m}) = \max_{1 \le m \le n} R(m). \tag{2.11}$$

Consequently, an optimal solution to the initial monopolist's pricing problem should be an optimal $\bar{m}$-policy. For ease of reference, we introduce the following definition:

Definition 2.4.1 *If $\bar{m}$ satisfies (2.11), then any optimal $\bar{m}$-policy is named as* **an optimal policy** *for the monopolist's pricing problem.*

The following theorem follows from Theorem 2.3.3 and Theorem 2.4.1.

Theorem 2.4.1 *Suppose that $\bar{m}$ satisfies (2.11). If the demand constraint (2.6) for restricted product is not binding for some optimal solution $\{q^*_{r,1}, \cdots, q^*_{r,\bar{m}}; q^*_{u,\bar{m}}, \cdots, q^*_{u,n}\}$*

in the BL-model, that is,

$$\beta \equiv \sum_{i=1}^{\bar{m}} \frac{q^*_{r,i}}{D_{r,i}} < 1,$$

or equivalently, the sales of restricted products according to the partial plan $\{q^*_{r,1}, \cdots, q^*_{r,\bar{m}}\}$ *do not exhaust the restricted demand at price level* $p_{\bar{m}}$*, then this optimal policy contains exactly* **one unrestricted price.**

Proof: We know that after the sales of the restricted products, the residual demand for the product at the *restricted price* p_j $(j \geq \bar{m})$ is given by

$$d^r_{\bar{m},j} = D_{r,j}(1 - \sum_{i=1}^{\bar{m}} \frac{q^*_i}{D_{r,i}}) = (1-\beta)D_{r,j};$$

and by Lemma 2.3.1 the residual demand for the product at the *unrestricted price* p_j is given by

$$d_{\bar{m},j} = D_j - D_{r,j} \sum_{i=1}^{\bar{m}} \frac{q^*_i}{D_{r,i}} = D_j - \beta D_{r,j}.$$

Since $\beta < 1$, it follows that $d^r_{\bar{m},j} > 0$ for all $j \geq \bar{m}$. Define

$$1 - \tilde{\alpha}_j = \frac{d^r_{\bar{m},j}}{d_{\bar{m},j}}, \text{ for } j = \bar{m}, \cdots, n.$$

Then we have

$$1 - \tilde{\alpha}_j = \frac{(1-\beta)(1-\alpha_j)}{1-\beta(1-\alpha_j)}.$$

It is easy to check that $\tilde{\alpha}_j$ is *strictly increasing* in j because α_j is strictly increasing in j and $\beta < 1$.

On the other hand, as demonstrated in Theorem 2.3.4, $\{q^*_{u,m}, \cdots, q^*_{u,n}\}$ solves the following reduced problem:

$$\begin{aligned} \text{Max} \quad & \textstyle\sum_{j=\bar{m}}^{n} p_j q_{u,j} \\ \text{s.t.} \quad & \\ & \textstyle\sum_{j=\bar{m}}^{n} \frac{q_{u,j}}{d_{\bar{m},j}} \leq 1 \\ & \textstyle\sum_{j=\bar{m}}^{n} q_{u,j} \leq k - k_r \\ & q_{u,j} \geq 0, \forall j, \end{aligned}$$

where $k_r = \sum_{i=1}^{\bar{m}} q_{r,i}$. Clearly, this is just Wilson's formulation for the residual market for the unrestricted product. More specifically speaking, this *subproblem* can be restated as follows: the market demand at price p_j is given by $d_{\bar{m},j} : j = \bar{m}, \cdots, n$, and the firm has the option of imposing the restriction on the product, where the impact of the restriction is characterized by $\tilde{\alpha}_j$, which is strictly increasing as proved above. Therefore we can use Theorem 2.3.3 to this subproblem, which says that *if the above reduced problem (Wilson-type) has an optimal solution consisting of two distinct prices, then the firm can do* **strictly better** *by properly using the restriction.* This implies that the optimal solution to the above reduced problem *must* consist of exactly one unrestricted price since otherwise it will lead to *a contradiction* to the assumption that $\{q^*_{r,i}, \cdots, q^*_{r,\bar{m}}; q^*_{u,\bar{m}}, \cdots, q^*_{rn}\}$ is an optimal solution. Therefore, the theorem is proved. □

We notice that the main difficulty in the formulation of the pricing model in Theorem 2.3.2 is that the first demand constraint (2.6) may not be binding, which will force us to deal with a nonlinear constraint in the model. But if the first demand constraint (2.6) is in fact binding, then the BL-model leads to the following linear programming problem:

$$\text{Max} \quad \sum_{i=1}^{m} p_i q_{r,i} + \sum_{j=m}^{n} p_j q_{u,j}$$

s.t.

$$\sum_{i=1}^{m} \frac{q_{r,i}}{D_{r,i}} = 1$$

$$\sum_{j=m}^{n} \frac{q_{u,j}}{D_j - D_{r,j}} \leq 1$$

$$\sum_{i=1}^{n} q_{r,i} + \sum_{j=m}^{n} q_{u,j} \leq k$$

$$q_{r,i}, q_{u,j} \geq 0, \ \forall i \text{ and } \forall j.$$

I use $\tilde{R}(m)$ to denote the optimal revenue value derived from above linear programming problem. I call it *the tight problem* since the first demand constraint (2.6) in BL model is forced to be tight (or equivalently, binding). The implication of the fact that (2.6) is tight is that there will be no residual demand for restricted product at price $p \geq p_{\bar{m}}$ after the sales of the restricted products according to the partial plan $\{q_{r,1}, \cdots, q_{r,m}\}$.

Before presenting the main theorem of this section, I need to clarify a simple technical issue here:

- $\tilde{R}(m)$ will not be well-defined at all m such that $D_{r,m} > k$ since there is no feasible solution for the above tight linear programming problem
- As a convention, I will let $\tilde{R}(m) \equiv 0$ at any m where the tight linear programming problem has no feasible solution.

The following theorem, the main result of this section, shows that a linear programming characterization for $R(\bar{m})$ is possible by using $\tilde{R}(m)$.

Theorem 2.4.2 *It is always true that*

$$\max_{1 \leq m \leq n} \tilde{R}(m) = \max_{1 \leq m \leq n} R(m).$$

Proof: Let $\bar{m}$ be such that

$$R(\bar{m}) = \max_{1 \le m \le n} R(m).$$

I now first prove that there exists some $\tilde{m}$ such that

$$\tilde{R}(\tilde{m}) \ge R(\bar{m}). \tag{2.12}$$

Let $\{q^*_{r,i}, \cdots, q^*_{r\bar{m}}; q^*_{u\bar{m}}, \cdots, q^*_{u,n}\}$ be any *basic* optimal $\bar{m}$-policy derived from the BL-model. If this optimal solution satisfies:

$$\beta \equiv \sum_{i=1}^{\bar{m}} \frac{q^*_{r,i}}{D_{r,i}} = 1,$$

then this solution is feasible to the tight problem associated with $\bar{m}$. Therefore (2.12) holds for $\tilde{m} = \bar{m}$.

I now prove (2.12) when $\beta < 1$. By Theorem 2.4.1, we can limit our discussion on the case that exactly one of $q^*_{u,j}, j = \bar{m}, \cdots, n$ is positive, say, $q^*_{u\bar{j}}$. By Theorem 2.3.5, the second constraint (2.7) must be binding, so it follows that

$$q^*_{u\bar{j}} = d_{\bar{m},\bar{j}} = D_{\bar{j}} - \beta D_{r,\bar{j}}.$$

Now let $\tilde{m} = \bar{j}$, and define:

$$\tilde{q}_{r,i} = \begin{cases} q^*_{r,i} & \text{if } 1 \le i \le \bar{m}, \\ 0 & \text{if } \bar{m}+1 \le i \le \bar{j}-1 \ , \\ (1-\beta)D_{r,\bar{j}} & \text{if } i = \bar{j}; \end{cases}$$

and

$$\tilde{q}_{u,j} = \begin{cases} D_{\bar{j}} - D_{r,\bar{j}} & \text{if } j = \bar{j}, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that the policy $\{\tilde{q}_{r,i}, 1 \le i \le \tilde{m}; \tilde{q}_{u,j}, \tilde{m} \le j \le n\}$ is a feasible solution to the tight problem associated with $\tilde{m}$. Then since $q^*_{u\bar{j}} = D_{\bar{j}} - \beta D_{r\bar{j}}$,

we have

$$\begin{aligned}\tilde{R}(\tilde{m}) &\geq \sum_{i=1}^{\tilde{m}} p_i \tilde{q}_{r,i} + \sum_{j=\tilde{m}}^{n} p_j \tilde{q}_{u,j} = \sum_{i=1}^{\bar{m}} p_i q^*_{r,i} + p_{\bar{j}}(\tilde{q}_{r,\bar{j}} + \tilde{q}_{u,\bar{j}}) \\ &= \sum_{i=1}^{\bar{m}} p_i q^*_{r,i} + p_{\bar{j}}(D_{\bar{j}} - \beta D_{r,\bar{j}}) = \sum_{i=1}^{\bar{m}} p_i q^*_{r,i} + p_{\bar{j}} q^*_{u,\bar{j}} = R(\bar{m}),\end{aligned}$$

which implies that

$$\tilde{R}(\tilde{m}) \geq R(\bar{m}).$$

So (2.12) is true. On the other hand, since $\tilde{R}(m) \leq R(\bar{m})$ for all $m : 1 \leq m \leq n$. Therefore it is shown that

$$\tilde{R}(\tilde{m}) = R(\bar{m}).$$

Hence the theorem is proved as required. □

Let me say a few words here on the computational issue on $\tilde{m}$. *The worst case* is to solve n linear programming problems associated with $\tilde{R}(m)$ for $m = 1, \cdots, n$. The natural approach is to study the curvature property of $\tilde{R}(m)$. A desirable property is that $\tilde{R}(m)$ is in fact *quasi-concave*, which will reduce the number of linear programming problems that are needed to obtain an optimal policy from the order of n to the order of $\ln(n)$. But at the this moment, I can not prove or disprove whether $\tilde{R}(m)$ is indeed quasi-concave.

Before concluding this section, I present another comparison result between the R^w — the optimal revenue derived from Wilson model — and $R(\bar{m})$ for the case that there exists an optimal solution in Wilson's model that consists of exactly one price. I first introduce the following notation:

$$i^*_r = \max\{i : p_i D_{r,i} = \max_{1 \leq j \leq n} p_j D_{r,j}\}; \tag{2.13}$$

$$i_u^* = \min\{i : p_i D_{u,i} = \max_{1 \le j \le n} p_j D_{u,j}\}. \tag{2.14}$$

I now first prove the following interesting lemma.

Lemma 2.4.3 *For any i^* such that*

$$p_{i^*} D_{i^*} = \max_{1 \le j \le n} p_j D_j,$$

it is true that

$$i_r^* \le i^* \le i_u^*.$$

Proof: I will prove the result by contradiction. First suppose that

$$i_r^* > i^*.$$

By definition, it follows that

$$p_{i_r^*} D_{r,i_r^*} = (1 - \alpha_{i_r^*}) p_{i_r^*} D_{i_r^*} \ge p_{i^*} D_{r,i^*} = (1 - \alpha_{i^*}) p_{i^*} D_{i^*},$$

which is impossible since, by the monotonicity assumption on α_i's and the definition of i_r^* (i.e., (2.13)), we have

$$1 - \alpha_{i_r^*} < 1 - \alpha_{i^*} \text{ and } p_{i_r^*} D_{i_r^*} \le p_{i^*} D_{i^*}.$$

Therefore we must have $i_r^* \le i^*$.

Now suppose that

$$i_u^* < i^*.$$

By definition, we have

$$p_{i_u^*} D_{u,i_u^*} = \alpha_{i_u^*} p_{i_u^*} D_{i_u^*} \ge p_{i^*} D_{u,i^*} = \alpha_{i^*} p_{i^*} D_{i^*},$$

which leads to a contradiction since, again by the monotonicity assumption on α_i's and the definition of i_u^* (i.e., (2.14)), we have

$$\alpha_{i_u^*} < \alpha_{i^*} \text{ and } p_{i_u^*} D_{i_u^*} \leq p_{i^*} D_{i^*}.$$

Then we must have $i^* \leq i_u^*$. Hence the lemma is proved. □

I now prove the following interesting result on the structure of optimal policies in Wilson's model that consist of only one price.

Lemma 2.4.4 *Suppose that there exists an optimal policy in Wilson's model that consists of one price, say, $\{p_{i_0}, q_{i_0}\}$, and such that the supply constraint (2.4) is not binding, then*

$$p_{i_0} q_{i_0} = \max_{1 \leq i \leq n} p_i D_i. \tag{2.15}$$

Proof: By Theorem 2.2.3, we know that the demand constraint (2.3) is binding, that is,

$$q_{i_0} = D_{i_0}. \tag{2.16}$$

By assumption, we also know that

$$D_{i_0} < k. \tag{2.17}$$

I now prove the result by contradiction. Suppose that (2.15) is false, that is,

$$p_{i_0} D_{i_0} < \max_{1 \leq i \leq n} p_i D_i. \tag{2.18}$$

Define

$$\bar{i}^* = \max\{i^* : p_{i^*} D_{i^*} = \max_{1 \leq i \leq n} p_i D_i\}.$$

Clearly, from (2.18), we know that $i_0 \neq \bar{i}^*$. So we only need to consider the following two cases.

Case 1: $i_0 < \bar{i}^*$

Under this case, it follows that

$$D_{i_0} > D_{\bar{i}^*},$$

which implies that the firm can sell $D_{\bar{i}^*}$ units at the price $p_{\bar{i}^*}$. By (2.18), the allocation plan $\{p_{\bar{i}^*}, D_{\bar{i}^*}\}$ is *strictly preferred* by the monopolist. This contradicts to the condition that $\{p_{i_0}., q_{i_0}\}$ is an optimal solution to the Wilson-model.

Case 2: $i_0 > \bar{i}^*$

First, by the optimality of the plan $\{p_{i_0}, D_{i_0}\}$, it follows that

$$k < D_{\bar{i}^*}. \tag{2.19}$$

Now let $\{q'_{\bar{i}^*}, q'_{i_0}\}$ satisfy the following system:

$$\begin{aligned} \frac{q'_{\bar{i}^*}}{D_{\bar{i}^*}} + \frac{q'_{i_0}}{D_{i_0}} &= 1 \\ q'_{\bar{i}^*} + q'_{i_0} &= k. \end{aligned}$$

Solving the above system leads to

$$\begin{aligned} q'_{\bar{i}^*} &= \frac{k - D_{i_0}}{D_{\bar{i}^*} - D_{i_0}} D_{\bar{i}^*} \\ q'_{i_0} &= \frac{D_{\bar{i}^*} - k}{D_{\bar{i}^*} - D_{i_0}} D_{i_0}, \end{aligned}$$

which generates the total revenue of

$$R = p_{\bar{i}^*} q'_{\bar{i}^*} + p_{i_0} q'_{i_0} = \frac{k - D_{i_0}}{D_{\bar{i}^*} - D_{i_0}} p_{\bar{i}^*} D_{\bar{i}^*} + \frac{D_{\bar{i}^*} - k}{D_{\bar{i}^*} - D_{i_0}} p_{i_0} D_{i_0}.$$

Then by (2.17), (2.18) and (2.19), we know that

$$R > p_{i_0} D_{i_0},$$

which contradicts to the fact that $\{p_{i_0}, D_{i_0}\}$ is an optimal solution.

In summary, both cases lead to a contradiction. Therefore, we conclude that (2.15) is true, as required. □

The following theorem investigates whether the monopolist still prefers to use the restriction when there is an optimal solution in Wilson's model that consists of only one price.

Theorem 2.4.5 *Suppose that Wilson's model has an optimal solution consisting of exactly one price, say, $\{p_{i_0}, D_{i_0}\}$, with the optimal revenue value of $R^w = p_{i_0} D_{i_0}$, which is such that the supply constraint (2.4) is not binding. Then* **a necessary and sufficient condition** *that there exists an integer m such that $R(m) > R^w$ is:*

$$i_r^* < i_u^*, \tag{2.20}$$

where i_r^* and i_u^* are given by (2.13) and (2.14) respectively.

Proof: By Lemmas 2.4.3 and 2.4.4, we know that

$$i_0 \in \{i^* : p_{i^*} D_{i^*} = \max_{1 \le i \le n} p_i D_i\}$$

and $i_r^* \le i_0 \le i_u^*$.

Sufficiency of (2.20): Here we need to show that (2.20) implies that $R(m) > R^w$ for some m. I will prove this through two cases.

Case 1: $i_r^* = i_0$.

Clearly, under this case, we must have that

$$i_0 < i_u^*.$$

Now let $m = i_0 = i_r^*$ and an allocation plan $\{q_{r,1}, \cdots, q_{r,m}; q_{u,m}, \cdots, q_{u,n}\}$ as follows:

$$q_{r,i} = \begin{cases} D_{r,i} & \text{if } i = m \\ 0 & \text{otherwise;} \end{cases}$$

and

$$q_{u,j} = \begin{cases} D_{u,j} & \text{if } j = i_u^* \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the above plan is feasible to the BL-model associated with m since

$$k > D_m = D_{r,m} + D_{u,m} \geq D_{r,i_0} + D_{u,i_u^*}.$$

By definition of i_u^*, we know that

$$p_{i_0} D_{i_0} < p_{i_u^*} D_{i_u^*},$$

which implies that

$$p_{r,i_0} D_{r,i_0} + p_{u,i_u^*} D_{u,i_u^*} > p_{r,i_0} D_{i_0}.$$

Therefore for this m, we have that $R(m) > R^w$.

Case 2: $i_0 > i_r^*$.

Let $m = i_0$. Then it is clear that $m \leq i_u^*$, Now define a policy such that the part of the allocation plan for the unrestricted product $\{q_{u,m}, \cdots, q_{u,n}\}$ is given as follows:

$$q_{u,j} = \begin{cases} D_{u,j} & \text{if } j = i_u^* \\ 0 & \text{otherwise.} \end{cases}$$

By definition of i_u^*, we know that

$$\sum_{j=m}^{n} p_j q_{u,j} \geq p_m D_{u,m}. \tag{2.21}$$

Denote $k_r = k - D_{u,i_u^*}$. Then by assumption, we know that

$$k > D_m = D_{r,m} + D_{u,m} \geq D_{r,m} + D_{u,i_u^*}$$

which leads to

$$k_r > D_{r,m}.$$

Now if, in addition, $k_r \geq D_{r,i_r^*}$, then define the allocation plan for the restricted product as follows:

$$q_{r,i} = \begin{cases} D_{r,i} & \text{if } i = i_r^* \\ 0 & \text{otherwise.} \end{cases}$$

Then by definition of i_r^* and the fact that $i_r^* < m$, we obtain

$$\sum_{i=1}^{m} p_i D_{r,i} = p_{i_r^*} D_{r,i_r^*} > p_{i_0} D_{r,i_0} = p_m D_{r,m}, \tag{2.22}$$

which implies that $R(m) > R^w$. We still need to prove the result for the following case:

$$D_{r,m} < k_r < D_{r,i_r^*}. \tag{2.23}$$

For this case, let $q_{r,i} = 0$ for $i \neq i_r^*, m$; and q_{r,i^*} and $q_{r,m}$ be the solution of the following system:

$$\begin{aligned} \frac{q_{r,i_r^*}}{D_{r,i_r^*}} + \frac{q_{r,m}}{D_{r,m}} &= 1 \\ q_{r,i_r^*} + q_{r,m} &= k_r. \end{aligned}$$

It is easy to check that the above system has the following unique solution:

$$\begin{aligned} q_{r,i_r^*} &= \frac{k_r - D_{r,m}}{D_{r,i_r^*} - D_{r,m}} D_{r,i_r^*} \\ q_{r,m} &= \frac{D_{r,i_r^*} - k_r}{D_{r,i_r^*} - D_{r,m}} D_{r,m}, \end{aligned}$$

which, according to (2.22) and (2.23), leads to the following:

$$\begin{aligned}
\sum_{i=1}^{m} p_i q_{r,i} &= p_{i_r^*} q_{r,i_r^*} + p_m q_{r,m} \\
&= \frac{k_r - D_{r,m}}{D_{r,i_r^*} - D_{r,m}} p_{i_r^*} D_{r,i_r^*} + \frac{D_{r,i_r^*} - k_r}{D_{r,i_r^*} - D_{r,m}} p_m D_{r,m} \\
&> \frac{k_r - D_{r,m}}{D_{r,i_r^*} - D_{r,m}} p_m D_{r,m} + \frac{D_{r,i_r^*} - k_r}{D_{r,i_r^*} - D_{r,m}} p_m D_{r,m} \\
&= p_m D_{r,m}. \qquad (2.24)
\end{aligned}$$

Therefore by (2.21) and (2.24), it follows that

$$R(m) \geq \sum_{i=1}^{m} p_i q_{r,i} + \sum_{j=m}^{n} p_j q_{u,j} > p_m D_{r,m} + p_m D_{u,m} = p_m D_m = R^w.$$

This proves the result under (2.23). In summary, we prove the sufficiency of (2.20).

Necessity of (2.20): We want to prove that $i_r^* = i_u^*$ implies that

$$\max_{1 \leq m \leq n} R(m) = R^w = p_{i_0} D_{i_0},$$

which, according to Theorem 2.4.2, is equivalent to

$$\max_{1 \leq m \leq n} \tilde{R}(m) = R^w. \qquad (2.25)$$

Note that by Lemmas 2.4.3 and 2.4.4, $i_r^* = i_u^*$ implies that

$$i_r^* = i_u^* = i_0. \qquad (2.26)$$

It is evident that $\tilde{R}(i_0) \geq R^w$. Therefore, to show (2.25), it suffices to prove that

$$\tilde{R}(m) \leq R^w. \qquad (2.27)$$

Note that for these m's such that $\tilde{R}(m) = 0$, (2.27) is true by default. Also, for these m's such that $\tilde{R}(m) > 0$, we know that $\tilde{R}(m)$ must be the optimal objective value of

the tight problem associated with m. Let $\{q^*_{r,1}, \cdots, q^*_{r,m}; q^*_{u,m}, \cdots, q^*_{u,n}\}$ be an arbitrary optimal solution to the tight problem associated with m. Define

$$\begin{aligned} \eta_i &= \tfrac{q^*_{r,i}}{D_{r,i}}, \text{ for } i = 1, \cdots, m, \\ \gamma_j &= \tfrac{q^*_{u,j}}{D_{u,j}}, \text{ for } j = m, \cdots, n. \end{aligned}$$

Then we know that

$$\begin{aligned} \sum_{i=1}^{m} \eta_i &= 1, \\ \sum_{i=1}^{m} \gamma_i &\leq 1. \end{aligned}$$

Now it is easy to see that by the definitions of i^*_r and i^*_u,

$$\begin{aligned} \tilde{R}(m) &= \sum_{i=1}^{m} p_i q^*_{r,i} + \sum_{j=m}^{n} p_j q^*_{u,j} \\ &= \sum_{i=1}^{m} \eta_i p_i D_{r,i} + \sum_{j=m}^{n} \gamma_j p_j D_{u,j} \\ &\leq \sum_{i=1}^{m} \eta_i p_{i^*_r} D_{r,i^*_r} + \sum_{j=m}^{n} \gamma_j p_{i^*_u} D_{u,i^*_u} \\ &\leq p_{i^*_r} D_{r,i^*_r} + p_{i^*_u} D_{u,i^*_u} \\ &= p_{i_0} D_{i_0} = R^w, \end{aligned}$$

since $i^*_r = i^*_u = i_0$. This proves that (2.27) is true. Consequently, (2.25) is proved. This finishes the proof of the necessity of (2.20). □

Remark: We know that when Wilson's model has an optimal solution that consists of only one price and there is excess capacity left, it means that the monopolist will choose the best value of $p_i D_i$. The significance of Theorem 2.4.5 is two-fold. First, it shows that the monopolist can always sell some restricted units and obtain a strictly larger value

of revenues by further exploiting the two different market segments. Second, it gives plausible evidence that the monopolist will utilize the limited capacity more effectively if he uses the restriction as a marketing mechanism. It is also interesting to note that the only case which makes the use of the restriction unattractive is when $i_r^* = i_u^*$. This simply says that when both markets reach the maximum revenue point at the *same price level* and the capacity is high, it makes no difference whether or not the monopolist uses the restriction.

2.5 An Application to Airline Fare Pricing

In this section, I will present a simple application of the pricing model developed in the last section to the airline fare pricing problem. It is well-known that it is a common practice for airlines to apply the restriction on some low-priced fares and that the availability of these restricted fares is controlled. Unfortunately, there has been no theoretical pricing model that can be used to justify such a widespread business practice. The purpose of this section is to show that the new pricing model developed in this chapter may be a useful framework that will fill that theoretical gap.

It is commonly recognized that there are at least two distinct consumer groups for air travel industry: the *leisure group* and the *business group* (sometimes called *must-go group*). For the purpose of illustration,[18] I only consider the case that the market demand consists of two segments — leisure and business.

Let $D^l(p)$ and $D^b(p)$ be the demand from leisure segment and business segment for the unrestricted tickets at price p respectively. Suppose that the airline has an option of a restriction on some fare classes. To simplify the analysis, I make the following assumptions:

[18]The actual number of consumer segments is not crucial in applying the new model.

Restrictions		*Percentage of market satisfying restrictions*	
Advance Bookings (Days)	Minimum Stay (Days)	Leisure Travellers $(1-\gamma_l)$	Business Travellers $(1-\gamma_b)$
7	None	82%	53%
14	None	68%	27%
30	None	49%	11%
7	7	53%	11%
14	7	46%	8%
30	7	35%	4%

Table 2.1: Effect of Fare Restrictions — Boeing Company (1982)

- The restriction on a ticket of price p will reduce the demand from the leisure segment from $D^l(p)$ to $(1-\gamma_l)D^l(p)$, where γ_l — *a fixed constant* — is the proportion of the leisure segment travellers who cannot fly because of the restrictions.

- The restriction on a ticket of price p will reduce the demand from the business segment from $D^b(p)$ to $(1-\gamma_b)D^b(p)$, where γ_b — *a fixed constant* — is the proportion of the business segment travellers who cannot fly because of the restriction.

- $\gamma_l < \gamma_b$, that is, a higher percentage of business segment travellers cannot accommodate the restriction.

The Boeing Company (1982) reported some empirical results on the effect of fare restrictions in the airline industry, which are given in Table 2.1. The table gives clear evidence that $\gamma_l < \gamma_b$.

Now it is clear that the demand for the restricted tickets at price p is given by:

$$D_r(p) = (1-\gamma_l)D^l(p) + (1-\gamma_b)D^b(p).$$

The demand for unrestricted tickets at price p is given by:

$$D(p) = D^b(p) + D^l(p).$$

Then $\alpha(p)$, the percentage of total demand at price p that will not buy the restricted ticket at price p, is specified by

$$1-\alpha(p)=\frac{D_r(p)}{D(p)}=\frac{(1-\gamma_l)D^l(p)+(1-\gamma_b)D^b(p)}{D^b(p)+D^l(p)}.$$

In the development of my pricing model, I require that $\alpha(p)$ be a strictly increasing function. In the airline case considered here, I am going to show that this assumption is equivalent to requiring that the leisure segment travellers are more price sensitive than the business segment travellers, which is of course a very plausible assumption. Basically, this is just saying that the market demand is indeed segmented.

Lemma 2.5.1 *Suppose that the demand functions $D^l(p)$ and $D^b(p)$ are differentiable and decreasing in p. If the impact of the restriction is such that $\gamma_l < \gamma_h$, then $\alpha(p)$ is strictly increasing if and only if*

$$\eta^l(p) > \eta^b(p) \text{ for all } p, \tag{2.28}$$

where η^l and η^b are the prices elasticities of the leisure segment demand and business segment demand respectively, that is,

$$\eta^l(p) \equiv -\frac{\mathrm{d}D^l(p)}{\mathrm{d}p}\times\frac{p}{D^l(p)}, \text{ and } \eta^b(p) \equiv -\frac{\mathrm{d}D^b(p)}{\mathrm{d}p}\times\frac{p}{D^b(p)}.$$

Proof: Note that

$$1-\alpha(p)=\frac{(1-\gamma_l)D^l(p)+(1-\gamma_b)D^b(p)}{D^b(p)+D^l(p)}=(1-\gamma_l)\frac{\frac{D^l(p)}{D^b(p)}+\frac{1-\gamma_b}{1-\gamma_l}}{\frac{D^l(p)}{D^b(p)}+1}.$$

Since $(1-\gamma_b)/(1-\gamma_l) < 1$, then $\alpha(p)$ is strictly increasing if and only if the function $D^l(p)/D^b(p)$ is strictly decreasing, which is equivalent to the condition that

$$\frac{\mathrm{d}D^l(p)}{\mathrm{d}p}D^b(p) < \frac{\mathrm{d}D^b(p)}{\mathrm{d}p}D^l(p);$$

which leads to the condition (2.28) immediately since both D^l and D^b are assumed to be decreasing. Therefore the lemma is proved. □

Remark: It is not quite right to claim that the above lemma is consistent with the discussions in previous sections because of the fact that D^l and D^h are assumed to be step functions in previous sections, rather than differentiable functions as required in Lemma 2.5.1. Nevertheless, it shows us that the monotonicity assumption on the α_i's is plausible in the airline case.

One of the main reasons that airlines impose restrictions on low-price tickets is to *discourage* travellers who are from the business segment from buying the discounted, restricted fares. As demonstrated in the above lemma, the existence of consumer groups with different prices elasticities, together with the condition that the restriction has a non-uniform impact across different consumer groups will provide *sufficient conditions* for a monopolist to consider using restrictions. Empirical results of Oum, Gillen and Nobel (1986, [172]) indicated that in U.S., the average price elasticity for the leisure group was 1.5 whereas for the must-go group it was only 1.15. Then according to the pricing model in the previous section, we may say that the practice of using restrictions in the airline passenger market is justifiable.

Let us now go through the following simple example on airline fare pricing by using one type of restriction.

Example 2.5.1 (*Airline Fare Pricing Problem*)

Consider that the demand function from leisure segment is given by

$$D^l(p) = \begin{cases} 500 & \text{if } p \leq 100 \\ 300 & \text{if } p \in (100, 150] \\ 100 & \text{if } p \in (150, 250] \\ 0 & \text{if } p > 250; \end{cases}$$

and that the demand function from the business segment is given by

$$D^b(p) = \begin{cases} 400 & \text{if } p \leq 100 \\ 300 & \text{if } p \in (100, 150] \\ 200 & \text{if } p \in (150, 250] \\ 50 & \text{if } p \in (250, 400] \\ 0 & \text{if } p > 400. \end{cases}$$

Further assume that $\gamma_l = 0$ and $\gamma_b = 0.5$, that is, the restriction will result in 50 percent of business segment travellers not willing to purchase a restricted ticket and have no impact on the leisure segment. Then the airline has four possible prices $p_1 = \$100, p_2 = \$150, p_3 = \$250$, and $p_4 = \$400$, with $\alpha_1 = 0.23, \alpha_2 = 0.27, \alpha_3 = 0.33$, and $\alpha_4 = 0.5$, which is strictly increasing. The corresponding demand for the restricted fares are:

$$D_{r,1} = 700, D_{r2} = 450, D_{r3} = 200, D_{r4} = 25;$$

and the total demand for unrestricted fares are:

$$D_1 = 900, D_2 = 600, D_3 = 300, D_4 = 50.$$

Table 2.2 summarizes the solutions to the Tight Problems for this example. It is clear from the table that $\tilde{m} = 3$ and there exists a *unique* optimal solution which is given by:

$$q^*_{r2} = 180, q^*_{r3} = 120, q^*_{u3} = 100,$$

Highest Restricted Fare	*An Optimal Solution*	*Derived Revenue*
$p_1 = \$100$ $(m = 1)$	no feasible solution	0
$p_2 = \$150$ $(m = 2)$	no feasible solution	0
$p_3 = \$250$ $(m = 3)$	$q_{r2} = 180, q_{r3} = 120, q_{u3} = 100$	82,000
$p_4 = \$400$ $(m = 4)$	$q_{r2} = 370.6, q_{r4} = 4.4, q_{u4} = 25$	67,350

Table 2.2: Solutions of the Tight Problem — A Numerical Example

with $\tilde{R}(3) = 82,000$.[19]

On the other hand, there is another optimal solution to the original BL-model. Note that in the above optimal solution, the airline will sell 120 restricted tickets and 100 unrestricted tickets at the same price level $p_3 = \$250$ after the sales of 180 restricted tickets at $p_2 = \$150$. We also know that at end of the sales at the price level p_3, there will be no residual demand for both the restricted product and the unrestricted product at $p \geq p_3$. This implies that after the sales of 180 restricted tickets at $p_2 = \$150$, the airline can actually allocate the remaining 220 tickets all *unrestricted* at the price level p_3. Consequently, the following policy:

$$q_{r2} = 180, q_{u3} = 220$$

is also optimal. It is worthwhile to mention that that this new optimal policy seems to be more appealing to the airline since it does not involve selling some restricted tickets and unrestricted tickets at the same price level.

Remarks: In conclusion to this example, we can make the following two general comments:

- The original BL-model may have multiple optimal solutions, which might allow the

[19]Just for the sake of comparison, if the airline decides not to use the restriction on any fare, then the optimal revenue derived from Wilson's model is 80,000.

firm to avoid using an optimal policy that is less appealing;

- The Tight Problem can be used for two purposes: (i) to find the maximum revenue value; and (ii) to *derive* a simple optimal pricing structure.

2.6 Summary – Pricing by Using Restriction

In this chapter, I have investigated the monopoly pricing problem for a unique type of perishable product, such as airline seats and hotel rooms. The model explicitly incorporates the use of artificial restrictions. It is shown that the monopolist's optimal pricing problem can be formulated as a mathematical programming problem. Furthermore, if the impact of the restriction on the demand is such that as price increases, the percentage of consumers who can accommodate the restriction is decreasing, then the monopolist will have a linear programming characterization on the optimal revenue and the optimal pricing policy, which substantially simplifies the original formulation.

I also give a simple application of the model to airlines. I show that the model assumption that $\alpha(p)$ is strictly increasing, in the airline case, is equivalent to saying that leisure travellers are more price sensitive than business travellers, which is exactly what the airlines have in mind when they use restrictions.

Chapter 3

General Optimality Results and Other Properties

3.1 Introduction

Chapter 2 developed a simple pricing model for a monopolist who uses restrictions as a mean of segmenting the demand market. It showed that the monopolist's pricing problem can be reduced to a series of linear programming problems. It demonstrated that by properly setting the level of the highest restricted price and rationing the sales at different prices, the monopolist needs to charge no more than three prices to maximize the revenue. On the other hand, there is a key assumption that the monopolist offers the restricted prices first, and then after all sales have been accommodated at the restricted prices, sales are offered at the unrestricted prices. I call this type of policy a *primary policy.* Even though a primary policy is very natural, it does limit the monopolist's choice on possible pricing structures.

Another implicit assumption was that the firm offers one kind of price at a time, which means that if the firm is selling restricted products, then unrestricted products are *not available.* In practice, for example airlines, both restricted fares and unrestricted fares are used *and* they are made available at the same time. I will call this *the simultaneous availability issue.*

The purpose of this chapter is to resolve these two important issues. In response to the first concern, I will introduce a *general pricing policy* which allows the firm to put restrictions at any possible prices. I am going to show that any general pricing policy

is *weakly dominated* by a primary policy, where the criterion for dominance is defined in terms of *realized revenue.* This consequently implies that any optimal policy in the class of primary policies remains optimal in the class of general policies.

To address the simultaneous availability issue, I will demonstrate that based an optimal policy derived from the Tight Problem, it is possible to construct another policy, which may not be primary, that has the property that making all allocated units available can still generate the same amount of the revenue realized by the primary optimal policy. In other words, the firm can always find an optimal policy that is consistent with existing pricing practice by use of artificial restrictions. To make the argument go through, a reasonable behavioral assumption is needed, that is, when some unrestricted units and some restricted units are offered at the same price level, consumers from the restricted market will first try to buy the unrestricted units.

This chapter is organized as follows. Section 3.2 establishes some preliminary results on general pricing policies. Section 3.3 will prove the general optimality results. Then section 3.4 discusses simultaneous availability issue for optimal policies. The last section is a summary.

3.2 Auxiliary Results on General Pricing Policies

3.2.1 Notation and Definitions

I will use the same model settings as in Chapter 2. That is, there is a monopoly firm which wants to sell a fixed number of units of a certain product. The product is perishable and not storable for consumers (for example, airline tickets and hotel rooms). Let $D_r(p)$ be the demand for the product at price p with the restriction; and let $D_u(p)$ be the demand for the product at price p *only when the restriction is not attached.* Denote the

total demand at price p by $D(p)$. Then

$$D(p) = D_r(p) + D_u(p).$$

Let $\alpha(p)$ be the percentage of those consumers with reservation prices of p or higher *who will be unable to buy the product because of the restriction*, which is given by:

$$\alpha(p) = \frac{D_u(p)}{D(p)} = 1 - \frac{D_r(p)}{D(p)}.$$

Further assume that $D_r(p)$ and $D_u(p)$ are step functions defined on a finite set of prices $\{P_1, \cdots, P_N\}$ such that $P_1 < P_2 < \cdots < P_N < \infty$[1] That is, for $l = r, u$,

$$D_l(p) = \begin{cases} D_{l,1} & \text{if } p \leq P_1; \\ D_{l,i} & \text{if } p \in (P_{i-1}, P_i] \text{ for } 2 \leq i \leq N; \\ 0 & \text{if } p > P_N. \end{cases}$$

Consequently $D(p)$ and $\alpha(p)$ are also step functions:

$$D(p) = \begin{cases} D_1 & \text{if } p \leq P_1 \\ D_i & \text{if } p \in (P_{i-1}, P_i] \text{ for } 2 \leq i \leq n \\ 0 & \text{if } p > P_n, \end{cases}$$

where $D_i = D_{r,i} + D_{u,i}$, and

$$\alpha(p) = \begin{cases} \alpha_1 & \text{if } p \leq P_1 \\ \alpha_i & \text{if } p \in (P_{i-1}, P_i] \text{ for } 2 \leq i \leq n \\ 1 & \text{if } p > P_n; \end{cases}$$

where $\alpha_i = \alpha(P_i)$. As in Chapter 2, I will make the following assumptions throughout this chapter:

- D_i is *strictly* decreasing in i;

[1]There is a minor notational change here from Chapter 2, where the price set is specified by $\{p_1, \cdots, p_n\}$. The change here is purely presentational.

- $D_{l,i}$ is decreasing for $l = r, u$;
- α_j is *strictly increasing* in j.

Since the monopolist has a choice of imposing restrictions on any possible prices and offering the product with and/or without restrictions at the same price, a *general pricing policy* must include

- the choice of a set of prices which will be attached with the restriction;
- the allocation plan.

Some observations are needed here before I formally introduce the definition of a general pricing policy.[2] First, a general policy should allow the firm to offer restricted and unrestricted product at the same price level in an arbitrary order in the sense that the firm can sell unrestricted product first and the restricted product later at the same price, and vice versa. Because of this, it is possible there are more than two allocations on the same price, for example, the firm may offer some restricted product at price p_1, then some unrestricted product at the same price, then again offering some additional restricted product at this price. Consequently, the number of initial prices, $P_1, \cdots, P_N$, has to be modified to capture this possibility. Therefore, by looking at the *order* of offered prices together with the option of attaching restrictions, we may have *a sequence of prices* $p_1 \leq p_2 \cdots \leq p_n$ such that $p_i \in \{P_1, \cdots, P_N\}$, for all $i = 1, \cdots, n$. Furthermore, at each of these offered prices, there is also a *tag* which indicates the *nature of this price.* For example, let δ_i is the tag variable on the price p_i, then $\delta_i = r$ implies that the price p_i be a restricted price; and $\delta_i = u$ indicates that the price is unrestricted. I now introduce the following definition of a general pricing policy.

[2]The following discussion also helps us to understand the reason why I need the above minor notational change in this chapter.

Definition 3.2.1: *A* **general pricing policy** *is a bundle* $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$ *such that*

- $p_i \in \{P_1, \cdots, P_N\}$, *for all* $i = 1, \cdots, n$ *such that* $p_1 \leq p_2 \leq \cdots \leq p_n$;
- $\delta_i = r$ *indicates that* p_i *is attached with restriction and* $\delta_i = u$ *implies that* p_i *is unrestricted;* and
- $q_{\delta_i,i}$ *is the quantity available for sale at the price* p_i *associated with the tag variable* δ_i.

Remarks:

- For ease of presentation, we will call the class of general pricing polices **G-class**; and any general pricing policy will be called a **G-policy**.
- It is clear that there is no point for the firm to offer several positive quantities with the the same characteristics (that is, restricted or unrestricted) *consecutively* at the same price level. For example, if $\delta_i = r$ and $q_{r,i} > 0$, then there is no need to consider any policy such that
  - $\delta_{i+1} = r$;
  - $p_{i+1} = p_i$;
  - $q_{r,i+1} > 0$.
- For the rest of this chapter, when we refer to G-policy, we require that if there are positive quantities allocated at the same price level, then the characteristics of the units must be *alternating*.

Recall that Chapter 2 limited the discussion to the following type of policies:

- $n = N + 1$;
- $p_i = P_i$ for $i = 1, \cdots, m$, and $p_j = P_{j-1}$ for $j = m+1, \cdots, n$; and
- $\delta_i = r$ for $i = 1, \cdots, m$ and $\delta_j = u$ for $j = m+1, \cdots, n$.

I have shown that under the assumptions: α_i is strictly increasing and the allocated restricted units are sold *before* the sales of the allocated unrestricted units, the monopolist's pricing problem can be formulated as the following two-stage process:

1. For any given $1 \leq m \leq N$, solve the following linear programming problem:

$$\begin{array}{ll} \text{Max} & \sum_{i=1}^{m} P_i q_{ri} + \sum_{j=m}^{N} P_j q_{uj} \\ \text{s.t.} & \\ & \sum_{i=1}^{m} \frac{q_{ri}}{D_{ri}} = 1 \\ & \sum_{j=m}^{N} \frac{q_{uj}}{D_j - D_{r,j}} = 1 \\ & \sum_{i=1}^{m} q_{ri} + \sum_{j=m}^{N} q_{uj} \leq q \\ & q_{ri}, q_{uj} \geq 0, \forall i \text{ and } \forall j, \end{array}$$

   which leads to a realized revenue of $\tilde{R}(m)$;

2. Find a value of $\tilde{m}$ such that

$$\tilde{R}(\tilde{m}) = \max_{1 \leq m \leq N} \tilde{R}(m).$$

It is the purpose of this chapter to show that there exists an optimal policy in the G-class that has the above special structure. For presentational purpose, I give a special name for the type of polices discussed in Chapter 2:

Definition 3.2.2: *We call a G-policy* $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$ *a* **primary** m**-policy** *if*

1. $\delta_i = r$, for $i = 1, \cdots, m$; and $\delta_i = u$, for $i = m+1, \cdots, n$; and
2. $p_m = p_{m+1}$,

where $m : 1 \leq m \leq n$.

Remark: Because of the special structure of a primary policy, we can represent any primary policy as follows:

$$\{(p_1, q_{r,1}), \cdots, (p_m, q_{r,m}); (p_m, q_{u,m}), \cdots, (p_n, q_{u,n})\};$$

or simply,

$$\{q_{r,1}, \cdots, q_{r,m}; q_{u,m}, \cdots, q_{u,n}\}.$$

I will use this representation if it causes no confusion. We should also keep in mind that under this notation, at the price level p_m, the restricted units are sold first.

Since the monopolist's goal is to maximize its revenue, it is natural to *rank* the pricing policies in term of the corresponding revenues. For this, I introduce the following concept:

Definition 3.2.3: *We say a feasible G-policy* $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$ **weakly dominates** *another feasible G-policy* $\{n'; (p'_i, q'_{\delta'_i,i}, \delta'_i), i = 1, \cdots, n'\}$ *if*

$$\sum_{i=1}^{n} p_i q_{\delta_i,i} \geq \sum_{i=1}^{n'} p'_i q'_{\delta'_i,i}.$$

If the above inequality is strict, we call it **strict dominance**. *And if it holds with equality, we say that the two pricing policies are* **equivalent**. *Consequently, we say a feasible G-policy is* **optimal** *if it weakly dominates all other feasible policies.*

Given any G-policy $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$, we introduce the following notation:

- For $i = 1, \cdots, n$ let $d_i^r = D_r(p_i)$, $d_i^u = D_u(p_i)$ and $d_i = d_i^r + d_i^u$.
- Let $\gamma_j = 1 - \frac{d_j^r}{d_j}$ be the percentage of consumers who will not purchase the product at price p_j if restrictions are attached.
- For $t = 1, \cdots, n$ and $j \geq t$, let $d_{t,j}$ be the total residual demand at the unrestricted price p_j *after* the sales according to the partial policy $\{(p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, t\}$. And let $d_{t,j}^r$ be the corresponding residual demand at the restricted price p_j. Define $d_{t,j}^u = d_{t,j} - d_{t,j}^r$.
- Let $\gamma_{t,j} = 1 - \frac{d_{t,j}^r}{d_{t,j}}$ be the percentage of residual consumers who will not purchase the product at the restricted price p_j after the sales according to $\{(p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, t\}$.

It is worthwhile to mention that

1. d_i is in general not equal to D_i;
2. $\{d_i\}$ is decreasing, but may not be strictly decreasing as D_i is;
3. γ_j is in general not equal to α_j; and
4. $\{\gamma_j\}$ is increasing, but may not be strictly increasing as α_j is.

3.2.2 A Lemma on the Property of Monotonicity

Let us now go back to the discussion of the G-policy. For a G-policy as defined in Definition 3.2.1, if the sales are at an unrestricted price, then there is no impact on the values of the γ's, which means that the portion of these consumers who can not accommodate the restriction remains unchanged due to the use of proportional-rationing rule to update

the residual demand. But if the sales are at a restricted price, the values of γ's will be changed. The following lemma shows that the monotonicity of γ's will be maintained.

Lemma 3.2.1: *For any given G-policy* $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$ *and given* t, $\gamma_{t,j}$ *is increasing in* j *on* $j \geq t$.

Proof: I prove the result by induction. First, consider $t = 1$. If $\delta_1 = u$, then the sales of unrestricted products makes no impact on the γ's, that is, $\gamma_{1,j} = \gamma_j$, which is increasing by the monotonicity assumption on α_i's. Now if $\delta_1 = r$, then

$$d^r_{1,j} = d^r_j(1 - \frac{q_{r,1}}{d^r_1}),$$

and

$$d_{1,j} = d_j - d^r_j \frac{q_{r,1}}{d^r_1}.$$

Therefore,

$$1 - \gamma_{1,j} = \frac{d^r_{1,j}}{d_{1,j}} = \frac{(1 - \frac{q_{r,1}}{d^r_1})(1 - \gamma_j)}{1 - (1 - \gamma_j)\frac{q_{r,1}}{d^r_1}}.$$

Since γ_j is increasing in j and $\frac{q_{r,1}}{d^r_1} \leq 1$, it follows immediately from the above expression that $\gamma_{1,j}$ is also *increasing*. This proves the result for $t = 1$.

Suppose that the result holds for t, that is, $\gamma_{t,j}$ is increasing. By the same argument, if $\delta_{t+1} = u$, then $\gamma_{t+1,j} = \gamma_{t,j}$; hence the result is true for $t + 1$ in this case. If $\delta_{t+1} = r$, since

$$1 - \gamma_{t+1,j} = \frac{d^r_{t+1,j}}{d_{t+1,j}} = \frac{(1 - \frac{q_{r,t+1}}{d^r_{t,t+1}})(1 - \gamma_{t,j})}{1 - (1 - \gamma_{t,j})\frac{q_{r,t+1}}{d^r_{t,t+1}}},$$

the monotonicity of $\gamma_{t+1,j}$ follows immediately from the monotonicity of $\gamma_{t,j}$ and the fact that $q_{r,t+1} \leq d^r_{t,t+1}$. This implies that the result also holds for $t + 1$ when $\delta_{t+1} = r$. Therefore by induction the result is true for $t = 1, 2, \cdots$ and the lemma is proved. □

The following is an immediate consequence of Lemma 3.2.1.

Corollary 3.2.2: *For any $t = 1, \cdots, n$, one and only one of the following two statements is true:*

1. $\gamma_{t,j} = 1$ *for all* $j \geq t$;
2. $\{\gamma_{t,j}\}$ *has a* **subsequence** *that is* **strictly increasing** *in* j.

Furthermore, we have:

1. *for a given t and any $j_1 < j_2$ such that $p_{j_1} < p_{j_2}$, if $\gamma_{t,j_1} < 1$, then*

$$\gamma_{t,j_1} < \gamma_{t,j_2};$$

2. *for all $j \geq t'' > t'$,*

$$\gamma_{t',j} \leq \gamma_{t'',j}.$$

Remark: The above result shows us that as long as the residual demand for the restricted product is positive, the γ-sequence is *strictly increasing* in terms of price levels. Note that the monotonicity condition on the original α-sequence is also in reference to price levels. Therefore, the result in Corollary 3.2.2 will enable us to use the BL-model in Chapter 2 for the *residual market* if we limit to primary policies for the residual market.

3.3 General Optimality Results

3.3.1 Three Fundamental Lemmas

Recall that the BL-model in Chapter 2 only considered a special type of pricing policy – one which only allows the firm to sell the unrestricted products at higher prices than the restricted products. Such a type of policy limits the firm's choices on possible pricing

structures. For example, it rules out the possibility that the firm may consider to sell some unrestricted units *before* the sales of some restricted units. The purpose of this section is to further investigate the firm's pricing problem under the *expanded* set of pricing policies — the set of general pricing policies as defined by Definition 3.2.1, which contains policies that allow the firm to impose the restrictions at any price level. In this section, I want to prove a very interesting general optimality result, which shows that any G-policy is in fact *weakly dominated* by a primary policy. Consequently, any optimal policy determined in the BL-model, which is thus primary, is also optimal in the G-class. This is a very important and useful result because:

- There exists an optimal pricing policy in the BL-model characterized by a series of linear programming problems; and
- It is very difficult, if not impossible, to discuss the nature of optimal polices in the G-class.

For ease of presentation, I introduce the following definition:

Definition 3.3.1: *For any G-policy* $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$, *the price* p_i *is said to be* **active** *if* $q_{\delta_i,i} > 0$. *If* $\delta_i = r$, *I call it* **an active restricted price***; and similarly, if* $\delta_i = u$, *I call it* **an active unrestricted price**.

By definition, a primary policy will not sell any unrestricted product at a price that is strictly less than an active restricted price. The following lemma shows that I can focus on the policies such that there is no unrestricted product sold at the level of the last active restricted price *prior to* the sales of the restricted product at this price.

Lemma 3.3.1: *There always exists an optimal policy in the G-class with the following properties:*

1. *At the* level *of the last active restricted price, there are at most two active prices; and*

2. *The first one of these two active prices is associated with the sales of restricted units.*

In words, at the last active restricted price level, the firm sells the allocated restricted units **before** *the sales of allocated unrestricted units at this price level, if any.*

Proof: Consider any optimal G-policy $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$. Let p_m be the last active restricted price and i_0 be the *largest index* for an active unrestricted price *before* the restricted price p_m, that is,

$$i_0 = \max\{i : i < m, \delta_i = u \text{ and } q_{u,i} > 0\}.$$

It is evident that $i_0 \leq m - 1$ and $p_{i_0} \leq p_m$.

Now if $p_{i_0} < p_m$, then Lemma 3.3.1 is true by default since it means that there are no unrestricted units that are sold at the price level p_m *before* the sales of the allocated $q_{r,m}$ restricted units (also) at p_m. So we only need to consider the case that $p_{i_0} = p_m$. Without loss of generality, we can take $i_0 = m - 1$, *i.e.*, $p_{m-1} = p_m$. Hence p_{m-1} is the largest unrestricted price that is *precedent to* p_m.

By definition of p_m, we know that after price p_m, the firm only sells the unrestricted products. Therefore, again without loss of generality, we can assume that $p_m \leq p_{m+1} < \cdots < p_n$. We need to discuss two cases here.

Case 1: $p_m < p_{m+1}$.

By Corollary 3.2.2, we know that $\gamma_{t-1,j}$ is strictly increasing in $j \geq m$ since p_j is strictly increasing in $j \geq m$. Then by Theorem 2.4.2 and Lemma 3.2.1, I know that for

the remaining market specified by $\{d_{m-1,j}, d^r_{m-1,j} : j = m, \cdots, n\}$ together with the price set $\{p_m, \cdots, p_n\}$, there is a new primary policy for the residual market such that

- it is at least as good as the original partial policy

$$\{(p_m, q_{r,m}, r), (p_{m+1}, q_{u,m+1}, u), \cdots, (p_n, q_{u,n}, u)\}; \text{ and}$$

- there is no residual demand for restricted product after the sales at the last active restricted price under this new primary policy for the residual market.

Because of this, *without loss of generality*, we will assume that $d^r_{m,j} = 0$ for all $j \geq m$, which implies that after the sales at the last active restricted price in the original policy, there is indeed *no residual demand left for the restricted product* at price level $p > p_m$. Consequently, since $p_{m-1} = p_m$,

$$q_{r,m} = d^r_{m-1,m} = (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}})d^r_{m-2,m-1} = (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}})d^r_{m-2,m}, \tag{3.1}$$

and the total residual demand at the (unrestricted) price p_j after the sales up to the restricted price p_m is given by:

$$d_{m,j} = d^u_{m,j} = d^u_{m-1,j} = (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}})d^u_{m-2,j} \text{ for } j \geq m. \tag{3.2}$$

By (3.1), it is easy to check that

$$q_{u,m-1} + q_{r,m} > d^r_{m-2,m} = d^r_{m-2,m-1}.$$

Let us now define a new G-policy as follows: $\{n; (p_i, q'_{\delta'_i,i}, \delta'_i), i = 1, \cdots, n\}$ such that

$$\delta'_i = \begin{cases} \delta_i & \text{if } 1 \leq i \leq m-2 \\ r & \text{if } i = m-1 \\ u & \text{if } i \geq m; \end{cases}$$

and

$$q'_{\delta'_i,i} = \begin{cases} q_{\delta_i,i} & \text{if } 1 \le i \le m-2 \\ d^r_{m-2,m} & \text{if } i = m-1 \\ q_{u,m-1} + q_{r,m} - d^r_{m-2,m} & \text{if } i = m \\ q_{u,i} & \text{if } i > m. \end{cases}$$

Intuitively speaking, the new G-policy has the following properties:

- it has the same price set as the original G-policy;
- we *switched* the tag variables on prices p_{m-1} and p_m in the original G-policy so that p_{m-1} becomes the last active restricted price;
- furthermore, the new policy is *equivalent* to the old policy since it is easy to check that
$$\sum_{i=1}^{n} p_i q'_{\delta'_i,i} = \sum_{i=1}^{n} p_i q_{\delta_i,i}; \text{ and}$$
- after the sales at the restricted price p_{m-1}, there is no residual demand for the restricted product at price p_{m-1} or higher.

Now we need to show that the new policy is in fact feasible. By (3.1),

$$\begin{aligned} q'_{u,m} &= q_{u,m-1} + q_{r,m} - d^r_{m-2,m} = q_{u,m-1} + (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}}) d^r_{m-2,m-1} - d^r_{m-2,m-1} \\ &= \frac{d^u_{m-2,m-1}}{d_{m-2,m-1}} q_{u,m-1}. \end{aligned} \tag{3.3}$$

Note that, after the sales of $q'_{r,m-1}$ units at the restricted price p_{m-1}, there is no demand for restricted product at any price p_j for $j \ge m$. As usual, let $d'_{m-1,j}$ be the total residual demand at price p_j after the sales according to the new *partial G-policy*

$$\{(p_i, q'_{\delta'_i,i}, \delta'_i) : i = 1, \cdots, m-1\}.$$

Then it is clear that

$$d'_{m-1,j} = d^u_{m-2,j}, \text{ for } j \geq m-1.$$

Since $p_{m-1} = p_m$, it follows from (3.3) that for the new policy the residual demand at the unrestricted price p_j $(j \geq m)$ after sales up to the unrestricted price p_m is given by

$$d'_{m,j} = (1 - \frac{q'_{u,m}}{d^u_{m-2,m}})d^u_{m-2,j} = (1 - \frac{q'_{u,m}}{d^u_{m-2,m-1}})d^u_{m-2,j} = (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}})d^u_{m-2,j},$$

which is identical to (3.2). This shows that the new policy is feasible. Therefore the new G-policy

$$\{n; (p_i, q'_{\delta'_i,i}, \delta'_i), i = 1, \cdots, n\}$$

is also an optimal policy and is such that at the last active restricted price level, the unrestricted product, if allocated with a positive amount, will be sold *after* the sales of the restricted product at this price.

Case 2: $p_m = p_{m+1}$.

Under this case, we know that at the level of the last active restricted price, the firm will sell some unrestricted units first $(q_{u,m-1})$, then some restricted units $(q_{r,m})$, and then some unrestricted units again $(q_{u,m+1})$. Note that after the sales according to the original policy up to p_{m-1}, the relevant part of γ-sequence is

$$\{\gamma_{m-1,m}, \gamma_{m-1,m+2}, \cdots, \gamma_{m-1,n}\},$$

which, by Corollary 3.2.2, is strictly increasing. And the partial policy

$$\{(p_m, q_{r,m}), (p_{m+1}, q_{u,m+1}), \cdots, (p_n, q_{u,n})\}$$

is a primary 1-policy for the residual market demand

$$\{(d^r_{m-1,m}, d_{m-1,m}), (d^r_{m-1,m+2}, d_{m-1,m+2}), \cdots, (d^r_{m-1,n}, d_{m-1,n})\},$$

together with the set of prices $\{p_m, p_{m+2}, \cdots, p_n\}$. By the same argument used in Case 1, without loss of generality, we can assume that $d^r_{m,j} = 0$. Therefore (3.1) and (3.2) are still valid. Now define another new G-policy

$$\{n''; (p''_i, q''_{\delta''_i,i}, \delta''_i), i = 1, \cdots, n''\},$$

such that $n'' = n - 1$ and

$$p''_i = \begin{cases} p_i & \text{if } 1 \le i \le m \\ p_{i+1} & \text{if } m+1 \le i \le n''; \end{cases}$$

$$\delta''_i = \begin{cases} \delta_i & \text{if } 1 \le i \le m-2 \\ r & \text{if } i = m-1 \\ u & \text{if } m \le i \le n''; \end{cases}$$

and

$$q''_{\delta''_i,i} = \begin{cases} q_{\delta_i,i} & \text{if } 1 \le i \le m-2 \\ d^r_{m-2,m} & \text{if } i = m-1 \\ q_{u,m-1} + q_{r,m} - d^r_{m-2,m} + q_{u,m+1} & \text{if } i = m \\ q_{u,i+1} & \text{if } m+1 \le i \le n''. \end{cases}$$

Again, we only need to show that this new G-policy is feasible. First, by (3.3), we get

$$q''_{u,m} = \frac{d^u_{m-2,m-1}}{d_{m-2,m-1}} q_{u,m-1} + q_{u,m+1}. \tag{3.4}$$

Similarly, let $d''_{m-1,j}$ be the total residual demand at price p_j after the sales according to the partial new policy $\{(p''_i, q''_{\delta''_i,i}, \delta''_i) : i = 1, \cdots, m-1\}$. As in Case 1, we know that

$$d''_{m-1,j} = d^u_{m-2,j}, \text{ for } j \ge m-1.$$

Under the new policy, by (3.4) and the fact that $p_{m-1} = p_m = p_{m+1}$, the residual demand at the unrestricted price p_j $(j \ge m)$ after the sales up to the unrestricted price p_m is

given by:

$$d''_{m,j} = (1 - \frac{q''_{u,m}}{d''_{m-1,m}})d''_{m-1,j} = (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}} - \frac{q_{u,m+1}}{d_{m-2,m}})d^u_{m-2,j}. \tag{3.5}$$

Now since $p_m = p_{m+1}$, then according to (3.2), for $j \geq m+1$,

$$\begin{aligned} d_{m+1,j} &= (1 - \frac{q_{u,m+1}}{d^u_{m,m+1}})d^u_{m,j} \\ &= d^u_{m,j} - q_{u,m+1}\frac{d^u_{m,j}}{d^u_{m,m+1}} \\ &= (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}})d^u_{m-2,j} - \frac{q_{u,m+1}}{d^u_{m-2,m+1}}d^u_{m-2,j} \\ &= (1 - \frac{q_{u,m-1}}{d_{m-2,m-1}} - \frac{q_{u,m+1}}{d_{m-2,m}})d^u_{m-2,j}, \end{aligned}$$

which implies that

$$d''_{m,j-1} = d_{m+1,j}, \text{ for } j = m+1, \cdots, n. \tag{3.6}$$

Note that for $j = m+2, \cdots, n$, $q''_{u,j-1} = q_{u,j}$. So (3.6) implies that the new G-policy is indeed feasible. Therefore the new G-policy is an optimal policy with the property that p_{m-1} is the last active restricted price.

In summary, we have shown that there always exists another optimal G-policy such that p_{m-1} is the last active restricted price. If there exists another active unrestricted price $p_{i'_0}$ such that

- $i'_0 < m-1$,
- $\delta_{i'_0} = u$,
- $p_{i'_0} = p_{m-1}$,

then we can *repeat* the above procedure. In the end, we will have an optimal policy such that $p_{i'_0}$ will be the active active restricted price. Consequently, by induction, this will lead an optimal policy with the following properties:

- if we let $p_{\bar{m}}$ be the active restricted price and define

$$l_0 = \min\{l : p_l \geq p_{\bar{m}}, \delta_l = u, q_{u,l} > 0\},$$

then $l_0 \geq m + 1$.

This implies that there are no unrestricted units at the price $p_{\bar{m}}$ that are sold before the sales of the allocated restricted units also at the price level $p_{\bar{m}}$. Therefore the lemma is proved. □

Remark: Simply speaking, Lemma 3.3.1 says that if the firm plans to sell some unrestricted units and some restricted units at the last active restricted price level, then without loss of generality the firm can limit its attention on these policies such that the restricted units are sold *first.*

The following two lemmas further explore this issue by considering cases where there are many (at least one) restricted prices between two active unrestricted prices.

Lemma 3.3.2: *Any optimal G-policy of the following form:*

$$\{(p_1, q_{u,1}, u), (p_2, q_{r,2}, r); (p_3, q_{u,3}, u), \cdots, (p_n, q_{u,n}, u)\}$$

is equivalent to a primary policy, where

$$p_1 \leq p_2 \leq p_3 < p_4 < \cdots < p_n \text{ and } q_{u,1} > 0, q_{r,2} = d^r_{1,2}.$$

In words, it says that if an optimal policy in the G-class is such that (1) there is only one active restricted price; (2) there is exactly one active unrestricted price that is smaller (not necessarily strict) than the only active restricted price; and (3) the allocated restricted

units at this active restricted price will exhaust all the residual demand for the restricted product, then the optimal policy is equivalent to a primary policy.

Proof: Clearly, under the given policy, I know that after the sales of the restricted product at price p_2, there will be no residual demand for restricted product at price p_j for $j \geq 2$. Therefore, the total residual demand at price p_j is given by

$$d_{2,j} = d^u_{1,j} = (1 - \frac{q_{u,1}}{d_1})d^u_j, j \geq 3. \tag{3.7}$$

Consequently, the feasibility condition for the remaining unrestricted market is

$$\frac{q_{u,1}}{d_1} + \sum_{j=3}^{n} \frac{q_{u,j}}{d^u_j} \leq 1.$$

I now construct a new policy as follows:

$$\begin{aligned} q'_{r,1} = q_{u,1} \wedge d^r_1, \quad q'_{u,1} &= q_{u,1} - q'_{r,1}; \\ q'_{r,2} = (1 - \frac{q'_{r,1}}{d^r_1})d^r_2, \quad q'_{u,2} &= q_{r,2} - q'_{r,2}; \\ q'_{r,j} = 0, \quad q'_{u,j} &= q_{u,j}, \forall j \geq 3; \end{aligned}$$

where $x \wedge y = \min(x, y)$. Some observations here will be helpful:

- the basic idea here is to *sell the restricted product as much as possible at the price level* p_1, where the upper bound is determined by the existing demand d^r_1 and the original allocation $q_{u,1}$ at the unrestricted price p_1;
- if at p_1, the allocation $q'_{r,1}$ cannot exhaust all the restricted demand, then $q'_{r,2}$ will take care of the residual market for restricted product at p_2;
- *the new policy leads to a primary policy* since either (i) $q'_{r,1} = d^r_1$, which leads to a following primary 1-policy:

$$\{q'_{r,1}; q'_{u,1}, q_{u,2}, \cdots, q_{u,n}\},$$

or (ii) $q'_{r,2} = (1 - q'_{r,1}/d^r_1)d^r_2 > 0$, which implies that $q'_{u,1} = 0$ and consequently leads to a following primary 2-policy:

$$\{q'_{r,1}, q_{r,2}; q'_{u,2}, q_{u,3}, \cdots, q_{u,n}\};$$

- it is easy to check that the new derived primary policy is equivalent to the original policy.

Let $\{q'_{r,1}, \cdots, q'_{r,t}; q'_{u,t}, q_{u,t+1}, \cdots, q_{u,n}\}$ be the derived primary t-policy from the above specification, where $t = 1$ or $t = 2$.

Case 1: $t = 1$.

This implies that

$$q'_{r,1} = d^r_1 \leq q_{u,1}, q'_{u,1} = q_{u,1} - d^r_1 q'_{u,2} = q_{r,2}, \text{ and } q'_{u,j} = q_{u,j}, \forall j \geq 3.$$

Let $d'_{2,j}$ be the total residual demand *after* the sales according to $(p_1, q'_{r,1})$, $(p_1, q'_{u,1})$ and $(p_2, q'_{u,2})$. Then it is easy to check that

$$d'_{2,j} = \left[1 - \frac{q_{u,1} - d^r_1}{d^u_1} - \frac{q_{r,2}}{d^u_2}\right] d^u_j.$$

Therefore, to prove the feasibility of the new primary policy, it suffices to show that,

$$d'_{2,j} \geq d_{2,j}, \forall j \geq 3$$

which, by (3.7), is equivalent to[3]

$$\begin{aligned} 1 - \frac{q_{u,1} - d^r_1}{d^u_1} - \frac{q_{r,2}}{d^u_2} \geq 1 - \frac{q_{u,1}}{d_1} \quad &\Longleftrightarrow \quad \frac{q_{u,1} - d^r_1}{d^u_1} + \frac{q_{r,2}}{d^u_2} \leq \frac{q_{u,1}}{d_1}, \\ &\Longleftrightarrow \quad \frac{q_{u,1} - d^r_1}{d^u_1} + (1 - \frac{q_{u,1}}{d_1})\frac{d^r_2}{d^u_2} \leq \frac{q_{u,1}}{d_1} \end{aligned}$$

[3]In what follows, the symbol $\Longleftrightarrow$ means "equivalent".

$$\begin{aligned}
&\Longleftrightarrow \quad \frac{q_{u,1}}{d_1^u} - \frac{d_1^r}{d_1^u} + \frac{d_2^r}{d_2^u} \leq \frac{q_{u,1}}{d_1}(1 + \frac{d_2^r}{d_2^u}) = \frac{q_{u,1}d_2}{d_1 d_2^u}, \\
&\Longleftrightarrow \quad \frac{q_{u,1}}{d_1}(\frac{d_1}{d_1^u} - \frac{d_2}{d_2^u}) \leq \frac{d_1^r}{d_1^u} - \frac{d_2^r}{d_2^u} = \frac{d_1}{d_1^u} - \frac{d_2}{d_2^u} \\
&\Longleftrightarrow \quad q_{u,1} \leq d_1,
\end{aligned}$$

which follows from the feasibility condition for the original policy. This proves Case 1.

Case 2: $t = 2$.

Under this case, we have that

$$\begin{aligned}
q'_{r,1} &= q_{u,1} \leq d_1^r; q'_{r,2} = (1 - \frac{q_{u,1}}{d_1^r})d_2^r \\
q'_{u,2} &= q_{r,2} - q'_{r,2} = (1 - \frac{q_{u,1}}{d_1})d_2^r - (1 - \frac{q_{u,1}}{d_1^r})d_2^r = q_{u,1}d_2^r(\frac{1}{d_1^r} - \frac{1}{d_1}) = \frac{q_{u,1}}{d_1}\frac{d_2^r}{d_1^r}d_1^u \\
q'_{u,j} &= q_{u,j}, \forall j \geq 3.
\end{aligned}$$

Again, let $d'_{2,j}$ be the total residual demand after the sales according to $(p_1, q'_{r,1})$, $(p_2, q'_{r,2})$ and $(p_2, q'_{u,j})$. Then

$$d'_{2,j} = (1 - \frac{q'_{u,2}}{d_2^u})d_j^u = (1 - \frac{q_{u,1}}{d_1}\frac{d_1^u}{d_1^r}\frac{d_2}{d_2^u})d_j^u.$$

I hope to show that

$$d'_{2,j} \geq d_{2,j},$$

which, by (3.7), is equivalent to

$$\frac{d_1^u}{d_1^r}\frac{d_2^r}{d_2^u} \leq 1 \Leftrightarrow \frac{d_2^r}{d_2^u} \leq \frac{d_1^r}{d_1^u} \Leftrightarrow \frac{d_2}{d_2^u} \leq \frac{d_1}{d_1^u} \Leftrightarrow \frac{d_1^u}{d_1} \leq \frac{d_2^u}{d_2} \Leftrightarrow \gamma_1 \leq \gamma_2,$$

which follows from the monotonicity condition on γ's. This shows that the new policy is feasible. This proves Case 2. In summary, the lemma is proved. □

The following lemma considers the case that there are two active restricted prices between the first two active unrestricted prices.

Lemma 3.3.3: *Any optimal G-policy of the following form:*

$$\{(p_1, q_{u,1}, u), (p_2, q_{r,2}, r), (p_3, q_{r,3}, r); (p_4, q_{u,4}, u), \cdots, (p_n, q_{u,n}, u)\}$$

is equivalent to a primary policy, where

$$p_1 \leq p_2 < p_3 \leq p_4 < \cdots < p_n \text{ and } q_{u,1} > 0, q_{r,2} > 0, q_{r,3} = d^r_{2,3} > 0.$$

In words, it says that if there exists an optimal policy in the G-class such that (1) it has two consecutive *active restricted prices, (2) there is* **one and only one** *active unrestricted price before these two active restricted prices, (3) there is no residual demand left for the restricted product at the end of sales of the restricted units at the second restricted price level, then this optimal policy is equivalent to a primary policy.*

Proof: First of all, it is easy to show that

$$q_{r,3} = d^r_{2,3} = (1 - \frac{q_{2,r}}{d^r_{1,2}})d^r_{1,3} = (1 - \frac{q_{u,1}}{d_1})d^r_3 - \frac{q_{r,2}}{d^r_2}d^r_3 = (1 - \frac{q_{u,1}}{d_1} - \frac{q_{r,2}}{d^r_2})d^r_3,$$

which is equivalent to

$$\frac{q_{u,1}}{d_1} + \frac{q_{r,2}}{d^r_2} + \frac{q_{r,3}}{d^r_3} = 1. \tag{3.8}$$

This also implies that there is no residual demand for restricted product after the sales up to $(p_3, q_{r,3})$. Therefore, the total residual demand after the sales up to $(p_3, q_{r,3})$ is given by

$$d_{3,j} = d^u_{3,j} = (1 - \frac{q_{u,1}}{d_1})d^u_j. \tag{3.9}$$

Define a new policy as follows

$$q'_{r,1} = q_{u,1} \wedge d^r_1, \quad q'_{u,1} = \quad q_{u,1} - q'_{r,1};$$

$$q'_{r,2} = \left[(1 - \frac{q'_{r,1}}{d_1^r})d_2^r\right] \wedge q_{r,2}, \quad q'_{u,2} = q_{r,2} - q'_{r,2};$$
$$q'_{r,3} = \left[(1 - \frac{q'_{r,1}}{d_1^r} - \frac{q'_{r,2}}{d_2^r})d_3^r\right], \quad q'_{u,3} = q_{r,3} - q'_{r,3};$$
$$q'_{u,j} = q_{u,j}, \forall j \geq 4.$$

Step 1: To show that the above new allocation plan corresponds to a primary policy.

First of all, at price p_1, if $q'_{r,1} = d_1^r$, it indicates that $q_{u,1} \geq d_1^r$. This implies that

- $q'_{u,1} \geq 0$, and $q'_{r,2} = q'_{r,3} = 0$,

which leads to a primary 1-policy. Secondly, if $q'_{r,1} < d_1^r$ and $q'_{r,2} = (1 - \frac{q'_{r,1}}{d_1^r})d_2^r$, then it implies that

- $q'_{r,1} = q_{u,1}$, $q'_{u,1} = 0$, $q'_{u,2} \geq 0$, and $q'_{r,3} = 0$,

which leads to a primary 2-policy. Finally, if $q'_{r,1} < d_1^r$, $q'_{r,2} < (1 - \frac{q'_{r,1}}{d_1^r})d_2^r$ and $q'_{r,3} = (1 - \frac{q'_{r,1}}{d_1^r} - \frac{q'_{r,2}}{d_2^r})d_3^r$, then it implies that

- $q'_{u,1} = q'_{u,2} = 0$, and $q'_{u,3} \geq 0$,

which leads to a primary 3-policy. In summary, the new plan leads to a primary policy.

Step 2: To show that the new derived primary policy is *feasible*.

Let the derived primary policy be in the form of

$$\{(p_1, q'_{r,1}), \cdots, (p_t, q'_{r,t}); (p_{t+1}, q'_{u,t+1}), \cdots, (p_n, q'_{u,n})\},$$

where $1 \leq t \leq 3$. I will finish this step by studying three cases.

Case 1: $t = 1$.

Under this case, I have that

$$q'_{r,1} = d_1^r \leq q_{u,1}, q'_{u,1} = q_{u,1} - d_1^r$$
$$q'_{u.2} = q_{r,2}, q'_{u,3} = q_{r,3}, q'_{u,j} = q_{u,j}, \forall j \geq 4.$$

Some further observations can be made here:

- the new plan is feasible for the restricted market;
- there is no residual demand for restricted product at any price p_j $(j \geq 1)$ after the sales of $q'_{r,1}$ units at the restricted price p_1.

Let $d'_{3,j}$ be the total residual demand after the sales according to $(p_1, q'_{r,1}), (p_1, q'_{u,1}), (p_2, q'_{u,2})$ and $(p_3, q'_{u,3})$. Then I get

$$d'_{3,j} = (1 - \frac{q'_{u,1}}{d^u_1} - \frac{q'_{u,2}}{d^u_2} - \frac{q'_{u,3}}{d^u_3})d^u_j = (1 - \frac{q_{u,1} - d^r_1}{d^u_1} - \frac{q_{r,2}}{d^u_2} - \frac{q_{r,3}}{d^u_3})d^u_j, \forall j \geq 4.$$

Therefore to prove the feasibility of the new policy, it suffices to show that

$$d'_{3,j} \geq d_{3,j},$$

which, by (3.9), is equivalent to requiring that

$$\frac{q_{u,1} - d^r_1}{d^u_1} + \frac{q_{r,2}}{d^u_2} + \frac{q_{r,3}}{d^u_3} \leq \frac{q_{u,1}}{d_1} \iff q_{u,1}(\frac{1}{d^u_1} - \frac{1}{d_1}) + \frac{q_{r,2}}{d^u_2} + \frac{q_{r,3}}{d^u_3} \leq \frac{d^r_1}{d^u_1};$$

$$\iff \frac{q_{u,1}}{d_1}\frac{d^r_1}{d^u_1} + \frac{q_{r,2}}{d^u_2} + \frac{q_{r,3}}{d^u_3} \leq \frac{d^r_1}{d^u_1} \iff \frac{q_{u,1}}{d_1} + \frac{q_{r,2}}{d^u_2\frac{d^r_1}{d^u_1}} + \frac{q_{r,3}}{d^u_3\frac{d^r_1}{d^u_1}} \leq 1. \tag{3.10}$$

On the other hand, by (3.8), I know that to show (3.10), it suffices to prove that

$$d^u_2\frac{d^r_1}{d^u_1} \geq d^r_2 \text{ and } d^u_3\frac{d^r_1}{d^u_1} \geq d^r_3 \iff \frac{d^r_1}{d^u_1} \geq \frac{d^r_2}{d^u_2} \text{ and } \frac{d^r_1}{d^u_1} \geq \frac{d^r_3}{d^u_3};$$

$$\iff \frac{d_1}{d^u_1} \geq \frac{d_2}{d^u_2} \text{ and } \frac{d_1}{d^u_1} \geq \frac{d_3}{d^u_3} \iff \frac{d^u_1}{d_1} \leq \frac{d^u_2}{d_2} \text{ and } \frac{d^u_1}{d_1} \leq \frac{d^u_3}{d_3} \iff \gamma_1 \leq \gamma_2 \text{ and } \gamma_1 \leq \gamma_3;$$

which follows from our monotonicity assumption on γ's. Therefore (3.10) is true. Hence the result is proved for Case 1.

Case 2: $t = 2$.

Under this case, I will have the following:

$$q'_{r,1} = q_{u,1} < d^r_1, q'_{r,2} = (1 - \tfrac{q_{u,1}}{d^r_1})d^r_2;$$

$$q'_{u,2} = q_{r,2} - q'_{r,2}, q'_{u,3} = q_{r,3}, q'_{u,j} = q_{u,j}, \forall j \geq 4.$$

I also know that

- p_2 is the last active restricted price;
- there is no residual demand for restricted product after the sales at restricted price p_2.

Again, let $d'_{3,j}$ be the total residual demand at price p_j ($j \geq 4$) after the sales according to $(p_1, q'_{r,1}), (p_2, q'_{r,2}), (p_2, q'_{u,2})$ and $(p_3, q'_{u,3})$. Then

$$d'_{3,j} = (1 - \frac{q'_{u,2}}{d^u_2} - \frac{q'_{u,3}}{d^u_3})d^u_j = (1 - \frac{q_{r,2} - q'_{r,2}}{d^u_2} - \frac{q_{r,3}}{d^r_3})d^u_j.$$

I now want to show that

$$d'_{3,j} \geq d_{3,j}, \forall j \geq 4,$$

which, by (3.9), is equivalent to

$$\begin{aligned} \frac{q_{r,2} - q'_{r,2}}{d^u_2} + \frac{q_{r,3}}{d^u_3} = \frac{q_{r,2}}{d^u_2} - (1 - \frac{q_{u,1}}{d^r_1})\frac{d^r_2}{d^u_2} + \frac{q_{r,3}}{d^u_3} \leq \frac{q_{u,1}}{d_1} \\ \iff (-\frac{1}{d_1}\frac{d^u_2}{d^r_2} + \frac{1}{d^r_1})q_{u,1} + \frac{q_{r,2}}{d^r_2} + \frac{q_{r,3}}{d^u_3 \frac{d^r_2}{d^u_2}} \leq 1. \end{aligned} \tag{3.11}$$

Again, using (3.8), we know that to show (3.11), it suffices to show that

$$\begin{aligned} -\frac{1}{d_1}\frac{d^u_2}{d^r_2} + \frac{1}{d^r_1} \leq \frac{1}{d_1} \text{ and } d^u_3\frac{d^r_2}{d^u_2} \geq d^r_3 &\iff \frac{1}{d^r_1} \leq (1 + \frac{d^u_2}{d^r_2})\frac{1}{d_1} \text{ and } \frac{d^r_2}{d^u_2} \geq \frac{d^r_3}{d^u_3}; \\ &\iff \frac{d_1}{d^r_1} \leq \frac{d_2}{d^r_2} \text{ and } \frac{d_2}{d^u_2} \geq \frac{d_3}{d^u_3} \\ &\iff \frac{d^r_1}{d_1} \geq \frac{d^r_2}{d_2} \text{ and } \frac{d^u_2}{d_2} \leq \frac{d^u_3}{d_3} \\ &\iff \gamma_1 \leq \gamma_2 \text{ and } \gamma_2 \leq \gamma_3, \end{aligned}$$

which again follows from the monotonicity assumption on γ's. Therefore (3.11) holds. So this proves result for Case 2.

Case 3: $t = 3$.

Similarly, I know that under this case,

$$q'_{r,1} = q_{u,1} < d_1^r, q'_{r,2} = q_{r,2} < (1 - \frac{q_{u,1}}{d_1^r})d_2^r, q'_{r,3} = (1 - \frac{q_{u,1}}{d_1^r} - \frac{q_{r,2}}{d_2^r})d_3^r;$$
$$q'_{u,3} = q_{r,3} - q'_{r,3}, q'_{u,j} = q_{u,j}, \forall j \geq 4.$$

Let $d'_{3,j}$ be the total residual demand at price p_j ($j \geq 4$) after the sales according to $(p_1, q'_{r,1}), (p_2, q'_{r,2}), (p_3, q'_{r,3})$ and $(p_3, q'_{u,3})$. Then

$$d'_{3,j} = (1 - \frac{q'_{u,3}}{d_3^u})d_j^u = (1 - \frac{q_{r,3}}{d_3^u} + \frac{d_3^r}{d_3^u}(1 - \frac{q_{u,1}}{d_1^r} - \frac{q_{r,2}}{d_2^r}))d_j^u.$$

I need to show that

$$d'_{3,j} \geq d_{3,j},$$

which, by (3.9), is equivalent to

$$\frac{q_{r,3}}{d_3^u} - \frac{d_3^r}{d_3^u}(1 - \frac{q_{u,1}}{d_1^r} - \frac{q_{r,2}}{d_2^r}) \leq \frac{q_{u,1}}{d_1} \Leftrightarrow (\frac{1}{d_1^r} - \frac{d_3^u}{d_1 d_3^r})q_{u,1} + \frac{q_{r,2}}{d_2^r} + \frac{q_{r,3}}{d_3^r} \leq 1. \tag{3.12}$$

Again using the fact (3.8), to show (3.12), it suffices to prove

$$\frac{1}{d_1^r} - \frac{d_3^u}{d_1 d_3^r} \leq \frac{1}{d_1} \Longleftrightarrow \frac{d_3}{d_3^r} \leq \frac{d_1}{d_1^r} \Longleftrightarrow \gamma_3 \geq \gamma_1,$$

which is again assumed. This proves (3.12). Then the result is proved for this case too.

Summarizing the above three cases, I have shown that the new derived primary policy is indeed feasible. Finally, it is easy to see that

$$\sum_{i=1}^{t} p_i q'_{r,i} + \sum_{j=t}^{n} p_j q'_{u,j} = p_1 q_{u,1} + \sum_{i=2}^{3} p_i q_{r,i} + \sum_{j=3}^{n} p_j q_{u,j},$$

which implies that the new primary policy $\{q'_{r,1}, \cdots, q'_{r,t}; q'_{u,t}, \cdots, q'_{u,n}\}$ is in fact equivalent to the original policy.

Therefore, we have shown that there exists a feasible primary policy that is equivalent to the original policy. So the lemma is proved. □

In summary, Lemma 3.3.1 resolves the issue on the *order of sales* of the unrestricted units and the restricted units at the last active restricted price level by showing that the firm can limit its attention on policies such that the restricted units are sold first at the level of the last active restricted price. Lemmas 3.3.2 and 3.3.3 demonstrate that any optimal policy with the following two properties:

- the first active unrestricted price is followed by one restricted price or two consecutive restricted prices; and
- there is no residual demand for the restricted product at the end of sales of the allocated restricted units at the last active restricted price,

will be equivalent to a primary policy.

3.3.2 The General Optimality Theorems

We now use Lemmas 3.3.1, 3.3.2 and 3.3.3 to prove the following theorem, which deals with the situation where following the first active unrestricted price, there are many restricted prices.

Theorem 3.3.4: *For $m : 2 \leq m \leq n$, any G-policy of the following form:*

$$\{(p_1, q_{u,1}, u); (p_2, q_{r,2}, r), \cdots, (p_m, q_{r,m}, r); (p_{m+1}, q_{u,m+1}, u), \cdots, (p_n, q_{u,n}, u)\}$$

is weakly dominated by a primary policy, where

$$p_1 \leq p_2 < \cdots < p_m \leq p_{m+1} < \cdots p_n.$$

Proof: Clearly, I only need to consider the case that $q_{u,1} > 0$. Note that, given the value of $q_{u,1}$, the total residual demand at price p_j is given by:

$$d_{1,j} = (1 - \frac{q_{u,1}}{d_1})d_j, j \geq 2;$$

and the residual demand for restricted product is given by:

$$d^r_{1,j} = (1 - \frac{q_{u,1}}{d_1})d^r_j, j \geq 2.$$

Given that after the sales of unrestricted product at price p_1 the firm plans to sell restricted products at prices $p_2, \cdots, p_m$ and unrestricted products at prices $p_{m+1}, \cdots, p_n$. We also know that the sales of $q_{u,1}$ units at the unrestricted price p_1 does not change γ_j's, so $\gamma_{1,j} = \gamma_j$ for all $j \geq 2$. By Corollary 3.2.2, if $p_m < p_{m+1}$, then $\gamma_{1,j}$ is strictly increasing in j. If $p_m = p_{m+1}$, then $\gamma_{1,m} = \gamma_{1,m+1}$ and

$$\{\gamma_{1,2}, \cdots, \gamma_{1,m}, \gamma_{1,m+2}, \cdots, \gamma_n\}$$

constitutes a strictly increasing sequence. Therefore, we can use the results in Chapter 2 to the residual market specified by $\{(d^r_{1,j}, d_{1,j}) : j = 2, \cdots, n\}$ together with the price set $\{p_2, \cdots, p_n\}$ By Theorem 2.4.2, we know that there will be a primary policy

$$\{(p'_2, q'_{r,2}), \cdots, (p'_m, q'_{r,m'}); (p'_m, q'_{u,m'}), \cdots, (p'_{n'}, q'_{u,n'})\},$$

which is defined on the residual market such that

1. $n' = n$ if $p_m < p_{m+1}$ and $n' = n - 1$ if $p_m = p_{m+1}$;

2. the set of prices $\{p'_2, \cdots, p'_{n'}\}$ is the same as the set of prices $\{p_2, \cdots, p_n\}$;

3. $d'^r_{m',m'+1} = 0$, that is, there is no residual demand for restricted product at price $p \geq p'_{m'}$ after the sales according to $\{(p_1, q_{u,1}), (p'_2, q'_{r,2}), \cdots, (p'_{m'}, q'_{r,m'})\}$;

4. *at most two* of $q'_{r,2}, \cdots, q'_{r,m'}$ are strictly positive;

5. it *weakly dominates* the partial policy:

$$\{(p_2, q_{r,2}), \cdots, (p_m, q_{r,m}); (p_{m+1}, q_{u,m+1}), \cdots, (p_n, q_{un})\}.$$

This consequently leads to the new G-policy:

$$\{(p_1, q_{u,1}, u); (p'_2, q'_{r,2}, r), \cdots, (p'_{m'}, q'_{r,m'}, r); (p'_m, q_{u,m'}, u), \cdots, (p'_n, q'_{u,m}, u)\},$$

which in turn is weakly dominating the original policy. Now if all of $q'_{r,2}, \cdots, q'_{r,m'}$ are zeros, then the theorem follows immediately. If there is exactly one of them is positive, then the theorem follows from Lemma 3.3.2. And finally, if there are exactly two positive allocations at restricted prices, then the theorem follows from Lemma 3.3.3. This proves the theorem. □

I now present the key result of this section.

Theorem 3.3.5: *Any G-policy is weakly dominated by a primary policy.*

Proof: Consider any G-policy $\{n; (p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, n\}$. Let p_m be the last active restricted price and let $p_{u_1} < \cdots < p_{u_s}$ be the unrestricted prices in the price set $\{p_1, \cdots, p_m\}$. By Lemma 3.3.1, without loss of generality, we may assume that $p_{u_s} < p_m$.

Now for the given partial G-policy

$$\{(p_i, q_{\delta_i,i}, \delta_i), i = 1, \cdots, u_s - 1\},$$

we can use Theorem 3.3.4 on the remaining policy:

$$\{(p_{u_s}, q_{u,u_s}); (p_i, q_{r,i}), i = u_s + 1, \cdots, m; (p_j, q_{u,j}), j = m + 1, \cdots, n\},$$

which will lead to a policy in the form of

$$\{(p_i, q'_{r,i}), i = u_s, \cdots, m_s; (p_j, q_{u,j}), j = m_s + 1, \cdots, n\}.$$

Therefore, after this application of Theorem 3.3.4, we know the followings:

- p_{m_s} becomes the last active restricted price and $p_{u_{s-1}}$ is the largest unrestricted price that is below p_{m_s}; and

- there are only $s-1$ active unrestricted prices among in the price set $\{p_1, \cdots, p_{m_s}\}$, which are $p_{u_1}, \cdots, p_{u_{s-1}}$.

As we can see here, every time we use Theorem 3.3.4 on the modified policy, we will *eliminate one active unrestricted price that is strictly less than the last active restricted price.* Therefore, after s times, we will end up with a policy that has no active unrestricted price that is less than the last active restricted price, or equivalently, we will get a primary policy, as required. Hence the theorem is proved. □

To conclude this section, I here present an example that demonstrates that there are situations which have optimal policies that are not primary.

Example 3.3.1 *Existence of Non-Primary Optimal Policy*

Consider the following simple demand structure:

$$P_1 = \$100, P_2 = \$140; D_{u,1} = 200, D_{u,2} = 100; D_{r,1} = 100, D_{r,2} = 0; k = 250.$$

It is very easy to check that there exists a unique primary optimal policy given by:

$$q^*_{r,1} = 100; q^*_{u,1} = 100, q^*_{u,2} = 50,$$

which generates total revenue of $\$27,000$. Let's consider the following policy in a form as given by Definition 3.2.1:

$$\{(p_1 = P_1, q_{u,1} = 150, u), (p_2 = P_1, q_{r,2} = 50, r), (p_3 = P_2, q_{u,3} = 50, u)\},$$

which says that the firm first sells 150 unrestricted units at $P_1 = \$100$, then sells 50 restricted units also at $P_1 = \$100$, and finally 50 unrestricted units at $P_2 = \$140$. It is clear that this policy also generates total revenue of $\$27,000$. Let's check that this policy is also *feasible.* First, note that $D_1 = 300$ and $D_2 = 100$. If the firm sells 150 unrestricted units at P_1, then we know the following:

- among these 150 units sold, the portion that is originally from the restricted market is given by:

$$\frac{D_{r,1}}{D_1} q_{u,1} = \frac{100}{300} \times 150 = 50,$$

  which implies that the residual demand for the restricted product at the price level P_1 is 50;

- there is residual demand for the restricted product at the price level P_2 since $D_{r,2} = 0$; and

- the residual demand for the unrestricted product at P_2 is

$$(1 - \frac{q_{u,1}}{D_1}) D_{u,2} = (1 - \frac{150}{300}) \times 100 = 50.$$

Therefore, the allocated 50 restricted units at the price level P_1 and the allocated 50 unrestricted units at the price level P_2 are indeed feasible.

Remark: The above example has some very interesting implications:

- it is possible to have optimal policies that are not primary;

- the use of the non-primary optimal policies will resolve the implementation difficulty of some primary optimal policies where the firm has to sell some restricted units *first* and some unrestricted units *later* at the same price level.[4]

These observations prove to be quite useful in the next section too.

[4]In the above example, it is not easy to sell 100 restricted units after knowing that there are still another 100 unrestricted units available later. This is what I mean *implementation difficulty*. The new non-primary policy does not have this difficulty.

3.4 Further Properties of Optimal Policies

In the above section, I have demonstrated that there exists a primary optimal pricing policy in the G-class. As pointed out at the beginning of this chapter, there is another implicit assumption in the BL-model, which assumes that the firm offers one type of the product at a time. In other words, the allocated unrestricted units will not be put on sale until the restricted units are sold out according to the plan. On the other hand, such an assumption is not consistent with the existing pricing practice by airlines, which use multiple prices and make all offered prices available at the same time. The purpose of this section is to resolve this issue by *arguing* that there always exists an optimal policy with the property that all offered quantities — both restricted and unrestricted — at different price levels can actually be made available at the same time. To finish the argument, I need the following *behavioral assumption*:[5]

- **Behavioral Assumption**: *If there are restricted units and unrestricted units allocated at the same price level and all these units are made available at the same time, then the unrestricted units will be sold first, or equivalently, when restricted units and unrestricted units are offered at the same price time at the same time, consumers from the restricted market will try to buy the unrestricted units first.*

Let us now start with the characterization results on primary optimal policies from Theorem 2.4.2. Clearly, we only need to focus on the *basic primary optimal policies* that are derived from the Tight Problem in Chapter 2. Recall that the Tight Problem is a linear programming problem with three structural constraints, which implies that any basic primary optimal policy consists of at most three positive quantities, of which at least one, but at most two, of them is at an unrestricted price level. Because of this, I need to discuss two scenarios.

[5]Even though I do make this assumption, but it is only used *once*.

Scenario I: *There is exactly one unrestricted price.*

Under this scenario, we can write any basic primary optimal policy as follows:

$$\{q_{r,i}, q_{r,j}; q_{u,k}\},$$

where $i \leq j \leq k$ with active prices P_i, P_j and P_k. We also know that

- $q_{u,k} = D_{u,k}$, which implies that there will be no residual demand for the unrestricted product at $p \geq P_k$ after the sales of the unrestricted units at P_k;
- $q_{r,j} = (1 - \frac{q_{r,i}}{D_{r,i}})D_{r,j}$, which implies that there will be no residual demand for the restricted product at price level $p \geq P_j$ after the sales of the restricted units according to $\{(P_i, q_{r,i}), (P_j, q_{r,j})\}$.

I now need to consider three cases. First, for the case that $P_j < P_k$, it is easy to see that if the two restricted prices P_i, P_j and the one unrestricted price P_k are all made available at the same time, then the firm still can sell $q_{r,i}$ units restricted product at P_i, $q_{r,j}$ units of restricted product at P_j, and $q_{u,k}$ units of unrestricted units at P_k due to the following fact:

- Consumers from two market segments will only buy the product targeted to their respective segment because a consumer from the restricted market will not only buy the unrestricted product because the restricted units are cheaper and a consumer from the unrestricted market will not buy a restricted unit because he cannot accommodate the restriction.

For the case that $i < j = k$, consider the following new policy:

$$q'_{r,t} = \begin{cases} q_{r,i} & \text{if } t = i \\ 0 & \text{otherwise;} \end{cases}$$

and

$$q'_{u,t} = \begin{cases} D_{u,k} + (1 - \frac{q_{r,i}}{D_{r,i}})D_{r,k} & \text{if } t = k \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this new policy is also a primary optimal policy with only one active restricted price P_i and one active unrestricted price P_k. We now argue that for this new optimal policy, if both the restricted units at P_i and the unrestricted units at P_k are available at the same, the firm still can *reach the projected sales* of $q'_{r,i}$ restricted units at P_i and $q'_{u,k}$ unrestricted units. Note that:

- at the beginning of the sales in the both markets, $D_{r,i}$ and $D_{u,k}$ are *exposed*;
- since consumers from the unrestricted market will buy the restricted product, the firm will capture all the demand for the unrestricted product at price P_k, which is $D_{u,k}$;
- for consumers from the restricted market, they will not buy the restricted product at price P_k until the allocated $q'_{r,i}$ restricted units are sold out;
- after the allocated $q'_{r,i}$ restricted units are sold out, according to the proportional-rationing rule, the residual demand from the restricted market at price P_k is $(1 - \frac{q_{r,i}}{D_{r,i}})D_{r,k}$, which implies that the firm can sell up to $(1 - \frac{q_{r,i}}{D_{r,i}})D_{r,k}$ units of the unrestricted product at price P_k to the restricted market.

Therefore, the firm will reach its projected sales in both markets.

At last, for the case that $i = j = k$, it must be true that $q_{r,i} = 0$, $q_{r,j} = D_{r,j}$. Then it is obvious that the firm just sells $D_{r,k} + D_{u,k} = D_k$ unrestricted units at the price level P_k without selling any restricted unit.

Scenario II: *There are exactly two unrestricted prices.*

Under this scenario, we can write an primary optimal policy derived from the Tight Problem as follows:

$$\{q_{r,i}; q_{u,k_1}, q_{u,k_2}\},$$

with the following properties:

- $i \leq k_1 < k_2$;
- $q_{r,i} = D_{r,i}$;
- $q_{u,k_2} = (1 - \frac{q_{u,k_1}}{D_{u,k_1}})D_{u,k_2}$.

We now need to discuss two cases. First, consider the case that $i < k_1$, that is, $P_i < P_{k_1}$. Using the same the argument as for the first case under Scenario I, we know that even making the restricted price P_i and the two unrestricted prices P_{k_1} and P_{k_2} available at the same time, the firm still can reach the projected sales of $q_{r,i}$ restricted units at P_i, q_{r,k_1} unrestricted units at P_{k_1} and q_{u,k_2} unrestricted units at P_{k_2}.

For the case that $i = k_1$, additional care is needed if the firm makes all offered prices available at the same time because of the situation that the firm now offers restricted units *and* unrestricted units at the same price level P_{k_1}. The dilemma is that if *only one* of the potential D_{r,k_1} consumers from the restricted market actually purchases an unrestricted unit at P_{k_1}, the firm will not be able to sell D_{r,k_1} restricted units. In fact, under the Behavioral Assumption, if q_{u,k_1} unrestricted units and q_{r,k_1} restricted units are available at the same price level P_{k_1}, then the allocated q_{u,k_1} unrestricted units will be sold *first*. Among these who have purchased the unrestricted product at price level P_{k_1}, there is a positive fraction of them who are from the restricted market. This implies that the firm definitely cannot sell *additional* D_{r,k_1} restricted units at the price level P_{k_1}. Consequently the firm cannot directly use the given primary optimal policy if it decides

to make all allocated units available at the same time. Therefore, we have to construct a new optimal policy. For this, consider the following policy:[6]

- the firm *first* sells $q'_{u,k} = (1 + \frac{D_{r,k_1}}{D_{u,k_1}})q_{u,k}$ *unrestricted units* at price level P_{k_1};
- the firm *then* offers $q'_{r,k_1} = (1 - \frac{q_{u,k_1}}{D_{u,k_1}})D_{r,k_1}$ *restricted units* at price level P_{k_1};
- the firm *finally* allocates $q'_{u,k_2} = q_{u,k_2} = (1 - \frac{q_{q,k_1}}{D_{u,k_1}})D_{u,k_2}$ *unrestricted units* at price level P_{k_2}.

To conclude our argument, we have to finish three steps: (1) this new policy is in fact feasible, (2) it is equivalent to the original primary optimal policy, and (3) if all offered prices in this new optimal policy are available at the same time, the firm still can reach its *projected total revenue.* Regarding the feasibility of this new policy, we observe that:

- the allocation q'_{u,k_1} is feasible at P_{k_1} since

$$q'_{u,k_1} = \frac{D_{u,k_1} + D_{r,k_1}}{D_{u,k_1}} q_{u,k_1} = \frac{q_{u,k_1}}{D_{u,k_1}}(D_{u,k_1} + D_{r,k_1}) < D_{u,k_1} + D_{r,k_1} = D_{k_1},$$

  which further implies that

$$\frac{q'_{u,k_1}}{D_{k_1}} = \frac{q_{u,k_1}}{D_{u,k_1}};$$

- after the sales of the q'_{u,k_1} unrestricted units at the price P_{k_1}, the residual demand for the restricted product is given by:

$$d'_{r,j} = (1 - \frac{q'_{u,k_1}}{D_{k_1}})D_{r,j} = (1 - \frac{q_{u,k_1}}{D_{u,k_1}})D_{r,j} \text{ for } j \geq k_1,$$

  which implies that $q'_{r,k_1} = (1 - \frac{q_{u,k_1}}{D_{u,k_1}})D_{r,k_1}$ restricted units at P_1 is indeed feasible;

[6]This construction of a non-primary optimal policy is motivated from the discussions in Example 3.3.1.

- after the sales of the q'_{u,k_1} unrestricted units at the price level P_{k_1}, the residual demand for the unrestricted product from the unrestricted market is specified by:

$$d'_{u,j} = (1 - \frac{q'_{u,k_1}}{D_{k_1}})D_{u,j} = (1 - \frac{q_{u,k_1}}{D_{u,k_1}})D_{u,j} \text{ for } j \geq k_1,$$

  which indicates that $q'_{u,k_2} = (1 - \frac{q_{u,k_1}}{D_{u,k_1}})D_{u,k_2}$ is also feasible.

So I have shown that the new policy is feasible. To see that it is also equivalent to the original primary policy optimal, note that

$$\begin{aligned} P_{k_1}(q'_{u,k_1} + q'_{r,k_1}) + P_{k_2}q'_{u,k_2} &= P_{k_1}(q_{u,k_1} + D_{r,k_1}) + P_{k_2}q_{u,k_2} \\ &= P_{k_1}(q_{u,k_1} + q_{r,k_1}) + P_{k_2}q_{u,k_2}. \end{aligned}$$

Now consider that for this new optimal policy, the firm makes q'_{u,k_1} unrestricted units at P_{k_1}, q'_{r,k_1} restricted units at P_{k_1}, and q'_{u,k_2} unrestricted units at P_{k_2} all available at the same time. Then we have the following observations:

- under the Behavioral Assumption, q'_{u,k_1} unrestricted units at the price level P_{k_1} will be sold *before* the sales of q'_{r,k_1} restricted units, which implies that at the beginning of the sales, only the unrestricted P_{k_1} is actually active;

- after the sales of the unrestricted units at P_{k_1}, the restricted P_{k_1} and the unrestricted P_{k_2} become active;

- since $P_{k_1} < P_{k_2}$, then by the same argument as used in the first case under Scenario I, we know that the firm can reach the projected sales levels in both residual markets, that is, selling q'_{r,k_1} restricted units and q'_{u,k_2} unrestricted units.

In conclusion, we claim that all allocated quantities in the new optimal policy can be made available without any adverse impact to total projected revenue.

As a last comment of this section, it is interesting to note that in order to be able to maintain the projected value of the optimal revenue and to be able to make all allocated quantities available at the same time, the firm might have to implement a non-primary optimal policy.

3.5 Summary – General Optimality Results

This chapter addresses two unsettled issues in Chapter 2. The first issue is the limitation of only using the type of pricing policy that sell restricted product at lower first at some lower prices and later unrestricted product at some higher prices, namely, the primary policies. The other issue is whether all offered prices, restricted and unrestricted, can be made available simultaneously and still generate the same amount of revenue realized by a primary optimal policy. First of all, this chapter has completely resolved the first issue by showing that even allowing the firm to attach restrictions at any possible price and at any order, an optimal primary policy remains to be optimal in this general class of pricing policies. The main implication of this generality result is that if a firm decides to use a marketing restriction in its pricing decision, then the firm can limit its pricing practice to primary policies, that is, selling restricted units *first* at a set of prices that are lower than these prices for unrestricted units.

This chapter also gives a satisfactory answer to the second issue by demonstrating that there always exists an optimal policy, which may be non-primary, with the property that making all allocated units — both restricted and unrestricted — available at the same time will not have any negative impact on the optimal revenue value derived from a primary optimal policy. This is important because it shows that some optimal policies are consistent with the existing pricing practices by firms like airlines, which (1) offer multiple prices — restricted and unrestricted, and (2) make all allocated quantities at

different price levels available the same time. Such a consistency makes the pricing model developed in the last chapter and this chapter much more useful to firms which use artificial restrictions in the process of pricing perishable inventories such as airline seats and hotel rooms.

Chapter 4

Pricing Models with Two Types of Restrictions

4.1 Introduction and Model Setting

Chapter 2 developed a simple pricing model for a monopolist who uses one restriction as a mean of segmenting the market demand. It showed that by properly setting the level of the highest restricted price and rationing the sales at lower prices, the monopolist needs to charge no more than three prices to maximize its revenue. Then Chapter 3 further analyzed the problem by considering a general class of pricing policies which allow firms to offer restricted units at any price level. It showed that the optimal policies characterized in Chapter 2 remain to be optimal in this general class of policies. Also in Chapter 3, I have addressed the issue of simultaneous availability problem for all prices that are planned to be offered. These two chapters together prove a very powerful result: as long as the restriction is effective, that is, as price increases, the percentage of these consumers who cannot accommodate the restriction is strictly increasing, *the optimal pricing practice is to sell restricted units at some lower prices and unrestricted units at some subsequent higher prices*, and *the firm only needs at most three prices to maximize its revenue.*

On the other hand, the discussions in the previous chapters are limited to the use of *one type of restrictions.* As a consequence of this, the firm can only use two types of product – restricted product and unrestricted product. But any realistic situation involves multiple restrictions and multiple types of product. Firms, such as airlines and

hotels, must deal with several product restrictions together with several other corporate restrictions.[1] It is important to note that each restriction is targeted to a unique market segment; and firms should be able to explore these market segments by dealing with all these restrictions at the same time. In this chapter, I will develop pricing models for perishable inventories when a firm uses two types of restrictions. As we will see later, the use of two types of restrictions will allow the firm to introduce three or four different types of products. This will substantially increase the flexibility of pricing practice for perishable inventories, such as airline seats and hotel rooms.

Consider that a monopoly firm has a limited quantity (k) of a certain perishable product, such as airline seats or hotel rooms. Let $D(p)$ be the market demand function for the product at price p. We can interpret $D(p)$ as the number of consumers in the market who are willing to buy *one unit* of the product at price p. The firm also considers to use two types of restrictions in its pricing decisions. Let $\gamma_t(p)$ be the percentage of consumers who can not purchase the product if type t restriction is attached to the product at price p, where $t = 1, 2$. Therefore the demand for type t restricted product at price p is given by

$$D^{rt}(p) = (1 - \gamma_t(p))D(p), \ t = 1, 2.$$

As in Chapters 2 and 3, I will assume that the demand functions $D(p)$, $D^{r1}(p)$ and $D^{r2}(p)$ are all step functions defined on the price set $\{p_1, \cdots, p_n\}$, that is,

$$D(p) = \begin{cases} D_1 & \text{if } p \leq p_1, \\ D_i & \text{if } p_{i-1} < p \leq p_i \text{ for } i = 2, \cdots, n, \\ 0 & \text{if } p > p_n; \end{cases}$$

[1] For example, airlines offer special fares to major corporations and government, which are not available to the general public. The existence of these fares can be analyzed by treating them as corporate restrictions.

and for $t = 1, 2$,

$$D^{rt}(p) = \begin{cases} D_1^{rt} & \text{if } p \leq p_1, \\ D_i^{rt} & \text{if } p_{i-1} < p \leq p_i \text{ for } i = 2, \cdots, n, \\ 0 & \text{if } p > p_n, \end{cases}$$

where $\infty > D_1 > D_2 > \cdots > D_n > 0$ and $\infty > D_1^{rt} > D_2^{rt} > \cdots > D_n^{rt} > 0$. For $i = 1, \cdots, n$, and $t = 1, 2$, denote

$$\gamma_{t,i} = 1 - \frac{D_i^{rt}}{D_i}.$$

Then it is easy to see that $\gamma_t(p)$ is also a step function.

This chapter is organized as follows. Section 4.2 presents an extension of the BL-model to the case that there are two types of *nested restrictions*, which is a quite common practice for airlines.[2] I show in this section that there exists optimal policies that can be characterized by a linear programming problem, which indicates that the firm needs to offer at most four price levels to maximize its revenue by using three different types of product. Section 4.3 will extend the basic BL-model to the case of two type of restrictions that are *mutually exclusive*. I also show that in this case the firm needs to offer no more than four different price levels with three different types of product. And then in Section 4.4 I will use the results developed in Sections 4.2 and 4.3 to study the pricing problem when a firm uses two general types of restrictions. I demonstrate that under certain reasonable conditions, the firm can limit itself to the type of pricing policies that consist of at most five different price levels with four different types of product. Finally, the last section is a summary.

[2]For example, airlines offer discount fares with *advance booking* and Saturday night conditions. In the meantime, the discount fares just with Saturday night condition are also available.

4.2 Pricing Problem by Using Two Nested Restrictions

I first introduce the following definition:

Definition 4.2.1: *We call type-2 restriction is* **nested into** *type-1 restriction if there exist* $\alpha_t(p)$, $t = 1, 2$ *such that*

1. $1 - \gamma_1(p) = 1 - \alpha_1(p)$; and
2. $1 - \gamma_2(p) = (1 - \alpha_1(p))(1 - \alpha_2(p))$.

From the definition, it follows that if type-2 restriction is nested into type-1 restriction, then I will have

$$D^{r2}(p) = (1 - \alpha_2(p))D^{r1}(p), \tag{4.1}$$

which, in fact, is the main reason why I call them nested since it implies that the group of those who can accommodate type-2 restriction is a *subset* of of those who can accommodate type-1 restriction. Hence, it is clear that type-2 restriction is *more restrictive* than type-1 restriction. Denote $\alpha_{ti} = \alpha_t(p_i)$ for $t = 1, 2$ and $j = 1, \cdots, n$.

The firm needs to address the following fundamental questions:

- Is it necessary to use restrictions? Two or just one?
- What are the optimal pricing policies?

Therefore a pricing policy should tell the firm which prices are restricted and how many units of the product available at each of these price are offered. Throughout this section, I will make the following assumptions:

- both $\{\alpha_{1i}\}$ and $\{\alpha_{2i}\}$ are *strictly increasing*;

- type-2 restricted units are sold first, then type-1, and finally the unrestricted;
- if type-2 restricted units and type-1 restricted units are sold at the same price level, then type-2 restricted units are sold first;
- if type-1 restricted units and unrestricted units are sold at the same price level, then type-1 restricted units are sold first; and
- the product is sold at prices in the order of price levels $p_1, \cdots, p_n$.

With these in mind, the following definition of a pricing policy under two nested restrictions is quite natural:

Definition 4.2.2: *Suppose that restriction 2 is nested into the restriction 1. Then* **a pricing policy** *for the firm is in a form of*

$$\{(p_i, q_{r2,i}) : i = 1, \cdots, m_2; (p_j, q_{r1,j}) : j = m_2, \cdots, m_1; (p_k, q_k) : k = m_1, \cdots, n\},$$

where p_{m_t} is the highest type-t restricted price, $q_{rt,i}$ is the quantity available for sale at the type-t restricted price p_i, q_k is the quantity available for sale at the unrestricted price p_k, and $1 \leq m_2 \leq m_1 \leq n$.

Remarks: From the definition, I have the following observations:

- If $m_2 = 1$ and $q_{r2,1} = 0$, then the above policy indicates that only type-2 restrictions will be used;
- If $m_2 = m_1 = m$ and $q_{r1,m} = 0$, then it indicates that only type-1 restriction will be used;
- If $m_2 = m_1 = 1$ and $q_{r2,1} = q_{r1,1} = 0$, then it means that all units are unrestricted, that is, no restriction is used.

Also, since the values of m_1 and m_2 are completely controlled by the firm, it is of the firm's interest to find the best possible combination of these two values. But first, I need to specify the set of *all feasible pricing policies* for any given pair (m_2, m_1). For this, I will use the same techniques used in Chapter 2 to derive these feasibility conditions.

First, consider that type-2 restricted units are put on sale according to

$$\{(p_1, q_{21}), \cdots, (p_{m_2}, q_{2,m_2})\}$$

with demands given by D_i^{r2} for $i = 1, \cdots, m_2$. Therefore, the feasibility condition for the type-t restricted product is given by

$$\beta_2 \equiv \sum_{i=1}^{m_2} \frac{q_{r2,i}}{D_i^{r2}} \leq 1.$$

After the sales of the type-2 restricted product, the *residual demand for type-1 restricted product* is given by

$$d_{m_2,j}^{r1} = D_j^{r1} - \beta_2 D_j^{r2}, \text{ for } j = m_2, \cdots, n.$$

Consequently, the feasibility condition for the plan $\{(p_{m_2}, q_{1,m_2}), \cdots, (p_{m_1}, q_{1,m_1})\}$ for the type-1 restricted product is

$$\beta_1 \equiv \sum_{j=m_2}^{m_1} \frac{q_j^{r1}}{d_{m_2,j}^{r1}} = \sum_{j=m_2}^{m_1} \frac{q_j^{r1}}{D_j^{r1} - \beta_2 D_j^{r2}} \leq 1.$$

Finally, let us consider selling unrestricted product according to the plan $\{(p_k, q_k) : k = m_1, \cdots, n\}$. It is easy to see that the residual demand for the unrestricted product *after* the sales of the product with restrictions is given by

$$d_{m_1,k} = \alpha_{1k} D_k + (1 - \beta_1) d_{m_2,k}^{r1} = D_k - \beta_1 D_k^{r1} - (1 - \beta_1)\beta_2 D_k^{r2}, \text{ for } k \geq m_1.$$

Hence, the feasibility condition for the remaining unrestricted product is given by:

$$\sum_{k=m_1}^{n} \frac{q_k}{d_{m_1,k}} = \sum_{k=m_1}^{n} \frac{q_k}{D_k - \beta_1 D_k^{r1} - (1 - \beta_1)\beta_2 D_k^{r2}} \leq 1.$$

Summarizing the above discussions, I obtain the following formulation of the firm's pricing problem when using two nested restrictions:

$$\text{Max} \quad \sum_{i=1}^{m_2} p_i q_{r2,i} + \sum_{j=m_2}^{m_1} p_j q_{r1,j} + \sum_{k=m_1}^{n} p_k q_k$$

s.t.

$$\sum_{i=1}^{m_2} \frac{q_{r2,i}}{D_i^{r2}} \leq 1 \tag{4.2}$$

$$\sum_{j=m_2}^{m_1} \frac{q_{r1,j}}{D_j^{r1} - \beta_2 D_j^{r2}} \leq 1 \tag{4.3}$$

$$\sum_{k=m_1}^{n} \frac{q_k}{D_k - \beta_1 D_k^{r1} - (1-\beta_1)\beta_2 D_k^{r2}} \leq 1 \tag{4.4}$$

$$\sum_{i=1}^{m_2} q_{r2,i} + \sum_{j=m_2}^{m_1} q_{r1,j} + \sum_{k=m_1}^{n} q_k \leq q$$

$$q_{r2,i}, q_{r1,j}, q_k \geq 0, \forall i, \forall j, \text{ and } \forall k.$$

I will call the above pricing model *N-model*, where N represents for *nested*. Let $R(m_2, m_1)$ be the *derived maximum revenue* from the above N-model. The following theorem proves that there is a linear programming characterization for the optimal revenue:

$$\max_{1 \leq m_2 \leq m_1 \leq n} R(m_2, m_1).$$

Theorem 4.2.1: *Let $\tilde{R}(m_2, m_1)$ be the optimal objective value of the following linear programming problem (named as the* **Tight N-model***):*

$$\text{Max} \quad \sum_{i=1}^{m_2} p_i q_{r2,i} + \sum_{j=m_1}^{n} p_j q_{r1,j} + \sum_{k=m_1}^{n} p_k q_k$$

s.t.

$$\sum_{i=1}^{m_2} \frac{q_{r2,i}}{D_i^{r2}} = 1$$

$$\sum_{j=m_2}^{m_1} \frac{q_{r1,j}}{D_j^{r1} - D_j^{r2}} = 1$$

$$\sum_{k=m_1}^{n} \frac{q_k}{D_k - D_k^{r1}} = 1$$

$$\sum_{i=1}^{m_2} q_{r2,i} + \sum_{j=m_2}^{m_1} q_{r1,j} + \sum_{k=m_1}^{n} q_k \leq q$$

$$q_{r2,i}, q_{r1,j}, q_k \geq 0, \forall i, \forall j, \text{ and } \forall k.$$

Then

$$\max_{1\le m_2\le m_1\le n} R(m_2,m_1) = \max_{1\le m_2\le m_1\le n} \tilde{R}(m_2,m_1). \tag{4.5}$$

Proof: It is evident that

$$\max_{1\le m_2\le m_1\le n} R(m_2,m_1) \ge \max_{1\le m_2\le m_1\le n} \tilde{R}(m_2,m_1).$$

Then to prove (4.5), it suffices to show that

$$\max_{1\le m_2\le m_1\le n} R(m_2,m_1) \le \max_{1\le m_2\le m_1\le n} \tilde{R}(m_2,m_1). \tag{4.6}$$

To prove (4.6), I only need to show that for any pair: $1 \le m_2 \le m_1 \le n$, there exists another pair: $1 \le \tilde{m}_2 \le \tilde{m}_1 \le n$ such that

$$R(m_2,m_1) \le \tilde{R}(\tilde{m}_2,\tilde{m}_1). \tag{4.7}$$

Consider any pair $1 \le m_2 \le m_1 \le n$ and let

$$\{(p_i, q_{r2,i}), i = 1,\cdots,m_2; (p_j, q_{r1,j}), j = m_2,\cdots,m_1; (p_k, q_k), k = m_1,\cdots,n\}$$

be an optimal solution to the N-model. Denote

$$q_2 \equiv \sum_{i=1}^{m_2} q_{r2,i},$$

which is the total sales for type-2 restricted units. Note that the residual demand for the type-1 restricted product after the sales of the type-2 restricted product is given by:

$$d^{r1}_{m_2,j} = D^{r1}_j - \beta_2 D^{r2}_j, \text{ for } j = m_2,\cdots,n,$$

and the total residual demand for unrestricted product at price p_j after the sales of the type-2 restricted product is given by:

$$d_{m_2,j} = D_j - \beta_2 D^{r2}_j, \text{ for } j = m_2,\cdots,n.$$

Therefore the residual percentage of consumers who can not accommodate the type-1 restriction is given by:

$$\bar{\alpha}_{1j} = 1 - \frac{d^{r1}_{m_2,j}}{d_{m_2,j}} = 1 - \frac{D^{r1}_j - \beta_2 D^{r2}_j}{D_j - \beta_2 D^{r2}_j} = \frac{\alpha_{1j}}{1 - \beta_2(1-\alpha_{1j})(1-\alpha_{2j})},$$

which is strictly increasing since both α_{1j} and α_{2j} are strictly increasing by assumption. Therefore, I can use Theorem 2.4.2 to the residual market for the type-1 restricted product and the unrestricted product. This implies that considering $q - q_2$ as the capacity limit, I know that there exists an integer $m_2 \leq \tilde{m}_1 \leq n$ and a *new optimal allocation plan* in the form of

$$\{(p_j, \tilde{q}_{r1,j}), j = m_2, \cdots, \tilde{m}_1; (p_k, \tilde{q}_k), k = \tilde{m}_1, \cdots, n\},$$

such that

$$\sum_{j=m_2}^{\tilde{m}_1} \frac{\tilde{q}_{r1,j}}{D^{r1}_j - \beta_2 D^{r2}_j} = 1 \tag{4.8}$$

$$\sum_{k=\tilde{m}_1}^{n} \frac{\tilde{q}_k}{D_k - D^{r1}_k} = 1 \tag{4.9}$$

$$\sum_{j=m_2}^{\tilde{m}_1} \tilde{q}_{r1,j} + \sum_{k=\tilde{m}_1}^{n} \tilde{q}_k \leq q - q_2; \text{ and} \tag{4.10}$$

$$\sum_{j=m_2}^{\tilde{m}_1} p_j \tilde{q}_{r1,j} + \sum_{k=\tilde{m}_1}^{n} p_k \tilde{q}_k \geq \sum_{j=m_2}^{m_1} p_j q_{r1,j} + \sum_{k=m_1}^{n} p_k q_k. \tag{4.11}$$

We should notice that the relationship (4.9) is independent of the value of β_2. In fact, as long as (4.8) holds, that is, the feasibility constraint for the type-1 restricted product is *tight*, (4.9) is automatically true!

Now let $\tilde{q}_0 \equiv \sum_{k=\tilde{m}_1}^{n} \tilde{q}_k$. Therefore $\tilde{q}_0$ is the total number of units of the product allocated for unrestricted units. Then with $\tilde{q}_0$ units *protected*, the firm needs to solve the following subproblem:

$$\begin{array}{ll} \text{Max} & \sum_{i=1}^{m_2} p_i q_{r2,i} + \sum_{j=m_2}^{\tilde{m}_1} p_j q_{r1,j} \\ \text{s.t.} & \\ & \beta_2 = \sum_{i=1}^{m_2} \frac{q_{r2,i}}{D_i^{r2}} \leq 1 \\ & \sum_{j=m_2}^{\tilde{m}_1} \frac{q_{r1,j}}{D_j^{r1} - \beta_2 D_j^{r2}} \leq 1 \\ & \sum_{i=1}^{m_2} q_{r2,i} + \sum_{j=m_2}^{\tilde{m}_1} q_{r1,j} \leq q - \tilde{q}_0 \\ & q_{r2,i}, q_{r1,j} \geq 0, \forall i, \text{ and } \forall j. \end{array}$$

This again leads to the basic model in Chapter 2. By Theorem 2.3.5, I know that (4.8) holds at any optimal solution. Furthermore, again using Theorem 2.4.2 , I conclude that there exists another integer $1 \leq \tilde{m}_2 \leq \tilde{m}_1$ and another allocation plan

$$\{(p_i, \tilde{q}_{r2,i}), i = 1, \cdots, \tilde{m}_2; (p_j, \bar{q}_{r1,j}), j = \tilde{m}_2, \cdots, \tilde{m}_1\},$$

such that

$$\sum_{i=1}^{\tilde{m}_2} \frac{\tilde{q}_{r2,i}}{D_i^{r2}} = 1, \tag{4.12}$$

$$\sum_{j=\tilde{m}_2}^{\tilde{m}_1} \frac{\bar{q}_{r1,j}}{D_j^{r1} - D_j^{r2}} = 1, \tag{4.13}$$

$$\sum_{i=1}^{\tilde{m}_2} \tilde{q}_{r2,i} + \sum_{j=\tilde{m}_2}^{\tilde{m}_1} \bar{q}_{r1,j} + \sum_{k=\tilde{m}_1}^{n} \tilde{q}_k \leq q; \text{ and} \tag{4.14}$$

$$\sum_{i=1}^{\tilde{m}_2} p_i \tilde{q}_{r2,i} + \sum_{j=\tilde{m}_2}^{\tilde{m}_1} p_j \bar{q}_{r1,j} \geq \sum_{i=1}^{m_2} p_i q_{r2,i} + \sum_{j=m_2}^{\tilde{m}_1} p_j \tilde{q}_{r1,j}, \tag{4.15}$$

since it is easy to check that $\{q_{r2,i}, i = 1, \cdots, m_2; \tilde{q}_{r1,j}, j = m_2, \cdots, \tilde{m}_1\}$ is also a feasible policy for the above subproblem. Finally, by (4.9), (4.12), (4.13) and (4.14), I know that the new allocation plan

$$\{(p_i, \tilde{q}_{r2,i}) : i = 1, \cdots, \tilde{m}_2; (p_j, \bar{q}_{r1,j}) : j = \tilde{m}_2, \cdots, \tilde{m}_1; (p_k, \tilde{q}_k) : k = \tilde{m}_1, \cdots, n\}$$

constitutes *a feasible policy for the Tight N-model* with respect to $\tilde{m}_2$ and $\tilde{m}_1$. Furthermore, by (4.11) and (4.15), it follows that

$$\begin{aligned}
\tilde{R}(\tilde{m}_2, \tilde{m}_1) &\geq \sum_{i=1}^{\tilde{m}_2} p_i \tilde{q}_{r2,i} + \sum_{j=\tilde{m}_2}^{\tilde{m}_1} p_j \bar{q}_{r1,j} + \sum_{k=\tilde{m}_1}^{n} p_k \tilde{q}_k \\
&\geq \sum_{i=1}^{m_2} p_i q_{r2,i} + \sum_{j=m_2}^{\tilde{m}_1} p_j \tilde{q}_{r1,j} + \sum_{k=\tilde{m}_1}^{n} p_k \tilde{q}_k \\
&\geq \sum_{i=1}^{m_2} p_i q_{r2,i} + \sum_{j=m_2}^{m_1} p_j q_{r1,j} + \sum_{k=m_1}^{n} p_k q_k = R(m_2, m_1).
\end{aligned}$$

So (4.7) is true; and therefore (4.6) is proved. □

As an immediate consequence of the above theorem, I have the following useful result on the optimal pricing policy by using two nested restrictions.

Corollary 4.2.2: *If two type of restrictions are nested, then there exists an optimal pricing policy that consists of at most* **four** *different prices, which is characterized by a linear programming.*

As a final remark of this section, the above discussion can be easily extended to the case of multiple nested restrictions.

4.3 Pricing Problem by Using Two Mutually Exclusive Restrictions

In this section, I will discuss another case of two types of restrictions. Before moving on, I need to introduce the following definition of *mutually exclusive restrictions*:

Definition 4.3.1: *Let $\gamma(p)$ be the percentage of consumers who cannot purchase the product if* **both** *restrictions are attached to the product. Then we say that these two*

restrictions are **mutually exclusive** *if* $\gamma(p) = 1$.

The mutual exclusiveness here means that those who can accommodate the type-1 restriction cannot accommodate the type-2 restriction, and vice versa. Or intuitively speaking, if two types of restrictions are mutually exclusive, then they are targeting two distinct consumer groups. In other words, the set of those consumers who do not mind the type-1 restriction is *disjoint* from the set of those consumers who are do not mind the type-2 restriction. This sounds very restrictive, but it is nevertheless an important case I need to discuss. In the next section, I will discuss the general case of two types of restrictions that are not mutually exclusive and not nested, by integrating the model developed in the last section and in this section.

If the two restrictions are mutually exclusive and the units with restrictions are sold first, then the following lemma shows that the firm can limit itself to policies that are such that both restrictions share the same highest price.

Lemma 4.3.1: *If two restrictions are mutually exclusive, then there exists an optimal policy of the following form:*

$$\{m; (q_{rt,1}, \cdots, q_{rt,m}), t = 1, 2; (q_m, \cdots, q_n)\},$$

where p_m *is the common highest restricted price for both restrictions,* q_{ti} *is the quantity available at price* p_i *with restriction* t *and* q_k *is the quantity available at the unrestricted price* p_k *after the sales of the product with restrictions.*

Proof: Since γ_{ti} is strictly increasing, then by Theorem 3.3.5 of Chapter 3, I know that for each restriction *alone*, the firm only needs to consider policies that sell restricted units first. Let $\{p_1, \cdots, p_{m_1}\}$ be the set of the type-1 restricted prices and $\{p_1, \cdots, p_{m_2}\}$ be the set of the type-2 restricted prices. Without loss of generality, I assume that $m_1 < m_2$.

So this leads to policies of the following form:

$$\{(p_i, q_{r1,i}, q_{r2,i}) : i = 1, \cdots, m_1; (p_j, q_{r2,j}, q_j) : j = m_1, \cdots, m_2; (p_k, q_k) : k = m_2, \cdots, n\},$$

where

- for $1 \leq i \leq m_1$, the firm allocates $q_{r1,i}$ and $q_{r2,i}$ units at price p_i with type-1 restriction and type-2 restriction respectively;
- for $m_1 \leq j \leq m_2$ and at the price level p_j, the firm sells $q_{r2,j}$ type-2 restricted units first and the q_j unrestricted units; and
- for $m_2 \leq k \leq n$, the firm sells unrestricted units only.

After the sales according to $\{(p_i, q_{r1,i}, q_{r2,i}) : i = 1, \cdots, m_1\}$ up to price level p_{m_1}, let

- $d^{rt}_{m_1,j}$ be the residual demand for type-t restricted units at price p_j for $j \geq m_1$ and $t = 1, 2$;
- $d_{m_1,j}$ be the total residual demand at the unrestricted price level p_j for $j \geq m_1$; and
- $\gamma_{m_1,tj}$ be the percentage of consumers in the residual market who can not accommodate type-t restriction at price p_j.

Then it is straightforward to check that for $t = 1, 2$,

$$d^{rt}_{m_1,j} = (1 - \eta_t) D^{rt}_j, \text{ with } \eta_t \equiv \sum_{i=1}^{m_1} \frac{q_{rt,i}}{D^{rt}_i}.$$

So

$$d_{m_1,j} = D_j - \eta_1 D^{r1}_j - \eta_2 D^{r2}_j.$$

Note that

$$1 - \gamma_{m_1,2j} = \frac{d^{r2}_{m_1,j}}{d_{m_1,j}} = \frac{(1-\eta_2) D^{r2}_j}{D_j - \eta_1 D^{r1}_j - \eta_2 D^{r2}_j} = \frac{(1-\eta_2)(1-\gamma_{2j})}{1 - (1-\eta_1)(1-\gamma_{1j}) - (1-\eta_2)(1-\gamma_{2j})}$$

is strictly decreasing if and only if

- $\frac{(1-\eta_1)\gamma_{1j}+\eta_1}{1-\gamma_{2j}}$ is strictly increasing,

which follows from the assumption that both γ_{1j} and γ_{2j} are strictly increasing. Therefore $\gamma_{m_1,2j}$ is strictly increasing in j; and similarly, $\gamma_{m_1,1j}$ is also strictly increasing in j.

Now, our pricing policy indicates that after the price level p_{m_1}, the firm will stop selling type-1 restricted units. In other words, the firm only sells type-2 restricted and unrestricted units in the residual market. But as illustrated above, for any given values of η_1 and η_2, $\gamma_{m_2,j}$ is strictly increasing in j. Therefore in the residual market only with type-2 restriction, we can use Theorem 3.3.5, which shows that there is another allocation plan of the following form:

$$\{(p_j, q'_{r2,j}) : j = m_1, \cdots, m'; (p_k, q'_k) : k = m', \cdots, p_n\},$$

which will perform as least as good as the original policy:

$$\{(p_j, q_{r2,j}, q_j) : j = m_1, \cdots, m_2; (p_k, q_k) : k = m_2, \cdots, n\}.$$

Therefore, we can treat $p_{m'}$ as the *common* highest price level for both restrictions.[3] This proves the lemma. □

With the help of Lemma 4.3.1, I know that the firm's pricing problem can be formulated as the following mathematical programming:

[3]The main purpose of using the common price levels for both restrictions is to simplify the formulation that leads to simple optimal pricing policies. This is purely for technical reasons. On the other hand, it is possible though that the two restrictions *may not have a common last active price level.*

$$\text{Max} \quad \sum_{i=1}^{m} p_i(q_{r1,i} + q_{r2,i}) + \sum_{j=m}^{n} p_j q_j$$

s.t.

$$\beta_1 \equiv \sum_{i=1}^{m} \frac{q_{r1,i}}{D_i^{r1}} \leq 1 \tag{4.16}$$

$$\beta_2 \equiv \sum_{i=1}^{m} \frac{q_{r2,i}}{D_i^{r2}} \leq 1 \tag{4.17}$$

$$\sum_{j=m}^{n} \frac{q_j}{D_j - \beta_1 D_j^{r1} - \beta_2 D_j^{r2}} \leq 1 \tag{4.18}$$

$$\sum_{i=1}^{m}(q_{r1,i} + q_{r2,i}) + \sum_{j=m}^{n} q_j \leq q$$

$$q_{r1,i}, q_{r2,i}, q_j \geq 0, \forall i \text{ and } \forall j.$$

I will call this formulation *ME-Model*, where "*ME*" stands for *mutually exclusive*. Again let $R(m; \gamma_1, \gamma_2)$, or $R(m)$ in brief, be the optimal objective value of the above ME-Model. The following theorem demonstrates that there also exists a linear programming characterization for the optimal revenue value — $\max_{1 \leq m \leq n} R(m)$.

Theorem 4.3.2: *Let $\tilde{R}(m)$ be the optimal objective value of the following linear programming (named the* **Tight ME-model***):*

$$\text{Max} \quad \sum_{i=1}^{m} p_i(q_{r1,i} + q_{r2,i}) + \sum_{j=m}^{n} p_j q_j$$

s.t.

$$\sum_{i=1}^{m} \frac{q_{r1,i}}{D_i^{r1}} = 1$$

$$\sum_{i=1}^{m} \frac{q_{r2,i}}{D_i^{r2}} = 1$$

$$\sum_{j=m}^{n} \frac{q_j}{D_j - D_j^{r1} - D_j^{r2}} = 1$$

$$\sum_{i=1}^{m}(q_{r1,i} + q_{r2,i}) + \sum_{j=m}^{n} q_j \leq q$$

$$q_{r1,i}, q_{r2,i}, q_j \geq 0, \forall i \text{ and } \forall j.$$

Then

$$\max_{1\le m\le n} R(m) = \max_{1\le m\le n} \tilde{R}(m). \tag{4.19}$$

Proof: To prove (4.19), it suffices to show that for any m, there exists another $\tilde{m}$ such that

$$R(m) \le \tilde{R}(\tilde{m}). \tag{4.20}$$

Let $\{(q_{rt,1},\cdots,q_{rt,m}), t=1,2; (q_m,\cdots,q_n)\}$ be an optimal solution to the ME-model with respect to m. If at this solution, constraints (4.16), (4.17), (4.18) are all binding, then (4.20) clearly holds for $\tilde{m}=m$.

By Theorem 2.4.1 of Chapter 2, I know that if *either* one of the constraints (4.16) and (4.17) is not binding, then exactly one of these $q_m,\cdots,q_n$ can be strictly positive. Let us call it $q_{\bar{j}}$, which, according to Theorem 2.3.5, must satisfy

$$q_{\bar{j}} = D_{\bar{j}} - \beta_1 D_{\bar{j}}^{r1} - \beta_2 D_{\bar{j}}^{r2}. \tag{4.21}$$

Now take $\tilde{m}=\bar{j}$ and define a new pricing policy as follows: for $t=1,2$,

$$\tilde{q}_{rt,i} = \begin{cases} q_{rt,i} & \text{if } 1\le i\le m, \\ 0 & \text{if } m+1\le i\le \bar{j}-1\ , \\ (1-\beta_t)D_{t\bar{j}} & \text{if } i=\bar{j}; \end{cases}$$

and

$$\tilde{q}_j = \begin{cases} D_{\bar{j}} - D_{\bar{j}}^{r1} - D_{\bar{j}}^{r2} & \text{if } j=\bar{j}, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that the policy $\{\tilde{q}_{ti}, 1\le i\le \tilde{m}, t=1,2; \tilde{q}_j, \tilde{m}\le j\le n\}$ is a feasible solution to the Tight ME-model associated with $\tilde{m}$. Furthermore, by (4.21) I have

$$\tilde{R}(\tilde{m}) \ge \sum_{i=1}^{\tilde{m}} p_i(\tilde{q}_{r1,i}+\tilde{q}_{r2,i}) + \sum_{j=\tilde{m}}^{n} p_j\tilde{q}_j = R(m);$$

therefore (4.20) is true. Hence the theorem is proved as required. □

As an immediate consequence of the above theorem, I have

Corollary 4.3.3: *If two types of restrictions are mutually exclusive, then there exists an optimal pricing policy that consists of at most* **four** *different prices, which is characterized by a series of linear programming problems.*

Finally, it is straightforward to extend the above model to the case of *multiple mutually exclusive restrictions.*

4.4 Pricing Problem by Using Two General Restrictions

Recall that $\gamma_t(p)$ is the percentage of consumers who cannot accommodate the type-t restriction, and $\gamma(p)$ is the percentage of consumers who cannot purchase the product if both restrictions are attached to the product. Then we should notice the followings:

- If $\gamma(p) = \gamma_1(p)$ or $\gamma(p) = \gamma_2(p)$), then it leads to the case of *two nested restrictions*;
- If $\gamma(p) = 1$, then it becomes the case of *two mutually exclusive restrictions.*

On the other hand, it is clear that $\gamma(p) \geq \gamma_t(p)$ for $t = 1, 2$. Therefore, it is always true that

$$\max(\gamma_1(p), \gamma_2(p)) \leq \gamma(p) \leq 1.$$

In the previous two sections I have discussed two extreme cases of two types of specific restrictions. In this section, I will discuss the case of two general product restrictions, that is,

$$0 < 1 - \gamma(p) < \min(1 - \gamma_1(p), 1 - \gamma_2(p)). \tag{4.22}$$

This implies that there are some consumers who can purchase the product even if both restrictions are attached. In this section, I will only consider the following type of pricing policies:

- the firm first sells some units with two restrictions attached;
- the firm then offers type-1 and type-2 restricted units at *certain orders*; and
- the firm finally offers the rest of units without any restrictions.

I need to introduce additional notation here:

- I will call the restriction consisting of type-1 and type-2 restrictions the *type-3* restriction;
- let $D^{r3}(p) = (1 - \gamma(p))D(p)$ and denote $D_i^{r3} = D^{r3}(p_i)$ for $i = 1, \cdots, n$; and
- define
$$1 - \alpha_{ti} = \frac{D_i^{r3}}{D_i^{rt}}, \text{ for } i = 1, \cdots, n \text{ and } t = 1, 2,$$
which measures the percentage of consumers in the market demand for type-t restricted product who can accommodate both restrictions.

Throughout this section, I will assume the following monotonicity conditions:

- γ_{ti} is strictly increasing in i for $t = 1, 2$; and
- α_{ti} is strictly increasing in i for $t = 1, 2$.

On the other hand, it is easy to check that

$$(1 - \gamma_{1i})(1 - \alpha_{1i}) = (1 - \gamma_{2i})(1 - \alpha_{2i}) = 1 - \gamma(p_i) = 1 - \gamma_i, \tag{4.23}$$

which implies that only four of these five sets of parameters can be independently given. As a consequence of this, I know that γ_i is also strictly increasing.

Let $\{(p_i, q_{r3,i}) : i = 1, \cdots, m_1\}$ be the allocation plan for the type-3 restricted units, where $1 \leq m_1 \leq n$. Denote $d^{r3}_{m_1,j}$ as the residual demand for type-3 restricted product at price p_j for all $j \geq m_1$ after the sales according to this allocation. Then I know that

$$d^{r3}_{m_1,j} = (1 - \sum_{i=1}^{m_1} \frac{q_{r3,i}}{D^{r3}_i}) D^{r3}_j, \text{ for all } j \geq m_1.$$

And it is clear that the feasibility condition for type-3 restricted units is given by

$$\beta_3 \equiv \sum_{i=1}^{m_1} \frac{q_{r3,i}}{D^{r3}_i} \leq 1.$$

Suppose that $\beta_3 = 1$. It implies that in the residual demand market for restricted product, each consumer can at most accommodate one type of restriction. Or equivalently speaking, after the sales of type-3 restricted units, all these who can accommodate both restrictions are satisfied. Therefore, type-1 restriction and type-2 restriction are *mutually exclusive in the residual demand market.* By the discussions in the previous section, I can specify the remaining part of a pricing policy as follows:

$$\{(p_j, q_{r1,j}, q_{r2,j}) : j = m_1, \cdots, m; (p_k, q_k) : k = m, \cdots, n\},$$

where $1 \leq m_1 \leq m \leq n$.

In order to use Theorem 4.3.2 in Section 4.3, I need to check the monotonicity conditions on the impact of type-1 and type-2 restrictions on the residual market. Let $\gamma_{m_1,tj}$ be the percentage of consumers in the residual demand market who can purchase type-t restricted product at price p_j ($j \geq m_1$) after the sales of type-3 restricted units. Since $\beta_3 = 1$, it follows that by (4.23)

$$1 - \gamma_{m_1,tj} = \frac{D^{rt}_j - D^{r3}_j}{D_j - D^{r3}_j} = (1 - \gamma_{tj}) \frac{\gamma_j}{\alpha_{tj}} = \frac{\gamma_j - \gamma_{tj}}{\gamma_j} = 1 - \frac{\gamma_{tj}}{\gamma_j}.$$

Therefore, for $t = 1, 2$, $\gamma_{m_1,tj}$ is strictly increasing in j if and only if

- γ_{tj}/γ_j is strictly increasing in j,

which means that as the price increases, the relative ratio of the percentage of consumers who cannot buy type-t restricted product to the percentage of consumers who can not buy type-3 restricted product is strictly increasing. A *sufficient condition* is that

- $\gamma_j - \gamma_{tj}$ is strictly decreasing in j,

or, equivalently, $\frac{D_j^{r1} - D_j^{r3}}{D_j}$ is strictly decreasing in j, which implies that the percentage of consumers who can accommodate type-t restricted product, but not type-3 restricted product, is strictly decreasing. This sufficient condition is stronger than required; but it has an intuitive interpretation.

Summarizing the above discussions, I have the following theorem.

Theorem 4.4.1: *Suppose that the firm only consider the type of policies such that $\beta_3 = 1$. Further assume that for $t = 1, 2$,*

- *γ_{tj}/γ_j is strictly increasing in j,*

then the firm's pricing problem can be formulated as the following linear programming (I will call it **Tight-G Model**, *where "G" stands for "general"):*

$$\text{Max} \quad \sum_{i=1}^{m_1} p_i q_{r3,i} + \sum_{j=m_1}^{m} p_j (q_{r1,j} + q_{r2,j}) + \sum_{k=m}^{n} p_k q_k$$

s.t.

$$\sum_{i=1}^{m_1} \frac{q_{r3,i}}{D_i^{r3}} = 1$$

$$\sum_{j=m_1}^{m} \frac{q_{r1,j}}{D_j^{r1} - D_j^{3r}} = 1$$

$$\sum_{j=m_1}^{m} \frac{q_{r2,j}}{D_j^{r2} - D_j^{3r}} = 1$$

$$\sum_{k=m}^{n} \frac{q_k}{D_k-(D_k^{r1}-D_k^{r3})-(D_k^{r2}-D_k^{r3})} = 1$$

$$\sum_{i=1}^{m_1} q_{r3,i} + \sum_{j=m_1}^{m} (q_{r1,j} + q_{r2,j}) + \sum_{k=m}^{n} q_k \leq q$$

$$q_{rt,i} \geq 0 \text{ and } q_k \geq 0.$$

Proof: It follows from the above discussion that if γ_{tj}/γ_j is strictly increasing, then $\gamma_{m_1,tj}$ is strictly increasing for $t = 1, 2$. On the other hand, since $\beta_3 = 1$, I know that in the residual demand market, type-1 and type-2 restrictions are mutually exclusive. Hence, the rest of the proof follows immediately from Theorem 4.3.2. □

Therefore, if $\beta_3 = 1$, then the firm needs to offer *at most five* prices to maximize its revenue. The Tight-G model provides a benchmark on the optimal revenue value for any other general models since it at least provides *a lower bound* for the optimal revenue value. In Chapter 2 and Chapter 3, I have demonstrated that for the case of one type of restriction, an optimal solution for a tight model remains to be optimal in the general class of pricing policies. It is not clear at this moment whether or not Tight-G model in fact provides an optimal solution in the general context.

The main issue is whether we can find an optimal solution such that $\beta_3 = 1$. This is not an easy task since if $\beta_3 < 1$, then *it is not even clear how to specify the remaining part of pricing policy* because

- $\beta_3 < 1$ implies that if both type-1 and type-2 restricted units are offered *at the same price level*, some consumers can buy *either one of them*;
- On the other hand, the impact on the residual demand market of selling type-1 restricted units first and type-2 restricted units thereafter at the same price level is *different* from the impact on residual demand market of selling type-2 restricted units first and type-1 restricted units later at the same price, which will complicate the process of updating the residual market.

So the key question is: *which type of restricted units will be offered first* if both restricted units are offered at the same price? One way to get around this difficulty is to consider policies of the form:

$$\{(p_j, q^1_{r1,j}, q_{r2,j}, q^2_{r1,j}) : j = m_1, \cdots, m\},$$

where $1 \leq m_1 \leq m \leq n$. This type of policy has the following technical properties:

- it allows the firm to sell some type-1 restricted units first, then some type-2 restricted units and finally some type-1 restricted units, all the same price level;
- if at price p_j the firm wants to sell type-2 restricted units *first*, it can do so by letting $q^1_{r1,j} = 0$; and
- it is possible possible for the firm to sell type-1 restricted units first on a price set and type-2 restricted units on the remaining price set, or vice versa.

Consequently, I have the following definition of a pricing policy when using two general specific restrictions:

Definition 4.4.1: *For two general restrictions, a pricing policy is specified by*

$$\{(p_i, q_{r3,i}) : i = 1, \cdots, m_1; (p_j, q^1_{r1,j}, q_{r2,j}, q^2_{r1,j}) : j = m_1, \cdots, m; (p_k, q_k) : k = m, \cdots, n\},$$

where $1 \leq m_1 \leq m \leq n$.

Let us now analyze a special case that leads to Tight-G model.

Theorem 4.4.2: *Assume that* γ_{1j}/γ_j *and* γ_{2j}/γ_j *are strictly increasing in* j. *If one of the following two conditions holds:*

(i) $\frac{\gamma_j - \gamma_{2j}}{\gamma_j - \gamma_{1j}}$ *is strictly increasing in* j,

(ii) $\frac{\gamma_j-\gamma_{1j}}{\gamma_j-\gamma_{2j}}$ *is strictly increasing in* j,

there exists an optimal policy that is characterized by the Tight-G model.

Proof: By Theorem 4.4.1, I know that it suffices to show that there exists an optimal policy such that $\beta_3 = 1$. It is evident that conditions (i) and (ii) are perfectly symmetric. I will prove the result by assuming (i).

I will prove the theorem through two steps. In the first step, I will formulate the pricing problem into a mathematically programming formulation. The main technique here is to show that I need only to consider a small class of policies which contains an optimal policy. I will use the main results in Chapters 2 and 3. And the second step is to further simplify the model formulation so that I can have a linear programming formulation, which will lead to a simple optimal pricing structure.

Technically speaking, I can start with policies specified by Definition 4.4.1. But because with the additional assumption in the theorem, I can actually focus on a smaller class of policies that makes the model formulation much simpler and easier. First of all, it is clear that any policy will start with the sales of type-3 restricted units.[4] Let the part of allocation for type-3 restricted units in *an optimal policy* be given by:

$$\{(p_i, q_{r3,i}) : i = 1, \cdots, m_1\},$$

where m_1 is an integer such that $1 \leq m_1 \leq n$. The key is to analyze the demand structure in the residual market after the sales of type-3 restricted units. As above, let

$$\beta_3 = \sum_{i=1}^{m_1} \frac{q_{r3,i}}{D_i^{r3}}.$$

I will use the following notation in the residual market: let $j \geq m_1$

[4]It is possible though that the firm may not sell any type-3 restricted units.

- $d^{rt}_{m_1,j}$ is the residual demand for type-t restricted units at price p_j for $t = 1, 2, 3$, which are given by:

$$d^{r3}_{m_1,j} = (1-\beta_3)D^{r3}_j; \text{ and } d^{rt}_{m_1,j} = D^{rt}_j - \beta_3 D^{r3}_j, \text{ for } t = 1, 2.$$

- $d^{r}_{m_1,j}$ is the *total* residual demand for restricted product at price p_j (including the demand type-1 restricted product and the demand for type-2 restricted product), which is given by:

$$\begin{aligned} d^{r}_{m_1,j} &= d^{r1}_{m_1,j} + d^{r2}_{m_1,j} - d^{r3}_{m_1,j} = (D^{r2}_j - D^{r3}_j) + d^{r1}_{m_1,j} \\ &= (D^{r1}_j - D^{r3}_j) + d^{r2}_{m_1,j} = D^{r1}_j + D^{r2}_j - (1+\beta_3)D^{r3}_j. \end{aligned}$$

- $d_{m_1,j}$ is the total residual demand for unrestricted product at price p_j, which is given by:

$$d_{m_1,j} = (D_j - D^{r1}_j - D^{r2}_j + D^{r3}_j) + d^{r}_{m_1,j} = D_j - \beta_3 D^{r3}_j.$$

- Let $1 - \gamma_{m_1,j} = \frac{d^{r3}_{m_1,j}}{d_{m_1,j}}$.
- Let $1 - \gamma_{m_1,tj} = \frac{d^{rt}_{m_1,j}}{d_{m_1,j}}$ for $t = 1, 2$.
- Let $1 - \alpha_{m_1,tj} = \frac{d^{rt}_{m_1,j}}{d^{r}_{m_1,j}}$.

Now if $\beta_3 = 1$, then the rest of the proof follows from Theorem 4.3.2. So I only need to focus on the optimal policies such that $\beta_3 < 1$. Let $\eta_{m_1,1j}$ be the percentage of consumers in the residual market for restricted product who cannot accommodate type-1 restriction. Then,

$$1 - \eta_{m_1,1j} = \frac{d^{r1}_{m_1,j}}{d^{r}_{m_1,j}} = \frac{D^{r1}_j - \beta_3 D^{r3}_j}{D^{r1}_j + D^{r2}_j - (1+\beta_3)D^{r3}_j} = \frac{D^{r1}_j - \beta_3 D^{r3}_j}{(D^{r1}_j - \beta_3 D^{r3}_j) + (D^{r2}_j - D^{r3}_j)}.$$

Similarly, let $\eta_{m_1,j}$ be the percentage of consumer in the residual demand market for unrestricted product who cannot accommodate *any restrictions*, then

$$1 - \eta_{m_1,j} = \frac{d^{r}_{m_1,j}}{d_{m_1,j}} = \frac{D^{r1}_j + D^{r2}_j - (1+\beta_3)D^{r3}_j}{D_j - \beta_3 D^{r3}_j}.$$

Claim 1: *Both $\eta_{m_1,1j}$ and $\eta_{m_1,j}$ are strictly increasing in j.*

Proof of Claim 1: Clearly, $\eta_{m_1,1j}$ is strictly increasing if and only if

- $\frac{D_j^{r1}-\beta_3 D_j^{r3}}{D_j^{r2}-D_j^{r3}}$ is strictly decreasing.

On the other hand, note that

$$\begin{aligned}
\frac{D_j^{r1}-\beta_3 D_j^{r3}}{D_j^{r2}-D_j^{r3}} &= \frac{D_j^{r1}-D_j^{r3}}{D_j^{r2}-D_j^{r3}} + \frac{(1-\beta_3)D_j^{r3}}{D_j^{r2}-D_j^{r3}} \\
&= \frac{\frac{D_j^{r1}}{D_j}-\frac{D_j^{r3}}{D_j}}{\frac{D_j^{r2}}{D_j}-\frac{D_j^{r3}}{D_j}} + \frac{(1-\beta_3)\frac{D_j^{r3}}{D_j^{r2}}}{1-\frac{D_j^{r3}}{D_j^{r2}}} \\
&= \frac{\gamma_j-\gamma_{1j}}{\gamma_j-\gamma_{2j}} + (1-\beta_3)\frac{1-\alpha_{2j}}{\alpha_{2j}},
\end{aligned}$$

which is strictly decreasing since

- $\frac{\gamma_j-\gamma_{1j}}{\gamma_j-\gamma_{2j}}$ is strictly decreasing by assumption; and
- $\frac{1-\alpha_{2j}}{\alpha_{2j}}$ is strictly decreasing because α_{2j} is strictly increasing.

For $\eta_{m_1,j}$, note that

$$\begin{aligned}
\frac{D_j^{r1}+D_j^{r2}-(1+\beta_3)D_j^{r3}}{D_j-\beta_3 D_j^{r3}} &= \frac{\frac{D_j^{1}}{D_j}+\frac{D_j^{r2}}{D_j}-(1+\beta_3)\frac{D_j^{r3}}{D_j}}{1-\beta_3\frac{D_j^{r3}}{D_j}} \\
&= \frac{(1-\gamma_{1j})+(1-\gamma_{2j})-(1+\beta_3)(1-\gamma_j)}{1-\beta_3(1-\gamma_j)} \\
&= \frac{(1-\beta_3(1-\gamma_j))+(\gamma_j-\gamma_{1j}-\gamma_{2j})}{1-\beta_3(1-\gamma_j)},
\end{aligned}$$

which is strictly decreasing if and only if

$$\frac{(\gamma_j-\gamma_{1j}-\gamma_{2j})}{1-\beta_3(1-\gamma_j)}$$

is strictly decreasing. But

$$\frac{(\gamma_j-\gamma_{1j}-\gamma_{2j})}{1-\beta_3(1-\gamma_j)} = \frac{(\gamma_j-\gamma_{1j}-\gamma_{2j})}{(1+\beta_3)\gamma_j-\beta_3}$$

$$
\begin{aligned}
&= \frac{\gamma_j}{(1+\beta_3)\gamma_j - \beta_3}(1 - \frac{\gamma_{1j}}{\gamma_j} - \frac{\gamma_{2j}}{\gamma_j}) \\
&= \frac{1}{1+\beta_3}(1 + \frac{\beta_3}{(1+\beta_3)\gamma_j - \beta_3})(1 - \frac{\gamma_{1j}}{\gamma_j} - \frac{\gamma_{2j}}{\gamma_j}),
\end{aligned}
$$

which is indeed strictly decreasing since

- γ_j is strictly increasing; and
- both $\frac{\gamma_{1j}}{\gamma_j}$ and $\frac{\gamma_{2j}}{\gamma_j}$ are strictly increasing.

This proves Claim 1.

Now by Theorem 3.3.5, the monotonicity properties of $\eta_{m_1,1j}$ and $\eta_{m_1,j}$ indicate,

- in the residual market, the firm only needs to consider policies such that type-1 restricted units, if allocated, should be sold before the sales of type-2 restricted units.

Therefore, the firm can limit itself on the following type of policies in the residual market:

$$\{(p_j, q_{r1,j}) : j = m_1, \cdots, m_2; (p_l, q_{r2,l}) : l = m_2, \cdots, m; (p_k, q_k) : k = m, \cdots, n\},$$

where $m_1 \leq m_2 \leq m \leq n$.

By using this type of policies, I know that the feasibility condition for type-1 restricted allocations is

$$\beta_1 \equiv \sum_{j=m_1}^{m_2} \frac{q_{r1,j}}{d^{r1}_{m_1,j}} = \sum_{j=m_1}^{m_2} \frac{q_{r1,j}}{D^{r1}_j - \beta_3 D^{r3}_j} \leq 1.$$

After the sales of type-1 restricted units, the residual demand for type-1 restricted product is

$$d^{r1}_{m_2,l} = (1-\beta_1)(D^{r1}_l - \beta_3 D^{r3}_l), l \geq m_2;$$

and the residual demand for type-3 restricted units is

$$d^{r3}_{m_2,l} = (1-\beta_1)(1-\beta_3)D^{r3}_l, l \geq m_2.$$

Consequently, the residual demand for type-2 restricted product is given by

$$\begin{aligned} d^{r2}_{m_2,l} &= (D^{r2}_l - D^{r3}_l) + d^{r3}_{m_2,l} \\ &= D^{r2}_l - (\beta_1 + \beta_3 - \beta_1\beta_3)D^{r3}_l. \end{aligned}$$

Therefore the feasibility condition for type-2 restricted units is characterized by:

$$\beta_2 \equiv \sum_{l=m_2}^{m} \frac{q_{r2,l}}{D^{r2}_l - (\beta_1 + \beta_3 - \beta_1\beta_3)D^{r3}_l} \leq 1.$$

Now after the sales of type-2 restricted units, the residual demand for type-3 restricted product is

$$d^{r3}_{m,k} = (1-\beta_2)(1-\beta_1)(1-\beta_3)D^{r3}_l, k \geq m; \text{ and}$$

the residual demand for type-2 restricted product is

$$\begin{aligned} d^{r2}_{m,k} &= (1-\beta_2)d^{r2}_{m_2,k} \\ &= (1-\beta_2)(D^{r2}_k - (\beta_1 + \beta_3 - \beta_1\beta_3)D^{r3}_k) \text{ for } k \geq m. \end{aligned}$$

Thus the total residual demand for type-1 *and* type-2 restricted units is given by:

$$\begin{aligned} d^{r}_{m,k} &= (d^{r1}_{m_2,k} - d^{r3}_{m_2,k}) + d^{r2}_{m,k} \\ &= (1-\beta_1)(D^{r1}_k - D^{r3}_k) + (1-\beta_2)(D^{r2}_k - (\beta_1 + \beta_3 - \beta_1\beta_3)D^{r3}_k); \end{aligned}$$

and the total residual demand for *unrestricted product* is:

$$\begin{aligned} d_{m,k} &= (D_k - D^{r1}_k - D^{r2}_k + D^{r3}_k) + d^{r}_{m,k} \\ &= D_k - \beta_1(D^{r1}_k - D^{r3}_k) - \beta_2 D^{r2}_k - (1-\beta_2)(\beta_1 + \beta_3 - \beta_1\beta_3)D^{r3}_k \\ &= D_k - \beta_1 D^{r1}_k - \beta_2 D^{r2}_k + (\beta_1\beta_2 - (1-\beta_1)(1-\beta_2)\beta_3)D^{r3}_k. \end{aligned}$$

So the feasibility condition for the final unrestricted units is

$$\sum_{k=m}^{n} \frac{q_k}{D_k - \beta_1 D^{r1}_k - \beta_2 D^{r2}_k + (\beta_1\beta_2 - (1-\beta_1)(1-\beta_2)\beta_3)D^{r3}_k} \leq 1.$$

Summarizing these discussions, I get the following formulation for the firm's pricing problem:

$$\text{Max} \quad \sum_{i=1}^{m_1} p_i q_{r3,i} + \sum_{j=m_1}^{m_2} p_j q_{r1,j} + \sum_{l=m_2}^{m} p_j q_{r2,l} + \sum_{k=m}^{n} p_k q_k$$

s.t.

$$\beta_3 \equiv \sum_{i=1}^{m_1} \frac{q_{r3,i}}{D_i^{r3}} \leq 1 \tag{4.24}$$

$$\beta_1 \equiv \sum_{j=m_1}^{m_2} \frac{q_{r1,j}}{D_j^{r1} - \beta_3 D_j^{3r}} \leq 1 \tag{4.25}$$

$$\beta_2 \equiv \sum_{l=m_2}^{m} \frac{q_{r2,l}}{D_l^{r2} - (\beta_1 + \beta_3 - \beta_1\beta_3) D_l^{3r}} \leq 1 \tag{4.26}$$

$$\sum_{k=m}^{n} \frac{q_k}{D_k - \beta_1 D_k^{r1} - \beta_2 D_k^{r2} + (\beta_1\beta_2 - (1-\beta_1)(1-\beta_2)\beta_3) D_k^{r3}} \leq 1 \tag{4.27}$$

$$\sum_{i=1}^{m_1} q_{r3,i} + \sum_{j=m_1}^{m_2} q_{r1,j} + \sum_{l=m_2}^{m} q_{r2,l} + \sum_{k=m}^{n} q_k \leq q$$

$$q_{rt,i} \geq 0 \text{ for } t = 1, 2, 3, \text{ and } q_k \geq 0.$$

I now show that there exists an optimal solution to the above formulation such that

$$\beta_3 = \beta_1 = \beta_2 = 1,$$

which will reduce the above formulation into a linear programming.

Claim 2: *There exists an optimal policy such that* $\beta_1 = \beta_2 = 1$.

Proof of Claim 2: Let

$$\{(p_i, q_{r3,i}) : i = 1, \cdots, m_1; (p_j, q_{r1,j}) : j = m_1, \cdots, m_2;$$
$$(p_l, q_{r2,l}) : l = m_2, \cdots, m; (p_k, q_k) : k = m, \cdots, n\}$$

be an arbitrary optimal solution to the above formulation. If $\beta_3 = 1$, then I know that our problem leads to the case studied in Theorem 4.4.1. I now consider that $\beta_3 < 1$.

First of all, the monotonicity properties of $\eta_{m_1,1j}$ and $\eta_{m_1,j}$ help us to formulate the pricing problem. On the surface, these properties should also lead to the model of *two nested restrictions*. This is not so since

- the firm sells type-2 restricted units *after* the sales of type-1 restricted units;

- the set of those consumers who can buy type-1 restricted product *is not a subset* of the set of these consumers who can buy type-2 restricted product.

On the other hand, if $\beta_3 < 1$, then there is a positive residual demand market for type-3 restricted product. Most importantly, observe that

- *type-3 restriction is nested into type-2 restriction*;
- *both $\alpha_{m_1,2j}$ and $\gamma_{m_1,2j}$ are still strictly increasing*; and
- *during the sales of type-1 restricted units, the residual demand for type-3 restricted product is proportionally reduced.*

Therefore, in the residual market in regard of type-3 and type-2 restrictions I can use the results in Theorem 4.2.1, which says that after the sales of type-1 restricted units that have included the residual demand for type-3 restricted product, there is no residual demand for type-3 restricted product. This implies that

$$d^{r3}_{m_2,l} = (1-\beta_1)(1-\beta_3)D^{r3}_j = 0,$$

which leads to $\beta_1 = 1$.

After the sales of type-1 restricted units with $\beta_1 = 1$, I know that the residual market consists of exactly two types of consumers – theoe who cannot purchase type-2 restricted units and those who can. Let $\gamma_{m_2,2l}$ be the percentage of consumers in the residual market who cannot accommodate type-2 restriction after the sales of type-3 and type-1 restricted units according to the allocation plan. Then since $\beta_1 = 1$, it follows that

$$\begin{aligned} 1-\gamma_{m_2,2l} &= \frac{d^{r2}_{m_2,l}}{d_{m_2,l}} = \frac{D^{r2}_j - D^{r3}_j}{D_j - D^{r1}_j} \\ &= \frac{(1-\gamma_{2j})-(1-\gamma_j)}{1-(1-\gamma_{1j})} = \frac{\gamma_j - \gamma_{2j}}{\gamma_{1j}} = \frac{\gamma_j}{\gamma_{1j}}(1-\frac{\gamma_{2j}}{\gamma_j}), \end{aligned}$$

which is strictly decreasing since $\frac{\gamma_{tj}}{\gamma_j}$ is assumed to be strictly increasing for $t = 1, 2$. Therefore $\gamma_{m_2,2l}$ is strictly increasing. Then by Theorem 3.3.5 and Theorem 2.4.2 , it follows that the firm only needs to consider policies such that $\beta_2 = 1$. In summary, I have shown that the firm can focus on the class of policies such that $\beta_1 = \beta_2 = 1$. This proves Claim 2.

Using the results in Claim 2, I know that the pricing problem can be reduced to the following form:

$$\text{Max} \quad \sum_{i=1}^{m_1} p_i q_{r3,i} + \sum_{j=m_1}^{m_2} p_j q_{r1,j} + \sum_{l=m_2}^{m} p_j q_{r2,l} + \sum_{k=m}^{n} p_k q_k$$

s.t.

$$\beta_3 \equiv \sum_{i=1}^{m_1} \frac{q_{r3,i}}{D_i^{r3}} = 1 \tag{4.28}$$

$$\sum_{j=m_1}^{m_2} \frac{q_{r1,j}}{D_j^{r1} - \beta_3 D_j^{3r}} = 1 \tag{4.29}$$

$$\sum_{l=m_2}^{m} \frac{q_{r2,l}}{D_l^{r2} - D_l^{3r}} = 1 \tag{4.30}$$

$$\sum_{k=m}^{n} \frac{q_k}{D_k - D_k^{r1} - D_k^{r2} + D_k^{r3}} = 1 \tag{4.31}$$

$$\sum_{i=1}^{m_1} q_{r3,i} + \sum_{j=m_1}^{m_2} q_{r1,j} + \sum_{l=m_2}^{m} q_{r2,l} + \sum_{k=m}^{n} q_k \leq q$$

$$q_{rt,i} \geq 0 \text{ for } t = 1, 2, 3, \text{ and } q_k \geq 0.$$

Note that constraints (4.30) and (4.31) are no longer related to β_3, the only parameter that causes a non-linear constraint (4.29). Furthermore, I have the following observations:

- The demand for type-1 restricted product will be exhausted after the sales according to the plan: $\{(p_i, q_{r3}) : i = 1, \cdots, m_1; (p_j, q_{r1,j}) : j = m_1, \cdots, m_2\}$;
- α_{1i}, the percentage of consumers in the market demand for type-1 restricted product who can also accommodate type-2 restriction, is strictly increasing; and
- The total sales for type-3 and type-1 restricted units is bounded by

$$\bar{q}_{r1} = q - \sum_{l=m_2}^{m} q_{r2,l} - \sum_{k=m}^{n} q_k,$$

which can be considered as the *capacity* for type-3 and type-1 restricted units since it is independent of the allocation plan for type-3 and type-1 restricted units.

Hence this leads to a well-defined subproblem that has the same structure as the problem addressed in Chapter 2. Then using Theorem 2.4.2 and Theorem 3.3.5, the firm again only needs to focus on policies having the property of $\beta_3 = 1$. Therefore I prove the theorem, since the rest follows from Theorem 4.4.1. □

Note that the above theorem is aimed at the case that it is advantageous to *delay* the sales for one type of restricted units. There may be other cases that also lead to Tight-G formulation.

4.5 Summary – Pricing by Using Two Types of Restrictions

This chapter addresses the issue of pricing perishable inventories by using two types of restrictions. After studying two extreme cases, which are two nested restrictions and two mutually exclusive restrictions, I have also discussed the general case. I here provide the following additional insights:

- *Extensions from the basic BL-model to multiple restrictions are not straightforward.*
- *A pricing model by using two general types of restrictions is capable of handling four different types of prices.*
- *Extra demand structures are needed in order to obtain simplified optimal pricing structures.*
- *Simplified optimal pricing structures can be characterized in a similar manner as in the basic BL-model.*

I have shown that for the two extreme cases, there exist optimal pricing structures that consist of at most *four prices.* And for the general case, I present two situations where there exist optimal pricing structures that consist of at most *five prices.* All of these optimal pricing structures are characterized by linear programming problems, which make these models tractable in application.

Chapter 5

Airline Pricing by Using Membership and Product Restrictions

5.1 Introduction

Chapter 2 and Chapter 4 developed a series of pricing models by using artificial restrictions. These models gave us conceptual tools for analyzing the use of restrictions as a mechanism. In this chapter, I will present an application, in a convincing way, in the context of airlines. It is well-known that all airlines have special arrangements with certain clientele-specific rates. It is important to note that the fares for these special consumer groups are not available to the general public through travel agents. In most cases, these rates are internally controlled by the airlines. These opportunities represent revenue potentials which the airline may be able to exploit. It is important to notice that any special clientele represents some kind of membership which covers only a small of portion of the general population. Because of this, from the pricing point of view, these memberships are in fact restrictions because their existence will prevent some consumers from being able to purchase the product at a price that is available to members only. On the other hand, the availability of the seats for these clientele needs to be controlled and a good yield management system must be capable of handling the presence of these special clientele. For ease of presentation, I will use the following convention on terminologies:

- Any consumer in a special clientele is called a *member*;
- I use *fares* and *tickets* interchangeably with *prices* and *units*;

- Restricted fares and unrestricted fares are understood as fares that are available to the general public, which means that all consumers are allowed to purchase these fares; and
- These fares that are specifically available to members are called either *restricted membership fares* or *unrestricted membership fares.*

Note that there is clearly an asymmetry between members and non-members on the access to the reservation information. A member can purchase a ticket either at a membership fare or at a public fare, whichever is less; but non-members can only purchase the public fares since the membership fares are not available to them. In this sense, we should treat the existence of membership as another restriction in addition to the product restrictions, such as, Advance Booking, Saturday Night Stay, and No Refund. As a result, the airline faces *two types of restrictions.* The purpose of this chapter is to use the results developed in the previous chapters to illustrate how they can be applied in this context and what the implications are. In particular, I will discuss several different situations where a special membership deal makes sense to the airline. As a result of this analysis, it will improve our understanding about the *operating environment* for each of these situations.

Consider that an airline has a scheduled flight with a fixed capacity. Assume that the airline has a private agreement with a special group under the following simple terms:

- As a member of the group, the traveller can purchase a ticket at a membership price that will be lower than the comparable price offered to the general public; and
- The availability of these special fares are limited and members are served on the first-come-first-serve basis.

When an airline decides to use both membership restriction and product restriction, the airline can offer at most *four* different types of fares: restricted membership fares, restricted fares, unrestricted membership fares and unrestricted fares. Because of the membership agreement, we know that:

- *Restricted membership fares, if offered, should not be higher than the planned restricted fares*; and
- *Unrestricted membership fares, if offered, should not be higher than the planned unrestricted fares.*

In this section I will consider three common arrangements in terms of membership privileges: (1) *Cheaper restricted fares only*; (2) *Cheaper unrestricted fares only*; and (3) *Cheaper restricted fares* and *cheaper unrestricted fares.* The aim of this chapter is to find out conditions that lead to a proper choice among these arrangements. This is of great importance since the airline needs to know under what kind of operating environment, a particular membership deal is worthwhile at the first place.[1]

This chapter is organized as follows. Section 5.2 presents the basic model setting and notation. Section 5.3 addresses the airline fare pricing problem when members are promised that they can by restricted tickets for less. I will give two examples that are consistent with this scenario. Section 5.4 discusses the airline fare pricing problem when members are offered cheaper unrestricted fares. In particular, I will examine the impact of corporate policy for interval travellers and conclude that such a corporate commitment can cost an airline a lot of money. Then Section 5.5 deals with the case that an airline gives members both cheaper restricted fares and cheaper unrestricted fares. Sufficient

[1]The readers should be warned that our model is purely tactical. When an airline negotiates a special deal with an interest group, there are many other issues involved which can not be directly addressed in our model. Hopefully, our model will be helpful for airline managers to understand the impact of different types of membership deals.

conditions are identified for simple optimal fare structures. Examples are also used to give further support. The last section is a summary.

5.2 Model Setting and Notation

Let $D(p)$ be the total market demand function at price p (unrestricted), among which $D^M(p)$ of them have the membership. Denote $D^N(p)$ to be the demand from the non-member group. Then,

$$D^N(p) = D(p) - D^M(p).$$

Naturally, since membership is a form of restriction, we let $\gamma_M(p)$ be the percentage of consumers in the demand market at price p who are *non-members.* Then clearly,

$$1 - \gamma_M(p) = \frac{D^M(p)}{D(p)}.$$

On the impact of product restriction, we note that membership restricted fares are *more restrictive* that the general restricted fares. Therefore I use the following notation:

- $\gamma_{M^r}(p)$ is the percentage of consumers in the *total demand market* who cannot purchase the product at the restricted membership fare p, that is

  $$1 - \gamma_{M^r}(p) = \frac{D^{M^r}(p)}{D(p)},$$

  where $D^{M^r}(p)$ is the demand for the product at the restricted membership fare p;

- $\gamma_r(p)$ be the percentage of consumers in the total demand market who cannot accommodate the product restriction at price p, that is

  $$1 - \gamma_r(p) = \frac{D^r(p)}{D(p)},$$

  where $D^r(p)$ is the demand for product at price p with product-restriction only.

Further assume that $D(p)$ and $D^M(p)$ are non-decreasing step function defined on the set of fares $\{p_1, \cdots, p_n\}$. As usual, for $i = 1, \cdots, n$, denote

$$\gamma_{M,i} = \gamma_M(p_i); \gamma_{Mr,i} = \gamma_{Mr}(p_i); \gamma_{r,i} = \gamma_r(p_i);$$

and

$$D_i = D(p_i); D_i^r = D^r(p_i); D_i^M = D^M(p_i); D_i^{Mr} = D^{Mr}(p_i).$$

Throughout this chapter, I will assume that

- $\gamma_{Mr,i}$ *and* $\gamma_{r,i}$ *are strictly increasing in* j.

Additional conditions may be assumed in each of the following three sections in order to obtain *simple* optimal fare structures.

5.3 Cheaper Restricted Membership Fares Only

Let us first analyze the case where an airline only offers the members lower restricted fares. As a consequence of this, as a member, the traveller has *three types of fares* to choose from: restricted membership fares, (public) restricted fares and (public) unrestricted fares. On the other hand, any non-member can only purchase the public restricted fares and the public unrestricted fares.

Now if I call the restriction consisting of membership restriction and product restriction as the *type-3 restriction* and the product restriction as the *type-1 restriction*, then it is clear that *type-3 restriction is nested into type-1 restriction.* This allows me to use the model discussed in Section 4.2 of Chapter 4 to handle the problem here. But in the presentation that follows I will avoid to use the notion of type-1 and type-3 restrictions here after. I put them here for the purpose of illustration only.

Let $\alpha_{Mr,i}$ be the percentage of consumers in the demand market at restricted price p_i who are non-members. Then $1-\alpha_{Mr,i} = \frac{D_i^{Mr}}{D_i^r}$, for $i \geq 1$. Hence $1-\gamma_{Mr,i} = (1-\gamma_{r,i})(1-\alpha_{Mr,i})$, which indicates that type-3 restriction is indeed nested into type-1 restriction.

On the other hand, since restricted membership fares are supposed not to be higher than the general restricted fares, I can define *a pricing policy* for the airline as follows:

$$\{(p_i, q_{Mr,i}) : i = 1, \cdots, m_2; (p_j, q_{r,j}) : j = m_2, \cdots, m_1; (p_k, q_k) : k = m_1, \cdots, n\},$$

where $q_{Mr,i}$ is the allocation to members at the restricted price p_i for $1 \leq i \leq m_2$, $q_{r,j}$ is the allocation to general public at the restricted price p_j for $m_2 \leq j \leq m_1$, q_k is the allocation to the general public at the unrestricted price p_k for $m_1 \leq k \leq n$, and $1 \leq m_2 \leq m_1 \leq n$. Furthermore, I also know from my discussions in Section 4.2 that the airline's pricing problem can be formulated as the following N-model:

$$\begin{aligned}
\text{Max} \quad & \textstyle\sum_{i=1}^{m_2} p_i q_{Mr,i} + \sum_{j=m_2}^{m_1} p_j q_{r,j} + \sum_{k=m_1}^{n} p_k q_k \\
\text{s.t.} \quad & \\
& \beta_2 \equiv \textstyle\sum_{i=1}^{m_2} \frac{q_{Mr,i}}{D_i^{Mr}} \leq 1 \\
& \beta_1 \equiv \textstyle\sum_{j=m_2}^{m_1} \frac{q_{r,j}}{D_j^r - \beta_2 D_j^{Mr}} \leq 1 \\
& \textstyle\sum_{k=m_1}^{n} \frac{q_k}{D_k - \beta_1 D_k^r - (1-\beta_1)\beta_2 D_k^{Mr}} \leq 1 \\
& \textstyle\sum_{i=1}^{m_2} q_{Mr,i} + \sum_{j=m_2}^{m_1} q_{r,j} + \sum_{k=m_1}^{n} q_k \leq q \\
& q_{Mr,i}, q_{r,j}, q_k \geq 0, \forall i, \forall j, \text{ and } \forall k.
\end{aligned}$$

Rephrasing Theorem 4.2.1 in Section 4.2, I get the following proposition:

Proposition 5.3.1: *Suppose that the airline only offers lower restricted membership fares, and that both $\gamma_{r,i}$ and $\alpha_{Mr,i}$ are strictly increasing, then*

- *The airline needs to offer at most four fares to the market;* and

- *The optimal fare structures are designed in such a way that the demand for restricted membership fares will be exhausted right after the sales of these restricted fares targeted to members.*

I now discuss the implications of the assumptions in the above proposition. Clearly the assumption that $\gamma_{r,i}$ is strictly increasing is a basic requirement for *an effective product restriction.* For the assumption that $\alpha_{Mr,i}$ is strictly increasing, it simply means that as the price increases, among these who can purchase restricted product, *the percentage of the members is strictly decreasing.* As demonstrated in Theorem 4.2.1, this assumption plays a key role in the process of obtaining *a simple pricing structure*, or *a simple fare structure* in our context here. Basically, it allows us to focus on the type of policies with the property that the demand for restricted membership fares will be exhausted right after the sales of these restricted fares targeted to members.

Offering lower restricted fares to special clienteles is a very common approach for airlines dealing with membership issues. I here present two examples that are consistent with this particular approach.

Example 5.3.1: *Tour Operators*

Consider vacation operators which sell tour packages or vacation packages with many predetermined destinations. One of the key ingredients in these packages is the air fare. Most packages announce that they obtain much cheaper fares than the general fares, which implies that the vacation operators must have reached an agreement with a particular airline. From an airline point of view, these vacation operators create substantial revenue opportunities since:

- A successful operator usually offers different packages all year round, which are planned many months in advance;

- It is relatively predictable on the size of each package;
- An airline needs not to deal each individual traveller in each package;
- In most cases, air fares are the main part of the cost for each package, which implies that these operators who can strike good deals with some airlines will more likely be the successful ones;
- An airline can play a crucial role on flight schedule arrangement of the package if the airline indeed makes concessions on air fares; and
- Almost all vacationers who arrange their vacations through operators ***will not fly at an unrestricted fare***, which is usually much higher than an ordinary restricted fare.

Therefore, for these airlines which are willing to take that extra step to negotiate with vacation operators and are willing to give additional discounts on air fares, additional revenue can be generated. But the airline must address the following issues:

- Given the time condition by the vacation operator, the airline needs to tell them *which flight may be available*;
- If there are many flights that satisfy the time constraint and capacity constraint, then the airline can either give its own best choice on the basis of most revenue or negotiate with the vacation operator on a common choice; and
- On the other hand, if the airline cannot find an appropriate flight, then it is possible for the airline to suggest an alternative flight.

All these issues involve delicate work on pricing and inventory control. Our above model will be helpful in this regard. First, since the airline only deals with the vacation operator

with a clearly defined capacity requirement, our model at least gives *a bottom-line value for the total revenue* for this particular part of the capacity if these seats are offered to the operator. It will also give the airline a base for any negotiation with the operator.

Example 5.3.2: *Corporate Retreat Programs*

Many corporations have annual retreat programs to reward their high performance employees. Every year, the management will select a group of employees whose contributions are better than most of the other employees. The company will send this group to a special place for a couple of days of retreat. Typical examples include companies from the insurance, lodging and real estate industries. The main characteristics for these types of programs are:

- The company will take care of all expenses and make all the arrangements;
- Air fare, if the program decides to use air transportation, is a major part of the total expense, which is tax deductible;
- These programs are usually arranged for weekends; and
- Some top executives in the company usually need to show up at the program; and these executives are in fact *captives* of the unrestricted fare or the first class fare since they simply cannot make an early commitment.

Many large corporations use charter service for their retreat programs because it is flexible and convenient. But major airlines can also capture a large market share of this business. Airline managers who understand the impact of these programs on the capacity and revenue should be able to explore this market segment.

5.4 Cheaper Unrestricted Membership Fares Only

I now discuss the case where an airline considers only to offer lower unrestricted membership fares. There are many situations that lead to this type of practice. For example, an airline's employees can fly wherever and whenever they want to for free subject to the avaibility of seats. Most airlines also have an internal policy that will give an employee *a guaranteed seat* and all the flexibility if the employee *pays a small fraction of the unrestricted fare.* This is of course a result of its internal corporate policy. But nevertheless, it has a direct revenue impact. Another example may be a big consulting corporation which constantly has its consultants on different assignments all the time. These travellers require the maximum flexibility. As a result of this, most of them must pay the substantially higher unrestricted fares. On the other hand, as modern telecommunication technology becomes more advanced and much cheaper, many companies have cut their travel budget to cope with the high cost of air travel. From an airline's point of view, these travellers are part of the frequent travellers. And increasing these travellers' frequency will create substantial revenue impact for the airline. Technically speaking, an airline may consider negotiating a special travel deal with such a corporation under certain provisions, such as a minimum commitment on the total number of trips. The general feature for this type of membership demand structure is that there is no demand for restricted fares from members. Therefore, I can directly use the ME-model developed in Section 4.3.

But the case for internal travellers is slightly different and in fact quite interesting. It is well-known that some internal travellers, even those who can fly free, are willing to buy *an unrestricted ticket* at a price only lower that the (public) restricted fare in order to get a guaranteed seat.[2] In other words, we have the following:

[2]Of course, the majority of internal travellers are free-riders. But technically speaking, these travellers have no impact in airline's revenue. They also have no negative impact on airline's operation since these

- $D_i^{Mr} = 0$, or equivalently, $\gamma_{Mr,i} = 1$ for all i; that is, there are no internal travellers who are willing to pay an restricted fare even if it is very cheap; and

- A pricing policy is in the form of

$$\{(p_i, q_{M,i}) : i = 1, \cdots, m_1; (p_j, q_{r,j}) : j = m_1, \cdots, m; (p_k, q_k) : k = m, \cdots, n\},$$

  where $q_{M,i}$ is the allocation to members at the (unrestricted) price level p_i for $1 \leq i \leq m_1$, $q_{r,j}$ is the allocation to the general public at the restricted price p_j for $m_1 \leq j \leq m$, and q_k is the allocation to the general public at the unrestricted price p_k for $m \leq k \leq n$.

In summary of the above discussions, we have the following result.

Proposition 5.4.1: *If there is no demand for restricted tickets from the membership group and the airline is committed to offer unrestricted membership fares that are lower than restricted fares, then the airline's pricing problem can be formulated as follows:*

$$\text{Max} \quad \sum_{i=1}^{m_1} p_i q_{M,i} + \sum_{j=m_1}^{m} p_j q_{r,j} + \sum_{k=m}^{n} p_k q_k$$

s.t.

$$\beta_2 \equiv \sum_{i=1}^{m_1} \frac{q_{M,i}}{D_i^M} \leq 1$$

$$\beta_1 \equiv \sum_{j=m_1}^{m} \frac{q_{r,j}}{D_j^r} \leq 1$$

$$\sum_{k=m}^{n} \frac{q_k}{D_k - \beta_1 D_k^r - \beta_2 D_k^M} \leq 1$$

$$\sum_{i=1}^{m_1} q_{M,i} + \sum_{j=m_1}^{m} q_{r,j} + \sum_{k=m}^{n} q_k \leq q$$

$$q_{M,i}, q_{r,j}, q_k \geq 0, \forall i, \forall j, \text{ and } \forall k.$$

Proof: Straightforward. The details are omitted here. □

---

traveller are called *stand-bys*, which means that their seats are not guaranteed. In other words, they may be dumped during their trip. I here only focus on paying internal travellers.

Even though the membership restriction and the product restriction are *mutually exclusive* in this context, the airline must address *an additional constraint* that requires the airline to offer the members unrestricted fares that are lower than the restricted fares. In the above formulation, it is easy to make $\beta_1 = 1$. A natural condition for this is that $\frac{D_i^r}{D_i^N}$ is strictly decreasing, which says that for non-members, as the price increases, the percentage of those who are willing to purchase restricted tickets is strictly decreasing. Since the restriction is targeted to non-members, this condition is just a condition for the restriction to be effective, as argued in Chapter 2.

In order to obtain a simple optimal fare structure, the key is to make β_2 equal to one. But the problem is that *there is no natural conditions that generically guarantee* $\beta_2 = 1$. On the other hand, $\beta_2 = 1$ is in fact *a corporate policy* since the airline will give any employee a guaranteed unrestricted ticket if the employee pays a small fraction of the public unrestricted fare. This is good news for the above formulation, which can accomplish a simple optimal fare structure because it reduces to a linear programming problem. But it may be bad news to the airline since it may cause a revenue loss. For these flights with low demand from internal travellers, the above formulation will likely lead to an optimal policy that is consistent with an optimal solution derived directly from ME-model. But for some high demand flights or flights at busy travelling seasons such as Christmas, the commitment that $\beta_2 = 1$ may prove to be costly. Therefore, the above analysis calls for *close monitoring* on these flights and some modification may be necessary. One possible modification is to satisfy *a fixed percentage* of those who request a booking on a flight. There are several benefits for this provision. First, it will still reduce the above formulation to a linear programming, since β_2 is a fixed constant and β_1 can always be taken as one. Second, it is not hard to implement because it involves the airline's own employees. For example, it can be implemented through *a lottery draw by a computer* among those employees who wish to book the same flight. Or it can be

implemented according to the seniority of employees.[3] Another possible modification is *to ask internal travellers to pay for an unrestricted ticket at a restricted price.* The best possible modification is to implement an optimal fare structure derived from the ME-model, which may *force some internal travellers to pay for an unrestricted ticket at a price that is higher than the restricted fares.* In summary, the main point here is that strictly committing $\beta_2 = 1$ may cost an airline a lot of money.

5.5 Cheaper Restricted Membership Fares and Cheaper Unrestricted Membership Fares

In the above two sections, I have discussed several cases which indicate that it is sensible for an airline to offer either lower restricted membership fares *or* lower unrestricted membership fares. But in some situations, with enough information about the demand behaviour from the members, the airline may consider offering both lower restricted membership fares and lower unrestricted membership fares. This leads to the pricing problem with two general restrictions – one membership restriction and one product restriction. To see that we can obtain the same results as in Section 4.4, we might use the following notation: [4]

- we can call the product restriction a *type-1* restriction, which implies that *a type-1 restricted ticket is a (public) restricted ticket*;
- we can call the membership restriction a *type-2* restriction, which means *a type-2 restricted ticket is an unrestricted membership ticket*;

[3]Some airlines practice the rationing for the internal travelers in term of *service seniority*, which means that these who are with the airline longer will have the first priority of obtaining confirmed seats.

[4]But, in fact, I will avoid these generic notation in our discussions that follow. I present them here just for the sake of consistency with the discussions in Section 4.4.

- we can call a restricted membership ticket a *type-3* restricted unit since if it is attached with both type-1 and type-2 restrictions.

It is clear that for the case here, the airline has the flexibility of offering four types of fares: restricted membership fares, (public) restricted fares, (unrestricted) membership fares, and (public) unrestricted fares. As argued at the beginning of this section, restricted membership fares, if offered, should not be higher than the public restricted fares; and similarly, the unrestricted membership fares, if offered, also should not be higher than the public unrestricted fares. Clearly, the airline should always consider selling restricted membership fares first, since they are most restricted among these four proposed fares. From Theorem 4.4.1 of Section 4.4 it follows that if *the membership deal between the airline and the members* requires that an airline *must give a guaranteed booking for a restricted ticket from any member who requests for it*, then the airline's pricing problem can be formulated as a simple linear programming, which shows that the airline needs *at most five fare levels* with four different types of ticket. Rephrasing Theorem 4.4.1 in our context, we have the following proposition:

Proposition 5.5.1: *Suppose that an airline plans to use a membership restriction and a product restriction. Assume that*

- *both* $\frac{\gamma_{M,j}}{\gamma_{Mr,j}}$ *and* $\frac{\gamma_{r,j}}{\gamma_{Mr,j}}$ *are strictly increasing in* j,
- *the airline will satisfy all bookings for restricted membership fares.*

Then the airline's pricing problem can be formulated as the following l.p.:

$$Max \quad \sum_{i=1}^{m_1} p_i q_{Mr,i} + \sum_{j=m_1}^{m} p_j(q_{r,j} + q_{M,j}) + \sum_{k=m}^{n} p_k q_k$$

$$s.t.$$

$$\sum_{i=1}^{m_1} \frac{q_{Mr,i}}{D_i^{Mr}} = 1$$

$$\sum_{j=m_1}^{m} \frac{q_{r,j}}{D_j^r - D_j^{Mr}} = 1$$

$$\sum_{j=m_1}^{m} \frac{q_{M,j}}{D_j^M - D_j^{Mr}} = 1$$

$$\sum_{k=m}^{n} \frac{q_k}{D_k - (D_k^r - D_k^{Mr}) - (D_k^M - D_k^{Mr})} = 1$$

$$\sum_{i=1}^{m_1} q_{Mr,i} + \sum_{j=m_1}^{m} (q_{r,j} + q_{M,j}) + \sum_{k=m}^{n} q_k \leq q$$

$$q_{Mr,i}, q_{r,j}, q_{M,j} \geq 0 \text{ and } q_k \geq 0,$$

where $q_{Mr,i}$ is the allocation to the members at the restricted price p_i, $q_{r,j}$ is the allocation to the general public at the restricted price p_j, $q_{M,j}$ is the allocation to the members at the unrestricted price p_j and q_k is the allocation to the general public at the unrestricted price p_k.

Let us analyze the two assumptions used in the above proposition. First, the condition that $\frac{\gamma_{M,j}}{\gamma_{Mr,j}}$ and $\frac{\gamma_{r,j}}{\gamma_{Mr,j}}$ are strictly increasing in j is equivalent to requiring that

- both $\frac{D_i^M - D_i^{Mr}}{D_i - D_i^{Mr}}$ and $\frac{D_i^r - D_i^{Mr}}{D_i - D_i^{Mr}}$ are strictly decreasing,

which implies that *in the residual market where no members will buy restricted tickets, as price increases, the percentage of members is strictly decreasing and the percentage of consumers who can buy restricted tickets is also strictly decreasing.* In essence, these conditions are *conditions of effectiveness* for both membership restriction and product restriction. They together provide sufficient information for an airline to decide whether or not to offer cheaper membership (unrestricted) fares and (public) restricted fares.

The second assumption is in fact *a corporate commitment* from the airline. It may involve issues other than pricing. A good example is a bilateral agreement between two *partner airlines*,[5] where each airline gives a special treatment to its partner's employees who use its passenger service. If an airline's partner has such a commitment, this airline

[5]Partner airlines usually involve agreements on code-sharing and traffic-feeding.

may have no choice but to give its partner's employees the same treatment. Our model shows that the airline will have a very simple optimal fare structure. On other hand, purely from revenue maximization point of view, such a commitment may have a negative impact on revenue. So it is really important for airline managers to understand that the best result is such that *the commitment is consistent with market behaviour.* Theorem 4.4.2 provides some additional conditions which guarantee that such a commitment will not cause any revenue loss. For this, I have the following proposition:

Proposition 5.5.2: *Suppose that an airline plans to use a membership restriction and a product restriction. Assume that*

- *both* $\frac{\gamma_{M,j}}{\gamma_{Mr,j}}$ *and* $\frac{\gamma_{r,j}}{\gamma_{Mr,j}}$ *are strictly increasing in* j.

Then

1. *if* $\frac{\gamma_{Mr,j}-\gamma_{M,j}}{\gamma_{Mr,j}-\gamma_{r,j}}$ *is strictly increasing in* j*, or equivalently, in the residual market where there is no members who can still buy restricted tickets, the relative size of the market for unrestricted tickets from members and the market for restricted tickets from general public is strictly increasing, the airline has an optimal fare structure having the following properties:*

   - *the airline's commitment of satisfying all bookings from members for restricted tickets is consistent with revenue maximizing;*
   - *the (unrestricted) membership fares will not be lower than the (public) restricted fares;*

2. *if* $\frac{\gamma_{Mr,j}-\gamma_{r,j}}{\gamma_{Mr,j}-\gamma_{M,j}}$ *is strictly increasing in* j*, or equivalently, in the residual market where there is no members who can still buy restricted tickets, the relative size of the market for restricted tickets and the market for unrestricted tickets from members*

is strictly increasing, the airline has an optimal fare structure having the following properties:

- *the airline's commitment of satisfying all bookings from members for restricted tickets is consistent with revenue maximizing;*
- *the (unrestricted) membership fares will not be higher than the (public) restricted fares.*

Furthermore, in both cases, the airline needs to offer no more than five different fares.

Proof: It follows from Theorem 4.4.2 and its proof. □

Again, let us look into the implications of the assumptions used in each of the two cases. Note that

$$\frac{\gamma_{Mr,j} - \gamma_{M,j}}{\gamma_{Mr,j} - \gamma_{r,j}} = \frac{D_j^M - D_j^{Mr}}{D_j^r - D_j^{Mr}}.$$

So intuitively speaking, if ignoring the group that consists of members who can buy restricted tickets, the first case says that the market demand for restricted tickets is *decreasing faster than* the market demand for unrestricted tickets from members; and the second case is vice versa.[6] With this in mind, it should not be surprising to see that in the first case, the airline will offer (unrestricted) membership fares *between the (public) restricted fares and the (public) unrestricted fares.* Similar argument can be made for the second case too. This is clearly consistent with the traditional idea of economic pricing discrimination: the firm should charge a higher price to the segment that is less sensitive to prices than to other segments.

I now give some examples that fit into the model. First, consider the market that consists of travellers from a partner airline. These travellers must pay for their tickets if

[6]Interestingly, I can easily state these assumptions in term of *price elasticities* as used in Section 5 of BL (1993a).

they want to fly on a flight from the partner airline. On the other hand, because of their affiliation with an airline and its internal benefit of free flying, these travellers are much more price sensitive than the general public for both restricted fares and unrestricted fares. I feel that this situation fits nicely into the second case in the above proposition.

Another example are government. It is well-known that there is an enormous amount of travelling activities from employees of government or government agencies. Some of this travelling can be planned in advance, but there is a substantial portion of it that cannot be planned ahead. In this case, an airline can give these clients a break on both restricted fares and unrestricted fares. It is evident that the government will be pleased to let its employees fly with unrestricted fare at a price that is between the public restricted fares and public unrestricted fares. It is a special treatment that may induce more travelling from government employees. In my view, the first case in the above proposition will be helpful for airlines to negotiate acceptable and rational fare prices with the government.

5.6 Summary – Membership and Product Restrictions

In the above three sections I have illustrated how the pricing models for two types of restrictions can be used by airlines when product restriction and membership restrictions are used at the same time. Unlike product restrictions, which are fairly difficult to design, membership restrictions are relatively easy to plan and execute. In my view, the use of membership restrictions will further enhance an airline's revenues. On the other hand, addressing all possible membership restrictions at the same time could be counter-productive because it is hard to monitor the interactions among many different member groups, in addition to the product restriction. A simple solution to this problem is to develop *a heuristic procedure* that has the following features:

- it can model the impact of each individual membership restriction on each flight;
- the airline can place a flag on a flight showing which member group(s) can book this flight; and
- the computer reservation system should be automated so that ordinary travel agents can authorize bookings for members.

The key ingredient here is *the tagging of flights.* As an example, for internal travelers, it may be prudent for airlines to limit employees on personal trips to off-peak flights only. To tag a flight several months ahead is not an easy task, but with more advanced forecasting tools, these pricing models in Chapter 4 will be helpful since

- they give optimal allocation plans in the short-run, which can help the airline tag these flights that are available for bookings from certain members;
- they specify the operating environment for an effective and rational fare structure; and
- they are consistent with inventory control tools used in yield management systems.

In summary, to enable an airline to take the full advantage of these revenue opportunities, the airline needs to have a much broader approach than what they do at present, since each of these opportunities represents a *market niche.* The airline must have a *systematic management tool* that is useful to specifically explore this potential source of revenues. A critical part of such a management system is the information system. The computer reservation system needs to be enhanced so that it is capable of providing real time recommendations to managers.

Chapter 6

Seat Allocation Game on Flights with Two fares

6.1 Introduction and Model Setting

Consider international flights between two countries which have a bilateral agreement, and where each carrier offers a few flights per week, such as a daily flight. Such a low frequency of flights between two destinations will make the issue of competition in the allocation decisions very important for the sake of market share and profit.

For example, Canadian Airlines International (CAI) and Japan Airlines (JAL) are sole carriers offering *direct* service between Vancouver and Tokyo. And both CAI and JAL offers daily flights. The schedules of both airlines are very close and their fares are actually identical. In particular, both airlines have scheduled flights on Saturdays and Sundays, which indicates that both airlines are competing heavily in the business market. The strategic interaction between two airlines involves the following decisions:

1. *how many discounts seats will be available?*

2. *what should an airline do when the other carrier stops selling discount tickets?*

In this chapter, I will discuss the seat allocation problem when each of two airlines is operating a single-leg flight with two given fares – a full fare and a discount fare. I will use the following notation:[1]

- The discount fare is ρ_B and the full fare is ρ_Y;

[1]The notation in this chapter is self-contained.

- Airline k's flight has a capacity of C_k, for $k = 1, 2$;
- The market demand for discount fare is given by B and the market demand for full fare is given by Y. B and Y are not assumed to be independent.
- Each airline chooses a *booking limit* for the discount fare as its decision variable and its objective is to achieve the highest revenue possible.

Let l_k be the booking limit for the discount fare set by airline k, $k = 1, 2$.[2] Since airline k's expected revenue is determined by the joint decision (l_k, l_j), we can write airline k's expected revenue function as $r_k(l_k, l_j)$ for $k = 1, 2$. More specifically speaking, if we let $B_k(l_k, l_j)$ be the airline k's *demand share for the discount fare* and $Y_k(l_k, l_j)$ be the airline k's *demand share for the full fare*, then the expected revenue function $r_k(l_k, l_j)$ is given by:

$$\begin{aligned} r_k(l_k, l_j) &= \rho_B E(B_k(l_k, l_j) \wedge l_k) + \rho_Y E(Y_k(l_k, l_j) \wedge [C_k - (B_k(l_k, l_j) \wedge l_k)]) \quad (6.1) \\ &\equiv E(\tilde{r}_k(l_k, l_j)); \end{aligned}$$

where E represents the *expectation* and $\tilde{r}_k(l_k, l_j)$ is the airline k's *random revenue* derived from the pair of booking limits (l_k, l_j). Therefore, this leads to a simple two-person game with *payoff functions* r_1 and r_2. With this in mind, we give the following definition of an equilibrium pair of booking limits.

Definition 6.1.1: *A pair of booking limits* (l_1^*, l_2^*) *is said to be an* **equilibrium pair of booking limits** *if for* $k = 1, 2$

$$r_k(l_k^*, l_j^*) \geq r_k(l_k, l_j^*) \text{ for all } l_k = 0, 1, \cdots, C_k. \quad (6.2)$$

[2]From now on, I will use k and j to identify the airlines. By convention, I always assume that $k \neq j$.

Note that there is no guarantee of the *existence* of an equilibrium pair of booking limits since it is of the *pure strategy* form. To extend this concept, I introduce the notion of *mixed strategy.* First of all, airline k's *pure strategy space* is given by

$$S_k = \{0, 1, \cdots, C_k\}, \text{ for } k = 1, 2.$$

And a *booking strategy* for airline k is a *probability distribution function* on S_k. Let F_k be the collection of all possible booking strategies for airline k. By definition, for any $\alpha_k \in F_k$, we have that $\sum_{i \in S_k} \alpha_k(l) = 1$, where $\alpha_k(l) \geq 0, \forall l \in S_k$. Now for any given booking strategy pair (α_k, α_j), define

$$R_k(\alpha_k, \alpha_j) = \sum_{l_k \in S_k} \sum_{l_j \in S_j} \alpha_k(l_k)\alpha_j(l_j) r_k(l_k, l_j).$$

I will call the game characterized by $(R_1, F_1; R_2, F_2)$ as the *seat allocation game.* I now give the following definition of an equilibrium booking strategy, which provides *a solution* for our seat allocation game.

Definition 6.1.2: An equilibrium booking strategy *is a pair* (α_1^*, α_2^*)*:* $\alpha_k^* \in F_k$ *for* $k = 1, 2$*, such that*

$$R_k(\alpha_k^*, \alpha_j^*) \geq R_k(\alpha_k, \alpha_j^*) \text{ for all } \alpha_k \in F_k.$$

Since the pure strategy spaces for both airlines (the players) are discrete and finite, our two-person game belongs to a special class of game, the so-called *bimatrix games.* By definition, a bimatrix game is a two-person non-zero sum game where each player has a finite number of pure strategies. For this type of games, the existence of an equilibrium booking strategy is guaranteed by the following famous theorem of Nash (1951, [163]):

Nash Theorem: *Every bimatrix game has at least one Nash equilibrium if we admit mixed strategy equilibrium as well as pure strategy equilibrium.*

It follows immediately from Definition 6.1.2 that our equilibrium booking strategies are Nash equilibria. Therefore, by the Nash Theorem, we always have at least one equilibrium booking strategy. In this section, my primary interest is to find an equilibrium pair of booking limits, that is, I want to characterize the pure strategy equilibria for the seat allocation game if they indeed exist.

So far, I have not discussed how to obtain the *derived revenue functions* r_1 and r_2. Without properly specifying these two functions, the seat allocation game is not even well defined. Since the two airlines face the same market demands (both of the full fare market and discount fare market), *the specification of the revenue functions are critically related to the way two airlines share market demands.* In this chapter, I will explore the seat allocation game under two types of market splitting rules: *the proportional splitting rule* and *the equal splitting rule.* By the proportional splitting rule, I mean that in the event that the total commitment for a certain fare class from two airlines exceeds the market demand, then the two airlines will split the market demand according to their proportions of the total commitment.[3] By the equal splitting rule, I mean that in the event that the total commitment for a certain fare class from two airlines exceeds the market demand, each airline will get a half of the market demand or reach its commitment level, whichever is less.[4]

This chapter is organized as follows. Section 6.2 investigates the seat allocation game when the proportional splitting rule is used to define each airline's derived revenue function. Section 6.3 studies the seat allocation game under the equal splitting rule with deterministic demands. And Section 6.4 is a summary.

[3]In fact, the intuition behind the proportional splitting rule comes from the *capacity share* or the *market share* consideration.

[4]In some sense, the equal splitting rule is build upon the premise that consumers are absolutely indifferent between the two carriers.

6.2 Seat Allocation Game Under Proportional Splitting Rule

Under the proportional splitting rule, we know that given a pair of booking limits (l_k, l_j), airline k's demand share for the discount fare is given by

$$B_k(l_k, l_j) = \frac{l_k}{l_k + l_j} B, \text{ for } k = 1, 2. \tag{6.3}$$

And the *actual sales* for airline k's discount fare should be the minimum of its booking limit and its demand share, that is,

$$s_k(l_k, l_j) = B_k(l_k, l_j) \wedge l_k = l_k(1 \wedge \frac{B}{l_k + l_j}). \tag{6.4}$$

After the booking for the discount fare is closed, airline k has a *residual capacity* of

$$L_k = C_k - s_k(l_k, l_j). \tag{6.5}$$

Consequently, airline k's demand share for the full fare, again according to the proportional splitting rule, now becomes

$$Y_k(l_k, l_j) = \frac{L_k}{L_k + L_j} Y. \tag{6.6}$$

Consequently, airline k's expected revenue function r_k is given by [5]

$$r_k(l_k, l_j) = \rho_B E(B_k \wedge l_k) + \rho_Y E(Y_k \wedge L_k) \equiv E(\tilde{r}_k(l_k, l_j)), \tag{6.7}$$

where $\tilde{r}(l_k, l_j)$ is the *random revenue* associated with the pair of booking limits (l_k, l_j).

By (6.3), (6.4) and (6.5), it is straightforward to verify the following facts:

- $s_k = l_k(1 \wedge (B/l))$;
- $s_k + s_j = B \wedge l$;
- $L_k + L_j = (C - (l \wedge B)$.

[5] For clarity of presentation, I will use B_k and Y_k to denote $B_k(l_k, l_j)$ and $Y_k(l_k, l_j)$ respectively.

We are now going to use the technique of Brumelle et al (1990, [37]) to look for any equilibrium pair of booking limits. Given that airline j's booking limit is fixed at l_j and that airline k has accepted $l_k - 1$ requests for its discount fare and there is an additional request for its discount fare, then airline k must decide to accept or to reject this particular request for the discount ticket. If airline k decides to reject it, its expected revenue is given by

$$E(\tilde{r}_k(l_k - 1, l_j)|B_k \geq l_k);$$

and if airline k decides to accept the request, its expected revenue becomes

$$E(\tilde{r}_k(l_k, l_j)|B_k \geq l_k).$$

The key is the revenue difference between these two decisions. For this, define the ***direct incremental gain function*** $G_k(l_k, l_j)$ for airline k as follows:

$$G_k(l_k, l_j) = E[\tilde{r}_k(l_k, l_j)|B_k \geq l_k] - E[\tilde{r}_k(l_k - 1, l_j)|B_k \geq l_k].$$

The following lemma gives a simple expression for the direct incremental gain functions.

Lemma 6.2.1: *For $k = 1, 2$,*

$$\begin{aligned} G_k(l_k, l_j) = \; & \rho_B - \rho_Y P(Y > C - l|B > l) - \rho_Y \tfrac{C_j - l_j}{(C-l)(C-l+1)} \times \\ & E(Y|Y \leq C - l, B \geq l)P(Y \leq C - l|B \geq l), \end{aligned} \tag{6.8}$$

where $C \equiv C_1 + C_2$ and $l \equiv l_1 + l_2$.

Proof: First of all, it is easy to check that

$$B_k \geq l_k \Longleftrightarrow B \geq l \Longleftrightarrow B_j \geq l_j.$$

Therefore,

$$G_k(l_k, l_j) \;=\; \rho_B + \rho_Y \frac{C_k - l_k}{C - l} E[Y \wedge (C - l)|B \geq l] +$$

$$
\begin{aligned}
& -\rho_Y \frac{C_k - l_k + 1}{C - l + 1} \times E[Y \wedge (C - l + 1) | B \geq l] \\
= \; & \rho_B + \rho_Y \frac{C_k - l_k}{C - l} \times \\
& \{E[Y \wedge (C - l) | Y > C - l, B \geq l] Pr(Y > C - l | B \geq l) + \\
& + E[Y \wedge (C - l) | Y \leq C - l, B \geq l] Pr(Y \leq C - l | B \geq l)\} + \\
& -\rho_Y \frac{C_k - l_k + 1}{C - l + 1} \times \\
& \{E[Y \wedge (C - l + 1) | Y > C - l, B \geq l] Pr(Y > C - l | B \geq l) + \\
& + E[Y \wedge (C - l + 1) | Y \leq C - l, B \geq l] Pr(Y \leq C - l | B \geq l)\} \\
= \; & \rho_B - \rho_Y Pr(Y > C - l | B > l) + \\
& -\rho_Y \frac{C_j - l_j}{(C - l)(C - l + 1)} E[Y | Y \leq C - l, B \geq l] Pr[Y \leq C - l | B \geq l].
\end{aligned}
$$

This proves the lemma. □

On the other hand, it is important to notice that if airline k declines a request for the discount fare, the passenger will make the same request to airline j. Airline j may or may not accept the request. The above direct incremental gain function is calculated under the assumption that airline j's decision is *fixed* at l_j. If airline j accepts the request, that is, airline j changes its initial booking limit for the discount fare from l_j to $l_j + 1$, then airline k must take into account the impact of this action by airline j on its revenue. Because of this, we introduce the notion of *indirect incremental gain function*:

$$g_k(l_k, l_j) = E[\tilde{r}_k(l_k, l_j) | B_k \geq l_k] - E[\tilde{r}_k(l_k - 1, l_j + 1) | B_k \geq l_k].$$

The following lemma gives us a simple formula for the indirect incremental gain function.

Lemma 6.2.2: *For* $k = 1, 2$,

$$g_k(l_k, l_j) = \rho_B - \rho_Y E(\frac{Y}{C - l} \wedge 1 | B \geq l).$$

Proof: By the definition of indirect incremental gain function, it follows that

$$\begin{aligned} g_j(l_k, l_j) &= E[\tilde{r}(l_k, l_j)|B_k \geq l_k] - E[\tilde{r}(l_k - 1, l_j + 1)|B \geq l_k] \\ &= [l_k\rho_B - \rho_Y \frac{C_k - l_k}{C - l} E(Y \wedge (C - l)|B \geq l)] + \\ &\quad -[(l_k - 1)\rho_B + \rho_Y \frac{C_k - l_k + 1}{C - l} E(Y \wedge (C - l)|B \geq l)] \\ &= \rho_B - \rho_Y \frac{1}{C - l} E(Y \wedge (C - l)|B \geq l), \end{aligned}$$

which proves the lemma. □

It is interesting to see that Lemma 6.2.2 leads to the following facts:

- $g_1(l_1, l_2) = g_2(l_2, l_1)$;
- $g_k(l_k, l_j)$ *is decreasing in* $l = l_1 + l_2$, *and consequently, is decreasing in* l_k *for any given* l_j, and vice versa.

On the other hand, it is easy to check that that

$$\begin{aligned} g_k(l_k, l_j) &= \rho_B - \rho_Y P(Y > C - l|B \geq l) \\ &\quad -\rho_Y \frac{1}{C - l} E(Y|Y \leq C - l, B \geq l) P(Y \leq C - l|B \geq l), \end{aligned}$$

which, together with (6.8), leads to

$$G_k(l_k, l_j) > g_k(l_k, l_j),$$

because $\frac{C_j - l_j}{C - l + 1} < 1$. Now because of this inequality, each airline will only focus on its direct incremental gain function. But it is not clear whether or not the direct incremental gain function G_k is decreasing in l_k for any given l_j. If G_k is indeed decreasing in l_k for any given l_j, then we can use each airline's *response function* to characterize an equilibrium pair of booking limits. In fact, we have the following proposition:

Proposition 6.2.3 *If for $k = 1, 2$, the gain function $G_k(l_k, l_j)$ is deceasing in l_k for any given l_j, there exists an equilibrium pair of booking limits (l_1^*, l_2^*) such that*

$$l_k^* = \max\{l_k : G_k(l_k, l_j^*) \geq 0\}. \tag{6.9}$$

Proof: Because $G_k(l_k, l_j)$ is decreasing in l_k for any given l_j, I can define airline k's *response function* to airline j's choice l_j as follows:

$$\begin{aligned}
\eta_k(l_j) &= \max\{l_k \in S_k : G_k(l_k, l_j) \geq 0\} \\
&= \max\{l_k \in S_k : P(Y > C - l | B \geq l) + \\
&\quad + \frac{C_j - l_j}{(C-l)(C-l+1)} E(Y | Y \leq C - l, B \geq l) P(Y \leq C - l | B \geq l) \leq \frac{\rho_B}{\rho_Y}\}.
\end{aligned}$$

First, it is clear that η_k is well-defined. To finish the proof, it suffices to show that, for l_1^* and l_2^* given by (6.9), I will have

$$\eta_k(l_j^*) = l_k^*, \text{ for } k = 1, 2. \tag{6.10}$$

Note that (6.8) and (6.9) imply that each airline *will protect the same number of seats for the high fare*, that is,

$$p^* \equiv C_1 - l_1^* = C_2 - l_2^*.$$

Upon agreeing on this, both airlines will face the *exactly same* decision on how to choose their booking limit for the discount fare so that there is no further possible revenue gain by allocating additional seats for the discount fare. This is in fact captured by the response function. Since in the end neither airline can unilaterally improve its revenue, the pair of booking limits (l_1^*, l_2^*) must be an equilibrium. This proves the result. □

As an immediate consequence of the above proposition, I get the following interesting corollary:

Corollary 6.2.4: *Under the same assumption as in Proposition 6.2.3, each airline will protect the same number of seats for the high fare at equilibrium.*

Proof: It follows directly from (6.8) and (6.9). □

The following corollary indicates that competition will collectively reduce the total number of seats that will be made available for the low fare.

Corollary 6.2.5: *Under the same assumption as in Proposition 6.2.3, at the equilibrium, the total number of seats available for the low fare class is less than the total number of seats that will be available for the low fare if the airlines completely cooperate.*

Proof: If the airlines completely cooperate, then they together will act as a monopoly. Then it follows from Brumelle et al (1990, [37]) that the optimal booking limit η^* for the low fare is given by:

$$\eta_* = \max\{0 \leq \eta \leq C : P(Y > C - \eta | B \geq \eta) \leq \frac{\rho_B}{\rho_Y}\}.$$

So I need to show that

$$l^* \equiv l_1^* + l_2^* < \eta_*. \tag{6.11}$$

Again, this is obvious since by Proposition 6.2.3 I know that

$$\begin{aligned} l_k^* &= \text{Max}\{l_k \in S_k : P(Y > C - l_k - l_j^* | B \geq l_k + l_j^*) + \frac{C_j - l_j^*}{(C - l_k - l_j^*)(C - l_k - l_j^* + 1)} \times \\ &\quad E(Y | Y \leq C - l_k - l_j^*, B \geq l_k + l_j^*) P(Y \leq C - l_k - l_j^* | B \geq l_k + l_j^*) \leq \frac{\rho_B}{\rho_Y}\}, \end{aligned}$$

which implies that

$$\begin{aligned} P(Y > C - l^* | B \geq l^*) \;+\; & \frac{C_j - l_j^*}{(C - l^*)(C - l^* + 1)} \times \\ & E(Y | Y \leq C - l^*, B \geq l^*) P(Y \leq C - l^* | B \geq l^*) \leq \frac{\rho_B}{\rho_Y}. \end{aligned}$$

Therefore, I must have

$$P(Y > C - l^* | B \geq l^*) \leq \frac{\rho_B}{\rho_Y}.$$

Then by definition of η^* (6.11) must be true. This proves the corollary. □

As a last comment for this section, I should say that the seat allocation game under the proportional splitting rule is not completely solved yet in the sense that we still need a complete characterization of the whole set of equilibrium booking strategies, rather than one equilibrium pair of booking limits only. On the other hand, the above results are indeed quite encouraging.

6.3 Seat Allocation Game Under Equal Splitting Rule with Deterministic Demand

In this section I will look at the seat allocation game from a different perspective. After many unsuccessful attempts to find equilibrium solutions to the seat allocation game under the equal splitting rule with uncertain demands, I am here only able to report some preliminary analysis for the case that both B and Y are in fact deterministic, that is, they are fixed constants. So the main purpose of this section is to see how the use of the equal splitting rule changes the way we handle the seat allocation game.

The main question faced by each airline is: *how many seats should each airline protect for its full fare?*[6] By definition, a *protection level* for full fare by each airline must be a *commitment* in the following sense: if p_k is the protection level, then airline k will *only* sell at most $C_k - p_k$ discount tickets even knowing that it can not sell p_k full fare tickets.

Let me now motivate the idea of *the equal splitting rule.* Given a pair of protection levels (p_1, p_2), airline k will only sell $C_k - p_k$ low fare tickets. Once the discount fare is

[6] I deliberately switch from booking limit to protection level. They are technically equivalent.

closed, both airlines start to sell the full fare. First of all, if $p_1 + p_2 \leq Y$, then each airline is able to sell the full fare tickets up to its respective protection level. Now consider that $p_1 + p_2 > Y$, that is, the joint commitment for full fare is too high. If $p_1 > Y/2$ and $p_2 > Y/2$, then we should *expect* that each airline will be able to sell $Y/2$ full fare tickets if travellers have *no preference over airlines* and travellers *arrive at a random order.* As a consequence of this, neither airline can sell enough seats to exhaust its commitment. If $p_1 \leq Y/2$ and $p_1 + p_2 > Y$, then airline 1 will sell p_1 discount tickets and airline 2 will sell $Y - p_1$ discount fares. Note that in this case airline 2 is unable to sell up to its protections level since $p_2 > Y - p_1$. The essence of this type of demand sharing is that both airlines will *share the market with a certain degree of equality*, or equivalently, they will *split the market evenly* whenever it is possible.Summarizing the above discussion, I have the following:

- If an airline's protection level is not greater than *half of the full market*, it will sell up to its protection level regardless of the other airline's commitment;
- If an airline's protection level is greater than the half of the full fare market, it will capture at least half of the market.

Now let $Y_k(p_k, p_j)$ be the airline k's share of the full fare demand, where $k = 1, 2$. Then the above motivation leads to the following formal specification: [7]

$$Y_k(p_k, p_j) = \begin{cases} Y - p_j & \text{if } p_k + p_j > Y,\ p_k > Y/2,\ p_j \leq Y/2; \\ \frac{Y}{2} & \text{if } p_k > Y/2,\ p_j > Y/2; \\ p_k & \text{otherwise.} \end{cases}$$

[7] Again, I will use convention that $k, j = 1, 2$ and $k \neq j$.

For the demand for low fare, denote $l_k = C_k - p_k$ for $k = 1, 2$. Then similarly, I will have

$$B_k(p_k, p_j) = \begin{cases} B - l_j & \text{if } l_k + l_j > B,\ l_k > B/2,\ l_j \leq B/2; \\ \frac{B}{2} & \text{if } l_k > B/2,\ l_j > B/2; \\ l_k & \text{otherwise.} \end{cases}$$

Since under the above splitting rule, the the number of sales for each fare class for each airline is in fact equal to the actual share of the demand, airline k's revenue function $r_k(p_k, p_j)$ is given by

$$r_k(p_k, p_j) = \rho_B B_k(p_k, p_j) + \rho_Y Y_k(p_k, p_j).$$

Since the goal is to find an equilibrium strategy for each airline, for completeness I give the following definition:

Definition 6.3.1: *A pair of protection levels (p_1^*, p_2^*) is said to be an equilibrium for the seat allocation game if*

$$r_k(p_k^*, p_j^*) \geq r_k(p_k, p_j^*), \text{ for all } p_k = 0, 1, \cdots, C_k \text{ and } k = 1, 2.$$

The following proposition gives a characterization for an equilibrium strategy for airlines when the demands are deterministic.

Proposition 6.3.1: *Under the deterministic demands, $(Y/2, Y/2)$ is always an equilibrium pair of protection levels if $C_k \geq Y/2$ for $k = 1, 2$..*

Proof: To prove that $(Y/2, Y/2)$ is an equilibrium, we need to prove that neither airline has any strict incentive to *deviate*.

Case 1: $C_1 + C_2 > B + Y$.

In this case, the joint capacity exceeds the total demand of the discount fare and the full fare. Given $p_k = Y/2$, airline k is guaranteed sales of $Y/2$ units of the full fare tickets *regardless what is the airline j's commitment for the full fare tickets.* Even though there may have many equally good responses from airline j, $p_j = Y/2$ is always one of these responses. As a consequence of this, each airline will also sell $B/2$ discount fare tickets.[8]

Case 2: $C_1 + C_2 \leq B + Y$.

Since $C_k \geq Y/2$, each airline can sell at least $Y/2$ at the full fare, which implies that at the equilibrium, for $k = 1, 2$,

$$p_k^* \geq Y/2.$$

Once $p_j^* = Y/2$, the airline k's best possible result on the sales of the full fare is also $Y/2$. Note the if $p_k > Y/2$, then two things will happen: (1) airline k can still only sell $Y/2$ full fare tickets; and (2) since a protection level is a commitment, airline k only allocate $C_k - p_k$ number of seats for the discount fare, which may reduce the chance to obtain its fair demand share for the discount fare. Therefore $p_k^* = Y/2$ must be one of the best choices. This proves that no airline can strictly improve its revenue by unilaterally deviating from $(Y/2, Y/2)$. This proves that the proposed pair of protection levels must be an equilibrium. □.

Remark: Technically speaking, the above proposition is not very interesting. One plausible aspect is the fact that both airlines will make the same commitment for the full fare class, which is consistent with the findings in the previous section. It remains to be seen whether this property is still true when the demands are actually random.

[8]This implies that when total capacity level is too high, the larger carrier has no clear advantage in terms of sales.

6.4 Summary – Seat Allocation Game

This chapter discusses the seat allocation problem in the presence of another airline. It is shown that under the proportional splitting rule, there exists an equilibrium booking policy such that each airline will protect the same number of seats for the full fare. I also demonstrate that at equilibrium the total number of seats that are available for the discount fare is smaller than the total number of seats that would be available if the two airlines cooperate. Under the equal splitting rule, it is shown that if the demands are deterministic, there is an equilibrium such that each airline will protect enough seats for high fare so that each airline will split the market demand for the high fare equally.

Chapter 7

A Note On Three Models for Multi-fare Seat Allocation Problem

7.1 Introduction

As we can see from the discussions in Chapter 1, there are many interesting problems in pricing and yield management that still need answers. The purpose of this chapter is to address the seat allocation problem for single-leg flights with multiple fares with independent random demands. There are three different models to solve the seat allocation problem for multi-fare flights, done independently by Curry (1990, [52]), Wollmer (1992, [269]), and Brumelle and McGill (1993, [38]).[1] After briefly reviewing these three models, I will present a proposition showing that their optimality conditions are analytically equivalent, which implies that none of them has computational advantage over others, as misleadingly claimed by Curry and Wollmer.

Throughout this chapter, I will use the following standard notation: [2]

- n is the total number of fare classes for a single-leg flight;
- f_k is the average revenue of a fare class k ticket, where $f_1 > f_2 > \cdots > f_n$;
- X_k is the random demand for fare class k;
- p_k is the *protection level* for fare classes $k, k-1, \cdots, 1$;

[1] The time difference on publication for three papers is not important since all three papers have been circulating for several years before they are published. In fact, all of them announced their findings in 1988, refer to Wollmer (1988, [268]), McGill (1988, [143]) and Curry (1988, [50]).

[2] All notation in this chapter is also self-contained, which bears no relationship with any other notation used in other part of this thesis.

- C is the capacity of the flight.

The following assumptions are used in three models:

- Low fares are booked first;
- Those fares, once closed, will not reopen;
- There are no cancellations and no no-shows; and consequently, there is no overbooking; and
- Random variables $X_1, X_2, \cdots, X_n$ are *independent.*

This chapter is organized as follows. Section 7.2 presents the seat allocation model due to Wollmer (1992, [269]), where the demands are assumed to be independent discrete random variables. Section 7.3 discusses the model due to Curry (1990, [52]) which addresses the problem for independent continuous random demands. Section 7.4 analyzes the model due to Brumelle and McGill (1993, [38]) which solves the seat allocation problem for *any* independent random demands. Section 7.5 proves an equivalence result for the optimality conditions from three models. And the last section is a summary.

7.2 Wollmer's Model

Wollmer (1988, [268]; 1992, [269]) studied the seat allocation problem for multi-fare flights by considering that the demands are *discrete random variables.* By formulating the seat allocation problem into a *Markov decision problem*, Wollmer derives optimality conditions that are natural extensions of Littlewood's optimal condition for two-fare flights with independent demands.

Here is a sketch of Wollmer's model. Let $R_k(s)$ be the expected revenue under an optimal policy if s seats are available for booking when fare class k is the lowest fare that

is open.[3] Define,

$$\triangle R_k(s) = R_k(s) - R_k(s-1),$$

which is the *incremental value of an additional seat.* By using simple facts on conditional probabilities, it is easy to derive the following recursive relationships:

$$R_1(s) = f_1 E(X_1 \wedge s), \triangle R_1(s) = f_1 P(X_1 > s);$$

and

$$\begin{aligned} R_k(p_{k-1}+j) &= R_{k-1}(p_{k-1}) + \sum_{i=0}^{j-1}[\triangle R_{k-1}(p_{k-1}+j-i)P(X_k \le i) + f_k P(X_k \ge i+1)] \\ \triangle R_k(p_{k-1}+j) &= \sum_{i=0}^{j-1} \triangle R_{k-1}(p_{k-1}+j-i)P(X_k = i) + f_k P(X_k \ge j), \end{aligned} \tag{7.1}$$

where $E(\cdot)$ represents the "expectation" or *expected value*, $P(\cdot)$ is the *probability* and $x \wedge y \equiv \min(x, y)$. By proving that $\triangle R_k(s)$ is decreasing in s, Wollmer gives the following characterization for optimal booking policies:

$$p_k^* = \max\{s | \triangle R_k(s) > f_{k+1}\}, \text{ for } k \ge 1. \tag{7.2}$$

Based on this characterization, Wollmer also presented algorithms to find the optimal booking policies.

7.3 Curry's Model

Curry (1988, [50]; 1990, [52]) studied the seat allocation problem for multi-fare single-leg flight by assuming that the demands are *continuous random variables.* His approach is in the spirit of *marginal seat value* since his analysis is based upon the concept of the *revenue slope*, or equivalently, the *marginal seat revenue curve.* Let $R_k(p_{k-1}, s)$ be the expected

[3]In other words, $R_k(s)$ is the total expected revenue for the remaining s seats for fare classes $i = k, k-1, \cdots, 1$ only. It simply means that fare classes $i = n, \cdots, k+1$ are closed.

revenue for the remaining s seats that are available for fare classes $i = k, k-1, \cdots, 1$ if the protection level for fare classes $i = k-1, \cdots, 1$ is given by p_{k-1}. Define the slope function of the expected revenue function as follows:

$$S_k(p_{k-1}, s) = \frac{\partial R_k(p_{k-1}, s)}{\partial s}.$$

The recursive relationship on R_k's is

$$\begin{aligned} R_k(p_{k-1}, s) &= \int_0^{s-p_{k-1}} (f_k x + R_{k-1}(p_{k-2}, s-x)) dF_k(x) + \\ &\quad + ((s - p_{k-1}) f_k + R_{k-1}(p_{k-2}, p_{k-1})) P(X_k > s - p_{k-1}), \end{aligned}$$

where F_k is the probability distribution of the random variable X_k. And the recursive relationship on S_k's takes the following form:

$$\frac{\partial R_k(p_{k-1}, s)}{\partial p_{k-1}} = \left(-f_k + \frac{\partial R_{k-1}(p_{k-2}, p_{k-1})}{\partial s} \right) P(X_k > s - p_{k-1});$$

and

$$S_k(p_{k-1}, s) = f_k P(X_k > s - p_{k-1}) + \int_0^{s-p_{k-1}} S_{k-1}(p_{k-1}, s-x) dF_k(x). \tag{7.3}$$

After showing that the revenue function $R_k(p_{k-1}, s)$ is concave in s, Curry uses *first order conditions* to characterize the optimal booking policies. More specifically speaking, he proves that optimal booking policies in terms of a vector of protection levels $p^* = (p_1^*, p_2^*, \cdots, p_n^*)$ is given recursively by:

$$p_0^* \equiv 0, \text{ and } S_k(p_{k-1}^*, p_k^*) = f_{k+1} \text{ for } k = 1, \cdots, n-1, \tag{7.4}$$

which says that the optimal protection level for fare classes $k, k-1, \cdots, 1$ is chosen in such a way that marginal value of an additional seat is equal to the value of a fare class $k+1$ ticket, that is, f_{k+1}. This is clearly consistent with Littlewood's condition for case of two-fare classes with independent demand.

7.4 Brumelle-McGill's Model

Brumelle and McGill (1993, [38]) investigated the seat allocation problem for multi-fare single-leg flights with independent demands. The key difference is that their analysis does not depend upon whether the demand is continuous or discrete. In fact, by using *non-smooth optimization* techniques and optimal stopping rules, they establish two important results:

- there is a closed-form solution for the optimal booking policies; and
- the best policy in the class of booking policies in terms of protection levels is in fact also *optimal* in the class of all possible booking policies.

I briefly sketch their model here. Let $r_k(s;p,x)$ be the revenue function generated by k highest fare classes, given that (a) there are s seats available, (b) the vector $p = (p_1, p_2, \cdots, p_n)$ is the protection level booking policy, and (c) the vector $x = (x_1, x_2, \cdots, x_n)$ is a *realization* of the vector of random demands $(X_1, X_2, \cdots, X_n)$. In their model, the recursive relationships are built directly on the revenue function rather than on the expected revenue function:

$$r_1(s;p,x) = \begin{cases} f_1 s, & \text{if } 0 \le s \le x_1, \\ f_1 x_1, & \text{if } x_1 < s; \end{cases}$$

and

$$r_{k+1}(s;p,x) = \begin{cases} r_k(s;p,x) & \text{if } 0 \le s \le p_k, \\ (s-p_k)f_{k+1} + r_k(p_k,p,x) & \text{if } p_k \le s < p_k + x_{k+1}, \\ x_{k+1}f_{k+1} + r_k(s - x_{k+1};p,x) & \text{if } s \ge p_k + x_{k+1}. \end{cases}$$

To use the non-standard optimization method, we have to deal with the notion of *left derivative* and *right derivative*. Let δ_- and δ_+ be the *operators* for left derivative and

right derivative of the revenue function with respect to s. Then it follows that

$$\delta_+ r_1(s;p,x) = \begin{cases} f_1 & \text{if } s < x_1 \\ 0 & \text{if } s \geq x_1 \end{cases}$$

$$\delta_- r_1(s;p,x) = \begin{cases} f_1 & \text{if } s \leq x_1 \\ 0 & \text{if } s > x_1. \end{cases}$$

In general,

$$\delta_+ r_k(s;p,x) = \begin{cases} \delta_+ r_k(s;p,x) & \text{if } 0 \leq s < p_k \\ f_{k+1} & \text{if } p_k \leq s < p_k + x_{k+1} \\ \delta_+ r_k(s - x_{k+1};p,x) & \text{if } p_k + x_{k+1} \leq s \end{cases}$$

$$\delta_- r_k(s;p,x) = \begin{cases} \delta_- r_k(s;p,x) & \text{if } 0 < s \leq p_k \\ f_{k+1} & \text{if } p_k < s \leq p_k + x_{k+1} \\ \delta_- r_k(s - x_{k+1};p,x) & \text{if } p_k + x_{k+1} < s. \end{cases}$$

The main analytical tools for non-standard optimization are the following two facts: for any continuous, piecewise-linear function $f(s)$,

1. $f(s)$ is *concave* on $s > 0$ if and only is $\delta_+ f(s) \leq \delta_- f(s)$ for all s; and

2. A concave $f(s)$ can be *maximized* at any point s^* such that $0 \in \delta f(s^*)$, where $\delta f(s^*)$ is the *subdifferential* defined as the *closed interval* determined by $\delta_+ f(s^*)$ and $\delta_- f(s^*)$.

Define the *expected* revenue function to be

$$R_k(s;p,X) = E(r_k(s,p,X)),$$

where $X = (X_1, X_2, \cdots, X_n)$ is the vector of random demands. It can be interpreted as the expected revenue for the remaining s seats if the sales are available to fare classes $k, k-$

$1, \cdots, 1$ and the booking policy $p = (p_1, p_2, \cdots, p_n)$ is used. By proving some concavity properties of R_k, they derive the following condition for optimal booking policies:

$$f_{k+1} \in \delta R_k(p_k^*; (p_0^*, p_1^*, \cdots, p_{k-1}^*), X). \tag{7.5}$$

After some analytical simplifications, they further prove that if *demand distribution functions are continuous*, then there is a closed-form characterization for optimal booking policies:

$$f_{k+1} = f_1 P(X_1 > p_1^*, X_1 + X_2 > p_2^*, \cdots, X_1 + \cdots + X_k > p_k^*), \text{ for } k = 1, \cdots, n-1. \tag{7.6}$$

Clearly, when $n = 2$, it reduces to Littlewood's formula.

7.5 Equivalence of Optimality Conditions

I am now going to show that the optimality conditions for all three models are in fact analytically equivalent, which therefore implies that they are also computationally equivalent.

First of all, it is clear that Wollmer's model is just a discrete version of Curry's model. This becomes more evident if we compare Wollmer's dynamic condition (7.2) with Curry's dynamic condition (7.3). Because of this, I only need to prove the equivalence of (7.4) and (7.6).

Proposition 7.4.1: *Conditions (7.4) and (7.6) are equivalent.*

Proof: (i) (7.4) $\Longrightarrow$ (7.6)

For simplicity, let $p = (p_1, p_2, \cdots)$ be an optimal policy derived from condition (7.4). First note that by (7.4),

$$S_1(p_0, s) = f_1 P(X_1 > s),$$

which implies that

$$f_2 = f_1 P(X_1 > p_1).$$

So (7.6) holds for $k = 1$. Now suppose that based on (7.4), we already have shown that (7.6) holds for $i = 1, \cdots, k$, that is,

$$f_{i+1} = f_1 P(X_1 > p_1, X_1 + X_2 > p_2, \cdots, X_1 + \cdots + X_i > p_i), \tag{7.7}$$

for $i = 1, \cdots, k$. We now need to show that (7.6) also holds for $i = k + 1$. To show this, I first show the following claim:

Claim 6.1: *If (7.6) holds for $i = 1, \cdots, k$, then for $i = 1, \cdots, k+1$ and all $s \geq 0$,*

$$S_i(p_{i-1}, s) = f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_{i-1} > p_{i-1}, X_1 + \cdots + X_{i-1} + X_i > s). \tag{7.8}$$

Proof of Claim 6.1: I will prove it by induction. For $k = 1$, I need to show that (7.8) holds for $i = 1$ and $i = 2$. Clearly, it holds for $i = 1$. I now show that if $f_2 = f_1 P(X_1 > p_1)$, then (7.8) also holds for $i = 2$. By (7.4),

$$\begin{aligned}
S_2(p_1, s) &= f_2 P(X_2 > s - p_1) + \int_0^{s-p_1} S_1(p_0, s - x) dF_2(x) \\
&= f_2 P(X_2 > s - p_1) + f_1 \int_0^{s-p_1} P(X_1 > s - x) dF_2(x) \\
&= f_1 P(X_1 > P_1) P(X_2 > s - p_1) + f_1 E(I_{(X_1+X_2>s)} I_{(X_2 \leq s-p_1)}) \\
&= f_1 P(X_1 > p_1, X_2 > s - p_1) + f_1 E(I_{(X_1+X_2>s, X_2 \leq s-p_1)}) \\
&= f_1 P(X_1 > p_1, X_2 > s - p_1) + f_1 P(X_1 + X_2 > s, X_2 \leq s - p_1) \\
&= f_1 P(X_1 > p_1, X_1 + X_2 > s),
\end{aligned}$$

where I_A represents the *indicator function* defined on the set A, that $I_A(x) = 1$ if $x \in A$; and 0 if $x \notin A$. This implies that (7.8) holds for $i = 2$. Therefore, the claim is proved for $k = 1$.

Now suppose that the claim is true for $k = 1, \cdots, m$, and that I want to show that it is also true for $k = m + 1$. First, I have shown that for $i = 1, \cdots, m + 1$;

$$S_i(p_{i-1}, s) = f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_{i-1} > p_{i-1}, X_1 + \cdots + X_{i-1} + X_i > s); \quad (7.9)$$

and the inductive assumption implies that

$$f_{i+1} = f_1 P(X_1 > p_1, X_1 + X_2 > p_2, \cdots, X_1 + \cdots + X_i > p_i), \text{ for } i = 1, \cdots, m + 1. \quad (7.10)$$

We only need to show that (7.8) holds for $i = m+2$. By (7.4), (7.9) and (7.10), it follows that

$$\begin{aligned}
& S_{m+2}(p_{m+1}, s) \\
= \; & f_{m+2} P(X_{m+2} > s - p_{m+1}) + \int_0^{s-p_{m+1}} S_{m+1}(p_m, s - x) dF_{m+2}(x) \\
= \; & f_1 P(X_1 > p_1, X_1 + X_2 > p_2, \cdots, X_1 + \cdots + X_{m+1} > p_{m+1}) P(X_{m+2} > s - p_{m+1}) + \\
& + f_1 \int_0^{s-p_{m+1}} P(X_1 > p_1, \cdots, X_1 + \cdots + X_m > p_m, X_1 + \cdots + X_{m+1} > s - x) dF_{m+2}(x) \\
= \; & f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_{m+1} > p_{m+1}, X_{m+2} > s - p_{m+1}) + \\
& + f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_m > p_m, X_1 + \cdots + X_{m+2} > s, X_{m+2} \leq s - p_{m+1}) \\
= \; & f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_{m+1} > p_{m+1}, X_1 + \cdots X_{m+1} + X_{m+2} > s),
\end{aligned}$$

This proves (7.8) for $i = m + 2$. Hence the claim is true for $k = m + 1$. Therefore, by induction, the claim is proved.

I now go back to the original proof. In fact, I want to show that (7.6) holds for $k+1$. Note that by inductive assumption, (7.7) is true for $i = 1, \cdots, k$, which, by Claim 6.1, will lead to (7.8), that is, for $i = 1, \cdots, k + 1$ and all $s \geq 0$,

$$S_i(p_{i-1}, s) = f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_{i-1} > p_{i-1}, X_1 + \cdots + X_{i-1} + X_i > s).$$

In particular, taking $i = k+1$, I obtain

$$S_{k+1}(p_k, s) = f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_k > p_k, X_1 + \cdots + X_k + X_{k+1} > s).$$

But (7.4) implies that $S_{k+1}(p_k, p_{k+1}) = f_{k+2}$, it follows that

$$f_{k+2} = f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_k > p_k, X_1 + \cdots + X_k + X_{k+1} > p_{k+1}),$$

which is exactly (7.6) for $i = k+1$. This shows that (7.6) is true for $k+1$. Then by induction, (7.6) is always implied by (7.4). This finishes the first of the proofs.

(ii) (7.6) $\Longrightarrow$ (7.4)

Again, I can prove this by induction. In fact, this part follows immediately from Claim 6.1 since it only uses the facts (7.4) and (7.6). For $k = 1$, (7.4) follows directly from (7.6). Suppose that (7.4) folds for $i = 1, \cdots, k$. I will show that (7.4) also holds for $i = k+1$. Since (7.6) is true for all $i = 1, \cdots, n$, then claim 6.1 implies that

$$S_{k+1}(p_k, s) = f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_k > p_k, X_1 + \cdots + X_k + X_{k+1} > s).$$

Taking $s = p_{k+1}$ leads to

$$\begin{aligned} & S_{k+1}(p_k, p_{k+1}) \\ = \; & f_1 P(X_1 > p_1, \cdots, X_1 + \cdots + X_k > p_k, X_1 + \cdots + X_{k+1} > p_{k+1}) = f_{k+2}, \end{aligned}$$

which means that (7.4) holds for $i = k+1$. Then by induction, (7.4) holds for all k. Therefore, the proposition is proved. □

Remark: The significance of the above proposition is that it gives a closed-form representation for the dynamic equation in Curry's formulation. This new representation makes it clear on that the basic idea behind Littlewood's result remains valid for the multi-fare case when demands are independent. It may also shed new light on a further extension to the case of certain dependent demands.

7.6 Summary — On Three Model for Multi-fare Seat Allocation Problem

This chapter provides a note on the seat allocation problem for multi-fare single-leg flights with independent demands. It proves that three existing models for the seat allocation problem for multi-fare flights with independent demands all have equivalent optimality conditions, which clarifies the issue on which model has the computational advantage over the others. This equivalency result may also shed new light on the seat allocation for multi-fare flight with dependent demands.

Chapter 8

Summary of the Thesis and Future Directions

8.1 Summary of the Thesis

This dissertation studies the pricing problem for perishable inventories for a capacity constrained monopolist. The main innovation is to explicitly incorporate the use of artificial restrictions. And the main goal is to develop useful pricing models for firms like airlines and hotels.

Chapter 2 begins with a detailed discussion on a pricing model due to Wilson, which leads to the notion of rationing sales at lower prices. I then present a monopoly pricing model for perishable inventories by using one type of restriction. A pricing policy is called *a primary policy* if the prices for restricted units are not larger than the prices for unrestricted units. Under the following conditions: (1) the impact of restriction on the demand market has the property that as price increases, the percentage of consumers who can accommodate the restriction is decreasing; (2) the demand function is a non-increasing step function; (3) the firm only uses primary policies; and (4) the firm sells the product at prices in an increasing order, then I prove that the monopolist's pricing problem can be formulated as a nonlinear mathematical programming problem with three constraints. I further show that the monopolist will have incentives to use the restriction and can maximize its revenue by using no more than three prices. I also apply the model to airline fare pricing. I demonstrate that if the demand market is divided into leisure travellers and business travellers, then the condition imposed on the impact of restriction

on demand market is equivalent to saying that leisure travellers are more price sensitive than the business travellers.

Chapter 3 further discusses the same pricing problem in a more general context. I first extend the class of pricing policies so that it is possible for the firm to place the restriction at any price level. I prove a general optimality theorem, which shows that any optimal pricing policies in the class of primary policies remain optimal in the general class of policies. This implies that, for perishable inventories, if a monopolist decides to use an artificial restriction, *an optimal pricing practice* is to sell restricted units at prices that are lower than the prices for unrestricted units. Motivated by the current practice of airlines, this chapter also investigates the issue of whether all active prices in an optimal primary policy can be made available at the same time. It is shown that: (1) as long as active restricted prices are different from active unrestricted prices, then the optimal pricing primary will sustain even if all active prices – both restricted and unrestricted – are offered at the same time; and (2) if there exists an active restricted price that is equal to an active unrestricted price, then the firm will be better off by selling restricted units first at this price rather than making both restricted units and unrestricted units available at the same time.

Chapter 4 extends the discussion to the case of using two types of restrictions. Three cases are analyzed. For two nested restrictions and two mutually exclusive restrictions, I show that there exist optimal pricing policies that consist of at most four prices by offering three types of product. For the case of two general restrictions, I prove that under certain additional conditions, there exists optimal policies that consist of no more than five prices by utilizing four types of product.

Chapter 5 presents an application to the airline fare pricing problem in the presence of membership and product restriction. In this chapter, I demonstrate that these pricing

models developed in previous chapters are useful tools for airlines to identify the *operating environment* for each one of three common membership privileges: (1) cheaper restricted fares only; (2) cheaper unrestricted fares only; and (3) cheaper restricted fares and cheaper unrestricted fares. Examples are used to provide further support.

Chapter 6 explores the seat allocation problem in the presence of another airline. The discussion is limited to two fare flights. It is shown that under the proportional splitting rule, there exists an equilibrium pair of booking limits for the discount fare such that each airline will protect the same number of seats for the full fare regardless of their respective capacities. Under the equal splitting rule, it is shown that, for deterministic demands, there exists an equilibrium such that each airline will protect enough seats to capture half of the demand market for the full fare.

Chapter 7 discusses three models for the multi-fare seat allocation problem with independent random demands. We have shown that the optimality conditions from three models are in fact equivalent. This clarifies an issue of which method has a computational advantage over the other two. It also provides an integrated approach to the multi-fare seat allocation problem, which could be an important step toward the development of a seat allocation model for multi-fare flights with *dependent demands.*

8.2 Future Directions

8.2.1 On Seat Allocation Problems

A key theoretical work that needs to be done along this direction is to solve the seat allocation problem for single-leg flights with dependent demands. There are many possible approaches. A natural approach is to extend the model of Brumelle and McGill (1993, [38]) to the case of dependent demands. In my view, a generic treatment of dependent demands could be very difficult because a general dependency structure among multiple

random variables is too complicated. A more delicate approach is to introduce more structural dependency among these random demands for fare classes. Recall that the optimality conditions in Brumelle-McGill model (refer to Section 6.2) actually can be rewritten as follows:

$$f_{k+1} = f_k P(X_1 + \cdots + X_k > p_k | X_1 + \cdots + X_{k-1} > p_{k-1}), \text{ for } k = 1, \cdots, n-1. \quad (8.1)$$

This is closely related to the original Littlewood formula and the result due to Brumelle et al. (1990, [37]) for two-fare flights with dependent demands. Note that $W_k \equiv X_1 + \cdots + X_k$ should be interpreted as the *total demand for fare class k*. As long as $\{X_k\}_{k=1}^n$ is a sequence of independent non-negative random variables, the resulting sequence $\{W_k\}_{k=1}^n$ is a *Markovian sequence*. We should notice that working directly with the stochastic sequence $\{W_k\}_{k=1}^n$ is consistent with the traditional view of *economic demand function with uncertainties*. It is also possible to introduce a more dedicate dependency structure so that the impact of restrictions can also be incorporated.

As mentioned in Chapter 1, there are many other important seat allocation problems. Two of them are highlighted here again. First, I feel that the seat allocation problem in the presence of connecting passengers is workable. A joint optimization model should be possible. Another key problem is to develop optimal seat allocation models for multi-leg flights. A concrete theoretical model that handles random demands and multiple fares remains to be a very major challenge in yield management research.

8.2.2 On Seat Allocation Games

The discussion in Section 6.3 is just a beginning of research along this direction. In my view, this should be a very fruitful area of research in the near future. It is a very important step to push the field of yield management to a more realistic situation. The main issues are:

- Under the proportional splitting rule, (a) find the necessary and sufficient conditions that lead to the existence of equilibrium booking limits; and (b) find the conditions that lead to equilibria of *mixed strategies.*

- Under the equal splitting rule, (a) solves the seat allocation problem under random demands; and (b) characterizes the equilibrium protection levels or the equilibrium booking strategies.

It is also important to discuss the seat allocation game for multiple fares. Note that for the case of two fares, each airline has only one strategic variable, either the booking limit for low fare, or the protection level for high fare. But when each airline operates a flight with n fares, then each airline will have $n-1$ strategic variables. This gain opens up two possible ways of dealing with this game, which will be the same when $n = 2$. One way is from the static point of view – treating all these $n-1$ variables as one strategic vector. This is equivalent to a one-shot game. Another is from the dynamic point of view – treating these $n-1$ strategic variables *sequentially*, which sounds more natural. But it is not clear at this moment which way will lead to more interesting results.

8.2.3 On the Pricing Problem by Using Restrictions

This thesis explores the pricing problem by using restrictions only for a special kind of product, that is, it is perishable, and not storable for consumers with fixed supply. From a different perspective, we may say that the results on this thesis show that, *at least for a non-storable perishable products*, a carefully designed restriction can be a very effective method to practice price discrimination. So this leads to the following general question:

- *What are the product characteristics required to induce firms to intoduce restrictions as a mechanism of price discrimination?*

On the other hand, firms have complete freedom to use or not to use restrictions during the pricing process. This leads to a very interesting issue: suppose a firm can *design* the restriction, so the question is:

- *Given the market demand structure for the unrestricted product, what is the monopolist's most desirable restriction?*

To put the above question in a more analytical way, recall that in Section 2.4, the aim is to characterize $R(\bar{m}, \alpha)$, where

$$\bar{m} = \text{argmax}_{m \in S} R(m, \alpha).$$

So the above question is equivalent to asking to find an optimal α-function: $\bar{\alpha}(p)$ such that

$$\bar{\alpha} = \text{argmax}_{\alpha \in \Gamma} R(\bar{m}, \alpha),$$

where Γ is the set of all increasing functions from the interval $[p_1, p_n]$ to unit interval $[0, 1]$. So this is an optimization problem on a functional space. In marketing terms, this may be termed *the problem of optimal design of restrictions.*

Recall that the pricing models developed in this thesis have a common assumption that the monopolist offers one price at a time. Even though Section 3.4 argues that there always exists an optimal policy that will realize the projected optimal revenue value from the BL-model, but the question still remains when the firm makes all allocated quantities available at the same time, but we do not have a *formal pricing model* that makes all prices — both restricted and unrestricted — available at the same time. Clearly, we must deal with a class of pricing policies that is different from the class of general policies discussed in Chapter 3. And questions related to the corresponding optimal pricing problem that are of great interests include:

- *What is the formulation for the monopolist's pricing problem?*
- *What are the optimal policies?*
- *What can we say about any relationship between the optimal policies and the optimal polices from the BL-model in Chapter 2?*

This issue remains a very interesting but open question.

The oligopoly pricing problem by using restrictions is another important issue that needs to be addressed in the near future. Since the pricing problem in the context of a step-wise demand function is, in fact, a problem of quantity allocations by rationing, when there are several firms, the pricing game becomes a *dynamic* quantity allocation game. It is possible to approach this problem by treating it as a static game, since in this case each firm will throw out *an allocation plan* at the table and the let the market decide the outcome. Some key questions are:

- *What are the equilibrium pricing strategies?*
- *What is the role of capacity?*

We should notice that the oligopoly pricing model is an important step toward a formal analysis of the seat allocation game for multi-fare flights with random demands.

8.2.4 On Further Applications

This thesis exclusively focuses on applications to airlines. It is well-known that many other industries, such as hotels, cruise lines, car rentals, and TV stations, also offer products that are characteristically similar to airline seats. The pricing models in this thesis should be useful to these industries too. I here identify several problems in hotels room pricing.

Bibliography

[1] Abraham, M and T.E. Keeler, "Market structure, pricing and service quality in the airline industry under deregulation", In W. Sichel and T. Gies (eds) *Application of Economic Principles in Public Utility Industries* Michigan Business Studies Vol. II(2), University of Michigan Press, pp. 103–120 (1981).

[2] Allen, B. and M. Hellwig, "Bertrand–Edgeworth oligopoly in large markets", *Review of Economic Studies* 53:175–204 (1986).

[3] Alstrup, J., S. Boas, O.B.G. Madsen, and R.V.V. Vidal, "Booking policy for flights with two types of passengers", *European Journal of Operational Research* 27:274–288 (1986).

[4] Anderson, S.E., "Operational planning in airline business – Can science improve efficiency? Experienced from SAS", *European Journal of Operational Research* 43:3–12 (1989).

[5] Anderson, S.P., A. de Palma and J. Thisse, "Demand for differentiated products, discrete choice models, and the characteristic approaches", *Review of Economic Studies* 56:21–35 (1989).

[6] Bailey, E.E., "Peak laod pricing under regulatory constraint", *Journal of Political Economy* 80:662–679 (1972).

[7] Bailey, E.E., D.R. Graham and D.P Kaplan, *Deregulating the Airlines* MIT Press, Cambridge, MA (1985).

[8] Bailey, E.E. and L.J. White, "Reversals in peak and off–peak prices", *Bell Journal of Economics and Management Science* 5:75–92 (1974).

[9] Beckmann, M.J., "Decision and team problems in airline reservations", *Econometrica* 26:134–145 (1958).

[10] Beckmann, M.J. and F. Bobkowski, "Airline demand: an analysis of some frequency distributions", *Naval Logistics Research Quarterly* 5:43–51 (1958).

[11] Beckmann, M.J., "Edgeworth-Bertrand duopoly revisited", in R. Henn, ed., *Operations Research-Verfahren, III*, Meisenheim: Verlag Anton Hein, pp. 55–68 (1965).

[12] Belobaba, P.P., *Air Travel Demand and Airline Seat Inventory Management.* Unpublished Doctoral Dissertation, MIT; Cambridge, MA (1987).

[13] Belobaba, P.P., "Airline yield management: an review of seat inventory control", *Transportation Science* 21:63–73 (1987).

[14] Belobaba, P.P., "Application of a probabilistic decision model to airline seat inventory", *Operations Research* 37:183–197 (1989).

[15] Belobaba, P.P., "Heuristic methods for airline O–D seat inventory control", Working Paper, Flight Transportation Lab, MIT (1991).

[16] Belobaba, P.P., "Optimal vs. heuristic methods for nested seat allocation", Presented at ORSA/TIMS Annual Meeting, San Francisco, CA (1992).

[17] Ben–Akiva, M. and S.R. Lerman, *Discrete Choice Analysis: Theory and Application to Travel Demand* The MIT Press, Cambridge, MA (1985).

[18] Benoit, J.P. and V. Krishna, "Dynamic duopoly: prices and quantities", *Review of Economic Studies* 54:23–35 (1987).

[19] Berchman, J. and O. Shy, "Airline deregulation and the choice of networks", Working Paper No.2–91, the Sackler Institute of Economic Studies, Tel–Aviv University, Ramat Aviv, Israel (1991).

[20] Bergstrom, T. and J.K. MacKie–Mason, "Some simple analytics of peak load pricing", *RAND Journal of Economics* 22:241–249 (1991).

[21] Bertrand, J., "Theorie Mathematique de la Richesse Sociale", *Journal des Savants* pp. 499-508 (1883).

[22] Bhatia, A.V. and S. Parekh, "Optimal allocation of seats by fare", In *AGIFORS Symp. Proc.* vol.13, (1973).

[23] Bitran, G.R. and S.M. Gilbert, "Managing hotel reservations with uncertain arrivals", Working Paper, Sloan School, MIT (1992)

[24] Bodily, S.E. and P.E. Pfeifer, "Overbooking decision rule." *OMEGA* 20:129–133 (1992).

[25] Bodily, S.E. and L.R. Weatherford, "Yield management: achieving full profit potential from your market", Darden School Working Paper No. 89–30, University of Virginia (1989).

[26] Bodily, S.E. and L.R. Weatherford, "Perishable asset revenue management: yield management and pricing", Darden School Working Paper, University of Virginia (1991).

[27] Boeing Company, "Service Quality – Discount Fare Survey", Company Report, Boeing Co. April, 1982.

[28] Bohm, V., E. Maskin, H. Polemarchakis and A. Postlewaite, "Monopolistic quantity rationing", *Quarterly Journal of Economics* 98:189–197 (1983).

[29] Bohnenblust, H.F., S. Karlin and L.S. Shapley, "Solutions of discrete, two–person games", *Annals of Mathematics Studies* 24:51–72 (1950).

[30] Boiteux, M., "Peak load pricing", *Journal of Business* 33:157–179 (1960).

[31] Borenstein, S., "Hubs and high fares: dominance and market power in the U.S. airline industry", *Rand Journal of Economics* 20:344–365 (1989).

[32] Borenstein, S. and N.L Rose, "Price discrimination in the US airline industry", Discussion paper 306, Institute of Public Policy Studies, University of Michigan (1989).

[33] Brander, J.A. and A. Zhang, "Market conduct in the airline industry: an empirical investigation." *Rand Journal of Economics* 21:567–583 (1990).

[34] Brown, J.H., "An economic model of airline hubbing–and–spoking", *The Logistics and Transportation Review* 27(3):225–239 (1991).

[35] Brueckner, J.K. and P.T. Spiller, "Competition and mergers in airline networks", *International Journal of Industrial Organization* 9:323–342 (1991).

[36] Brueckner, J.K., N.J. Dyer and P.T. Spiller, " Fare determination in airline hub–and–spoke networks", Unpublished Working Paper, University of Illinois (1990).

[37] Brumelle, S.L., J.I. McGill, T.H. Oum, K. Sawaki, and M.W. Tretheway, "Allocation of airline seat between stochastically dependent demands", *Transportation Science* 24:183–192 (1990).

[38] Brumelle, S.L. and J.I. McGill, "Airline seat allocation with multiple nested fare classes" *Operations Research* 41(1):127–137 (1993).

[39] Bulow, J., "Durable–goods monopolies", *Journal of Political Economy* 90:314–332 (1982).

[40] Bulow, J.I., J.D. Geanakoplos and P.D. Klemperer, "Multimarket oligopoly: strategic substitutes and complements", *Journal of Political Economy* 93:488–511 (1985).

[41] Buhr, J., "Optimal sales limits for two–sector flights", *AGIFORS Symp. Proc.* 22:291–304 (1982).

[42] Call, G.D. and T.E. Keeler, "Airline deregulation, fares, and market behaviour", in *Analytic Studies in Transportation Economics* 221, edited by A. Daughety (1985).

[43] Caplin, A. and B. Nalebuff, "Aggregation and imperfect competition: on the existence of equilibrium", *Econometrica* 59:25–59 (1991).

[44] Caplin, A. and B. Nalebuff, "Aggregation and social choice: a mean voter theorem", *Econometrica* 59:60– (1991).

[45] Chin, H., T. Parthasarathy and T.E.S. Raghavan, "Structure of equilibria in N–person non–cooperative games", *International Journal of Game Theory* 3: 1–19 (1974).

[46] Chow, Y.S., H. Robbins, and D. Siegmund, *Great Expectation: The Theory of Optimal Stopping.* Houghton Mifflin Company, Boston (1971).

[47] Cremer, H., M. Marchand, and J.F. Thisse, "Mixed ologopoly with differentiated products", *International Journal of Industrial Organization* 9(1):43–53 (1991).

[48] Crew, M.A. and P.R. Kleindorfer, "Marshall and Turvey on peak load or joint product pricing", *Journal of Political Economy* 79:1369–1377 (1971).

[49] Crew, M.A. and P.R. Kleindorfer, "Peak load pricing with diverse technology", *Bell Journal of Economics* 7:207–231 (1976).

[50] Curry, R.E., "Optimum seat allocation with fare classes nested on segments and legs", Technical Report Note 88–1, Aeronomics Incorporated, Fayetteville, Georgia (1988).

[51] Curry, R.E., "A dynamic–systems approach to the optimal allocation of seat inventory", *AGIFORS Symp.Proc.* 29:299–318 (1989).

[52] Curry, R.E., "Optimal airline seat allocation with fare classes nested by origins and destinations", *Transportation Science* 24:193–204 (1990).

[53] Curry, R.E., "Parallel nesting and overallocation", Presented at ORSA/TIMS Annual Meeting, San Francisco (1992).

[54] Curry, R.E., M. Jaul, A.Storey, P.S. Swope Jr., and L.J. Whale, "E.R.O.S.: putting pleasure into yield management", *AGIFORS Symp. Proc.* 30:41–60 (1990).

[55] Dansby, R.E., "Capacity constrained peak load pricing", *Quarterly Journal of Economics* 92:387–398 (1978).

[56] Dasgupta, P. and E. Maskin, "The existence of equilibrium in discontinous economic games, I: theory", *Review of Economic Studies* 53:1–26 (1986).

[57] Dasgupta, P. and E. Maskin, "The existence of equilibrium in discontinous economic games, II: applications", *Review of Economic Studies* 53:27–41 (1986).

[58] d'Aspremont, C., J.J. Gabszewicz and J.K. Thisse, "On Hotelling's stability in competition", *Econometrica* 47:1145–1150 (1979).

[59] Davidson, C. and R. Deneckere, "Long–run competition on capacity, short–run competition in price, and the Cournot model", *Rand Journal of Economics* 17:404–415 (1986).

[60] DeSalvia, D.N., "An application of peak load pricing", *Journal of Business* 42:458–476 (1969).

[61] Dixit, A.K. and J.E. Stiglitz, "Monopolistic competition and optimum product diversity", *American Economic Review* 67:297–308 (1977).

[62] Dobson, G. and S. Kalish, "Positioning and pricing a product line", *Marketing Science* 7:107–124 (1988).

[63] Dror, M., P. Trudeau and S.P. Ladany, "Network models for seat allocation on flights", *Transportation Research B* 22:239–250 (1988).

[64] Dubey, P., "Price–quantity strategic market game", *Econometrica* 50:111–126 (1982).

[65] Dunn, K.D. and D.E. Brooks, "Profit analysis: beyond yield management", *Cornell Hotel and Restaurant Administration Quarterly* 31:80–90 (1990).

[66] Eaton, B.C. and R.G. Lipsey, "Product Differentiation", Chapter 12 of *Handbook of Industrial Organization* Vol.1 edited by R. Schmalensee and R.D. Willig. Elsenvier Science Publisher, B.V. (1989).

[67] Economides, N., "Symmetric equilibrium existence and optimality if differentiated product market", *Journal of Economic Theory* 47:178–194 (1989).

[68] Edgeworth, F., "La Theoria Pura del Monopolio", *Giornale degli Economisti* 40:13-31 (1897). In English: "The Pure Theory of Monopoly", in *Papers Relating to Political Economy*, volume 1, ed. F. Edgeworth, London: MacMillan (1925).

[69] Ekelund, R.B., "Price discrimination and product differentiation in economic theory: An early analysis", *Quarterly Journal of Economics* 84:268–278 (1970).

[70] Etschmaier, M. and M. Rothstein, "Operations research in the management of airlines", *OMEGA* 2:157–179 (1972).

[71] Economides, N., "Symmetric equilibrium, existence and optimality in differentiated product market", *Journal of Economic Theory* 47:178–194 (1989).

[72] Gale, D. and S. Sherman, "Solutions of finite two–person games", *Annals of Mathematics Studies* 24:37–49 (1950).

[73] Gale, I.L. and T.J Holmes, "The efficiency of advance-purchase discounts in the presence of aggregate demand uncertainty", *International Journal of Industrial Organization* 10(4):413-437 (1992).

[74] Gale, I.L. and T.J. Holmes, "Advance-purchase discounts and monopoly allocation of capacity", *American Economic Review* 83:135–146 (1993).

[75] Gallego, G. and G. van Ryzin, "Optimal dynamic pricing of inventories with stochastic demand over finite horizons", Working Paper, Department of IE/OR, Columbia University, NY (1992).

[76] Gerstner, E., "Peak load pricing in competitive markets", *Economic Inquiry* 24:349–361 (1986).

[77] Gerstner, E. and D. Holthausen, "Profitable pricing when market segments overlap", *Marketing Science* 5:55–69 (1986).

[78] Gerchak, Y.; M. Parlar and T.K.M. Yee, "Optimal rationing policies and production quantities for products with several demand classes", *Canadian Journal of Administrative Sciences* 2:161–176 (1985).

[79] Gillen, D.W., T.H. Oum and M.W. Tretheway, *Airline Cost and Performance: Implications for Public and Industry Policies* Centre for transportation Studies, University of British Columbia, Vancouver, Canada (1985).

[80] Glover, F., R. Glover, J. Lorenzo, and C. McMillian, "The passenger mix problem in the scheduled airlines", *Interfaces* 12:73–79 (1982).

[81] Goldman, M., H. Leland and D. Sibley, "Optimal nonuniform pricing", *Review of Economic Review* 51:305–320 (1984).

[82] Graham, D.R., D.P. Kaplan and D.S. Sibley, "Efficiency and competition in airline industry", *Bell Journal of Economics* 14:118–138 (1983).

[83] Hanks, R.D., R.G. Cross and R.P. Noland, "Discounting in the Hotel industry: a new approach.", *Cornell Hotel and Restaurant Administration Quarterly* 32:15–23 (1992).

[84] Hansen, Mark M., *A Model of Airline Hub Competition.* Unpublished Doctoral Dissertation, University of California, Berkeley, CA (1988).

[85] Hansen, Mark M., "Airline competition in a hub–dominated environment: an application of noncooperative game theory", *Transportation Research B.* 24(1):27–43 (1990).

[86] Harris, M. and A. Raviv, "A theory of monopoly pricing schemes with demand uncertainty", *American Economic Review* 71:347–365 (1981).

[87] Hart, O.D. , "Monopolistic competition in a large exoncomy with differentiated commodities", *Review of Economic Studies* 42:353–382 (1979).

[88] Heal, G., "Rational rationing and microeconomic references", *Economic Letters* 8:18–27 (1981).

[89] Hersh, M. and S.P Ladany, "Optimal seat allocation for flights with intermediate stops", *Computers and Operations Research* 5:31–37 (1978).

[90] Heuer, G.A., "On completely mixed strategies in bimatrix games", *Journal of London Mathematical Society* 11: 17–20 (1975).

[91] Heuer, G.A., "Uniqueness of equilibrium points in bimatrix games", *International Journal of Game Theory* 8:13–25 (1979).

[92] Heuer, G.A. and C.B. Millham, "On Nash subsets and mobility chains in bimatrix games", *Naval Logistics Research Quarterly* 23: 311–319 (1976).

[93] Jansen, M.J.M., "Regularity and stability of equilibrium points of bimatrix games", *Mathematics of Operations Research* 6: 530–550 (1981).

[94] Jansen, M.J.M., "Maximal Nash subsets for bimatrix games", *Naval Logistics Research Quarterly* 28: 147–152 (1981).

[95] Johnson, R., (1985) "Networking and market entry in the airline industry", *Journal of Transport Economics and Policy* 19:299–304 (1985).

[96] Jones, P. and D. Hamilton, "Yield management: putting people in the big picture", *Cornell Hotel and Restaurant Administration Quarterly* 32:89–95 (1992).

[97] Jordan, W.J., "Heterogeneous users and the peak load pricing model", *Quarterly Journal of Economics* 97:127–138 (1983).

[98] Joskow, P.J., "Contributions to the theory of marginal cost pricing", *Bell Journal of Economics* 7:197–206 (1976).

[99] Kalish, S., "Monopolistic pricing with dynamic and production cost", *Marketing Science* 2(2):135–159 (1983).

[100] Kanafani, A. and A.A. Ghobrial, "Airline hubbing – some implications for airport economics", *Transportation Research A* 19:15–27 (1985).

[101] Kaplan, A., "Stock rationing", *Management Science* 15:260–267 (1969).

[102] Katz, M., "Firm–specific differentiation and competition among multiproduct firms", *Journal of Business* 57: S149–S166 (1984).

[103] Kendall, W.R. and C. Jordan, "Fare and income elasticities in the North Atlantic air travel market: economic and policy implications", *Transportation Journal*, Summer (1989).

[104] Kimes, S.E., "Yield management: a tool for capacity–constrained service firms", *Journal of Operations Management* 4:348–363 (1989).

[105] Kimes, S.E., "The basics of yield management", *Cornell Hotel and Restaurant Administration Quarterly* 30:14–19 (1989).

[106] Kinberg, Y. and A.G. Rao, "Stochastic models of a price promotion", *Management Science* 21(8):897–907 (1975).

[107] Klemperer, P. and M. Meyer, "Price competition vs. quantity competition: the role of uncertainty", *Rand Journal of Economics* 17:618–638 (1986).

[108] Kling, J.A., C.M. Grimm and T.M. Corsi, "Hub–dominated airports: an empirical assessment of challenger strategies", *The Logistics and Transportation Review* 27(3):203–223 (1991).

[109] Kraft, D.G.H., T.H. Oum and M.W. Tretheway, "Airlines seat management", *Logistics and Transportation Review* 22:115–130 (1986).

[110] Kreps, D.M. and J.A. Scheinkman, "Quantity precommitment and Bertrand competition yield Cournot outcome", *Bell Journal of Economics* 14:326–337 (1983).

[111] Kreps, V.L., "Bimatrix games with unique equilibrium points", *International Journal of Game Theory* 3:115–118 (1974).

[112] Kuhn, H.W., "An algorithm for equilibrium points in bimatrix games", *Proceedings, National Academy of Science U.S.A.* 47:1656–1662 (1961).

[113] Kunreuther, H. and J.F. Richard, "Optimal pricing and inventory decisions for non–seasonal items", *Econometrica* 39:173–175, 1971.

[114] Kunreuther, H. and L. Schrage, "Joint pricing and inventory decisions for constant priced items", *Management Science* 19:732–738, 1973.

[115] Ladany, S.P., "Dynamic operating rules for motel reservations", *Decision Sciences* 7:827–840 (1976).

[116] Ladany, S.P., "Bayesian dynamic operating rules for optimal hotel reservations", *Zeitschrift Opns. Res.* 21: B165–B176 (1977).

[117] Ladany, S.P and D.N. Bedi, "Dynamic booking rules for flight with intermediate stop", *OMEGA* 5:721–730 (1977).

[118] Ladany, S.P and M. Hersh, "Non–stop versus one–stop flights", *Transportation Research* 11:151–159 (1977).

[119] Lambert, C.U. and J.M. Lambert and T. P. Cullen, "The overbooking question: a simulation", *Cornell Hotel and Restaurant Administration Quarterly* 30(2):14–20 (1990).

[120] Lee, C.H. and E.H. Warren, "Rationing by seller's preference and racial price discrimination", *Economic Inquiry* 14:36–44 (1976).

[121] Leland, H.E., "Theory of the firm facing uncertain demand", *American Economic Review* 62:278–291 (1971).

[122] Leland, H.E. and R.A. Meyer, "Monopoly pricing structures with imperfect information", *Bell Journal of Economics* 7(2):449–462 (1976).

[123] Lemke, C.E. and J.T. Howson, Jr., "Equilibrium points of bimatrix games", *Journal of the Society of Industrial and Applied Mathematics* 12:413–423 (1964).

[124] Levine, M.E., "Airline competition in deregulated markets: theory, firm strategy, and public policy", *Yale Journal of Regulation* 4:393–494 (1987).

[125] Levitan, R. and M. Shubik, "Price duopoly and capacity constraints", *International Economics Review* 13:111–122 (1972).

[126] Lewis, R.C., "Consumer–based Hotel pricing", *Cornell Hotel and Restaurant Administration Quarterly* 26(3):18–21 (1986).

[127] Liberman, V. and U. Yechiali, "Hotel overbooking problem – inventory system with stochastic cancellations", *Advances in Applied Probability* 9:220–??? (1977).

[128] Liberman, V. and U. Yechiali, "On the hotel overbooking problem – an inventory system with stochastic cancellations", *Management Science* 24:1117–1126 (1978).

[129] Lieberman, W., "Revenue enhancement opportunities through yield management", Discussion Document, Arthur D. Little, Inc., San Francisco, CA (1992).

[130] Lieberman, W., "Implementing yield management", Presented at ORSA/TIMS Annual Meeting, San Francisco (1992).

[131] Littlewood, K., "Forecasting and control of passenger bookings", *AGIFORS Symp. Proc.* 12:95–117 (1972).

[132] Lott, J.R. Jr. and R.D. Roberts, "A guide to the pitfalls of identifying price discrimination", *Economic Inquiry* 29:14–23 (1991).

[133] Lovell, C.A. and K.L. Wertz, "Price discrimination in related markets", *Economic Inquiry* 19:488–494 (1981).

[134] Lovell, C.A. and K.L. Wertz, "Third degree price discrimination in imperfectly sealed markets", *Atlantic Economic Journal* 13:1–11 (1985).

[135] Luce, R.D. and H. Raiffa, *Games and Decisons* John Wiley and Sons, NY (1957).

[136] Luenburger, D.G., *Linear and Nonlinear Programming* Second Edition, Addison-Wesley Publishing Company; Reading, MA (1984).

[137] Mangasarian, O.L., "Equilibrium points of bimatrix games", *Journal of the Society of Industrial and Applied Mathematics* 12:778–780 (1964).

[138] Mas-Colell, A., "A model of equilibrium with differentiated commodities", *Journal of Mathematical Economics* 2:263–295 (1975).

[139] Maskin, E. and J. Riley, "Monopoly with imperfect information", *Rand Journal of Economics* 15:171–196 (1984).

[140] Mayer, M., "Seat allocation, or a simple model of seat allocation vis sophisticated ones", *AGIFORS Symp. Proc.* 16:103–135 (1976).

[141] McCall, J.J., "Probabilistic microeconomics", *Bell Journal of Economics* 2(2):403–433 (1971).

[142] McCleary, K.W. and P.A. Weaver, "Are frequent–guest programs effective?" *Cornell Hotel and Restaurant Administration Quarterly* 31(3):39–45 (1991).

[143] McGill, J., "Airline multiple fare class seat allocation", Presented at Fall ORSA/TIMS Joint National Conference, Denver, Colorado (1988).

[144] McGill, J.I., *Optimization and Estimation Problems in Airline Yield Management.* Unpublished Doctoral Dissertation, Faculty of Commerce and Business Administration, the University of British Columbia, Vancouver, Canada (1989).

[145] Mendelson, T., J.S. Pliskin and U. Yechiali, "A stochastic allocation problem", *Operations Research* 28:687–693 (1980).

[146] Meyer, J.R. and J.S. Strong, "From closed set to open set deregulation: an assessment of the U.S. airline industry", *The Logistics and Transportation Review* 28(1):1–21 (1992).

[147] Millham, C.B., "On the structure of equilibrium points in bimatrix games", *SIAM Review* 10:447–449 (1968).

[148] Millham, C.B., "Constructing bimatrix games with special properties", *Naval Research Logistics Quarterly* 19:709–714 (1972).

[149] Millham, C.B., "On Nash subsets of bimatrix games", *Naval Logistics Research Quarterly* 21:307–317 (1974).

[150] Mirman, L.J. and D. Sibley, "Optimal nonlinear prices for multiproduct monopolies", *Bell Journal of Economics* 11:659–670 (1980).

[151] Monroe, K.B. and A.J. Della Bitta, "Models for pricing decisions", *Journal of Marketing Research* 15:413–428 (1978).

[152] Moorthy, K.S., "Market segmentation, self–selection, and production line design", *Marketing Science* 3:288–307 (1984).

[153] Moorthy, K.S., "Cournot competition in a differentiated oligopoly", *Journal of Economic Theory* 36:86–109 (1985).

[154] Moorthy, K.S., "Product and price competition in a duopoly", *Marketing Science* 7:141–168 (1988).

[155] Morrison, S.A. and C. Winston, "Intercity transportation route structures under deregulation", *American Economic Review* 75:57–61 (1985).

[156] Morrison, S.A. and C. Winston, *The Economic Effects of Airline Deregulation.* Brookings Institute, Washington, D.C. (1987).

[157] Morrison, S.A. and C. Winston, "The Dynamics of airline pricing and competition", *American Economics Review* 80(2):389–393 (1990).

[158] Mussa, M. and S. Rosen, "Monopoly and product quality", *Journal of Economic Theory* 18:301–317 (1978).

[159] Nagle, T., "Economic foundations for pricing", *Journal of Business* 57:S3–S35 (1984).

[160] Nahmias, S., "Perishable inventory theory: a review", *Operations Research* 30(4):680–708 (1982).

[161] Nahmias, S. and W.S. Demmy, "Operating characteristics of an inventory system with rationing", *Management Science* 11:1236–1245 (1981).

[162] Nash, J.F., "Equilibrium points in n–person games", *Proceedings, National Academy of Science U.S.A.* 36:48–49 (1950).

[163] Nash, J.F., "Non–cooperative games", *Annals of Mathematics* 54:286–295 (1951).

[164] Nguyen, D.J., "The problems of peak loads and inventories", *Bell Journal of Economics* 7:242–248 (1976).

[165] Okada, A., "Strictly perfect equilibrium points of bimatrix games", *International Journal of Game Theory* 13:45–154 (1984).

[166] Oren, S.S., S. Smith, and R.B. Wilson, "Pricing a product line", *Journal of Business* 57:S73–S100 (1984).

[167] Orkin, E.B., "Boosting your bottom line with yield management", *Cornell Hotel and Restaurant Administration Quarterly* 29(2):52–55 (1988).

[168] Orkin, E.B., "Strategies for managing transient rates", *Cornell Hotel and Restaurant Administration Quarterly* 30:34–39 (1990).

[169] Osborne, W. and C. Pitchik, "Price competition in a capacity contrained duopoly", *Journal of Economic Theory* 38:238–260 (1986).

[170] Oum, T.H. and D.W. Gillen, "A study of inter–fareclass competition in airline market", *Transportation Research Forum Proceedings*, 599–609 (1980).

[171] Oum, T.H. and D.W. Gillen, "Structure of intercity demands in Canada: theory, tests, and empirical result", *Transportation Research B* 17:175–191 (1983).

[172] Oum, T.H., D. Gillen and D. Nobel, "Demand for fareclasses and pricing in airline markets", *Logistics and Transportation Review* 23:195–222 (1986).

[173] Oum, T.H. and M.W. Tretheway, *Airline Economics*. Published by Institute of Transportation Studies, University of British Columbia, Vancouver, Canada (1992).

[174] Oum, T.H., A. Zhang and Y. Zhang, "Inter–firm rivalry and firm–specific demand in the deregulated airline markets", Working Paper 91–TRA–008, Faculty of Commerce, University of British Columbia, Vancouver, Canada (1991).

[175] Owen, G., "Optimal threat strategies of bimatrix games", *International Journal of Game Theory* 1:3–9 (1971).

[176] Panzar, J., "Equilibrium and welfare in unregulated airline markets", *American Economic Review (Papers and Proceedings)* 69:92–95 (1979).

[177] Parthasarathy, T. and T.E.S. Raghavan, *Some Topics in Two–person Games.* American Elsevier Publishing Company, Inc., NY (1971).

[178] Perloff, J.M. and S.C Salop, "Equilibrium with product differentiation", *Review of Economic Studies* LII:107–120 (1985).

[179] Perry, M., "Sustainable positive profit multiple-price strategies in contestable markets", *Journal of Economic Theory* 32:246-265 (1984).

[180] Pfeifer, P.E., "The airline discount fare allocation problem", *Decision Sciences* 20:149–157 (1989).

[181] Phillps, L., *The Economics of Price Discrimination* Cambrudge University Press, Cambridge (1983).

[182] Pigou, A.C., *Economics of Welfare* MacMillan, London (1920).

[183] P'ng, I.P.L., "Reservations: customers insurance in marketing of capacity", *Marketing Science* 8:248–264 (1989).

[184] P'ng, I.P.L and D. Hirshleifer, "Price discrimination through offers to match price", *Journal of Business* 60:365–383 (1987).

[185] Pressman, I., "A mathematical formulation of the peak load problem", *Bell Journal of Economics and Management Science* 1:304–326 (1970).

[186] Pustay, M.W., "Toward a global airline industry: prospects and impediments", *The Logistics and transportation Review* 28(1):103–128 (1992).

[187] Pustay, M.W., "Six myths of contemporary airline industry", *Transportation Research Forum* 32(2):286–291 (1992).

[188] Raghavan, T.E.S., "Completely mixed strategies in bimatrix games", *Journal of London Mathematics Society* 2:709–712 (1970).

[189] Rajan, A.R. and R. Steinberg, "Dynamic pricing and ordering decisons by a monopolist", *Management Science* 38(2):240–262 (1992).

[190] Rao, V.R., "Pricing research in marketing: the state of the art", *Journal of Business* 57:S38–S64 (1984).

[191] Reagan, P.B., "Inventory and price behaviour", *Review of Economic Studies* 47:137–142 (1982).

[192] Reiss, P.C. and P.T. Spiller, "Competition and entry in small airline markets", *Journal of Law and Economics* 32:S179–S202 (1989).

[193] Relihan, W.J.III., "The yield management approach to hotel–room pricing", *Cornell Hotel and Restaurant Administration Quarterly* 30(1):40–45 (1989).

[194] Richter, H., "The Differential revenue method to determine optimal seat allotments by fare type", *AGIFORS Symp. Proc.* 22:339–362 (1982).

[195] Robinson, L.W., "Optimal and approximate control policies for airline booking with sequential fare classes", Working Paper, Johnson Graduate School of Management, Cornell University (1991).

[196] Rockafellar, R.T., *Convex Analysis* Princeton University Press (1970).

[197] Ross, S.M., *Stochastic Process* John Wiley and Sons, New York (1982).

[198] Ross, S.M., *Introduction to Stochastic Dynamic Programming* Academic Press, New York (1983).

[199] Rothstein, M., "An airline overbooking model", *Transportation Science* 5:180–192 (1971).

[200] Rothstein, M., "Airlines overbooking: the state of art", *Journal of Transport Economics and Policy* 5:96–99 (1971).

[201] Rothstein, M., "Hotel overbooking as a markovian sequential decision process", *Decision Sciences* 5:389–404 (1974).

[202] Rothstein, M., "Airlines overbooking: fresh approaches are needed", *Transportation Science* 9:169–173 (1975).

[203] Rothstein, M., "O.R. and the airline overbooking problem", *Operations Research* 33:392–435 (1985).

[204] Salop, S., "The noisy monopolist: imperfect information, price disperson and price discrimination", *Review of Economic Studies* 44:393–406 (1977).

[205] Salop, S., "Monopolistic competition with outside goods", *Bell Journal of Economics* 10:141–156 (1979).

[206] Salop, S. and J.E. Stiglitz, "Bargains and ripoffs: A model of monopolistically competitive price disperson", *Review of Economic Studies* 44:493–510 (1977).

[207] Salop, S. and J.E. Stiglitz, "The theory of sales: A simple model of equilibrium price diperson with identical agents", *American Economic Review* 72(5):1121–1130 (1982).

[208] Shaked, A. and J. Sutton, "Relaxing price competition through product differentiation", *Review of Economic Studies* 49:3–13 (182).

[209] Shaked, A. and J. Sutton, "Natural oligopolies", *Econometrica* 51(5): 1469–1483 (1983).

[210] Shapiro, C., "Theories of Oligopoly Behaviour", Chapter 6 of *Handbook of Industrial Organization* Vol.1, 329–414; edited by R. Schmalensee and R.D. Willig. Elsenvier Science Publisher, B.V. (1989).

[211] Shapley, L.S., "A note on the Lemke–Howson algorithm", *Mathematical Programming Study* 1:175–189 (1974).

[212] Shapley, L.S. and R.N. Snow, "Basic solutions of discrete games", *Annals of Mathematics Studies* 24:27–35 (1950).

[213] Shaw, S., *Airline Marketing and Management* 3rd Edition, Pitman Publishing, London (1990).

[214] Shlifer, R. and Y. Vardi, "An airline overbooking policy", *Transportation Science* 9:101–114 (1975).

[215] Simon, J., "An almost practical solution to airline overbooking", *Journal of Transport Economics and Policy* 2:201–202 (1968).

[216] Simon, J., "Airline overbooking: the state of art – a reply", *Journal of Transport Economics and Policy* 6:254–256 (1972).

[217] Simpson, R.W., "Setting optimal booking levels for flight segments with multi–class, multi–market traffic", *AGIFORS Symp. Proc.* 25:263–279 (1985).

[218] Simpson, R.W., "Theoretical concepts for capacity/yield management", *AGIFORS Symp. Proc.* 25:281–293 (1985).

[219] Singh, N. and X. Vives, "Price and quantity competition in a differentiated duopoly", *Rand Journal of Economics* 15:546–554 (1984).

[220] Smith, B.C., "A group decision support model", *AGIFORS Symp. Proc.* 30:27–39 (1990).

[221] Smith, B.C., J.F. Leimkuhler and R.M. Darrow, "Yield management at American Airlines", *Interface* 22:8–31 (1992).

[222] Smith, B.C. and C.W. Penn, "Analysis of alternate origin–destination control strategies", *AGIFORS Symp. Proc.* 28:123–143 (1988).

[223] Sorenson, N., *Airline Competitive Strategy: A Spatial Perspective.* Unpublished Doctoral Dissertation, Department of Geography, University of Washington, Seattle, WA. (1990).

[224] Spence, A.M., "Nonlinear prices and welfare", *Journal of Public Economics* 8:1–18 (1977).

[225] Spiller, P.T., "Pricing of hub–and–spoke networks", *Economics Letters* 30:165–169 (1989).

[226] Stadie, W., "A full information pricing problem for sales of serveral identical commodities", *Z. Operations Research* 34:161–169 (1990).

[227] Stein, A.A., "Pricing and differentiation strategies", *Planning Review* 17(5):30–34 (1989).

[228] Steiner, P., "Peak loads and efficient pricing", *Quarterly Journal of Economics* 71:585–610 (1957).

[229] Stephenson, F.J. and R.J. Fox, "Corporate attitudes toward frequent–flier programs", *Transportation Journal* 27:10–22 (1987).

[230] Stokey, N., "Intertemporal price discrimination", *Quarterly Jounarl of Economics* 93:355–371 (1979).

[231] Stokey, N., "Rational expectations and durable goods pricing", *Bell Journal of Economics* 12:112–128 (1981).

[232] Stone, R. and M. Diamond, "Optimal inventory control for a single flight leg", Staff Paper, Operations Research Division, Northwest Airlines (1992).

[233] Strassmann, D.L., "Potential competition in the deregulated airlines", *Review of Economics and Statistics (Netherlands)* 72(4):696–702 (1990).

[234] Taylor, C.J., "The determination of passenger booking levels", *AGIFORS Symp.Proc.* 2:93–116 (1962).

[235] Thompson, H.R., "Statistical problems in airline reservations control", *Operational Research Quarterly* 12:167–185 (1961).

[236] Tirole, J., *The Theory of Industrial Organization* The MIT Press, Cambridge, MA (1988).

[237] Titze, B. and R. Griesshaber, "Realistic passenger booking behaviours and simple low-fare/high-fare seat allotment model", *AGIFORS Symp. Proc.* 23:197–223 (1983).

[238] Toh, R., "Airline revenue yield protection: Joint reservation control over full and low fare sales", *Transportation Journal* Winter 74–80 (1979).

[239] Toh, R., "An inventory depletion overbooking for the hotel industry", *Journal of Travel Research* 24:24–30 (1985).

[240] Toh, R.S. and M.Y. Hu, "Frequent–flier programs: passenger attributes and attitudes", *Transportation Journal* 29:11–22 (1988).

[241] Toh, R.S., M.K. Kelley and M. Y. Hu, "An approach to the determination of optimal airline fares: some useful insights on price elasticities, monopoly power and competitive factors in the airline industry", *Journal of Travel Research* 25(1):26–33 (1986).

[242] Toh, R.S. and M.J. Rivers, "Frequent–guest programs: do they fly?" *Cornell Hotel and Restaurant Administration Quarterly* 31(3):46–52 (1991).

[243] Topkis, D.M., "Optimal ordering and rationing policies in a non–stationary dynamic inventory model with n demand classes", *Management Science* 15:160–176 (1968).

[244] Tretheway, M.W., "Frequent flyer programs: marketing bonanza or anti–competitive tool", *Proceedings of Canadian Transportation Research Forum*, University of Saskatchewan Printing Service (1989).

[245] Tretheway, M.W., "The characteristics of modern post–deregulation air transport", Working Paper, Faculty of Commerce and Business Administration, University of British Columbia; Vancouver, Canada (1991).

[246] Tretheway, M.W. and T.H. Oum, *Airline Economics: Foundations for Strategy and Policy* Centre for transportation Studies, University of British Columbia, Vancouver, Canada (1992).

[247] Turvey, R., "Peak load pricing" *Journal of Political Economy* 76:101–113 (1968).

[235] Thompson, H.R., "Statistical problems in airline reservations control", *Operational Research Quarterly* 12:167–185 (1961).

[236] Tirole, J., *The Theory of Industrial Organization* The MIT Press, Cambridge, MA (1988).

[237] Titze, B. and R. Griesshaber, "Realistic passenger booking behaviours and simple low-fare/high-fare seat allotment model", *AGIFORS Symp. Proc.* 23:197–223 (1983).

[238] Toh, R., "Airline revenue yield protection: Joint reservation control over full and low fare sales", *Transportation Journal* Winter 74–80 (1979).

[239] Toh, R., "An inventory depletion overbooking for the hotel industry", *Journal of Travel Research* 24:24–30 (1985).

[240] Toh, R.S. and M.Y. Hu, "Frequent–flier programs: passenger attributes and attitudes", *Transportation Journal* 29:11–22 (1988).

[241] Toh, R.S., M.K. Kelley and M. Y. Hu, "An approach to the determination of optimal airline fares: some useful insights on price elasticities, monopoly power and competitive factors in the airline industry", *Journal of Travel Research* 25(1):26–33 (1986).

[242] Toh, R.S. and M.J. Rivers, "Frequent–guest programs: do they fly?" *Cornell Hotel and Restaurant Administration Quarterly* 31(3):46–52 (1991).

[243] Topkis, D.M., "Optimal ordering and rationing policies in a non–stationary dynamic inventory model with n demand classes", *Management Science* 15:160–176 (1968).

[244] Tretheway, M.W., "Frequent flyer programs: marketing bonanza or anti–competitive tool", *Proceedings of Canadian Transportation Research Forum*, University of Saskatchewan Printing Service (1989).

[245] Tretheway, M.W., "The characteristics of modern post–deregulation air transport", Working Paper, Faculty of Commerce and Business Administration, University of British Columbia; Vancouver, Canada (1991).

[246] Tretheway, M.W. and T.H. Oum, *Airline Economics: Foundations for Strategy and Policy* Centre for transportation Studies, University of British Columbia, Vancouver, Canada (1992).

[247] Turvey, R., "Peak load pricing" *Journal of Political Economy* 76:101–113 (1968).

[248] Varian, H.R., *Microeconomic Analysis* Second Edition, W.W. Norton and Company, New York (1984).

[249] Varian, R., "Price Discrimination", Chapter 11 of *Handbook of Industrial Organization* Vol.1 edited by R. Schmalensee and R.D. Willig. Elsenvier Science Publisher, B.V. (1989).

[250] Vickrey, W., "Airline overbooking: some further solutions", *Journal of Transport Economics and Policy* 6:257–270 (1972).

[251] von Neumann, J. and O. Morganstern, *Theory of Games and Economic Behavior, 3rd Edition* Princeton University Press, Princeton, N.J. (1953).

[252] Vorob'ev, N.N., "Equilibrium points in bimatrix games", *Theory of Probability and its Applications* 3:297–309 (1958).

[253] Wang, K., "Optimum seat allocation for multi–leg flights with multiple fare types", *AGIFORS Symp. Proc.* 23:225–246 (1983).

[254] Weatherford, L.R., *Perishable Asset Revenue Management in General Business Situations* Unpublished Doctoral Dissertation, Darden Graduate School of Business, University of Virginia; (1991).

[255] Weatherford, L.R. and S.E. Bodily, "A taxonomy and research overview of perishable–asset revenue management: yield management, overbooking, and pricing", *Operations Research* ??:???–??? (1992).

[256] Weatherford, L.R., S.E. Bodily and P.E. Pfeifer, "Modelling the customer arrival process and comparing decision rules in perishable asset revenue management situations", to appear in *Transportation Science*, (1993).

[257] Wenders, J.T., "Peak load pricing in Electric Utility Industry", *Bell Journal of Economics* 7:232–241 (1976).

[258] Wernerfelt, B., "A special case of dynamic pricing policy", *Managament Science* 32:651–659 (1986).

[259] William, F.E., "Decision theory and innkeeper: an approach for setting hotel reservation policy", *Interfaces* 7:18–30 (1977).

[260] Williamson, E.L. and P.P. Belobaba, "Optimization techniques for seat inventory control", *AGIFORS Symp. Proc.* 28:153–170 (1988).

[261] Williamson, O.E., "Peak load pricing and optimal capacity under indivisibility constraint", *American Economic Review* 56:810–827 (1966).

[262] Wilson, C.A., "On the optimal pricing of a monopolist", *Journal of Political Economy* 96:164–176 (1988).

[263] Wilson, R., "Efficient and competitive rationing", *Econometrica* 57:1–40 (1989).

[264] Wolinsky, A., "Product differentiation with imperfect information", *Review of Economic Studies* 51:53–61 (1984).

[265] Wollmer, R.D., "A hub–spoke seal management model", Unpublished company report, Douglas Aircraft Company, McDonnell Douglas Corporation, Long Beach, CA (1986a).

[266] Wollmer, R.D., "An airline reservation model for opening and closing fare classes", Unpublished Company Report, Douglas Aircraft Company, Long Beach, CA (1986b).

[267] Wollmer, R.D., "A seat management model for a single leg route", Unpublished Company Report, Douglas Aircraft Company, Long Beach, CA (1987).

[268] Wollmer, R.D., "A seat management model for a single leg route when lower fare classes book first", Presented at Fall ORSA/TIMS Joint National Conference. Denver, Colorado (1988).

[269] Wollmer, R.D., "An airline seat management model for single leg route when low fare classes book first", *Operations Research* 40:26–37 (1992).

[270] Wong, J.T., *Airline Network Seat Allocation.* Unpublished Doctoral Dissertation, Northwestern University, Evanston, IL (1990).

[271] Wong, J.T., F.S. Koppleman and M.S. Daskin, "Flexible assignment approach to itinerary seat allocation", *Transportation Research Ser. B* ??:??–?? (1992).

[272] Zhang, A., *Essays on Strategic and Contractual Relationships in Oligopoly* Unpublished Doctoral Dissertation, Faculty of Commerce, the University of British Columbia; Vancouver, Canada (1990).