UNIVERSALITY CLASSES OF MATRIX MODELS IN 4-ϵ DIMENSIONS

by

SEBASTIAN JAIMUNGAL

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Abstract

The role that matrix models, in $(4-\epsilon)$ dimensions, play in quantum critical phenomena is explored. We begin with a traceless Hermitean scalar matrix model and add operators that couple to fermions, and gauge fields. Through each stage of generalization the universality class of the resulting theory is explored. We also argue that chiral symmetry breaking in $(2+1)$ dimensional QCD can be identified with Néel ordering in two dimensional quantum antiferromagents. When operators that drive the phase transition are added to these theories, we postulate that the resulting quantum critical behavior lies in the universality class of gauged Yukawa matrix models. As a consequence of the phase structure of this matrix model, the chiral transition is typically of first order with computable critical exponents.

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Chapter 0

Introduction

This thesis is organized into five chapters. The first chapter is a review of universality and spontaneous symmetry breaking. We review how radiative corrections to the classical potential can introduce a global minima away from zero field, as first shown by Coleman and Weinberg [1]. The onset of spontaneous symmetry breaking can also be understood through the beta functions of the theory alone, this work was carried out by Amit [2] and Yamagishi [3]. We apply his work to a matrix valued field theory, and derive conditions necessary for first order behavior. The second chapter introduces the matrix field theories that are to be studied in chapter five, the beta functions and critical exponents are also calculated here. Chapters three and four discuss some other models of interest to us. In chapter three a four fermion theory with gauged color, is introduced, and it's critical exponents calculated. Chapter four reviews the connection between two lattice models, a generalized quantum Heseinberg antiferromagnet and lattice quantum chromodynamics (QCD). In chapter five the phase structure of the matrix models are discussed, and the connections between these models and those found in chapter three and four is explored. A more detailed discussion of the connection between the various chapters is given below.

In the second chapter we introduce a traceless Hermitean matrix scalar theory with internal $SU(N_F)$ symmetry. This theory contains two relevant couplings, $tr\phi^4$ and $(tr\phi^2)^2$. A generic feature of field theories with more than one coupling is that the phase transitions are typically of first order. The multidimensionality of the coupling constant space is responsible for this, simply because there are more directions for the fixed point to be unstable in. The classic example of this is the massless scalar electrodynamics studied by Coleman and Weinberg [1]. There, in the two dimensional plane of the Higgs self-coupling there are no infrared (IR) stable fixed points, and hence all phase transitions are of first order. Another example occurs in the complex matrix scalar field theory with $SU(N_L) \times SU(N_R)$ symmetry, which represents the universality class of the finite temperature chiral transition in QCD [4]. There are two coupling constants for the renormalizable interactions $tr(M^\dagger M)^2$ and $(trM^\dagger M)^2$, and when the matrices are larger than 2×2 the phase transition is first order. When the matrices are 2×2 the model is equivalent to a vector theory with one coupling constant and the phase transition is second order. Similar reasoning has been used to argue that the $(4 - \epsilon)$ dimensional traceless Hermitean matrix scalar field theory has a second order phase transition only when $N_f = 2$ and has a fluctuation-induced first order phase transition when $N_f > 2$ [5].

Two generalizations of the traceless Hermitean matrix scalar theory are also introduced in chapter two. The first stage is the introduction a Yukawa interaction to fermions. We calculate the beta functions of the theory, however, the analysis of the phase structure is left until chapter five, where we show that the existence of a non-trivial Yukawa fixed point tends to make the fixed points of the theory IR stable. This theory is shown to have the same symmetries as a four fermi theory with gauged flavor, and hints that they might lie in the same universality class. This suspicion is confirmed when one compares the critical exponents of the two theories. This work was originally carried out by [6]. One of the goals of that work was to develop a theory that lies in the universality class of $(2 + 1)$ dimensional QCD. Unfortunately this matrix model does not have the same symmetries, and hence can not describe it's critical behavior. Once a $U(N_C)$ gauge field is introduced into this generalization of the traceless Hermitean matrix theory, the symmetries are identical and the resulting theory can lie in the same universality class as QCD.

In chapter four we discuss an intriguing feature of quantum spin systems - their relationship to gauge theories. This connection was originally used to study chiral symmetry breaking in QCD, where the strong coupling limit resembles a spin system [7]. More recently, the analogy has been exploited to prove that certain gauge theories break chiral symmetry in the strong coupling limit [8, 9, 10]. It has also been used to formulate mean field theories for magnetic systems [11]. For the most part, these works use the formal similarity between a gauge theory and a spin system at the lattice distance scale. Recently it has been suggested that the analogy is much broader in that it can account for the quasi-particle spectrum and other infrared features of the two systems [12]. In this thesis evidence for the latter will be presented by discussing a common feature of the phase diagrams of 2-dimensional quantum antiferromagnets and 3-dimensional QCD. The dependence of the chiral symmetry breaking pattern on the number of flavors and colors of quarks in QCD is similar to that of the antiferromagnet where the rank of the spin algebra and the size of its representation play the same role as the number of flavors and colors, respectively. We shall also study the critical behavior associated with a chiral or Néel phase transition. Such a transition must be driven by operators which are added to the QCD or antiferromagnet Hamiltonian and which have the appropriate symmetries. We argue that these transitions fall into a universality class which can be analyzed using the epsilon expansion. In particular we argue that the universality class is that of the gauged traceless Hermitean matrix scalar theory with Yukawa coupling to fermions. In chapter five we show that in many cases the phase transitions are fluctuation induced first order ones.

Chapter 1

The Renormalization Group and Spontaneous Symmetry Breaking

1.1 Universality

Phase transitions are abundant in physics. One example is the phase transition that occurs in hot quantum chromodynamics (QCD) with more than two flavors of fermions [4]. Here chiral symmetry is broken at finite temperature as the fermions pick up a mass. This type of transition, where the free energy contains a discontinuity is known as a first order phase transition. If the free energy is continuous but has a singular derivative [1] the theory is said to undergo a second order phase transition at that point. An example of a second order transition occurs in the 2 dimensional Ising model. The Onsager[13] solution shows that the phase transition associated with this model is of second order. The phase transitions in the above two examples occurred as the result of tunning thermodynamic parameters of the theory, namely the temperature. We are, however, interested in quantum critical phenomena where one deals with transitions, typically at zero temperature, which results from the tunning of mechanical parameters in the theory, such as coupling constants or particle masses.

An important difference between first and second phase transitions is that the latter exhibit a property known as *universality*. This appears in second order transitions because near the critical point one can approximate the scaling behavior of the relevant functions as power laws. These powers depend upon the particular Hamiltonian under

[1] This need not be the first derivative, as long as some derivative contains a singularity the phase transition is known as a second order one.

study, yet if two systems have identical critical exponents (as these powers are called) then the theories exhibit the same critical behavior.

To see how universality arises consider a general Hamiltonian with a single field variable ϕ,

$$\mathcal{H}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^D x_1 \dots d^D x_n \; \phi(x_1) \times \dots \times \phi(x_n) \times \mathcal{H}_n(x_1, \dots, x_n) \tag{1.1}$$

The n-point functions of this theory are defined as,

$$W^{(n)}(x_1, \dots, x_n) \equiv \int D[\phi] \; \phi(x_1) \times \dots \times \phi(x_n) \times e^{-[\phi]} \tag{1.2}$$

The renormalization group method maps $\mathcal{H}[\phi]$ to a scale dependent Hamiltonian $\mathcal{H}_\lambda[\phi]$ such that the respective n-point functions are related as follows,

$$W_\lambda^{(n)}(x_1, \dots, x_n) = Z^{-n/2}(\lambda) W^{(n)}(\lambda x_1, \dots, \lambda x_n) \tag{1.3}$$

where $Z(\lambda)$ is a renormalization factor.

This mapping is interesting if it has a fixed point $\mathcal{H}_\lambda[\phi] \to \mathcal{H}_*[\phi]$ as $\lambda \to \infty$. Assuming such a map the n-point functions will also have a fixed point,

$$W^{(n)}(\lambda x_1, \dots, \lambda x_n) = \lim_{\lambda \to \infty} Z^{+n/2}(\lambda) W_*^{(n)}(x_1, \dots, x_n) \tag{1.4}$$

A curious relation between W_* at two different scales can be seen by introducing another scale parameter,

$$\begin{aligned} W^{(n)}(\lambda\mu x_1, \dots, \lambda\mu x_n) &= \lim_{\lambda \to \infty} Z^{+n/2}(\lambda) W_*^{(n)}(\mu x_1, \dots, \mu x_n) \\ &= \lim_{\lambda \to \infty} Z^{+n/2}(\lambda\mu) W_*^{(n)}(x_1, \dots, x_n) \end{aligned} \tag{1.5}$$

which then urges one to write the following equality,

$$W_*^{(n)}(\mu x_1, \dots, \mu x_n) = Z_*^{+n/2}(\mu) W_*^{(n)}(x_1, \dots, x_n) \tag{1.6}$$

where the fixed point renormalization constant has been introduced,

$$Z_*(\mu) \equiv \lim_{\lambda \to \infty} Z(\lambda\mu)/Z(\lambda) \tag{1.7}$$

Since the above equations are valid for all scales μ the $Z_*(\mu)$ must obey the dilatation operation, and hence form a representation of the dilatation group. Thus we can immediately infer,

$$Z_*(\mu) = \mu^{-2d_\phi} \tag{1.8}$$

for some positive constant d_ϕ. The correlation functions then have a very simple scaling behavior parametrized by d_ϕ,

$$W^{(n)}(\mu x_1, ..., \mu x_n) = \mu^{-nd_\phi} W_*^{(n)}(x_1, ..., x_n) \qquad \mu >> 1 \tag{1.9}$$

This is a remarkable result since the R.H.S. depends only on the fixed point Hamiltonian. This is what is meant by universality, after renormalization flow to a fixed point the behavior of the system depends only on the fixed point. So if two different theories flow to the same fixed point, then they are in the same universality class and hence have the same critical exponents (in this single variable example that exponent would be d_ϕ).

Recall that (1.3) leads to the Callan-Symanzik [14] equation,

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_{i=1}^{n}\beta_i\frac{\partial}{\partial g_i} + n\gamma\phi_c\frac{\partial}{\partial\phi_c}\right) W^{(n)}(x_1, ..., x_n) = 0 \tag{1.10}$$

where,

$$\frac{\mathrm{d}}{\mathrm{d}\ \ln\lambda} g_i = \beta_i(g_1(\lambda), ..., g_n(\lambda)) \tag{1.11}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\ \ln\lambda} \ln Z(\lambda) = \gamma(g_1(\lambda), ..., g_n(\lambda)) \tag{1.12}$$

These equation actually tells us under what conditions second order phase transitions can occur. Since the renormalization flow is governed by the beta functions of the theory a necessary condition is that the beta functions have an infra red (IR) fixed point. This can also be seen by integrating (1.11) to obtain,

$$\frac{\lambda}{\lambda_0} = exp\left(\int_{g_0}^{g} \frac{dg'}{\beta(g')}\right) \tag{1.13}$$

for the case of one coupling constant. This tells us that if the scale parameter , λ, is to diverge (so that a second order phase transition can occur) a necessary condition is that the beta function has a zero. The zero must be IR stable, otherwise the couplings would flow away from that point and never cause a divergence in λ. This concludes our discussion of universality.

1.2 Spontaneous Symmetry Breaking and the Coleman Wienberg Phenomena

Spontaneous symmetry breaking is a well known phenomena. In the classical world these processes are observed daily. For example, holding a pencil vertically with it's point on the table and then releasing it causes the pencil to fall. The direction in which the top points to as it falls is arbitrary, since all directions have the same free energy. However, we observe only one direction in reality. The universe has picked one state from a set of states that all correspond to the same free energy - this is what is known as spontaneous symmetry breaking. Even though the theory posses some symmetry, the ground state does not.

The situation is analogous in the quantum realm. A system of spins on a lattice with Ising type interactions have a degeneracy in the magnetization direction. But as the temperature is cooled below some critical temperature T_c the spins all align in a **particular** direction. Coleman and Weinberg found another mechanism for generating spontaneous symmetry breaking in field theories. They noticed that loop (or radiative) corrections to the classical potential can produce a minima away from the origin, and hence introduce states that lie below the classical ground state. Even at the level of one-loop corrections the onset of symmetry breaking can be realized. Phase transitions that occur in this manner are known as fluctuation induced first order transitions. Such transitions occur if the effective potential contains a global minima at a non-zero value of the classical field. If the beta functions of the theory does not vanish then any phase transition that occurs must be of first order. This is a direct consequence of the argument in the last section, which showed that a necessary condition for second order behavior is that the beta functions contain a zero.

One can see this explicitly by following the work of Coleman and Weinberg [1]. In this section we will review how the ϕ field in the massless scalar ϕ^4 theory picks up a mass, as shown by [1]. Consider the ϕ^4 Euclidean action,

$$S = \int d^D x \left\{ \frac{1}{2}(1+z_\phi)\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}z_m\phi^2 + \frac{\lambda}{4!}(1+z_\lambda)\phi^4 \right\} \tag{1.14}$$

where z_ϕ, z_m and z_λ are the wave-function, mass and coupling constant renormalization constants respectively. To obtain the effective potential we will perform a loop-wise expansion of the functional integral. Classically the ϕ^4 potential is given by,

$$V = \frac{\lambda}{4!}\phi_c^4 \tag{1.15}$$

This also corresponds to the tree level approximation to the effective potential, as it should. The next order in the semiclassical or loopwise expansion consists of polygons with arbitrary number of sides with zero momenta entering the ring, see fig.(1.1). The renormalization terms also appear here, so that the effective potential becomes,

$$V_{eff} = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}z_m\phi_c + \frac{1}{4!}z_\lambda\phi_c^4 + \int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\phi_c^2}{k^2+i\epsilon}\right)^n \tag{1.16}$$

Performing the formal sum in the integral takes care of the apparently dangerous infrared divergent terms,

$$V_{eff} = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}z_m\phi_c^2 + \frac{1}{4!}z_\lambda\phi_c^4 + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\phi_c^2}{2k^2}\right) \tag{1.17}$$

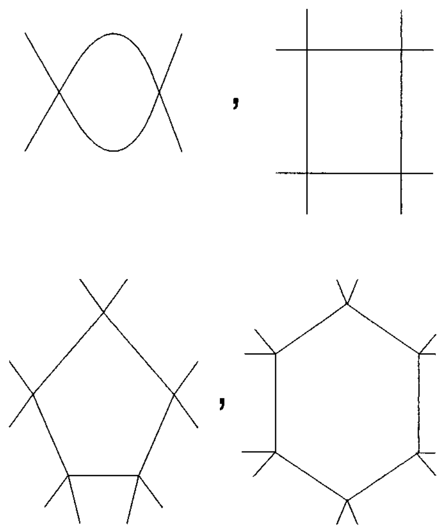

Figure 1.1: Typical diagrams occurring in the one-loop expansion of the effective potential.

A cutoff at the high momentum scale, implemented at $k^2 = \Lambda^2$, is introduced so that the still ultraviolet divergent integral can be performed. The resulting expression is,

$$V_{eff} = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}z_m\phi_c^2 + \frac{1}{4!}z_\lambda\phi_c^4 + \frac{\lambda\Lambda^2}{64\pi^2}\phi_c^2 + \frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\frac{\lambda\phi_c^2}{2\Lambda^2} - \frac{1}{2}\right) \tag{1.18}$$

As usual terms that vanish as $\Lambda \to \infty$ are ignored. It now remains to find the normalization constants. The action (1.14) contains no mass term at the tree level potential, thus, if the theory is to remain massless at the current level of the loop-wise expansion, the second derivative of the effective potential must vanish at zero field. This forces the mass normalization constant to be,

$$z_m = -\frac{\lambda\Lambda^2}{32\pi^2} \tag{1.19}$$

exactly what one would calculate using diagrammatic methods. At tree level the coupling constant is found by taking the fourth derivative of the potential at $\phi = M$, where M is some arbitrary mass scale. This does not change as the order of the loop-wise expansion increases, i.e. the coupling normalization constant is found by setting,

$$\left.\frac{d^4V}{d\phi_c^4}\right|_{\phi=M} = \lambda \qquad \Rightarrow \quad z_\lambda = -\frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda M^2}{2\Lambda^2} + \frac{11}{3}\right) \tag{1.20}$$

The effective potential with all normalization constants inserted becomes,

$$V_{eff} = \frac{\lambda}{4!}\phi_c^4 + \frac{\lambda^2\phi_c^4}{256\pi^2}\left(\ln\frac{\phi_c^2}{M^2} - \frac{25}{6}\right) \tag{1.21}$$

At tree level, the minima in the potential occurred at zero field, now there is a global minima located at a new non-zero field value $<\phi>$ given by the expression,

$$<\phi>^2 = -\frac{32}{2}\pi^2 + O(\lambda) \tag{1.22}$$

$\phi = <\phi>$ is now the ground state of the system, and at this point the second derivative does not vanish. $d^2V/d\phi^2|_{<\phi>}$ is proportional to the mass of the ϕ field thus a mass is spontaneously generated. Coleman and Weinberg argue that this minima is in fact outside the perturbative regime, hence there is no reason to believe that a first order phase transition actually occurs. However, this is attributed to the simplicity of the model. In fact they show, using the renormalization group method, that the phase transition is of second order. They also show that in a more realistic model - massless scalar electrodynamics, the new minima developed there is indeed in the perturbative regime, and a first order phase transition occurs.

1.3 Renormalization Group Flow and First Order Phase Transitions

In the previous section, first order phase transitions were shown to occur as the effective potential develops a minima away from zero field. It is, however, useful to find conditions for first order behavior which depends solely on the beta functions of the theory without directly obtaining the effective potential. Amit[2] and Yamagishi[3] noticed that a connection between the beta functions and first order behavior existed. This approach will be applied to a matrix valued field theory so that the results here can be quoted later on. We have already pointed out that if the beta functions support no IR stable fixed points, then a second order phase transition cannot take place. The still open question is then: "Does the existence of IR stable fixed points guarantee second order behavior?" We will see that there is no guarantee, instead the resulting phase transition depends upon the initial conditions of the coupling constants and where the fixed points lie in coupling constant space.

In later chapters we will be concerned with matrix valued field theories. In particular we will like the scalar matrix field, ϕ, to be traceless and Hermitean. Also the theories under consideration will have the symmetry $\phi \to U^\dagger \phi U$, with $U \in SU(N)$. In addition the possibility of interactions with other fields is left open. With these considerations there are only two renormalizable self-interactions of the ϕ field. Here is the part of the action that contains only ϕ fields,

$$S_\phi = \int d^D x \left[\frac{1}{2} \mathrm{Tr}(\partial_\mu \phi \partial_\mu \phi) + \frac{8\pi^2 \mu^\epsilon}{4!} \left\{ \frac{g_1}{N^2} (\mathrm{Tr}\phi^2)^2 + \frac{g_2}{N} \mathrm{Tr}\phi^4 \right\} \right] \tag{1.23}$$

where we have normalized the terms so that planar diagrams are suppressed in the large N limit. In determining whether spontaneous symmetry breaking occurs the symmetry breaking pattern must first be found. The first question to be answered is what class of traceless Hermitean matrices minimize the effective potential. To answer this, we must consider the form of the effective potential at the minimum point. Since flows generated by the renormalization group cannot alter the form of the self interactions, i.e. g_1 and g_2 can only pick up some dependence on a dimensionless combination of ϕ and the mass scale μ, say $t \equiv ln(\phi/\mu^{\frac{D-2}{2}})$, the effective potential at some scale is given by,

$$U(\phi, \mu, g_i) = \left\{ \frac{g_1(t)}{N^2} (\mathrm{Tr}\phi^2)^2 + \frac{g_2(t)}{N} \mathrm{Tr}\phi^4 \right\} e^{\eta(t, g_i(t))} \tag{1.24}$$

where $\eta(t, g_i)$ is some dimensionless function. Let $\bar{\phi}\,\vec{a}$ contain the eigenvalues of the ϕ field, where $\bar{\phi}$ is some fixed value, and the N-dimensional unit vector $\vec{a}$ is variable. The effective potential can then be written as,

$$U(\phi, \mu, g_i) = \bar{\phi}^4 \left\{ \frac{g_1(t)}{N^2} + \frac{g_2(t)}{N} \sum_{i=1}^{N} a_i^4 \right\} e^{\eta(t, g_i(t))} \tag{1.25}$$

The configuration which minimizes U depends on the sign of g_2: i) if $g_2 > 0$ then we must minimize $\sum_{i=1}^{N} a_i^4$. ii) if $g_2 < 0$ then $\sum_{i=1}^{N} a_i^4$ must be maximized. Notice that

$$\begin{aligned} \sum_{i=1}^{N} a_i^4 &= \left(\sum_{i=1}^{N} a_i^2 \right)^2 - 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} a_i^2 a_j^2 \\ &= 1 - 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} a_i^2 a_j^2 \\ &\equiv 1 - F(\vec{a}) \end{aligned} \tag{1.26}$$

Case i) then corresponds to maximizing $F(\vec{a})$, the configuration with minimum energy points in the "diagonal" direction,

$$\vec{a} = \begin{cases} \frac{1}{\sqrt{N-1}}(+1, -1, +1, -1, ..., 0, ..., +1, -1), & N \in \text{odd}\ ; \\ \frac{1}{\sqrt{N}}(+1, -1, +1, -1, ..., +1, -1), & N \in \text{even}. \end{cases} \tag{1.27}$$

When $g_2 < 0$ the symmetry breaking configuration corresponds to minimizing $F(\vec{a})$ and occurs along the "flat" direction,

$$\vec{a} = \frac{1}{\sqrt{2}}(+1, 0, 0, 0, ..., 0, -1), \quad \forall N > 1 \tag{1.28}$$

When looking for spontaneously broken symmetry the ϕ field is chosen to be in one of the above configurations even before flowing through the RG equations. The reason is simple: any other configuration would end up in a higher energy state, even if such a configuration develops a non-zero minima.

The renormalization group equation for the effective potential of the theory can be written in terms of the beta functions in the following manner,

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial g_i} + \gamma \phi_c \frac{\partial}{\partial \phi_c} \right) U(\phi(\mu), g_i(\mu)) = 0 \tag{1.29}$$

A dimensionless form of this equation can be formed from knowing that the mass dimension of the ϕ field is $[\phi] = (D-2)/2$, so that the effective potential has form,

$$U(\phi(\mu), g_i(\mu)) = \phi^{\frac{2D}{(D-2)}} V(t, g_i(t)) \tag{1.30}$$

where the dimensionless function $V(t, g_i(t))$ is unknown. Substituting (1.30) into (1.29) gives the desired result,

$$\left(\frac{2D}{D-2}\bar{\gamma} + \sum_{i=1}^{n} \bar{\beta}_i \frac{\partial}{\partial g_i} - \frac{\partial}{\partial t}\right) V(t, g_i(t)) = 0 \tag{1.31}$$

where,

$$\bar{\gamma} \equiv \frac{\gamma}{D/2 - (1+\gamma)} \qquad \text{and,} \;\; \bar{\beta}_i \equiv \frac{\beta_i}{D/2 - (1+\gamma)} \tag{1.32}$$

The general solution to (1.31) is well known and is given by,

$$V(t, g_i(t)) = f(g_i'(t, g_i)) \exp\left\{\frac{2D}{D-2} \int_0^t \bar{\gamma}(g_i'(x, g_i))dx\right\} \tag{1.33}$$

where $g_i'(t, g_i)$ are the solutions to the set of coupled differential equations,

$$\frac{dg_i'}{dt} = \bar{\beta}_i(g_i') \qquad \text{with i.c.} \;\; g_i'(0, g_i) = g_i \tag{1.34}$$

The function $f(g_i'(t, g_i))$ can be fixed with the knowledge of what the symmetry breaking pattern is. The boundary $t = 0$ is simply

$$f(g_i'(0, g_i)) = \frac{8\pi^2}{4!N^2}(g_1 + \alpha(g_2, N)g_2) \qquad \text{where,} \;\; \alpha = \begin{cases} 1, & g_2 > 0, \; N \in \text{even}; \\ \frac{N}{N-1}, & g_2 > 0, \; N \in \text{odd}; \\ \frac{N}{2}, & g_2 \leq 0, \; \forall N > 1. \end{cases} \tag{1.35}$$

which is seen by comparing (1.33) with (1.25). The function $f(g_i'(t, g_1))$ is then the natural extension from the $t = 0$ case, namely replace g_i with $g_i'(t, g_i)$ in (1.35). The effective potential in terms of the renormalized couplings is then,

$$U(\phi, \mu, g_i(t)) = \frac{8\pi^2}{4!N^2}\phi^{\frac{2D}{D-2}}(g_1'(t) + \alpha(g_2', N)g_2'(t))e^{\frac{2D}{D-2}\int_0^t \bar{\gamma}(g_i'(x))dx} \tag{1.36}$$

The dependence of $g_i'(t)$ on g_i is to be understood from here on. In order for a first order phase transition to occur, a global minima must occur at some non-zero ϕ. Since we have tuned the effective potential to be exactly zero at $\phi = 0$, as long as there is only one other minima it will be a global one if $U|_{<\phi>} < 0$. This implies that

$$g_1' + \alpha(g_2', N)g_2' < 0 \tag{1.37}$$

The first derivative vanishes at the minima so another condition is,

$$\frac{\partial U}{\partial \phi} = 0, \tag{1.38}$$

and finally the second derivative must be positive,

$$\frac{\partial^2 U}{\partial \phi^2} > 0. \tag{1.39}$$

so that the resulting extrema is a local minima rather than a maxima. The result is that if the flow crosses the surface (this surface will be called the stability surface),

$$\mathcal{P} = 0 \qquad \text{where } \mathcal{P} \equiv D(g_1' + \alpha(g_2', N)g_2') + \beta_1 + \alpha(g_2', N)\beta_2 \tag{1.40}$$

in the region,

$$\begin{aligned} g_1' + \alpha(g_2', N)g_2' &< 0 \\ D(\beta_1 + \alpha(g_2', N)\beta_2) + \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial g_i}(\beta_1 + \alpha(g_2', N)\beta_2) &> 0 \end{aligned} \tag{1.41}$$

the theory will undergo a first order phase transition, as the ϕ field will pick up a mass at this new non-zero global minima. These results will prove to limit the region of second order behavior even further than requiring the existence of IR stable fixed points.

1.4 The Stability Wedge and Restrictions on Fixed Points

We have found the criteria under which a matrix model displays first order behavior as the couplings are flowed from some initial values along their renormalization trajectories.

Thus far our analysis leave the initial choices of the couplings arbitrary. This is in error - the effective potential, U given by (1.24), must be bounded from below in the UV limit, and hence the initial couplings are restrained. Obviously to bound the potential 1.36 from below the couplings must lie within the region,

$$g_1 + \alpha(g_2, N)g_2 \geq 0 \tag{1.42}$$

notice that the primes on the g's are removed since this is an initial restraint. As long as the couplings start in this regime, called the stability wedge, there will be some flows that reach a fixed point if one exists. Even more can be said along these lines. If a fixed point lies outside of the stability wedge then it is possible to show that all flows must cross the stability surface. To see this notice that the stability curves are all of the form $\mathcal{P} \equiv D(g_1 + \alpha g_2) + \beta_1 + \alpha\beta_2 = 0$ and $\beta_i \approx -\epsilon g_i$ for small couplings. Then in the UV limit $\mathcal{P} \approx (D - \epsilon)(g_1 + \alpha g_2) > 0$ since the couplings must begin within the stability wedge. However at the fixed point the beta functions vanish so that $\mathcal{P} = D(g_1 + \alpha g_2)$ and if this is negative the flow must have hit the stability surface, but this is negative if and if only the fixed point lies outside of the stability wedge. This restriction of the fixed points conspire to reduce the size of the conformal window [2] in the (N_F, N_C) plane further than requiring the existence of IR stable fixed points.

[2] The conformal window refers to that region in which second order behavior is observed, as then the theory admits a massless and hence conformally invariant limit.

Chapter 2

Matrix Models

2.1 A Traceless Hermitean Scalar Matrix Model

The basic matrix model in $D = 4 - \epsilon$ dimensions is a simple generalization of the scalar ϕ^4- theory,

$$S = \int d^{4-\epsilon}x \left\{ \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{g}{4!}\mu^\epsilon\phi^4 \right\} \tag{2.1}$$

The standard one-loop calculation of the ϕ^4 beta function gives,

$$\beta = -\epsilon g + \frac{3}{(4\pi)^2}g^2 + \mathrm{O}(g^3) \tag{2.2}$$

It is easy to see that the zeros are given by,

$$g^o = 0 \qquad\qquad g^* = \frac{(4\pi)^2}{3}\epsilon \tag{2.3}$$

Obviously g^o corresponds to an ultraviolet (UV) fixed point and tells us nothing about the critical behavior of the theory. However, the slope of the beta function at the second fixed point, g^*, is positive so that g^* corresponds to an infra red (IR) stable fixed point. In the case of a single coupling constant the region of instabilities that lead to first order behavior is $g < 0$ (in analogy with (1.37)). However, since $g^* > 0$ and the sign of the beta function does not change in the interval $[0, g^*]$ one can conclude that a second order transition will occur. A massless ϕ^4 theory is then allowed where the propagators become scale independent. When this situation is generalized to matrix fields only a very limited subset of the theories will be seen to have a good conformal limit.

To generalize the action (2.1), first notice that the scalar ϕ^4 theory has the symmetry $\phi \to U\phi U^\dagger$ for $U \in U(1)$. An obvious generalization of this symmetry that should accompany the introduction of a $N \times N$ matrix field, ϕ, is $\phi \to U\phi U^\dagger$ where $U \in U(N)$. Such a theory would have the action,

$$\begin{aligned} S &= \int d^{4-\epsilon}x\Big[\frac{1}{2}\mathrm{Tr}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}\partial_\mu\Big(\mathrm{Tr}\phi\Big)\partial_\mu\Big(\mathrm{Tr}\phi\Big) + \frac{8\pi^2}{4!}\Big\{g_1\mathrm{Tr}(\phi^2)^2 + \\ &\qquad g_2(\mathrm{Tr}\phi^2)^2 + g_3\mathrm{Tr}\phi^2(\mathrm{Tr}\phi)^2 + g_4\mathrm{Tr}\phi^3\mathrm{Tr}\phi + g_5(\mathrm{Tr}\phi)^4\Big\}\Big] \qquad (2.4) \end{aligned}$$

where all terms renormalizable in four dimensions that are consistent with the symmetry are included. Although one can study this action, we do not need to consider such a complication to obtain interesting results. In addition, later on we will see that if we restrict the matrix fields to be traceless and introduce couplings to fermions the theory lies in the universality class of certain four fermi theories. Such a restriction of (2.4) has Euclidean action,

$$S = \int d^{4-\epsilon}\left[\frac{1}{2}\mathrm{Tr}\ \partial_\mu\phi\partial_\mu\phi + \frac{8\pi^2}{4!}\left\{\frac{g_1}{N^2}(\mathrm{Tr}\ (\phi^2))^2 + \frac{g_2}{N}\mathrm{Tr}\ (\phi^4)\right\}\right] \qquad (2.5)$$

Again, all terms renormalizable in four dimensions that are in harmony with the prescribed symmetry are included. Notice that in contrast to (2.1) there are now two coupling constants. The introduction of just one more degree of freedom had a drastic affect on massless scalar electrodynamics model as shown by Coleman and Weinberg [1]. It was noted in their work, that due to the existence of two coupling constants all fixed points (within the perturbative regime) were IR unstable. Generically, when the coupling constant space is multidimensional, fixed points of the flow are not IR stable and the phase transitions , if any, are first order. This applies to the present situation as well, as we will see.

As discussed in section 1.3, knowledge of the beta functions tells us about the critical behavior of the theory through the Yamagashi analysis. As such, we would like to obtain them here. However, using diagrammatics for the evaluation of the beta functions in this theory is quite cumbersome, as there are many contributions and it is difficult to know if all diagrams were included or not. Instead, we will generate the terms that appear in the effective action which are infinite as $\epsilon \to 0$, and read off the normalization constants from there.

Let $\phi = \phi^a T^a$ where the T^a's are generators of $SU(N)$ normalized so that Tr $T^a T^b = (1/2)\delta^{ab}$ and the ϕ^a's are scalar fields. By perturbing around the constant classical solution $\phi \to \phi + \phi_c$ the action can be written as,

$$S = \int d^{4-\epsilon}x \Bigg[\frac{1}{2}\phi^a(-\delta^{ab}\partial_\mu\partial_\mu \quad + \quad M^{ab})\phi^b + \frac{8\pi^2}{4!}\left\{ \frac{g_1}{N^2}(\mathrm{Tr}\ (\phi_c^2))^2 + \frac{g_2}{N}\mathrm{Tr}\ (\phi_c^4) \right\} \\ + O(\phi^4) \Bigg] \tag{2.6}$$

where,

$$\begin{aligned} M^{ab} \quad &= \quad \frac{2\pi^2}{9N}\Bigg\{ \frac{1}{N}g_1 \left(2\mathrm{Tr}[\phi_c T^a]\mathrm{Tr}[T^b\phi_c] + \frac{\delta^{ab}}{2}\mathrm{Tr}{\phi_c}^2 \right) \\ &\quad + g_2 \left(\mathrm{Tr}[{\phi_c}^2 T^{(a}T^{b)}] + \mathrm{Tr}[\phi_c T^a \phi_c T^b] \right) \Bigg\} \end{aligned} \tag{2.7}$$

and the notation $T^{(a}T^{b)} \equiv T^aT^b + T^bT^a$ is used. Integrating out the scalar ϕ-fields from (2.6) leaves the effective action,

$$S_{eff} \quad = \quad N\mathrm{TR}\ln\left[-\delta^{ab}\partial_\mu\partial_\mu + M^{ab} \right] + \frac{8\pi^2}{4!}\left\{ \frac{g_1}{N^2}(\mathrm{Tr}\ (\phi_c^2))^2 + \frac{g_2}{N}\mathrm{Tr}\ (\phi_c^4) \right\} \tag{2.8}$$

where TR means a trace in function space and indices. To one-loop the only contribution from the expansion of (2.8) is the second order term,

$$\frac{1}{2}M^{ab}\cdot M^{ba}\int \frac{d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}}\frac{1}{(p^2)^2} = \frac{1}{2}\frac{\Gamma(\epsilon/2)}{(4\pi)^{(2-\epsilon/2)}}M^{ab}M^{ba} \tag{2.9}$$

To evaluate the contraction the Fierz identity is of use,

$$T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \tag{2.10}$$

A simple application of (2.10) produces the identities,

$$\begin{aligned}\mathrm{Tr}[AT^a]\mathrm{Tr}[BT^a] &= \frac{1}{2}\left(\mathrm{Tr}[AB] - \frac{1}{N}[\mathrm{Tr}A][\mathrm{Tr}B]\right)\\ \mathrm{Tr}[AT^aBT^a] &= \frac{1}{2}\left([\mathrm{Tr}A][\mathrm{Tr}B] - \frac{1}{N}\mathrm{Tr}[AB]\right)\end{aligned} \tag{2.11}$$

for arbitrary matrices A and B. Applying these identities to the expression $M^{ab}M^{ba}$ immediately produces the normalization constants of the theory,

$$\begin{aligned}Z_1 &= 1 + \left\{\frac{N^2+7}{6N^2}g_1 + \frac{2N^2-3}{3N^2}g_2 + \frac{N^2+3}{2N^2}\frac{g_2^2}{g_1}\right\}\frac{1}{\epsilon}\\ Z_2 &= 1 + \left\{\frac{2}{N^2}g_1 + \frac{N^2-9}{3N^2}g_2\right\}\frac{1}{\epsilon}\\ Z_\phi &= 1\end{aligned} \tag{2.12}$$

The beta functions are then obtained by writing the bare couplings in terms of the coupling at some scale μ through the normalization constants,

$$g_i^o = \mu^\epsilon g_i \frac{Z_i}{Z_\phi^2} \tag{2.13}$$

Since a change of scale does not alter the bare coupling, applying the operator $\mu\partial/\partial\mu$ yields the beta functions in terms of the normalization constants:

$$0 = \mu^\epsilon\left\{\epsilon g_i\frac{Z_i}{Z_\phi^2} + \beta_1\frac{\partial}{\partial g_1}\left(g_i\frac{Z_i}{Z_\phi^2}\right) + \beta_2\frac{\partial}{\partial g_2}\left(g_i\frac{Z_i}{Z_\phi^2}\right)\right\} \tag{2.14}$$

In calculating β_i, $\beta_{j\neq i}$ are set to their $O(\epsilon)$ term, i.e. $\beta_{j\neq i} = -\epsilon g_j$. Then, keeping only terms up to $O(\epsilon)$ in expression (2.14) we obtain the beta functions,

$$\begin{aligned}\beta_1 &= -\epsilon g_1 + \frac{N^2+7}{6N^2}g_1^2 + \frac{2N^2-3}{3N^2}g_1g_2 + \frac{N^2+3}{2N^2}g_2^2\\ \beta_2 &= -\epsilon g_2 + \frac{2}{N^2}g_1g_2 + \frac{N^2-9}{3N^2}g_2^2\end{aligned} \tag{2.15}$$

The critical behavior of the action (2.5) can now be studied through the fixed points of these beta functions. A necessary condition for second order behavior is that the fixed point must be IR stable, this is achieved if the stability matrix,

$$w_{ij} \equiv \frac{\partial \beta_i}{\partial \beta_j} \tag{2.16}$$

has all positive eigenvalues at the fixed points. Pisarski [15] noted that these matrix models have IR stable fixed points only if $N \leq \sqrt{5}$. 2×2 matrices are the only matrices that satisfies this requirement , but a theory with such fields are known to be equivalent to a two-dimensional vector theory (since a 2×2 traceless matrix has only two degrees of freedom.) Thus any "real" traceless matrix theory can have only first order phase transitions. We will se that by adding a Yukawa coupling to fermions the theory is stabilized and a window of second order phase transitions is opened.

2.2 Yukawa Coupling to Fermions

We now consider the first non-trivial matrix valued field theory. As was shown above, the matrix model by itself has only a very narrow range of values in which second order behavior is observed. In this section and beyond the dimension of the matrix field will be denoted by N_F (rather than N as in the last section) indicating the number of quarks flavors in the theory. By introducing a Yukawa coupling to an $N_C \times N_F$ fermionic field, ψ^a_α, we will show that due to a non-trivial fixed point in the Yukawa coupling constant, the range of second order behavior is extended tremendously (here N_C indicates the number of colors.) The most general renormalizable theory in harmony with the symmetries, $\phi \to U\phi U^\dagger$, $\psi \to U\psi$ and $\bar{\psi} \to \bar{\psi} U^\dagger$ with $U \in SU(N_F)$ has Euclidean action,

$$\begin{aligned} S &= \int d^{4-\epsilon}x \Bigg\{ \bar{\psi}^a_\alpha \left(\delta_{\alpha\beta}\gamma_\mu\partial_\mu + \frac{\pi\mu^{\epsilon/2}}{\sqrt{N_F N_C}} y \phi_{\alpha\beta} \right) \psi^a_\beta \\ &\quad + \frac{1}{2}\mathrm{Tr}\partial_\mu\phi\partial_\mu\phi + \frac{8\pi^2\mu^\epsilon}{4!}\left[\frac{g_1}{N_F^2}(\mathrm{Tr}\phi^2)^2 + \frac{g_2}{N_F}\mathrm{Tr}\phi^4 \right] \Bigg\} \end{aligned} \tag{2.17}$$

The factors that appear along side the coupling constants are placed there in hind site so that planar graphs dominate and fermion loops are suppressed in the large N_F limit.

Notice that the color symmetry has been kept intact $\psi \to V\psi$ and $\bar{\psi} \to \bar{\psi}V^{\dagger}$.

Determining the beta functions goes much the same as before. In fact there is no alteration in any of the matrix self couplings, however some new diagrams which alter the normalization constants do appear. Previously the contribution to the normalization constants were read off through the effective potential using algebraic methods. Now, however, it is easier to obtain the new contributions directly from the diagrammatics. Since we had no need for diagrams previously, the Feynmann rules appear here for the first time. The vertices are shown in fig.(2.1), and the propagators in fig.(2.2). It is not difficult to convince oneself that the contribution to the matrix self coupling corrections alter only the g_2 vertex, and is given diagrammatically in fig.(2.3). The other contribution to the old beta functions is introduced from the scalar matrix wavefunction corrections. The relevant diagram is shown in fig.(2.4). The Yukawa coupling is also corrected by a one-loop diagram, see fig.(2.5). The fermion propagator, although not important for β_1 and β_2, is required for the calculation of β_y and also has a one-loop correction. Fig. (2.6) shows the relevant diagrams.

Relegating the details of the loop integrals to the reader, one finds that the normalization factors for this model are:

$$
\begin{aligned}
Z_1 &= 1 + \left\{ \frac{N^2+7}{6N^2} g_1 + \frac{2N^2-3}{3N^2} g_2 + \frac{N^2+3}{2N^2}\frac{g_2^2}{g_1} - \frac{3}{4N_F N_C}\frac{y^4}{g_2} \right\} \frac{1}{\epsilon} \\
Z_2 &= 1 + \left\{ \frac{2}{N^2} g_1 + \frac{N^2-9}{3N^2} g_2 \right\} \frac{1}{\epsilon} \\
Z_y &= 1 - \frac{y^2}{8N_F^2 N_C}\frac{1}{\epsilon} \\
Z_\phi &= 1 - \frac{y^2}{4N_F}\frac{1}{\epsilon} \\
Z_\psi &= 1 - \frac{N_F^2-1}{16N_F^2 N_C} y^2 \frac{1}{\epsilon}
\end{aligned} \tag{2.18}
$$

As before the matrix self coupling beta functions are found by writing the bare coupling

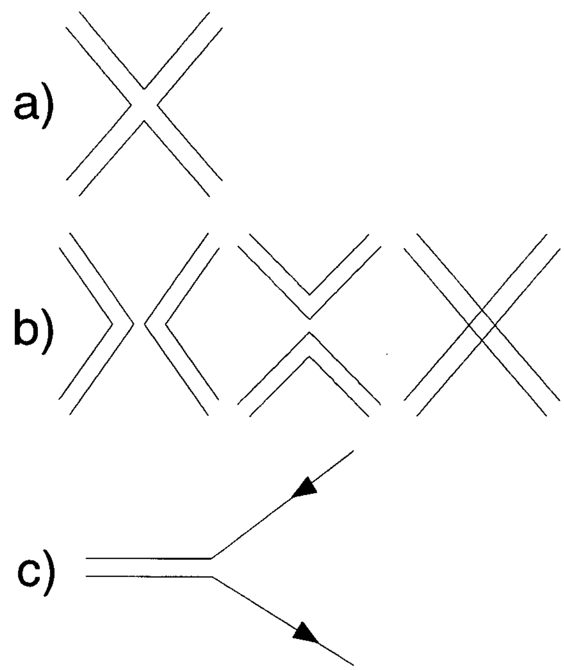

Figure 2.1: Typical vertices for the Yukawa coupled model.

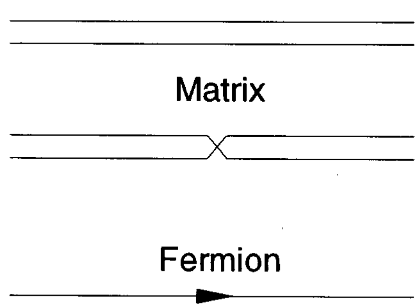

Figure 2.2: Propagators in the theory.

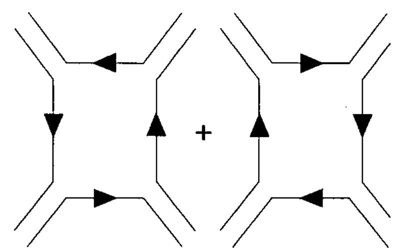

Figure 2.3: Yukawa correction to the g_2 vertex.

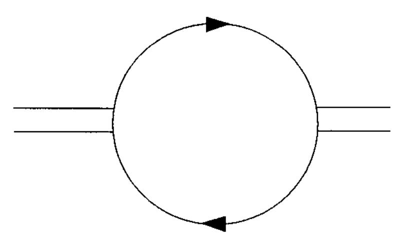

Figure 2.4: Fermion bubble correction to the scalar wavefunction.

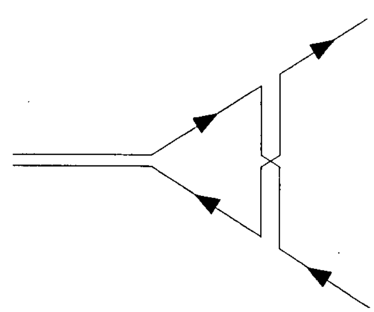

Figure 2.5: Corrections to the Yukawa interaction.

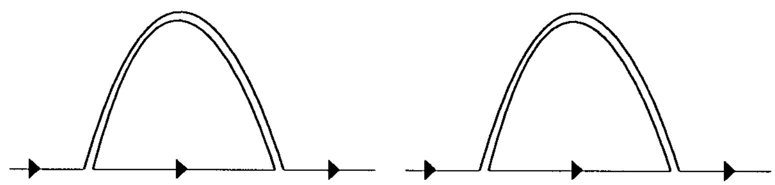

Figure 2.6: Matrix correction to the fermion wave function normalization.

in terms of the coupling at some scale μ as in (2.13), and applying the operator $\mu\partial/\partial\mu$,

$$0 = \mu^{\epsilon}\left\{\epsilon g_i \frac{Z_i}{Z_\phi} + \beta_y \frac{\partial}{\partial y}\left(g_i \frac{Z_i}{Z_\phi}\right) + \beta_1 \frac{\partial}{\partial g_1}\left(g_i \frac{Z_i}{Z_\phi}\right) + \beta_2 \frac{\partial}{\partial g_2}\left(g_i \frac{Z_i}{Z_\phi}\right)\right\} \tag{2.19}$$

β_i is then found by setting $\beta_{j\neq i} = -\epsilon g_j$, we now have a new beta to set as well, $\beta_y = -(\epsilon/2)y$. Then all goes as before keeping only up to $O(\epsilon)$ terms. The bare Yukawa coupling scales as,

$$y^o = \mu^{\epsilon/2} y \frac{Z_y}{Z_\phi^{1/2} Z_\psi} \tag{2.20}$$

and yields β_y by the above procedure. The results are,

$$\begin{aligned}
\beta_1 &= -\epsilon g_1 + \frac{N_F^2+7}{6N_F^2} g_1^2 + \frac{2N_F^2-3}{3N_F^2} g_1 g_2 + \frac{N_F^2+3}{2N_F^2} g_2^2 + \frac{1}{2N_F} y^2 g_1 \\
\beta_2 &= -\epsilon g_2 + \frac{2}{N_F^2} g_1 g_2 + \frac{N_F^2-9}{3N_F^2} g_2^2 - \frac{3}{8N_C N_F} y^4 + \frac{1}{2N_F} y^2 g_2 \\
\beta_y &= -\frac{\epsilon}{2} y + \frac{N_F^2 + 2N_F N_C - 3}{16 N_F^2 N_C} y^3
\end{aligned} \tag{2.21}$$

The second order behavior is again obtained through a study of the fixed points of the beta functions. As this analysis is much richer than that appearing in the basic matrix model a discussion will be delayed until chapter 5 where we discuss the renormalization group flows, and stability in epsilon as well.

Assuming that there is some regime of second order behavior it is possible to calculate the anomalous dimensions of the fields. Using (1.12) and writing the wave function normalization constant as $Z = 1 + \sum_{k=1}^{\infty}(a_k/\epsilon^k)$ we have,

$$\begin{aligned}
2\gamma &= \frac{\partial}{\partial ln(\mu)} lnZ = \frac{1}{1+\sum_{k=1}^{\infty}(a_k/\epsilon^k)} \sum_{k=1}^{\infty} \frac{\partial}{\partial ln(\mu)} \frac{a_k}{\epsilon^k} \\
&\approx \left(1 - \frac{a_1}{\epsilon}\right) \sum_{k=1}^{\infty} \left\{ \beta_i \frac{\partial}{\partial g_i} + \beta_y \frac{\partial}{\partial y} \right\} \frac{a_k}{\epsilon^k} \\
&= \sum_{k=1}^{\infty} \left\{ \beta_i \frac{\partial}{\partial g_i} + \beta_y \frac{\partial}{\partial y} \right\} \frac{a_k}{\epsilon^k} - \sum_{k=1}^{\infty} \frac{a_1}{\epsilon^{k+1}} \left\{ \beta_i \frac{\partial}{\partial g_i} + \beta_y \frac{\partial}{\partial y} \right\} a_j
\end{aligned} \tag{2.22}$$

However since the anomalous dimension must be finite it is easy to see that,

$$\gamma = \frac{1}{2} \left\{ s_{g_i} g_i \frac{\partial}{\partial g_i} + s_y y \frac{\partial}{\partial y} \right\} a_1 \ , \tag{2.23}$$

where $\beta_i = s_{g_i} g_i \epsilon + \ldots$ and $\beta_y = s_y y \epsilon + \ldots$. With this tool in hand the anomalous dimensions are,

$$\begin{aligned}
\gamma_\phi &= +\frac{1}{8N_F}(y^*)^2 \quad \text{and,} \\
\gamma_\psi &= +\frac{1}{32} \frac{N_F^2 - 1}{N_F^2 N_C} (y^*)^2
\end{aligned} \tag{2.24}$$

Notice there are no two pairs of (N_F, N_C) values with identical anomalous dimensions, hence every choice of N_C and N_F corresponds to a different universality class.

This theory has the possibility of being solved in the $1/N_C$ expansion, as can be seen from the fact the color indices are only contracted over the fermions. In chapter 5 we will show that there exists a four fermi theory that has $1/N_C$ expansion with identical critical exponents as the one-loop analysis carried out here. In the next section a gauge field interaction is added to the present model in the name of generalization and in hopes of stabilizing the theory further.

2.3 The Introduction of a Color Gauge Field

This will be the last stage of generalization of the ϕ^4 action (2.1). The action is once again the most general renormalizable action consistent with the same symmetries in the last section, but now a $U(N_C)$ gauge field interaction to the fermions is included. Here is the relevant action,

$$\begin{aligned} S &= \int d^{4-\epsilon}x\Big\{\bar{\psi}^a_\alpha\Big(\delta^{ab}\delta_{\alpha\beta}\gamma_\mu\partial_\mu + \frac{\pi\mu^{\epsilon/2}}{\sqrt{N_F N_C}}y\delta^{ab}\phi_{\alpha\beta} \\ &\quad + i\mu^{\epsilon/2}e_1\gamma_\mu\hat{A}^{ab}_\mu\delta_{\alpha\beta} + i\mu^{\epsilon/2}e_2\delta_{\alpha\beta}\delta^{ab}\gamma_\mu\mathrm{Tr}\hat{A}_\mu\Big)\psi^b_\beta \\ &\quad + \frac{1}{2}\mathrm{Tr}\partial_\mu\phi\partial_\mu\phi + \frac{8\pi^2\mu^\epsilon}{4!}\left[\frac{g_1}{N_F^2}(\mathrm{Tr}\phi^2)^2 + \frac{g_2}{N_F}\mathrm{Tr}\phi^4\right] \\ &\quad + \frac{1}{4}\mathrm{tr}F^2_{\mu\nu}\Big\} \end{aligned} \tag{2.25}$$

where $\hat{A}_\mu$ is the traceless part of $A_\mu \equiv A^a_\mu T^a$ with T^a's the generators of $U(N_C)$. The beta function for the non-abelian field is the just the standard QCD result, as all relevant normalization factors can be read off directly from ghost contributions and the gauge field normalization, neither of which detect the matrix field at the one-loop level. Thus (see e.g. [16]),

$$\beta_{e_1} = -\epsilon e_1 - \frac{11N_C - 2N_F}{48\pi^2}e_1^3 \tag{2.26}$$

To one loop the this gauge field does not affect the matrix self coupling normalization constants. However, the gauge field does have an affect on the Yukawa vertex through the new interactions shown in fig.(2.7). This introduces the contributions in fig.(2.8) to the normalization constant. The last beta function needed is the $U(1)$ part of the gauge field. For this calculation the diagrams shown in figures (2.9), (2.10) and (2.11) are required.

Leaving the details of the loop calculations to the reader, the set of normalization constants for the theory (excluding that for the non-abelian part of the gauge field, since we already have it's beta function) are displayed here:

$$Z_1 = 1 + \left\{\frac{N^2+7}{6N^2}g_1 + \frac{2N^2-3}{3N^2}g_2 + \frac{N^2+3}{2N^2}\frac{g_2^2}{g_1} - \frac{3}{4N_F N_C}\frac{y^4}{g_2}\right\}\frac{1}{\epsilon}$$

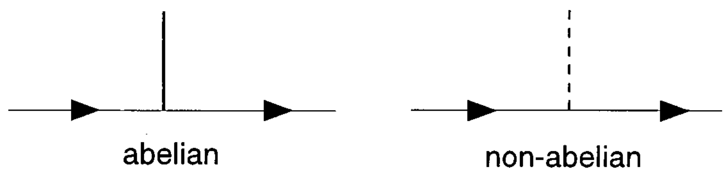

Figure 2.7: New diagrams appearing in the model due to the gauge field interactions.

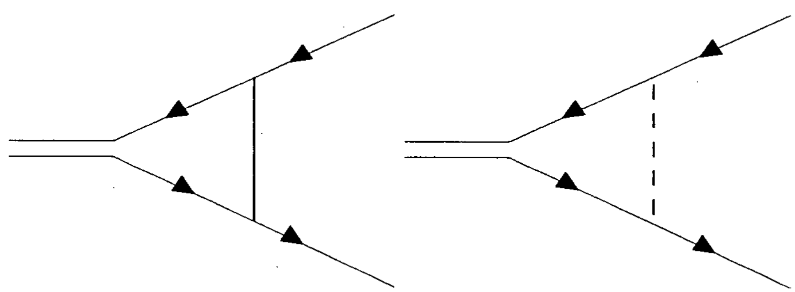

Figure 2.8: Additional corrections to the Yukawa vertex.

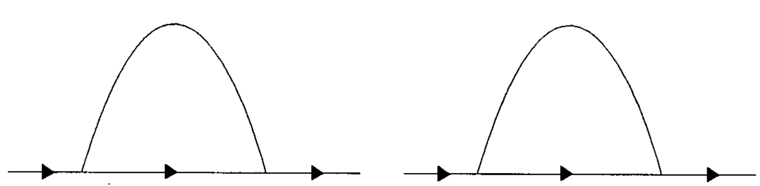

Figure 2.9: New corrections to the Fermion propagator.

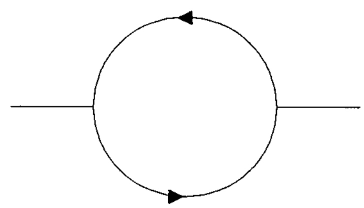

Figure 2.10: Corrections to the $U(1)$ gauge field propagator.

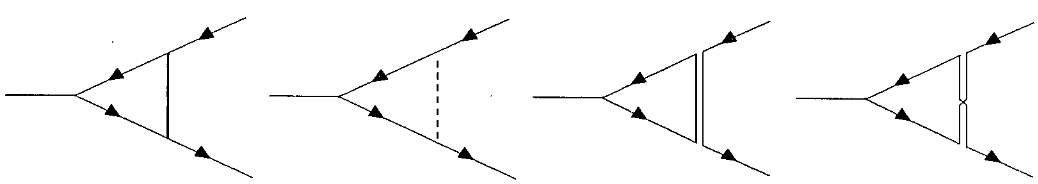

Figure 2.11: Corrections to the abelian gauge field interaction.

$$
\begin{aligned}
Z_2 &= 1+\left\{\frac{2}{N^2}g_1+\frac{N^2-9}{3N^2}g_2\right\}\frac{1}{\epsilon} \\
Z_y &= 1-\left(\frac{y^2}{8N_F^2N_C}+\frac{N_C^2-1}{2N_C}\frac{e_1^2}{4\pi^2}+\frac{e_2^2}{4\pi^2}\right)\frac{1}{\epsilon} \\
Z_{e_2} &= 1-\left(\frac{1}{16\pi^2}e_2^2+\frac{1}{16\pi^2}\frac{N_C^2-1}{2N_C}e_1^2+\frac{1}{32N_FN_C}y^2\right)\frac{1}{\epsilon} \\
Z_\phi &= 1-\frac{y^2}{4N_F}\frac{1}{\epsilon} \\
Z_\psi &= 1-\left(\frac{N_F^2-1}{16N_F^2N_C}y^2+\frac{N_C^2-1}{2N_C}\frac{e_1^2}{8\pi^2}+\frac{e_2^2}{8\pi^2}\right)\frac{1}{\epsilon} \\
Z_{A^0} &= 1-\frac{N_FN_C}{12\pi^2}e_2^2\frac{1}{\epsilon}
\end{aligned}
\tag{2.27}
$$

The beta functions are calculated by following the procedure in the previous section with the additional operator,

$$
\sum_{i=1}^{2}\beta_{e_i}\frac{\partial}{\partial e_i}
\tag{2.28}
$$

added to (2.19) to accommodate the new degrees of freedom. This yields the beta functions for the gauged Yukawa matrix model,

$$
\begin{aligned}
\beta_1 &= -\epsilon g_1+\frac{N_F^2+7}{6N_F^2}g_1^2+\frac{2N_F^2-3}{3N_F^2}g_1g_2+\frac{N_F^2+3}{2N_F^2}g_2^2+\frac{1}{2N_F}y^2g_1 \\
\beta_2 &= -\epsilon g_2+\frac{2}{N_F^2}g_1g_2+\frac{N_F^2-9}{3N_F^2}g_2^2-\frac{3}{8N_CN_F}y^4+\frac{1}{2N_F}y^2g_2 \\
\beta_y &= -\frac{\epsilon}{2}y-\frac{3}{16\pi^2}\frac{N_C^2-1}{N_C}e_1^2y-\frac{3}{8\pi^2}e_2^2y+\frac{N_F^2+2N_FN_C-3}{16N_F^2N_C}y^3 \\
\beta_{e_1} &= -\frac{\epsilon}{2}e_1-\frac{11N_C-2N_F}{48\pi^2}e_1^3 \\
\beta_{e_2} &= -\frac{\epsilon}{2}e_2+\frac{N_CN_F}{12\pi^2}e_2^3
\end{aligned}
\tag{2.29}
$$

The analysis of the phase transitions of this model will also be left until chapter 5 where a detailed discussion is given.

As a final point, the anomalous dimensions of the theory can be calculated through a generalized form of (2.23) namely,

$$\gamma = \frac{1}{2}\left\{ s_{g_i} g_i \frac{\partial}{\partial g_i} + s_y y \frac{\partial}{\partial y} + s_{e_i} e_i \frac{\partial}{\partial e_i} \right\} a_1 \ , \tag{2.30}$$

where $\beta_{e_i} = s_{e_i} e_i \epsilon + \ldots$. Assuming that an IR fixed point of the coupled beta functions exists, the critical exponents are,

$$\begin{aligned}
\gamma_\psi &= \left\{ \frac{N_F^2 - 1}{16 N_F^2 N_C} (y^*)^2 + \frac{N_C^2 - 1}{2N_C} \frac{(e_1^*)^2}{8\pi^2} + \frac{(e_2^*)^2}{8\pi^2} \right\} \qquad \text{and,} \\
\gamma_\phi &= \frac{(y^*)^2}{8 N_F}
\end{aligned} \tag{2.31}$$

where the stared couplings are evaluated at the fixed points. Notice that the boson field exponents dependence on y^* has not been altered by the introduction of the gauge field - this is a direct consequence of the lack of interaction between the matrix and gauge fields. However, the numerical value of the critical exponents differ from the non-gauged model for any choice of N_F and N_C showing that they lie in different universality classes. Also within the present model alone, each choice of (N_F, N_C) leads yields different exponents and hence lie in different universality classes.

Chapter 3

Four Fermi Models

3.1 The Gross-Neveu Model

The Gross-Neveu model consists of N_C fermions, ψ^a , $a = 1, ..., N_C$, interacting through a four-fermion vertex which obeys a $U(N_C)$ symmetry. The action is,

$$S = \int d^D x \left\{ \bar{\psi}^a \gamma_\mu \partial_\mu \psi^a - \frac{\lambda}{N_C} (\bar{\psi}^a \psi^a)^2 \right\} \tag{3.1}$$

and was first studied by Gross and Neveu [17]. It was introduced to study fermion mass generation, and also exhibits a number of other interesting phenomena such as dimensional transmutation, however we are only interested in the first.

Notice that the model has C, P, T and discrete chiral symmetry in addition to the global symmetry $\psi \to U\psi$ where $U \in U(N_C)$. These symmetries will survive even after the generalization of the next section, and hints towards the type of matrix model to study later on.

This model in solvable in the $1/N_C$ expansion. To see this firstly introduce an auxiliary field ϕ to "replace" the four fermi interaction,

$$S = \int d^D x \left\{ \bar{\psi}^a \left(\gamma_\mu \partial_\mu + \phi \right) \psi^a + \frac{N_C}{2\lambda} \phi^2 \right\} \tag{3.2}$$

This model is seen to be equivalent to the original one through the equations of motion of the field ϕ, or by carrying out the gaussian integration in the functional integral representation of the partition function. In this form the fermions are easily integrated out of the theory, leaving the effective action,

$$S_{eff} = -N_C \mathrm{TR}\, \ln(\gamma_\mu \partial_\mu + \phi) + \int d^D x \frac{N_C}{2\lambda}\phi^2, \tag{3.3}$$

here TR means a trace in function space. In eq.(3.3) $1/N_C$ takes over the roll of the Plank constant. All quantities in the theory are expandable in this parameter. In particular we are interested in scalar and fermionic propagators, as these objects lead directly to a computation of the critical exponents of the theory.

To $O(1)$ the scalar field ϕ is corrected by the fermionic bubble diagrams depicted in fig. 3.1. These diagrams contribute at this order since each bubble is a closed fermi loop and a factor of N_C originates there, while each internal ϕ propagator introduces a factor of $1/N_C$, so that the resulting contribution is $O(1)$. Summing up all of these diagrams

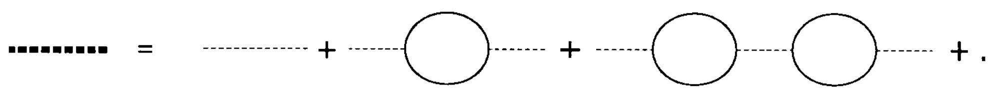

Figure 3.1: Bubble corrections to the scalar field propagator

yields the dressed up propagator,

$$\begin{aligned}
\Delta_0(p,\Lambda) &= \frac{\lambda}{N_C}\sum_{n=0}^{\infty}\left\{-(i)^2\lambda\int\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\frac{\hat{k}(\hat{k}-\hat{p})}{k^2(k-p)^2}\right\}^n \\
&= \frac{\lambda}{N_C}\left[1-\lambda\int\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\frac{\hat{k}(\hat{k}-\hat{p})}{k^2(k-p)^2}\right]^{-1} \\
&= \frac{1}{N_C}\left[\frac{1}{\lambda}-\frac{8}{(4\pi)^{D/2}}\frac{1}{(D-2)\Gamma(D/2)}\Lambda^{D-2}\right. \\
&\qquad \left. -\frac{4}{(4\pi)^{D/2}}\frac{\Gamma(1-D/2)[\Gamma(D/2)]^2}{\Gamma(D-1)}(p^2)^{\frac{D}{2}-1}\right]^{-1}
\end{aligned} \tag{3.4}$$

where in the last line we have introduced a naive UV cutoff at momentum scale $p^2 = \Lambda^2$.

From the above it is obvious that at the critical point,

$$\lambda = \lambda_c \equiv \frac{(4\pi)^{D/2}}{8}(D-2)\Gamma(D/2)\Lambda^{2-D} \tag{3.5}$$

the dressed propagator becomes scale independent, and takes on the form,

$$\Delta_0(p)^{-1} = -\frac{4\, N_C}{(4\pi)^{D/2}}\frac{\Gamma(1-D/2)[\Gamma(D/2)]^2}{\Gamma(D-1)}(p^2)^{\frac{D}{2}-1} \equiv \mathcal{A}(D)(p^2)^{\frac{D}{2}-1} \tag{3.6}$$

The first correction to the fermion propagator occurs at the next order in the $1/N_C$ ex-

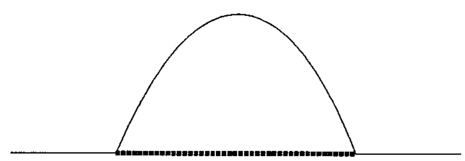

Figure 3.2: Loop correction to the Fermion self energy

pansion, and is shown diagrammatically in fig.(3.2). The correction consists of a dressed scalar connecting the external fermion lines. The dressed scalar appears since all subdiagrams in the dressed propagator are $O(1)$, so all of them must be included in the fermion self-energy to obtain the exact $O(1/N_C)$ contribution. Computing the loop integral gives the inverse propagator,

$$\begin{aligned} S^{-1}(p) &= i\hat{p} - \frac{i}{\mathcal{A}(D)}\int\frac{d^Dk}{(2\pi)^D}\frac{\hat{k}+\hat{p}}{(p^2)^{\frac{D}{2}-1}(k+p)^2} \\ &= i\hat{p}\left\{1 - \frac{\Gamma(D-1)}{2N_C D\Gamma(D/2-1)\Gamma(1-D/2)[\Gamma(D/2)]^2}\ln\left(\frac{\Lambda^2}{p^2}\right) + \text{finite}\right\} \end{aligned} \tag{3.7}$$

Once again the momentum integral was cutoff off at a scale $p^2 = \Lambda^2$. Exponentiating the R.H.S. of eq.(3.7) immediately yields the scaling dimension of the fermion propagator. Since both the scalar and fermion propagators are scale independent once the coupling is tuned to it's critical, the theory must undergo a second order phase transition if the field remains massless. The scaling dimensions[1] at this conformal point are given by,

$$\begin{aligned} S(p) &\propto \frac{1}{-i\hat{p}}|p|^{2\Delta_F+1-D} \\ \Delta_0(p) &\propto |p|^{2\Delta_B-D} \end{aligned} \tag{3.8}$$

where we have found,

$$\begin{aligned} \Delta_B &= 1+O(1/N_C) \\ \Delta_F &= \frac{D-1}{2} - \frac{\Gamma(D-1)}{2N_C D\Gamma(D/2-1)\Gamma(1-D/2)[\Gamma(D/2)]^2} + O(1/N_C^2) \end{aligned} \tag{3.9}$$

Notice that the fermionic exponent is N_C dependent, thus each choice of N_C corresponds to a theory in a different universality class.

3.2 A Four Fermi Theory with Gauged Flavor

In this section a flavor index is introduced in the Gross-Neveu model and the simple fermion coupling is changed to an $SU(N_F)$ isovector coupling. The action is,

$$S = \int d^D x \left\{ \bar{\psi}^a_\alpha \gamma_\mu \partial_\mu \psi^a_\alpha - \frac{\lambda}{N_C} \bar{\psi}^a_\alpha T^A_{\alpha\beta} \psi^a_\beta \bar{\psi}^b_\gamma T^A_{\gamma\delta} \psi^b_\delta \right\} \tag{3.10}$$

where ψ is a $N_C \times N_F$ matrix with spinor entries, and T^A are the generators of $SU(N_F)$ with normalization $\mathrm{tr} T^A T^B = \delta^{AB}/2$. This theory has all of the same symmetries of the basic one: C, P, T , and discrete chiral symmetry but has a larger internal symmetry: $SU(N_F) \times U(N_C)$. As before one can introduce an auxiliary field ϕ, this time a $N_F \times N_F$ matrix field, which decouples the four fermi interaction. The action can then be written as,

$$S = \int d^D x \left\{ \bar{\psi}^a_\alpha (\delta_{\alpha\beta} \gamma_\mu \partial_\mu + \phi_{\alpha\beta}) \psi^a_\beta + \frac{N_C}{2\lambda} \mathrm{tr} \phi^2 \right\} \tag{3.11}$$

[1]The scaling dimension, Δ, is related to the anomalous dimension, γ, via $\Delta = \mathcal{D} + \gamma$ where $\mathcal{D}$ is the canonical dimension of the field. The canonical dimensions are $\mathcal{D}_\phi = 1 - \epsilon/2$ and $\mathcal{D}_\psi = (3-\epsilon)/2$.

Notice that the ϕ field must also be traceless and Hermitean to reproduce the $SU(N_F)$ flavor interaction. Once again the fermions can be integrated out leaving the effective action,

$$S = -N_C \mathrm{TR} \ln (\gamma_\mu \partial_\mu + \phi) + \int d^D x \, \frac{N_C}{2\lambda} \mathrm{tr}\phi^2 \tag{3.12}$$

where this time TR means a trace in matrix indices as well as in function space. Since we

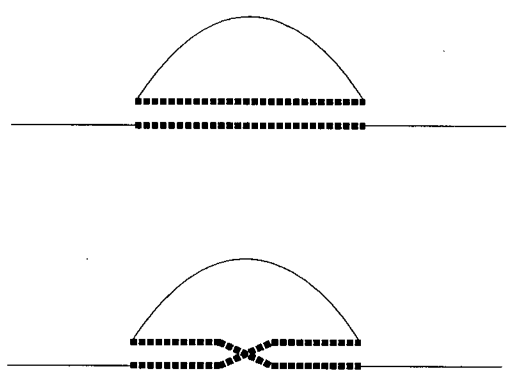

Figure 3.3: New corrections to the Fermion propagator. The lower graph is suppressed by a factor of $-1/N_F$ due to the $SU(N_F)$ group structure.

are now dealing with matrix valued fields the double line representation of propagators and vertices will be useful. The diagrams that correct the the scalar propagator are then exactly as in the Gross-Neveu model. However, the fermion propagator picks up an extra factor due to the occurrence of the two diagrams in fig.(3.3). This introduces a factor of $N_F - 1/N_F$ into it's correction (see section 2.1 for details). Thus we immediately obtain the scaling dimensions of the fields,

$$\begin{aligned} \Delta_B &= 1 + O(1/N_C) \\ \Delta_F &= \frac{D-1}{2} - \frac{N_F^2 - 1}{2N_C N_F} \frac{\Gamma(D-1)}{D\Gamma(D/2-1)\Gamma(1-D/2)[\Gamma(D/2)]^2} + O(1/N_C^2) \end{aligned} \tag{3.13}$$

If we expand this expression around four dimensions, by setting $D = 4 - \epsilon$, Δ_F becomes,

$$\Delta_F = \frac{3-\epsilon}{2} + \frac{N_F^2 - 1}{8 N_C N_F}\epsilon + O(\left(\frac{\epsilon}{N_C}\right)^2) \tag{3.14}$$

We will show in a later chapter that this four fermi theory lies in the universality class of one of the matrix valued field theories studied in chapter 2.

Chapter 4

Spin Systems and Lattice Gauge Theories

4.1 Spin systems

We will consider a generalization of the quantum Heisenberg antiferromagnet in D dimensions on a square lattice. The Hamiltonian is

$$H = \frac{1}{2g^2} \sum_{<x,y>} J(x)J(y) \tag{4.1}$$

where $J(x)$ are quantum spin operators operating on a irreducible representation of $SU(N_F)$ at sites $x, y, ...$ on the lattice and with $< x, y >$ the link between nearest neighbors. They have the algebra

$$\left[J^A(x), J^B(y)\right] = if^{ABC}J^C(x)\delta_{xy} \tag{4.2}$$

The Hamiltonian (4.1) does not have a coupling constant. The constant g^2 simply sets the units in which one measures the energies of the quantum states (or one could consider it as a unit of time). For $SU(N_F)$ spin systems large N_F corresponds to the quantum limit, as opposed to the limit of large representations which is the classical limit. In the former limit, the ground state of the antiferromagnet has an $SU(N_F)$ version of Néel order. To get a handle on this theory we will construct the spin operators explicitly.

It is convenient to construct the spin operators for a given algebra in an irreducible representation using oscillators. These could be either fermionic or bosonic. Here, we shall use fermionic oscillators, with destruction and creation operators $\psi_a^{\alpha\dagger}(x)$ and $\psi_a^{\alpha}(x)$, respectively, and the algebra

$$\left\{\psi_a^\alpha(x), \psi_b^{\beta\dagger}(y)\right\} = \delta_{ab}\delta^{\alpha\beta}\delta(x-y) \tag{4.3}$$

We shall call the indices $a, b, \ldots = 1, \ldots N_C$ "color" indices and $\alpha, \beta, \ldots = 1, \ldots, N_F$ the "flavor" indices. The reason for this nomenclature will become clear shortly when we discuss lattice gauge theory. The spin operators in the Lie algebra of $SU(N_F)$ are

$$J^A(x) \;=\; \psi_a^{\alpha\dagger}(x) T_{\alpha\beta}^A \psi_a^\beta(x) \tag{4.4}$$

where the fundamental representation Hermitean matrix generators T^A of $SU(N_F)$ obey the algebra

$$\left[T^A, T^B\right] = i f^{ABC} T^C \tag{4.5}$$

and have the normalization condition

$$\mathrm{Tr}\left(T^A T^B\right) \;=\; \frac{1}{2}\delta^{AB} \tag{4.6}$$

The spin operators, $J^A(x)$, then obey the commutation relations given in (4.2), and the space on which these generators operate is the Fock space which is created by operating creation operators $\psi_a^{\alpha\dagger}(x)$

$$\psi_{a_1}^{\alpha_1\dagger}(x_1)\psi_{a_1}^{\alpha_1\dagger}(x_1)\ldots|0\rangle \tag{4.7}$$

on the empty vacuum which obeys

$$\psi_a^\alpha(x)\;|0\rangle \;=\; 0 \;\;,\; \forall x, a, \alpha \tag{4.8}$$

For any N_C, this Fock space carries a reducible representation of the algebra. An irreducible representation is obtained by projecting onto a subspace of the Fock space. This is accomplished by imposing a constraint. The representation with a rectangular Young tableau with k rows of N_C boxes is gotten by imposing the condition of gauge invariance under the $U(N_C)$ transformation which is generated by

$$\mathcal{G}_{ab}(x)\ |\text{phys.}\rangle \equiv \left(\sum_{\alpha=1}^{N_F} \psi_a^{\alpha\dagger}(x)\psi_b^\alpha(x)\right)\ |\text{phys.}\rangle = \delta_{ab}\ k\ |\text{phys.}\rangle \tag{4.9}$$

The "physical states", $|\text{phys.}\rangle$ span the irreducible representation of the spin algebra given by the Young Tableau with N_C columns and k rows of boxes. The constraint operators obey the local algebra of $U(N_C)$,

$$[\mathcal{G}_{ab}(x), \mathcal{G}_{cd}(y)] = (\delta_{ae}\delta_{bc}\delta_{df} - \delta_{ad}\delta_{ce}\delta_{bf})\,\mathcal{G}_{ef}(x)\ \delta_{xy} \tag{4.10}$$

and commute with the Hamiltonian,

$$[\mathcal{G}_{ab}(x), H]\ =\ 0 \tag{4.11}$$

and the "observables",

$$\left[\mathcal{G}_{ab}(x), J^A(x)\right]\ =\ 0 \tag{4.12}$$

They generate the gauge transformation,

$$\psi_a^\alpha(x)\ \rightarrow\ U_{ab}(x)\psi_b^\alpha(x) \tag{4.13}$$

and the Heisenberg antiferromagnet in this formalism is gauge invariant.

Antiferromagnets of this type were studied by Read and Sachdev [18] using semiclassical methods. Their analysis was in two dimensions ($D = 2$) and the only free parameters are the integers N_C and N_F. $N_C >> N_F$ is the classical limit of large representations, where the classical Néel ground state is stable with the staggered spin order parameter

$$\mu_{ab}(x) = (-1)^{\sum_i x_i} < \sum_{\alpha=1}^{N_C} \psi_{a\alpha}^\dagger(x)\psi_{b\alpha}(x) > \tag{4.14}$$

On the other hand, the limit $N_F >> N_C$ is the quantum limit where fluctuations are important and the system is in a spin disordered state. For both N_C and N_F large, they find a line of second order phase transitions in the (N_C, N_F) plane at $N_F = \text{const.} \cdot N_C$

where the constant is a number of order one. These results are somewhat insensitive to the value of k.[1]

4.2 Hamiltonian lattice gauge theory

In this Section, we shall review the Hamiltonian lattice formulation of QCD in $D+1$ dimensions. We shall see that the relationship between the antiferromagnet and QCD is a very close one. This is a summary of the work reported in ref. [9] which maps the strong coupling limit of lattice QCD onto the antiferromagnet with Hamiltonian (4.1) and in particular irreducible representations of the flavor algebra. The degrees of freedom of lattice QCD are the gauge fields which are unitary operators $U_{ab}(xy)$ and color electric fields which are Hermitean operators $E^{ab}(xy)$, both which live on links $<xy>$ of the lattice and transform under the adjoint action of the gauge group as

$$U(xy) \to V_x U(xy) V_y^\dagger \quad , \quad E(xy) \to V_x E(xy) V_x^\dagger \tag{4.15}$$

and obey the reflection conditions,

$$U(yx) = U^\dagger(xy) \quad , \quad E(yx) = -U^\dagger(xy)E(xy)U(xy) \tag{4.16}$$

We shall assume that the gauge group is $U(N_C)$. There are also quark fields $\psi_a^\alpha(x)$ which live on sites, x and transform under the fundamental representation of the gauge group

$$\psi_\alpha(x) \to V(x)\psi_\alpha(x) \tag{4.17}$$

and which also transform under the fundamental representation of a global $SU(N_F)$ flavor group

$$\psi_a^\alpha(x) \to g_{\alpha\beta}\psi_a^\beta(x) \quad , \quad g \in SU(N_F) \tag{4.18}$$

[1]In fact, they considered a slightly more general case than we have described here where there are different representations on even and odd sublattices. Even in that case, their results are insensitive to the representations on each sublattice.

The QCD Hamiltonian contains three terms, the quark kinetic energy and the electric and magnetic energies:

$$H = \sum_{<x,y>} \left(\psi^{\dagger}_{a\alpha}(x) U^{ab}(xy) \psi_{b\alpha}(y) + h.c. + \frac{e^2}{2} \sum_{A=1}^{N_C^2} \left(E^A(xy) \right)^2 \right) + \\ + \frac{1}{2e^2} \sum_{\square} \mathrm{tr} \left(\prod_{\square} U + \prod_{\square} U^{\dagger} \right) \tag{4.19}$$

where the first sum over links $< xy >$ is the quark kinetic and total electric energies, respectively and the second over plaquettes $\square$ is the magnetic energy. The lattice regularization of the quark kinetic energy uses staggered fermions [19]. The electric fields $E(xy)$ are Lie algebra valued operators and can be expanded in terms of the generators of $U(N_C)$,

$$E(xy) = \sum_{A=0}^{N_C^2-1} E^A(xy) T^A \tag{4.20}$$

with T^0 the (unit matrix) generator of U(1) and T^A are the generators of $SU(N_C)$. The electric energy in the Hamiltonian is the sum over gauge group laplacians which act on the color group degrees of freedom associated with each link. The magnetic term is the Wilson energy function for a gauge field.

The gauge fields and electric field operators satisfy the algebra

$$\left[E^A(xy), E^B(zw) \right] = i f^{ABC} E^C(xy) \delta(xy, zw) \tag{4.21}$$

$$\left[E^A(xy), U(wz) \right] = U(xy) T^A \, \delta(xy, zw) \quad . \tag{4.22}$$

The Hamiltonian is supplemented by the Gauss' law constraint, which we impose as a physical state condition,

$$\tilde{\mathcal{G}}^{ab}(x)|\text{phys.}\rangle \equiv \left[\sum_{y \in \mathcal{N}(x)} E^A(xy) T^A_{ab} + \sum_{\alpha=1}^{N_F} \psi^{\dagger}_{a\alpha}(x) \psi_{b\alpha}(x) \right] \ |\text{phys.}\rangle \\ = \delta_{ab} N_F/2 \ |\text{phys.}\rangle \tag{4.23}$$

and which enforces gauge invariance of the physical states. Here the first summation is over all links one of whose endpoints is the site x. This term is a latticization of the covariant divergence of the electric field.

Staggered fermions have a relativistic continuum limit when their density is 1/2 of the maximum that is allowed by Fermi statistics, in this case $N_C N_F/2$ per site. Furthermore, the quark kinetic energy term in the Hamiltonian must have phases which produce an effective $U(1)$ magnetic flux π per plaquette [19, 9]. That the latter fact is necessary in order to obtain a relativistic spectrum in the continuum limit is easy to see if one considers the naive latticization of the Dirac Hamiltonian which has the correct lattice spectrum, but also has fermion doubling,

$$h = \sum_{x,i} \left(i\psi^{\dagger}(x)\alpha_j\psi(x+j) - i\psi(x)\alpha_j\psi(x-j) \right) \tag{4.24}$$

where α_i are the Dirac matrices. This operator describes a spinor with energy spectrum

$$E(k) = \sum_i \sin k_i \tag{4.25}$$

This dispersion relation has small energies where $k_i \sim 0$ and where $k_i \sim \pi$. There are 2^D such combinations, resulting in a fermion multiplicity of 2^D for each component of the spinor in (4.24). This multiplicity can be reduced by the spin diagonalization method. This method begins with the observation that the naively latticized Dirac Hamiltonian (4.24) resembles an ordinary lattice hopping problem where the Dirac matrices can be thought of as a background gauge field. This background gauge field can be diagonalized by a "gauge" transformation. This gives a number of independent copies of the staggered fermion Hamiltonians, one for each of the original components of the spinor. Choosing one component gives staggered fermions. These still have a multiplicity 2^D in the continuum limit.

That diagonalization is possible is a result of the fact that the curvature of the Dirac matrix background gauge field is a constant and is diagonal[10],

$$\alpha_i \alpha_j \alpha_i^{-1} \alpha_j^{-1} = -1 \tag{4.26}$$

and there is a gauge in which the "gauge field" α_i is diagonal. In order to obtain the appropriate phases in the weak coupling continuum limit, we have chosen the sign of the third, magnetic term in the Hamiltonian so that it is minimized by the configuration of $U(N_C)$ gauge fields with the property $< \prod_\Box U >= -1$. The constraint of half-filling is enforced by (4.23). If the lattice is 2 dimensional, the naive continuum limit yields 2+1-dimensional QCD with gauge group $U(N_C)$ and N_F species of massless four component fermions. In 3 dimensions, there are 4 species of 4 component fermions. In both cases, the (chiral) flavor symmetry of the continuum limit is greater than that of the lattice theory.

On the lattice, for example, staggered fermions do not have any continuous chiral symmetry. On the other hand, they do have a discrete remnant of chiral symmetry (translation by one site) which forbids explicit fermion mass terms [9, 10]. A fermion mass term is a staggered density operator.

$$\bar{\psi}(x) T^A \psi(x) \sim (-1)^{\sum x} \psi^{\alpha\dagger}(x) T^A_{\alpha\beta} \psi^\beta_a(x) = \sum_{\alpha\beta} T^A_{\alpha\beta} \mu_{\alpha\beta}(x) \tag{4.27}$$

Thus, the antiferromagnetic order parameter and the order parameter for chiral symmetry breaking with a flavor-vector condensate are identical.

In fact, in the strong coupling limit, the problem of finding the ground state of lattice QCD is identical to that of solving the generalized antiferromagnet with Néel order playing the role of chiral symmetry breaking. The argument of [9] can be summarized as follows: The strong coupling limit, $e^2 \to \infty$ suppresses fermion propagation (since the fermion kinetic term in the Hamiltonian is subdominant). In the leading approximation, the Hamiltonian is minimized by the states which contain as little electric field as possible and which are compatible with the gauge constraint (4.23). When N_F is even [20], it is possible to solve Gauss' law with $E^A = 0$. The occupation number of each site is $N_C N_F/2$ and $< (-1)^x \psi^\dagger_{\alpha a} \psi_{\alpha a} >= 0$ in this state. This is a highly degenerate state - any gauge invariant state with $N_F N_C/2$ fermions has the same energy. Because they are required to be color singlets, this is the same set of states as occurs in the antiferromagnet when $k = N_F/2$, i.e. in the representation of $SU(N_F)$ whose Young tableau has N_C columns and $N_F/2$ rows. Furthermore, to resolve the degeneracy, one must diagonalize the matrix of perturbations. These are non-zero only at second order and the diagonalization problem is equivalent [9, 10] to solving for the ground state of the antiferromagnet Hamiltonian (4.1) with $\frac{1}{g^2} = t^2/e^2$. Finally, since the order parameters are identical, the Néel ordered states of the antiferromagnet correspond to chiral symmetry breaking states of QCD.

Thus, the infinite coupling limit of QCD is identical to the antiferromagnet. A main difference between QCD with finite coupling and the antiferromagnet is that QCD contains electric and gauge fields which allow a fermion kinetic energy and still retain gauge invariance, whereas in the antiferromagnet, the fermions are not allowed to move. One could regard the corrections to the strong coupling limit of QCD as the addition of degrees of freedom and gauge invariant perturbations in the antiferromagnet which allow fermion propagation. In fact, ref. [12] suggests even a stronger correspondence, that the additional degrees of freedom are generated dynamically.

Chapter 5

Nature of the Phase Transitions

5.1 The Yukawa Matrix Model and Four Fermion Theories

In this section we will show that when a Yukawa interaction to fermions is introduced in the Hermitean matrix theory of section 2.1, the existence of a non-trivial fixed point tends to make the phase transitions second order. The Yukawa matrix model action is given by (2.17) and is re-displayed here,

$$\begin{aligned} S &= \int d^{4-\epsilon}x \Bigg\{ \bar{\psi}^a_\alpha \left(\delta_{\alpha\beta} \vec{\gamma}.\vec{\nabla} + \frac{\pi \mu^{\epsilon/2}}{\sqrt{N_F N_C}} y \phi_{\alpha\beta} \right) \psi^a_\beta \\ &\quad + \frac{1}{2} \mathrm{Tr} \vec{\nabla}\phi . \vec{\nabla}\phi + \frac{8\pi^2 \mu^\epsilon}{4!} \left[\frac{g_1}{N_F^2} (\mathrm{Tr}\phi^2)^2 + \frac{g_2}{N_F} \mathrm{Tr}\phi^4 \right] \Bigg\} \end{aligned} \tag{5.1}$$

The beta functions for the theory were derived in section (2.2) and are given in (2.21). A study of the second order behavior requires knowledge of the fixed points of the beta functions. The Yukawa coupling fixed points are easily found,

$$y^0 = 0 \qquad (y^*)^2 = \frac{8 N_F^2 N_C}{N_F^2 + 2 N_F N_C - 3} \epsilon \tag{5.2}$$

Second order behavior can be observed if the beta functions support an IR stable fixed point. This is turn requires that the stability matrix $w_{ij} \equiv \partial\beta_i / \partial g_j$ ($g_3 \equiv y$) has all positive eigenvalues at the relevant point. A fixed point of the coupled system (2.21) using y^0 could be UV stable but never IR stable, since β_y has negative slope there. The non-trivial fixed point y^* must then be used when looking for IR stable fixed points. The beta functions for the matrix self couplings are then reduced to,

$$\beta_1 = \left(\frac{(y^*)^2}{2N_F} - \epsilon \right) g_1 + \frac{N_F^2 + 7}{6 N_F^2} g_1^2 + \frac{2N_F^2 - 3}{3 N_F^2} g_1 g_2 + \frac{N_F^2 + 3}{2 N_F^2} g_2^2$$

$$\beta_2 = \left(\frac{(y^*)^2}{2N_F} - \epsilon\right) g_2 + \frac{2}{N_F^2} g_1 g_2 + \frac{N_F^2 - 9}{3N_F^2} g_2^2 - \frac{3}{8N_C N_F}(y^*)^4 \tag{5.3}$$

at this point. The existence of a non-trivial Yukawa fixed point has introduced a constant term in β_2 (with respect to the Hermitean matrix model, see (2.15) for the beta functions.) This serves to push the gaussian fixed point $(g_1^o, g_2^o) = (0, 0)$, of the basic matrix model, to some non-trivial value. The constant term steamed from the introduction of a box diagram (fig.(2.3)) to the correction of the g_2 vertex. This diagram is not a planar graph and is therefore suppressed by factors of N_F. Consequently in the limit $N_F \gg 1$ [1] the Yukawa matrix model reduces to the basic model since,

$$\lim_{N_F \to \infty} \frac{(y^*)^2}{2N_F} = 0 \qquad \text{and,} \quad \lim_{N_F \to \infty} \frac{3}{8N_C N_F}(y^*)^4 = 0 \tag{5.4}$$

In the basic model second order behavior was found to occur only if $N_F \leq \sqrt{5}$, as a result the large N_F limit produces a theory that undergoes a first order phase transition. However, in the opposite limit [2], $N_C \gg 1$, the Yukawa interaction adds new structure. Under such circumstances the non-gaussian fixed points of the simple matrix theory are stabilized by the Yukawa term. It is trivial to show that in the large N_C limit, an IR stable fixed point of the coupled system, (5.3), occurs at,

$$\begin{aligned} g_1^* &= -\frac{18(N_F^2+3)}{N_C^2}\epsilon + O(1/N_C^3) \\ g_2^* &= +\frac{6N_F}{N_C}\epsilon + O(1/N_C)^2) \\ (y^*)^2 &= 4N_F\epsilon + O(1/N_C) \end{aligned} \tag{5.5}$$

The matrix self couplings are suppressed by factors of N_C and are consequently consistent with a one-loop calculation. In this limit the scaling dimensions[3] of the bosonic and

[1] This is known as the quantum limit in the case of the generalized Heisenberg antiferromagnet

[2] This is the limit of large representations in the antiferromagnet, and corresponds to the classical limit

[3] The scaling dimension, Δ, is related to the anomalous dimension, γ, via $\Delta = \mathcal{D} + \gamma$ where $\mathcal{D}$ is the canonical dimension of the field. The functions γ were calculated previously and are given by (2.24).

fermionic fields are,

$$\begin{aligned}
\Delta_\phi &= 1 + O(1/N_C) \\
\Delta_\psi &= \frac{3-\epsilon}{2} + \frac{N_F^2 - 1}{8N_C N_F}\epsilon + O(1/N_C^2)
\end{aligned} \tag{5.6}$$

We now understand the quantum and classical limits of this theory, it is possible to obtain results for intermediate number of colors and flavors only through computer calculations. In [21] the analysis was carried out, however the yamagishi requirements for first order behavior were neglected, as such the region of second order behavior as claimed by [21] is slightly larger than the actual regime. As the arguments in section 1.4 show, a theory can show second order behavior only if the coupled system of equations (2.21) admit a IR stable fixed in the stability wedge given by (1.42),

$$\begin{aligned}
g_1 + g_2 \geq 0 \ , &\quad g_2 > 0, \ N_F \in \text{even} \\
g_1 + \frac{N_F}{N_F - 1} g_2 \geq 0 \ , &\quad g_2 > 0, \ N_F \in \text{odd} \\
g_1 + \frac{N_F}{2} g_2 \geq 0 \ , &\quad g_2 \leq 0, \ \forall N_F > 1
\end{aligned}$$

Once this is kept in mind the numerical results of [21] are modified slightly, but the remaining arguments are valid. The critical relation between N_C and N_F for second order behavior is shown in fig.(5.1).

In addition we have found that varying epsilon does not alter the character of the renormalization flow. The flow diagrams scale with epsilon (as dictated by a one-loop calculation), however the stability surface does not. Figure 5.2 show some flows of all three couplings from the UV limit near the origin to either the IR stable point shown, or off to infinity. Only the projection down to the (g_1, g_2) coupling space is shown so that lines that appear to cross in the diagram do not do so in the full space. The stars, circles and crosses show the points where that particular flow crossed the stability surface in the region that indicates first order behavior (see (1.41) and related ones) for

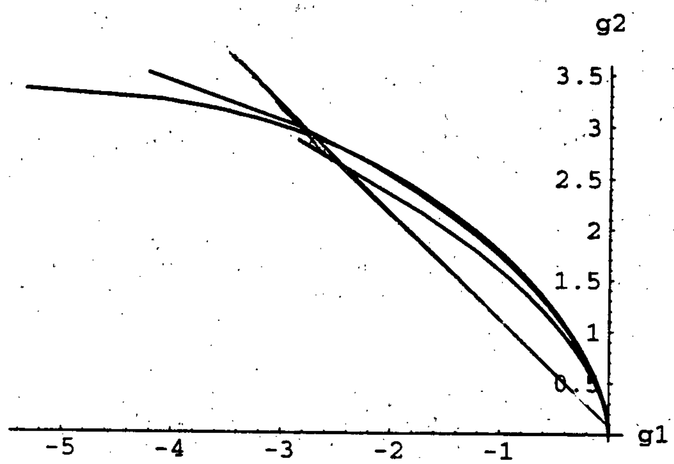

Figure 5.1: Each line corresponds to a fixed choice of N_C, as one moves from the lower region to the top N_F is decreased from infinity down to it's lowest allowable value, $N_F^{crit}(N_C)$ for a second order phase transition to occur.

$\epsilon = 1$, 0.5 and 0.25 respectively. Notice that all flows that lead to first order behavior do so regardless of the value of epsilon, the points at which the flows hit the stability surface simply move in closer to the stability wedge as $\epsilon \to 0$, but no curve that leads to second order behavior suddenly hits the stability surface as epsilon is varied. This suggests that the critical behavior is insensitive to the choice of ϵ.

From this point on choose the pair (N_C, N_F) such that the corresponding theory allows second order phase transitions. With this aside we will to discuss the universality class of this matrix valued field theory. This model was built as a generalization of ϕ^4 theory, where the symmetry $\phi \to U\phi U^\dagger$, with $U \in U(1)$ was generalized to $U \in SU(N_F)$ and a Yukawa interaction to fermions was introduced. in the present model there are alot more symmetries: C, P, T, discrete chiral symmetry and global $SU(N_F) \times U(N_C)$ symmetry. These symmetries are shared by another theory, the four fermi theory described by the

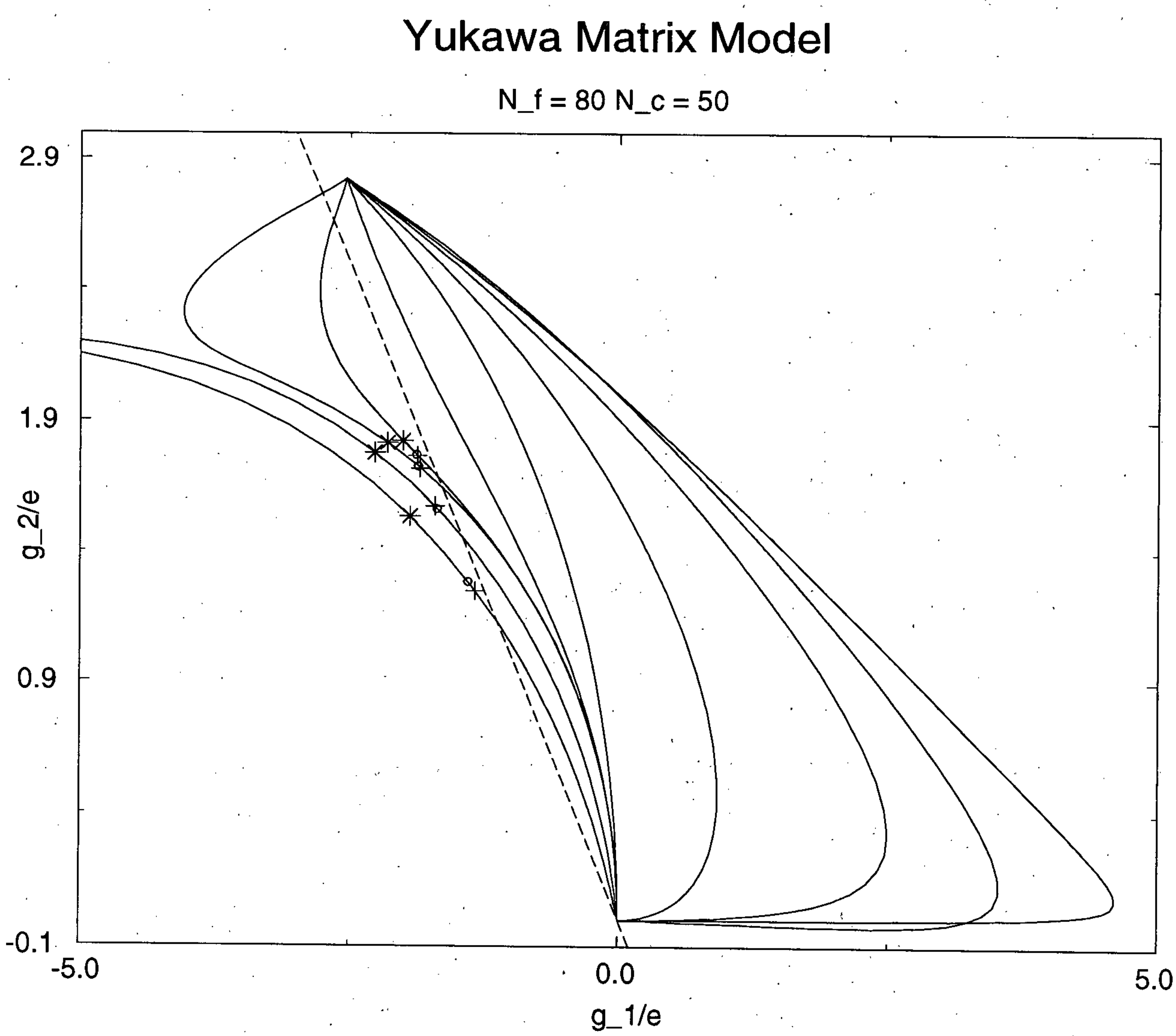

Figure 5.2: These are the renormalization trajectories projected down to the coupling constant space (g_1, g_2). The dotted line represents the boundary of the stability wedge. The stars, circles, and crosses indicates where that particular flow hit the stability surface with $\epsilon = 1$, 0.5, and 0.25 respectively.

action,

$$S = \int d^D x \left\{ \bar{\psi}^a_\alpha \bar{\gamma}_\mu \partial_\mu \psi^a_\alpha - \frac{\lambda}{N} \bar{\psi}^a_\alpha T^A_{\alpha\beta} \psi^a_\beta \bar{\psi}^b_\gamma T^A_{\gamma\eta} \psi^b_\eta \right\} \tag{5.7}$$

where $\alpha, \beta = 1, ..., N_F$, a, b $= 1, ..., N_C$, ψ is a $N_C \times N_F$ matrix of four component Dirac spinors and T^A are the generators of $SU(N_F)$ with the standard normalization. The fact that this theory obeys the same symmetries as the Yukawa matrix model suggests that they might lie in the same universality class. As the study in section 3.2 showed, the critical exponents coincided exactly with those of the present theory, to this order in N_C (compare (5.6) with (3.13 and (3.14).) Thus the connection between the matrix model and the four-fermi theory has been established.

Numerical calculations have shown that for N_F large, the critical number of colors is given by $N_C^* \approx 0.27 N_F$. For intermediate values the results are consistent with those obtained from equivalent four-fermi theories [22]. One can speculate whether the upper critical value of $N_F \approx 3.7 N_C$ for the existence of chiral symmetry breaking in the four-fermi theory is related to the upper critical $N_F \sim$ conts. $\times N_{color}$ for the existence of chiral symmetry breaking in $(2+1)$ dimensional QCD [23]. A correspondence of this type was the intention of [24]. However, one cannot naively identify $U(N_C)$ with the color group, since if QCD is confining $U(N_C)$ symmetry is absent and if it is not confining then massless glouns should contribute to the critical behavior. In the next section a theory in which the identification can be made is discussed.

5.2 The Chiral Phase Transition and The Gauged Yukawa Matrix Model

It has been observed that, in both 2+1-dimensional QED [25] and QCD [23], there exists a critical number of flavors such that if $N_F < N_F^{\text{crit.}}$ the model breaks chiral symmetry spontaneously and if $N_F > N_F^{\text{crit.}}$ the theory is in a chirally symmetric, deconfined phase. For large N_F and for large number of colors N_C the equation of the critical line is approximately

$$N_F - \frac{128}{3\pi^2} N_C = 0 \tag{5.8}$$

A heuristic argument for this behavior is that when $N_F >> N_C$ internal gluon exchanges and the gluon self-coupling are suppressed by factors of N_C/N_F. Resummation of leading order diagrams, which are chains of bubbles, produces an effective interaction which falls off like $1/r$, rather than the tree level $\ln |r|$. The weak coupling of order N_C/N_F and mild infrared behavior of this resummed theory result in a chirally symmetric, de-confined phase. When N_F is small, the effective coupling is large and can generate a condensate, which is already seen in QED [25]. In fact, in QCD, when $N_F << N_C$ all planar diagrams contribute to processes, making the effective interaction string-like [26] and the theory is in a confining and chiral symmetry breaking phase [1]. Numerical simulations [27] of 3-dimensional QED support this scenario with $N_F^{\text{crit.}} \sim 4$.

Mass operators for basic 2-component fermions in 2+1 dimensions are pseudoscalars and break parity explicitly [28]. Massless 2+1 dimensional QCD with an odd number of flavors of 2-component fermions is afflicted with the parity anomaly [29] which generates a parity violating Chern-Simons term and also fermion mass term by radiative corrections. With an even number of flavors, there exists a parity and gauge invariant regularization and QCD is the 2+1-dimensional analog of a vector-like gauge theory in 3+1 dimensions. In particular a kind of chiral symmetry can be defined. It is known that, in this case, parity cannot be broken spontaneously [30] and therefore to study chiral symmetry breaking it is necessary to seek parity conserving mass operators. Following [25, 23, 5] we shall use N_F species of 4-component fermions. The flavor symmetry of massless QCD in this case is actually $SU(2N_F)$. We will add operators to the action which reduce the symmetry to $SU(N_F)$, for example, the gauged Nambu-Jona Lasinio (NJL) model with four-fermion interaction,

$$S = \int d^3x \left(\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}\gamma_\mu D_\mu \psi + \frac{\lambda}{2} \left(\bar{\psi} T^A \psi \right)^2 \right) \tag{5.9}$$

where T^A is a generator of $SU(N_F)$ in the fundamental representation. Notice this is just the gauged form of the four-fermi theory (5.7). The 4-fermi operator, which is renormalizable in the $1/N_C$ expansion [6], can drive the chiral phase transition with condensate $\phi^A =< \bar{\psi} T^A \psi >$. The results of [25, 23] indicate that if $N_F < N_F^{\rm crit}$ the order persists even when $\lambda = 0$. Gauged NJL models, when analyzed by solving the gap equation [31], exhibit second order behavior at a surface in the space (N_F, N_C, λ). Our analysis will indicate that for a large range of parameters fluctuations make this transition first order. Our results do not apply to the hypothetical case where N_F or N_C are varied to drive the transition [32].

In chapter 4 we have reviewed the arguments that show that the strong coupling limit of (2+1) dimensional QCD is equivalent to the generalized Heisenberg antiferromagnet on a 2 dimensional lattice. One common feature of these theories is that besides the external parameters, N_F and N_C, they have no free parameters. In these theories once N_F and N_C are given, either the vacuum is symmetric or the symmetry is spontaneously broken. One could imagine adding operators such that if their coupling constants are varied, it can induce the chiral transition. We conjecture that these transitions fall into a universality class which can take into account all such modifications, as long as they respect the symmetries of the theory. We argue that the universality class is described by the $4 - \epsilon$ dimensional Euclidean field theory given by 2.25 and redisplayed here,

$$\begin{aligned} S &= \int d^{4-\epsilon}x \Big\{ \bar{\psi}^a_\alpha \Big(\delta^{ab}\delta_{\alpha\beta}\gamma_\mu\partial_\mu + \frac{\pi\mu^{\epsilon/2}}{\sqrt{N_F N_C}} y \delta^{ab}\phi_{\alpha\beta} \\ &\quad + i\mu^{\epsilon/2} e_1 \gamma_\mu \hat{A}^{ab}_\mu \delta_{\alpha\beta} + i\mu^{\epsilon/2} e_2 \delta_{\alpha\beta}\delta^{ab}\gamma_\mu \mathrm{Tr}\hat{A}_\mu \Big) \psi^b_\beta \\ &\quad + \frac{1}{2}\mathrm{Tr}\partial_\mu\phi\partial_\mu\phi + \frac{8\pi^2\mu^\epsilon}{4!} \left[\frac{g_1}{N_F^2} (\mathrm{Tr}\phi^2)^2 + \frac{g_2}{N_F}\mathrm{Tr}\phi^4 \right] \end{aligned}$$

$$+\frac{1}{4}\mathrm{tr}F_{\mu\nu}^2\Big\} \tag{5.10}$$

The evidence that (5.10) describes the universality class comes from the work of the last section, where the gauge fields are absent. Recall that the anomalous dimensions of the fermion and matrix fields were identical to leading order in $1/N_C$ and ϵ to those of a $2 < D < 4$ dimensional four-fermi theory. However, that model suffered from a lack of $U(N_C)$ gauge invariance, and as such could not represent the universality class of $2+1$ dimensional QCD. The model (5.10) does however have all the required symmetries, and we conjecture that it represents the universality class of lower-dimensional four-fermi theories with $U(N_C)$ gauge invariance. We will show that, as a consequence, the chiral phase transition is a fluctuation induced first order transition for a large range of values of (N_C, N_F). When it is second order, the critical exponents are computable and are presented here.

In the last section we showed that the introduction of a Yukawa coupling to the Hermitean matrix model opened a window in (N_C, N_F) space for second order phase transitions. When two more couplings, to a gauge field, is introduced we will show that the window closes (but not completely). This is to be expected since at each fixed point there is now an entire plane of directions in which instabilities can develop. This result can be shown explicitly by repeating the analysis of the last section.

The fixed points of the beta functions for this model have a similarity to the Yukawa model. In particular, all but the matrix self couplings can be trivially removed from the coupled system (2.29). The zeros of β_{e_1} and β_{e_2} can be solved independently and allows the potentially IR stable non-zero solutions,

$$\begin{aligned}
(e_1^*)^2 &= \frac{24\pi^2}{2N_F - 11N_C}\epsilon \qquad \text{and,} \\
(e_2^*)^2 &= \frac{6\pi^2}{N_C N_F}\epsilon
\end{aligned} \tag{5.11}$$

Using these zeroes in β_y gives the potentially IR stable Yukawa coupling fixed point,

$$(y^*)^2 = \left(1 - 9\frac{N_C - \frac{11}{2N_F}}{11N_C - 2N_F}\right)\frac{8N_F^2 N_C}{N_F^2 + 2N_F N_C - 3}\epsilon \tag{5.12}$$

Notice that in the large N_F limit the result obtained without the gauge couplings is reproduced, however large N_C introduces an extra numerical factor of $\frac{2}{9}$. This difference is a result of the gauge fields coupling to the color indices of the matrix field and not the flavor. Since the beta functions for the matrix self couplings are identical in the gauged model and non-gauged model, the IR fixed points are obtained through the solution of the coupled equations (5.3) where the y^* is now the one appearing in (5.12).

In 4 dimensional QCD if $N_F < 5.5N_C$ the theory is asymptotically free, however in this regime e_1^* corresponds to an UV stable fixed point and not an IR one. The entire system is then IR unstable in the asymptotically free region. To obtain IR stable solutions, one must move into the non-asymptotically free region. This sets a lower bound for our conformal window. Recall that the arguments in section 1.4 forced the fixed point to remain in the stability wedge,

$$g_1 + \alpha(N_F, g_2)g_2 > 0 \tag{5.13}$$

for second order transitions to occur (where $\alpha(N_F, g_2)$ is given by (1.35).) Let us consider the case $\alpha(N_F \in \text{even}, g_2 > 0) = 1$ so that the symmetry breaking pattern is the one where half the eigenvalues of the $< \phi >$ groundstate all have equal magnitude but half are positive and the other half negative. With this symmetry breaking pattern in mind we must impose the condition $g_1 > -g_2$. By setting $g_1^* = -g_2^*$ exactly, (5.3) can be used to obtain a critical relation between N_F and N_C which separates those theories that allow second order transitions from those that do not. The two equations for g_1^* are,

$$\begin{aligned} 0 &= \left(\frac{(y^*)^2}{2N_F} - \epsilon\right) g_1^* + \frac{11}{3N_F^2}(g_1^*)^2 \\ 0 &= -\left(\frac{(y^*)^2}{2N_F} - \epsilon\right) g_1^* + \frac{N_F^2 - 15}{3N_F^2}(g_1^*)^2 - \frac{3}{8N_C N_F}(y^*)^4 \end{aligned} \tag{5.14}$$

Writing $\gamma = N_F/N_C$ and taking the limit $N_C \to \infty$, one obtains the constraint,

$$\gamma < 8.3 \tag{5.15}$$

for a solution to exist. Figure 5.3 shows how the fixed points in the (g_1, g_2) space vary with N_C and N_F. Notice that even though IR fixed points exists for $N_F \geq 8.3N_C$, they correspond to fixed points that have no flows which miss the stability surface, and all lead to first order phase transitions. Also, the flows in this model, like the Yukawa model, are insensitive to ϵ in the domain $0 < \epsilon < 1$. Near $\epsilon = 1$ the coupling constants are large and one would not expect the one-loop approximation that we have used to be accurate there.

One can also find the critical exponents for large N_F and N_C,

$$\begin{aligned} \Delta_S &= 1 - \frac{2\gamma^2 - 11\gamma - 18}{(2\gamma - 11)(\gamma + 2)}\frac{\epsilon}{2} \\ \Delta_F &= \frac{3}{2} - \frac{2\gamma^2 - 15\gamma - 50}{(2\gamma - 11)(\gamma_2)}\frac{\epsilon}{4} \end{aligned} \tag{5.16}$$

where the bound $5.5 < \gamma < 8.3$ is imposed by asymptotic freedom and the above arguments. An important note can be made here, in the asymptotic free regime of the theory, where $N_F < 11N_C/2$, there should be nonperturbative behavior associated with confinement which is inaccessible to our computation. So that second order behavior in the region $\gamma < 5.5$ is not entirely ruled out.

If is possible to obtain some physical results from this analysis. In particular, the physical quantum spin j antiferromagnet corresponds to $N_F = 2$ and $N_C = 2j$. With $N_C = 1$ ($j = 1/2$) the non-abelian field must be removed, and the resulting theory has no IR fixed point, indicating a first order transition. For $N_F = 2$, $N_C \geq 2$ (i.e. $j \geq 1$) the theory is in the asymptotically free regime and hence these antiferromagnets cannot be analyzed by these techniques. We speculate that confinement is associated with a nonperturbative IR fixed point of the gauge coupling. In that case, since we have noticed

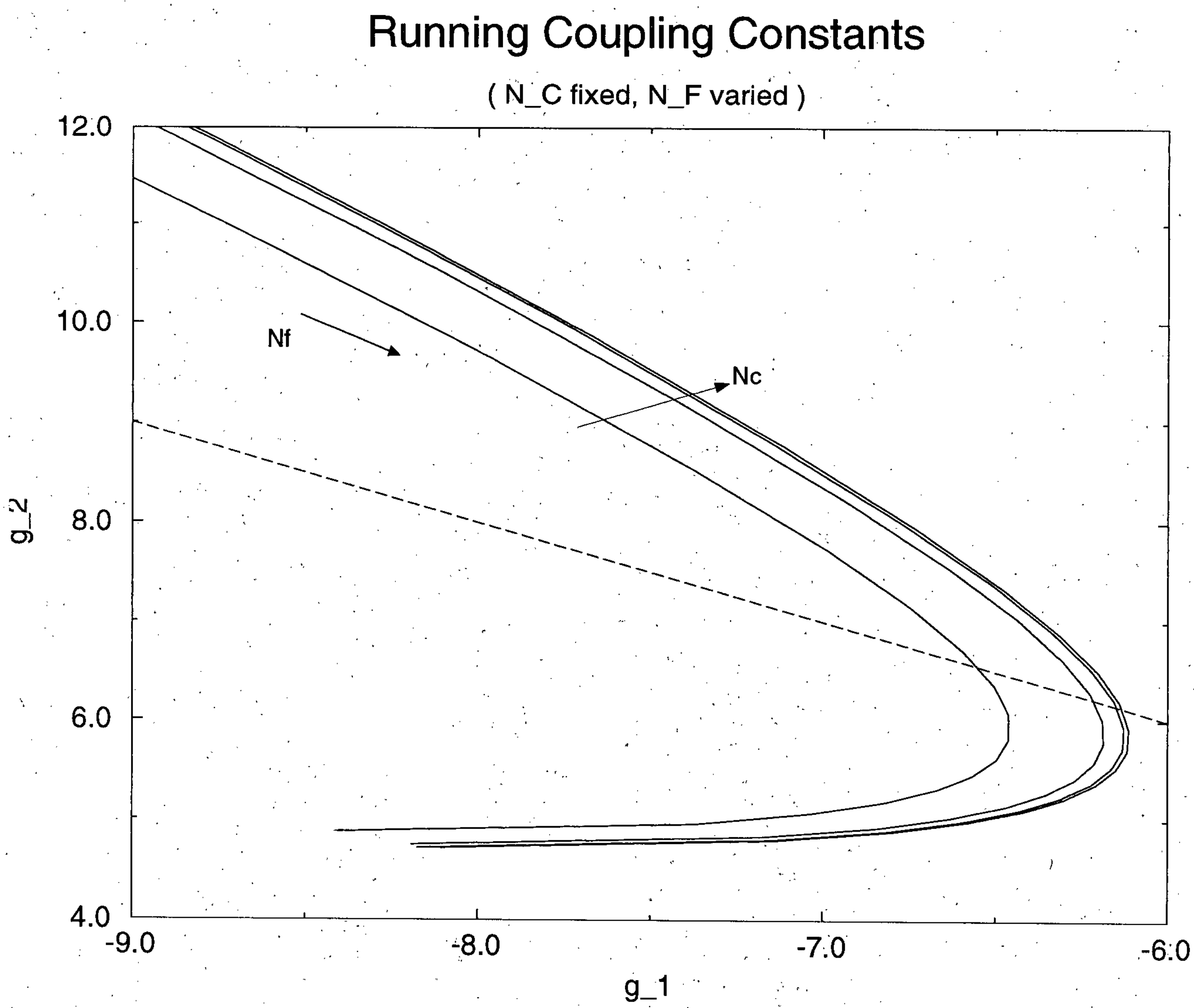

Figure 5.3: Each line corresponds to a fixed choice of N_C, as one moves from the top left region to the bottom, N_F is increased from $11/2N_C$ up to it's lowest allowable value. The dotted line indicates where $g_1 + g_2 = 0$, and also specifies the upper critical $N_F^{crit}(N_C)$ for a second order phase transition to occur.

that non-trivial fixed points tends to stabilize the fixed points, it is likely that these antiferromagnets would have a second order transition.

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