Prepayment and the Valuation of Canadian Mortgage-Backed Securities: A Proportional Hazards Approach

by

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Abstract

This paper estimates both parametric and non-parametric proportional hazards models for a subset of Canadian mortgage-backed security data. The estimated parametric hazard function is then used to drive exogenous prepayments within an arbitrage-free model of the term structure of interest rates. Theoretical prices as well as option-adjusted spreads (OAS) are obtained for three different mortgage-backed securities using a Monte-Carlo simulation. Though no formal test is done to compare the ability of the different hazard models to explain observed market prices, the non-parametric baseline hazard is more consistent with the age-dependent prepayment provisions typical of most mortgage contracts in Canada.
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I. Introduction

This study presents a prepayment and valuation model for Canadian mortgage backed securities (MBS). The importance of unscheduled prepayment to the valuation and risk of mortgage-backed securities in the US has been well documented in the literature. However, these issues have received comparatively little attention with regard to the Canadian market, where MBS have been actively issued since 1987. Although the US research provides a helpful starting point for analyzing Canadian MBS, we need to be careful about extrapolating results from the US experience to the Canadian situation due to differences between the two markets. The typical Canadian mortgage is a five year balloon loan amortized over 25 years, and is subject to a host of prepayment restrictions and penalties, though an array of other mortgage types are also available. Ostensibly similar mortgages issued by one institution may exhibit quite different prepayment characteristics from those issued by another, in contrast to their more generic US counterparts. Finally, mortgage interest payments in Canada are not tax-deductible, which all else equal should increase the incentive to prepay one's mortgage.

The valuation of mortgage-backed securities is considerably more complex than it is for most other fixed income securities that have embedded option features.¹ This is because the cash flows on a MBS are determined by the payment decisions of a pool of heterogeneous borrowers, who hold options to prepay and default on their loans. These option-like features are driven by a variety of factors, many of which are not purely

¹ When valuing a callable bond, for example, it is usually sufficient to model the term structure of interest rates as a function of one or two stochastic state variables. The exercise condition for the call option is then endogenous to the model, so valuation is relatively straightforward.
financial. For example, prepayment, which is often ascribed to mortgage refinancing, can also occur due to housing turnover, which could depend on a range of demographic and macro-economic variables. It is not generally practicable, however, to model the number and type of state variables that drive MBS prepayment and default within the standard option-pricing framework, particularly the macro-economic, demographic, and other non-financial variables. Furthermore, borrower heterogeneity complicates the application of standard option-pricing methods. Borrowers within a mortgage pool likely face different transaction costs, and in many cases different strike prices as well (consider housing equity, for example). If one were to apply the optimal exercise condition for a default or prepayment option at the aggregate level of an MBS, borrowers with different strike prices and transaction costs would appear to behave in a less than optimal manner. To overcome the limitations of the option pricing model, more recent US studies separate the estimation of MBS cash flows from the valuation model itself, an approach we follow here.

In this paper, we estimate an empirical prepayment function for Canadian mortgage-backed securities using a proportional hazards model. The hazards model allows us to estimate the conditional probability of a mortgage being prepaid as a function of a variety of explanatory variables, and thus allows a high degree of flexibility. We follow Schwartz and Torous (1989) and Boyle (1989) in estimating a log-logistic hazard, which assumes a highly stylized prepayment function. To better account for the influence of prepayment restrictions and penalties on the shape of the prepayment

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2 Of course, it is also possible that borrowers themselves do not exercise their options in the most efficient manner, perhaps because housing has consumption as well as investment components.
function in Canada, we subsequently estimate a non-parametric hazard model in which the form of the prepayment function is determined by the data. The estimated prepayment function is then integrated into the Ho and Lee (1986) arbitrage-free model of the term structure. Using Monte Carlo simulation methods, we compare the model’s estimated values to market prices of actual MBS.

It is hoped that the type of approach presented here will help lead to a better understanding of value and risk in the Canadian MBS market. More efficient valuation of Canadian MBS has implications beyond the limited scope of this paper, however. For example, more efficient pricing in the secondary market will be importantly related to issues such as the cost of NHA insurance premiums, and securitization’s effect on the homeowner’s cost of credit in the primary mortgage market. This type of approach could also be applied to the valuation of the large individual mortgage portfolios held by Canadian financial institutions, which are generally not marked to market on a regular basis.

The paper is organized as follows. In the next section, we present an overview of the history and size of the Canadian MBS market, as well as the different types of securities available. The third section presents a review of the literature on MBS valuation, with some reference to the extensive literature related to individual mortgage loans. Section four develops the proportional hazards models and presents empirical results for the two different model specifications. Section five presents the valuation model and results, followed by concluding remarks in section six. Appendices present derivations of some of the key valuation equations.
II. The Canadian Mortgage-Backed Security Market

National Housing Act (NHA) insured mortgage-backed securities have been issued in Canada since January, 1987. A decade later, as of March 1997, $30.2 billion face-value had been issued, though in recent years growth of the market has slowed considerably from its pace in the early 1990s (Figure 1). In contrast to the US, the total dollar value of securitized mortgages in Canada is a relatively small fraction of the total single-family mortgage debt outstanding. For example, of the approximately $300 billion of residential mortgage debt outstanding on at the end of 1993, approximately $21 billion had been securitized as mortgage-backed securities. At the same time in the US, approximately $1.7 trillion of the $4.2 trillion market of residential mortgage debt had been securitized (DeLiban and Lancaster 1995).

Figure 1: NHA Mortgage-Backed Securities Issued by Year
The holder of a Canadian mortgage-backed security receives a pro-rata share in the scheduled and unscheduled payments of principal made by the underlying borrowers, or *mortgagors*. This is in contrast to a mortgage bond, for example, where the underlying mortgage payments act as collateral for a stipulated principal amount and coupon rate. While principal payments are passed through to the mortgage-backed security holder directly, interest payments on an MBS are not. MBS holders instead receive interest payments calculated at the mortgage-backed security coupon rate, which the issuer is obligated to set at least 50 basis points below the lowest mortgage rate in the pool. The spread between the rates on the underlying mortgages and the mortgage-backed security coupon is retained by the issuer as a servicing fee. In most cases, prepayment penalties are passed through directly to the MBS investor.

**Types of MBS**
The NHA mortgage-backed securities program consists of five pool types, the characteristics of which are listed in Table 1. The focus of this paper is on Prepayable single-family pools, of which two types are available. The first of these, denoted by the prefix 964, passes penalty interest payments (PIP) through to the MBS investor. As Figure 1 shows, over $15 billion of Prepayable 964 pools has been issued since the inception of the NHA program. Although the mortgages are prepayable, various provisions in the mortgage contract restrict when prepayments can occur. More recently, issuers who believed that the market was not adequately valuing the pass-through of penalty interest began to issue a new class of single-family prepayable pool in which the issuer retains the penalty charges. The pools are otherwise identical to the single-family
964 pools, though they trade at wider spreads. Approximately $3 billion of these pools, denoted by the prefix 967 and commonly referred to as no-PIP, has been issued.

<table>
<thead>
<tr>
<th>Pool Type</th>
<th>ID Prefix</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prepayable with PIP</td>
<td>964</td>
<td>Market-rate single family residential mortgages, penalties (PIP) passed through to investors</td>
</tr>
<tr>
<td>Prepayable without PIP</td>
<td>967</td>
<td>Market-rate single family residential mortgages, penalties (PIP) retained by issuer.</td>
</tr>
<tr>
<td>Non-Prepayable Multi-Family</td>
<td>966</td>
<td>Market-rate multi-family mortgages</td>
</tr>
<tr>
<td>Non-Prepayable Social Housing</td>
<td>990</td>
<td>Single-family social housing loans</td>
</tr>
<tr>
<td>Mixed</td>
<td>965</td>
<td>Any combination of the other 4 pool types</td>
</tr>
</tbody>
</table>

Table 1: Types of NHA Mortgage-Backed Securities

The next largest type of pool, with $9.7 billion issued since 1987, is the 990 series of social housing pools. These pools consist of mortgages on single-family subsidized housing loans, which do not permit prepayment. The two remaining pool types make up a smaller fraction of the mortgage-backed securities market. Market-rate multi-family pools, denoted 966, of which about $1.5 billion have been issued, consist of non-prepayable loans on multi-family housing. Mixed pools, denoted 965, can contain any combination of the other four loan types.

**NHA Insurance**

Under authority of the National Housing Act, the Canada Mortgage and Housing Corporation (CMHC), a government agency, guarantees the payment of interest and principal on NHA mortgage-backed securities. If a borrower is in arrears, for example, the CMHC ensures that the mortgage-backed security holder receives the scheduled monthly payment. Similarly, if a borrower defaults, the mortgage-backed security holder
receives from the CMHC an early payment of the outstanding balance on that mortgage. Though it does nothing for prepayment risk, this payment guaranty eliminates credit risk, allowing NHA MBS to receive the highest (AAA) credit rating from the Canadian credit rating agencies.

**NHA Prepayment Provisions**

To be eligible for securitization under the NHA mortgage-backed securities program, single family mortgages (except for social housing loans) must permit minimum levels of prepayment and levy no more than specified maximum penalties. A five year NHA-insured single family mortgage, for example, is largely closed to prepayment in its first three years, permitting the full prepayment of principal in years four and five upon payment of a penalty. The penalty is typically the lesser of three scheduled monthly interest payments and the present value of the interest differential, which is defined as the difference between the remaining interest payments calculated at the stated contract rate, and calculated at the (lower) current market rate. The penalty provides compensation to the lender for lost interest. In addition, borrowers are allowed to prepay up to 10% of the original principal amount without penalty in each of the first three years.

Competition has lead issuers to relax the NHA provisions, and many permit partial prepayments in the first three years of 15% of the original issue amount, and some permit up to 20%. Additionally, lenders may often permit the monthly payment to be increased by a similar percentage once each year for the remainder of the amortization schedule. More importantly, most lenders will permit the full prepayment of principal during the first three years under certain conditions, generally when the borrower refinances with the
same institution or sells her house. Penalties are higher for full prepayment during the
closed period, however, typically the maximum of three months' interest and the interest-
rate differential. Although the preceding is a typical scenario, competition has resulted in
a variety of mortgages being available, from the largely closed to the fully open, all
eligible for NHA securitization.

**NHA Securitization Guidelines**

The NHA sets forth guidelines as to which mortgages can be grouped in the same pool.
The original guidelines, intended to provide a relative degree of homogeneity to the pool,
are listed in Table 2.

<table>
<thead>
<tr>
<th>Mortgage Rate</th>
<th>Fixed rate mortgages only. All mortgage interest rates within a 100 basis point band.</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBS Coupon</td>
<td>At least 50 basis points below the lowest mortgage rate in the pool.</td>
</tr>
<tr>
<td>Amortization</td>
<td>Longest loan amortization schedule cannot exceed 130% of the shortest.</td>
</tr>
<tr>
<td>Issue Date</td>
<td>All mortgages must be issued within 6 months of each other, none can be more than 6 months old.</td>
</tr>
<tr>
<td>Diversification</td>
<td>No single mortgage can make up more than 25% of the MBS pool.</td>
</tr>
<tr>
<td>Mortgage Type</td>
<td>NHA-insured single-family residential mortgages.</td>
</tr>
<tr>
<td>Pool Size</td>
<td>Minimum pool balance is $2 million.</td>
</tr>
</tbody>
</table>

**Table 2: Old NHA Securitization Guidelines for Prepayable Pools**

Recently, the CMHC, after consulting with the mortgage-backed securities issuers and
dealers, instituted new securitization guidelines giving more flexibility to issuers in the
characteristics of the loans they could combine in the same pool. The new guidelines are
intended to increase the liquidity and breadth of the market by increasing supply and
making the highly issuer-specific securities more generic. To assist dealers and investors
in the evaluation of prepayment and prices, the new guidelines also call for better reporting, particularly with regard to the distribution of individual loans in the pool.

**Issuers and Investors**
Issuers of MBS have tended to be smaller financial institutions like trust and life insurance companies rather than the charter banks. For non-bank institutions which may be unable to issue high-grade paper, securitization has been a comparatively inexpensive source of financing. However, as the major banks begin to compete more with mutual funds for their retail deposit base, and spreads on mortgage-backed securities decline, the banks could potentially become much more significant issuers of mortgage-backed securities. Investors in Canadian mortgage-backed securities are divided almost exclusively by product line. Retail investors buy primarily non-prepayable MBS, which have certain cash flows, while institutional investors like pension funds, life companies, and mutual funds focus more on prepayable MBS.

**Canadian vs. US Mortgage-Backed Securities**
There are a number of differences between the US and Canadian markets that can be expected to affect the rate of prepayment observed in Canada. First, Canadian mortgages are of much shorter term than their US counterparts. The typical mortgage is a five year balloon loan amortized over 25 years. At the end of five years, borrowers typically roll over the balloon at the prevailing market rate. Loans of three years or less are also common, and some longer terms are also available. Because of the shorter term, the incentive to refinance is lower than it would be on a comparable US mortgage like a 30-year GNMA. Second, full prepayments on Canadian mortgage-backed securities are
subject to penalties, often throughout the life of the loan, which tends to reduce the incentive to prepay, particularly for the purpose of refinancing.

Another important difference is that in the US, mortgage interest payments are tax deductible, whereas in Canada they are not. Thus, when choosing between repaying one’s mortgage and investing that dollar elsewhere, the US mortgagor has an incentive to invest in the alternate asset rather than forego the effective tax subsidy on the debt. In contrast, the Canadian tax environment increases the incentive to repay one’s mortgage.

Finally, mortgage-backed securities in Canada are not generic securities, unlike their relatively homogenous US counterparts. That is to say, prepayment behavior may be quite idiosyncratic from one issuer to the next. For example, mortgage-backed securities issued by one issuer may experience lifetime prepayments averaging 30% annually, whereas otherwise identical securities issued by another may only experience lifetime prepayment rates of 15%. This is primarily due to different prepayment provisions across issuers, though could also be caused by a clientele effect among borrowers. Certain issuers offer a variety of prepayment privileges to their clients, from fully open to closed. For these issuers, prepayments can vary significantly from pool to pool, depending on the dominant prepayment provision on the mortgages in the pool.
III. Review of the Literature on MBS Valuation

The literature review is organized as follows. It begins with a brief overview of the prepayment and default decisions facing the individual borrower which drive the aggregate prepayments observed on mortgage-backed securities. After this overview, the remainder of the literature review is a survey of the main developments in MBS pricing, organized in roughly chronological order. It begins with a review of the application to MBS pricing of one and two-factor models of the term structure of interest rates, and early efforts to account for the so-called “sub-optimal” prepayments observed in practice that could not be accounted for within a pure option-pricing framework. It then examines the estimation of empirical prepayment functions, which allow the observed prepayment rate to be a function of a variety of explanatory variables. Following these approaches, which focus mainly on the effect of refinancing, a recent attempt to incorporate default as well as prepayment is examined. A brief discussion of the existing Canadian MBS literature follows. The review concludes with a discussion of attempts to model borrower heterogeneity more explicitly, as well as risk.

It should be noted that there exists a large body of literature related to individual mortgage loans, much of which is unfortunately beyond the scope of the present paper. However, within the chronological discussion of MBS pricing, literature that focuses on individual mortgages is presented where relevant. Some studies of individual mortgages are also referred to in the discussion of causes of MBS prepayment at the individual mortgage level.
**Causes of Prepayment on Mortgage-Backed Securities**

Prepayment on mortgage-backed securities is typically couched in terms of refinancing. However, borrowers may repay their mortgages for other reasons, such as when moving house, or when the expected return on other assets is low. In addition, default on an insured mortgage will appear as an unscheduled prepayment of principal to the MBS holder. This sub-section provides a brief overview of the factors motivating prepayment and default on individual mortgage loans.

**Refinancing**

When market interest rates decline, a borrower can reduce his or her mortgage interest costs by refinancing an existing high-rate mortgage with a new mortgage at the lower market rate. In this way, the wealth-maximizing borrower minimizes the market value of his or her mortgage liability. From an option-pricing perspective, refinancing can thus be viewed as a call option on the market value of the mortgage, with a strike equal to the mortgage’s outstanding principal balance. The value minimizing policy is thus to refinance once the loan’s market value exceeds the outstanding principal balance by an amount sufficient to cover transaction costs and the value of the live refinancing option.

In practice, individual mortgage borrowers do not exercise their refinancing options as ruthlessly as the option model predicts. Quigley and Van Order (1990), for example, find that it could take several years for borrowers to prepay given a 200 basis point decline in interest rates. In part, the poor performance of the option model could arise because it does not properly account for transaction costs. A second, potentially
more important explanation is that consideration of refinancing in isolation fails to account for the presence of other options like default, discussed below. At the level of a MBS, the option model is found to be considerably less successful in explaining observed prepayments, as discussed later in the literature review.

In most valuation studies, it is assumed that refinancing is driven by a single rate. For example, in US studies, borrowers are assumed to decide between holding their existing 30 year mortgage, or taking out a new 30 year fixed rate mortgage (FRM) at the market-wide 30-year rate. This is clearly a simplification, for US borrowers could also opt for a 15-year FRM, an adjustable rate mortgage (ARM), as well as different blends of up-front points and mortgage rates. In Canada, where borrowers can choose between an array of mortgage terms, between fixed and variable rates, and between substantial rate differences from one lender to the next, the assumption of a single refinancing rate is an even greater simplification. In a US study of individual mortgage data, Brueckner and Follain (1988) find that the probability of a US borrower choosing an ARM over a long-term FRM increases significantly under an environment of high overall interest rates and a steep yield curve, such that short-term rates, which determine ARM rates, are substantially lower than long-term rates, which determine the FRM rate (The authors report somewhat more mixed results for the influence of demographic and mobility variables).

**Household Mobility**

In situations where a mortgage cannot be assumed by a new purchaser or transferred by the owner to a new property, household mobility may result in the early
repayment of principal. Mobility rates depend on economic factors like the variation in regional economies, particularly regional changes in the demand for labour, and on the costs of searching for and moving into a new home. Mobility also depends on changes in household demographics, for example a young family moving into a larger house to accommodate new children, one spouse moving after a divorce, or a retiring couple moving to smaller accommodations. Quigley (1987) finds that variables like household size and age of the household head affect the rate of household mobility. Few of these economic and demographic variables are easily observable.

Interest rates are also relevant to household mobility. Green and Shoven (1986) and Quigley (1987) identify a financial disincentive to move in situations where mortgages are due-on-sale, or where full capitalization of financial terms does not occur. A mortgagor with a below-market fixed-rate mortgage has a valuable commodity, since the same amount of financing at the higher market rates would be more expensive. If the benefit of the lower rate is not fully capitalized into the property value, the mortgagor may be dissuaded from moving and is said to be locked-in. Similarly, if mortgages are due on sale, the only way to retain the benefit of the low-rate loan is to stay in the home rather than move. Further, lenders place restrictions on assumability by requiring purchasers to meet credit conditions, so that mortgages are frequently not assumed even

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3 It is fairly well understood that regional differences in mobility can affect mortgage pricing in the secondary market via the influence on prepayment. Rosenthal and Zorn (1992), however, find that regional differences in mobility could affect mortgage pricing more directly by causing differences in the mortgage rates offered by lenders at origination.
when it is economically advantageous and legally permissible to do so. In this situation, the mortgagor is locked-in, even if full capitalization occurs.\textsuperscript{4}

\textbf{Default}

Default is relevant to the study of prepayment on MBS because, when a borrower defaults on an insured loan in the MBS pool, the default insurance pays the MBS investors the outstanding principal balance of the loan. To the MBS investor, this payment is reported as a prepayment, and is indistinguishable from other sources of prepayment like refinancing, though it tends to occur under different economic environments. In addition, default and prepayment may substitute for one another to some extent.

Default is typically analyzed in terms of the level of a borrower's housing equity, and thus can also be cast in terms of option pricing theory: default is an option to put the property to the lender at a strike price equal to the present value of the remaining debt cash flows (the market value of the mortgage). The value-minimizing policy is to default once the mortgage value exceeds the value of the property as well as the value of retaining the option to default and any transaction costs. Equity would thus generally be negative before default becomes optimal.\textsuperscript{5} Unfortunately, equity levels and transaction costs are not easy to observe, and tend to be highly idiosyncratic. For example, both components of the strike price of the default option, housing equity and transaction costs,

\textsuperscript{4} The mortgage lock-in is effectively the same variable as the refinancing incentive, which suggests that a mortgage valuation model should allow for a non-linear response to the lock-in/refinancing variable, depending on whether the mortgage is priced at a premium or a discount.

\textsuperscript{5} Note that if the house value exceeds the face value of the mortgage yet equity is negative, it will likely be optimal to prepay rather than default, since prepayment transaction costs tend to be lower.
are likely to be borrower-specific. Housing is already a heterogeneous good, and differing mortgage terms will further increase the heterogeneity of housing equity. Similarly, a poor credit rating may have different costs to different borrowers, depending on anticipated capital requirements in the future, and psychological costs of default may be particularly idiosyncratic.

Quigley and Van Order (1992) find that default rates are highly sensitive to the level of housing equity, though less so than would be predicted by the ruthless option-pricing model. In particular, it is found that a borrower with an initial loan-to-value ratio of 95% is approximately four times more likely to default than is a borrower with a loan-to-value ratio on the order of 80%. Quigley and Van Order (1992) employ an interesting way of incorporating housing heterogeneity into the study of mortgage default. As is often the case with mortgage data, their data on mortgage default includes no information about house values. To overcome this limitation, the authors take a second data set of house prices. While it is known that the mortgages on these houses were sold into Federal Housing Authority (FHA) pools, there is no way to assign the houses in this data set to the mortgages in the first data set. Quigley and Van Order then follow Case and Schiller (1987) in constructing a weighted repeat sales house price index from the housing data, which provides an estimate of both the mean and distribution of house prices over time. By sampling from this distribution, it is thus possible to incorporate the heterogeneity of housing equity into the model, though of course not without potentially sizable error.\(^6\)

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\(^6\) It is worth noting that this general approach, where a second data set is used to gain more insight into the primary data, underlies much of the recent work in MBS pricing to account for default and mobility effects, as well as borrower heterogeneity. This is discussed further at the end of the literature review.
Aside from the heterogeneity question, analyzing default from the pure equity based option approach overlooks the fact that a borrower's ability to meet scheduled debt payments may also affect default. Note that the payment burden is only likely to be relevant when equity is positive, since default is optimal regardless of one's ability to pay when equity is sufficiently negative. High payment burdens could give rise to default or prepayment. Because there are high real and psychological costs to default, a mortgagor with positive equity who is unable to meet debt payments may choose to sell the home and use the proceeds to repay the mortgage, rather than to default. In this case, default-related factors motivate a default-avoidance kind of prepayment. For this to be feasible, markets must be relatively free of frictions. In particular, it is assumed that the mortgage can be prepaid without penalty, that the house can be readily sold, and that the bank will not foreclose before the sale is complete. To the extent there are market frictions, increasing payment burdens can be expected to result in defaults rather than default-avoiding prepayments. Again, various idiosyncratic economic and demographic variables, for example illness, divorce, or unemployment, may affect a mortgagor's ability to pay. Zorn and Lea (1989), for example, in a study of Canadian mortgage loans, find that both housing equity and ability-to-pay variables influence default rates.

Default also depends on the legal environment, and thus, like mobility, it has a regional component. Clauretie and Herzog (1989) analyze the significance of deficiency judgments and foreclosure laws on default probabilities in the US. Jones (1993) extends the analysis of Clauretie and Herzog to the Canadian situation, and finds that, after controlling for the presence or type of deficiency judgment legislation in place, individual
borrower characteristics can be identified at loan origination which tend to influence future default behavior.

**Expected Returns on Alternate Investments**

Minimizing the value of a mortgage by refinancing when market rates fall, or by defaulting when house values fall, is optimal if the mortgage is separable from other items on one’s balance sheet. However, if markets are integrated the wealth-maximizing borrower will minimize the total value of all liabilities, not one in particular. De Rosa (1978) finds that mortgage credit is not used exclusively for housing purchases. Rather, people consider their total asset consumption when considering the size of downpayment and the level of mortgage financing. Thus, someone with an excess demand for automobile financing might take out a larger than “optimal” mortgage. Other studies, for example Muth (1986), find support for the integrated-markets hypothesis. If this is the case, it may have implications for how we think about mortgage refinancing and default. For example, Zorn and Lea (1989) find evidence that borrowers will repay their mortgages if the effective return from doing so exceeds the expected return on alternate investment opportunities, behaviour that may appear sub-optimal when analyzing each option separately.\(^7\)

This is probably a more relevant hypothesis for the Canadian mortgage-backed security market than the US. In Canada, investments earn taxable income, while mortgage interest payments are made from after-tax income. Hence, there is an incentive to prepay.

\(^7\) Note that so-called “sub-optimal” prepayments associated with household mobility could also appear optimal when viewed from the perspective of the household’s overall balance sheet.
to pay down one’s mortgage. For example, a homeowner with a 10% mortgage and 50% marginal tax rate would have to earn 20% on other investment opportunities before prepayment would become undesirable. This would seem to be a likely cause of the partial prepayments observed in Canada. However, the influence on prepayment of the expected return on other assets has not been widely applied in the pricing literature. This is likely because measuring expected return is not a trivial matter, even without the problem of accounting for differing marginal tax rates (see, for example, Merton (1980) for a discussion). Also note that simple interest rate proxies for expected return are not helpful in a refinancing model, since the second interest rate variable will be highly correlated with the refinancing variable. Additionally, option-type valuation models quickly become overly cumbersome as additional state variables are included.

While this has been at best a cursory overview of the factors that can cause unscheduled payments on individual mortgage loans, it does help illustrate the nature of the problems faced when valuing MBS based on aggregate pool data alone. In particular, as noted in the introduction, the number of possible state variables needed to adequately describe prepayment and default at the individual borrower level, combined with the fact that borrowers are not homogeneous, in large part precludes the application of standard option-pricing models to MBS valuation. Because of this limitation, emphasis in MBS pricing shifted to separating the cash flow estimation model from the valuation model. We trace the development of this general approach in the remainder of the literature.

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* Comparison should be made between investments of similar risk. The effective return on the mortgage should be penalized for the risk of under-diversification.
review, beginning with the work of Dunn and McConnell (1981), who modify the standard single-factor model to include an exogenous payment term.

**Single-Factor Term Structure Models: Dunn & McConnell**

In the pioneering work of Dunn and McConnell (1981a, 1981b), borrowers are assumed to hold pure refinancing options which they exercise in a rational wealth-maximizing manner. That is, they are assumed to prepay their loans when the market value of the loan exceeds the loan’s face value by an amount sufficient to cover the live option cost. In addition, to account for the observed tendency for some borrowers to prepay in states when the mortgage would be priced at a discount and refinancing does not appear to be optimal, Dunn and McConnell assume prepayments follow an exogenous payment function. We begin by showing how MBS are priced in the Dunn and McConnell model with ruthless refinancing only, and then show how “sub-optimal” refinancing is incorporated via the exogenous prepayment function.

Dunn and McConnell assume that MBS prices depend on a single state variable, the short-term interest rate, which follows the Cox, Ingersoll, and Ross (1985) mean-reverting square root process:

\[ dr = k(m - r)dt + \sigma \sqrt{r} \, dz \]  

*Equation 1*

where \( m \) is the long run value of the short rate, \( k \) is a speed of adjustment parameter, \( \sigma \) is a volatility parameter, and \( dz \) is a Brownian motion. The Cox, Ingersoll, and Ross

---

*In discrete time, the Brownian motion \( \Delta z \) is (approximately) equal to \( \epsilon \sqrt{\Delta t} \), where \( \epsilon \) is a random draw from the standard normal distribution. See Hull (1993) pp. 192-196 for an intuitive discussion.*
process has several desirable properties for the dynamics of the short rate. The specification of the drift, $k(m-r)$, allows the short rate to revert to its long run value, $m$, at rate $k$. When rates are above $m$, the drift term is negative and thus pulls rates back down. Conversely, when rates are low, the drift term tends to pull them back up. The specification of the instantaneous variance, $\sigma^2 r dt$, prevents rates from becoming negative. When the short rate is zero, for a positive value of $m$ the drift term ensures that it will be pulled back into positive territory.

An equation for valuing a security that depends on the short rate process in [1] and time can be obtained through the standard hedging arguments developed by Merton (1973) and Black and Scholes (1973). From Ito's lemma, it is possible to obtain the process for the price of any security that depends on [1]. If two such securities are available, a riskless portfolio can be constructed from a long position in the first security and a short position in the other. Because this portfolio is riskless, its return, which consists of coupon payments plus capital gains, must equal the risk-free return; otherwise arbitrage opportunities would exist. Following this line of reasoning, it can be shown that the price of a security which depends on [1] must satisfy the following non-stochastic partial differential equation (see derivation in Appendix A):

$$\frac{\partial M}{\partial t} + (k(m-r) - qr) \frac{\partial M}{\partial r} + \frac{1}{2} \frac{\partial^2 M}{\partial r^2} \sigma^2 r + c = rM$$

Equation 2

where $q$ is a parameter proportional to the market price of short term interest rate risk.$^{10}$ Because this is a general valuation equation, to price a particular security, we need to

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$^{10}$ In contrast to partial equilibrium approaches, the general equilibrium model of Cox, Ingersoll, and Ross provides a complete description of a simplified economy, including an expression for the price of short-term interest rate risk.
specify appropriate boundary conditions. For a fully-amortizing mortgage (or MBS), the boundary conditions are as follows:

\[ M(r, T) = F(T) = 0; \]
\[ \lim_{r \to \infty} M(r, t) = 0; \]
\[ M(r_c, t) = F(t); \]
\[ \frac{\partial M}{\partial r} = 0 \text{ at } r \leq r_c; \] \hspace{1cm} \text{Equation 3}

where \( r_c \) is the critical rate at which the mortgage is called. The first condition states that the market and face values of a fully amortizing mortgage are zero at maturity. The second states that mortgage values approach zero as interest rates become infinitely large. The third condition states that the value of the mortgage equals its outstanding balance when the mortgage is called (i.e. refinanced), while the fourth states that mortgage values are insensitive to interest rate changes at rates below the critical rate at which the mortgage is called.

Sub-optimal prepayments are introduced into the model via a function \( \pi \), which measures the probability of a borrower prepaying when the contract rate on the mortgage is below the prevailing market rate, i.e. when the mortgage is at a discount. The function \( \pi \) is estimated from historical FHA prepayment experience, and is assumed to follow a Poisson process. When a borrower prepays a discount mortgage, the holder of the discount MBS receives a capital gain equal to the amount by which the face value of the underlying loan exceeds its market value. This unscheduled capital gain can be represented by the expression \( \pi (F-M) \). By incorporating this extra capital gain into the

\[ \lambda = \frac{qr}{\sigma \sqrt{r}}. \]
standard hedging argument (see Appendix A), Dunn and McConnell obtain the following partial differential equation for valuing prepayable mortgage-backed securities:

$$\frac{\partial M}{\partial t} + (k(m-r)-qr)\frac{\partial M}{\partial r} + \frac{1}{2} \frac{\partial^2 M}{\partial r^2} \sigma^2 r + c + \pi(F(t)-M) = rM$$

Equation 4

The previous boundary conditions apply. Thus, mortgagors follow a ruthless call policy when the mortgage is priced above par, and prepay according to the exogenous prepayment function $\pi$ when the mortgage is priced at a discount and ruthless refinancing is not optimal. The introduction of sub-optimal prepayments makes the mortgage cash flows path dependent, and thus analytic solutions to the partial differential equation are unavailable. Dunn and McConnell therefore use an implicit finite difference method to solve [4].

Dunn and McConnell compare prices for two hypothetical 30 year fixed rate MBS, one with ruthless refinancing only and one with ruthless as well as sub-optimal refinancing. As expected, the authors find that a discount security with exogenous sub-optimal prepayments is more valuable than a discount security with optimal prepayments only.

While the Dunn and McConnell approach makes important contributions in that it introduces a sophisticated interest-rate option pricing model to the problem of MBS valuation, and also endeavors to account for the "sub-optimal" prepayments that are observed in practice, it has several shortcomings. First, the assumption of a ruthless call policy has several drawbacks. It necessitates the assumption that borrowers are homogeneous, and will all find it optimal to prepay at the same time, and that borrowers face no transaction costs (or at least, that transaction costs are homogenous). Further, it
necessitates the assumption that borrowers only hold a single option, the option to refinance, when in fact borrowers also hold the option to default, which may substitute for refinancing to some extent, and may hold other options like household mobility that are not purely financial. As a result, the imposition of an optimal, value-minimizing call policy will tend to overstate the incidence of prepayment on a premium-priced MBS.

Second, the exogenously specified prepayment function has a number of limitations. In particular, it is not a function of the state variable, r, but only of the remaining term to maturity, and thus does not reflect the sensitivity of prepayments to the interest rate environment. Furthermore, π is estimated from historical FHA data, which includes both optimal and so-called sub-optimal prepayments. However, since π is not a function of the interest rate environment, it is not possible to partial-out optimal prepayments in the estimation of π. As a result, optimal refinancings are counted twice, once when MBS are at a discount via the term T, and again when they are at a premium via the optimal call policy.

A third potential criticism is the assumption that the term structure can be described by a single state variable, the short term interest rate. The main consequence of the use of a single-factor model like that of Cox, Ingersoll, and Ross is that interest rates of different maturities are instantaneously perfectly correlated. Accordingly, it may be advantageous to use a two-factor model of the term structure in the valuation of MBS, particularly given that the decision to refinance tends to be based on the level of longer term rates. This approach is examined in more detail in the next section.
Brennan and Schwartz (1985) extend the Dunn and McConnell approach to valuing prepayable MBS in two important ways. First, they allow mortgage values to be a function of two state variables, the short term interest rate, and the long term rate on a consol, or perpetual, bond. Second, they contrast the Dunn and McConnell approach, in which prepayments are driven by a combination of an exogenous payment function and a ruthless call policy, with the case where the optimal call policy is abandoned, and prepayments are driven by an exogenous payment function exclusively.

The contingent claims model is that of Brennan and Schwartz (1979), in which the short rate, r, and the long consol rate, R, follow the log-normal processes:

\[ dr = (a_1 + b_1(R - r))dt + \sigma_1 r dz_1 \]
\[ dR = (a_2 + b_2 r + c_2 R)R dt + \sigma_2 R dz_2 \]

where the \( \sigma_i \) are volatility parameters, and the \( dz_i \) are correlated Brownian motions, such that \( E[dz_1 dz_2] = \rho_{12} dt \), with \( \rho_{12} \) denoting the coefficient of correlation. Here, the short rate reverts at rate \( b_1 \) to a long run level, \( R \), which itself is stochastic. In turn, the drift of the long rate depends on the level of both the long and short rates (The parameters \( a_1 \) and \( a_2 \) have no obvious interpretation).

Following the standard hedging arguments, it can be shown that the price of a prepayable MBS dependent on the two state variables, \( r \) and \( R \), must satisfy the following partial-differential pricing equation (see derivation in Appendix B):
\[
\frac{\partial M}{\partial \alpha} + (m_1 - \lambda_1 \sigma_1) \frac{\partial M}{\partial r} + \frac{1}{2} \frac{\partial^2 M}{\partial r^2} \sigma_1^2 + \left(\frac{\sigma_2}{R} + \frac{R^2}{R^2} - R\right) \frac{\partial M}{\partial R} + \frac{1}{2} \frac{\partial^2 M}{\partial R^2} \sigma_2^2
\]

\[
+ \frac{1}{2 \partial \partial R} \rho \sigma_1 \sigma_2 + c + \pi(X, t)(F(t) - M) = rM
\]

Equation 6

The boundary conditions for valuing a prepayable mortgage or MBS under an optimal call policy are analogous to those in Dunn and McConnell, except that Brennan and Schwartz allow for the presence of transaction costs. It is thus optimal to refinance the loan when its market value equals its face value plus transaction costs, denoted TC:

\[M(r_c, t) = F(t) + TC;\]

where, again, \(r_c\) is the critical rate at which the mortgage is called.

As in Dunn and McConnell, exogenous prepayments enter via the term \(\pi(F-M)\). Brennan and Schwartz observe that the term \(\pi\) could be estimated using a proportional hazards model (discussed below), citing the work of Green and Shoven (1986) as an example of the approach. However, in their empirical work, Brennan and Schwartz follow Dunn and McConnell in estimating the exogenous payment function, \(\pi\), from FHA historical prepayment experience and dependent only on time. Thus, the previous criticism of Dunn and McConnell applies.

Brennan and Schwartz compare prices from three prepayment scenarios. In the first, prepayments are assumed to follow a ruthless call policy exclusively, thus the exogenous payment function, \(\pi\), equals zero. The second case is that of Dunn and McConnell, where prepayments occur due to both optimal calls and via the exogenous payment function (i.e. \(\pi(t)>0\)). Finally, in the third case, Brennan and Schwartz abandon the value-minimizing call policy entirely, and allow prepayments to occur based on
empirical experience only. Prices differ substantially under each prepayment scenario. The mixed Dunn and McConnell scenario results in prices higher than those under the ruthless call scenario. This is reasonable because “sub-optimal” prepayments provide the MBS holder with an unscheduled capital gain, and thus raise the price of the security. The exclusively empirical scenario, however, where the optimal call policy is abandoned, results in higher prices still. Again, this is consistent with intuition. Because the optimal call policy fails to account for transaction costs and the heterogeneity of borrowers, it will tend to overstate prepayment. Since prepayment on a premium-priced MBS results in an unscheduled capital loss, overstating prepayments will tend to lower the value of the MBS relative to the case where prepayments on premium-priced pools follow empirical experience.

Brennan and Schwartz also compare their model, in which prices depend on two state variables, to prices generated under the single state-variable models of Dunn and McConnell (1981) and Black and Scholes (1973), and again find substantially different results. In particular, the single-factor models are found to produce consistently higher mortgage-backed security prices. Brennan and Schwartz argue that a single-factor model understates risk by ignoring the stochastic nature of long-term rates, and hence will tend to under-price call and put options relative to a two-factor model. Thus, because the mortgage-backed security holder is effectively short a call option, a single-factor model will tend to over-price mortgage-backed securities and other instruments with embedded call options relative to a two-factor model.
**Exogenous Prepayment: The Proportional Hazards Approach**

The work Dunn and McConnell (1981) and Brennan and Schwartz (1985) laid the foundation for applying sophisticated option-pricing methods to MBS valuation by abandoning the optimal endogenous call policy in favour of an exogenous payment function estimated from MBS prepayment data. One of the most successful and theoretically appealing approaches to estimating an empirical prepayment function for mortgages and mortgage-backed securities is the proportional hazards model. These models are amenable to a wide range of problems that are naturally analyzed as a series of conditional probabilities. Hazard models were first used in the bio-sciences to model the survival time of hospital patients, and in engineering to model the failure-time of machine parts. More recently, hazard rate models have been applied to problems in labour economics to model the probability of ending a strike, or of an unemployed person finding work (Kiefer 1988).

Full prepayment of a mortgage is equivalent to the termination of the mortgage contract, and thus falls naturally within the hazard framework. When applied to mortgage prepayment, the hazard measures the conditional probability that a mortgage will be prepaid at some time t, given that it has not been prepaid prior to this time. Generally, the model allows for a baseline or neutral-case hazard function, which is then affected by a variety of explanatory variables. This baseline hazard can either follow a particular parametric distribution that is specified in advance, or, in the non-parametric case, its form can simply be determined by the data. Both parametric and non-parametric hazard models are developed more formally in the estimation section of this paper.
Green and Shoven (1986) was one of the first applications of the proportional hazard framework to mortgage prepayment. Using a large data set of some 5000 individual mortgage loans, Green and Shoven fit a non-parametric hazard function to assess the sensitivity of prepayment to the level of market interest rates and the enforcement of due-on-sale clauses. In another key study of individual loan data that is related to the question of mortgage prepayment, Quigley (1987) uses a non-parametric hazard model to estimate the impact of various economic and demographic variables on the rate of household mobility. One of the key findings of Quigley is that, in cases where a below market rate loan is neither fully capitalized into the house value, nor easily assumed by a potential buyer, households have a lower conditional probability of moving than would otherwise be the case.

The over-riding problem when we turn to modeling a prepayment function for mortgage-backed securities is that only aggregate data are available. Variables like house values, for example, let alone age and education of the household head, cannot readily be observed in the MBS data. The only data that are available are aggregate pool statistics like the MBS coupon and the pool’s weighted average remaining term to maturity, as well as the historical mortgage refinancing rates that were available in the market. As a result, when modeling mortgage-backed security prepayment, age-dependent patterns tend to proxy for many of the unobservable variables.

One of the most complete examples of the application of the proportional hazards model to the valuation of mortgage-backed securities is the 1989 study by Schwartz and Torous, who estimate a parametric hazard function for 30-year fixed rate GNMA mortgage-backed securities. The resulting hazard function is then used to drive
exogenous prepayments in the Brennan and Schwartz (1979) two-state variable model discussed above.

Schwartz and Torous (1989) assume that completed mortgage durations are distributed log-logistically, which leads to the following functional form for the baseline, or neutral-case, hazard function:

\[ \pi_0 = \frac{\gamma \alpha t^{\alpha-1}}{1 + \gamma t^\alpha} \]

where \( t \) is mortgage age, and \( \alpha \) and \( \gamma \) are parameters to be estimated. Of the known family of tractable parametric distributions for the hazard function, the log-logistic is the only one which will permit the hazard rate first to increase with duration (i.e. mortgage age), then to decrease. The choice of the log-logistic parameterization is thus intended to capture the so-called \textit{seasoning} or aging effect observed in US mortgage-backed securities data, in which prepayments typically accelerate early in the security's life, say out to 6 or 7 years, and subsequently level off or decline. This pattern is usually ascribed to the influence of housing turnover on prepayment in cases where a loan cannot be readily assumed on sale of the house (see Richard and Roll 1989 for a discussion).

The baseline hazard function \( \pi_0 \) is shifted up or down by four explanatory variables, including the incentive to refinance and the season of the year. The incentive to

\[ 11 \text{ The log-logistic hazard is derived formally in the estimation section.} \]

\[ 12 \text{ Length of housing tenure is the true variable of interest when considering the impact of mobility on prepayment. A new homeowner is unlikely to sell her house shortly after moving in. However, as family and employment conditions change, the probability of moving increases with the length of housing tenure. At some point, however, the probability of moving reaches a peak, as households who have already lived in their houses for a long time become increasingly less likely to move. Clearly, mortgage age is not a perfect substitute for length of housing tenure: consider the case of a borrower who refinances the mortgage on an existing home. However, mortgage age is not an unreasonable proxy for length of housing tenure in the case of 30 year mortgages. Pools of shorter-term mortgages, however, probably include} \]
refinance is modeled as the difference between the contract rate on the mortgage-backed security grossed up by 50 basis points, and the long (30 year) Treasury rate. In the valuation stage, Schwartz and Torous use the long consol rate to proxy for the Treasury rate. Due to transaction costs and the value of the live refinancing option, it is not usually profitable to refinance as soon as market rates fall below the contract rate on the mortgage. To capture this effect, Schwartz and Torous also cube the refinancing incentive variable, which allows estimated prepayment rates to accelerate for larger rate differentials. To capture the tendency for people to move in the summer rather than the winter, Schwartz and Torous also include a seasonal indicator variable.

The final variable in the model is intended to capture a phenomenon observed in US mortgage-backed security prepayment data known as burnout. The pattern of burnout is typically explained as a process whereby borrowers with low refinancing costs remove themselves from a pool soon after the refinancing incentive becomes positive, leaving behind a pool of more homogenous borrowers with higher refinancing costs and lower incentives to prepay (see Roll and Ross 1989 for a discussion). Thus, for the same refinancing incentive, prepayments on a burned out pool would be lower than prepayments on an otherwise identical pool that has not yet burned out. Burnout is modeled by taking the natural log of the actual balance over the scheduled balance at time \( t \). Thus, a pool that has experienced a lot of prepayments in the past will have a much lower actual balance than scheduled, and is more likely to be burned out. This implies a

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homeowners with quite different lengths of tenure, and thus may not exhibit as pronounced a seasoning pattern.
lower hazard of prepayment, and, since the sign of the burnout variable itself will be negative, a positive coefficient on the burnout variable.

Schwartz and Torous find that all estimated coefficients are of the expected sign, and, except for the summer indicator variable, are statistically significant. The coefficient on the refinancing variable, for example, indicates that, for a 200 basis point decline in market rates, a borrower is more than twice as likely to prepay than is a borrower under the neutral or average refinancing environment. In the valuation stage, Schwartz and Torous compare prices for a hypothetical mortgage-backed security under four different assumptions: that the security is not prepayable, that prepayments follow the ruthless call policy, that prepayments follow FHA experience, and that prepayments follow the estimated proportional hazards function (the partial differential equation is similar to that in Brennan and Schwartz (1985) shown above). The authors find that the scenarios where prepayments occur according to FHA experience and according to the estimated prepayment function are more consistent with the pattern of pricing observed in the market, though they do not compare the simulated results to actual market data. They also find that prices using the prepayment model differ significantly from those under the FHA experience case for more extreme starting values for the short rate and the long rate, reflecting the sensitivity of prepayments to the interest rate environment in the estimated proportional hazards model.

**Combined Prepayment and Default**

As mentioned previously, default can affect the level of observed prepayment on an insured MBS, because the insurance pays the MBS holder the outstanding principal
balance of any loans that default. At least on the surface, the insurance payout from
default is indistinguishable from a prepayment due to refinancing, though the two events
tend to occur under different economic environments. As a result, default could
potentially explain the "sub-optimal" prepayments observed on MBS. However, in the
models discussed thus far, which focus on the refinancing incentive, default is not
explicitly modeled. Rather, default (as well as any other age-dependent effects) is lumped
into "sub-optimal" prepayments, or into the estimate of the baseline prepayment function.
In this section, we begin with a brief overview of the combined default and prepayment
literature with regard to individual mortgage loans, before examining efforts to include
both default and prepayment behavior in MBS valuation.

There is an extensive literature on the study of default as it relates to individual
mortgage loans, some of which was discussed at the start of the literature review. Far
fewer studies exist that consider the interaction of prepayment and default. Leung and
Sirmans (1990) use Boyle's (1988) two state-variable lattice approach to price a
hypothetical mortgage subject to both prepayment and default risk. Both are modeled as
ruthlessly exercised options, though Leung and Sirmans make the analysis more realistic
by presenting results for different (static) assumptions about transaction costs. Kau,
Keenan, Muller, and Epperson (1992) take a similar approach, valuing an individual
mortgage with both default and prepayment options, though instead of modeling the
evolution of the state variables directly in a lattice, Kau et al use a finite difference
method to solve the partial differential pricing equation numerically. More importantly,
Kau et al also consider the value of default insurance, and incorporate the Dunn and
McConnell (1981) approach of hybrid prepayments, where optimal prepayments follow
the option model, while suboptimal prepayments occur according to empirical
experience.

Due to the fact that only aggregate payment data are available for mortgage-
backed securities, the influence of default on mortgage-backed security prepayment and
pricing has received comparatively little attention. One of the few examples is that of
Schwartz and Torous (1992), who illustrate how exogenous prepayment and default
functions could be incorporated into a two state-variable model to value simultaneously
an insured mortgage-backed security, the underlying risky mortgage, and the insurance
the specification of the underlying state variables, the short term interest rate \( r \), and the
house price, \( H \). The short rate is assumed to follow the Cox, Ingersoll, and Ross (1985)
process:

\[
dr = k(m-r)dt + \sigma_r \sqrt{r}dz_r,
\]

with parameters defined as before. House prices are assumed to follow the log-normal
process:

\[
dH = (\mu - b)Hdt + \sigma_H Hdz_H
\]

Equation 7

where \( \mu \) measures the expected return to housing, and \( \sigma_H \) is a volatility parameter. Since
housing has a consumption as well as an investment component, the payout rate \( b \) is used
to represent the flow of housing services. The instantaneous correlation coefficient
between increments to the short rate and house price processes is given by a parameter \( \rho \),
and, as before, \( dz_H \) is a Brownian motion.
Given the processes for $r$ and $H$, as well as default and prepayment functions $\delta$ and $\pi$, the partial differential equations for pricing the risky mortgage, the insurance, and the insured MBS can be derived using standard hedging arguments (though some rather heroic assumptions need to be made about the ability to continuously trade houses in a frictionless market). The value of a MBS in the model is thus given by the following equation (see derivation in Appendix B):

$$
\frac{1}{2} \frac{\partial^2 M}{\partial r^2} \sigma_r^2 r + \frac{1}{2} \frac{\partial^2 M}{\partial H^2} \sigma_H^2 H^2 + \frac{\partial^2 M}{\partial r \partial H} \rho \sigma_r \sigma_H \sqrt{r} H + \left[ k(m - r) + qr \right] \frac{\partial M}{\partial r} + (\mu - b) H \frac{\partial M}{\partial H} + \frac{\partial M}{\partial t} + C - F(c - p) + (\delta + \pi)(F - M) = rV
$$

Equation 8

where, as before, $q$ is a parameter proportional to the market price of risk. The term $c$ is the mortgage rate on the underlying loan, and $p$ is the coupon rate on the MBS. The term $F(c - p)$ thus represents the spread retained by the issuer as a servicing fee. Since the underlying loan has a level payment (denoted by $C$), the MBS payment, $P = C - F(c - p)$, gradually increases over time. Note that the unscheduled gain/loss in the partial differential equation for the insured MBS is given by the sum of the default and prepayment rates: to the MBS holder, both are sources of unscheduled principal payments. Though not shown here, Schwartz and Torous derive similar partial differential equations for valuing the risky mortgage and the insurance, and then solve all three equations simultaneously.

For illustrative purposes, Schwartz and Torous assume relatively simple prepayment and default hazards:
Prepayment function: 
\[ \pi = \begin{cases} 
\pi_0(t) \cdot \exp\left( \frac{\beta(M - F)}{H} \right) & \text{if } \delta = 0; \\
0 & \text{if } \delta > 0.
\end{cases} \]

In the prepayment function, the covariate term \( \exp(\beta) \) is from Green and Shoven (1986).

Schwartz and Torous (1992) do not fit the model to actual data, but instead use Green and Shoven's estimates for \( \beta \), and assume that the baseline \( \pi_0 \) follows the PSA curve.\(^{13}\)

Default function: 
\[ \delta = \begin{cases} 
\frac{M - H}{H} \cdot \exp\left( \eta \frac{M - H}{H} \right) & \text{if } H < M \text{ and } H < F; \\
0 & \text{otherwise.}
\end{cases} \]

Note the interaction of prepayment and default in the equations above. For default to occur, not only must housing equity, \( H - M \), be negative, but the house value must also be less than the face value of the loan, otherwise it would be preferable to prepay than to default. The interaction of the prepayment and default rates in the pricing equation for MBS shows how, at least in theory, default may explain the "suboptimal" prepayments that are observed in practice. For example, if house values are low, a borrower could potentially default on a below market-rate loan, which would then appear as a prepayment of principal on the discount MBS.

For low house values, Schwartz and Torous find that the value of the mortgage backed security exceeds that of the underlying risky mortgage, reflecting the value of the default insurance. In particular, when the MBS is priced at a discount (interest rates are high) and house values decline, the higher probability of a default insurance payout causes the price of the MBS to increase, consistent with the "sub-optimal" prepayments

\(^{13}\) The Public Securities Administration (PSA) curve assumes that annualized MBS prepayment rates increase linearly from zero to 6% over the first 30 months an MBS is outstanding, and thereafter remain
of Dunn and McConnell (1981). The authors also find that, for higher values of the speed of refinancing parameter $\beta$, MBS values behave more like those under an optimal call policy, while insurance values are lower due to the reduced probability of default when prepayments are high.

Schwartz and Torous also present results for equilibrium mortgage rates and insurance fees under different assumptions about house values and prepayment speeds. They find that the equilibrium mortgage rate declines for higher house values due to the lower probability of default, while it increases for higher values of the speed of refinancing parameter in order to compensate the lender for the more valuable refinancing option. Insurance fees are found to increase significantly for loan-to-value ratios over 80%, and to be minimal for ratios below 66%.

The paper by Schwartz and Torous (1992) is a useful illustration of how simultaneous default and prepayment functions could potentially be incorporated into mortgage-backed security valuation. However, Schwartz and Torous do not attempt to fit the prepayment and default functions to actual data, so it remains to be seen how well such a model would explain traded mortgage-backed security prices. In particular, Schwartz and Torous leave a key question unanswered: how should the two hazard functions be simultaneously estimated from aggregate payment data that does not distinguish between the two sources of unscheduled principal? Presumably, one could allow the prepayment function to hold when the security is priced at a premium, and the default function to hold when the security is at a discount. Regardless of the method, it

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constant at 6%. In the US MBS market, the convention is to price Collateralized Mortgage Obligations (CMO) and some MBS at a percentage of the PSA vector when quoting prices for trading purposes.
would probably be necessary to follow the approach of Quigley and Van Order (1992), and use a micro-level set of housing data to estimate the parameters for the house price process and the default function. Yet doing so would require the implicit assumption that the houses in the chosen micro-level data set and those in the underlying MBS pool are homogeneous, when we know that housing, and particularly housing equity, is a very heterogeneous good. It is thus not yet clear how applicable the approach might be for mortgage-backed security valuation in practice. However, one advantage of the approach suggested by Schwartz and Torous (1992) could be in the valuation of mortgage insurance, given observations on individual mortgage loans and house prices.

**Canadian MBS Literature**

Few published studies of Canadian MBS prepayment and valuation exist. Boyle (1989), in the first application to the Canadian mortgage-backed security market, applies the Schwartz and Torous (1989) model to valuing Canadian MBS. Boyle fits a log-logistic baseline hazard function to the Canadian data. However, due to the limited data at the time of the study, Boyle uses Schwartz and Torous' US estimate for the refinancing coefficient in the hazard function. Hypothetical securities are then valued using the Brennan and Schwartz (1979) two-factor model in a Monte Carlo simulation.

Cooperman, et al (1995) discuss a prepayment model developed for Canadian MBS by Goldman, Sachs. The authors present a good overview of the Canadian MBS market, and the types of issues one needs to consider when modeling prepayments in Canada. In particular, they note that one needs to be concerned with modeling partial as
well as full prepayments, and also prepayment penalties. However, in common with the
US studies published by Wall Street firms, details of the model are not disclosed.

Other Developments and Conclusions

One of the implicit themes in the studies discussed thus far is that an MBS can in
large measure be viewed for valuation purposes as a single mortgage. Certainly this is the
case in the pure option framework. And it is also the case to some extent in the studies
that estimate exogenous payment functions, though the use of an estimated payment
function is in part driven by the recognition that the mortgagors underlying a MBS pool
are not homogenous, as is the attempt to model burnout. But having made these
allowances for heterogeneity, it is then assumed that all borrowers respond to a given
refinancing incentive in the same way. A natural extension would seem to be to allow the
underlying borrowers to respond differently to the same observable refinancing incentive.
Patruno (1994) describes in general terms a model of this type in use at Goldman, Sachs,
where borrowers are assumed to be drawn from an underlying distribution of different
types: fast prepayers, normal prepayers, and slow prepayers. Over time, the fast prepayers
will remove themselves from the pool, leaving behind a more homogenous pool that
responds more slowly to the same refinancing incentive. This approach is thus
particularly amenable to modeling the borrower self-selection that is thought to underlie
the burnout phenomenon observed in the US. As an alternative to the model suggested by
Patruno, borrower heterogeneity could also be modeled by allowing some component of
transaction costs to be stochastic.
Another area of research not really discussed so far is the analysis of risk. It is by now fairly standard to calculate risk measures such as effective duration and convexity for fixed income instruments like MBS that have embedded option-like features. These are based on the numerical approximation to the first and second order derivatives of price with respect to a small change in the simulated discount rates. From these relatively standard measures, other authors have turned to examining the sensitivity of MBS prices and hedge statistics to changes in prepayment model assumptions (see, for example, Sparks and Sung (1995)). Hayre and Chang (1997) consider empirical duration measures, which are obtained by regressing changes in MBS prices on changes in Treasury yields over time, and compare these to the effective duration measures from the theoretical term-structure models.

Before turning to the estimation section, it is worth noting that, in studies which estimate an exogenous payment function, the baseline or neutral-case prepayment function is always a deterministic function of mortgage age. But as noted above, the estimated shape of the baseline typically picks up effects such as the influence of household mobility on prepayment, which are not purely deterministic. Though beyond the scope of the present paper, a fruitful avenue for further research could well be the incorporation of a stochastic component in the estimated baseline function. This would be a natural extension of the efforts described above to measure risk with respect to changes in prepayment model assumptions, for it allows one to model explicitly the uncertainty inherent in a baseline prepayment estimate.
IV. Estimating the Proportional Hazards Prepayment Function

In this section, we use proportional hazards models to estimate prepayment functions for a subset of Canadian mortgage-backed security data. These estimated functions will then be used in the following section to drive exogenous prepayments within an arbitrage-free valuation model. To begin, we follow Schwartz and Torous (1989) and Boyle (1989) in estimating a proportional hazards model with a log-logistic baseline prepayment function. As an alternative to the log-logistic hazard, which may be too stylized to capture adequately the prepayment patterns observed in Canada, we then estimate a non-parametric proportional hazards model for the same data. The non-parametric hazards model may be better suited to capturing the influence on baseline prepayments of the age-dependent prepayment restrictions in most Canadian mortgage contracts.

The proportional Hazards Model

As noted in the literature review, the hazard rate measures the conditional probability of an event occurring at some point in time, given that it has not occurred prior to this time. In the context of mortgage prepayment, the hazard rate, $h(t)$, measures the probability that a mortgagor will prepay his mortgage at time $t$, given that he or she has not prepaid up until this time:
\[ h(t) = \lim_{\Delta t \to 0} \frac{\Pr(t \leq T < t + \Delta t | T \geq t)}{\Delta t} \]

\[ = \frac{f(t)}{1 - F(t)} \quad \text{Equation 9} \]

where

\[ F(t) = \Pr(T \leq t) \]

is the cumulative distribution function of a duration of length \( t \),

and

\[ f(t) = \frac{dF}{dt} \]

is the associated probability density of a duration of length \( t \).

Defining the survivor function, \( S(t) = 1 - F(t) \), the hazard can be written:

\[ h(t) = \frac{f(t)}{S(t)} \]

Therefore, we can obtain the following relationship which will be useful in the estimation stage:

\[ -\frac{dS}{dt} = \frac{h(t)}{S} \]

\[ = \frac{-d\ln S}{dt} \quad \text{Equation 10} \]

In the proportional hazards framework, it is assumed that there is a baseline hazard rate, \( h(t) \), that depends only on the duration of the spell, in this case mortgage age.

The baseline hazard is then shifted up or down by some function of the explanatory variables, or covariates. The usual form of the covariate function is the exponential, \( \exp(X, \beta) \). Thus, the total hazard, \( h(t, X, B) \), is written:

\[ h(t, X, \beta) = h(t) \cdot \exp(X, \beta) \quad \text{Equation 11} \]
Since the explanatory variable or covariate function $exp(X, \beta)$ does not depend on the duration of the spell, $t$, the covariates have the same proportionate effect on the baseline hazard at all durations. It is this property that gives the proportional hazards model its name. Although other functional forms of the covariate function are certainly possible, the exponential form ensures that the conditional probability is non-negative. Also, use of the exponential specification has the further advantage that the coefficients, $\beta$, of the explanatory variables have the same type of partial-derivative interpretation as the coefficients in ordinary linear regression (Kiefer 1988 p. 664):

$$\frac{\partial \ln h(\cdot)}{\partial X} = \beta$$

It is usual practice to measure the covariates or explanatory variables as deviations from their means. In this way, the baseline function can be interpreted as the conditional probability of prepayment that would prevail under a neutral economic environment.

**Parametric Specification of Baseline Hazard**

Several parametric distributions for the observed spells (i.e. completed mortgage durations) are well documented, such as the exponential, weibul, and log-logistic, among others (Cox and Oakes 1984 pp. 16-28). Of these, the only one which allows the estimated hazard function first to increase, then to decrease, consistent with the observed behavior of Canadian MBS prepayments, is the log-logistic distribution. The distribution and density functions, respectively, for the log-logistic distribution are the following:
\[ F(\alpha, \gamma, t) = 1 - \frac{1}{1 + \gamma^a} \]

and

\[ f(\alpha, \gamma, t) = \frac{\alpha \gamma^{a-1}}{(1 + \gamma^a)^2} \]

where \( t \) is the duration of the mortgage in months, and \( \alpha \) and \( \gamma \) are parameters to be estimated. From equation [9] above, the proportional log-logistic hazard is therefore:

\[ h(t, \alpha, \gamma, X, B) = \frac{\alpha \gamma^{a-1}}{1 + \gamma^a} \exp(X, B) \]  

\textbf{Equation 12}

A function that will be useful in the estimation stage which follows is the integrated hazard, \( H(t) \):

\[ H(t) = \int_0^t h(t, \alpha, \gamma, X, B) dt \]

\[ = \exp(X, B) \ln(1 + \gamma^a) \]  

\textbf{Equation 13}

From equation [10] above, the integrated hazard is related to the survivor function as follows:

\[ H(t) = -\ln(S(t)). \]  

\textbf{Equation 14}

\[ ^{15} \text{We make use of the following rule:} \]

\[ \int A \frac{g'(u)}{g(u)} du = A \int \frac{g'(u)}{g(u)} du = A \ln |g(u)| \]

where \( A \) is a constant. To obtain the expression for \( H(t) \), we substitute \( A = \exp(X'B) \) and \( g(u) = 1 + \gamma^a \).
**Maximum Likelihood Estimation**

The hazards model in [12] is estimated by maximum likelihood. Under the assumption that each individual mortgagor’s spell is independent, the likelihood function for a sample with \( n \) completed spells and no censored spells can be written as follows:

\[
L^*(\theta) = \prod_{i=1}^{n} f(t_i, \theta)
\]

As before, \( f(t) \) is the probability density of duration, and \( \theta \) is a vector of parameters. As written, this likelihood function assumes that all individuals’ spells are observed from origination to completion, and that only one mortgagor’s spell is completed at each \( i \).\(^{16}\) A more realistic situation is one in which multiple spells are completed at each point in time, and where not all spells are observed in their entirety. In particular, it is frequently the case in economic data that not all spells will be completed by the end of the observation period, a situation known right-censoring.\(^{17}\) Failure to account for right-censoring will tend to over-state the hazard, since incomplete spells are erroneously assumed to have ended.

To illustrate, we will consider the likelihood function for a right-censored sample with no ties. The survivor function, \( 1-F(t) = P(T>t) \), gives the contribution of a right-censored spell to the likelihood:

---

\(^{16}\) In survival analysis, a spell refers to the length of time for which the object under study survives.

\(^{17}\) The other possibility is that some spells will have already begun prior to the start of the observation period, a situation known as left-censoring. Depending on whether the hazard increases or decreases with duration, failure to account for left-censoring in a sample will under- or over-state the true hazard. In the present study, where we can observe MBS prepayments from the origination of each security up to the observation cut-off date, left-censoring is not relevant.
\[ L' (\theta) = \prod_{i=1}^{n} f(t_i, \theta)^{d_i} (1 - F(t_i, \theta))^{1-d_i} \]

where

\[ d_i = 1 \text{ if the observation is not right-censored, } 0 \text{ otherwise.} \]

Letting \( L(\theta) = \ln L^*(\theta) \), we can write the log-likelihood function for the right-censored sample as follows:

\[ L(\theta) = \sum_{i=1}^{n} d_i \ln f(t_i, \theta) + \sum_{i=1}^{n} (1-d_i) \ln(1-F(t_i, \theta)) \]

From equations [9] and [14] above, the log-likelihood function can be re-written in terms of the proportional hazards function using the relationships (Kiefer 1988):

\[ f(t_i, \theta) = h(t_i, \theta)S(t_i, \theta) \]

\[ \ln S(t_i, \theta) = -H(t_i, \theta) \]

to obtain

\[ L(\theta) = \sum_{i=1}^{n} d_i \ln h(t_i, \theta) - \sum_{i=1}^{n} H(t_i, \theta) \]  \hspace{1cm} \text{Equation 15} \]

Substituting the expressions for \( h \) and \( H \) in [12] and [13], the log-likelihood function for the proportional hazards model with right-censoring and log-logistic baseline is given by the following:

\[ L(\theta) = \sum_{j=1}^{J} \left[ \sum_{i=1}^{n} d_i \left[ \ln \alpha + \ln \gamma + (\alpha - 1) \ln t_i - \ln(1 + \eta^\alpha) + X'B \right] - \exp(X'B) \ln(1 + \eta^\alpha) \right] \]  \hspace{1cm} \text{Equation 16} \]

where
\[ t_i^j = \text{duration in months of } i^{th} \text{ observation in the } j^{th} \text{ mortgage-backed security pool}. \]

Since there is no analytical solution to the maximization of [16], a numerical maximization procedure must be used. In the results reported below, we maximize the log-likelihood function using the quasi-Newton method from Judge et al (1985, pp. 958-960) provided in the NL function in Shazam 7.0.

**Data**

The data are monthly observations on 10 five-year MBS pools issued by Royal Trust between January 1, 1990 and October 1, 1991, and were supplied by Scotia Capital Markets Fixed Income Research. Each pool contained between 166 and 684 individual loans at issue. The data thus contain observations on 3,571 individual loans over time. For each month, we observe the number of mortgages that prepay in full, as well as the number of loans remaining in the pool. In the hazard framework, this is the number of mortgages at risk. We also observe aggregate statistics each month such as the weighted average mortgage rate on the underlying loans, the weighted average remaining amortization period, the remaining principal balance, as well as scheduled and unscheduled principal, interest, and penalty cash flows. The refinancing rate is represented by the conventional five year mortgage rate.

Table 3 presents summary statistics for the MBS pools. Note that the average mortgage rate on the underlying loans is high relative to the average market rate over the sample period. The refinancing option for most of these loans was thus well in the money for much of the sample period. All else equal, the estimated baseline hazard function is
likely to overstate the “true” baseline hazard in a more representative sample. Similarly, the sensitivity of prepayments to a given refinancing incentive is likely to be higher here than in a large sample that includes both premium and discount mortgages.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 & WAC & Market Mortgage Rate & Original Amortization (months) & Number Mortgages at Issue \\
\hline
Average & 11.47 & 9.81 & 287 & 357 \\
Min & 11.14 & 7.25 & 274 & 166 \\
Max & 11.88 & 14.25 & 300 & 684 \\
Total & & & & 3,571 \\
\hline
\end{tabular}
\caption{Summary Statistics}
\end{table}

\textbf{Results}

We use a very simple measure of the incentive to refinance, C-R, the difference between the weighted average rate on the underlying mortgages and the 5 year mortgage rate available in the market.\textsuperscript{18} This specification is particularly convenient when it comes to interpreting the results, and is somewhat less prone to extrapolation problems than other more complex variables, which is an important consideration when the goal is valuation within a stochastic model of the term structure. However, the choice of C-R clearly sacrifices realism for simplicity. Particularly in the case of a short-term balloon mortgage, the savings from refinancing for a given decrease in market rates declines.

\textsuperscript{18} Several authors have suggested using the ratio of these rates, C/R, arguing that this measure can better reflect the effect of transaction costs, which tend to be proportional to the outstanding balance on the loan (see, for example, Richard and Roll 1989). Alternately, rather than approximate the refinancing incentive by comparing the relative level of interest rates, one could measure the “true” refinancing incentive by calculating the present value of the MBS at the market rate, and comparing this to the current balance of the security.
rapidly as the mortgage approaches its interest rate reset date. However, we will leave the use of a time-dependent refinancing variable, perhaps the present value of the mortgage cash flows under the simulated interest rates, to future research. Additionally, a refinancing model in which borrowers can choose from a menu of mortgage rates and terms is beyond the scope of this paper. Note that the refinancing variable is measured as the deviation from its sample mean, so that the baseline hazard measures prepayment under a neutral refinancing environment.

In Table 4 presented below, we also report results estimated for a model that includes indicator variables, $SUMMER$ and $FALL$, to capture the seasonal tendency for prepayments to rise when people move. Although intuition would suggest that household mobility is probably at its highest in the summer, we use a fall indicator as well to take account of administrative lags in the processing and reporting of prepayments. The variable $SUMMER$ is defined to be equal to one if the observation occurs during the months June to August inclusive, and zero otherwise. The variable $FALL$ is defined analogously, equal to one over the period September to November inclusive, and zero otherwise.

We do not attempt to model a Canadian counterpart to the US burnout phenomenon. This is because the slowing of prepayments as the pool ages following the peak after the third year can be largely explained by the reduced savings from refinancing as the pool approaches its maturity date. In addition, the two year prepayment window is a rather limited amount of time for borrowers to self-select and remove themselves from the pool. Thus, while it is conceivable that a burnout effect could be present, effort spent
modeling the time-dependent nature of the refinancing incentive is likely to be more rewarding, at least initially, than efforts to detect borrower burnout in Canada.

Similarly, although one would expect prepayments to be positively related to cycles in the housing market, we do not try to model the kind of age-dependent mobility patterns discussed in the US MBS literature. There are two reasons for this. First, as we noted above, in the case of an MBS made up of relatively short-term rollover mortgages, mortgage age is unlikely to be a good proxy for length of housing tenure, the true variable of interest when analyzing mobility-related prepayment. As a result, the kind of age-dependent mobility patterns we expect to see when examining length of housing tenure are likely to be far less pronounced when we turn to a Canadian MBS pool. Second, the shape of the baseline hazard is dominated by prepayment restrictions and penalties, making any age-dependent mobility effects difficult to discern.\textsuperscript{19}

As shown in Table 4, the parameters of the baseline hazard are of the expected sign, and are highly significant in all cases. The estimated value for $\alpha$ is greater than one, indicating that the baseline prepayment rate initially rises with mortgage duration, then falls. This is consistent with expected prepayment behavior in a market with duration-dependent prepayment provisions. The parameter $\gamma$ is close to zero, but again highly significant. The maximum likelihood estimation is sensitive to the starting value of $\gamma$, insofar as large positive values for $\gamma$ will tend to produce negative values in subsequent iterations, which are inadmissible in the log-logistic distribution. Otherwise, the

\textsuperscript{19} An alternative way to pick-up mobility-related prepayment is to include the rate of housing turnover as an explanatory variable. While this might be useful for inference purposes, it is, as Richard and Roll (1989) point out, somewhat like having the same variable on both sides of a regression equation. Additionally, it
parameter and coefficient estimates reported in the tables are robust to a range of starting values.

### A. Log-Logistic Proportional Hazards

<table>
<thead>
<tr>
<th>Variable</th>
<th>parameter/ coefficient</th>
<th>exp(coefficient)</th>
<th>z-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>alpha</td>
<td>1.8158</td>
<td>-</td>
<td>44.623</td>
</tr>
<tr>
<td>gamma</td>
<td>0.000546</td>
<td>-</td>
<td>6.653</td>
</tr>
<tr>
<td>C - R</td>
<td>0.42298</td>
<td>1.5265</td>
<td>29.437</td>
</tr>
</tbody>
</table>

Likelihood Ratio Test 1093.54 on 3 df

### B. Log-Logistic Proportional Hazards with Seasonal Indicators.

<table>
<thead>
<tr>
<th>Variable</th>
<th>parameter/ coefficient</th>
<th>exp(coefficient)</th>
<th>z-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>alpha</td>
<td>1.8406</td>
<td>-</td>
<td>44.408</td>
</tr>
<tr>
<td>gamma</td>
<td>0.000377</td>
<td>-</td>
<td>6.558</td>
</tr>
<tr>
<td>C - R</td>
<td>0.34981</td>
<td>1.419</td>
<td>18.898</td>
</tr>
<tr>
<td>SUMMER</td>
<td>0.39614</td>
<td>1.486</td>
<td>6.709</td>
</tr>
<tr>
<td>FALL</td>
<td>1.3584</td>
<td>3.890</td>
<td>20.339</td>
</tr>
</tbody>
</table>

Likelihood Ratio Test 731.18 on 5 df

Table 4: Maximum Likelihood Estimates for Log-Logistic Prepayment Function

The coefficients on the explanatory variables are also all highly significant, and of the expected sign. The coefficient on the refinancing incentive C-R in panel A, 0.423, is interpreted as follows: for a 100 basis point decline in market interest rates, borrowers are over 1.5 times more likely to prepay at any time $t$ than are borrowers in the baseline or neutral refinancing environment. The coefficients on the SUMMER and FALL indicators are interpreted similarly. Note that the FALL variable has the larger coefficient, likely would require modeling an additional (untraded) macro-economic state variable, which is a problem in
reflecting administrative lags in the reporting of prepayments due to the sale of houses in the summer. The coefficient on FALL, 1.36, implies that borrowers are almost four times as likely to prepay in the fall than at other times. Note that the estimate of \( \gamma \) is substantially lower when the seasonal indicators are included, since the baseline now reflects the "neutral" lower-prepayment environment of December to May.

**Non-Parametric Hazard: The Cox Model**

Cox (Cox and Oakes 1984, Kiefer 1988) provides a so-called partial-likelihood method of estimating the sensitivity of prepayment to a vector of explanatory variables without specifying a parametric distribution for the underlying hazard. Given an estimate of the coefficients on the relevant explanatory variables, it is then possible to extract from the data a baseline hazard function, without imposing any particular form on the baseline ahead of time. This approach could therefore provide an alternative method of estimating a prepayment function if parametric models like that presented above are thought to be too stylized. The Cox proportional hazards model is of the form:

\[
    h(x, \beta) = h_0(t) \cdot \exp(x, \beta)
\]

Equation 17

As equation [17] shows, at each \( t \), the baseline hazard function is the same for all individuals, \( i \).

Cox shows that, if completed durations are ordered from shortest to longest, and the hazard is as specified in [17], the conditional probability that individual \( j \) concludes a spell at \( t \), given that any of the \( n \) spells could have ended, is given by:
In contrast to the log-logistic hazard model in [12] and the other parametric hazards, the baseline hazard does not appear in the conditional probability [18]. The log-likelihood function is then:

\[
L(\beta) = \sum_{j=1}^{n} \left( x_j, \beta \right) - \ln \left( \sum_{i=j}^{n} \exp(x_i, \beta) \right)
\]

Equation 19

Results from estimating this model using the built-in CoxReg function in S-Plus are shown in Table 5 below. The estimated coefficient on the refinancing incentive from the Cox model with just the one explanatory variable is 0.472, of the correct sign and highly significant. It is encouraging that the refinancing coefficient estimated from the Cox model is comparable to that estimated for the log-logistic model, even though the baselines are different. As shown in panel B of the table, the signs on the seasonal indicators are also of the expected sign and highly significant, although only the coefficient on the FALL variable is similar in magnitude to that under the parametric hazard. The coefficient on the SUMMER variable in the Cox model indicates a much higher probability of prepayment in the summer than was the case for the parametric log-logistic hazard.
### A. Cox Regression

<table>
<thead>
<tr>
<th>Variable</th>
<th>coefficient</th>
<th>exp(coefficient)</th>
<th>z-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>C - R</td>
<td>0.472</td>
<td>1.603</td>
<td>19.749</td>
</tr>
</tbody>
</table>

Number alive: 1900  Number dead: 1667  Likelihood Ratio Test 323 on 1 df, $p=0$

### B. Cox Regression with Seasonal Indicators.

<table>
<thead>
<tr>
<th>Variable</th>
<th>coefficient</th>
<th>exp(coefficient)</th>
<th>z-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>C - R</td>
<td>0.499</td>
<td>1.65</td>
<td>19.341</td>
</tr>
<tr>
<td>SUMMER</td>
<td>1.199</td>
<td>3.32</td>
<td>19.983</td>
</tr>
<tr>
<td>FALL</td>
<td>1.624</td>
<td>5.07</td>
<td>25.495</td>
</tr>
</tbody>
</table>

Likelihood Ratio Test 1068 on 3 df, $p=0$

Table 5: Partial Likelihood Estimates for Non-Parametric Prepayment Function

Given estimates of the coefficients $\beta$ from the Cox model, it is then possible to estimate the baseline hazard rate. For a non-parametric discrete hazard $h(t_j)$, the survivor function for a sample of $n$ ordered durations is given by:

$$S(t_j) = \prod_{i=j}^{n} (1 - h(t_i))$$

It can then be shown that the maximum likelihood estimator for $h(t_j)$ is:

$$h(t_j) = \frac{m(t_j)}{\sum_{i=j} \exp(x_i, \beta)}$$

Equation 20

where $m(t_j)$ is the number of events completed at $t_j$. As before, spells censored up to and including time $t_j$ are included in the denominator. This approach to estimating the
baseline hazard in the Cox model is analogous to the product-limit estimator of Kaplan-Meier for the survivor function, except that the denominator, the number at risk, is adjusted for the covariate function. See Kiefer (1988) and Cox and Oakes (1984) for a discussion.

Figure 2: Non-Parametric Baseline Hazard Function

Figure 2 shows the estimated non-parametric baseline hazard for the 10 MBS pools. In contrast to the highly stylized log-logistic hazard function, the non-parametric baseline reflects the influence of prepayment restrictions and penalties. In particular, it captures the low probability of prepayments in years one to three, when the mortgages are in large part closed to prepayment, and the spike in prepayments at year three when mortgages
enter their prepayable or open period. Note also that there is a decline in the probability of prepayment around month 30 in anticipation of the upcoming open period. Though not reported here, this pattern of low initial prepayments followed by a spike at year three is consistent with prepayment patterns observed in NHA pools issued by other lenders, although the pick-up in prepayments very late in the security's life seems unique to Royal Trust over this sample period.
V. Valuation

In this section we show how the prepayment functions estimated previously can be incorporated into a stochastic model of the term structure of interest rates for valuation purposes, and compare the simulated model prices to actual market levels for three different mortgage-backed securities. In contrast to models discussed in the literature, we use an arbitrage-free model which provides an exact fit to the initial term structure of interest rates, and is thus guaranteed to price default-free pure discount bonds correctly.

Term Structure Model

In order to value Canadian mortgage-backed securities with the prepayment functions estimated in the previous section, we use the Ho-Lee short-rate model of the term structure of interest rates. Ho and Lee (1986) was the first of the so-called arbitrage-free models, which are structured so that they are guaranteed to price the underlying option-free, default-free securities correctly. This arbitrage-free feature is particularly important in a trading operation where one is concerned with hedging a book of options or other securities with option features against short-term movements in the market, though is not particularly helpful if one is more concerned with forecasting the future level of interest rates, since the initial shape of the term structure constrains how future-period rates can evolve.
In the Ho and Lee model, the short-term interest rate, \( r \), is assumed to follow the normal process:

\[
dr = \theta(t) \, dt + \sigma \, dz
\]

Equation 21

where \( dz \) is a Brownian motion with mean zero and variance equal to the time-step, \( dt \), and \( \sigma \) is a volatility parameter. The variance of changes in the short rate over some time interval \( \Delta t \) is thus \( \sigma^2 \Delta t \). The term \( \theta(t) \) is a time-varying parameter which ensures the arbitrage-free property. This is achieved by fitting the parameter \( \theta(t) \) to the term structure of default free pure-discount bond prices (or spot interest rates) via the following recursive relationship:

\[
\theta_k = \frac{-\ln[P((k + 1) \cdot \Delta t)]}{\Delta t^2} \left( \frac{k + 1}{\Delta t} \cdot r_0 + \frac{1}{2} (k^2 + (k - 1)^2 + \ldots + 1) \cdot \sigma^2 \cdot \Delta t \right) - \theta_1 - (k - 1) \theta_2 - \ldots - \theta_{k-1} \]

Equation 22

where \( P(k \Delta t) \) is the price of a pure discount bond of maturity \( k \cdot \Delta t \) (i.e. \( k \) intervals of length \( \Delta t \)), and \( r_0 \) is the continuously compounded short-term interest rate known at time \( t = 0 \). This equation is derived in Appendix C.

Because in practice the Government of Canada does not issue pure-discount bonds beyond the money market maturities, the parameter \( \theta(t) \) must be matched to a set of hypothetical zero-coupon bond prices estimated from the Government of Canada coupon bond curve. See, for example, McEnally and Jordan (1991) for a discussion of some of the different methods of fitting a zero-coupon or spot rate yield curve. For the

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20 The continuous-time specification of the Ho and Lee (1986) model was pointed out by Heath, Jarrow, and Morton (1990). In the original paper by Ho and Lee, the model is presented as a discreet-time binomial tree.
purposes of this report, hypothetical zero-coupon Canada prices were supplied by Scotia Capital Markets / ScotiaMcLeod Fixed Income Research.

Like most models of the term structure, the Ho and Lee model has several drawbacks worth noting. First, as with all single-factor models of the term structure, it makes the false assumption that returns on bonds of different maturities are instantaneously perfectly correlated. Second, it permits interest rates to become negative. And third, it assumes that the volatility of long term yields and short-term yields is the same, when even casual empiricism makes it clear that interest rate volatility is a (broadly) declining function of term to maturity.

**Valuation Equation**

In choosing the Ho & Lee term structure model for pricing purposes, we have implicitly assumed that the price of a mortgage-backed security depends on a single stochastic variable, the short term interest rate. If there is a second default-free security available, such as a Government of Canada bond, whose price also depends on this single stochastic variable, these two securities can be combined to form a portfolio that is instantaneously riskless. In the absence of arbitrage opportunities, this portfolio must earn the instantaneous risk-free rate of return. Thus, from the standard hedging arguments, it can be shown that the price of a prepayable mortgage-backed security under the
assumption that the short rate follows the Ho-Lee process is given by the solution to the following partial differential equation:\textsuperscript{21}

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial r} \theta(t) + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \sigma^2 + c + \pi(t) \cdot (F(t) - V) = rV$$ \hspace{1cm} \text{Equation 23}

where $V$ is the price of the security, $c$ is a (continuous) coupon payment, $F(t)$ is the face value of the security at time $t$, and $h(t)$ is the estimated prepayment hazard function. The other terms are as defined for the Ho and Lee continuous time process \cite{Ho_Lee}. Except for the fully-amortizing condition, boundary conditions would be as previously defined for Dunn and McConnell (1981). As was discussed in the literature review, the term $h(t)(F(t)-V)$ measures the capital gain or loss resulting from early prepayment. When the security is priced at a premium, exogenous prepayments driven by $h(t)$ cause the security holder to experience a capital loss, and thus drive the price of the security down closer to par. Conversely, when the security is priced at a discount, prepayments result in a capital gain, bringing the security’s price up toward par. The introduction of the exogenous payment function $h(t)$ causes mortgage-backed security cash flows to be path dependent.

As a result, equation [23] cannot be solved analytically, and instead a numerical method like finite difference must be used (for a discussion of finite difference methods, see, for example, Hull 1993 pp. 352-362). However, in the results presented here, rather than solve the partial differential equation numerically, we use a simple Monte Carlo method to simulate the stochastic short-rate process \cite{Ho_Lee} directly.

\textsuperscript{21} In calibrating the drift term $\theta$ to the initial term structure, the short rate process \cite{Ho_Lee} is already a risk-adjusted process, thus it is not necessary to adjust the drift by the market price of risk, $\lambda$, and as a result, $\lambda$ does not appear in the resulting partial differential equation. Otherwise, the derivation of the partial differential equation for valuing contingent claims in the Ho and Lee model is analogous to that presented in Appendix A.
Monte Carlo Simulation

To illustrate the Monte Carlo approach for valuing a mortgage-backed security or other security, consider the discrete-time version of the Ho and Lee process:

\[ \Delta r = \theta \Delta t + \sigma \varepsilon \sqrt{\Delta t} \]

where

\[ \varepsilon \sim N(0,1) \]

Denoting the short term rate known today by \( r_0 \), the forward rate next period is given by \( r_0 + \Delta r \), while the forward rate \( k \) periods ahead is given by \( r_0 + \Sigma_k \Delta r \). Given values for \( \theta \) and \( \sigma \), we can simulate one possible evolution for the path of short rates over time by randomly drawing values for \( \varepsilon \) from the standard normal distribution. If the future cash flows for a particular security are fully determined by the forward rates and time, then we can calculate the present value of these cash flows using the path of simulated rates. In so doing, we obtain the price of the security under one possible evolution of the short rate process. By repeating this procedure \( n \) times, we can obtain \( n \) security prices, each corresponding to one possible future path for the short-term interest rate. Since by construction each path is equally likely (probability =1/n), the estimated price of the security is a simple arithmetic average of the \( n \) simulated prices. In the results presented below, we held the number of simulation paths to 1000; though preliminary results

---

22 Rates are expressed in annual terms with continuous compounding. Note that market risk preferences are already built into \( \theta \) through the process of matching the initial term structure. If instead one were to simulate a process such as Cox, Ingersoll, and Ross (1985), the drift term would have to be adjusted by the market price of interest rate risk.
indicated that convergence to the theoretical model price is quite good after only 650 paths.\textsuperscript{23}

For the prepayment function, we use the parametric log-logistic hazard from the previous section, with parameters $\alpha = 1.8158$ and $\gamma = 0.000546$. We use a single explanatory variable, the refinancing incentive, with coefficient $\beta = 0.42298$.\textsuperscript{24} In the estimation stage, the prepayment function was assumed to be driven by the 5 year mortgage rate. For simplicity rather than realism, we approximate this in the valuation model by using a 175 basis point spread over the 5-year default-free rate. A more realistic and considerably more complex model would specify a second stochastic process for the mortgage spread, and potentially another state variable such that short term and long term default-free rates are not instantaneously perfectly correlated.\textsuperscript{25}

\textbf{Results}

Table 6 shows valuation results for three 5 year MBS originated by Royal Trust, one priced at par, the other two at a discount or premium of approximately $2.00. Prices are for settlement April 15, 1994, at which time the securities had approximately three years remaining to maturity, and thus had not yet entered their fully open period. Due to limitations in the model and the data, these results are largely illustrative, intended

\textsuperscript{23} Depending on the convergence properties of a particular security’s price, this type of procedure could take some time. There are, however, variance reduction techniques and pseudo-random number sequences that can be used to reduce the number of simulated paths.

\textsuperscript{24} Actually, for the simulation results reported here, we re-estimate the model using effective monthly rates rather than annual rates for the weighted average mortgage rate, C, and the market rate, R.

\textsuperscript{25} One could argue that the refinancing incentive should be estimated with respect to the default-free Government of Canada rate rather than the five year mortgage rate, to be consistent with the choice of a one-factor valuation model.
primarily to motivate a qualitative discussion of the model. First, we do not attempt to model partial prepayments or penalties, and estimate the full prepayment function from a relatively limited data set. Second, in order to illustrate the application of the model to premium as well as discount market prices, we value securities within sample. The results must therefore be interpreted with some caution.

The first column of the table shows the security ID as well as its coupon and maturity date. The second column shows the market price supplied by Scotia Capital Markets / ScotiaMcLeod Fixed Income Research. The remainder of the table shows results from the model under three different assumptions about the volatility of the short-term interest rate. For each volatility assumption, we report the option-adjusted spread (OAS), as well as the theoretical price from the prepayment valuation model under an OAS of zero. The OAS results converged in either three or four iterations.

<table>
<thead>
<tr>
<th>Security</th>
<th>Market Price</th>
<th>Volatility Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>100 bps</td>
</tr>
<tr>
<td>96406319</td>
<td>7.00% 8/1/97</td>
<td>98.171</td>
</tr>
<tr>
<td>96406582</td>
<td>7.75% 10/1/97</td>
<td>100.039</td>
</tr>
<tr>
<td>96405154</td>
<td>8.625% 4/1/97</td>
<td>102.196</td>
</tr>
</tbody>
</table>

Table 6: Valuation Results for MBS Under Different Volatility Assumptions

26 The option-adjusted spread is defined to be a constant spread which, when added to the short-rates in the Monte-Carlo simulation, produces a theoretical model price equal to the market price. It is measured in basis points.
At the time - April 1994 - market rates were quite volatile, since the US Federal Reserve Board was in the midst of a series of short-term interest rate hikes. For example, the absolute volatility of three month Canadian T-Bill yields over the past 30 trading days was on the order of 250 basis points, compared to an average over the previous 12 months of closer to 120 basis points. This, in and of itself, might suggest that the results reported for $\sigma = 200$ are the best representation of market conditions. However, historical volatility measures are not necessarily the best measures to use for option pricing. One reason for this is that expected future volatility need not equal historical volatility, despite the fact that most models assume volatility parameters to be constant. The second reason is that a single-factor model does not fit all areas of the yield curve equally well. A single-factor option model calibrated to the short end of the curve may not price long-dated options correctly, because it will tend to over-state the volatility of the longer part of the yield curve. The fact that an MBS is a medium term security - and that the prepayment window is largely closed until the security has been outstanding three years - suggests that the model should be calibrated to fit best in the medium-term part of the curve. Though we do not attempt to definitively answer the question of the best parameter value for $\sigma$ here, these qualitative observations do indicate that a volatility level in the 150 basis point range was probably more appropriate for MBS valuation in April, 1994 than a higher value like 200 basis points.

Setting aside the question of term-structure model calibration, the results in Table 6 can nonetheless yield some interesting insights. As expected, the OAS declines as volatility increases. The reason for this is that the option to refinance, like other interest
rate options, becomes more valuable as interest rates become more volatile. Since the holder of an MBS is effectively short the refinancing option, the theoretical value of the MBS declines, and along with it the OAS, as the value of the option increases.

In most cases, the theoretical value is greater than the market price, even though the model makes no allowance for penalty payments, which all else equal would tend to increase the theoretical value of the security. This would seem to provide support for the view that the market was not adequately valuing the pass-through of penalty interest (under the assumption that the prepayment model is correctly specified).

The third observation worth noting from the Table is that, for any given volatility assumption, as we move from the discount to the premium security, the OAS declines considerably. This suggests that, in April 1994, the market may have been undervaluing discount MBS, and overvaluing premium securities. This can also be seen by comparing theoretical and market prices across the three securities. The difference between the market prices of the premium and discount securities is just over $4.00, whereas the difference between the theoretical prices for these securities is about $2.90. One explanation for this is as follows. At the time under consideration -April 1994 - the market convention was to use a relatively static system of pricing. Prepayment assumptions would remain constant over moderately small yield movements, though would be adjusted to reflect larger market moves. As a result, prepayment assumptions were less sensitive to interest rates, at least over short horizons. Since a mortgage-backed security includes an embedded short option position, this static nature of the prepayment assumption would tend to make MBS prices more sensitive to interest rates than they would be under a more dynamic prepayment pricing scenario, which may explain why
the model shows the discount security to be relatively undervalued, and the premium
security relatively overvalued. Alternately, the relatively poor fit of the highly stylized
parametric hazard model, in particular its tendency to overstate prepayments over the first
part of a pool's life, could account for the perceived relative value of the different
securities, as could the fact that the model was estimated from relatively high coupon
pools which had their refinancing options in the money over most of the sample period.
VI. Conclusion and Possible Extensions

This paper illustrates a methodology for valuing Canadian mortgage-backed securities that explicitly takes into account the cash flow uncertainty caused by borrowers' prepayment and default behavior. It follows US studies such as Schwartz and Torous (1989) in using the proportional hazards model to estimate prepayments as a function of several explanatory variables, and then integrating this empirical prepayment function into a term structure model for valuation purposes. The approach in this paper extends the previous US studies of this subject, insofar as both parametric and non-parametric hazards are estimated, and the resulting cash flows are valued within an arbitrage-free model of the term structure.

Due to limitations in the data and the prepayment functions estimated, the model presented here is primarily an illustration of the approach, lacking sufficient realism for trading and hedging purposes. However, a much more realistic model, at least for full prepayments, could be obtained through relatively straightforward refinement of the model presented here. To begin, the model would need to be estimated for a much larger data set, covering a range of pools and refinancing environments. Issuer-specific prepayment patterns could easily be captured by including appropriate indicator variables in the covariate vector. A time-dependent refinancing variable could also be included, such as the simulated market value of the MBS in each state relative to its face value. Time-dependent prepayment penalties could be incorporated directly into the refinancing variable as a spread over the market rate. Similarly, the model could allow for a different refinancing coefficient between years one to three and years four to five in order to pick...
up the effective prepayment lock-out in the first three years. These are all relatively modest extensions to the model presented here. Estimating empirical functions for penalties and partial prepayments is a somewhat larger task.\textsuperscript{27}

The valuation results presented in this paper are within sample, and thus should not be given too much weight, though qualitatively they are consistent with expectations. Given a more complete prepayment model, it would be worth examining in a more robust fashion the ability of the prepayment model to explain market prices. In particular, given that the Canadian market is still relatively young, it would be worth examining whether the market is valuing MBS efficiently, including the pass-through of penalty interest payments. It will also be worth examining whether a relatively simple one-factor term structure model can adequately explain market prices of these medium-term securities, or whether a more complex two-factor model is required, as suggested by Brennan and Schwartz (1985). Another broad area for future research is examining how effective duration measures implied by prepayment models relate to the empirical durations of Canadian MBS derived from market data, and what this implies for hedging strategies.

\textsuperscript{27} Note that hazard models are not suited to modeling partial prepayments and penalty charges. Unlike the case of full prepayment, it is not possible to define unambiguously for partial prepayments and penalties the number of spells at risk, nor the event that terminates a spell.
Bibliography


Appendix A - Derivation of the PDE in a Single-Factor Model

In this appendix, we derive the general partial differential equation for valuing a derivative security that depends on a single stochastic variable, as well as the specific equation in Dunn and McConnell (1981) for valuing a prepayable mortgage-backed security. In the general single-factor term structure model, fixed income security prices are a function of the short term interest rate, \( r \), which follows the stochastic process:

\[
dr = m_r dt + s_r dz,
\]

Equation A-1

where \( m_r \) and \( s_r \) are the drift and volatility of the process, and could be specified as functions of \( r \) or \( t \), or simply constant. The term \( dz \) is a Brownian motion with variance equal to the time step, \( dt \).

Suppose there exist two assets whose prices depend on the short rate process above, say a bond of price \( B(r,t) \), and a mortgage of price \( M(r,t) \). It is possible to combine a long position in one with a short position in the other to form a portfolio that is instantaneously riskless, and which, in the absence of arbitrage opportunities, must earn the risk-free return.

By Ito’s lemma,\(^1\) prices for the bond and mortgage follow the processes:

\[
\begin{align*}
\frac{dB}{B} &= \left( \frac{\partial B}{\partial r} m_r + \frac{\partial^2 B}{\partial r^2} \frac{1}{2} s_r^2 \right) dt + \frac{\partial B}{\partial r} s_r dz, \\
\frac{dM}{M} &= \left( \frac{\partial M}{\partial r} m_r + \frac{\partial^2 M}{\partial r^2} \frac{1}{2} s_r^2 \right) dt + \frac{\partial M}{\partial r} s_r dz,
\end{align*}
\]

Equation A-2

The rates of return on the two securities are thus given by the following:

\[
\begin{align*}
\frac{dB}{B} &= m_B dt + s_B dz, \\
\text{and} \\
\frac{dM}{M} &= m_M dt + s_M dz,
\end{align*}
\]

Equation A-3

where

\(^1\) For an accessible discussion of Ito’s lemma, see, Hull (1993).
\[ m_B = \left( \frac{\partial B}{\partial r} m_r + \frac{\partial B}{\partial \sigma} + \frac{1}{2} \frac{\partial^2 B}{\partial \sigma^2} s_r^2 \right) / B \]

\[ s_B = \frac{\partial B}{\partial \sigma} s_r / B \]

The drift and volatility terms for the mortgage price, \( M \), are defined analogously.

A portfolio, \( P \), consisting of \( w \) units of \( B \) and \( (1-w) \) units of \( M \) earns the following return:

\[ \frac{dP}{P} = w \cdot \frac{dB}{B} + (1-w) \cdot \frac{dM}{M} \]

\[ = w \cdot (m_B dt + s_B dz) + (1-w) \cdot (m_M dt + s_M dz) \]

Equation A-4

It is possible to eliminate the risk of this portfolio by choosing \( w^* \) such that the coefficient on \( dz \) is zero:

\[ w \cdot s_B + (1-w) \cdot s_M = 0 \]

Equation A-5

Solving for \( w^* \):

\[ w^* = \frac{s_M}{s_M - s_B} \]

\[ 1 - w^* = \frac{-s_B}{s_M - s_B} \]

Since risk is (instantaneously) eliminated, the portfolio of \( B \) and \( M \) must earn the risk-free return, \( r dt \):

\[ w \cdot (m_B dt + s_B dz) + (1-w) \cdot (m_M dt + s_M dz) = r \cdot dt \]

Equation A-6

By substitution the expression for \( w^* \) into [A-6] and re-arranging, we obtain the following relationship:

\[ \frac{m_B - r}{s_B} = \frac{m_M - r}{s_M} \]

This states that the excess return per unit of risk is the same for both securities. Since these two securities were chosen arbitrarily, we can generalize to say that the excess return per unit of risk is the same for any securities whose prices are a function of the
short-rate process [A-1]. This excess return is known as the market price of (short-term) interest rate risk, and is usually denoted \( \lambda \). Thus, for any security \( i \):

\[
\frac{m_i - r}{s_i} = \lambda
\]

Equation A-7

By re-arranging, we can state the following no-arbitrage condition for any asset price that depends on the state variable \( r \):

\[
m_i - s_i \cdot \lambda = r
\]

Equation A-8

By substituting the expressions for \( m_M \) and \( s_M \) from above, we obtain the following partial differential equation for valuing the mortgage-backed security:

\[
\frac{\partial M}{\partial r}(m_r - s_r, \lambda) + \frac{\partial M}{\partial r} + \frac{1}{2} \frac{\partial^2 M}{\partial r^2} s_r^2 = rM
\]

Equation A-9

In order to obtain the specific form of the general valuation equation used by Dunn and McConnell, we need to specify functional forms for the drift and volatility parameters in [A.1]. For the Cox, Ingersoll, and Ross (1985) model of the short rate process, the drift and volatility terms \( m_r \) and \( s_r \) have the following specifications:

\[
m_r = k(\mu - r)
\]
\[
s_r = \sigma_r \sqrt{r}
\]

Equation A-10

In addition, the general equilibrium model of CIR (1985) includes a specification for the market price of risk, \( \lambda \):

\[
\lambda = \frac{qr}{\sigma \sqrt{r}}
\]

Equation A-11

Dunn and McConnell derive the partial differential equation for pricing a mortgage security under the assumption that exogenous payments are driven by a Poisson process. Brennan and Schwartz (1985) show that the same result can be achieved without the assumption that exogenous payments are stochastic. In both approaches, the rate of exogenous payment is given by \( \pi(t) \), and the unscheduled capital gain \( \pi(t)(F(t) - M)/M \), where \( F(t) \) is the face value or outstanding balance at time \( t \), enters as an additional term in the rate of return equation [A-3]. Thus, including unscheduled capital gains as well as a coupon cash flow, \( c \), the rate of return on the prepayable MBS is given by:
\[
\frac{dM + c + \pi(t) \cdot (F(t) - M)}{M}
\]

Equation A-12

The derivation of the partial differential equation then proceeds as described above. By using the return equation [A-12] and substituting equations [A-10] and [A-11] from Cox, Ingersoll, and Ross into [A-1], we obtain the Dunn and McConnell (1981) partial differential equation for valuing a prepayable MBS:

\[
\frac{\partial M}{\partial t} + (k(m - r) - qr) \frac{\partial M}{\partial r} + \frac{1}{2} \frac{\partial^2 M}{\partial r^2} \sigma^2 r + c + \pi(t) \cdot (F(t) - M) = rM
\]

Equation A-13

---

2 Aside from the unscheduled capital gain term, the resulting partial differential equation is the same as that in CIR (1985). However, its derivation presented here follows a simple partial-equilibrium argument based on zero-arbitrage between bonds, and thus differs from, or is less complete, than the general equilibrium model of CIR (1985). For a discussion, see Brown and Dybvig (1986).
Appendix B - Derivation of the PDE in a 2 State-Variable Model

In the first part of this appendix, we derive the partial differential equation for pricing a mortgage subject to default and prepayment risk. We then show how the same kind of approach can be used to obtain the partial differential equation in the Brennan and Schwartz (1985) two-state variable model. This section draws from Brennan and Schwartz (1979).

In the models of Schwartz and Torous (1992), Kau et al. (1992) and Titman and Torous (1989), it is assumed that asset prices are a function of two state variables, the short term interest rate, \( r \), and house values, \( H \). The short term rate follows the CIR (1985) process:

\[
dr = m_r dt + s_r dz_r,
\]

where

\[
m_r = k(m - r) \\
s_r = \sigma_r \sqrt{r}
\]

House prices are assumed to follow the log-normal process:

\[
dH = m_H dt + s_H dz_H
\]

Equation B-1

where

\[
m_H = (\mu - b) \cdot H \\
s_H = \sigma_H H
\]

Assume that we can trade in three assets: the house, \( H \), a discount bond, \( B = B(r,t) \), and the mortgage-backed security, \( V = V(r,H,t) \). The process for the house price is given by the equation above. From Ito's lemma, the processes for the bond, \( B \), and MBS, \( V \), are given by the following:

\[
dB = \left( \frac{\partial B}{\partial r} m_r + \frac{\partial B}{\partial r_H} m_H + \frac{1}{2} \frac{\partial^2 B}{\partial s_r^2} s_r^2 \right) dt + \frac{\partial B}{\partial s_r} s_r dz_r,
\]

\[
dV = \left( \frac{\partial V}{\partial r} m_r + \frac{\partial V}{\partial H} m_H + \frac{1}{2} \frac{\partial^2 V}{\partial s_r^2} s_r^2 + \frac{1}{2} \frac{\partial^2 V}{\partial H^2} s_H^2 + \frac{\partial^2 V}{\partial r \partial H} s_r s_H \right) dt + \frac{\partial V}{\partial s_r} s_r dz_r + \frac{\partial V}{\partial s_H} s_H dz_H,
\]

which can be written in terms of rates of return as:
\[
\frac{dB}{B} = m_B dt + s_B dz,
\]

where

\[
m_B = \left( \frac{\partial B}{\partial r} m_r + \frac{\partial B}{\partial H} m_H + \frac{\partial^2 B}{\partial^2 r^2} s_r^2 + \frac{\partial^2 B}{\partial^2 H^2} s_H^2 + \frac{\partial^2 B}{\partial H \partial r} \rho_r s_H + v \right) / B
\]

\[
s_B = \frac{\partial B}{\partial r} s_r / B
\]

\[
\frac{dV}{V} = m_v dt + s'_V dz_x + s''_V dz_H
\]

where

\[
m_v = \left( \frac{\partial V}{\partial r} m_r + \frac{\partial V}{\partial H} m_H + \frac{1}{2} \frac{\partial^2 V}{\partial^2 r^2} s_r^2 + \frac{1}{2} \frac{\partial^2 V}{\partial^2 H^2} s_H^2 + \frac{\partial^2 V}{\partial H \partial r} \rho_r s_H + v \right) / V
\]

\[
s'_V = \frac{\partial V}{\partial r} s_r / V
\]

\[
s''_V = \frac{\partial V}{\partial H} s_H / V
\]

Form a new portfolio, \( P \), by shorting \( dV/dH \) units of the house for each unit of the mortgage held such that house price risk is eliminated: \(^1\)

\[
dP = dV - \frac{\partial V}{\partial H} dH
\]

\[
= \left( \frac{\partial V}{\partial r} m_r + \frac{\partial V}{\partial H} m_H + \frac{1}{2} \frac{\partial^2 V}{\partial^2 r^2} s_r^2 + \frac{1}{2} \frac{\partial^2 V}{\partial^2 H^2} s_H^2 + \frac{\partial^2 V}{\partial H \partial r} \rho_r s_H \right) dt + \frac{\partial V}{\partial r} s_r dz_r
\]

which can be written in terms of rates of return as:

---

\(^1\) Without loss of generality, we could reverse the signs to obtain the more realistic situation of shorting the mortgage against a long position in the house.
\[
\frac{dP}{P} = m_P dt + s_P dz, \\
m_P = \left( \frac{\partial V}{\partial r} + m_r + \frac{\partial^2 V}{2 \partial r^2} s^2_r + \frac{1}{2} \frac{\partial^2 V}{\partial H^2} s^2_r + \frac{\partial^2 V}{\partial r \partial H} \alpha_r s_H \right) / p \quad \text{Equation B-4} \\
s_P = \frac{\partial V}{\partial r} s_r / p
\]

We now have two portfolios, portfolio P and the bond B, which are exposed to a single source of risk, that of the short term interest rate. Thus, we can now proceed as we did in the single state variable case of Appendix A. We construct a new portfolio Q consisting of w units of portfolio P and (1-w) units of the bond, B. The return on this new portfolio is given by the following process:

\[
\frac{dQ}{Q} = w \cdot \frac{dP}{P} + (1-w) \cdot \frac{dB}{B} \\
= w \cdot (m_P dt + s_P dz_r) + (1-w) \cdot (m_B dt + s_B dz_r) \quad \text{Equation B-5}
\]

Again, we choose w* such that the coefficient on the Brownian motion dz, is zero:

\[
w \cdot s_P + (1-w) \cdot s_B = 0
\]

to obtain:

\[
w = \frac{s_B}{s_B - s_P} \quad \text{and} \quad 1-w = \frac{-s_P}{s_B - s_P}
\]

Since risk is eliminated, the new portfolio must earn the risk-free return r dt in the absence of arbitrage opportunities:

\[
w \cdot (m_P dt + s_P dz_r) + (1-w) \cdot (m_B dt + s_B dz_r) = r \cdot dt \quad \text{Equation B-6}
\]

Substituting in the expressions for w and the drift and volatility parameters and re-arranging, we obtain the relationship from Appendix A, which states that the excess return per unit of (short-rate) risk is constant for all securities, and is denoted by \( \lambda \), the market price of interest rate risk.

\[
\frac{m_P - r}{s_P} = \frac{m_B - r}{s_B} = \lambda
\]
By substituting the expressions for $m_v$ and $s_v$ and rearranging, we obtain the following partial differential equation for valuing the mortgage-backed security, $V$:

$$
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial H} m_H + \frac{\partial V}{\partial r} + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} s_r^2 + \frac{1}{2} \frac{\partial^2 V}{\partial H^2} s_H^2 + \frac{\partial^2 V}{\partial r \partial H} s_r s_H = r V
$$

Equation B-7

For the case of exogenous prepayment and default functions $\pi$ and $\delta$, the unscheduled capital gain term $(\pi + \delta)(F(t) - V)/V$ can be incorporated in the same manner as in Appendix A. Similarly, by substituting in the expressions for $m_r$, $s_r$, and $\lambda$ given in Appendix A, we obtain the Schwartz and Torous (1992) partial differential equation.

The derivation of the partial differential equation for pricing contingent claims in the Brennan and Schwartz (1979) model is analogous to that presented above. It is assumed that prices are a function of two state variables, the short term rate, $r$, and the perpetual consol rate, $R$, which follow the stochastic processes:

$$
dr = m_r dt + s_r dz_r.
$$
$$
dR = m_R dt + s_R dz_R.
$$

Equation B-8

where

$$
m_r = (a_1 + b_1 (R - r))
$$

$$
s_r = \sigma r
$$

$$
m_R = (a_2 + b_2 r + c_2 R) \cdot R
$$

$$
s_R = \sigma R R
$$

For a consol bond, there is simple relation between price and yield:

$$
C = \frac{c}{R}
$$

Therefore, by Ito's lemma, the process for the price of a consol is given by the following:

$$
dC = \left( \frac{\partial C}{\partial R} m_R + \frac{\partial C}{\partial r} + \frac{1}{2} \frac{\partial^2 C}{\partial R^2} s_R^2 \right) dt + \frac{\partial C}{\partial R} s_R dz_R.
$$

Equation B-9

Since the partial derivatives of the consol price with respect to the state variables are known, this reduces to:

$$
dC = \left( \frac{-c}{R^2} m_R + \frac{c}{R^3} s_R^2 \right) dt - \frac{c}{R^2} s_R dz_R.
$$
The derivation proceeds as before, substituting the consol bond price, C, for the house price, H, as appropriate. As was the case above, because the second state variable is effectively a traded asset whose derivatives with respect to all the state variables are known, both the drift and price of risk for the second state variable drop out of the valuation equation. With some additional algebra, it can thus be shown that the price of a prepayable mortgage dependent on \( r \) and \( R \) must satisfy the following partial differential equation (including coupons and exogenous unscheduled capital gains) from Brennan and Schwartz (1985):

\[
\frac{\partial M}{\partial \alpha} + (m - \lambda_1 \sigma_1) \frac{\partial M}{\partial \alpha} + \frac{1}{2} \frac{\partial M^2}{\partial \alpha^2} + \frac{1}{2} \frac{\partial^2 M^2}{\partial \sigma_1^2} + \frac{1}{2} \frac{\partial^2 M^2}{\partial \sigma_2^2} + \frac{1}{2} \frac{\partial M}{\partial \alpha \partial \sigma_1} \rho_{\alpha \sigma_1} \sigma_1 \sigma_2 + \pi(X,t)(F(t) - M) = rM
\]

Equation B-10
Appendix C - Fitting the Ho & Lee Model to the Initial Term Structure of Interest Rates

In discrete-time notation, the short-term rate in the Ho & Lee model follows the stochastic process

\[ \Delta r = \theta(t) \cdot \Delta t + \sigma \Delta z \]  

\[ \text{Equation C-1} \]

Where \( \Delta z \) is a Brownian motion with mean zero and variance equal to the time step, \( \Delta t \). The Ho & Lee model is arbitrage-free, in the sense that prices of default-free securities generated by the model match quoted prices of default-free securities in the market. The is achieved via the following no-arbitrage condition, which requires that the model be able to match market prices of default-free zero-coupon bonds:

\[ E \left[ e^{-\sum_{i=1}^{k} r_i \cdot \Delta t} \right] = P((k + 1) \cdot \Delta t) \]

or, taking logs of both sides:

\[ -E \left[ \sum_{i=1}^{k} r_i \cdot \Delta t \right] = \ln(P((k + 1) \cdot \Delta t)) \]  

\[ \text{Equation C-2} \]

In order to match the initial term structure, whether of discount bond prices or spot (i.e. zero coupon) rates, we need to find an expression for the left hand side of [C-2] in terms of the parameters \( \theta(t) \) and \( \sigma \). We then equate this expression to the observed market spot rates (or prices) on the right hand side of [C-2], and solve for \( \theta(t) \). Along the way, it will be necessary find expressions for the mean and variance of the spot rate process, as explained below. We begin by developing a recursive expression for the forward rates, \( r_k \):

\[ r_0 = \frac{-\ln(P(\Delta t))}{\Delta t} \]

\[ r_1 = r_0 + \theta_1 \Delta t + \sigma \Delta z_1 \]

\[ r_2 = r_1 + \theta_2 \Delta t + \sigma \Delta z_2 \]

\[ r_k = r_0 + \left( \sum_{i=1}^{k-1} \theta_i \right) \cdot \Delta t + \left( \sum_{i=1}^{k-1} \Delta z_i \right) \cdot \sigma \]

\[ \text{Equation C-3} \]

Next, we obtain a recursive expression for the process \( \sum_k r_i \Delta t \), the bracketed term on the left hand side of C-2.
\[
\sum_{i=0}^{1} r_i = r_0 + r_1 \\
= 2r_0 + \theta_1 \Delta t + \sigma \Delta z_i \\
\sum_{i=0}^{2} r_i = r_0 + r_1 + r_2 \\
= 3r_0 + (2\theta_1 + \theta_2) \Delta t + (2\Delta z_1 + \Delta z_2) \sigma
\]

therefore

\[
\sum_{i=0}^{k} r_i \Delta t = (k + 1)r_0 \Delta t + (k \theta_1 + (k - 1) \theta_2 + \ldots + \theta_k) \Delta t^2 + (k \Delta z_1 + (k - 1) \Delta z_2 + \ldots + \Delta z_k) \sigma \Delta t
\]

Equation C-4

Before taking expectations of the process \( \sum_{k} r_i \Delta t \), we note that

\[
E(e^x) = e^{(\mu \sigma^2)} \text{ if } x \sim N(\mu, \sigma^2)
\]

Equation C-5

Thus, the key to taking the expectation of \( \sum_{k} r_i \Delta t \) is finding the mean (\( \mu \)) and variance (\( \sigma^2 \)) of the process in [C-4].

Denote the mean of \( \sum_{k} r_i \Delta t \) by \( \mu_\Sigma \), and the variance \( \sigma^2_\Sigma \).

\[
\mu_\Sigma = E\left[ \sum_{i=0}^{k} r_i \Delta t \right] \\
= (k + 1)r_0 \Delta t + (k \theta_1 + (k - 1) \theta_2 + \ldots + \theta_k) \Delta t^2
\]

Equation C-6

To derive the expression for the variance, we first define three new variables to simplify the notation:

\[
R_0 = (k + 1)r_0 \Delta t \\
\Theta_0 = (k \theta_1 + (k - 1) \theta_2 + \ldots + \theta_k) \Delta t^2 \\
Z_k = (k \Delta z_1 + (k - 1) \Delta z_2 + \ldots + \Delta z_k) \sigma \Delta t
\]

Equation C-7

The variance of process [C-4] is thus:
\[
\sigma_z = E \left[ \left( \sum_{i=0}^{k} r_i \Delta t \right)^2 \right] - \left( E \left[ \sum_{i=0}^{k} r_i \Delta t \right] \right)^2
\]

\[
= E[(R_0 + \Theta_k + Z_k)^2] - (E[R_0 + \Theta_k + Z_k])^2
\]

\[
= E[R_0^2 + 2R_0\Theta_k + 2R_0Z_k + 2\Theta_kZ_k + \Theta_k^2 + Z_k^2] - (R_0^2 + 2R_0\Theta_k + \Theta_k^2)
\]

The terms in \( R_0 \) and \( \Theta_k \) drop out of the equation, leaving:

\[
\sigma_z = E[Z_k^2]
\]

Substituting back in the expression for \( Z_k \) from [C-7]:

\[
\sigma_z = E[((k\Delta z_1 + (k - 1)\Delta z_2 + ... + \Delta z_k) \sigma \Delta t)^2]
\]

Since \( E[a\Delta z_i \cdot b\Delta z_j] = 0 \) for all \( i \neq j \), the cross-product terms drop out of the expansion of the equation above, and we obtain the final expression for the variance the process C-4:

\[
\sigma_z = (k^2 + (k-1)^2 + ... + 1)\sigma^2 \Delta t^3 \quad \text{Equation C-8}
\]

Substituting [C-6] and [C-8] into [C-5]:

\[
E \left[ e^{-\sum_{i=0}^{k} r_i \Delta t} \right] = e^{-\mu \sigma^2 \Delta t / 2}
\]

\[
= \exp(-(k + 1)r_0 \Delta t - (k\theta_1 + (k - 1)\theta_2 + ... + \theta_k)\Delta t^2
\]

\[
+ (k^2 + (k - 1)^2 + ... + 1)\sigma^2 \Delta t^3 / 2)
\]

\[
= P((k + 1) \cdot \Delta t)
\]

by the no-arbitrage condition [C.1]. Taking logs of both sides of [C.9], we solve for \( \theta_k \) to obtain a recursive expression for fitting the Ho & Lee continuous time process for the short rate to the initial term structure:

\[
\theta_k = -\ln P((k + 1) \cdot \Delta t) \Delta t^2 \frac{(k + 1)r_0}{\Delta t} + \frac{(k^2 + (k - 1)^2 + ... + 1)\sigma^2 \Delta t}{2} - (k\theta_1 + (k - 1)\theta_2 + ... + \theta_k)
\]

\[
\text{Equation C-10}
\]