ONE-DIMENSIONAL SOLITON DYNAMICS IN THE PRESENCE OF A PINNING POTENTIAL

By

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Abstract

The dissipative effect of (what could be modeled as) a pinning center on the low-amplitude motion of a quasi-one-dimensional ferromagnetic soliton (domain wall) at temperature $T = 0$ is investigated. The method of collective coordinates is used to eliminate the problem of the zero-frequency eigenmode and the question is, subsequently, cast into a Caldeira-Leggett form to calculate the parametrized spectral density function. It is shown that this function equals zero below a specific frequency with the consequence that magnons, in the presence of a domain wall, have no dissipative effect at $T = 0$ on a slow moving wall trapped by a soft pinning center.

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Benoit Leduc

Feast of Saint John of the Cross,

December 14th, 1995.

To Jeannie and Dr. Eu

Apart from me you can do nothing.

John 15:5

If any one shall not be ashamed to assert that, except for matter, nothing exists; let him be anathema.

The Vatican Council, Session 3, Canon 2, April 24th, 1870.

Chapter 1

Introduction

Ever since Descartes hinted at a world view based on a materialistic and mechanical reductionistic model,[1] natural philosophers of the modern era have attempted to explain complex phenomena as being nothing *more* than the sum of more fundamental self-existent ones occurring at a microscopic level regulated by some laws, inferred or simply given axiomatically. In contrast with most Scholastics of the Middle-Ages, for whom, Aristotelian as they were, four general kinds of causes were recognized — namely: 1) the material cause (the scholastic *causa materialis*), which provided the passive receptacle on which the causes act; 2) the formal cause (*causa formalis*), which contributed the essence, idea, or quality of the thing concerned; 3) the motive force of efficient cause (*causa efficiens*) which was the external compulsion that bodies had to obey; and 4) the final cause (*causa finalis*) as the goal to which everything strove and which everything served — modern science would be concerned only with the efficient causes as the ontological pattern of being and becoming. Although the sciences in the nineteenth and twentieth centuries came to recognize this restriction as inappropriate for the faithful description and explanation of natural phenomena — by introducing, for instance, the notions of

[1] As is well known, Descartes's secret philosophy, expounded in *Le Monde* (1633), which he did not dare publish, was almost entirely materialistic; it was only his public philosophy that was rationalistic to the extent to which reason did not conflict with religious dogma [1, 2]. Descartes considered the living as machines which could entirely be described by matter and motions [2, 3]. Motion, as confirmed by Newton's laws, was for him - as for Spinoza, Bruno, and Leibniz - self-caused and self-moving. Indeed, Spinoza went even further by asserting categorically that *substance* was not only self-subsistent but also self-existent, self-moving, and self-caused.

reciprocal interaction, fields,[2] and statistical determined events — very few scientists did, and still to this day, do support the existence of organizational levels whose behaviors are different from and *inherently* irreducible to the superposition of their constitutive elements and their accompanying laws.

Materialistic reductionism is appealing to reason owing to its simplicity and its exclusion of supernumerary transcendental principles. Physicists were among the first to wholeheartedly sell their souls over to this model permitting them to have control over phenomena which, until then, were clouded in an awe-inspiring mist of mystery. Poets would therefore be relegated to singing the odes of love and leaving Nature to those new mechanists.

At the end of the nineteenth century, various results, such as the blackbody ultraviolet catastrophe, based on the established mathematical models of classical physics of the time, were in outright conflict with the experiments which they proposed to explain. These eventually paved the way for a new theory, essentially microscopic: quantum mechanics. As this latter model dealt with more fundamental units of matter, many physicists, imbued as they were with reductionistic ideas, suggested that the classical world should entirely be explained within its framework. However, that was without reckoning with the powerful personality of Niels Bohr and his idealistic Copenhagen interpretation of quantum mechanics which quickly quelled the ambition of these reductionists.

[2] Although Newton did introduce those last two notions in *Principia* with his third law of motion and theory of gravitation, they were hardly ever recognized or understood by the scientific community until much later. (However, one must note that his notion of a "field" was different from that in which it will be understood in the nineteenth century.) The fact that the agent can be influenced by the so-called passive object on which it acts, which then acts on the latter in a new way so that the cycle repeats itself, was hardly present in the scientific thought of the seventeenth century.[1]

1.1 The quantum measurement paradox

This interpretation, maintained by Bohr throughout his life, can be subsumed in four major points: [4]

1. Microscopic entities (such as electrons and atoms) are not even to be thought of as possessing properties in the absence of specification of the macroscopic experimental arrangement.
2. Macroscopic experimental arrangements, and the results of experiments, are to be expressed in classical, realistic terms.
3. There exists what Bohr repeatedly refers to as an "unanalyzable link" between the microsystem and the macroscopic measuring apparatus.
4. The principle of complementarity: different experimental arrangements exclude one another, and the measurement of one property may therefore be "complementary" to the measurement of another.

It can be inferred from his numerous papers [5, 6] that Bohr himself did not rule out the possibility of obtaining macroscopic quantum phenomena in which the principle of complementarity is retained and merely extended to a new level, the macroscopic one.[3] Still, very few physicists hoped to see quantum phenomena at the macroscopic level owing to an argument developed by Bohr. First, he stressed the fact that quantum effects are important only when the action S is on the order of the quantum of action $\hbar$. Second, if E is the energy of the macroscopic system in question and $\tau_{cl} \approx \omega_{cl}^{-1}$ the order of magnitude of its classical period of motion, then in a typical experiment we have $S \gtrsim E\tau_{cl}$ so that the condition $S \lesssim \hbar$ implies $E \lesssim \hbar\omega_{cl}$. Third, since for a macroscopic system the relevant classical frequencies are certainly not greater than $10^{16}s^{-1}$, this in

[3]Namely, that macrophysical objects have objective existence and intrinsic properties in one set of circumstances such as when used as a measuring apparatus, and have properties relative to the observer in another set of circumstances.

turn implies that the characteristic energy scale for the macroscopic motion can be no greater than a few eV (the order of ionization energy of a single atom). Bohr concluded that it was simply unlikely that any macroscopic variable should be associated with so tiny an energy.

Since the time of Bohr, however, physics with the help of technology has brought to our attention cases where the motion of a *macroscopic* variable can be controlled by a *microscopic* energy on the order of the thermal energy of an atom at room temperature or even lower (e.g., a macroscopic variable such as the current or trapped flux in a bulk superconducting ring).

Meanwhile, the reductionistic compulsion resurfaced with the publication of von Neumann's book [7] on the axiomatization of quantum mechanics and his interpretation of measurements, and its popularized adaptation by London and Bauer [8] of the problem of the quantum measurement as exemplified by the Schrödinger's Cat Paradox. Let us briefly expose this paradox. For simplicity, consider an ensemble of microscopic systems which have only two available microscopically different states ψ_1 and ψ_2. In order to measure which of these states is realized for a particular system of the ensemble, we couple it to a macroscopic measuring apparatus assumed to be entirely described by quantum mechanics. This measuring device is such that, upon coupling, the state ψ_1 of the system will induce final state Ψ_1 of the apparatus, while ψ_2 will induce the macroscopically distinguishable final state of the apparatus Ψ_2. Schematically, we have

$$\psi_1 \Psi_o \rightarrow \psi_1 \Psi_1 \qquad \text{and} \qquad \psi_2 \Psi_o \rightarrow \psi_2 \Psi_2$$

where Ψ_o corresponds to the initial state of the apparatus.

Now, suppose we prepare an ensemble for the microsystems which is described by the *linear superposition* state $a\,\psi_1 + b\,\psi_2$, $a, b \neq 0$. When coupled to a measuring apparatus, what is its description after the interaction? In order to find this out, we consider the

quantum evolution of the "universe" consisting of the apparatus plus the microsystem which, by making good use of the linearity of quantum mechanics, is given by

$$(a\,\psi_1 + b\,\psi_2)\,\Psi_o \;\rightarrow\; a\,\psi_1\,\Psi_1 \;+\; b\,\psi_2\,\Psi_2. \tag{1.1}$$

We come to the astounding conclusion that we obtain a linear superposition of states of the "universe" corresponding to *macroscopically distinct* behavior of the measuring apparatus.

Various solutions have been proposed in the past to account for the lack of such macroscopic linear superpositions. Among the staunch supporters of the universality of quantum mechanics in the physical world (who seem to form a majority in the physics community), it is argued that it will be impossible in real life to discriminate between the experimental predictions made by equation (1.1) and those made by a classical "mixture" description, in which the universe is simply assigned a probability $p_i = \mid c_i \mid^2$ of being in the macroscopic state $\psi_i\,\Psi_i$, $i = 1, 2$ ($c_1 = a$ and $c_2 = b$). In the technical language of the quantum mechanical density matrix ρ, it is claimed that its off-diagonal elements in a representation corresponding to the macroscopically distinguishable states i are unobservable in any realistic experiment so that the correct ρ, which has elements

$$\rho_{ij} \;=\; c_i^*\,c_j,$$

may be safely replaced by the one corresponding to the classical mixture, namely

$$\rho_{ij}^{cl} \;= \mid c_i \mid^2\;\delta_{ij}.$$

This reduction is claimed to be effected by the dissipative interaction that the so-called universe (which is never totally isolated) has with its environment, so that possible quantum macroscopic phenomena are washed out; using their jargon, the environment would decohere the wavefunction (1.1) with macroscopic distinct states. However, as was so

well pointed out by Leggett [9], the conceptual problem is not whether at the macroscopic level the object behaves *as if* it were in a definite macroscopic state, but whether it *is* in such a state. In other words, can we simply assume that the linear superposition (1.1) ever exist at the classical level?

It seems evident from the foregoing discussion that a crucial test for the pretension of quantum mechanics as the all-encompassing physical law, extending its dominion right up to the classical world, is the observation of a macroscopic quantum coherence (MQC) which presupposes the existence of such macroscopic states (1.1). Until now, no such conclusive test has been shown. On the other hand, quantum tunnelling of a macroscopic variable has been observed! [10, 31, 33, 34]

Before discussing at length this promising vindication of quantum mechanics in the classical regime, we return to the decoherence effect of the environment discussed above. It was thought (on the implicit assumption of the universality of quantum mechanics) that this decoherence effect could be more or less severe so as to permit us to expect that, under some specific circumstances, a remnant of quantum macroscopic behavior could be made manifest. To that end, it seemed imperative to construct a quantum theoretical model which would take into account the dissipative effect of the environment; the Caldeira-Leggett formalism [12], to be introduced below, is a response to such a need. By a greater understanding of the environment's effect on a system, one could certainly devise ways in which the disastrous decoherence would be brought low enough to salvage a possible quantum character to a macroscopic object.

This study will bear on one macroscopic quantum effect: the macroscopic quantum tunnelling, abbreviated as MQT, in the presence of dissipation. As Leggett [11] rightly observed, in most discussions of macroscopic quantum phenomena, the "system" in question is microscopic, and it is the the interaction with a *macroscopic* system, the measuring apparatus, which destroys the coherence of its wave function. By contrast, below, we

consider a "system" which is itself *macroscopic*, the coherence of whose wavefunction may be destroyed by its dissipative interaction with a host of *microscopic* degrees of freedom representing the "environment."

Let us briefly review the elementary features of microscopic quantum tunnelling in the following section.

1.2 The microscopic quantum tunnelling

In figure (1.1(a)), we show the first energy levels of a particle in a finite square potential well of width a and arbitrary depth U, along with their respective probability curves given by the modulus squared of the wavefunctions.

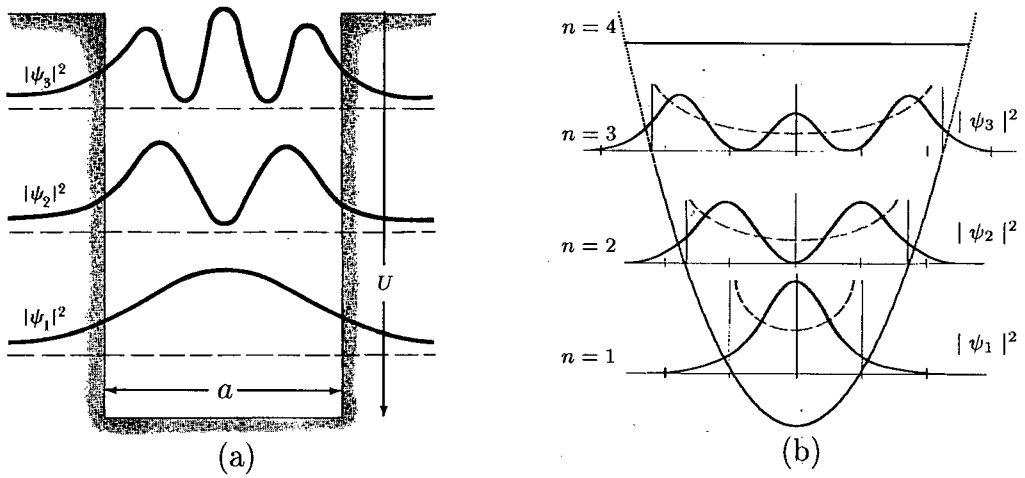


Figure 1.1: Probability densities of low-lying quantum energy levels for a finite square (a) and harmonic (b) potential well. In (a), the probability densities for the lowest three energy levels of a square well of height U and width a are shown with a non-zero probability of finding a particle with energy less than U outside the well. The same is shown in (b) for a harmonic potential well; the vertical bars indicate the classical turning points and the dashed curves represent the classical probability densities [14]. Only the lowest three probability densities are explicitly shown.

Classically, as long as the energy E of the particle is less than U, it is impossible for the particle to be present outside the potential well, the barriers on both sides preventing it from crossing beyond the classical turning points $x = 0$ and $x = a$. However, quantum mechanics allows the wavefunctions to leak through by a small amount into the classically forbidden zone within which they adopt an exponentially decaying form. Therefore, by the orthodox interpretation of quantum mechanics, there should be a non-zero probability of finding the particle in that region. Figure (1.1(b)) shows the same for a quadratic potential well.

This is the origin of the α-decay in nuclei; although nucleons are totally confined classically, some nucleons will make it through the potential barrier thanks to their wavefunctions extending beyond it. It is important to note that what distinguishes MQT from the α-particle decay and other microscopic tunnelling effects is that the classically accessible regions which are separated by the barrier are effectively *macroscopically* distinct.

Now, consider a metastable potential well $V(Q)$ as shown in figure (1.2). The stability

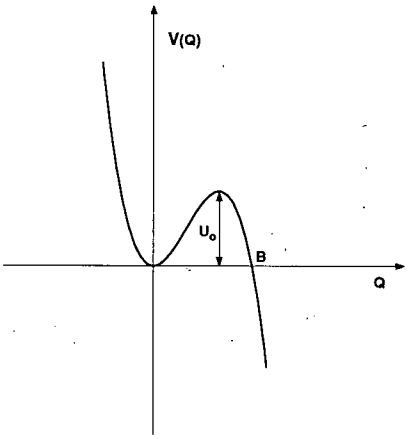


Figure 1.2: A metastable potential. This metastable potential has a barrier of height U_o and width B.

frequency at the bottom of this well located at $Q = 0$ is given by

$$\omega_o = \left\{ \frac{1}{M} \left. \frac{d^2V}{dQ^2} \right|_{Q=0} \right\}^{1/2} \tag{1.2}$$

for a particle of mass M. The low-lying quantum levels associated with this well with energy less than U_o will, also, see their wavefunctions leaking through beyond the point $Q = B$. By construction of $V(Q)$, we see that, once the particle has tunnelled through the barrier to the right, it will simply roll off and move away from point B with a very small likelihood of tunnelling right back in.

Let the Lagrangian of this system be given by

$$L(Q, \dot{Q}) = \frac{1}{2} M \dot{Q}^2 - V(Q) \tag{1.3}$$

By the method of Wentzel-Kramers-Brillouin (WKB), one can calculate the probability per unit time that the particle escape from the well. The result is[4]

$$P \;=\; A\, e^{-B/\hbar} \qquad \text{with} \qquad A \;\triangleq\; C\,\omega_o\,(B/2\pi\hbar)^{1/2} \tag{1.4}$$

where

$$B \;=\; 2\int_0^B \sqrt{2\,M\,V(Q)}\,dQ \tag{1.5}$$

and C is a dimensionless constant of order unity depending on the shape of the potential $V(Q)$ [12].

This is the standard result for a particle, be it microscopic or macroscopic, subjected to a conservative potential $V(Q)$. However, we noted above that the difficulty to observe quantum tunnelling at the macroscopic level had to do with the dissipative effect of the environment on the macroscopic object. Therefore, one should modify the simple Lagrangian (1.3) in order to take this into account. This is discussed in the next section.

1.3 The Caldeira-Leggett formalism

This treatment will be rather brief; we suggest the reader consult the proper references for more details [12, 13]. In the Caldeira-Leggett paper [12], the authors sought to answer the questions of how and to what degree dissipation can affect the tunnelling probability of a particle whose quasi-classical equation of motion is given by

$$M\ddot{Q} \;+\; \eta\,Q \;+\; \frac{dV}{dQ} \;=\; F_{ext}(t)\,, \tag{1.6}$$

with η being the phenomenological friction coefficient which may be frequency- and amplitude-dependent [13], as compared with

$$M\ddot{Q} \;+\; \frac{dV}{dQ} \;=\; F_{ext}(t)\,, \tag{1.7}$$

[4]The symbol $\triangleq$ stands for "by definition."

the quasi-classical equation of motion derived from the dissipation-free Lagrangian (1.3) (complemented with the required term $Q\,F_{ext}(t)$ which accounts for the forced motion). We stress the fact that both potentials $V(Q)$ in equations (1.6) and (1.7) are exactly the same regardless of possible potential-renormalization effects [13].[5] By the term quasi-classical, we mean that the above equations of motion apply for the expectation value $Q = \langle \hat{Q} \rangle$ where we assume that it is legitimate to replace

$$\langle \frac{dV(\hat{Q})}{d\hat{Q}} \rangle \qquad \text{by} \qquad \left. \frac{dV(Q)}{dQ} \right|_{Q=\langle \hat{Q} \rangle} .$$

Such a substitution is valid whenever the potential $V(Q)$ varies appreciably *only* over macroscopic scales *and* the wavefunction associated with the system has negligible uncertainties or fluctuations about its expectation values of the coordinate Q, $\langle \hat{Q} \rangle$, and its conjugate momentum P, $\langle \hat{P} \rangle$.

Caldeira and Leggett showed that the most general Lagrangian we need ever be concerned with in modeling the interaction between a system and its environment is

$$\begin{aligned} L \;=\; & \frac{1}{2} M\dot{Q}^2 \;-\; V(Q) \;+\; Q\,F_{ext}(t) \;+\; \frac{1}{2} \sum_j m_j \left(\dot{x}_j^2 - \omega_j^2 x_j^2\right) \\ & - Q \sum_j C_j\, x_j \;-\; Q^2 \sum_j \frac{|\, C_j \,|^2}{2\, m_j\, \omega_j^2} \end{aligned} \tag{1.8}$$

under the express assumption of three provisos, namely, 1) any one degree of freedom of the environment, x_j, is sufficiently *weakly* perturbed so as to neglect nonlinear effects [15], 2) the interaction Lagrangian coupling the system to its environment contains terms either (i) linear in the system coordinate Q and its conjugate momentum P or (ii) quadratic in Q and P, but *not* containing the environment's coordinates x_j or their conjugate momenta, and 3) condition of time-reversal invariance. We note that M, $V(Q)$, m_j, ω_j, and

[5] The dissipative interaction with the environment may cause the potential $V(Q)$ appearing in (1.3) to be different from the one in (1.6). For purposes of comparison, we suppose, then, that the potential $V(Q)$ in (1.3) already includes this possible interaction and enquire about the sheer effect of the term $\eta\,\dot{Q}$ on the tunnelling rate.

the environment coordinates x_j in (1.8) may carry renormalization factors arising from the interaction itself. The stability frequencies ω_j are those associated with the *small* oscillations of the environment coordinates x_j consequent to their *small* perturbation as required under proviso 1 above. In other words, we can represent a system's environment, insofar as its effect on the system is concerned, by a bath of harmonic oscillators. This idea is not new, already Feynman and Vernon in 1963 [15] came to the same conclusion.

In reference [12], only proviso 1 is explicitly retained plus some other restrictions so that the general Lagrangian takes the form

$$L = \frac{1}{2} M\dot{Q}^2 - V(Q) + Q\,F_{ext}(t) + \frac{1}{2}\sum_j m_j\,(\dot{x}_j^2 - \omega_j^2\,x_j^2) \qquad (1.9)$$
$$- \sum_j F_j(Q)\,x_j - \sum_j \frac{|\,F_j(Q)\,|^2}{2\,m_j\,\omega_j^2}$$

Here, we note that the interaction is more general than that in (1.8) with a coupling linear in the environment coordinates x_j only. Two cases are to distinguished: the quasi-linear and the the strictly linear dissipation mechanisms. The former applies when, for every fixed typical amplitude Q, it is possible to find a frequency $\omega(Q)$ small enough such that equation (1.6) holds, and the latter when, for any typical amplitude Q, it is possible to find such a frequency ω *independent* of Q. It is then shown in reference [12] that, in order that the Lagrangian (1.9) be consistent with the quasi-classical damped equation of motion for the system's coordinate Q in the quasi-linear case, we should have the condition

$$\eta = \frac{\pi}{2}\sum_j \frac{1}{m_j\,\omega_j^2}\left|\frac{\partial F_j(Q)}{\partial Q}\right|^2 \delta(\omega - \omega_j). \qquad (1.10)$$

For the more specific case of strict linearity, we should have $F_j(Q) = C_j\,Q$ so that condition (1.10) reduces to

$$\eta = \frac{\pi}{2}\sum_j \frac{1}{m_j\,\omega_j^2}\,|\,C_j\,|^2\,\delta(\omega - \omega_j)\,,$$

or, cast in a different form, to

$$J(\omega) \;=\; \eta\,\omega \tag{1.11}$$

with

$$J(\omega) \;\triangleq\; \frac{\pi}{2}\sum_j \frac{1}{m_j\,\omega_j} \mid C_j \mid^2 \;\delta(\omega-\omega_j) \tag{1.12}$$

denoted as the spectral density function. The condition of strict linearity, therefore, requires that the coupling be linear in both the system's and environment coordinates, Q and $\{x_j\}$.

Equipped with these theoretical tools and making good use of the instanton technique, Caldeira and Leggett can calculate the quantum tunnelling rate at temperature $T = 0$ of a system coupled to a bath of harmonic oscillators and deduce an expression of the form (1.4) with the various parameters carrying an integration of the environment's effect.

However, in this study, we are not directly concerned with the calculation of the quantum tunnelling rate. Instead, we seek to calculate a piece of information which gives an idea of the dissipation involved in a ferromagnetic system at $T = 0$ and is used for the evaluation of the MQT rate itself: the spectral density function $J(\omega)$. It is time, now, to introduce our macroscopic object that will be associated with the macroscopic coordinate Q; to this, the next section is devoted.

1.4 The domain wall and its dynamics

In this study, our macroscopic objects will be of a magnetic nature: domain walls in magnetic wires [16, 17]. A domain wall is the transition region of finite length separating magnetic domains with different orientations of the magnetization. Here, we will focus on a wall separating domains with a 180^o reversal of magnetization called a Bloch wall; a sketch of which is shown in figure (1.3) [18].

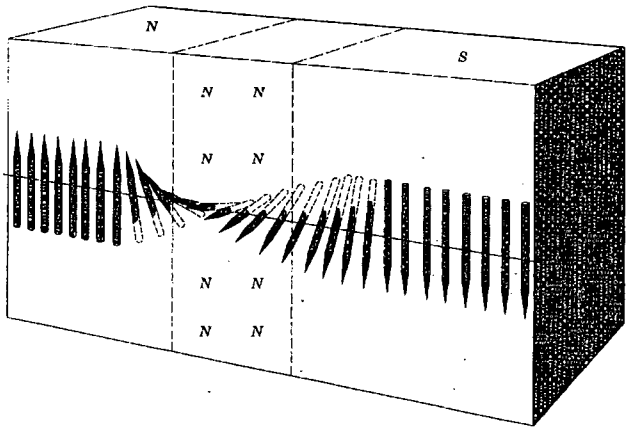


Figure 1.3: A 180^o domain wall, also called Bloch wall.

Consider the coordinate system for a 180^o domain wall shown in figure (1.4) with the magnetization vector pointing in the x-direction for $z \to -\infty$ and in the negative x-direction for $z \to +\infty$. We suppose that in any x-y plane, all magnetization vectors have the same relative orientation so that the magnetization varies only along the z-axis.

The isotropic exchange energy for nearest-neighbour interaction is given by

$$E_{ex} = -J \sum_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j \tag{1.13}$$

where $\mathbf{S}_i$ is the total magnetization vector, or spin, of the i^{th} ion and $J > 0$ for ferromagnetic substances. By construction of our domain wall, it is more useful to consider the energy per unit area ϵ_{ex}. If we assume for simplicity that we deal with a simple hexagonal

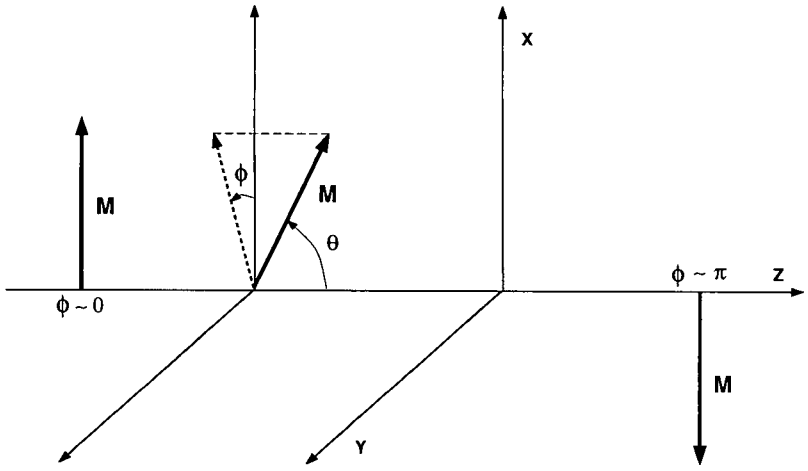


Figure 1.4: The coordinate system. The angles θ and ϕ are, respectively, the polar and azimuthal coordinates of the magnetization vector M. For a Bloch wall, the polar angle has value $\pi/2$ for each magnetization vector along the z-axis and the azimuthal angle takes the value of zero at $z \to -\infty$ and of π at $z \to +\infty$.

lattice whose primitive vectors are of equal length a, then

$$\epsilon_{ex} = -\frac{J}{\sqrt{3}\,a^2} \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} \tag{1.14}$$

with the index n running in one dimension only, the z-axis. Expression (1.14) can be rewritten as

$$\epsilon_{ex} = -\frac{JS^2}{\sqrt{3}\,a^2} \sum_n \cos\alpha_n \tag{1.15}$$

where S is the magnitude of the magnetization, i.e., the total spin quantum number of each ion, and α_n the angle subtending the magnetization vectors $\mathbf{S}_n$ and $\mathbf{S}_{n+1}$. For small α_n, we can expand $\cos\alpha_n$ to quadratic order to obtain

$$\epsilon_{ex} = +\frac{JS^2}{\sqrt{3}\,a^2} \sum_n \alpha_n^2 + \text{constant.} \tag{1.16}$$

In the continuum limit, the angle between neighbouring magnetization vectors is given by

$$a\left(\frac{\partial\phi}{\partial z}\right)$$

with ϕ being the azimuthal angle because the spins are totally confined to x-y planes. Then, equation (1.16) transforms as

$$\epsilon_{ex} = \frac{JS^2}{\sqrt{3}\,a} \int_{-\infty}^{+\infty} dz \left(\frac{\partial \phi}{\partial z}\right)^2, \tag{1.17}$$

where we have discarded the arbitrary constant and scale z in units of a. In chapter 2, we will consider a model for ferromagnetic wires with an anisotropy energy term

$$K_y \sum_n (S_n^y)^2$$

which makes the x-axis the easy axis. Again, in the continuum limit, this anisotropy term per unit area for our previous simple hexagonal lattice becomes

$$\epsilon_a = \frac{KS^2}{\sqrt{3}\,a^3} \int_{-\infty}^{+\infty} dz \, \sin^2 \phi(z) \tag{1.18}$$

for spin vectors confined to x-y planes so that $\theta = \pi/2$ for each of them. The total energy per unit area is thus

$$\epsilon = \frac{S^2}{\sqrt{3}\,a} \int_{-\infty}^{+\infty} dz \left\{ J \left(\frac{\partial \phi}{\partial z}\right)^2 + \frac{K}{a^2} \sin^2 \phi(z) \right\}. \tag{1.19}$$

In order to determine the stable wall configuration, we must minimize the energy expression ϵ by the usual variational procedure which yields the following Euler equation

$$\frac{\partial^2 \eta}{\partial z^2} = \frac{K}{Ja^2} \sin \eta(z) \tag{1.20}$$

where $\eta \triangleq 2\,\phi$.

Expression (1.20) is the static Sine-Gordon equation. We will have more to say about the Sine-Gordon (SG) equation in the next section; for the time being, suffice it to say that the non-trivial solutions to (1.20) are

$$\eta(z) = 4 \tan^{-1} \left(e^{\pm m(z)}\right) \qquad \text{with} \tag{1.21}$$

$$m^2 \triangleq \frac{K}{J\,a^2}.$$

The $\pm$ signs in (1.21) correspond to the two possible helicities of the rotation of the magnetization vector **S** within the wall (cf. sec. 4). Also, it is supposed in (1.21) that the spin vector **S** points *exactly* in the positive y-direction at $z = 0$.[6] For simplicity, we shall choose the helicity corresponding to the positive sign. In terms of the original azimuthal angle ϕ, we have from (1.21) that

$$\sin\phi(z) \;=\; \operatorname{sech}(mz). \tag{1.22}$$

The energy of the wall per unit area, equation (1.19), can also be rewritten in terms of the angle η as

$$\epsilon \;=\; \frac{JS^2}{2\sqrt{3}\,a}\int_{-\infty}^{+\infty} dz\,\left\{\frac{1}{2}\left(\frac{\partial\eta}{\partial z}\right)^2 + m^2\,(1-\cos\eta)\right\}, \tag{1.23}$$

so that upon using (1.21) we find

$$\begin{aligned}\epsilon \;&=\; \frac{4\,JS^2 m}{\sqrt{3}\,a}\\ &=\; \frac{4\,JS^2\sqrt{KJ}}{\sqrt{3}\,a^2}.\end{aligned} \tag{1.24}$$

Döring [19] in 1948 was the first to discover that a domain wall exhibits an inertia, despite the lack of any mass displacement. The mass of a domain wall has its origin in the angular momenta of the spins forming the wall. We recall that a spin will adopt a precession motion about the direction of the applied magnetic field in the same fashion that a spinning top would when placed in a gravitational field. Hence, when we apply a magnetic field in the x-direction, each spin of the wall will start precessing about the x-axis resulting in a rotation out of the x-y plane. This rotation will simply induce the appearance of magnetic free poles giving rise to a small demagnetization field in the z-direction,

$$H_z \;=\; -\frac{M_z}{\mu_o}, \tag{1.25}$$

[6] Should the spin vector point in the positive y-direction at $z = z_o$, the exponential in (1.21) would have as its argument $\pm m(z - z_o)$ instead.

where M_z is the z-component of the induced magnetization and μ_o the permeability of the vacuum. This demagnetization field then, in turn, acts on each spin so as to induce a precession motion in the x-y plane which results in the displacement of the wall in the z-direction.

The rotational velocity of this precession motion about the z-axis is given by

$$\dot{\phi} \;=\; \tilde{g}\, H_z \tag{1.26}$$

where $\tilde{g}$ is the gyromagnetic constant such that $\tilde{g} = g\,\mu_b$ with μ_b being the Bohr magneton and g the gyromagnetic ratio. But,

$$\frac{d\phi}{dt} \;=\; \frac{\partial\phi}{\partial z}\,\frac{dz}{dt} \;=\; v\,\frac{\partial\phi}{\partial z} \tag{1.27}$$

where v is the translational velocity of the wall as it moves along the z-axis. Using equations (1.26) and (1.27), we find

$$M_z \;=\; -\frac{\mu_o\, v}{\tilde{g}}\,\frac{\partial\phi}{\partial z}\,. \tag{1.28}$$

Thus, the moving wall will have the additional surface energy

$$\begin{aligned}\epsilon_k \;&= -\frac{1}{2}\int_{-\infty}^{+\infty} dz\; M_z\, H_z \\ &= \frac{1}{2\,\mu_o}\int_{-\infty}^{+\infty} dz\; M_z^2 \;\;= \frac{\mu_o\, v^2}{2\,\tilde{g}^2}\int_{-\infty}^{+\infty} dz\; \left(\frac{\partial\phi}{\partial z}\right)^2 .\end{aligned} \tag{1.29}$$

From (1.22), we find that

$$\left(\frac{\partial\phi}{\partial z}\right)^2 \;=\; m^2\,\mathrm{sech}^{\,2}(mz)\,,$$

so that substituting the latter into (1.29), ϵ_k becomes

$$\begin{aligned}\epsilon_k \;&=\; \frac{\mu_o\, m}{\tilde{g}^2}\, v^2 \\ &=\; \frac{M}{2}\, v^2,\end{aligned} \tag{1.30}$$

where

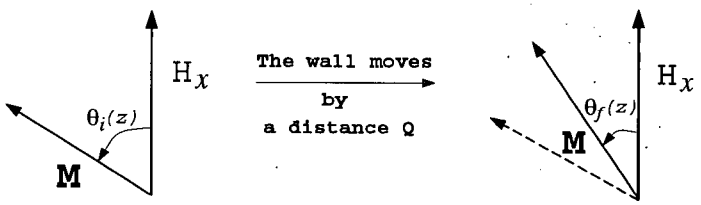
$$M \triangleq \frac{\mu_o}{2\,\tilde{g}^2\,a}\sqrt{\frac{K}{J}} \tag{1.31}$$

can be interpreted as a virtual mass per unit area for the wall.

What is the energy supplied by the applied magnetic field H_x which makes the wall move? For low fields and velocities, assume that the wall has moved by a distance Q, in units of the lattice constant a, from its original center's position at $z = 0$. The work done per unit area by the field H_x is given symbolically by

$$\begin{aligned} W &= \Delta\text{Energy} \\ &= -\mathbf{H}_x \cdot \int dz \,\{\, \text{Wall configuration at } z = Q \,-\, \text{Wall configuration at } z = 0 \,\}. \end{aligned}$$

On each spin vector, we have the following change in direction



From (1.22), we deduce for a wall that

$$\cos\phi(z) \;=\; -\,\mathrm{tgh}\,(mz). \tag{1.32}$$

Thus, when the wall's center has moved to position $z = Q$, we have

$$\begin{aligned} \cos\phi_f(z) &= -\,\mathrm{tgh}\,(m(z-Q)) \\ &\approx \cos\phi_i(z) \;+\; m\,Q\,\mathrm{sech}^{\,2}(mz)\,, \end{aligned} \tag{1.33}$$

where $\phi_f(z)$ and $\phi_i(z)$ are, respectively, the azimuthal angle for the initial and final configurations, and the last approximation is valid for small displacements. Since, for

each spin, the work done by the field is given by

$$\begin{aligned} -H_x\, S_{x,f}(z) \;+\; H_x\, S_{x,i}(z) \;&=\; -H_x\, S\,(\cos\phi_f(z) \;-\; \cos\phi_i(z)) \\ &\approx\; -H_x\, S\, m\, Q\, \mathrm{sech}^{\,2}(mz)\,, \end{aligned}$$

we obtain by integration

$$\begin{aligned} \Delta\mathrm{Energy} \;&=\; -H_x\, S\, m\, Q \int_{-\infty}^{+\infty} dz\; \mathrm{sech}^{\,2}(mz) \\ &=\; -2\, H_x\, S\, Q \end{aligned} \tag{1.34}$$

as the energy per unit area supplied by the magnetic field H_x in order to push the wall by a *small* distance Q.

Therefore,

$$\begin{aligned} -\frac{\Delta\mathrm{Energy}}{Q} \;&=\; 2\, H_x\, S \\ &\triangleq\; P \end{aligned}$$

can be thought of as the pressure P exerted on the wall by the applied field. In general, this pressure will cause the wall to swell and a radius of curvature is therefore established. In quasi-one-dimensional ferromagnetic systems, we assume that the area is small enough to neglect this curvature effect.

As mentioned previously, Döring showed that the domain wall has dynamic properties analogous to a particle of mass M. Indeed, one can write down a phenomenological equation of motion for a wall as

$$M\ddot{Q}(t) \;+\; \eta\dot{Q}(t) \;+\; \frac{dV(Q)}{dQ} \;=\; P_{ext}\,, \tag{1.35}$$

where $Q(t)$ is the domain wall's center coordinate, M the virtual mass of the wall given in (1.31), η the friction coefficient, $V(Q)$ a conservative surface potential, and, finally, P_{ext} an external forcing pressure term. Note that equation (1.35) is given in terms of quantities

per unit area. In general, the magnetic material will contain impurities, defects, or voids; all of which contribute to a non-uniform background for the domain wall in *addition* to the magnetic anisotropy. This means that the wall will see its surface energy vary as it moves from one position to the other. In this study, we will show in chapter 4 that pinning centers can be thought of, qualitatively, as giving rise to a potential of the form

$$V(Q) \;=\; 2\,\alpha\,\mathrm{sech}\,(mQ) \;+\; \epsilon \tag{1.36}$$

where ϵ is the surface energy of the wall (1.24) and α a negative small coupling constant. Adding the potential energy arising from the application of the external magnetic field H_x, equation (1.34), we obtain, by discarding the constant term ϵ,

$$\mathcal{V}(Q) \;=\; 2\,\alpha\,\mathrm{sech}\,(mQ) \;-\; 2\,H_x\,S\,Q\,. \tag{1.37}$$

Both potentials are shown in figure (1.5).

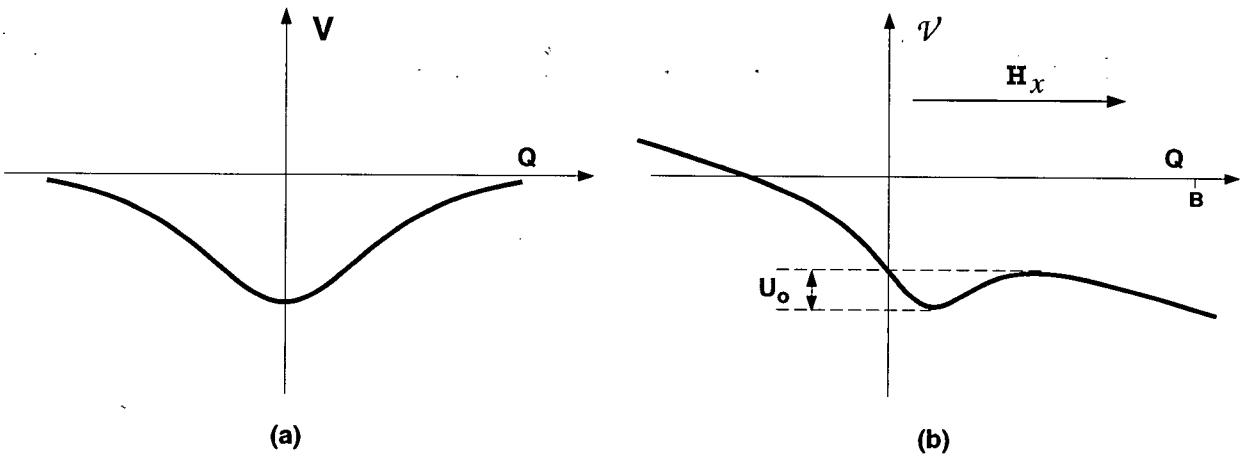


Figure 1.5: The physical pinning potential in the absence (a) and presence (b) of a magnetic force H_x. A pinning impurity in a quasi-one-dimensional ferromagnet generates the above potentials for a Bloch domain wall whose center is parametrized by coordinate Q. In (a), the potential is of a hyperbolic secant form. The application of a magnetic field H_x in (b) effectively tips potential (a) asymmetrically and gives rise to a metastable potential with a barrier of height U_o and width B. Compare with fig.(1.2).

In this case of an applied magnetic field H_x, equation (1.35) reads

$$M\ddot{Q}(t) + \eta\dot{Q}(t) + \frac{dV(Q)}{dQ} = 2\,S\,H_x\,. \tag{1.38}$$

One can immediately notice the parallel between the equation of motion for the wall's coordinate $Q(t)$ (1.35) and figure (1.5(b)) and the quasi-classical equation of motion (1.6) and figure (1.2) in section 2. In other words, $Q(t)$ becomes the coordinate for the macroscopic object we are interested in: the domain wall. The phenomenological friction coefficient η carries the dissipative effect of the environment on the motion of a domain wall. One would then be curious to know how this dissipative interaction affects, at temperature $T = 0$, the tunnelling of a domain wall under a barrier of height U_o as shown in figure (1.5(b)). This is exactly the question the Caldeira-Leggett formalism in section 3 was tailored to answer. This study will be mainly concerned with one source of dissipation, namely, the magnons. We will discuss in the last section the other sources of dissipation likely to be present at $T = 0$ and $T \neq 0$. But first, we will turn to the Sine-Gordon equation and some of its properties.

1.5 The Sine-Gordon equation

The Sine-Gordon equation [20, 21] a non-linear wave-equation of the form

$$\frac{\partial^2\psi(x,t)}{\partial t^2} - c_o^2\frac{\partial^2\psi(x,t)}{\partial x^2} + \omega_o^2\sin\psi(x,t) = 0 \tag{1.39}$$

where c_o is a characteristic velocity and ω_o a characteristic frequency. This is the equation of motion obtained from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\{\dot{\psi}^2 - c_o\,\psi'^2\} - \omega_o^2\,(1-\cos\psi)\,. \tag{1.40}$$

We note that in the case of very small amplitude $|\,\psi\,| \ll 1$, one can replace $\sin\psi$ by ψ so that the Sine-Gordon equation reduces to the ordinary Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} - c_o^2\frac{\partial^2\psi}{\partial x^2} + \omega_o^2\,\psi = 0\,.$$

It has been shown by Faddeev and Takhtadzhian [22] that the classical Sine-Gordon system represents a completely integrable Hamiltonian system whose spectrum is totally exhausted by free oscillation modes (or magnons), solitons, and breathers. The last two represent bounded large-amplitude solutions of the SG equation (1.39) for which the soliton has the form

$$\psi_s^v(x,t) \;=\; 4\ \mathrm{tgh}^{-1}\left\{\exp[\pm\frac{\omega_o}{c_o}\,\gamma\,(x-vt)]\right\}, \tag{1.41}$$

where

$$\gamma \;\triangleq\; \left(1\;-\;\frac{v^2}{c_o^2}\right) \qquad\qquad |\,v\,| < c_o\,, \tag{1.42}$$

and the $\pm$ signs refer respectively to a soliton and anti-soliton of opposite helicities. In the literature, one also calls a soliton a kink. A soliton and an anti-soliton in their rest frames are shown in figure (1.6).

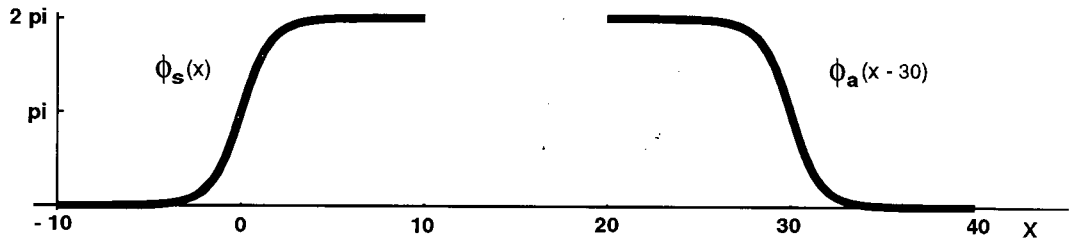


Figure 1.6: A soliton and anti-soliton at rest.

These are bounded since $\psi_s^v \to 0 \pmod{2\pi}$ as $|\,x\,| \to \infty$. Furthermore, these solutions are Lorentz invariant, as are the Lagrangian density and its resulting equation of motion ((1.39), (1.40)) where c_o plays the role of a limiting speed.[7]

Moving in the rest frame of a soliton with speed v, one obtains

$$\psi_s(x,t) \;=\; 4\ \mathrm{tgh}^{-1}\left\{e^{(\omega_o/c_o)\,x}\right\}. \tag{1.43}$$

[7]In magnetic systems, this limiting value is simply the Walker velocity.

Before finding the excitations of the field $\psi(x,t)$ in the presence of a soliton $\psi_s(x,t)$, it might be useful to digress for a brief moment and offer a mechanical analogy whose equation of motion is of the Sine-Gordon form. A mechanical model is shown below in figure (1.7).

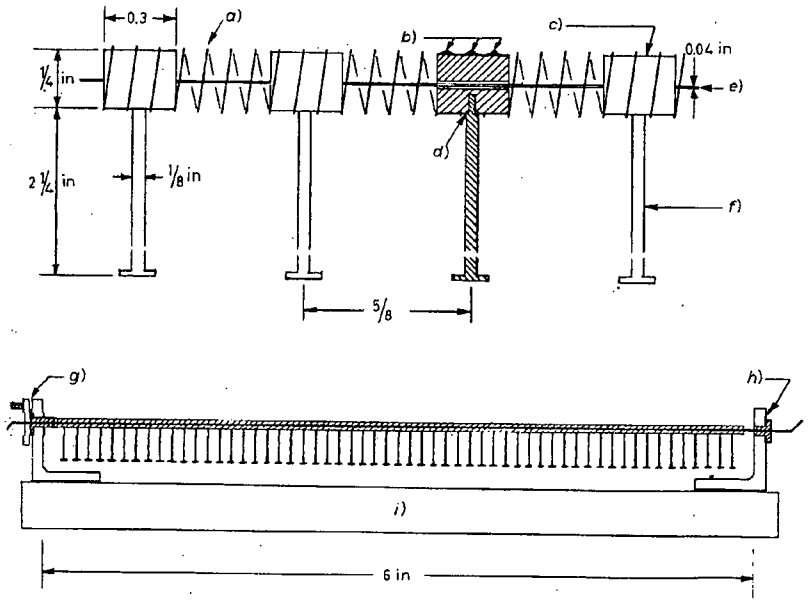


Fig. 3. – Mechanical model of the SGE: *a*) spring 0.2 diameter, *b*) solder, *c*) brass, *d*) tap and thread, *e*) piano wire, *f*) nail, *g*) and *h*) ball bearings, *i*) wooden base.

Figure 1.7: A mechanical model for a soliton obeying a Sine-Gordon equation [21].

Its motion is governed by the difference differential equation

$$M\frac{d^2\phi_i}{dt^2} = K\left[\phi_{i+1} - 2\phi_i + \phi_{i-1}\right] - T\sin\phi_i\,, \tag{1.44}$$

where ϕ_i is the angle between the direction of the gravitational field and the i^{th} pendulum, M the moment of inertia of a single pendulum, K the constant torque of a section of spring between two pendula, and $T\sin\phi_i$ the gravitational restoring torque of the i^{th} pendulum. For waves which vary slightly over the distance Δx separating two pendula, one can turn the difference equation (1.44) into a scaled continuum partial differential

equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin\phi = 0, \tag{1.45}$$

where distance is measured in units of $\gamma_o = \sqrt{K/T}\,\Delta x$ and time in units of $t_o = \sqrt{M/T}$. Expression (1.45) is simply the normalized SG equation (1.39). A strobe photograph of a soliton travelling from right to left is shown in figure (1.8).

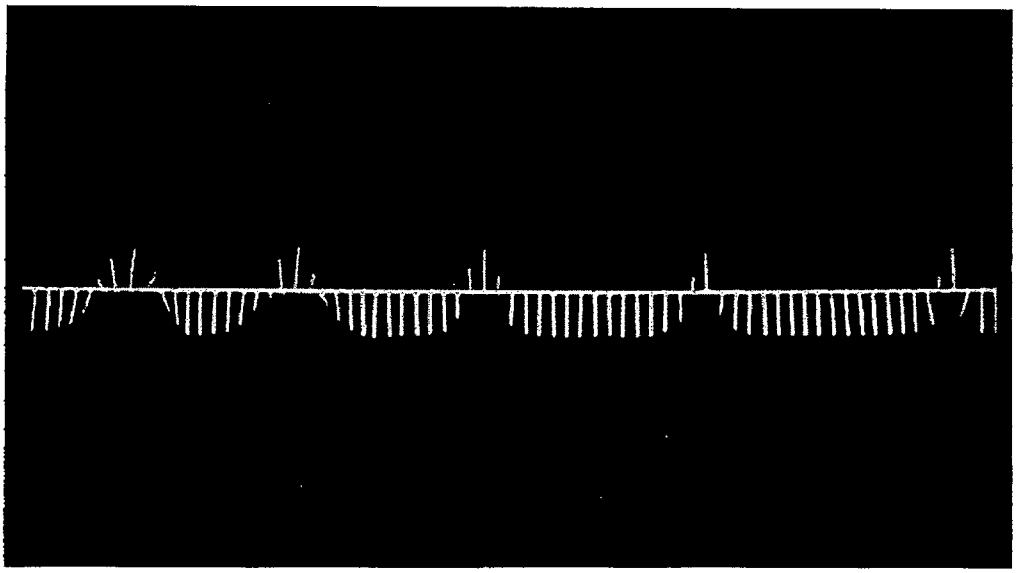


Figure 1.8: The propagation of a kink. A single kink is moving from right to left for the mechanical model shown in figure (1.7). The strobe photographs are taken at time intervals of 0.6 second. The kink slows down because of friction effects which are not accounted for in the equation of motion (1.45). Notice that the kink becomes wider as its velocity decreases, which demonstrates the "Lorentz" contraction effect we expect from the "Lorentz" invariant continuum equation of motion (1.46). The limiting velocity is 50 cm/s for this mechanical system as the length and time scales, γ_o and τ_o, are 5 cm and 0.1 s respectively [21]. (cf. text)

Now, back to our Sine-Gordon Lagrangian. It is well known that, in condensed-matter physics, many properties of materials are explained simply by the notion that small–amplitude waves (phonons, magnons, etc ...) are the fundamental entities that enter the description of their thermodynamics and response to external probes. One, therefore,

would like to find the excitations of the field ψ about a stationary soliton solution ψ_s in the form of small amplitudes ϕ.

Thus, consider

$$\psi(x,t) \;=\; \psi_s(x) \;+\; \phi(x,t) \tag{1.46}$$

with

$$|\,\phi(x,t)\,| \ll 1\,. \tag{1.47}$$

We seek to find a solution for $\phi(x,t)$ such that $\psi(x,t)$ obeys the Sine-Gordon equation (1.39). Substituting (1.46) in (1.39) and making use of the expansion

$$\sin(\psi(x,t)) \;=\; \sin(\psi_s(x)) \;+\; \cos(\psi_s(x))\,\phi(x,t)\,, \tag{1.48}$$

which is sufficient on account of (1.47), we obtain

$$\ddot{\phi} \;-\; c_o^2\,\phi'' \;+\; \omega_o^2\left(1 \;-\; 2\,\mathrm{sech}^{\,2}\left(\frac{\omega_o}{c_o}x\right)\right)\phi \;=\; 0\,. \tag{1.49}$$

In deriving (1.49), we used the identity

$$\cos(\psi_s(x)) \;=\; 1 \;-\; 2\,\mathrm{sech}^{\,2}\left(\frac{\omega_o}{c_o}x\right)$$

and equation (1.39) for the static solution $\psi_s(x)$.

In order to diagonalize equation (1.49), we seek its eigenfunctions by assuming a solution $\phi(x,t)$ with harmonic time-dependence

$$\phi(x,t) \;=\; f(x)\,e^{-i\omega t}, \tag{1.50}$$

and substitute in (1.49) to obtain the following equation for $f(x)$:

$$-c_o^2\,\frac{d^2 f(x)}{dx^2} \;+\; \omega_o^2\left(1 \;-\; 2\,\mathrm{sech}^{\,2}\left(\frac{\omega_o}{c_o}x\right)\right) f(x) \;=\; \omega^2\,f(x)\,. \tag{1.51}$$

The above equation has the form of a Schrödinger's equation with potential

$$V(x) \;=\; \omega_o\left(1 \;-\; 2\,\mathrm{sech}^{\,2}\left(\frac{\omega_o}{c_o}x\right)\right).$$

The solutions $f(x)$ are combinations of hypergeometric functions [23] which generate exactly one bound state with

$$\omega_b^2 \;=\; 0$$

and eigenfunction

$$f_b(x) \;=\; 2\,\frac{\omega_o}{c_o}\;\mathrm{sech}\,\left(\frac{\omega_o}{c_o}x\right)\,, \tag{1.52}$$

and oscillatory states with a continuum of eigenvalues

$$\omega_l^2 \;=\; c_o^2\,l^2 \;+\; \omega_o^2 \tag{1.53}$$

and corresponding eigenfunctions

$$f_l(x) \;=\; \frac{1}{\sqrt{2\pi}}\,\frac{c_o}{\omega_l}\,e^{ilx}\,\left\{\frac{\omega_o}{c_o}\;\mathrm{tgh}\,\left(\frac{\omega_o}{c_o}x\right)\;-\;i\,l\right\}. \tag{1.54}$$

The zero-frequency eigenmode $f_b(x)$ corresponds roughly, as we will see in more details in chapter 3, to the translational excitation of the soliton in the *limit* of *small* displacements; whereas $f_l(x)$ are the small-amplitude oscillatory excitation waves for a magnetic system (the magnons) but in the PRESENCE of the stationary soliton and NOT built around the ground-state of a ferromagnetic material, in which all spins are aligned pointing in the direction of the applied field. For this latter case, we recover the usual magnons with which we are all familiar. The different magnons, represented by $f_l(x)$, correspond to precessional modes in the domain wall, or magnetic soliton, configuration.

Because the functions $\{f_b(x), f_l(x)\}$ are eigenmodes of the self-adjoint operator

$$D \;=\; -c_o^2\,\frac{d^2}{dx^2} \;+\; \omega_o^2\,\left(1\;-\;2\;\mathrm{sech}^{\,2}\left(\frac{\omega_o}{c_o}x\right)\right)\,,$$

they form a *complete* basis-set of orthogonal functions with

$$\langle f_b, f_b\rangle \;=\; 8\,\frac{\omega_o}{c_o}\,, \tag{1.55}$$

$$\langle f_l, f_m\rangle \;=\; \delta(l-m)\,, \tag{1.56}$$

$$\langle f_l, f_b\rangle \;=\; 0\,, \tag{1.57}$$

where

$$\langle f, g \rangle \triangleq \int_{-\infty}^{+\infty} dx \; f(x)^* \, g(x) \, .$$

As a final note on the Sine-Gordon equation, we mention that there exist solutions $\psi(x,t)$ which, under some circumstances, can be thought of as the linear superposition of various solitons and anti-solitons, and are classified by a topological index N defined as

$$N \; = \; \frac{1}{2\pi} \left\{ \psi(\infty, t) \; - \; \psi(-\infty, t) \right\} . \tag{1.58}$$

For instance, two $N_s = 1$ solitons are shown in figure (1.9) so that the total index N_T is given by $2N_s = 2$.

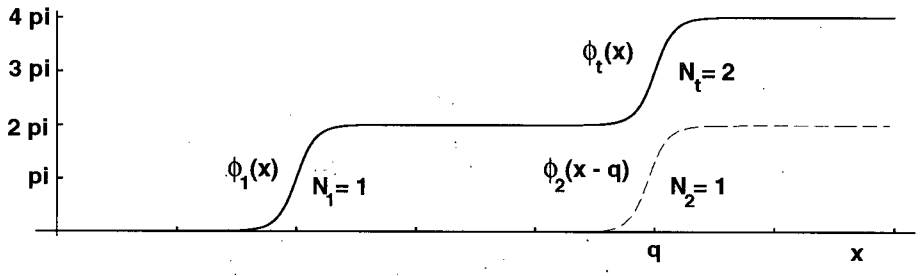


Figure 1.9: Two kinks of same helicity. Solitons ϕ_1 and ϕ_2 have the same topological index $\mathsf{N}_s = 1$ so that the total function $\phi_t(x) = \phi_1(x) + \phi_2(x - q)$ has a total index of $\mathsf{N}_t = 2\,\mathsf{N}_s = 2$.

In the mechanical analogy of the pendula, this corresponds to two successive revolutions of the pendula separated by a distance $Q = Q_1 + Q_2$ so that the total angular displacement adds up to 4π.

1.6 Sources of dissipation

In addition to the dissipative effect of magnons on the quantum motion of a domain wall, this latter can also interact with phonons *via* magnetoelastic coupling, photons, other impurities and defects from a distance, nuclear spins *via* the hyperfine coupling, and free electrons. For magnetic insulators, we may disregard this last dissipative interaction. At temperature $T = 0$, only the nuclear spin interaction should survive with an appreciable effect in the *absence* of impurities. As pointed out by Stamp [24], magnons and phonons should have no dissipative effect on the wall motion at $T = 0$ in the absence of such defects. Besides, several workers, such as Chudnosky and Stamp [24, 26], have shown that the coupling to photons and other impurities gives negligible dissipation on the motion of the domain wall. One is thus left with the interactions of phonons, magnons, and nuclear spins as the major sources of dissipation in the *presence* of impurities. The case of phonons was first addressed by Garg and Kim [27, 28] who showed that the dissipation might be weak for some *single-domain* ferromagnetic materials on account of a superohmic spectral density $J(\omega) \sim \omega^3$.

At this point, we may digress slightly to acquaint the reader with the various types of spectral density functions. For $\omega \to 0$, the behavior of $J(\omega)$ can be classified as sub-Ohmic, Ohmic, or super-Ohmic according as the exponent s in

$$J(\omega) \sim \omega^s \tag{1.59}$$

is $s < 1$, $s = 1$, or $s > 1$. This classification is coarse and needs refinement, but does serve our purpose for the time being. The reader may consult Leggett *et al.* [29] for more details. As a rule of thumb, the dissipation can be considered small for the super-Ohmic case and large for the sub-Ohmic case.[8] The normal Ohmic case, $s = 1$, strikes a middle position between these two extremes and corresponds to the usual decomposition of the

[8] Actually, the sub-Ohmic case is pathological and leads to serious problems of divergence.

spectral density as $J(\omega) = \eta\omega$, where η is the phenomenological friction coefficient in expression (1.6).

Now, the application of phonon-related dissipation in the presence of impurities on a domain wall motion may require more care and, according to the few studies which have been made so far [24, 26], may give rise to an Ohmic dissipation. The dissipation caused by a spin bath, such as nuclear spins and magnetic impurities, was studied by Stamp and Prokof'ev [30] where a model different from the Caldeira-Leggett type must be considered with the consequence that such a bath can have disastrous dissipative effects on macroscopic quantum phenomena.

The interaction with magnons in the presence of pinning centers is the object of this study. To be specific, our goal is to resolve the dissipative effect of magnons at $T = 0$ on the quantum motion of a domain wall *bound* to a pinning center in the limit of a one-dimensional ferromagnetic system such as a magnetic wire. We will have the domain wall oscillate back and forth at the bottom of the potential well $V(Q)$ in (1.36), generated by the pinning defect, and extract the spectral density $J(\omega)$ which gives a qualitative measure of the dissipation to be expected from magnons.

This whole calculation is motivated by the fact that some experiments might provide the evidence for domain wall tunnelling; the reader is urged to consult the works of Uehara and Barbara *et al.* [31, 32, 33, 34]

This thesis will be divided as follows: in chapter 2, we justify the use of solitons and their attendant Sine-Gordon equation for the study of domain walls in a 1-D ferromagnet; in chapter 3, we review in detail the effect of a soliton colliding with some form of potential barrier and ,thus, prepare the stage for the manner with which we will deal with a soliton bound to a pinning center in chapter 4; finally, in chapter 5, a discussion of the results found in the previous chapter is given and possible refinements are proposed.

Chapter 2

A theoretical model for ferromagnetic wires with pinning centers

In the first and third sections of this chapter, we give a general theoretical model for ferromagnetic wires with impurities which display a soliton-like structure, the Bloch wall. In the second section, we introduce the scheme of quantization about classical solutions which will be further developed in the next chapter.

2.1 A model for some types of easy-axis ferromagnetic quasi-one-dimensional magnets

We use a one dimensional spin-chain having the following Hamiltonian

$$\hat{H} = -J \sum_n \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1} + K_z \sum_n (\hat{S}_n^z)^2 + K_y \sum_n (\hat{S}_n^y)^2 \tag{2.1}$$

where $K_y \ll K_z$. This Hamiltonian corresponds to a ferromagnet with isotropic exchange supplemented by a form of uniaxial anisotropy in which the symmetry in the x-y plane is broken by the weak anisotropy constant K_y in the y-direction. We will show below that such a Hamiltonian in the continuum and classical limit admits soliton solutions for which our spins are virtually confined to the x-y plane, this confinement being the result of the relatively high K_z. The soliton solution for the time-independent case is the classical Bloch Wall configuration separating domains with a 180^o reversal of magnetization. (cf. figure (1.3))

We note that equation (2.1) is one-dimensional whereas magnetic wires are obviously 3-dimensional objects. However, for domain walls, we can always assume the behavior

within the two-dimensional cross-section of the wire as uniform for the following reason. In (1.21), for a planar domain wall with the same anisotropy $K_y S^2 \sin^2\phi$ in the y-direction as above, m^{-1} gives a measure of the width of the wall in units of the lattice constant a:

$$m^{-1} = a\sqrt{\frac{J}{K_y}}.$$

We assume, then, that the cross-section of a magnetic wire is much less than $a^2 K_y/J$ so that we can, with confidence, consider the magnetization as uniform across any cross-sectional plane of the wire. Indeed, *ignoring* the demagnetization field, this rules out any wall configuration in directions other than the z-axis and any curvature of the domain wall.

In order to derive the equations of motion from the Hamiltonian (2.1) in the continuum and classical limit (i.e., loosely speaking, treating the magnetization S as a field and letting $\hbar$ go to zero), two methods are at our disposal : either we apply the continuum approximation to the equations of motion on the discrete lattice, or we work with the continuum limit of the Hamiltonian (2.1) using the Poisson brackets derived from the quantum commutation relations. We shall choose the first method [35].

In the Heisenberg picture of quantum mechanics, the operators carry the full burden of the dynamical evolution of the system's observables. We define the following decomposition of the magnetization vector $\mathbf{S}$:

$$\hat{\mathbf{S}}_n = \sqrt{S(S+1)}\,(\sin\hat{\theta}_n(t)\cos\hat{\phi}_n(t),\ \sin\hat{\theta}_n(t)\sin\hat{\phi}_n(t),\ \cos\hat{\theta}_n(t)) = (\hat{S}_n^x,\ \hat{S}_n^y,\ \hat{S}_n^z) \quad (2.2)$$

where $\{\hat{\theta}_n, \hat{\phi}_n\}$ are the polar and azimuthal angles of the magnetization vector for the n^{th} atom in the spin chain. We obviously have the usual operations on an eigenvector $|S, M_n\rangle$ such as $\hat{S}_n^z\,|S, M_n\rangle = \hbar\, M_n\,|S, M_n\rangle = \hbar\sqrt{S(S+1)}\,\cos\theta_n\,|S, M_n\rangle$ (the angles $\{\theta_n, \phi_n\}$ are quantized by virtue of the respective quantization of the magnetization components) as

well as the general commutation relationships for angular momentum observables such as $[\,\hat{S}_n^i, \hat{S}_m^j\,] = i\hbar\,\epsilon_{ijk}\,\hat{S}_n^k\,\delta_{n,m}$.

The Hamiltonian (2.1) can be rewritten as

$$\hat{H} = -J\sum_n \left\{\hat{S}_n^z\cdot\hat{S}_{n+1}^z + \frac{1}{2}(\hat{S}_{n+1}^+\cdot\hat{S}_n^- + \hat{S}_{n+1}^-\cdot\hat{S}_n^+)\right\} + K_z\sum_n(\hat{S}_n^z)^2 + K_y\sum_n(\hat{S}_n^y)^2, \quad (2.3)$$

the time evolution of the $\hat{S}_n^z(t)$ being given by

$$\begin{aligned}\frac{d\hat{S}_m^z}{dt} &= -\frac{i}{\hbar}\,[\,\hat{S}_m^z, \hat{H}\,] \\ &= \frac{i\,J}{2}\,\{\hat{S}_m^+\,(\hat{S}_{m-1}^- + \hat{S}_{m+1}^-) - \hat{S}_m^-\,(\hat{S}_{m-1}^+ + \hat{S}_{m+1}^+)\} + i\hbar\,K_y\,\hat{S}_m^z - 2K_y\,\hat{S}_m^x\,\hat{S}_m^y\,.\end{aligned}$$

Writing the above using our decomposition of the magnetization vector, we obtain:

$$\begin{aligned}\sqrt{S(S+1)}\frac{d}{dt}\cos\hat{\theta}_m \;=\; & \\ \frac{i\,J\,S(S+1)}{2}\Big\{&\sin\hat{\theta}_m\sin\hat{\theta}_{m-1}e^{i(\hat{\phi}_m-\hat{\phi}_{m-1})} \\ &- \sin\hat{\theta}_{m+1}\sin\hat{\theta}_m e^{i(\hat{\phi}_{m+1}-\hat{\phi}_m)} - \sin\hat{\theta}_m\sin\hat{\theta}_{m-1}e^{-i(\hat{\phi}_m-\hat{\phi}_{m-1})} \\ &+ \sin\hat{\theta}_{m+1}\sin\hat{\theta}_m e^{-i(\hat{\phi}_{m+1}-\hat{\phi}_m)}\Big\} \\ + i\hbar\sqrt{S(S+1)}\,K_y\,\cos\hat{\theta}_m \;-\; & 2\,S(S+1)\,K_y\,\sin\hat{\theta}_m\cos\hat{\theta}_m\sin\hat{\theta}_m\sin\hat{\phi}_m\,.\end{aligned} \quad (2.4)$$

By the Ehrenfest theorem, we know that the time evolution of the expectation value of an observable is given by

$$\frac{d\,\langle\hat{\Omega}(t)\rangle}{dt} = \frac{-i}{\hbar}\,\langle\,[\,\hat{\Omega}(t), \hat{H}\,]\,\rangle\,. \quad (2.5)$$

Since we are interested by the classical limit, we may always consider a quantum state $|\Psi\rangle$ which has the characteristics we would ascribe to a classical one, chiefly, that of being able to specify conjugate dynamical coordinates within the experimental uncertainty. Such a state can always be constructed in principle by the use of gaussian-like wavefunctions

which saturate the Heisenberg uncertainty inequality. For instance, for a free particle, the wavefunction

$$\psi_c(x) = \left(\frac{1}{\pi\Delta^2}\right)^{1/4} e^{\frac{ip_o x}{\hbar}} e^{\frac{-(x-x_o)^2}{2\Delta^2}} \tag{2.6}$$

will give uncertainties $\Delta X = \Delta$ and $\Delta P = \hbar/\Delta$, both of which can be made small in the classical regime by choosing Δ appropriately and consonant with the experimentally detectable range. For instance, by choosing $\Delta = 10^{-13}\ cm$,which is the size of a proton, we obtain $\Delta P = 10^{-14}\ g\ cm/sec$ which translates for a particle of mass 1 g to a mere uncertainty in speed Δv of about $10^{-14}\ cm/sec$! Hence, in the classical scale, such a state can always be said to have well defined values of X and P, namely, x_0 and p_0, since the fluctuations around these values are truly negligible. In the same manner, we assume that we have a state $|\Psi_s\rangle_c$ such that we can ascribe to it angular momentum with precise components in each directions in the sense understood above. Then, the angles $\{\theta, \phi\}$ will have definite values with small fluctuations which, in any case, will be negligible as we let $\hbar \to 0$ to mimic the classical behavior. Therefore, with impunity, we will consider such a state and sandwich (2.5) with it leading to our particular form of the Ehrenfest theorem[1]:

$$\begin{aligned}
\hbar\sqrt{S(S+1)}\,\frac{d}{dt}\cos\theta_m \;=\; & \\
\hbar^2\,\frac{iJ\,S(S+1)}{2}\Big\{ & \sin\theta_m \sin\theta_{m-1} e^{i(\phi_m - \phi_{m-1})} \\
& - \sin\theta_{m+1}\sin\theta_m e^{i(\phi_{m+1}-\phi_m)} - \sin\theta_m \sin\theta_{m-1} e^{-i(\phi_m - \phi_{m-1})} \\
& + \sin\theta_{m+1}\sin\theta_m\, e^{-i(\phi_{m+1}-\phi_m)}\Big\} \\
+\, i\hbar^2\sqrt{S(S+1)}\,K_z\cos\theta_m \;-\; & 2\hbar^2\,S(S+1)\,K_y \sin\theta_m\cos\phi_m \sin\theta_m \sin\phi_m\,.
\end{aligned} \tag{2.7}$$

We want a solution where the magnetization vector is nearly confined to the x-y plane;

[1]This state is $\prod_m \otimes \mid \psi_s\rangle_{c,m}$ where $\mid \psi_s\rangle_{c,m}$ is the nearly classical state constructed above for the lattice point m. Also, note that we rule out possible linear combinations of such general state vectors since we know, by experience, that such combinations for the classical realm have never been observed.

so we expand the functions of θ_m around the value $\theta'_m = \pi/2$ and consider, instead of θ_m, the deviations $\theta^s_m = \theta'_m - \theta_m$. Hence, we obtain[2]

$$\cos\theta_m \simeq \theta^s_m \tag{2.8}$$

and

$$\sin\theta_m \simeq 1 - (\theta^s_m)^2. \tag{2.9}$$

Taking the discrete result over to the continuum case involves the use of expansions about the "field points" $\theta(z,t)$ and $\phi(z,t)$ and the assumption that those fields are weakly varying over z and t. The details are left in appendix A and we merely state the result reminding the reader that we disregard any small variations of order 2 or higher.

$$\begin{aligned}\hbar\sqrt{S(S+1)}\,\frac{\partial\theta^s(z,t)}{\partial t} &= \hbar^2 S(S+1)\,a^2 J\,\frac{\partial^2\phi(z,t)}{\partial z^2} \\ &\quad + i\hbar^2\sqrt{S(S+1)}\,K_z\,\theta^s - 2\hbar^2 S(S+1)\,K_y\cos\phi\sin\phi\,.\end{aligned} \tag{2.10}$$

The classical limit is obtained by having $\hbar \to 0$ and $S \to \infty$ such that the product $\hbar S$ remains finite and gives the classical magnetization s in the limiting process. Thus, by taking the limit on both sides of equation (2.10), we obtain

$$\frac{\partial\theta^s(z,t)}{\partial t} = a^2 s\,J\,\frac{\partial^2\phi(z,t)}{\partial z^2} - 2\,K_y\cos\phi\,\sin\phi\,, \tag{2.11}$$

the second term disappearing since

$$\lim_{\substack{\hbar\to 0\\ S\to\infty}} \hbar^2\sqrt{S(S+1)} = \lim_{\substack{\hbar\to 0\\ S\to\infty}} \hbar^2 S = s\lim_{\substack{\hbar\to 0\\ S\to\infty}} \hbar = 0\,.$$

The value a is the lattice spacing of our near-dense spin chain.

Let $\eta \triangleq 2\,\phi$. Then, the above equation reduces to

$$\frac{\partial\theta^s(z,t)}{\partial t} = \frac{a^2 J\,s}{2}\,\frac{\partial^2\eta(z,t)}{\partial z^2} - K_y\,s\,\sin\eta(z,t)\,. \tag{2.12}$$

[2] We could also have made this expansion for the operators $\cos\hat{\theta}_m$ and $\sin\hat{\theta}_m$ resulting, respectively, in $\hat{\theta}^s_m$ and $1-(\hat{\theta}^s_m)^2$ where $\cos\hat{\theta}_m$ is the operator measuring the deviation from the x-y plane.

pp In like manner, we can proceed with the time evolution of operator $S_m^+(t)$ and obtain the following equation in the continuum and classical limit:

$$\frac{\partial\,\phi(z,t)}{\partial t} \;\approx\; Ja^2\,s\,\frac{\partial^2\,\theta(z,t)}{\partial z^2} \;+\; 2\,K_z\,s\,\theta_s\,. \tag{2.13}$$

In this derivation, which is left in appendix A, it is important to note that we explicitly made use of the fact that $K_y \ll K_z$.

Differentiating equation (2.13) with respect to time, discarding the resulting term $\dfrac{\partial^3\theta}{\partial t\partial z^2}$ as being negligible, and using equation (2.12), we get

$$\frac{\partial^2\phi(z,t)}{\partial t^2} \;\approx\; 4\,K_z\,s\left(\frac{a^2J\,s}{2}\,\frac{\partial^2\,\eta(z,t)}{\partial z^2} \;-\; K_y\,s\,\sin\eta\right). \tag{2.14}$$

Rearranging the terms and dividing through by $a^2s^2\,K_z\,J$ leads to

$$\frac{\partial^2\,\eta(z,t)}{\partial z^2} \;-\; \frac{1}{c^2}\frac{\partial^2\,\eta(z,t)}{\partial t^2} \;=\; m^2\,\sin\eta(z,t) \tag{2.15}$$

where $c = a\,s\,(2K_zJ)^{1/2}$, a characteristic velocity, and $m = (2K_y/a^2J)^{1/2}$ playing the role of a mass. The above equation is none other than the Sine-Gordon equation we met in chapter 1!

Hence, from the requirement that the magnetization vector be confined to the x-y plane, we have shown that the field-like classical limit of our ferromagnetic Heisenberg spin chain admits a soliton solution. Incidentally, the Poisson bracket derived from the quantum commutator relation $\sqrt{S(S+1)}\,[\cos\hat{\theta}_n(t), \hat{\phi}_m(t')] = -i\hbar\,\delta_{nm}\,\delta_{tt'}$, or using the small fluctuations approximation about the x-y plane

$$\sqrt{S(S+1)}\,[\,\hat{\theta}_n^s(t), \hat{\phi}_m(t')\,] \;=\; -i\hbar\,\delta_{nm}\,\delta_{tt'}, \tag{2.16}$$

is $s\,\{\theta_n^s(t), \phi_m(t')\} = -\delta_{nm}\,\delta_{tt'}$ which transforms in the continuum case to

$$\{s\,\theta^s(z,t),\ \phi(z',t')\} = -\delta_{nm}\,\delta_{tt'}\,.$$

Therefore, $s\,\theta(z,t)$ becomes the conjugate momentum to the angular position $\phi(z,t)$ so that, upon using equation (2.13) and discarding again the small term $Js\dfrac{\partial^2\theta^s}{\partial z^2}$ we have[3]

$$\frac{\partial\,\phi(z,t)}{\partial t} \;=\; 2\,K_z\,s\,\theta^s(z,t)\,. \tag{2.17}$$

We note that the Sine-Gordon equation is Lorentz invariant with respect to the characteristic velocity c. As shown above, this characteristic velocity is proportional to the lattice constant a, which is consistent with a reduction of intermediate lattice sites between any two points thereby increasing the response time of the spin chain.

For the time-independent case, equation (2.15) reduces to the fixed Bloch Wall solution

$$\frac{\partial^2\eta}{\partial z^2} \;=\; m^2\,\sin\eta\,, \tag{2.18}$$

and θ_s equals zero by virtue of

$$\frac{\partial\,\eta(z,t)}{\partial t} \;=\; 4\,K_z\,s\,\theta^s(z,t)\,; \tag{2.19}$$

confirming that the magnetization is truly confined to the x-y plane. With the soliton moving, $\theta_s(z,t)$ no longer equals zero and thus provides a magnetization in the z-direction to sustain this motion (as explained in chapter 1).

One might wonder why we bothered deriving the classical limit of the Heisenberg spin chain (2.5). The reason for doing so is to provide us with the platform for the quantization of the magnetization field by a method which treats quantum fluctuations about a classical solution. One may seriously objects to our starting with a perfectly quantum Hamiltonian (equation (2.5)), deriving the classical limit, and then trying to quantize the system all over again! "Should you not have stuck with the quantum mechanical Hamiltonian that you had and worked with it directly taking good care with the transition towards a quantum field?," such an astute reader may retort. Very well, but the

[3]Recall that the transformation from the Poisson brackets of dynamical quantities $A(t)$, $B(t)$ to the quantum commutators is given by the prescription $\{A,B\}_{PB} \to -(i/\hbar)[\hat{A},\hat{B}]$.

Heisenberg equations of motion derived from this Hamiltonian are intractable and so must be supplemented by some classical intuition where we *suppose* that the quantum phenomenon we are interested in manifests itself in the form of small quantum fluctuations (which may have nonetheless important observable effects such as the α-decay) about a stable configuration in classical phase space (such as a classical particle to sit quietly at the bottom of a harmonic potential well).

After, we hope, dispelling some misgivings, it seems appropriate to introduce this quantization scheme about a classical solution. The exposition of the subject follows closely that given in a book by Rajamaran [36], and we refer the reader to this book for more details.

2.2 Quantization about classical solutions

In order to give a qualitative flavor of the method and help to build up our intuition, we consider a non-relativistic unit-mass particle under the influence of a potential $V(x)$ in one dimension as shown in figure (2.1).

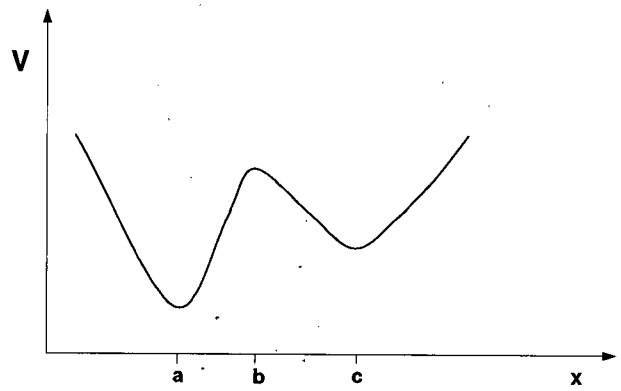


Figure 2.1: A general potential $V(x)$. The point **a** can be thought as the absolute minimum of the potential whereas point **b** constitutes a relative minimum. The point **b**, although an extremum of the potential function, is obviously an unstable one.

In the classical regime, the particle obeys Newton's law of motion given by $\frac{d^2x}{dt^2} = -\frac{dV}{dx}$. Quantum mechanically, the particle is assigned a state vector $|\psi\rangle$ in a one-particle Hilbert space whose projection in the x-basis is the familiar wavefunction $\psi(x)$. The energy eigenstates are given by solving the time-independent Schrödinger equation:

$$\hat{H}\,\psi_n \;=\; \frac{1}{2}\,(\hat{p}^2 \,+\, V(x))\,\psi_n \;=\; E_n\,\psi_n \tag{2.20}$$

represented in the x-basis with $\hat{p} = -i\hbar\frac{d}{dx}$. We will now draw some relationships between the classical and quantum cases.

Classically, the stable time-independent solutions for the particle with the potential $V(x)$ shown are $x = a$ and $x = c$, the first case being an absolute minimum while the second a relative one (the point $x = b$ is obviously an unstable point). The absolute minimum corresponds to the the classical ground-state of the particle with energy $E_g^{cl} = V(a)$. In quantum mechanics, such a tranquil state is forbidden by the Heisenberg uncertainty principle; small fluctuations exist about this classical ground state. For energy states close enough to E_g^{cl}, the potential $V(x)$ may not differ much from $V(a)$ so that a truncated Taylor expansion about $x = a$ is legitimate

$$V(x) \;\approx\; V(a) \,+\, \frac{1}{2}\omega^2\,(x-a)^2 \,+\, \frac{1}{3!}\,\lambda_3\,(x-a)^3 \,+\, \frac{1}{4!}\,\lambda_4\,(x-a)^4, \tag{2.21}$$

where ω^2 corresponds to the curvature of $V(x)$ at $x = a$ and the linear term is absent as $V(a)$ is an extremum. Now, for those state vectors $|\psi\rangle$ that satisfy the condition

$$\lambda_r\,\langle\psi|\,(\hat{\mathbf{X}}-a)^r\,|\psi\rangle \;\ll\; \omega^2\langle\psi|\,(\hat{\mathbf{X}}-a)^2\,|\psi\rangle \tag{2.22}$$

for $r = 3, 4$, and higher, the anharmonic terms in (2.15) will be negligible so as to leave us with a quadratic potential for all intents and purposes. Furthermore, we exactly know how to treat a harmonic oscillator quantum mechanically; hence, for those energy levels

close to E_g^{cl} and states obeying inequality (2.22), we have the approximate energies

$$E_n \;=\; V(a) \;+\; (n+\frac{1}{2})\,\hbar\,\omega \;+\; \mathcal{O}(\lambda_r). \tag{2.23}$$

Note that the first term corresponds to the classical energy, the second, the first order quantum correction with ω being the classical stability frequency, and the last, higher order corrections which may be treated by standard perturbation techniques. Already, in this simple equation, the right- hand side of which contains classically obtainable values, we can appreciate the many connections between the quantum and classical regimes (recall that even $\lambda_r'^s$ are classical values, they represent successive derivatives of the potential $V(x)$ at the *classical* position $x = a$). Equation (2.23) is valid only up to some quantum number n at which point it is unwise any longer to consider E_n as a low-lying state. Note that inequality (2.22) implies state vectors whose wavefunctions are localized around $x = a$, once again the classical solution.

Now, if we return to our classical particle, we can easily determine by inspection of $V(x)$ another static solution: the particle sits at the bottom of the well located at $x = c$, the local minimum. The energy of the particle is then $E_{ex}^{cl} = V(c)$ which is higher than that at the absolute minimum. The energy E_{ex}^{cl} corresponds to an excited state above the classical ground state energy E_g^{cl} about which one can also construct a set of quantum energy levels using an expansion of $V(x)$ around $x = c$ instead. Those energy levels will be higher than any of those built around E_g^{cl} in (2.23). Running quickly through the previous procedure, we obtain :

$$V(x) \;\approx\; V(c) \;+\; \frac{1}{2}\,(\omega')^2\,(x-c)^2 \;+\; \frac{1}{3!}\,\lambda_3'\,(x-c)^3 \;+\; \frac{1}{4!}\,\lambda_4'\,(x-c)^4 \tag{2.24}$$

for the state vectors $|\psi\rangle$ such that

$$\lambda_r'\,\langle\psi|\,(\hat{\mathbf{X}}-c)^r\,|\psi\rangle \;\ll\; (\omega')^2\,\langle\psi|\,(\hat{\mathbf{X}}-c)^2\,|\psi\rangle \tag{2.25}$$

holds for all $r \geq 3$. Here, ω' is the classical stability frequency *at* $x = c$ and λ'_r the corresponding derivatives of $V(x)$ at the same point. Once again, those state vectors have wavefunctions strongly localized about the local minimum $x = c$. The energy levels built around this excited classical solution is then

$$E_{ex,n} \;=\; V(c) \;+\; (n + \frac{1}{2})\,\hbar\,\omega' \;+\; \mathcal{O}(\lambda'_r). \tag{2.26}$$

The same limitations apply for this expression as for the previous one.

Now, here lies the full force of our analogy: for a classical field $\phi(x,t)$ there also exist static solutions, say $\phi_1(x,t)$ and $\phi_2(x,t)$, for which the energy of $\phi_2(x,t)$ may, for instance, be higher than that of $\phi_1(x,t)$. This latter solution may therefore be thought of as the classical ground state of the system in the same way as $x = a$ was for the discrete case. Thus, one can also build approximate quantum energy levels about those respective classical field solutions by following the same method outlined above but generalized to fields; namely, by "Taylor expanding" the potential functional $V[\phi]$ and keeping only the quadratic part. The important feature to bear in mind is that the energetically higher classical field solution $\phi_2(x,t)$ may very well be a *non-perturbative* solution in the sense that as $\lambda \to 0$ (λ being a parameter of the Hamiltonian which is present in $\phi_2(x,t)$) this solution fails to make sense ...it simply blows up! In other words, the solution $\phi_2(x,t)$ cannot be continuously deformed to the classical ground state solution $\phi_1(x,t)$ and corresponds, in à more sophisticated jargon, to a completely different topological sector of the Hamiltonian spectrum. Before giving the details concerning the transition to fields, the perspicacious reader may wonder what happens if ω equals zero as inequality (2.22) can never be fulfilled. In this case, the curvature is zero, so that the wavefunction is less localized that it would be in the presence of a harmonic-like potential well, and our procedure completely breaks down insofar as a series of harmonic-like low-lying energy states cannot be found (for those vanishing curvatures cases only). One might want to

construct states keeping only the more significant stable anharmonic term (for instance, $\tilde{\lambda}_4\, x^4$); however, we assume that most minima we are interested in will be of a harmonic type.

A more interesting case arises when $V(x)$ is independent of x in some region; classically, the particle can be at rest at any of those points within this region for which all derivatives ω, $\{\lambda_r\}$ of the potential equal zero. In other words, there is absolutely no constraint to localize the wavefunction, and consequently, it will tend to spread within this region and adopt some form of wavefunction resembling e^{ikx} (the momentum eigenfunction). The extreme limit would be a potential independent of the coordinate x_1 over its *entire* range even though it may be constrained along the other coordinates x_i (we consider a multi-dimensional potential $V(\mathbf{x})$). This independence of $V(\mathbf{x})$ for one dimension corresponds to a translational symmetry of the system along this direction. Therefore, our procedure needs be modified for those coordinates absent in the potential $V(\mathbf{x})$; we will deal with this particular problem as the case arises later in our treatment of soliton dynamics. [Indeed, even for fields, there exist symmetries for which the corresponding frequencies ω vanish (these are called zero-frequency or translational modes).] In the meantime, in order to give a brief idea of how we will accomplish this, note that for a quantum particle in a constant potential $V(x)$, its energy levels are given by $E_n = V + \frac{1}{2}\,(p_n)^2$ where p_n are the *conserved* momenta; the conservation of which arises precisely from the translational symmetry of the problem. Hence, for zero-frequency modes, one might expect to deal with *conserved* momenta conjugate to those coordinates displaying the various symmetries (in the Lagrangian formalism, it is useful to think about these as the cyclic coordinates).

We are now ready to move on to the case of fields. Consider for simplicity a scalar field $\phi(\mathbf{x}, t)$, vector fields being easy to generalize to. The dynamics of the field is governed

by a Lagrangian functional

$$L[\phi] = \int d\mathbf{x} \left\{ \frac{1}{2} \left(\frac{\partial \phi(\mathbf{x},t)}{\partial t} \right)^2 - \frac{1}{2} (\nabla\phi)^2 - U(\phi) \right\} \tag{2.27}$$

$$= T[\phi] - V[\phi], \tag{2.28}$$

with kinetic energy

$$T[\phi] = \frac{1}{2} \int d\mathbf{x} \left(\frac{\partial \phi(\mathbf{x},t)}{\partial t} \right)^2$$

and potential energy

$$V[\phi] = \int d\mathbf{x} \left\{ \frac{1}{2} (\nabla\phi)^2 - U(\phi) \right\}. \tag{2.29}$$

The classical solutions for this Lagrangian is obviously given by the Euler-Lagrange equations of motion

$$\frac{\partial^2 \phi(\mathbf{x},t)}{\partial t^2} = -\frac{\delta V[\phi]}{\delta \phi(\mathbf{x},t)}, \tag{2.30}$$

where $\frac{\delta}{\delta\phi(\mathbf{x},t)}$ is understood as a functional derivative. If we enquire about the static solutions, these are simply given by the generalization of the extremum condition for the functional potential $V[\phi]$:

$$\frac{\delta V[\phi]}{\delta \phi(\mathbf{x},t)} = 0. \tag{2.31}$$

From equation (2.31) (and the second order version), one may find the field minima $\phi_i(\mathbf{x})$ which can be arranged in ascending order of energy so that $\phi_0(\mathbf{x})$ is considered as the absolute minimum or ground-state of the system (also referred to as the virtual vacuum). Consider one such minimum $\phi_n(\mathbf{x})$. One may then expand the functional $V[\phi]$ about such a minimum in a functional Taylor expansion

$$V[\phi] = V[\phi_n] + \int d\mathbf{x} \, \frac{1}{2} \left\{ \eta(\mathbf{x}) \left[-\nabla^2 + \left. \frac{d^2U}{d\phi^2} \right|_{\phi=\phi_n} \right] \eta(\mathbf{x}) \right\} + \ldots \tag{2.32}$$

where $\eta(\mathbf{x}) \triangleq \phi(\mathbf{x}) - \phi_n(\mathbf{x})$, the dot representing higher terms in $\eta(\mathbf{x})$. The second term is the second order or harmonic variation in $\eta(\mathbf{x})$ in which the term between brackets

corresponds to

$$\left.\frac{\delta^2 V[\phi]}{\delta \phi^2}\right|_{\phi(\mathbf{x})=\phi_n(\mathbf{x})}$$

(note that integration by parts has been used) in an obvious generalization of $\dfrac{d^2 V(x)}{dx^2}$ for the one-dimensional case discussed above. The linear term is absent by the very fact that we expand about a minimum so that

$$\left.\frac{\delta V[\phi]}{\delta \phi}\right|_{\phi=\phi_0} = 0.$$

What we want to do next is to diagonalize the operator

$$-\nabla^2 + \left.\frac{d^2U}{d\phi^2}\right|_{\phi(\mathbf{x})=\phi_n(\mathbf{x})}$$

so as to express the harmonic term in equation (2.32) in terms of normal modes $f_i(\mathbf{x})$; these being found by solving the eigenvalue equations

$$\left[-\nabla^2 + \left.\frac{d^2U}{d\phi^2}\right|_{\phi(\mathbf{x})=\phi_n(\mathbf{x})}\right] f_i(\mathbf{x}) = \omega_i^2 f_i(\mathbf{x}). \tag{2.33}$$

These modes constitute an orthonormal basis in terms of which one may write the variation $\eta(\mathbf{x})$ about the minimum $\phi_n(\mathbf{x})$ as

$$\eta(\mathbf{x}) = \sum_i c_i f_i(\mathbf{x}), \tag{2.34}$$

so that the harmonic term in (2.32) becomes simply a sum over the amplitudes of the modes times their corresponding energies : $(1/2)\sum_i (c_i)^2 \omega_i^2$.

Setting

$$\begin{aligned}\eta(\mathbf{x},t) &= \phi(\mathbf{x},t) - \phi_n(\mathbf{x}) \\ &= \sum_i c_i(t) f_i(\mathbf{x})\end{aligned}$$

and assuming that the fluctuation $\eta(\mathbf{x},t)$ remains small enough over time to neglect the higher order terms in (2.32) represented by the dots, one may write the Lagrangian L as

$$L = \frac{1}{2}\sum_i [\dot{c}_i(t)]^2 - \left(V[\phi_n] + \frac{1}{2}[c_i(t)]^2 \omega_i^2\right) + \mathcal{O}(\eta^3(\mathbf{x},t)), \tag{2.35}$$

where the correction $\mathcal{O}(\eta^3(\mathbf{x},t))$ can always be treated by the usual perturbation techniques. The most significant terms in (2.35) correspond then to harmonic oscillations which (as the one-dimensional case discussed above) we perfectly know how to quantize.

Therefore, one constructs a set of low-lying energy states around the classical static field solutions $\phi_n(\mathbf{x})$ in the same manner as before,

$$E_{n_i} \;=\; V[\phi_n] \;+\; \hbar \sum_i \left(n_i + 1/2\right) \omega_i \;+\; \text{higher order terms}, \tag{2.36}$$

owing to the quantization of the normal modes coefficients c_i. Once again, we remind the reader that the higher order terms can be treated perturbatively and that $\phi_n(\mathbf{x})$ may turn out to be a non-perturbative result in the sense outlined above. The classical expression (2.35) is valid as long as the fluctuation $\eta(\mathbf{x},t)$ around a given minimum $\phi_m(\mathbf{x})$ is negligible. But, it may happen, as before, that a symmetry exists for the Lagrangian L so that $V[\phi]$ has a minimum $\phi_n(\mathbf{x})$ which is translational invariant, i.e. one for which $\phi_n(\mathbf{x}+\mathbf{a})$, for any vector $\mathbf{a}$, are also minima with the same energy $V[\phi_n(\mathbf{x})]$. In such a case, we will obtain eigenfunctions $f_i(\mathbf{x})$ with eigenvalues $\omega_i = 0$ for which the procedure given above also breaks down. These translational modes[4] are the exact equivalent of those symmetries encountered before, for which $V(\mathbf{x})$ was independent of some coordinates x_i, if one is willing to view $\phi_n(\mathbf{x})$ as a point in the field space $\phi(\mathbf{x})$ and consider $\phi_n(\mathbf{x}+\mathbf{a})$ as a ray or line in this space. As promised, these anomalies will be treated in due course. Note from equation (2.36) that the energy levels are built around a classical static solution $\phi_n(x)$ with the leading quantum correction given by the harmonic oscillators. For the absolute minimum of $V[\phi]$, $\phi_o(\mathbf{x})$ which is space-independent on account of the term $(\nabla^2\phi)^2$ in $V[\phi]$ and which may, incidentally, be degenerate, this corresponds to the vacuum state and its familiar associated quanta of the field.

[4]The word translational may convey the wrong impression that symmetries may exist only for "rectilinear dimensions", which is not true. One may also have angular-like symmetries. Perhaps, one should adopt the more appropriate term zero-frequency mode.

In our case, however, we will build quantum energy levels around a classical static solution of the Euler-Lagrange equations of motion which corresponds to a soliton with higher energy than the ground-state solution $\phi_o(\mathbf{x}) \equiv 0$.

After this introduction to the quantization procedure about classical solutions, we discuss in the next section the toy-model we will be considering in this thesis.

2.3 Toy-model for pinning centers in magnetic wires

The mathematical model we will consider is the one-dimensional Sine-Gordon Hamiltonian density plus a perturbation term of the form $\alpha \frac{\partial \psi}{\partial x} V(x)$ where α is a small coupling constant:

$$\mathcal{H} = \mathcal{H}_o + \mathcal{H}_p \quad \text{with} \tag{2.37}$$

$$\mathcal{H}_o = \frac{1}{2}\pi^2 + \frac{c_o^2}{2} + \omega_o^2 (1 - \cos\psi), \tag{2.38}$$

$$\mathcal{H}_p = \alpha \frac{\partial \psi}{\partial x} V(x), \tag{2.39}$$

c_o and ω_o being the characteristic velocity and frequency of the system. The coupling with the gradient of the field itself has been chosen so as to eliminate possible ambiguity with regards to the different values of the SG soliton $\psi_s(x)$ at $x = \pm\infty$ ($\psi_s(-\infty) = 0$ and $\psi_s(+\infty) = 2\pi$ for $N = 1$). The field $\psi(x,t)$ is dimensionless since it corresponds to angular values and the above Hamiltonian density can be made totally so by introducing the following scaling

$$t^* \triangleq \omega_o t, \quad x^* \triangleq (\frac{\omega_o}{c_o}) x, \quad \text{and} \quad \alpha^* \triangleq \frac{\alpha}{c_o \omega_o}.$$

The reason why we choose this Hamiltonian density is clear from the discussion at the beginning of this chapter: we found that the spin chain modeling a ferromagnetic wire admits in the continuum limit a Sine-Gordon soliton solution as exemplified by equation

(2.15). The correspondence with equation (2.38) is achieved by setting m in equation (2.15) to ω_o/c_o. From now on, we will deal exclusively with the dimensionless Hamiltonian density and drop the asterisks on the space-time coordinates and the coupling constant for convenience.

As mentioned in the introduction, we want to investigate the effect of pinning centers on the motion of solitons. Thus, how do we model those pinning potentials? We know by experience that those pinning potentials are very strong within a very narrow range. Qualitatively, such potentials can be approximated by a Dirac-delta functions $\delta(x)$. In any case, the real pinning potential $V(x)$ can always be decomposed as a linear superposition of such delta functions,

$$V(x) \;=\; \int dx_o \, V(x_o) \; \delta(x - x_o) \,,$$

so that, since we will consider a *linear* approximation about the soliton solution, the response of the system to such a potential can be inferred from that of a single delta function. One may have some difficulties to make sense of the coupling of $\delta(x)$ to the gradient $\dfrac{\partial \psi}{\partial x}$ which results in a a force-like term of the form $\delta'(x)$ in the Euler-Lagrange equations of motion. After some reasoning, the reader may convince himself that such an idealized force $\delta'(x)$ (which carries only the qualitative aspect of what the real force might be) will indeed constrain a moving soliton between two points once we recall our mechanical analogy with weights and springs discussed in the introduction for which the wave propagating is of an *angular* sort and *not* of the usual linear one, which we have come to think about whenever the term wave is evoked. Indeed, such a reasoning will be confirmed later by the derivation of a hyperbolic secant potential given in terms of the soliton's center coordinate.

As a preliminary to the study of solitons bound to impurities, we will examine in the next chapter the response of a soliton to a potential $V(x) = \Theta(x - x_o)$ in equation (2.39). This will introduce us to the notion of emission of radiation consequent to a soliton hitting such a step potential and acquaint us to the way with which we will similarly tackle the pinning potential case. This emission of radiation is an example of excitation of the quantum field built around the soliton solution.

Chapter 3

Soliton bremsstrahlung and the method of collective coordinates

The treatment in the first section of this chapter is largely based on a paper by Eilenberger published in 1977 [37]. This will be divided in two sections; the first one treats the subject at a classical level where we seek small-amplitude solutions about the soliton solution, the second and third go beyond Eilenberger and attempts the jump to the quantized version.

3.1 Classical treatment of the soliton bremsstrahlung

The Hamiltonian density is given by equation (2.43) and (2.44) with $V(x) = -\Theta(x-x_o)$. An integration by parts for

$$H_p \;=\; \alpha \int dx\; \mathcal{H}_p(x,t)$$

allows us to rewrite it as

$$H_p \;=\; \alpha \int dx\; \psi(x,t)\, \delta(x-x_o) \;-\; K$$

with $K = \lim_{x\to+\infty} \psi(x,t)$ which we consider as being equal to 2π, the limiting angular value for a soliton with topological index N equal to one (cf. introduction). As a constant appearing in a Hamiltonian is benign insofar as the dynamics of the system is concerned, we may disregard it totally. Hence, the perturbation Hamiltonian density can be considered as being equal to $+\alpha\, \psi(x,t)\, \delta(x-x_o)$ where $-\alpha\, \delta(x-x_o) = F(x)$ plays the role of a Dirac-delta force acting on the system. Since the Euler-Lagrange equations of motion for $\mathcal{H}_o$ are linearized about the soliton solution (cf. introduction), the linear response to a general force $F(x)$ can be determined by superposition.

We consider a *fast* soliton moving with speed v to the right with the step potential positioned at x_o, the location of the impurity or defect. Intuitively, the soliton begins to feel the step potential only when its center moves in its vicinity. At this point, a force acts on the soliton which has it slow down until it has gone beyond the potential's immediate range. Thus, we expect the soliton to emerge from its interaction with the potential with a speed v' which is less than the initial speed v. However, since we suppose a very fast soliton, v' should not be so different from v in the presence of a soft potential owing to the smallness of the coupling constant α. By conservation of energy, the impurity should in principle also gain a small speed; however, we assume that the restoring forces in the material are strong enough to counter this motion so as to leave the impurity stationary. These forces in the material are not accounted for in the above Hamiltonian as we isolate the relevant part of the universe and "ignore" the surrounding whose effect, in any case, manifests itself in the renormalized masses of the various parts of the system under consideration. The important point to note, here, is that radiation can also be emitted as a result of the collision in the same way as we expect heat to be given off when a fast ball grazes the surface of a large massive block. A still better analogy is the case of a charged particle which, being decelerated by some force, gives off radiation (for instance, an electron colliding with a nucleus surrounded by its electrical field). One may then substitute the charged particle for our soliton and the decelerating force for that provided by the step potential; this is the origin of the term "bremsstrahlung" to describe this process of a soliton collision.

Solving for the equation of motion derived from our Hamiltonian density $\mathcal{H}$, we obtain

$$\ddot{\psi} - \psi'' + \sin\psi = -\alpha\,\delta(x - x_o)\,. \tag{3.1}$$

Linearizing about the small-amplitude $\phi(x,t)$ as outlined in the introduction leads to

$$\ddot{\phi}(x,t) - \phi''(x,t) + (1 - 2\mathrm{sech}^2[\gamma(x - vt)])\,\phi(x,t) = -\alpha\,\delta(x - x_o)\,. \tag{3.2}$$

For $t \to -\infty$, the initial condition where the soliton is far away to the left of the step potential, the hyperbolic secant term in (3.2) virtually vanishes to leave us with

$$\ddot{\phi}(x,t) \; - \; \phi''(x,t) \; + \; \phi(x,t) \; = \; -\alpha \, \delta(x - x_o) \, . \tag{3.3}$$

We suppose that the behavior of ϕ for this initial condition is relatively independent of time so that a Fourier transform of equation (3.3) can be performed with respect to the position only,

$$(k^2 + 1) \, \hat{\phi}(k) \; = \; -\alpha \, ,$$

where $\hat{\phi}(k)$ is the Fourier transform of $\phi(x)$ in the vicinity of "$t \approx -\infty$." One can easily solve this equation, transform back to the x-space, and obtain

$$\phi_o(x) \; = \; -\frac{\alpha}{2} \, e^{-|x - x_o|} \tag{3.4}$$

as the initial small-amplitude field generated by the sole presence of the step potential. This can be referred to as the "dressing" of the impurity located at x_o.

We expect, then, to recover the same dressing at $t \to -\infty$ with the soliton moving with a slightly different speed to the right of the impurity. It is convenient to transform to a different inertial frame in which the soliton is stationary. In this reference frame, we have instead the impurity or step potential coming from the right with speed v (cf. fig. (3.1)).

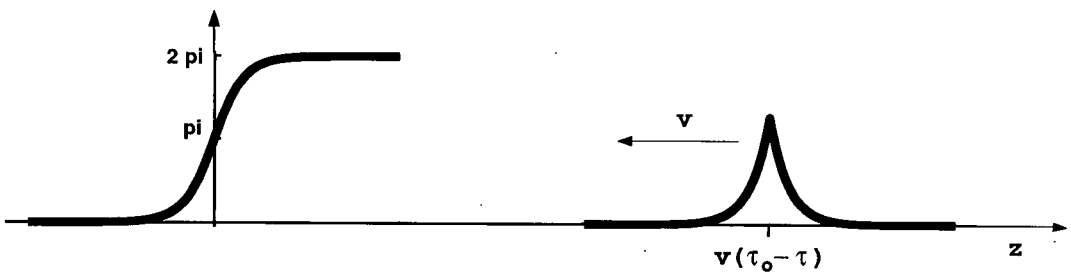


Figure 3.1: The impurity wavefunction "moving" towards a stationary soliton. In the rest frame of the soliton, whose center is located at coordinate $\mathbf{z} = 0$, it is the impurity center, located at $\mathbf{z} = \mathbf{v}(\tau_o - \tau)$ where τ is the soliton's proper time, that moves towards it with speed $\mathbf{v}$. The impurity dressing assumes an even decaying exponential form at large distances from the soliton's center.

After performing the required Lorentz transformation [38]

$$x = \gamma(z + v\tau)\,, \quad t = \gamma(\tau + vz)\,, \quad v\gamma\tau_o = x_o\,, \tag{3.5}$$

the last equality making the impurity collide with the stationary soliton located at $z = 0$ at time $\tau \approx \tau_o$, we obtain

$$\ddot{\phi}(z,t) \; - \; \phi''(z,t) \; + \; (1 - 2\,\mathrm{sech}^2[\gamma(z - vt)])\,\phi(z,t) \; = \; -\frac{\alpha}{|\gamma|}\,\delta(z + v(\tau - \tau_o))\,, \tag{3.6}$$

the partial derivatives being with respect to the new coordinates (z, τ). Since we consider a soliton with topological index N equal to one, i.e., a kink, $\gamma \geq 0$ and we may drop the absolute sign in equation (3.6).

Expanding $\phi(z,\tau)$ in the orthonormal basis $\{f_b(z), f_k(z)\}$ that we found earlier,[1]

$$\phi(z,\tau) \; = \; \frac{c_b(\tau)}{8}\,f_b(z) \; + \; \int dk\; c_k(\tau)\,f_k(z) \tag{3.7}$$

where $(c_k(\tau))^* = c_{-k}(\tau)$ ensuring thereby that $\phi(z,\tau)$ remains a *real* field, we get

$$\frac{1}{8}\,\{[\ddot{c}_b(\tau) \; + \; c_b(\tau)\,(1 - 2\mathrm{sech}^2 z)]\,f_b(z) \; - \; c_b\,f_b''(z)\}$$
$$+ \int dk\;\{[\ddot{c}_k(\tau) \; + \; c_k(\tau)\,(1 - 2\mathrm{sech}^2 z)]\,f_k(z) \; - \; c_k\,f_k''(z)\} = -\frac{\alpha}{\gamma}\,\delta(z + v(\tau - \tau_o)). \tag{3.8}$$

In order to obtain the equations of motion for the normal modes' coefficients, we simply multiply the above equation by the complex conjugate of a normal mode $f_p(z)$, integrate the resulting equation with respect to z, and use the orthonormality relations. Doing so with the zero-frequency mode, $f_b(z)$, knowing the following relations

$$\int_{-\infty}^{+\infty} dz\;[(1 - 2\,\mathrm{sech}^2 z)\,f_b^2(z) \; - \; f_b''(z)\,f_b(z)] \; = \; 0$$
$$\int_{-\infty}^{+\infty} dz\;[2\,\mathrm{sech}^2 z\,f_k(z)\,f_b(z) \; + \; f_k''(z)\,f_b(z)] \; = \; 0\,,$$

[1]Because of the scaling we performed on our Hamiltonian, the normal modes (1.52), (1.54) become $f_b(x) = 2\,\mathrm{sech}\,(z)$, $f_k(x) = \frac{1}{\sqrt{2\pi\omega_k}}e^{ikz}(\mathrm{tgh}\,(z) - ik)$ with eigenvalues $\omega_k^2 = k^2 + 1$, $k = (\omega_o/c_o)l$, and $\langle f_b, f_b\rangle = 8$, $\langle f_k, f_p\rangle = \delta(k - p)$.

leads to

$$\begin{aligned} \ddot{c}_b(\tau) &= -\frac{2\alpha}{\gamma}\,\mathrm{sech}(v(\tau-\tau_o)) \\ &= -\frac{\alpha}{\gamma}\,f_b(v(\tau-\tau_o))\,. \end{aligned} \tag{3.9}$$

One can infer a good deal of physics from equation (3.9). The mode $f_b(z)$ provides the translational motion of the soliton for small displacements since then

$$\begin{aligned} \psi_s(z+\frac{c_b(\tau)}{8}) &\approx \psi_s(z) + \frac{c_b(\tau)}{8}\,\frac{d\,\psi_s(z)}{dz} \\ &= \psi_s(z) + \frac{c_b(\tau)}{8}\,f_b(z)\,. \end{aligned} \tag{3.10}$$

Thus, equation (3.9) can be viewed (at least for small displacements) as some form of Newton's equation of motion where the normalized[2] soliton's center coordinates $c_b(\tau)/8$ responds to a force in the immediate vicinity of the impurity (realized when $\tau \to \tau_o$). This is even more so by the fact that v is large since we deal with very fast solitons (in the lab frame). Furthermore, as the coupling constant α makes the force small, $c_b(\tau)$ will only acquire a small speed by the time the force becomes virtually unimportant. This smallness of $\dot{c}_b(\tau)$ guarantees the approximation of a moving soliton $\psi_s(\gamma\,(z + c_b(\tau)/8))$ by $\psi_s(z + c_b(\tau)/8)$ with $\gamma \approx 1$. Hence, this confirms our earlier expectation of a soliton emerging from a collision with the impurity with a speed slightly less than initially (in the moving reference frame, this translates into a soliton acquiring a small velocity in the negative direction). Note, however, that this interpretation breaks down once $c_b(\tau)/8$ becomes important since then the approximate equation (3.10) is no longer valid; this will occur, though, only after the soliton leaves the immediate vicinity of the impurity at which point the force is vanishingly small and the soliton cruises with a nearly constant slow speed (in the moving reference frame).

[2]It is normalized with respect to the the mass of the soliton which has the value of eight in dimensionless units.

Now, tackling the other modes $f_p(z)$, we obtain with the help of the following relations

$$\begin{aligned} f_k''(z) &= -f_k(z)\,[k^2 + 2\,\mathrm{sech}^2 z]\,, \\ f_b''(z) &= f_b(z)\,(\,\mathrm{tgh}^{\,2}(z) - \mathrm{sech}^2(z))\,, \\ \int_{-\infty}^{+\infty} dz\; f_b(z)\, f_p^*(z)\,(\mathrm{sech}^2(z) + \mathrm{tgh}^{\,2}(z)) &= 0\,, \end{aligned}$$

the expression

$$\ddot{c}_p(\tau) + \omega_p^2\, c_p(\tau) = -\frac{\alpha}{\gamma}\, f_p^*(v(\tau - \tau_o))\,. \tag{3.11}$$

This equation is rather difficult to solve unless we resort to a trick proposed by Eilenberger. We decomposed the small field amplitudes $c_p(\tau)$ as

$$c_p(\tau) = \tilde{c}_p(\tau) + c_{op}(\tau) \tag{3.12}$$

with

$$c_{op}(\tau) = \frac{-\gamma\,\alpha}{\sqrt{2\pi} w_p (p^2 + \gamma^2)}\; e^{ipv(\tau - \tau_o)}\;(-\,\mathrm{tgh}\,[v(\tau - \tau_o)] + ip)\,. \tag{3.13}$$

What does this decomposition correspond to ? One may find out what the terms $c_{op}(\tau)$ generate in the (z, τ) space:

$$\begin{aligned} \phi_o(z,\tau) &= \int_{-\infty}^{+\infty} dk\; c_{ok}(\tau)\, f_k(z) \\ &= \frac{(-\alpha)\,\gamma}{2\pi} \int_{-\infty}^{+\infty} dk\; \frac{e^{ikv(\tau - \tau_o)}}{w_k(k^2 + \gamma^2)}\; \{k^2 + ik\,(\,\mathrm{tgh}\, z + \mathrm{tgh}\,[v(\tau - \tau_o)]) \\ &\qquad - \mathrm{tgh}\, z\, \mathrm{tgh}\,[v(\tau - \tau_o)]\} \\ &= \frac{(-\alpha)}{2}\; \Big\{ e^{-\gamma\,|\, z + v(\tau - \tau_o)\,|}\; \frac{(\gamma + \mathrm{tgh}\, z)(\gamma + \mathrm{tgh}\,[v(\tau - \tau_o)])}{(\gamma^2 + 1)} \\ &\qquad + \gamma\; \frac{(1 + \mathrm{tgh}\, z)(1 + \mathrm{tgh}\,[v(\tau - \tau_o)])}{(\gamma^2 + 1)} \Big\}. \end{aligned}$$

Note that for $|\, v(\tau - \tau_o)\,| \to +\infty$, $\phi_o(z,\tau) \to -\,(\alpha/2)\; e^{-\gamma\,|\, z + v(\tau - \tau_o)\,|}$ which is exactly the Lorentz-transformed expression of the impurity dressing (3.4) and corresponds

to the expected situation of the impurity being virtually unaffected a long time after its collision with the soliton. So, making use of decomposition (3.12) and substituting equation (3.13) in (3.11), we subtract the impurity dressing contribution and obtain

$$\ddot{\tilde{c}}_p(\tau) + \omega_p^2 \tilde{c}_p(\tau) = \frac{-\alpha\gamma}{\omega_k(k^2+\gamma^2)} e^{ikv(\tau-\tau_o)} [\frac{2ikv^2}{\cosh^2 v(\tau-\tau_o)} + v\frac{d}{d\tau}(\frac{1}{\cosh^2 v(\tau-\tau_o)})].$$

This equation is easier to handle in frequency-space; so taking the Fourier transform on both sides and using the integral

$$\int_{-\infty}^{+\infty} dx\, e^{ikx}\, \mathrm{sech}^2(ax) = \frac{\pi}{a^2}\, k\, \mathrm{cosech}(\frac{\pi}{2}\frac{k}{a})$$

we get

$$\hat{\tilde{c}}_p(\omega) = -\frac{i\alpha\pi e^{i\omega\tau_o}}{(\omega+i\epsilon)^2-\omega_p^2} \frac{\gamma(\omega-kv)}{\omega_p(\gamma^2+k^2)} \frac{(\omega/v)+k}{\sinh[(\pi/2)((\omega_k/v)+k)]}, \qquad (3.14)$$

where we added an infinitesimal imaginary part to the frequency denominator to place the poles in the lower half of the complex plane. Transforming back in the τ-space and setting ω as a complex variable u, we have

$$\tilde{c}_p(\tau) = \frac{-i\alpha\pi\gamma}{2\pi(\gamma^2+k^2)} \lim_{\epsilon\to 0}\int_{\Gamma_R} du\, e^{(-iu(\tau-\tau_o))} \frac{(u-kv)((u/v)+k)}{\omega_k((u+i\epsilon)^2-\omega_p^2)\sinh[(\pi/2)((u/v)+k)]}$$

where Γ_R is the path of integration in the complex plane and the subscript R reminds us that we will let the radius of a semicircle tend to infinity in order to evaluate $\tilde{c}_p(\tau)$.

For $\tau < \tau_o$, we need to close the contour of integration in the upper half-plane (cf. fig. (3.2)); since there is no pole, $\tilde{c}_p(\tau) = 0$, which is consistent with what we expect before the soliton appreciably begins to feel the impurity at $\tau \approx \tau_o$. Therefore, our choice for putting the poles below the real line turns out to satisfy the initial conditions of the problem.

For $\tau > \tau_o$, we close the contour in the lower half-plane, make use of the Residue theorem and the fact that

$$\frac{\gamma\, v\, (\omega_p^2/v^2 - p^2)}{(\gamma^2+k^2)} = \frac{1}{v\,\gamma}$$

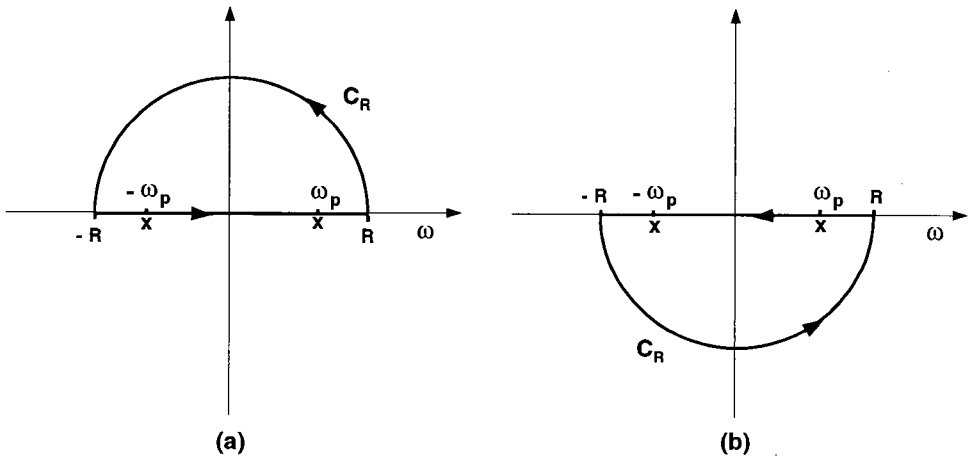


Figure 3.2: The integration path. The poles, designated by x on the graphs, are barely located in the lower half-plane because of an infinitesimal imaginary part added to the frequencies $-\omega_{\mathbf{p}}$ and $\omega_{\mathbf{p}}$. The integration path is made up of the line from $-\mathbf{R}$ to $\mathbf{R}$ on the real axis and a semicircle $\mathbf{C_R}$ of radius $\mathbf{R}$ in the upper half-plane in (a) and in the lower in (b) to accommodate the initial conditions of the problem. This complex integration is necessary to evaluate the inverse Fourier transformation of the Fourier coefficients $\widehat{\tilde{c}}_p(\omega)$ in equation (3.14). For that purpose, we let $\mathbf{R}$ tend to infinity.

to find

$$\tilde{c}_p\tau \;=\; -\frac{\alpha\,\pi}{2\,\omega_p^2\,\gamma\,v}\left(\frac{e^{-i\omega_p(\tau\,-\,\tau_o)}}{\sinh[(\pi/2)(\omega_k/v\,+\,k)]}\;-\;\frac{e^{i\omega_p(\tau\,-\,\tau_o)}}{\sinh[(\pi/2)(-\omega_k/v\,+\,k)]}\right). \qquad (3.15)$$

So, subtracting off the intrinsic impurity dressing and its effect during the collision, equation (3.15) gives us wave-packets

$$\tilde{\phi}(z,\tau) \;=\; \int_{-\infty}^{+\infty} dz\;\tilde{c}_k\tau\;f_k(z)$$

which move with a dispersion relation $\omega_k = \sqrt{1+k^2}$. To second order in α, we can calculate the energy of the system in the small-amplitude approximation using the following decomposition of $\mathcal{H}_o$

$$\begin{aligned}
\mathcal{H}_o \;&=\; \mathcal{H}_s \;+\; \mathcal{H}_{osc} \;+\; \mathcal{H}_I \qquad\qquad \text{with}\\
\mathcal{H}_s \;&=\; \frac{1}{2}(\dot{\psi}_s^2 + \psi_s'^2) \;+\; (1-\cos\psi_s)\,,\\
\mathcal{H}_{osc} \;&=\; \frac{1}{2}(\dot{\phi}_s^2 + \phi_s'^2) \;+\; \frac{1}{2}\phi^2\cos(\phi_s)\,,
\end{aligned}$$

$$\mathcal{H}_I \;=\; \dot{\psi}_s\,\dot{\phi} \;+\; \psi_s'\,\phi' \;+\; (\sin\psi_s)\,\phi\,,$$

where the field is written as

$$\psi(z,\tau) \;=\; \psi_s(z) \;+\; \phi(z,\tau)\,,$$

$\psi_s(z)$ being the static soliton located at $z=0$ and $\phi(z,\tau)$ given by equation (3.7). Integrating with respect to the coordinate z leads to

$$\begin{aligned}
H_s &= \int_{-\infty}^{+\infty} dz\,\mathcal{H}_s \;=\; M_o \;=\; 8, && \text{the rest mass of the soliton,}\\
H_{osc} &= \int_{-\infty}^{+\infty} dz\,\mathcal{H}_{osc} \;=\; \frac{1}{2}\{\frac{\dot{c}_b^2(\tau)}{8} + \int_{-\infty}^{+\infty} dk\;\mid \dot{c}_k^2(\tau)\mid^2 + \omega_k^2 \mid c_k^2(\tau)\mid^2\} && (3.16)\\
H_I &= \int_{-\infty}^{+\infty} dz\,\mathcal{H}_I \;=\; 0, && \text{and}\\
H_p &= \int_{-\infty}^{+\infty} dz\,\mathcal{H}_p \;=\; \alpha\,\psi_s(x_o) \;+\; \alpha\,\phi(x_o,\tau) \;\approx\; \psi_s(x_o)
\end{aligned}$$

since we assume $\mid \phi \mid \ll 1$.

After a sufficiently long time, the two wave packets, moving in opposite directions in the soliton rest frame, will become entirely detached from each other and, presumably, from the impurity dressing as well on account of the impurity's great speed. Therefore, we can ignore the mixed terms in $\mid c_k(\tau)\mid^2$ and isolate the terms pertaining to the wavepackets alone

$$\tilde{c}^{\,r}_{k,\sigma}(\tau) \;=\; -\frac{\alpha\pi}{2\omega_k^2\gamma v}\;\frac{e^{-i\sigma\omega_k(\tau-\tau_o)}}{\sinh[(\pi/2)((\omega_k/v)+k)]} \qquad (3.17)$$

with $\sigma=\pm 1$; the sign referring respectively to a right- or a left-moving wavepacket and the superscript r standing for "radiation" which is how we coin this particular excitation of the small-amplitude field.

Plugging equation (3.17) into (3.16), we obtain for the energy of this radiation

$$\begin{aligned}\mathcal{H}_r(\tau \gg \tau_o) &= \sum_{\sigma=1}^{2} \int_{-\infty}^{+\infty} dk\ \omega_k^2\ |\ \tilde{c}^r_{k,\sigma}\ |^2 \\ &= (\frac{\alpha\pi}{2\gamma v})^2 \sum_{\sigma=1}^{2} \int_{-\infty}^{+\infty} dk\ \frac{1}{\sinh^2[(\pi/2)((\sigma\omega_k/v)+k)]} \\ &= \frac{1}{2}(\frac{\alpha\pi}{2\gamma v})^2 \int_{-\infty}^{+\infty} dk\ \mathrm{cosech}^2[(\pi/2)((\omega_k/v)+k)]\end{aligned} \tag{3.18}$$

Hence, $\tilde{c}^r_{k,\sigma}$ can be interpreted as the amplitude of the excited normal mode k with equation (3.18) being the usual sum of individual mode energies.

A general step potential $V(x)$ can always be constructed using the decomposition

$$V(x) = \int_{-\infty}^{+\infty} dx_o\ \tilde{V}(x_o)\ \Theta(x - x_o)\,,$$

from which we obtain the corresponding force

$$F(x) = \int_{-\infty}^{+\infty} dx_o\ F(x_o)\ \delta(x - x_o)$$

with

$$F(x_o) = -\tilde{V}(x_o)\,.$$

It is then a simple matter to calculate the radiation amplitudes by linear superposition.

Naively, one would assume that equation (3.17) is the correct result at the classical level; one for which the amplitudes $\tilde{c}^r_{k,\sigma}$ simply carry over to the quantum world as the amplitudes of the corresponding quantum oscillators. However, this classical result is incorrect, and the reason has to do with the warning we issued when we talked about the quantization scheme about a classical solution: the presence of a translational mode, namely in our case, $f_b(x)$ with its eigenvalue $\omega_b = 0$. In classical *or* quantum mechanics, the decomposition of the harmonic Hamiltonian density (3.16) is insufficient since the translational mode has no *confinement* to approximate it as a harmonic mode. This simply means that, in the $f_b(x)$ "direction", the anharmonic terms in a functional Taylor

expansion of a potential V (cf. equ. (2.32)) are likely to be as important as the harmonic term, whether the state is classical or quantal. As promised, we give in the next section a method to handle those zero-frequency modes. With this method, we will see that equation (3.16) is nonetheless valid only after taking into account some facts specific to our physical situation of a fast soliton's colliding with a soft impurity.

3.2 The method of collective coordinates

As pointed out in chapter 2, eigenfunctions of the operator

$$\left[-\nabla^2 + \left(\frac{d^2U}{d\phi^2}\right)\Big|_{\phi=\phi_n}\right]$$

with eigenvalues ω_o equal to zero may exist for an expansion about some minimum $\phi_n(\mathbf{x})$. This arises from continuous symmetries displayed by a Lagrangian

$$L[\phi] \;=\; \int d\mathbf{x}\, \left\{\frac{1}{2}\left(\frac{\partial\phi(\mathbf{x},t)}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 \;-\; U(\phi)\right\},$$

and in particular from the translational invariance of minima $\phi_n(\mathbf{x})$ for which

$$V[\phi_n(\mathbf{x}+\mathbf{a})] \;=\; \phi_n(\mathbf{x}),\; \forall\, \mathbf{a}\,.$$

A minimum $\phi_j(\mathbf{x})$ which is independent of $\mathbf{x}$ will not produce a zero-frequency mode for the reason that $\phi_j(\mathbf{x}+\mathbf{a}) = \phi_j(\mathbf{x})$ and, therefore, can be considered as a point in "field space". The zero-frequency eigenfunctions $f_o(\mathbf{x})$ correspond to the infinitesimal translational generators of the minima $\phi_n(\mathbf{x})$ in the sense that

$$\begin{aligned}\delta\phi_n(\mathbf{x}) \;&=\; \phi_n(\mathbf{x}+\delta\mathbf{a}) - \phi_n(\mathbf{x}) \\ &=\; (\delta\mathbf{a})\; f_o(\mathbf{x})\end{aligned}$$

In our case, this is the function $f_b(x) = 2\,\mathrm{sech}\,x$ which is exactly the derivative $d\psi_s(x)/dx$ of the soliton solution. In chapter 2, we gave an idea of how symmetries in a classical

Lagrangian translate over to the quantum world: their respective conjugate momenta operators turned out to be conserved. Before stepping into relativistic field space, let us deal with a simple example in non-relativistic one-particle quantum mechanics. This will provide the reader with the basic physics behind the collective coordinates method applied to fields.

Consider a particle in two dimensions with the following Lagrangian

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - V(\sqrt{x_1^2 + x_2^2}) . \tag{3.19}$$

The potential enjoys an obvious continuous symmetry: it is rotationally invariant. Let $\mathbf{x}_r \neq 0$ be a minimum of V. This is not the only classical solution since $V(\mathbf{x}_p)$ such that $\mid \mathbf{x}_p \mid = \mid \mathbf{x}_r \mid$ will also be a minimum. If we naively apply our method of quantization about the classical solution $\mathbf{x}_r$, we obtain

$$E_{n_1,n_2} = V(\mathbf{x}_r) + \sum_{1=1}^{2} (n_i + 1/2)\,\hbar\,\omega_i + \text{ higher orders} \tag{3.20}$$

where ω_i are the eigenvalues of

$$\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)\Bigg|_{\mathbf{x}=\mathbf{x}_r} .$$

One finds that one eigenvalue, say ω_1, equals zero on account of the rotational symmetry of the potential V. This zero eigenvalue implies that no confinement exists for the wavefunction in the angular direction. Therefore, it fails to be localized on the circle of radius $\mid \mathbf{x}_r \mid = R$ although, radially, it is so owing to the nearly quadratic potential for each of those points $\mathbf{x}_p$ in the radial direction. This shows that the quantum stationary states of the system are not localized around any specific classical solution, with the consequence that the approximate equation for their energies (3.20) is incorrect.

One simple way to deal with this problem is to change from cartesian to polar coordinates, which change is naturally dictated by the symmetry of the potential. The

Lagrangian (3.19) transforms then to

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - V(r). \tag{3.21}$$

One immediately notices the absence of the angular coordinate which makes it *ipso facto* a cyclic one, so that its conjugate momentum $l = r^2\dot{\theta}$ becomes a conserved quantity. Therefore, one may readily reduce the problem to a one-dimensional case involving the radial coordinate r and the momentum conjugate to θ, l, considered as a given constant. Thus, we can write the Lagrangian as

$$\begin{aligned} L_l &= \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} + V(r) \\ &= \frac{1}{2}\dot{r}^2 + V_{eff}(r;l) \end{aligned} \tag{3.22}$$

where $V_{eff}(r;l)$ becomes an effective potential acting radially on the particle with a given angular momentum l. The term $l^2/(2\,r^2)$ simply corresponds to a fictitious centrifugal potential. Now, we can apply the usual procedure to this Lagrangian by finding first, a static solution $r(t) = R_l$ given by the extrema of the effective potential $V_{eff}(r;l)$,

$$\begin{aligned} \left.\frac{V_{eff}(r;l)}{dr}\right|_{r=R_l} &= 0 \\ \left.\frac{dV}{dr}\right|_{r=R_l} - \frac{l^2}{R_l^3} &= 0, \end{aligned} \tag{3.23}$$

and second, by building a series of low-lying quantum energy levels about this solution in the harmonic oscillator approximation,

$$\begin{aligned} E_{n,l} &\approx V_{eff}(r;l) + \frac{1}{2}(n+1/2)\,\hbar\,\tilde{\omega} \\ &= V(R_l) + \frac{1}{2}(n+1/2)\,\hbar\,\tilde{\omega} + \frac{l^2}{R_l^2} \end{aligned} \tag{3.24}$$

whose eigenvalues $\tilde{\omega}$ are given by

$$\tilde{\omega}^2 = \left.\frac{d^2V_{eff}}{d^2r}\right|_{r=R_l}.$$

Note the difference between equations (3.20) and (3.24) where the indices in the former, $\{n_1, n_2\}$, have been replaced in the latter by $\{n, l\}$ with l being a number associated with the coordinate responsible for the symmetry of the Lagrangian. This difference is a manifestation of our changing from one set of coordinates, the cartesian, to another, the polar. In the field case, we will encounter the same difference resulting from a *change* of coordinates.

A more illuminating way of looking at this reduction insofar as it bears on its generalization to fields is to view the construction of quantum levels in equation (3.24) as being built around a classical *time-independent* solution instead of a static one. Indeed, the equations of motion derived from (3.21) are

$$\ddot{r} = r\dot{\theta}^2 + \frac{dV}{dr} = 0$$

and

$$\frac{d(r^2\dot{\theta}^2)}{dt} = 0.$$

If we set $\mathbf{r}_{cl}(t) = (r_{cl}(t), \theta_{cl}(t)) = (R_l, \nu t)$ with $\nu = l/R_l^2$ a constant, the second equation is automatically solved while the first reduces to equation (3.23). Therefore, the approximate energy equation (3.24) can be thought of as an expansion about a time-dependent rotating solution which samples uniformly all points on the circle R and which ought to give a better expansion than that about a *specific* point $\mathbf{x}_\mathbf{r}$ on this circle (which rotational symmetry does not favour in any way).

After this short excursion in the discrete non-relativistic realm, we will deal first, with relativistic but classical fields and then, move on to the quantized format.

In order to keep things simple, we will confine ourselves to a scalar field in one dimension with one non-degenerate static solution. The generalization to fields in multiple dimensions is straightforward.

Consider the Lagrangian

$$L \;=\; \int_{-\infty}^{+\infty} dx\, \frac{1}{2}\,(\dot{\phi}^2(x,t) - (\phi'(x,t))^2) \;-\; U(\phi)\,, \tag{3.25}$$

which is the one-dimensional version of equation (2.27), and a static minimum which we designate by $\phi_p(x)$. We recall that this minimum is a solution to

$$\left.\frac{\delta V[\phi]}{\delta \phi}\right|_{\phi=\phi_p} \;=\; 0\,,$$

namely, it satisfies

$$\frac{d^2\phi_p}{dx^2} \;-\; \frac{dU(\phi_p)}{d\phi_p} \;=\; 0\,. \tag{3.26}$$

However, by translational invariance, $\phi_p(x)$ is *not* the only solution; indeed, $\phi_p(x+X)$ for any X is also a minimum. This invariance signals that we will run into problems if we attempt a harmonic expansion about a *specific* solution $\phi_p(x+X)$, where X is fixed arbitrarily, just as we did about the specific solution $\mathbf{x}_r$ for the rotational symmetric Lagrangian. We exhort the reader to be vigilant because we will use the exact same methods that we used before in order to get out of the fix!

The naive expansion would be to express $\phi(x)$ about the classical static solution $\phi_p(x+X)$ in the orthonormal basis $\{f_b(x+X), f_n(x+X)\}$ which consists of the eigenmodes (cf. equ. (2.34)) that diagonalize the functional potential $V[\phi]$ with $f_b(x+X)$ being a zero-frequency mode (i.e., one whose eigenvalue equals zero),

$$\phi(x,t) \;=\; \phi_p(x+X) \;+\; c_b(t)\, f_b(x) \;+\; \sum_{n=0}^{\infty} c_n(t)\, f_n(x+X)\,. \tag{3.27}$$

The approximate energy equation is then given by

$$E_{n_i} \;=\; V[\phi_n] \;+\; \hbar \sum_i (n_i + 1/2)\,\omega_i \;+\; \text{higher order terms.} \tag{3.28}$$

We have just recapitulated what we already mentioned in section 2.2. Due to the zero-frequency mode $f_b(x)$ we know that this expression is incorrect. Now, we entreat the reader to bear closely with us.

In function space, the basis $\{f_b(x), f_n(x)\}$ acts as a reference system in the same way as the basis $\{\hat{\imath}, \hat{\jmath}\}$ does for a 2-dimensional space with decomposition $\mathbf{x}(t) = x_1(t)\hat{\imath} + x_2(t)\hat{\jmath}$. Hence, $c_b(t)$ and $c_n(t)$ act as the *coordinates* of the system with respect to a reference frame whose origin is at the "point" $\phi_p(x+X)$ in field space. In order to fix our problem in the non-relativistic case, we recall that we made a change of coordinates from $x_1(t), x_2(t)$ to the polar representation $r(t), \theta(t)$; which representation naturally lends itself to on account of the rotational symmetry the Lagrangian enjoyed. Furthermore, we pointed out that the expansion was made about a time-dependent classical solution instead; for which the symmetry-related coordinate $\theta(t)$ somehow sampled uniformly all the points connected by the rotational symmetry. The reader may have guessed already that one would have to *change* the set of coordinates $\{c_b(t), c_n(t)\}$ to one including the coordinate responsible for the symmetry of the Lagrangian. We know what this coordinate is: this is the value X in $\phi_n(x + X)$. Therefore, our new coordinates would be something like $\{X(t), q_n(t)\}$ where $q_n(t)$ should be associated with the usual basis functions $f_n(x)$ as the wavefunction is perfectly confined in these "directions" (as it was in the radial direction for the previous one-particle case). The proper expansion is then as follows:

$$\phi(x,t) \;=\; \phi_n(x - X(t)) \;+\; \sum_{n=0} q_n(t)\, f_n(x - X(t))\,, \tag{3.29}$$

where $f_n(x - X(t))$ is as appropriate as $f_n(x)$ due to the translational invariance of the Lagrangian L (equ. (3.25)). Notice two things: the zero-frequency mode $f_b(x)$, which caused so much trouble, has TOTALLY disappeared; and the expansion is made about a *time-dependent* solution $\phi_n(t)(x - X(t))$ which samples all minima related to each other by the translational symmetry. This sampling will be uniform as we discover later that the momentum conjugate to the coordinate $X(t)$ is a conserved quantity in the same way the angular momentum $r^2\dot{\theta}$ previously was. Expression (3.29) is what we refer to as the method of collective coordinates [36, 41, 42, 43, 44] where $X(t)$ is called the collective

coordinate associated with the corresponding translational symmetry.

For convenience, let us denote $q_b(t) \triangleq X(t)$. As mentioned earlier, the potential term $V[\phi]$ is translation invariant and, therefore, does not depend on $X(t)$. Hence, using expansion (3.29), we obtain

$$\begin{aligned} V[\phi] &\triangleq \int_{-\infty}^{+\infty} dx \left\{ \frac{1}{2} (\phi')^2 + U(\phi) \right\} \\ &= V(M_o, \{q_n\}) \\ &\approx V[\phi_n] + \frac{1}{2} \sum_n q_n^2 w_n^2 + \text{higher order terms} \end{aligned} \tag{3.30}$$

in the small-amplitude harmonic approximation with $V[\phi_n]$ and $\{w_n\}$ being the same as in equation (3.28).

We note that

$$\begin{aligned} V[\phi_n] &= \int_{-\infty}^{+\infty} dx \left\{ \frac{1}{2} \left(\frac{d\phi_n}{dx} \right)^2 + U(\phi_n) \right\} \\ &\triangleq M_o \end{aligned}$$

corresponds to the "rest mass" of the field for this particular classical excitation, $\phi_n(x)$. Then the kinetic energy term

$$T[\phi] = \int_{-\infty}^{+\infty} dx \, \frac{1}{2} (\dot{\phi})^2$$

becomes

$$\begin{aligned} T &= \frac{1}{2} \left\{ \dot{q}_b(t) \, D_{bb} \, \dot{q}_b(t) + \int_{-\infty}^{+\infty} dk \; \dot{q}_b(t) \, D_{bk} \, \dot{q}_k(t) \right. \\ &\qquad \left. + \int_{-\infty}^{+\infty} dk \; \dot{q}_k(t) \, D_{kb} \, \dot{q}_b(t) + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk \, dp \; \dot{q}_k(t) \, D_{kp} \, \dot{q}_p(t) \right\} \\ &\triangleq \frac{1}{2} \sum_{i,j=b,-\infty}^{+\infty\,\prime} \dot{q}_i(t) \, D_{i,j} \, \dot{q}_j(t) \, , \end{aligned} \tag{3.31}$$

where

$$\sum_{i=b,-\infty}^{+\infty\,\prime} a_i \triangleq a_b + \sum_{i=-\infty}^{+\infty} a_i$$

for any indexed quantity a_i, and

$$D_{i,j} = D_{j,i}\,, \tag{3.32}$$

$$D_{b,b} = \int_{-\infty}^{+\infty} dx\, \left\{\phi_n'(x) + \int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x)\right\}^2, \tag{3.33}$$

$$\begin{aligned} D_{b,k} &= -\int_{-\infty}^{+\infty} dx\, \left\{\phi_n'(x) + \int_{-\infty}^{+\infty} dp\; q_p(t)\, f_p'(x)\right\} f_k(x) \\ &= -\int_{-\infty}^{+\infty} dp\; q_p(t) \int_{-\infty}^{+\infty} dx\; f_p'(x)\, f_k(x)\,, \end{aligned} \tag{3.34}$$

$$D_{k,p} = \delta(k+p)\,, \tag{3.35}$$

in which we made use of the fact that $\phi_n'(x)$ and $f_k(x)$ are orthonormal to each other by virtue of $\phi_n'(x) = f_b(x)$. Also, we have $(D_{b,-k})^* = D_{b,k}$. Equation (3.35) is the simple generalization to continuous case of the Kronecker function $\delta_{k,-p}$ in the discrete case. (If the minus sign bothers, recall that expansion (3.29) is made up of complex eigenfunctions whose coefficients are related by $(q_k(t))^* = q_{-k}(t)$ to ensure a real field.) In equations (3.33) and (3.34), ϕ_n and the eigenfunctions f_k appear with their arguments set to x instead of $x + X(t)$ since under the integral sign it does not matter.

The field Lagrangian written in terms of variables $\{q_b(t), q_k(t)\}$ is

$$L = \frac{1}{2}\left[\sum_{i,j=b,-\infty}^{+\infty\,\prime} \dot{q}_i(t)\, D_{i,j}(\{q_k\})\, \dot{q}_j(t) - \tilde{V}(M_o, \{q_k\})\right] + \mathcal{O}(\phi^3) \tag{3.36}$$

where $\tilde{V}(M_o, \{q_k\}) \triangleq 2\, V(M_o, \{q_k\})$.

Note in the above equation that the coordinate $q_b(t) = X(t)$ nowhere appears, though its speed, $\dot{q}_b(t) = \dot{X}(t)$, does.

In order to derive the Hamiltonian, we proceed as usual by finding the canonical momenta π_i conjugate to $q_i(t)$

$$\pi_i(t) = \frac{\partial L}{\partial \dot{q}_i(t)} = \sum_{j=b,-\infty}^{+\infty\,\prime} D_{i,j}\, \dot{q}_j(t)$$

so that formally the inversion reads as

$$\dot{q}_i(t) \;=\; \sum_{j=b,-\infty}^{+\infty\,\prime} (D)^{-1}_{i,j}\; \pi_j(t)\,, \tag{3.37}$$

and so obtains

$$\begin{aligned} H \;&=\; \sum_{i=b,-\infty}^{+\infty\,\prime} \pi_i\, \dot{q}_i \;-\; L \\ &=\; \frac{1}{2} \sum_{i,j=b,-\infty}^{+\infty\,\prime} \pi_i\, (D)^{-1}_{i,j}\, \pi_j \;+\; \tilde{V}(M_o, \{q_j\})\,. \end{aligned} \tag{3.38}$$

In appendix B, we derive the inverse of the matrix D and, therefore, merely state the results:

$$\begin{aligned} (D)^{-1}_{b,b} \;&=\; \frac{-1}{\mathcal{D}}\,, \\ (D)^{-1}_{b,k} \;&=\; \frac{D_{b,-k}}{\mathcal{D}}\,, \qquad\qquad \text{and} \\ (D)^{-1}_{k.p} \;&=\; \delta(k+p) \;-\; \frac{D_{b,-k}\, D_{b,-p}}{\mathcal{D}} \qquad \text{for } \; k,p \neq b \end{aligned}$$

where

$$\begin{aligned} \mathcal{D} \;&\triangleq\; \text{determinant of } D \\ &=\; \frac{-1}{M_o}\,(\alpha + M_o)^2 \qquad \text{with} \\ &\qquad\qquad \alpha \;=\; \int_{-\infty}^{+\infty} dk\; q_k(t)\; \left(\int_{-\infty}^{+\infty} dx\; \phi'_n(x)\, f'_k(x) \right). \end{aligned} \tag{3.39}$$

It is very illuminating to see to what π_b, the momentum conjugate to $X(t)$, corresponds. The total momentum for a given field ψ is defined as

$$P \;\triangleq\; -\int_{-\infty}^{+\infty} dx\; \frac{\partial \psi}{\partial t}\, \frac{\partial \psi}{\partial x}\,.$$

In our case, this total field momentum is

$$
\begin{aligned}
P &= -\int_{-\infty}^{+\infty} dx\, \frac{\partial\phi}{\partial t}\,\frac{\partial\phi}{\partial x} \\
&= \int_{-\infty}^{+\infty} dx\, \left\{\left(\phi_n'(x) + \int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x)\right)\, \dot{X}(t) - \int_{-\infty}^{+\infty} dk\; \dot{q}_k(t)\, f_k(x)\right\} \\
&\qquad\qquad \times \left\{\phi_n'(x) + \int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x)\right\} \\
&= \int_{-\infty}^{+\infty} dx\, \left(\phi_n'(x) + \int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x)\right)^2 \dot{X}(t) \\
&\qquad - \int_{-\infty}^{+\infty} dk\; \dot{q}_k(t) \int_{-\infty}^{+\infty} dx\, \{\phi_n'(x) + \int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x)\}\, f_k(x) \\
&= D_{b,b}\, \dot{X}(t) + \int_{-\infty}^{+\infty} dk\; D_{b,k}\, \dot{q}_k(t) \\
&= D_{b,b}\, \dot{q}_b(t) + \int_{-\infty}^{+\infty} dk\; D_{b,k}\, \dot{q}_k(t) \\
P &= \pi_b\,.
\end{aligned}
$$

Hence, π_b is none other than the total field momentum of the system; which is quite reassuring as we know that its conjugate coordinate X(t) can be thought of as the "center-of-mass" of the field $\phi(x,t)$ giving an idea of its location in space. For example, for the Sine-Gordon Lagrangian, $\phi_n(x - X(t)) = \psi_s(x - X(t))$ where $X(t)$ locates the center of the soliton (even though the Lorentz factor γ, which acts as a scaling factor, is absent). Furthermore, $q_b(t)$ or $X(t)$ is a cyclic coordinate in Hamiltonian (3.38) and, therefore, we expect P to be a conserved quantity in the same manner as the angular momentum $l = r^2\dot{\theta}$ was for the discrete case. One can surely appreciate, by now, the many analogies we can draw between the discrete and field cases.

Using $\pi_b = P$, we can rewrite the classical field-relativistic Hamiltonian as

$$
\begin{aligned}
H &= -\frac{P^2}{2\mathcal{D}} + \frac{P}{\mathcal{D}} \int_{-\infty}^{+\infty} dk\; D_{b,-k}\, \pi_k + \frac{1}{2} \int\int dk\, dp \left\{\delta(k+p) - \frac{D_{b,-k}\, D_{b,-p}}{\mathcal{D}}\right\} \pi_k\, \pi_p \\
&\qquad + \tilde{V}(M_o, \{q_n(t)\})\,. \qquad (3.40)
\end{aligned}
$$

Now, we must quantize this Hamiltonian. We will use a method developed by Christ and Lee [41] which makes use of the usual canonical quantization scheme based on the Hamiltonian. This latter consists in promoting the field $\phi(x,t)$ and its conjugate field momentum $\pi(x,t)$ into operators obeying the equal-time commutation rule ($\hbar = 1$)

$$[\,\hat{\phi}(x,t), \hat{\pi}(y,t')\,] \;=\; i\,\delta(x-y)\,\delta_{tt'}\,. \tag{3.41}$$

The quantum Hamiltonian is simply derived from the classical one by elevating the fields to the rank of operators

$$\hat{H} \;=\; \frac{1}{2}\int_{-\infty}^{+\infty} dx\;\{\hat{\pi}^2(x,t) \;+\; \tilde{V}(\hat{\phi}(x,t))\} \tag{3.42}$$

with

$$\tilde{V}(\hat{\phi}(x,t)) \;=\; \frac{d\hat{\phi}(x,t)}{dx} \;+\; 2\,U(\hat{\phi})$$

and

$$\hat{\pi}(x,t) \;=\; \frac{d\hat{\phi}(x,t)}{dt}.$$

Formally, the commutator (3.41) is fulfilled by defining

$$\hat{\pi}(x,t) \;=\; -i\,\frac{\delta}{\delta\hat{\phi}(x,t)}$$

in the sense of a functional derivative operator on the appropriate Hilbert space. However, what we want to do is to perform a change of variables from $\hat{\phi}(x,t)$ to $\{\hat{X}(t), \hat{q}_k(t)\}$ which can be executed unambiguously. In order to see things in a more illuminating way, the reader can always think of $\hat{\phi}(x,t)$ as $\hat{\phi}_x(t)$, an operator with a continuous index x, that acts on "functions" $\psi(\mathbf{y})$ with $\mathbf{y} \equiv (y_x)$; in other words, functionals. Thus, $\hat{\pi}_x(t)$ is correspondingly viewed as $\hat{\pi}_x(t)$ which translates as the operator $-i\delta/\delta\phi_x(t)$ on this space of "wavefunctionals." Hence, in effect, we want to achieve the change of variables

$$\{\hat{\phi}_x(t)\} \;\rightarrow\; \{\hat{X}(t), \hat{q}_k(t)\} \tag{3.43}$$

Under this transformation , how do our operators $\hat{\pi}_x(t) = -i\,\delta/\delta\hat{\phi}_x(t)$ look like ?
We stress the fact that the only difficulty in the quantum Hamiltonian (3.42) under this change of variables lies in the momentum operator $\hat{\pi}(x,t)$ since the potential operator term $\tilde{V}(\hat{\phi}(x,t))$ can be handled in the same manner as before,

$$\tilde{V}[\hat{\phi}(x,t)] \;=\; \int_{-\infty}^{+\infty} dx\; \tilde{V}(\hat{\phi}(x,t)) \;=\; \tilde{V}(M_o, \{\hat{q}_n(t)\})$$

with the simple prescription of elevating all the $q_n(t)$ in $\tilde{V}(M_o, \{q_n(t)\})$ into operators.[3] This is obviously due to the quantum mechanical decomposition of the field operator $\hat{\phi}(x,t)$ as

$$\begin{aligned} \hat{\phi}(x,t) \;&=\; \phi_n(x - \hat{X}(t)) \;+\; \int_{-\infty}^{+\infty} dk\; \hat{q}_k(t)\, f_k(x - \hat{X}(t)) \\ &\triangleq\; \phi_n(x - \hat{X}(t)) \;+\; \hat{\rho}(x) \end{aligned} \tag{3.44}$$

which lends itself to the same treatment of $\tilde{V}[\phi(x,t)]$ on account of the integration over the x coordinate.

Going back to the transformation of $\hat{\pi}_x(t)$ under the change of variables (3.43), we, once more, turn to classical concepts to give us a hint of how to handle this problem in the continuous case; this time, the classical concept is the standard rules of calculus. Were we to have only a finite number of variables $(\alpha_1, ..., \alpha_N)$ instead of the continuum-indexed (ϕ_x), the Laplacian $\int dx\, \delta^2/\delta\phi_x^2$ would be given by

$$\sum_{i=1}^{N} \frac{\delta^2}{\delta\alpha_i^2}.$$

This Laplacian would act on wavefunctions of the form $\psi(\alpha_1, ..., \alpha_N)$. By the standard rules of calculus, a change of variables $\{\alpha_i\} \to \{\beta_i\}$ would have the Laplacian take the form

$$\sum_{i=1}^{N} \frac{\delta^2}{\delta\alpha_i^2} \;=\; \sum_{i,j=1}^{N} \frac{1}{\sqrt{\mathcal{B}}}\, \frac{\delta}{\delta\beta_i}\, B^{-1}_{i,j}\, \sqrt{\mathcal{B}}\, \frac{\delta}{\delta\beta_j}, \tag{3.45}$$

[3] Pragmatically, this involves covering the coordinates $q_n(t)$ with a nice little hat "$\hat{}$".

where

$$B_{i,j} = \sum_{k=1}^{N} \frac{\delta\alpha_k}{\delta\beta_i} \frac{\delta\alpha_k}{\delta\beta_j} \tag{3.46}$$

and

$$\mathcal{B} \triangleq Det(B)_{i,j} .$$

It is now easy to see the parallel with the continuum case; the following equivalences can be made

$$\alpha_i \longleftrightarrow \phi_x \qquad \sum_{i=1}^{N} \longleftrightarrow \int dx ,$$

$$\beta_i \longleftrightarrow q_i \qquad \sum_{i,j=1}^{N} \longleftrightarrow \sum_{i,j=b,-\infty}^{+\infty\,\prime} ,$$

so that the generalization of equations (3.46) and (3.45) amounts to

$$B_{i,j} = \int_{-\infty}^{+\infty} dx \, \frac{\delta\phi_x}{\delta q_i} \frac{\delta\phi_x}{\delta q_j} ,$$

$$B_{i,j} = \int_{-\infty}^{+\infty} dx \, \frac{\delta\phi(x)}{\delta q_i} \frac{\delta\phi(x)}{\delta q_j} ,$$

$$\begin{aligned} \int_{-\infty}^{+\infty} dx \; \hat{\pi}(x,t) &= \int_{-\infty}^{+\infty} dx \; \hat{\pi}_x(t) \\ &= \int_{-\infty}^{+\infty} dx \, \frac{\delta^2}{\delta_x^2(t)} \\ &= \int_{-\infty}^{+\infty} dx \, \frac{\delta^2}{\delta^2(x,t)} \\ &= \sum_{i,j=b,-\infty}^{+\infty\,\prime} \frac{1}{\sqrt{\mathcal{B}}} \frac{\delta}{\delta q_i} B_{i,j}^{-1} \sqrt{\mathcal{B}} \frac{\delta}{\delta q_j} . \end{aligned}$$

One can easily convince himself that the matrix B corresponds *exactly* to the matrix D we encountered before. Hence, writing the full Hamiltonian in the coordinate basis $\{q_b(t), q_k(t)\}$, we obtain

$$\begin{aligned}\hat{H}_c \;=\; & -\frac{1}{2}\int_{-\infty}^{+\infty} dk\,dp\;\frac{1}{\sqrt{\mathcal{D}}}\,\frac{\delta}{\delta q_k(t)}\,(D^{-1})_{k,p}\,\sqrt{\mathcal{D}}\,\frac{\delta}{\delta q_p(t)} \\ & -\frac{1}{2}\,\frac{1}{\sqrt{\mathcal{D}}}\,\frac{\delta}{\delta X(t)}\,(D^{-1})_{b,b}\,\sqrt{\mathcal{D}}\frac{\delta}{\delta X(t)} \\ & -\frac{1}{2}\int_{-\infty}^{+\infty} dk\,\left\{\frac{\delta}{\delta X(t)}\,(D^{-1})_{b,k}\,\sqrt{\mathcal{D}}\,\frac{\delta}{\delta q_k(t)} \;+\; \frac{\delta}{\delta q_k(t)}\,(D^{-1})_{k,b}\,\sqrt{\mathcal{D}}\,\frac{\delta}{\delta X(t)}\right\} \\ & \qquad\qquad + \tilde{V}(M_o,\{q_k(t)\})\,,\end{aligned} \tag{3.47}$$

where we chose to unmask $q_b(t)$ to reveal its true nature, $X(t)$, and affixed to $\hat{H}$ the subscript c to remind us that such an operator is represented in a basis. Setting

$$-i\,\frac{\delta}{\delta X(t)} \;=\; \hat{\pi}_b(t) \;\triangleq\; \hat{P} \quad\text{and}\quad -i\,\frac{\delta}{\delta q_k^*(t)} \;=\; \hat{\pi}_k(t)\,,$$

consistent with the commutation rules

$$\begin{aligned}&[\,\hat{q}_k(t),\hat{\pi}_p(t')\,] \;=\; i\,\delta(k+p)\,\delta_{tt'} \qquad [\,\hat{X}(t),\hat{P}(t')\,] \;=\; i\,\delta_{tt'} \\ &\qquad\qquad [\,\hat{X}(t),\hat{\pi}_k(t')\,] \;=\; 0 \;=\; [\,\hat{q}_k(t),\hat{P}(t')\,]\end{aligned} \tag{3.48}$$

(which corresponds exactly to the usual canonical quantization scheme applied on coordinates $\{\hat{X}(t),\hat{q}_k(t),\hat{P}(t),\hat{\pi}(t)\}$), we can finally write down the representation-free quantum Hamiltonian as

$$\begin{aligned}\hat{H} \;=\; & -\frac{1}{2}\,\frac{1}{\sqrt{\hat{\mathcal{D}}}}\left(\frac{1}{\hat{\mathcal{D}}}\right)\sqrt{\hat{\mathcal{D}}}\,\hat{P}^2 \;+\; \frac{1}{2}\int_{-\infty}^{+\infty} dk\;\frac{1}{\hat{\mathcal{D}}}\left\{\hat{P}\,\frac{\hat{D}^+_{b,k}}{\sqrt{\hat{\mathcal{D}}}}\,\hat{\pi}^+_k \;+\; \hat{\pi}^+_k\,\frac{\hat{D}^+_{b,k}}{\sqrt{\hat{\mathcal{D}}}}\,\hat{P}\right\} \\ & +\frac{1}{2}\int\!\!\int dk\,dp\;\frac{1}{\sqrt{\hat{\mathcal{D}}}}\,\hat{\pi}_k\left\{\delta(k+p) \;-\; \frac{\hat{D}^+_{b,k}\,\hat{D}^+_{b,p}}{\hat{\mathcal{D}}}\right\}\sqrt{\hat{\mathcal{D}}}\,\hat{\pi}_p \;+\; \tilde{V}'(M_o,\{\hat{q}_k(t)\})\,,\end{aligned} \tag{3.49}$$

where

$$\begin{aligned}\hat{\mathcal{D}} \;&=\; -\hat{D}_{b,b} \;+\; \int_{-\infty}^{+\infty} dk\;\hat{D}^+_{b,k}\,\hat{D}_{b,k} \\ &=\; -\frac{1}{M_o}\{\hat{\alpha}+M_o\}^2\end{aligned} \tag{3.50}$$

with

$$\hat{\alpha} \;=\; \int_{-\infty}^{+\infty} dx\; \phi_n'(x)\; \left(\int_{-\infty}^{+\infty} dk\; \hat{q}_k(t)\, f_k'(x) \right)$$

$$(\hat{D}^{-1})_{b,b} \;=\; -\frac{1}{\hat{\mathcal{D}}}\,, \tag{3.51}$$

$$(\hat{D}^{-1})_{b,k} \;=\; \frac{\hat{D}_{b,-k}}{\hat{\mathcal{D}}} \;=\; \frac{\hat{D}^{+}_{b,k}}{\hat{\mathcal{D}}} \;=\; (\hat{D}^{-1})_{k,b}\,, \tag{3.52}$$

$$(\hat{D}^{-1})_{p,k} \;=\; \delta(k+p) - \frac{\hat{D}^{+}_{b,k}\, \hat{D}^{+}_{b,p}}{\hat{\mathcal{D}}}\,. \tag{3.53}$$

Note that all the various operators $\hat{D}_{i,j}$ and $\hat{\mathcal{D}}$ are made so by virtue of $\{\hat{q}_k(t)\}$ of which they are function (cf. equs. (3.32) - (3.35)). Also, as before, the operator $\hat{X}(t)$ is completely absent from equation (3.47); only its conjugate momentum $\hat{P}$ comes along for the ride. Finally, we bring the reader's attention to the fact that the above Hamiltonian is *exact* in the sense that no approximation whatsoever has been made in its derivation. Had we carelessly quantized the Hamiltonian (3.40) by using the commutation relations (3.48), we would have been stuck with the ordering ambiguity of the second and third terms since $\hat{D}_{i,j}$ and $\hat{\pi}_k$ do not commute. Our expedient to treating $\hat{\pi}_k$ as differential operators has just naturally taken care of this ordering.

By virtue of the absence of $\hat{X}(t)$ in the Hamiltonian (3.49), we obtain

$$\frac{d\hat{P}}{dt} = i\,[\,\hat{H}, \hat{P}\,] = 0$$

and thus, $\hat{P}$, the total field momentum operator, is conserved. Therefore, we can choose simultaneous eigenstates of $\hat{H}$ and $\hat{P}$. Let us consider the eigenvalue $P = 0$. In this case, the Hamiltonian reduces to

$$\hat{H}_{(P=0)} \;=\; \frac{1}{2} \int\int dk\, dp\, \frac{1}{\sqrt{\hat{\mathcal{D}}}}\, \hat{\pi}_k \left\{ \delta(k+p) \;-\; \frac{\hat{D}^{+}_{b,k}\, \hat{D}^{+}_{b,p}}{\hat{\mathcal{D}}} \right\} \sqrt{\hat{\mathcal{D}}}\, \hat{\pi}_p \tag{3.54}$$
$$+\; \tilde{V}'(M_o, \{\hat{q}_k(t)\})\,.$$

The above is an exact result ($\tilde{V}'$ is just expressed in terms of M_o and $\{\hat{q}_k(t)\}$ through the decomposition (3.45); in principle, this can be an intractable expression except in the harmonic approximation). Now, we can start our expansion and keep only the leading terms since $\mid \hat{\rho}(x) \mid \ll \mid \phi_n(x - \hat{X}(t)) \mid$; this translates in keeping low powers of $\hat{q}_k(t)$ and bearing in mind that

$$M_o = \int_{-\infty}^{+\infty} dx \; \phi_n(x - \hat{X}(t))$$

is huge compared with any expressions including only $\{\hat{q}_k(t)\}$.

Let us deal first with the determinant operator $\hat{\mathcal{D}}$. The leading term is $-M_o$ as

$$\hat{\mathcal{D}} \;=\; -\frac{1}{M_o}(\hat{\alpha} + M_o)^2 \;=\; -M_o - 2\,\hat{\alpha} - \frac{(\hat{\alpha})^2}{M_o} \;\approx\; -M_o\,. \tag{3.55}$$

Therefore,

$$\sqrt{\hat{\mathcal{D}}} \;=\; i\sqrt{M_o} + \frac{i\alpha}{\sqrt{M_o}} \tag{3.56}$$

and

$$\begin{aligned} \frac{1}{\sqrt{\hat{\mathcal{D}}}} \;&=\; \frac{1}{i\sqrt{M_o}} + \frac{i\hat{\alpha}}{\sqrt{M_o}} \\ &\approx\; \frac{1}{i\sqrt{M_o}} + \frac{i\hat{\alpha}}{M_o\sqrt{M_o}}\,. \end{aligned} \tag{3.57}$$

Substituting equations (3.55)- (3.57) into (3.54) and dropping terms of order $1/M_o$, $(\hat{q}_k)^3$ and higher, we find

$$\begin{aligned} \hat{H}_{(P=0)} \;\approx\; & \frac{1}{2}\int\int dk\,dp\;\; \hat{\pi}_k^+\;\delta(k+p)\,\hat{\pi}_p^+ \;+\; \tilde{V}_h(M_o,\{\hat{q}_k(t)\}) \\ & + \frac{1}{2M_o}\int\int dk\,dp\;\hat{\pi}_k^+\,\{\hat{D}_{b,k}^+\hat{D}_{b,p}^+\}\,\hat{\pi}_p^+ \;-\; \frac{1}{2M_o^2}\int\int dk\,dp\;\; \hat{\pi}_k^+\;\delta(k+p)\,(\hat{\alpha})^2\,\hat{\pi}_p^+ \end{aligned} \tag{3.58}$$

where

$$\tilde{V}_h(M_o,\{\hat{q}_k(t)\}) \;=\; M_o + \frac{1}{2}\int dk\;\hat{q}_k^+(t)\,\hat{q}_k(t)\,\omega_k^2 \tag{3.59}$$

with ω_k being the eigenvalue of the mode $f_k(x)$.

If we discard terms involving order of $1/M_o$ and $\hat{\pi}_k\,\hat{q}_l(t)\,\hat{q}_m(t)\,\hat{\pi}_p$ or higher and treat them as negligible with respect to terms of order M_o, $\hat{q}_k(t)\,\hat{q}_p(t)$,and $\hat{\pi}_k\,\hat{\pi}_p(t)$, we obtain the leading quantum correction to the classical result $M_o = V[\phi_n(x)]$:

$$\hat{H}_{(P=0)} \;\approx\; M_o\,\hat{I} \;+\; \frac{1}{2}\int_{-\infty}^{+\infty} dk\;\{\hat{\pi}_k^{+}\,\hat{\pi}_k \;+\; \omega_k^2\,\hat{q}_k^{+}(t)\,\hat{q}_k(t)\}\,. \tag{3.60}$$

The above Hamiltonian in the rest frame of the total field is the result we tried so much to achieve where the zero-frequency ω_b no longer appears. This expression, therefore, is entirely correct insofar as the small-amplitude and harmonic approximations are concerned; the wavefunction is truly confined in the "directions" of the eigenfunctions $f_k(x)$.

For eigenvalues P different from zero, we need not go back to equation (3.49) but simply invoke the Lorentz invariance so that

$$\hat{H}_P \;=\; \sqrt{\hat{P}^2 \;+\; \hat{H}_{P=0}}\,. \tag{3.61}$$

According to Hamiltonian (3.60), the approximate eigenstates in the rest frame ($P = 0$) will have energies

$$E^{*}_{rest,\{n_k\}} \;\approx\; M_o \;+\; \frac{1}{2}\sum_k\,(n_k \;+\; 1/2)\,\omega_k\,, \tag{3.62}$$

where the asterisk reminds the reader that we deal with quantum "excited" states. For field momenta $P \neq 0$, once again, Lorentz invariance will give us

$$P = \gamma\,E^{*}_{rest,\{n_k\}}V \tag{3.63}$$

and

$$T = \gamma\,E^{*}_{rest,\{n_k\}} \tag{3.64}$$

as the total field momentum and energy, where

$$\gamma = \frac{1}{\sqrt{1-V^2}} \tag{3.65}$$

and V is the "speed" of the total field as determined by

$$\frac{P}{E^*_{rest,\{n_k\}}} = \frac{V}{\sqrt{1-V^2}}\,. \tag{3.66}$$

Practically, the above relations lead us to a situation where one would be unable to isolate the different variables, even though the relations remain self-consistent. In principle, for $P \neq 0$, one can always move to an inertial frame where $P = 0$, determine $E^*_{rest,\{n_k\}}$, and finally boost the result in order to get the answer in the original inertial frame in which $P \neq 0$.

However, for slow moving classical solutions $\phi_n(x - \hat{X}(t))$, one can extract a Hamiltonian in leading orders where the field operators $\hat{P}$ stands by itself. Indeed, we can make the following approximation[4]

$$\begin{aligned} T &= \gamma\, E^*_{rest} \\ &= \frac{1}{\sqrt{1-V^2}}\, E^*_{rest} \\ &\approx \left(1 + \frac{V^2}{2}\right) E^*_{rest} \\ &= E^*_{rest} + \frac{E^*_{rest}\, V^2}{2} \\ &= E^*_{rest} + \frac{E^*_{rest}\, P^2}{2\,(E^{*2}_{rest} + P^2)} \end{aligned} \tag{3.67}$$

by use of equation (3.66).

Furthermore,

$$\begin{aligned} \frac{E^*_{rest}}{(E^{*2}_{rest} + P^2)} &= \frac{1}{[E^*_{rest} + (P^2/E^*_{rest})]} \\ &\approx \frac{1}{[(M_o + \delta) + (P^2/E^*_{rest})]} \end{aligned} \tag{3.68}$$

where

$$E^*_{rest} = M_o + \delta \quad \text{with} \quad \delta \ll M_o\,.$$

[4]We have remove $\{n_k\}$ from the expression $E^*_{rest,\{n_k\}}$ for simplicity.

Now, we can assume for low velocity $\dot{X}(t)$ that

$$P^2 \approx M_o^2 \dot{X}^2 , \tag{3.69}$$

so that

$$\begin{aligned} \frac{P^2}{E^*_{rest}} &\approx \frac{M_o^2 \dot{X}^2}{M_o + \delta} \\ &\approx M_o \dot{X}^2 - \frac{M_o^2 \dot{X}^2 \delta}{M_o^2} \\ &= M_o \dot{X}^2 + \dot{X}^2 \delta . \end{aligned} \tag{3.70}$$

Substituting equation (3.70) into (3.68) leads to

$$\begin{aligned} \frac{E^*_{rest}}{(E^{*2}_{rest} + P^2)} &\approx \frac{1}{M_o + \delta + M_o \dot{X}^2 + \dot{X}^2 \delta} \\ &\approx \frac{1}{M_o + \delta + M_o \dot{X}^2} \\ &\approx \frac{1}{M_o + M_o \dot{X}^2} - \frac{\delta}{(M_o + M_o \dot{X}^2)^2} \\ &\approx \frac{1}{M_o + M_o \dot{X}^2} \\ &\approx \frac{1}{M_o}(1 - \dot{X}^2) . \end{aligned} \tag{3.71}$$

With this approximation, the total energy T becomes

$$\begin{aligned} T &\approx E^*_{rest} + \frac{P^2}{2M_o}(1 - \dot{X}^2) \\ &\approx E^*_{rest} + \frac{P^2}{2M_o} - \frac{P^2 \dot{X}^2}{2M_o} . \end{aligned} \tag{3.72}$$

We shall *suppose* that the last term in the above equation is negligible compared to the other terms, because of the smallness of $\dot{X}^2$, and verify *a posteriori* if such an assumption is reasonable.

Hence, we assume, for small $P \neq 0$, the following Hamiltonian in the small-amplitude and harmonic expansion:

$$\hat{H}_P^s \;=\; \frac{\hat{P}^2}{2M_o} + \frac{1}{2}\int_{-\infty}^{+\infty} dk\, \{\hat{\pi}_k^+\, \hat{\pi}_k + \omega_k^2\, \hat{q}_k^+(t)\, \hat{q}_k(t)\} \;+\; M_o\, \hat{I}\,. \tag{3.73}$$

The above Hamiltonian concludes our program for quantization of fields about a classical solution. It will be considered thereafter as the unperturbed Hamiltonian $\hat{H}_o$ to which small perturbation Hamiltonians $\hat{H}_p$ are grafted.

3.3 Quantization of the bremsstrahlung problem

For a classical field $\psi(x,t)$, the approximations we made to derive the low P-valued quantum Hamiltonian (3.73) from the exact one (3.49) can also be made to the classical Hamiltonian (3.40) to obtain

$$H_P^s \;=\; \frac{P^2}{2M_o} + \frac{1}{2}\int_{-\infty}^{+\infty} dk\, \{\mid \pi_k \mid^2 + \omega_k^2 \mid c_k \mid^2\} \tag{3.74}$$

for low values of the total field momentum P with $\pi_k(t) = \dot{c}_k(t)$.
The proper expansion in the moving reference frame attached to the impurity is

$$\psi(z,\tau) \;=\; \psi_s(z - X(\tau)) \;+\; \phi(z,\tau) \tag{3.75}$$

where

$$\phi(z,\tau) \;=\; \int_{-\infty}^{+\infty} dk\; c_k(t)\; f_k(z - X(\tau))\,. \tag{3.76}$$

In section 3.1, we recall that we used instead the expansion

$$\psi(z,\tau) \;=\; \psi_s(z) \;+\; \phi(z,\tau)$$

with

$$\phi(z,\tau) \;=\; \frac{c_b(\tau)}{8}\, f_b(z) \;+\; \int_{-\infty}^{+\infty} dk\; c_k(t)\; f_k(z)\,,$$

which proves a problem for the correct evaluation of the classical or quantum energy of the system.

With expansion (3.75), the perturbation Hamiltonian becomes

$$\begin{aligned} H_p &= \frac{\alpha}{\gamma}\int_{-\infty}^{+\infty} dz\ \psi(z,\tau)\,\delta(z+v(\tau-\tau_o)) \\ &= \frac{\alpha}{\gamma}\,\psi_s(v(\tau-\tau_o)-X(\tau)) + \frac{\alpha}{\gamma}\int_{-\infty}^{+\infty} dk\ c_k(t)\,f_k(v(\tau-\tau_o)-X(\tau)) \end{aligned} \quad (3.77)$$

where the term $\alpha\,\psi(x,t)\,\delta(x)$ making up the perturbation Hamiltonian density has been transformed to $(\alpha/\gamma)\,\psi(z,\tau)\,\delta(z+v(\tau-\tau_o))$ under the Lorentz transformation (3.5). Using approximation (3.69), $P \approx M_o\dot{x}(\tau)$, valid for slow moving solitons, the full Hamiltonian reads as

$$\begin{aligned} H &= M_o + \frac{M_o}{2}\,(\dot{X}(\tau))^2 + \frac{1}{2}\int_{-\infty}^{+\infty} dk\ \{\mid \pi_k \mid^2 + \omega_k^2 \mid c_k \mid^2\} \\ &\quad + \frac{\alpha}{\gamma}\,\psi_s(v(\tau-\tau_o)-X(\tau)) + \frac{\alpha}{\gamma}\int_{-\infty}^{+\infty} dk\ c_k(t)\,f_k(v(\tau-\tau_o)-X(\tau))\,. \end{aligned} \quad (3.78)$$

It is a simple matter to derive the equations of motion for $X(t)$ and $c_k(t)$ from the above Hamiltonian; these are

$$M_o\,\ddot{X}(\tau) = \frac{\alpha}{\gamma}\,f_b(v(\tau-\tau_o)-X(\tau)) + \frac{\alpha}{\gamma}\int_{-\infty}^{+\infty} dk\ c_k(\tau)\,f_k'(v(\tau-\tau_o)-X(\tau)) \quad (3.79)$$

and

$$\ddot{c}_p(\tau) + \omega_p^2 c_p(\tau) = -\frac{\alpha}{\gamma}\,f_p^*(v(\tau-\tau_o)-X(\tau))\,. \quad (3.80)$$

These equations are different from the corresponding ones found previously (equs. (3.9) and (3.11)) by the presence of the additional terms $-X(\tau)$ in the functions' arguments and $(\alpha/\gamma)\int dk\,c_k(\tau)\,f_k'(v(\tau-\tau_o)-X(\tau))$ in (3.79).[5] Those additional terms destroy the simplicity we had previously. It seems that the only way to solve those mixed equations

[5]Expression (3.79), apart from the difference pointed above, is for small $X(\tau)$ approximately the same as equation (3.9) once we make the identification $c_b(\tau)/M_o \triangleq -X(\tau)$, where $M_o = 8$, so that $\psi_s(x-X(\tau)) \approx \psi_s(x) - X(\tau)f_b(x)$.

is by self-consistent iteration. However, there is a lucky break. This is provided by the fact that the soliton moves very slowly after its impact with the impurity so that we can expect

$$| \dot{X}(\tau) | \ll v$$

in such a way that

$$v(\tau_o - \tau) - X(\tau) \approx v(\tau_o - \tau) \tag{3.81}$$

for all time τ.

With this approximation, our equations of motion become

$$M_o \ddot{X}(\tau) = \frac{\alpha}{\gamma} f_b(v(\tau - \tau_o)) + \frac{\alpha}{\gamma} \int_{-\infty}^{+\infty} dk \; c_k(\tau) \, f_k'(v(\tau - \tau_o)) \tag{3.82}$$

$$\ddot{c}_p(\tau) + \omega_p^2 \, c_p(\tau) = -\frac{\alpha}{\gamma} f_p^*(v(\tau - \tau_o)) \tag{3.83}$$

This last expression has the exact same *form* as equation (3.11) and, therefore, solving for $c_p(\tau)$, gives the same answer as for $c_p(\tau)$ in section 3.1. Hence, we have not laboured in vain in deriving that section's results. We know that those $c_p(\tau)$ contain α to linear order so as to make the second term in (3.82) proportional to α^2. Since we are interested only in effects linear in α because of its smallness, we can neglect this second term and recover essentially the previous expression (3.9). Thus, we have

$$M_o \ddot{X}(\tau) \approx \frac{\alpha}{\gamma} f_b(v(\tau - \tau_o)) \, . \tag{3.84}$$

In the quantum mechanical domain, we make use of expansion

$$\hat{\psi}(z, \tau) = \psi_s(z - \hat{X}(\tau)) + \int_{-\infty}^{+\infty} dk \; \hat{q}_k(\tau) \, f_k(z - \hat{X}(\tau))$$

so that the perturbed Hamiltonian $\hat{H}_p$ is

$$\begin{aligned} \hat{H}_p &= \frac{\alpha}{\gamma} \int_{-\infty}^{+\infty} dz \; \hat{\psi}(z, \tau) \, \delta(z + v(\tau - \tau_o)) \\ &= \frac{\alpha}{\gamma} \psi_s(v(\tau - \tau_o) - \hat{X}(\tau)) + \frac{\alpha}{\gamma} \int_{-\infty}^{+\infty} dk \; \hat{q}_k(\tau) \, f_k(v(\tau - \tau_o) - \hat{X}(\tau)) \, . \end{aligned} \tag{3.85}$$

The Hamiltonian of the problem is therefore

$$\hat{H} = \hat{H}_o + \hat{H}_p$$

with $\hat{H}_o = \hat{H}_p^s$ as given by equation (3.73).

We can then determine the equations of motion for the operators $\hat{X}(\tau)$ and $\hat{q}_k(\tau)$ in the usual way by making good use of the commutation relations (3.48). For $\hat{X}(\tau)$, we have

$$\begin{aligned} \frac{d\hat{X}(\tau)}{d\tau} &= -i\,[\,\hat{X}(\tau), \hat{H}(\tau)\,] \\ &= -\frac{i}{2M_o}\,[\,\hat{X}(\tau), \hat{P}^2\,] \\ \frac{d\hat{X}(\tau)}{d\tau} &= \frac{\hat{P}}{M_o} \end{aligned} \tag{3.86}$$

and

$$\begin{aligned} \frac{d\hat{P}}{d\tau} &= -i\,[\,\hat{P}, \hat{H}(\tau)\,] \\ &= -\frac{i\alpha}{\gamma}\,[\,\hat{P}, \psi_s(v(\tau-\tau_o) - \hat{X}(\tau))\,] - \frac{i\alpha}{\gamma}\int_{-\infty}^{+\infty} dk\; \hat{q}_k(\tau)\,[\,\hat{P}, f_k(v(\tau-\tau_o) - \hat{X}(\tau))\,] \\ &= -\frac{\alpha}{\gamma}\,\frac{\partial \psi_s}{\partial \hat{X}}(v(\tau-\tau_o) - \hat{X}(\tau)) - \frac{\alpha}{\gamma}\int_{-\infty}^{+\infty} dk\; \hat{q}_k(\tau)\,\frac{\partial f_k}{\partial \hat{X}}(v(\tau-\tau_o) - \hat{X}(\tau)) \\ \frac{d\hat{P}}{d\tau} &= \frac{\alpha}{\gamma}\left\{ f_b(v(\tau-\tau_o) - \hat{X}(\tau)) + \int_{-\infty}^{+\infty} dk\; \hat{q}_k(\tau)\, f_k'(v(\tau-\tau_o) - \hat{X}(\tau)) \right\}, \end{aligned} \tag{3.87}$$

so that substituting the former expression in the latter yields

$$M_o\,\frac{d^2\hat{X}(\tau)}{d\tau^2} = \frac{\alpha}{\gamma}\left\{ f_b(v(\tau-\tau_o) - \hat{X}(\tau)) + \int_{-\infty}^{+\infty} dk\; \hat{q}_k(\tau)\, f_k'(v(\tau-\tau_o) - \hat{X}(\tau)) \right\}. \tag{3.88}$$

For $\hat{q}_p(\tau)$, we have

$$\begin{aligned} \frac{d\hat{q}_p(\tau)}{d\tau} &= -i\,[\,\hat{q}_p(\tau), \hat{H}(\tau)\,] \\ &= \hat{\pi}_p \end{aligned} \tag{3.89}$$

and

$$\begin{aligned}\frac{d\hat{\pi}_p}{d\tau} &= -i\,[\,\hat{\pi}_p, \hat{H}\,] \\ &= -i\,\omega_p^2\,[\,\hat{\pi}_p,\, \hat{q}_p^+(\tau)\,\hat{q}_p(\tau)\,] \;-\; \frac{i\alpha}{\gamma} \;-\; i\,[\,\hat{\pi}_p, \hat{q}_{-p}(\tau)\,]\, f_{-p}(v(\tau-\tau_o) - \hat{X}(\tau)) \\ &= \omega_p^2\,\hat{q}_p(\tau) \;-\; \frac{\alpha}{\gamma}\, f_p^*(v(\tau-\tau_o) - \hat{X}(\tau))\,, \end{aligned} \tag{3.90}$$

so that

$$\frac{d^2\hat{q}_p(\tau)}{d\tau^2} + \omega_p^2\,\hat{q}_p(\tau) \;=\; -\frac{\alpha}{\gamma}\, f_p^*(v(\tau-\tau_o) - \hat{X}(\tau))\,. \tag{3.91}$$

We note that the form of these operator equations is exactly the same as that of their classical counterparts, equations (3.79) and (3.80). In a like manner, we can make the same approximations on $\hat{X}(\tau)$ and the second term in equation (3.82) to obtain

$$M_o\,\frac{d^2\hat{X}(\tau)}{d\tau^2} \;\approx\; \frac{\alpha}{\gamma}\, f_b(v(\tau-\tau_o)) \tag{3.92}$$

$$\frac{d^2\hat{q}_p(\tau)}{d\tau^2} + \omega_p^2\,\hat{q}_p(\tau) \;=\; -\frac{\alpha}{\gamma}\, f_p^*(v(\tau_o-\tau))\,. \tag{3.93}$$

We remind the reader that the above equations and their respective classical analogues are valid in the limit of a slow-moving soliton as viewed in the frame we designated as the soliton rest frame (whose term assumes its true meaning only for $\tau \to -\infty$).

In conclusion, by virtue of Ehrenfest theorem, the above shows that the evolution of the expectation value of the Heisenberg state $|\,\psi\rangle$, which corresponds as closely as possible to our initial conditions of a soliton at rest and an impurity dressing moving with speed v at $\tau \to -\infty$, will be governed for all intents and purposes by the classical equations of motion (3.9) and (3.11). Therefore, we can expect that, by comparing with the classical energy expression (3.74), the *average* energy for the *quantum* excitation of the field to be also given by the classical bremsstrahlung energy $H_r(\tau \gg \tau_o)$, equation (3.18). In order to prove this assertion, we had to take a lengthy detour in the direction of the collective coordinates land; a trip, nonetheless, that will prove most profitable in the next chapter.

Chapter 4

Oscillations of a pinned soliton

The previous two chapters cleared the ground for the main problem we seek to solve and which constitutes the *raison d'être* of this study. The perturbation Hamiltonian density we are interested in is given by equation (2.39) with

$$V(x) \;=\; \delta(x) \tag{4.1}$$

for the impurity (or pinning center) located at the origin. As in the last chapter, we divide the discussion between a classical and a quantum treatment.

4.1 Classical treatment

The equation of motion flowing from the Hamiltonian density $\mathcal{H} = \mathcal{H}_o + \mathcal{H}_p$ is

$$\ddot{\psi} \;-\; \psi'' \;+\; \sin\psi \;=\; \alpha\,\delta'(x)\,. \tag{4.2}$$

As expected, we use the expansion of the field $\psi(x,t)$ as follows:

$$\psi(x,t) \;=\; \psi_s(x - X(t)) \;+\; \phi(x,t) \tag{4.3}$$

where

$$\phi(x,t) \;=\; \int_{-\infty}^{+\infty} dk\; c_k(t)\; f_k(x - X(t))\,. \tag{4.4}$$

With expansion (4.3), the perturbation Hamiltonian yields

$$
\begin{aligned}
H_p &= \alpha \int_{-\infty}^{+\infty} dx \, \frac{\partial \psi(x,t)}{\partial x} \, \delta(x) \\
&= \alpha \left\{ \int_{-\infty}^{+\infty} dx \, \frac{\partial \psi_s(x - X(t))}{\partial x} \, \delta(x) \; + \; \int_{-\infty}^{+\infty} dk \; c_k(t) \int_{-\infty}^{+\infty} dx \, \frac{\partial f_k(x - X(t))}{\partial x} \, \delta(x) \right\} \\
H_p &= \alpha \, f_b(X(t)) \; + \; \alpha \int_{-\infty}^{+\infty} dk \; c_k(t) \, f_k'(-X(t)) \, . \qquad (4.5)
\end{aligned}
$$

If we assume from the outset that we deal with slow solitons, then, upon using the unperturbed Hamiltonian H_P^s (3.74) and the approximation (3.69), the total Hamiltonian becomes

$$
\begin{aligned}
H &= M_o \; + \; \frac{M_o}{2} \, \dot{X}^2(t) \; + \; \frac{1}{2} \int_{-\infty}^{+\infty} dk \; \{ \mid \pi_k \mid^2 \; + \omega_k^2 \; \mid c_k \mid^2 \} \\
&\qquad + \; 2 \, \alpha \, \mathrm{sech}(X(t)) \; + \; \alpha \int_{-\infty}^{+\infty} dk \; c_k(t) \, f_k'(-X(t)) \, . \qquad (4.6)
\end{aligned}
$$

The equations of motion for $X(t)$ and $c_p(t)$ from the above Hamiltonian are found to be

$$
\begin{aligned}
M_o \, \ddot{X}(t) &= -\alpha \, f_b'(X(t)) \; + \; \alpha \int_{-\infty}^{+\infty} dk \; c_k(t) \, f_k''(-X(t)) \qquad (4.7) \\
\ddot{c}_p(t) \; + \; \omega_p^2 \, c_p(t) &= -\alpha \, f_p^{*\,\prime}(-X(t)) \qquad (4.8)
\end{aligned}
$$

in the limit of a slow soliton.

Equations (4.7) and (4.8) couple the coordinates $X(t)$ and $c_p(t)$. We must therefore resort to a self-consistent iterative method. For the start of our iteration process, we consider first solving for $X(t)$ by assuming $c_k(t) = 0$ in (4.7). Under some assumptions to be defined later, the resulting solution $X(t)$, upon substitution into equation (4.8), will generate solutions $c_p(t)$ containing α to linear order which in turn, upon feeding back into (4.7) for the second iteration step, would make the term $\alpha \int dk c_k(t) f_k''(-X(t))$ quadratic in α. Since we consider, as a first approximation, equations of motion only linear in the small coupling constant α, we may as well ignore the higher order term in equation (5.7) and obtain instead

$$M_o\,\ddot{X}(t) \;=\; -\alpha\, f_b'(X(t))\,. \tag{4.9}$$

Note that the Caldeira-Leggett formalism, which is taken up below, will permit us to tackle the full equation (5.7) in the limit of low amplitudes for $X(t)$.

Once again, (5.7) gives some form of Newton's law of motion for the center-of-mass coordinate of the soliton $X(t)$ with a force term $-\alpha\, f_b'(X)$. Indeed, this is exactly what we would expect from a potential $\mathsf{V}(X)$ felt by a stationary soliton determined as follows: Consider the terms

$$\mathcal{V}(x;X) \;=\; \frac{1}{2}\,(\psi_s')^2 \;+\; (1-\cos\psi_s) \;+\; \alpha\,\frac{\partial\psi_s}{\partial x}\,\delta(x)$$

given by the Hamiltonian density $\mathcal{H}$ for a stationary soliton $\psi_s(x-X)$ whose center is located at $x=X$, a distance $\mid X \mid$ from the impurity. We may always consider $\mathcal{V}(x;X)$ as the total potential density term as a function of the parameter X, the soliton's center. The corresponding potential is therefore

$$\begin{aligned}
\mathsf{V}(X) \;&=\; \int_{-\infty}^{+\infty} dx\;\mathcal{V}(x;X) \\
&=\; \int_{-\infty}^{+\infty} dx\;\left\{\frac{1}{2}\,(\psi_s'(x-X))^2 \;+\; (1-\cos\psi_s(x-X)) \;+\; \alpha\,\frac{\partial\psi_s(x-X)}{\partial x}\,\delta(x)\right\} \\
&=\; \int_{-\infty}^{+\infty} dx\;\{\frac{1}{2}\,(\psi_s'(x-X))^2 \;+\; (1-\cos\psi_s(x-X))\} \\
&\qquad\qquad + \alpha\int_{-\infty}^{+\infty} dx\;\frac{\partial\psi_s(x-X)}{\partial x}\,\delta(x) \\
&=\; M_o \;+\; \alpha\int_{-\infty}^{+\infty} dx\;f_b(x-X)\,\delta(x) \\
&=\; M_o \;+\; \alpha\, f_b(-X) \\
\mathsf{V}(X) \;&=\; M_o \;+\; 2\,\alpha\,\mathrm{sech}(X)\,.
\end{aligned} \tag{4.10}$$

This potential is drawn in figure (4.1).

Notice that the force term in equation (4.9), $-\alpha\, f_b'(X)$, is exactly the negative of the gradient of $V(X)$, $-dV(X)/dX$, reinforcing even further the identification of (4.9) as a

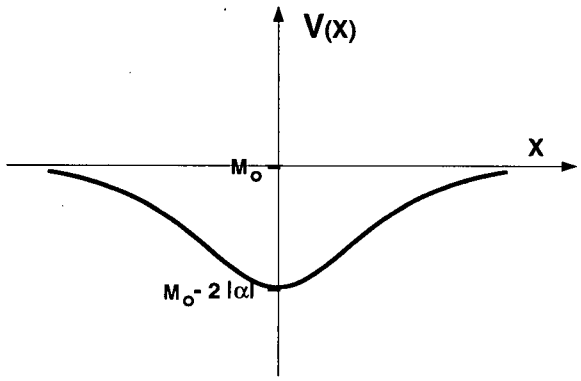


Figure 4.1: The physical potential provided by the pinning center. The potential for the domain wall, whose center is parametrized by coordinate X, is of hyperbolic secant form with M_o being the rest mass of the wall and α a negative small coupling constant.

Newton's law of motion for the soliton's coordinate X:

$$M_o \ddot{X} = -\frac{dV(X)}{dX}.$$

Hence, insofar as quantum fluctuations are set apart, the effect of the pinning center is to provide a potential well of hyperbolic secant form to a moving soliton which may become trapped if it does not possess sufficient kinetic energy.

In order to solve equation (4.9), we *suppose* that we have a small amplitude $X(t)$ about the minimum of the potential $V(X)$. In this case, we may expand the potential $V(X)$ and keep only the quadratic order so that

$$M_o \ddot{X}(t) = -\frac{dV(X)}{dX},$$

$$8\ddot{X}(t) \approx 2\alpha X, \tag{4.11}$$

$$\ddot{X}(t) = \frac{\alpha}{4} X(t), \tag{4.12}$$

where we used the mass of the soliton $M_o = 8$.

A simple solution to equation (4.12) is given by

$$X(t) = X_o \sin(\omega t) \tag{4.13}$$

with $\omega^2 = -\alpha/4$ (recall that α is negative). Owing to the fact that α is small, ωX_o, which gives the maximal speed of the oscillating soliton, will have this one move slowly. Equipped with this approximation for $X(t)$, we can hope to solve for the coefficients $c_p(t)$ by keeping terms only up to $\mathcal{O}(X_o)$. Let us see what we get for $f_p^{*\prime}(X(t))$ in this soliton's low-amplitude approximation.

$$\begin{aligned} f_p^{*\prime}(X) &= -ip\, f_p^* + \frac{\operatorname{sech}^2 X\, e^{-ipX}}{\sqrt{2\pi}\,\omega_p} \\ &= \frac{e^{-ipX}}{\sqrt{2\pi}\,\omega_p}\{p^2 + \operatorname{sech}^2 X - ip \operatorname{tgh} X\}. \end{aligned} \tag{4.14}$$

For small X, we have

$$e^{-ipX} \approx 1 - ipX - \frac{p^2X^2}{2}, \tag{4.15}$$

$$\operatorname{tgh} X \approx X,$$

$$\operatorname{sech}^2 X \approx 1 - X^2. \tag{4.16}$$

Note that the approximation for the exponential e^{ipX} is valid only for sufficiently low values of p. If we look at our problem from the physics point of view, we would expect only the low-lying quantum energy states to be excited by the *slow* motion of the soliton at the bottom of the hyperbolic secant potential. Therefore, only states with low momenta should make an important contribution to the quantum fluctuation. We remind the reader that this whole calculation is done at temperature $T \approx 0$ so that the higher energy states will hardly be populated at all. Therefore, we consider the approximation (4.15) as valid under those circumstances.

Hence, with the above approximations, we obtain

$$f_p^{*\prime}(-X(t)) \approx \frac{1+ipX(t)}{\sqrt{2\pi}\,\omega_p}\{(p^2+1) - X^2(t) + ipX(t)\}$$

$$\approx \frac{\omega_p}{\sqrt{2\pi}} + \frac{ip\,(p^2+2)\,X(t)}{\sqrt{2\pi}\,\omega_p} \tag{4.17}$$

to order $\mathcal{O}(X(t))$, so that equation (4.8), to this order, becomes

$$\ddot{c}_p(t) + \omega_p^2\, c_p(t) = -\frac{\alpha\,\omega_p}{\sqrt{2\pi}} - \frac{i\,\alpha\,p\,(p^2+2)\,X(t)}{\sqrt{2\pi}\,\omega_p}\,. \tag{4.18}$$

Note that for a stationary soliton positioned right at the pinning center, equation (4.18) becomes, with $\ddot{c}_p(t) = 0 = \dot{X}(t)$ and $X(t) \to 0$,

$$\begin{aligned} \omega_p^2\, c_p^o &= -\frac{\alpha\,\omega_p}{\sqrt{2\pi}}\,, \\ c_p^o &= -\frac{\alpha}{\sqrt{2\pi}\,\omega_p}\,. \end{aligned} \tag{4.19}$$

This value simply corresponds to the screening of the soliton caused by the presence of the impurity; in other words, this is the impurity "dressing" in the presence of the soliton. Therefore, the screening is given by

$$\begin{aligned} \phi_{scr}(x) &= \int_{-\infty}^{+\infty} dk\; c_k^o\, f_k(x) \\ &= -\frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\; \frac{f_k(x)}{\omega_k}\,. \end{aligned} \tag{4.20}$$

We can subtract off this impurity contribution from equation (4.18) by setting

$$c_p(t) = c_p^o + \tilde{c}_p(t)\,. \tag{4.21}$$

Thus, we obtain

$$\ddot{\tilde{c}}_p(t) + \omega^2\, \tilde{c}_p(t) = -\frac{i\,\alpha\,p\,(p^2+2)\,X(t)}{\sqrt{2\pi}\,\omega_p}\,, \tag{4.22}$$

or

$$\ddot{\tilde{c}}_p(t) + \omega^2\, \tilde{c}_p(t) = -\frac{i\,\alpha\,p\,(p^2+2)}{\sqrt{2\pi}\,\omega_p}\, X_o\, \sin(\omega t) \tag{4.23}$$

where we used equation (4.13) for $X(t)$.

Solving for $\tilde{c}_p(t)$ is easy and the result is

$$\tilde{c}_p(t) \;=\; a\, e^{-i\omega_p t} \;+\; B\, e^{i\omega_p t} \;-\; \frac{i\,\alpha\, X_o\, p\,(\omega_p^2+1)}{\sqrt{2\pi}\,\omega_p}\,\frac{\sin(\omega t)}{(\omega_p^2-\omega^2)}\,, \tag{4.24}$$

with A and B being two constants of integration.

In order to fix those constants, we need the initial conditions. These will be determined by the physical situation at time $t = -\pi/(2\omega)$ of a soliton at its maximal amplitude, $x = -X_o$, with no velocity, about to be pulled in towards the minimum of the potential $V(X)$. For this case, obviously,

$$\dot{c}_p(-\pi/(2\omega)) \;=\; 0\,, \tag{4.25}$$

and from equation (4.18)

$$\begin{aligned} \omega_p^2\, c_p(-\pi/(2\omega)) \;&=\; -\frac{\alpha\,\omega_p}{\sqrt{2\pi}} \;+\; \frac{i\,\alpha\, X_o\, p\,(\omega_p^2+1)}{\sqrt{2\pi}\,\omega_p}\,, \\ c_p(-\pi/(2\omega)) \;&=\; c_p^o \;+\; \frac{i\,\alpha\, X_o\, p\,(\omega_p^2+1)}{\sqrt{2\pi}\,\omega_p^3}\,. \end{aligned} \tag{4.26}$$

With initial conditions (4.25) and (4.26), we find

$$A \;=\; \frac{\Upsilon_p}{2}\,\frac{e^{-i\pi\omega_p/(2\omega)}\,\omega^2}{(\omega_p^2-\omega^2)\,\omega_p^2}\,, \qquad\qquad B \;=\; A\, e^{i\pi\omega_p/\omega} \tag{4.27}$$

with

$$\Upsilon_p \;\triangleq\; -\frac{i\,\alpha\, X_o\, p\,(\omega_p^2+1)}{\sqrt{2\pi}\,\omega_p}\,.$$

Pulling everything together, we obtain

$$\begin{aligned} c_p(t) \;&=\; c_p^o \;+\; \tilde{c}_p(t) \qquad\qquad \text{with} \\ \tilde{c}_p(t) \;&=\; \frac{\Upsilon_p}{(\omega_p^2-\omega^2)}\left\{\frac{\omega^2}{\omega_p^2}\cos\left(\frac{\pi\,\omega_p}{2\,\omega} \;+\; \omega_p\, t\right) \;+\; \sin(\omega t)\right\}. \end{aligned} \tag{4.28}$$

We need not worry about the vanishing of the denominator $(\omega_p^2 - \omega^2)$ since it will never occur as $\omega_p^2 \geq 1$ whereas $\omega^2 = -\alpha/4 \ll 1$.

Therefore,

$$\tilde{\phi}(x,t) \;=\; \int_{-\infty}^{+\infty} dk\; \tilde{c}_k(t)\; f_k(x - X(t))$$

represents the classical small-amplitude fluctuation generated by the presence of a pinning center *apart* from the screening effect. The energy fluctuation is given by the classical expression in equation (3.74),

$$\begin{aligned}
E_{fluc}^{cl} &= H_P^S \;-\; (M_o \;+\; \frac{P^2}{2\,M_o}) \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} dk\; \{\mid \dot{c}_k(t)\mid^2 \;+\omega_k^2 \mid c_k(t)\mid^2\} \qquad (4.29) \\
&= \frac{\alpha^2}{4\pi}\int_{-\Lambda}^{+\Lambda} dk \;+\; \frac{\alpha^2 X_o^2}{4\pi}\int_{-\Lambda}^{+\Lambda} dk\; \frac{k^2\,(\omega_k^2+1)^2}{\omega_k^2\,(\omega_k^2-\omega^2)^2}\left\{\frac{\omega^4}{\omega_k^4} \;+\; (\omega_k^4+\omega^2)\,\sin\omega t \qquad (4.30)\right. \\
&\qquad\qquad \left. +\; 2\,\omega^2 \sin\omega t\left[\cos\left(\frac{\pi\,\omega_k}{2\,\omega}+\omega_k t\right) - \frac{\omega}{\omega_k}\sin\left(\frac{\pi\,\omega_k}{2\,\omega}+\omega_k t\right)\right]\right\},
\end{aligned}$$

where a momentum cut-off Λ has been introduced in keeping with our previous discussion of a limited number of modes involved in the limit of $T \to 0$ and a slowly oscillating soliton. The first term in (4.30) corresponds to the impurity screening effect alone. Also, for a slow soliton, we have

$$\frac{P^2}{2\,M_o} \;\approx\; \frac{M_o\,(\dot{X})^2}{2}$$

and

$$\begin{aligned}
H_p &= \int_{-\infty}^{+\infty} dx\; \mathcal{H}_p \\
&= \alpha\, f_b(X(t)) \;+\; \alpha\int_{-\infty}^{+\infty} dk\; c_k(t)\; f_k'(-X(t)) \\
&\approx \alpha \;+\; \frac{\alpha^2}{2\pi}\int_{-\Lambda}^{+\Lambda} dk\; \left\{-1 + \frac{k^2\,(\omega_k^2+1)^2\,X_o^2}{\omega_k^2\,(\omega_k^2-\omega^2)}\,\sin\omega t\left[\sin\omega t + \frac{\omega^2}{\omega_k^2}\cos\left(\frac{\pi\omega_k}{2\omega}+\omega_k t\right)\right]\right\}
\end{aligned} \qquad (4.31)$$

as the energy of the perturbation proper.

Classically, we can speak of the eigenmodes $f_k(x - X(t))$ in $\phi(x,t)$ of equation 4.4 as being subjected to a time-dependent potential $\bar{V}(X(t)) = 2\,\alpha$ sech $(X(t))$, which destroys the simple harmonic motion that they would otherwise enjoy (in the limit of a *slow* soliton), as can be inferred from equation (4.18) with $\alpha = 0$ and $\dot{X}(t) \approx 0$.

The above discussion was entirely classical; the next section deals with the quantum treatment.

4.2 Quantum treatment

In the last chapter, we expanded at length on the transition from the classical to the quantum world. We recall that such a transition needed a special treatment on account of the zero-frequency eigenmode $f_b(x)$ associated with the translation invariance of the classical static solution $\psi_s(x - X)$. However, here, in the presence of the perturbation term

$$\mathcal{H}_p \;=\; \alpha \frac{\partial \psi}{\partial x}\,\delta(x)\,,$$

we found that this led to a potential well 4.10 for which, classically, we expect the soliton to sit at the bottom motionless. (cf. fig. (4.2 (a)))

Therefore, the classical solution is no longer $\psi_s(x-X)$ for any X, but rather $\psi_s(x)$ which carries NO translational invariance. This implies that we could attempt to diagonalize the operator

$$\left[-\nabla^2 \;\left(\frac{d^2U_\alpha}{d\phi^2}\right)\Big|_{\phi=\psi_s(x)}\right], \tag{4.32}$$

with

$$U_\alpha(\phi) \;=\; 1 \;-\; \cos\phi \;+\; \alpha \frac{\partial \phi}{\partial x}\,\delta(x)\,,$$

with no risk of obtaining a zero-frequency mode. The classical picture (figure 1.2a) of a soliton at rest at the bottom of the potential well is forbidden in quantum mechanics and

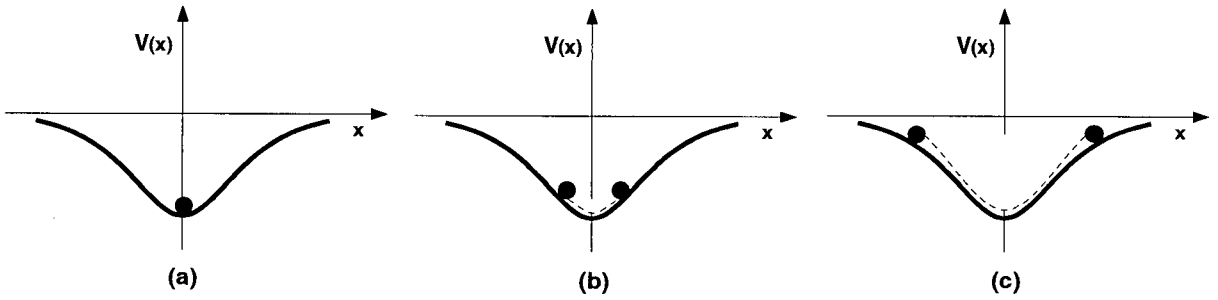


Figure 4.2: Quantum and classical oscillations of a soliton. The little filled dot is the virtual particle representing a magnetic soliton, the domain wall. In (a), the soliton is expected classically to be at the bottom of the potential well V(x); an effaced state that is strictly forbidden in quantum mechanics which, therefore, imparts the soliton in (b) with quantum oscillations about the minimum. In case of high energy states, which may come about in many different ways, the oscillations in (c) may assume a macroscopic character as befit classical states.

we expect, therefore, to have small quantum oscillations about the minimum. Furthermore, as the soliton is very massive, these quantum fluctuations are believed to be minute. Since the potential well $2\alpha\,\mathrm{sech}\,(x)$ is somewhat shallow, because of the small coupling constant α, we anticipate the eigenmodes $\{\tilde{f}_b(x), \tilde{f}_k(x)\}$ of operator(4.32) to resemble our previous eigenmodes $\{f_b(x), f_k(x)\}$ obtained with the potential $U(\phi) = 1 - \cos\phi$. Therefore, it is reasonable to expect the coordinate $\tilde{c}_b(t)$ associated with $\tilde{f}_b(x)$ to reflect the position of the soliton in the same manner as $X(t)$ did. In addition to this quantum effect for coordinate X (or if you prefer $\tilde{c}_b$), there also will be quantum fluctuations for the coordinate $\tilde{c}_k$ in the "directions" of the eigenmodes $\tilde{f}_k(x)$.

All this is very fine insofar as the exact eigenmodes $\{\tilde{f}_b(x), \tilde{f}_k(x)\}$ flowing from the operator (4.32) can easily be found. Unfortunately, this might not be the case. However, once again, because of the smallness of α, we can think that expansion (4.3) with eigenmodes $f_k(x - X(t))$ will not be a bad choice; the price to pay being the sad consequence of obtaining equations of motion mixing the coordinates $X(t)$ and $c_k(t)$, as can easily be seen by inspection of equation (4.7) and (4.8), away from the simple harmonic form they

enjoy with the proper coordinates $\{\tilde{c}_b(t), \tilde{k}(t)\}$.[1] Perhaps, this a better trade-off as the diagonalization of operator (4.32) can prove most difficult.

There is another way in which the original expansion (4.3) is preferable. The soliton may very well perform *classical* oscillations in the sense that the amplitude is large enough to be measured with arbitrarily experimental precision (cf. fig. (4.2 (c))). Such a motion could be initiated by the application of a force for a very short interval of time even at $T = 0$. For a magnetic soliton, the Bloch wall, this force is obviously magnetic. In our case, the amplitude X_o for the harmonic motion (4.13) *could* be large compared with that associated with the quantum fluctuations (cf. fig.(4.2 (b))), and yet, small enough, on the length scale as determined in section 2.2, to develop the real potential sech (X) up to quadratic order only. In *that* case, expansion 4.3 can be regarded as one about a *classical* time-dependent solution $\psi_s(x - X(t))$ with $X(t)$ obeying the equation of motion (4.9).

In keeping with our special quantization scheme of collective coordinates in chapter 3, we consider, therefore, the unperturbed Hamiltonian

$$\hat{H}_o \;=\; \frac{\hat{P}^2}{2\,M_o} \;+\; \int_{-\infty}^{+\infty} dk\; \{\hat{\pi}_k^+\,\hat{\pi}_k \;+\; \omega_k^2\,\hat{c}_k^+\,\hat{c}_k\} \;+\; M_o\,\hat{I}\,, \tag{4.33}$$

with the perturbation term

$$\hat{H}_p \;=\; \alpha \int_{-\infty}^{+\infty} dx\; \frac{\partial\hat{\psi}(x,t)}{\partial x}\;\delta(x) \tag{4.34}$$

where

$$\hat{\psi}(x,t) \;=\; \psi_s(x - \hat{X}(t)) \;+\; \int_{-\infty}^{+\infty} dk\; \hat{c}_k(t)\; f_k'(-\hat{X}(t))\,. \tag{4.35}$$

The equations of motion for $\hat{X}(t)$ and $\hat{c}_k$ can be found as follows:

[1] This is after all what diagonalization is all about !

$$\begin{aligned}
\frac{d\hat{X}(t)}{dt} &= -i\,[\,\hat{X}(t),\hat{H}(t)\,] \\
&= -\frac{i}{2\,M_o}\,[\,\hat{X}(t),\hat{P}^2\,] \\
&= \frac{\hat{P}}{M_o}\,,
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{P}}{dt} &= -i\,[\,\hat{P},\hat{H}\,] \\
&= -i\,\alpha\,[\,\hat{P},f_b(\hat{X})\,] \;-\; i\,\alpha\int_{-\infty}^{+\infty} dk\;\hat{c}_k(t)\,[\,\hat{P},f_k'(-\hat{X}(t))\,] \\
&= -\alpha\,\frac{\partial f_b(\hat{X})}{\partial\hat{X}} \;-\; \alpha\int_{-\infty}^{+\infty} dk\;\hat{c}_k(t)\,\frac{\partial f_k'(-\hat{X})}{\partial\hat{X}}\,,
\end{aligned}$$

so that

$$M_o\,\frac{d^2\hat{X}(t)}{dt^2} \;=\; -\alpha\,\frac{\partial f_b(\hat{X})}{\partial\hat{X}} \;+\; \alpha\int_{-\infty}^{+\infty} dk\;\hat{c}_k(t)\,\frac{\partial^2 f_k(\hat{X})}{\partial\hat{X}^2}\,, \tag{4.36}$$

and

$$\begin{aligned}
\frac{d\hat{c}_k(t)}{dt} &= -i\,[\,\hat{c}_k(t),\hat{H}(t)\,] \\
&= i\,[\,\hat{c}_k(t),\hat{\pi}_k^+\,\hat{\pi}_k\,] \\
&= \hat{\pi}_k\,,
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{\pi}_k(t)}{dt} &= -i\,[\,\hat{\pi}_k(t),\hat{H}(t)\,] \\
&= -i\,[\,\hat{\pi}_k,\omega_k^2\,\hat{c}_k^+\,\hat{c}_k] \;-\; i\,\alpha\,[\,\hat{\pi}_k,\hat{c}_{-k}\,f_{-k}'(-\hat{X}(t))\,] \\
&= -\omega_k^2\,\hat{c}_k(t) \;-\; \alpha\,f_k^{*\,\prime}(-\hat{X}(t))\,,
\end{aligned}$$

so that

$$\frac{d^2\hat{c}_k(t)}{dt^2} \;+\; \omega_k^2\,\hat{c}_k(t) \;=\; -\alpha\,f_k^{*\,\prime}(-\hat{X}(t))\,. \tag{4.37}$$

We note immediately that the operator equations for $\hat{X}(t)$ and $\hat{c}_k(t)$ ((4.36) and (4.37)) correspond exactly to the form for the equations of motion of $X(t)$ and $c_p(t)$ found above, equations (4.7) and (4.8). Under the same assumptions of low amplitudes for $X(t)$, so that

$$f_k^{*\prime}(-\hat{X}(t)) \approx \frac{\omega_k}{\sqrt{2\pi}} + \frac{ik\,(\omega_k^2+1)}{\sqrt{2\pi}\,\omega_k}\,\hat{X}(t)\,,$$

and of omission of terms higher than linear order in α, we can approximate (4.36) by

$$M_o \frac{d^2\hat{X}(t)}{dt^2} = -\alpha \frac{\partial f_b(\hat{X})}{\partial \hat{X}}\,. \tag{4.38}$$

Thus, in the limit of a slow soliton and small coupling constant α, the time evolution of the operators $\hat{X}(t)$ and $\hat{c}_k(t)$ mimic that of their classical counterparts $X(t)$ and $c_k(t)$. As discussed at the end of section 3.3, we expect, then, the *average* energy of the quantum fluctuation to be given by the classical result for the small-amplitude approximation, equation (4.30). Formally, the quantum fluctuation energy is given by the harmonic-oscillator form of Hamiltonian (4.33) (minus the classical energy M_o),

$$\begin{aligned}\Delta E_{\{\dot{X},n_k\}} &\approx \frac{M_o\,\dot{X}^2}{2} + \sum_k (n_k+1/2)\,\omega_k \\ &= \frac{M_o\,\dot{X}^2}{2} + \int_{-\infty}^{+\infty} dk\;(n_k+1/2)\,\omega_k\,,\end{aligned} \tag{4.39}$$

where $\dot{X}$ is the average velocity of the oscillating soliton.[2] Expressions (4.30) and (4.39) do not include the energy of the perturbation proper, equation (4.31). It is important to point out that some of those energies may contain infinities which should be handled with the greatest care to obtain a meaningful physical result. Here, for $T = 0$, the introduction of momentum cut-offs and rejection of the zero-point energy would seem to be sufficient for our purpose. In order to carry out the perturbation to higher orders, one would have to deal with the usual techniques of normal ordering, counter terms, and the like, to handle the troublesome divergences.

[2] Recall that we use the convention $\hbar = 1$.

The next section deals with the mapping of our problem onto the Caldeira-Leggett formalism and the calculation of the spectral density function $J(\omega)$, the key result of this study.

4.3 Mapping to the Caldeira-Leggett formalism

In the introduction, we mentioned that a general Lagrangian, which expresses the physical situation of a system coupled to its environment and whose quasi-classical equation of motion is given by

$$M\,\ddot{Q} \;+\; \eta\,\dot{Q} \;+\; \frac{dV(Q)}{dQ} \;=\; F_{ext}(t)\,, \tag{4.40}$$

is

$$L \;=\; \frac{1}{2}M\,\dot{Q}^2 \;-\; V(Q) \;+\; Q\,F_{ext}(t) \;+\; \frac{1}{2}\sum_j\,(m_j\,\dot{x}_j^2 \;-\; m_j\,\omega_j^2\,x_j^2) \tag{4.41}$$

$$-\;\sum_j\,F_j(Q)\,x_j \;-\;\sum_j\,\frac{F_j^2(Q)}{2\,m_j\,\omega_j^2}\,.$$

For the case of strictly linear dissipation, we have

$$F_j(Q) \;=\; Q\,C_j\,,$$

a linear coupling in the system's coordinate Q.

It is a simple exercise to derive the equations of motion for the harmonic oscillators x_j in the strictly linear dissipation case:

$$m_j\,\ddot{x}_j \;+\; m_j\,\omega_j^2\,x_j \;=\; -Q\,C_j\,, \tag{4.42}$$

or for unit-mass oscillators,

$$\ddot{x}_p \;+\; \omega_p^2\,x_p \;=\; -Q(t)\,C_p\,, \tag{4.43}$$

showing the time-dependence of Q explicitly and changing the index for later convenience.

Recall the equation of motion we obtained for the coordinates $c_p(t)$ in the classical treatment of the pinning potential in the limit of a slow oscillating soliton in section 4.1 (cf. equation (4.18))

$$\ddot{c}_p(t) \; + \; \omega_p^2 \, c_p(t) \; = \; -\frac{\alpha \, \omega_p}{\sqrt{2\pi}} \; - \; \frac{i \, \alpha \, p \, (\omega_p^2 + 1)}{\sqrt{2\pi} \, \omega_p} \, X(t) \, . \tag{4.44}$$

Apart from the screening contribution $-\alpha \, \omega_p / \sqrt{2\pi}$, expressions (4.43) and (4.44) would be somewhat equivalent should the identification of $Q(t)$ with the soliton's center coordinate $X(t)$, $x_p(t)$ with $c_p(t)$, and C_p with $-(i \, \alpha \, p \, (\omega_p^2+1))/(\sqrt{2\pi} \, \omega_p)$ be made [3]. One's first reaction would be to regard as suspicious the identification of a *whole* system parametrized by the coordinates $\{X(t), c_p(t)\}$, whose interaction with the surroundings is not accounted for by the Hamiltonian (2.37), with the Caldeira-Leggett universe consisting, respectively, of the system *and* the environment coordinates, $Q(t)$ *and* $\{x_p(t)\}$. However, we must admit that the partition of the universe is always arbitrary and consonant with the kinds of questions one may wish to address. Furthermore, we showed in section 4.1 that, under some specific circumstances, the soliton's coordinate might be considered as a classical one subjected to a harmonic forced motion. It seems therefore reasonable to entertain the idea of the soliton's constituting the system (parametrized by $X(t)$) coupled to a bath of harmonic oscillators (parametrized by $\{c_p(t)\}$) which may be viewed as the "environment." In our specific case, those oscillators correspond to magnons, the elementary quantum excitations of the magnetization field *about* the soliton solution.

Above, we promised to deal with equation 4.7 without omitting the second term

$$\alpha \int_{-\infty}^{+\infty} dk \; c_k(t) \; f_k''(-X(t)).$$

Indeed, it is the presence of this term which makes it possible to think of the coordinate

[3] The minus sign arises from the fact that we deal with complex coefficients c_p.

$X(t)$, the soliton's center, as being subjected to viscous forces reflected by the phenomenological friction parameter η in equation (4.40). In order to see this more clearly, let us write down the Lagrangian derived from the Hamiltonian 4.6 in the small-amplitude approximation,

$$\begin{aligned} L &= M_o \frac{\dot{X}^2(t)}{2} - V(X(t)) + \frac{1}{2}\sum_k \{| \dot{c}_k(t) |^2 - \omega_k^2 \mid c_k(t) |^2\} \\ &\quad - \alpha \sum_k c_k(t)\, f_k'(-X(t))\,, \end{aligned} \tag{4.45}$$

where

$$\begin{aligned} V(X(t)) &= \alpha\, f_b(X(t)) + M_o = 2\,\alpha\, \mathrm{sech}\,(X(t)) + M_o\,, \\ \pi_k(t) &= \dot{c}_k(t)\,, \end{aligned}$$

and a discrete summation has been chosen instead of the integral sign for convenience. In the limit of low amplitudes for $X(t)$, approximation (4.17) is valid and so L transforms to

$$\begin{aligned} L &= M_o \frac{\dot{X}^2(t)}{2} - V(X(t)) + \frac{1}{2}\sum_k \{| \dot{c}_k(t) |^2 - \omega_k^2 \mid c_k(t) |^2\} \\ &\quad - \alpha \sum_k c_k(t)\, f_k^o - \sum_k c_k(t)\, C_k\, X(t) \end{aligned} \tag{4.46}$$

where

$$f_k^o \triangleq \frac{\omega_k}{\sqrt{2\pi}} \quad \text{and} \quad C_k = -\frac{i\,\alpha\, k\,(\omega_k^2+1)}{\sqrt{2\pi}\,\omega_k}. \tag{4.47}$$

The terms $(1/2)\sum_k \{\omega_k^2 \mid c_k(t) \mid^2 + 2\,\alpha\, c_k(t)\, f_k^o\}$ lend themselves to completion of a square by the addition of a constant to the Lagrangian; which is immaterial insofar as energy differences and equations of motion are concerned.[4] Using the fact that $c_k^* = c_{-k}$, we find that

$$-\frac{\alpha^2}{2}\sum_k \left(\frac{f_k^o}{\omega_k^2}\right)^2$$

[4]This simply amounts to a canonical transformation for which the coordinates suffer a translation by specific constants [40].

does the trick so that

$$L = M_o \frac{\dot{X}^2(t)}{2} - V(X(t)) + \frac{1}{2}\sum_k \{| \dot{q}_k(t) |^2 - \omega_k^2 | q_k(t) |^2\} \qquad (4.48)$$
$$- \sum_k q_k(t)\, C_k\, X(t) + \alpha^2 \sum_k \frac{f_k^o}{\omega_k^2}\, C_k\, X(t)$$

where

$$q_k(t) \triangleq c_k(t) + \frac{\alpha\, f_k^o}{\omega_k^2} = c_k(t) + \frac{\alpha}{\sqrt{2\pi}\,\omega_k}. \qquad (4.49)$$

The last term in (4.48) can always be combined with $V(X)$ to give an effective potential which takes directly into account the screening effect of the impurity [5]

$$V_{eff}(X) = V(X(t)) - \frac{\alpha^2}{\sqrt{2\pi}} \sum_k \frac{C_k}{\omega_k} X(t). \qquad (4.50)$$

Thus, we obtain

$$L = M_o \frac{\dot{X}^2(t)}{2} - V_{eff}(X(t)) + \frac{1}{2}\sum_k \{| \dot{q}_k(t) |^2 - \omega_k^2 | q_k(t) |^2\} \qquad (4.51)$$
$$- \sum_k q_k(t)\, C_k\, X(t),$$

which fits a Lagrangian of the Caldeira-Leggett type (4.41) in the strictly linear case, with $m_k = 1$ and no potential renormalization term

$$\sum_k \frac{F_k^2(X)}{2\,\omega_k^2}.$$

The equation of motion for the coordinate $q_p(t)$ is given by

$$\ddot{q}_p(t) + \omega_p^2\, q_p(t) = C_p\, X(t),$$
$$\ddot{q}_p(t) + \omega_p^2\, q_p(t) = -\frac{i\,\alpha\, p\,(\omega_p^2 + 1)}{\sqrt{2\pi}\,\omega_p} X(t), \qquad (4.52)$$

which is exactly the equation for $\tilde{c}_p(t)$ (4.23) that we found previously in the limit of a slow soliton and low momentum p, as could have easily been inferred from a comparison of equations (4.21) and (4.49).

[5] In the limit of low amplitudes, $V(x) \approx \omega X^2$ so that $V_{eff}(X)$ simply corresponds to a displaced harmonic potential centered around a value of $X \neq 0$ because of the constant force $(\alpha^2/\sqrt{2\pi})\sum_k c_k/\omega_k$.

Similarly, should we derive the equation of motion for $X(t)$ from Lagrangian (4.51), we would get the original equation 4.7 in which $c_k(t)$ has been decomposed as (4.49) and $f_k''(-X(t))$ *derived* from the approximation for $f_k'(-X(t))$ in (4.17).

Thus, having cast our problem into a Caldeira-Leggett form, we can proceed with the application of various results associated with this formalism; in particular, the calculation of the spectral density given by

$$J(\nu) \triangleq \frac{\pi}{2} \sum_j \frac{|\, C_j \,|^2}{m_j\, \omega_j}\, \delta(\nu - \omega_j)\,. \tag{4.53}$$

In our particular case, we have C_p assuming the form in (4.47) with $m_p = 1$ and the summation replaced by an integral on account of the continuous variable p. Hence, we derive

$$\begin{aligned}
J(\nu) &= \frac{\pi}{2} \int_{-\infty}^{+\infty} dp\, \frac{|\, C_p \,|^2}{\omega_p}\, \delta(\nu - \omega_p) \\
&= \frac{\alpha^2}{4} \int_{-\infty}^{+\infty} dp\, \frac{p^2\, (\omega_p^2 + 1)^2}{\omega_p^3}\, \delta(\nu - \omega_p) \\
&= \frac{\alpha^2}{2} \int_{1}^{+\infty} d\omega_p\, \frac{\omega_p}{\sqrt{\omega_p^2 - 1}}\, \frac{(\omega_p^2 - 1)\, (\omega_p^2 + 1)^2}{\omega_p^3}\, \delta(\nu - \omega_p) \\
&= \frac{\alpha^2}{2} \int_{1}^{+\infty} d\omega_p\, \frac{\sqrt{\omega_p^2 - 1}\, (\omega_p^2 + 1)^2}{\omega_p^2}\, \delta(\nu - \omega_p) \\
J(\nu) &= \begin{cases} \dfrac{\alpha^2}{2}\, \dfrac{(\nu^2 - 1)^{1/2}\, (\nu^2 + 1)^2}{\nu^2} & \nu > 1 \\ 0 & \nu < 1\,. \end{cases}
\end{aligned} \tag{4.54}$$

Let $\nu = 1 + x$. For $\nu \to 1$, we have $x \to 0$ so that

$$J(\nu) \equiv \bar{J}(x) \to 2\sqrt{2}\, \alpha^2\, x^{1/2}. \tag{4.55}$$

Equation (4.54) is the major result we sought to derive in this study. A graph of the normalized spectral function is shown in figure (4.3) along with its limiting behavior in

the region of $\nu \approx 1$ as reflected by (4.55). It is very important to bear in mind that this spectral density is for the case $T = 0$. This function gives the key to discussions of dissipation and its effect on the motion of a magnetic soliton, a Bloch wall, at temperature $T \approx 0$. This dissipation, we recall, is generated by the pinning potential which, indirectly, couples the magnetic harmonic oscillators (the magnons) to the quasi-classical motion of the Bloch wall.

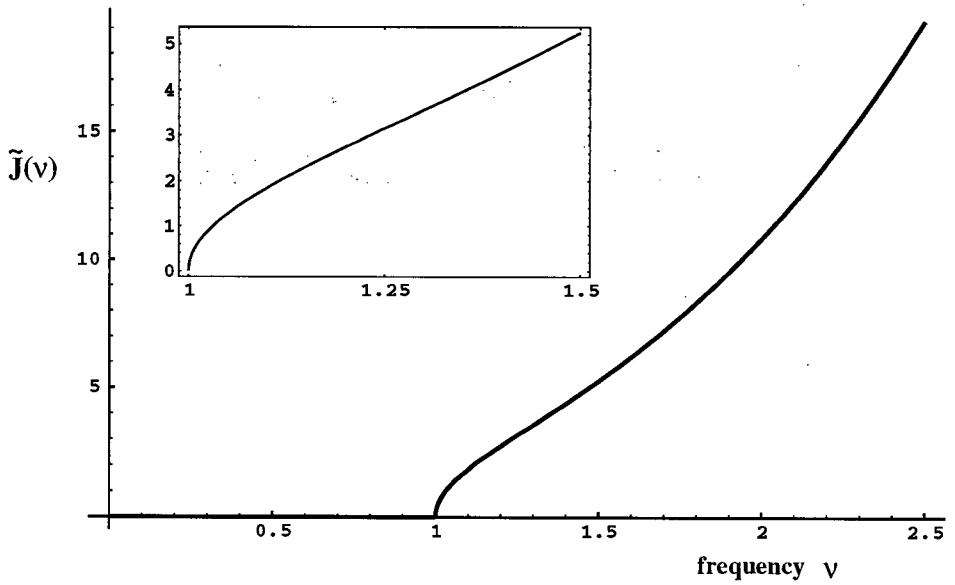


Figure 4.3: The normalized spectral density function $\tilde{J}(\nu)$. This is the normalized spectral density function at $T = 0$ for the interaction between magnons and an oscillating Bloch wall at low amplitudes. Those magnons are not the typical spin-waves, but rather the elementary quantum excitations built in the *presence* of a domain wall. The function given above is normalized in the sense that the factor $\alpha^2/2$ in the real spectral density function $J(\nu)$ (equation (4.54)), where α is a small coupling constant parameter, has been dropped in $\tilde{J}(\nu)$. In the inset, the behavior of the function near the frequency $\nu = 1$ is shown magnified.

In passing, we mention that, should the dissipative mechanism be quasi-linear in the sense given by Caldeira-Leggett [12], then the phenomenological friction coefficient η would have the following relation

$$\eta = \frac{\pi}{2} \sum_j \frac{1}{m_j \omega_j^2} \left| \frac{\partial F_j(Q)}{\partial Q} \right|^2 \delta(\nu - \omega_j), \tag{4.56}$$

using the notation of expression (4.41). Looking at our specific Lagrangian (4.45), where $Q \leftrightarrow X$, $x_j \leftrightarrow c_k$, and $m_k = 1$, we note that $F_k(X) = \alpha\, f_k'(-X)$ so that

$$\eta \;=\; \frac{\pi}{2}\int_{-\infty}^{+\infty} dk\; \frac{\alpha^2}{\omega^2}\;|\; f_k''(-X)\;|^2\;\; \delta(\nu-\omega_k)\,. \tag{4.57}$$

Knowing that $f_k''(X) = -(k^2 + 2\,\mathrm{sech}^{\,2}x) f_k(X)$, we obtain

$$\begin{aligned}
\eta(\nu) \;&=\; \frac{\alpha^2}{4}\left\{\int_{-\infty}^{+\infty} dk\;\frac{k^4}{\omega_k^2}\;\delta(\nu-\omega_k) \;+\; 4\;\mathrm{sech}^{\,4}X\;\mathrm{tgh}^{\,2}X\int_{-\infty}^{+\infty} dk\;\frac{\delta(\nu-\omega_k)}{\omega_k^4}\right.\\
&\qquad\left. +\;\mathrm{sech}^{\,2}X\int_{-\infty}^{+\infty} dk\;\frac{k^2\,(4\,\omega_k^2-k^2)}{\omega_k^4}\;\delta(\nu-\omega_k)\right\}\\
&=\; \frac{\alpha^2}{2}\left\{\int_1^{+\infty} d\omega_k\;\frac{(\omega_k^2-1)^{3/2}}{\omega_k}\;\delta(\nu-\omega_k) \;+\; 4\;\mathrm{sech}^{\,4}X\;\mathrm{tgh}^{\,2}X\int_1^{+\infty} d\omega_k\;\frac{\delta(\nu-\omega_k)}{\omega_k^3\sqrt{\omega_k^2-1}}\right.\\
&\qquad\left. +\;\mathrm{sech}^{\,2}X\int_1^{+\infty} d\omega_k\;\frac{(\omega_k^2-1)^{1/2}\,(3\,\omega_k^2+1)}{\omega_k^3}\;\delta(\nu-\omega_k)\right\}\\
\eta(\nu) \;&=\; \begin{cases}
\dfrac{\alpha^2}{2}\;\dfrac{1}{\nu\,(\nu^2-1)^{1/2}}\left[(\nu^2-1)^2 \;+\; 4\;\mathrm{sech}^{\,4}X\;\mathrm{tgh}^{\,2}X\;\dfrac{1}{\nu^2}\right. & \nu>1\\
\qquad\qquad\left. +\;\mathrm{sech}^{\,2}X\dfrac{(\nu^2-1)\,(3\nu^2+1)}{\nu^2}\right] & \\
0 & \nu<1\,.
\end{cases}
\end{aligned} \tag{4.58}$$

This calculation of η only assumes a slow moving soliton in contrast with that of $J(\nu)$ (4.54) which, in addition, also assumed low amplitudes for $X(t)$ and low momenta for $c_p(t)$. In the limit of low value for $X(t)$, expression (4.58) takes the form

$$\eta(\nu) \;\rightarrow\; \begin{cases}
\dfrac{\alpha^2}{2}\;\dfrac{(\nu^2-1)^2}{\nu^3}\;\left[(\nu^2+1)^2 \;+\; 2\,\nu^2\right] & \nu>1\\
0 & \nu<1\,.
\end{cases} \tag{4.59}$$

The key point to note is that the friction coefficient η still vanishes for $\nu < 1$ as $J(\nu)$ does. Hence, the conclusion to be drawn in the next chapter will remain unaffected.

Chapter 5

Discussion

5.1 Interpretation of the spectral density function $J(\nu)$

The interpretation of our specific form of $J(\nu)$ is very interesting. For driving frequencies ν less than one, we expect the oscillating soliton to experience *no* friction as $J(\nu) = 0$ in this domain. In other words, the magnons do not interact with the soliton at such low frequencies. As the natural frequency of the system can be thought of as $\omega = \sqrt{-\alpha}/2$ (the stable frequency of the potential $V(X) = 2\,\alpha\,\mathrm{sech}\,(X)$) and α is already a very small coupling constant, we think likely that the driving frequency ν, which would cause resonance, will fall within the dissipation-free domain. In practice, as $\nu \to \omega$, we should expect a resonance peak at $\nu \approx \omega$ with virtually no linewidth !

Indeed, in appendix C, we show that Lagrangian (4.51) admits the lifetime τ_n of an energy level E_n for the harmonic oscillator $X(t)$ coupled to a bath of harmonic oscillator $\{q_k\}$ to be expressed as

$$n\,J(\omega_o) \;=\; \frac{1}{2\,\tau_n} \tag{5.1}$$

where ω_o is the natural frequency of the oscillator $X(t)$. For our specific case, $\omega_o = \omega < 1$ so that, in principle, the lifetime of each energy level of the soliton, parametrized by $X(t)$, is infinite. In other words, whatever small oscillatory motion the soliton may possess, it will not be damped by the magnon environment at $T = 0$.

In section 1.5, we discussed the various sources of dissipation beside that of magnons. We have suggested that, at $T = 0$, only the dissipative effect of a spin bath and possibly

of phonons could be considered important on the motion of a domain wall. Our result for the spectral density (4.54) arising from magnons at $T = 0$ vindicates this suggestion by ruling out any dissipation whatsoever for walls oscillating at a sufficiently low frequency. Therefore, the quantum tunnelling of domain walls at zero temperature under a pinning potential barrier can only be possibly suppressed by nuclear spins, magnetic impurities or phonons in the presence of defects.[1] Hence, magnons, at $T = 0$ to the very least, should *not* be singled out as the main cause for the suppression of quantum tunnelling for states that are macroscopically distinct from each other, such as the case is for a domain wall pinned at one location and depinned at another on a macroscopic scale.

5.2 What could be done next?

The first obvious extension of this study would be to investigate the dissipation for the system at non-zero temperature. In so doing, one might want to adopt a treatment of the zero-frequency modes that would be amenable to the path-integral formalism since the latter offers a straightforward generalization to quantum field theory at temperature $T \neq 0$ and may, in some cases, be more gentle on the well-known ordering ambiguity in going from classical to quantum formalism. There exist adaptations of the collective co-ordinates method to the functional-integral formalism; the reader may refer, for instance, to Rajamaran and Weinberg [46] and Lee [41].

At $T \neq 0$, we may have to worry about an additional degree of freedom besides the direction of the magnetization vector $\mathbf{M}$, namely, its magnitude M which may very well differ form its saturation value M_o expected at $T \approx 0$. Also, one could mind taking into account the scattering of solitons among themselves; a behavior which is unlikely to be present at $T \approx 0$.[2] Furthermore, one could consider higher-order kinks or breather modes,

[1] The effect of nuclear spins is important despite the presence of defects.

[2] Even though the scattering amplitude is time-independent, there is a higher likelihood of collision

all of which translates into more massive, and therefore more energetic, solitons which a non-zero temperature could "produce" and support. In any event, unless one considers relatively high temperature T in the sense that $e^{-E'/kT}$ is non-negligible for energies E' associated with the above phenomena, these would constitute higher-order corrections to the magnons' effect on the motion of a simple soliton.

5.3 Conclusion

We recall that the goal of this study was to give a qualitative idea of the dissipative effect of magnons on the macroscopic quantum tunneling of a Bloch wall pinned a defect. We have effectively shown above that magnons do NOT interact dissipatively with such a domain wall which oscillates at a low frequency at temperature $T = 0$. In order to demonstrate this, we needed a method, the collective coordinates, to eliminate the zero-frequency eigenmode $f_b(x)$ which rendered invalid, in the classical or quantum description, the approximation of the low-lying energy levels of a domain wall by a set of harmonic oscillators which included this mode. Yet, the eigenmode $f_b(x)$ was shown to play a crucial role in the establishment of a Newton-like law governing the center-of-mass motion of a domain wall in the presence of impurities. Also, we cast our problem in the Caldeira-Leggett formalism in order to apply the usual techniques related to the suppression of macroscopic quantum tunnelling by a microscopic environment.

One obvious extension of this study was shown to be the case at non-zero temperature which corresponds to the real situation in which experiments are conducted. Unless the dissipation arising from a spin bath and from phonons in the presence of impurities are resolved in detail, the relative importance of magnons on the quantum tunnelling of a domain wall will be difficult to assess. For the same reasons, a comparison between the theoretical predictions and experimental results cannot be fully realized.

events at higher temperature. This new feature considerably complicates the statistical behavior of the quasi-one-dimensional system; preferably, we strive to obtain magnetic wires with one domain wall only.

In closing, we mention that the existence of quantum tunnelling at a level as macroscopic as a domain wall provides a strong argument for the validity of quantum mechanics in the classical world. Although a more convincing test in this regards would be the observation of MQC, a greater understanding of the effect of dissipation on quantum phenomena could surely indicate us on how "macroscopic" a scale we should ever hope to observe a quantum tunnelling event. Should experimentalists repeatedly fail to observe the quantum tunnelling of a macroscopic object which theory allows, after all factors have been exhaustively taken into account; we might witness either the breakdown of quantum mechanics at this object's macroscopic level or the inadequacy of our theoretical description of quantum dissipative processes. In any case, this would certainly rekindle the controversy around the quantum measurement problem and, perhaps, prompt physicists to entertain the idea that entirely new irreducible levels of reality may emerge at certain stages of complexity of a system.

However, reductionism have served us so well for the understanding of Nature that it truly deserves our vote of confidence for her investigation. The author have nevertheless the solid impression that Nature, somehow, will never allow to be so easily contained within the confines of a reductionistic receptacle; her true beauty lying in her mysteries.

Bibliography

[1] M. Bunge, *Causality*, The Place of the Causal Principle in Modern Science (The World Publishing Company, Cleveland, 1963).

[2] R. Descartes, in *Oeuvres Philosophiques*, edited by F. Alquié (Garnier Frères, Paris, 1963), tome I, p. 315; ibid., pp. 479-480.

[3] *Encyclopædia Universalis* (Encyclopædia Universalis France, Paris, 1989), corpus 19.

[4] A. J. Leggett, in *The Lesson of Quantum Theory*, edited by J. de Boer, E. Dal, and O. Ulfbeck (Elsevier Science Publishers, Amsterdam. 1986), p. 35.

[5] M. Jammer, *The Philosophy of Quantum Mechanics* (John Wiley & Sons, New York, 1974).

[6] N. Bohr, in *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp (Library of Living Philosophers, Evanston, Illinois,1949), pp. 221-222.

[7] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932). [translated in English by R. T. Beyer in *Mathematical foundations of quantum mechanics* (Princeton University Press, Princeton, N. J., 1955)]

[8] F. London and E. Bauer, in *Actualités scientifiques et industrielles: Exposés de physique générale*, edited by P. Langevin (Hermann, Paris, 1939), **775**.

[9] A. J. Leggett, in *Quantum implications: essays in honour of David Bohm*, edited by B. J. Hiley and F. David Peat (Routledge & Kegan Paul, London, 1987), p. 85.

[10] J. M. Martinis, M. H. Devoret, and J. Clarke, Phys. Rev. Lett., **55**, 1543 (1985); S. Washburn *et al.*, Phys. Rev. Lett., **54**, 2712 (1985); D. B. Schwartz *et al.*, Phys. Rev. Lett., **55**, 1547 (1985).

[11] A. J. Leggett, in *Quantum Tunnelling in Condensed Media*, edited by Y. Kagan and A. J. Leggett (Elsevier Science Publishers, Amsterdam. 1992), p. 1.

[12] A. O. Caldeira, A. J. Leggett, Ann. Phys. (USA), **149**, 374 (1984).

[13] A. J. Leggett, Phys. Rev. B, **30**, 1208 (1984).

[14] R. A. Serway, C. J. Moses, and C. A. Moyer, *Modern Physics*, Saunder Golden Sunburst Series (Saunders College, Philadelphia, 1989).

[15] R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (USA), **24**, 118 (1963).

[16] C. Kittel and J. K. Galt, in *Solid-State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1956), vol. 3, p. 437.

[17] A. P. Malozemoff and J. C. Slonczewski, *Magnetic Domain Walls in Bubble Materials* (Academic Press, Inc., New York, 1979).

[18] C. Kittel, *Introduction to Solid State Physics* (John Wiley & Sons, New York, 1966), 3rd edition.

[19] W. Döring, Z. Naturforsch., **3a**, 373 (1948).

[20] J. Rubinstein, J. Math. Phys., **11**, 258 (1970).

[21] A. Barone *et al.*, Riv. Nuovo Cimento, **1**, 227 (1971).

[22] L. D. Faddeev and L. A. Takhtadzhyan, Teor. Mat. Fiz. (Russia), **21**, 160 (1974). [Theor. Math. Phys., **21**, 1046 (1974)]

[23] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part II, International Series in Pure and Applied Physics (McGraw-Hill, New York, 1953), p. 1651.

[24] P. C. E. Stamp, Phys. Rev. Lett., **66**, 2802 (1991).

[25] P. C. E. Stamp, E. M. Chudnovsky, and B. Barbara, Int. J. Mod. Phys. B, **6**, 1355 (1992).

[26] O. Iglesias, E. M. Chudnovsky, and P. C. E. Stamp, Phys. Rev. B, **46**, 5392 (1992).

[27] A. Garg and G.-H. Kim, Phys. Rev. Lett., **63**, 2512 (1989).

[28] A. Garg and G.-H. Kim, J. Appl. Phys., **67**, 5669 (1990).

[29] A. J. Leggett *et al.*, Rev. Mod. Phys., **59**, 1 (1987).

[30] N. V. Prokof'ev and P. C. E. Stamp, unpublished.

[31] B. Barbara *et al.*, J. Appl. Phys., **73**, 6703 (1993); M. Uehara and B. Barbara, J. Physique, **47**, 235 (1986).

[32] M. Uehara *et al.*, Phys. Lett. A, **114**, 23 (1986).

[33] E. M. Chudnovsky, J. Appl. Phys., **73**, 6697 (1993).

[34] K. Hong and N. Giordano, unpublished.

[35] H. J. Mikeska, J. Phys. C: Solid State Phys., **11**, L29 (1978).

[36] R. Rajamaran, *Solitons and Instantons*, An Introduction to Solitons and Instantons in Quantum Field Theory (North-Holland, Amsterdam, 1982).

[37] G. Eilenberger, Z. Phys. B, **27**, 199 (1977).

[38] M. B. Fogel *et al.*, Phys. Rev. B, **15**, 1578 (1977).

[39] R. P. Feynman, *Statistical Mechanics: A Set of Lectures*, Frontiers in Physics, Lecture Note Series (Addison-Wesley, Redwood City, California, 1972).

[40] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Massachusetts, 1980), 2nd edition.

[41] N. H. Christ and T. D. Lee, Phys. Rev. D, **12**, 1606 (1975).

[42] M. Creutz, Phys. Rev. D, **12**, 3126 (1975).

[43] J. F. Janak, Phys. Rev., **134**, A411 (1964).

[44] J. M. Winter, Phys. Rev., **124**, 452 (1961).

[45] D. Esteve, M. H. Devoret, and J. M. Martinis, Phys. Rev. B, **34**, 158 (1986).

[46] T. D. Lee, *Non-Topological Solitons* (Columbia University Preprint Co., 1976), p. 2271, Lectures given at the Symposium on Frontier Problems in High Energy Physics, Pisa; R. Rajamaran and E. J. Weinberg, Phys. Rev. D, **11**, 2950 (1975).

Appendix A

Detailed derivation of equation (2.10) and (2.13)

In this appendix, we provide the details for the derivation of equation (2.13) in chapter 2. In equation (2.8), we have the following terms

$$\sin\theta_m \sin\theta_{m+1}, \tag{A.2}$$

$$\sin\theta_m \sin\theta_{m-1}, \tag{A.3}$$

$$\phi_{m+1} - \phi_m, \text{ and} \tag{A.4}$$

$$\phi_m - \phi_{m-1}. \tag{A.5}$$

On account of the assumption that the direction of the magnetization vector varies weakly over the lattice spacing a, we would like to express the above equations in terms of quantities defined at the m^{th} site only. Starting with (A.3), we have

$$\phi_{m+1} - \phi_m \approx \frac{\partial\phi}{\partial z}a,$$

where the derivative is taken at the point $z = ma$, i.e., the coordinate of the m^{th} ion. For (A.4), we have to take into account the second derivative at the point $z = ma$ as shown below:

Let $\phi_m - \phi_{m-1} \triangleq \Delta\phi_{m-1}$. Then,

$$\begin{aligned}
\Delta\phi_{m-1} &= \Delta\phi_m - \frac{(\Delta\phi_m - \Delta\phi_{m-1})}{\Delta z}\Delta z, \qquad \text{where } \Delta z = ma - (m-1)a, \\
&= \Delta\phi_m + \frac{(\phi_{m-1} - \Delta\phi_m)}{\Delta z}a \\
&= \Delta\phi_m - \left.\frac{\Delta(\Delta\phi)}{\Delta z}\right|_{z=ma}.
\end{aligned} \tag{A.6}$$

And so, we obtain

$$\begin{aligned}\Delta\phi_{m-1} &= \frac{a\,\Delta\phi_{m-1}}{m\,a\,-\,(m-1)\,a} \\ &\approx a\,\frac{\Delta\phi_m}{m\,a\,-\,(m-1)\,a}\,-\,\frac{a^2}{m\,a\,-\,(m-1)\,a}\,\frac{\Delta(\Delta\phi)}{\Delta z}\bigg|_{z=m\,a}\,,\end{aligned}$$

using equation $(A.5)$,

$$\phi_m(t)\,-\,\phi_{m-1}(t)\;\rightarrow\;a\,\frac{\partial\,\phi}{\partial z}\,-\,a^2\,\frac{\partial^2\phi}{\partial z^2}\,. \tag{A.7}$$

In like manner, proceeding with (A.1), we get

$$\sin\theta_{m+1}\;\approx\;\sin\theta_m\;+\;\cos\theta_m\,(\theta_{m+1}-\theta_m)\,,$$

so that

$$\begin{aligned}\sin\theta_m(t)\,\sin\theta_{m+1}(t) &\approx \sin^2\theta_m(t)\;+\;\sin\theta_m(t)\,\cos\theta_m(t)\,(\theta_{m+1}(t)\;-\;\theta_m(t)) \\ &\rightarrow \sin^2\theta(z,t)\;+\;\sin\theta(z,t)\,\cos\theta(z,t)\,\frac{\partial\theta(z,t)}{\partial z}\,a \\ &\approx 1\;-\;(\theta^s(z,t))^2\;+\;a\,\theta^s(z,t)\,\frac{\partial\theta(z,t)}{\partial z}\,,\end{aligned} \tag{A.8}$$

where we used approximations (2.8) and (2.9) applied to the continuous case,

$$\cos\theta(z,t)\;\approx\;\theta^s(z,t) \tag{A.9}$$

$$\sin\theta(z,t)\;\approx\;1\;-\;(\theta^s(z,t))^2, \tag{A.10}$$

and discarded variations in (A.7) of order two and higher. For $\sin\theta_{m-1}$ in (A.4), we obtain in the continuum limit

$$\sin\theta_{m-1}(t)\;\rightarrow\;\sin\theta(z,t)\;-\;a\,\cos\theta(z,t)\,\frac{\partial\theta(z,t)}{\partial z}\;+\;a^2\cos\theta(z,t)\,\frac{\partial^2\theta(z,t)}{\partial z^2}\,,$$

so that

$$\begin{aligned}\sin\theta_m(t)\,\sin\theta_{m-1}(t) \;&\rightarrow\; \sin^2\theta(z,t) \;-\; a\,\sin\theta(z,t)\,\cos\theta(z,t)\,\frac{\partial\theta(z,t)}{\partial z} \\ &\qquad + a^2\,\sin\theta(z,t)\,\cos\theta(z,t)\,\frac{\partial^2\theta(z,t)}{\partial z^2} \\ &\approx\; 1 \;-\; (\theta^s(z,t))^2 \;+\; \theta^s(z,t)\left(-a\,\frac{\partial\theta(z,t)}{\partial z} \;+\; a^2\,\frac{\partial^2\theta(z,t)}{\partial z^2}\right),\end{aligned}$$

where we neglected $(\theta^s)^3$.

Also, using (A.8) and (A.9), we obviously have

$$\sin\theta_m(t)\,\cos\phi_m(t)\,\sin\theta_m(t)\,\sin\phi_m(t) \;\rightarrow\; \cos\phi(z,t)\,\sin\phi(z,t)\,(1 \;-\; 2\,(\theta^s(z,t))^2)\,.$$

Substituting all those relations into (2.8) yields in the continuum limit

$$\begin{aligned}&\hbar\sqrt{S(S+1)}\frac{\partial\theta^s}{\partial t} \;= \\ &\frac{\hbar^2 iJS(S+1)}{2}\left\{(1-(\theta^s)^2)\left[e^{i(a\frac{\partial\phi}{\partial z}-a^2\frac{\partial^2\phi}{\partial z^2})} - e^{-i(a\frac{\partial\phi}{\partial z}-a^2\frac{\partial^2\phi}{\partial z^2})} + e^{-i\,a\frac{\partial\phi}{\partial z}} - e^{i\,a\frac{\partial\phi}{\partial z}}\right]\right. \\ &\qquad + a\,\theta^s\frac{\partial\phi}{\partial z}\left[e^{-i\,a\frac{\partial\phi}{\partial z}} - e^{i\,a\frac{\partial\phi}{\partial z}}\right] \\ &\qquad \left. + \theta^s\left(-a\frac{\partial\phi}{\partial z} + a^2\frac{\partial^2\phi}{\partial z^2}\right)\left[e^{i(a\frac{\partial\phi}{\partial z}-a^2\frac{\partial^2\phi}{\partial z^2})} - e^{-i(a\frac{\partial\phi}{\partial z}-a^2\frac{\partial^2\phi}{\partial z^2})}\right]\right\} \\ &\qquad - 2\,\hbar^2 S(S+1)\,K_y\cos\phi\,\sin\phi\,(1-2(\theta^s)^2) \;+\; i\hbar^2\sqrt{S(S+1)}\,K_z\,\theta^s \qquad \text{(A.11)} \\ &\approx\; \frac{i\hbar^2 J}{2}\,S(S+1)\left\{(1-(\theta^s)^2)\left[2i\left(a\frac{\partial\phi}{\partial z} - a^2\frac{\partial^2\phi}{\partial z^2}\right) - 2i\left(a\frac{\partial\phi}{\partial z}\right)\right]\right\} \\ &\qquad - 2\hbar^2 S(S+1)\,K_y\cos\phi\,\sin\phi\,(1-2(\theta^s)^2) \;+\; i\hbar^2\sqrt{S(S+1)}\,K_z\,\theta^s \\ &=\; \hbar^2 a^2 J\,S(S+1)\left\{(1 \;-\; 2\,(\theta^s)^2)\,\frac{\partial^2\phi}{\partial z^2}\right\} \qquad \text{(A.12)} \\ &\qquad - 2\hbar^2 S(S+1)\,K_y\,\cos\phi\,\sin\phi\,(1-2(\theta^s)^2) \;+\; i\hbar^2\sqrt{S(S+1)}\,K_z\,\theta^s\,,\end{aligned}$$

where we discarded in (A.11) the last two terms within the parentheses for they involve variations of order two and higher, which which we neglect.

If we also neglect the term $(\theta^s)^2$ with respect to one in expression (A.12), we obtain (2.10)

$$\hbar\sqrt{S(S+1)}\,\frac{\partial\theta^s}{\partial t} \;\approx\; \hbar^2 J\,a^2\,S(S+1)\,\frac{\partial^2\phi}{\partial z^2} \;-\; 2\,\hbar^2 S(S+1)\,K_y\cos\phi\,\sin\phi + i\hbar^2\sqrt{S(S+1)}\,K_z\,\theta^s\,. \quad (2.10)$$

Let us now calculate the time evolution of the operator $\hat{S}^+_m(t)$.

$$\begin{aligned}
\frac{d\hat{S}^+_m}{dt} &= -\frac{i}{\hbar}\,[\,\hat{S}^+_m, \hat{H}\,] \\
&= \frac{i\,J}{\hbar}\left\{[\,\hat{S}^+_m, \hat{S}^z_m\hat{S}^z_{m+1}\,] + [\,\hat{S}^+_m, \hat{S}^z_{m-1}\hat{S}^z_m\,] + \frac{1}{2}[\,\hat{S}^+_m, \hat{S}^-_m\hat{S}^+_{m-1}\,] + \frac{1}{2}[\,\hat{S}^+_m, \hat{S}^+_{m+1}\hat{S}^-_m\,]\right\} \\
&\qquad + K_z\,i\,\{\hat{S}^+_m\,\hat{S}^z_m + \hat{S}^z_m\,\hat{S}^+_m\} + K_y\,\{\hat{S}^y_m\,\hat{S}^z_m + \hat{S}^z_m\,\hat{S}^y_m\} \qquad (A.13) \\
&= i\,J\,\{-\hat{S}^+_m\,\hat{S}^z_{m+1} - \hat{S}^z_{m-1}\,\hat{S}^+_m + \hat{S}^z_m\,\hat{S}^+_{m-1} + \hat{S}^+_{m+1}\,\hat{S}^z_m\} \\
&\qquad + K_z\,i\,\{\hat{S}^+_m\,\hat{S}^z_m + \hat{S}^z_m\,\hat{S}^+_m\} + K_y\,\{\hat{S}^y_m\,\hat{S}^z_m + \hat{S}^z_m\,\hat{S}^y_m\}\,,
\end{aligned}$$

$$\begin{aligned}
&\sqrt{S(S+1)}\,\frac{d}{dt}\sin\hat{\theta}_m\,e^{i\hat{\phi}_m} = \\
&\qquad iJ\,S(S+1)\left\{-\sin\hat{\theta}_m\,e^{i\hat{\phi}_m}\cos\hat{\theta}_{m+1} - \sin\hat{\theta}_m\cos\hat{\theta}_{m-1}\,e^{i\hat{\phi}_m}\right. \qquad (A.14) \\
&\qquad\qquad \left. + \cos\hat{\theta}_m\sin\hat{\theta}_{m-1}\,e^{i\hat{\phi}_{m-1}} + \sin\hat{\theta}_{m+1}\,e^{i\hat{\phi}_{m+1}}\cos\hat{\theta}_m\right\} \\
&\qquad + i\,K_z\,S(S+1)\,\{\sin\hat{\theta}_m\,e^{i\hat{\phi}_m}\cos\hat{\theta}_m + \cos\hat{\theta}_m\sin\hat{\theta}_m\,e^{i\hat{\phi}_m}\} \\
&\qquad + K_y\,S(S+1)\,\{\sin\hat{\theta}_m\sin\hat{\phi}_m\cos\hat{\theta}_m + \cos\hat{\theta}_m\sin\hat{\theta}_m\sin\hat{\phi}_m\}.
\end{aligned}$$

Sandwiching equation (A.14) with a classical state vector as outlined in chapter 2, we obtain:

$$\hbar\sqrt{S(S+1)}\,\frac{d}{dt}(\sin\theta_m\, e^{i\phi_m}) \;=$$

$$\begin{aligned} & i\hbar^2 J\, S(S+1)\,\Big\{-e^{i\phi_m}\,\sin\theta_m\,(\cos\theta_{m+1}\,+\,\cos\theta_{m-1}) \\ & \quad +\,\cos\theta_m\,(e^{i\phi_{m-1}}\,\sin\theta_{m-1}\,+\,e^{i\phi_{m+1}}\,\sin\theta_{m+1})\Big\} \\ & \quad +\,2\,i\,\hbar^2\,K_z\,S(S+1)\,\sin\theta_m\,\cos\theta_m\,e^{i\phi_m} \\ & \quad +\,2\hbar^2\,K_y\,S(S+1)\,\sin\theta_m\,\cos\theta_m\,\sin\phi_m\,. \end{aligned} \tag{A.15}$$

Concentrating on the first term on the right-hand side of (A.15), denoted as $\mathcal{I}$, we make use of the following expansions

$$\begin{aligned} \cos\theta_{m+1} \;&\approx\; \cos\theta_m \;-\; \sin\theta_m\,(\theta_{m+1}\,-\,\theta_m) \\ &\to\; \cos\theta(z,t) \;-\; \sin\theta(z,t)\,\frac{\partial\theta}{\partial z}\,a\,, \\ \cos\theta_{m-1} \;&\approx\; \cos\theta_m \;-\; \sin\theta_m\,(\theta_{m-1}\,-\,\theta_m) \\ &\to\; \cos\theta(z,t) \;-\; \sin\theta(z,t)\,\left(-a\,\frac{\partial\theta}{\partial z}\,+\,a^2\,\frac{\partial^2\theta}{\partial z^2}\right), \end{aligned} \tag{A.16}$$

to obtain in the continuum limit

$$\begin{aligned} \mathcal{I} \;=\; & i\hbar^2 J\, S(S+1)\,\Big\{-e^{i\phi(z,t)}\,\sin\theta(z,t)\,\left(2\cos\theta(z,t)\,-\,\sin\theta(z,t)\,a^2\,\frac{\partial^2\theta}{\partial z^2}\right) \\ & +\,\cos\theta(z,t)\,\Bigg[\left(e^{i(\phi(z,t)\,-\,a\,\frac{\partial\phi}{\partial z}\,+\,a^2\,\frac{\partial^2\phi}{\partial z^2})}\,+\,e^{i(\phi(z,t)\,+\,a\,\frac{\partial\phi}{\partial z})}\right)\sin\theta(z,t) \\ & +\,e^{i(\phi(z,t)\,-\,a\,\frac{\partial\phi}{\partial z}\,+\,a^2\,\frac{\partial^2\phi}{\partial z^2})}\,\left(-a\,\cos\theta(z,t)\,\frac{\partial\phi}{\partial z}\,+\,a^2\,\cos\theta(z,t)\,\frac{\partial^2\phi}{\partial z^2}\right) \\ & +\,e^{i(\phi(z,t)\,+\,a\,\frac{\partial\phi}{\partial z})}\,\left(a\,\frac{\partial\phi}{\partial z}\,\cos\theta(z,t)\right)\Bigg]\Big\} \\ =\; & i\hbar^2 J\, S(S+1)\,e^{i\phi}\,\Big\{-2\,\sin\theta\,\cos\theta\,+\,\sin^2\theta\,a^2\,\frac{\partial^2\theta}{\partial z^2} \\ & +\,\cos\theta\,\Bigg[\left(e^{i(-a\,\frac{\partial\phi}{\partial z}\,+\,a^2\,\frac{\partial^2\phi}{\partial z^2})}\,+\,e^{ia\,\frac{\partial\phi}{\partial z}}\right)\,\sin\theta \end{aligned}$$

$$+ e^{i(-a\frac{\partial\phi}{\partial z} + a^2\frac{\partial^2\phi}{\partial z^2})}\left(-a\cos\theta\frac{\partial\theta}{\partial z} + a^2\cos\theta\frac{\partial^2\theta}{\partial z^2}\right) + e^{ia\frac{\partial\phi}{\partial z}}\, a\frac{\partial\theta}{\partial z}\cos\theta\Bigg]\Bigg\}.$$

Once more, we discard variations of order two or higher that would result by expanding the exponentials and obtain

$$\begin{aligned}\mathcal{I} &\approx i\hbar^2 J S(S+1)\, e^{i\phi}\left\{-2\sin\theta\cos\theta + \sin^2\theta\, a^2\frac{\partial^2\theta}{\partial z^2}\right. \\ &\qquad \left. + \cos\theta\left[(2 + a^2\frac{\partial^2\phi}{\partial z^2})\sin\theta + a^2\cos\theta\frac{\partial^2\theta}{\partial z^2}\right]\right\} \qquad \text{(A.17)} \\ &\approx i\hbar^2 J S(S+1)\, e^{i\phi}\left\{\theta_s\, a^2\frac{\partial^2\phi}{\partial z^2} + a^2\frac{\partial^2\theta}{\partial z^2}\right\},\end{aligned}$$

where we used (A.8) and (A.9) and neglected θ_s^3.
We assume that a term like $\theta_s\dfrac{\partial^2\phi}{\partial z^2}$ constituted of two small quantities is negligible so that (A.17) reduces to

$$i\hbar^2 J S(S+1)\, e^{i\phi(z,t)}\, a^2\frac{\partial^2\theta}{\partial z^2}. \qquad \text{(A.18)}$$

In the continuum limit, the second and third terms on the right-hand side of equation (A.15) simply transform to

$$\begin{aligned}&2\,i\hbar^2 K_z S(S+1)\, e^{i\phi(z,t)}\sin\theta(z,t)\cos\theta(z,t) \\ &+ 2\hbar^2 K_y S(S+1)\sin\theta(z,t)\cos\theta(z,t)\sin\phi(z,t) \approx \\ &\qquad 2\,i\hbar^2 K_z S(S+1)\, e^{i\phi(z,t)}\,\theta_s(z,t) \qquad \text{(A.19)} \\ &\qquad + 2\hbar^2 K_y S(S+1)\,\theta_s(z,t)\sin\phi(z,t).\end{aligned}$$

We mentioned in chapter 2 that K_y is much smaller than K_z; using this fact and knowing that $\sin\phi(z,t)$ have the same order of magnitude as $e^{i\phi(z,t)}$, it is legitimate, therefore, to approximate (A.19) with

$$2\,i\hbar^2 K_z S(S+1)\, e^{i\phi(z,t)}\,\theta_s(z,t). \qquad \text{(A.20)}$$

The left-hand side of equation (A.15) yields then the following in the continuum limit:

$$\begin{aligned}
\hbar\sqrt{S(S+1)}\,\frac{d}{dt}(\sin\theta_m\, e^{i\phi_m}) &\rightarrow \hbar\sqrt{S(S+1)}\,\frac{\partial}{\partial t}\,(\sin\theta(z,t)\, e^{i\phi(z,t)}) \\
&= \hbar\sqrt{S(S+1)}\left\{\cos\theta(z,t)\,\frac{\partial\theta}{\partial t} + i\sin\theta(z,t)\, e^{i\phi(z,t)}\,\frac{\partial\phi}{\partial t}\right\} \\
&\approx \hbar\sqrt{S(S+1)}\left\{-\theta_s(z,t)\,\frac{\partial\theta_s}{\partial t} + i\,(1-(\theta_s)^2)\, e^{i\phi}\,\frac{\partial\phi}{\partial t}\right\} \\
&\approx i\hbar\sqrt{S(S+1)}\, e^{i\phi(z,t)}\,\frac{\partial\phi}{\partial t}, \qquad \text{(A.21)}
\end{aligned}$$

where we neglect terms like $\theta_s\,\dfrac{\partial\theta_s}{\partial t}$ and $\theta_s^2\,\dfrac{\partial\phi}{\partial t}$ because they are composed of two small quantities.

Therefore, using (A.20) and (A.22), equation (A.15) in the continuum limit reads as

$$\begin{aligned}
i\hbar\sqrt{S(S+1)}\,\frac{\partial\phi}{\partial t}\, e^{i\phi} &\approx i\hbar^2 J\, S(S+1)\, a^2\,\frac{\partial^2\theta}{\partial z^2}\, e^{i\phi} + 2\, i\hbar^2 K_z\, S(S+1)\,\theta_s(z,t)\, e^{i\phi} \\
\hbar\sqrt{S(S+1)}\,\frac{\partial\phi}{\partial t} &\approx \hbar^2\, S(S+1)\left\{J\, a^2\,\frac{\partial^2\theta}{\partial z^2} + 2\, K_z\,\theta_s\right\}. \qquad \text{(A.22)}
\end{aligned}$$

Taking the classical limit ($\hbar \rightarrow 0, S \rightarrow \infty$) on both sides of (A.24) leads to equation (2.13)

$$\frac{\partial\phi(z,t)}{\partial t} \approx J\, s\, a^2\,\frac{\partial^2\theta}{\partial z^2} + 2\, K_z\, s\,\theta_s\,,$$

where s is the value of the classical spin, i.e., the magnitude of the magnetization.

Appendix B

Inverse of the velocity matrix D

This appendix is not meant to be rigorous, but gives a heuristic derivation of the inverse of the matrix D introduced in chapter 3. One should realize that matrix D is one indexed by a continuum, the variable k, and therefore, it may not obvious, or, for that matter, even true, to generalize the method we adopt in this appendix (valid for the usual matrix with a discrete index) to a continuum.

A discrete indexed version of matrix D is the following

$$A_n \triangleq \begin{pmatrix} a_{bb} & a_{b,-n} & \cdots & a_{b,0} & \cdots & a_{b,n} \\ a_{-n,b} & & & & & 1 \\ \vdots & & \bigcirc & & & \\ a_{0,b} & & & 1 & & \\ \vdots & & & & \bigcirc & \\ a_{n,b} & 1 & & & & \end{pmatrix}, \tag{B.1}$$

where $a_{i,j} = a_{j,i}$, $(a_{b,-k})^* = a_{b,k}$, and $a_{k,p} = \delta_{k,-p}$. We let i, j denote the total index set including the index "b", whereas we reserve k, p for the numeral indices proper. The special case $n = 1$ is shown below where, for simplicity, we denote A_1 by A only;

$$A = \begin{pmatrix} a_{bb} & a_{b,-1} & a_{b,0} & a_{b,1} \\ a_{-1,b} & 0 & 0 & 1 \\ a_{0,b} & 0 & 1 & 0 \\ a_{1,b} & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{bb} & a_{b,-1} & a_{b,0} & a_{b,1} \\ a_{b,-1} & 0 & 0 & 1 \\ a_{b,0} & 0 & 1 & 0 \\ a_{b,1} & 1 & 0 & 0 \end{pmatrix}. \tag{B.2}$$

The real matrix D in chapter 3 is obtained from two successive limiting transformations: first, the index k running from $-n$ to n has $n \to \infty$ to obtain an infinite countable index set, and second, is made a continuum to obtain an uncountable index set. We will not show the generalization of our method for either of these two limiting transformations.

We know that the inverse of a matrix C is given by

$$C^{-1} = \frac{[C_{ij}]^T}{DetC} \tag{B.3}$$

where

$$C_{ij} \triangleq (-1)^{i+j} \, Det \, M_{ij}$$

and M_{ij} is the matrix obtained from C by deleting the i^{th} row and j^{th} column.
Let us compute the inverse of matrix A_n with the help our special case, matrix A . First, let us count the numeral indices as

$$\begin{array}{rcl} k & & k' \\ -n & \leftrightarrow & 1 \\ -n+1 & \leftrightarrow & 2 \\ \vdots & & \vdots \end{array}$$

so that for (B.2) we have

$$\begin{array}{ccc} k & & k' \\ -1 & \leftrightarrow & 1 \\ 0 & \leftrightarrow & 2 \\ 1 & \leftrightarrow & 3 \end{array}$$

and obtain respectively

$$A_n = \begin{pmatrix} a_{bb} & a_{b,1} & \cdots & a_{b,n+1} & \cdots & a_{b,2n+1} \\ a_{b,1} & & & & & 1 \\ \vdots & & \bigcirc & & & \\ a_{b,n+1} & & & 1 & & \\ \vdots & & & & \bigcirc & \\ a_{b,2n+1} & 1 & & & & \end{pmatrix} \tag{B.4}$$

and

$$A = \begin{pmatrix} a_{bb} & a_{b,1} & a_{b,2} & a_{b,3} \\ a_{b,1} & 0 & 0 & 1 \\ a_{b,2} & 0 & 1 & 0 \\ a_{b,3} & 1 & 0 & 0 \end{pmatrix}. \tag{B.5}$$

Now, denoting A_n by C for convenience, we have

$$C_{bb} = Det \begin{pmatrix} & & & 1 \\ & \bigcirc & & \\ & & 1 & \\ & & & \bigcirc \\ 1 & & & \end{pmatrix} = -1.$$

Also, $C_{b,k'} = (-1)^{1+(k'+1)}\, Det\, M_{b,k'}$ with

$$M_{b,k'} \;=\; M_{b,k} \;=\; \begin{pmatrix} a_{b,-n} & 0 & 0 & \dots & 0 & 1 \\ \vdots & & & & & \\ a_{b,-k} & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ a_{b,n} & 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where the resulting matrix has its first column made up of all the $a_{b,p}$ with a string of zero following *only* the row starting with the specific value $a_{b,-k}$. It is, then, a simple matter to see that

$$\begin{aligned} Det\, M_{b,k'} &= (-1)^{k'+1}\, a_{b,p'}\, (-1) \\ &= (-1)^{k'+2}\, a_{b,p'} \end{aligned}$$

with p' being the value corresponding to $p = -k$.

Thus,

$$\begin{aligned} C_{b,k'} &= (-1)^{(2+k')}\, (-1)^{(k'+2)}\, a_{b,p'} \\ &= a_{b,p'}\,, \\ C_{b,k} &= a_{b,-k}\,. \end{aligned} \tag{B.6}$$

Now, for $C_{k,p}$ of numeral indices k, this involves a bit more bookkeeping. For this reason, we will explicitly deal with the specific case of $n = 1$, i.e. the matrix A in (B.2), in the hope of finding a general pattern.

We have

$$A_{k,p} \;=\; A_{k',p'} \;=\; (-1)^{k'+p'+2}\, Det\, M_{k',p'}.$$

We need calculate only the six different sub-matrices $M_{k',p'}$ derived from A.

$$
\begin{aligned}
Det\, M_{1,1} &= Det \begin{pmatrix} a_{bb} & a_{b,2} & a_{b,3} \\ a_{b,2} & 1 & 0 \\ a_{b,3} & 0 & 0 \end{pmatrix} \\
&= -(a_{b,3})\,(a_{b,3}) \\
&= -(a_{b,3})^2, \qquad \text{or} \\
Det\, M_{-1,-1} &= -(a_{b,1})^2 \qquad \text{with the original indices } k.
\end{aligned}
$$

$$
\begin{aligned}
Det\, M_{1,2} &= Det \begin{pmatrix} a_{bb} & a_{b,1} & a_{b,3} \\ a_{b,2} & 0 & 0 \\ a_{b,3} & 1 & 0 \end{pmatrix} \\
&= a_{b,2}\, a_{b,3}\,, \qquad \text{or} \\
Det\, M_{-1,0} &= a_{b,0}\, a_{b,1}
\end{aligned}
$$

$$
\begin{aligned}
Det\, M_{1,3} &= Det \begin{pmatrix} a_{bb} & a_{b,1} & a_{b,2} \\ a_{b,2} & 0 & 1 \\ a_{b,3} & 1 & 0 \end{pmatrix} \\
&= -a_{bb} + a_{b,1}\, a_{b,3} + a_{b,2}\, a_{b,2}\,, \qquad \text{or} \\
Det\, M_{-1,1} &= -a_{bb} + a_{b,-1}\, a_{b,1} + (a_{b,0})^2 \\
&= -a_{bb} + |\, a_{b,1}\,|^2 + (a_{b,0})^2
\end{aligned}
$$

$$
Det\, M_{2,2} = Det \begin{pmatrix} a_{bb} & a_{b,1} & a_{b,3} \\ a_{b,1} & 0 & 1 \\ a_{b,3} & 1 & 0 \end{pmatrix}
$$

$$= -a_{bb} + a_{b,1}\, a_{b,3} + a_{b,3}\, a_{b,1}\,, \qquad \text{or}$$

$$Det\, M_{0,0} = -a_{bb} + a_{b,-1}\, a_{b,1} + a_{b,1}\, a_{b,-1}$$

$$= -a_{bb} + 2 \mid a_{b,1} \mid^2$$

$$Det\, M_{2,3} = Det \begin{pmatrix} a_{bb} & a_{b,1} & a_{b,2} \\ a_{b,1} & 0 & 0 \\ a_{b,3} & 1 & 0 \end{pmatrix}$$

$$= a_{b,1}\, a_{b,2}\,, \qquad \text{or}$$

$$Det\, M_{0,1} = a_{b,-1}\, a_{b,0}$$

$$Det\, M_{3,3} = Det \begin{pmatrix} a_{bb} & a_{b,1} & a_{b,2} \\ a_{b,1} & 0 & 0 \\ a_{b,2} & 0 & 1 \end{pmatrix}$$

$$= -a_{b,1}\, a_{b,1}$$

$$= -(a_{b,1})^2\,, \qquad \text{or}$$

$$Det\, M_{1,1} = -(a_{b,-1})^2.$$

We note that the determinant of matrix A is given by

$$Det\, A = -a_{bb} + a_{b,1}\, a_{b,3} + a_{b,2}\, a_{b,2} + a_{b,3}\, a_{b,1}\,, \qquad \text{or}$$

$$Det\, A = -a_{bb} + a_{b,-1}\, a_{b,1} + a_{b,0}\, a_{b,0} + a_{b,1}\, a_{b,-1}$$

$$= -a_{bb} + (a_{b,0})^2 + 2 \mid a_{b,1} \mid^2,$$

so that $Det\, M_{-1,1}$ and $Det\, M_{0,0}$ could be rewritten as

$$\begin{aligned} Det\, M_{-1,1} &= Det\, A - a_{b,-1}\, a_{b,1} \\ &= Det\, A - |\, a_{b,1}\,|^2 \end{aligned}$$

$$Det\, M_{0,0} = Det\, A - (a_{b,0})^2$$

Thus, we obtain for $A_{k,p}$:

$$\begin{aligned} A_{-1,-1} &= -(a_{b,1})^2, \\ A_{-1,0} &= -a_{b,0}\, a_{b,1}\,, \\ A_{-1,1} &= Det\, A - a_{b,-1}\, a_{b,1}\,, \\ A_{0,0} &= Det\, A - (a_{b,0})^2, \\ A_{0,1} &= -a_{b,-1}\, a_{b,0}\,, \\ A_{1,1} &= -(a_{b,-1})^2. \end{aligned}$$

We immediately notice that the above formulae can be described by a general expression like

$$A_{k,p} = \delta_{k,-p}\, Det\, A - a_{b,-k}\, a_{b,-p}\,.$$

Therefore, from (B.3), we obtain, for our specific example, the inverse A^{-1} where

$$\begin{aligned} (A^{-1})_{bb} &= \frac{-1}{Det\, A}, \\ (A^{-1})_{b,k} &= \frac{a_{b,-k}}{Det\, A}, \qquad \text{and} \\ (A^{-1})_{k,p} &= \delta_{k,-p} - \frac{a_{b,-k}\, a_{b,-p}}{Det\, A}\,. \end{aligned}$$

Hence, the "non-trivial" generalization to a continuous index set yields

$$(D^{-1})_{bb} = \frac{-1}{\mathcal{D}},$$

$$
\begin{aligned}
(D^{-1})_{b,k} &= \frac{D_{b,-k}}{\mathcal{D}}, \\
(D^{-1})_{k,p} &= \delta(k+p) - \frac{D_{b,-k}\, D_{b,-p}}{\mathcal{D}},
\end{aligned}
$$

which is the result we sought to show.

On the other hand, the determinant of $D\,(\mathcal{D})$ can easily be determined for an infinite countable index set. Indeed, let

$$
D_n \triangleq \begin{pmatrix}
D_{bb} & D_{b,-n} & \cdots & D_{b,0} & \cdots & D_{b,n} \\
D_{b,-n} & & & & & 1 \\
\vdots & & \bigcirc & & & \\
D_{b,0} & & & 1 & & \\
\vdots & & & & \bigcirc & \\
D_{b,n} & 1 & & & &
\end{pmatrix}, \tag{B.7}
$$

where $\begin{pmatrix} & & & & 1 \\ & \bigcirc & & & \\ & & 1 & & \\ & & & \bigcirc & \\ 1 & & & & \end{pmatrix}$ is a $(2n+1)\times(2n+1)$ matrix whose only non-zero entries are numbers one on the diagonal running from the top right to the bottom left corners.

One can easily show that

$$\begin{aligned}\mathcal{D}_n &= Det\, D_n \\ &= D_{bb}(-1) + \sum_{i=-n}^{n} (-1)^{(i+1+n)}\, D_{b,i}\, Det \begin{pmatrix} D_{b,-n} & 0 & 0 & \dots & 0 & 0 & 1 \\ D_{b,-n+1} & 0 & 0 & \dots & 0 & 1 & 0 \\ \vdots & & & & & & \\ D_{b,-i} & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \\ D_{b,n} & 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{B.8}$$

where the matrix within the summation possesses only one row with a complete string of zeros following the first element, namely, the row that starts with element $D_{b,-i}$.

Hence,

$$\mathcal{D}_n = -D_{bb} + \sum_{i=-n}^{n} (-1)^{(i+1+n)}\, D_{b,i}\, (-1)^{(-i+n)}\, D_{b,-i}\, (-1)\,,$$

since by crossing out the row containing the element $D_{b,-i}$, we are left with a $(2n) \times (2n)$ matrix of the same form as (B.5) and whose determinant is -1.

Thus, we obtain, making good use of $(D_{b,-i})^* = D_{b,i}$,

$$\begin{aligned}\mathcal{D}_n &= -D_{bb} + \sum_{i=-n}^{n} (-1)^2 \mid D_{b,i} \mid^2 \\ &= -D_{bb} + \sum_{k=-n}^{n} \mid D_{b,k} \mid^2, \qquad \text{(we changed the index from } i \text{ to } k\text{)}\end{aligned}$$

and taking n to infinity

$$\begin{aligned}\tilde{\mathcal{D}} &= \lim_{n\to\infty} \mathcal{D}_n \\ &= -D_{bb} + \sum_{k=-\infty}^{\infty} \mid D_{b,k} \mid^2 . \end{aligned} \tag{B.9}$$

We *suppose* that for a continuous index set, we have the obvious generalization

$$\mathcal{D} = -D_{bb} + \int_{-\infty}^{+\infty} dk \mid D_{b,k} \mid^2 \tag{B.10}$$

Now, we would like to prove expression (3.39) for the determinant $\mathcal{D}$. According to (B.10), (3.36), and (3.38), we have

$$\mathcal{D} \;=\; I_1 \;+\; I_2$$

where

$$I_1 \;=\; -\int_{-\infty}^{+\infty} dx\, \left\{\phi_n'(x) \;+\; \int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x)\right\}^2, \tag{B.11}$$

$$I_2 \;=\; \int_{-\infty}^{+\infty} dk\, \left\{\int_{-\infty}^{+\infty} dp\; q_p(t) \int_{-\infty}^{+\infty} dx\; f_p'(x)\, f_k(x)\right\} \tag{B.12}$$

$$\times\, \left\{\int_{-\infty}^{+\infty} dv\; q_v^*(t) \int_{-\infty}^{+\infty} dy\; f_v'^*(y)\, f_k^*(y)\right\}.$$

The term I_1 can be expanded as

$$I_1 \;=\; I_1^a \;+\; I_1^b \qquad \text{with} \tag{B.13}$$

$$I_1^a \;=\; -\int_{-\infty}^{+\infty} dx\; (\phi_n'(x))^2 \;-\; 2\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dk\; q_k(t)\, \phi_n'(x)\, f_k'(x)\,, \tag{B.14}$$

$$I_1^b \;=\; -\int_{-\infty}^{+\infty} dx\; \left\{\int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x)\right\} \left\{\int_{-\infty}^{+\infty} dp\; q_p(t)\, f_p'(x)\right\}. \tag{B.15}$$

We recall that $\phi_n(x) \triangleq f_b(x)$ with $\{f_b(x), f_k(x)\}$ forming an orthonormal basis where

$$\int_{-\infty}^{+\infty} dx\; \phi_n^2(x) \;=\; \int_{-\infty}^{+\infty} dx\; f_b^2(x)$$

$$\triangleq\; M_o\,. \tag{B.16}$$

Therefore, one can express

$$\int_{-\infty}^{+\infty} dk\; q_k(t)\, f_k'(x) \;=\; c_b(t)\, f_b(x) \;+\; \int_{-\infty}^{+\infty} dr\; c_r(t)\, f_r(x)\,. \tag{B.17}$$

Using decomposition (B.17), we have for I_1^b

$$I_1^b \;=\; -\left[M_o\, c_b^2(t) \;+\; \int\int dr\, ds\; c_r(t)\, c_s(t)\; \delta(r+s)\right]$$

$$=\; -\left[c_b^2(t)\, M_o \;+\; \int_{-\infty}^{+\infty} dr\; |\; c_r(t)\; |^2\right], \tag{B.18}$$

and, from I_2,

$$\begin{aligned}\int_{-\infty}^{+\infty} dp\; q_p(t) \int_{-\infty}^{+\infty} dx\; f'_p(x)\, f_k(x) &= \int_{-\infty}^{+\infty} dx\; f_k(x) \int_{-\infty}^{+\infty} dp\; q_p(t)\, f'_p(x) \\ &= \int_{-\infty}^{+\infty} dr\; c_r(t)\; \delta(r+k) \\ &= c_{-k}(t)\,, \end{aligned} \tag{B.19}$$

and, similarly,

$$\begin{aligned}\int_{-\infty}^{+\infty} dv\; q^*_v(t) \int_{-\infty}^{+\infty} dy\; f'^*_v(y)\, f^*_k(y) &= \int_{-\infty}^{+\infty} dy\; f^*_k(y) \int_{-\infty}^{+\infty} dv\; q^*_v(t)\, f'^*_v(y) \\ &= \int_{-\infty}^{+\infty} dr\; c^*_r(t)\; \delta(r+k) \\ &= c^*_{-k}(t)\,. \end{aligned} \tag{B.20}$$

Using (B.19) and (B.20), I_2 takes the simple form

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty} dk\; c_{-k}(t)\, c^*_{-k}(t) \\ &= \int_{-\infty}^{+\infty} dk\; |\, c_k(t)\,|^2\,. \end{aligned} \tag{B.21}$$

Therefore, adding I_1^b and I_2 in their simplified expressions (B.18) and (B.21), we obtain

$$\begin{aligned} I_1^b + I_2 &= -\int_{-\infty}^{+\infty} dx\; \left\{\int_{-\infty}^{+\infty} dk\; q_k(t)\, f'_k(x)\right\} \left\{\int_{-\infty}^{+\infty} dp\; q_p(t)\, f'_p(x)\right\} + I_2 \\ &= -c_b^2(t)\, M_o - \int_{-\infty}^{+\infty} dr\; |\, c_r(t)\,|^2 + \int_{-\infty}^{+\infty} dr\; |\, c_r(t)\,|^2 \\ &= -c_b^2(t)\, M_o\,. \end{aligned} \tag{B.22}$$

But,

$$\left[\int_{-\infty}^{+\infty} dx\ \phi_n'(x)\left(\int_{-\infty}^{+\infty} dk\ q_k(t)\ f_k'(x)\right)\right]^2 =$$

$$\left[\int_{-\infty}^{+\infty} dx\ f_b(x)\left(c_b(t)\ f_b(x) + \int_{-\infty}^{+\infty} dr\ c_r(t)\ f_r(x)\right)\right]^2$$

$$=$$

$$\left[c_b(t)\int_{-\infty}^{+\infty} dx\ f_b^2(x) + \int_{-\infty}^{+\infty} dr\ c_r(t)\underbrace{\int_{-\infty}^{+\infty} dx\ f_b(x) f_r(x)}_{0}\right]^2$$

$$= [c_b(t)\ M_o]^2$$

$$= c_b^2(t)\ M_o^2, \qquad \text{(B.23)}$$

so that equation (B.22) can be expressed as

$$\begin{aligned} I_1^b + I_2 &= -c_b^2(t)\ M_o \\ &= -\frac{1}{M_o}\left(c_b^2(t)\ M_o^2\right) \\ &= -\frac{1}{M_o}\left[\int_{-\infty}^{+\infty} dx\ \phi_n'(x)\left(\int_{-\infty}^{+\infty} dk\ q_k(t)\ f_k'(x)\right)\right]^2. \end{aligned}$$

Hence, the determinant $\mathcal{D}$ becomes

$$\begin{aligned} \mathcal{D} &= I_1^a + I_1^b + I_2 \\ &= -\int_{-\infty}^{+\infty} dx\ (\phi_n'(x))^2 - 2\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dk\ q_k(t)\ \phi_n'(x)\ f_k'(x) \\ &= -\frac{1}{M_o}\left[\int_{-\infty}^{+\infty} dx\ \phi_n'(x)\left(\int_{-\infty}^{+\infty} dk\ q_k(t)\ f_k'(x)\right)\right]^2 \\ &= -\left\{M_o + 2\int_{-\infty}^{+\infty} dx\ \phi_n'(x)\left(\int_{-\infty}^{+\infty} dk\ q_k(t)\ f_k'(x)\right)\right. \\ &\qquad \left. + \frac{1}{M_o}\left[\int_{-\infty}^{+\infty} dx\ \phi_n'(x)\left(\int_{-\infty}^{+\infty} dk\ q_k(t)\ f_k'(x)\right)\right]^2\right\}, \\ \mathcal{D} &= -\{M_o + 2\,\alpha + \frac{1}{M_o}\,\alpha^2\} \qquad \text{where} \end{aligned}$$

$$\alpha \triangleq \int_{-\infty}^{+\infty} dx\ \phi_n'(x) \left(\int_{-\infty}^{+\infty} dk\ q_k(t)\ f_k'(x) \right).$$

Therefore,

$$\begin{aligned} \mathcal{D} &= -\frac{1}{M_o} \{\alpha^2 + 2\, M_o\, \alpha + M_o^2\} \\ &= -\frac{1}{M_o} (\alpha + M_o)^2, \end{aligned}$$

which is exactly equation (3.39).

Appendix C

Lifetime of an environment-coupled harmonic oscillator

In this appendix, we show that the Hamiltonian, derived from the Lagrangian (4.51) in the limit of low amplitudes for the domain wall, makes the lifetime of a domain wall's energy level E_n take the form

$$n\, J(\omega_o) \;=\; \frac{1}{2\,\tau_n} \tag{C.1}$$

where $J(\nu)$ is the spectral density function and ω_o the natural frequency of the oscillating domain wall parametrized by the coordinate $X(t)$.

For low amplitudes of the domain wall, $V(X)$ in (4.50) can be approximated by a harmonic potential so that

$$V_{eff}(X) \;\approx\; \tilde{V}(X) \;=\; \omega_o^2\, X^2 \;-\; \frac{\alpha^2}{\sqrt{2\pi}} \sum_k \frac{C_k}{\omega_k}\, X\,, \tag{C.2}$$

with ω_o^2 reflecting the curvature at the bottom of the well $V(X)$.

As explained in the footnote of page 99 in chapter 4, expression (C.2) can be thought of as a displaced harmonic potential well centered around a value of X different from zero owing to the presence of the constant force term $(\alpha^2/\sqrt{2\pi}) \sum_k (C_k/\omega_k)$. Let us denote the new center by $X = X'$ (cf. fig. (C.1)). The energy eigenstates for the Hamiltonian

$$H_X \;=\; \frac{M_o}{2}\, \dot{X}^2(t) \;+\; \tilde{V}(X) \tag{C.3}$$

are the same as those for the ordinary harmonic oscillator except for their eigenvalues being all offset by the same constant term $\tilde{V}(X') \;=\; -(\alpha^4/8\pi\omega_o^2)\,[\sum_k (C_k/\omega_k)\,]^2$.

In the limit of low amplitudes for a domain wall, we have the total Hamiltonian

$$H \;=\; H_{DW} \;+\; \frac{1}{2}\sum_k \{\,|\, \dot{q}_k(t)\,|^2 \;+\omega_k^2\,|\, q_k(t)\,|^2\} \;+\; X(t) \sum_k q_k(t)\, C_k\,, \tag{C.4}$$

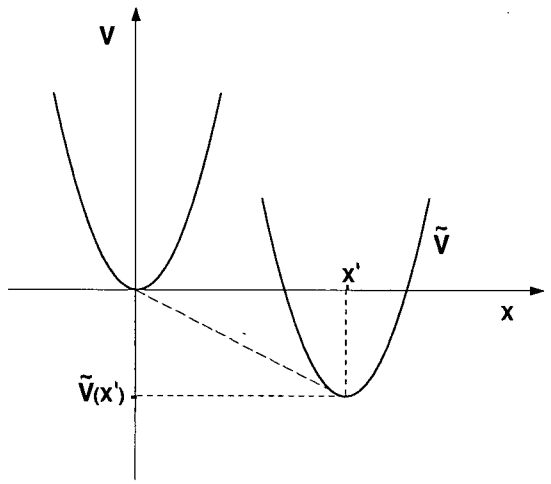


Figure C.1: A shifted harmonic potential in the presence of a constant force. This figure shows how a harmonic potential is shifted by the presence of a constant force f_c. The new minimum for the shifted potential $\tilde{V}(X) = \omega\, x^2 - f_c\, X$ is now given by $X' = f_c/\omega$.

where $H_{DW} \equiv H_X$ with the subscript DW reminding us that the classical object is a domain wall.

This Hamiltonian is composed of the usual harmonic oscillator term for the domain wall, H_{DW}; a (infinite) series of harmonic oscillator Hamiltonians[1] representing the environment; and a coupling Hamiltonian for the last term.

We suppose that

$$H_o \;=\; H_{DW} \;+\; \frac{1}{2}\sum_k \{\mid \dot{q}_k(t) \mid^2 + \omega_k^2 \mid q_k(t) \mid^2\} \tag{C.5}$$

represents the unperturbed Hamiltonian, whose energy eigenstates are denoted by $\mid n\rangle_{DW} \mid n\rangle_b$ where

$$\begin{aligned} \mid n\rangle_b \;&\triangleq\; \bigotimes_{k=-\infty}^{+\infty} \mid n_k\rangle \\ &=\; \mid n_0\rangle \otimes \mid n_1\rangle \otimes \mid n_{-1}\rangle \otimes \cdots, \end{aligned}$$

[1]The fact that we have complex coordinates $q_k(t)$ does not invalidate our procedure, although this may appear odd. These coordinates are not totally independent of each other so as to ensure that we deal after all with a real field. This is analogous to going from a real basis $\{\cos(kx), \sin(kx)\}$ with *real* coefficients to a complex basis $\{e^{k\,x}\}$ with *complex* coefficients.

and

$$H_1 \;=\; X \sum_k q_k(t)\, C_k$$

represents the perturbation Hamiltonian.

According to the standard perturbation theory, the second order correction to the energy level $E_{n,\{n_k\}}$ is given by

$$_2\Delta E_{n,\{n_k\}} \;=\; \sum_{m,\{m_k\}} \frac{|\, \langle n\,|_{DW}\, \langle n\,|_b\; H_1 \;|\; m\rangle_{DW} \;|\; m\rangle_b\,|^2}{(E_{n,\{n_k\}} \;-\; E_{m,\{m_k\}})} \tag{C.6}$$

where $\{n_k\}$ represents the quantum number for each oscillator "$q_k(t)$" of the bath.

Now,

$$\begin{aligned}
\langle n\,|_{DW}\, \langle n\,|_b\; H_1 \;|\; m\rangle_{DW} \;|\; m\rangle_b \;&=\; \langle n\,|_{DW}\, \langle n\,|_b \left(X \sum_k q_k\, C_k\right) |\; m\rangle_{DW} \;|\; m\rangle_b \\
&=\; (\langle n\,|_{DW}\; X \;|\; m\rangle_{DW}) \sum_k C_k\, \langle n\,|_b\; q_k \;|\; m\rangle_b \\
&=\; (\langle n\,|_{DW}\; X \;|\; m\rangle_{DW}) \sum_k C_k\, \langle n_k \mid\; q_k \;|\; m_k\rangle\; \tilde{\delta}_{n_k,m_k}\,,
\end{aligned}$$

where

$$\tilde{\delta}_{n_k,m_k} \;=\; \delta_{n_0,m_0}\, \delta_{n_{-1},m_{scmin1}}\, \delta_{n_1,m_1} \cdots \hat{\delta}_{n_k,m_k} \cdots \delta_{n_{51},m_{51}} \cdots\,,$$

with a hat over a particular Kronecker symbol signifying that this symbol is to be omitted from the expression in which it appears.

For the harmonic oscillator eigenstates, we know that

$$\begin{aligned}
\langle n_k \mid\; q_k \;|\; m_k\rangle \;&=\; \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \{\langle n_k \mid\; a_k \;|\; m_k\rangle \;+\; \langle n_k \mid\; a_k^+ \;|\; m_k\rangle\} \\
&\qquad \text{using the usual lowering and raising operators, } a_k \text{ and } a_k^+, \\
&\qquad \text{for the } k^{\text{th}} \text{ oscillator,} \\
&=\; \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \{\sqrt{m_k}\, \langle n_k \mid m_k - 1\rangle \;+\; \sqrt{m_k+1}\, \langle n_k \mid m_k + 1\rangle\} \\
&=\; \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \{\sqrt{m_k}\, \delta_{n_k,m_k-1} \;+\; \sqrt{m_k+1}\, \delta_{n_k,m_k+1}\,.\}
\end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
&|\,\langle n\,|_{DW}\,\langle n\,|_b\;H_1\;|\,m\rangle_{DW}\,|\,m\rangle_b\,|^2\;=\\
&\left\{\langle n\,|_{DW}\;\;X\;\;|\,m\rangle_{DW}\sum_k C_k\left(\frac{\hbar}{2\omega_k}\right)^{1/2}\{\sqrt{m_k}\delta_{n_k,m_k-1}+\sqrt{m_k+1}\delta_{n_k,m_k+1}\}\tilde{\delta}_{n_k,m_k}\right\}\\
&\times\left\{(\langle n\,|_{DW}\;\;X\;\;|\,m\rangle_{DW})^*\sum_p C_p^*\left(\frac{\hbar}{2\omega_k}\right)^{1/2}\{\sqrt{m_p}\delta_{n_p,m_p-1}+\sqrt{m_p+1}\delta_{n_p,m_p+1}\}\tilde{\delta}_{n_p,m_p}\right\}\\
&\qquad\qquad=\;\;|\,\langle n\,|_{DW}\;\;X\;\;|\,m\rangle_{DW}\,|^2\times\\
&\sum_{k,p}C_k\,C_p^*\,\frac{\hbar}{2}\frac{1}{\sqrt{\omega_k\omega_p}}\left\{\sqrt{m_pm_k}\delta_{n_k,m_k-1}\,\delta_{n_p,m_p-1}+\sqrt{(m_k+1)(m_p+1)}\delta_{n_k,m_k+1}\,\delta_{n_p,m_p+1}\right.\\
&\qquad\left.+\sqrt{m_p(m_k+1)}\delta_{n_p,m_p-1}\;\delta_{n_k,m_k+1}+\sqrt{m_k(m_p+1)}\delta_{n_p,m_p+1}\,\delta_{n_k,m_k-1}\right\}\tilde{\delta}_{n_k,m_k}\,\tilde{\delta}_{n_p,m_p}\,.
\end{aligned}
$$

Thus, we have for the summation over the indices $\{m_r\}$,

$$
\begin{aligned}
\sum_{m_r}\frac{|\,\langle n\,|_{DW}\,\langle n\,|_b\;H_1\;|\,m\rangle_{DW}\,|\,m\rangle_b\,|^2}{(E_{n,\{n_r\}}-E_{m,\{m_r\}})}\;&=\;\sum_{m_1,m_2,m_3,\ldots}\frac{|\,\langle n\,|_{DW}\,\langle n\,|_b\;H_1\;|\,m\rangle_{DW}\,|\,m\rangle_b\,|^2}{[(E_n-E_m)+\sum_r(E_{n_r}-E_{m_r})]}\\
&=\;|\,\langle n\,|_{DW}\;\;X\;\;|\,m\rangle_{DW}\,|^2\times\\
&\qquad\sum_{m_1,m_2,m_3,\ldots}\frac{\sum_{k,p}a_{k,p}(\{m_r\})}{[(E_n-E_m)+\sum_r(E_{n_r}-E_{m_r})]}\\
&=\;|\,\langle n\,|_{DW}\;\;X\;\;|\,m\rangle_{DW}\,|^2\times\qquad\qquad(C.7)\\
&\qquad\sum_{k,p}\sum_{m_1,m_2,m_3,\ldots}\frac{a_{k,p}(\{m_r\})}{[(E_n-E_m)+\sum_r(E_{n_r}-E_{m_r})]}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
a_{k,p}(\{m_r\})\;=\;C_k\,C_p^*\,\frac{\hbar}{2}\,\frac{1}{\sqrt{\omega_k\,\omega_p}}\Big\{&\sqrt{m_pm_k}\;\delta_{n_k,m_k-1}\,\delta_{n_p,m_p-1}\,\tilde{\delta}_{n_k,m_k}\,\tilde{\delta}_{n_p,m_p}\qquad(C.8)\\
&+\sqrt{(m_k+1)(m_p+1)}\,\delta_{n_k,m_k+1}\;\delta_{n_p,m_p+1}\,\tilde{\delta}_{n_k,m_k}\,\tilde{\delta}_{n_p,m_p}\;+\\
\sqrt{m_p(m_k+1)}\,\delta_{n_p,m_p-1}\,\delta_{n_k,m_k+1}\,\tilde{\delta}_{n_k,m_k}\,\tilde{\delta}_{n_p,m_p}\;+\;&\sqrt{m_k(m_p+1)}\,\delta_{n_p,m_p+1}\,\delta_{n_k,m_k-1}\,\tilde{\delta}_{n_k,m_k}\,\tilde{\delta}_{n_p,m_p}\Big\}.
\end{aligned}
$$

For the "r^{th}" oscillator, the Kronecker symbol δ_{n_r,m_r} is present for each product $\tilde{\delta}_{n_k,m_k}\tilde{\delta}_{n_p,m_p}$ unless $k=p=r$, in which case we have

$$\tilde{\delta}_{n_k,m_k}\tilde{\delta}_{n_k,m_k} \;=\; (\delta_{n_0,m_0})^2\,(\delta_{n_{-1},m_{-1}})^2\,(\delta_{n_1,m_1})^2\,\cdots\,(\widehat{\delta_{n_k,m_k}})^2\,\cdots\,(\delta_{n_{123},m_{123}})^2\cdots .$$

What this means is that, when we perform our summation $\sum_{m_r} = \sum_{m_1,m_2,m_3,\ldots}$, only the terms for which $k=p$ in (C.8) may possibly survive since all other terms for which $k\neq p$ vanish because $\tilde{\delta}_{n_k,m_k}\tilde{\delta}_{n_p,m_p}$ would require $n_r=m_r$ for all r in order to be non-zero, but in such a case the Kronecker symbols $\delta_{n_k,m_k-1}\,\delta_{n_p,m_p-1}$, $\delta_{n_k,m_k+1}\,\delta_{n_p,m_p+1}$, $\delta_{n_p,m_p+1}\,\delta_{n_k,m_k-1}$, and $\delta_{n_p,m_p-1}\,\delta_{n_k,m_k+1}$ in $a_{k,p}(\{m_r\})$ would all vanish causing $a_{k,p}(\{m_r\})$ itself to have the same fate; and if any of those latter Kronecker symbols were non-zero, then it would mean that for some r, $n_r\neq m_r$ making $\tilde{\delta}_{n_k,m_k}\tilde{\delta}_{n_p,m_p}$ and, therefore, $a_{k,p}(\{m_r\})$ both zero. Hence, we obtain

$$\sum_{m_r}\frac{|\,\langle n\,|_{DW}\,\langle n\,|_b\;H_1\;|\,m\rangle_{DW}\,|\,m\rangle_b\,|^2}{(E_{n,\{n_r\}}-E_{m,\{m_r\}})} \;=\; |\,\langle n\,|_{DW}\;X\;|\,m\rangle_{DW}\,|^2\;\times$$

$$\sum_{k,p}\sum_{m_1,m_2,m_3,\ldots}\frac{a_{k,p}(\{m_r\})}{[(E_n-E_m)+\sum_r(E_{n_r}-E_{m_r})]}$$

$$=\; |\,\langle n\,|_{DW}\;X\;|\,m\rangle_{DW}\,|^2\;\times$$

$$\sum_{k}\sum_{m_1,m_2,m_3,\ldots}\frac{a_{k,k}(\{m_r\})}{[(E_n-E_m)+\sum_r(E_{n_r}-E_{m_r})]}$$

$$=\; |\,\langle n\,|_{DW}\;X\;|\,m\rangle_{DW}\,|^2\;\times$$

$$\sum_k\frac{|\,C_k\,|^2\,\hbar}{2\omega_k}\sum_{m_1,m_2,m_3,\ldots}\Big\{\,m_k\,(\delta_{n_k,m_k-1})^2\,\tilde{\delta}_{n_k,m_k}\tilde{\delta}_{n_k,m_k}\;+\;(m_k+1)\,(\delta_{n_k,m_k+1})^2\,\tilde{\delta}_{n_k,m_k}\tilde{\delta}_{n_k,m_k}$$

$$+\;2\sqrt{m_k(m_k+1)}\;\delta_{n_k,m_k-1}\,\delta_{n_k,m_k+1}\,\tilde{\delta}_{n_k,m_k}\tilde{\delta}_{n_k,m_k}\,\Big\}\Big/[(E_n-E_m)+\sum_r(E_{n_r}-E_{m_r})]$$

$$= \mid \langle n \mid_{DW} \; X \; \mid m \rangle_{DW} \mid^2 \times$$

$$\sum_k \frac{\mid C_k \mid^2 \hbar}{2\omega_k} \left\{ \frac{n_k + 1}{(E_n - E_m) + (E_{n_k} - E_{n_k+1})} + \frac{n_k}{(E_n - E_m) + (E_{n_k} - E_{n_k-1})} \right\},$$

since the term containing $\delta_{n_k,m_k-1}\, \delta_{n_k,m_k+1}\, \tilde{\delta}_{n_k,m_k}\, \tilde{\delta}_{n_k,m_k}$ always vanishes no matter what $m_1, m_2, m_3, \ldots$ are.

Therefore, making use of the fact that

$$E_{n_k} \;=\; (n_k + 1/2)\, \hbar\, \omega_k$$

for each oscillator k, expression (C.6) yields

$$_2\Delta E_{n,\{n_k\}} \;=\; \sum_m M_{nm} \sum_k \frac{\mid C_k \mid^2 \hbar}{2\,\omega_k} \left\{ \frac{(2\,n_k + 1)\,\Delta E \;+\; \hbar\,\omega_k}{(\Delta E)^2 \;-\; (\hbar\,\omega_k)^2} \right\}, \qquad \text{(C.9)}$$

where

$$\begin{aligned} M_{nm} &\triangleq \mid \langle n \mid_{DW} \; X \; \mid m \rangle_{DW} \mid^2, \qquad \text{and} \\ \Delta E &\triangleq E_n \;-\; E_m \\ &= \hbar\,(\omega_n \;-\; \omega_m) \\ &= \hbar\,\omega_{nm}. \end{aligned}$$

The second order correction to the energy level where all oscillators of the bath would be at their ground states ($n_k = 0$ for all k), were it not for the perturbation, is given by

$$\begin{aligned} _2\Delta E_{n,\{n_k=0\}} &= \sum_m M_{nm} \sum_{k=-\infty}^{\infty} \frac{\mid C_k \mid^2 \hbar}{2\,\omega_k} \frac{1}{(\Delta E \;-\; \hbar\,\omega_k)} \\ &= \sum_m M_{nm} \sum_{k=-\infty}^{\infty} \frac{\mid C_k \mid^2}{2\,\omega_k} \frac{1}{(\omega_{nm} \;-\; \omega_k)}. \end{aligned} \qquad \text{(C.10)}$$

However, by definition, we have [13, 45]

$$J(\nu) \;=\; \frac{\pi}{2}\sum_{k}\frac{\mid C_k \mid^2}{\omega_k}\,\delta(\nu-\omega_k) \tag{C.11}$$

for a bath of unit-mass oscillators.

Substituting (C.11) into (C.10) yields

$$_2\Delta E_{n,\{n_k=0\}} \;=\; \sum_{m}\frac{M_{nm}}{\pi}\int_0^\infty d\nu\;\frac{J(\nu)}{(\omega_{nm}-\nu)}\,. \tag{C.12}$$

In order to compute this integral, we add an infinitesimal imaginary part to the denominator and split the integral as follows

$$\begin{aligned}
_2\Delta E_{n,\{n_k=0\}} \;&=\; \sum_{m}\frac{M_{nm}}{\pi}\lim_{\epsilon\to 0+}\int_0^\infty d\nu\;\frac{J(\nu)}{(\omega_{nm}-\nu+i\,\epsilon)} \\
&=\; \sum_{m}\frac{-M_{nm}}{\pi}\lim_{\epsilon\to 0+}\int_0^\infty d\nu\;\frac{J(\nu)}{(\nu-(\omega_{nm}-i\,\epsilon))} \\
&=\; \sum_{m<n}\frac{-M_{nm}}{\pi}\lim_{\epsilon\to 0+}\int_0^\infty d\nu\;\frac{J(\nu)}{(\nu-(\omega_{nm}-i\,\epsilon))} \\
&\qquad\qquad + \sum_{m>n}\frac{-M_{nm}}{\pi}\lim_{\epsilon\to 0+}\int_0^\infty d\nu\;\frac{J(\nu)}{(\nu-(\omega_{nm}-i\,\epsilon))}\,, \\
_2\Delta E_{n,\{n_k=0\}} \;&=\; I_1 \;+\; I_2\,,
\end{aligned} \tag{C.13}$$

since $M_{nn} \;=\mid \langle n \mid_{DW}\; X \mid n\rangle_{DW} \mid^2 \;=\; 0.$

In the classical regime, the macroscopic coordinate $X(t)$ obeys a general phenomenological equation of motion of the form

$$\ddot{X}(t) \;+\; \hat{K}\{X(t)\} \;=\; -\frac{\partial V(X)}{\partial X}$$

where $\hat{K}$ is a linear operator subject to the requirement of causality [45]. For our case, we have $V(X) \;=\; \tilde{V}(X)$ in expression (C.3). The Fourier transform of $\hat{K}$, $K(\nu)$, is analytical in the lower half of the complex plane,[2] $K(0) = 0$, and we have the following

[2] As a consequence of causality.

relation [13, 45]:

$$\begin{aligned} K(\nu) &= i\,J(\nu) \qquad \text{for } \nu \in \mathbf{IR} \\ \text{so that} \qquad \mathrm{Im}\,K(\nu) &= J(\nu)\,. \end{aligned} \tag{C.14}$$

Now, in order to solve

$$\begin{aligned} I_1 &= \sum_{m<n} \frac{(-M_{nm})}{\pi} \lim_{\epsilon\to 0^+} \int_0^\infty d\nu\, \frac{J(\nu)}{(\nu - (\omega_{nm} - i\,\epsilon))} \\ &= \sum_{m<n} \frac{(-M_{nm})}{2\,\pi} \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\nu\, \frac{J(\mid \nu \mid)}{(\mid \nu \mid - (\omega_{nm} - i\,\epsilon))} \end{aligned} \tag{C.15}$$

where $\omega_{nm} > 0$, we transform ν into a complex variable z and choose the integration path Γ_R by closing the contour in the lower half-plane, where the pole $\omega_{nm} - i\epsilon$ is located, and let $R \to \infty$ (cf. fig. (C.2)) to obtain

$$\sum_{m<n} \frac{(-M_{nm})}{2\pi} \lim_{\epsilon\to 0^+} \left\{ \lim_{R\to\infty} \int_{\Gamma_R} \frac{J(\mid z \mid)}{(\mid z \mid -(\omega_{nm} - i\epsilon))} \right\} = \sum_{m<n} \frac{(-M_{nm})}{2\pi} \lim_{\epsilon\to 0^+} 2\pi i J(\mid \omega_{nm} - i\epsilon \mid)$$

by the Residue theorem. We *suppose* that $J(\mid z \mid)$ is sufficiently well-behaved so that the integrand in the above integral vanishes on the semicircle C_R as $R \to \infty$.

Thus,

$$\begin{aligned} I_1 &= \sum_{m<n} (-M_{nm})\, i \lim_{\epsilon\to 0^+} J(\mid \omega_{nm} - i\,\epsilon \mid) \\ &= \sum_{m<n} (-M_{nm})\, i\, J(\omega_{nm})\,, \\ I_1 &= \sum_{m<n} M_{nm}\, K^*(\omega_{nm}) \end{aligned} \tag{C.16}$$

according to (C.14).

For the term I_2, we do not have to resort to a complex integral on account of the fact that for $m > n$, $\omega_{nm} < 0$. Indeed, we have

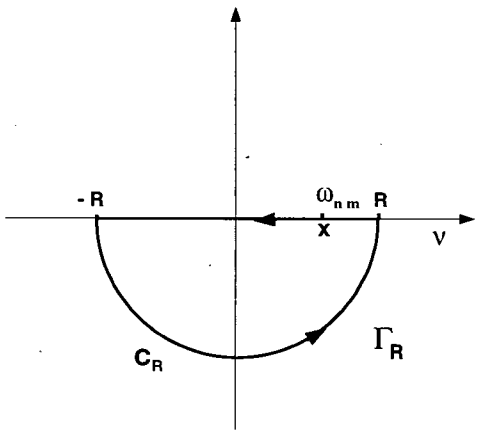


Figure C.2: The integration path Γ_R for the partial evaluation of the second order perturbation energy term. The pole, denoted by x in the figure, is barely located in the lower half-plane on account of an infinitesimal imaginary part added to the harmonic oscillator's frequency difference ω_{nm}. The path Γ_R is made up of the line extending from $-\mathbf{R}$ to $\mathbf{R}$ on the real line and a semi-circle $\mathbf{C_R}$ of radius $\mathbf{R}$ in the lower half-plane. For the proper evaluation of our expression of interest, we let $\mathbf{R}$ tend to infinity.

$$
\begin{aligned}
I_2 &= \sum_{m>n} \frac{(-M_{nm})}{\pi} \lim_{\epsilon \to 0^+} \int_0^\infty d\nu \, \frac{J(\nu)}{(\nu - (\omega_{nm} - i\,\epsilon))} \\
&= \sum_{m>n} \frac{(-M_{nm})}{\pi} \lim_{\epsilon \to 0^+} \int_0^\infty d\nu \, \frac{J(\nu)}{(\nu + \mid \omega_{nm} \mid + i\,\epsilon)} \\
&= \sum_{m>n} \frac{(-M_{nm})}{\pi} \int_0^\infty d\nu \, \frac{J(\nu)}{(\nu + \mid \omega_{nm} \mid)} \\
I_2 &= \sum_{m>n} \frac{(-M_{nm})}{\pi} \int_0^\infty d\nu \, \frac{\mathrm{Im}\, K(\nu)}{(\nu + \mid \omega_{nm} \mid + i\,\epsilon)}
\end{aligned}
\tag{C.17}
$$

using expression (C.14) once again.

Of the terms I_1 and I_2 in (C.13), only I_1 is complex-valued. According to the standard interpretation [39], one relates the imaginary part of ${}_2\Delta E_{n,\{n_k=0\}}$ with the lifetime of the energy level E_n of the particle (parametrized by the coordinate $X(t)$) *for which*, were it not for the perturbation, all the bath's oscillators would normally be in their ground states; a situation which is likely to be realized at very low temperatures and consistent

with our requirement that each oscillator be weakly perturbed.[3] The relation is as follows

$$\mathrm{Im}\,({}_2\Delta E_{n,\{n_k\}}) \;=\; \mathrm{Im}\,(I_1) \;=\; -\frac{1}{2}\,\frac{1}{\tau_n} \tag{C.18}$$

where τ_n is the lifetime of energy level E_n for the system's harmonic oscillator.

We showed previously that

$$I_1 \;=\; \sum_{m<n} M_{nm}\; K^*(\omega_{nm})$$

with $M_{nm} = |\,\langle n\,|_{DW}\; X\;|\,m\rangle_{DW}\,|^2$. Now,

$$M_{nm} \;=\; \{m\,(\delta_{n,m-1})^2 \;+\; (m+1)\,(\delta_{n,m+1})^2\}$$

for a harmonic oscillator so that

$$\begin{aligned} I_1 \;&=\; \sum_{m<n} M_{nm}\; K^*(\omega_{n,m}) \\ &=\; n\; K^*(\omega_{n,n-1})\,, \\ I_1 \;&=\; -i\; n\; J(\omega_{n,n-1}) \end{aligned} \tag{C.19}$$

using expression (C.14).

Substituting equation (C.19) into (C.18), we obtain

$$\begin{aligned} n\; J(\omega_{n,n-1}) \;&=\; \frac{1}{2}\,\frac{1}{\tau_n} \\ n\; J(\hbar\omega_o) \;&=\; \frac{1}{2}\,\frac{1}{\tau_n}\,, \end{aligned} \tag{C.20}$$

[3] In reality, $\mathrm{Im}({}_2\Delta E_{n,\{n_k\}})$ relates to the lifetime of the energy level $E'_{n,\{n_k\}}$ which corresponds to the unperturbed energy level $E_{n,\{n_k\}}$, with the harmonic oscillator $X(t)$ having quantum number n and the bath's oscillators, quantum numbers n_k, in the sense that, by slowly turning off the coupling between the particle and the bath, $E'_{n,\{n_k\}}$ would approach the value of $E_{n,\{n_k\}}$. We implicitly suppose that the interaction does not create additional quantum states for a *complete description* of the universe (system plus bath). Obviously, this is always a matter of semantics when we assign the full thrust of the perturbation effect to the system's particle alone, as we did above.

since, for a harmonic oscillator, all energy levels are separated by the same difference $\hbar\,\omega_o$ where ω_o is the oscillator's natural frequency. Using the convention $\hbar = 1$, we obtain expression (5.1)

$$n\ J(\omega_o) \ = \ \frac{1}{2}\,\frac{1}{\tau_n},$$

which is the result we wished to demonstrate in this appendix.