The Optimal Taxation of Families

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ABSTRACT

This thesis presents an analysis of two classical problems in the theory of optimal taxation: commodity tax reform and nonlinear income taxation. Economic behavior is modeled as arising out of a family decision making process rather than owing to individual utility maximization. The taxation authority is assumed to have no direct control over intra–family allocations of resources. In this way, family interactions change the nature of the second-best constraints the planner faces. The analysis focuses on the impact of these constraints on optimal policy choices. Attention is focused on families with two members, whom the planner can (in most situations that are modeled) tell apart.

In the chapters dealing with commodity tax reform, behaviour is modeled as the Pareto–efficient outcome of a family decision process. Conditions for the existence of a feasible, Pareto–improving tax change are presented and contrasted with those that obtain in the individualistic case. The consequences of treating households as a single individual are also discussed. It is shown that treating families as if they were individuals can lead to misleading conclusions. An example is presented to demonstrate that the traditional analysis may go wrong even when families behave as if they are individuals. Moreover, it is shown that household budget data alone is insufficient to address this issue. The model is then put to use to address question of temporary inefficiencies in tax reform. I present how the circumstances under which temporary inefficiencies can arise vary with the structure of poll taxes.

The problem faced by a planner choosing an income tax schedule for families is modeled as a multi–dimensional screening problem. Families are described by a two–dimensional vector of characteristics, interpreted as the labour productivities of their members. The planner cannot observe these characteristics directly. Furthermore, families are free to redistribute the after–tax incomes of their members. The planner must take this behaviour into account when choosing the tax schedule. A description of the possible Pareto–efficient mechanisms is given. The implications of a standard redistributive assumption on the sign of marginal tax rates are explored. In contrast to unidimensional taxation models, the redistributive assumption does not imply that marginal tax rates are everywhere non–negative. For much of the analysis, the usual assumption of quasi–linear preferences is jettisoned, allowing an exploration of the implications of this additional structure. The qualitative features of optimal tax schedules are discussed. It is concluded that neither individual–based taxation nor taxation based solely on total family income is optimal.

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CHAPTER 1: Family Economics and Family Taxation

1.1. Introduction

An understanding how individuals interact with each other for the provision of their wants and needs is among the primary goals of economic analysis. Many of these interactions take the form of market transactions. Indeed, the detailed study of individual decisions on the basis of market signals (prices) and of the concomitant equilibria has become the basis upon which both modern welfare economics and the theory of optimal taxation rests. Individual interactions are not limited to exchanges in anonymous markets. Often, agents agree to bring (at least some of) their resources into group relationships and to engage in activities governed by forces other than the market. One need look no further than the family for an example of this type of arrangement. This analysis represents an attempt to bring family decision–making into models of optimal taxation. It has two complementary aims: uncovering the aspects of individual–based models of taxation that are robust to introducing family interactions, and uncovering some shortcomings of these theories in the family context.

Families engage in many activities: child–rearing, the care of seniors, home production, and consumption. In this study, I make no attempt to capture all behaviours or to discern the effects of policy on family formation and composition. The objective of this study is more modest. In line with the literature on optimal taxation spawned

by Ramsey (1927), I consider the problem of a taxation authority wishing to design a tax system for a set of agents who act as both consumers and suppliers of labour. Unlike much of the literature, I take the family as the unit of decision making, not individuals.

Even in the restricted setting proposed in this analysis, the problem of designing optimal policies for families raises many questions that cannot arise when all decision–making agents are individuals, along with some issues that have solutions that are not easily transferred to the family setting. One would expect families to behave differently than individuals. That is, individual–based consumer theory may be an inappropriate way to view how consumption decisions are made within a family. Moreover, the process of deriving statements about individual well–being from family behaviour is rather involved. Indeed, there may even be a notion of family well–being that is an important consideration in the problem of taxing families. These three sets of issues – behavioural, informational and normative – are central in this study.

The remainder of this introduction is devoted to expounding the special features of models of family behaviour and family taxation. The next section surveys models of family behaviour and their links to the analysis of tax policy. Section 3 provides a discussion of normative and informational issues. The remainder of this thesis is sketched in Section 4.

1.2. Models of Family Decision–Making

While substantial agreement exists among economists on how to model the behaviour of individual consumers (at least in the absence of uncertainty), numerous ways to think about families have been suggested. One approach is to simply treat families as individuals, positing that total family consumption is the consumption of some aggregate agent. Hoddinott and Haddad (1993) have termed this approach the "unitary" model of family behaviour. Unlike the treatment of consumption goods, the labour supplies of individuals within families are often dealt with as separate goods in this framework. Killingsworth (1983) provides a survey of attempts to use this model in the analysis of family labour supply behaviour, showing how its empirical implications are rejected by most studies. These findings are hardly surprising, given the strong assumptions on individual preferences required for a collective to behave as if it were an individual (cf. Gorman (1953), Deaton and Muellbauer (1980)).

A natural way to abandon the unitary model is to endow each family member with their own preferences over consumption and leisure. However, once this is done, it is necessary to make statements about how these possibly differing objectives are reconciled in family decisions. A wide range of possibilities now emerges.

One class of models uses non–cooperative game theory to describe interactions among family members. Leuthold (1968) introduced a simple Cournot–type model of labour supply decisions for families with two members. Each member is assumed

to maximise his or her own utility subject to the family budget constraint, taking the choices of the other as given. The resulting decisions give rise to a pair of labour supply functions that depend on prices (including wages) and the actions of the other. The family is at an equilibrium if these labour supplies are compatible; that is, if both members are actually maximising their preferences simultaneously, given the actions of the other. Ashworth and Ulph (1981) have tested the unitary model against the Leuthold model. Their data support a rejection of the unitary model. Woolley (1992) extended this model to study the provision of a household public good and the impact of income taxation on these decisions. She reports the possibility of a tradeoff between reducing intra-family inequality and negative effects on the provision of the household public good.

When family members engage in a non–cooperative game, there is no guarantee that the equilibrium outcome is Pareto–efficient for the family. That is, there may a rearrangement of family resources that makes both members better off. This feature of non–cooperative models has been criticised (cf. Kooreman and Kapteyn (1990)). Given the repeated nature of family interactions and the degree of communication that it possible among partners, it seems reasonable to expect efficient outcomes to emerge. Of course, efficient outcomes may arise from an ostensibly non–cooperative decision process. Even in a one–shot game, Becker (1974) shows how the actions of

a benevolent patriarch may induce self–interested family members to choose actions that result in efficient outcomes.[1]

Many studies treat within–family efficiency as a maintained hypothesis. Among this class of models are those that make explicit use of (cooperative) bargaining theory. The early work in applying bargaining theory to analyse family decisions was carried out by Manser and Brown (1980) and McElroy and Horney (1981). In both of these works, labour–consumption outcomes depend on the feasible set of allocations for the family and a pair of threat points. Threat points are meant to capture the utilities of the family members in the absence of a cooperative agreement. There remains some debate over the appropriate specification of disagreement outcomes. Should they correspond to outside options (divorce) or to some sub–optimal (non–cooperative) outcome within the family?[2] As McElroy (1990) has pointed out, if outside options influence threat points, there is scope for variables that reflect the state of the marriage market or divorce settlements to influence the intra–family distribution of resources.

The bargaining approach has gained acceptance, in part, because it allows for family behaviour to violate the income–pooling hypothesis, which states that only total family (exogenous) income matters in family decisions. That is, for a fixed total family income, behaviour is invariant to redistribution of that total among

[1] This is the essence of Becker's famous "rotten kid" theorem. For a detailed discussion of the importance and limitations of this theorem see Bergstrom (1989).

[2] See Lundberg and Pollak (1993) for a discussion of how the sort of actions to which family members can credibly commit affects the way threat points ought to be interpreted.

family members. To see that income pooling need not hold for a bargaining solution, consider an increase in the exogenous income of one family member in a two–member household. This expands the consumption possibilities of both members. It may also improve the threat position of the person with the increased income.[3] In general, changes in behaviour reflect both of these effects. Suppose, instead, that the same increase in exogenous income had fallen to the other family member. The effects on the family budget would be the same, while threat–point effects would be different. Notice that in the unitary model changes in exogenous incomes influence the budget constraint alone, so that income pooling is satisfied. Income pooling is, in principle, a testable hypothesis. Among the empirical studies that have rejected it are those by Thomas (1990) and Phipps and Burton (1993). Indeed, this series of evidence is one of the major motivations for abandoning the unitary model of family behaviour.

The scope of theories of efficient family decision–making reaches beyond cooperative bargaining models. Indeed, it has already been mentioned that ostensibly non–cooperative procedures may lead to efficient outcomes. Forms of cooperative decision–making other than the bargaining models usually employed are also conceivable. It is interesting to ask, then, if there are features common to *all* decision–making procedures that generate efficient allocations within families. This is the research agenda of the "collective" school of modeling the family.

[3] Lundberg and Pollak (1993) argue that this need not be the case. It depends on the form of the income gain and how the marriage market responds to such changes.

An early contribution to the collective paradigm is the work of Samuelson (1956). He posits the existence of a fixed social welfare function, defined over the utilities of family members. Abstracting from the details of budgeting decisions, he assumes that there is a "family consensus" that gives rise to the family objective. Families are then assumed to choose consumption bundles as if to maximise this social welfare function, subject to the family budget constraint. He shows how such a family can be viewed as engaging in a two–stage budgeting procedure. In the first stage, the family budget is allocated among the family members. In the second stage, each individual maximises his or her own utility, subject to the constraint that he or she can spend no more than the "allowance" granted him or her in the first stage. The division of resources depends on prices and total family income, so that Samuelson's model generates behaviour consistent with the income pooling hypothesis.[4]

More recent work in the collective approach has jettisoned the notion of an agreed–upon family social welfare function. Instead, the "sharing–rule" approach has been adopted. In his analysis of household labour supply decisions, Chiappori (1988) was the first to make explicit use of household sharing rules. He shows that when family members have preferences over bundles of own–consumption, all efficient decision–making procedures can be modeled as two–stage budgeting procedures. Unlike the situation that obtains in Samuelson's model, the income sharing rule may

[4] It is more usual to group Samuelson's contribution with the unitary model, because there is a well–defined family objective, and income pooling holds. However, it has two important features of the collective model: family members have their own preferences; and there is a notion of income–sharing in the model.

depend on income sources and the state of the marriage market. This interpretation also holds for families in which individuals show altruism in the form of caring about the utility of other family members, not about their consumption *per se*.

Besides the generality of the model, the collective approach has a number of empirical advantages. First, it leads to testable restrictions on household labour supply and consumption behaviour. (Chiappori (1988), Browning and Chiappori (1994)) Moreover, the tests that have been carried out to date show that there is insufficient evidence to reject these restrictions, at least not those restrictions placed on consumption (Bourguignon *et al.* (1992), Browning *et al.* (1994), Browning and Chiappori (1994)). Second, a good deal of information about the sharing rule can be recovered from family budget data alone. In particular, Chiappori (1988, 1992) has shown that the derivatives of the sharing rule with respect to prices and individual incomes are identifiable. From the viewpoint of tax theorists, this means that information is available on how changes in consumption and income taxes affect sharing with the family. This represents a potentially powerful tool to be in used in analysing the effects of taxes on families and on their constituent members.

1.3. Issues of Ethics and Information

The link between applied welfare economics, including the theory of optimal taxation, is both well–known and often–exploited. For an individual, well–being is usually identified with preferences. Once this notion of well–being is accepted, net market

transactions contain quite a bit of welfare information. Roy's Identity, a standard result in consumer theory, states that the effects on welfare of changes in consumer prices are negatively proportional to an individual's net demand vector. The consequences of this result for the theory of taxation are profound. It says that a taxation authority need observe no more than net market transactions to decide if a change in commodity taxes leads to a local welfare improvement for an individual. It is not surprising, then, that much of the modern theory of tax reform rests on Roy's Identity.[5]

Unfortunately for the applied welfare economist, market transactions are not always recorded at the individual level. Instead, most data sources contain, at best, records of family transactions. The direct link between observed transactions and individual welfare is broken. One might ask: Is there any welfare information in net family transactions? Because this is one of the central questions of Chapter 2 of this thesis, I provide only preliminary remarks on it here.

If the unitary model of family decisions holds, Roy's Theorem can be used along with market data to identify changes in consumer prices that increase the optimised value of the family criterion function. The normative significance of such changes is unclear. If there is an ethical notion of making families better off, then one can substitute the word "family" for the word "individual" and carry out the standard

[5] Guesnerie (1977) is the classic reference. This theory is expounded in greater detail in Guesnerie (1995). There is another branch of the tax reform literature, based on compensated demand functions and Shephard's Lemma, owing much to Hatta (1977) and Diewert (1978). The two approaches yield equivalent results.

procedure.[6] Such a substitution is in violation of the individualistic principles usually held by economists. Multi–person families can be viewed as mini–societies. As such, they have no more claim to being units of ethical account than countries do.

A more generous interpretation is available within the Samuelson framework. It can be argued that a family social welfare function summarises a set of agreed–upon ethics that govern the intra–family allocation of goods. A taxation authority interested in increasing the value of household welfare may be said to be respecting the ethics of families. Two caveats to this interpretation are worth keeping in mind. First, respecting the ethics of the family may involve sacrificing the welfare of one family member for the benefit of another. Second, it is often difficult to distinguish between agreed–upon family ethics and patently unjust relationships within families.[7]

Family decisions result in informational problems more profound than just those faced by researchers or policy analysts wishing to identify and uncover relevant data. When families possess more tax–relevant information than the planner does, it is often necessary that the planner take account of this asymmetry. This is especially true when the tax system features non–linearities, or different rates for different individuals. A common example of a non–linear tax structure is income taxation. Mirrlees

[6] The somewhat ambiguous term "households" is often used by economists to indicate decision–making units, be they individuals or families. I would argue that it is important to account for which usage of the term "households" is more appropriate in specific settings.

[7] This caveat was recognised by Mill (1859, p. 238), who presents an argument against accepting the notion family ethics based on the "almost despotic power of husbands over wives."

(1971) was the first to recognise the importance of asymmetric information in designing non–linear income tax schedules. He envisions a world in which, if it could, the taxation authority would like to tax on the basis of innate ability. Individuals alone know their ability. The best the planner can do is to tax earned income. If the planner chooses to ignore the fact that individuals have an informational advantage, workers may choose to "hide" their ability by working less than they otherwise might. This approach to modeling the effects of income taxation on work effort has been used often, and has generated a form of conventional wisdom on the qualitative features of optimal income tax schedules.[8] Much of this work has relied heavily on the assumption that there is a single tax–relevant characteristic the planner does not know. Even if one accepts the notion that ability is the only such individual trait, it is difficult to imagine that there is some notion of "family ability" that can take the place of individual ability in these models. It seems more natural to allow family members to differ in ability, resulting in a model of decision–making units that differ along more than one dimension. *A priori*, it is not clear how much of the conventional wisdom applies to this more complex setting.

1.4. An Overview of This Study

I have already stated that the focus of this thesis is to study the optimal taxation of families as units of consumption and labour supply. This is a natural point from which

[8] Chapter 4 of this thesis gives an account of this literature.

to begin adding models of family decisions to the traditional analysis of commodity tax reform and optimal non–linear taxation. The remainder of this thesis is devoted to exactly this task. The assumption that all families consist of two members, each of whom may participate in the decision–making process, is maintained throughout. It is often maintained that these two individuals can be distinguished on the basis of some demographic characteristic, like gender.

The analysis of commodity tax reform begins with Chapter 2. In it, I assume that each member of the family has preferences over own–consumption of a set of private goods. In line with the collective approach, I assume that family decisions are Pareto–efficient for the family. The taxation authority is assumed to have at its disposal a full set of linear commodity taxes, including a tax on leisure time, and poll taxes. It cannot, however, effect lump–sum redistributions across families, nor impose the division of resources within families. A simple aggregate production sector is posited, with all pure profit taxed away. Under these assumptions, I characterise the directions of policy reform that are both feasible and Pareto–improving at the individual level. This characterisation is compared to the results that would obtain if family budget data were treated as if the family were actually an individual. It is shown that the necessary and sufficient conditions for such pseudo–Pareto–improving changes are necessary for an actual Pareto–improving direction to exist, but not sufficient. Thus, changes in policy that appear to be Pareto–improving when family interactions are ignored may fail to be actual Pareto–improvements. It is shown

by means of any example that this may be the case even when family consumption behaviour is observationally equivalent to that of an individual. Moreover, I show that family budget data alone is insufficient to identify actual Pareto–improving directions of policy reform.

Chapter 3 continues the analysis of tax reform, showing how the characterisation of Pareto–improving directions of reform is influenced by the types of demogrants available to the planner. Three demogrant structures are compared: poll taxes varying by the demographic characteristic, identical poll taxes for each individual, and redistribution of a fixed total family demogrant between the two family members. The chapter also contains a discussion of the related issue of temporary inefficiencies in tax reform procedures. In the context of an individual–based model of tax reform, Guesnerie (1977) noticed that under some circumstances, all feasible Pareto–improving directions of policy reform may require the economy be moved inside its production frontier. I present conditions under which this somewhat anomalous occurrence cannot arise. These conditions are restrictions on aggregate demand behaviour and the derivatives of the family sharing rule. Thus, they can be checked using family budget data.

The focus shifts to the question of optimal non–linear income taxes for families in Chapter 4. In line with the work of Mirrlees (1971), I allow the individuals within families to differ in labour productivity. I focus attention on the case of only two productivity types. The taxation authority can observe before–tax labour income,

but cannot separate the contributions of hours worked and skill to the total. Labour–consumption decisions are made by families, assumed to maximise a weighted sum of their members' utilities. The weights are assumed to be independent of incomes. In this way, the family decision process can be described by a single parameter, the relative weight of person 2. This parameter is known to the planner. Because families are assumed to maximise a fixed social welfare function, an income pooling result holds. The allocation of consumption within the family depends on total family after–tax income alone.

With this characterisation of family behaviour, it is possible to view the taxation problem as a particular mechanism design problem. The planner offers bundles of goods composed of the before–tax incomes of the two family members and total family after–tax income. The planner is assumed to maximise some social objective function subject to an economy–wide materials balance constraint and a set of self–selection constraints. The self–selection constraints are formulated in such a way that *families* have no incentive to mis–report the types of their members. Because each family has two members (possibly of different individual productivity), a two–dimensional screening problem arises. In order to highlight the role of the added dimension to the analysis, I assume that the social objective is defined over the family criterion functions, not over individual utilities. That is, despite the caveats mentioned in the previous section, I allow the planner to respect the ethics of the family. Were the

planner to care about individual utility, some qualitative features of optimal non–linear income taxes in the individual setting would not hold merely because of a desire to "correct" for family decisions.[9] I wish to explore the robustness of the individual–based results in a more comparable situation – where the planner respects the criterion functions of the decision–making agents (here, families).

A variety of qualitative features of all Pareto–efficient tax structures are outlined in the chapter. It is shown that self–selection requires that, when two individuals in distinct families have equally productive partners, the individual with higher productivity must receive at least as much before–tax income. This is, however, the only strong monotonicity result owing to self–selection alone. In particular, it cannot be shown that self–selection implies that both members of a family of highest type be allocated more before–tax income than the corresponding members of other families. Further results are available once conditions of optimality are added to the analysis. It is shown that, at an optimum, the materials balance constraint must always bind, that no two families receive the same allocation, and that there exists a family that faces no marginal distortions.

Chapter 5 contains a discussion of two issues related to the problem of non–linear income taxation: redistributive taxation, and the influence asymmetries in family decision–making. It is common in non–linear income taxation to assume that, in the

[9] Seade (1980) shows how optimal tax schedules with non–standard features can arise when the planner has non–welfarist objectives.

absence of self–selection constraints, the planner would like to redistribute consumption from the more able to the less able. Assumptions of this form are often sufficient to guarantee that marginal tax rates must lie between zero and one (Guesnerie and Seade (1982), Röell (1985)). I formulate the analogue to this criterion in the family context, stating that whenever two families can be unambiguously ordered on the basis of ability, the planner would like to redistribute after–tax income from the more able to the less able. It is shown that this assumption is not sufficient to guarantee non–negative marginal tax rates. Negative marginal tax rates are shown to arise when there is a tension between the redistributive assumption and the ability of the planner to distinguish between the (different) families with one member of each productivity type. It is also shown that when families face linear tradeoffs between the labour supplies of their members, knowledge of the parameter of the family decision–making process can be used to infer which families have least incentive to imitate others. This highlights both the importance of the special assumptions on family behaviour maintained throughout the analysis and the usefulness of information about family decisions for setting tax policy.

CHAPTER 2: Tax Reform and Collective Family Decision–Making

2.1. Introduction

Welfarist evaluations of tax policy are concerned with the effects of these policies on the well–being of the individuals that comprise a society. The standard literatures on both optimal taxation in private good economies (Atkinson (1977), Stiglitz (1987)) and tax reform (Guesnerie (1977), Diewert (1978), Weymark (1979)) identify well–being with preferences as revealed by market behaviour. There is an important, and rarely addressed, issue to be resolved when implementing second–best taxation models. Market data often presents itself at the family level, whereas welfarist evaluations are carried out at the individual level. Standard results in normative taxation theory can be reinterpreted to suit market data only if family preferences are well defined and family well–being is an appropriate ethical concept.

In order to bring the analysis back to the level of the individual we must be able to assess the well–being of each household member *separately.* If we wish to maintain the family as the basic unit of consumption decisions then we must make some statements about how the possibly conflicting interests of family members are reconciled within the household. I draw on the literature spawned by Becker (1974) on efficient household decision–making processes to make these statements. I find that household budget data are insufficient to calculate Pareto–improving directions of tax change, even under circumstances in which these data can be used to identify changes

in the intra–family allocation of resources. Moreover, I show that the necessary and sufficient conditions for a Pareto–improving change found by acting as if families were a single person fail to be sufficient in the family setting, although they remain necessary. Indeed, as long as there is at least one good for which *individual* net demands are not observable, a rote application of traditional tax reform formulae may lead to erroneous policy prescriptions.

The approach of this analysis is very much in the spirit of second–best taxation problems. The planner cannot impose the division of resources within the household. It can control the family only indirectly by effecting changes in the environment it faces. Control over the environment is described by the set of available policy instruments: a full set of linear commodity taxes, including a tax on leisure time, and poll taxes. Lump–sum redistribution between and within families is not feasible. Policy is thus constrained by both the material balance constraints for the economy and the behavioural responses of individuals. Unlike the economies described in traditional second–best problems, these behavioural responses occur on two fronts: the allocation of resources within families and the interaction of families in competitive markets. I also choose to consider the problem of tax reform rather than that of tax design. From an initial, possibly sub–optimal tax policy I seek small changes in the rates that are both feasible and Pareto improving at the *individual* level. If no such directions exist then this initial tax structure is a local second–best optimum.

I focus attention on an economy made up of two–person families. It is assumed that each member of the household has preferences over own–consumption of a set of private goods, and that household decisions are Pareto–efficient for the family with respect to these preferences. In order to derive a characterisation of Pareto improving policy changes, I assume the social planner can observe the intra–family allocation of all commodities. The planner can also see how this allocation varies with changes in the economic environment. This is more information than is included in standard budget surveys, but this assumption is made so that I can focus on the normative question of how one ought to use information on intra–family allocation apart from the question of how one might obtain this information. I show that some of this additional information is actually needed to calculate Pareto–improving directions of tax change, demonstrating the deficiency of household budget data alone.

It should be pointed out that the families considered in this study conform to the Chiappori (1988,1992) collective model of decision–making. Moreover, there is no household production. The work of Apps and Rees (1996) and Chiappori (1994) has underlined the difficulty in identifying the parameters of the household decision process when household production is incorporated into the analysis. The addition of household production to the current model can only limit the information available to the planner, strengthening the conclusions of this analysis.

This chapter is organised as follows. Section 2 presents the model of family decision–making. Section 3 provides a description of the production and government

sectors of the economy and the basic structure of general equilibria. Section 4 provides a characterisation of directions of tax reform that are both feasible and Pareto–improving. This furnishes *a fortiori* a characterisation of second–best optima. The consequences of ignoring family interactions are considered in Section 5.

2.2. Collective Family Decision–Making

I begin by setting out the notation. There are H households in the economy, indexed by $h = 1, \ldots, H$. Each family consists of two members, indexed by $i = 1, 2$. Individuals have preferences over their own consumption, x^{ih}, of vectors of n private goods. Consumer prices are given by the vector q. I assume that the preferences of person i in household h can be represented by a continuous, increasing and quasi-concave function $U^{ih} : I\!R^n_+ \rightarrow I\!R$, all i, h. I allow family members to have an initial endowment of goods. Let these endowments of goods be denoted by $\omega^{ih} \in I\!R^n_+$. The total resources available to any family are its total endowment plus any transfers from the planner. There is no household production process. Because the planner must take the endowments as parametric, they play a limited role in computing optimal directions of tax change. I assume that each family allocates its resources so that the final allocation is Pareto efficient for the family. It is helpful to think of the family members as the agents in a two–person exchange economy with the final allocation lying somewhere on the contract curve.

Let us now consider a typical family in more detail. For notational convenience I omit the superscript h for the remainder of this section. Let m^i denote the total income of person i. The assumption of Pareto efficiency implies that the family allocates resources as if to solve

$$(P) \qquad \begin{aligned} &\max_{x^1,x^2} U^1(x^1) \; s.t. \; U^2(x^2) \geq \bar{u}(q, m^1, m^2), \\ &q(x^1 + x^2) \leq m^1 + m^2. \end{aligned}$$

Note that the utility level $\bar{u}$ is not exogenous, but may change with incomes and prices. Later on, I allow the incomes of the individuals to be under the control of the planner via tax instruments. Problem (P) models the decision process of a family conditional on its tax environment. I follow the lead of Chiappori (1988, 1992) and give a sharing rule interpretation of this decision process. For this I need the following result, due to Chiappori (1988, p. 68).

Lemma 2.1. *Let $\hat{x}^1(q, m^1, m^2), \hat{x}^2(m^1, m^2)$ solve (P). Then there exists a function $\varphi(q, m^1, m^2)$ such that*

i.) $\hat{x}^1(q, m^1, m^2)$ solves:

$$(P1) \qquad \max_{x^1} U^1(x^1) \; s.t. \; qx^1 \leq \varphi(q, m^1, m^2).$$

ii.) $\hat{x}^2(q, m^1, m^2)$ solves:

$$(P2) \qquad \max_{x^2} U^2(x^2) \; s.t. \; qx^2 \leq m^1 + m^2 - \varphi(q, m^1, m^2).$$

Total demand of the family is denoted $\hat{x}(q, m^1, m^2) := \hat{x}^1(q, m^1, m^2) + \hat{x}^2(q, m^1, m^2)$. The function φ may be interpreted as a household sharing rule, indicating the value of household resources spent on the consumption of goods by person 1. With this interpretation in mind, denote by μ^1, μ^2 the *effective* incomes of family members 1 and 2, respectively. That is,

$$\begin{aligned} \mu^1(q, m^1, m^2) &:= \varphi(q, m^1, m^2); \\ \mu^2(q, m^1, m^2) &:= m^1 + m^2 - \varphi(q, m^1, m^2). \end{aligned} \tag{2.1}$$

Lemma 2.1 is the analogue in the family context to the second theorem of welfare economics. Under this interpretation $\varphi = m^1 - t^1$, where t^1 is the lump–sum transfer of income from person 1 to person 2 required to decentralise the Pareto efficient allocation $(\hat{x}^1, \hat{x}^2)$ at prices q. Notice that the sharing rule depends on the incomes of the individuals rather than on total family income. That is, I am not imposing the income pooling assumption. I am allowing individual incomes to affect household decisions in ways other than their effects on total income. This may be due to bargaining effects, as in McElroy and Horney (1981), or compensating transfers, as in Chiappori (1992). I remain agnostic on this account.

The levels of well–being attained by the family members are given by the value functions for the programs (P1) and (P2). Specifically, let V^i be the indirect utility function dual to $U^i, i = 1, 2$. Then

$$\begin{aligned} u^1 &= U^1(\hat{x}^1(q, m^1, m^2)) = V^1(q, \mu^1(q, m^1, m^2)) \\ u^2 &= U^2(\hat{x}^2(q, m^1, m^2)) = V^2(q, \mu^2(q, m^1, m^2)). \end{aligned} \tag{2.2}$$

Expression (2.2) displays the role of price and income changes on the well–being of the individuals. Consumer price changes play two roles. The usual price effects are present, and changes in consumer prices serve to change the effective incomes of the family members through sharing–rule effects. Income changes affect well–being in a complicated way. A one–unit increase in own–income does not necessarily increase indirect utility by the marginal utility of income $V^i_{\mu^i}$. It is true that a one–unit change in *effective* income has this impact on utility. However, changes in money income need not be translated one–for–one into changes in effective income. It is also important to notice that an increase in the income of another family member can have a positive or negative effect on an individual's welfare.

2.3. General Equilibria

I now describe the environment the families face. There is a production sector characterised by an aggregate technology set Y. The aggregate firm faces a vector of producer prices p, and acts so as to maximise profit at these prices. The assumption of an aggregate firm can be made without loss of generality under competitive conditions (Bliss (1975, p. 68)). I assume that the solution to the profit maximisation problem defines a net supply function $y = \eta(p)$, where $\eta : I\!R^n_+ \setminus \{ 0 \} \to I\!R^n$. Negative supplies are interpreted as input demands. I assume that η is differentiable. In addition, I make the following assumption:

Assumption F (Full rank). $\nabla_p \eta(p)$ *is of rank* $n - 1$.

Homogeneity of the supply function implies that $\nabla_p \eta(p)$ is singular, so that this is a maximal rank assumption. Assumption F rules out the possibility of kinks or ridges in the aggregate production frontier. At any kink or ridge in the production frontier the set of supporting prices is non–unique. Thus, sufficiently small changes in producer prices may fail to have any effect on production. When Assumption F is satisfied the planner has full local control over the production sector via changes in producer prices. Among the formal consequences of Assumption F is that the mapping $\nabla_p \eta(p) : \mathbb{R}^n_+ \setminus \{0\} \rightarrow \mathbb{R}^n$ is invertible in the subspace orthogonal to p (Guesnerie (1995)).

The government sector is characterised by a set of available tax instruments. For simplicity, there is no public sector production. The planner has at its disposal a full set of per–unit commodity taxes $t_1, \ldots, t_n$. Consumer prices are thus the sum of producer prices and taxes. The planner can also use a poll tax or subsidy. I allow this demogrant to vary by the index i, but not by household. This requires that this index corresponds to some easily observable characteristic. For a 'traditional' family, gender would be such a characteristic. Denote these transfers R^1 and R^2. This formulation of the demogrant structure admits two important restrictions on the powers of the planner as special cases. When the planner cannot make changes in the demogrant conditional on the index i, I can write $dR^1 = dR^2$. The restriction $dR^1 + dR^2 = 0$ corresponds to the redistribution of a fixed total demogrant between persons of different indicies. I also assume that the planner taxes away all pure

profit, so that the profits of the production sector do not affect the consumption sector through distributed profit. With this policy structure, the income of a typical individual is given by

$$m^{ih} = R^i + q\omega^{ih}. \tag{2.3}$$

An examination of (2.3) reveals that the family decision process described by the programme (P) is not the most general process that generates Pareto–efficient outcomes. I could have permitted the utility level $\bar{u}$ to depend on endowments and lump–sum incomes separately, instead of on total incomes. There are two justifications for this simplification. First, it is quite reasonable to expect whatever influence the ownership of endowments has on family decision–making to depend on their value. Second, the possibility arises of non–homogeneity of net demands when endowments exert effects independent of their values.

I am now in a position to describe the equilibria in this economy. First, aggregate demand for goods is given by

$$x(q, R^1, R^2) := \sum_{h=1}^{H} x^h(q, R^1, R^2) = \sum_{h=1}^{H} \hat{x}^h(q, q\omega^{1h} + R^1, q\omega^{2h} + R^2). \tag{2.4}$$

The dependence of aggregate demand on the endowment is suppressed from the notation. The aggregate endowment is denoted by ω, and is given by $\omega := \sum_{i,h} \omega^{ih}$. An equilibrium for this economy exists when aggregate demand is satisfied by the

combination of production and aggregate endowment. Thus, an equilibrium is said to exist at (q, R^1, R^2, p) if

$$x(q, R^1, R^2) \leq \eta(p) + \omega. \tag{2.5}$$

An equilibrium is said to be *tight* if (2.5) holds with equality. If families exhaust their budgets on consumption, Walras' Law guarantees that the government budget is in balance at any tight equilibrium. It should be pointed out that, contrary to the situation that obtains in a general equilibrium setting with no distortions, non–tight equilibria are consistent with all goods having positive prices. The existence of a government sector implies that the value of private excess demand (measured at either consumer or producer prices) need not be zero. Indeed, the government never runs a deficit at any equilibrium (Guesnerie (1995)). Thus, the value of private excess demand can be negative, so that non–tight equilibrium are possible.

2.4. Optimal Policy Changes

Assume that the economy is initially in a tight equilibrium at $(\bar{q}, \bar{R}^1, \bar{R}^2, \bar{p})$. In order to avoid boundary problems, I assume $(\bar{q}, \bar{R}^1, \bar{R}^2, \bar{p}) >> 0$. Optimal policy changes are defined to be marginal changes $[dq^\top, dR^1, dR^2, dp^\top]^\top$ in the tax instruments that are both feasible and Pareto improving.[10] Feasibility can be given two equivalent interpretations. The first is common in public economics: the government budget position is not worsened by the change. The second is that the change is equilibrium

[10] As a notational convention, all vectors (both row and column) are enclosed in square brackets.

preserving. The equivalence follows from the fact that moves from a tight equilibrium to any other equilibrium cause the government budget to change from balance to a non–deficit.

Definition: A direction of policy change $[dq^\top, dR^1, dR^2, dp^\top]^\top$ is said to be *equilibrium preserving* if for initial policy $(\bar{q}, \bar{R}^1, \bar{R}^2, \bar{p})$ it satisfies

$$\nabla_q x(\bar{q}, \bar{R}^1, \bar{R}^2)dq + \nabla_{R^1} x(\bar{q}, \bar{R}^1, \bar{R}^2)dR^1 + \nabla_{R^2} x(\bar{q}, \bar{R}^1, \bar{R}^2)dR^2 \leq \nabla_p \eta(\bar{p})dp, \qquad (2.6)$$

where $\nabla_q x(q, R^1, R^2)$ is the Jacobian of aggregate demand with respect to consumer prices.

The assumptions made on the aggregate technology provide an alternate representation of feasible directions of policy change. This is given in the following result, due to Guesnerie (1977, p. 187).

Lemma 2.2. *Suppose that the aggregate technology satisfies the full rank assumption. Let*

$(\bar{q}, \bar{R}^1, \bar{R}^2, \bar{p})$ *be an initial tight equilibrium. Then for any* $[dq^\top, dR^1, dR^2]^\top$ *that satisfies*

$$\bar{p}^\top \left(\nabla_q x(\bar{q}, \bar{R}^1, \bar{R}^2)dq + \nabla_{R^1} x(\bar{q}, \bar{R}^1, \bar{R}^2)dR^1 + \nabla_{R^2} x(\bar{q}, \bar{R}^1, \bar{R}^2)dR^2\right) \leq 0, \qquad (2.7)$$

there exists a direction of producer price change dp *such that* $[dq^\top, dR^1, dR^2, dp^\top]^\top$ *is equilibrium preserving.*

Lemma 2.2 states simply that a policy reform is feasible if and only if it induces a marginal change in demand that remains inside the production set. I use (2.7) as the characterisation of equilibrium preserving directions of change. It is helpful to rewrite (2.7) as

$$[\Phi_q, \Phi_{R^1}, \Phi_{R^2}] \begin{bmatrix} dq \\ dR^1 \\ dR^2 \end{bmatrix} \geq 0, \tag{2.8}$$

where $\Phi_q := -\bar{p}^\top \nabla_q x(\bar{q}, \bar{R}^1, \bar{R}^2)$ and $\Phi_{R^i} := -\bar{p}^\top \nabla_{R^i} x(\bar{q}, \bar{R}^1, \bar{R}^2)$, $i = 1, 2$.

Consider welfare–improving directions of policy change. I search for directions that are strictly Pareto–improving; that is, changes that make everyone better off. In what follows let f_z denote the partial derivative of the function f with respect to the argument z. Differentiating the top line of (2.2), and using (2.1) and (2.3), I obtain

$$\begin{aligned} du^{1h} &= \nabla_q^\top V^{1h} dq + V^{1h}_{\mu^{1h}} \nabla_q^\top \varphi^h dq + V^{1h}_{\mu^{1h}} \varphi^h_{m^1} \omega^{1h} dq + V^{1h}_{\mu^{1h}} \varphi^h_{m^2} \omega^{2h} dq \\ &\quad + V^{1h}_{\mu^{1h}} \varphi^h_{m^1} dR^1 + V^{1h}_{\mu^{1h}} \varphi^h_{m^2} dR^2. \end{aligned} \tag{2.9}$$

Now let $\alpha^{1h} := V^{1h}_{\mu^{1h}}$, the marginal utility of effective income for person $1h$. By Roy's Identity,

$$-\alpha^{1h} x^{1h^\top} = \nabla_q^\top V^{1h}. \tag{2.10}$$

Substituting (2.10) into (2.9) and rewriting in matrix notation yields

$$du^{1h} = \alpha^{1h} \left[-x^{1h^\top} + \nabla_q^\top \varphi^h + \varphi^h_{m^1} \omega^{1h} + \varphi^h_{m^2} \omega^{2h},\ \varphi^h_{m^1},\ \varphi^h_{m^2} \right] \begin{bmatrix} dq \\ dR^1 \\ dR^2 \end{bmatrix}. \tag{2.11}$$

Because α^{1h} is positive, the directions of change $[dq^\top, dR^1, dR^2]^\top$ that make person $1h$ better off are exactly those changes for which

$$\left[-x^{1h^\top} + \nabla_q^\top \varphi^h + \varphi^h_{m^1} \omega^{1h} + \varphi^h_{m^2} \omega^{2h},\ \varphi^h_{m^1},\ \varphi^h_{m^2} \right] \begin{bmatrix} dq \\ dR^1 \\ dR^2 \end{bmatrix} > 0. \tag{2.12}$$

This is the characterisation that I need in what follows.

It is possible to rearrange (2.12) to reinterpret it in terms of the change in the budget of person $1h$ caused by the policy change. Notice that

$$du^{1h} > 0 \leftrightarrow x^{1h^\top} dq < \nabla_q^\top \varphi^h dq + (\varphi_{m^1}^h \omega^{1h} + \varphi_{m^2}^h \omega^{2h}) dq + \varphi_{m^1}^h dR^1 + \varphi_{m^2}^h dR^2. \quad (2.13)$$

The left–hand side of the second inequality of (2.13) is the change in the cost of the initial consumption bundle. The right–hand side is the change in effective income brought about by two sources. The first three terms capture the effect of relative price changes on intra–family allocation. The fourth and fifth terms give the portion of the changes in the demogrants that are added to (or subtracted from) effective income. Hence, the consumer is made better off if the net increase in the cost of the initial consumption bundle is less than the net increase in effective income brought about in the family decision–making process.

Next, consider the effects of a policy change on person $2h$. I follow the same procedure and notational conventions as in the analysis of person $1h$ to conclude that the condition for $[dq^\top, dR^1, dR^2, dp^\top]^\top$ to bring about a welfare improvement for person $2h$ is

$$du^{2h} > 0 \leftrightarrow$$

$$\left[-x^{2h^\top} - \nabla_q^\top \varphi^h - \varphi_{m^1}^h \omega^{1h} - \varphi_{m^2}^h \omega^{2h} + \omega^{1h} + \omega^{2h},\ 1 - \varphi_{m^1}^h,\ 1 - \varphi_{m^2}^h\right] \begin{bmatrix} dq \\ dR^1 \\ dR^2 \end{bmatrix} > 0. \quad (2.14)$$

An equivalent condition is

$$x^{2h^\top} dq < -\nabla_q^\top \varphi^h dq$$
$$+ (-\varphi^h_{m^1}\omega^{1h} - \varphi^h_{m^2}\omega^{2h})dq + (\omega^{1h} + \omega^{2h})dq + (1 - \varphi^h_{m^1})dR^1 + (1 - \varphi^h_{m^2})dR^2. \tag{2.15}$$

The interpretation of (2.15) is exactly the same as that of (2.13), keeping in mind that φ^h gives the effective income of person $1h$. Person $2h$ spends the remainder of the family budget.

I am now in a position to state a simple, but important, proposition.

Proposition 2.1. *A necessary condition for the welfare of both members of any family h to be improved by a policy change is that the cost of total initial family consumption increase by less than household income.*

Proof: Suppose that $[dq^\top, dR^1, dR^2]^\top$ improves the welfare of both members of household h. Then both (2.13) and (2.15) are satisfied for this household. Adding these inequalities yields

$$(x^{1h^\top} + x^{2h^\top})dq < (\omega^{1h} + \omega^{2h})dq + dR^1 + dR^2. \tag{2.16}$$

But the left–hand side of (2.16) is just the change in the cost of the initial aggregate consumption bundle of the family, while the right–hand side is the net change in full family income. □

Proposition 2.1 has an important (and immediate) corollary.

Corollary 2.1.1 *There are no directions of policy change* $[dq^\top, dR^1, dR^2]^\top$ *satisfying* $dq = 0$ *and* $dR^1 + dR^2 = 0$ *which make both members of any family better off.*

Corollary 2.1.1 speaks to the debate on the identity of the recipient of income transfers from the government to households. It says: when household decisions are made efficiently, a marginal redistribution of a fixed lump–sum between family members cannot make both members better off. Note that this conclusion holds even in the absence of income–pooling as a behavioural hypothesis. Thus any arguments in favour of marginal redistributions between family members must presuppose inefficiencies in the household, or be based on distributional judgments.

Note that the condition expressed in the Proposition 2.1 is not sufficient for the improvement of the welfare of *both* family members. Because intra–family allocations are efficient, ameliorating the family budget position implies that at least one family member is made better off. However, this improvement may come at the expense of the other member of the family. The following example makes this point clear.

Example 2.1

Consider a household whose members have utility functions

$$U^1(x^1, y^1) = \ln x^1 + \ln y^1; \quad U^2(x^2, y^2) = \ln x^2 + 2 \ln y^2. \tag{2.17}$$

Suppose that the family acts so as to maximise the sum of the utilities of its members. Assume also that the family has no endowment, so that all income is lump–sum. It is easy to deduce that the demand functions of the family are:

$$\begin{aligned} x^1(q_x, q_y, R^1, R^2) &= (R^1 + R^2)/5q_x; \quad y^1(q_x, q_y, R^1, R^2) = (R^1 + R^2)/5q_y \\ x^2(q_x, q_y, R^1, R^2) &= (R^1 + R^2)/5q_x; \quad y^2(q_x, q_y, R^1, R^2) = 2(R^1 + R^2)/5q_y. \end{aligned} \tag{2.18}$$

(2.18) entails that the indirect utility functions of the individuals can be written as

$$\begin{aligned} V^1(q_x, q_y, R^1, R^2) &= \ln((R^1 + R^2)/5q_x)) + \ln((R^1 + R^2)/5q_y)); \\ V^2(q_x, q_y, R^1, R^2) &= \ln((R^1 + R^2)/5q_x)) + 2\ln(2(R^1 + R^2)/5q_y)). \end{aligned} \tag{2.19}$$

Now consider a direction of policy reform (dq_x, dq_y, dR^1, dR^2). It follows from (2.19) that

$$\begin{aligned} du^1 &= -\frac{dq_x}{q_x} - \frac{dq_y}{q_y} + \frac{2(dR^1 + dR^2)}{R^1 + R^2} \\ du^2 &= -\frac{dq_x}{q_x} - \frac{2dq_y}{q_y} + \frac{3(dR^1 + dR^2)}{R^1 + R^2} \end{aligned} \tag{2.20}$$

If the initial state of the economy is characterised by $R^1 + R^2 = 5$, $q_x = q_y = 1$, then the initial consumption vector is

$$(x^1, y^1, x^2, y^2) = (1, 1, 1, 2). \tag{2.21}$$

Furthermore, (2.20) becomes

$$\begin{aligned} du^1 &= -dq_x - dq_y + 2(dR^1 + dR^2)/5 \\ du^2 &= -dq_x - 2dq_y + 3(dR^1 + dR^2)/5 \end{aligned} \tag{2.22}$$

Now consider the direction of policy reform $\gamma := (dq_x, dq_y, dR^1, dR^2) = (-1.9, 1, 0, 0)$.[11] The direction γ clearly satisfies (2.16). Evaluating (2.22) for this direction yields $du^1 = 0.9$ and $du^2 = -0.1$.

[11] The length of the vector γ is of no consequence in this example. Any vector pointing in the same direction is feasible and induces changes in the utilities of the individuals of the same signs.

Example 2.1 brings out an important point. Even if aggregate household behaviour is identical to that of a single person, using that behaviour for normative analysis may be inappropriate. The family depicted in the example behaves like a single consumer with Cobb–Douglas preferences. However, directions of policy change that improve the welfare of this 'constructed' consumer may reduce the welfare of one of the actual consumers in the family. Alternatively, we may view the family as having Utilitarian ethics, but the planner does not necessarily respect these ethics when making decisions concerning policy change.

This conclusion is similar in spirit to a finding of Apps and Rees (1988). In their model, the planner may wish to use a linear income tax to redistribute income within a family if the ethics of the planner and of the family differ. In Example 2.1 it is changes in relative consumer prices that are acting as redistributive tools rather than changes in demogrants. This demonstrates the importance of considering the intra–family effects of changes in all possible tax instruments.

I am now in a position to characterise the feasible strictly Pareto–improving directions of change. First some notation is introduced that allows one to consider (2.8),(2.12) and (2.14) jointly.

$$\Gamma^{1h} := \left[-x^{1h^\top} + \nabla_q^\top \varphi^h + \varphi^h_{m^1}\omega^{1h^\top} + \varphi^h_{m^2}\omega^{2h^\top},\, \varphi^h_{m^1},\, \varphi^h_{m^2}\right]^\top;$$

$$\Gamma^{2h} := \left[-x^{2h^\top} - \nabla_q^\top \varphi^h - \varphi^h_{m^1}\omega^{1h^\top} - \varphi^h_{m^2}\omega^{2h^\top} + \omega^{1h^\top} + \omega^{2h^\top},\, 1-\varphi^h_{m^1},\, 1-\varphi^h_{m^2}\right]^\top. \tag{2.23}$$

Then a direction $\gamma := [dq^\top, dR^1, dR^2]^\top$ is both feasible and Pareto–improving if and only if

$$\begin{aligned} &\Gamma^{ih^\top}\gamma > 0,\ h = 1, \ldots, H,\ i = 1, 2; \\ &\Phi\gamma \geq 0, \end{aligned} \tag{2.24}$$

where $\Phi := [\Phi_q, \Phi_{R^1}, \Phi_{R^2}]^\top$. I make some use of the mathematics of cones in the sequel. Hence, I require the following definition.

Definition: Let $\langle x^i \rangle$ be a collection of vectors in $\mathbb{R}^k$. Then the cone generated by $\langle x^i \rangle$, denoted $K(\langle x^i \rangle)$, is defined by:

$$K(\langle x^i \rangle) := \left\{ z \in \mathbb{R}^k | z = \sum \lambda^i x^i, \lambda^i \geq 0 \right\}. \tag{2.25}$$

Before stating a central proposition, I introduce two assumptions.

Assumption A: *There exists a γ such that $\Gamma^{ih^\top}\gamma > 0,\ h = 1, \ldots, H,\ i = 1, 2$.*

That is, there exists a strictly Pareto–improving direction of policy change, ignoring feasibility constraints. This assumption is generally maintained in the literature (Diewert et al. (1989), Guesnerie (1977)), and is a minimal condition for making the problem interesting.

Assumption B: $\Phi \neq 0$.

A typical component of the vector Φ is the marginal cost, measured at initial producer prices, of meeting the changes in demand induced by a change in the corresponding policy instrument. Assumption B simply states that not all of these marginal costs are zero. This assumption rules out the possibility that all directions of policy reform are tight–equilibrium preserving. If this assumption were violated, the requirement of Assumption A renders the problem uninteresting. The planner could implement the change γ mentioned therein (or any other for that matter) while maintaining tight equilibrium.

The next proposition gives a characterisation of the local second–best optima in this environment. Moreover, read contrapositively, it characterises the feasible strictly Pareto–improving directions of policy reform.

Proposition 2.2. *Let Assumptions A and B hold. Then $\Phi \in K(\langle -\Gamma^{ih} \rangle)$ if and only if there exist no feasible strictly Pareto–improving directions of policy change.*

Proof: γ is feasible and Pareto–improving if and only if it satisfies (2.24). But (2.24) is satisfied if and only if

$$\nexists \beta^{ih} \geq 0, \text{ not all } \beta^{ih} = 0 \text{ and } \lambda \geq 0 \text{ such that } \sum_{i,h} \beta^{ih}\Gamma^{ih} + \lambda\Phi = 0_{n+2}, \tag{2.26}$$

by Motzkin's transposition theorem.[12] Equivalently, there are no feasible strictly Pareto–improving directions exactly when there are $\beta^{ih} \geq 0, \forall i, \forall h$, (not all equal

[12] Motzkin's Theorem states: for given matricies A, B and C, with A nonvacuous, exactly one of the systems of relations a) and b) below has a solution: a)$\{Ax >> 0, Bx \geq 0, Cx = 0\}$; b) $\{A^\top y_1 + B^\top y_2 + C^\top y_3 = 0, y_1 \geq 0 \text{ (but } y_1 \neq 0), y_2 \geq 0\}$ (Mangasarian (1969, pp. 28–29)).

zero) and $\lambda \geq 0$ such that

$$\sum_{i,h} \beta^{ih}\Gamma^{ih} + \lambda\Phi = 0_{n+2}. \tag{2.27}$$

Suppose that $\lambda = 0$. Then $\sum_{i,h} \beta^{ih}\Gamma^{ih} = 0$. Then, by Motzkin's Theorem, the top half of (2.24) has no solution. This contradicts Assumption A. Thus, in view of Assumption B, $\lambda > 0$. Hence, I may rearrange (2.27) to give that $\Phi \in K(\langle -\Gamma^{ih}\rangle)$. It is clear that $\Phi \in K(\langle -\Gamma^{ih}\rangle)$ implies the existence of a solution to (2.27). □

Although a somewhat technical condition, $\Phi \in K(\langle -\Gamma^{ih}\rangle)$ states that the vector Φ can be written as a negative linear combination of the vectors $\langle \Gamma^{ih}\rangle$.

A revealed preference version of the argument may shed some light on the condition (2.26). Consider the indirect utility functions of the agents defined in terms of policy variables

$$\tilde{V}^{ih}(q, R^1, R^2) := V^{ih}(q, \mu_i^h(q, q\omega^{1h} + R^1, q\omega^{2h} + R^2)). \tag{2.28}$$

Employing the notation of (2.23), write

$$\Gamma^{ih} = \xi^{ih}\left[\tilde{V}_q^{ih^\top}, \tilde{V}_{R^1}^{ih}, \tilde{V}_{R^2}^{ih}\right]^\top; \quad \xi^{ih} := 1/\alpha^{ih}. \tag{2.29}$$

Suppose there are no feasible Pareto–improving directions. Then the equation in (2.26) *is* satisfied. Then for any $\gamma = [dq^\top, dR^1, dR^2]^\top$

$$\gamma^\top\Big(\sum_{i,h} \beta^{ih}\Gamma^{ih} + \lambda\Phi\Big) = 0_{n+2}. \tag{2.30}$$

Expanding (2.30) using (2.29) yields

$$\sum_{i,h}\Big(\beta^{ih}\xi^{ih}\left\{\tilde{V}_q^{ih}dq+\tilde{V}_{R^1}^{ih}dR^1+\tilde{V}_{R^2}^{ih}dR^2\right\}\Big)+\lambda\gamma^\top\Phi=0. \tag{2.31}$$

Now, assume γ is equilibrium preserving so that it satisfies (2.7). Then $\lambda > 0$ implies

$$\sum_{i,h}\Big(\beta^{ih}\xi^{ih}\left\{\tilde{V}_q^{ih}dq+\tilde{V}_{R^1}^{ih}dR^1+\tilde{V}_{R^2}^{ih}dR^2\right\}\Big)\leq 0. \tag{2.32}$$

Because there is least one positive β^{ih}, the term in braces in (2.32) must be non–positive for at least one individual i, h. However, these terms correspond to the changes in utilities brought about by the policy change γ. When this quantity is non–positive a consumer cannot be made better off. Hence, there must be at least one individual who is not made better off by the change.

2.5. Implementation

One of the most attractive features of the standard literature on optimal tax changes is that the information requirements of implementing the procedure are not prohibitive. Knowledge of net market transactions and aggregate demand and supply elasticities suffice.[13] In the family context, once one has the information needed to calculate the vectors Γ^{ih} and Φ the problem of computing optimal policy changes reduces to finding a solution to (2.24). This can be done by standard linear programming techniques. The vector Φ can be constructed with knowledge of producer prices

[13] Guesnerie (1977) and Wibaut (1987) have excellent discussions of information requirements in some specific settings.

and aggregate demand elasticities with respect to consumer prices, male income and female income.[14] It is important to note that the two sets of income elasticities may differ. The calculation of the vectors Γ^{ih} is more difficult. One needs to know the derivatives of each family sharing rule, and the initial transactions of each *individual.* Bourguignon *et.al.* (1992) demonstrate how the derivatives of the sharing rule can be calculated from demand systems estimated on family–level data. Family budget data is not sufficient, however, to tell one how much of each commodity each member of the household consumes.

2.5.1. Consequences of Ignoring Family Interactions

In light of the difficulty in obtaining sufficient data to compute optimal policy changes in the family setting, it is worth asking what penalty is paid when family interactions are ignored in applied work. One source of error is inaccurate calculation of aggregate demand elasticities, because ignoring family interaction amounts to imposing income pooling in the aggregate. There is, however, a more serious problem. Suppose that, in line with the standard literature, one takes the amelioration of the family budget position as a necessary and *sufficient* condition for improvement in the welfare of each family member. Rearrange (2.16) for each household and form the system of inequalities

$$\begin{aligned} &\Psi^{h^\top}\gamma > 0,\ h = 1, \ldots, H, \\ &\Phi^\top\gamma \geq 0, \end{aligned} \tag{2.33}$$

[14] I interpret the index i as gender to facilitate discussion.

where

$$\Psi^h := \left[-(x^{1h^\top} + x^{2h^\top}) + (\omega^{1h} + \omega^{2h}),\ 1,\ 1\right]^\top. \tag{2.34}$$

Now, the condition for γ to be an 'optimal' direction is that it satisfy (2.33).

In view of Example 2.1, satisfying (2.33) is not sufficient for a direction to be Pareto–improving. Thus, policy based on such a recommendation may send the economy in the wrong direction. If the analysis finds no solutions to (2.33), one reports that the economy is at a local second–best optimum. This conclusion is not in error, because, under the regularity conditions assumed throughout this analysis, satisfying (2.33) is a necessary condition for a direction to be truly optimal. It is also interesting to note that (2.33) is exactly the system that characterises feasible directions of welfare improvement if the family has a well–defined value function and the planner respects family ethics. It is not surprising that family level data suffices for such a planner to make decisions.

The limitations of family–level data can be best exemplified by considering the search for Pareto–improving directions of consumer prices alone, ignoring feasibility constraints. Once again employing Motzkin's theorem, such a direction can be found if there is no solution to

$$\sum_{i,h} \beta^{ih} \Gamma_Q^{ih} = 0; \quad \beta^{ih} \geq 0 \text{ (not all zero)}. \tag{2.35}$$

One sufficient condition for (2.35) to have no solution is that each Γ_Q^{ih} has a strictly positive entry in (for example) the first position.[15] Recalling (2.23), this is the case when, for all h,

$$\begin{aligned} -x_1^{1h} + \frac{\partial \varphi^h}{\partial q_1} + \varphi_{m^1}^h \omega_1^{1h} + \varphi_{m^2}^h \omega_1^{2h} > 0, \\ -x_1^{2h} - \frac{\partial \varphi^h}{\partial q_1} + (1 - \varphi_{m^1}^h)\omega_1^{1h} + (1 - \varphi_{m^2}^h)\omega_1^{2h} > 0. \end{aligned} \tag{2.36}$$

Combining the two inequalities in (2.36) yields

$$(\omega_1^{1h} + \omega_1^{2h}) - (x_1^{1h} + x_1^{2h}) > -x_1^{2h} - \frac{\partial \varphi^h}{\partial q_1} + (1 - \varphi_{m^1}^h)\omega_1^{1h} + (1 - \varphi_{m^2}^h)\omega_1^{2h} > 0. \tag{2.37}$$

(2.37) can be compared to the Diamond–Mirrlees (1971) sufficient condition for the existence of a strictly Pareto–improving direction of price change in the individual–based model. The latter holds when there exists a good that is in net supply by all individuals. Thus, increasing its consumer price leads to a budget amelioration for each individual. Notice that (2.37) is satisfied when the combined net supply of good 1 is sufficiently greater than zero for each family, so that a rote application of the Diamond-Mirrlees condition to combined family net supplies is inappropriate. Although increasing the consumer price of good 1 leads to an amelioration of the family budget position, Example 2.1 indicates that this need not bring about a Pareto–improvement.

Two words of caution are in order at this stage. First, the magnitude by which net supply must exceed zero varies with h. Furthermore, calculation of the middle term of (2.37) requires that the division of consumption and endowments of good 1

[15] A similar argument can be made for the case of strictly negative entries.

within the household be known. For most goods this is clearly more information than is contained in a family budget survey.

CHAPTER 3: Temporary Inefficiencies and Demogrants

3.1. Introduction

One of the curious features of the standard tax reform model is that under certain circumstances, feasible Pareto–improving directions must fail to be tight equilibrium preserving. (See Guesnerie (1977).) That is, the planner may have to choose directions of price change that move the economy inside the production frontier. However, the Diamond–Mirrlees (1971) result indicates that full optimality requires production efficiency. For this reason the above phenomenon is termed temporary inefficiency. Smith (1983) has pointed out that temporary inefficiencies disappear if a poll tax or subsidy is among the instruments available to the planner and if aggregate demand satisfies the Hatta (1977) normality conditions.

The planner considered here has potentially two lump–sum transfers available, so we might expect temporary inefficiencies to be ruled out on similar grounds. However, the agents in the present model interact differently than they do in the standard model. In a sense, pairs of agents are forced to behave cooperatively. It is conceivable that this may result in non–standard responses to demogrants. In this Chapter, I turn my attention to this issue.

It is also helpful at this point to recall that a change is non–tight equilibrium preserving if the bottom line of (2.24) holds with strict inequality.

Suppose that the planner can make independent changes in the poll subsidies. Then the following restatement of the theorem of Guesnerie (1977, Proposition 4, pp. 189–190) on this matter obtains in the present context.

Proposition 3.1 *Under the Assumptions A and B, the following statements hold:*

i) $\Phi \in K(\langle \Gamma^{ih} \rangle)$ *if and only if there exist strictly Pareto–improving directions, all of which are non–tight equilibrium–preserving.*

ii) $\Phi \in K(\langle \Gamma^{ih} \rangle)^C \cap K(\langle -\Gamma^{ih} \rangle)^C$ *if and only if there exist strictly Pareto–improving directions that are tight equilibrium–preserving.*

Proof: i) $\Phi \in K(\langle \Gamma^{ih} \rangle)$ exactly when there exist $\beta^{ih} \geq 0$ satisfying

$$\sum \beta^{ih} \Gamma^{ih} - \Phi = 0. \tag{3.1}$$

Assumption B ensures that at least one β^{ih} is positive. By Motzkin's theorem (with $A = [\Gamma^{11}, \ldots, \Gamma^{2H}]^\top$ and $B = -\Phi$), (3.1) implies the following has no solution

$$\Gamma^{ih^\top} \gamma > 0, \forall i, \forall h; \quad \Phi^\top \gamma \leq 0. \tag{3.2}$$

In particular, there are no strictly Pareto–improving tight equilibrium–preserving directions of reform. In order for Assumption A to be satisfied, there must exist a γ for which

$$\Gamma^{ih^\top} \gamma > 0, \forall i, \forall h. \tag{3.3}$$

By (3.2), $\Phi^\top \gamma > 0$ for any such γ. The 'only if' part of Statement i) follows.

Conversely, let

$$\Gamma^{ih^\top} \gamma > 0, \forall i, \forall h; \quad \Phi^\top \gamma > 0 \tag{3.4}$$

have a solution and let

$$\Gamma^{ih^\top} \gamma > 0, \forall i, \forall h; \quad \Phi^\top \gamma = 0 \tag{3.5}$$

have no solution. Apply Motzkin's Theorem to (3.5) to conclude that there exists $\beta^{ih} \geq 0$ (not all equal zero) and a $\lambda \in I\!R$ such that

$$\sum \beta^{ih} \Gamma^{ih} + \lambda \Phi = 0. \tag{3.6}$$

By (3.4) and Proposition 2.2, $\Phi \notin K(\langle -\Gamma^{ih} \rangle)$. Hence, $\lambda \leq 0$. By Assumption A, $\lambda \neq 0$. (The argument is identical to the one used in the proof of Proposition 2.2.) Thus, $\lambda < 0$. Rearranging (3.6) yields $\Phi \in K(\langle \Gamma^{ih} \rangle)$.

ii) This follows from Statement i) and Proposition 2.□

Smith (1983) rules out case (i) of Proposition 3.1 by assuming that the Hatta (1977) conditions are satisfied; that is, an increase in a lump–sum transfer leads to a positive change in the cost of (net) demand, evaluated at the original producer prices.[16] This assumption is obviously satisfied when producer and consumer prices coincide, provided that consumers are nonsatiated. The intuition behind this result is clear. Suppose that the planner changes consumer prices in such a way that everyone

[16] The original Hatta conditions impose this restriction on compensated aggregate demand. In the sequel, I impose similar conditions on uncompensated aggregate demand.

is better off and the resulting equilibrium is non–tight. Then there exists an excess supply of goods. Suppose now that the planner redistributes this surplus with a lump–sum transfer. Everyone is made still better off, and by the restriction placed on aggregate demand, some of the surplus is consumed. The planner could choose an increase in the poll subsidy large enough to get rid of the entire surplus.

In the present context, it is not guaranteed that an increase in a specific demogrant makes all consumers better off. Something must be said about sharing within families before such a conclusion can obtain.

Assumption C: $0 < \varphi^h_{m^1} < 1,\ \forall h\ and\ 0 < \varphi^h_{m^2} < 1,\ \forall h.$

That is, additions to the lump–sum grants are 'split' in the usual sense of the word. This property need not hold in general, due to the presence of endowments. As long as the planner may increase the demogrant afforded to each person, Assumption C entails Assumption A of the preceding section because an increase in either demogrant makes all consumers better off.

In what follows I have occasion to use the following assumptions, each of which is in the spirit of Smith's restriction.

Assumption N1: $\bar{p}^\top \nabla_{R^1} x(\bar{q}, \bar{R}^1, \bar{R}^2) > 0.$

Assumption N2: $\bar{p}^\top \nabla_{R^2} x(\bar{q}, \bar{R}^1, \bar{R}^2) > 0.$

Like the Hatta conditions, Assumptions N1 and N2 are a form of normality conditions on aggregate demand. It turns out that temporary inefficiencies may be ruled out when either N1 or N2 holds. This is the content of the next proposition.

Proposition 3.2 *Let Assumptions B and C hold. Then strictly Pareto–improving directions of policy change with temporary inefficiencies cannot arise if either Assumption N1 or N2 holds.*

Proof: Proposition 3.1 states that temporary inefficiencies occur exactly when there exist $\beta^{ih} \geq 0$ such that $\Phi = \sum_{i,h} \beta^{ih}\Gamma^{ih}$. The last two rows of this equality are:

$$\begin{aligned} -\bar{p}^{\top}\nabla_{R^1}x(\bar{q}, \bar{R}^1, \bar{R}^2) &= \sum_h \beta^{1h}\varphi^h_{m^1} + \sum_h \beta^{2h}(1-\varphi^h_{m1}), \\ -\bar{p}^{\top}\nabla_{R^2}x(\bar{q}, \bar{R}^1, \bar{R}^2) &= \sum_h \beta^{1h}\varphi^h_{m^2} + \sum_h \beta^{2h}(1-\varphi^h_{m^2}). \end{aligned} \tag{3.7}$$

Assumption C allows me to conclude that the right–hand sides of each equation in (3.7) is nonnegative. N1 precludes the top line of (3.7) from holding. N2 precludes the second.□

That only one of N1 and N2 is required to rule out temporary inefficiencies is not surprising. The planner needs but one instrument to redistribute the surplus to individuals. Note the role played by Assumption C in this framework. It ensures that an increase in either poll subsidy is unanimously preferred by everyone, thereby allowing the surplus to be distributed in a Pareto–improving way.

It may be argued that the planner cannot make lump–sum transfers contingent on the index i. This may be because there is no easily observed characteristic to which it corresponds, or it may be deemed inappropriate to 'discriminate' on the basis of that characteristic. One may also take the view that it is difficult to enact new policies that aim at reducing existing differences in lump–sum payments. Either of these circumstances can be viewed as imposing the restriction $dR^1 = dR^2$ on the directions of policy change available to the planner. It is, therefore, interesting to investigate the conditions under which temporary inefficiencies may arise in this context.

It is necessary to introduce some new notation at this point. Let $\mathbf{0}$ denote the $2N-$ dimensional zero vector. Define also the following sets:

$$\hat{\ell} := \left\{ x \in I\!R^{2N+2} | x = \nu \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix}, \nu \in I\!R \right\}, \tag{3.8}$$

$$\hat{K} := K\left(\langle -\Gamma^{ih} \rangle, \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ -1 \\ 1 \end{bmatrix} \right) \backslash \hat{\ell}, \tag{3.9}$$

$$\check{K} := K\left(\langle \Gamma^{ih} \rangle, \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ -1 \\ 1 \end{bmatrix} \right) \backslash \hat{\ell}. \tag{3.10}$$

$\hat{\ell}$ is the negative 45–degree line in the subspace of $I\!R^{2N+2}$ spanned by the last two elements of the standard basis. Given Assumption C, the sets $\hat{K}$ and $\check{K}$ are supersets of $K(\langle -\Gamma^{ih} \rangle)$ and $K(\langle \Gamma^{ih} \rangle)$, respectively. The first $2N$ components of any vector in $\hat{K}$ must be generated as a semi–positive linear combination of the first $2N$ components of vectors $\langle -\Gamma^{ih} \rangle$. Furthermore, the projection of any vector in $\hat{K}$ onto its last two

components lies off the negative 45–degree line. $\check{K}$ bears an analogous relation to $K(\langle\Gamma^{ih}\rangle)$. Bearing these notations in mind, the following obtains.

Proposition 3.3 *Let Assumptions B and C hold. Then the following hold.*

i) $\Phi \in \hat{K}$ *if and only if there are no feasible strictly Pareto–improving directions of policy reform satisfying* $dR^1 = dR^2$.

ii) $\Phi \in \check{K}$ *if and only if there exist strictly Pareto–improving directions of policy change satisfying* $dR^1 = dR^2$, *all of which are necessarily non–tight equilibrium–preserving.*

iii) $\Phi \in \hat{K}^C \cap \check{K}^C$ *if and only if there exists tight equilibrium–preserving directions of policy change satisfying* $dR^1 = dR^2$.

Proof: i) There are no feasible strictly Pareto–improving directions of change satisfying $dR^1 = dR^2$ if and only if the following has no solution

$$\Gamma^{ih^\top}\gamma > 0, \forall i, h; \quad \Phi^\top\gamma \geq 0; \quad [\mathbf{0}^\top, 1, -1]\gamma = 0. \tag{3.11}$$

But, by Motzkin's theorem, (3.11) has no solution exactly when

$$\sum_{i,h} \beta^{ih}\Gamma^{ih} + \lambda\Phi + \kappa \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix} = 0;\ \beta^{ih} \geq 0, \text{ some}\, \beta^{ih} > 0,\ \lambda \geq 0,\ \kappa \in I\!R \tag{3.12}$$

has a solution. Suppose, by way of contradiction, that $\lambda = 0$. Then the last two rows of (3.12) become

$$\begin{aligned} \sum_h \beta^{1h}\varphi^h_{m^1} + \sum_h \beta^{2h}(1 - \varphi^h_{m^1}) + \kappa = 0, \\ \sum_h \beta^{1h}\varphi^h_{m^2} + \sum_h \beta^{2h}(1 - \varphi^h_{m^2}) - \kappa = 0. \end{aligned} \tag{3.13}$$

Now, by Assumption C, the top line of (3.13) implies $\kappa < 0$, whereas the bottom line of (3.13) implies $\kappa > 0$. A contradiction ensues. Therefore, $\lambda > 0$. Rearranging (3.12) yields that $\Phi \in \hat{K}$. It is a matter of straightforward computation to show that $\Phi \in \hat{K}$ implies the existence of a solution to (3.12) .

ii) $\Phi \in \check{K}$ if and only if there exist $\beta^{ih} \geq 0$ (not all zero), and $\kappa \in I\!R$ satisfying

$$\Phi = \sum_{i,h} \beta^{ih} \Gamma^{ih} + \kappa \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix}. \tag{3.14}$$

Now, by Motzkin's Theorem, it must be the case that

$$\Gamma^{ih^\top} \gamma > 0, \forall i, h; \quad \Phi^\top \gamma \leq 0; \quad [\mathbf{0}^\top, 1, -1]\gamma = 0 \tag{3.15}$$

has no solution. In particular, there is no solution to

$$\Gamma^{ih^\top} \gamma > 0, \forall i, \forall h; \quad \Phi^\top \gamma = 0; \quad [\mathbf{0}^\top, 1, -1]\gamma = 0. \tag{3.16}$$

Note that Assumption C implies that there is a solution to

$$\Gamma^{ih^\top} \gamma > 0, \forall i, h; \quad [\mathbf{0}^\top, 1, -1]\gamma = 0. \tag{3.17}$$

(Pick $\gamma = [\mathbf{0}^\top, 1, 1]^\top$; that is, increase both poll subsidies by the same amount.) Hence, there must be a solution to

$$\Gamma^{ih^\top} \gamma > 0, \forall i, \forall h; \quad \Phi^\top \gamma > 0; \quad [\mathbf{0}^\top, 1, -1]\gamma = 0. \tag{3.18}$$

The 'only if' part of Statement ii) follows.

Conversely, suppose there is a solution to (3.18), but that there is no solution to (3.16). Apply Motzkin's Theorem to (3.16) to conclude that there exists $\beta^{ih} \geq 0$ (not all zero), and $\kappa, \lambda \in I\!R$ satisfying

$$\sum_{i,h} \beta^{ih}\Gamma^{ih} + \lambda\Phi + \kappa \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix} = 0. \tag{3.19}$$

When $\lambda = 0$, the last two rows of (3.19) reduce to (3.13). This violates Assumption C. $\lambda > 0$ implies $\Phi \in \hat{K}$. In view of Statement i), this violates condition (3.18). Thus, $\lambda < 0$. Rearranging (3.19) now yields that $\Phi \in \check{K}$.

iii) This statement follows directly from i) and ii). □

It is interesting to compare Proposition 3.3 with Propositions 2.2 and 3.1 when Assumptions B and C (and, hence, A) hold. Because $K(\langle -\Gamma^{ih}\rangle)$ is contained in $\hat{K}$, statement (i) of Proposition 3.3 indicates that there are fewer values of the vector Φ for which feasible strictly Pareto–improving directions of policy reform exist when poll taxes are restricted. This is hardly surprising. Because $K(\langle \Gamma^{ih}\rangle)$ is contained in $\check{K}$, statement (ii) of Proposition 3.3 indicates that there are more values of the vector Φ for which temporary inefficiencies arise when the planner operates within the restricted set of poll taxes.

Some insight into circumstances giving rise temporary inefficiencies is afforded by considering a necessary condition for temporary inefficiencies. Suppose $\Phi \in \check{K}$. Then it must be the case that

$$\bar{p}^\top \nabla_{R^1} x(\bar{q}, \bar{R}^1, \bar{R}^2) \neq \bar{p}^\top \nabla_{R^2} x(\bar{q}, \bar{R}^1, \bar{R}^2). \tag{3.20}$$

In the presence of Assumption N1 (or N2), it is clear that temporary inefficiencies cannot arise when (3.20) is violated. In that case, the population would act as two sub–populations, each inducing the same change in the value (measured at producer prices) of aggregate demand to changes in the demogrant. The intuition underlying Proposition 3.2 would apply.

Given that the restricted planner still has some effective means of using poll subsidies, one might suspect that temporary inefficiencies may be easily ruled out. This can be shown by appealing to the following assumption.

Assumption N3: $\bar{p}^\top(\nabla_{R^1}x(\bar{q},\bar{R}^1,\bar{R}^2)+\nabla_{R^2}x(\bar{q},\bar{R}^1,\bar{R}^2))>0.$

Notice that Assumption N3 implies that one of N1 or N2 must hold, but not necessarily both. It is also consistent with condition (3.20) . The following proposition may come as no surprise.

Proposition 3.4 *Let Assumptions B, C and N3 hold. Then strictly Pareto–improving directions of policy reform with temporary inefficiencies cannot arise when* $dR^1 = dR^2$.

Proof: By Proposition 3.3, temporary inefficiencies can hold only when there exist $\beta^{ih} \geq 0$ (not all equal zero),$\kappa \in I\!R$ satisfying (3.14). Adding the last two lines of (3.14) yields

$$-\bar{p}^{\top}(\nabla_{R^1}x(\bar{q},\bar{R}^1,\bar{R}^2)+\nabla_{R^2}x(\bar{q},\bar{R}^1,\bar{R}^2)) = \sum_h \beta^{1h}(\varphi^h_{m^1}+\varphi^h_{m^2})+\sum_h \beta^{2h}(2-\varphi^h_{m^1}-\varphi^h_{m^2}). \tag{3.21}$$

Assumption N3 implies that the left–hand side of (3.21) is negative, whereas Assumption C implies that the right–hand side of (3.21) is nonnegative. A contradiction ensues. □

The role played by assumption N3 in Proposition 3.4 is clear. It is sufficient to ensure that any surplus generated by a price change will be 'eaten up' if each poll subsidy is increased by the same amount.

I also wish to investigate the power of a planner who can merely redistribute a fixed lump–sum between the family members. Corollary 2.1.1 indicates that the intuitive argument for the elimination of temporary inefficiencies breaks down, since redistribution of a fixed total alone cannot achieve strict Pareto–improvements. However, a planner endowed with such a power is not identical to one who has no lump–sum taxation power at all. In order to state the analogue to Proposition 3.3 in this context, I require some additional notation:

$$\tilde{\ell} := \left\{(x \in I\!R^{2N+2} | x = \nu \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix}, \nu \in I\!R\right\}. \tag{3.22}$$

$$\tilde{K} := K\left(\langle -\Gamma^{ih}\rangle, \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ -1 \\ -1 \end{bmatrix}\right) \backslash \tilde{\ell}. \tag{3.23}$$

$$\bar{K} := K\left(\langle \Gamma^{ih}\rangle, \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ -1 \\ -1 \end{bmatrix}\right) \backslash \tilde{\ell}. \tag{3.24}$$

The line $\tilde{\ell}$ is the 45–degree line in the plane spanned by the last two elements of the standard basis. The sets $\tilde{K}$ and $\bar{K}$ are analogous to $\hat{K}$ and $\check{K}$, respectively, and can be given similar interpretations. Let Φ_Q, Γ_Q^{ih} denote the vectors Φ, Γ^{ih}, respectively, with their last two components deleted.

Proposition 3.5 *Let Assumptions A, B and C hold. Then the following statements hold.*

i) There are no feasible strictly Pareto–improving directions of policy reform that satisfy $dR^1 + dR^2 = 0$ *if and only if* $\Phi \in \tilde{K}$ *or* $\begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix} \in K(\langle \Gamma^{ih}\rangle)$.

ii) There are strictly Pareto–improving directions of policy reform that satisfy $dR^1 + dR^2 = 0$, *all of which are non–tight equilibrium preserving, if and only if* $\Phi \in \bar{K}$ *and* $\begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix} \notin K(\langle \Gamma^{ih}\rangle)$.

iii) There exist tight equilibrium preserving strictly Pareto–improving directions of policy reform that satisfy $dR^1 + dR^2 = 0$ *if and only if* $\Phi \in \tilde{K}^C \cap \bar{K}^C$ *and* $\begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix} \notin K(\langle \Gamma^{ih}\rangle)$.

iv) Moreover, if there exists $\bar{h}$ *such that* $\varphi_{\mu^1}^{\bar{h}} \neq \varphi_{\mu^2}^{\bar{h}}$, *then whenever* $p^\top \nabla_{R^1} x(q, R^1, R^2)$ *and* $p^\top \nabla_{R^2} x(q, R^1, R^2)$ *are distinct, there exists* $\Phi_Q \in K(\langle \Gamma_Q^{ih}\rangle)$ *such that* $\begin{bmatrix} \Phi_Q \\ p^\top \nabla_{R^1} x(q, R^1, R^2) \\ p^\top \nabla_{R^2} x(q, R^1, R^2) \end{bmatrix} \in \bar{K}$.

Proof: i) There are no feasible strictly Pareto–improving directions of change satisfying $dR^1 + dR^2 = 0$ exactly when there is no solution to

$$\Gamma^{ih^\top}\gamma > 0, \forall i, h; \quad \Phi^\top\gamma \geq 0; \quad [\mathbf{0}^\top, 1, 1]\gamma = 0. \tag{3.25}$$

By Motzkin's Theorem, (3.25) has no solution if and only if there exist $\beta^{ih} \geq 0$ (some $\beta^{ih} > 0$), $\lambda \geq 0$,$\kappa \in I\!R$ satisfying

$$\sum_{i,h} \beta^{ih}\Gamma^{ih} + \lambda\Phi + \kappa \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix} = 0. \tag{3.26}$$

When (3.26) holds with $\lambda > 0$, $\Phi \in \tilde{K}$. If $\lambda = 0$, the last line of (3.26) becomes

$$\sum_h \beta^{1h}\varphi^h_{m^2} + \sum_h \beta^{2h}(1 - \varphi^h_{m^2}) + \kappa = 0 \tag{3.27}$$

It follows from Assumption C that $\kappa < 0$. Hence, $[\mathbf{0}^\top, 1, 1]^\top \in K(\langle\Gamma^{ih}\rangle)$. Direct calculation confirms that either $[\mathbf{0}^\top, 1, 1]^\top \in K(\langle\Gamma^{ih}\rangle)$ or $\Phi \in \tilde{K}$ implies the existence of a solution to (3.26).

ii) Let $\Phi \in \bar{K}$ and $[\mathbf{0}^\top, 1, 1]^\top \notin K(\langle\Gamma^{ih}\rangle)$. $\Phi \in \bar{K}$ exactly when there exist $\beta^{ih} \geq 0$ (not all zero) and $\kappa \in I\!R$ such that

$$\Phi = \sum_{i,h} \beta^{ih}\Gamma^{ih} + \kappa \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix} \tag{3.28}$$

By Motzkin's Theorem, (3.28) implies that there is no solution to

$$\Gamma^{ih^\top}\gamma > 0, \forall i, h; \quad \Phi^\top\gamma \leq 0; \quad [\mathbf{0}^\top, 1, 1]\gamma = 0. \tag{3.29}$$

In particular, there is no solution to

$$\Gamma^{ih^\top}\gamma > 0, \forall i, h; \quad \Phi^\top\gamma = 0; \quad [\mathbf{0}^\top, 1, 1]\gamma = 0. \tag{3.30}$$

$[\mathbf{0}^\top, 1, 1]^\top \notin K(\langle\Gamma^{ih}\rangle)$ implies that there does not exist $\beta^{ih} \geq 0$ (not all zero) and $\kappa < 0$ satisfying

$$\sum_{i,h} \beta^{ih}\Gamma^{ih} + \kappa \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix} = 0. \tag{3.31}$$

By Assumption A, (3.31) has no solution with semi–positive β^{ih} and $\kappa = 0$. The last row of (3.31) is exactly (3.27), so that Assumption C rules out the possibility of a solution to (3.31) with semi–positive β^{ih} and $\kappa > 0$. Thus, by Motzkin's Theorem, there exists a solution to

$$\Gamma^{ih^\top}\gamma > 0, \forall i, h; \quad [\mathbf{0}^\top, 1, 1]\gamma = 0. \tag{3.32}$$

Then, by (3.29), there is a solution to

$$\Gamma^{ih^\top}\gamma > 0, \forall i, h; \quad \Phi^\top\gamma > 0; \quad [\mathbf{0}^\top, 1, 1]\gamma = 0. \tag{3.33}$$

Conversely, let (3.33) have a solution, but let there be no solution to (3.30). Apply Motzkin's Theorem to the first and third components of (3.33) to conclude that $[\mathbf{0}^\top, 1, 1]^\top \notin K(\langle\Gamma^{ih}\rangle)$. Because (3.30) has no solution, Motzkin's Theorem implies the existence of $\beta^{ih} \geq 0$ (not all zero) and real numbers λ and κ satisfying (3.26). In view of (the proof of) Statement i), $\lambda \geq 0$ contradicts (3.33). Thus, (3.26) holds with $\lambda < 0$. Hence, $\Phi \in \bar{K}$.

iii) This statement follows from i) and ii).

iv) Take arbitrary $p^\top \nabla_{R^1} x(q, R^1, R^2) \neq p^\top \nabla_{R^2} x(q, R^1, R^2)$. The existence of an $\bar{h}$ as described in the statement ensures that

$$K\left(\langle \begin{bmatrix} \varphi^h_{m^1} \\ \varphi^h_{m^2} \end{bmatrix} \rangle, \langle \begin{bmatrix} 1-\varphi^h_{m^1} \\ 1-\varphi^h_{m^2} \end{bmatrix} \rangle, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = I\!R^2, \tag{3.34}$$

since the set of generators contains at least three noncollinear vectors. Then there exist $\beta^{ih} \geq 0$, at least one nonzero and $\kappa \in I\!R$ such that

$$\begin{aligned} -p^\top \nabla_{R^1} x(q, R^1, R^2) &= \sum_h \beta^{1h} \varphi^h_{m^1} + \sum_h \beta^{2h}(1-\varphi^h_{m^1}) + \kappa \\ -p^\top \nabla_{R^2} x(q, R^1, R^2) &= \sum_h \beta^{1h} \varphi^h_{m^2} + \sum_h \beta^{2h}(1-\varphi^h_{m^2}) + \kappa \end{aligned} \tag{3.35}$$

Now select $\Phi_Q := \sum_{i,h} \beta^{ih} \Gamma^{ih}$, and the result follows. □

A few words of comment on Proposition 3.5 are in order. First of all, suppose that there are no Pareto–improving directions of change in consumer prices alone, ignoring feasibility. Then, by Motzkin's Theorem, $\mathbf{0} \in K(\langle \Gamma^{ih}_Q \rangle)$. When Assumption C holds, this condition is equivalent to $[\mathbf{0}^\top, 1, 1]^\top \in K(\langle \Gamma^{ih} \rangle)$. This is a strengthening of Corollary 2.1.1: Whenever there are no strictly Pareto–improving directions of change in consumer prices alone, adding purely redistributive transfers affords no new Pareto–improving directions of reform. Moreover, $\Phi \in \tilde{K}$ implies $\Phi_Q \in K(\langle -\Gamma^{ih} \rangle)$. Hence, whenever the planner considered in Proposition 3.5 can find no feasible strictly Pareto–improving directions of policy reform, neither can a planner who does not have the power to use any demogrants. This conclusion should come as no surprise, since $dR^1 + dR^2 = 0$ is satisfied whenever $dR^1 = dR^2 = 0$. However, purely redistributive transfers can be used to induce demand responses that make feasible some changes in consumer prices that would otherwise be infeasible.

Statement (iv) of Proposition 3.5 indicates that the restricted planner may face the prospect of temporary inefficiencies regardless of the demand responses to lump–sum taxation. Purely redistributive transfers may induce demand responses that lead the economy toward the production frontier (indeed such a transfer can be found whenever the conditions expressed in statement iv) are satisfied), but, in view of Corollary 2.1.1, these transfers by themselves can bring about no Pareto–improvements.

Notice that Assumption A must be included in the hypothesis of Proposition 3.5, because it is not implied by Assumption C when only purely redistributive transfers are available. To see this, let $\Phi \in \tilde{\ell}$. Then $[dq^\top, dR^1, dR^2] \cdot \Phi = 0$ for all dq and for all purely redistributive changes in lump–sum transfers dR^1, dR^2. There exists a feasible Pareto–improving direction of policy change if and only if there exists a Pareto–improving change in consumer prices alone. Moreover, whenever $p^\top \nabla_{R^1} x(\bar{q}, \bar{R}^1, \bar{R}^2) = -p^\top \nabla_{R^2} x(\bar{q}, \bar{R}^1, \bar{R}^2)$ purely redistributive changes in the demogrants have no effect on the value of aggregate demand (measured at initial producer prices). Hence, the feasibility condition reduces to feasibility of directions of price changes alone. Assumption C also fails to imply Assumption A when there is income pooling in the presence of purely redistributive taxation. Indeed, as Apps and Rees (1988) have shown, when households act as if they are maximising a price and income independent social welfare function, marginal purely redistributive changes

in demogrants have no effect at all on intra–family allocations as long as boundary constraints do not bind.

CHAPTER 4: Optimal Non–linear Taxes for Families

4.1. Introduction

Since the work of Mirrlees (1971) it has been recognised that the design of income tax policy must take into account the asymmetry of information between agents and the planner. When agents have private information about their characteristics upon which the planner wishes to base a taxation scheme, they may have an incentive to mis–report these characteristics to the planner. The planner must design the taxation scheme to prevent this sort of misrepresentation. It is customary in the literature to assume that the hidden differences among agents can be summarised by a single parameter.[17] This assumption has been questioned as a complete description of individuals. It seems even more dubious when decision–making units are comprised of more than one individual.

An example of a planner designing optimal policies for heterogeneous groups is the problem of family income taxation. Within a family, individuals may differ in labour productivity or they may have unequal say in family decisions. These differences influence labour supply behaviour, which, in turn, has consequences for the design of a tax system. In this study, I trace the effects of family interactions

[17] Guesnerie and Seade (1982) provide a characterisation of optimal tax schedules for a finite economy under this assumption. Weymark (1986a, 1986b, 1987) shows how the problem can be decomposed into simpler sub–problems when preferences are quasi–linear, and provides a more detailed description of optimal tax schedules.

on optimal tax schedules, paying particular attention to the role played by diversity within the family.

Some classic questions of public finance can be addressed by considering the non–linear tax problem for families. For instance, the analysis of this problem can contribute to the debate over whether the base for income taxation ought to be family income or individual income. Indeed, the issue of whether all members of the same family should be taxed at the same rate is one of the central questions of this study. I show that taxing distinct members of a family at the same rate is not always optimal. This can also be viewed as a contribution to the debate over the desirability of a uniform "flat" tax.

I consider an economy inhabited by two–person families. Attention is restricted to the case of workers of two productivity types. This results in four possible family compositions. Each family member has preferences over leisure and a consumption good. Families are assumed to act so as to maximise a weighted sum of their members' utilities. The weights are assumed to be independent of incomes. In this way, the family decision process can be described by a single parameter. This parameter is known to the planner.

The planner designs a tax schedule for these families. Families are then free to re–allocate their after–tax incomes to maximise their objectives. Because decisions are made at the family level, self–selection constraints are formulated in such a way that *families* have no incentive to mis–report the types of their members. That is,

families are viewed as decision–making units, differing along two dimensions, namely the productivities of their members. Under the maintained hypothesis that there exists a family social welfare function, there is no loss of generality in considering tax schedules that specify the total tax liability for a family as a function of the pair of before–tax incomes of its members. Hence, the model economy studied here has three goods in it. This contrasts with the work of Guesnerie and Seade (1982), who consider non–linear taxation in a two–good economy.

Because there are three goods in the economy, family indifference surfaces are two–dimensional. Thus, if the indifference surfaces of distinct families cross, their intersection usually occurs along a curve. In particular, it is unreasonable to expect a standard single–crossing property, such as the one posited by Guesnerie and Seade (1982, Assumption B, p. 168), to hold. Nevertheless, the family objectives specified in this analysis possess some special geometric properties. When two families differ along exactly one dimension, certain projections of their indifference surfaces satisfy a single–crossing property. Moreover, the indifference surfaces of any such pair such families intersect along a line parallel to one of the coordinate axes. I demonstrate that these two features of preferences have implications for the structure of optimal tax schedules.

A minimal amount of structure is placed on the objectives of the taxation authority. Only Pareto efficiency with respect to family objectives is posited. Family objectives are assumed to be additively separable, but not quasi–linear. Even with

this minimal amount of structure, it can be shown that individuals in the same family may face different marginal tax rates, so that using total family income as a tax base is not optimal.

The formal analysis is an example of mechanism design with two–dimensional uncertainty, of which the work of Rochet (1995) is the most complete to date.[18] He considers the problem faced by a monopolist wishing to maximise profits by offering a non–linear price schedule for two goods. He derives all the possible optimal pricing mechanisms for the case of linear preferences and quadratic costs. Mechanisms with the qualitative features he describes are possible in the present context. Given the relatively unstructured environment considered here, some additional possibilities arise. Drawing attention to these possibilities demonstrates the strength of the assumption of linearity.

Like the non–linear pricing problem, the taxation problem can be described as a choice among alternatives that satisfy self–selection criteria. There are, however, fundamental differences between the two problems. There is a natural group of consumers whom a monopolist wishes to identify and extract surplus from, those who have a greater taste for its product. *A priori*, there is no group of workers that a planner wishes to tax more heavily than others. It is common to assume that the planner wishes to redistribute income from more able workers to less able workers.

[18] Among the contributions to multi–dimensional income tax problems in the continuous case are Mirrlees (1976) and Seade (1979). Wilson (1995) provides an overview of the mechanism design problem and the associated computational issues.

Nevertheless, it is important to recognise that such an assumption is a value judgment. Moreover, non–linear pricing problems are usually presented in a partial equilibrium context, with the amount of total surplus extracted by the monopolist constrained by the voluntary participation of consumers. An economy–wide materials balance constraint limits the scope of the taxation authority.

Another important distinction between optimal taxation problems and monopoly pricing models is that the objective function of the taxation authority is usually assumed to be increasing in the welfare of each agent, whereas monopolists are modeled as being concerned about profits rather than the welfare of their customers.[19] As Brito *et al.* (1990) have shown, many qualitative features of optimal non–linear tax schedules follow directly from the planner giving positive consideration to the utility of *every* agent. Their analysis is not restricted to the two–good world, nor to the case of agents who differ in only one characteristic. Indeed, they do not use any parameterisation of the differences among agents to derive their results. By exploiting the special structure of the family objectives in the present model, I can make some statements about the tax rates faced by specific families. These kinds of statements cannot be derived in the more general framework of Brito *et al.* (1990).

The remainder of the chapter is organised as follows. Section 2 gives an outline of the economy. The implications of the self–selection constraints are presented in

[19] Another related class of problems, regulation design, features a planner with an objective function that takes both consumers' and producers' surplus into account (cf. Dana (1993)).

Section 3. Section 4 summarises the implications of the Pareto-efficiency assumption. Proofs for this chapter and the following one are collected in an Appendix.

4.2. The Model

Individuals in the economy are assumed to differ according to their productivity. Specifically, there are two types of individuals, indexed by w_L,w_H with $w_L < w_H$. This index corresponds to the efficiency units of labour per unit of labour time supplied by the individual. I assume constant returns to scale in the production sector and a perfectly competitive labour market so that the before-tax income of an individual is given by

$$y_i := w_i l_i, \tag{4.1}$$

where l_i is the labour supplied by person i. There is a single consumption good, c. Individuals have preferences over the consumption good and labour supply given by

$$u_i(c_i, l_i) := U(c_i) - h(l_i). \tag{4.2}$$

The function $U(\cdot)$ is assumed to be continuously differentiable, increasing and strictly concave. It is also assumed that $U'(c)$ tends to positive infinity as c tends to zero. The function $h(\cdot)$ is assumed to be continuously differentiable, increasing and strictly convex.

Families consist of two individuals. There are four types of families: LL, HL, HH and LH. Let $\mathcal{F}$ denote the set of families. Family decisions are assumed to be

consistent with the maximisation of a weighted utilitarian household social welfare function

$$W(u_1, u_2) := u_1 + \gamma u_2, \quad \gamma > 0. \tag{4.3}$$

Note that $W(\cdot)$ is not a symmetric function, so that a family of type HL is not identical to one of type LH. The function $W(\cdot)$ is symmetric only when $\gamma = 1$. Even in this special case, it is not possible to identify families HL and LH *a priori*. As long as the index i attached to an individual is observable, the planner can use this information in setting the tax schedule. One would need to show whether it is indeed optimal to ignore this information by allocating the same bundle of goods to, say, individual 1 in family HL and individual 2 in family LH.

Let x_1, x_2 denote the after–tax incomes of the two individuals in the family and let $x := x_1 + x_2$. Then consumption decisions for a typical household arise as the solution to

$$(P) \qquad \begin{aligned} &\max_{c_1, c_2} \left[U(c_1) - h\left(\frac{y_1}{w_1}\right)\right] + \gamma\left[U(c_2) - h\left(\frac{y_2}{w_2}\right)\right] \\ &\textit{subject to} \quad c_1 + c_2 \leq x_1 + x_2. \end{aligned}$$

I now turn to a description of the important features of the solutions to (P).[20]

Proposition 4.1 *Let $\hat{c}_1(x_1, x_2)$ and $\hat{c}_2(x_1, x_2)$ be the solution functions for (P). Then*

$$\hat{c}_1(x_1, x_2) = \tilde{c}_1(x); \quad \hat{c}_2(x_1, x_2) = \tilde{c}_2(x). \tag{4.4}$$

[20] Throughout this analysis it is assumed that the non–negativity conditions (which are not stated explicitly) are satisfied.

Condition (4.4) states that for a given total family after–tax income, the allocation of consumption within the family does not depend on the identity of the family member who receives the income. This is know as the income pooling condition on family behaviour. It holds when families maximise any Bergson–Samuelson social welfare function, not just for the additive form specified here. Its major implication for this study is the reduction of the number of planner's choice variables per family from four to three. Because the choice variables are consumption levels, the formulation of (P) may lead one to believe that the family is not making any labour supply decisions. This suspicion is untrue. The problem (P) describes how after–tax income is allocated between family members. Labour supply choices are modeled as the choice from the tax menu offered by the planner. Families take the division of consumption and its effects on family welfare into account when choosing how much to work.

Using (4.4), it is possible to define the function

$$V(x) := U(\tilde{c}_1(x)) + \gamma U(\tilde{c}_2(x)), \tag{4.5}$$

the component of family welfare owing to after–tax income. With this notation, I can now state some further properties of solutions to the problem (P).

Proposition 4.2 *Let $\tilde{c}_1(\cdot)$ and $\tilde{c}_2(\cdot)$ be as defined in Proposition 4.1. Then*

i.) $\tilde{c}_1'(x) + \tilde{c}_2'(x) = 1$.

ii.) $\tilde{c}_1(\cdot)$ *and* $\tilde{c}_2(\cdot)$ *are increasing.*

iii.) $V(\cdot)$ is increasing and strictly concave.

Condition i) is a direct result of the increasingness of $U(\cdot)$. It states that a one–unit increase in after–tax income increases total family consumption by one unit. Statement ii) indicates that the family considers the consumption of each member to be a normal good. This follows from the separability of preferences. Increasingness of $V(\cdot)$ follows from the increasingness of individual utility in consumption. While the function $V(\cdot)$ is the sum of concave transformations of the optimal consumption choices, the choice functions need not be concave. Indeed, condtion i) of the proposition implies that no more than one of the choice functions can be strictly concave. However, the same condition prevents both of the choice functions from being convex, and results in the strict concavity of $V(\cdot)$.

In what follows I need to consider the value function for (P), which depends on the before–tax incomes of the family members and on joint family after–tax income. Denote the value function for a family of type i by W^i. Then

$$W^i(x^i, y_1^i, y_2^i) = V(x^i) - h\left(\frac{y_1^i}{w_1^i}\right) - \gamma h\left(\frac{y_2^i}{w_2^i}\right). \tag{4.6}$$

I take the consumption good as numeraire and assume that the producer price of effective labour is one. Let π^i be the proportion of families of type i in the population. Then the materials balance constraint for the economy can be written as

$$(F) \qquad \sum_i \pi^i x^i \leq \sum_i \pi^i y_1^i + \sum_i \pi^i y_2^i.$$

The lack of complete information prohibits the use of optimal lump–sum taxation. Instead, the planner must design an allocation of goods that satisfies the self–selection conditions

$$(SS) \qquad W^i(x^i, y_1^i, y_2^i) \geq W^i(x^j, y_1^j, y_2^j), \ \forall i \neq j.$$

That is, each family must (weakly) prefer the bundle of goods designed for it to the bundle intended for any other family. The relations (SS) represent the natural incentive–compatibility constraints in this environment, as decisions are made by families.

It follows from the taxation principle (cf. Guesnerie (1981)) that the problem of designing a tax schedule for these families is equivalent to offering a menu of alternatives satisfying (SS). Because of income pooling, there is no loss of generality in considering tax functions that specify the total tax liability of a family for a given *ordered* pair of before–tax incomes. For fixed before–tax incomes it is merely a transformation of variables to consider total family after–tax income rather than total tax liability as the decision variable of the taxation authority. Such a tax schedule must be anonymous in that each family faces the same budget set. It need not be anonymous at the individual level, because the tax paid by one member of a family may depend on the choices of her partner. Clearly, when families choose labour–consumption bundles to maximise their welfare from an anonymous tax schedule the outcomes satisfy (SS).

Furthermore, it has been shown by Guesnerie (1981) that for any allocation that satisfies (SS) and (F) a tax schedule can be constructed that induces the families to choose that allocation. The tax schedule constructed by Guesnerie requires the possibility of offering an infinitely negative amount of after–tax income to a family with before–tax incomes other than those that arise from a truth–telling game. When negative after–tax incomes are infeasible (as they are assumed to be here), these "punishment" allocations cannot be used. In the current model, it is possible to support any allocation which satisfies (SS) and (F) with a tax schedule that does not require negative consumption. To see this, take an allocation that satisfies (SS) and (F). Let $\mathcal{S}$ denote the set of family bundles in that allocation. Define the sets

$$\mathcal{L} := \{(y_1, y_2) | (y_1, y_2, x) \in \mathcal{S}\}; \quad \chi(y_1, y_2) := \{x | (y_1, y_2, x) \in \mathcal{S}\}. \tag{4.7}$$

For any ordered pair (y_1, y_2), $\chi(y_1, y_2)$ is a singleton, so with a slight mis–use of notation, I consider $\chi(\cdot)$ to be a function from $\mathcal{L}$ into $\mathbb{R}_+$. Now, construct a tax function $T\colon \mathbb{R}^2_+ \to \mathbb{R}$ by

$$T(y_1, y_2) := \begin{cases} y_1 + y_2 - \chi(y_1, y_2), & \text{if } (y_1, y_2) \in \mathcal{L}; \\ y_1 + y_2, & \text{if } (y_1, y_2) \notin \mathcal{L}. \end{cases} \tag{4.8}$$

Given the tax function $T(\cdot)$, the budget set faced by each family is

$$\mathcal{B} := \mathcal{S} \cup \{(y_1, y_2, 0) | (y_1, y_2) \notin \mathcal{L}\}. \tag{4.9}$$

Given that $U'(c)$ tends to positive infinity as c tends to zero, family indifference surfaces do not cross the (y_1, y_2)–plane. Thus, families choose only the bundles from $\mathcal{B}$ that are contained in $\mathcal{S}$. Moreover, because the bundles in $\mathcal{S}$ satisfy (SS), each

family chooses the bundle intended for it in the allocation (SS). Hence, the resulting choices from the tax schedule also satisfy (F).

The planner seeks to maximise a social objective. There are two possible sets of arguments for a welfare–based objective: individual utilities and family welfare levels. The former is more closely related to the standard value judgments of individualism and welfarism. However, the arguments of the planner's objective function then fail to coincide with the criterion functions of the agents (here, families) in the economy. Hence, the formal analysis of this case is closely related to the work on income taxation with non–welfarist objectives (Seade (1980), Kanbur, Keen and Tuomala (1994)). The latter class of planner's objectives arise inevitably from treating a family as a homogeneous unit. Using family welfare levels as the arguments of a planner's objective can also be interpreted as the planner respecting the ethics embodied in the family social welfare function. The resulting analysis is formally related to the recent work on optimal non–linear policies in environments of two–dimensional uncertainty. (Dana (1993), Rochet (1995), Armstrong (1996)).

In order to place this study in the more familiar context of multi–dimensional screening I choose to take the family as the basic unit of welfare analysis. Specifically, I analyse the solutions to the problem

$$(PF) \qquad \max_{\mathbf{y_1},\mathbf{y_2},\mathbf{x}} Z(W^{LL}, W^{HL}, W^{HH}, W^{LH}) \textit{ subject to } \text{(SS) and (F)},$$

where $\mathbf{x} := (x^{LL}, x^{HL}, x^{HH}, x^{LH})$, $\mathbf{y_i} := (y_i^{LL}, y_i^{HL}, y_i^{HH}, y_i^{LH})$, $i = 1, 2$ and the function $Z(\cdot)$ is assumed to be increasing, continuously differentiable and concave.[21]

4.3. Self–Selection

The structure of family objectives, summarised by (4.6), influences the nature of the self–selection constraints. I now turn to an elucidation of the important features of family objectives and the implications these features have for allocations that satisfy the constraints (SS).

Consider, first, the marginal rates of substitution between before–tax incomes and after–tax income, given by

$$MRS^i_{y_1,x} := \frac{h'(\frac{y_1^i}{w_1^i})}{w_1^i V'(x^i)}, \quad MRS^i_{y_2,x} = \frac{\gamma h'(\frac{y_2^i}{w_2^i})}{w_2^i V'(x^i)}. \tag{4.10}$$

It is clear from (4.10) that any differences among the marginal rates of substitution of different families arise from the structure of the function $h(\cdot)$. For this analysis, two properties of $h(\cdot)$, both of which are direct consequences of its convexity, are crucial. These properties are given in the following two lemmas.

Lemma 4.1 *For all* y, $\frac{1}{w_L}h'(\frac{y}{w_L}) > \frac{1}{w_H}h'(\frac{y}{w_H})$.

[21] The problem (PF) may also be interpreted as the problem faced by a planner who wishes to design a tax scheme for individuals with multiple characteristics and preferences represented by (4.6).

An important implication of Lemma 4.1 is that holding y_1, say, constant, preferences in (y_2, x)–space satisfy the single–crossing property. That is, the projections of indifference surfaces onto (y_2, x)–space are flatter for families with a person of higher ability in position 2. It is natural to expect such a property to hold, as individuals of low type must give up more leisure time than their high–type counterparts to gain an equal amount of before–tax income. Hence, families with low–type individuals would require more additional consumption to compensate them for increases in before–tax income.

Lemma 4.2 *For any $(\tilde{y}, \hat{y})$, $\tilde{y} \geq \hat{y}$ if and only if*

$$h\left(\frac{\tilde{y}}{w_L}\right) - h\left(\frac{\hat{y}}{w_L}\right) \geq h\left(\frac{\tilde{y}}{w_H}\right) - h\left(\frac{\hat{y}}{w_H}\right). \tag{4.11}$$

Lemma 4.2 states that, viewed as a function of y and w, $h(\cdot)$ has decreasing–differences in y (cf. Topkis (1978)). It is important to note that no such property holds for $W(\cdot)$.

A standard feature of optimal income tax models is that more able individuals receive both a higher before–tax and a higher after–tax income (Weymark (1986a)). One would not expect this result to obtain in the present context given that the notion of a "more able" family is not immediate. At least, the immediate concept of a more able family does not completely order the families. It is, however, possible to define a partial ordering on the set of families in a natural way.

Definition: The relation $\geq_F$ on $\mathcal{F}$, the set of families, is defined by: $i \geq_F j$ if and only if $w_1^i \geq w_1^j$ and $w_2^i \geq w_2^j$. The relation $>_F$ is defined to be the asymmetric component of $\geq_F$.

Figure 1 provides a diagrammatic representation of the set of family types. The relation $\geq_F$ orders all families but the pair $\{HL, LH\}$, located at cross–corners and off the 45–degree line.

The self–selection constraints place some structure on the pattern of before–tax incomes, especially for those families that can be compared using the partial order $\geq_F$. This is the content of the next two propositions.

Proposition 4.3 *Any allocation that satisfies (SS) also satisfies:*

i.) $y_1^{HH} \geq y_1^{LH}$. *Moreover, if* $y_2^{HH} > y_2^{LH}$, *then* $x^{HH} > x^{LH}$ *and if* $y_2^{HH} = y_2^{LH}$, *then* $x^{HH} \geq x^{LH}$ *(with equality if and only if* $y_1^{HH} = y_1^{LH}$*).*

ii.) $y_1^{HL} \geq y_1^{LL}$. *Moreover, if* $y_2^{HL} > y_2^{LL}$, *then* $x^{HL} > x^{LL}$ *and if* $y_2^{HL} = y_2^{LL}$, *then* $x^{HL} \geq x^{LL}$ *(with equality if and only if* $y_1^{HL} = y_1^{LL}$*).*

iii.) $y_2^{HH} \geq y_2^{HL}$. *Moreover, if* $y_1^{HH} > y_1^{HL}$, *then* $x^{HH} > x^{HL}$ *and if* $y_1^{HH} = y_1^{HL}$, *then* $x^{HH} \geq x^{HL}$ *(with equality if and only if* $y_2^{HH} = y_2^{HL}$*).*

iv.) $y_2^{LH} \geq y_2^{LL}$. *Moreover, if* $y_2^{LH} > y_2^{LL}$, *then* $x^{LH} > x^{LL}$ *and if* $y_2^{LH} = y_2^{LL}$, *then* $x^{LH} \geq x^{LL}$ *(with equality if and only if* $y_2^{LH} = y_2^{LL}$*).*

Proposition 4.3 states that given an equally productive partner, a person of high productivity will earn at least as much before–tax income as a low–productivity person. Abstracting from differences among individuals 2 (by, say, considering an economy populated entirely by families of types HH and LH) produces a three–good economy with one–dimensional uncertainty. That is, when focusing on families along an edge of the type space depicted in Figure 1, the planner is facing what is essentially a one–dimensional screening problem with multiple instruments. Proposition 4.3 states that self–selection imposes a monotonicity property on the allocation of y_1, the good over which agents differ in a way that is unknown to the planner. Such worlds have been studied in an environmental regulation context by van Egteren (1996). He reports that self–selection requires a monotonicity property on pollution control standards, the cost of which is the only source of private information in his model.

Proposition 4.3 has a geometric interpretation, which I now give for clause i). Denote by $(\bar{y}_1^{LH}, \bar{y}_2^{LH}, \bar{x}^{LH})$ the bundle designed for family LH. Consider the indifference surface of family LH through this point. In order for the LH–HH self–selection constraint to be satisfied, the bundle offered to family HH must lie on or below this surface. To satisfy the HH–LH self–selection constraint, the bundle designed for family HH must lie on or above the indifference surface of family HH passing through the point $(\bar{y}_1^{LH}, \bar{y}_2^{LH}, \bar{x}^{LH})$. According to Proposition 4.3, all points lying between these two surfaces have the feature that $y_1 \geq \bar{y}_1^{LH}$. In fact, the intersection of these surfaces is the line with equation: $y_1 = \bar{y}_1^{LH}$.

Figures 2 and 3 provide a two–dimensional representation of Proposition 4.3. To understand these figures, it is useful to think in terms of "pseudo–indifference curves" for the families. A pseudo–indifference curve is a level set of a partial welfare function, showing points that give equal amounts of welfare derived from two of the three goods (say, y_1 and x). Given the additively separable form of family objectives, it may also be interpreted as a slice of a family indifference surface (drawn for a constant y_2).

First, I consider Figure 2. Let $\bar{u}^{LH}$ denote the welfare level of family LH at the allocation designed for it. The point D represents $(\bar{y}_1^{LH}, \bar{x}^{LH})$. Now let

$$\tilde{u}^{LH} := V(\bar{x}^{LH}) - h\left(\frac{\bar{y}_1^{LH}}{w_L}\right) = \bar{u}^{LH} + \gamma h\left(\frac{\bar{y}_2^{LH}}{w_H}\right). \tag{4.12}$$

$\tilde{u}^{LH}$ is the label on a pseudo–indifference curve for family LH in (y_1, x)–space, namely the one that depicts the set of bundles that give family LH welfare level $\bar{u}^{LH}$, holding y_2^{LH} at $\bar{y}_2^{LH}$. This curve is denoted by LH in Figure 2. The value in parentheses next to the label LH, $\bar{y}_2^{LH}$, serves as a reminder that the curve LH is the $\bar{y}_2^{LH}$–slice of the indifference surface of family LH through $(\bar{y}_1^{LH}, \bar{y}_2^{LH}, \bar{x}^{LH})$. Clearly, the curve LH passes through the point D. To analyse self–selection, it important to consider the welfare of family LH at the allocation designed for some *other* family (here, family HH). With this purpose in mind, denote the allocation of family HH by $(\bar{y}_1^{HH}, \bar{y}_2^{HH}, \bar{x}^{HH})$ and define

$$\hat{u}^{LH} := \bar{u}^{LH} + \gamma h\left(\frac{\bar{y}_2^{HH}}{w_H}\right). \tag{4.13}$$

The LH–HH self–selection condition implies

$$\hat{u}^{LH} \geq V(\bar{x}^{HH}) - h(\frac{\bar{y}_1^{HH}}{w_L}). \tag{4.14}$$

That is, $\hat{u}^{LH}$ is the label of a pseudo–indifference curve for family LH that all (y_1^{HH}, x^{HH}) pairs that satisfy the LH–HH constraint must lie below. This curve is the slice of the indifference surface of family LH through $(\bar{y}_1^{LH}, \bar{y}_2^{LH}, \bar{x}^{LH})$ at $y_2 = \bar{y}_2^{HH}$. Note that $\tilde{u}^{LH} \geq \hat{u}^{LH}$ if and only if $\bar{y}_2^{LH} \geq \bar{y}_2^{HH}$. The curve labeled LH$'$ in Figure 2 corresponds to the $\hat{u}^{LH}$ indifference curve under the assumption that $\bar{y}_2^{HH}$ is greater than $\bar{y}_2^{LH}$.

Consider now the possibility of family HH mimicking family LH. Let $\bar{u}^{HH}$ denote the welfare level of family HH at $(\bar{y}_1^{LH}, \bar{y}_2^{LH}, \bar{x}^{LH})$. Define

$$\tilde{u}^{HH} := V(\bar{x}^{LH}) - h(\frac{\bar{y}_1^{LH}}{w_H}) = \bar{u}^{HH} + \gamma h(\frac{\bar{y}_2^{LH}}{w_H}). \tag{4.15}$$

$\tilde{u}^{HH}$ is the label on the pseudo–indifference curve for family HH in (y_1, x)–space depicting the (y_1^{HH}, x^{HH}) pairs that provide family HH the welfare level $\bar{u}^{HH}$, holding y_2^{HH} at $\bar{y}_2^{LH}$. This curve is denoted by HH in Figure 2. Notice that it passes through the point D. But the before–tax income of person 2 in family HH is $\bar{y}_2^{HH}$, not $\bar{y}_2^{LH}$. Thus, when analysing the behaviour of family HH it is more appropriate to consider the pseudo–indifference curve given by the equation

$$\hat{u}^{HH} := \bar{u}^{HH} + \gamma h(\frac{\bar{y}_2^{HH}}{w_H}). \tag{4.16}$$

The HH–LH self–selection constraint requires:

$$V(\bar{x}^{HH}) - h(\frac{\bar{y}_1^{HH}}{w_H}) - \gamma h(\frac{\bar{y}_2^{HH}}{w_H}) \geq \bar{u}^{HH}. \tag{4.17}$$

Conditions (4.16) and (4.17) imply

$$\hat{u}^{HH} \leq V(\bar{x}^{HH}) - h(\frac{\bar{y}_1^{HH}}{w_H}). \tag{4.18}$$

Thus, we may interpret $\hat{u}^{HH}$ as the label on the pseudo–indifference curve for family HH above which all (y_1^{HH}, x^{HH}) pairs satisfying the HH–LH constraint must lie. From (4.15) and (4.16) it follows that $\tilde{u}^{HH} \geq \hat{u}^{HH}$ if and only if $\bar{y}_2^{LH} \geq \bar{y}_2^{HH}$. Under the assumption that $\bar{y}_2^{HH}$ is greater than $\bar{y}_2^{LH}$, we may draw the $\hat{u}^{LH}$ indifference curve in Figure 2 as the curve HH$'$. Comparing (4.13) with (4.16), we can see that the vertical distance between the curves LH and LH$'$ is identical to the vertical distance between the curves HH and HH$'$. Thus, LH$'$ and HH$'$ intersect at a point like D$'$, directly above D.

The discussion of the preceding two paragraphs has shown that when y_2^{HH} is equal to $\bar{y}_2^{HH}$, all combinations of y_1^{HH} and x^{HH} that satisfy both the LH–HH and HH–LL self–selection constraints lie in the wedge between the curves LH$'$ and HH$'$, assuming $\bar{y}_2^{HH} > \bar{y}_2^{LH}$. That is, this wedge is the $\bar{y}_2^{HH}$–slice of the region in 3-space corresponding to the bundles for family HH that satisfy both the HH–LH and LH–HH self–selection constraints, given the bundle of family LH. All points in that wedge are to the right of the point D and have at least as much y_1 and more x than the point D.

Figure 3 illustrates the argument for $\bar{y}_2^{HH} < \bar{y}_2^{LH}$. The point of intersection of the relevant pseudo–indifference curves is directly below the point D, with the self–selection slice depicted by the shaded region. Notice that there are points in this

region that have a lower x than the allocation D has. Hence, by itself, Proposition 4.3 does not provide enough information to order after–tax incomes.

In order to derive the validity of Proposition 4.3 from these diagrammatic arguments, it is enough to note that for any choice of $\bar{y}_2^{HH}$, the intersection of the $\bar{y}_2^{HH}$–pseudo–indifference curves for families LH and HH lies on the vertical line in (y_1, x)–space through the point D.[22]

Proposition 4.4 *Let (SS) hold. Then*

i.) $y_1^{HH} < y_1^{LL}$ *implies* $y_2^{HH} > y_2^{LL}$. *Moreover,* $y_1^{HH} = y_1^{LL}$ *implies either: a)* $y_2^{HH} > y_2^{LL}$ *and* $x^{HH} > x^{LL}$, *or b)* $y_2^{HH} = y_2^{LL}$ *and* $x^{HH} = x^{LL}$. *Furthermore,* $y_1^{HH} > y_1^{LL}$ *and* $y_2^{HH} \geq y_2^{LL}$ *implies* $x^{HH} > x^{LL}$.

ii.) $y_2^{HH} < y_2^{LL}$ *implies* $y_1^{HH} > y_1^{LL}$. *Moreover,* $y_2^{HH} = y_2^{LL}$ *implies either: a)* $y_1^{HH} > y_1^{LL}$ *and* $x^{HH} > x^{LL}$, *or b)* $y_1^{HH} = y_1^{LL}$ *and* $x^{HH} = x^{LL}$. *Furthermore,* $y_2^{HH} > y_2^{LL}$ *and* $y_1^{HH} \geq y_1^{LL}$ *implies* $x^{HH} > x^{LL}$.

iii.) $y_1^{HL} < y_1^{LH}$ *implies* $y_2^{LH} > y_2^{HL}$ *and* $x^{LH} > x^{HL}$. *Moreover,* $y_1^{HL} = y_1^{LH}$ *implies either: a)* $y_2^{LH} > y_2^{HL}$ *and* $x^{LH} > x^{HL}$, *or b)* $y_2^{LH} = y_2^{HL}$ *and* $x^{LH} = x^{HL}$.

iv.) $y_2^{LH} < y_2^{HL}$ *implies* $y_1^{HL} > y_1^{LH}$ *and* $x^{HL} > x^{LH}$. *Moreover,* $y_2^{LH} = y_2^{HL}$ *implies either: a)* $y_1^{HL} > y_1^{LH}$ *and* $x^{HL} > x^{LH}$, *or b)* $y_1^{HL} = y_1^{LH}$ *and* $x^{HL} = x^{LH}$.

[22] This is also true for the slice at $\bar{y}_2^{HH} = \bar{y}_2^{LH}$, the case for which the pseudo–indifference curves through $(\bar{y}_1^{LH}, \bar{x}^{LH})$ need not be shifted.

Statements i.) and ii.) of Proposition 4.4 are consistent with a general notion of more productive individuals receiving a higher before–tax income, as at least one member of family HH must earn more before–tax income than the corresponding member of family LL. The only exception to this tendency occurs when families HH and LL receive exactly the same bundle.

The geometric intuition behind statement ii.) of Proposition 4.4 is presented in Figure 4. Let $(\bar{y}_1^{LL}, \bar{y}_2^{LL}, \bar{x}^{LL})$ denote the bundle designed for family LL and let the point A be the projection of this point onto the (y_1, x) plane. The curves LL and HH are the pseudo indifference curves through the bundle designed for family LL at $\bar{y}_2^{LL}$ for families LL and HH, respectively. Suppose $\bar{y}_2^{HH} < \bar{y}_2^{LL}$. Following the notational conventions used in the discussion of Proposition 4.3, define the number

$$\hat{u}^{LL} := \bar{u}^{LL} + \gamma h\left(\frac{\bar{y}_2^{HH}}{w_L}\right). \tag{4.19}$$

The LL–HH self–selection condition implies

$$\hat{u}^{LL} \geq V(\bar{x}^{HH}) - h\left(\frac{\bar{y}_2^{HH}}{w_L}\right). \tag{4.20}$$

That is, the pseudo–indifference curve with the label $\hat{u}^{LL}$ is the upper boundary of the region of points in (y_1, x)–space satisfying the LL–HH self–selection constraint, fixing y_2^{HH} at $\bar{y}_2^{HH}$. This curve is labeled LL$'$ in Figure 4. Now, let $\bar{u}^{HH}$ denote the welfare level of family HH at $(\bar{y}_1^{LL}, \bar{y}_2^{LL}, \bar{x}^{LL})$. With this reinterpretation of the symbol $\bar{u}^{HH}$, equation (4.16) describes the pseudo–indifference curve above which all points satisfying the HH–LL constraint must lie, given $y_2^{HH} = \bar{y}_2^{HH}$. In Figure 4,

this curve is denoted HH$'$. In view of the relation $w_L < w_H$, (4.16) and (4.19) imply that the vertical distance between the curves LL and LL$'$ is greater than the vertical distance between HH and HH$'$. Thus, the intersection of LL$'$ and HH$'$ occurs to the right of the point A. Notice that the point A$'$ may lie below the point A, so that no conclusion may be drawn about the ordering of x^{LL} and x^{HH}.

Some high–productivity individuals may earn less than corresponding low–productivity individuals at an allocation consistent with self–selection. This possibility is illustrated in Figure 5. Let point A denote the projection of the allocation of family LL onto (y_1, x)–space. Suppose that $\bar{y}_2^{LL} < \bar{y}_2^{HH}$. Given $y_2^{HH} = \bar{y}_2^{HH}$, all (y_1^{HH}, x^{HH}) pairs that satisfy the LL–HH self–selection constraint lie below the curve labeled LL$'$. The curve HH$'$ delimits the lower boundary of the region in (y_1, x)–space that all (y_1^{HH}, x^{HH}) pairs satisfying the HH–LL self–selection constraint must lie in, assuming $y_2^{HH} = \bar{y}_2^{HH}$. (The argument is identical to that used in the discussion of Figure 4, save that curves shift in the opposite direction.) As Figure 5 illustrates, the intersection of these two curves may occur at a point to the southwest of the point A. When this is the case, no conclusion can be drawn about the ordering of x^{LL} and x^{HH}, nor can y_1^{LL} and y_1^{HH} be ordered by self–selection considerations alone. Intuitively, an increase in y_2^{HH} is less costly in terms of labour time for family HH than it is for family LL viewed as a mimicker of family HH. Hence, such a change induces a tightening of the HH–LL self–selection constraint of a magnitude smaller than its slackening effect on the LL–HH self–selection constraint.

Statements iii.) and iv.) of Proposition 4.4 indicate that there are cases for which the self–selection constraints place restrictions on the ordering of the after–tax incomes of families LH and HL. There is, however, a caveat to this apparently strong result about after–tax incomes. The conditions under which this ordering is available appear to be rather strong. For instance, when y_2^{LH} is less than y_2^{HL}, it is the case that an individual of lower ability is working less than a high–ability individual.

Figure 6 gives a graphical illustration of statement iv.). Let the point D denote the projection of family LH's bundle onto the (y_1, x)–plane. The curve LH is a slice of the indifference surface for family LH through $(\bar{y}_1^{LH}, \bar{y}_2^{LH}, \bar{x}^{LH})$, drawn for $y_2^{LH} = \bar{y}_2^{LH}$. The $\bar{y}_2^{LH}$–slice of family HL's indifference surface through $(\bar{y}_1^{LH}, \bar{y}_2^{LH}, \bar{x}^{LH})$ is denoted by HL. Suppose $\bar{y}_2^{LH} < \bar{y}_2^{HL}$. Then the corresponding $\bar{y}_2^{HL}$ pseudo–indifference curves are shown by the curves LH$'$ and HL$'$ in Figure 6. Like family LL, family HL has a person of lower ability in position 2, while (like family HH) family LH has a person of higher ability in position 2. Thus, the vertical distance between the curves HL$'$ and HL is greater than the vertical distance between the curves LH$'$ and LH. Consequently, the intersection of HL$'$ and LH$'$ must occur to the northeast of the point D. Hence, family HL receives more y_1 and more x than family LH.

The indeterminacy of the ordering of after–tax incomes at this stage of the analysis is perhaps the most important obstacle to overcome in characterising the optimal solution. It is also one of the most important distinctions between multi–dimensional optimal tax mechanisms and their single–dimensional counterparts. This

indeterminacy is not the direct result of the two–dimensional uncertainty faced by the planner. Rather, as Proposition 4.3 illustrates, it results from the multiplicity of instruments. Nevertheless, when hidden information can be summarised by a one–dimensional characteristic, the use of multiple instruments leads to solutions that bear striking resemblance to the standard unidimensional problem with two goods as long as sufficient structure is imposed to ensure that agents of higher type receive more of all goods.[23]

Among the contributions of Dana (1993) and Rochet (1995) has been to draw attention to self–selection constraints between agents that differ in more than one dimension. Both present screening models in which such constraints can bind at an optimum. Neither devotes attention to which properties of the solution are directly attributable to these binding constraints. As a step in that direction, I now turn to the problem of describing the characteristics of allocations that satisfy (SS) with one of these constraints binding.

Proposition 4.5 *Let (SS) hold.*

i.) If, in addition, the HH–LL self–selection constraint holds with equality, then $y_1^{LH} \leq y_1^{LL}$ *(with equality only if the HH–LH and LH–LL self–selection constraints also hold with equality) and* $y_2^{HL} \leq y_2^{LL}$ *(with equality only if the HH–HL and HL–LL self–selection constraints also hold with equality).*

[23] This property is known as "attribute ordering" (cf. Matthews and Moore (1987)). Besley and Coate (1995) provide a model in which a monotonicity property different from, but in the same spirit as, attribute ordering is sufficient to render the problem "well–behaved."

ii.) If, in addition, the LL–HH self–selection constraint holds with equality, then $y_1^{HH} \leq y_1^{HL}$ *(with equality only if the LL–HL and HL–HH self–selection constraints also hold with equality) and* $y_2^{HH} \leq y_2^{LH}$ *(with equality only if the LL–LH and LH–HH self–selection constraints also hold with equality).*

The geometric intuition behind clause i.) of Proposition 4.5 is illustrated in Figure 7. Let the point A represent the projection of the allocation of family LL onto (y_1, x)–space. The curve LL is the pseudo–indifference curve, drawn at $y_2^{LL} = \bar{y}_2^{LL}$, for family LL through A. When the HH–LL self–selection constraint is binding, given $y_2^{HH} = \bar{y}_2^{HH}$, family HH must have an allocation along the pseudo–indifference curve labeled $\tilde{u}^{HH}$, where

$$\tilde{u}^{HH} := V(\bar{x}^{LL}) - h\Big(\frac{\bar{y}_1^{LL}}{w_H}\Big) + \gamma\Big[h\Big(\frac{\bar{y}_2^{HH}}{w_H}\Big) - h\Big(\frac{\bar{y}_2^{LL}}{w_H}\Big)\Big]. \tag{4.21}$$

When $\bar{y}_2^{HH} > \bar{y}_2^{LL}$, this pseudo–indifference curve lies above the pseudo–indifference curve of family HH through A (which is drawn for $y_2^{HH} = \bar{y}_2^{LL}$). Let HH$'$ denote the $\tilde{u}^{HH}$–curve. Now suppose that $y_2^{LH} > y_2^{LL}$. ($y_2^{LH} \geq y_2^{LL}$ by Proposition 4.3.) Then, in order for the LH–LL self–selection constraint to be satisfied, it must be the case that the (projection of the) allocation of family LH must lie on or above a pseudo–indifference curve such as the one labeled LH, assuming $y_2^{LH} = \bar{y}_2^{LH}$. Notice that the curve LH is above the curve LL, due to the fact that family LH must be compensated for the higher before–tax income of its person 2. The label on the curve LH is given

by

$$\tilde{u}^{LH} := V(\bar{x}^{LL}) - h\Big(\frac{\bar{y}_1^{LL}}{w_L}\Big) + \gamma\Big[h\Big(\frac{\bar{y}_2^{LH}}{w_H}\Big) - h\Big(\frac{\bar{y}_2^{LL}}{w_H}\Big)\Big]. \tag{4.22}$$

Now consider family HH faced with the possibility of mimicking family LH. The HH–LH self–selection constraint is satisfied for any (y_1^{LH}, x^{LH}) pair on or below the pseudo–indifference curve $\overline{\mathrm{HH}}''$ (a $\bar{y}_2^{LH}$–slice). $\overline{\mathrm{HH}}''$ has utility label

$$\check{u}^{HH} := V(\bar{x}^{LH}) - h\Big(\frac{\bar{y}_1^{LH}}{w_H}\Big) + \gamma\Big[h\Big(\frac{\bar{y}_2^{LH}}{w_H}\Big) - h\Big(\frac{\bar{y}_2^{HH}}{w_H}\Big)\Big]. \tag{4.23}$$

The vertical distance between the curves $\overline{\mathrm{HH}}''$ and LL is equal to the sum of the last term in (4.21) and the last term in (4.23). The final term of (4.22) is the vertical distance between LH and LL. Comparing (4.22) with (4.21) and (4.23), it can be seen that the intersection of the curves LH and $\overline{\mathrm{HH}}''$ lies directly above the point A.[24] Thus, all points that satisfy (SS) for $y_2^{LH} = \bar{y}_2^{LH}$ must lie in the shaded area of Figure 7, and, hence, $\bar{y}_1^{LH} \leq \bar{y}_1^{LL}$. Note that this shaded region contains both points above A and points below A. Thus, Proposition 4.5 is silent about the ordering of after–tax incomes.

Proposition 4.6 *Let (SS) hold.*

i.) If, in addition, the LH–HL self–selection constraint holds with equality then $y_1^{HL} \leq y_1^{HH}$ (with equality only if the LH–HH and HH–HL self–selection constraints also hold with equality) and $y_2^{LL} \leq y_2^{HL}$ (with equality only if the LH–LL and LL–HL self–selection constraints also hold with equality).

[24] The argument given here implicitly assumes $\bar{y}_2^{LH} > \bar{y}_2^{HH}$. When $\bar{y}_2^{LH} < \bar{y}_2^{HH}$, the curve $\overline{\mathrm{HH}}''$ lies below the curve HH′. It is still the case, however, that $\overline{\mathrm{HH}}''$ and LH intersect directly above the point A.

ii.) If, in addition, the HL–LH self–selection constraint holds with equality then $y_1^{LL} \leq y_1^{HL}$ *(with equality only if the HL–LL and LL–LH self–selection constraints also hold with equality) and* $y_2^{LH} \leq y_2^{HH}$ *(with equality only if the HL–HH and HH–LH self–selection constraints also hold with equality).*

The geometric intuition of Proposition 4.6 is identical to the intuition underlying Proposition 4.5, except for some relabeling of indifference curves. Indeed, the symmetry in the statements of Propositions 4.5 and 4.6 is striking. This may be somewhat surprising, given that the families HH and LL are ordered by the relation $\geq_F$, while the families LH and HL are not. Nevertheless, there are reasons to expect some symmetry between the two propositions. Like Proposition 4.5, each statement of Proposition 4.6 uses information contained in the self–selection constraints relating three distinct families. However, while only "downward" self–selection conditions are used in deriving Proposition 4.5, Proposition 4.6 requires the use of "upward" conditions. Another view of Figure 1 shows that the pair $\{\mathrm{HH}, \mathrm{LL}\}$ has something in common with the pair $\{\mathrm{LH}, \mathrm{HL}\}$. One may always proceed along the edges of the type space given in Figure 1 from the family LH to the family HL if we allow ourselves to travel "up the partial order $\geq_F$." It is the symmetric treatment of upward and downward self–selection constraints that results in the similarity between Propositions 4.5 and 4.6.

A comparison of Propositions 4.5 and 4.6 reveals that the presence of certain binding constraints implies orderings on components of allocations that cannot be

ordered by appealing to Proposition 4.3. Note, however, that a binding HH–LL self–selection constraint and a binding HL–LH constraint have almost opposite implications for the ordering of y_2^{HL} and y_2^{LL}. Indeed, these two constraints can bind together only if y_2^{HL} and y_2^{LL} are equal.

4.4. Properties of Optimal Tax Schedules

The results of the last section followed directly from the self–selection constraints. In particular, none of the analysis relied on the assumption that the planner seeks to maximise a social objective function. In this section I incorporate optimising behaviour on the part of the taxation authority into the analysis to derive some additional properties of solutions to the problem (PF). The first of these is standard in optimal income tax theory.

Proposition 4.7 *At any solution to (PF), the constraint (F) binds with positive multiplier.*

Proposition 4.7 can be interpreted as stating that any surplus output can always be distributed among the families in such a way that self–selection constraints are not violated.

Although simple, Proposition 4.7 is an important building block for further analysis of the problem. In particular, it allows us to conclude that if, starting from an

initial allocation, a change can be made that slackens the resource constraint, does not violate the self–selection constraints and makes no family worse off, then the initial allocation is not a solution to (PF). This is exactly the type of reasoning behind the next proposition.

Proposition 4.8 *Suppose that at a solution to (PF),* $W^i(x^i, y_1^i, y_2^i) = W^i(x^j, y_1^j, y_2^j)$ *for a pair of families* i, j. *Then*

$$y_1^i + y_2^i - x^i \geq y_1^j + y_2^j - x^j. \tag{4.24}$$

Proposition 4.8 is the restatement of a result of Brito *et al.* (1990, Proposition 1, p. 66) in the current context. One would expect this result to hold here, for it states simply that a family never wishes to mimic a family that has a higher total tax bill than itself. In particular, the statement of Proposition 4.8 contains no reference to the labeling of families adopted in this analysis. It is well–known that, in the two–good model with unidimensional differences among agents, the pattern of binding self–selection constraints determines the qualitative features of optimal marginal tax rates (Röell (1985)). Proposition 4.8 tells us that the pattern of binding self–selection constraints can also be used to make statements concerning total tax liabilities.

When discussing the relationship between the self–selection constraints and tax functions, I considered a wide class of tax functions. In particular, the tax functions were allowed to be non–differentiable. At kinks in the tax schedule, it is impossible

to define marginal tax rates as (partial) derivatives of the tax function. It is possible, however, to define implicit marginal tax rates at any allocation by:

$$t_1^i := 1 - MRS^i_{y_1,x} = 1 - \frac{h'(\frac{y_1^i}{w_1^i})}{w_1^i V'(x^i)}, \quad t_2^i := 1 - MRS^i_{y_2,x} = 1 - \frac{\gamma h'(\frac{y_2^i}{w_2^i})}{w_2^i V'(x^i)}. \tag{4.25}$$

It is clear from their definitions that implicit marginal tax rates are positive when the marginal rate of substitution between labour and consumption is less than one, which is the producer wage for an efficiency unit of labour in this model. Marginal wage subsidies (negative marginal tax rates) correspond to marginal rates of substitution in excess of the producer wage.

Marginal tax rates serve as an important summary statistic of the distortions arising from the planner's lack of information. In the current context, the following proposition demonstrates that the sign of marginal tax rates depends on the pattern of binding self–selection constraints.

Proposition 4.9

i.) For any family i and any family member k, if $w_k^i = w_L$, then k faces a non–negative marginal tax rate t_k^i. Furthermore, $t_k^i > 0$ if and only if there is a family j with $w_k^j = w_H$ and the j–i self–selection constraint binds with a positive multiplier.

ii.) For any family i and any family member k, if $w_k^i = w_H$, then k faces a non–positive marginal tax rate t_k^i. Furthermore, $t_k^i < 0$ if and only if there is a

family j with $w_k^j = w_L$ and the j–i self–selection constraint binds with a positive multiplier.

It follows directly from Proposition 4.9 that if, for all j, the j–i self–selection constraint does not bind at a solution to (PF), then both members of family i face a zero marginal tax rate. This is a direct analogue of a result due to Guesnerie and Seade (1982, Proposition 2, p. 164), which states that if there is an individual with a bundle that all other individuals view as strictly inferior to their own, then that individual faces a zero marginal tax rate. However, Proposition 4.9 is more than a restatement of the result of Guesnerie and Seade (1982) in the current context. It says: no distortions are introduced on the labour supply behaviour of individual 1, say, *when a self–selection constraint binds* between families with identical persons 1. Because two such families view trade–offs between y_1 and x identically, any such trade–offs can be made without violating the self–selection constraints involving these two families. Were it not for the presence of other families, the planner could vary the allocations of y_1 and x for these two families until an undistorted bundle is achieved.

Proposition 4.9 speaks to the debate over the choice of tax base. In particular, it suggests that a tax based solely on total family income will fail to be optimal in many circumstances. Notice that the implied marginal tax rates for the individuals in family HL are, in general, different. Indeed, these rates can coincide only if both are zero. The person of lower productivity faces a higher marginal tax rate at the optimum. An analogous result holds for family LH. It seems reasonable that the planner might want

to apply different tax rates within a family, given that total production in the economy is determined by how much *individuals* decide to work. Like one–dimensional taxation models, the choice of marginal tax rates for higher ability individuals is dominated by efficiency considerations. The planner wishes to extract effort from these individuals, since they are the ones whose work effort provides the most output. By using different marginal tax rates within the household, the planner is able to identify which of the two individuals in a "mixed" household is the one with higher productivity, enabling it to provide sufficient incentives for that individual to provide work effort. If the planner is forced to use a "flat" tax, then its power to identify high–productivity individuals is limited.

Proposition 4.9 has an important corollary.

Proposition 4.10 *No two families of different type receive the same allocation at a solution to (PF).*

Phrased in the language of screening models, there is no bunching at the optimum. Proposition 4.10 is an analogue to the no–pooling result of Stiglitz (1982) for economies with two types of consumers. It shows that it is not the existence of only two types of consumers *per se* that is important in deriving the no–pooling result. As long as there are only two families along each dimension of uncertainty there must be separation at the optimum. It should be stressed at this point that the argument establishing Proposition 4.10 relies on the assumption that all individuals

receive a positive before–tax income at the solution to (PF). Later on, I present a special case of the model for which some individuals receive zero before–tax income. In that environment, some families are pooled at the optimum. (See Proposition 5.5.)

Further progress in understanding the solutions to (PF) requires that something be said about the pattern of binding self–selection constraints at the optimum. One can expect this pattern to be influenced by the form of the social objective function $Z(\cdot)$. The question arises: Can some patterns may be ruled out at any solution? I now turn to the task of answering this question.

In order to make some of the discussion easier, I adopt some terminology of Wilson (1995).

Definition: Family i *attracts* the family j if the j–i self–selection constraint is binding.

Proposition 4.11 *At a solution to (PF), any pair of families i and j, except possibly the pair $\{ HH, LL \}$, are mutually attracted to each other only if they receive the same allocation.*

The one–dimensional analogue to Proposition 4.11 is an immediate consequence of the usual single–crossing property, and holds for all allocations. In the present context indifference surfaces intersect at more than one point. Hence, it is important to emphasise that the conclusion of Proposition 4.11 holds at the optimum. The

exclusion of the pair $\{ HH, LL \}$ at this point is due to the fact that there is insufficient structure on the general form of (PF) to ensure that the solution is attribute ordered (cf. Matthews and Moore (1987)). In particular, we cannot conclude that both members of family HH receive more before–tax income than the corresponding members of family LL.

The intuition behind Proposition 4.11 can be illustrated by considering families HH and LH once more. Given the specification of family objectives adopted here, mutual attraction of these two families is equivalent to $y_1^{HH} = y_1^{LH}$, so that self–selection requires they be placed on the same indifference curve in (y_2, x)–space. Figure 8 illustrates the case of different allocations for the two families, drawn at the same level of y_1. The curve LL is the $\bar{y}_1^{HH}$ –slice of the indifference surfaces of families LL and HL through the bundle designed for family HH. The curve HH is the analogous slice for families HH and LH. The points C and D correspond to the (y_2, x) pairs offered to families HH and LH, respectively, under the assumption that $y_2^{HH} < y_2^{LH}$. For this pair of allocations it turns out that neither LL nor HL can be attracted to LH, for otherwise the LL–HH and HL–HH constraints would be violated. To see this, notice that for *any* pseudo–indifference curve for the families HH and LH, points such as C must lie to the left of points such as D. If the pseudo–indifference curve for family LL defining the upper boundary of the set of points that satisfy the LL–LH constraint passes through a point such as D, then the point corresponding to C must be above that boundary. Then, the LL–HH self–selection constraint must be

violated. Now, by Proposition 4.9, $MRS^{LH}_{y_2,x} = 1$. Then $MRS^{HH}_{y_2,x} < 1$. Consider a move along the indifference curve from point C toward point D. This has no effect on the HH–LH and LH–HH constraints. Furthermore the move is resource saving and can only slacken the LL–HH and HL–HH constraints.

Proposition 4.11 is closely related to a result of Brito *et al.* (1990, Proposition 2, p. 67). They show that when there are cycles of self–selection constraints, the planner can do no worse by pooling the agents involved in the cycle. The additional structure of this model allows this conclusion to be strengthened for most pairs of families to say that the planner can always do better by pooling. However, in view of Proposition 4.10, the planner can do better still. This statement is formalised in the next proposition.

Proposition 4.12 *No pair of families is mutually attracted at a solution to (PF).*

For all pairs save $\{\mathrm{HH}, \mathrm{LL}\}$ Proposition 4.12 follows directly from Propositions 4.10 and 4.11. For the pair $\{\mathrm{HH}, \mathrm{LL}\}$ the proposition follows from Proposition 4.10 and the aforementioned result of Brito *et al.* (1990). Indeed, Proposition 4.12 is a special case of a more general result, itself a direct consequence of Proposition 4.10 and the work of Brito *et al.* (1990). In order to state this result, I need the following definition.

Definition: A self selection cycle is said to exist when there is a set of families $\{i_1, \ldots, i_n\}$ such that family i_1 attracts family i_n and for all $k < n$, family i_{k+1} attracts family i_k.

Proposition 4.13 *There are no self–selection cycles at a solution to (PF).*

Another standard feature of optimal non–linear tax schedules is the existence of an agent who faces no distortions. In one–dimensional models with single–crossing and two goods, this is often the agent of highest ability, and, for some continuous models, the agent of lowest ability as well (cf. Seade (1977)). A version of this result also holds in the present context.

Proposition 4.14 *For any family i, if*

$$y_1^i + y_2^i - x^i \geq y_1^j + y_2^j - x^j, \text{ for all } j \neq i, \tag{4.26}$$

then family i faces no marginal distortions at a solution to (PF).

It follows from Proposition 4.14 that there is a family whose allocation is not subject to marginal distortions. Moreover, it identifies undistorted families : those that pay the highest taxes. This is exactly the finding of Brito *et al.* (1990). Of course, a family that faces no marginal distortions is not pooled with any other family, just as was found by Guesnerie and Seade (1982).

At this point, it is useful to define two classes of constraints.

Definition: The HH–LL and LL–HH self–selection constraints are called *diagonal*. The HL–LH and LH–HL self–selection constraints are called *transverse*.

Combining Propositions 4.5 and 4.6 shows that when diagonal and transverse self–selection constraints bind simultaneously there are allocations which are identical in some components. Given this tendency toward pooling, at least for some goods, the following proposition may come as little surprise.

Proposition 4.15 *At a solution to (PF), it cannot be the case that a diagonal constraint and a transverse constraint bind simultaneously.*

Proposition 4.15 greatly simplifies the search for solutions to (PF). Moreover, it shows that it is not by coincidence that no solutions discovered by Dana (1993) and by Rochet (1995) in their models feature a binding diagonal constraint and a binding transverse constraint. Note that quasi–linearity (which is assumed in these earlier studies) is not needed to generate this result.

CHAPTER 5: Optimal Non–linear Taxes: Some Special Cases

5.1. Introduction

The requirement of Pareto–efficiency has yielded quite a bit of information about the properties of optimal tax schedules. Further progress in characterising the solutions to the problem (PF) requires that some structure be placed on the objectives of the planner. There are two ways to provide this structure. One is to posit a form for the objective function of the planner, say a weighted sum of family objectives. The other method involves making a redistributive assumption. In the next section, I formulate an analogue to a standard redistributive assumption. Unlike the situation that usually occurs in one–dimensional income tax problems (cf. Guesnerie and Seade (1982)), it is shown that the redistributive assumption is not strong enough to rule out negative marginal tax rates for some individuals. It does, however, have an implication for the ordering of the optimal before–tax incomes of some families.

The general characterisations provided in the preceding chapter do not make the role of asymmetries within the family transparent. In the final section of this chapter, I specialise the model to the case of families who trade off the labour supplies of their members in a linear fashion. In this case, differences within the family are reflected in labour–force participation decisions. An analysis of these decisions provides a clearer picture of the role of differences between individuals within the same family.

5.2. Redistributive Taxation

It is usual in the literature to assume that, when incentive effects are ignored, the planner wishes to transfer consumption from the more able to the less able (Guesnerie and Seade (1982), Chambers (1989)). In the spirit of the literature, I employ the following redistributive assumption.

Assumption R *Suppose that $i >_F j$ for a pair of families (i, j). Then there exists some sufficiently small $\epsilon > 0$ such that, at the optimum, it is socially desirable to transfer ϵ units of after–tax income from family i to family j if the constraints (SS) are ignored.*

Assumption R is not equivalent, in general, to a desire to increase the welfare of families with less able individuals (Dixit and Seade (1979)). It is also important to note that Assumption R places no restriction on how the planner views the consumption of family HL *vis–a–vis* the consumption of family LH. Despite this, Assumption R has implications for the structure of solutions to (PF). It is not sufficient, however, to ensure that all marginal tax rates are non–negative.

It is also necessary to say something about the distribution of families in order to classify more succinctly the optimal taxation mechanisms. Employing Assumption R is much more straightforward when all of the redistributions it declares to be desirable

are production–feasible. This condition is guaranteed by the following assumption about the distribution of families.

Assumption U $\pi^{LL} = \pi^{HL} = \pi^{HH} = \pi^{LH}$.

Notice that Assumption U is stronger than the assumption of statistical independence between the two components of the space of family types. The analysis can also be carried out under the independence assumption, with additional care taken to assess which redistributions of after–tax income can be made without violating the materials balance constraint.

Unlike the situation that obtains in the model of Dana (1993), Assumption U does not rule out asymmetric solutions in this model. Asymmetries may arise due to the form of the planner's objectives or due to the asymmetry of the family decision process.

The remainder of this section is devoted to cataloguing the qualitative features of possible optimal configurations. In order to compare the current results with others in the literature, it is necessary at times to impose additional structure on family objectives. The following assumption will sometimes be cited.

Assumption Q $V(x) = x$.

Assumption Q is consistent with the model of family decision making only if $U(x) = x$.[25] This specification can lead to one member of the family consuming everything. Assumption Q may be more palatable in other screening contexts. The main reason for its use in this study is to compare the more general results obtained without it to the optimal mechanisms that arise when it is satisfied. In this way, the full force of this common assumption can be ascertained.

One more piece of notation is required.

Definition: The set $\mathcal{C}$ is defined by saying that an ordered pair of indices (i, j) is an element of $\mathcal{C}$ if and only if the i–j self–selection constraint binds at the solution to (PF).

5.2.1. The usual cases

As emphasised by Dana (1993) and Rochet (1995), much of the analytical difficulty in multi–dimensional screening problems arises from the possibility of binding diagonal or transverse constraints. In this subsection, patterns of binding constraints that can occur when diagonal constraints are assumed not to bind are presented. There is a surprisingly large number of configurations that are compatible with Assumption R. The main qualitative implication of Assumption R is given by the next proposition.

[25] Strictly speaking, Assumption Q is inconsistent with the boundary conditions required on preferences required to guarantee that, for any allocation satisfying (SS), the planner can construct an equivalent tax schedule.

Proposition 5.1

i.) Let Assumptions R and U hold and let $(HH, LL) \notin \mathcal{C}$. *Then*

$\{(HH, HL), (LH, LL)\} \subseteq \mathcal{C}$ *or* $\{(HH, LH), (HL, LL)\} \subseteq \mathcal{C}$.

ii.) Let Assumptions R, U and Q hold and let $(HH, LL) \notin \mathcal{C}$. *Then*

$\{(HH, HL), (HL, LL), (LH, LL), \} \subseteq \mathcal{C}$ *or* $\{(HH, LH), (LH, LL), (HL, LL)\} \subseteq \mathcal{C}$. *Moreover,* $(LL, HH) \notin \mathcal{C}$.

The implication of Proposition 5.1 for tax rates is that either all low–productivity individuals occupying position 1 in a family or all low–productivity individuals in position 2 in a family face a strictly positive marginal tax rate.[26] Assumption R implies that family HH is attracted to some other family, say family HL. Then individual 2 of family HL has its labour supply distorted downward. Also by Assumption R, family LL must attract some other family. If family HL is attracted to family LL (while family HL attracts family HH), satisfying the HH–LL self–selection constraint requires that $y_2^{HL} \geq y_2^{LL}$.[27] Then, individual 2 of family LL must also have its labour supply distorted downwards. Statement ii.) of Proposition 5.1 shows the force of Assumption Q. When Assumption Q is maintained, both members of family LL face a strictly positive marginal tax rate at the optimum.

[26] For the remainder of the discussion, it is assumed that all binding constraints bind with positive shadow values. A proof of this assertion requires a slight strengthening of Assumption R to state that the social gains from a downward transfer of after–tax income are of the first order. Guesnerie and Seade (1982, p. 174) provide the argument.

[27] This assertion can be checked using a calculation similar to the one used to prove Proposition 4.5, without imposing that the C–A constraint binds.

Proposition 5.1 is silent about the pattern of binding constraints when family LL attracts family HH. However, Proposition 4.12 ensures that family LL cannot be attracted to family HH when the HH–LL self–selection constraint binds. Then, statement ii.) of Proposition 5.1 permits the conclusion that the LL–HH self–selection constraint can never bind under Assumptions R, U and Q. Notice, however, that the combination of Assumptions R and Q is not strong enough to rule out a binding LL–HH constraint at the optimum.

Given Assumption R, any redistribution of after–tax income to family LL from any other family is socially desirable. A binding HH–LL self–selection constraint is an impediment to this kind of redistribution. However, transfers to family LL can only slacken the LL–HH self–selection constraint. Although it is possible for the LL–HH self–selection constraint to bind at the optimum, the planner can, for a given pattern of other binding self–selection constraints, carry out the same downward redistributions regardless of whether family HH attracts family LL or not. Hence, there is no interesting analogue to Proposition 5.1 for the case of $(LL, HH) \notin \mathcal{C}$.

5.2.2. The unusual cases

Up to this point, little has been said about families who are unordered by $\geq_F$. It is the presence of these families that makes the current model substantively different from a one–dimensional model. The possibility of the LH–HL or HL–LH constraint binding implies the possibility of negative marginal tax rates for high–productivity

individuals in families HL and LH. This is a major departure from the standard results of the literature on non–linear redistributive taxation. In this subsection, I show that Assumption R does not provide sufficient structure to rule out such phenomena. It does, however, place some restrictions on the pattern of marginal tax rates when a transverse constraint binds.

The next proposition shows that a negative marginal rate to person 1 in family HL (*resp.* person 2 in family LH) arising from a binding transverse constraint must be accompanied by a negative marginal tax rate for an individual 1 (*resp.* 2) in family HH.

Proposition 5.2 *Let Assumptions R and U hold. Then*

i.) $(LH, HL) \in \mathcal{C}$ *implies* $\{ (HH, HL), (LH, LL), (LH, HH) \} \subseteq \mathcal{C}$.

ii.) $(HL, LH) \in \mathcal{C}$ *implies* $\{ (HH, LH), (HL, LL), (HL, HH) \} \subseteq \mathcal{C}$.

Proposition 5.2 implies that when one of the transverse self–selection constraints binds, the pattern of binding constraints for family HH is completely determined. For instance, when the LH–HL self–selection constraint binds, family HH is attracted to family HL and family LH is attracted to family HH. Because a transverse constraint binds, it follows from Proposition 4.15 that neither of family HH or family LL is attracted to the other. Notice, however, that the pattern remains undetermined for family LL. This asymmetry is a direct result of Assumption R and the concomitant tendency for high–ability families to be tempted to under–report their productivity at

the optimum. This indeterminacy vanishes, however, when Assumption Q is satisfied. This is the content of the next proposition.

Proposition 5.3 *Let Assumptions R, U and Q hold. Then*

i.) $(LH, HL) \in \mathcal{C}$ *implies* $\mathcal{C} = \{ (HH, HL), (HL, LL), (LH, LL), (LH, HH) \}$.

ii.) $(HL, LH) \in \mathcal{C}$ *implies* $\mathcal{C} = \{ (HH, LH), (HL, LL), (LH, LL), (HL, HH) \}$.[28]

The only possibility not considered thus far is that the HH–LL self–selection constraint may bind. In previous work (Dana (1993), Rochet (1995)), this case has arisen only in the so–called "separable" context, in which family i attracts family j if and only if $j >_F i$. In the present model, separability means that, at the optimum, all low–productivity individuals receive the same before–tax income. Income effects once again destroy the simplicity of the solution. However, in view of Proposition 4.5, it is clear that the "separable" case can occur only if the j–i self–selection constraint binds for all pairs of families with $j >_F i$.

5.2.3. A summary

At first glance, Assumption R appears to provide only limited structure on the nature of optimal tax mechanisms. While it is true that the situation is much more

[28] It is interesting to compare Propositions 5.2 and 5.3 with the "singular" case of Rochet (1995, Proposition 3.4). The introduction of income effects leads to some additional indeterminacy. For example, Rochet's analogue to statement i) of Proposition 5.2 features the LH–LL self–selection constraint binding as well. This is neither ruled out nor required in the absence of Assumption Q. Proposition 5.3 demonstrates that the assumption of quadratic cost functions employed by Rochet is not essential in deriving the singular case, but that the singular case is the only possible outcome in this class under Assumption Q.

complex than is usually found in the literature, some basic properties are common to all solutions. First, as the next proposition shows, when high–ability individuals occupy the same position in a household, the individual with a low–productivity partner works at least as hard as the person with a high–productivity partner.

Proposition 5.4 *Let Assumptions R and U hold. Then, at any solution to (PF),* $y_1^{HL} \geq y_1^{HH}$ *and* $y_2^{LH} \geq y_2^{HH}$.

In view of Proposition 5.4, any marginal wage subsidies offered to individuals in family HH must not be large enough to induce them to work more than high–productivity individuals in other types of families. Phrased in the language of screening models, the solution fails to be attribute ordered. Moreover, it seems natural for this violation to occur. We would expect part of the gain to having a high–productivity partner to be consumed as leisure.

Second, in view of Proposition 5.2, whenever family LL is attracted to no other family, the use of wage subsidies for high–productivity individuals is concentrated on individuals with high–productivity partners. When no family attracts family LL, high–productivity individuals with low–productivity partners receive wage subsidies only if individuals in the same position in a family with high–productivity partners also receive marginal subsidies. Suppose, for the sake of concreteness, that family HL attracts family LH, so that individual 1 in family HL faces a negative marginal tax

rate. In the quasi–linear context it is clear that this can happen only if marginal increases in the welfare of family HL are weighted more heavily than marginal increases in the welfare of family LH at the optimum. This desire to redistribute resources away from family LH conflicts with the need to prevent family LH from masquerading as family HH. The only way to prevent this occurring, it turns out, is to distort the labour supply of family HH upward. In this case, it is precisely the desire to make "transverse" redistributions that upsets the standard non–negative marginal tax rate result. Notice that Assumption R is silent about the desirability of such redistributions. A characterisation of the class of social evaluation functionals for which both downward and transverse redistributions are desirable at the optimum is unknown to me.

Third, status in the household can exert an independent influence on optimal tax policy. That is, two equally–productive individuals with equally productive partners can face different marginal tax rates on the basis of observed demographics that have a *known* effect on household decisions. Once again, let family HL attract family LH. It follows from Proposition 5.2 that the low–productivity individual in family HL faces a positive marginal tax rate, while, because no family is attracted to family LH, the low–productivity individual in family LH faces a zero marginal tax rate.

5.3. The Consequences of Asymmetric Family Decisions

In this section, I present a special case of the model in which the role played by the

relative weights each individual has in family decision making is made transparent. The special case corresponds to the following assumption on preferences.

Assumption SQ $h(l) = l$.

Given weighted–utilitarian decision making in the household, Assumption SQ implies that families trade off the labour supplies of their members in a linear fashion. Thus, it is impossible to assume that all individuals will supply a strictly positive amount of labour at the optimum. Indeed, it is the case that only one person per family will work for most values of γ. The identity of the person who does not work in a family depends on γ. It is this extreme behavioural response to variations in relative say in family decisions that provides a stark picture of the role of the internal workings of the family.

Proposition 5.5 *Let Assumption SQ hold, and let* $\gamma \in (w_L/w_H, 1)$. *Then, at the solution to (PF),* $y_1^{LH} = y_2^{HL} = 0$.

If in addition, $y_1^{HH} = 0$ *, then*

i.) $y_1^{LL} = 0$.

ii.) Families HH and LH receive the same allocation.

iii.) $(HH, LL) \in \mathcal{C}$ *if and only if* $(LH, LL) \in \mathcal{C}$.

Proposition 5.5 states how families sort themselves according to participation decisions. When $\gamma < 1$, person 1 's utility gets more weight in family decisions. In particular, the labour supply of person 1 is considered more distasteful to the family. For family LH, the effect of relative say in family decisions is exacerbated by the fact that person 1 is less productive, so the family has an additional reason to concentrate work effort with person 2. For family HL, productivity considerations lead the family to prefer that person 1 work. When $\gamma > w_L/w_H$, the extra weight given to the leisure of person 1 is not sufficient to undo the productivity effect. This has two important consequences for the way the planner trades off the before–tax incomes of individuals within certain families. First, because the objective function of the planner is increasing in the welfare of family HL, it wishes to substitute the effort of person 1 in family HL for that of person 2. Second, a one–unit increase in y_2^{LH}, combined with a one–unit decrease in y_1^{LH}, reduces the welfare of family HL at the allocation designed for family LH, slackening the HL–LH self–selection constraint. Hence, self–selection considerations and concerns for the welfare of family LH work in the same direction.

The importance of the assumption that the planner knows the value of γ ought to be stressed at this point. Proposition 5.5 indicates exactly how the planner uses this information. For the range of γ considered in the proposition, the planner knows that family HL is the only one that prefers for its individual 1 to supply labour. Without prior knowledge of γ, the planner would not be able to disentangle the effects

of "pure" efficiency considerations within the family from the consequences of family distributional ethics. In particular, it would not be able to assess how rearrangements of before–tax incomes affect the welfare of family HL viewed as a mimicker of family LH.

For families with equally productive members (LL and HH), the preference for the work of person 2 is enough to make the family want to substitute the effort of person 2 for that of person 1. In the absence of binding self–selection constraints, this process can be stopped only when person 1 is supplying no labour at all. Consequently, if no family is attracted to family HH, then person 1 in family HH supplies zero labour. Thus, the assumption that $y_1^{HH}=0$ can be interpreted as saying that it is not optimal to distort the labour supply behaviour of family HH enough to induce its individual 1 to participate in the labour market.

Statement ii.) of Proposition 5.5 appears to contradict Proposition 4.10, the no–pooling result. It should be noted, however, that the no–pooling conclusion is derived under the assumption that all individuals have a positive before–tax income. A binding non–negativity constraint is a source of upward distortion on labour supply. For an individual of low–type, it is the only possible source of upward distortion. The argument for Proposition 4.10 rests on a non–negative marginal tax rate for all individuals of low–ability. The presence of a binding non–negativity constraint destroys this argument.

Statement iii.) of Proposition 5.5 has implications for how the planner can feasibly redistribute consumption to family LL. Suppose that the planner wishes to increase the consumption of family LL at the expense of families HH and LH. Given the assumptions underlying statement iii.), $y_1^{LL} = y_1^{LH} = y_1^{HH}$. Hence, the problem reduces to the one–dimensional case with two agents of high type (HH and LH) and one of low type (LL). If, instead, the planner would rather carry out redistributions from family HL to family LL, then it is possible that the process of redistribution could be stopped by concern over family HL mimicking family LL, say, before families HH and LH would be tempted to masquerade as family LL. In that case, neither the HH-LL nor the LH-LL constraints binds.

CONCLUSION

There is mounting evidence that families do not behave as if they are a single person (see, for example, Thomas (1990), Phipps and Burton (1992)), but that their decisions are compatible with household efficiency (Browning *et.al.*(1994) Browning and Chiappori (1994)). This presents a challenge to applied welfare economists: how can we use the new insights in the economics of family behaviour to improve our normative analyses? This thesis has presented a view on how we might consider tax policy in a setting that takes the interaction of family members seriously.

This analysis of tax reform has also uncovered some formal similarities between the problem of tax reform in a family setting and tax reform in an individual–based model. For example, in both cases temporary inefficiencies may be ruled out by suitable normality conditions and sufficiently flexible powers of lump–sum taxation. Despite these similarities it may be a grave error to apply the individual–based model to the family setting. Misleading policy prescriptions may follow, even if the family is behaviourally equivalent to an individual.

The preceding analysis has also put the debate on identifying household sharing rules in some perspective by showing that, although useful in descriptive analysis of household behaviour, knowledge of the derivatives of the sharing rule is not sufficient for calculating Pareto–improving directions of tax reform. Thus, the model proposed here is not implementable with family budget data alone. The analysis can be seen

as reinforcing an idea that ought to be self–evident: In order to accurately assess the welfare impact of policies on individuals it is necessary to have individual–level data.

This study has also presented a model of family income taxation based on the notion of multi–dimensional screening. Viewing the problem in this way gives arguments for members of the same family to face different marginal income tax rates, casting doubt on total family income as appropriate income tax base. The results show that using the individual as the basis of income taxation is also not sufficient. Both the productivity of the partner and the relative position of a person in the household have some bearing on the marginal tax rate faced by an individual.

It is interesting to reflect upon the possibility of negative marginal tax rates in this model. It was noted in the discussion following Proposition 5.4 that, when Assumptions R and U hold, this important difference from standard one–dimensional tax models arises when the planner has a reason to transfer consumption among families that differ in both characteristics. In the current model, the desirability of such redistributions depends on the social welfare function. When Assumptions R and U are violated, asymmetries in the distribution of the population or a desire to offset "undesirable" effects of family interactions may provide motivations for such redistributions. When there is no reason to use differences along both dimensions to identify families to whom it would be desirable to transfer consumption, there is no reason (at least not in the quasi–linear context) to use marginal wage subsidies.

The biggest obstacle to a more specific characterisation of optimal solutions is the income effects embodied in family objectives. It seems unnatural to rule these out. However, the analysis of this study could be made more sharp in environments where the quasi–linearity assumption is more palatable. For instance, the work of Besley and Coate (1995) on work requirements in income maintenance programmes could be extended to cover the case where market work and required work are qualitatively different and individuals may be more or less productive at one or the other.

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APPENDIX A

In this appendix I present the proofs of the propositions and lemmas contained in Chapters 4 and 5. First, I introduce some notation that will be used throughout. Relabel the families so that the symbols A, B, C and D correspond to families LL, HL, HH and LH, respectively.

Proof of Proposition 4.1:

The statement follows immediately from the way that x_1 and x_2 enter the constraint of problem (P) and only the constraint. □

Proof of Proposition 4.2:

i) follows from the fact that the resource constraint binds at any optimum for the family, since $U(\cdot)$ is increasing. Differentiating this equality constraint gives i).

The first–order necessary conditions for the problem (P) are:

$$\begin{aligned} U'(c_1) - \kappa &= 0, \\ \gamma U'(c_2) - \kappa &= 0, \\ x - c_1 - c_2 &= 0, \end{aligned} \tag{A.1}$$

where κ is the multiplier associated with the budget constraint. Differentiating the system (A.1) yields

$$\begin{aligned} U''(c_1)dc_1 - d\kappa &= 0, \\ \gamma U''(c_2)dc_2 - d\kappa &= 0, \\ dx - dc_1 - dc_2 &= 0. \end{aligned} \tag{A.2}$$

Apply Cramer's rule to the system (A.2) to conclude

$$\begin{aligned} \frac{\partial c_1(x)}{\partial x} &= \frac{\gamma U''(c_2)}{U''(c_1) + \gamma U''(c_2)}; \\ \frac{\partial c_2(x)}{\partial x} &= \frac{U''(c_1)}{U''(c_1) + \gamma U''(c_2)}. \end{aligned} \tag{A.3}$$

Strict concavity of $U(\cdot)$ implies that in each line of (A.3) both numerator and denominator are negative. Statement ii.) follows.

Increasingness of $V(\cdot)$ is established by the calculation:

$$V'(x) = U'(\tilde{c}_1(x))\tilde{c}_1'(x) + \gamma U'(\tilde{c}_2(x))\tilde{c}_2'(x) > 0. \tag{A.4}$$

It follows from (A.4) that

$$V''(x) = U''(\tilde{c}_1(x))[\tilde{c}_1'(x)]^2 + U'(\tilde{c}_1(x))\tilde{c}_1''(x) + \gamma U''(\tilde{c}_2(x))[\tilde{c}_2'(x)]^2 + \gamma U'(\tilde{c}_2(x))\tilde{c}_2''(x). \tag{A.5}$$

By (A.1), we may group the second and fourth terms of (A.5) to yield

$$V''(x) = U''(\tilde{c}_1(x))[\tilde{c}_1'(x)]^2 + \gamma U''(\tilde{c}_2(x))[\tilde{c}_2'(x)]^2 + U'(\tilde{c}_1(x))\left[\tilde{c}_1''(x) + \tilde{c}_2''(x)\right]. \tag{A.6}$$

By statement i.), the part of the last term of (A.6) in square brackets is zero, so that

$$V''(x) = U''(\tilde{c}_1(x))[\tilde{c}_1'(x)]^2 + \gamma U''(\tilde{c}_2(x))[\tilde{c}_2'(x)]^2 < 0, \tag{A.7}$$

establishing the strict concavity of $V(\cdot)$. □

Proof of Lemma 4.1:

$w_H > w_L$ implies that for all y, $y/w_L > y/w_H$. It follows from strict convexity of $h(\cdot)$ that $h'\left(\frac{y}{w_L}\right) > h'\left(\frac{y}{w_H}\right)$. Lemma 4.1 follows immediately. □

Proof of Lemma 4.2

By the first fundamental theorem of calculus, (4.11) holds if and only if

$$\int_{\hat{y}}^{\tilde{y}} \frac{1}{w_L} h'(\frac{y}{w_L}) dy \geq \int_{\hat{y}}^{\tilde{y}} \frac{1}{w_H} h'(\frac{y}{w_H}) dy. \tag{A.8}$$

$h(\cdot)$ strictly increasing implies that the integrands on each side of (A.8) are positive. Lemma 4.1 implies that the integrand on the left–hand side of (A.8) is everywhere greater than the integrand on the right–hand side. Thus, (A.8) holds if and only if $\tilde{y} \geq \hat{y}$. □

Proof of Proposition 4.3:

The proof of statement i) is presented. Statements ii)–iv) are proven analagously to statement i).

Let the C–D and D–C self–selection constraints be satisfied. Then

$$V(x^C) - h(\frac{y_1^C}{w_H}) - \gamma h(\frac{y_2^C}{w_H}) - V(x^D) + h(\frac{y_1^D}{w_H}) + \gamma h(\frac{y_2^D}{w_H}) \geq 0, \tag{A.9}$$

and

$$V(x^D) - h(\frac{y_1^D}{w_L}) - \gamma h(\frac{y_2^D}{w_H}) - V(x^C) + h(\frac{y_1^C}{w_L}) + \gamma h(\frac{y_2^C}{w_H}) \geq 0. \tag{A.10}$$

Adding (A.9) and (A.10) yields

$$h(\frac{y_1^C}{w_L}) - h(\frac{y_1^D}{w_L}) + h(\frac{y_1^D}{w_H}) - h(\frac{y_1^C}{w_H}) \geq 0. \tag{A.11}$$

Appy Lemma 4.2 to conclude that $y_1^C \geq y_1^D$.

Now let $y_2^C > y_2^D$. Because $y_1^C \geq y_1^D$, (A.9) implies $V(x^C) > V(x^D)$. But $V(\cdot)$ is increasing, so $x^C > x^D$.

Suppose, instead, that $y_2^C = y_2^D$. Then $y_1^C > y_1^D$ and (A.9) imply $V(x^C) > V(x^D)$, and, hence, $x^C > x^D$. On the other hand, $y_1^C = y_1^D$, (A.9) and (A.10) imply $V(x^C) = V(x^D)$. Then $x^C = x^D$. □

Proof of Proposition 4.4:

The proof of statements i) and iii.) are presented. Statement ii) can be proved in an anlagous manner to statement i.); statement iv.), to that of statement iii.).

Let the C–A and A–C self-selection constraints be satisfied. Then

$$V(x^C) - h(\frac{y_1^C}{w_H}) - \gamma h(\frac{y_2^C}{w_H}) - V(x^A) + h(\frac{y_1^A}{w_H}) + \gamma h(\frac{y_2^A}{w_H}) \geq 0, \tag{A.12}$$

and

$$V(x^A) - h(\frac{y_1^A}{w_L}) - \gamma h(\frac{y_2^A}{w_L}) - V(x^C) + h(\frac{y_1^C}{w_L}) + \gamma h(\frac{y_2^C}{w_L}) \geq 0. \tag{A.13}$$

Adding (A.12) and (A.13) yields

$$\Big[h(\frac{y_1^C}{w_L}) - h(\frac{y_1^A}{w_L}) + h(\frac{y_1^A}{w_H}) - h(\frac{y_1^C}{w_H})\Big] + \gamma\Big[h(\frac{y_2^C}{w_L}) - h(\frac{y_2^A}{w_L}) + h(\frac{y_2^A}{w_H}) - h(\frac{y_2^C}{w_H})\Big] \geq 0. \tag{A.14}$$

Now let $y_1^C < y_1^A$. Apply Lemma 4.2 to conclude that the first term in square brackets in (A.14) is negative. Hence, the second term is positive. Apply Lemma 4.2 once more to conclude $y_2^C > y_2^A$.

Suppose $y_1^C = y_1^A$. Then, by (A.14) and Lemma 4.2, $y_2^C \geq y_2^A$. If $y_2^C = y_2^A$ then (A.12) and (A.13) can both be satisfied only if $V(x^A) = V(x^C)$. Given increasingness of $V(\cdot)$, this implies $x^A = x^C$. If $y_2^C > y_2^A$ then (A.12) can be satisfied only if $V(x^C) > V(x^A)$; that is, when $x^C > x^A$. The final sentence of i.) follows directly from (A.12).

Let the B–D and D–B self-selection constraints be satisfied. Then

$$V(x^B) - h(\frac{y_1^B}{w_H}) - \gamma h(\frac{y_2^B}{w_L}) - V(x^D) + h(\frac{y_1^D}{w_H}) + \gamma h(\frac{y_2^D}{w_L}) \geq 0, \tag{A.15}$$

and

$$V(x^D) - h(\frac{y_1^D}{w_L}) - \gamma h(\frac{y_2^D}{w_H}) - V(x^B) + h(\frac{y_1^B}{w_L}) + \gamma h(\frac{y_2^B}{w_H}) \geq 0. \tag{A.16}$$

Adding (A.15) and (A.16) yields

$$\Big[h(\frac{y_1^B}{w_L}) - h(\frac{y_1^D}{w_L}) + h(\frac{y_1^D}{w_H}) - h(\frac{y_1^B}{w_H})\Big] + \gamma\Big[h(\frac{y_2^D}{w_L}) - h(\frac{y_2^B}{w_L}) + h(\frac{y_2^B}{w_H}) - h(\frac{y_2^D}{w_H})\Big] \geq 0. \tag{A.17}$$

Now let $y_1^B < y_1^D$. By Lemma 4.2, the first term in square brackets in (A.17) is negative. Then, the second term must be positive. Apply Lemma 4.2 to conclude that $y_2^D > y_2^B$. Furthermore, by (A.16), $V(x^D) > V(x^B)$. Hence, $x^D > x^B$.

Suppose $y_1^B = y_1^D$. Then, by Lemma 4.2 and (A.17), $y_2^D \geq y_2^B$. If $y_2^D > y_2^B$, then (A.16) can be satisfied only if $x^D > x^B$. If $y_2^D = y_2^B$, then (A.15) and (A.16) imply $x^D = x^B$. □

Let $\Psi^{ij} := W^i(x^i, y_1^i, y_2^i) - W^i(x^j, y_1^j, y_2^j)$.

Proof of Proposition 4.5:

Once again, the proof of statement i.) is presented. ii.) can be proved in analagous fashion.

Notice that

$$\begin{aligned}\Psi^{CA} - \Psi^{CD} - \Psi^{DA} = &- V(x^A) + h(\frac{y_1^A}{w_H}) + h(\frac{y_2^A}{w_H}) + V(x^D) - h(\frac{y_1^D}{w_H}) - h(\frac{y_2^D}{w_H}) \\ &- V(x^D) + h(\frac{y_1^D}{w_L}) + h(\frac{y_2^D}{w_H}) + V(x^A) - h(\frac{y_1^A}{w_L}) - h(\frac{y_2^A}{w_H}).\end{aligned} \tag{A.18}$$

From (A.18) it follows that

$$\Psi^{CA} - \Psi^{CD} - \Psi^{DA} = h(\frac{y_1^D}{w_L}) - h(\frac{y_1^A}{w_L}) + h(\frac{y_1^A}{w_H}) - h(\frac{y_1^D}{w_H}). \tag{A.19}$$

Let the C–A self–selection constraint bind. Then $\Psi^{CA} = 0$, so that the left–hand side of (A.19) must be non–positive when (SS) holds. Apply Lemma 4.2 to the right–hand side of (A.19) to conclude that $y_1^D \leq y_1^A$, with $y_1^D = y_1^A$ only if $\Psi^{CD} = \Psi^{DA} = 0$.

The argument establishing $y_2^B \leq y_2^A$ is analogous. □

Proof of Proposition 4.6:

The proof of i.) is presented. Statement ii.) has a similar proof

It can be shown that

$$\Psi^{DB} - \Psi^{DC} - \Psi^{CB} = h(\frac{y_1^B}{w_L}) - h(\frac{y_1^C}{w_L}) + h(\frac{y_1^C}{w_H}) - h(\frac{y_1^B}{w_H}). \tag{A.20}$$

Now let the D–B self–selection constraint bind, so $\Psi^{DB} = 0$. By (SS), $\Psi^{DC} \geq 0$ and $\Psi^{CB} \geq 0$. Thus, the left–hand side of (A.20) is non–positive. Apply Lemma 4.2 to conclude that $y_1^C \geq y_1^B$, with equality only if $\Psi^{DC} = \Psi^{CB} = 0$.

The argument establishing $y_2^B \geq y_2^A$ is analogous. □

Let μ_{ij} denote the Lagrange multiplier associated with the i–j self–selection constraint, and let λ denote the Lagrange multiplier associated with the constraint (F). The first order necessary conditions for a solution to (PF) can be written as:

$$\Big[Z_{W^i} + \sum_{j \neq i} (\mu_{ij} - \mu ji)\Big] V'(x^i) = \lambda \pi^i \quad \forall i; \tag{A.21}$$

$$\Big[Z_{W^i} + \sum_{j \neq i} \mu_{ij}\Big] \frac{1}{w_1^i} h'(\frac{y_1^i}{w_1^i}) - \sum_{j \neq i} \mu_{ji} \frac{1}{w_1^j} h'(\frac{y_1^i}{w_1^j}) = \lambda \pi^i \quad \forall i; \tag{A.22}$$

$$\Big[Z_{W^i} + \sum_{j \neq i} \mu_{ij}\Big] \frac{\gamma}{w_2^i} h'(\frac{y_2^i}{w_2^i}) - \sum_{j \neq i} \mu_{ji} \frac{\gamma}{w_2^j} h'(\frac{y_2^i}{w_2^j}) = \lambda \pi^i \quad \forall i. \tag{A.23}$$

Proof of Proposition 4.7:

Consider the first order necessary conditions for a solution to (PF). Suppose $\lambda = 0$. Then, the first–order necessary condition associated with y_2^A becomes

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA}\Big] \frac{\gamma}{w_L} h'(\frac{y_2^A}{w_L}) - \Big[\mu_{CA} + \mu_{DA}\Big] \frac{\gamma}{w_H} h'(\frac{y_2^A}{w_H}) = 0. \tag{A.24}$$

Suppose that $\mu_{CA} + \mu_{DA} > 0$. It follows from Lemma 4.1 that

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA} - \mu_{CA} - \mu_{DA}\Big] \frac{\gamma}{w_L} h'(\frac{y_2^A}{w_L}) < 0. \tag{A.25}$$

Because $h(\cdot)$ is increasing,

$$Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA} - \mu_{CA} - \mu_{DA} < 0. \tag{A.26}$$

However, the first–order condition associated with x^A is

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} - \mu_{BA} - \mu_{CA} - \mu_{DA}\Big] V'(x^A) = 0. \tag{A.27}$$

Hence, $V'(x^A) = 0$, a contradiction. Thus, $\mu_{CA} + \mu_{DA} = 0$.

A similar argument, using the first–order necessary condition associated with y_1^A, establishes that $\mu_{CA} + \mu_{BA} = 0$. Because all the multipliers are non–negative, $\mu_{CA} = \mu_{DA} = \mu_{BA} = 0$. Then (A.27) becomes

$$\Big[Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD}\Big] V'(x^A) = 0. \tag{A.28}$$

But $Z_{W^A} + \mu_{AB} + \mu_{AC} + \mu_{AD} > 0$, since $Z(\cdot)$ is increasing. Thus, $V'(x^A) = 0$, a contradiction. □

Proof of Proposition 4.8:

The argument given here is essentially due to Brito *et al.* (1990), except that they analyse a "dual" problem that any solution to (PF) must also be a solution to. The proof below makes reference to (PF) alone.

Suppose, contrary to the statement of the proposition, that at a solution to (PF) there exists families i and j such that $W^i(x^i, y_1^i, y_2^i) = W^i(x^j, y_1^j, y_2^j)$ and

$$y_1^i + y_2^i - x^i < y_1^j + y_2^j - x^j. \tag{A.29}$$

Now construct a new allocation identical to the original solution, except replace the allocation of family i by that of family j (leaving family j with its original allocation). Clearly, $\Psi^{ij} = \Psi^{ji} = 0$ at the new allocation. Furthermore, because $\Psi^{kj} \geq 0$ for all $k \neq i, j$ at the old allocation, $\Psi^{ki} \geq 0$ for all $k \neq i, j$ in the new allocation. In addition, $\Psi^{ik} \geq 0$ for all $k \neq i, j$ at the new allocation because in the original allocation family i was indifferent between its own allocation and that of family j and weakly preferred its own bundle to that of other families. Thus, the new allocation satisfies all self–selection constraints, makes no family worse off than in the original allocation and, by (A.29), slackens the feasibility constraint. Hence, the original allocation could not have been a solution to (PF). □

Proof of Proposition 4.9:

The proof of statement i) is presented. The proof of statement ii.) is almost identical.

Pick a family i with $w_1^i = w_L$. Let J be the set of indices not equal to i. Partition J into $\hat{J} := \{j \in J : w_1^j = w_H\}$ and $\tilde{J} := \{j \in J : w_1^j = w_L\}$. The first–order necessary conditions for a solution to (PF) include

$$\Big[Z_{W^i} + \sum_{j \in J} \mu_{ij} - \sum_{j \in \tilde{J}} \mu_{ji}\Big] \frac{1}{w_L} h'\Big(\frac{y_1^i}{w_L}\Big) - \sum_{j \in \hat{J}} \mu_{ji} \frac{1}{w_H} h'\Big(\frac{y_1^i}{w_H}\Big) = \lambda \pi^i, \tag{A.30}$$

and

$$\Big[Z_{W^i} + \sum_{j \in J} (\mu_{ij} - \mu_{ji})\Big] V'(x^i) = \lambda \pi^i. \tag{A.31}$$

If we replace each term in the sum over $\hat{J}$ in (A.30) by $\mu_{ji}\Big(h'\big(\frac{y_1^i}{w_L}\big)/w_L\Big)$, then we may use Lemma 4.1 to conclude

$$\Big[Z_{W^i} + \sum_{j\in J}(\mu_{ij} - \mu_{ji})\Big]\frac{1}{w_L}h'\big(\frac{y_1^i}{w_L}\big) \leq \lambda\pi^i, \tag{A.32}$$

with (A.32) holding with equality if and only if $\mu_{ji} = 0$ for all $j \in \hat{J}$. By Proposition 4.7, λ is positive. Hence we may divide (A.32) by (A.31) to yield

$$\frac{h'(\frac{y_1^i}{w_L})}{w_L V'(x^i)} \leq 1, \tag{A.33}$$

with (A.33) holding with equality if and only if $\mu_{ji} = 0$ for all $j \in \hat{J}$. □

Proof of Proposition 4.10:

Suppose, by way of contradiction, that there exist two families i and j of different type that receive the same allocation, $(\bar{x}, \bar{y}_1, \bar{y}_2)$, at a solution to (PF). There exist $k \in \{1, 2\}$ such that $w_k^i = w_H$ and $w_k^j = w_L$. Then $MRS^j_{y_k,x}(\bar{x}, \bar{y}_1, \bar{y}_2) > MRS^i_{y_k,x}(\bar{x}, \bar{y}_1, \bar{y}_2)$. That is, person k in family j faces a lower marginal tax rate than person k in family i. But, by Proposition 4.9, the former faces a non–negative marginal tax rate while the latter faces a non–positive marginal tax rate. A contradiction ensues. □

Proof of Proposition 4.11

The result is proven for the pair of families (C,D) and for the pair of families (B,D). The proofs for the pairs (A,B), (A,D) and (C,B) are analogous to the demonstration for the pair (C,D).

Let the C–D and D–C constraints bind simultaneously. Then (A.9), (A.10), and (A.11) all hold with equality. By Lemma 4.2, $y_1^C = y_1^D$. Thus, (A.10) reduces to

$$V(x^D) - \gamma h(\frac{y_2^D}{w_H}) - V(x^C) + \gamma h(\frac{y_2^C}{w_H}) = 0. \tag{A.34}$$

By definition,

$$\begin{aligned}\Psi^{AD} &= V(x^A) - h(\frac{y_1^A}{w_L}) - \gamma h(\frac{y_2^A}{w_L}) - V(x^D) + h(\frac{y_1^D}{w_L}) + \gamma h(\frac{y_2^D}{w_L}) \\ &= V(x^A) - h(\frac{y_1^A}{w_L}) - \gamma h(\frac{y_2^A}{w_L}) - V(x^D) + h(\frac{y_1^C}{w_L}) + \gamma h(\frac{y_2^D}{w_L}).\end{aligned} \tag{A.35}$$

Adding (A.34) and (A.35) yields

$$\begin{aligned}\Psi^{AD} &= V(x^A) - h(\frac{y_1^A}{w_L}) - \gamma h(\frac{y_2^A}{w_L}) - V(x^C) + h(\frac{y_1^C}{w_L}) + \gamma h(\frac{y_2^C}{w_L}) \\ &\quad + \gamma\Big[h(\frac{y_2^D}{w_L}) - h(\frac{y_2^C}{w_L}) + h(\frac{y_2^C}{w_H}) - h(\frac{y_2^D}{w_H})\Big] \\ &= \Psi^{AC} + \gamma\Big[h(\frac{y_2^D}{w_L}) - h(\frac{y_2^C}{w_L}) + h(\frac{y_2^C}{w_H}) - h(\frac{y_2^D}{w_H})\Big].\end{aligned} \tag{A.36}$$

Analogous calculations can be used to show

$$\Psi^{BD} = \Psi^{BC} + \gamma\Big[h(\frac{y_2^D}{w_L}) - h(\frac{y_2^C}{w_L}) + h(\frac{y_2^C}{w_H}) - h(\frac{y_2^D}{w_H})\Big]. \tag{A.37}$$

Suppose that $y_2^D > y_2^C$. Then Lemma 4.2, (A.36) and (A.37) imply that both the A–D and B–D constraints are slack. Applying Proposition 4.9 yields that $MRS_{y_2,x}^D = 1$ and $MRS_{y_2,x}^C \geq 1$. However, (A.34) implies $x^D > x^C$, so that $MRS_{y_2,x}^D > MRS_{y_2,x}^C$, a contradiction. The case of $y_2^C > y_2^D$ is ruled out by a symmetric argument. Thus, $y_2^C = y_2^D$. In order for (A.34) to hold, it must be the case that families C and D receive the same bundle.

Suppose that the D–B and B–D constraints both bind. Then

$$V(x^D) - h(\frac{y_1^D}{w_L}) - \gamma h(\frac{y_2^D}{w_H}) - V(x^B) + h(\frac{y_1^B}{w_L}) + \gamma h(\frac{y_2^B}{w_H}) = 0, \tag{A.38}$$

and

$$V(x^B) - h(\frac{y_1^B}{w_H}) - \gamma h(\frac{y_2^B}{w_L}) - V(x^D) + h(\frac{y_1^D}{w_H}) + \gamma h(\frac{y_2^D}{w_L}) = 0. \tag{A.39}$$

Adding (A.38) and (A.39) gives

$$h(\frac{y_1^B}{w_L}) - h(\frac{y_1^D}{w_L}) + h(\frac{y_1^D}{w_H}) - h(\frac{y_1^B}{w_H}) = \gamma\Big[h(\frac{y_2^B}{w_L}) - h(\frac{y_2^D}{w_L}) + h(\frac{y_2^D}{w_H}) - h(\frac{y_2^B}{w_H})\Big]. \tag{A.40}$$

It follows from Lemma 4.2 that $y_1^B \geq y_1^D$ if and only if $y_2^B \geq y_2^D$.

Assume that $y_1^B > y_1^D$. Then $y_2^B > y_2^D$, and self–selection requires $x^B > x^D$. Furthermore, $y_2^B/w_L > y_2^D/w_H$. Thus, $MRS^B_{y_2,x} > MRS^D_{y_2,x}$. However, Proposition 4.9 implies that at a solution to (PF), $MRS^B_{y_2,x} \leq 1$ and $MRS^D_{y_2,x} \geq 1$. A contradiction ensues.

The case of $y_1^D > y_1^B$ can be ruled out by similar arguments, involving $MRS^B_{y_1,x}$ and $MRS^D_{y_1,x}$. Hence, $y_1^B = y_1^D$ and $y_2^B = y_2^D$. Self–selection requires $x^B = x^D$, so that families B and D receive the same bundle. □

Proof of Proposition 4.12:

For all pairs of families except the pair {C,A} the proposition follows directly from Propositions 4.10 and 4.11. Suppose that families C and A are mutually attracted at a solution to (PF). Then, by Proposition 4.8,

$$y_1^A + y_2^A - x^A = y_1^C + y_2^C - x^C. \tag{A.41}$$

Now construct a new allocation by giving family A the allocation of family C, leaving the bundles received by all other families unchanged. By the argument outlined

in the proof of Proposition 4.8, this new allocation satisfies all of the self–selection constraints. (A.41) ensures that the new allocation is production–feasible. Moreover, all families are indifferent between the new allocation and the original. Because the original allocation solves (PF), the new one does as well. That is, there is a solution to (PF) at which families A and C receive the same allocation, contradicting Proposition 4.10. □

Proof of Proposition 4.13:

Suppose, by way of contradiction, that there is a self–selection cycle at a solution to (PF). In view of Proposition 4.8, all families in the cycle have the same total tax liability. Now select a pair of families, (i, j), from the cycle for which family j attracts family i. Then, we may replace the bundle of family i by the bundle of family j without making any family worse off and without violating the materials balance constraint. This results in a solution to (PF) in which families i and j are pooled, contradicting Proposition 4.10. *square*

Proof of Proposition 4.14:

Suppose that relation (4.26) of the text holds for family i. If no families are attracted to family i, then the result follows immediately from Proposition 4.9. By Proposition 4.8, any families attracted to family i must pay the same total tax bill as family i. Let family j be such a family. Replicating the argument of Proposition 4.12 allows one to conclude that replacing the allocation of family i by that of family j

results in another solution to (PF) that features pooling. This contradicts Proposition 4.10. □

Proof of Proposition 4.15:

I show that the C–A and D–B self–selection constraints cannot bind simultaneously. All other cases have identical proofs.

Suppose, by way of contradiction, that the C–A and D–B self–selection constraints both bind at a solution to (PF). By Proposition 4.5, $y_2^B \leq y_2^A$. But, by Proposition 4.6, $y_2^A \leq y_2^B$. Hence, $y_2^B = y_2^A$. Now apply Propositions 4.5 and 4.6 once more to conclude that the B–A and A–B self–selection constraints must both bind. This contradicts Proposition 4.12. □

Proof of Proposition 5.1:

The proof employs the following claim:

Claim 1: If $(C, A) \notin \mathcal{C}$ and if $\{ (C, B), (B, A) \} \subseteq \mathcal{C}$, then $(D, A) \in \mathcal{C}$.

Suppose, by way of contradiction, that the D–A constraint is slack. It can be shown that

$$\Psi^{CA} - \Psi^{CB} - \Psi^{BA} = \gamma\Big[h(\frac{y_2^B}{w_L}) - h(\frac{y_2^A}{w_L}) - h(\frac{y_2^B}{w_H}) + h(\frac{y_2^A}{w_H})\Big]. \tag{A.42}$$

Now, by the hypothesis of the claim, $\Psi^{CB} = \Psi^{BA} = 0$. Thus, by Lemma 4.2, the C–A self–selection constraint can be satisfied only if $y_2^B \geq y_2^A$. By Proposition

4.3, $y_1^B \geq y_1^A$. Then self–selection requires that $x^B \geq x^A$. Hence, $MRS^A_{y_2,x} \leq MRS^B_{y_2,x}$. Moreover, $x^B > x^A$ if either $y_1^B > y_1^A$ or $y_2^B > y_2^A$. When $x^B > x^A$, $MRS^A_{y_2,x} < MRS^B_{y_2,x}$ (because $y_2^B \geq y_2^A$). Hence, if $MRS^A_{y_2,x} = MRS^B_{y_2,x}$, then $y_1^B = y_1^A$ and $y_2^B = y_2^A$. But then self–selection implies $x^B = x^A$, violating the no–pooling condition. Consequently, $MRS^A_{y_2,x} < MRS^B_{y_2,x}$. Because the C–A constraint is slack, Proposition 4.9 implies $MRS^A_{y_2,x} = 1$. But, by Proposition 4.9, $MRS^B_{y_2,x} \leq 1$ at a solution to (PF), a contradiction. Hence, the D–A self–selection constraint must bind. This establishes the claim.

Suppose that Assumptions R and U hold. Consider a pair of families (i, j) with $i >_F j$. By Assumption R, a sufficiently small transfer of x from family i to family j is socially desirable and does not lead to a violation of the materials balance constraint (by Assumption U). Moreover, such a redistribution will not lead to a violation of the constraints (SS) as long as, at the before–transfer allocation, the "donor" family is not attracted to any other family and the "recipient" family attracts no other family.

Now, let the C–A constraint be slack. Suppose that the C–D constraint is slack as well. I will show that the C–B constraint must bind. Suppose otherwise. Then a small redistribution of x from family C to family B would not violate the materials balance constraint (by Assumption U) and would be socially desirable (by Assumption R). Because family C is not attracted to any family at the original candidate solution, the posited redistribution can be infeasible only if either the D–B constraint or the A–B constraint binds at the candidate solution.

Case 1: D–B binds and A–B binds.

The D–A self selection constraint is slack, for otherwise there would be a cycle, contradicting Proposition 4.13. By Proposition 4.12, the B–A self–selection constraint is slack. Then a small redistribution from family C to family A is both feasible and desirable. Thus, the original allocation is not a solution to (PF).

Case 2: D–B binds and A–B does not bind.

By Proposition 4.12, the B–D self–selection constraint is slack. Suppose the A–D constraint is slack as well. Because the C–D constraint is slack by assumption, a small redistribution of x from family C to family D is feasible and socially desirable, contradicting the optimality of the original allocation. Now suppose the A–D constraint binds. Then the B–A constraint must not bind, for otherwise there is a cycle ($\{(D, B), (B, A), (A, D)\}$). By Proposition 4.12, the D–A constraint must be slack. Consequently, a small redistribution from family C to family A is feasible and socially desirable. Hence, the candidate solution is not optimal.

Case 3: D–B does not bind and A–B binds.

The B–A constraint does not bind (by Proposition 4.12). If the D–A constraint is slack, a small redistribution of x from family C to family A is both feasible and socially desirable. If, instead, the D–A constraint is binding, then Proposition 4.12 ensures that the A–D constraint is slack. Now, a binding B–D constraint produces the cycle $\{(B, D), (D, A), (A, B)\}$. Hence, the B–D constraint is slack. Thus, a small

redistribution from family C to family D is feasible and socially desirable, violating the optimality of the original allocation.

Thus, it has been shown that the C–B constraint must bind whenever the C–A and C–D constraints are both slack.

It is also the case that either the D–A constraint or the B–A constraint must bind (possibly both). Otherwise a small redistribution of x from family C to family A would be socially desirable (by Assumption R), production–feasible (by Assumption U) and will not lead to a violation of the self–selection constraints (because the C–A constraint is slack). If the D–A constraint binds, we have immediately that $\{ (C, B), (D, A) \} \subseteq \mathcal{C}$. Now, suppose that the D–A constraint does not bind. Then the B–A constraint must bind. Employ the claim to conclude that the D–A constraint must bind as well, a contradiction.

Now suppose that the C–D constraint is binding. Again, we must have either the D–A or the B–A constraint (or both) binding. If the B–A constraint binds, we are done. If the D–A constraint binds, an argument symmetric to the one establishing Claim 1 applies, and the proof of statement i) is complete.

Now suppose that Assumptions R, U and Q hold and that the C–A and C–D constraints do not bind. Then, by statement i) of the proposition, the C–B and D–A constraints must bind. Suppose that the B–A constraint does not bind. Then, by Assumption Q, we can transfer sufficiently small and *equal* amounts of x from C to

D and from B to A without affecting the C–B or D–A constraints. By Assumption U, this redistribution does not affect the production constraint, and it is socially desirable (by Assumption R). There are now two cases to consider.

Case 1: B–D does not bind.

By Proposition 4.12 and the D–A constraint binding, the A–D constraint is slack. Thus, no family is attracted to family D. Because family A attracts family D alone, the posited redistribution does not violate (SS).

Case 2: B–D binds.

When $\Psi^{BD} = 0$, it can be shown that

$$\Psi^{CD} = \Psi^{CB} + \gamma\Big[h\big(\frac{y_2^B}{w_L}\big) - h\big(\frac{y_2^B}{w_H}\big) + h\big(\frac{y_2^D}{w_H}\big) - h\big(\frac{y_2^D}{w_L}\big)\Big]. \tag{A.43}$$

Given that the C–B constraint binds, Lemma 4.2 implies $\Psi^{CD} \geq 0$ if and only if $y_2^B \geq y_2^D$. Because the C–D constraint is slack, $y_2^B > y_2^D$. From Assumption Q, it follows that $MRS^B_{y_2,x} > MRS^D_{y_2,x}$, violating Proposition 4.9. Hence, this case cannot arise.

Now suppose the C–D constraint binds. We may replicate the argument in the last paragraph of the proof of statement i) to conclude that the B–A constraint must bind as well. If, in addition, the C–B constraint binds, it follows from Claim 1 that the D–A constraint is binding, establishing statement ii). If the C–B constraint does

not bind, then an argument symmetric to the one used when it was assumed that the C–D constraint is slack can be employed.

Now suppose that the A–C constraint binds. Then, if the C–D constraint binds, $\{(C, D), (D, A), (A, C)\}$ is a self–selection cycle. If the C–D constraint is slack, the C–B constraint must bind. Hence, $\{(C, B), (B, A), (A, C)\}$ is a self–selection cycle. Thus, in both cases, Proposition 4.13 is violated. □

Proof of Proposition 5.2:

Only the proof of statement i) is presented. Proposition 4.15 implies that we may assume that the C–A and A–C self–selection constraints are slack throughout this proof.

Assume that the D–B constraint binds. Then $\Psi^{CB} = \Psi^{CB} - \Psi^{DB}$, or

$$\begin{aligned}\Psi^{CB} = V(x^C) - h(\frac{y_1^C}{w_H}) - \gamma h(\frac{y_2^C}{w_H}) - V(x^B) + h(\frac{y_1^B}{w_H}) + \gamma h(\frac{y_2^B}{w_H}) \\ + V(x^B) - h(\frac{y_1^B}{w_L}) - \gamma h(\frac{y_2^B}{w_H}) - V(x^D) + h(\frac{y_1^D}{w_L}) + \gamma h(\frac{y_2^D}{w_H}).\end{aligned} \tag{A.44}$$

Add and subtract $h(\frac{y_1^D}{w_H})$ to the left–hand side of (A.44) to conclude that

$$\Psi^{CB} = \Psi^{CD} + h(\frac{y_1^D}{w_L}) - h(\frac{y_1^D}{w_H}) + h(\frac{y_1^B}{w_H}) - h(\frac{y_1^B}{w_L}). \tag{A.45}$$

Case 1: $y_1^D > y_1^B$.

Lemma 4.2 and (A.45) imply that the C–B constraint must be slack, so, by Proposition 5.1, the C–D and B–A constraints bind. It follows from Proposition 4.12

that the D–C and A–B constraints are slack. Thus, by Proposition 4.9, $MRS^C_{y_1,x} = 1$ and $MRS^B_{y_1,x} \geq 1$. However, by Proposition 4.3, $y_1^C \geq y_1^D$, so that $y_1^C > y_1^B$. Apply the MRS conditions to conclude that $x^B > x^C$. But, by Proposition 4.3, $y_2^C \geq y_2^B$. Then the C–B constraint must be violated, a contradiction.

Case 2: $y_1^D = y_1^B$.

Then, by Lemma 4.2, (A.45), and Proposition 5.1, both the C–D and C–B constraints bind. Thus, $MRS^C_{y_1,x} = 1$ and $MRS^B_{y_1,x} \geq 1$. From this, we may replicate the argument of Case 1 to deduce a contradiction.

Case 3: $y_1^B > y_1^D$.

Then, by Lemma 4.2 and (A.45), the C–D constraint is slack. Thus, by Proposition 5.1, both the C–B and D–A constraints bind. Now, it must be the case that $\Psi^{DC} = \Psi^{DC} - \Psi^{DB} + \Psi^{CB}$, or

$$\Psi^{DC} = h(\frac{y_1^C}{w_L}) - h(\frac{y_1^B}{w_L}) + h(\frac{y_1^B}{w_H}) - h(\frac{y_1^C}{w_H}). \tag{A.46}$$

Hence, the D–C constraint is satisfied only if $y_1^C \geq y_1^B$. Suppose $y_1^C > y_1^B$. By Proposition 4.3, $y_2^C \geq y_2^B$. Hence, the C–B constraint can be satisfied only if $x^C > x^B$. Thus, $MRS^C_{y_1,x} > MRS^B_{y_1,x} \geq 1$, where the second inequality follows from Proposition 4.9 and D–B binding. Because the A–C constraint is slack, we may apply Proposition 4.9 to conclude that the D–C constraint must bind. So that, by (A.46), $y_1^C = y_1^B$, contradicting $y_1^C = y_1^B$. Thus, we may conclude that $y_1^C = y_1^B$ and, in view of (A.46), that the D–C constraint binds.

The result follows from Case 3 being the only possibility that is not contradictory.

□

Proof of Proposition 5.3:

In view of Proposition 4.15, this proposition follows immediately from Propositions 5.1 and 5.2. □

Proof of Proposition 5.4:

I give the proof of the first inequality. The second is proven in a symmetric fashion.

I first show that if $MRS^B_{y_1,x} \geq MRS^C_{y_1,x}$ at a solution to (PF), then $y_1^B \geq y_1^C$. Suppose, to the contrary, that $MRS^B_{y_1,x} \geq MRS^C_{y_1,x}$ and $y_1^B < y_1^C$. Then, $x^B > x^C$. But, by Proposition 4.3, $y_2^C \geq y_2^B$. Thus, the C–B constraint is violated.

Now suppose $MRS^B_{y_1,x} < MRS^C_{y_1,x}$. (Otherwise, we are done.) By Proposition 4.9, $MRS^B_{y_1,x} \geq 1$. Thus, $MRS^C_{y_1,x} > 1$. Hence, by Proposition 4.9, it must be the case that either the A–C constraint binds or the D–C constraint binds (or both). But, by Proposition 4.5, when the A–C constraint binds, $y_1^B \geq y_1^C$.

It remains to consider the case of D–C binding. Two possibilities need to be considered.

Case 1: C–A does not bind.

By Proposition 4.12, the C–D constraint is slack. In view of Proposition 5.1, this implies that the C–B constraint binds. By the argument used to establish (A.46), we know that

$$\Psi^{DC} - \Psi^{DB} + \Psi^{CB} = h(\frac{y_1^C}{w_L}) - h(\frac{y_1^B}{w_L}) + h(\frac{y_1^B}{w_H}) - h(\frac{y_1^C}{w_H}). \tag{A.47}$$

Because the C–B and D–C constraints bind, (A.47) and Lemma 4.2 imply that $\Psi^{DB} \geq 0$ if and only if $y_1^B \geq y_1^C$.

Case 2: C–A binds.

It can be shown that

$$\Psi^{CA} + \Psi^{DC} = \Psi^{DA} + h(\frac{y_1^C}{w_L}) - h(\frac{y_1^A}{w_L}) + h(\frac{y_1^C}{w_H}) - h(\frac{y_1^A}{w_H}). \tag{A.48}$$

Because both the C–A and D–C constraints are assumed to bind, the left–hand side of (A.48) is zero. Apply Lemma 4.2 and (A.48) to conclude that $\Psi^{DA} \geq 0$ if and only if $y_1^A \geq y_1^C$. However, by Proposition 4.3, $y_1^B \geq y_1^A$. Thus, $y_1^B \geq y_1^C$. □

Proof of Proposition 5.5:

In addition to the notation already used, let η_i^j denote the multiplier on the constraint $y_i^j \geq 0$, for $j = A, B, C, D$ and $i = 1, 2$.

Under Assumption SQ, the first–order necessary conditions associated with the choices of before–tax incomes for family D are

$$(Z_{W^D} + \sum_{j \neq D} \mu_{Dj})/w_L - \sum_{j \neq D} \mu_{jD}/w_1^j - \eta_1^D = \lambda \pi^D, \tag{A.49}$$

$$\gamma\big(Z_{W^D} + \sum_{j\neq D} \mu_{Dj}\big)/w_H - \gamma \sum_{j\neq D} \mu_{jD}/w_2^j - \eta_2^D = \lambda\pi^D. \tag{A.50}$$

Eliminating $\lambda\pi^D$ from (A.49) and (A.50) yields

$$\eta_1^D - \eta_2^D$$

$$= \left[\left(\frac{1}{w_L} - \frac{\gamma}{w_H}\right)\big(Z_{W^D} + \sum_{j\neq D} \mu_{Dj}\big)\right] + \left[\frac{\gamma-1}{w_H}\right]\mu_{CD} + \left[\frac{\gamma-1}{w_L}\right]\mu_{AD} + \left[\frac{\gamma}{w_L} - \frac{1}{w_H}\right]\mu_{BD}. \tag{A.51}$$

Next, I show that the right–hand side of (A.51) is positive, so that $\eta_1^D > \eta_2^D \geq 0$, and, hence, that $y_1^D = 0$. Now, $\gamma > w_L/w_H$ implies

$$\frac{1}{w_L} - \frac{\gamma}{w_H} > 0. \tag{A.52}$$

Thus, the last term of (A.51) is non–negative. Consequently, it suffices to show that the first term, which is non–negative, exceeds the second and third terms in absolute value. But,

$$\left[\frac{1-\gamma}{w_H}\right]\mu_{CD} + \left[\frac{1-\gamma}{w_L}\right]\mu_{AD} < \left[\frac{1-\gamma}{w_L}\right]\big(\mu_{CD} + \mu_{AD}\big), \tag{A.53}$$

since $w_L < w_H$. However, $w_L < w_H$ also implies that

$$\frac{1-\gamma}{w_L} < \frac{1}{w_L} - \frac{\gamma}{w_H}. \tag{A.54}$$

The first–order necessary condition associated with the choice of after–tax income for family D is

$$\Big[Z_{W^D} + \sum_{j\neq D} \mu_{Dj} - \sum_{j\neq D} \mu_{jD}\Big]V'(x^D) + \eta_x^D - \lambda\pi^D = 0, \tag{A.55}$$

where the η_x^D is the multiplier associated with the non–negativity constraint on x^D. Because $U'(c)$ tends to positive infinity as c tends to zero, $\eta_x^D = 0$. By Proposition 4.7, $\lambda > 0$. Hence,

$$\Big[Z_{W^D} + \sum_{j \neq D} \mu_{Dj} - \sum_{j \neq D} \mu_{jD}\Big] V'(x^D) > 0. \tag{A.56}$$

Because $V'(x^D) > 0$, it may be canceled from (A.56) without changing the sense of the inequality. Using $\mu_{BD} \geq 0$, it now follows that

$$Z_{W^D} + \sum_{j \neq D} \mu_{Dj} > \mu_{CD} + \mu_{AD}. \tag{A.57}$$

Combining (A.53), (A.54), and (A.57) yields the result that $y_1^D - 0$.

Using the analogues to (A.49), (A.50), and (A.51) for family B, we can show that

$$\eta_1^B - \eta_2^B$$
$$= \left[\left(\frac{1}{w_H} - \frac{\gamma}{w_L}\right)\left(Z_{W^B} + \sum_{j \neq B} \mu_{Bj}\right)\right] + \left[\frac{\gamma - 1}{w_H}\right]\mu_{CB} + \left[\frac{\gamma - 1}{w_L}\right]\mu_{AB} + \left[\frac{\gamma}{w_H} - \frac{1}{w_L}\right]\mu_{DB}. \tag{A.58}$$

But the first term on the right–hand side of (A.58) is negative, while all the others are non–positive. Hence, $\eta_1^B < \eta_2^B$. Thus, $\eta_2^B > 0$, and $y_2^B = 0$.

From the assumption $y_1^C = 0\,(= y_1^D)$, along with (A.9) and (A.10), it follows that

$$\Psi^{DC} = -\Psi^{CD} = V(x^D) - \gamma\frac{y_2^D}{w_H} - V(x^C) + \gamma\frac{y_2^C}{w_H} = 0. \tag{A.59}$$

Substituting (A.59) into the definition of Ψ^{CA}, and using $y_1^D = 0$, yields

$$\Psi^{CA} = \Psi^{DA} + \frac{y_1^A}{w_H} - \frac{y_1^A}{w_L}. \tag{A.60}$$

Suppose that $y_1^A > 0$. Then, by (A.60), the D–A constraint is slack. Using the analogues to (A.49), (A.50), and (A.51) for family A and the fact that $\mu_{DA} = 0$,

$$\begin{aligned}&\eta_1^A - \eta_2^A \\ &= \left[\frac{(1-\gamma)}{w_L}\left(Z_{W^A} + \sum_j \mu_{Aj}\right)\right] + \left[\frac{\gamma}{w_L} - \frac{1}{w_H}\right]\mu_{BA} + \left[\frac{\gamma - 1}{w_H}\right]\mu_{CA}.\end{aligned} \tag{A.61}$$

The first term in (A.61) is positive, while the second is non–negative. Hence, if $\mu_{CA} = 0$, we have $\eta_1^A > \eta_2^A \geq 0$. Now suppose $\mu_{CA} > 0$. In this case, $(1-\gamma)/w_H < (1-\gamma)/w_L$, so that we have

$$\left[\frac{1-\gamma}{w_H}\right]\mu_{CA} < \left[\frac{1-\gamma}{w_L}\right]\mu_{CA}. \tag{A.62}$$

But the first–order necessary condition for the allocation of after–tax income to family A implies

$$Z_{W^A} + \sum_{j \neq A} \mu_{Aj} > \mu_{CA}. \tag{A.63}$$

Then, the first term on the right–hand side of (A.61) exceeds the final term in absolute value, so that $\eta_1^A > \eta_2^A \geq 0$ in this case as well. Thus, $y_1^A = 0$, a contradiction. It follows that $y_1^A = 0$. This proves i.).

Statement iii.) now follows directly from (A.60).

By (A.59), both the C–D and D–C constraints bind. Suppose, by way of contradiction, that $y_2^D > y_2^C$. Then, by the argument given in the proof of Proposition 4.11, both the A–D and B–D constraints are slack. Also, $y_2^D > 0$, so that $MRS_{y_2,x}^D = 1$. Notice that η_2^C enters the analogue to (A.50) for family C with a negative sign, just

like μ_{jC} does for $j \neq C$. Thus, by an argument similar to the one used to establish Proposition 4.9, $MRS^C_{y_2,x} \geq 1$. Self–selection requires $x^D > x^C$. But then, $MRS^C_{y_2,x} < MRS^D_{y_2,x}$, a contradiction. A symmetric argument can be used to rule out $y_2^C > y_2^D$.

Thus, $y_2^C = y_2^D$, and, because $y_1^C = y_1^D$, Proposition 4.3 implies $x^C = x^D$. This establishes ii.). $\square$

APPENDIX B

Figure 1. The Space of Family Types

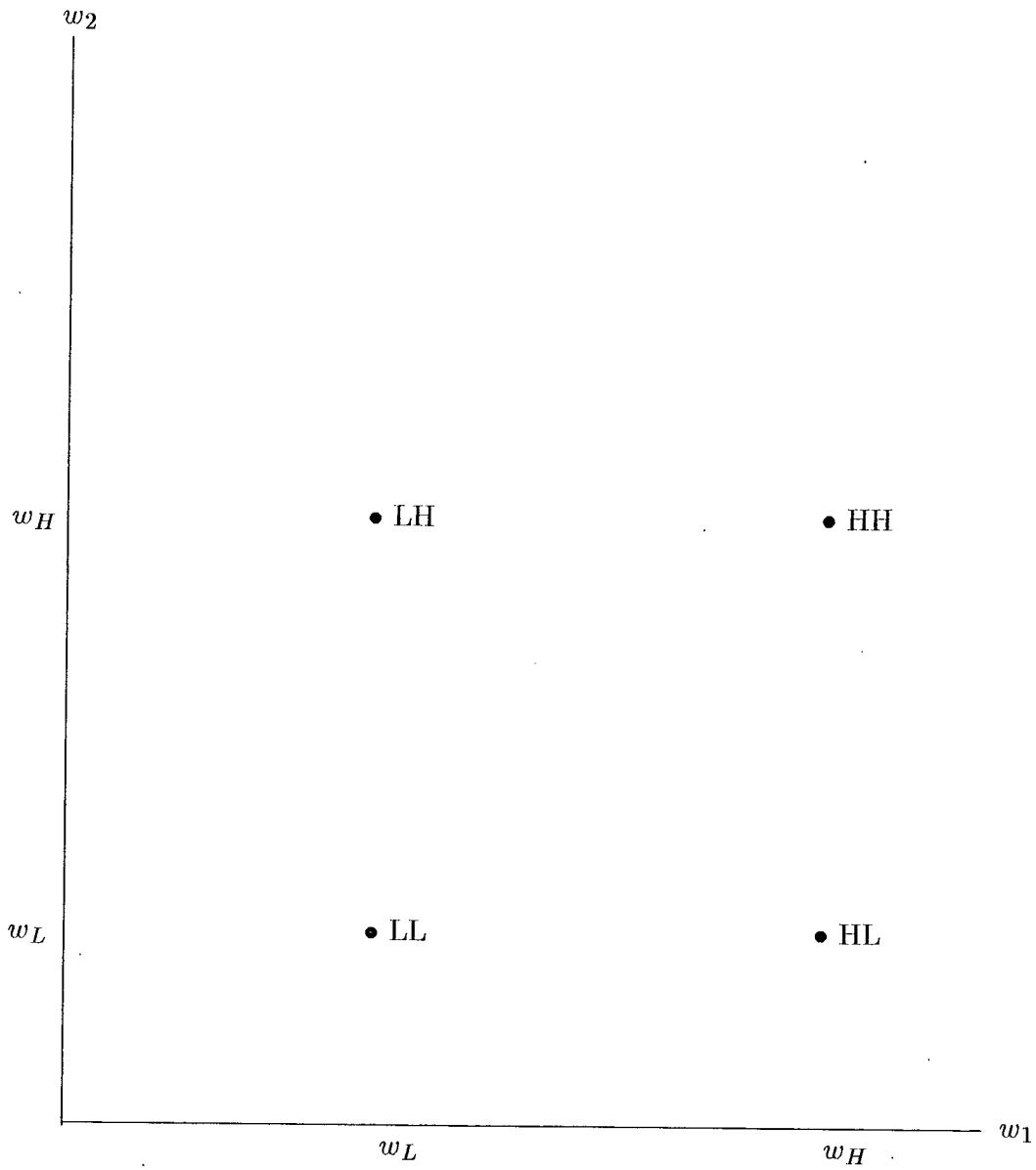

Figure 2. Monotonicity Properties Implied By Self-Selection

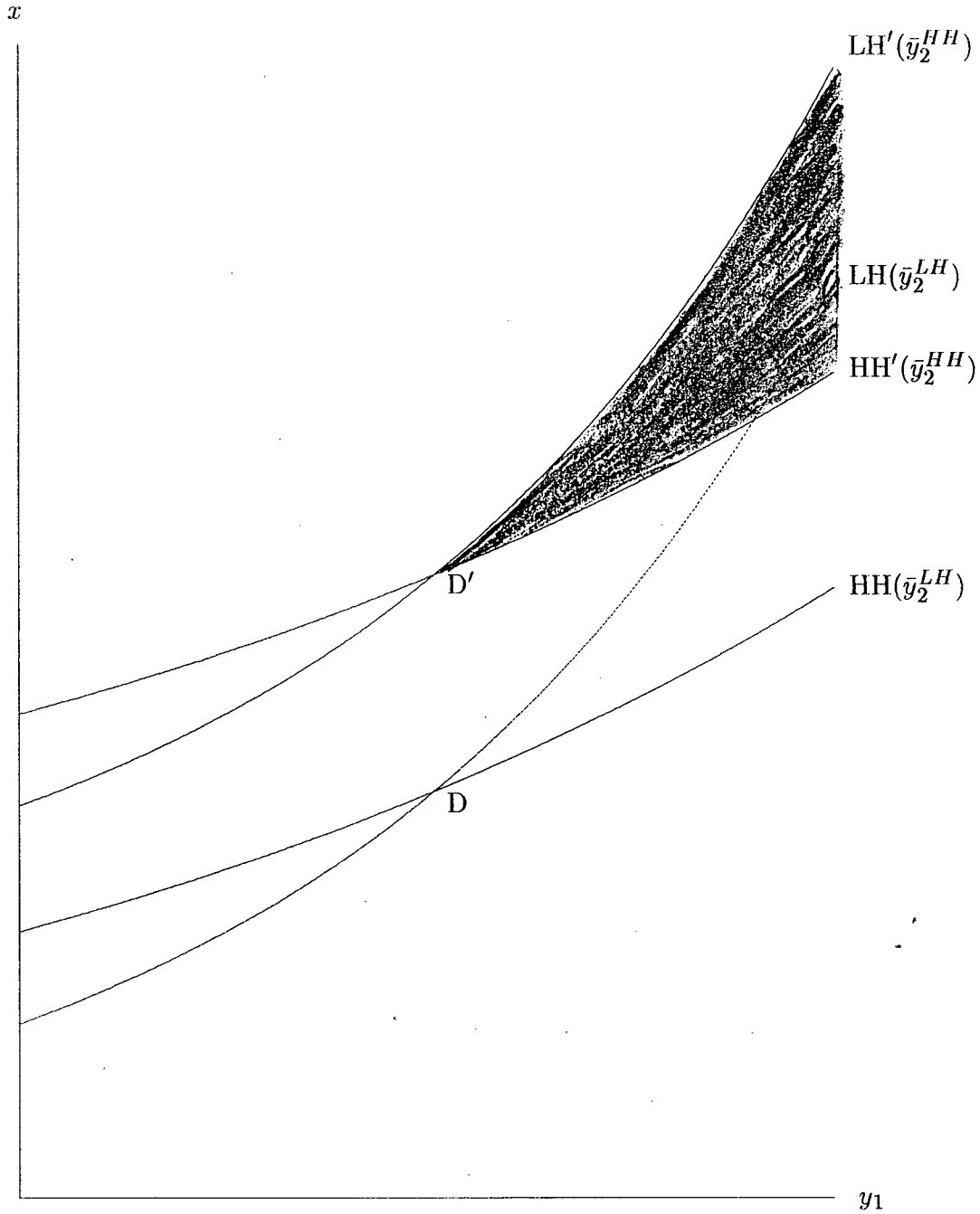

Figure 3. Monotonicity Properties Implied By Self-Selection

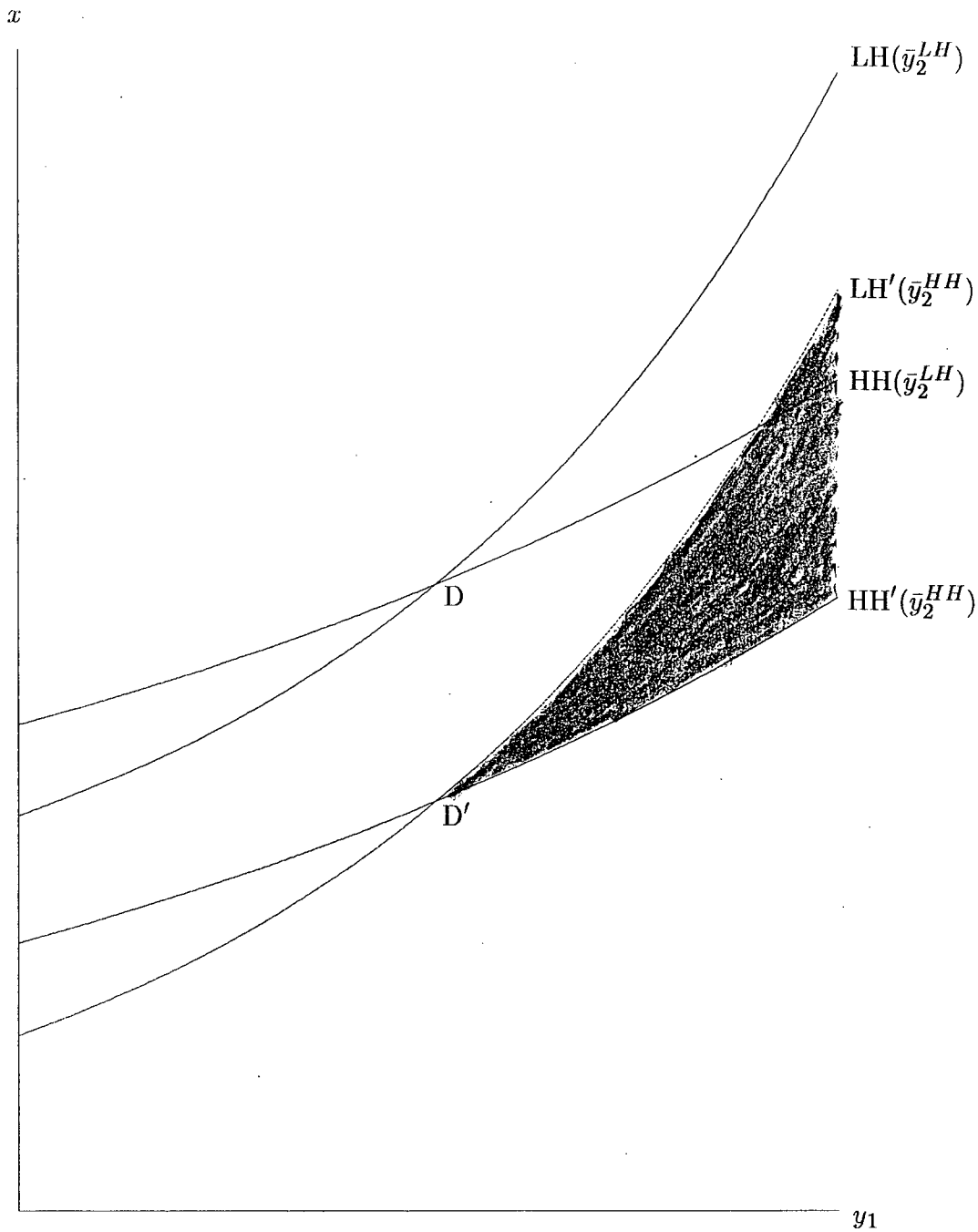

Figure 4. Partial Monotonicity for Families HH and LL

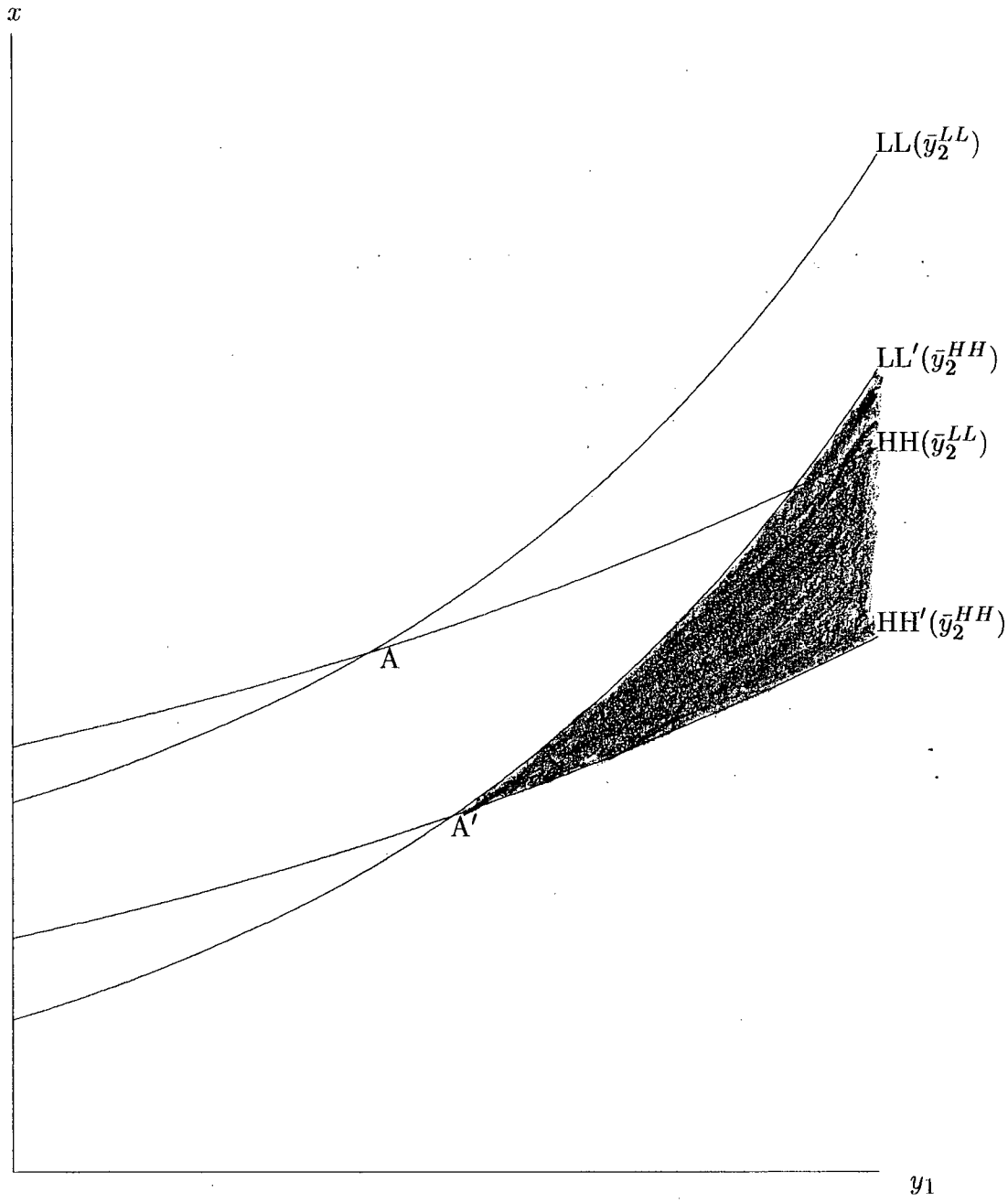

Figure 5. The Lack of Attribute Ordering

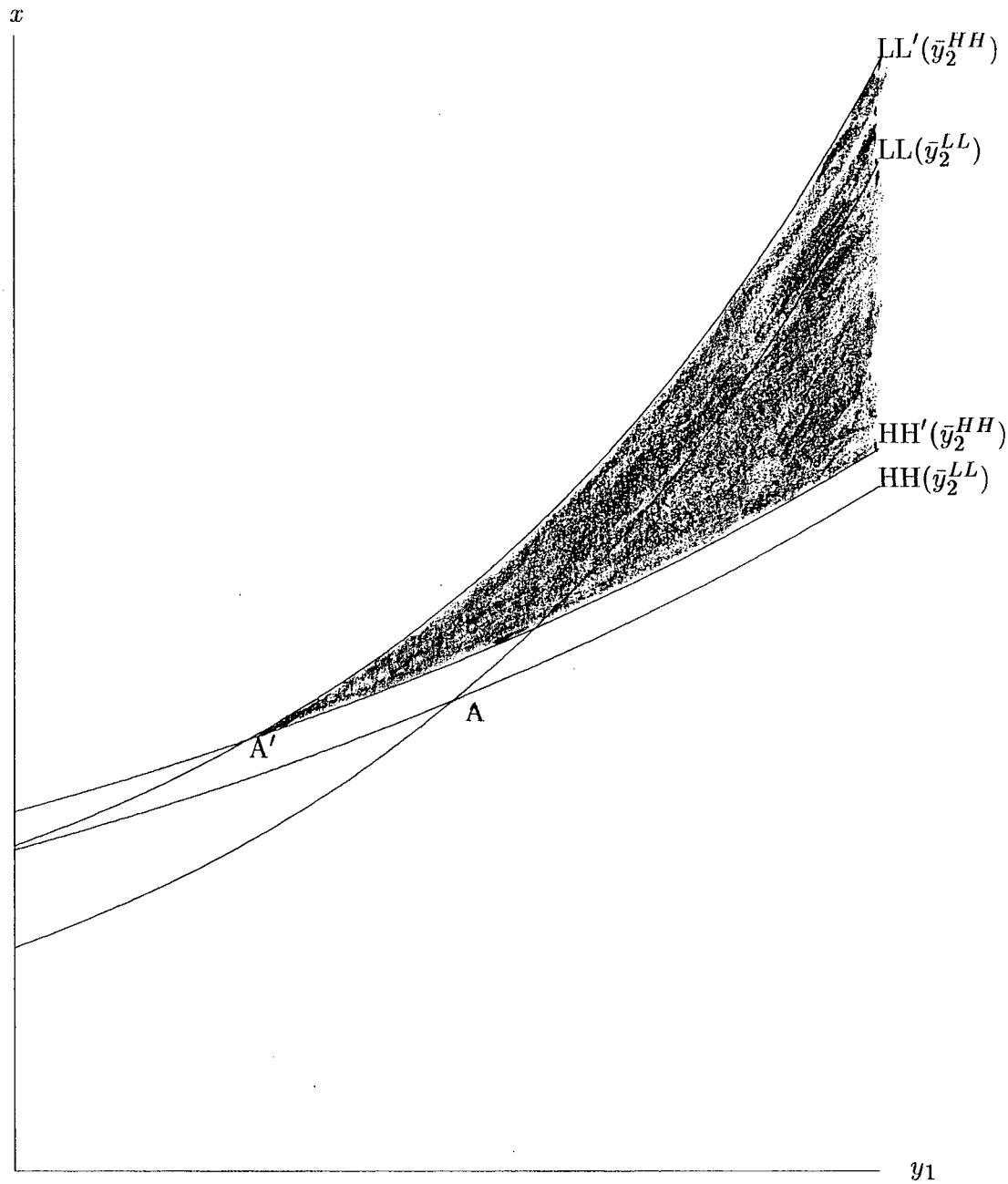

Figure 6. Allocations for Families not Ordered by $\geq_F$

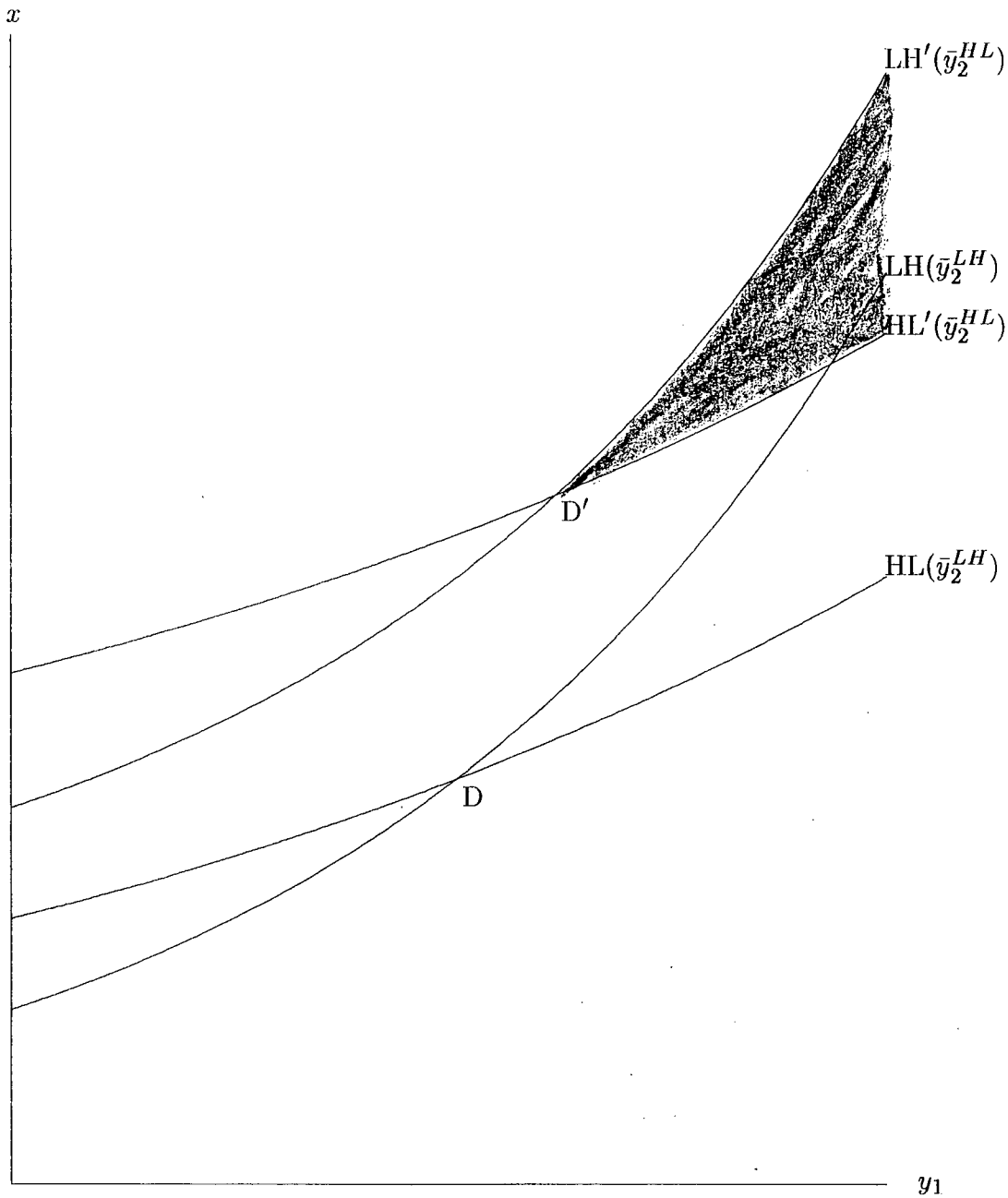

Figure 7. An Implication of a Binding Self-Selection Constraint

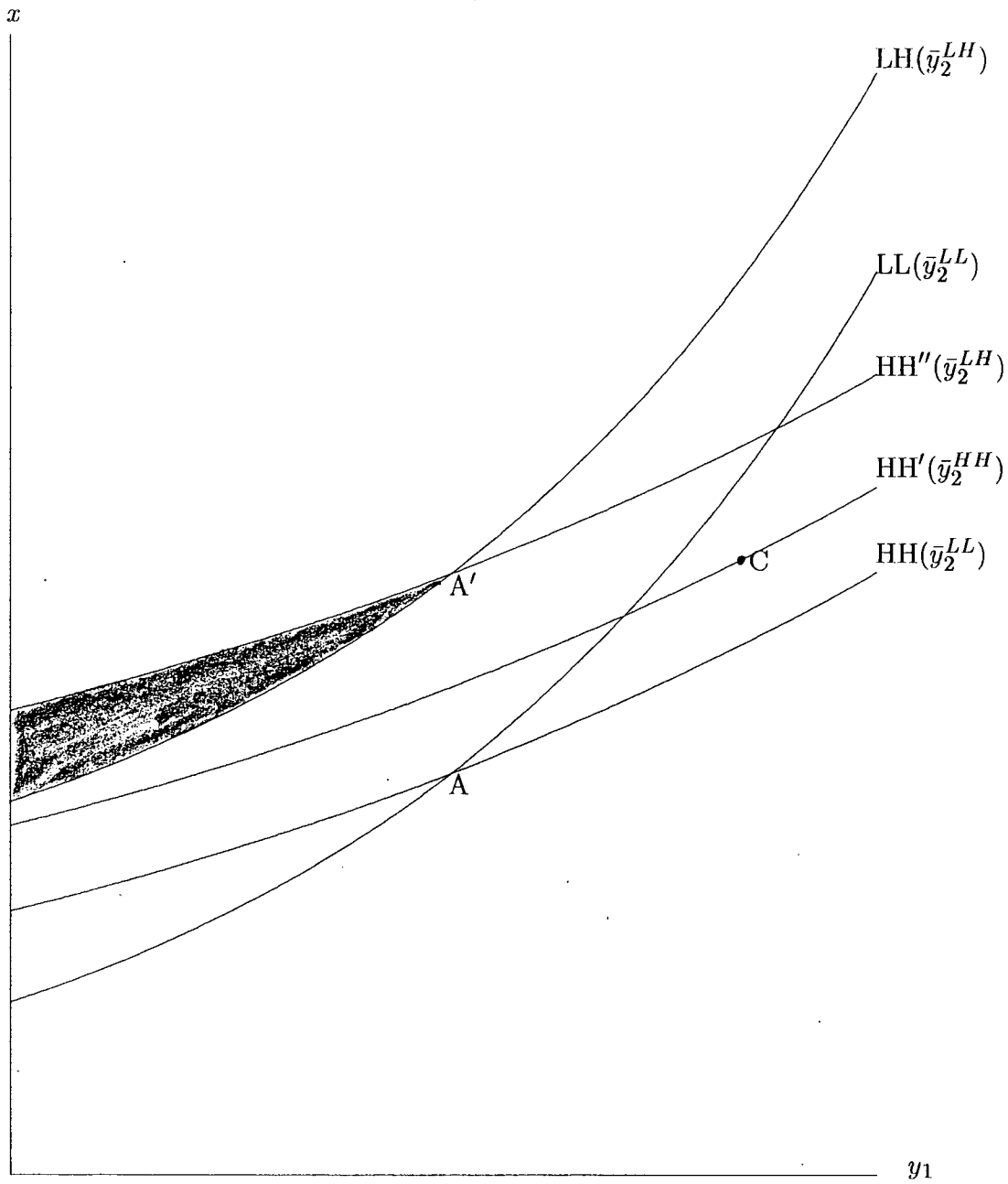

Figure 8. An Example of When A Zero Marginal Tax Rate is Optimal