AN EXPLORATORY STUDY OF STUDENTS' REPRESENTATIONS OF UNITS AND UNIT RELATIONSHIPS IN FOUR MATHEMATICAL CONTEXTS

by

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ABSTRACT

This study explores characteristics of students' repertoires of representations in two mathematical contexts: whole number multiplication and the comparison of common fractions. A repertoire of representations refers to a set of representations which a student can reconstruct as needed. Of particular interest are (1) how multiplicative relationships among units were represented, and (2) whether continuous measurement was an underlying conceptual framework for their representations. In addition, the characteristics of students' representations and interpretation of units of linear and area measurement were explored. Data were collected through a series of interviews with Grade 5 and Grade 7 students.

Some results of the study were as follows. Each repertoire of representations was exemplified by a dominant form of units, either discrete or contiguous. Within a repertoire, all forms of units were related, first through a common system of measurement (either numerosity or area), and second through their two-dimensional characteristic.

In the multiplication context, some repertoires were comprised only of representations with discrete units, but others also included some representations with contiguous units. Students sought characteristics in their representations which reflected those based on continuous measurement, however linear or area measurement was not used as a conceptual framework. Instead, all representations were based on the measurement of numerosity. Also, students exhibited different limits in their representation of multiplicative relationships among units. Some represented no multiplicative relationships, but most represented at least a multiplicative relationship between two units. Relationships among three units were seldom constructed and difficult to achieve.

Common fraction repertoires were based on the measurement of either numerosity or area, but the physical characteristics of the units varied. Some repertoires had only contiguous representations of units, others also included representations with discrete units, and a few did not represent fractional units at all. Students' representations reflected characteristics of area-based representations, however

area measurement was not necessarily a conceptual framework. In addition, students' beliefs about what constituted units of area measurement were variable. As a result, they either represented no multiplicative relationships among units, or fluctuated between representing two-unit and three-unit relationships.

Linear measurement was notably absent as a basis for representations in both mathematical contexts. The one-dimensional characteristic of linear measurement did not fit students' dominant framework for constructing mathematical representations.

With respect to measurement, students represented linear units in terms of discrete points or line segments. Counting points and interpreting the count in terms of the numerosity of line segments was problematic for nearly all students. When partitioning regions into units of area, a few students also equated the number of lines with the number of parts. The direct relationship of action and result in counting discrete objects was generalized without consideration of other geometric characteristics.

When comparing quantities having linear or area units, numerical reasoning was not always used. Alternatively, either quantities were transformed to facilitate a direct comparison, or only perceptual judgements were made. No students consistently used numerical reasoning to compare fractional units of area. In the latter situations, the part-whole relationship among units seldom was observed.

In general, there was no direct relationship between the forms of representations used by students in the two mathematical contexts and the characteristics of their representations of units of the measurement contexts. The development of repertoires of representations appears to be context specific. The repertoires were strictly limited in terms of the forms of representations of which they were comprised.

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CHAPTER 1

OVERVIEW OF THE STUDY

There is an extensive history of mathematics educators advocating the use of physical representations as a means of facilitating children's learning of mathematics. McLellan and Dewey (1895), Montessori (1912/1964), and Brownell (1928) considered experiences with manipulative and pictorial representations of numerical relationships to be central to meaningful learning of mathematical relationships at the elementary level. More recently, Piaget's formulation of concrete operational and formal operational stages of cognitive development, Bruner's (1960,1968) formulation of enactive, iconic, and symbolic representational stages, and Dienes' (1963, 1964) advocacy of the use of multiple embodiments based on perceptual variability and mathematical variability principles have lent support to previously held beliefs about the importance of children's experiences with physical representations of mathematical relationships in their learning process (Post, 1980). In British Columbia during the 1970's elementary teachers and other mathematics educators generally concurred with this view (Province of British Columbia, Ministry of Education, 1978; Robitaille & Sherrill, 1977). Most recently, the importance placed on representations in the learning of mathematics has not diminished. In the Mathematics Curriculum Guide 1-8 (Province of British Columbia, 1987) it is stated that, in the early years "all [students] require extensive experiences in concrete manipulations in order to form sound, transferable mathematical concepts," and in later years "the use of models is beneficial when introducing a topic" (p. 2). For the later elementary grades, the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989), emphasized the importance of the use of mathematical representations by students, stating that:

> the study of mathematics should include opportunities to communicate so that students can model situations using oral, written, <u>concrete</u>, <u>pictorial</u>, graphical, and algebraic methods. (p.78) [Emphasis added.]

Despite general beliefs about the importance of representations in mathematics learning, concerns remain about the efficacy of different representations to exemplify mathematical ideas, and about how students might make sense of the mathematics being represented (e.g., Dufour-Janvier, Bednarz, & Belanger, 1987; Fischbein, 1977; Hart, 1987; Hunting, 1984a, Hunting, 1984b; McLellan & Dewey, 1895; Payne, 1975, 1984). In this regard, Fischbein (1977) argues that many representations such as graphs or Venn diagrams are themselves representations of conceptual structures which are defined independently of the mathematics they represent. He further contends that an incomplete understanding of the conceptual structure and "language" of the representation will result in misunderstandings of the mathematics being represented. As well, Dufour-Janvier, Bednarz, and Belanger (1987) argue that some representations in elementary instruction might be "inaccessible" to students if they are too far removed from the child's "internal representations" of the situation or the representations envisaged by the child of the problem situation" (p. 118). They contend that,

> the use of representations that are just as abstract for the child as that which is studied brings the child to manipulate rules and symbols that are meaningless to her. (p. 118)

In the elementary school curriculum, representations of mathematical relationships commonly are based on conceptual structures which are "defined independently of the mathematics they represent." For instance, relationships between factors and product might be represented as the sum of a number of equal groups of discrete objects, as the length of a regular number of jumps of linear units along the number line, or as the number of units within a rectangle when units are defined by the partitioning of each dimension by the factors. These three forms of representation can be differentiated by the systems of measurement which underlie them. The first representation is based on measures of the numerosity. The other two are based on measures of length and area respectively. In the first representation units are discrete, and in the other two the units are contiguous. As a consequence, they differ in terms of the physical properties which define units, the limits on the spatial configurations of the units, and the procedures governing the representation of unit relationships.

The primary motivation for using different forms of representations in elementary instruction is to facilitate the students' learning of the mathematics being represented. The students' learning about a conceptual structure underlying a particular form of representation is, at best, secondary, and may never be addressed explicitly by the teacher. Yet, as Fischbein (1977) suggests, the ways in which students make sense of the form of the representation might influence how students construe the mathematics represented, as well as how the students use such representations to solve mathematical problems.

How students might make sense of different types of mathematical representations, the forms of which are assumed by others to be based on different systems of measure, may depend on a number of factors. These may include: (1) the associations students make between forms of representation of mathematical relationships and measurement systems, (2) the nature of students' conceptions of units and unit relationships in the measurement system used by students as a framework through which to make sense of the units and unit relationships in the mathematical representation, and (3) the complexity of the unit relationships represented. It would not be sufficient for students, given a representation based on a particular measure system, simply to conceive of the units of this measure system independently of their use in the representation. They would also need to conceive of the representation as being based on that system of measure. Thus, students' conceptions regarding what constitutes a representation of different mathematical ideas, what the salient features of a mathematical representation are, and what attributes of a representation are mathematically significant, would be critical to the ways in which students interpret and construct mathematical representations.

There are two purposes to this study, given that students' interpretations and use of different forms of representations might depend on: (1) their prior knowledge of the conceptual structure underlying a form of representation, (2) their previous experiences with mathematical representations, and (3) the complexity of the unit relationships represented or to be represented. One purpose is to characterize the repertoires of representations that students use to explain relationships among units. A repertoire of representations refers to a set of representations which a student can reconstruct as needed. Do students have different forms of representations in their repertoires to explain mathematical relationships? Are there forms of representations commonly used in instruction which are not included in students' repertoires of representations? How do students represent multiplicative relationships between or among different units? The second purpose of this study was to characterize students' representations of units of length and area measurement as well as multiplicative relationships between different units of length or different units of area.

As Resnick (1983) states, "Learners try to link new information to what they already know in order to interpret the new material in terms of established schemata....[They] construct understanding,...look for meaning and will try to find regularity and order in the events around them." The events may include experiencing a variety of different forms of mathematical representations. Students, in making sense of these experiences, may interpret new forms of mathematical representations in terms of their prior knowledge about mathematical representations, and such interpretations may not conform to those intended by mathematics educators. Students' conceptions of representations and the representational process, as characterized by their repertoires of representations, would have implications for how they might construe mathematical ideas embodied in different forms of representations.

Research Questions

The following research questions were used to guide the investigation:

1. What are the characteristics of students' repertoires of representations of whole number multiplication?
2. What are the characteristics of students' repertoires of representations of common fractions?
3. What are the characteristics of students' representations and interpretations of units of length measurement?
4. What are the characteristics of students' representations and interpretations of units of area measurement?

Significance of the Study

Whole number multiplication and common fraction concepts and relationships were selected as the mathematical contexts within which to investigate the characteristics of students' repertoires of representations for a number of reasons: (1) because of their importance as components in the intermediate curriculum (Grades 4 to 7); (2) because they require a different way of thinking about units and representing relationships between units, compared to students' earlier, extensive experience with representations of additive relationships; and (3) because of the increasing significance of continuous measurement as a basis for representing these mathematical concepts and operations.

In the primary grades additive relationships of units constitute a major part of the curriculum. Comparing the numerosity of collections of objects, or adding and subtracting with whole numbers involve additive relationships of quantities measured with only one unit. However, in the intermediate grades, the curriculum shifts to an emphasis on multiplicative relationships (Hiebert & Behr, 1988; Vergnaud, 1983a). Both whole number multiplication and common fractions involve <u>multiplicative relationships</u> between different units. Two or more units are related proportionally or through a simple ratio. It is in this sense that Vergnaud (1983a) considers both multiplication and common fractions to be connected within a conceptual field of multiplicative structures. Instruction in whole number multiplication, which precedes formal instruction in common fraction concepts and relationships, is one of the first formal experiences children have of multiplicative relationships between different units. Common fraction concepts, relationships, and operations are another major part of the curriculum at the intermediate level.

In both of these mathematical contexts, representations based on discrete sets, on measurement of length, and on measurement of area are used in instruction. At least some of these forms of representations would have been used previously to explain simpler, additive relationships. However, with additive relationships, representations embodying number concepts and operations involve units which are all of the same order of magnitude. Representations used to embody additive relationships in the primary grades are extended to explain multiplicative relationships between different units, where one unit is an aggregate of another or, in other words, one unit measures a different amount than the other.

In addition to the increasing complexity of unit relationships inherent in the shift from additive relationships to multiplicative relationships, there is the increasing importance of instructional representations which rely on properties of continuous measurement for embodying these multiplicative relationships. As Osborne (1975) stated:

> ... measurement is ubiquitous, pervading all we do in mathematics and science. Within mathematics this is reflected by the fact that measure concepts are used so frequently as the intuitive base for instruction for nonmeasure concepts and skills. ... But the purpose of using these measure based embodiments is not to teach about measure; rather the instruction is directed towards nonmeasure teaching. (p. 37)

A critical assumption behind the use of representations based on continuous measurement is that students' conceptions of linear or area measurement are sufficient for these representations to provide an intuitive understanding of the mathematics being represented. In questioning this assumption, Osborne (1975) argues for the need to focus on the "nature of the measure base possessed by the students" (p. 39) as this might relate directly to their interpretations and use of such representations.

Studies of students' conceptions of length and area measurement suggest significant variability in the nature of students' understanding of these concepts in the intermediate grades (Bailey, 1974; Beilen & Franklin, 1962; Beilen, 1964; Hirstein, 1974; Hirstein, Lamb and Osborne 1975; Wagman, 1975). The collective evidence suggests that, for some students, limitations in understanding of linear or area measurement concepts persist throughout the elementary grades.

The nature of students' linear and area measurement concepts may play a role in the extent to which students use properties of linear and area measurement to represent mathematical ideas. On the other hand, regardless of their conceptions of linear or area measurement, students simply may not consider properties of linear or area measurement as a framework for thinking about and using representations of numerical relationships. Their previous experiences with discrete representations in which measurement of numerosity played a singular role in defining the form of representations may persist as their framework for thinking about all representations. However, with the introduction and development of rational number concepts and relations, the dichotomy between representations based on the measurement of numerosity and representations based on the measurement of continuous quantities is highly significant to the learning process. Representations of fractions based on continuous measurement would no longer convey the intended meaning of fractions if interpreted as discrete representations. At this stage, students who think about all representations as only collections of units without special space-filling characteristics would construct different meanings for common fractions than students who incorporate properties of linear or area measurement within their conception of the form of the representations.

Until recently, little research had been conducted which directly investigated students' knowledge and use of different forms of representations. Many studies were conducted which compare the effectiveness of representations within different instructional treatments (Fennema, 1972; Suydam & Higgins, 1977; Sowell, 1985); but none was designed directly to examine the students' knowledge of different forms of representations. A few studies have reported on students' interpretation or use of specific forms of representations in the common fractions context (Behr, Lesh, Post, & Silver, 1983; Bright, Behr, Post, & Wachsmuth, 1988; Hart, 1987; Hasemann, 1981; Novillis Larson, 1980, 1987; Payne, 1975), but these investigations did not explore students' personal repertoires of representation. Instead, they focused on the difficulties students have with interpreting forms of representations determined by others in a test or interview situation.

That different forms of representations might not be equally understood by students has been recognized. The number line has been identified as one form of representation which is particularly problematic for students to interpret and use in a wide variety of mathematical contexts, including the common fractions context. However, representations whose form is based on the area of regions or the numerosity of sets also may present interpretive problems to students, particularly when multiple units are embodied within the representation. (Behr et al., 1983; Bright et al., 1988; Dufour-Janvier et al., 1987; Hiebert and Tonneson 1978; Hunting 1984a; Novillis Larson, 1980, 1987; Payne, 1975; Post et al., 1985; Sowder, 1976: Vergnaud, 1983b)

As a result of observing the difficulties and misinterpretations associated with children's use of instructionally-imposed representations, Dufour-Janvier et al. (1987) assert that an instructional representation which is too distant from a child's internal representations renders the instructional representation inaccessible to the child, and "brings the child to manipulate rules and symbols that are meaningless to him." They propose that "there is a need to examine how children use objects, how they act, and the representations that they construct." (pp. 118-119) The exploration into the characteristics of students' repertoires of representations contributes to an illumination of the issues related to mathematical representations.

A premise which guided the formulation of this study was that students construct their own conceptions about representations, and that the nature of their representations is not well understood by researchers and teachers. The intention therefore was to identify characteristics of students' repertoires of representations which they use to explain whole number multiplication and common fraction relationships and thereby contribute to an understanding of students' conceptions of representations and the representational process. With the introduction of common and decimal fraction concepts and operations as major components of the curriculum during the intermediate grades, students' interpretations of representations based on continuous measurement become critical to the learning process. Students' representations of units of linear and area measurement and relationships between units were also investigated because of this increasing importance of continuous measurement as a basis for mathematical representations.

Definition of Terms

Beginning Diagram refers to an imposed framework, such as a partitioned line or region, within which the student constructs a final diagrammatic representation of mathematical relationships. (see Figures 3.01 and 3.02)

Dominant Form of Representation refers to the form of representation generated most frequently by a student.

Conception refers to the internal representations of a mathematical relationship.

Extensiveness of a Repertoire of Representations refers to the number of different forms of representations included in a student's repertoire or representations. The extensiveness was defined positively in terms of the forms of representations generated by a student, and negatively in terms of the forms of representations overtly rejected by a student.

Intermediate Level refers to students in Grades 4 to 7.

Interview, Generative Students were required to generate as many different representations as they could for any particular interview task.

Interview, Generative, Cued A collection of materials was available from which students independently selected materials with which to construct representations of the mathematical tasks. It was anticipated that the presence of the materials might cue some students to construct different forms of representations from those constructed in the uncued generative interview.

Interview, Generative, Uncued Students had no materials to suggest different types of representations which could be used to explain the mathematical tasks. They were presented with a mathematical task and asked to explain the meaning of the task through the drawing of diagrams or rough pictures. The questioning was designed to elicit as many different kinds of diagrams as the students could think of to explain the same task. All of the representations generated in this way represent a student's primary repertoire of representations of whole number multiplication.

Interview, Interpretive Students were required to interpret and evaluate a series of specific beginning diagrams as to their appropriateness as a framework for constructing a representation to explain the mathematical ideas. Students also were asked to construct representations of the mathematical task, using those beginning diagrams which they deemed to be appropriate.

Interview, Interpretive, Area An interview designed to investigate the extent to which students attend to properties of area measurement and other geometric characteristics as critical features in their representations. To this end, geometric and measurement properties associated with area-based representations were identified and systematically distorted within a series of beginning diagrams.

Interview, Interpretive, Linear An interview designed to investigate the extent to which students attend to properties of linear measurement and other geometric characteristics as critical features in their representations. To this end, geometric and measurement properties associated with linear-based representations were identified and systematically distorted within a series of beginning diagrams.

Mathematical Context refers to the general mathematical domains of interview tasks. The four mathematical contexts were whole number multiplication, common fractions, linear measurement, and area measurement.

Repertoire is defined in the Shorter Oxford English dictionary on historical principles as, "A stock of dramatic or musical pieces which a company or player is accustomed or prepared to perform; one's stock of parts, tunes, songs, etc." The term repertoire underscores the notion of a set of active productions, that which you may re-create and perform, not that which you simply may recognize in the performance of others. Applying the term in this mathematical context, repertoire has the sense of a stock of representations which a person is accustomed to reconstructing or prepared to reconstruct. Repertoires can have set-subset relationships. A person's repertoire of representations of common fractions is a sub-set of a person's repertoire of mathematical representations.

Repertoire, General..A general repertoire of representations refers to all forms of representation that a person evokes with or without external cues and prompting. The general repertoire is defined in both positive and negative terms, that is in terms of those forms of representations a person evokes as well as in terms of those forms of representations that a person rejects as a means of explaining the mathematics.

Repertoire, Primary A primary repertoire of representations of multiplication or common fractions refers to the variety of forms of mathematical representations that a person evokes spontaneously, in the absence of external cues or suggestions.

Representation When used without modifiers the term refers to what others have called "external representations, or embodiments" such as manipulative models and diagrams used to embody mathematical ideas.(see for example, Janvier, 1987) Written symbols also are considered to be external representations of mathematical ideas, but in this study they would be referred to as "symbolic representations," not "representations" alone. Internal representations" are referred to as conceptions.

Representation, Form of refers to the physical characteristics of a representation which distinguish it from other representations of the same mathematical ideas. Forms of representations are differentiated by the spatial organization and geometric characteristics of the units.

Representation, Function of refers primarily to students' interpretations of relationships between different units as expressed through their representations: that is, the extent to which a representation served to express relationships between different units.

Setting, Interview refers to material and procedural characteristics of an interview. It does not refer to the mathematical content of the interviews. The mathematical content was relatively consistent across all interviews, but the setting of the interviews differed.

Setting, Material refers to the objects, materials or diagrams available to the student with which to construct mathematical representations. This changed from one interview to another.

Spatial Framework By spatial framework of a representation is meant the general structure within which the units are represented, whether geometric regions, lines, or sets. Within a spatial framework, units may be represented in different forms. For example, a line may be used as spatial frameworks to represent discrete or contiguous units, or may be used to represent quantities which were not defined explicitly with units. Units within the spatial framework of sets are necessarily discrete.

Unit, Aggregate refers to a unit which is also a collection of smaller units. For example, in the place value context 1 ten would be an aggregate of 10 ones. This term was adopted in this study from Gal'perin and Georgiev (1960/69). Aggregate unit is synonymous to derived unit as used by McLellan and Dewey (1895), and composite unit as used by Steffe and von Glasersfeld (1983).

Unit, Contiguous refers to a unit which is touching or adjoining another unit.

Unit, Discrete refers to a unit which is spatially separate from other units.

Units, Bi-relational Whole number multiplication as well as rational number concepts minimally involve a relationship between two different units. These units are related one to the other by a simple ratio; one unit is either an aggregate or a part of the other. Such a relationship between two units has been termed bi-relational.

Units, Mono-relational A representation of unit relationships in which all units are treated as equivalent. Additive situations, such as comparing, adding, or subtracting whole numbers involve measures of only one unit.

Units, Tri-relational A representation of units relationships among three different units. In such a representation the three units would be "double-nested." That is, of the form A groups of B groups of C. In the multiplication context it would occur as the representation of the product of three factors. In the common fraction context it would occur as the representation of two fractional units nested within a whole unit, such as the representation of thirds and ninths within a single region.

Overview of the Plan of the Study

To identify the characteristics of repertoires of representations which Grade 5 and Grade 7 students use to explain whole number multiplication and common fraction concepts and relations, clinical interview methods were employed. Two pilot studies were conducted to develop and refine the interview tasks and procedures.

Six Grade 5 students and nine Grade 7 students from two schools were selected by their classroom teachers for the main study. The students within each grade were considered by their teachers to represent a range of school mathematics achievement.

Four interviews were designed to explore the characteristics of students' repertoires of representations. The interviews differed in their material setting. In the first interview the students had only paper and pencil available with which to construct their representations. In the second interview a variety of materials was available. In the third interview the students were presented with beginning diagrams related to properties of length measurement. In the fourth interview the students were presented with beginning diagrams related to properties of area measurement. In each of these interviews, students produced representations to explain some numerical mathematical tasks. It was assumed that by changing the material setting within which the students were asked to construct representations of mathematical tasks, forms of representations which students might not use in one setting might be used by them in another.

Data to explore the nature of students' representations of units of linear and area measurement were derived primarily from two interviews: one for linear measurement and the other for area measurement. One of these two measurement interviews directly followed the third or fourth interviews on representations.

Limitations of the Study

As an exploratory study, 15 subjects from Grade 5 and Grade 7 were selected to represent a range in achievement and a range in formal instructional experience in order to characterize variations in students' knowledge and use of mathematical representations of units and multiple unit relationships. It was not the purpose of this study to determine the extent to which such conceptions of mathematical representations are held by students in general. Rather, the study was designed to characterize possible conceptions of mathematical representations held by students in the intermediate grades. Thus, subjects were not selected as a representative sample of a population.

The characterization of students' conceptions of representations of units and unit relationships of measurement was derived from a categorization of students' responses to a limited number of tasks. The study investigated characteristics related only to students' translations of symbolic representations of whole number multiplication or common fractions into manipulative or diagrammatic representations. Additional characteristics of students' representations may have been derived if the tasks had involved real world problems rather than only symbolic expressions. The results therefore are limited to a characterization of these students' responses within the constraints of the tasks used and are not considered to be exhaustive.

Justification of the Study

Teachers use a variety of representations during mathematics instruction which are assumed to illustrate and clarify abstract mathematical relationships for students. During instruction it is assumed that the interpretations imposed on the representations by teachers are accessible to the students. That is, it is assumed that teachers and students share a common knowledge about the nature of the units and unit relationships in the representations such that the meaning of the mathematics being represented is interpreted appropriately by the students. If students hold conceptions of units which differ from those assumed by the teacher, the students' interpretation of the mathematics being represented may differ from the meanings assumed to be communicated by the teacher. This study characterizes students' representations of units and unit relationships in four mathematical contexts, each of which is an important component in the elementary mathematics curriculum. The nature of students' representations of units is likely to play a significant role in the ways in which they interpret representations commonly used in instruction. This study raises questions about the ways in which different representations, which are assumed to communicate ideas about mathematical relations, might be interpreted by students.

These questions are particularly critical at the intermediate grade level. When the number of relationships between units contained within representations increases, and when properties of linear or area measurement are used more extensively as the framework for constructing representations, the structure of instructional representations becomes more complex. These sources of complexity may decrease the accessibility of instructional representations for the students.

Organization of the Dissertation

There are five remaining chapters to this dissertation. Literature related to the study is reviewed in Chapter 2. The plan and implementation of the study is presented in Chapter 3. The results and discussion are reported in two chapters. Chapter 4 explores the the characteristics of students' representations of whole number multiplication and common fractions. Students' representations of linear and area measurement are explored in Chapter 5. The conclusions and implications for future research and instruction are presented in Chapter 6.

CHAPTER 2

REVIEW OF RELATED LITERATURE

The unifying focus of this study is the investigation of students' representations of units and unit relationships in a number of different mathematical contexts: whole number multiplication, common fractions concepts and comparisons, length measurement, and area measurement. As little research has been directed toward the issue of students' personal representations of mathematical relationships, the review of the related literature is divided into four main themes of discussion, each of which raised issues which contributed to the formulation of the study. Some of these themes draw upon theoretical discourses in the literature, while others draw upon some empirical studies. The four themes are:

1. The construct of repertoires of representations.
2. The nature of mathematical representations.
3. The role of measurement as a framework for representing mathematical relationships. Included in this theme are two issues:

   (1) representations of multiplicative relationships between units, and

   (2) students' conceptions of units of continuous measurement.
4. Students' representations of whole number multiplication and common fractions. Included in this theme are two issues:

   (1) The nature of student-generated representations.

   (2) Students' interpretations of instructional representations.

Repertoires of Representations

Although little was said a decade ago about students' representations of mathematics, much attention was paid to the need to use manipulative and diagrammatic representations during instruction. Core features of elementary instruction were that teachers should use "a wealth of manipulative experiences through which concepts and relations are understood at an intuitive level," and that "mathematics as a discipline, as a formal structure, must be built upon a sound foundation of concrete experiences" (Province of British Columbia, 1978, p. 1). The National Council of Teachers of Mathematics (1989) stated as part of their "Standard 2: Mathematics as Communication" that:

> In grades 5 to 8, the study of mathematics should include opportunities to communicate so that students can model situations using oral, written, concrete, pictorial, graphical and algebraic methods. (p.78)

Where, in 1978, the British Columbia Ministry of Education spoke of manipulative and concrete experiences as simply a means to attain an objective, the objective being the formal study of mathematics, in 1989 the N.C.T.M. speaks of students' use of concrete, pictorial or graphical representations of mathematics as objectives in themselves. The N.C.T.M.'s statement about the importance of these forms of communication closely resembles the model of "representation systems" proposed by Lesh (1979). Lesh argued that students should be able to translate mathematical ideas between and within all modes of representations, namely real world situations, oral and written language, manipulative-concrete, pictorial or graphical, and algebraic or symbolic methods. The assumption behind this model is that students should be able to both interpret and construct mathematical representations in and between these modes of representation. The representational modes are to be considered both expressive and receptive (Clements & Lean, 1988).

In this study the focus is on the nature of students' expressions of mathematical ideas as they are represented with concrete materials, pictures or diagrams. As was explained in Chapter 1, the term representations is used in this report to mean concrete or diagrammatic expressions unless otherwise qualified. The term "repertoire of representations" was defined by the investigator of this study to refer to a set of representations which a student could construct as a way to explain a mathematical concept, relationship or operation.

Proponents of instruction of mathematical concepts or relationships with multiple forms of representations, such as Bruner (1968) and Dienes (1967), were some of the sources which influenced the thinking about students' representations and questions about possible repertoires of representations in this study. Bruner (1968) proposes that "by giving the child multiple embodiments of the same general idea expressed in a common notation we lead him to "empty" the concept of specific sensory properties until he is able to grasp its abstract properties" (p. 65). The "Perceptual Variability Principle" elaborated by Dienes (1967) expressed the same belief in the role of multiple embodiments in the learning of abstract mathematical concepts. Dienes argued that "the same conceptual structure should be presented in the form of as many perceptual equivalents as possible" in order "to allow as much scope as possible for individual variations in concept-formation, as well as to induce children to gather the mathematical essence of an abstraction." (p. 32). The goal motivating these proposals for instruction with multiple forms of representations was to create an environment which would facilitate children's abstract mathematical thinking. However, both Bruner and Dienes observed other phenomena in relation to children's thinking with and about different forms of representations.

Bruner (1968) and Bruner and Kenney (1965) report that, when children explored mathematical relationships with multiple forms of representations, they were observed to have constructed "a store of concrete images that served to exemplify the abstractions." They reached the tentative conclusion that, "it is probably necessary for a child learning mathematics not only to have as firm a sense of the abstraction underlying what he was working on but, also, a good stock of visual images for embodying them. For without the latter, it is difficult to track correspondences and to check what one is doing symbolically." (Bruner, 1968, pp. 65-66; Bruner & Kenney, 1965, pp. 56-57) The "stock of visual images" is analogous to a "repertoire of representations" as used in this study.

Bruner and Kenney (1965) also reported that children were observed to have "'equated' concrete features of one form [of representation] with concrete features of another" as a means of making sense of new forms of representations (p. 57). This would suggest that analogies between familiar and new forms of representations may play an important role in the way in which children interpret different forms or representations, regardless of critical differences in the physical attributes of each form of representation. Furthermore, Dienes (1964) suggested that, through such analogous thinking about different forms of representations, students might come to conceive of one situation as a prototype of all others in their previous experience; that is, that one representation might come to serve as a "set of symbols" for their repertoire of representations (p. 142).

Dienes' association of students' multiple representations with a prototype or representative exemplar is similar to Vinner's and Hershkowitz' construct of a concept image (Vinner & Hershkowitz 1980; Hershkowitz & Vinner 1984; Hershkowitz, 1987). The construct of a concept image was derived and elaborated through a number of studies which explored students' and teachers' construction or identification and interpretation of representations of geometric concepts.

Vinner and Hershkowitz argue that a concept image develops from a collection of real diagrammatic or manipulative experiences. A student's interpretation of the significance to be attached to features common to these collective experiences serves to construct a generalized concept image. In other words, a process is involved whereby salient features among a variety of representations are defined, attended to, and used by a student to construct general concept images. Furthermore, they suggest that a student's perception of which features are salient to the task at hand may include non-critical features or exclude critical features, and thereby result in a concept image which is distorted or incomplete compared to the intended objectives of instruction. For example, for many students, the concept image of obtuse angles included only angles with a horizontal ray. The horizontal characteristic of the ray, though a non-critical feature of obtuse angles, was interpreted as a critical attribute of these students' conceptions of obtuse angles. Even when a formal definition of a concept such as the altitude of a triangle was provided, both elementary teachers and students were found to respond on the basis of a concept image rather than on the basis of the defined attributes of the concept. The predominant concept image of altitudes of triangles was limited to images in which the altitude fell within the interior of a triangle. A concept image appeared to be unrelated or indirectly related to a formal definition of the attributes of a concept. (Vinner & Hershkowitz, 1980; Hershkowitz & Vinner, 1984)

Hershkowitz (1987) related concept images to that of Rosch's (1977; 1978) formulation of prototypes of categories. Hershkowitz stated,

> Every concept has a set of critical attributes and a set of examples. In the set of concept examples there are the "super" examples: - the prototypes; that is the popular examples. In other words, all the concept-examples are mathematically equal, because they conform to the concept definition and contain all its critical attributes, but they are different one from the other psychologically. (p. 240)

In these terms, the set of examples were those defined from a mathematical point of view. From the students' point of view, often only a subset of the examples was associated with a concept. This subset was considered to contain the "super" examples or the prototypes of the concept.

A distinction needs to be made between people interpreting someone else's representation and people constructing their own representation. Tasks in which a person need only recognize or identify representations are less complex than those in which a person must reconstruct representations (Clements, & Del Campo, 1987; Sinclair, 1971b). Vinner and Hershkowitz used both forms of data to support their construct of concept images. However, they relied more heavily upon data which derived from students' and teachers' interpretations and judgements of the correctness of a series of predetermined representation rather than data derived from the students' and teachers' own representations of a concept. There was some evidence of concurrence between the image most commonly associated with a concept in the interpretive and constructive condition. The distinction between interpretation and construction of representations is significant if one is concerned about a repertoire of representations. Physical images which people may recognize as a representation of a mathematical idea still may not be evoked by them independently. What a teacher's or student's repertoire of physical images would be was not investigated.

Vinner and Hershkowitz focused primarily but not solely upon the interpretation of representations rather than their construction. However, a concept image is defined as "all mental pictures and associated properties and processes of a concept built up by a person over the years through experiences of all kinds" (Tall & Vinner, 1981). As such a concept image can seen to include the notion of a repertoire of representations of some mathematical idea. A repertoire, rather than a single representation, is assumed in this study to illuminate more completely a student's conception of the mathematical concept under consideration.

There are a variety of other terms used to denote constructs similar to that of a concept image, as well as variations in the meanings associated with these terms. Janvier (1987) speaks of schematizations within a representation. Schematizations are the variety of ways in which a mathematical relationship may be expressed externally, and "representation" is used as synonymous with the term "conception" in this study. Included within the schematizations would be a repertoire of physical representations evoked to explain a mathematical relationship. Dufour-Janvier, Bednarz and Belanger (1987) make a distinction between external representations and internal representations in the context of discussing manipulative and diagrammatic instructional representations. Their external/internal dichotomy would parallel the use of representations and conceptions in this study. Further, Lesh, Post, & Behr (1987) use a model of modes of representations (written representations, symbolic representations, representation through spoken language, static pictorial representations, and manipulative representations). They define conceptual understanding in terms of translations within and between these modes. In such a case, "representation" is used to refer to all expressions of a person's conceptions. Despite apparent differences in these uses of terms, there are a number of assumptions shared by these constructs. These include a constructivist view of cognitive functioning, a dichotomy between internal and external representational forms, and variabilities within internal and external representations. None would assume that there would be a single representation which might capture all facets of a person's conception of a mathematical idea or which would serve as a concept definition. In all of these cases, repertoires of manipulative and diagrammatic representations, explored in this study, would fit as a sub-structure within the proposed models.

From the observations of Dienes, and Bruner and Kenney, there are a number of elements which might characterize students' repertoires of representations. First, a repertoire might consist of a variety of "concrete images" of a particular mathematical relationship which students have constructed through their experiences with different forms of representations. Second, the variety of "concrete images" might be equated by the students in terms of their analogous features. Third, there might be a particular form of representation which could be considered to act as a prototype of their repertoire of representations. Such prototypes might be derived by students analogously equating common features of the representations in their repertoire. The prototypes would then serve as a generalized form for representing the mathematical relationship.

Where Bruner, Kenney and Dienes were concerned with the learning of mathematical concepts or relationships which were not intrinsically geometric but for which geometric representations might be constructed, Vinner and Hershkowitz were concerned with geometric concepts which were directly representable as objects or diagrams. The differences in their formulation of "concrete images" as opposed to "concept images" appear to be more a question of levels of abstraction rather than a question of competing premises about representations.

The Nature of Mathematical Representations

Post (1980) defines manipulative materials and diagrams as partial isomorphisms or isomorphic structures which represent the more abstract mathematical notions children are to learn. He states that,

> ...if a parallel structure that was more accessible and perhaps manipulable could be identified having the same properties as the set of whole numbers, then it would be possible to operate within this accessible (and isomorphic) structure and subsequently make conclusions about the more abstract system of number. (p. 113)

The assumption is that the intervention of physical representations in the learning situation provides a concrete environment within which children may think and learn about mathematical relationships. The other assumption is that such concrete representations are accessible to children.

On the other hand, Post (1980) also acknowledges that these physical interpretations or embodiments of abstract mathematical notions are also "artificially constructed systems" (p. 113). The artificial construction of physical representations as models or embodiments of mathematical relationships implies that they are themselves abstractions relative to real world situations. They are meant to "simplify and generalize" mathematical relationships interpreted from real world situations (Lesh, 1979). In other words, they are "generally a simplified version of the original, which permits an easier and more complete control of a set of variables" which may serve to facilitate the interpretation of certain given facts (Fischbein, 1977, p. 155). Physical representations could be considered one level of abstraction from which more abstract representations of concepts and operations are derived by children. As Von Glasersfeld (1983) stated,

> Concepts and operations involved in mathematics are not merely abstractions, but most of them are the product of several levels of abstraction. (p. 64)

How accessible different physical representations are to children is a question considered by numerous authors. (E.g., Dufour-Janvier, Bednarz, & Belanger, 1987; Ernest, 1985; Fischbein, 1977;

Bell & Janvier, 1981; Sowder, 1976; Vergnaud, 1983b, 1984) Underlying this question is a recognition that some forms of physical representations are more abstract than others. Furthermore, there is a recognition that different types of physical representations are based on different assumptions, rules, and conventions regarding what and how attributes of the representations embody and convey mathematical meanings. There is also a recognition that children must understand the assumptions, rules, and conventions associated with a particular form of representation in order to interpret or use the physical representation appropriately.

As an example, Fischbein (1977) states that visual images used to represent mathematical relationships, such as Venn diagrams, tree diagrams, and graphs, are not primitive images which map directly to real world situations. Instead, he terms them "conceptualized images, controlled by abstract meaning" (p. 154). For example, the image of a curve on a graph is at least a third-order abstraction. Its production involves translations from the physical situation of the function to pairs of numbers, to the plotting of points, to the drawing of the curve. Fischbein argues that "the child has to learn the language of the image, not relating it *directly* to the physical process, but indirectly to the conceptual language of coordinates. A short circuit between the graph and the original, physical phenomenon, will result in misunderstandings" (p. 154).

Doufour-Janvier et al. (1987) presented a similar argument about using an abacus to represent the place value system. An abacus does not directly represent the image of the successive groupings in the place value system. Attributes such as colour and position are used to distinguish the fact that one disc represents "one" and another disc represents "ten." The basis for the grouping rule is not apparent in the representation, and unless its meaning is understood the operations illustrated on the abacus would not be understood in terms of the place value system. The child would need to interpret the form of the representation in terms of the meaning of its structure and language in order to apprehend the place value relationships and operations represented. In conclusion, Dufour-Janvier et al. (1987) stated,

> The use of representations that are just as abstract for the child as that which is studied brings the child to manipulate rules and symbols which are meaningless to him....The child is forced to "learn" the representation that is submitted to him: the rules of usage,

> the conventions, the symbols, and the language linked to the representation....The use of such nonaccessible representations encourages a play on symbols, puts the emphasis on the syntactical manipulation of symbols without reference to the meaning. (pp. 118-119)

A physical representation, based on a framework such as a number line, an a x b rectangular model, an abacus, or base ten blocks, can be thought of as a duality comprised of (1) the representation's form with associated attributes, structure and language, and (2) the representation's function as a potential representation of diverse mathematical relationships. Similar dualities have been proposed in the analysis of symbolic representations of mathematics (Byers & Erlwanger, 1984; Hiebert, 1984; Resnick & Omanson, 1987; Skemp, 1982). Byers and Erlwanger (1984), analyzing mathematics in terms of its form and content, state that "the content of mathematics consists of ideas embodied in its methods and results; mathematical form includes symbolic notation and chains of logical arguments (Byers & Herscovics, 1977)" (p. 260).

Byers and Erlwanger (1984) acknowledge a further, subtle distinction in the analysis of mathematics in terms of form and content. Using place value notation as an example, they argue that, on the surface, this notation would be considered to belong to the form of mathematics in contrast to operations which would be considered to be the content. Yet when learning two-digit addition, place value notation becomes the content to be learned. They conclude that mathematical forms have content of their own. The understanding of the content of mathematical form is important to be able to use the mathematical forms appropriately to represent other mathematical content. In other words, there is a duality within a duality.

Byers and Erlwanger analyzed symbolic notation in terms of form and content, but their arguments parallel that of Fischbein's (1977) analysis of the nature of physical representations. Physical representations also have their own content independent of the mathematics which is represented. This content needs to be understood by children in order for the mathematical relationships embodied in the representations to be accessible. In this context, accessibility refers to children being able to interpret the meaning of attributes of the form of the representation in order to interpret the mathematics represented. Accessibility also refers to children being able to use these physical representations to embody their own mathematical ideas. The duality of physical representations has been used as a major framework for considering the characteristics of students' representations in this study. This duality is expressed in terms of the form of a representation and the function of a representation as an embodiment of a mathematical relationship. However, that the form of a representation has content independent of the mathematics represented is also a significant analytical element within the framework of the current study.

Measurement as Content of Forms of Mathematical Representations

> The concept of number would not even exist if man had not met problems of measurement;...Numerical situations usually deal with quantities that are not pure numbers but magnitudes of various kinds....Numbers are undoubtedly central in mathematics but it is impossible to understand what difficulties children meet with numbers if you do not look at them as magnitudes of different sorts, transformations, or relationships.
> (Vergnaud, 1980, pp. 264, 265)

Implied in Vergnaud's assertion is a broader meaning of measurement than that which is normally associated with measurement in elementary mathematics teaching. It implies not only measured magnitudes of continuous quantities, but also measures of discrete quantities. That the common distinction between counting objects and measuring continuous quantities is a false dichotomy has been argued by others such as Blakers (1967) and McLellan and Dewey (1895). Blakers refers to "the measurement of numerosity," stating that,

> the ideas involved in the empirical measurement of numerosity ... are so simple, and the structure of the most appropriate value set (the set of positive integers, whose prehistorical development was undoubtedly due to the empirical properties of the measurement operation known as 'counting') is so closely related to the attribute that we wish to measure, that numerosity measurement is ignored in many treatments of the general subject of measurement. (pp. 77-78).

In a similar vein, McLellan and Dewey (1895) asserted that

> all counting is measuring, and all measuring is counting....When we count up the number of particular books in a library, we measure the library....The only way to measure weight is by counting so many units of density; distance by counting so many particular units of length;... (p. 48)

It has long been recognized that "the character of a child's first encounters with numerousness is in the sense of measure applied to discrete sets of objects" (Osborne, 1975, p. 38). The use of continuous measure systems as bases for representations of mathematical relationships and operations is qualitatively different from children's earlier experience with discrete measures because of the additional attributes involved in defining the properties of the units. For example, the space-filling characteristic of units of area, the definition of units of length as congruent line segments, as well as the contiguity of units of length and area are particular attributes which differ from discrete units (Osborne, 1975). Nonetheless, measurement of discrete objects, length, and area are used as a basis for representing numerical operations and relationships in elementary mathematics curricula without necessarily attending to the qualitative differences between properties of the units and the attributes being measured.

The lack of explicit attention to the qualitative differences between the attributes and properties of the units in different forms of representations is to be found in elementary school textbooks. In Eicholz, O'Daffer, & Fleenor, (1974), both the number line and discrete sets are used to represent the operation of multiplication. The distributive property of multiplication is introduced with representations based on the area of rectangles, then represented with discrete rectangular arrays at the Grade 3 level. Yet, each of these forms of representations is associated with a common, abstract language, "X of Y," without referring to other attributes of the units. Similarly, at the Grade 4 level, common fraction concepts and relations are introduced primarily with representations based on area measurement, and secondarily with discrete sets. Yet, both forms of representations are associated with the same language, "X out of Y parts." Later, representations of common fractions on the number line are used extensively. Notions of distance on the number line is associated cursorily with the ruler and running races, but in the exercises only the language of points is used: explicit reference to distances from the origin, or units of length do not occur. In the previous examples, the assumption appears to be that the student will discriminate intuitively between the properties of units in different forms of representations. As Osborne stated:

> It is assumed that measure is intuitive and sufficiently possessed and understood by the learner to serve as an intuitive representation for explaining numerical operations. This assumption should be questioned. (Osborne, 1975, p. 42)

There are several issues, related to this study, which arise from the fact that both discrete and continuous measurement are used as a basis for representations of whole number multiplication and common fractions. One issue is how students conceive of one unit to be a multiple or a part of another unit regardless of the basis of a representation. A second issue is whether students have an intuitive understanding of the measurement of length or area with which they might interpret and construct continuous measurement representations appropriately. The literature related to each of these issues will be discussed in turn.

<u>Multiplicative Relationships Between Units</u>.

As early as 1895, McLellan and Dewey argued that number represents a ratio or measure of a unit in relationship to a quantity or "unity." Furthermore, the unit should be conceived of also as a quantity measurable by smaller units.

> More than half of the difficulty of the teacher in teaching, and the learner in learning, is due to the misconception of what the unit really is. It is not a single unmeasured object; it is not even a single defined or measured thing; it is *any measuring part* by which a quantity is numerically defined;.... (p. 158)
>
> As necessary to the growth of the true conception of unit as a measuring part, the idea of the unit as a unity of measured parts must be clearly brought out. The given quantity is measured by a certain unit; this unit itself is a quantity, and so is made up of measuring parts. This idea must be used from the beginning; it is absolutely essential to the clear idea of the unit, and of number as measurement of quantity. (p.160)

They expressed the view that, by limiting representations of numbers to ones in which one object within a set always represented the unit, the opportunity for children to construct a comprehensive understanding of the meaning of units, relationships between units, and subsequently the meaning of number is artificially restricted.

> Count by ones, but not necessarily by single things; in fact, to avoid the fixed unit error, <u>do not begin with counting single things</u>. (McLellan & Dewey, 1895, p. 160)

From an experience of variable representations of units used to express measures of quantities numerically, McLellan and Dewey argued that children would conceive of numerical concepts in both additive and multiplicative terms. <u>Derived units</u> were made up of a number of smaller units, called <u>primary units</u>. These units were to to be central to children's earliest, formal experiences of number concepts and relationships. From these experiences of the meaning of number, the operation of multiplication, therefore, was represented as "the complete expression of any measured quantity [through] (1) the derived unit of measure, (2) the number of such units, and (3) the number of primary units in the derived unit of measure" (p. 112). Also, from these initial experiences of units and number, the "measure meaning" of common fractions would follow as an expression of a relationship between two variable, designated units of measure. It should be noted that McLellan and Dewey's formulation of common fractions as measures is more general than that defined by Kieren (1975). Kieren refers to one of seven possible interpretations of rational numbers as "measures or points on a number line" (p. 103), McLellan and Dewey refer to any measures of discrete and continuous quantities.

Even Thorndike (1924) gave explicit support to McLellan and Dewey. He argued that "desirable bonds" would be neglected if the numbers were not

> connected ... each with the appropriate amount of some continuous quantity like length or volume or weight, as well as with the appropriate sized collection of apples, counters, blocks and the like.... Otherwise the pupil is likely to limit the meaning of, say, four to four sensibly discrete things and to have difficulty with multiplication and division. (p. 75)

During the 1950's McLellan and Dewey's (1895) ideas on the role of units and unit relationships on the child's development of number concepts and relationships was a significant influence on the work of Gal'perin., & Georgiev (1969) in the Soviet Union. In that study, McLellan and Dewey's analyses are seen to be directly relevant to students' representations of relationships among units.

In the educational setting of this study, whole number concepts and relationships, as well as addition and subtraction, are introduced primarily, if not solely, through the counting of individual units. Experience with units as aggregates arises for students, at the primary level, in the context of teaching place value notation, and the operations of whole number multiplication and division, and at the intermediate level, with common fraction concepts and relations. (Province of British Columbia, 1987)

That students at the intermediate grade level have difficulty interpreting units as aggregates in the common fractions context is attested to by several reports in the literature. Behr, Lesh, Post and Silver (1983) report results of a study in which Grade 4 students, after normal classroom instruction, were tested on the representation of common fractions with regions, number lines, or discrete sets.

Students had more difficulty completing the task when the number of units presented was a multiple of the denominator, than when the number of units presented was equal to the denominator, or when the presentation of the units was incomplete. For example, given three rectangles, (a) one with four equal parts marked, (b) one with one part equal to a fourth marked, and (c) one with eight equal parts marked and asked to represent 3/4, there was an order of increasing difficulty from a to c. The idea of a unit, which results in a measure of four, being an aggregate of two smaller units, was considerably more difficult for students to conceive.

Behr et al. (1983), reporting on responses to similar tasks in clinical interview settings during their teaching experiment, found similar difficulties in the ability of students to interpret representations in terms of aggregate units. For example, some students were unable to consider a fourth of a circle partitioned into three as a representation of one-fourth even though they were able to explain that an equivalent unpartitioned sector in the same circle was one-fourth.

Bright, Behr, Post and Wachsmuth (1988), reporting on a part of the same teaching experiment project directed towards the identification and ordering of common fractions on the number line with students in Grades 4 and 5, concluded that "when a representation is given in unreduced form, students have difficulty choosing the correct reduced symbolic fraction" (p. 227). Similar results were reported for Grade 7 students by Novillis-Larson (1980). Comparing subtest scores of different types of number line tasks, Novillis-Larson found that unreduced representations were more difficult to interpret than reduced representations.

Collectively, these studies suggest that some intermediate grade students may conceive of units generally as only single objects or parts; that is, students may not conceive of units as aggregates in other numerical contexts besides the common fraction context. No studies were located which explored students' representations of units in the whole number multiplication context. The question remains whether some students construct a "fixed unit error," as McLellan and Dewey (1895) termed it, based on their extensive experience of number represented with units as only single objects or parts.

Conceptions of Units of Length and Area.

Carpenter (1975a) in his review of research on measurement, observed that most studies had focused on the development of the pre-measurement concepts of conservation and transitivity, and that fewer studies had dealt with measuring (p. 47). Furthermore, many of the studies which investigated children's conceptions of measurement and units of measure, or the acquisition of measurement concepts, have been conducted at the primary grade level (Bailey, 1974; Beilen & Franklin, 1962; Carpenter, 1975b; Carpenter & Lewis, 1976; Daehler, 1972; Heraud, 1987; Hiebert, 1981; Sinclair, 1971a). Few studies have been identified, at the intermediate grade level, which investigated students' conceptions of units of measurement and their thinking about the measurement process.

Beilen and Franklin (1962), conducted a Piagetian training study involving the acquisition of logical operations in area and length measurement, with students from Grades 1 and 3. On the basis of the full results of the study, they concluded that length and area measurement are achieved in successive order and that, "the constituent operations of measurement (transitivity, subdivision, change in position etc.) are applied more easily first to a single dimension, then to two dimensions, and then to three" (p. 617). At the Grade 3 level, 27 percent of the students were at a concrete operational level for area measurement, and 82 percent were at a concrete operational level for linear measurement. These findings would suggest that at the intermediate level, students would be more likely to have a conception of linear measurement that would be consistent with properties of that measurement system than would be the case with area measurement.

Linear Measurement.

Beilen and Franklin's (1962) results suggest it is likely that students in the intermediate grades would conceive of linear measurement in terms of unit iteration. On the other hand, the linear measurement tasks in their study required students to iterate a single unit to compare quantities of length. They did not require students to use compensatory reasoning about the inverse relationship between the size of the unit and the value of a measure of a quantity of length, or identify and interpret units on a line. In contrast, there are two studies, Bailey (1974) and Hirstein, Lamb, and Osborne (1978), which indirectly suggest that students at the intermediate grade level may not consistently conceive of units as congruent line segments in a length measurement context, or may not conceive of the logical, inverse relationship between the size of the unit and the value of a measure of a quantity of length.

Bailey (1974) conducted a study in which children from Grades 1 to 3 were required to compare polygonal paths in which the size and number of line segments were varied. Four tests were individually administered to students judged to be in the top two thirds of Grades 1, 2, and 3. Test 1 involved two symmetric paths with an equal number of segments but unequal segment length; Test 2 involved two paths with equal segment lengths but unequal numbers of segments; Test 3 involved two paths in which one path had longer but fewer segments than the other, and Test 4 involved two paths with segments of equal number and length. None of the students was able to come to a logical conclusion in Test 3. With respect to the other tests, Bailey reports that only three of the 90 students, all of them Grade 3 students, indicated an ability to use the dimensions of length of units and number of iterations simultaneously to establish the length relationship between two polygonal paths. The other children centred either on the number of segments, or the length of the segments to explain their comparisons. He concludes that "perhaps children older than nine years will experience the same problem" (p. 524).

Hirstein, Lamb and Osborne (1978) conducted a study in which 106 children from Grades 3 to 6 were interviewed on a variety of area-measurement tasks designed to investigate how children incorporate number into their area measurement judgements. In one set of tasks, the rectangles were marked along the dimensions to indicate a grid. In this context, they observed that students frequently counted points rather than line segments along the dimensions. Although they did not report whether this behaviour occurred more frequently with younger children, or was observed with equal frequency throughout the grades, they concluded that these children had no sense of a linear unit.

These studies suggest that some students at the intermediate grade level are likely to have restricted conceptions of units of length, and have difficulty logically coordinating the inverse relationship between size and number of units. These factors would influence the way in which students interpret and use mathematical representations based on length measurement.

Area Measurement.

There are two aspects of area measurement and students' conceptions of units of area which relate to students' interpretations and use of representations based on area. The first aspect is how students interpret and use units to compare the area of region, and the second is how students construct equal units within a region. The literature related to these two issues will be discussed separately.

Comparing Areas of Regions. Beilen (1964) conducted a study with students from Kindergarten to Grade 4, in which he evaluated their ability to make equality and inequality judgements of areas inscribed with square units. The equality tasks involved the comparison of a 2 X 2 or 3 X 3 square with an irregular polygon of the same area. To resolve the task, the student would have to either compare the number of units in each region or imagine the transformation of one region into another. Some students who failed to determine the equalities established that the numbers of "boxes" were equal but still judged the space to be unequal because a piece was "sticking out." Only 50 percent of the Grade 4 students determined the equality of the regions with four square units, and only 38 percent did so with the regions with nine square units.

Wagman (1975) found, with a unit-area task in which the measure of a quantity was determined by tiling different units, that 42 percent of 11-year olds (Grade 5) were able to understand consistently the notion of the unit, take the differences in the size of the units into account, and predict the measure with a unit given its relationships to another larger or smaller unit. The percentage of younger students who were able to reason consistently about the unit relationships decreased with age.

In Wagman's study the two units were a square and a triangle which was half the size of the square. Once the area measure was determined by the square, it was possible for children to reason that two triangles were equal to a square, and therefore the measure would be double with the triangular units. In contrast to this task, Hirstein, in an unpublished pilot study in 1974 (Steffe & Hirstein, 1976) used a task in which two units, which were equal in area but not congruent, were tiled within congruent rectangles, that is, six of each unit tiled a rectangle. This task would require the student to either reason from the premise that an equal number of measures of the same quantity implies equal units of area, regardless of incongruency, or use imagined transformations to compare the two units. With this task no students below the age of 10 were able to determine that the non-congruent units were equal in area.

Bergeron and Herscovics (1987) interviewed students from Grades 3 to 6 with a task similar to the one used by Hirstein (Steffe & Hirstein, 1976). The students were asked to compare the area of one fourth of two congruent squares when the parts in one square were not congruent to the parts in the other. Even after students had established that one part in each square was one fourth of the square, some students in all grades (7 out of 13 in Grade 6) later used visual transformations of one unit into another to justify their judgements that the parts in both squares were equal in area. The numerical reasoning used to name the fraction of each square was not related to reasoning about the comparison of the area of the non-congruent parts.

Hirstein, Lamb and Osborne (1978) report a number of alternative strategies which students from Grades 3 to 6 used to compare quantities of area. They interviewed 106 students to study how children incorporate number into their comparative judgements of areas of regions. They observed that centering on only one dimension was a strategy used by some students at all grade levels. Others used primitive compensation methods, reasoning that one region is longer but the other region is wider. Partitioning and re-combining was a third alternative strategy used which did not involve units.

These studies all indicate that in the intermediate grades, children's notions about units of area, and the relationship between the number of units, the size of the units, and the measure of a quantity might vary greatly. That students incorporate number into their reasoning about comparisons of area, particularly when units are not congruent, cannot be assumed.

Partitioning Regions. Investigations into the development of children's conceptions of equal subdivisions of areas have revealed a complex process in which differences in the number being considered, and differences in the shape of the regions being partitioned, affect the success of young children's partitions. Rectangles were found to be easier to subdivide than squares and both were easier to subdivide than circles. Subdivisions of powers of two (e.g 2, 4, 8 ...) are generally easier to construct than subdivisions of odd numbers. (Hiebert & Tonnessen, 1978; Piaget, Inhelder & Szeminska, 1948/1964; Pothier & Sawada, 1983)

Piaget, Inhelder and Szeminska (1948/1964) found that subdivisions of six were constructed successfully by children at a much later age than subdivisions of three and subdivisions of powers of two, a result which they found surprising (p. 326). This delay may be explained by the findings of Pothier and Sawada (1983). They found that "number-theoretic" concepts such as even and odd are critical factors in children's differentiation of algorithms for partitioning with different values. Odd numbers eventually are recognized as "hard," become disassociated from the algorithm of successive halving, and children anticipate the need to use different partitioning strategies with these odd values. On the other hand, the even-odd differentiation is insufficient to accommodate the subdivision of an area into an even number of parts which is not a power of two. For example, although six is an even number, one of its factors is odd; successive halving therefore will not result in six equal parts. Not only must children eventually differentiate between even and odd number, but also they must differentiate between even numbers that are powers of two and those that are not. Pothier and Sawada hypothesized that at a later level children, to become efficient partitioners, would have to construct a multiplicative algorithm for partitioning with factors. This algorithm would allow for the efficient partitioning of areas by all composite numbers which are not powers of two. Whether students at the intermediate grade level use number-theoretic concepts as a basis for partitioning, along with other partitioning strategies, is unclear.

A distinction must be made between knowing or not knowing techniques for achieving any number of equal partitions and believing that parts are equal when they are not (Streefland, 1978). Students may believe that the parts are equal in area when they are not, and by extension believe that that a technique which results necessarily in unequal parts is adequate. On the other hand, they may be aware of the inadequacy of their technique and the resulting inequality of the area of the parts, but may be unable to devise an alternative technique to achieve their goal. Hence, there are two factors involved in successfully partitioning regions into parts of equal area. The first is the understanding of what constitutes parts of equal area in different types of regions. The second is the understanding of techniques for constructing any number of parts of equal area in different types of regions. Both involve conceptions of geometric characteristics of different regions. But "number-theoretic" concepts are related primarily to the development of partitioning techniques. Students' conceptions of what constitutes parts of equal area may not be reflected fully in the partitioning techniques available to them.

Students' Representations of Whole Number Multiplication and Common Fractions

There are few studies which have explored, either directly or indirectly, the forms of representations students generate independently to illustrate or explain symbolic mathematical expressions. The studies that have been located deal with students' representations of common fraction concepts or operations, not with whole number multiplication. More common are studies in the common fractions context which report ways in which students interpret and use different forms of representations when the representation used was determined by someone else in a test or interview situation.

Student-Generated Representations.

Previous studies in which students were asked to represent common fraction concepts or operations suggest that parts of regions, particularly parts of circles, is a pervasive image. Peck and Jencks (1981) report the results of interviews with 20 Grade 6 students in which the students were asked to explain their ideas about common fraction concepts, relations, and operations with the use of diagrams. They also report general observations based on a large number of interviews of students in Grades 6, 7 and 9. The forms of representations constructed by students were invariably based on parts of a region, and the "large majority of students used pies (circles) as their only model for fractions" (p. 347). Clements and Lean (1988) report that, of the 59 students from Grades 4, 5 and 6 interviewed about the meaning of unit fractions, all constructed representations based on regions and all but one used either circles or squares. These results are similar to those found by Silver (1983) when interviewing community college students. Fifteen of the 20 college students first represented a common fraction with circles, and 10 of these students could not extend their thinking to other images. In addition, Silver found that the more restricted the student's representations of common fractions were, the less able the student was to comprehend numerical relationships between and operations with common fractions.

The students in Peck and Jencks' study based common fractions representations on parts of regions, but over half did not interpret their representation in terms of measures of area. For these students, the number of parts determined by the denominator was the criterion used for comparison. In these cases, Peck and Jencks observed that many students had consistent but incorrect strategies for partitioning the circle which resulted in unequal parts. They also did not believe that the parts ought to be equal. A lack of adequate partitioning techniques alone did not account for the form of their representations. Clay and Kolb (1983) reported that some Grade 4 students, even when they constructed representations of common fractions with equal parts of a region, did not believe in the need for equal parts. One student, when asked about why he drew equal parts after he agreed that unequal parts were adequate in a representation, replied, "that's the way my teacher does it, ... because it is easier or it looks better" (p. 245).

In the context of comparing common fractions, Peck and Jencks (1981) report that, considering the interviews across all grades, fewer than 10 percent were able to provide an appropriate rationale based upon a diagram. Among the 20 Grade 6 students interviewed, 13 were unable to do so. Some students had constructed representations of common fractions with equal parts, but when using such representations to compare common fractions only the number of "pieces left over" were compared.

In these studies, two characteristics of the forms of representations students constructed are notable. The most obvious characteristic is the exclusive use of regions, particularly circles by students from the intermediate grades up to college level. However, only in Silver's study were students asked for alternative forms of representations. It therefore is uncertain in the other cases whether parts of regions is their only or simply their most common form of representation. The other notable characteristic is that measures of numerosity rather than measures of area appear to provide the basis for many of these students' representations, despite the fact that some students constructed equal parts of regions to explain the meaning of a common fraction. This would suggest that the "equal parts" requirement in representations with regions may be conceived by students to be unconnected to a conception of units as measures of area.

Recognition memory and memory which requires the evocation of situations already encountered but not present are distinguishable in terms of their cognitive complexity (Sinclair, 1971b). The limitations on the forms of representations of common fractions generated by the students in the previous studies do not imply that other forms of representation would not be recognized by the students as constituting physical embodiments of fractions. However, alternative forms of representations of common fractions may not be part of students' repertoires of evokable representations.

Students' Interpretations of Instructional Representations.

At the elementary and lower secondary levels, the number line is a form of representation which students have difficulty interpreting and using appropriately in a wide range of mathematical contexts from whole number addition to operations with integers (Behr, et al., 1983; Bright et al., 1988; Dufour-Janvier et al., 1987; Ernest, 1985; Hart, 1981; Novillis-Larson, 1980, 1987; Payne, 1975, 1984; Sowder, 1976: Vergnaud, 1983b). Nonetheless, it is a common form of representation for the instruction of whole number multiplication and common fraction concepts and relations.

Dufour-Janvier et al. (1987) describe some students' naive conceptions of the number line. In one case some primary students thought of the dots on the line to be stepping stones which are counted in a manner similar to the counting of discrete objects. In another case, students interpreted curved arrows marking the jumps of units along a number line in an addition question as boundaries marking off sets of dots (see Figure 2.01). The dots which were counted for the addition question were those which were beneath the curved arrow. The beginning point and end point of the arrows were not counted in the representation of the addends.

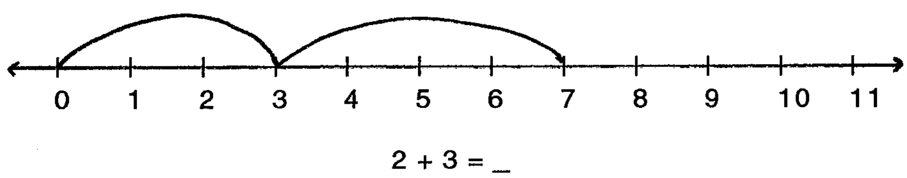


Figure 2.01 Counting sets of dots beneath the arrow on a number line.

Similar misinterpretations are reported by Sowder (1976). Whether such interpretations of representations of whole number operations on a number line persist with students in the intermediate grades is not known.

In the common fraction context, number line representations have been reported to be more difficult than representations based on the area of regions. Payne (1975), reporting on a series of teaching studies conducted at the University of Michigan, summarized the findings that students from Grades 1 to 5 had difficulty with number lines. Behr, et al. (1983) report that two to three times as many students made errors when representing proper fractions on a number line than with sets, regardless of the number of consistent or inconsistent cues embodied on the line or in the sets. Discrete sets were only as difficult as the number line when students were required to represent improper fractions. Novillis-Larson (1987) also found, from results of a test administered to Grade 5 and 6 students on their ability to associate proper fractions, improper fractions, and mixed numerals with representations based on the number line, ruler, and regions, that the mean score for number line items was much lower than for regions or rulers. In general, these results suggest that the number line is the most difficult form of representation for students to interpret and use in a common fraction context.

Representations of common fractions based on discrete sets have also been identified as more difficult for students to interpret and use than area-based representations (Behr, et al., 1983). Similarly, Payne (1975), reported that, when teaching initial fraction concepts and multiplication of fractions, the use of instructional representations based on sets resulted in lower achievement than the use of representations based on the area of regions. Partitioning a discrete set is easier than partitioning continuous quantities because it can be done through a partitive division process without having to plan and anticipate the full partitioning process (Hiebert & Tonnessen, 1978). But Post, Wachsmuth, Lesh, & Behr (1985) argue that one distinction between the construction of area based and discrete representations of common fraction relationships is that the region predetermines the whole unit, whereas with discrete sets no whole unit is predetermined. To construct representations with discrete sets to compare common fractions with different denominators, knowledge about common denominators has to be used in the planning process. Hart (1987), observing 10- to 11-year olds' failures to derive unknown equivalent fractions with sets, concluded that the task seems "impossible, since one does not know how many bricks to select for the demonstration" (p. 7). Hart concluded, "Although we may use materials and diagrams to make the rule seem reasonable, if we expect children to reconstruct these concrete models we must probably explicitly teach them a method for doing so" (p. 7).

Rules and conventions related to the structure and use of different forms of representations differ, and appear to influence the facility with which students are able to use them to interpret and represent common fraction relations. Bright et al. (1988), in their analysis of the differences between representations of common fractions on the number line and those based on sets or regions, illustrate some of the ways that different physical representations each have unique norms and conventions. First, the measurement constructs of each form of representation differ: length, numerosity and area. Second, the spatial characteristics of units differ in terms of their visual discreteness: with discrete sets all units are separate, with regions units representing the whole generally are separate, with number lines none of the units are separate. With each form of representation different physical attributes are significant in representing the meaning of common fraction relationships.

The nature of students' interpretation and use of physical representations is recognized as a current problem in mathematics education. Even the "simple" matching of a pictorial representation with a symbolic expression has been found to be not a "simple" process. Among fraction and ratio tasks which required students to match one mode of representation to another, it was found that matching pictures to symbols was the most difficult task (Lesh, Landau, & Hamilton, 1983; Lesh, Behr & Post, 1987). Lesh, Post, and Behr (1987), conclude that:

> Not only do most fourth through eighth grade students have seriously deficient understandings in the context of "word problems" and "pencil and paper computations," many have equally deficient understanding about the models and language(s) needed to represent (describe and illustrate) and manipulate these ideas. Furthermore, we have found that these "translation (dis)abilities" are significant factors influencing both mathematical learning and problem solving performance.... (p. 36)

This conclusion and the research cited previously suggest that the nature of students' interpretations of instructional representations cannot be taken for granted. Furthermore, there is reason to believe that as students attempt to construe meaning from instructional representations, their own conceptions as to the significance of attributes in the representation are important to their thinking process. In this regard, Dufour-Janvier et.al (1987) have asserted that:

> The imposition of representations which are too distant from the child results in the child reacting negatively or causes him difficulties. If one wants to use an external representation in teaching, he needs to take into consideration that it should be as close as possible to the children's internal representations. (p. 117-118)

One premise of this assertion appears to be that teachers need to know about the ways in which students might represent mathematical ideas in order to make instructional decisions about how to represent mathematics for the students.

A premise of this study is that the distance between children's personal, intuitive representations of mathematical relationships and forms of physical representations used in instruction has implications for the learning process. It is widely held that students are not passive receptors of "knowledge" but are actively involved in their own construction of knowledge based upon the reconstructed meanings they draw from their present and past experiences (Sinclair, 1987; Vergnaud, 1983b; Von Glasersfeld, 1983). This study was designed to explore the nature of students' representations of units and unit relationships in the context of whole number multiplication and common fractions, as well as length and area measurement, in order to contribute further to understanding of the ways in which students might represent their mathematical ideas.

CHAPTER 3

PLAN AND IMPLEMENTATION OF THE STUDY

The study was designed to explore the characteristics of students' representations of multiplicative relationships of units, and the extent to which properties of length or area measurement were used as a framework for representations within their repertoires. The representations constructed by students were translations of numerical expressions of whole number multiplication and common fraction concepts and relations. The students constructed these representations during a series of four clinical interviews.

The interview tasks included whole number multiplication and common fractions concepts and comparisons. Because both are multiplicative in structure (Vergnaud, 1983a), they were selected as the domain of study in order to explore the students' representations of relationships between different units. They were also selected because it was likely that students would have encountered measurement-based representations in one or both of these mathematical contexts, and because measurement-based representations play a critical role in the instruction of the part-whole meaning of common fractions.

Students' representations of units of measurement were also investigated in order to interpret the extent to which students incorporated properties of linear or area measurement in light of their conceptions of units of measurement. Their representations of units of measurement were investigated through their responses to a series of measurement tasks in a written test and two interviews.

The plan of the study, along with a description of the interview tasks and procedures designed to investigate students' representations of whole number multiplication, common fractions, and units of length and area are described in this chapter. The analytical categories used to characterize students' responses during the interviews are defined in Chapters 4 and 5, immediately preceding the presentation and discussion of the results of the analyses of the data.

Two pilot studies were conducted before the plan of the study was finalized. These were designed to evaluate or redefine elements of the plan such as the appropriateness of selecting subjects of particular grade and achievement levels, the interview tasks and procedures, and the means for investigating students' representations of units of measurement. The plan of the study described in the balance of this chapter was derived from evaluations of the pilot studies.

Subjects in the Study

Students were selected to represent a range of achievement in order to maximize the likelihood of variations in their responses to the interview tasks. Teacher assessments were used to select students to represent a range of achievement. Criteria used by each teacher varied, but this was not considered to be a limitation of the study since questions about relationships between levels of achievement and students' conceptions of representations were not addressed. A range of achievement was sought only to increase the likelihood that students would generate a range of responses to the interview tasks.

Grades 5 and 7 were selected to represent a variation in students' formal mathematical experiences. According to the prescribed mathematics curriculum for all school districts in the Province of British Columbia, students in Grade 5 should have received the first systematic unit of instruction on the comparison and equivalence of common fractions in Grade 4. For the Grade 7 students, systematic instruction should have occurred with common factions during each of the previous three years.

The prescribed curriculum was followed in both schools that participated in the pilot studies, which were in two different school districts. The students who participated in the final study were from schools in a third school district. Teachers of the Grade 5 students in the final study had decided informally to delay instruction on common fractions. This was not known prior to the commencement of the study. The assumptions about students' previous formal instruction with common fraction concepts and relations at different grade levels, therefore, did not apply to the subjects in the study. The students in Grade 5 had received no systematic unit of instruction on common fraction concepts or the comparison of common fractions. The students in Grade 7 had received at least one systematic unit of instruction in Grade 6.

Fifteen students from two schools participated in the study: six in Grade 5 and nine in Grade 7. These students represented a range of achievement in each grade. The original plan to have an equal number of students per grade was not met because of technical failures with the video equipment.

Plan of the Study

The plan of the study is presented in four parts. The first part is an overview of the study. This is followed by more detailed descriptions of (1) the generative interviews, (2) the interpretive interviews, and (3) the measurement interviews and measurement concepts test.

Overview

In this section only the general characteristics of the different interviews and a general outline of the plan of the study are described. A more detailed description of interview procedures and tasks follows in subsequent sections of the chapter.

There were two areas of investigation in this study. The nature of students' repertoires of representations of multiplicative relationships was the primary area of investigation. The secondary area of investigation was the nature of students' conceptions of units of linear and area measurement. The repertoires of students' representations were investigated through a series of four different interviews, the characteristics of which are summarized in Table 3.01. Students' conceptions of units of linear and area measurement were investigated primarily through two interviews. One dealt with linear measurement tasks, and the other dealt with area measurement tasks. A measurement concepts test also was administered to all students prior to the interviews. Hence, there were six different interviews planned for the study. These interviews are summarized in this section in two parts. The first summary deals with the four interviews designed to investigate students' repertoires of representations in the whole number multiplication and common fraction contexts. The second summary deals with the two interviews designed to investigate students' conceptions of linear and area units.

Repertoires of Representations

There were four facets of students' repertoires of representations which four different interviews were designed to investigate (see Table 3.01). The first facet was the nature of the spontaneous repertoires students generated, that is the representations that students are likely to generate when nothing in the material setting cues them to consider a particular form of representation. The second

Table 3.01

Characteristics of the Interviews Designed to Investigate Students' Repertoires of Representations

	Generative Interviews		Interpretive Interviews	
Characteristics	Uncued	Cued	Linear	Area
Order	1st	2nd	3rd	4th
Material Setting	Paper & pencil	Variety of materials	Distorted linear-based diagrams	Distorted area-based diagrams
Student Action	Generate variety of represent-ations for each task	Generate variety of represent-ations for each task	Evaluate diagrams & represent tasks	Evaluate diagrams & represent tasks
Facets Explored	Spontaneous, primary repertoires	Extension of repertoires	Attention to properties of length as critical features?	Attention to properties of area as critical features?

facet was the nature of the repertoires of representations students generated when the material setting provided cues which might lead them to consider the construction of representations which they did not generate spontaneously. The third and fourth facets were the nature of students' attention to properties of linear and area measurement as critical features of representations.

In all four interviews students constructed representations of whole number multiplication and common fraction concepts and relations. But the two generative interviews are distinguished from the two interpretive interviews by particular differences in the demands placed on the students. The first two interviews were called generative interviews because the students were required primarily to generate as many different representations as they could for any particular task. The last two interviews were called interpretive interviews because students were required to interpret and evaluate a series of specific beginning diagrams (see Figures 3.01 and 3.02 later in this chapter) as to their appropriateness as a framework for constructing a representation to explain the mathematical ideas. The students imposed their own interpretation on these beginning diagrams and identified which physical features were critical to their evaluation.

The exhaustive generation of different representations required in the generative interviews was not a part of the procedures for the interpretive interviews. The interpretive interviews were designed to investigate students' attention to properties of linear or area measurement. As such, representations were constructed by students as part of their explanations of why and how particular beginning diagrams were or were not appropriate for explaining the mathematics. On the other hand, the direction imposed by the diagrams in the interpretive interviews was not a part of the generative interview procedures. The only direction involved in the generative interviews was that students construct representations, the nature of which was left up to them.

The order of the four interviews for investigating students' repertoires of representations was established to minimize the possibility that responses in some interviews would be confounded by students' experiences with materials in other interviews. Hence, the generative interviews, which were non-directive or less directive with regard to students' consideration of length or area-based representations, preceded the interpretive interviews. The uncued generative interview preceded the cued generative interview in order to elicit forms of representations students would be more likely to consider spontaneously without the benefit of cues.

Conceptions of Linear and Area Measurement.

There were two separate interviews: one on the measurement of length and another on the measurement of area. The length and area measurement interviews were conducted directly after each of the linear and area interpretive interviews, respectively. The measurement interviews were placed after rather than before each interpretive interview so that the tasks of the measurement interviews did not sensitize the students to consider measurement attributes during the interpretive interviews.

Data Collection

The students were interviewed individually in a private room within their school. Each interview was recorded on videotape. The order of the data collection was as follows:

1	2.	3.	4.	5.
Measurement Concepts Test	Uncued Generative Interview	Cued Generative Interview	a Linear Interpretive Interview	a Area Interpretive Interview
			b Linear Measurement Interview	b Area Measurement Interview

Each of the four interview periods took about two weeks to complete with all students. A break of approximately five weeks occurred in the schedule during October-November. This break occurred directly after the cued generative interviews. For each student, at least two weeks elapsed between interview periods.

Generative Interviews

In the uncued generative interview the students had no materials to suggest different types of representations which could be used to explain the mathematical tasks. They were presented with a mathematical task and asked to explain its meaning through the use of diagrams or rough pictures. During the cued generative interview, a collection of materials was available from which students independently selected materials with which to construct representations of the mathematical tasks. It was anticipated that the presence of the materials might cue some students to construct different forms of representations from those constructed in the uncued generative interview. During both interviews, the student was asked to produce a "completely different" representation upon completion of a previous representation. This request deliberately left undefined the meaning of "completely different" in order to allow the students to determine their own meaning of differences between representations.

Students were asked to adopt the role of teacher because it made more explicit the request for them to explain the meaning of the task and not simply to derive an answer. The teacher's role also provided a rationale for the repeated requests for completely different diagrams or for completely different ways of using the materials. It was explained that teachers often use a variety of diagrams or rough pictures to help to make the mathematical ideas clear to the students and that they, therefore, would draw diagrams or rough pictures to explain the mathematics clearly. At times, the interviewer played the role of a confused student to encourage the student to clarify his or her meaning or to construct different explanatory diagrams. The teaching role has been used to motivate students to explain their mathematical meanings by other investigators such as Vergnaud (1983a), and Hiebert and Wearne (1986).

The request for different diagrams to explain a task was repeated until one of two conditions was obtained: (a) the student declared that he or she could not think of any more "completely different" diagrams, or (b) the student generated a series of representations which were structurally the same but perceptually different. An example of the latter case would be a situation in which a student generated a series of discrete representations with the differences between representations being only changes in the objects within the sets, for example only changing from triangles to stars.

In addition, in the cued generative interview the student was asked to indicate which of the unused materials could not be used to explain the task and why. The student also was asked to explain how remaining materials not classified by him or her as inappropriate would be used. In the latter case the student was asked whether the materials would be used to construct representations which were different or similar to those already generated. The purpose of these questions was to explore the limits of the forms of representations within a students' repertoire of representations.

Generative Interview Tasks

Three sets of mathematical tasks were devised for the generative interviews: a set of whole number multiplication tasks, a set of common fraction concept tasks, and a set of comparison of common fraction tasks. The results of the second pilot study indicated that a student had to be sufficiently familiar with the mathematics for any response to be forthcoming. In order to anticipate differences in students' mathematical experience, the tasks within each set varied in terms of the complexity of unit relationships. It was assumed that, by Grade 7, students would be sufficiently familiar with comparisons of common fractions to offer an explanation, but that not all Grade 5 students might be as familiar with these tasks. Hence, the set of common fraction concept tasks was designed for the Grade 5 students as a less complex set of tasks.

Tasks from these sets were presented individually to a student, and the student was asked to explain the meaning of an individual task by constructing representations with diagrams or concrete materials. Procedures for selecting tasks from within a set are described in the next section.

Multiplication Tasks (Grade 5 and Grade 7)

1. 4 X 5 = __
2. 18 X 3 = _
3. 2 X 3 X 4 = __
4. 3 X (4 + 5) = (3 X 4) + (3 X 5)

Concept of common fractions (Grade 5 only)

The planned set of fraction concept tasks were:

1. 3/4
2. 4/5
3. 1 1/6
4. 7/6
5. 17/5

Unit fractions were not planned as part of this set but were used with some students because of their inability to respond to any of the tasks above.

Comparison of common fractions (Grade 5 and Grade 7)

1. 1/3 vs 2/6
2. 2/4 vs 4/8
3. 2/3 vs 3/4
4. 5/9 vs 2/3
5. 5/8 vs 7/12

Procedures for Selecting Tasks.

It was not intended that all tasks would be used with each student in both generative interviews. A selection procedure was devised to determine which tasks within a set would be presented to a student. The objectives of the procedure were twofold:

1. to select tasks which were the most complex within a set to which a student was able to respond,
2. to limit the number of tasks from each set presented to a student in order to keep the length of the interview within reasonable bounds.

It was not possible to anticipate the ability of a student to respond to more or less complex tasks. For example, in the second pilot study one "low achieving" student was able to respond more fully to the tasks than some "high achieving" students. Therefore the interviewer made decisions regarding which tasks to present to a student on the basis of the extent to which the student was able to respond to previous tasks.

The only time that this flexible selection procedure was not followed was when the first set of tasks, whole number multiplication, were used in the uncued generative interview. Since this was the first interaction with a student, the student's reaction to the interview situation or individual interview tasks could not be anticipated. A simple to complex order was used to provide general information about student's ability to respond to more or less complex tasks. Thereafter, the tasks for each student were selected flexibly to maximize the chance of presenting the most complex task to which a student could respond and minimize the number of tasks required to determine this threshold. In the cued generative interviews the first task of each set was the most complex task within that set to which the student was able to respond in the uncued generative interview. In both generative interviews, when the flexible selection procedure was used, second tasks may have been more complex or less complex, depending on the responses of the student to the first task. As a result, the interview tasks and their order of presentation varied among students because of the range of achievement represented by the students in the study.

Materials Used in the Cued Generative Interview

The materials available to the students during the cued generative interview were as follows:

Multilink blocks	Coloured Disks
Yellow wooden hexagons	Base ten blocks
Centimetre dotted paper	Centimetre squared paper
Circular filter paper	Inch squared paper
Coloured string loops	Coloured felt pens
Ruler with solid coloured unit markings	Blank line on a sheet of paper

These materials were either common in elementary classrooms or were used as illustrations in a prescribed textbook. Some also had a potential for multiple use. For example, the multilink blocks could be used to construct discrete representations in random, linear or rectangular configurations, or contiguous representations in linear, rectangular or cubic configurations. Particular care was taken to select a variety of materials which might cue students to construct measurement-based representations.

Interpretive Interviews

The linear interpretive interview was designed to investigate the extent to which students attend to properties of linear measurement, and the area interpretive interview was designed to investigate the extent to which students attend to properties of area measurement as critical features in their representations. In both interviews students' attention to measurement properties was investigated by exploring students' tolerance for features which did not conform to properties of linear or area measurement, for example, their tolerance for unequal line segments marked as units along a line. As well, the interviews explored other geometric characteristics of linear-based and area-based representations that may or may not have been considered as critical features by the students. To this end, geometric and measurement properties associated with either linear or area-based representations were identified and systematically distorted within a series of diagrams (see Figures 3.01 and 3.02). For example, with the number line, properties such as equality of units, horizontality, straightness, or continuity were subject to distortion. With the rectangular representations, properties such as equality of units and squareness of units were subject to distortion.

The Interpretive Interview Tasks

All of the mathematical tasks in the interpretive interviews were parallel or identical to tasks used in the generative interviews. In order to accommodate the different achievement levels of the students, each mathematical context contained two tasks which differed in complexity. For example, multiplication of whole numbers was represented by the following tasks:

a) 2 X 3 X 4 =_

b) 7 X 3 =_

In general only one of a pair of mathematical tasks was used with a student, and the selection of a task for a particular student depended on their responses in previous interviews.

As can be seen in Figures 3.01 and 3.02, five diagrams were designed for each of the linear and area interpretive interview. Each of these diagrams was associated with each mathematical task. For example, when 2 X 3 X 4 =_ was a task used with a student, then the student was asked to evaluate each of the linear or area diagrams with respect to its appropriateness for constructing a representation to explain 2 X 3 X 4 =_.

Interpretive Interview Procedures

The introductory phase of these interviews was the same as that of the generative interviews except that students were told that they would be shown some different diagrams and would be asked to

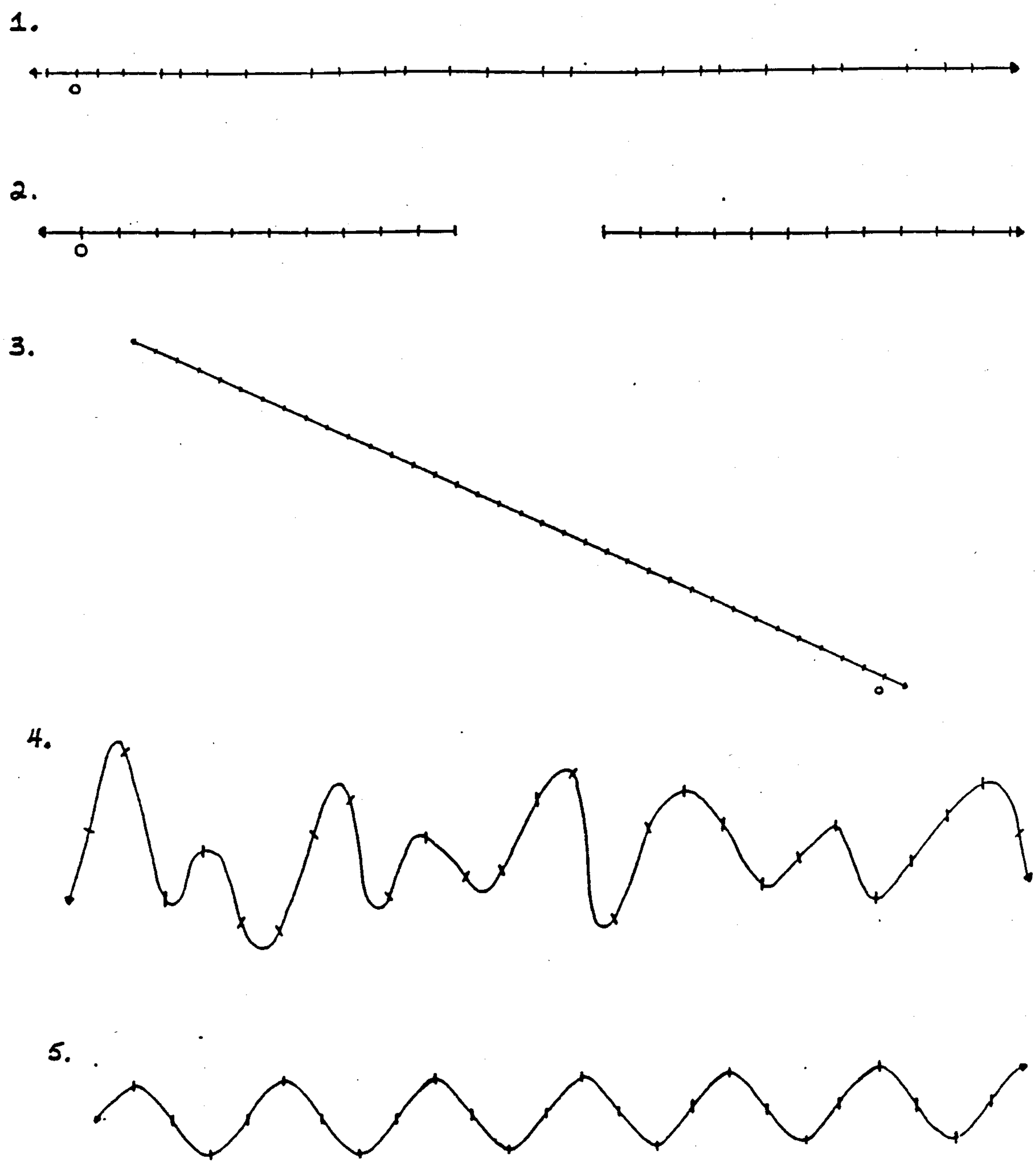


Figure 3.01 Beginning diagrams used in the linear interpretive interviews (Reduced to 48 percent).

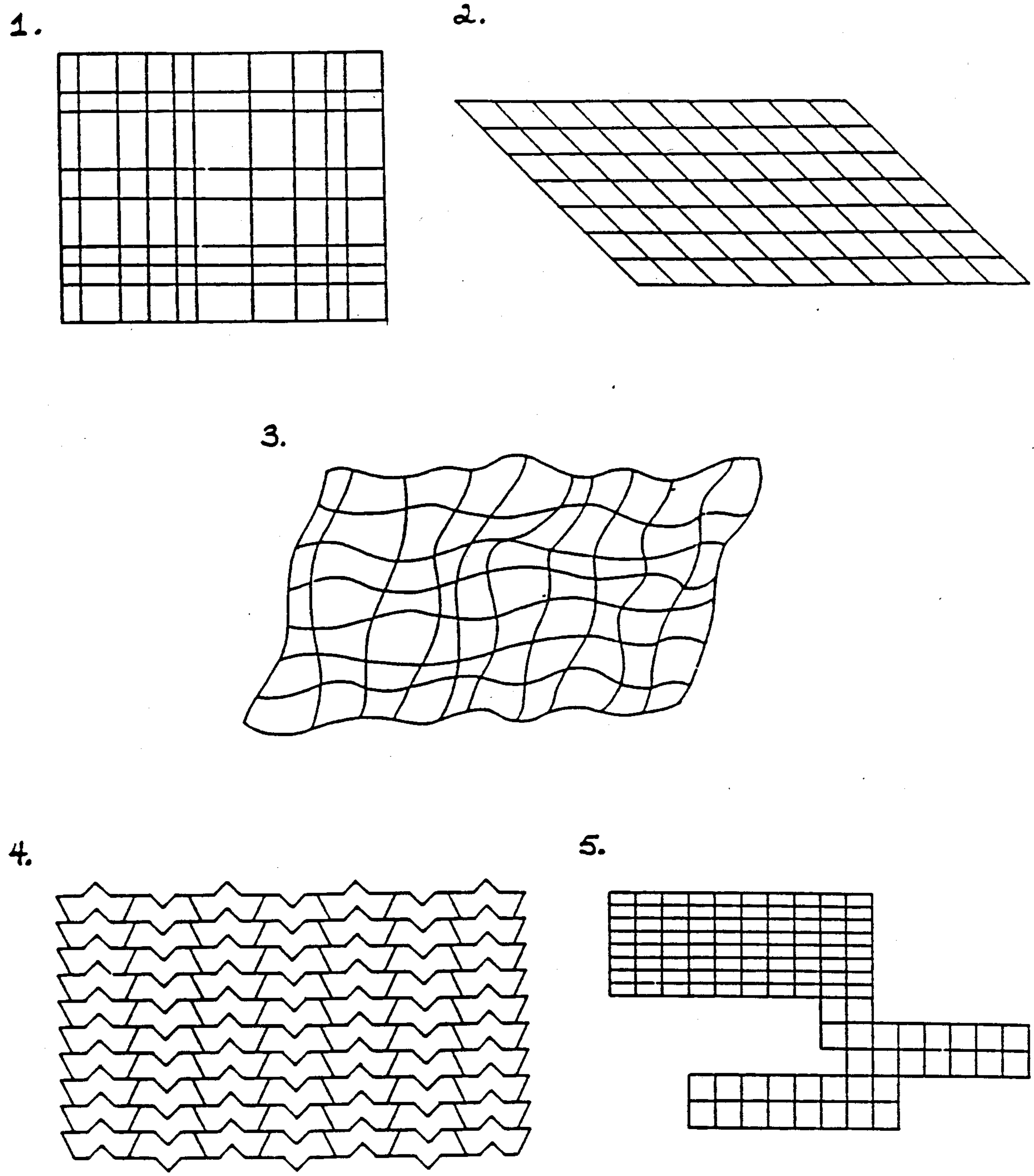


Figure 3.02 Beginning diagrams used in the area interpretive interviews (Reduced to 45 percent).

decide if each one would be a "good beginning diagram" to use for explaining the mathematics. An emphasis was placed upon the notion of a "beginning" diagram so that it is clear that they were free to add to or alter the diagram as they saw fit.

When a student responded positively to the question as to whether the diagram was a "good beginning diagram", he or she was requested to show how he or she would use the diagram to explain the mathematics. When a negative response occurred the student was asked for the reasons why the beginning diagram would not be good for constructing a representation of the mathematics. The student also was asked to explain what features should be changed in order for the beginning diagram to be used. Furthermore, because the reasons for rejecting a diagram could be based upon either mathematical or aesthetic criteria, the students were asked directly whether such changes were mathematically necessary or only preferred for other reasons.

Measurement Interviews and Test

The Measurement Concepts Test was designed before the second pilot study. Test items from Babcock's (1978) "Test of Basal Measurement Concepts" were used as the initial framework for the construction of the test. Some test items, such as questions related to the use of the ruler, were added, while others were modified. (See Appendix B.) From the results of the second pilot study, it was found that students' reasoning about the nature of and relationships between units of measurement could not be inferred from the results of a paper and pencil test. It was decided that students' representations of units of linear and area measurement would be investigated primarily through two individualized interviews: one for linear measurement and the other for area measurement.

Items from the Measurement Concepts Test were selected as tasks for the linear and area measurement interviews. Students' responses to these interview tasks constituted the primary source of data regarding students' representations of units of length and area measurement. However, the students' test responses to a few of the tasks which were amenable to analysis with regard to inferred strategies also were used as a secondary source of data. The test and interview items used to investigate students' representations of units of linear measurement are presented in Figures 3.03 and

3.04. The test and interview items used to investigate students' representations of units of area measurement are presented in Figures 3.05 to 3.07.

The linear measurement tasks were designed to investigate students' representations and use of units from a number of points of view. All of the tasks required students to construct or identify units of length to resolve the tasks, but the problem contexts differed. With the tasks in Figure 3.03, direct reference was made to the use of units and number, but relationships between units differed. These tasks were designed to investigate two different aspect of students' thinking about units of length: (1) the consistency with which they would identify or construct units of length as line segment, and (2) the extent to which they would reason appropriately about relationships between different sized units. With the tasks in Figure 3.04, no direct reference was made to the use of units or number. These tasks were designed to investigate whether students would identify or construct units of length and use appropriate numerical reasoning to resolve the comparisons.

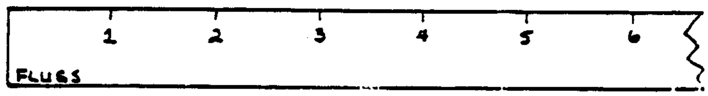
The irregular path tasks are modifications of items in Babcock's (1978) test. Babcock's test items were derived from interview tasks used by Bailey (1974) to investigate the extent to which primary school children coordinated length and numerical cues, and used compensatory reasoning about differences in the number and sizes of units when comparing lengths.

In Bailey's study the differences in the lengths of line segments in the paths were perceptually less obvious than in the tasks in the current study. Also, in Bailey's study the children had movable units which they could use to evaluate the equality of the line segments in the paths. In this study no measuring device, apart from fingers and pencils, was available to students. During the interviews in the second pilot study students were able to perceive the differences in lengths of line segments and use compensatory reasoning about differences in the number and sizes of units when comparing the lengths of the paths. It therefore was expected that students in the final study would be able to perceive the differences in the lengths of line segments within the paths, and make comparative judgements about the overall length of the paths without the assistance of measuring devices.

A. <u>Ruler task</u>

(Interview & Test Task)

This ancient ruler measures length in "FLUGS". One "FLUG" is the same as two centimetres. Draw a line above the ruler that is 6 centimetres long.



B. <u>Aggregate unit task</u>

(Interview & Test Task)

The line below is 4 units long. Draw a line that is 12 units long.

C. <u>Partitioning tasks</u>

1. (Test Task)

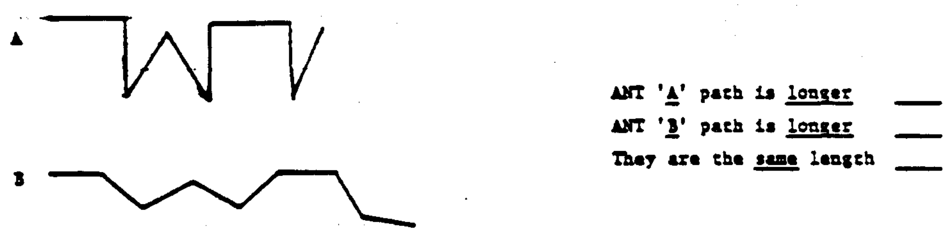
This path is <u>six units</u> long.
a) Mark the six units on the path.
b) Draw another path <u>5 units</u> long.

2. (Interview & Test Task)

This path is <u>5 units</u> long.
a) Mark the 5 units on the path.
b) Draw another path <u>3 units</u> long.

<u>Figure 3.03</u> Linear measurement tasks with explicit reference to units and number. (Reduced by 35 percent)

D. Irregular path task (Interview only)



E. Irregular path task (Interview only)

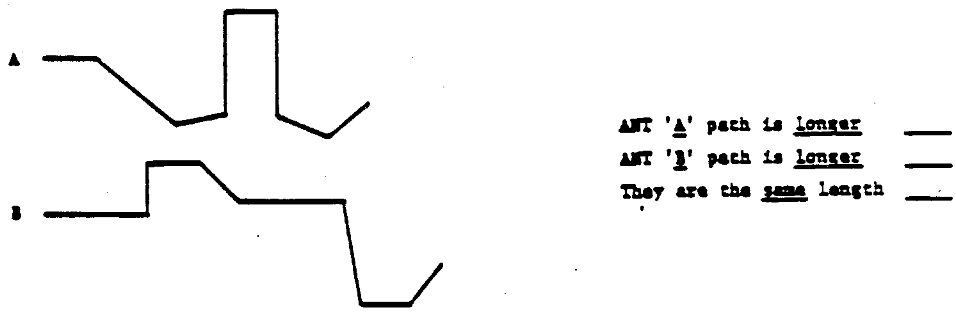


Figure 3.04 Linear measurement tasks without explicit references to units and number. (Reduced by 35 percent)

The area measurement tasks in Figures 3.05 to 3.07 were designed to investigate several aspects of students' representations and use of units of area. With the partitioning tasks in Figure 3.05, the students were asked to construct six equal parts in order to investigate the extent to which students relied on a successive halving algorithm, an algorithm which would not result in 6 equal parts. Different geometric regions were included in order to investigate how the geometry of the regions might influence the way in which students approached the partitioning problem.

Both the cake and tile tasks in Figures 3.06 and 3.07 were designed to investigate the extent to which students identified appropriate units and used numerical reasoning with different units to resolve the comparisons. In particular, the cake tasks in Figure 3.06 were designed to investigate the extent to which students considered parts of each region to be fractional units of area and used part-whole relationships to evaluate the comparison of the fractional units. The tile tasks in Figure 3.07 were designed to investigate the extent to which students attended to differences in the space-filling quality of units or simply counted units unequal in area as equivalent.

(Interview & Test Tasks)

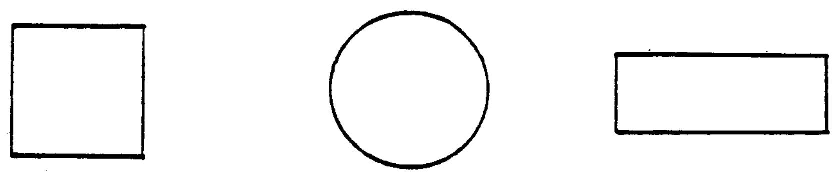


Figure 3.05 Partitioning tasks: divide each figure into 6 equal parts. (Reduced by 35 percent)

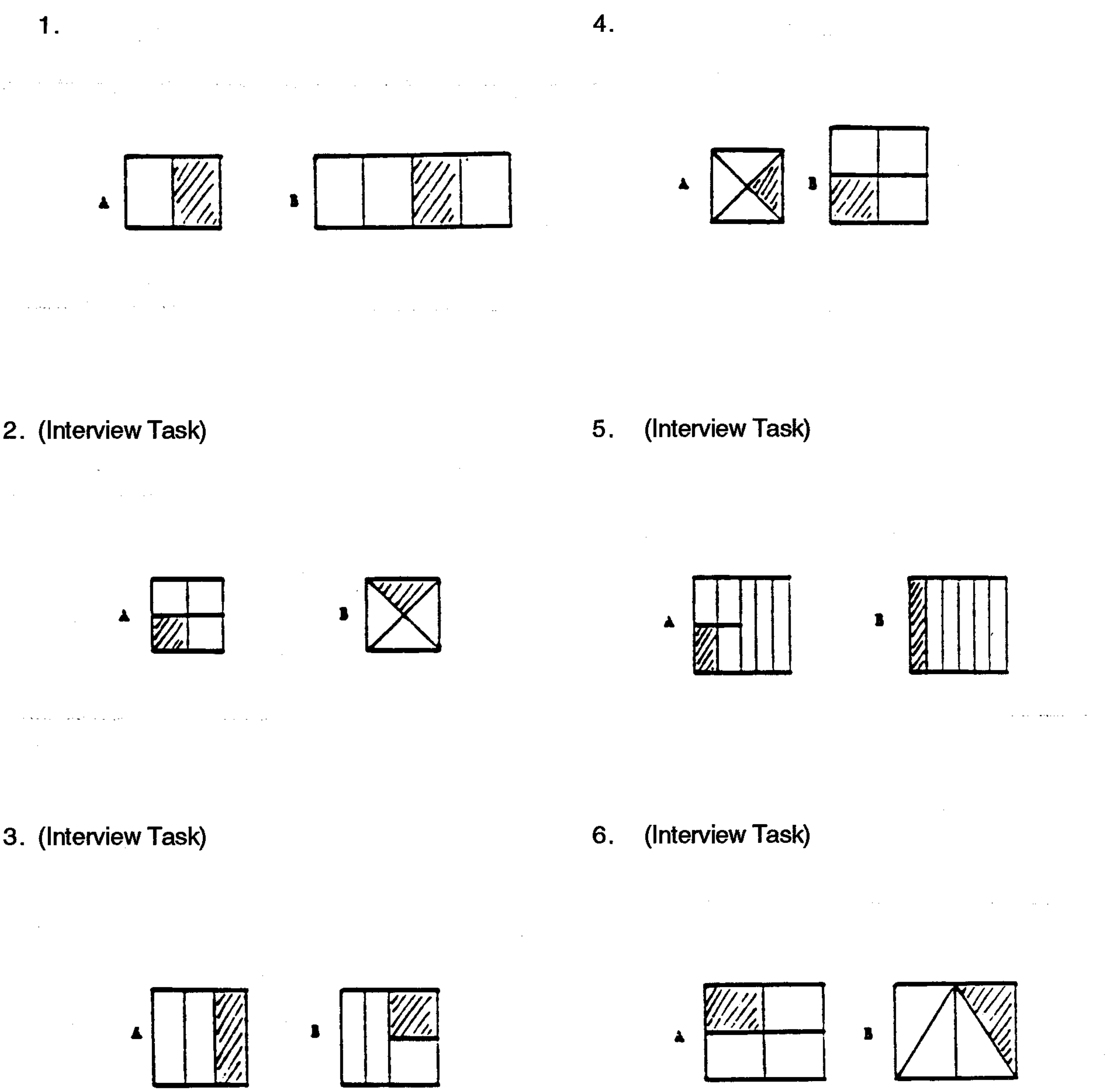
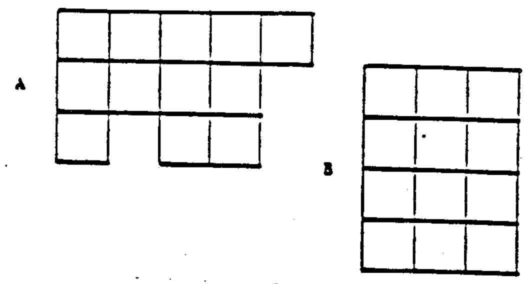
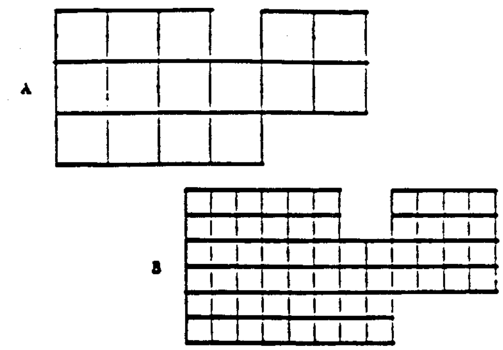


Figure 3.06 Cake tasks: compare the sizes of each shaded piece of cake. (Reduced by 35 percent)

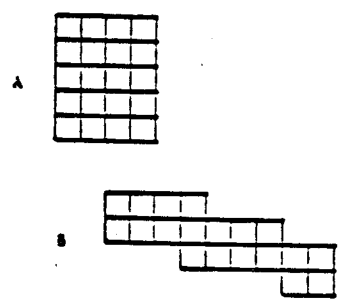
1. (Test task)



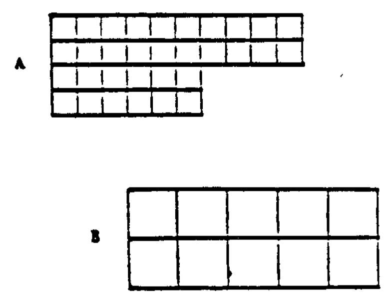
2. (Test task)



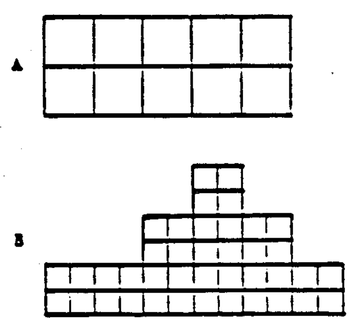
3. (Test Task)



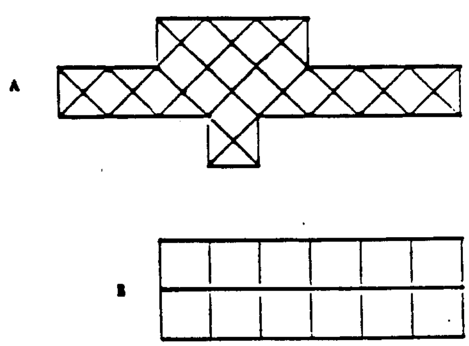
4. (Test Task)



5. (Interview & Test Task)



6. (Interview & Test Task)



<u>Figure 3.07</u> Tile tasks: compare the amount of space in each playroom. (Reduced by 40 percent)

Data Analysis

The principal sources of data for this study were transcripts of the six different interviews: the uncued generative interview, the cued generative interview, the linear interpretive interview, the area interpretive interview, the linear measurement interview and the area measurement interview. These interviews were designed to explore students' representations of units and unit relationships in four mathematical contexts: whole number multiplication, common fractions, linear measurement, and area measurement at the Grade 5 and Grade 7 levels.

Categories used to analyse the form and function of students' representations were not all determined before the implementation of the study. Some categories were suggested from the literature or became apparent during the pilot studies, but others were identified from the data of the final study. The identification and definition of categories with regard to students' conceptions of linear and area measurement followed a similar progression over time.

The final sets of categories are defined and illustrated at the beginning of each results section. Categories used to characterize students' representations of units in the whole number multiplication and common fractions context are defined and illustrated at the beginning of Chapter 4. In the case of linear and area measurement, the categories are defined and illustrated at the beginning of their respective sections of Chapter 5.

CHAPTER 4

STUDENTS' REPRESENTATIONS OF UNITS IN MULTIPLICATIVE RELATIONSHIPS

The characteristics of the repertoires of representations students constructed to explain whole number multiplication, common fractions, and comparisons of common fractions are explored in this chapter. The analysis of the characteristics of students' repertoires of representations in each of these mathematical contexts (whole number multiplication and common fractions) was guided by the following subset of questions:

1. What are the characteristics of students' primary repertoires of representations?
2. To what extent are the characteristics of representations in students' repertoires influenced by the material settings within which representations are constructed?
3. To what extent do students' repertoires of representations include forms of representations based on attributes of length or area?
4. To what extent do students' representations have the function of representing relationships between different units?

Relationship of research questions to interview data. The characterizations of students' representations of whole number multiplication and common fractions were based on an analysis of the representations they constructed during four different interviews. These were the uncued generative, cued generative, linear interpretive, and area interpretive interviews described in Chapter 3. The four interviews focused, in different ways, on particular questions related to the characteristics of students' repertoires of representations of whole number multiplication and common fractions.

1. The uncued generative interview was designed to elicit the representations which the students were most likely to construct spontaneously (see Question 1). A primary repertoire of representations of multiplication or common fractions refers to the variety of forms of representations that a person evokes spontaneously, in the absence of external cues or prompts.

2. The <u>cued generative and both interpretive interviews</u>, in which materials or different beginning diagrams were presented to the students, were designed to explore the influence of changes in the material setting on the characteristics of students' representations (see <u>Question 2</u>).

3. The <u>linear and area interpretive interviews</u> were designed to focus directly on the question of whether attributes of length or area are used by students as a basis for their construction of representations (see <u>Question 3</u>). In addition, characteristics of students' representations constructed in the <u>generative interviews</u> relate to Question 3.

4. <u>Question 4</u>, which refers to students' representation of relationships between different units, is addressed through the analysis of <u>all interview data</u>.

Collectively, the different forms of representations constructed by students during these interviews to explain whole number multiplication, or common fraction concepts and relationships are considered to represent their <u>general repertoires</u> of representations of these multiplicative relationships.

<u>Plan of the chapter</u>. This chapter is comprised of four major sections. First, the analytical categories used to classify students' representations are defined and illustrated. Second, the analyses of the characteristics of students' repertoires of representations of whole number multiplication are presented and discussed. Third, the analyses of the characteristics of students' repertoires of representations of common fraction relationships are presented and discussed. The last section consists of a discussion of common patterns and related themes in students' representations of units in both mathematical contexts.

Analytical Categories to Classify Representations

Two general elements are considered to constitute a representation: one is its form, and the other is its function. These elements, the form and the function of the representations, are used as major constructs to characterize students' representations.

The form of a mathematical representation refers to general physical characteristics which distinguish it from other representations of the same idea, just as a comic strip and an animated cartoon, which have different characteristics of form, may be used to tell the same story. For example, a representation of multiplication constructed with collections of objects differs in its physical characteristics from one constructed on the number line. The spatial relationships of the units within these two forms of representations and the formalities governing their construction are based upon different conceptual structures: measurement of numerosity in the first instance, and measurement of length in the second. However, both forms of representation may be used to communicate the same mathematical idea.

The function of a representation refers to the mathematical ideas communicated through the representation. Just as two comic strips, which share characteristics in their form, may have the function of communicating different meanings of an event, so also two representations which share characteristics in their form may have the function of communicating different mathematical meanings in terms of relationships among units. For example, Diagram A may have the function of representing 2 groups of 3 units, 3 groups of 2 units or simply 6 units. Each may represent an interpretation of 2 X 3. Diagram B is also similar in form to Diagram A, but may have the function of representing an operation on a number of single units such as 6 - 2, or a relationship between two different units such as 2/6 of the rectangle. Just as the function of the comic strip reflects the artist's interpretation of essential relationships in the event, the function of the representation reflects the student's interpretation of essential relationships in a mathematical situation.

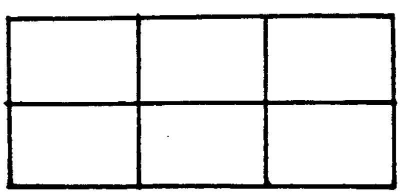


Diagram A

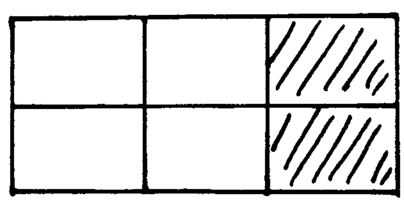


Diagram B

Analytical categories used to classify students' representations are discussed in two parts: (1) categories used to classify the form of students' representations, and (2) categories used to classify the function of students' representations. The distinction between the form and function of representations has not altered since the inception of the study, but the development of categories by which to analyze the interview data evolved over time. Some categories were identified during the planning and piloting stage and refined further during the transcription of the interview protocols, while others were defined during the initial analysis of the interview data. These analytical categories are defined in the next two sections. They are illustrated with selections from students' interview protocols.

Analytical Categories of Forms of Representations

There are two levels of categories related to the form of a representation. The first level is the spatial framework of the representation, and the second level is the units within the spatial framework (see Figure 4.01). By spatial framework of a representation is meant the general structure within which the units are represented, whether geometric regions, lines, or sets. Within a spatial

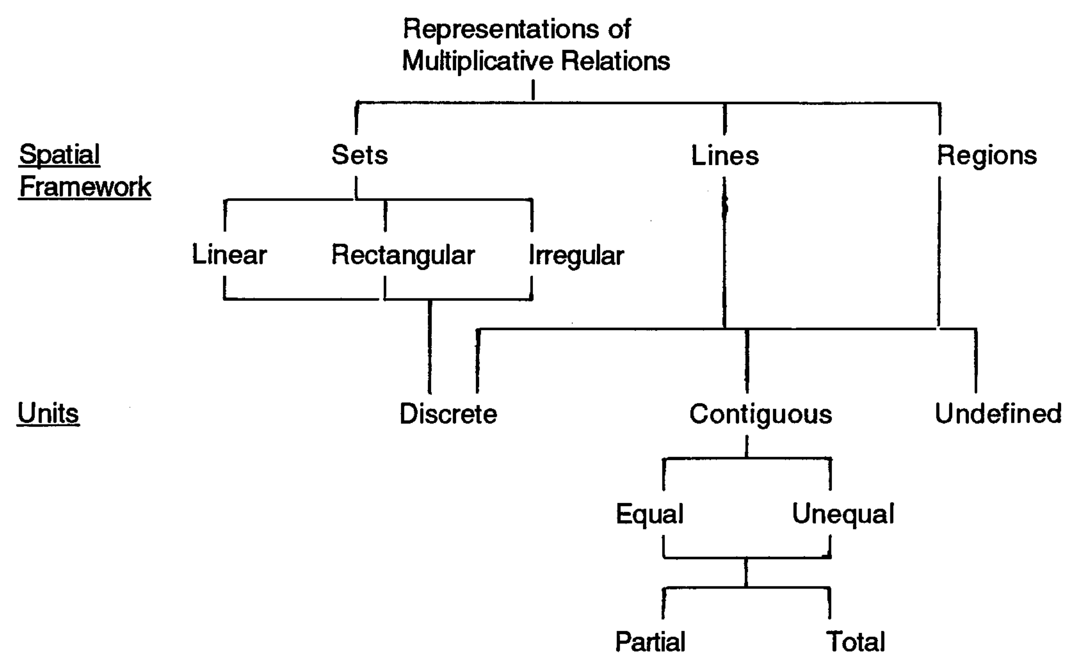


Figure 4.01 Categories for classifying the forms of representations.

framework, units may be represented in different forms. For example, a line may be used as spatial frameworks to represent discrete or contiguous units, or may be used to represent quantities which were not defined explicitly with units. Units within the spatial framework of sets are necessarily discrete.

The explanations of the categories which follow are organized in three sections by types of units: discrete, contiguous, and undefined. Within each section, examples of the variations in spatial frameworks are presented. Also, in the section on contiguous units, the other characteristics associated with this form of units are explained (i.e., equal, unequal, partial, and total).

Discrete Units

The over-riding characteristic of representations with discrete units is the complete separation of all units within the representation. This form of units was organized in every spatial framework observed in this study. These different spatial frameworks are explained and illustrated as they pertained to discrete units.

Discrete units with sets as spatial frameworks: linear, rectangular, or irregular. In a linear configuration, the discrete units were in an ordered sequence as shown in Examples 1 and 2. In a rectangular configuration, the discrete, primary units or aggregate units were arranged to create or approximate a rectangular array as shown in Example 3. "Irregular" applied to configurations of the units in sets which did not follow an obvious pattern.

Example 1: Discrete units in sets linear configuration.

Task: Compare 1/3 and 1/2

"They'd be the same." (i.e., 1/3 = 1/2)

"Well we take 4 for this one [drew 4 for 1/3] and 3 for this one [drew 3 for 1/2]"

"and I take 2 of these [from 1/2] and 3 of these [from 1/3]."

[The value of each set is derived by adding numerator and denominator, hence 1/2 = set of 3 objects. The equality of 1/2 and 1/3 is derived by comparing what is left in each set when the values of the denominators are subtracted from the set.]

James (Grade 5)

Example 2: Discrete units in sets with a linear configuration.

Task: 4 X 5 =

"Well, maybe you'd go like this. Maybe going this way."

△ △ △ △ △ + △ △ △ △ △ + △ △ △ △ △ + △ △ △ △ △

Edwin (Grade 5)

Example 3: Discrete units in sets with a rectangular configuration.

The student in this example organized the four sets into a 2 X 2 array, and approximated a rectangular form to represent five in each set.

Task: 4 X 5 =

```
X   X              X   X
       X                  X
X   X              X   X


X   X              X   X
       X                  X
X   X              X   X
```

"Okay, take 4 groups with 5 in it, and then you would count all of them, and when you've finished counting, you'd see what's the number, and then it should be the answer."

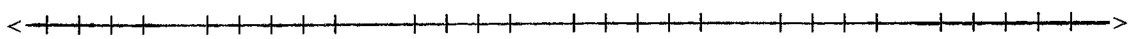
Dahlia (Grade 7)

Discrete units with a line or regions as spatial frameworks. When a line was a spatial framework, the discrete units invariably were marks along the line as shown in Example 4. The discrete units in regions were lines which generally gave the appearance of a partitioned region. However, the lines rather than the spaces were counted by the student as units as shown in Example 5.

Example 4: Discrete units on a line.

Task: 3 X (4 + 5) =

"Four lines, then a little ways apart 5 lines, then make two more groups the same."



Pete (Grade 7)

Example 5: Discrete units in regions.

Task: Compare 1/3 and 2/6

"There is the two triangles right there .... They are the same, and you draw a line between both of them."

"One, ... one, two, three and one," [drew 3 horizontal lines, then 1 horizontal line in the first triangle to represent 1/3]; "two ... one, two, three, four, five, six" [drew 2 horizontal lines, then 6 horizontal lines in the second triangle to represent 2/6]

"Now, this one is just the same as this one here [i.e the triangles are the same] but this has one instead of two [comparing the number of horizontal lines in the right side of each triangle] and this has three instead of six [comparing the number of horizontal lines in the left side of each triangle]. That [i.e., 1/3] equals 4 halves and that [i.e., 2/6] equals 8 halves."

Marlene (Grade 5)

Contiguous Units

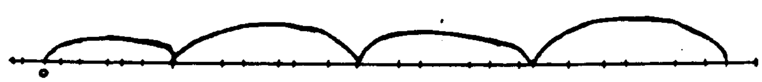
The over-riding characteristic of representations with contiguous units is that some, if not all, units within the representation "touch" each other or share a common boundary. These units were organized with a line as a spatial framework (see Example 1) or with a region as a spatial framework (see Example 2). Since not all units in a representation were necessarily contiguous, the extent of contiguity was classified as partial (see Example 2) or total (see Examples 3 and 4). In a partially contiguous representation aggregate units are discrete.

Representations of contiguous units were also classified with regard to the equality of the units. The units in the representations were classified as equal in length or area either when units appeared to be approximately equal, or when a student had difficulty constructing equal units but stated that the units should be equal.

Example 1: Contiguous units along a line.

Task: 2 X 3 X 4 =

[reduced photocopy of her number line]

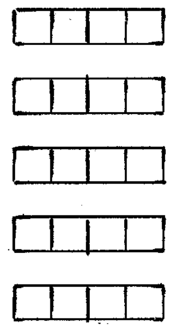


"To have one group of six you have to go up like that." [drew a jump from zero to the end of the sixth space] That would be one group of six, another group, another group, and another group - and then you leave it and when you are all finished there are 24 spaces."

Tammy (Grade 7)

Example 2: Partially contiguous units in regions.

Task: 4 X 5 =

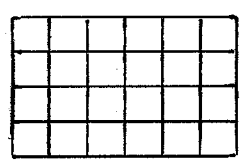


"There are five groups of one, two, three, four blocks."

Coran (Grade 7)

Example 3: Totally contiguous units in a region.

Task: 2 X 3 X 4 =



"Here, these things are all six and 1, 2, 3, 4." [counting the rows] "There are 4 here and four times six ... twenty-four."

"Just using squares and all that- well it's got sets right up to here and then it seems like adding." [indicating the rows in his representation]

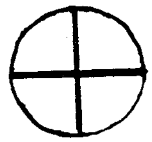
"Or you can do a different way of adding. Like 2 times 4, and then add them together makes 8 ... times 3 is 16, 24." [indicating columns and pairs of columns in his representation]

Edwin (Grade 5)

Example 4: Totally contiguous units in a region.

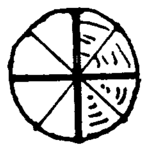
Task: Compare 2/4 & 4/8

Step 1.



"This is a whole and there's 1, 2, 3, 4 in that..."
"and that says 8" [i.e., 4/8] "well, that one says 4, so you need four-"

Step 2.



"and that one says 8, so you make like that and that." [partitions the fourths with two diagonals to make eighths] "and if it says 4, well 2, well you'd have - well say these lines aren't there - 2, you've got these two big pieces" [imagine removal of the additional diagonals think of 2/4] - "and 4, you've got 1, 2, 3, 4." [indicates four of the eighths within the 2/4] "It's just the same."

Tammy (Grade 7)

Units are Undefined

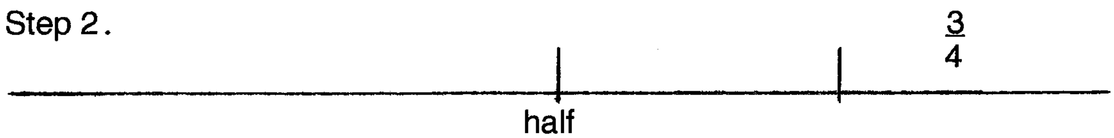
Units were classified as undefined when a student represented a value as a global quantity. The student did not construct units to represent a relationship between the numbers in the task. In the following examples, the student's general notion about the relative size of common fractions was represented as a global quantity of a line (see Example 1) or within a region (see Example 2). Neither the values of numerators or denominators, nor the relationship between numerators and denominators are represented with units.

Example 1: Units are undefined in a line.

Task: Explain meaning of 3/4

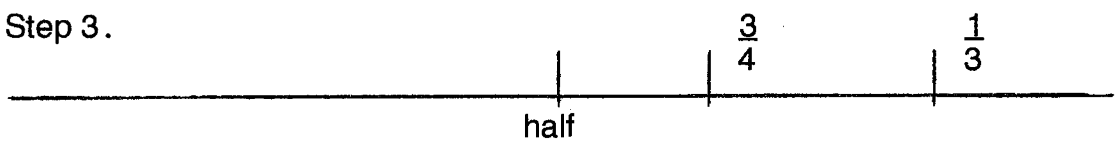
Step 1.

Step 2.



"That's three quarters right there." [between right-hand end and the first mark to the left] "and that's a half." [between the first mark to the left and the middle mark]
"Oh! That's wrong!"

Step 3.



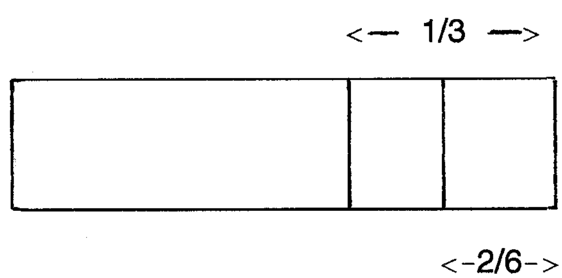
"This has got to be on this side ...." [Points to a position closer to but still to the right of the half mark] "and that would be one-quarter." [Indicates the space previous called three-quarters. Wrote 1/3.]

Brock (Grade 5)

Example 2: Units are undefined in a region.

Task: Compare 1/3 and 2/6

This would be smaller (2/6) so ... one third and two sixths.



Brock (Grade 5)

Analytical Categories of Functions of Representations

The categories related to the function of students' representations classify the extent to which relationships between different units were represented (see Figure 4.02).

	Representations of Multiplicative Relationships		
Unit Relations	Mono-	Bi-	Tri-
		Relational	

Figure 4.02 Categories for classifying the function of representations

Mono-relational Representation of Units

"Mono-relational" is used in this study to describe situations which involve quantities measured with only one unit. For example, when comparing, adding, or subtracting whole numbers all quantities are represented by only one unit. Within any particular case of these operations, all of the numerals represent a measure of the same unit. Hence, for $4 + 5 = 9$ to be true, the addends and the sum are measures of the same unit.

By definition, units in a multiplicative relationship are not all equivalent. However, some students constructed mono-relational representations of the units to explain the tasks. All units were represented equivalently. Such representations were classified as mono-relational. In the multiplication context, mono-relational representations generally were of the form presented in Example 1. All factors were represented by sets of equivalent units. In Example 2, the final comparison of the common fractions was based on measures of the numerosity of the sets. Each object in the sets was treated as equivalent units.

Example 1: Mono-relational representation of units.

Task: 4 X 5 =

"I already know what to do. You draw .... Hmm. And you count them out and it equals that."

o o o o x o o o o o = 20

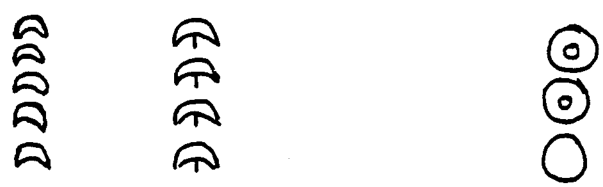
"Twenty."
"Well you go four times five and it equals twenty." [Points to the set of 4 circles and then the set of 5 circles]

Connie (Grade 7)

Example 2: Mono-relational representation of units.

Task: Compare 2/3 and 5/9

"Well, I'd draw holes.... holes in cheese and mouses live, and mouses live inside ... only four mouses came and there are five cheese left."



"Two thirds. Well, these are all donuts ... these are the two thirds that have holes and this one doesn't have a hole. See there's 9 cheeses and here's the 3 donuts and these two are the ones that have holes [2 donuts with holes], and these ones the five things that are left alone [cheeses without mice] .... that would be 5 and 2; 9 and 3." [comparing numerators and then denominators]

Edwin (Grade 5)

Bi-relational Representations of Units

Whole number multiplication as well as rational number concepts minimally involve a relationship between two different units. These units are related by a simple ratio; one unit is either an aggregate or a part of the other. Such a relationship between two units has been termed bi-relational.

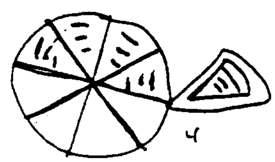
Although comparisons of common fractions with unlike denominators are minimally tri-relational (whole unit, 1st fractional unit, and 2nd fractional unit) bi-relational representations which omitted the relationship of one of the units were constructed as shown in Example 1. In this example the fractional units of ninths and thirds are inter-related, but they are not related to a consistent representation of a whole unit.

Other students constructed representations which appeared to account for relationships between three rather than two units, but interpreted their representation only in terms of two units. In Example 2, the student represented the part-whole relationship of eighths and twelfths but did not attend to the difference between eighths and twelfths. In Example 3, the student constructed a representation in which 2 groups of 3 groups of 4 are discernible, but to the student it only represented 6 groups of 4. These representations were classified as bi-relational.

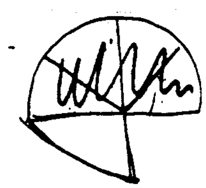
<u>Example 1: Bi-relational representation of units.</u>

<u>Task: Compare 2/3 and 5/9</u>

"I could use a pie like this. Then I'd have nine pies and whoever was eating the five pies, five pieces of pies out of the nine pieces of pie, would have eaten this much of the pie" [shaded pieces below] "...except for the person who has to eat four pieces of pie, okay, the leftovers."



"And then that [2/3] would be something like that" [see diagram below]. "This is a jumbo pie and the person out of two-thirds pie would only eat two out of three pies. Those two" [shaded pieces below]

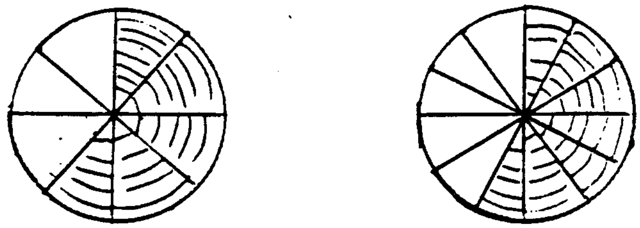


"...and then he'd take these and he'd ... those [thirds] are bigger than these [ninths]. So I'd ask him to half the pie and he'd go up ..." [partitioning of thirds into two] "...and those would be equal to two of the smaller pies, and he'd do the same for this one, so he's eaten four ..." [i.e., two thirds means four-ninths are eaten] "...and he'd only eaten half the pie over here ..." [4 ninths is half of the ninths pie] "but the person who ate five ninths of the pie had one more slice."

Pete (Grade 7)

Example 2: Bi-relational representation of units.

Task: Compare 5/8 and 7/12



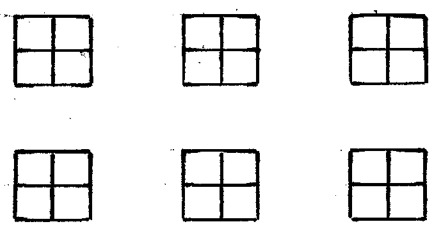
"and then you can see, like half of this circle would be four, 1, 2, 3, 4; so there would be half and a little bit - and then half of this one would be six, 1, 2, 3, 4, 5, 6, and then you shade in this one - so they would be equal. There is a little bit of each here."

Lara (Grade 7)

Example 3: Bi-relational representation of units.

Task: 2 X 3 X 4 = ___

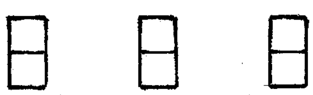
"You could put boxes, four in each ... four six times ... and count up all the boxes."



Interviewer: Where in the picture is two times three?

[Fanya was unable to indicate the units associated with these factors. Instead she drew a separate representation for this part of the numerical expression.]

"Well two sections in three so that equals up to six."



Fanya (Grade 7)

Tri-relational Representation of Units

Relationships between three different units underlies some complex whole number multiplication and the comparison of common fractions. In such a representation in the whole number multiplication context, the three units would be "double-nested." That is, of the form such as A groups of B groups of C. In the multiplication context it would occur as the representation of the product of three factors. In Example 1, a single representation embodies the relationships between all units representing the factors in 2 X 3 X 4 = _. In the common fraction context double-nested units would occur as the representation of two fractional units nested within a whole unit, such as the representation of thirds and ninths within a single region. However, tri-relational representations in the whole number context did not necessarily need double-nest units. In example 2, the student constructed a tri-relational representation of units without having the units double-nested.

Example 1: Tri-relational representation of units.

Task: 2 X 3 X 4 = ___

```
O | O   O
---------
O | O   O
  | +          12
O | O   O
O | O   O
    +
O | O   O
---------
O | O   O
  | +          12
O | O   O
O | O   O
```

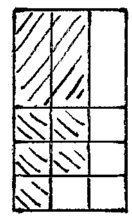
You could say there is 4 two times threes, and two times three is six, so four of those. You could say two times twelve, ... four times three is twelve there and four times three is twelve there.

Kit (Grade 5)

<u>Example 2: Tri-relational representation of units</u>.

<u>Task: Compare 2/3 and 5/9</u>

[Drew two squares, one above the other. Partitioned one into thirds and the other into ninths. Shaded 2/3 and 5/9, then compared the shaded amounts in both squares.]



"That is cut into thirds, and that is cut into ninths. So you have all of that shaded in - two-thirds and five ninths."

Lara (Grade 7)

In summary, the major elements or constructs used to analyze the characteristics of students' representations are their form and their function. Categories related to the form of representations are defined according to (1) the spatial framework of the representations, and (2) the physical characteristics of the units constructed within the spatial framework. Categories related to the function of representations are defined according to the number of different units explicitly inter-related by students in their representations of the mathematical tasks.

Students' Repertoires of Representations: Whole Number Multiplication

The four interviews (uncued generative, cued generative, linear interpretive, and area interpretive) focused, in different ways, on questions related to the characteristics of students' repertoires of representations of whole number multiplication. The uncued generative interview was designed to explore the characteristics of students' <u>primary repertoire</u> of representations. The first part of this section is devoted to an analysis of the characteristics of students' primary repertoires of representations.

In the second part, students' primary repertoires are compared and contrasted with their representations constructed in the subsequent interviews. Of particular interest was whether, by successively changing the interview setting, alternative forms of representation might be used that were not generated in the uncued interview, or whether their primary repertoire characterizes their general repertoire of representations.

There are two general characteristics of the forms of representations in students' repertoires which are explored through the sequential analyses of the interview data.

1. Whether there is a dominant form of representations in their repertoires.
2. The extensiveness of their repertoires. Extensiveness is considered in two ways: (1) in terms the variety of forms of representations in students' repertoires, and (2) in terms of the forms of representations rejected by students as a means for representing whole number multiplication.

The general organization of the analysis is as follows. Two general characteristics of repertoires, the form of representations as well as the function of the representations, provide the general framework within which students' repertoires of representations of whole number multiplication are characterized. Within this framework, the results of the analysis of the data from the uncued interview are considered to represent the characteristics of students' primary repertoires of representations. The primary repertoires are then compared and contrasted with students' responses in the other interview settings with regard to the forms and function of students' representations. Through the successive comparisons of students' primary repertoires with their responses in the other interview setting, the issue of the extent of their general repertoires will be addressed. Finally, the over-all analysis will be drawn upon to address the general research question, "What are the characteristics of students' repertoires of representations of whole number multiplication?"

Characteristics of Primary Repertoires of Representations: Whole Number Multiplication Context

The questions addressed in this section are:

1. What are the characteristics of students' primary repertoires of representations used to explain the meaning of whole number multiplication? In particular:
   a. Is there a form of representation which dominates within their primary repertoires?
   b. How extensive are their primary repertoires with regard to the variety of forms of representations generated to explain whole number multiplication tasks.
2. To what extent do the representations of whole number multiplication in students' primary repertoires have the function of representing relationships between different units?

Table 4.01 was designed to illustrate the nature of students' primary repertoires of representations of whole number multiplication. The major categories for classifying the form of their representations are represented in the columns of this table. These categories include the spatial framework of the representations, (sets, lines and regions), and the form of units within the spatial frameworks, (discrete, contiguous, and undefined). For example the first column represents discrete units (D) within a spatial framework of sets (S). For each different form of representation constructed by a student, the function of their representations with regard to unit relations (M, B, or T) is coded within the table. In addition, a dot in the body of the table indicates that the form of representation was not generated by the student. Blanks in the body of the table indicate that the task was not presented to the student.

Most students constructed more than one representation to explain one or more tasks. However, when the form and function of their multiple representations were the same, only the general pattern of their responses is codified. For example, Connie generated two representations for each task, representations which she considered to be completely different. All of her representations consisted of discrete mono-relational units in sets. Hence, there is one entry per task for Connie (M in column DS) which codes the form and function of all of her representations per task.

Table 4.01

Forms and Functions of Representations of Multiplication: Uncued Generative Interview.

	Interview tasks by form of representation																				
	A X B =							A X B X C =							A X (B + C) = ...						
Dominance of Forms	DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR
Exclusively Discrete																					
1 Connie	M	.	.	.	.	.	.	M	.	.	.	.	.	.							
Marlene	M	.	.	.	.	.	.	M	.	.	.	.	.	.							
2 Brock	B	.	.	.	.	.	.	M	.	.	.	.	.	.							
Derek	B	.	.	.	.	.	.	BM	.	.	.	.	.	.							
Edwin	B	.	.	.	.	.	.	M	.	.	.	.	.	.	B	.	.	.	.	.	.
James	B	.	.	.	.	.	.	MT	.	.	.	.	.	.							
3 Dahlia	B	.	.	.	.	.	.	B	.	.	.	.	.	.	B	.	.	.	.	.	.
Kasey	B	.	.	.	.	.	.	B	.	.	.	.	.	.							
Pete	B	.	.	.	.	.	.	B	.	.	.	.	.	.	B	.	.	.	.	.	
Tammy	B	.	.	.	.	.		B	.	.	.	.	.	.	B	.	.	.	.	.	
Lara	B	.	.	.	.	.	.	B	.	.	.	.	.	.	B	.	.	.	.	.	
Dominantly Discrete																					
Lolande	B	.	.	.	B	.	.	B	.	.	.	.	.	.	B	.	.	.	.	.	
Coran	B	.	.	.	B	.	.	BT	.	.	.	.	.	.	B	.	.	.	.	.	
Kit	B	.	.	.	B	.	.	T	.	.	.	.	.	.	MT	.	.	.	.	.	
Dominantly Contiguous																					
Fanya	B	.	.	.	B	.	.	B	.	.	.	B	.	.							

Note. Names of grade 5 students are underlined.
Dots within the table mean that the form was not used.
Blanks within the table mean that the task was not given.

Spatial Framework
S = sets
L = lines
R = regions

Units
D = discrete
C = contiguous
U = undefined

Unit Relations
M = mono-relational
B = bi-relational
T = tri-relational

There were two criteria used to position the students within the table. First they were grouped according to the dominance of a form of representation within their repertoire. A form of representations was considered to be dominant when a majority of the representations generated by a student were of that form. Second, they are ordered within the group according to the extent to which they represented relations between different units.

Characteristics of the Forms of Representations in Primary Repertoires.

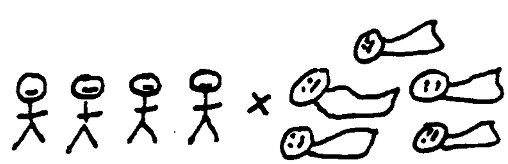
As can be seen in Table 4.01, in all but one case (Fanya), students' primary repertoires were dominated by representations based on a spatial framework of sets in which all units were discrete.[1] Despite the exclusive use of discrete forms of representations by two-thirds of the students, all but one student (Brock) constructed at least two representations for a task which they considered to be "completely different." Students' criteria for defining "completely different" were not related necessarily to differences between the proximity of contiguous and discrete units.

Characteristics of discrete forms of representations. There were three ways in which students constructed what they considered to be "completely different rough pictures or diagrams" to explain a multiplication task, while still using only sets as the spatial framework.

1. Change the type of objects within the sets. For example, Marlene represented 4 X 5 = first as

$$\Delta\,\Delta\,\Delta\,\Delta \quad \times \quad O \; O \; O \; O \; O \; = \; 20$$

then as



1. Descriptions of a student's responses serve only to illustrate patterns in responses. They are not intended to be read as part of individual case studies. Students' names are included only to assist the reader to locate data in the tables. Appendix A contains sample transcripts of each type of interview used in this study. Included in the appendix are selections from 10 of the 15 students' transcripts.

2. Change the spatial organization of the sets. For example, James' two representations of 4 X 5 = were "four groups," then "four rows with 5 in each."

A.

```
| |     | |
| |     | |
 |       |

| |     | |
| |    | | |
 |
```

then as

B.

X X X X X

X X X X X

X X X X X

X X X X X

3. Change the function of the representation by using the commutative or associative principles. For example, Lara represented 6 X 4 =_ first as

A.

△ △ △ △

△ △ △ △

△ △ △ △

△ △ △ △

△ △ △ △

△ △ △ △

then as

B.

△ △ △ △ △ △

△ △ △ △ △ △

△ △ △ △ △ △

△ △ △ △ △ △

These three variations in changes to discrete representations with sets, that is, (1) changes in type of objects, (2) changes in spatial organization of units, and (3) changes in the function of the representation, were used in different combinations by students to construct "completely different" representations of a task. Most students changed at least two of these three features to construct a "completely different" representation.

A rectangular rather than a linear or irregular spatial configuration was used exclusively or dominantly by 11 of the 15 students when they constructed representations based on sets (see Table 4.02). The students used factors in an A X B array to organize primary and aggregate units, or approximated a rectangular organization of units when the number of units was prime or odd. A linear configuration was used only with mono-relational representations, not to represent a multiplicative relationship between two or more different units. Hence, when students were representing unit relationships as bi-relational or tri-relational, they tended to represent the product as a rectangular configuration with factors as the dimensions, or to represent the cardinal values of the factors in a rectangular configuration. In the latter case, 4 groups of 5 would be represented with the 5 primary units as a (2 X 2 array) +1, and the 4 aggregate units as a 2 X 2 array.

Table 4.02

Extent to which Spatial Organizations were used to construct Representations of Multiplication with Sets (Uncued Generative Interview).

		Spatial Organizations		
Extent[1]	n[2]	Linear	Rectangular	Irregular
Exclusive	9	2	7	
Dominant	4		4	
Secondary	6	4	2	2

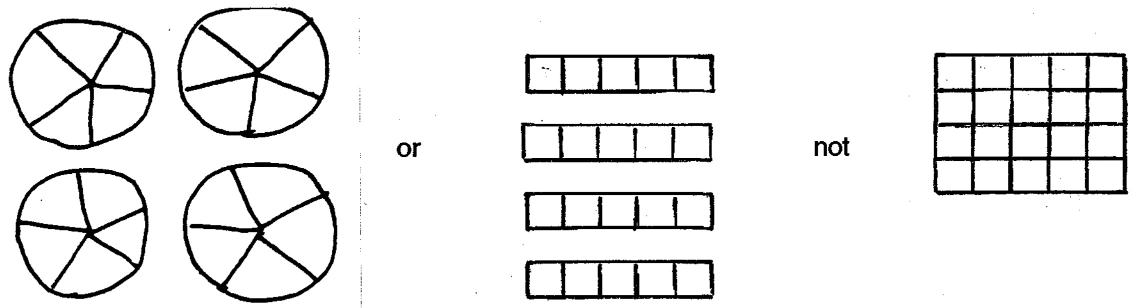
Note. 1. Exclusive means all, dominant means more than half, secondary means half or less of a student's discrete representations used the configuration.
2. Total n = 15 Students are counted more than once in the "dominant" and "secondary" categories.

Characteristics of contiguous forms of representations. Four students generated primary repertoires which included both discrete and contiguous representations. There were two common characteristics of their contiguous representations.

1. Only regions were used as the spatial framework, not lines (see CL & CR in Table 4.01).

2. The aggregate units were only partially contiguous. Each aggregate unit was represented as a separate partitioned region, not as a contiguous unit within a partitioned region. For example,

4 X 5 = was represented as:



The spatial separation of aggregate units meant that the features of the representation which embodied each of the two factors were visually distinct. To interpret the final representation in terms of both factors, less analysis would be required than if the units were fully contiguous.

The discrete and contiguous forms of representations in their primary repertoires, which they considered to be "completely different," were distinguished by the fact that the units within an aggregate unit were either discrete of contiguous. Aggregate units were invariably discrete.

Common spatial characteristics of discrete and contiguous forms of representations. There were two spatial characteristics common to both discrete and contiguous representations which appeared to serve to reduce the perceptual complexity of the representations.

1. A tendency for most students to organize their discrete and contiguous representations systematically within two dimensions rather than one dimension.

2. The consistent spatial separation of the aggregate units.

Characteristics of contiguous and discrete representations were analogous in an number of ways. Aggregate units were constructed as separate, partitioned regions or separate sets. The visual distinctiveness of each contiguous or discrete aggregate unit sometimes was emphasized by the insertion of operation signs between aggregate units. It therefore was unnecessary to reconstruct or imagine the aggregation of units from a collection of primary units. The multiplicative relationships among the units in the representation were visually explicit. Within a general, two-dimensional spatial organization, primary units within an aggregate unit were either contiguous or discrete.

Functions of Representations in Primary Repertoires.

As can be seen in Table 4.01, mono-relational representations of A X B tasks were constructed by only two students (Connie & Marlene), but the incidence of mono-relational representations increased with the more complex tasks. A majority of the students constructed a series of bi-relational representations for some or all of the more complex tasks. The bi-relational representations of these more complex tasks illustrated separate steps in a calculation procedure, not the inter-relationship of all units involved in the task. Only three students managed to construct any tri-relational representation.

Students who constructed only mono-relational representations were able either to determine some products with a skip counting or counting on procedures. They showed no indication that these procedures were associated with a representation of consecutive groups of units. For example, when Marlene "knew" a skip counting sequence, a representation such as IIII X IIIII = 20 provided a way to keep track of count. When counting by five's, she marked each item in the set of four with each count. When counting sequences were not known she did not use sets in her representation to keep track of her counting-on sequence. Instead she laboriously and often unsuccessfully used a verbal counting-on procedure while keeping track of the count on her fingers. The procedures (skip counting or repeated counting on) were not associated with a concrete representation of consecutive aggregate units. Instead, these mono-relational representations were, at most, procedural tools. There was no explicit representation of their skip counting or repeated addition procedures in which the product was a synthesis of the procedures.

There were two circumstances under which other students constructed mono-relational representations with only the more complex tasks. In the first circumstance, some students could not interpret the symbolic notation and determine the final product numerically or with mental arithmetic. They constructed mono-relational representations as a minimal response. In the second circumstance, some students determined the product, and appeared to establish the representation of relationships among all factors as a final goal. However, they had difficulty translating their thinking processes into a tri-relational representation. Their tri-relational representations were preceded by either mono-relational or bi-relational representations of the task. The preceding mono- or bi- relational representation assisted the student in thinking through the problem of representing units relationships among all factors when the product was already known. Reflection on their earlier mono- or bi-relational representations eventually resulted in the construction of a tri-relational representation.

In summary, there were different limits in the complexity of unit relations which students could represent. Two students were limited to mono-relational representations. Most other students were limited to bi-relational representations of units. Nearly all tri-relational representations were preceded by one or more trials.

Summary of Characteristics of Primary Repertoires of Representations of Whole Number Multiplication.

Considering the representations constructed by the students during the uncued generative interview, a number of features characterized their primary repertoires of representations of multiplication. With regard to the form of representations, characteristics were:

1. The general dominance of discrete units in sets, and the more limited use of partitioned regions as spatial frameworks.

2. The general use of a 2-dimensional spatial organizations of units.

3. The spatial separation of aggregate units.

With regard to the functions of the representations, characteristics were:

1. Limits in the complexity of the unit relations fluently represented by students. With few exceptions this was either mono- or bi-relational representations of the units.

2. Tri-relational representations were not constructed directly. Students seemed to derive strategies for constructing such representations from a reflection on preliminary mono-relational or bi-relational representations.

In general, the spatial frameworks and organization of the form of the representation appear to function to reduce the perceptual complexity of the representations. In this regard, the spatial organization of contiguous representations was analogous to that of discrete representations in that aggregate units were spatially separated, and organized within a two-dimensional configuration.

Characteristics of General Repertoires: Representations of Whole Number Multiplication in Other Material Settings

The questions addressed in this section are:

1. To what extent are the characteristics of representations in students' repertoires influenced by the material setting within which representations are constructed?
2. How extensive are their general repertoires with regard to the variety of forms of representations used to explain whole number multiplication tasks.
    a. To what extent are particular forms of representations excluded from students' general repertoires?
    b. To what extent do students' general repertoires of representations include forms of representations based on attributes of length or area?

In this section, the students' responses to the multiplication tasks during the other three interviews (cued generative, linear interpretive, and area interpretive) are compared and contrasted to the characteristics of the students' primary repertoires of representations generated during the uncued generative interview. The forms of representations of multiplication tasks constructed during the cued generative interview which differed from those spontaneously generated during the uncued interview were considered to indicate extensions within students' repertoires. As well, materials which students rejected during the cued generative interview were considered to be one indication of limits in their repertoires. The students' acceptance or rejection of beginning diagrams during the linear and area interpretive interviews was considered to be another indication of limits in their repertoires.

This section is organized in two parts. In the first part, the primary repertoires generated during the uncued generative interview are compared to the forms and functions of the representations constructed during the cued generative interview. In the second part, the responses in all of the interview settings are compared and contrasted.

Representations Constructed in the Cued Generative Interview Compared to the Uncued Generative Interview.

Table 4.03 was designed to illustrate students' representations constructed during the cued generative interview. Also included is a summary of their responses during the uncued generative interview which represent their primary repertoires. The format of the table is similar to that of Table 4.01. The major categories for classifying the form of their representations are represented in the columns of this table. These categories include the spatial framework of the representations, (sets, lines and regions), and the form of units within the spatial frameworks, (discrete, contiguous, and undefined). For each form of representation constructed by a student during the cued generative interview, the function of the representation with regard to unit relations is coded within the table.

As with the previous uncued generative interview, most students constructed more than one representation for one or more tasks. When the form and function of these multiple representations were the same, only the general pattern of the responses is indicated. In addition, a dot in the body of the table indicates that the form of representation was not generated by the student, and blanks indicate that the task was not presented to the student.

The summary of the characteristics of students' primary repertoires (as reflected in the uncued interview) is recorded to the far left with the students' names. The functions of their representations from the uncued generative interview precede each student's name. The students are grouped according to the dominance of a form of representation within their primary repertoire as in Table 4.01. Within these groups they are ordered according to the extent to which they represented relations between different units during the cued generative interview.

Influence of material setting on forms of representations generated. For most students, the patterns of response during the cued generative interview mirrored those of the uncued generative interview. Despite the presence of materials that might have cued students to construct contiguous forms of representations, only four students (Dahlia, Pete, Lara, and Fanya) constructed contiguous representations as their most frequent form of representation during the cued generative interview. Discrete units were still the dominant or only form of representation for 11 of the 15 students. However,

Table 4.03

Form and Function of Representations of Multiplication Tasks During the Cued Generative Interview with a Summary of the Primary Repertoires.

Summary of Primary Repertoires (Uncued)		Cued interview tasks by form of representation																				
		A X B =							A X B X C =							A X (B + C) =						
		DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR
Exclusively Discrete																						
M	Connie	M	.	.	.	.	.	.	M	M[1]	.	.	.	.	.							
M	Marlene	M	.	.	.	.	.	.	M	.	.	.	.	.	.							
BM	Brock	B	.	.	.	.	.	.	M	.	.	.	.	.	.							
BM	Derek	MB	.	.	.	.	.	.	M	.	.	.	.	.	.							
BMT	James	B	.	.	.	.	.	.	NR	.	.	.	.	.	.							
B	Tammy															B	.	.	.	B	.	.
B	Dahlia								.	.	.	.	B	.	.	.	.	.	.	B	.	.
B	Pete															.	B[1]	.	.	B	.	.
BM	Edwin								T	B[1]	.	.	BT[1]	.	.	B	.	.	.	.	.	.
B	Lara								.	.	.	.	BT	.	.	.	.	.	.	B	.	.
B	Kasey								T	.	.	.	B	.	.							
Dominantly Discrete																						
B	Lolande								B	.	.	.	.	.	.							
BT	Coran								BT	.	.	.	.	.	.							
BTM	Kit															T	B[1]	.	.	T	.	
Dominantly Contiguous																						
B	Fanya								.	.	.	.	B	.	.							

Note. Names of Grade 5 students are underlined.
Dots mean the form was not used; blanks mean the task was not given.
1. Representation constructed only when questioned directly about materials.

Spatial framework
S = sets
L = lines
R = regions

Units
D = discrete
C = contiguous
U = undefined

Unit Relations
M = mono-relational
B = bi-relational
T = tri-relational
NR = no response

six students (Tammy to Kasey in Table 4.03), whose primary repertoires included only representations constructed with sets of discrete units (DS), constructed at least some representations with contiguous units in regions (CR). Hence, there was an increase in the incidence of contiguous representations during the cued generative interview, but most students continued to favour discrete representations based on sets.

As was the case with the responses in the uncued generative interview, most students used a rectangular configuration for the organization of sets and units within sets during the cued generative interview (see Table 4.04). Similarly, the bi-relational and tri-relational representations were invariably in a rectangular configuration, whereas linear or irregular configurations were used only for mono-relational representations.

Table 4.04

Extent to which Spatial Organizations were used to Construct Representations of Multiplication with Sets (Cued Generative Interview).

		Spatial Organization		
Extent[1]	n[2]	Linear	Rectangular	Irregular
Exclusive	10		8	2
Dominant	2		2	
Secondary	2	2		

Note. 1. Exclusive means all, dominant means more than half, secondary means half or less of a student's discrete representations used the configuration.
2. Total n = 12 Students are counted more than once in the "dominant" and "secondary" categories.

Relationship between discrete and contiguous representations. As was the case with contiguous representations generated during the uncued generative interview, the contiguous representations constructed with materials were limited in a number of ways. All contiguous representations were based on regions. Generally aggregate units were separated, and the spatial organization of the aggregate units was rectangular.

Students constructed contiguous aggregate units with the materials by putting a set of separate units together rather than by partitioning regions. Aggregate units which the students had represented previously as a discrete set now were pushed together and represented contiguously. This suggested a close relationship between students' discrete and contiguous representations. The contiguous representations appeared to be derived through simple transformations in the proximity of primary units within each aggregate unit. For example:

<u>Task: 3 X (4 + 5) = (3 X 4) + (3 X 5)</u>

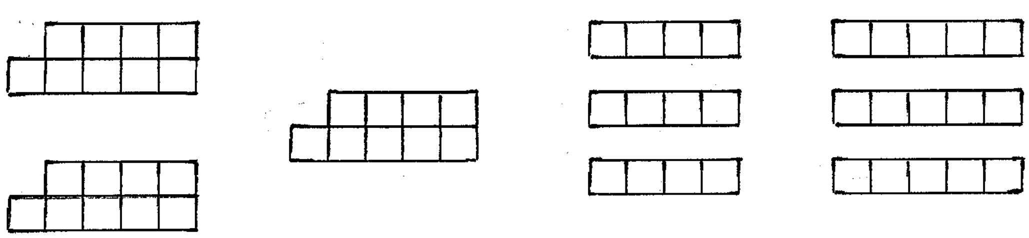
<u>Uncued Interview</u>

3 X (4 + 5)	(3 X 4) + (3 X 5)
O O O O O O O O O	Δ Δ Δ Δ Δ Δ Δ Δ Δ
O O O O O O O O O	Δ Δ Δ Δ Δ Δ Δ Δ Δ
O O O O O O O O O	Δ Δ Δ Δ Δ Δ Δ Δ Δ
=	=
2 7	2 7

Lara (Grade 7)

<u>Cued Interview: Partially Contiguous</u>

3 X (4 + 5)   (3 X 4) + (3 X 5)



Lara (Grade 7)

Extent of repertoires: attributes of area or length. Two-thirds of the students constructed discrete and contiguous representations during the two generative interviews. All contiguous representations were based on parts of regions, not line segments. Linear measurement was not a framework used for the construction of any representations. There are four factors which suggest that attributes of area measurement also were not considered by students as a basis for their contiguous representations.

1. As described above, students appeared simply to have altered the proximity of discrete units so that some units were contiguous.

2. The representations were partially contiguous. In all but one instance, A X B was not represented as a multiplicative relationship between units within a single region.

3. The units in contiguous representations were described by students as being analogous to units with discrete materials rather than analogous to units constructed by partitioning regions.

4. None indicated that parts within a region should be equal or adjusted their representations in a manner which would suggest such a concern during the uncued generative interview. During the cued generative interview, none of the students partitioned regions into units.

The line was not used independently as a spatial framework with which to construct a representation of a multiplication task. Four students (Connie, Edwin, Pete, Kit), when later asked if the line could be used to explain a task, agreed that they could do so. However, units of length were not used by these students when they represented multiplication on the line. Instead, they represented units as discrete marks along the line. For example:

Task: 3 X (4 + 5) = (3 X 4) + (3 X 5)

"Four lines, then a little ways apart 5 lines, then make two more groups the same."

Pete (Grade 7)

Other students either rejected the blank line altogether, or defined the whole line or the whole sheet on which the line was drawn as one among many discrete units.

The absence of representations based upon linear measurement within students' repertoires was further emphasized by students' responses to questions about the usefulness of a centimetre ruler. Despite the fact that students and interviewer all used the term "ruler", the three students (Tammy, Dahlia, & Edwin) who attempted to use the ruler perceived it to be a set of discrete green units, ignoring the unpainted spaces (see diagram). The ruler was then rejected because there were an insufficient number of units. No explicit association of the ruler with units of length was made by any students in this mathematical context.

Ruler used during Cued Generative Interview (Reduced by 50 Percent).

In summary, measurement of numerosity appeared provide the sole basis for defining units regardless of whether the units were contiguous or discrete. Alternative materials which could have provided a framework for representations based on linear or area measurement were evaluated in terms of units of numerosity alone.

Function of representations in both generative interviews. As can be seen in Table 4.03, most students constructed representations during the cued generative interview which had the same function as those they constructed during the uncued generative interview. However, three students (Edwin, Lara & Kasey), who previously had constructed either mono-relational or bi-relational representations, constructed tri-relational representations of A X B X C =_ during the cued generative.

The process of constructing a tri-relational representation was a complex problem for those students who were successful. The most common interpretation of a numerical expression of multiplication was that A X B =_ was read from left to right as "A groups of B." With this interpretation, the task 2 X 3 X 4 =_ was read as [(2 X 3) X 4] "2 groups of 3", then "6 groups of 4." This interpretation resulted in two bi-relational representations: one that represented 2 groups of 3, and one that represented 6 groups of 4. This interpretation follows normal procedures for the calculation of the product.

To construct a tri-relational representation of 2 X 3 X 4 =_ the associative or commutative properties have to be applied in ways which conflict with the common interpretation described above. In one thinking strategy the commutative principle was applied to interpret the expression as (a) (2 X 3) X 4 then (b) 4 X (2 X 3), or "4 groups of 2 groups of 3" or "2 groups of 3, repeated 4 times."

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| | |     | | |     | | |     | | |
| | |     | | |     | | |     | | |
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In another thinking strategy the associative principle was applied to interpret 2 X 3 X 4 as 2 X (3 X 4) or "2 groups of 3 groups of 4" constructed as 3 groups of 4, repeated twice.

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| | | |     | | | |
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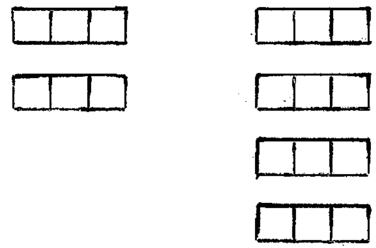
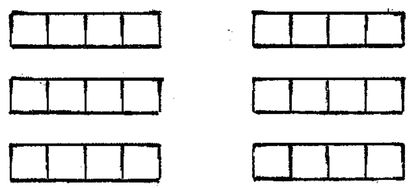
With the use of the associative and commutative principle these cases are generalizable to any combination of factors in the form X groups of Y repeated Z times.

Difficulties encountered by students who tried to represent the relationships among three units, are exemplified by Lara's responses during the cued generative interview. She began by trying to construct a representation of 4 X (2 X 3) to explain 2 X 3 X 4. She finished by constructing a representation of 2 X (3 X 4).

"You would have 2 times 3 and then that is a group of 6" (made 2 groups of 3). "And then you multiply that 2 times 3 by 4 more groups" (added 4 more groups of 3)



"I blew it! That will be two times three. It will equal 6 and then it is multiplied by 4. One has to equal that first to the 6 and then you have 4 more groups of that. .... Okay, I'm going to multiply that first" (i.e 3 X 4) (made 3 groups of 4) "and that equals 12. and you have to ... have to .... Okay, the 2 groups of 3 times 4. You have 2 groups, so there is that one and that one and you add them together and you get 24."



Lara's purpose from the outset was to construct one representation which would function to explain the relationship between three factors, having already determined that the product was 24. Her interpretation of "X 4" as "four more", and her association of this action with the group of 3 rather than the group of 6 led to a representation of (2 X 3) + (4 X 3).

Lara had to change her thinking strategies in order finally to construct a representation of this multiplicative process. Other students followed an even more protracted series of trials before achieving their goal of a tri-relational representation. All but one student who constructed a tri-relational representation had occasion to construct a series of bi-relational representations as steps towards representing the product. They then either reinterpreted their representation of 6 groups of 4 in terms of the three factors, or changed their thinking strategy to construct another representation in which the units were tri-relational.

Summary of characteristics of repertoires from both generative interviews. Some differences were observed with respect to the forms or functions of representations constructed in the uncued and cued generative interviews. Some of the students whose primary repertoires were exclusively discrete generated contiguous representations during the cued generative interview. In addition, some students constructed tri-relational representations whose representations previously had been only bi-relational.

Besides the differences between interviews in the form or function of some individual students' representations mentioned above, the general characteristics of the variety of representations generated remained the same. The students' bi-relational or tri-relational representations of whole number multiplication still were organized around rectangular spatial configurations within which units were more or less discrete. Discrete representations continued to be dominant for a majority of the students and a third of the students persisted in constructing only discrete representations with sets. Furthermore, the complexity with which different students were able to explain the multiplication tasks fluently continued to be limited to either mono-relational or bi-relational representations of units.

The extent of repertoires of representations appear to be limited as follows.

1. Representations based on regions were only partially contiguous with the aggregate units represented as separate regions.

2. Attributes of area measurement did not appear to be associated with contiguous representations. Instead, students described contiguous representations as being analogous to representations with discrete units in sets.

3. Attributes of length were not used as a basis for representing units and none of the students independently generated representations with a line as the spatial framework.

4. Measures of numerosity appeared to provide the quantitative framework for all forms of representations.

Comparison of Repertoires Reflected in the Generative Interviews With Responses in the Interpretive Interview Settings

The interpretive interviews were designed to explore more directly whether attributes of length or area measurement might be considered by students as salient features in their representations of whole number multiplication. To this end, attributes associated with representations based on linear or area measurement were distorted in beginning diagrams (see Figures 3.01 & 3.02). These distortions were designed to identify attributes students might consider to be critical for constructing a representation of whole number multiplication. Students constructed representations with the beginning diagrams they judged to be appropriate, and justified their rejection of other beginning diagrams.

Table 4.05 was designed to illustrate the general forms and function of the students' representations constructed in response to the interpretive interview tasks and allows for a comparison of these responses with students' responses during the generative interviews. The format of the table is similar to that used for Tables 4.01 and 4.03. Students constructed more than one representation during the linear or area interpretive interviews. When the form and function of these multiple representations were the same, only the general pattern of the responses is indicated.

The summary of the characteristics of students' primary repertoires is recorded to the far left along with the students' names. The functions of their representations from the uncued generative interview precede each student's name. The responses indicated in the table for the cued generative interview are a summary of the responses in Table 4.03. The responses indicated for the interpretive interviews are a summary of students' responses during those interviews. In the interpretive interviews only one of two forms of multiplication tasks were used: either A X B = __ or A X B X C = _.

The students are grouped according to the dominance of a form of representation within their primary repertoire as in Table 4.01. Within these groups they are ordered according to the extent to which they represented relations between different units during all interviews. Numbered groups indicate students with common characteristics in the function of their representations, regardless of their forms of representations. For example, students in Group 3 (Dahlia and Fanya) constructed only bi-relational representations throughout the four interviews. Students in Groups 4a to 4c constructed both bi- and tri-relational representations, but these groups differ in the frequency with which tri-relational representations were constructed.

Forms of representations in the interpretive interview setting. During the linear interpretive interview only one student (Tammy) represented the multiplication task with line segments as units (CL). All of the other students circled or marked off sets of points to represent units. The most salient feature of these linear beginning diagrams was the marks along the line and not the spaces between the marks.

Table 4.05

Forms and Functions of Representations of Multiplication in All Interview Settings.

		Interview settings by forms of representation																				
Summary of Primary Repertoire		Cued Generative							Linear Interpretive							Area Interpretive						
(Uncued)		DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR
Discrete																						
1.																						
mM	Connie	mM	M	.	.	.	.	.	m	m	.	.	.	.	.	m	.	.	.	m	.	.
mM	Marlene	mM	.	.	.	.	.	.	.	m	.	.	.	.	.	.	.	.	.	m	.	.
2.																						
bM	Brock	bM	.	.	.	.	.	.	.	b	.	.	.	.	.	.	.	.	.	b	.	.
bM	Derek	mbM	.	.	.	.	.	.	b	mb	.	.	.	.	.	.	.	.	.	b	.	.
3.																						
bB	Dahlia	.	.	.	.	BB	.	.	B	B	.	.	.	.	.	B	.	.	.	.	.	.
4a.																						
bB	Lara	.	.	.	.	BTB	.	.	.	B	.	.	.	.	.	.	.	.	.	B	.	.
bMB	Edwin	TB	B	.	.	BT	.	.	.	B	.	.	.	.	.	.	.	.	.	B	.	.
4b.																						
bB	Kasey	T	.	.	.	B	.	.	B	B	.	.	.	.	.	.	.	.	.	T	.	.
bB	Pete	.	B	.	.	B	.	.	.	T	.	.	.	.	.	.	.	.	.	T	.	.
bB	Tammy	B	.	.	.	B	.		.	.	.	T	.	.	.	.	.	.	.	TB	.	.
4c.																						
bMT	James	b	.	.	.	.	.	.	.	BT	.	.	.	.	.	BT	.	.	.	.	.	.
Discrete Dominant																						
4a.																						
bB	Lolande	B	.	.	.	.	.	.	.	B	.	.	.	.	.	.	.	.	.	T	.	.
4c.																						
bBT	Coran	BT	.	.	.	.	.	.	.	B	.	.	.	.	.	.	.	.	.	T	.	.
bMT	Kit	T	B	.	.	T	.	.	T	B	.	.	.	.	.	.	.	.	.	B	.	.
Contiguous Dominant																						
3.																						
bB	Fanya	.	.	.	.	B	.	.	.	B	.	.	.	.	.	.	.	.	.	B	.	.

Note. Names of Grade 5 students are underlined
Dots within the table means that the form was not used
Unit relations: lower-case indicates task A X B, upper-case indicates a complex task.

Framework of units	Units	Unit Relations
S = sets	D = discrete	M = mono-relational
L = lines	C = contiguous	B = bi-relational
R = regions	U = undefined	T = tri-relational

Four students, (Derek, Dahlia, Kasey, & Kit) initially rejected the segmented line as a good beginning diagram in general. They argued for representations based on discrete sets in a rectangular configuration (DS), even though they eventually represented units as points on a line (DL). Dahlia, the most adamant of these students, stated unambiguously, "I don't like lines," and rejected all subsequent diagrams after her first representation using points on the line as units. The dominance of two-dimensional configurations in their repertoires conflicted with the task of constructing a one-dimensional representation, and made the use of these linear beginning diagrams strongly improbable to the students.

During the area interpretive interview, the dominance of contiguous representations contrasts markedly with the dominance of discrete representations during all other interviews. Only three students (Connie, James & Dahlia) constructed representations with discrete units. All others used partially or totally contiguous units in their representations. However the use of contiguous units did not imply necessarily that the students were basing their representations on notions of area measurement. This issue is addressed in a later section.

Functions of representations in all interview settings. Considering the function of the representations constructed by students in all interviews, some patterns are discernible which suggest different limits in the extent to which students represent relationships between different units. As can be seen in Table 4.05, some students were relatively stable in the manner in which unit relationships were represented. Some constructed only mono-relational representations (see Group 1) regardless of the tasks, or only mono-relational representations of the more complex tasks (Group 2) Others constructed only bi-relational representations (Group 3). However, the balance of the students were unstable in their representations of unit relationships when explaining the more complex tasks, fluctuating between bi-relational and tri-relational representations (Groups 4a, 4b, & 4c).

Students' fluctuations between bi- and tri-relational representations reflect two factors associated with the representational process. The first factor was that both tri-relational and bi-relational representations of units were valid in terms of different objectives. One objective was to explain a procedure for calculating A X B C. This would result in bi-relational representations of A X B = D and D X C = E. Another objective was to explain the relationships among all factors and the product. This would result in a tri-relational representation. In some cases, fluctuations were associated with a student's change in objective.

The second factor was that, when students set the construction a representation of the relationships among all factors and product as their objective, tri-relational representations were difficult for most students to achieve. To plan and construct a tri-relational representation requires an analysis of the relationships among the factors which does not follow that of normal calculating procedures. Instead of the calculating steps (A X B) X C, either A X (B X C) or C X (A X B) has to be constructed. As a result, bi-relational representations at times preceded or substituted for attempts to construct tri-relational representations. Only three to five students constructed tri-relational representations in any one interview and no student was consistent in the construction of tri-relational representations across all interviews.

Characteristics of the form of representations and students' attention to attributes of length and area measurement. The extent to which units were represented as contiguous and the extent to which students attended to attributes of length or area measurement are illustrated in Table 4.06. The columns code three degrees of contiguity in the students' representations: none, partial and total contiguity. In the body of the table students' attention to the equality or inequality of the area of contiguous regions in students' representations in the generative interviews are coded (E, U, #). The latter code (#) indicates that the attributes of equality or inequality were not determinable. The students are grouped within the table first according to their tendency to require that beginning diagrams have line segments or regions which were equal in length or area, and second, according to their use of equal contiguous units during the generative interviews.

All students rejected some beginning diagrams and gave aesthetic rather than quantitative reasons for doing so. However, students expressed different tolerances for attributes in the beginning diagrams which are not associated with either a number line or a rectangular grid of square units. The beginning diagrams in which attributes were most distorted were rejected by more than two-thirds of the students (see Figure 3.01, #4, and Figure 3.02, #3). They were "messy", "too hard to see", "too

Table 4.06

Extent of Contiguity of Units and Form of Contiguous Units: Multiplication

	Interview contexts by extent of contiguity of units											
	Uncued			Cued			Linear			Area		
Groups	N	P	T	N	P	T	N	P	T	N	P	T
1.												
Connie	#	.	.	#	.	.	U	.	.	#	U	.
Marlene	#	.	.	#	.	.	U	.	.	.	U	.
Brock	#	.	.	#	.	.	U	.	.	.	.	U
Derek	#	.	.	#	.	.	U	.	.	.	.	U
2.												
Kasey	#	.	.	#	E	E	#	.	.	.	U	.
Tammy	#	.	.	#	E	.	.	.	U	.	.	U
3.												
Dahlia	#	.	.	.	U	.	E	.	.	U	.	.
Fanya	#	#	.	.	U	.	U	.	.	.	.	E
Lolande	#	#	.	#	.	.	U	.	.	.	.	E
4.												
Coran	#	E	.	#	.	.	U	.	.	.	.	E
Pete	#	.	.	#	E	.	U	.	.	.	.	E
Edwin	#	.	.	#	.	E	U	.	.	.	.	E
5.												
James	#	.	.	#	.	.	E	.	.	E	.	.
Lara	#	.	.	.	E	.	E	.	.	.	E	.
Kit	#	E	.	#	E	.	E	.	.	.	.	E

Note. Names of grade 5 students are underlined.

Extent of Contiguity	Units
N = none contiguous	E = equal space
P = partially contiguous	U = unequal space
T = totally contiguous	# = equality indeterminable

bumpy," "too squiggly," or "confusing." Whereas, fewer students questioned the appropriateness of other beginning diagrams which had less distortion. In addition, regularity in vertical/horizontal orientation of units was sought by some students. There is less distortion between squares and rhombi than between squares and irregular polygons (see Figure 3.02, #2 & #4). Yet, the sense of imbalance of the rhombi off the vertical was tolerated less than the irregularity of polygons (Figures 3.02, #1 & #4). For some, the congruency of units was necessary but not sufficient; for others horizontal/vertical orientation took precedence over congruency. In general, perceptual orderliness appeared to be a critical characteristic sought in different ways by students.

The use of contiguous units did not imply that continuous measurement was used as a framework for the representations. First, students constructed partially or totally contiguous representations without the congruency of the units being of importance to them (see Groups 1 & 2). The only student to use contiguous units during the linear interpretive interview (Tammy) did so with unequal line segments. Congruency was not a basis on which these students accepted or rejected beginning diagrams. Second, students in Groups 3, 4 & 5 generally were more consistent in their need for equal parts when using contiguous units, but the congruency was not related to ideas about equal measures of space. Instead, the need for perceptual orderliness was used to justify their rejection of beginning diagrams in which parts were not congruent.

The need for perceptual orderliness did not imply that the students' were making judgements on the basis of properties of length or area measurement. Even students who insisted that parts be equal in both interviews used only discrete units during the linear interpretive interview (see Group 5). In the most extreme case, James constructed only discrete representations during all interviews, yet accepted only beginning diagrams with equal parts during both interpretive interviews. They were not associated with representations based on the comparisons of measures of one or two-dimensional space. Perceptual orderliness appeared to be a characteristics of importance to students in representations in general. However, the importance placed on perceptual orderliness at times resulted in beginning diagrams with congruent parts being selected in preference to ones without congruent parts. Applying the most restricted criteria for perceptual orderliness, students could construct unwittingly representations of whole number multiplication which would be identical to instructional representations based on area measurement.

In summary, students had clearly defined criteria regarding appropriate characteristics of representations of whole number multiplication. Their criteria often resulted in a replication characteristics of units in instructional representations based on length or area measurement. However, the correspondence of students' judgements with attributes of units of length or area measurement did not imply that students used length or area measurement as a framework for representing whole number multiplication. Students evaluated beginning diagrams in terms of aesthetic criteria alone. Some students' may have developed ideas about appropriate physical characteristics of a representation from experiences with instructional representations based on linear and area measurement. However, in the context of whole number multiplication, these physical characteristics are not associated by them with linear or area measurement.

Summary of the Characteristics of Students' Repertoires of Representations of Whole Number Multiplication.

The students' primary repertoires, with few exceptions, also characterized the representations which students constructed when materials were present. Students generally used or evaluated materials in terms of their dominant form of representation in their primary repertoire. In all but one case, the dominant form of representation was discrete units within sets.

Contiguous representations included in students' repertoires were based on regions. For the most part, the extent to which the units within a representation were contiguous was limited. These representations were more closely associated with characteristics of discrete representations than characteristics of representations based explicitly on area measurement. Through a simple transformation of the proximity of units, one form of representations could be derived from the other. There was no direct evidence that area measurement was thought to provide a framework for defining units in contiguous representations. Instead, attributes of units of area measurement such as congruency of units were associated by students with perceptual orderliness rather than associated with measures of two-dimensional space. Regardless of the regularity of units in a representation, contiguity seems to be an incidental characteristic of discontinuous units which measure only numerosity.

Students' repertoires were limited in at least one of three ways.

1. Mono-relational or bi-relational representations were limits at which students could fluently construct representations of unit relationships. No student consistently constructed tri-relational representations, and only one student was fluent in her construction of such representations

2. The extent to which units were contiguous was limited in some students' repertoires. In one case units were always discrete, in five cases the units were only discrete or partially contiguous.

3. Forms of representations were excluded from some or all students' repertoires. Representations based on linear measurement were excluded from all repertoires, and contiguous representations were excluded from individual repertoires.

Students' Repertoires of Representations: Common Fractions and Comparisons of Common Fractions

The same general research questions regarding the characteristics of students' repertoires of representation of whole number multiplication guided the analysis of the characteristics of repertoires of representations of common fractions and their comparisons. These are:

1. What are the characteristics of students' primary repertoires of representations?
2. To what extent are the characteristics of representations in students' repertoires influenced by the material setting within which representations are constructed?
3. To what extent do students' repertoires of representations include forms of representations based on attributes of length or area?
4. To what extent do students' representations have the function of representing relationships between different units?

The general organization of the analysis of students' repertoires of representations in the common fraction context is the same as that which was followed in the whole number multiplication context. The two general characteristics of repertoires, the form of the representations, and the function of the representations within repertoires, provide the general analytical framework. This framework guided the analysis of the characteristics of students' repertoires of representations of common fractions and the comparisons of common fractions. Within this framework, the results of the analysis of the data from the uncued interview are considered to represent the characteristics of students' primary repertoires of representations. The primary repertoires are compared and contrasted with students' responses in the other interview settings with regard to the forms and function of students' representations. Through the successive comparisons of students' primary repertoires with their responses in the other interview setting, the issue of the extent of their general repertoires will be addressed. Finally, the over all analysis will be drawn upon to address the question of the characteristics of students' repertoires of representations of common fractions and the comparison of common fractions.

Students in both grades were presented with tasks which required them to compare two common fractions with unlike denominators. Because Grade 5 students would have had less formal experience with relationships between common fractions than Grade 7 students, they were given tasks to explain the meaning of a common fraction before being presented with comparative common fraction tasks. The selection of tasks used in these interviews are in Chapter 3, p. 46-47.

A variety of interpretations of common fractions were used by students in their explanations of their representations. These interpretations are used along with the categories related to the form and function of representations to characterize students' representations of common fraction concepts and comparisons. The interpretations are:

1. Cardinal values interpretation (Cv): Students represented only the cardinal values of numerators or denominators. There were two ways in which students expressed this interpretation. One way was that "a/b "was represented as the value of "a" as well as the value of "b." The other way this interpretation was expressed was that "a/b "was either represented as the value of "a" or represented as the value of "b". Comparisons were based on the direct comparisons of cardinal values, or on comparison of the sums of the values of numerators and denominators. In the latter case, a/b vs c/d would translate into (a+b) vs (c+d).

2. Half of a number interpretation (Hn): The student expressed 1/b as half of any number. For example 1/2 might mean 3 out of 6, and 1/4 might mean 2 out of 4, or half of 4. Regardless of the denominator, the reference was to represent a half.

3. Relative size interpretation (Rs): The student compared common fractions with notions about relative sizes of common fractions. There was no explicit reference to any relationship involving numerators or denominators.

4. Inverse size of the denominators interpretation (Id): The student made comparisons only by reasoning about the denominators. The premise of the representations was that the larger the value of the denominator, the smaller the size of the part. Hence, the common fraction with the larger denominator was the smaller fraction regardless of the values of the numerator. For example, 3/4 was less than 2/3 because 1 part would be a smaller piece than the other.

5. Multiplication interpretation (Mu): The student expressed "a/b" as "a x b" and compared the values of the products. For example, 2/6 was 2 groups of 6, or 2/3 was 3 groups of 2. Therefore 2/6 was greater than 2/3 because 12 was greater than 6.

6. Take away interpretation (Ta): The student expressed "a/b" as "b-a" or as "b-c=a" and related the meaning to a removal action. For example, 3/4 might be explained by "4 pieces of pie and 3 were eaten" or "4 chairs, one was broken, and 3 good chairs were left." The use of this explanation did not preclude the representation of part-whole relationships of bi- or tri-relational units.

Characteristics of Primary Repertoires of Representations: Common Fraction Context

The questions addressed in this section are:

1. What are the characteristics of students' primary repertoires of representations used to explain the meaning and comparisons of common fractions? In particular:
   a. Is there a form of representation which dominates within their primary repertoires?
   b. How extensive are their primary repertoires with regard to the variety of forms of representations generated to explain common fraction tasks and comparison of common fractions tasks.
2. To what extent do the representations of common fractions and the comparison of common fractions in students' primary repertoires have the function of representing relationships between different units?

Table 4.07 was designed to illustrate the nature of students' primary repertoires of representations of common fractions and comparisons of common fractions. The general format of this table is the same as that used for Tables 4.01, 4.03 and 4.05. The major categories for classifying the form of their representations are represented in the columns of this table. These categories include the

Table 4.07

Forms and Functions of Representations of Common Fractions: Uncued Generative Interview

Dominance of Forms & Interpretations		Interview tasks by form of representation													
		A/B							A/B vs C/D						
		DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR
Undefined Units															
Rs	Brock	.	.	.	.	.	.	N	.	.	.	.	.	.	N
Id	Lolande								.	.	.	.	.	.	N
Discrete & Contiguous															
Cv	Marlene	.	.	.	.	M	.	.	.	.	M	.	.	.	.
Hn	James	M	.	.	.	M	.	.	NR	.	.	.	.	.	.
TaCv	Edwin	M	.	.	.	M	.	.	M	.	.	.	M	.	.
Ta	Pete								B	.	.	.	B	.	.
TaMu	Dahlia	M	.	.	.	M	.	.	B	.	.	.	BT	.	.
Exclusively Contiguous															
Cv	Derek	.	.	.	.	M	.	.	NR	.	.	.	.	.	.
CvRs	Connie								.	.	.	.	M	.	.
Mu	Fanya								.	.	.	.	B	.	.
Ta	Kit	.	.	.	.	B	.	.	NR	.	.	.	.	.	.
Ta	Kasey								.	.	.	.	B	.	.
Ta	Lara								.	.	.	.	TB	.	.
Ta	Coran								.	.	.	.	T	.	.
Ta	Tammy								.	.	.	.	T	.	.

Note. Names of grade 5 students are underlined
Dots mean the form was not used; blanks mean the task was not given.

Units
D = discrete
C = contiguous
U = undefined

Spatial Framework
S = sets
L = lines
R = regions

Unit Relations
M = mono-relational
B = bi-relational
T = tri-relational
N = no units
NR = no response

Interpretations
Cv = cardinal values (numerator or denominator)
Hn = 1/2 of a number
Rs = relative sizes
Id = inverse denominator
Mu = multiplication, a/b = a x b
Ta = take-away, a/b = b - a

spatial framework of the representations, (sets, lines and regions), and the form of units within the spatial frameworks, (discrete, contiguous, and undefined). For each form of representation constructed by a student, the function of the representation with regard to unit relations (M, B, T, N) is coded within the table. The latter symbol (N) indicates representations in which units are undefined. The interpretations used to explain the meaning of common fractions during the uncued generative interview are coded in the first column of the table to the left of the students' names.

Students constructed more than one representation for one or more tasks. When the form and function of these multiple representations were the same, only the general pattern of the responses is indicated. In addition, a dot in the body of the table indicates that the form of representation was not generated by the student. Blanks indicate that the task was not presented to the student.

There are two criteria used to position the students within the table. First, they are grouped according to the dominance of a form of representation within their repertoire. Second, they are ordered within the group according to the extent to which they represented relations between different units.

Characteristics of the Forms of Representations in Primary Repertoires.

As can be seen in Table 4.07, more than half of the students generated repertoires of representations in which units were represented only as contiguous parts. Two students generated repertoires of representations in which no units were defined. The balance of the students generated repertoires in which no single form of units dominated. Instead they generated discrete and contiguous representations of units with equal frequency.

Regions were the most common spatial framework for representations of common fractions (DR, CR, & UR), regardless of the form of the units represented (whether discrete or contiguous). They were used by all students in at least one instance, and exclusively by two-thirds of the students. Even with students who generated a representation based on sets, their first representation of common fraction tasks was generally with regions. No students used a line as a spatial framework for their representations. Hence, for nearly all students, regions played a prominent role as a primary spatial framework for constructing representations of common fractions or comparisons of common fractions.

Table 4.08

Extent to which Different Regions were used to Construct Representations of Common Fractions (Uncued Generative Interview).

		Types of Regions		
Extent[1]	n[2]	Circle	Rectangle	Irregular
Exclusive	7	6	1	
Dominant	6	4	1	1
Secondary	8	4	6	2

Note. 1. Exclusive means all, dominant means more than half, secondary means half or less of a student's discrete representations used the configuration.
2. Total n = 15 Students are counted more than once in the "dominant" and "secondary" categories.

The types of regions used as a spatial framework for representations were limited. As can be seen in Table 4.08, circles were used most frequently when representations were based on a spatial framework of regions. Furthermore, of the 14 students who used circles, thirteen used one as the spatial framework for their first representation of common fractions. For most students, the response pattern during the interview was: (1) represent a task with circles and (2) in some cases, construct representations with other regions (generally a rectangle) or with sets.

Characteristics of forms of representations based on regions. Students encountered considerable difficulty in partitioning regions into equal parts. Efficient partitioning procedures require two forms of preliminary analysis. First is an analysis of the geometric characteristics of the region which determines what kinds of configuration will partition a region into equal parts (see Figure 4.03). Second is an analysis of number-theoretic characteristics of the task. No students consistently evidenced preliminary analyses of all relevant characteristics, particularly when partitioning circles. The successive halving algorithm was used with the radial configuration to partition circles, regardless of the value of the denominator (see Lara, Pete, and Kasey in Figure 4.04). Others displayed difficulties in anticipating the outcome with the hatched configuration when applied to rectangles (see Dahlia and Pete, Figure 4.04). These configurations required students to use factors when denominators were neither prime nor powers of two. However, some students had a variety of partitioning techniques including multiplicative partitioning with factors, but these techniques were not well defined in terms of the conditions in which they could be applied most effectively.

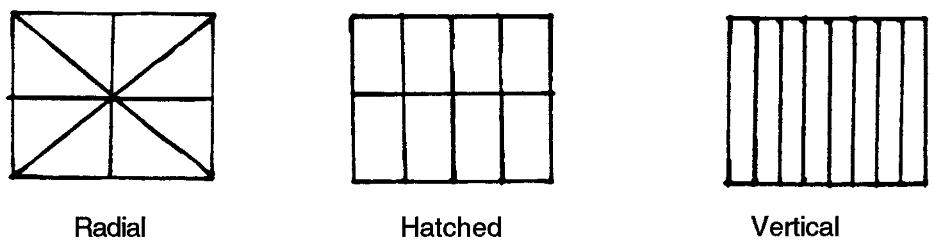


Figure 4.03 Examples of standard configurations used to partition regions for common fraction representations.

Although students' partitioning techniques might have been an important factor in the extent to which students constructed equal parts within a region, this was generally not the case. In the following three cases, the students encountered similar partitioning difficulties, yet "resolved" the problem differently.

Case 1. Number of parts more critical than equal parts: first partitioned the circle with the successive halving algorithm into 8 instead of 9 equal parts, then one of the eighths was halved to make 9 parts altogether (See Kasey, Figure 4.04). No attempts made to adjust the partitions. The comparison of thirds and ninths was confidently based on the actual space shaded in each circle, including the 2/16ths as 2/9ths.

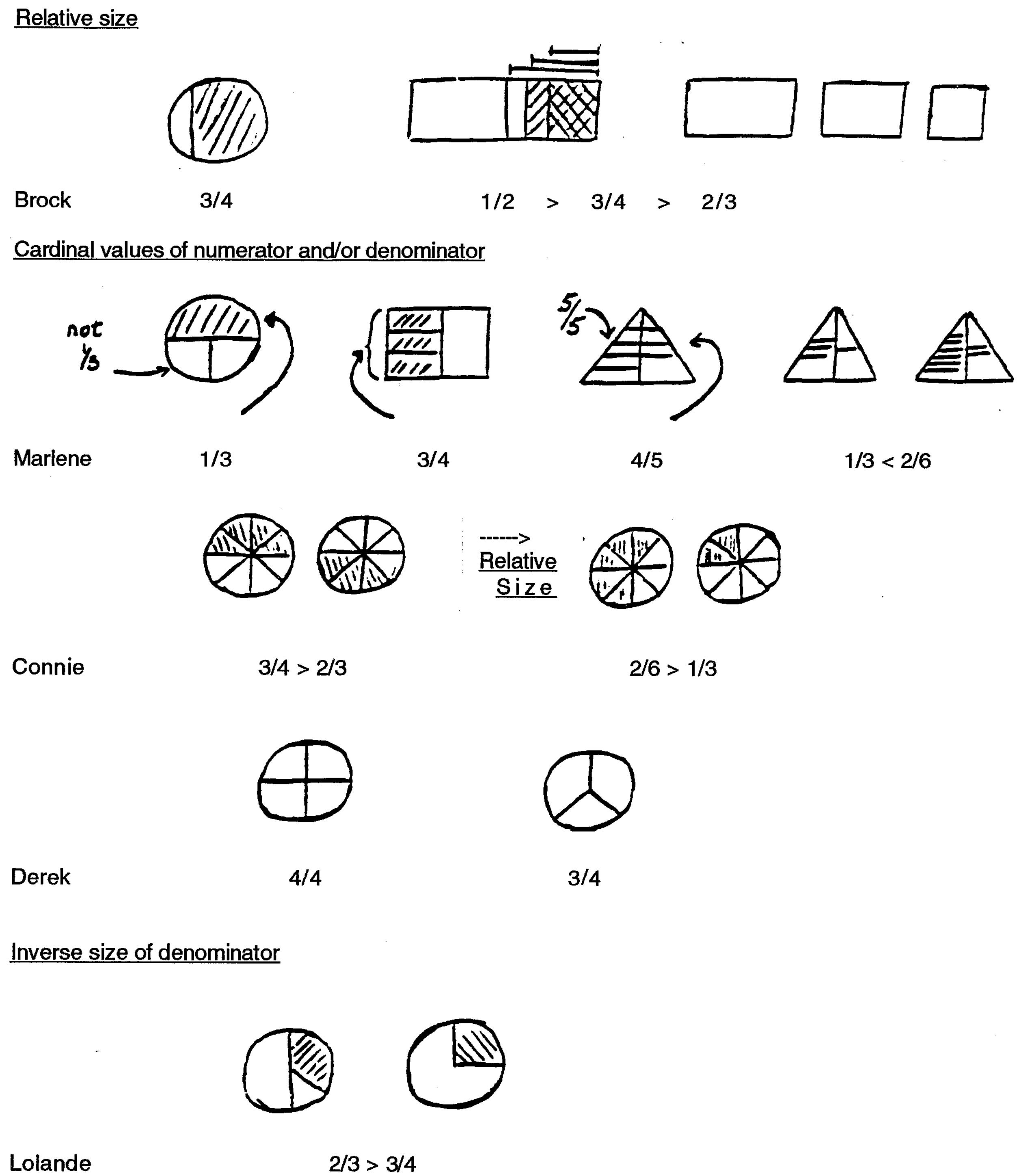


Figure 4.04 Examples of common fraction representations with regions as they relate to interpretations. (Continued on next page)

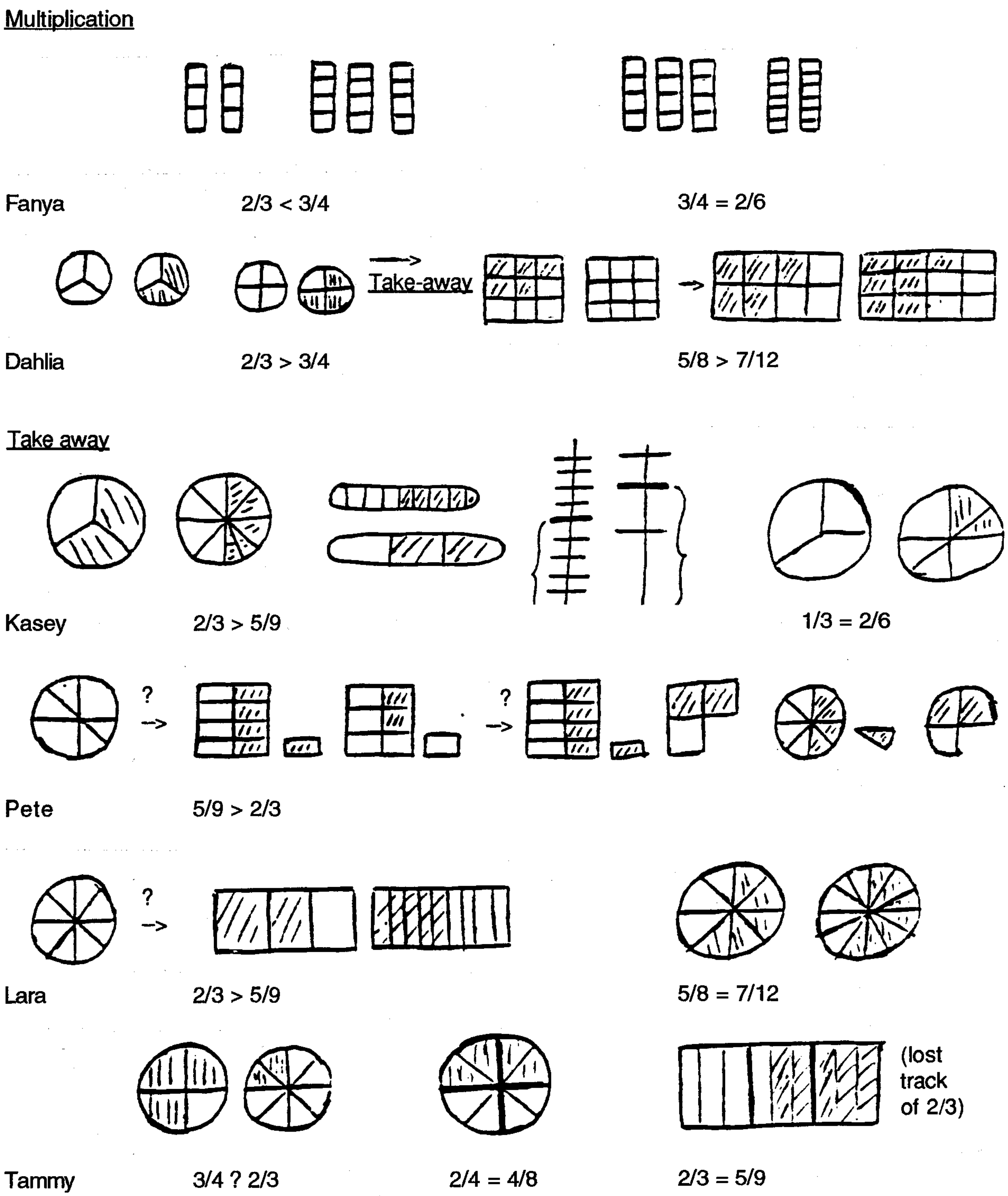


Figure 4.04 (Continued) Examples of common fraction representations with regions as they relate to interpretations.

Case 2. Equal parts are critical, but equal regions are not: partitioned a rectangle into 8 instead of 9 parts, so an additional but separate ninth piece was drawn the same size as those within the rectangle. The parts were then compared proportionally, but not the regions (See Pete, Figure 4.04).

Case 3. Equal parts and equal regions are critical: partitioned a circle with the successive halving algorithm into 8 instead of 9 parts, or a square with the hatched configuration into 9 instead of 8 parts (See Lara & Dahlia Figure 4.04). Either the type of region or the partitioning strategy was then altered in order to achieve the desired number of equal parts.

The students in the first two examples did not attempt to alter the partitions further. Nor did they comment on the inequality of the parts or the regions. They simply did not appear to consider their solution to be problematic. Differences in students' solutions suggested differences in their beliefs about which characteristics of units were critical in a representation of common fraction comparisons, rather than differences in technical abilities to partition the regions.

The ways in which regions were used to represent common fractions varied widely (see Figure 4.04). However, individual students appeared to hold relatively consistent beliefs about critical attributes of representations based on regions. These beliefs seemed to result in representational "procedures" or algorithms. For example:

1. Students had consistent procedures regarding how regions were used as a spatial framework for representing units (E.g., Marlene, Connie, Derek & Fanya, Figure 4.04).
2. Students expressed a belief that equal regions were important when comparing common fractions. However, units within the regions were interpreted as measures of numerosity (E.g., Marlene & Fanya).
3. Students' expressed beliefs that equal parts were important when representations were interpreted as measures of numerosity (see Connie & Derek). As well, equal parts were compared in terms of quantities of area without including comparisons of equal regions (see Pete).

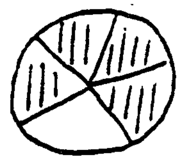
None of the students in the previous examples constructed a standard interpretation of common fractions based on measurement of area. Yet each student imposed some procedures on the form of their representations that harkened to attributes emphasized in area-based instructional representations. Beliefs about the form of representations were disassociated from the measurement framework on which instructional representations were based, yet their beliefs mirrored selective characteristics of such instructional representations.

Whether students thought in terms of measures of numerosity or measures of area was not always explicit, particularly when individual common fraction was represented with the take-away interpretation of a common fraction. The language of the take-away interpretation is sufficiently ambiguous that the verbal explanation of the representation could imply measures of numerosity or area. Likewise, a representation might contain all physical characteristics consistent with a standard area-based representation, but be thought of by the student in terms of units of numerosity alone.

Edwin's contiguous representations of individual common fractions and comparisons of common fractions provide an example of the ambiguities associated with representations based on regions and the take-away interpretation. He consistently expressed a take-away interpretation in a context of sharing pie (see Figure 4.05). Food was shared equally, and pieces were eaten and left over. However, his apparent belief in the need for all parts to be equal when representing a single common fraction (4/5) was not reflected in the comparison task. Neither the size of the left-overs nor the equality of the original pies was a concern. However, the notion that people in a group receive fair shares was consistent. The fair shares of pies were not related to his final comparison of 3/4 and 2/3. Instead, 3/4 was larger simply because of its greater numerosity. His social meaning of common fractions was disassociated from his mathematical meaning of common fractions. This could not have been anticipated from his take-away explanation and contiguous representation of four fifths. His representation of 4/5 and the language he used to explain the representation had the appearance of a standard representation and interpretation of a common fraction.

Task: Explain 4/5

"... you have five pieces of a pie and you take four out.

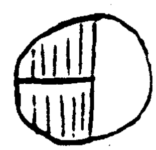


But one piece is like big (i.e., he drew it too big), but it is just the same size. You have five pieces here and one guy eats this piece, so this one is gone (shades pieces as he talks). Another guy eats this piece so this one goes. Another one eats this so this one goes. Another one eats this one and another one eats this one. All the people have aten (sic) it so they give it to the cat (i.e., the left-over piece)."

Task: Compare 2/3 and 3/4

"...they only got a half a piece of pie, like the man made a mistake at the wrong house, so they gave him only a half a pie. ....There was only four people in the family so they had to make it like this. So they go and eat three and a guy wasn't hungry so he didn't eat his piece. That would be three fourths.

Three fourths



Two thirds

Here is the two thirds. They delivered a whole pie to the other house. So they just split it in half and then two thirds. So they ate this piece (amount shaded)."

Figure 4.05 Representations of common fractions with alternative beliefs of equal parts: Edwin (Grade 5)

In summary, it could not be assumed that area measurement played a role in representing quantitative relationships between units even when a student's contiguous representation of a common fraction conformed to standard area-based representations. Students expressed beliefs that the parts, the regions, or both ought to be equal in size even when measurement of area played no quantifying role. These ideas about the form of the representation mirrored characteristics of contiguous representations probably stressed during instruction. Only when comparisons of common fractions were represented was the quantitative role of measurement of numerosity or area explicit. In the comparative context, some students integrated characteristics of units of area measurement as attributes critical to their representations of common fractions, but the consistency with which they attended to these attributes varied from task to task.

Characteristics of discrete forms of representations. Discrete representations of a common fraction generally mirrored characteristics of the student's contiguous representations (see Figure 4.06). When representing a common fraction with a take-away interpretation, the language, the actions, and the results of discrete and contiguous representations are the same. For example, in Edwin's representations of 3/4 in Figure 4.06, the action associated with the meaning of the units was the same: 4 pieces of pie or 4 chairs, 3 were eaten or used, and one remained. These actions were expressed by the same phrases: "3 of the 4 __ were __." Whether a student considered measures of numerosity and measures of area was of no immediate consequence when explaining the meaning of an individual common fraction with a take-away interpretation. The same could be said of these forms of representations applied to comparisons of common fractions with like denominators. Changes in the proximity of the units still would not alter the result of the comparison.

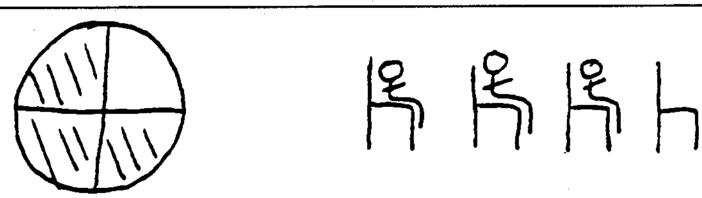
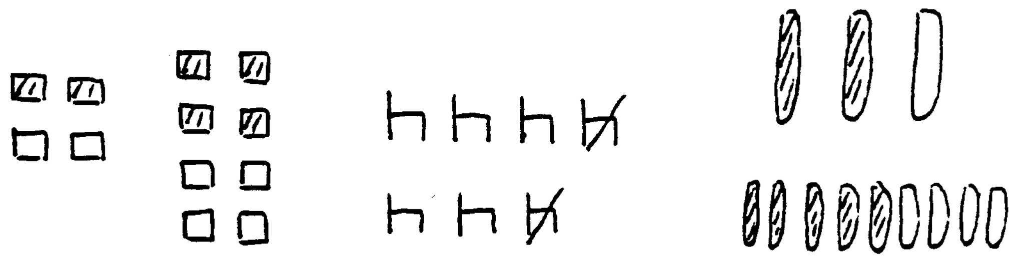


Figure 4.06 Discrete and contiguous representations of three fourths: (Edwin, Grade 5)

Students' discrete and contiguous representations of comparisons of common fractions with unlike denominators also differed only in the proximity of the units. No students represented comparisons with discrete sets by establishing the whole unit as a denominator common to both fractions (see Figure 4.07). Considering those who used a take-away interpretation, each student was able to rationalize their comparisons with discrete representations without facing any conceptual conflict. Edwin simply compared the numerosity of units, regardless of the form of the representation. Pete incorporated attributes of area in the discrete units to maintain a 2:1 part-part comparison in his representations. Dahlia interpreted both representations as equivalent to a half and thereby ignored quantitative disparities in her representation.



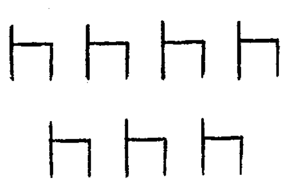
2/4 = 4/8 Dahlia | 3/4 > 2/3 Edwin | 2/3 < 5/9 Pete

Figure 4.07 Discrete representations of comparisons of common fractions.

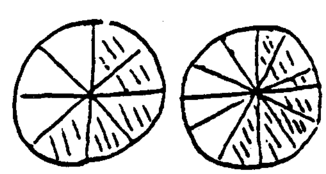
Functions of Representations of Common Fractions in Primary Repertoires.

The function of representations in students' primary repertoires was categorized in Table 4.07 by (a) the extent to which a representation expressed relationships between different units (M, B, T or N), and by (b) the interpretation students gave to their representations. There was no singular relationship between some interpretations students used and the extent to which students represented relationships between multiple units. Instead, a one-to-many relationship existed between some interpretations and representations of unit relationships. For example:

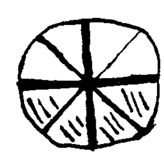
1. Students' cardinal values and multiplication interpretations were explained necessarily with mono-relational and bi-relational representations respectively. In contrast, comparisons with a take-away interpretation were explained with either mono-, bi-, or tri-relational representations, as in the following examples:



Mono-relational
3/4 > 2/3
(Edwin)

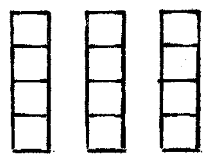


Bi-relational
5/8 = 7/12
(Lara)

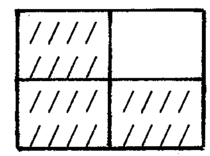


Tri-relational
2/4 = 4/8
(Tammy)

2. Similarly, tri-relational representations necessarily expressed students' comparisons with a take-away interpretation. In contrast, mono-relational representations expressed cardinal values, or take-away interpretations. Bi-relational representations expressed either a multiplication or take-away interpretation as in the following examples.



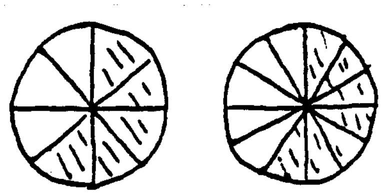
Bi-relational
Multiplication
3/4 (Fanya)



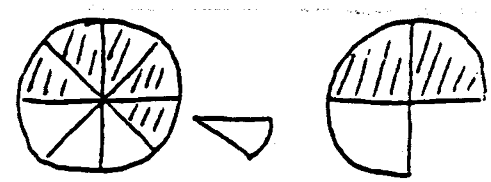
"4/4 equals 1 whole"
Bi-relational
Take-away
3/4 (Kit)

Six students constructed <u>mono-relational representations of units</u>, five of whom did so regardless of the task. For most of these students their mono-relational representations were associated with a cardinal values interpretation of common fractions. However, two students (Edwin and Dahlia, Table 4.07) also constructed mono-relational representations with the take-away interpretation by comparing the number of pieces removed or left over.

<u>Bi-relational representations</u> of the comparison of common fractions with the take-away interpretation took on one of two forms: (a) the student did not attend to the relationship between different sizes of the unit fractions, or (b) the student did not attend to the relationship between the two <u>wholes</u>, as in the following examples.



a. "Equal, both 1/2 plus a bit more"
(Lara)



b. 1/3 = 2/9, 5/9 > 2/3
(Pete)

For two students, the bi-relational representations were associated with a multiplication interpretation. The explanation from these students was insufficient to determine a genesis for this interpretation. Possibilities are that it might derive from their experience with (1) the multiplicative algorithm for generating equivalent fraction or (2) the cross-multiplication algorithm for comparing common fractions.[2]

When students constructed tri-relational representations for the comparison of fractions, generally each of the common fractions was represented with a separate region. The approximate equality of two regions and units within each region established a basis for judging the comparative difference in the area representing each common fraction. This method limited the need to represent units nested in units, and the need to relate fractional units to a common denominator. Only one student (Tammy) represented fractional units nested in fractional units within a single region.

Not all students represented common fractions with units. Instead, they (Brock, & Lolande) used gross quantities within a region to express ideas about the relative sizes of common fractions. In each case there was a rudimentary notion about common fractions being part of something. However, the representation of the whole unit was not a critical feature for expressing the differences in the sizes of common fractions. Instead separate regions of different sizes were sufficient to represent ideas about the comparisons (see Brock, Figure 4.04).

Summary of Characteristics of Primary Repertoires of Representations of Common Fractions.

Considering the forms and the functions of the representations constructed by students during the uncued generative interview, a number of features characterized students' primary repertoires of representations of common fractions.

With regard to the form of representations, characteristics were:

1. The exclusive or prominent use of regions, in particular circles as a spatial framework.
2. Wide variations in the ways in which regions were used as a framework to represent comparative quantities of common fractions, including the representation of discrete or contiguous units as well as representations without units.

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2. A similar procedures was used by a student in Hunting (1983, pp. 193-194). However, Hunting does not identify the procedures as an alternative interpretation of common fractions derived from some previous experience. Instead he explains the behaviour as a "freely invented algorithm."

3. Physical attributes such as equal regions or parts were believed to be important regardless of the interpretations of common fractions and the measurement system referenced in the representation.

4. Selective and variable attention to attributes of units area measurement when constructing and interpreting representations of common fractions.

5. Similarities between a student's discrete and contiguous representations such that the only substantial difference was in the proximity of the units.

With regard to the functions of the representations constructed:

1. Interpretations of common fractions influenced but did not always determine the extent to which students represented multiple relationships between units.

2. The implicit relationship of fractional units to the whole unit was not made explicit in their explanations when using a take-away interpretation. Explanations of the meaning of single common fractions were given generally with mono-relational representations.

3. Comparisons of common fractions were explained with take-away interpretation by just over half of the students. However, mono-, bi- and tri-relational representations were constructed to explain comparisons with this interpretation.

4. Explanations with other interpretations necessarily involved mono-relational or bi-relational representations, or representations without explicit units.

Characteristics of General Repertoires: Representations of Common Fractions in Other Material Settings

The questions addressed in this section are:

1. To what extent are the characteristics of representations of common fractions in students' repertoires influenced by the material setting within which representations are constructed?
2. How extensive are their general repertoires with regard to the variety of forms of representations used to explain common fraction tasks.
    a. To what extent are particular forms of representations excluded from their repertoires?

b. To what extent do students' general repertoires of representations include forms of representations based on attributes of length or area?

The other three interview settings (cued generative, linear interpretive, and area interpretive interviews) were designed to explore whether or not the primary repertoires of representations generated in the uncued interview represented the extent of students' repertoires. In this section, the students' responses to the common fraction tasks during the other three interviews are compared and contrasted to the characteristics of the students' primary repertoires of representations generated during the uncued generative interview.

The representations of common fraction tasks constructed during the cued generative interview which differed from those generated during the uncued interview were considered to indicate extensions of students' repertoires. As well, materials which students rejected during the cued generative interview were considered to be one indication of limits of repertoires. Students' acceptance or rejection of beginning diagrams (see Figures 3.01 and 3.02) during the linear and area interpretive interviews was considered to be another indication of limits of repertoires.

This section is organized in two parts. In the first part, the primary repertoires are compared to the forms and functions of the representations constructed during the cued generative interview. In the second part, the responses in all of the interview settings are compared and contrasted.

<u>Representations Constructed in the Cued Generative Interview Compared to the Uncued Generative Interview</u>.

Table 4.09 was designed to illustrate students' representations constructed during the cued generative interview. Also included is a summary of their responses during the uncued generative interview which represent their primary repertoires. The format of the table is similar to that of Table 4.07. The major categories for classifying the form of their representations are represented in the columns of this table. These categories include the spatial framework of the representations, (sets, lines and regions), and the form of units within the spatial frameworks, (discrete, contiguous, and undefined). For each form of representation constructed by a student during the <u>cued generative interview</u>, the function of the representation with regard to unit relations is coded within the table.

Table 4.09

<u>Form, Function and Interpretation of Representations of Common Fractions: Cued Generative Interview with a Summary of Primary Repertoires</u>

Primary Repertoires (Uncued) Forms, Functions & Interpretations			Cued Interview Interpretations	Cued Interview tasks by form of representation: A/B DS	A/B DL	A/B DR	A/B CL	A/B CR	A/B UL	A/B UR	A/B vs C/D DS	A/B vs C/D DL	A/B vs C/D DR	A/B vs C/D CL	A/B vs C/D CR	A/B vs C/D UL	A/B vs C/D UR
<u>Undefined Units</u>																	
	Rs	<u>Brock</u>	Rs	.	.	.	.	.	N	N	.	.	.	.	.	.	N
	Id	Lolande	Id								.	.	.	.	.	.	N
<u>Discrete & Contiguous</u>																	
M	Hn	<u>James</u>	HnTaCv	M	.	.	.	M	.	.	M	.	.	.	.	.	.
M	Cv	<u>Marlene</u>	Cv	M	.	.	.	M	.	.	M	.	.	.	M	.	.
M	TaCv	<u>Edwin</u>	TaCv	M	.	.	.	M	.	.	M	.	.	.	.	.	.
B	Ta	Pete	Ta								B	.	.	.	B	.	.
BT	TaMu	Dahlia	TaMu								B	.	.	.	TB	.	.
<u>Exclusively Contiguous</u>																	
M	CvRs	Connie	Cv	M	.	.	.	.	.	.							
M	Cv	<u>Derek</u>	Cv	M	.	.	.	M	.	.	M	.	.	.	M	.	.
B	Mu	Fanya	Mu?								.	.	.	.	M	.	.
B	Ta	<u>Kit</u>	Ta	.	.	.	.	B	.	.							
B	Ta	Kasey	Ta								.	.	.	.	B	.	.
TB	Ta	Lara	Ta								.	.	.	.	BT	.	.
T	Ta	Coran	Ta								.	.	.	.	TB	.	.
T	Ta	Tammy	Ta								.	.	.	.	T	.	.

<u>Note</u>. Names of grade 5 students are underlined.
Dots mean that the form was not constructed; blanks that the task was not given.

<u>Units</u>
D = discrete
C = contiguous
U = undefined

<u>Spatial Framework</u>
S = sets
L = lines
R = regions

<u>Unit Relations</u>
M = mono-relational
B = bi-relational
N = no unit relationship

<u>Interpretations</u>
Cv = cardinal values (numerator or denominator)
Hn = 1/2 of a number
Rs = relative sizes
Id = inverse denominator
Mu = multiplication, a/b = a x b
Ta = take-away, a/b = b - a

The summary of the characteristics of students' primary repertoires is recorded to the far left of the students' names. The function and interpretations of their representations from the uncued generative interview precede each student's name.

The students are grouped according to the dominance of a form of representation in their primary repertoire as in Table 4.07. Within these groups they are ordered according to the extent to which they represented relations between different units during the cued generative interview.

Influence of material setting on forms of representations generated. As can be seen in Table 4.09, most students used forms of representations during the cued generative interview which were consistent with the forms of representations in their primary repertoires. Only 3 students (Brock, Connie, & Derek) generated a form of representations which was not used in their primary repertoire.

With few exceptions, the importance of regions as a spatial framework continued despite the presence of a variety of materials which might have suggested alternative forms of representations. Furthermore, of the 14 students who constructed representations with regions as the spatial framework, 12 students used circles at least once and 7 students used circles exclusively or dominantly (see Table 4.10). These response patterns are similar to those observed during the uncued generative interview.

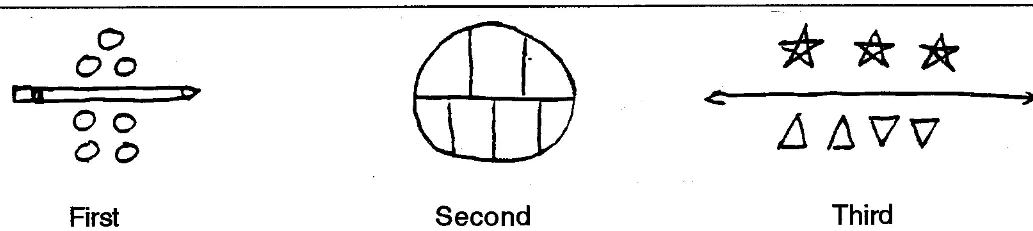
Table 4.10

Extent to which Different Regions were used to Construct Representations of Common Fractions (Cued Generative Interview).

		Types of Regions		
Extent[1]	n[2]	Circle	Rectangle	Irregular
Exclusive	7	5	2	
Dominant	2	2		
Secondary	7	5	7	

Note. 1. Exclusive means all, dominant means more than half, secondary means half or less of a student's discrete representations used the configuration.
2. Total n = 15 Students are counted more than once in the "dominant" and "secondary" categories.

When different forms of representations were generated by a student to explain a task, the representations generally shared a common measurement framework within which some spatial features changed, but the essential attributes attended to by the students did not change. Students who attended only to number and not to sizes of parts in contiguous representations, defined discrete sets only numerically. For example, Marlene independently generated the first two representations in Figure 4.08. When asked later if she could use the line, Marlene proceeded to construct the third representation. These representations are built around a common framework, and are distinguished only by changes in the spatial arrangement of units or the objects used as units, not by differences in attributes of measurement. Conversely, those who attended to differences in the size of the parts with contiguous representations, include this property in their discrete representations. Larger objects were used in the set which represented the fraction with the smaller denominator. There was no measured relationship between the sizes of objects which would represent a ratio between the two fractional units, but the attribute of differences in sizes of fractional units still was critical to the discrete representations. Discrete representations could be derived from the separation of units which were previously contiguous. In the similar vein, Brock's inclusion of representations based on a line related also to a framework common to all of his other representations based on regions. Quantities within regions invariably were compared by judging the distance along the horizontal dimension. Quantities of horizontal length provided a common framework for his representations whether based on regions or lines.



<u>Figure 4.08</u> Examples of different representations of three-fourths using a common framework: (Marlene, Grade 5)

<u>Extent of repertoires</u>. There were several ways in which students' repertoires were limited in the sense that particular spatial frameworks and forms of representations were excluded by students. Most students positively rejected the line as a spatial framework when directly questioned. Those who said they could use the line constructed a representation either with discrete units on the line, or with the line as a unit in a set of lines, or with the line as a boundary of a region. No students used line segments as units.

Materials were rejected or accepted in terms of the extent to which attributes of the materials could be used to fit the forms of representations students already had generated independently. If previous representations were discrete then most materials were viewed as potential discrete units. However, if previous representations were only contiguous, then little of the materials were "suitable" except the circular filter papers. In the latter case both discrete representations and representations based on on a linear framework were excluded by the students.

<u>Function of representations in both generative interviews</u>. Most students were consistent in their interpretations of common fractions during both interviews (see Table 4.09). They also

constructed representations which expressed relationships between different units to the same extent during both interviews. The extent to which students' constructed either mono- or bi-relational representations were relatively stable. Only the incidence of tri-relational representations varied somewhat from one interview to the other.

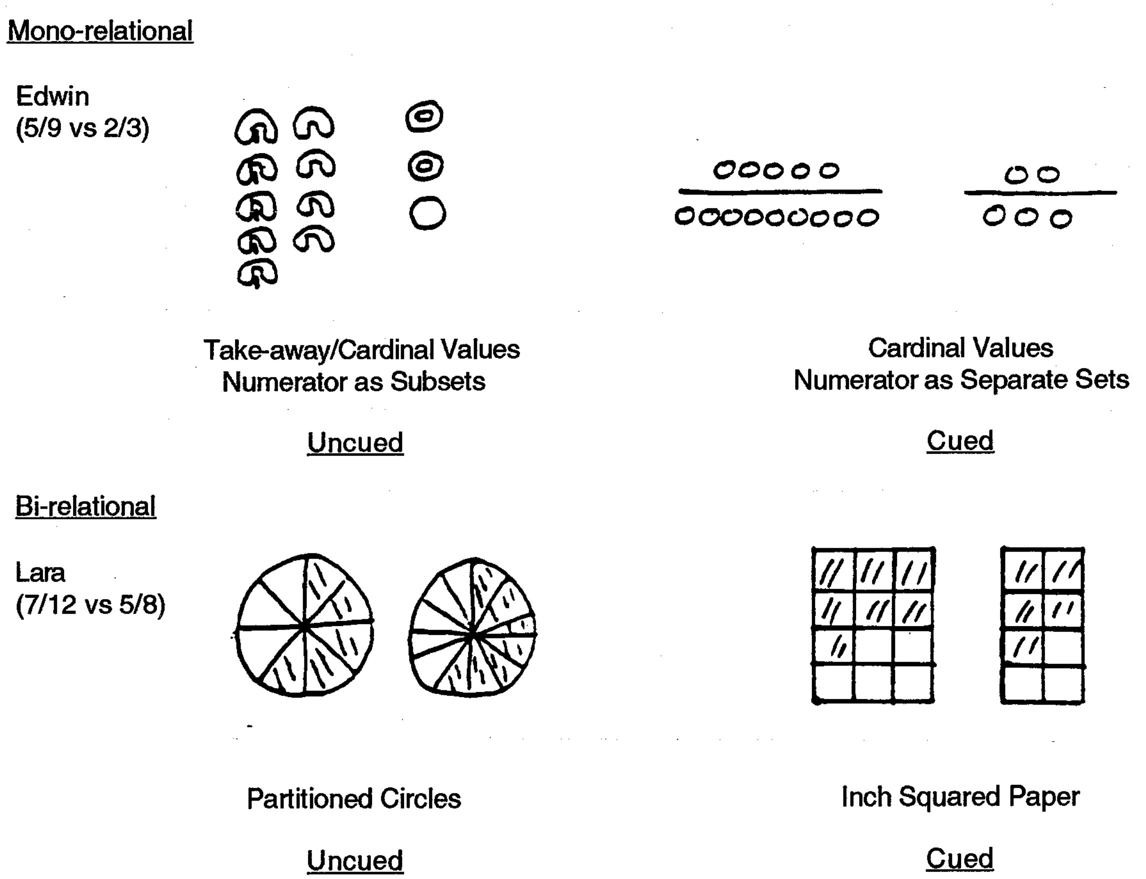


Figure 4.09 Examples of differences in mono- and bi-relational representations constructed in each generative interview

Some characteristics of students' representations of unit relationships differed from one interview to the next even when the general unit relationships did not differ. Differences related either to the instability of a student's interpretations of common fractions (see Edwin, Figure 4.09), or to characteristics of materials which placed constraints on the results of the representational process (see Lara, Figure 4.09).

1. <u>Instability of a student's interpretations</u>. In the uncued generative interview Edwin used the take-away interpretation to represent each common fraction then based his comparisons on a cardinal values interpretation (See Figure 4.09). However, the take-away interpretation was not used during the cued generative interview. Instead, the numerators and denominators of each fraction to be compared were represented as separate sets. These shifts in interpretations, reflected in the structures of the representations, did not mean that the function of the representations differed. In both cases, the units were represented mono-relationally and final comparisons were based on the same rationale.

2. <u>Characteristics of materials</u>. The processes of partitioning regions into units and combining discrete units into regions were assumed by students to be equivalent and interchangeable when representing comparisons of common fractions with unlike denominators. Other experience of representing whole number relationships, and of representing individual common fractions would support the student's suppositions about the equivalence of the two processes. However, these materials were governed by properties of discrete representations in common fraction contexts. The size of each part is pre-determined, not the size of the regions and therefore a cube or square could not represent two different unit fractions. Unless a common denominator was represented, all such representations would be bi-relational (see Lara's cued generative representation in Figure 4.09).

<u>Comparison of Repertoires Reflected in the Generative Interviews With Responses in the Interpretive Interview Settings</u>

Of particular interest in this section is the ways attributes of length or area measurement might be considered by students to be critical features in representations of common fractions. During the generative interviews representations based on length measurement were notable by their absence from students' repertoires. Furthermore, attributes of area-based representations were referenced by most students but were attended to inconsistently. The question remains as to how students would interpret and use beginning diagrams (see Figures 3.01 and 3.02) which were related to length or area. What would be the critical features of the beginning diagrams to which students would attend when evaluating their usefulness for representing common fractions and their comparisons, and would the students interpret and use the beginning diagrams with reference to attributes of length and area?

Table 4.11 was designed to illustrate the forms and functions of the students' responses to the interpretive interview tasks. It also allows for a comparison of these responses with summaries of students' responses during the generative interviews. The format of the table is similar to that used in Table 4.09. The summary of characteristics of students' primary repertoires, including the dominance of forms of representations as well as the functions of their representations, is recorded to the far left of the students' names. The interpretations used in all interviews are summarized after each student's name.

Responses to the common fractions tasks from the cued generative and the two interpretive interviews are presented in the body of the table. Responses to the single common fraction tasks are indicated by lower case letters, and responses to the comparison of common fraction tasks are indicated by upper case letters.

The students are grouped according to the dominance of a form of representation in their primary repertoire as in Table 4.07. Within these groups they are ordered according to the extent to which they represented relations between different units during the four interviews.

Forms of representations in the linear interpretive interview setting. As can be seen in Table 4.11, 8 of the 15 students accepted linear beginning diagrams as a possible spatial framework for constructing a representation of the common fraction or comparison of common fraction tasks (DL, CL & UL). Most students who used linear beginning diagrams interpreted units on the line to be discrete points (DL). In addition, one student (Pete) shifted between defining units as points and line segments (CL).

The presence of the line itself seemed to conflict with some students' beliefs about common fraction representations. Regardless of how students used the beginning diagrams, only two students (James & Fanya) did so without question. Some used the line to construct a discrete representation while expressing a preference for regions as a spatial framework; others constructed alternative representations without a line at some point during the interview. In addition, seven students either transformed the lines into regions (see Figure 4.10), or drew all representations on other parts of the

Table 4.11

Form and Function of Representations of Common Fractions: Cued Generative, Linear Interpretive, and Area Interpretive Interviews with a Summary of Interpretations Over All Interviews and a Summary of Primary Repertoires.

Summary of Primary Repertoire (Uncued)		Summary of Interpretations	Interview setting by forms of representations: Cued Generative							Linear Interpretive							Area Interpretive						
			DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR	DS	DL	DR	CL	CR	UL	UR
Undefined Units																							
	Brock	Rs	.	.	.	.	.	n	nN	.	.	.	.	.	.	n	.	.	.	.	.	.	n
	Lolande	Id	.	.	.	.	.	.	N	.	.	.	.	.	N	N	.	.	.	.	.	.	N
Discrete & Contiguous																							
m	James	HnCvTa	mM	.	.	.	m	.	.	.	m	.	.	.	.	.	m	.	.	.	mM	.	.
mM	Marlene	Cv	mM	.	.	.	mM	.	.	mM	.	.	.	m	.	.	.	.	m	.	M	.	.
M	Edwin	TaCv	mM	.	.	.	m	.	.	.	mbM	.	.	b	.	.	.	.	.	.	mMT	.	.
mBT	Dahlia	TaMuId	B	.	.	.	TB	.	.	.	.	.	.	B	.	.	.	.	.	.	B	.	.
B	Pete	Ta	B	.	.	.	B	.	.	.	B	.	T	T	.	.	.	.	.	.	BT	.	.
Contiguous																							
M	Connie	Cv	m	.	.	.	.	.	.	M	M	.	.	.	.	.	.	.	M	.	.	.	.
mM	Derek	Cv	mM	.	.	.	mM	.	.	.	.	.	.	M	.	.	.	.	.	.	mM	.	.
B	Fanya	MuTa	.	.	.	.	M	.	.	.	mM	.	.	.	.	.	.	.	.	.	B	.	.
B	Kasey	Ta	.	.	.	.	B	.	.	.	B	.	.	B	.	.	.	.	.	.	T	.	.
T	Coran	Ta	.	.	.	.	TB	.	.	.	B	.	.	.	.	.	.	.	.	.	B	.	.
TB	Lara	Ta	.	.	.	.	BT	.	.	.	.	.	.	T	.	.	.	.	.	.	T	.	.
b	Kit	Ta	.	.	.	.	b	.	.	.	.	.	.	bT	.	.	.	.	.	.	T	.	.
T	Tammy	Ta	.	.	.	.	T	.	.	.	.	.	.	T	.	.	.	.	.	.	T	.	.

Note. Names of Grade 5 students are underlined.
Unit Relations: upper case indicates comparison tasks, lower case indicates single fraction tasks.

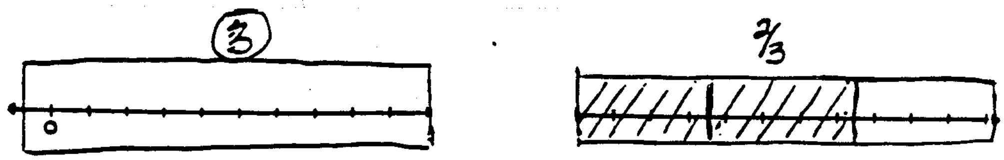
Units
D = discrete
C = contiguous
U = undefined

Spatial Framework
S = sets
L = lines
R = regions

Unit Relations
M(m) = mono-relational
B(b) = bi-relational
T(t) = tri-relational
N(n) = no units

Interpretations
Cv = cardinal values (numerator or denominator)
Hn = 1/2 a number
Rs = relative sizes
Id = inverse denominator
Mu = multiplication, a/b = a x b
Ta = take-away, a/b = b - a

paper. These responses confirmed a general absence of length as a spatial framework for representations of common fraction in some students' repertoires.



3 2/3 (Kit)

1/3 compared to 1/4 (Derek)

Figure 4.10 Examples of student's transformations of lines into regions.

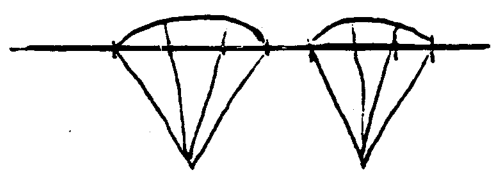


Figure 4.11 Deriving units of length analogously from sectors of a circle (Pete, Grade 7)

The general absence of length measurement as a form of representation in students' repertoires is underscored by some students' responses to linear beginning diagrams compared to the forms of representations they previously generated. First, the students who constructed only contiguous representations based on regions during the generative interviews responded to the linear beginning diagrams in one of three ways. They either (1) used the marks along the line as discrete units, (2) transformed the line into a region by drawing boxes around it (Figure 4.10), or (3) drew partitioned regions elsewhere on the paper. They did not associate a contiguous unit of two-dimensional space analogously to a contiguous unit of one-dimensional space. Second, the only student (Brock) to use a direct comparison of length as the critical feature in his representations of comparisons of common fractions during the generative interviews rejected the linear beginning diagrams altogether. His representations of common fractions did not include a notion of units, hence, length measurement as opposed to direct comparisons of length did not fit his repertoire of representations. Third, the one student (Pete) who eventually identified units as line segments used an indirect argument to do so. The line segments described as analogous to sectors of a circle rather than described solely on the basis of units of linear measurement (see Figure 4.11). Pete also related discrete units analogously to parts of regions, incorporating attributes of area in his discrete units. Hence, no students directly considered units of length to be a form of representation for common fractions. Their responses to the linear beginning diagrams were grounded on the characteristics of their repertoires of representations generated in the previous interviews.

The patterns of responses during the linear interpretive interview, collectively, confirmed the evidence from the generative interviews that suggested that forms of representations based on length measurement were not part of students' repertoires, and that lines as a spatial framework similarly were excluded from the repertoires of nearly half of the students.

Forms of representations in the area interpretive interview setting. Turning now to the students' responses during the area interpretive interview, all students accepted some beginning diagrams and constructed representations with regions as a spatial framework (Table 4.11, DR, CR, & UR). However, the use of regions as a spatial framework for common fraction representations did not imply necessarily that the representations were based on area measurement. Measures of numerosity formed the basis for mono-relational representations of the cardinal values and take-away interpretations of common fractions, and featured in bi-relational representations of comparisons of common fractions. In most instances these representations were constructed with regions as the spatial framework.

There were four criteria used by students to support their judgements of the appropriateness of the beginning diagrams for constructing representations of common fractions. The first two criteria related to limits students imposed specifically on characteristics of the whole unit. The second two criteria related to limits students imposed on the characteristics of units in general, whether used as whole units or parts of whole units. Patterns of responses related to these criteria as well as the form and function of representations constructed during the area interpretive interview are illustrated in Table 4.12.

1. Whole units must be separate, not be regions nested in regions (S in Table 4.12). The beginning diagrams were judged to be inadequate for the comparison of common fraction tasks because there were not two separate regions, one for each common fraction. There were two ways in which students circumvented this problem. Some students partitioned a whole diagram to represent a common fraction, ignoring the potential units already in the diagrams then drew a second region to represent the other fraction in the comparison task. Others removed parts of the beginning diagram which were extraneous to the representation.

2. Whole units must initially be empty regions (Em in Table 4.12). Representations of common fractions could only be constructed by partitioning an empty region into fractional units not by combining fractional units to make a whole unit. Students argued that the lines in the diagram should be removed to create an empty region, or they partitioned a small region within the beginning diagram to represent a common fraction.

3. Units should be orderly for aesthetic reasons (A in Table 4.12). Students rejected beginning diagrams with reasons such as "it was too wiggly," or "it needs to be squared up," or "it could be used but it's a really bad idea." However, in these cases, no direct reference was made to problems of representing quantities with unequal units. Perceptual orderliness was being sought.

4. Units should be equal for quantitative reasons (Qu in Table 4.12). Students rejected beginning diagrams with unequal parts with statements which directly referenced problems in comparing quantities such as, "Should be even in order to compare," or "You couldn't really compare them because ...they would be all different sizes."

As can be seen in Table 4.12, some students used none of these criteria. They simply accepted all beginning diagrams as a reasonable framework for constructing their representation of common fractions (N). Others used aesthetic criterion intermittently for evaluating general units

Table 4.12

Criteria Used to Evaluate Beginning Diagrams and the Form and Function of Representations (Area Interpretive Interview) with a Summary of Interpretations and Repertoires over all Interviews.

General Repertoire	Summary of Interpre-tations	Evaluative Criteria					Representations						
		S	Em	N	A	Qu	DS	DL	DR	CL	CR	UL	UR
Undefined Units													
Brock	Rs	#	.	#	.	.	.	.	.	.	.	.	n
Lolande	Id	.	.	#	#	.	.	.	.	.	.	.	N
Discrete & Contiguous													
Connie	Cv	.	.	#	.	.	.	.	M	.	.	.	.
Marlene	Cv	.	.	#	.	.	.	.	m	.	M	.	.
James	HnTaCv	.	.	#	#	.	m	.	.	.	mM	.	.
Edwin	TaCv	.	#[1]	#	#	.	.	.	..	.	mMT	.	.
Derek	Cv	#	.	.	#	.	.	.	.	.	mM	.	.
Fanya	MuTa	.	#[1]	.	.	#	.	.	.	.	B	.	.
Dahlia	TaMuId	.	#	.	.	#	.	.	.	.	.B	.	.
Coran	Ta	#[1]	.	.	.	#	.	.	.	.	B	.	.
Pete	Ta	.	#[1]	.	.	#	.	.	.	.	BT	.	.
Kasey	Ta	.	.	.	.	#	.	.	.	.	T	.	.
Contiguous													
Kit	Ta	.	.	.	.	#	.	.	.	.	T	.	.
Lara	Ta	.	.	.	.	#	.	.	.	.	T	.	.
Tammy	Ta	#	#[1]	.	.	#	.	.	.	.	T	.	.

Note. Names of Grade 5 students are underlined.
1. Criteria was not applied throughout interview.
Lower case indicates single fraction tasks, Upper case indicates comparison tasks.
# indicates the student used the evaluative criteria.

Criteria: Whole Units
S = separate regions
Em = empty regions

Criteria: General Units
N = not reject unequal units
A = aesthetic criteria
Qu = quantitative criteria

Units
D = discrete
C = contiguous
U = undefined

Spatial Framework
S = sets
L = lines
R = regions

Unit Relations
M(m) = mono-relational
B(b) = bi-relational
T(t) = tri-relational
N(n) = no units

Interpretations
Cv = cardinal values (numerator or denominator)
Hn = 1/2 a number
Rs = relative sizes
Id = inverse denominator
Mu = multiplication, a/b = a x b
Ta = take-away, a/b = b - a

(e.g., James). As well, the criterion which limited the whole unit to an empty region was imposed by some students only during part, not all of the interview.

The evaluative criteria applied to general units (N, A & Qu) seemed to relate to students' interpretations of common fractions and the functions of their representations. Those who did not use a quantitative criterion for evaluating the beginning diagrams constructed either mono-relational representations or representations which did not incorporate units. They interpreted comparisons of common fractions in terms of the cardinal values of numerators or denominators, or in terms of their relative size. The equality of regions or parts of regions were not critical to their interpretations. Hence, their evaluations of beginning diagrams were consistent with their representations and interpretations of common fractions. They either did not discriminate between physical characteristics, or discriminated only from an aesthetic point of view.

Students who used a quantitative criterion to support their selection and use of beginning diagrams constructed bi- or tri-relational representations. These students explained and represented the comparison tasks generally as a comparison of quantities of area. All but one student (Fanya) represented a take-away interpretation of common fractions during this interview. However, as the bi-relational representations indicate, not all of these students were consistent in their attention to attributes of area measurement. Some students attended to equal regions but not equal parts, while others attended to equal parts but not equal regions in their representations or interpretations of the relationships between the different units.

Attributes of Area Measurement in General Repertoires of Representations of Common Fractions.

Table 4.13 was designed to examine the extent and consistency with which attributes of area measurement were critical characteristics of representations in students' general repertoires of representations of comparative common fraction tasks. For this purpose, students' responses to the comparative tasks in each of the four interviews were coded in terms of the degree to which all units in their representations were contiguous (Columns N, P, T). The extent to which contiguous units were approximately equal in area is coded in the body of the table. The degree to which students maintained the equality of whole units is indicated by upper case letters, and the degree to which students maintained the equality of fractional units is indicated by lower case letters. In addition, the criteria students used to evaluate the characteristics of beginning diagrams with regard to the equality of units during the area interpretive interview are summarized in the right hand column. Summaries of students' interpretations of common fractions and their general repertoires of representations also are included in the table.

Students are grouped in the table on the basis of two criteria: first, according to the extent to which contiguous representations characterize their repertoire of representations of comparative tasks evidenced during the generative interviews; and second, according to the extent to which students persisted in constructing contiguous representations during the two interpretive interviews. For example, during the generative interviews students in Group 3 independently constructed both discrete and contiguous representations, whereas those in Group 4 constructed only contiguous representations. On the other hand, students in Group 4 constructed discrete representations in one of the interpretive interviews whereas those in Group 5 persisted in constructing contiguous representations in all interview settings.

Table 4.13 serves to illustrate two general patterns related to the the extent and consistency with which attributes of area measurement were critical characteristics of representations of comparative tasks in students' repertoires. The first pattern concerns the association of interpretations of common fractions and characteristics of the forms of representations in students' general repertoires. The second concerns the consistency with which students constructed equal contiguous units to represent both whole and fractional units.

Interpretations of common fractions and characteristics of the forms of representations in general repertoires. All students included representations based on regions while some also used sets during the uncued generative interview. However, considering general repertoires, there were students whose representations tended to be based on discrete units in sets rather than contiguous units in regions (see Group 2, Table 4.13). When these students did construct representations with contiguous units, attributes of area measurement were generally ignored or attended to inconsistently

Table 4.13

Contiguity and Equality of Units: Representations of Comparison of Common Fractions Tasks in All Interviews

General Repertoire	Summary of Interpre-tations	Uncued N	Uncued P	Uncued T	Cued N	Cued P	Cued T	Linear On Line N	Linear On Line P	Linear On Line T	Linear Off Line N	Linear Off Line P	Linear Off Line T	Area N	Area P	Area T	Criteria for Equality
Undefined Units																	
1. Brock	Rs	N	.	.	N	.	.	.	.	.	N	.	.	N	.	.	N
Lolande	Id	N	.	.	N	.	.	N	.	.	N	.	.	N	.	.	NA
Discrete & Contiguous																	
2. Connie	Cv	.	eE	.	#[1]	.	.	#	.	.	#	.	.	#	.	.	N
Marlene	Cv	E	.	.	#	uU	.	.	.	.	#	.	.	.	uU	.	N
James	HnTaCv	#[1]	u[1]	.	#	.	.	#[1]	.	.	.	.	.	.	iU	.	NA
Edwin	TaCv	#	iU	.	#	.	.	#	.	.	.	.	.	.	iI	.	NA
3. Derek	Cv	.	eE	.	#	iI	.	.	.	.	.	e[1]	.	.	iE	.	A
Dahlia	TaMuId	.	iE	.	eU	eE	.	.	.	.	.	eI	.	.	uE	.	Qu
Pete	Ta	eU	eU	.	eU	eE	.	eU	eE	.	.	.	.	.	eI	.	Qu
4. Fanya	MuTa	.	eU	.	.	eU	.	#	.	.	.	.	.	.	eU	.	Qu
Kasey	Ta	.	iI	.	.	iE	.	iE	.	.	.	iU	.	.	eE	.	Qu
Coran	Ta	.	eE	.	.	eE	.	eI	.	.	.	.	.	.	eI	.	Qu
Contiguous																	
5. Kit	Ta	.	e[1]	.	.	e[1]	.	.	.	.	.	iE	.	.	eE	.	Qu
Lara	Ta	.	eE	.	.	eI	.	.	.	.	.	eE	.	.	eE	.	Qu
Tammy	Ta	.	.	eE	.	.	iE	.	.	.	.	eE	.	.	.	eE	Qu

Note. Names of Grade 5 students are underlined.
1. Common fraction task only, no response to comparative tasks.

Contiguity of Units
N = none contiguous
P = partially contiguous
T = totally contiguous

Criteria for equal units
(Area Interpretive)
N = no attention
A = aesthetic
Qu = quantitative

Characteristics of Units
E = equal whole unit
U = unequal whole unit
I = inconsistent
e,u,i = fractional units
# = no area reference
N = no units

Interpretations
Cv = cardinal values
Id = inverse denominator
Hn = 1/2 a number
Rs = relative sizes
Mu = multiplication
Ta = take-away

(see codes i, u, I, U). In addition, when equal units were sought by these students, aesthetic rather than quantitative criteria were used to support their judgements (see code NA). In nearly all instances, these students interpreted the comparison of common fractions as comparisons of the numerosity of units in the representations of numerators, denominators, or differences between numerators and denominators. Units of area played no quantitative role in their representations. When contiguous units were represented they were, in essence, discrete units which were incidentally contiguous. Hence, characteristics of the forms of representations in their general repertoires were consistent with their interpretations of common fractions.

In contrast, students who consistently used a take-away or multiplication interpretation of common fractions (see Groups 3, 4, & 5) more consistently constructed representations based on contiguous units in regions. As well, they used quantitative rather than aesthetic criteria to justify their rejection of diagrams with unequal parts (see code Qu), and were more consistent in their attention to attributes of area measurement. Among these groups (3, 4, & 5), however, the extent to which representations with contiguous units dominated in their general repertoires varied. Only the students in Group 5 excluded discrete representations from their general repertoires. The others did include some discrete representations in their general repertoires.

Contrasts between the general repertoires of students in Group 2, those in Groups 3 and 4, and those in Group 5, suggests a differentiation in students' beliefs about forms of representations for comparing of common fractions. The students in Group 2 essentially represented common fractions with discrete units since contiguous representations were interpreted through attributes of discrete measures. Measures of numerosity underlay all their forms of representations of units even though some of these students explicitly stated that units in contiguous representations should be equal. There was generally no differentiation between systems of measurement regardless of the discreteness or contiguity of units.

In direct contrast, students in Group 5 used only regions as a spatial framework for area-based representations of comparisons of common fractions. They clearly differentiated their representations from other possible forms of representations. Discrete as well as linear representations were explicitly excluded from their general repertoires.

With the students who primarily used a take-away interpretation in Groups 3 and 4, their major form of representation was one based on comparisons of area in partitioned regions. As well, images of differences in sizes of contiguous units were used in discrete representations of comparisons to rationalize unit relationships (E.g. Pete, Table 4.12). These students interpreted representations, regardless of their form, through some notions about measures of area. For them, units were essentially represented as quantities of area which at times were incidentally discrete. This pattern of responses parallels and contrasts with repertoires of students in Group 2. In both cases, their repertoires contained different forms of representations most of which were interpreted through a single system of measurement, numerosity in one case and area in the other.

In summary, students' repertoires included different forms of representations but these were not differentiated clearly in terms of units and unit relations appropriate to each system of measurement. Instead, students essentially represented and interpreted comparisons of common fractions through a single measurement system, either numerosity or area. Even though most general repertoires included both discrete and contiguous representations of comparisons of common fractions, both forms of representations generally were quantified in terms of the dominant system of measurement in a student's repertoire.

Equality of contiguous units (whole and fractional units). The second patterns concerns the consistency with which students constructed equal contiguous units to represent both whole and fractional units. As can be seen in Table 4.13, none of the students were totally consistent in constructing contiguous representations in which all units conformed to properties of area measurement. Nonetheless, there were differences among the students in the extent to which they attended to ideas about equal measures. There are two different cases to be considered in this regard: (1) students whose representations were based essentially on measures of numerosity, and (2) students whose representations were based essentially on notions about measures of area.

First, considering the students whose representations were based on measures of numerosity, all of these students stressed that regions, parts of regions, or both should be equal, and included these characteristics in their representations at some instance during the four interviews. However, none of these students justified the need for equal parts quantitatively, and all but Derek attended seldom or very inconsistently to these characteristics in their contiguous representations of comparisons. Nonetheless, they all had some notions about characteristics of contiguous representations which reflected attributes stressed in instructional representations based on area measurement. Students adopted and attended to characteristics of this form of representation without associating them with the quantitative meaning assumed to be represented by educators.

Considering the case of students whose representations were based on some notions of area measurement, there were three ways in which students differed in their attention to the equality of units. First there were differences in the general consistency with which students attended to the equality of whole and fractional units. The students in Group 5 were most consistent. Each constructed unequal units in only one instance. Nearly all of the other students were inconsistent in at least three interview settings.

Second, there were different patterns in the manner in which students were inconsistent. For example, Coran's inconsistency in equating whole units was associated only with his representations of the algorithm for generating equivalent common fractions, Dahlia was inconsistent about the equality of fractional units with the take-away representations and about the equality of whole units with the multiplication interpretation, and Kasey was more generally inconsistent with regard to whole and fractional units. These students appeared to have integrated elements of area measurement into their forms of representations based on regions, but had not fully coordinated the relationships between measures of the whole units and measures of the fractional units when comparing common fractions.

Third, some students did not construct parts of equal area but still attended to equal measures. When partitioning circles with a vertical configuration, students in Group 5 focused on equal distances between the vertical lines as the representation of equal measures. With students in the other groups, their inconsistent attention to equal whole or fractional units reflected different beliefs about the characteristics of representations necessary for comparisons of common fractions.

Functions of Representations in Students' General Repertoires.

Turning again to Table 4.11, there are patterns in the functions of students' representations across all interviews. These suggest limits in the complexity of unit relations which different students might consider and represent for comparisons of common fractions.[3] There were eight students whose representations fell invariably into one of three categories: no units, mono-relational units, & tri-relational units. The other seven students varied in the extent to which they represented unit relationships.

The six students whose repertoires were limited to mono-relational representations or representations without units, represented units in a manner consistent with their interpretations of common fraction. Their interpretations were either relative values, half a number, inverse denominator, or cardinal values interpretation. The two students who constructed only tri-relational representations to explain the comparison tasks did so within a restricted repertoire of representations based only on partitioned regions. Neither discrete nor linear units were accepted by them. This meant that they did not face the problem of rationalizing representations of comparisons of common fractions with discrete units with representations based on measures of area. Whether these students would have constructed tri-relational representations with discrete units is not known.

The other seven students were unstable in their representations of unit relationships. One shifted between mono- and bi-relational representations as she shifted between a take-away and a multiplication interpretation. Another shifted from mono-relational representations to a tri-relational representation. The remaining five students shifted between constructing bi- and tri-relational representations of the comparative tasks.

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3. Only the comparison of common fraction tasks are discussed in this section because they are less ambiguous as a bench-mark of students' representations and interpretations of unit relations.

There were four general features in students' approaches to representations of comparisons of common fractions associated with the instability of students' representations of units relations. These were (1) changes in interpretations of common fractions, (2) notions about the algorithm for generating equivalent common fractions, (3) different complexities in the denominators of comparison tasks, as well as (4) shifts in between measures of area and measures of numerosity. Each of these will be discussed in turn.

1. Shifts in interpretations of common fractions. The multiplication interpretation invariably implied a bi-relational representation, whereas the take-away interpretation was associated with all levels of unit relationships. All of Fanya's and some of Dahlia's bi-relational representations were associated with the multiplication interpretation. Their mono- or tri-relational representations expressed a take-away interpretation of common fractions.

2. Notions about the multiplicative algorithm for generating equivalent fractions. There were two ways in which multiplication was incorporated in representations of comparisons of common fractions. One was with the multiplication interpretation, and the other was with the representation of the multiplicative algorithm for generating equivalent fraction. Attempts to represent the equivalent fraction algorithm resulted in a distortion in representation of relationships between fractional units. The multiplication of numerators and denominators was associated with the repeated addition of units rather than the repeated partitioning of units. As a result, the second equivalent common fractions compared to the first was larger in terms of both the numerousness of its fractional units and the actual size of its whole unit. For example, in Coran's comparison of 5/9 and 2/3 during the area interpretive interview, he represented 5/9 and 2/3 followed by a representation of 6/9 as equivalent to 2/3 (see Figure 4.12). The 2/3 was transformed into 6/9 by multiplying each numeral by three. He did not perceive as a conflict the fact that 2/3 and 6/9 were represented as different amounts of area while he stated that the fractions were the same. In contrast, when not considering the algorithm, he generally constructed tri-relational representations based on measures of area.

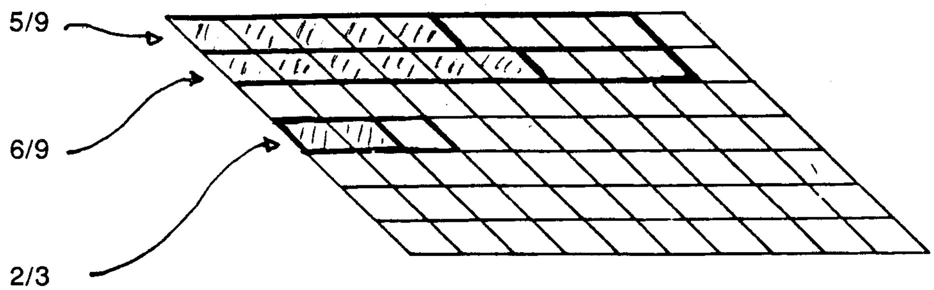

Figure 4.12 Representation of multiplicative algorithm for generating equivalent fractions (Coran, Grade 7)

3. Different complexities in the denominators of comparison tasks. Some representation strategies applied to tasks in which one denominator was a multiple of the other (e.g. 2/3 vs 5/9) or tasks in which denominators were not so related (e.g. 7/12 vs 5/8) were associated with some students' shifts between bi- and tri-relational representations. When students explicitly constructed representations with common denominators, they did so only with tasks in which one denominator was a multiple of the other. There were two procedures used to explain the part-part relationship (3:1 for ninths and thirds, 2:1 for eighths and fourths). First, there was the procedure through which the part-part relationship was explained indirectly with a numerical algorithm which resulted in a bi-relational representation. Second, there was the procedure through which the part-part relationship was explained directly with the partitioning of one or two equal regions which resulted in a tri-relational representation. With comparisons of the form a/b vs c/d, students directly compared representations of each common fraction. When differences in sizes of fractional units were relatively small, reliance on only perceptual evaluations of the representations led to bi-relational interpretations of units. With other tasks, perceptual evaluations led more easily to tri-relational interpretations of units. As well, when direct perceptual judgements were accompanied by reasoning about relative sizes of denominators and relative sizes of parts, students' comparisons were more reliable.

4. <u>Shifts between measures of numerosity and measures of area</u>. The most stark example of the association of shifts between systems of measurement and shifts in the representation of unit relationships is provided by Edwin. After constructing a mono-relational representation which supported his judgement that 4/8 was "much bigger" than 2/4 because the units are more numerous, he then determined that they were the "same amount of pie". When questioned about the two conclusions the following dialogue ensued:

I: But you just told me that 4 eighths is bigger than 2 fourths?

E: It is, but if the pie is the same size it is a quantity - well you have more of the quantity of 8 - well, you have 8 pieces but if the pie is the same size and 8 pieces it really doesn't matter.

I: It doesn't matter?

E: Yah. It could be like 4 in a pie or 8 in a pie. It really doesn't matter if the pies are the same - that's all.

I: But could you still say that 4 eighths is bigger than 2 fourths?

E: Yah.

I: So it just depends, eh?

E: Yah, if it's a pie - but if it's different it would be bigger.

I: It would be bigger?

E: Yah, if its not food.

I: So what kinds of things would you use to show when its bigger?

E: Okay, you could use lines (drew 8 lines) and take away 4 of them to make something - so you have 4 here. And 1, 2, 3, 4, (drew 4 lines) and then you would take away two because you made something.

The mono-relational representation comparing measures of numerosity was as intelligible to Edwin as the tri-relational representation comparing measures of area. In each case the take-away interpretation was applied, but differences in the situations were sufficient to justify the different results. The meaning rested for Edwin in characteristics unique to each situation, not in general characteristics of common fractions' multiple unit relationships which apply regardless of the system of measure.

Changes in interpretations of common fractions, notions about the algorithm for generating equivalent common fractions, different complexities in the denominators of comparison tasks, as well as shifts between measures of numerosity and area were inter-independent features of students' approaches to comparisons of common fractions. For example, shifts in attention between measures of area and numerosity occurred when multiplicative notions of the equivalent fractions algorithm were included in representations. As well, such shifts occurred directly when students simply changed the form of their representations from contiguous to discrete units, or changed the materials used to construct contiguous representations from empty regions to squared paper or multilink blocks. Indeed, shifting attention between measures of area and numerosity often was a consequence of the other three features, and underlay much of the instability in the representation of unit relations evidenced by these students. Students' beliefs about the characteristics of units of area as a basis for representing the comparison tasks alone are insufficient to explain the instability in the representation of units relationships. The other features in students' approaches to the representational tasks also contributed to the instability.

In summary, more than half of the students were stable in the extent to which they represented units relationship over all interviews. However, for those who were unstable in their representation of unit relationships, the instability was generally associated with shifts in attention between measures of area and numerosity, as well as with other beliefs about interpretations and representations of comparisons of common fractions.

Summary of the Characteristics of Students' Repertoires of Representations of Common Fractions and Comparisons of Common Fractions

A number of general patterns were evident in students' repertoires of representations. These are presented in three parts. The first concerns the extent of students repertoires of representations in general. The second concerns the inclusion of representations based on length or area measurement. The third concerns the extent to which representations functioned to express multiple relationships among units.

First, regarding the extent of their repertoires, the following patterns were evident:

1. Prominence of regions in primary and general repertoires. All students' primary repertoires included representations based on regions, regardless of whether units were discrete, contiguous or undefined. More particularly, circles were generally the region of choice. In addition, representations based on sets were included by some students as an alternative to their first representations with regions. During the other interviews, discrete representations were included by students whose primary repertoires had been based exclusively on regions. However, there were still three students whose general repertoires were based completely on regions partitioned into contiguous units.

2. Explicit limits were imposed on the variety of forms of representations in students' repertoires. All students imposed some limits on which forms of representations they deemed acceptable as a means to explain the common fractions. Most commonly, students rejected the line or transformed it into regions. Less commonly, students rejected discrete materials, required empty regions in which to construct units, or needed separate regions to represent each common fraction. Students reacted to alternative materials in a manner consistent with the forms of representations generated independently. The forms of representations finally included in their general repertoires represented a limit in the types of situations which students associated with common fraction representations.

3. A student's discrete and contiguous representations were related one to the other in terms of a common quantitative framework. Students who primarily focused on measures of numerosity constructed representations of comparisons with contiguous units. However, the contiguous units were treated simply as units of numerosity which were incidentally contiguous. Similarly, students who primarily focused on measures of area constructed representations with discrete units. However, the discrete units were treated simply as units of area which were incidentally discrete. In general, differences between discrete and contiguous representations lay only in the spatial features of the units. Students tended to interpret representations of units through a single measurement system, either numerosity or area.

Second, regarding the extent to which attributes of linear or area measurement were integrated in students' contiguous representations, the following patterns were evident:

1. Representations were not one-dimensional. Units of length were not used directly to represent common fractions. However, they were used indirectly to equate fractional units along a dimension in two-dimensional representations. Similarly, direct comparisons of the lengths of a dimension in regions were also the basis on which common fractions were compared. However, the spatial framework of the representations was two-dimensional.

2. Beliefs about the importance of units of equal area were not necessarily associated with quantitative characteristics. All students who constructed representations based on contiguous units at times expressed beliefs about and attended to the need for units of equal area. However, this did not mean that measures of area necessarily played a quantitative role in their representations of comparisons of common fractions. For some students, quantitative comparisons in their contiguous representations were based on measures of numerosity. Their need for equal regions or parts of regions was justified by aesthetic criteria alone. These students imposed procedures on the form of their representations which conformed to some characteristics of area-based instructional representations, but which were disassociated from an area measurement framework.

3. No student consistently attended to the equality of the area of fractional and whole units. The extent to which students were consistent appeared to be influenced by one or more of the following:

a. The measurement framework of interpretations of common fractions Some students primarily interpreted comparisons of common fractions in terms of the numerosity of the parts. Because measurement of numerosity was the framework for their representation, their comparisons were internally consistent regardless of the equality of the area of regions or parts of regions. Others who interpreted these comparisons in terms of quantities of area were more likely to attend to the equality of the area of regions or parts of regions.

b. Beliefs about critical physical characteristics of contiguous representations. There was a variety of reasons associated with students' beliefs that units should be equal in area. Even among students who used quantitative criteria to justify the need for these characteristics, some emphasized equal fractional units but not regions and vice versa. Students had different levels of tolerance with respect to inequalities of units in contiguous representations.

c. Interaction between partitioning problems and strengths of beliefs about the importance of equal units of area. Students' algorithmic approaches to the partitioning of regions (particularly circles) inhibited their ability to achieve equal parts with particular denominators. This, combined with variable beliefs in the importance of equal measures, led some students to violate properties of area measurement in some but not all instances.

Third, regarding the function of students' representations of the comparisons of common fractions, the following patterns were evident:

1. Students for whom the function of their representations was consistent did not construct bi-relational representations of units. Their representations were either all with mono-relational units, all with tri-relational units, or all without defined units. In each case, the function of a student's representations were associated consistently with their interpretation of common fractions and the measurement system on which the representations were based.

2. Students who represented unit relations inconsistently generally shifted between bi-relational and tri-relational representations. Regardless of how students shifted in the representations of unit relationships, one or more of the following features were associated with these shifts:

a. Shifts in the interpretations of the comparisons.

b. Shifts from direct to indirect comparison of common fractions through a representation of the multiplicative algorithm for generating equivalent fractions.

c. Shifts in representational strategies associated with different pairs of denominators which presented different complexities in the tasks.

d. Shifts between a focus on equal parts to a focus on the number of parts.

Students' Representation of Unit Relationships in the
Whole Number Multiplication and Common Fraction Contexts:
Common Patterns and Related Themes.

In this section general patterns and themes are discussed concerning students' repertoires of representations of units and unit relationships in the contexts of whole number multiplication and common fractions.

Repertoires of Representations

In both mathematical contexts, students demonstrated a variety of ways in which they could represent the mathematical tasks. They also clearly indicated limits in the forms of representations which they considered could be used to represent the tasks. It was in the sense of the forms of representations a student generated as well as the forms of representations a student rejected that the construct of a repertoire of representations was defined. Students' repertoires of representations in both mathematical contexts can be characterized by the six groups presented in Table 4.14.[4]

Central to a student's repertoires was a dominant form of representation. This was the form which the student used most frequently, and the form through which the student often interpreted other materials and beginning diagrams. In the case of whole number multiplication the dominant form of

representations was most commonly that of discrete sets. In the case of common fractions the dominant form was most commonly that of regions with contiguous or no defined units.

In addition to the dominant form in a student's repertoire, most often there was a secondary form of representation. The most common secondary form in repertoires of multiplication representations was that of contiguous units in regions. For repertoires of common fraction representations it was discrete units in sets. There was commonly an asymmetrical relationship between the dominant and secondary forms of representations of each mathematical context. The

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4. In order to focus on major patterns of response, the incidental references to line segments as units have not been presented graphically. Only 2 students referred to line segments and then not more than more than once each.

Table 4.14

General Patterns in the Form of Representations in Students' General Repertoires in the Multiplication and Common Fractions Contexts.

	Multiplication (Complex Tasks)			Common Fractions (Comparison Tasks)				
Groups	Disc	Part Contig	All Contig	Disc	Part Contig	All Contig	No Units	Inter-pret-ations
1								
James	*&**	[1]		&&*	&*			HnCvTa[2]
2								
Connie	**&&	*		*&*	*			Cv
Marlene	**&	&		***	**			Cv
Edwin	***		**	***	**			TaCv
3								
Brock	**&	&					****	Rs
Lolande	***		*				****	Id
4								
Derek	**&	&		*	**&*			Cv
Dahlia	***	*		*	****			TaMuId
Kasey	***	**	*	*	****			Ta
Pete	***	*	*	***	****			Ta
Coran	***		*	*	***			Ta
5								
Kit	***	*	*		&&**			Ta
Lara	**	**			****			Ta
Tammy	**	*	**		*	***		Ta
6								
Fanya	**	**	*	*	***			MuTa

Note: The number of symbols (*/&) indicates the number of interviews in which the form was used by the student.
Response only to the less complex task is indicated with an ampersand.
The order of the common fraction interpretations reflects the frequency of use.

Form of Units
Disc = discrete units in sets
Contig = contiguous units in regions
No units = regions without units

Common Fraction Interpretations
Hn = half of a number
Cv = cardinal value of numerals
Id = inverse denominator
Mu = multiplication
Rs = relative size
Ta = take-away

most common secondary form in one mathematical context was the most common dominant form in the other, and vice versa (see Groups 3 to 5 in Table 4.14).

There were some students whose repertoires in both mathematical contexts were based dominantly on discrete units (see Groups 1 & 2). The number of units, not the area of units was the critical attribute to which they attended in their representations. A child's dominant informal experience would be with discrete units in sets, whether a single or aggregate unit, with the fingers probably being the most common representations of units. The preponderance of discrete representations of units with a whole number operation fits the more general experiences of children's representations of whole numbers from early childhood to the intermediate grades. For these students it these discrete numerical experiences appear still to provide the basic framework for representing about numerical relationships in general. There was little or no differentiation between their two repertoires in terms of the dominant form of the representations, or the measurement framework through which the units were defined.

The contiguous-discrete dichotomy of dominant forms of representations in common fraction repertoires appears to distinguish the more proficient from the less proficient students in representing comparisons of common fractions. However, as the cases of Derek and Fanya indicate, the dominance of contiguous forms of representations does not necessarily mean that the students considered parts of a region to represent units of area. The relationship between forms of units and measurement systems the students referenced was equivocal.

<u>The Discrete-Continuous Dichotomy</u>.

Conventionally, whole number multiplication and common fractions are seldom represented explicitly in a continuous form even though units may be based on measures of continuous quantities. As Ohlsson (1988) stated,

> Partitioning replaces a [continuous] quantity with a set, namely the set of its parts. By reducing the continuous to the discrete case, partitioning enables us to assign numerical values to continuous quantities through counting. (P.74)

Once regions are partitioned, the continuity of the quantity is not the most salient characteristics of the representation. Instead, the contiguity of the units and their potential separateness are the more pronounced characteristics.[5] However, when instructional representations are based on partitioned regions, it is assumed that contiguous units should be interpreted to mean measures of the continuous quantities.

The physical characteristics of units, whether discrete or contiguous, do not automatically mean that units must be interpreted as measures of numerosity or measures of area in order for the unit relationships embodied in the representation to be mathematically valid. This is universally the case in the whole number multiplication context. Contiguous representations need not have units of equal area to embody the unit relations between factors and product. It would only be when other characteristics of the problem situation impose a continuous measurement context that units in a representation need conform to particular attributes of a measurement system.

Likewise, when representing a common fraction with contiguous units, the unit relationships embodied in the partitioned region may be based either on numerosity or on area. "A out of B pieces of pie" may mean <u>A/Bths of the pieces</u> (of pie) or may mean <u>A/Bths of the pie</u>. The contiguous units may be measuring discrete attributes or continuous attributes. Both interpretations of the representation are valid mathematically. The contiguity of the units does not, in itself, define the mathematical situation represented.

Which measurement systems are associated with discrete or contiguous units is inconsequential when representing whole number multiplication, single common fractions, and comparisons of common fractions with like-denominators. For example, the comparison of 5/9ths and 6/9ths could as easily be a difference of 1/9th <u>of the pieces</u> as 1/9th <u>of the pie</u>. It is only because the representation of comparison of common fractions with unlike denominators cannot be achieved in the

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5. Ohlsson (1988) associates the term discrete with all units regardless of their form. In this report a distinction is made between contiguous, referring to units which share a boundary, and discrete, referring to units which are spatially separate. These differences in definitions probably reflect the fact that Ohlsson was writing from a theoretical point of view rather than from the view of students mathematical thinking.

discrete case that units of equal area become a necessary characteristic. Except when common fractions with unlike denominators are compared, there would be no conceptual conflict regardless of whether students represented or interpreted the units as measures of continuous quantities or measures of numerosity. As such, to assume that a student who attends to the equality of units is also considering area as the quantitative attribute may be invalid in these mathematical situations.

The ambiguity concerning which system of measurement defined different forms of units was reflected in the manner in which students considered discrete and contiguous representations to be analogous. Considering either mathematical context, a student's discrete and contiguous representations essentially differed only in the proximity of their units. The student defined critical attributes of the units and interpreted the unit relationships through the same measurement framework. Even when the problem of representing a comparison of common fractions with unlike denominators was encountered, students who based their contiguous representations on units of area represented denominators with different sizes of discrete units. Their "discrete" representations were simply representations based on an approximation of quantities of area in which the units were no longer contiguous. In this way, there was a sense in which the contiguity and discreteness of units were considered to be reversible without altering the mathematical relationships being represented.

The ambiguity regarding which system of measurement is represented by contiguous units also was reflected in the students' use of aesthetic and quantitative criteria for evaluating contiguous beginning diagrams. In the multiplication context, that units be of equal area was incidental to the meaning of the mathematical relationships, even though some students imposed this characteristic on their representations. In that case, students used aesthetic criteria in the multiplication context to justify their rejection of diagrams with unequal units. In contrast, when presented with the comparison of common fractions, some used quantitative criteria to justify their rejection of diagrams with unequal units. These students were reflecting a subtle distinction between normal, preferred characteristics of units in one mathematical context and necessary characteristics of units in another. As would be expected, students who continued to base their comparisons of common fraction on measures of numerosity sought equal sized units for aesthetic reasons alone.

Students' repertoires of representations in either mathematical context generally were composed of discrete and contiguous representations with one or other form dominant. These forms of representations could be considered to be simply variations of representations of units within a fundamentally two-dimensional world. From the earliest experiences of counting discrete object to pushing discrete units together or fracturing a quantity into parts, the informal and formal experiences are essentially with two-dimensional or three-dimensional materials.

The general absence of linear measurement as a framework for representations should be considered in the context of the associations between discrete sets and contiguous regions in repertoires of representations in both mathematical contexts. The one-dimensional framework of the line stands in direct contrast to the two-dimensional characteristics discrete sets or contiguous regions. Thus, students transformed the line into two-dimensional representations, identified and used the dots on the line as analogous to discrete objects in sets, or rejected the line altogether in favour of sets or regions. In all cases, their avoidance or rejection of a line as a spatial framework and line segments as units could be related to a general two-dimensional characteristic of their repertoires of representations, regardless of the mathematical context.

The forms of representations in students repertoires were circumscribed in all cases. Students did not consider units in the general case in which different systems of measurement were potential bases for representations. They did not use a multiplicity of measurement frameworks between which the physical attributes of the units changed. Instead, their representations were formulated around a core image of units which were incidentally contiguous or discrete, but which primarily referenced a single system of measurement, either that of numerosity or area.

Multiple and Nested Relationships of Units

The representation of double-nested units was clearly a difficult task. Double-nested units were of the form A groups of B groups of C. In the multiplication context it would occur as the representation of the product of three factors. In the common fraction context it would occur as the representation of two fractional units nested within a whole unit, such as the representation of thirds and ninths within a single region. In each case the representations would be categorized as tri-relational. In the multiplication context, most of such representations were achieved only after some preliminary trials. In the common fraction context, the comparative relationships among the whole unit and two fractional units could be achieved without recourse to double nesting. Students generally did not encounter the double-nesting problem because they directly compared separate representations for each common fraction. There was only one instance when a student compared two common fractions with a tri-relational representations in which both fractional units were nested within a single region.

There were situations in both contexts in which students constructed representations which were potentially double-nested, but they did not "see" the units nested in units. In the occasional instances when thirds were transformed into ninths, few students saw aggregates of three-ninths also as thirds. Similarly, with multiplication, some students did not perceive two three's in representations of 2 X 3 X 4 as four sixes. In the most extreme example, a student placed two discrete marks in each of six circles to represent 2 X 2 X 3, then interpreted the representation as a product of 18. The meaning of his own nesting procedure was lost in his interpretation that all marks and circles represented equivalent units.

Table 4.15 presents a graphic summary of the unit relationships the students represented in both mathematical contexts. As can be seen in this table, in the multiplication context students were generally as successful or more successful in representing multiple unit relationships than in the common fractions context. There were students whose representations of unit relationships were relatively consistent in both mathematical contexts. For some, units were consistently mono-relational. It would appear that these students do not have a general representation of units as aggregates. For others, unit relationships fluctuated between being bi- and tri-relational in both mathematical contexts. As well, there were students for whom unit relationships were mono-relational in the common fractions context but whose representations were bi- or tri-relational in the multiplication context.

Table 4.15

General Patterns in Students Representations of Unit Relationships in the Multiplication and Common Fraction Contexts.

	Multiplication (Complex Tasks)			Common Fractions (Comparison Tasks)				Inter-pretations
	Mono	Bi	Tri	None	Mono	Bi	Tri	
1								
Connie	**&&				*&**			Cv
Marlene	**&&				****			Cv
Derek	**	&&			****			Cv
Brock	**	&&		**&&				Rs
2						1		
Fanya		****			**	**		MuTa
Dahlia		****				****	**	TaMuId
Edwin	*	****	*		****		*	TaCv
Lara		****	*			**	****	Ta
Lolande		***	*	****				Id
Kasey		***	**			***	*	Ta
Pete		**	**			****	**	Ta
Tammy		***	**				****	Ta
James	*	**	***		**&*			HnCvTa
Coran		***	***			**	**	Ta
Kit	*	**	***			&&	**	Ta

Note: The number of symbols (*,&) indicates the number of interviews in which the unit relationship was represented at least once by the student.
Response only to the less complex task is indicated with an ampersand.

1. The bi-relational representations occurred only with the multiplication interpretation.

Unit Relations
None = units were not defined
Mono = mono-relational units
Bi = bi-relational units
Tri = tri-relational units

Common Fraction Interpretations
Hn = half of a number
Cv = cardinal value of numerals
Id = inverse denominator
Rs = relative size
Mu = multiplication
Ta = take-away

In summary, students' repertoires of representations in both mathematical contexts were characterized in terms of the forms of representations which were included and excluded. As well, within a student's repertoire of representations the discrete and contiguous representations were analogously related one to the other through a single measurement framework. Which characteristics are quantitatively necessary to represent unit relationships is ambiguous in many mathematical situations including whole number multiplication and common fractions. As a result, students who interpret contiguous representations in terms of numerosity are not challenged until faced with the problem of representing comparisons of common fractions with unlike denominators. In this specific mathematical context, a student's beliefs about units of area measurement become critical to their representations with contiguous units. It is no longer adequate for a student to consider unit relationships with common fractions as mono-relational.

CHAPTER 5

STUDENTS' REPRESENTATIONS AND INTERPRETATIONS OF UNITS OF LENGTH AND AREA

The general questions which guided the analysis in this chapter were:

1. What are the characteristics of students' representations and interpretations of units of length?
2. What are the characteristics of students' representations and interpretations of units of area?

It was expected that students at Grades 5 and 7 would be more likely to operate appropriately with units of linear measurement than with units of area measurement given that "the order of achievement is a function of added dimensions" (Beilen & Franklin, 1962, p.617). On the other hand, it was not assumed that all students would operate appropriately with units of linear measurement. No clinical studies have been located that directly investigated children's conceptions of linear measurement beyond the age of 9 years.

Compared with linear measurement tasks, area measurement tasks present greater perceptual and conceptual complexities to the student. These complexities not only derive from the need to coordinate relationships between two dimensions, but also derive from the variations in geometric properties of different regions which must be attended to when measuring areas. There is evidence that students in Grades 5 and 7, when presented with area measurement tasks, exhibit a wide range of behaviours suggesting a variety of conceptions about area measurement (Hirstein, 1974 unpublished pilot study cited in Steffe & Hirstein, 1976; Hirstein et al., 1978; Wagman, 1975).

The chapter is divided into three main parts. In the first part, the analysis of the students' responses to the linear measurement tasks is presented. In the second part, the analysis of the students' responses to the area measurement tasks is presented. In the third part, an overview of all measurement tasks is presented.

Students' Representations and Interpretations of Units of Length

The research question concerning the characteristics of students' representations and interpretations of units of length was particularized by the following questions:

A. To what extent do students interpret or construct units of length in a manner consistent with properties of linear measurement?

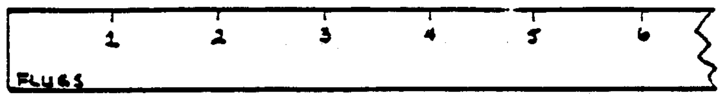
B. What are the characteristics of the reasoning strategies students use to compare quantities of length?

Each of the questions is addressed through the analysis of different subsets of the tasks on the test which were used during the interview. The ruler, aggregate unit and partitioning tasks (see Figure 5.01) were designed primarily to address Question A. All of these tasks directed students to construct and use units of linear measurement. The ways in which students did so provided evidence from which their representations of units were characterized. Question B was explored through students' responses to the irregular path tasks (see Figure 5.02). The irregular path tasks did not contain explicit reference to units or number.

The balance of this section on linear measurement is organized in the following manner. First, the analytical categories used to classify students' responses to the linear measurement tasks are presented and defined. Second, the analysis of the responses to the ruler, partitioning and aggregate unit tasks is presented. The responses to these tasks are directly relevant to Question A. Third, the analysis of the responses to the irregular path tasks is presented. The responses to these tasks are directly relevant to Question B.

A. <u>Ruler task</u>

This ancient ruler measures length in "FLUGS". One "FLUG" is the same as two centimetres. Draw a line above the ruler that is 6 centimetres long.



B. <u>Aggregate unit task</u>

The line below is 4 units long. Draw a line that is 12 units long.

C. <u>Partitioning task</u>

This path is <u>5 units</u> long. a) Mark the 5 units on the path.
b) Draw another path <u>3 units</u> long.

<u>Figure 5.01</u> Linear measurement tasks with explicit reference to units and number.

D. Irregular path task

E. Irregular path task

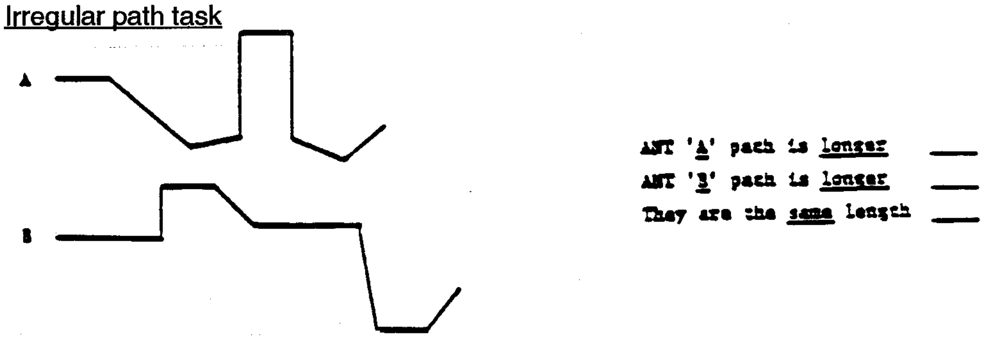


Figure 5.02 Linear measurement tasks without explicit references to units and number.

Analytical Categories to Classify Responses to Linear Measurement Tasks

The categories described in this section and outlined schematically in Figure 5.03 were derived from an integrated analysis of the tasks and the students' responses to the tasks during the interviews in the second pilot study and final study.

I. Definition of Units

A. Undefined Students directly compared the total lengths of two lines and did not define a unit.

B. Discrete points Students drew or counted points as the units along a line; or counted beginning, end-points and junctions between line segments along a path of line segments.

C. Line segment Students counted line segments as the unit.

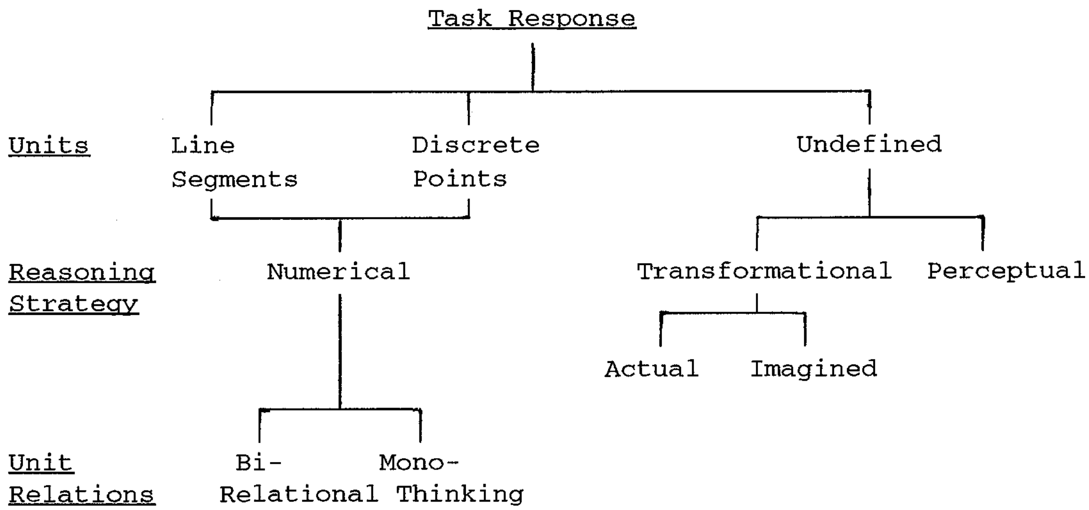


Figure 5.03 Analytical categories used to classify student responses to the linear measurement tasks.

II. Reasoning Strategies

A. Perceptual strategies Students did not describe imagined actions or use external actions to justify their judgements. Their judgements were based on the appearance of the paths and were derived by:

1. comparing the relative positions of the ends of the paths, or

2. evaluating the shapes of the paths with regard to the degree to which each path "goes up and down" to estimate the comparison of the total lengths

B. Transformational strategies Students transformed the paths in the following ways:

1. Imagined The students imagined actions to:

    a. stretch each path straight "in their mind" and compare the results, or

    b. "shrink" one path "in their mind" and compare the shapes of the paths, that is imagine one path squeezed up thereby reducing the angles at the junctions of the line segments.

2. Actual The student transformed the configuration of the paths by:

    a. redrawing one path into the shape of the other, or

    b. redrawing each path into a single line segment and comparing the total lengths.

C. Numerical strategies Students identified or constructed units. Mono-relational and bi-relational reasoning strategies were used to interpret the units. The relationships between units were expressed within their reasoning strategy in the following ways.

1. Mono-relational thinking

   a. Students defined units to be equivalent regardless of differences in their length and counted them accordingly.

   b Students used only one of the units to measure off the length of both paths and compared the counts of the measures.

2. Bi-relational thinking

   a. Students matched units in one path in one-to-one correspondence with units in the other, then reasoned about the comparison in terms of the numerosity and size of units.

   b. Students counted the number of short and long units and used compensatory reasoning with regards to the differences in the sizes and counts of the units.

Not all categories were applicable to all tasks. Distinctions between the students' use of reasoning strategies (numerical, transformational and perceptual) only applied to the irregular path tasks because number was an explicit part of the other tasks. Also, distinctions in students' reasoning with multiple units (mono-relational versus bi-relational thinking) did not apply to the partitioning task. Only one unit was used in this task; all of the responses were necessarily mono-relational.

Students' Representations of Units of Length.

This section addresses the question of the extent to which the students' representations of units were consistent with properties of linear measurement. For this purpose student responses (in both the interview and linear measurement test) to the following tasks were analyzed: the ruler tasks, the aggregate unit task and the partitioning task. These data are presented in Table 5.01.

Table 5.01 is designed to explore students' representations of units in several ways. First it permits us to determine the general extent to which students constructed units which were either discrete points or line segments. Second, it allows us to compare each of the tasks with regard to the extent to which discrete points or line segments were constructed by students. And finally it reveals the extent to which their reasoning about the relationship between different units was mono or bi-relational.

Table 5.01

<u>Students' Representations of Units of Length and Relationships Between Units</u>.

	Ruler Task						Aggregate Unit Task						Partitioning Task					
	Test			Interview			Test			Interview			Test			Interview		
Response Groups	D	Mi	L	D	Mi	L	D	Mi	L	D	Mi	L	D	Mi	L	D	Mi	L
<u>Dominantly Discrete</u>																		
<u>James</u>	M	.	.	B	.	.	B	.	.	.	.	B	M	.	.	.	M	.
<u>Marlene</u>	.	.	?	.	.	?	.	.	?	.	.	M	M	.	.	.	M	.
Lolande	M	.	.	.	.	B	B	.	.	B	.	.	.	.	?	M	.	.
Connie	NR	.	.	M	-->	B	.	.	?	B	.	.	M	.	.	.	M	.
Fanya	B	.	.	.	.	B	.	.	B	.	B	.	.	.	?	.	M	
<u>Derek</u>	?	.	.	.	.	B	B	.	.	.	.	MB	M	.	.	M	.	.
<u>Dominantly Line Segments</u>									.	.	.							
<u>Edwin</u>	B	.	.	B	.	.	.	.	B	.	.	B	.	.	M	.	.	M
Dahlia	B	.	.	.	.	B	.	.	B	.	.	MB	.	.	M	.	.	M
Tammy	B	.	.	.	.	B	.	.	B	.	.	B	.	.	M	.	.	M
Lara	B	.	.	.	.	B	.	.	B	.	.	B	.	.	M	.	.	M
Kasey	B	.	.	.	.	B	.	.	B	.	.	B	.	.	M	.	.	M
<u>Line segment</u>																		
<u>Brock</u>	.	.	B	.	.	B	.	.	B	.	.	B	.	.	?	.	.	M
<u>Kit</u>	.	.	B	.	.	B	.	.	B	.	.	B	.	.	M	.	.	M
Coran	.	.	B	.	.	B	.	.	B	.	.	B	.	.	M	.	.	M
Pete	.	.	B	.	.	B	.	.	B	.	.	B	.	.	M	.	.	M

<u>Note</u>. The names of the students in Grade 5 are underlined.

<u>Units</u>
D = discrete points
Mi = mixed, points & line segments
L = line segments

<u>Unit relations</u>
M = mono-relational
MB = mono- then bi-relational
B = bi-relational
? = no defined units
NR = no response

The students have been grouped in Table 5.01 according to the extent to which they represented units as discrete points or line segments: (a) dominantly discrete points, (b) dominantly line segments, and (c) consistently line segments.[1] All students who represented units dominantly as discrete points, did so in response to the partitioning task. With the aggregate unit and ruler tasks, the form of units was more variable. In contrast, other students used discrete points as units only with the ruler task. All of the students who dominantly or consistently used line segments as units used bi-relational reasoning for their final responses to the ruler and aggregate unit task. They accounted for the differences in size of units within their solutions. Of the students whose units were dominantly discrete points, four constructed a mono-relational representation with one or other of the tasks, but only one student (Marlene) did so in all instances. Marlene (Grade 5) was exceptional in that she generally did not reference units at all in the tasks which required the inter-relationship of two different units.

In summary, most students represented some units as discrete points. However, none did so exclusively, and a few students never did so. Instead, they consistently represented units as a line segment. The ruler task was the context in which the greatest number of students were inconsistent in their representation of units. But for those students who dominantly defined units as discrete points, the partitioning tasks was the context in which they did so most consistently. Nearly all students demonstrated an ability to reason appropriately about the relationship between two different linear units, although some students were inconsistent. In the balance of this section the manner and circumstances in which units were represented as discrete points are explored in more detail.

Variations in Students' Representations of Units of Length

Figure 5.04 illustrates ways in which discrete units were constructed for the partitioning task. The first two students marked five points and three points along each line. However, James gave a mixed response. He partitioned the line into five line segments, then counted three points to

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1. Students in the "predominantly discrete" group used discrete points as a unit or did not define units in more than half of the tasks in the test and interview setting.

determine the length of the second line. Similar variations occurred with the aggregate unit task when students used discrete points as the unit.

1. (Lolande, Grade 7)

2. (Derek, Grade 5)

3. (James, Grade 5)

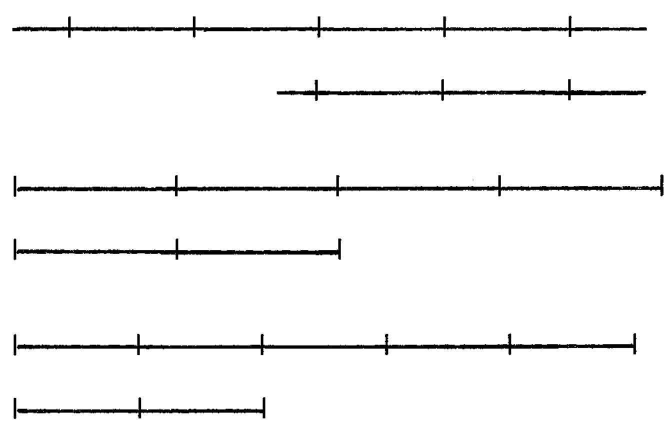

Figure 5.04 Examples of students' use of discrete points as units to partition a line into five units then drawing a line of three units.

Figure 5.05 illustrates ways in which discrete units were used with the ruler task. The reasoning behind these responses differed. In the first example, the relationship between the size of the centimetre and flug was ignored. The points with each numeral determined the length of the line drawn. In the second example, the student attended to the 2:1 relationship between centimetres and flugs but counted the beginning and end points of the line segments as the units, beginning with the point associated with the 1 on the ruler. In the third example, the student converted centimetres to flugs using mental arithmetic and then represented the 3 flugs to correspond with the numerals on the ruler and not the length of units.

The ways in which students used line segments to represent the units with the ruler task are illustrated in Figure 5.06. The only difference in the two cases was whether or not the line was extended from the beginning of the ruler or from the point associated with the numeral "1."

Point/Line Segment Conflict: Differences Among Tasks and Solution Strategies

It was observed earlier that there were differences among the tasks with regard to the extent to which students represented units as contiguous line segments or discrete points. The ruler task was the one with which the largest number of students were inconsistent in their representation of units. In the partitioning task, discrete units were used most extensively by the students in the

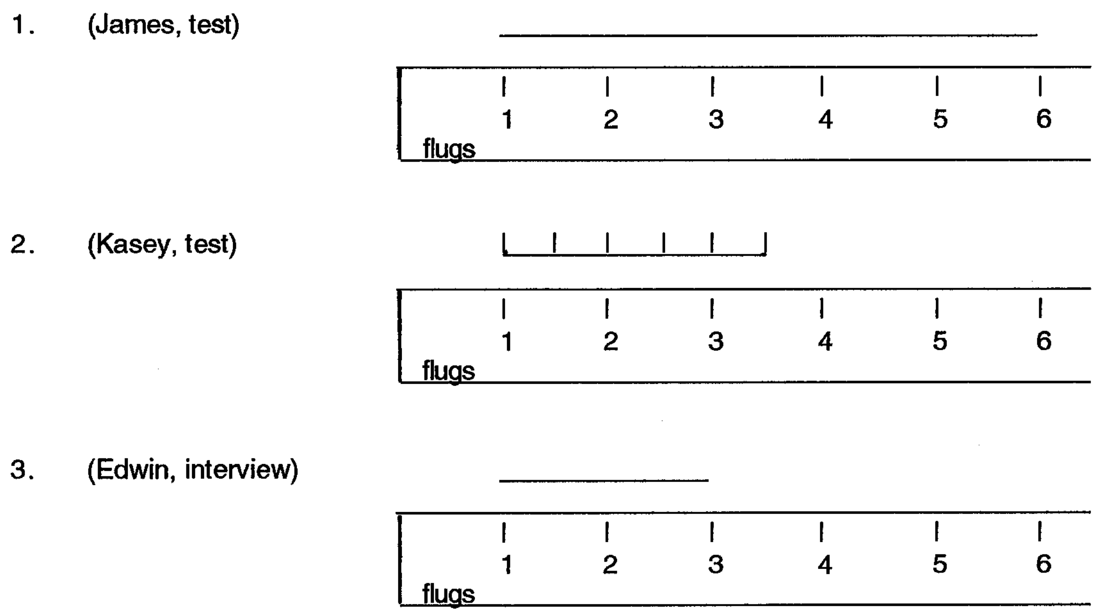

Figure 5.05 Students' use of discrete points as units with the ruler task.

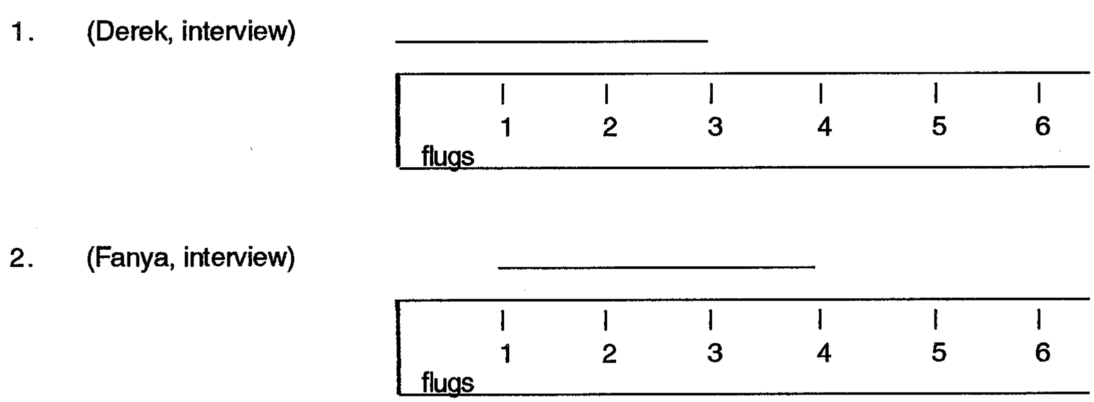

Figure 5.06 Students' use of line segments as units with the ruler task.

"dominantly discrete" group. There were a number of characteristics in students' responses which suggest that counting actions associated with different procedures for drawing a unit might have influenced the manner in which some students attended to line segments or discrete points as units.

When students who represented units dominantly as discrete points used a partitioning process to resolve the aggregate units task, they defined units as points. However, those who resolved the aggregate unit task by iterating line segments rather than partitioning a line faced no ambiguity about whether to determine the measure by the count of the points or the line segments. The contrast in the two approaches to the aggregate unit task suggests that the different counting actions influenced these students' attention on line segments or discrete points as units. It would appear that the process of partitioning a line led these students to attend more to the points rather than the line segments.

With the ruler task there was the additional feature that points were juxtaposed with numerals. This juxtaposition further emphasized a counting relationship between points and numerals. In this regard, students interpreted the "1" as the beginning marker of their representations of 6 centimetres, not as the end marker of the first "flug" unit (see Figures 5.05 & 5.06, 2) In all instances when students used discrete points as units and some when students used line segments as units, their representation of 6 centimetres began from the "1." The general structure of the ruler and the meaning of the numerals implied by that structure did not guide the students' representation of 6 centimetres.

The point/line segment conflict, which nearly all students demonstrated to some extent, appears to be influenced by perceptual and conceptual factors. The points are perceptually salient to the ruler and partitioning tasks, and attention is centred on them during their resolution. They are the component of the representation acted on synchronously with the verbal count, exactly the same actions as counting discrete units. However, it is indirectly through the points that line segments are defined as linear units. One has to attend to the points, think about line segments, and keep track of the relationship between the count of points and the number of line segments.

There is not a single, direct relationship between the count of points and the number of line segments. Depending on whether one counts all beginning and end-points, only end-points, or only internal points between line segments, X line segments would be represented by a count of $X + 1$, X, or X - 1 points. However, earlier experiences of counting discrete units establishes that the value of the count always equals the number of units. In addition, the common use of the ruler reinforces the notion that there is a direct relationship between the count of points and the number of units. Having placed a ruler correctly, only the points and numerals have to be attended to to "read" the length. As a result, the need to attend to other factors besides the count of the points when representing or interpreting units in some linear measurement situations does not appear to be recognized universally by the students.

In summary, only four students consistently represented units as line segments in the linear measurement context. Among the rest of the students, the extent to which they considered the units to be line segments varied with each task. For some, discrete units were used only with the ruler task, but for others inconsistencies in what constituted the unit occurred with all of the tasks. Some students do not appear to have a clear sense of the significance of the invariance of the attribute of length being considered. For these students their earlier experiences with number and counting as a measure of discrete units still appears to influence their representation and interpretation of units in a linear measurement context. From an instructional point of view, it cannot be assumed that students, even at the upper intermediate grades, will represent or interpret units in a linear measurement context invariably as line segments.

Reasoning with Units and Number with the Irregular Path Tasks

This section addresses the question of the characteristics of the reasoning strategies students use to compare quantities of length. For this purpose student responses to the irregular path tasks (see Figure 5.02) were analyzed. A secondary question is whether there appears to be any consistency between students' representations of units (dominantly discrete, dominantly line segments, or consistently line segments) and the types of reasoning strategies they used to compare the lengths of the paths. These data are presented in Table 5.02.

Table 5.02 presents the responses of students to each irregular paths task during the interview. The characterization of each student's representations of units as derived in the analysis of the previous section are also included: dominantly discrete points, dominantly line segments, and solely line segments. The intent of the table is threefold. First, it illustrates the extent to which different reasoning strategies were used by the students for each task. Second, it permits a comparison of the

Table 5.02

Students' Reasoning Strategies With the Irregular Path Tasks and a Summary of Their Representations of Units of Length

	Representations of units			Tasks by reasoning strategies											
				Task D						Task E					
Students	D	PL	L	P	TI	TA	ND	N1	N2	P	TI	TA	ND	N1	N2
Connie	#	.	.	#	.	.	.	.	.	.	.	.	#	.	.
Fanya	#	.	.	.		#	.	.	.	.	.	#	.	.	.
Marlene	#	.	.	.	.	#	.	.	.	.	#	.	.	.	.
James	#	.	.	.	#	.	.	.	.	.	#	.	.	.	.
Lolande	#	.	.	.	#	.	.	.	.	.	#	.	.	.	.
Derek	#	.	.	.	#	.	.	.	.	.	#	.	.	.	.
Dahlia	.	#	.	.	#	.	.	.	.	.	#	.	.	.	.
Brock	.	.	#	.	#	.	.	.	.	.	#	.	.	.	.
Tammy	.	#	.	.	#	.	.	.	.	.	.	.	.	.	#
Kasey	.	#	.	.	.	.	.	#	.	.	.	.	.	.	#
Lara	.	#	.	.	.	.	.	.	#	.	.	.	.	#	.
Edwin	.	#	.	.	.	.	.	.	#	.	.	.	.	.	#
Kit	.	.	#	.	.	.	.	.	#	.	.	.	.	.	#
Coran	.	.	#	.	.	.	.	.	#	.	.	.	.	.	#
Pete	.	.	#	.	.	.	.	.	#	.	.	.	.	.	#

Note The names of students in Grade 5 are underlined
Asterisk indicates the category of a student's responses

Representations of units
D = dominantly discrete points.
PL = dominantly line segments.
L = line segments only.

Reasoning strategies
P = perceptual.
TI = transformational (imagined)
TA = transformational (actual).
ND = numerical (discrete units)
N1 = numerical (with one linear unit)
N2 = numerical (with both linear units)

consistency in the reasoning strategies used by a student to resolve both comparisons. Finally, it allows for a comparison between the students' general representations of units and the reasoning strategies they used with these comparative tasks.

As can be seen in Table 5.02, the same number of students (eight) used a numerical reasoning strategy as used a transformational reasoning strategy to compare the lengths of the paths. Whether students used numerical or transformational reasoning strategies, nearly all were consistent in the strategies they used for both comparative tasks. In addition, there were two general response patterns: (a) students who compared the lengths of these irregular paths with appropriate numerical reasoning strategies also represented units dominantly as line segments during the first set of tasks, and (b) most of the students who used transformational strategies had represented units dominantly as discrete points with the first tasks.. Students in the latter case did not count discrete points as a final strategy, even though this strategy would have been consistent with their representations of units in other tasks. It is possible that these students did not consider numerical strategies to be reliable and used the strategy on which they attached the greatest confidence, namely a direct comparison of lengths without recourse to units.

Numerical reasoning strategies. Generally, when numerical reasoning strategies were used, the students identified and used units appropriately. There were three ways in which the students approached the comparative reasoning with units as line segments: (a) they measured both paths with only one unit and compared the measures, (b) they counted the long and short units separately and reasoned about the inverse relationship between the numerosity and sizes of the units, and (c) they evaluated the relative numerosity of the different units through one-to-one correspondence before reasoning about the inverse relationship between the numerosity and sizes of the units. The frequency with which each of these approaches was used for the different tasks is presented in Table 5.03. The approaches used most frequently to resolve Task D and Task E were the most efficient ones, given the complexities of the tasks.

Table 5.03

Frequency of the Different Approaches to Numerical Reasoning with Line Segments for Each Irregular Path Task

	n	Count discrete units	Measure with one unit	Count units separately and compare	One-to-one correspondence and compare
Task D	6		1	5	
Task E	8	1	1	1	5

Transformational reasoning strategies. Students imagined (and a few drew) the effect of changing the configuration of the paths. In so doing they sought to compare the lengths of the paths directly. In some cases, the differences in the heights and drops of paths were mentioned, or the configurations of the paths compared, while in others the "stretching" action was the only verbal rationale. They believed that a path would be longer or shorter if each were straightened, or that such was the case as they straightened each in their imagination. They appeared to be reasoning from the premise that the transformed length would be equivalent to the length of the original path. The manner in which students used imagined transformational strategies implied a belief in their own ability to make comparative judgements on the basis of very rudimentary approximations of each path's length.

Summary of the Characteristics of Students' Representations and Interpretations of Units of Length

There was little difficulty encountered when tasks were solved by iterating a line segment as a unit to define a length. However, tasks were less easily solved when they involved representing or interpreting units within a pre-defined length. The relationship between action, language, and number is more complex when representing or interpreting units within a pre-defined length. When interpreting the relationship between the count of points and the number of line segments, the orientation of points to line segments also must be attended to. There were students for whom the number of points on a line was often associated as the number of units determining the measure of the line, regardless of the orientation of points to line segments. Their assumption of a one-to-one relationship between counting action and number probably derives from the extensive experience of counting in a discrete context in which action, language and number are universally synchronous.[2] A belief that the number of points universally equals the number of line segments matches the numerous situations in which this condition is true in a linear measurement context.

When not directed to consider units as a means of comparing lengths, over half of the students compared lengths directly through perceptual or transformational reasoning strategies rather than indirectly through numerical reasoning strategies. These students either (1) were more confident with decisions based on transformational rather than numerical strategies, or (2) did not interpret the problem situation as one involving units of measure. Most students who used direct comparison strategies were also less consistent in representing units as line segments in other linear measurement contexts. For most of these students, measurement with units of length was not associated generally with the enumeration of congruent line segments. In contrast, most students who compared lengths indirectly through numerical reasoning strategies were more consistent in their representation of units of length as line segments and associated measurement of length with the enumeration of congruent line segments.

Students' Representations and Interpretations of Units of Area

The research question concerning the characteristics of students' representations and interpretations of units of area was particularized by the following questions:

A. To what extent do students interpret or construct units of area in a manner consistent with properties of area?

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2. The importance of the synchronous relationship of action, language, and number in the counting of discrete units, and the difficulties younger children encounter in this regard is well documented (e.g., Gelman & Gallistel, 1978). In linear and area measurement contexts this synchronous pattern can no longer be assumed. Hence, different relationships between partitioning actions, counting actions and the units have to be accommodated in the representation of linear and area units.

B. What are the characteristics of the reasoning strategies students use to compare quantities of area?

These questions led to the identification of a number of sub-questions related to particular aspects of students' responses to the area measurement tasks. Some of these sub-questions are outlined in this introduction, while others are referenced within the sections specific to different tasks.

Area measurement is geometrically more complex than linear measurement. The additional dimension introduces perceptual and conceptual variations into the characteristics of the unit. Regions or part of regions of equal area are not necessarily congruent. As a consequence, direct comparisons of areas are difficult if not impossible to achieve effectively. In addition, units themselves are subject to the many geometrical variations encountered in the regions to be compared. While only squares are used in the formal measurement of area, a more general definition of units is any polygon that can tessellate a region. This latter definition applies more generally to area-based representations of common fractions. Regions are partitioned into triangles, squares, rectangles or sectors of a circle to represent the same fractional relationships. Students are expected to comprehend the general case that when a region is partitioned into n parts of equal measure, the measure of each part is 1/n, regardless of the shape of the part (i.e. unit fraction piece). Whether students' conceptions of units of area might include such a generalization is a question explored in this section.

The area measurement tasks were designed to explore a number of aspects of students' representations of area units. The first aspect was how students partition different regions when asked to construct equal parts (see Figure 5.07). Students' representations of equal parts have direct implications on the ways in which they might represent and interpret area-based representations of common fractions. It also has implications for their general conception of units of area. The second aspect was how students would compare fractional units of regions (see Figure 5.08). Of particular interest was whether students would perceive a part-whole relationship between units as a basis for reasoning about these comparisons. While the third aspect was whether students would compare irregular regions by using numerical reasoning strategies with units which were consistent with properties of area measurement (see Figure 5.09).

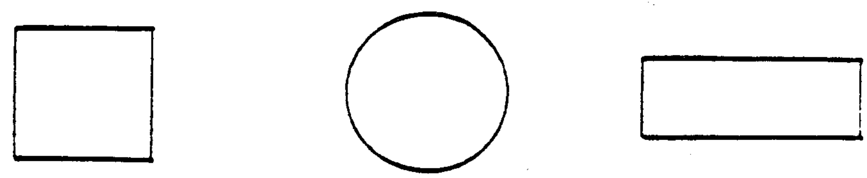


Figure 5.07 Partitioning tasks: divide each figure into 6 equal parts.

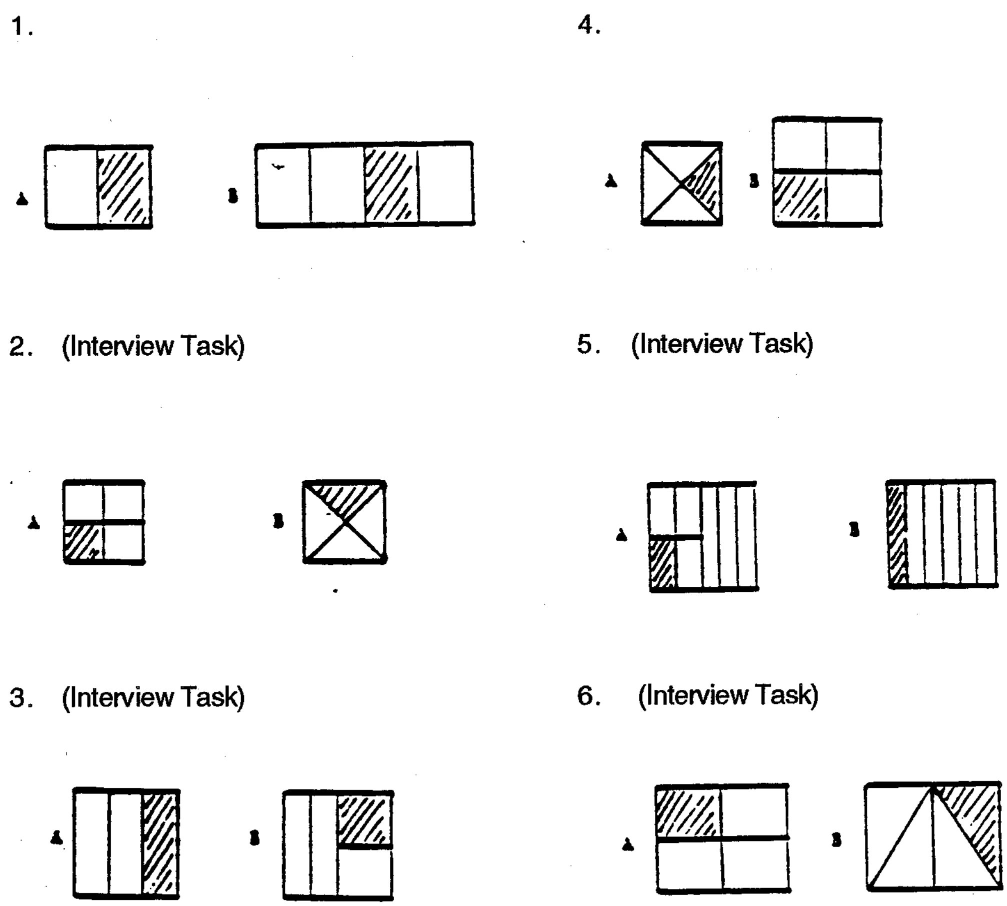


Figure 5.08 Cake tasks: compare the sizes of the shaded pieces of cake.

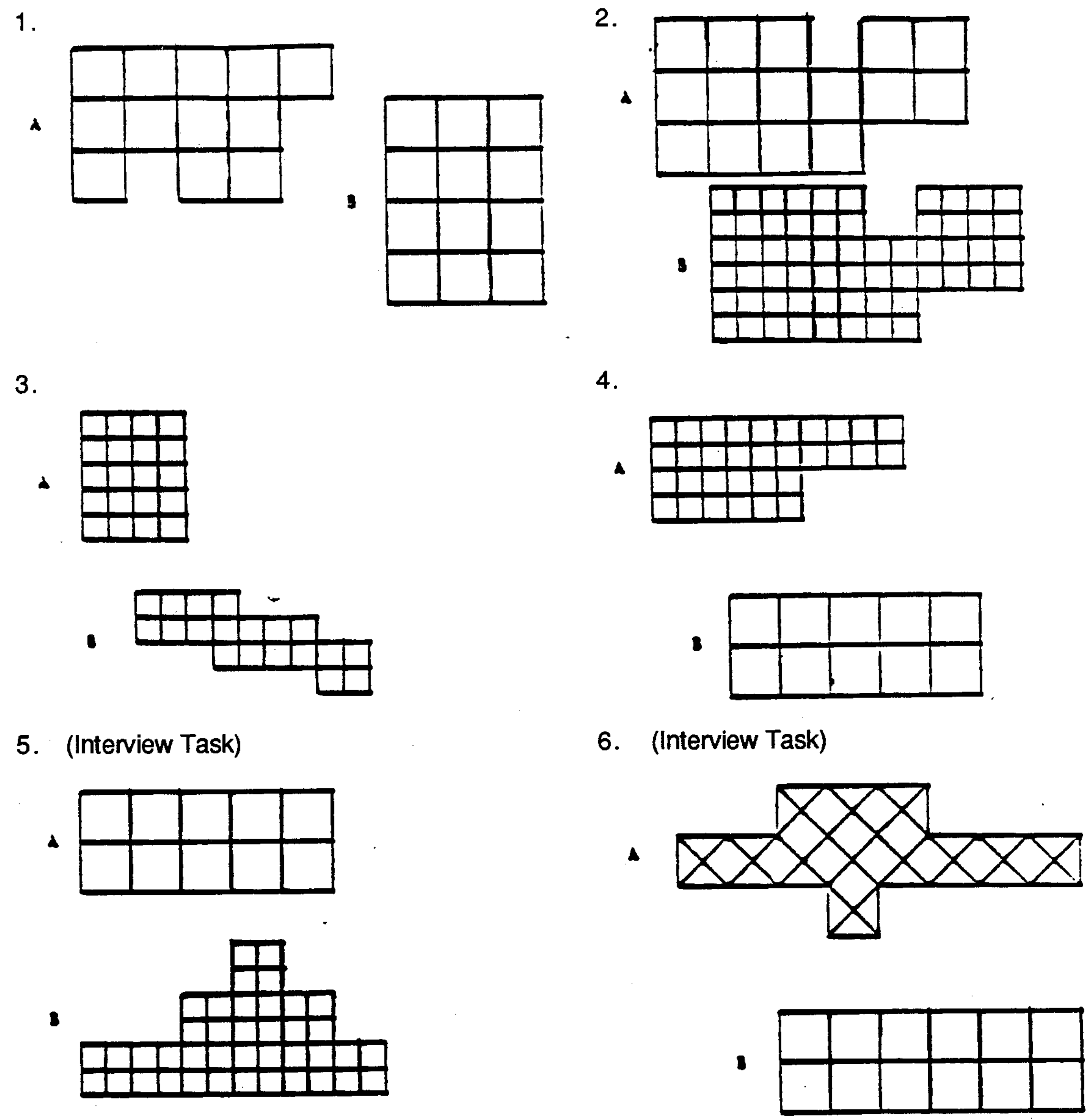


Figure 5.09 Tile tasks: compare the amount of space in each playroom.

Analytical Categories to Classify Responses to Area Measurement Tasks

The analytical categories described in this section and schematically represented in Figure 5.10 were derived from an analysis of the tasks and an analysis of the students' responses to the tasks during the second pilot study and the final study. The categories are analogous to those used to classify the responses to the linear measurement tasks. As was the case with the linear measurement categories, not all categories apply to all tasks because of differences in the demands of the tasks.

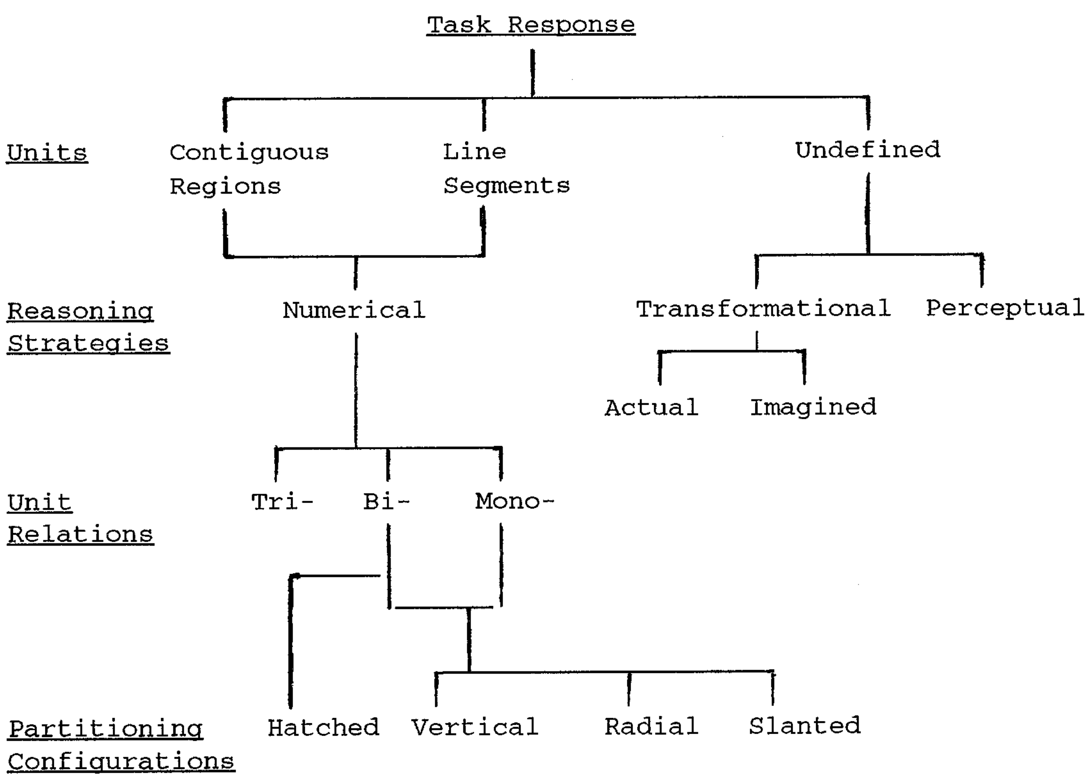


Figure 5.10 Analytical categories used to classify the students' responses to the area measurement tasks

I. Definition of Units

A. Undefined Students did not construct or use units to compare areas.

B. Line segments Students used discrete line segments as units when they partitioned a region.

C. Regions Students constructed or used regions as units.

II. Reasoning Strategies

A. Perceptual strategies Students' judgements were based on either

1. evaluations of only one of the dimensions in each region, such as stating that one region is taller therefore it is bigger than the other, or
2. global evaluations of the size of the regions that were not substantiated further, such as stating that one region "just looks bigger."

B. Transformational strategies The students transformed one or more regions in the following ways:

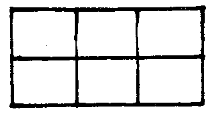
1. Imagined transformations Students imagined primitive compensations of the two dimensions such as describing one dimension stretching as the other shrinks or by implying compensations by stating that "one shape is wider but the other is longer."
2. Actual transformations Students performed actions which transformed regions by:
    a. overlaying (drawing) one region on the other,
    b. cutting and moving parts of one region in order to approximate the shape of the other region or to equalize both regions along one dimension, or
    c. partitioning each region into different sized subregions and comparing each subregion, one to one.

C. Numerical strategies Students identified or constructed units. The relationships between units were expressed within their reasoning strategy in the following ways.

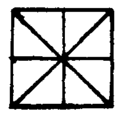
1. Mono-relational thinking Students defined the units as equivalent regardless of their size and counted them accordingly.
2. Bi-relational thinking Students identified two different units and included the relationship between the two units within their reasoning strategy by:
    a. translating one unit into the other with reference to the ratio between the units and counting the equivalent units, or
    b. counting the number of small and large units and using compensatory reasoning with regards to the differences in the size of the units and the counts.
3. Tri-relational thinking Students identified three different units and included the relationship between the three units within their reasoning strategy.

III. Partitioning configurations

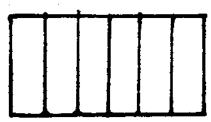
A. Hatched Configuration: involved the crossing of vertical and horizontal lines to produce a rectangular grid of subregions.



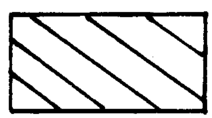
B. Radial configurations involved the crossing of diagonals at a midpoint or the extension of radii to or from a midpoint, or the crossing of diagonals and bisectors of sides of the square at a midpoint, such that all lines radiate from the midpoint of the region.



C. Vertical configurations involved the drawing of vertical parallel lines to produce six consecutive parts along one dimension of the region.



D. Slanted configuration involved the drawing of slanted, parallel lines to produce 6 consecutive parts.



Both the test and interview responses were analyzed with regard to the types of strategies used by the student to solve the problems and the consistency of a student's responses to similar tasks. In most cases strategies could not be inferred from the test papers; therefore, except where indicated, the classification of a student's strategies refers only to those used during the interview.

Equal Parts of a Region: The Partitioning Tasks

This section addresses the question of the extent to which students represent units of area in a manner consistent with properties of area. For this purpose, students' responses to the partitioning tasks were analyzed. Two secondary questions are addressed in this analysis.

1. Are there differences in students' success at partitioning each of the three regions?

2. If there are differences in the success with which students partition different regions into six equal parts, what might account for the differences?

Differences in Success with Different Regions

Table 5.04 illustrates the frequency with which students were successful at partitioning each of the regions into 6 approximately equal parts in both the test and interview setting. As can be seen in this table, the rectangle was consistently partitioned into 6 equal parts by the largest number of students and the circle by the least number of students. There was little difference in the number of students who were successful at partitioning the square and the rectangle at least once, but fewer students were consistent in their partitioning of the square.

Table 5.04

Frequency of Students' Constructions of Six Equal Parts of Each Region in the Test and Interview.

	Number of settings		
Regions	Both	One	Neither
Rectangle	10	5	0
Square	6	8	1
Circle	5	6	4

Strategies Associated with Partitioning Problems

Table 5.05 was designed to explore patterns in students' partitioning strategies which might influence the partitioning results. The results of student's partitioning strategies with each region in the test and interview setting are presented in this table. In the body of the table, first and final trials have been coded, indicating the type of configuration used in each instance: hatched, vertical, radial or slanted. The students are grouped within the table according to the number of different regions which

Table 5.05

Consistency of Students' Partitioning of Each Region and a Breakdown of the Results of the Initial and Final Partitions of Each Region.

	Circle						Square						Rectangle					
	Test			Interview			Test			Interview			Test			Interview		
	N	U	E	N	U	E	N	U	E	N	U	E	N	U	E	N	U	E
0 Regions																		
Marlene	H	.	.	.	H	.	H	.	.	.	.	H	H	.	.	*V	.	V
Dahlia	*R	H	.	.	.	R	*HH	.	.	.	.	H	H	*V	.	*V	.	H
Derek	R	.	.	*R	.	R	R	.	.	.	.	H	V	.	.	*V	.	V
James	.	.	R	.	*RH	.	H	.	.	*H	.	H	V	.	.	*V	.	H
1 Region																		
Edwin	R	.	.	.	R	.	R	.	.	.	R	.	.	.	H	.	.	H
Connie	.	H	.	.	R	.	.	.	H	.	.	V	.	.	V	.	S	.
Lolande	.	H	.	.	*R	R	.	R	.	.	*R	H	.	*V	H	.	.	V
Tammy	.	.	R	.	V	.	H	.	.	.	.	V	.	.	V	.	*V	V
2 Regions																		
Brock	.	H	.	.	H	.	.	.	H	.	.	H	.	.	H	.	.	H
Kit	.	*R	R	.	V	.	.	.	V	.	.	V	.	.	V	.	*V	V
Kasey	.	*R	R	.	*R	R	H	.	.	.	.	H	.	*H	H	.	.	H
Fanya	.	.	R	.	.	R	.	*R	V	.	R	.	.	.	V	.	*V	V
3 Regions																		
Coran	.	.	R	.	.	R	.	.	V	.	*R	V	.	*V	H	.	.	V
Lara	.	.	R	.	*R	R	.	.	H	.	.	H	.	*V	V	.	.	V
Pete	.	*R	R	.	*R	R	.	.	H	.	.	H	.	.	H	.	.	V
Totals																		
Initial	4	6	5	1	11	3	7	2	6	1	4	10	3	5	7	4	4	7
Final	3	4	8	0	7	8	7	1	7	0	2	13	4	0	11	0	1	14

Note. The names of the students in Grade 5 are underlined.

Result of Partition
N = number not equal to six.
U = 6 unequal parts
E = 6 equal parts

Configurations
R = radial configuration
H = hatched configuration
V = vertical configuration
S = slanted configuration
* = first partition

they successfully partitioned both on the test and in the interview. These groups are labelled "0 regions", "1 region", "2 regions", and "3 regions" and indicate the number of regions with which the students were consistently successful.

A number of patterns are discernible in this table. First, some problems of planning and executing partitions with different regions were encountered by students in all groups (see asterisked entries in table). However, production problems were not associated simply with regions with which students were less successful. Half of the students who successfully partitioned the rectangle during the interview required more than one trial to do so. The inconsistency with which some students partitioned regions arose not only because of technical production problems, but also because of characteristics in their solution strategies.

Students who failed to achieve six approximately equal parts for a region (see columns N & U) proceeded to partition the region in one of two ways.

1. Students used a configuration with which they could not partition the region into six equal parts (e.g., the radial configuration applied to the square).
2. Students used an appropriate configuration in ways which did not result in 6 equal parts (e.g., successive halving with the radial configuration applied to the circle).

Both patterns occurred with circles and squares, but only the second generally applied to rectangles. Each of these solution patterns will be discussed in turn.

Configurations applied inappropriately to a region. As can be seen in Table 5.05, nearly all students used standard configurations which had the potential to result in equal parts if applied appropriately. However, some students appeared to assume that a standard configuration would produce equal parts regardless of the geometric properties of a region to which it was applied. Two-thirds of the students applied the radial configuration to the square, or the vertical or hatched configuration to the circle (e.g. Lolande in Table 5.05). Five of these students applied one configuration to all three regions (e.g. Tammy & Brock). In these cases, notions of what constitutes equal parts for different regions appear not to be founded on measures of area in the usual sense.

Instead, the configurations appeared to function as over-generalization algorithms for partitioning regions.

Students applied configurations to different regions in a relatively systematic pattern. In Table 5.06, it can be seen that the initial configuration used most frequently by students differed with each region: radial with the circle, hatched with the square, and vertical with the rectangle. This suggests that variations in geometric properties of regions might influence students' partitioning procedures. In particular, the rotational symmetry of each region, and the dominance or lack of dominance of a dimension in the quadrilaterals might have had a bearing on students' solution strategies.

Table 5.06

Frequency of Students' use of Configurations to Partition Each Region: Initial and Final Partitions.

	Circle		Square		Rectangle	
	Test	Interview	Test	Interview	Test	Interview
Initial						
Radial	11	11	4	4		
Hatched	4	2	9	8	5	3
Vertical		2	2	3	10	11
Slanted						1
Final						
Radial	10	10	3	2		
Hatched	5	3	9	9	8	5
Vertical		2	3	4	7	9
Slanted						1

The hatched and radial configurations, which themselves have rotational symmetry, were the dominant configurations used with the square and circle in turn. Furthermore, the hatched configuration was also the second most frequent configuration used with the circle and the radial configuration was the second most frequent configuration used with the square. The more extensive rotational symmetries of the square and the circle may account for students' over-generalization of the radial configuration to the square and the hatched configuration to the circle. The limited rotational symmetry of the rectangle might account for the absence of the radial configuration, while the dominance of the horizontal dimension of the rectangle might account for the prominence of the vertical configuration for initial partitions of the rectangle. It would appear that, as a result of the differences in the salience of such geometric characteristics, some students treated the rectangle and the square as separate partitioning problems.

In summary, standard configurations appear to have functioned as algorithms which some students over-generalized. Furthermore, geometric properties of regions which are not necessarily related to the partitioning problem might have influenced the types of configurations students use with different regions, and the ways in which configurations are over-generalized.

Unsuccessful partitions with appropriate configurations. There were different problems associated with the ways in which students applied each of the standard configurations. First, the successive halving algorithm was used frequently as the means of applying the radial configuration. It accounted for all unsuccessful cases (see Table 5.05). There were few instances where the application of the successive halving algorithm did not precede a successful application of this configuration. Even three of the four students, who used a multiplicative partitioning procedure with the factors of 6, initially used the successive halving algorithm. At the outset, students generally did not differentiate between partitioning strategies for powers of two and other even numbers.

The hatched configuration, requires a multiplicative procedure to partition a region into six parts, but this multiplicative procedure was not invariant for some students. Instead, they used an additive relationship of 3 + 3 = 6 to plan the partitions. This was so in most cases when the square was partitioned with the hatched configuration into more than six parts. The result was either 9 or 16 parts, depending on whether the students counted lines or spaces.

With the vertical configuration major difficulties occurred only when students counted 6 lines rather than 6 spaces, resulting in 7 parts. This problem of count lines rather than spaces was most common among the students who did not partition any of the regions consistently. These students appeared to assume a one-to-one relationship between the count of the lines and the number of spaces in a manner analogous to the counting of discrete points with the linear measurement tasks.

Multiplicative procedures based on factors were used in few instances except with the hatched configuration. Three students also used it with the radial configuration, and one with both the radial and vertical configurations. Even so, three of these students applied this procedure with the radial configuration only after the failure of the successive halving algorithm. There was little evidence that this multiplicative procedure was treated by students as a general algorithm for partitioning regions.

In summary, the inconsistencies in the success with which the different students partitioned the regions into six equal parts appeared to be related to one of several beliefs about procedures for determining equal parts, all of which could be considered to be algorithmic in nature. These commonly were: (a) the count of the lines is equivalent to the number of regions, (b) successive halving with the radial configuration results in equal parts, and (c) a configuration which will partition one region into equal parts will do so with others regardless of their different geometric characteristics. For final partitions, the most common source of error amongst these was the latter, the over-generalization of a configuration. In this regard, the square appears to be more difficult to partition successfully than the rectangle because the two quadrilaterals were treated as different partitioning problems. Instead, the salience of geometric properties shared by the square and the circle probably influenced students' to differentiate between the two quadrilaterals.

Algorithmic configurations provide a means to circumvent the global problem of directly estimating and comparing the areas of the parts. Given a belief that the configuration will result in parts of equal area, judgements could be based on spatial relationships of the lines alone. For example, James used the hatched configuration to partition the circle. He adjusted the position of the two parallel lines so that the three segments of the intersecting line were equal, with the belief that this would establish the equality of the parts. In this sense, linear measures within a configuration indirectly determine the relative areas of parts. However, until a student relates the geometric properties of the regions to the choice of configuration, and relates the properties of the number to be partitioned to the procedure for partitioning with a configuration, inconsistencies in the student's partitioning of regions will occur.

Comparing Parts of Regions: The Cake Tasks

This section addresses the question of the characteristics of the reasoning strategies students use to compare the area, in particular the area of fractional units. Of particular interest was whether students would consider a part-whole relationship between units as a basis for reasoning about these comparisons. For this purpose student responses to the cake tasks were analyzed. The students' responses to the four cake tasks presented during the interview were clustered because of differences in the complexities of the tasks (see Figure 5.11). The data for each pair of tasks are presented in Table 5.07.

Table 5.07 was designed to reveal a number of response patterns. First, it allows the extent to which students used different reasoning strategies to be examined with reference to each pair of tasks and all tasks in general. Students with similarities in their responses to the Cake Tasks have been grouped within the table in order to assist the discussion of patterns in students' responses to both pairs of tasks. And finally, it is designed to explore the relationship between the consistency with which students partitioned regions into equal parts and the extent to which they used reasoning strategies which referenced attributes of area to compare parts of regions. For this purpose the consistency with which each student partitioned the three regions into 6 equal parts, that is whether the student partitioned 0, 1, 2, or 3 regions consistently is also entered in the table.

Table 5.07

Summary of the Reasoning Strategies used to Solve the Cake Tasks and the Consistency with Which Students Partitioned Regions into Six Equal Parts.

	Consistent partitions	No. correct tasks		Cake tasks by reasoning strategies											
				Tasks 2 & 6						Tasks 3 & 5					
Groups	No. regions	2&6	3&5	P	P1	Ti	T1	T2	N	P	P1	Ti	T1	T2	N
1.															
Derek	0	0	1	#	.	.	.	.	.	.	#	.	.	.	.
Connie	1	0	2	#	#	.	.	.	.	.	#	.	.	.	.
Marlene	0	0	1	#	.	.	.	.	.	#	#	.	.	.	.
2.															
Lolande	1	0	0	#	#	.	.	.	.	.	.	#	.	.	.
Brock	2	0	1	#	.	.	.	.	.	.	.	.	.	#	.
3.															
Dahlia	0	1	0	#	.	.	.	.	#	.	.	#	.	.	.
Tammy	1	2	0	.	.	.	.	.	#	.	.	#	.	.	.
4.															
Pete	3	2	0	.	.	.	.	#	.	.	.	.	#	.	.
Coran	3	2	0	.	.	.	.	#	.	.	.	.	#	.	.
James	0	1	1	.	.	.	.	#	.	.	.	.	#	#	.
5.															
Kasey	2	2	2	.	.	#	.	#	.	.	.	.	.	#	.
Fanya	2	1	2	.	.	.	.	#	.	.	.	.	.	#	.
Edwin	1	2	2	.	.	.	.	#	.	.	.	.	.	#	.
6.															
Lara	3	2	2	.	.	.	.	.	#	.	.	.	.	#	.
Kit	2	2	1	.	.	.	.	.	#	.	.	.	.	#	#

Note. The names of the students in Grade 5 are underlined.

Reasoning strategies

P = perceptual strategy with a global judgement.
P1 = perceptual strategy comparing one dimension.
Ti = transformational strategy (imagined) compensating on two dimensions.
T1 = transformational strategy (actual) comparing only one dimension.
T2 = transformational strategy (actual) comparing two dimensions.
N = numerical strategy.

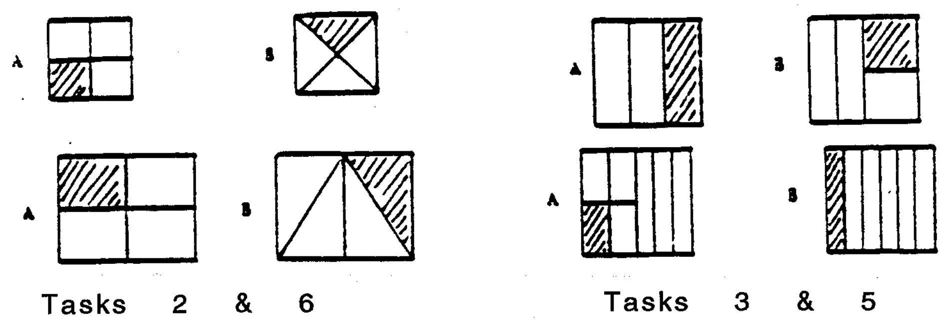


Figure 5.11 Comparisons of shaded fractional units: Pairs of cake tasks.

Comparing Non-congruent Fractional Units.

Part-whole relationships among units of area were seldom used as a basis for comparing the areas of the "pieces of cake." More than two-thirds of the students did not interpret any of the tasks to be problems of part-whole relationships of units. This was the case even with the common configurations of fourths in Tasks 2 and 6. Instead, the problems generally were interpreted as ones in which pieces of cake were compared directly without regard to the regions of which the pieces were a part.

Only a third of the students were relatively successful at comparing fractional parts with reasoning strategies which effectively addressed the two dimensional character of the task (see Groups 5 and 6). The other students' comparative judgements were subject to varying degrees of perceptual biases unrelated to the space-filling quality of the parts. The most extreme case of such biases occurred with the students in Group 1 who used only perceptual strategies.

Some students responded in a relatively stable manner to all four tasks. They used either perceptual reasoning strategies alone, or only transformational strategies based on two-dimensional reasoning(see Groups 1 and 5). However, for a majority of the students, the reasoning strategies generally differed between the pairs of tasks. This suggested that changes in strategies might be related to differences in the characteristics of the pairs of tasks. There appears to have been an inter-play between geometric properties of the tasks and ways in which students' reasoned about area.

The comparisons of rectangles (Tasks 3 & 5) were resolved by most students through some form of a transformational reasoning strategy (see Figure 5.12). In contrast, the comparisons of rectangles to triangles were resolved more frequently through perceptual or numerical reasoning strategies. The configurations in these latter tasks were less complex, which accounts for the larger number of students using numerical strategies. However, the transformation of triangles into rectangles is geometrically more complex. This may account for the smaller number of students using transformational strategies with these tasks.

Students in Group 2, 3, 4 and 6 changed their reasoning strategies in a manner consistent with the complexity of the tasks. Tasks 3 and 5 were more complex for students who used numerical reasoning strategies, hence the shift generally to transformational strategies with these task. Tasks 2 and 6 were more complex for students who did not use any numerical reasoning strategies, hence their shift to perceptual reasoning strategies with these tasks.

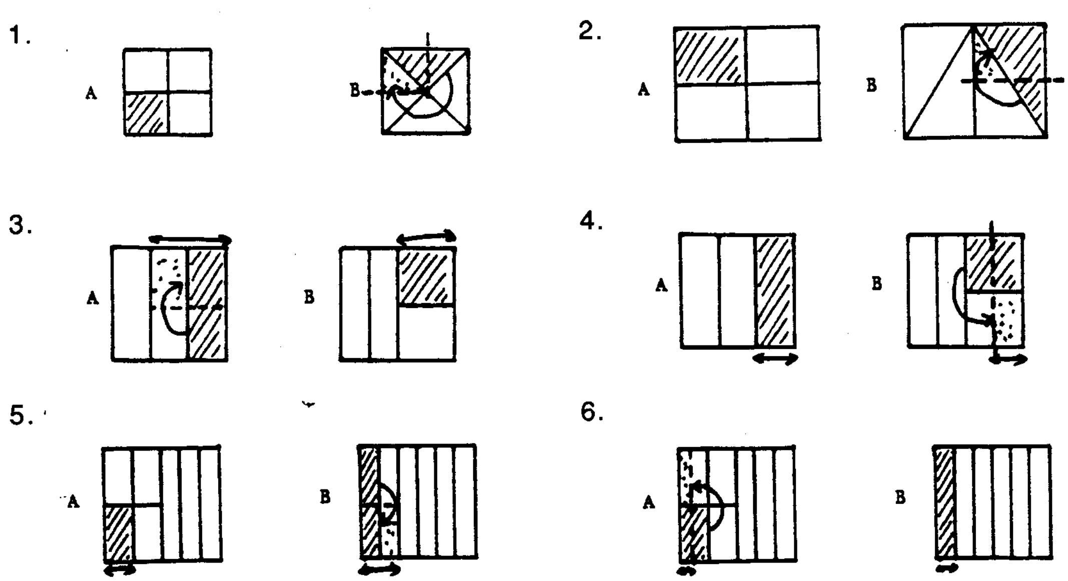


Figure 5.12 Examples of students' transformations of triangles into rectangles and rectangles into rectangles.

An exception to these general patterns was presented by the students in Group 4. These students successfully compared triangles and rectangles in Tasks 2 and 4, yet failed to compare the rectangles in Tasks 3 and 5 correctly. That transforming rectangles was easier than transforming a triangle into a rectangle did not seem, on the surface, to apply to these students. However, the actual transformations still were easier with the rectangles, but the act of transforming the rectangle appeared to fix their attention on the dimension on which they were equalizing the rectangles (see Figure 5.12). Their final judgements therefore were based on comparisons of only one dimension.

In summary, there were different limitations in the types of reasoning strategies used by students, and the problem contexts in which different reasoning strategies were applied. Geometric characteristics of the parts to be compared, as well as characteristics of the partitioning configurations appeared to affect the complexity of the tasks, and influence the types of reasoning strategies to which students resorted.

Variations in Reasoning Strategies.

There were two characteristics of some students' comparative strategies which might occur more generally than the interview data suggest. The first involved differences in the unit relationships considered in students' numerical reasoning strategies, and the second involved the application of a perceptual reasoning strategy to compare a triangle and rectangle.

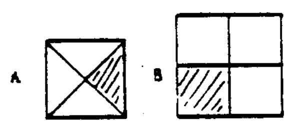


Figure 5.13 Comparisons of fractional units: Cake Task 4.

Numerical reasoning strategies. There were two numerical reasoning strategies used by students. One involved tri-relational thinking about units, and the other involved bi-relational thinking about units. In the tri-relational case, the student reasoned that if two regions were equal in area, and both regions were partitioned into 4 equal parts, then the shaded parts within each region would be equal regardless of their shape. They also recognized conditions when the fractional units were not equivalent, even though they were all fourths of a region. In the bi-relational case, the student reasoned that each subregion was equal because they both represented 1/4 of a region, regardless of the relative sizes of the regions. This restrictive bi-relational reasoning about units might have been used by other students to compare the subregions of Task 4 on the test (see Figure 5.14). Four of the 5 students who incorrectly compared these areas, judged the parts to be equal despite the fact that the partitioned regions were different sizes. Equality in such cases might have referred only to an abstract relationship between fractional numbers, not to measures of the space within each part.

Perceptual reasoning strategies. Perceptual strategies were used most often to compare a triangle with a rectangle, not to compare two rectangles. In such cases, students invariably concluded that the triangle was larger. A number of other students who eventually resolved the comparisons with a transformational strategy, also made this perceptual judgement initially, but on reflection compared the areas more systematically. In addition, on the test 9 students judged the triangle to be larger in Task 2, and 12 students did so with Task 6. This suggests that most of these students might consider this perceptual judgement to be a reliable comparison of areas of the regions unless prompted to reflect further on their initial judgement.

There are two possibilities which might account for the rather general view that a triangle, equal in area to a rectangle is judged to be larger. First, there is the sense that a triangle appears to be more extensive than a rectangle. There is a greater distance between the geometric centre and the corners of the triangle than of the rectangle. Without attention to other geometric characteristics of the two regions, this "appearance" would led to the conclusion that the triangle is larger.

A second explanation might rest in procedures followed to compare regions with both perceptual (P1) and transformational strategies. In both cases, final judgements of the comparisons generally were based on the comparison of the horizontal or vertical dimensions of the fractional parts. What differentiated strategies which were successful and unsuccessful in addressing the two-dimensional character of the tasks was the actions or reasoning which preceded this final comparison. Students either (1) ensured that the regions were equal on the other dimension, (2) assumed that regions were equal on the other dimension, or (3) did not attend to the question of the equality of the other dimension before they made the final comparative judgement. With the triangle-rectangle comparisons, students might have (1) attended only to the comparison of the differences in one horizontal or vertical "dimension," or (2) assumed that a horizontal or vertical "dimension" of both regions was already "equal" and concluded that the triangle was larger because it was longer on the "other dimension.". Procedures of equalizing and comparing horizontal or vertical dimensions, which applies to comparisons of rectangles, might be over-generalized without regard to other spatial characteristics of the regions.

Partitioning Regions and Comparing Parts of a Region

A feature common to most students' reasoning about the comparative problems was that they did not consider the relationship of parts to the region as a whole even when each region was partitioned into congruent parts. Regions simply were not considered to be a unit that had been partitioned into fractional units of equal area. This was the case regardless of the extent to which these students consistently partitioned regions into equal parts. For example, 2 of the 3 students (Coran & Pete) who consistently partitioned the three regions into 6 equal parts used transformational strategies for all of the cake tasks. The other student (Lara) was the one student who, when using a numerical reasoning strategy, used bi-relational reasoning without full regard to the relative size of regions.

One critical difference between these two types of tasks is the congruent/non-congruent characteristics of the parts being considered. Partitioning a region into X equal, congruent parts does not require a student to consider the more general case, that is, that parts associated with all configurations which partition that region into X equal parts would be equal even when non-congruent. Students who consistently partitioned 2 or more regions into congruent parts used reasoning strategies with the cake tasks which expressed one or more of the following beliefs:

1. Belief in the reliability of perceptual comparisons of non-congruent regions based on characteristics unrelated or only partially related to comparison of areas.

2. Belief in a compensatory relationship between dimensions and areas of rectangular regions without further means to act on their beliefs.

3. Belief that areas of regions are constant under transformations. Comparisons of non-congruent regions involved trials designed to reduce or eliminate the non-congruent characteristics of the fractional units by transforming one into a region more congruent with the second.

4. Belief that non-congruent regions are equal in area under specific conditions of unit relationships. The extent to which this belief was applied depended on the geometric and arithmetic complexity of the unit relations.

Except in the last instance, the two types of tasks, partitioning a region into congruent fractional units and comparing non-congruent fractional units, appear to have been treated as independent problems.

The Comparison of Irregular Regions: Tile Tasks

The third point of view from which students representations of units of area was explored, was whether students would compare irregular regions by reasoning with units of area measurement. For this purpose, student responses to the tile tasks were analyzed. These data are presented in Table 5.08.

Table 5.08 allows for a comparison of the reasoning strategies students used to compare areas in both interview tasks along with the students' success at comparing the areas in these tasks on the test. The reasoning strategy used by a student for each task is indicated by a hatch mark except in the case where a student used a numerical reasoning strategy to resolve a task. These numerical reasoning strategies are coded in the body of the table as mono or bi-relational. Also included in the left hand column are the test scores for all tile tasks. The students have been placed into three groups. The first group contains students who used reasoning strategies during the interview which did not focus explicitly on two dimensions, or whose test results strongly suggested that they treated units as equivalent regardless of their size. The second group contains students who used transformational reasoning strategies which focused on both dimensions of the regions. The third group contains students who used a numerical reasoning strategy.

In general, there is a consistency between the test results and the students' responses during the interview. Students who were more successful on the test also reasoned numerically with

Table 5.08

<u>Students' Interview Responses and Test Scores for the Tile Tasks.</u>

		Interview tasks by reasoning strategies									
		Tile Task 5					Tile Task 6				
Groups	Test score Max=6	P	P1	T1	T2	N	P	P1	T1	T2	N
1.											
Connie	3	#	.	.	.	.	#	.	.	.	.
Lolande	*2	.	#	.	.	.	.	#	.	.	.
<u>Brock</u>	2	.	.	#	.	.	.	.	#	.	.
<u>Marlene</u>	*2	.	.	.	.	MB	.	.	.	.	B
<u>James</u>	*2	.	.	.	.	B	.	.	.	.	B
2.											
<u>Edwin</u>	4	.	.	.	#	.	.	.	.	#	.
<u>Derek</u>	5	.	.	.	#	.	.	.	.	#	.
3.											
Fanya	4	.	.	.	.	B	.	.	.	.	B
Dahlia	5	.	.	.	.	B	.	.	.	.	B
Kasey	5	.	.	.	.	B	.	.	.	.	B
Pete	5	.	.	.	.	B	.	.	.	.	B
Tammy	6	.	.	.	.	B	.	.	.	.	B
Coran	6	.	.	.	.	B	.	.	.	.	B
Lara	6	.	.	.	.	B	.	.	.	.	B
<u>Kit</u>	6	.	.	.	.	B	.	.	.	.	B

<u>Note</u>. The names of the students in Grade 5 are underlined.
* indicates that the error pattern on test suggests units were treated as equivalent regardless of their size.
\# indicates the strategy was used by the student.

<u>Reasoning strategies</u>
P = perceptual strategy with a global judgement.
P1 = perceptual strategy comparing one dimension.
T1 = transformational strategy (actual) comparing one dimension
T2 = transformational strategy (actual) comparing two dimensions.
N = numerical strategy.

<u>Unit Relationships</u>
MB = mono-relational then bi-relational; counted units as equivalent then used 4:1 ratio.
B = bi-relational; related the units in a 4:1 ratio.

appropriate units of area, or who used transformational reasoning strategies which referenced two dimensions during the interview. Exceptions to this pattern occurred with students who, on the test, treated units as equivalent regardless of their size, yet during the interview reasoned with bi-relational units. On the test these students gave a correct response only to tasks in which the units were equivalent, and judged the region with the smaller units to be greater in all other tasks (see Figure 5.09). It was inferred that they were comparing regions on the basis of measures of numerosity. Their interpretations of units in an area measurement context were relatively unstable.

The students who reasoned with units aggregated the smaller units into a larger unit, then followed one of two procedures: (1) directly counted this larger unit within each region, or (2) matched larger units within each region in a one-to-one correspondence to compare the numerosity of the units in each case. When difficulties were encountered with these procedures, they invariably occurred with Task 6 which was perceptually more complex.

In general, more students were successful with these tasks than with the partitioning or the cake tasks. Two-thirds of the students interpreted the tile tasks as problems involving the comparison of measures of congruent units of area. The facility with which most students solved these tasks was probably due to the fact that the tasks more closely mirror curricular activities on area measurement. However, as can be seen by the strategies used and the difficulties encountered by the students in Group 1, bi-relational reasoning about units of area is not necessarily followed by students even at the Grade 7 level. Not only did students interpret units mono-relationally, but some still relied on gross perceptual judgements as a means of comparing regions already partitioned into units.

Students' Reasoning with Linear and Area Measurement: an Overview

The purpose of this chapter was to explore the characteristics of students' representations and interpretations of units in two measurement contexts, linear and area measurement. In order to address these questions, students' responses to a variety of linear and area measurement tasks have been analyzed in previous sections of the chapter. In this section an overview of the students' responses to these tasks serves to illustrate patterns which characterize variations in students' representations and interpretations of units in both contexts. Table 5.09 was designed for this purpose.

In Table 5.09 a summary of students' representations and interpretations of linear units is presented in two parts. In the first column are coded the students' representations of linear units summarized from their responses to the first three of the linear measurement tasks. In the second column are coded the reasoning strategies students used to resolve the comparisons of the irregular path tasks. These linear measurement data are a summary of the data in Table 5.02.

The area measurement data are summarized in three sections of the table. The consistency with which students partitioned three regions into 6 approximately equal parts is summarized in the third column. The reasoning strategies used by students to compare areas in the cake or tile tasks are summarized to indicate the occurrence of a reasoning strategy, but not the frequency of its occurrence. For both the cake and tile tasks, the two perceptual reasoning strategies (global and one dimensional) have been combined into one perceptual reasoning category (P). The students are grouped within the table to assist the discussion of patterns in their responses to the tasks.

There were considerable differences in the extent to which students used reasoning strategies which were consistent with properties of length or area measurement. At one extreme Connie (Grade 7) exhibited little understanding of length or area relations. She compared lengths or areas with perceptual strategies or with a numerical strategy in which units were discrete. At the other extreme, Kit (Grade 5) used appropriate numerical reasoning strategies to resolve all but one of the comparison measurement tasks (i.e., length and area). When the number of <u>sets of tasks</u> in which a student's responses were consistent with properties of length or area measurement (with or without units) are considered, three groups of students emerged. These groups are indicated in Table 5.09.

Students in Group 1 demonstrated limitations in their responses in at least 4 of the 5 sets of measurement tasks. For comparing lengths and areas they generally relied on gross perceptual judgements. Besides perceptual strategies, they used transformational strategies in ways which were subject to perceptual bias. In general, these students did not use units to compare quantities of length or area indirectly. Instead, they directly compared quantities with reasoning strategies which were

Table 5.09

Summary of Responses to Linear and Area Measurement Tasks.

				Area reasoning strategies								
				Cakes					Tiles			
Groups	Linear units/ strategies		Consistent partitions, No. regions	P	Ti	T1	T2	N	P	T1	T2	N
1 a.												
Connie	D	PNd	1	#	.	.	.	.	#	.	.	.
b.												
Lolande	D	Tr	1	#	#	.	.	.	#	.	.	M
Marlene	D	Tr	0	#	.	.	.	.	.	.	.	MB
Derek	D	Tr	0	#	.	.	.	.	.	.	#	.
James	D	Tr	0	.	.	#	#	.	.	.	.	MB
2 a.												
Dahlia	PL	Tr	0	#	#	.	.	T	.	.	.	B
Tammy	PL	TrN	1	.	#	.	.	T	.	.	.	B
Brock	L	Tr	2	#	.	.	#	.	.	#	.	.
b.												
Fanya	D	Tr	2	.	.	.	#	.	.	.	.	B
3.												
Edwin	PL	N	1	.	.	.	#	.	.	.	#	.
Lara	PL	N	3	.	.	.	#	B	.	.	.	B
Pete	L	N	3	.	.	#	#	.	.	.	.	B
Coran	L	N	3	.	.	#	#	.	.	.	.	B
Kasey	PL	N	2	.	.	.	#	.	.	.	.	B
Kit	L	N	2	.	.	.	#	T	.	.	.	B

Note. The names of the students in Grade 5 are underlined.

Reasoning strategies
P = perceptual strategy
Ti = transformational strategy (imagined)
T1 = transformational strategy comparing one dimension
T2 = transformational strategy comparing two dimensions
N = numerical strategy
Nd = numerical strategy with discrete units, linear measurement only
Tr = transformational strategies (combined), linear measurement only

Linear units
D = dominantly discrete points
PL = dominantly line segments
L = consistently line segments

Area Unit Relationships
# = units not considered
M = mono-relational
B = bi-relational
T = tri-relational

subject to perceptual bias. Furthermore, when attention to units was required explicitly in tasks, the units were represented and interpreted discretely. When units were represented in the linear context, they were defined generally as discrete points. Unequal units of area were treated as equivalent. As well, some of these students assumed that the count of the lines determined the number of units when partitioning regions.

Students in Group 2 were more unstable with regard to the limitations in their responses over all tasks. They demonstrated different degrees of sophistication or naivety in their representation and interpretation of units of measurement and in their strategies used to compare lengths or areas. For example, Brock consistently defined units of length to be line segments in the first set of linear measurement tasks. Yet, with the comparisons of the length he relied on transformational reasoning strategies which were subject to perceptual bias. His interpretation of units in the first instances were not applied in the second. Similarly, Dahlia and Tammy effectively resolved comparison of area tasks with numerical reasoning strategies. However, when numerical reasoning strategies were not used, they resorted to perceptual strategies or imagined transformational strategies which were subject to perceptual bias. They did not express confidence in the effectiveness of these alternative reasoning strategies, but they could think of no alternative strategy to apply.

Students in Group 3 demonstrated some but fewer limitations in their responses to the five sets of measurement tasks. They generally represented and interpreted linear units as congruent line segments and used numerical reasoning strategies appropriately to compare lengths. In the area context, numerical or transformational reasoning strategies generally accounted for the two-dimensional character of the tasks. However, even though students in this group were able generally to compare areas, the use of units as a basis for such comparisons clearly was limited. Most students in this group used numerical reasoning strategies only when the areas to be compared were partitioned into units, rather than when the areas to be compared were themselves units. Comparisons of fractional units generally were resolved without numerical reasoning strategies.

Students who were most consistent in applying properties of area measurement appropriately to resolve the area tasks generally were among those who most consistently represented and interpreted linear units appropriately. However the converse was not the case. This pattern mirrors the increasing conceptual complexity which results as the number of dimensions increase.[3]

Students' Beliefs About the Representation and Interpretation of Linear and Area Units.

Differences in the manner in which students finally resolved comparisons of length or area did not rest solely on distinction between those students who used perceptually biased strategies and those who used transformational actions or numerical reasoning. Most students initially made perceptual judgements for some of the comparison tasks. Some students believed their initial perceptual judgements to be correct without further question, but others, upon reflection, doubted these judgements. Some students appeared to have no alternative way to approach the task, while others were able to use alternative strategies to verify or disprove their initial judgements. When contradictions arose from the use of alternative strategies, students appeared to select the strategy and solution with which they were most confident. In some cases, beliefs lay most strongly with the perceptually biased judgements, in other cases, they lay most strongly with judgements based on transformational or numerical reasoning. Hence, there was often a tension between students' perceptual judgements and other judgements derived through transformational actions or reasoning with units, and differences among these students rested on the ways in which they resolved this tension rather than on the absolute absence or presence of perceptually biased judgements.

Similarly, during the partitioning tasks differences with the representation of units of length and area often lay, not in students' initial responses to the tasks, but in their reflection on the consequences of their first responses. Initial partitions of a line often were based on an assumption that the number of points determines the number of units. Likewise, students initially applied configurations inappropriately to different regions, or partitioning procedures which could not result in six equal parts.

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3. One student (Fanya) did not fit this pattern. She was among those most consistent in their use of appropriate strategies to resolve area measurement tasks, yet she represented linear units predominantly as discrete points, and lengths were compared only through imagined transformational strategies.

However, there were those who reflected beyond their initial belief in the appropriateness of their actions and those who did not.

Few students consistently anticipated all relevant variables which would determine appropriate strategies for successfully resolving measurement tasks. Attributes which differentiated problem situations were not considered before acting. In consequence, a procedure which was applicable in one case was over-generalized to other cases. However, some students observed contradiction in their initial responses and introduced alternative strategies for resolving the tasks, whereas others were not aware of the inappropriate consequences of their actions. Beliefs allowed students to act algorithmically unless some contradiction became apparent to the student during the solution process. Their subsequent reflection on the contradictory results led the student to act on the basis of the most strongly held belief.

CHAPTER 6

CONCLUSIONS, AND IMPLICATIONS

This was an exploratory study of students' representations of units and unit relationships in four mathematical contexts. The planning and implementation of the study, and the analyses of the data were guided by four research questions, each related to one of the four mathematical contexts. These were:

1. What are the characteristics of students' repertoires of representations of whole number multiplication?
2. What are the characteristics of students' repertoires of representations of common fractions?
3. What are the characteristics of students' representations and interpretations of units of length?
4. What are the characteristics of students' representations and interpretations of units of area?

The conclusions related to these research questions are followed by a discussion which explores issues arising from the student's representations of units and unit relationships across mathematical contexts. Included are conjectures regarding relationships between a student's representations of units in the arithmetical and measurement contexts. Finally, implications for future research are discussed.

Conclusions

The conclusions related to each of the four research questions will be summarized in four separate sections.

What are the Characteristics of Students' Repertoires of Representations of Whole Number Multiplication?

The conclusions presented are drawn from the analysis of the representations of whole number multiplication which the students constructed in four interview settings: the uncued and cued generative interviews and the linear- and area-based interpretive interviews. The findings from these data are to be found in Chapter 4. The dichotomy of form and function, which was used to make sense of the students' representations, is used as a framework within which to focus the conclusions characterizing students' repertoires of representations in the whole number multiplication context.

<u>Forms of Representations of Whole Number Multiplication in Students' Repertoires.</u>

All spontaneous, primary repertoires were comprised of representations of multiplication in which units were more or less discrete. These repertoires included only the first or both of the following forms of representations: (1) only discrete units, and (2) discrete aggregate units in which the primary units were contiguous regions. When forms of representations in students' general repertoires differed from those in their primary repertoires, the forms varied only in terms of the contiguity of units. However, for some students, the representations in their general repertoires were never fully contiguous (aggregate units remained separate). In the most limited case, all representations in the student's general repertoire were based on discrete units alone. In summary, the following generalizations can be made about the characteristics of students' general repertoires of representations of whole number multiplication:

1. The dominant form of representation in a general repertoire was based on discrete units in sets; the secondary form, when present, was based on more or less contiguous units in regions.
2. Representations constructed by students were grounded generally in a two-dimensional framework within which some students varied the form of units in terms of their contiguity. The line was not used as a spatial framework for generating a representation.
3. Physical attributes associated with instructional representations based on linear or area measurement, such as congruent line segments or regions, were considered to be important characteristics for representations of multiplication for aesthetic reasons alone. Neither the measurement of length nor the measurement of area served as a means for interpreting quantitative relationships among contiguous units.
4. The units in all forms of representations were interpreted quantitatively through a common measurement framework, namely measurement of numerosity.

In conclusion, representations in a student's general repertoire were closely related through two characteristics of their form: (1) the two-dimensional framework within which the units were spatially organized, and (2) the common measurement framework which underlay the quantitative meaning of the units. When discrete and contiguous forms of representations were differentiated by a student, the differentiation was not made in terms of systems of measurement. The essential difference between discrete and contiguous forms of representations lay only in the spatial proximity of the units.

Functions of Representations of Whole Number Multiplication

In general repertoires, representations of unit relationships for the less complex tasks (A X B) were consistent. Inconsistencies in the manner in which unit relationships were represented occurred in one of two ways: (1) mono-relational representations of units were constructed for the more complex tasks (e.g., A X B X C) in contrast to a student's bi-relational representations of the less complex tasks, or (2) a student explained the more complex tasks with bi-relational and tri-relational representations of units. In summary, the different patterns of responses regarding students' explicit representations of relationships among units constructed to explain the multiplication tasks were as follows:

Pattern 1 All tasks were explained with mono-relational representations of units.

Pattern 2 A X B tasks were explained with bi-relational representations of units. More complex tasks were explained with mono-relational representations of units.

Pattern 3 All tasks were explained with bi-relational representations of units.

Pattern 4 A X B tasks were explained with bi-relational representations of units. More complex tasks were explained with bi-relational and tri-relational representations of units.

In conclusion, the extent to which students explicitly represented unit relationships among factors and product varied greatly. For some students, multiplicative relationships among units were not represented explicitly. Even though skip counting as a process of repeated addition was their framework for the operation, the repeated aggregation of units implied by the skip counting process was not a concrete phenomenon to them. For the students who were most facile at representing multiplicative relationships among units, the multiple nesting of primary and aggregate units required to represent unit relationships in the more complex tasks demanded careful analysis before the students were able to reconstruct the relationships concretely. Being able to represent the multiplicative relationship among the units in A X B did not imply that a student was able to extend the multiplicative representation to a case such as A X B X C which involves the multiple nesting of units.

What are the Characteristics of Students' Repertoires of Representations of Common Fractions?

These conclusions are drawn from the analysis of the representations of common fractions which the students constructed in four interview settings: the uncued and cued generative interviews and the linear- and area-based interpretive interviews. The findings from these data are to be found in Chapter 4. The dichotomy of form and function, which was used to make sense of the the students' representations, is used as a framework within which to focus the conclusions characterizing students' repertoires of representations in the common fractions context.

Forms of Representations of Common Fractions in Students' Repertoires.

Primary repertoires of representations of common fraction comparisons were comprised of only the first or both of the following: (1) representations based on regions as the spatial framework, and (2) representations based on discrete units in sets. Regions were used to represent different forms of units (discrete or contiguous), and to represent quantities undefined by units. When forms of representations in students' general repertoire differed from those in their primary repertoire, the forms varied only in terms of the spatial proximity of units. However, for nearly a third of the students, all representations of common fractions in their primary and general repertoires were based only on contiguous units within regions or on regions without defined units. In summary, the following generalizations can be made about the characteristics of students' general repertoires of representations of common fractions and comparisons of common fractions:

1. The characteristic common to all students' repertoires was the use of regions, particularly circles, as the spatial framework for representations. As such, the dominant form in a repertoire

generally was based on regions as a spatial framework, and the secondary form, when present, was based on sets of discrete units.

2. Representations generally were grounded in a two-dimensional framework within which some students varied the form of units in terms of their contiguity. The line was not selected independently by students as a spatial framework for generating a representation. It also was not used by nearly half of the students when they were directly asked if such a spatial framework were possible.

3. A student used a common measurement framework to interpret all representations of units, regardless of whether the units were discrete or contiguous. Either all forms of representations of units were interpreted as measures of area or they were all interpreted as measures of numerosity.

4. All students placed some importance on characteristics common to instructional representations of common fractions based on area measurement, such as the equality of regions and parts within regions. However, students did not necessarily associate these characteristics with units of area measurement. Students who based their representations on measures of numerosity sought perceptual orderliness in their representations. The result of these aesthetic consideration was the equality of regions or parts of regions .

Area measurement as a basis for representing common fractions. To varying degrees, students who interpreted their representations of units as measures of area constructed units which did not conform to standard units of area. However, these students believed the units in their representations to be measures of area. They compared common fractions on that assumption. The lack of conformity of their units of area with standard units of area was not a result of technical difficulties in constructing equal parts of regions. Instead, students appeared to hold different beliefs about what constituted units of area, and about the limitations associated with the comparisons of quantities of area.

In conclusion, representations of common fractions in a student's general repertoire was closely related through two characteristics of their form: (1) the two-dimensional framework, within which the units were spatially organized, and (2) a common measurement framework which underlay the quantitative meaning of the units, either measures of area or measures of numerosity. When discrete and contiguous forms of representations differentiated by a student, the differentiation was not made in terms of systems of measurement. The essential difference between discrete and contiguous forms of representations lay only in differences in the spatial proximity of the units. In addition, students whose representations of common fractions were based on area measurement held different beliefs about what constituted units of area measurement. As a result, their representations did not always conform to standard area-based representations of common fractions.

Functions of representations

There were marked differences in the extent to which students explicitly represented relationships between units as an explanation of comparisons of common fractions.[1] In summary, students' patterns of responses regarding the representation of units relationships to explain the comparison of common fraction tasks were as follows:

Pattern 1 Students' representations of unit relationships were consistent in one of three ways: (1) all comparisons were explained through mono-relational representations of units, (2) all comparisons were explained through tri-relational representations of units, or (3) all comparisons were explained through representations without units.

Pattern 2 Students' representations of unit relationships in their general repertoires were inconsistent. Most often the inconsistencies in responses involved a shifting between bi-relational and tri-relational representations to explain the comparisons.

In conclusion, the extent to which students' representations of comparisons of common fractions functioned to express relationships among units was influenced by a diverse number of factors. Foremost among these were a student's interpretation(s) of common fractions, the measure system upon which the representations were based, and the manner in which a student attended to attributes of units relevant to their system of measure. Even for students who used the take-away interpretation

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1. The function of students' representation of a single common fraction cannot be determined in all cases. For this reason, these conclusions regarding the function of students' representations are based only on the findings regarding their representations of the comparisons of common fractions.

and considered units to be units of area, the need to maintain the invariance of the whole units as well as construct approximately equal fractional units was not obvious always to them.

What are the Characteristics of Students' Representations and Interpretations of Units of Length?

These conclusions are drawn from the analysis of students' representations and interpretations of units of linear measurement based on students' responses during the linear measurement interview as well as their responses to selected items on the measurement concepts test.. The linear measurement tasks were of four kinds: (1) constructing linear units with a specific ratio to a given unit (ruler task), (2) partitioning a line segment (partitioning task), (3) iterating an aggregate unit (aggregate units task), and (4) comparing irregular paths made up of line segments (irregular path task). The findings from these data are to be found in Chapter 5.

Representations of Units of Length

All students represented or interpreted linear units as line segments, but less than a third did so consistently. Alternatively, units were represented as or interpreted to be discrete points. Shifts between considering units in terms of line segments and discrete points were associated with differences between tasks and differences in the strategies used to resolve them. With regard to differences in the tasks, those which involved representing units or interpreting units within a pre-defined length necessarily focused students' attention on counting points. Whereas with those which involved iterating a unit to define a length counting of points was not an issue. Students were most inconsistent in their interpretation of units with the ruler task. The juxtaposition of points and numerals on a ruler may increase students' attention to the points without considering the role of the numerals as a record of the count of line segments from zero. With regard to strategies, the ways in which students attended to and coordinated the relationship between the number of points and the number of line segments influence the consistency with which students represented units as line segments. There were students for whom the number of points marked on a line most often determined the number of units, regardless of the orientation of points to line segments. They assumed a direct relationship between their counting action and number of units, an assumption which probably derives from the extensive experience of counting in a discrete context in which action, language and number are universally synchronous. Their belief that the number of points defines measures of linear units mirrors numerous continuous measurement situations in which this condition is true.

Comparing Lengths and Reasoning with Units of Length

Over half of the students compared lengths directly through perceptual or transformational reasoning strategies rather than indirectly through numerical reasoning strategies associated with units of length. The students who used direct comparison strategies were also less consistent in representing units as line segments. For most of these students, measurement with units of length was not associated generally with the enumeration of congruent line segments. In contrast, students who compared lengths indirectly through numerical reasoning strategies (1) were more consistent in their representation of units of length as line segments, (2) were more consistent in using bi-relational reasoning with regard to different sized units, and (3) generally associated measurement of length with the enumeration of congruent line segments.

What are the Characteristics of Students' Representations and Interpretations of Units of Area?

These conclusions are drawn from the analysis of students' representations and interpretations of units of area measurement based on students' responses during the area measurement interview as well as their responses to selected items on the measurement concepts test. The area measurement tasks were of three kinds: (1) partitioning regions (partitioning tasks), (2) comparing fractional units within a region (cake tasks), and (3) comparing the area of regions partitioned into units (tiles task). The findings from these data are to be found in Chapter 5.

Representations of Units of Area by Partitioning Regions.

Few students consistently partitioned the three regions (a square, a circle and a rectangle) into six equal parts. Among these regions, the rectangle was partitioned into equal parts by the largest number of students, and the circle by the least number of students. Geometric properties of regions which were not relevant to the partitioning problem, appeared to influence the partitioning strategies students adopted. As a result, (1) a different configuration was associated more frequently with each of the three regions, and (2) configurations used to partition a square often were related more closely to those applied to the circle than to the rectangle. The square, therefore, appears to be more difficult to partition successfully than the rectangle because the two quadrilaterals were treated by some students as different partitioning problems. In addition, configurations normally used in instructional materials to partition some of the regions were believed by some students to partition all regions into equal parts. For final partitions, the most common source of error was the over-generalization of a configuration as a means of partitioning a square or a circle. The configurations were over-generalized as partitioning algorithms.

When an appropriate configuration was used to partition a region, some followed procedures which could not result in six equal parts. For example, (1) the count of lines was assumed to be equivalent to the number of spaces with hatched and vertical configurations in a manner similar to the counting of points in the linear measurement context, or (2) a successive halving procedure was applied with the radial configuration, a procedure which applies only to powers of two not to all even numbers, or (3) an additive relationship of 3 + 3 = 6 rather than a multiplicative relationship of 2 X 3 = 6 was used to plan the partitioning of a square with the hatched configuration.

Limitations, which arise from the geometric and number-theoretic properties of different partitioning contexts, determine which configurations and procedures could be applied to achieve parts of equal area. For most students, these limitations were not differentiated clearly and consistently. Instead, the configurations and the procedures for their construction appeared to function as algorithms which some students over-generalized in the belief that their application would result in six equal parts.

<u>Comparisons of Area: the Interpretations of and Reasoning with Units</u>

Students' recognition of units of area generally was influenced by the structure and relationship of the units within the tasks. When students were asked to compare areas of regions partitioned further into sub-regions, two-thirds of the students interpreted the tasks as ones involving units and used numerical reasoning strategies. Most of these students consistently focused on space-filling attributes of the units, and accounted for differences in unit size within their reasoning strategy. In contrast, when students were asked to compare fractional units within a region, the tasks were seldom interpreted as ones in which a part-whole relationship among units would provided a basis for comparing the areas of the parts. Generally, the fractional units were compared directly through transformational or perceptual reasoning strategies without regard to the regions of which they were a part. Students either (1) did not consider the general case that when a region is partitioned into n parts of equal measure, the measure of each part is 1/n, regardless of the shape of the part (i.e., fractional unit), or (2) only observed this case when the regions were partitioned into fourths with different standard configurations. Only one student consistently observed unit relationships even with the standard configurations. In general, students did not appear to integrate the representation of unit fractions with the representation of units in an area measurement context.

Considering both problem contexts, two-thirds of the students' responses were unstable in their attention to two-dimensional, space-filling characteristics when comparing areas of regions. However there was considerable variation in the ways in which these students' reasoning strategies were subject to perceptual biases. Students' perceptual biases occurred either (1) in both problem contexts, or (2) only when fractional units were compared. Within each of these response patterns, further variation was observable, as shown in the following four cases.

<u>Pattern 1</u> <u>Perceptual biases in both problem contexts</u>

a. Perceptual strategies were used with all tasks except when comparing rectangular fractional units, in which case an imagined transformational strategy was applied.

b. Perceptual strategies were applied when triangles were involved, and when transformational strategies were applied student's attention to dimensions was inconsistent.

<u>Pattern 2</u> <u>Perceptual biases in fractional unit context only</u>

a. Perceptual strategies or imagined transformational strategies were applied to fractional unit tasks when numerical strategies were not applied.

b. Transformations, applied to all fractional unit tasks, were evaluated on only one dimension when comparing rectangles, but on two dimensions when comparing rectangles and triangles.

The comparison of non-congruent fractional units was the problem context in which more students were variable in their attention to attributes of area measurement. When numerical reasoning strategies were not applied, students responses either (1) were dependent on perceptual strategies, or (2) varied in the extent to which attention was maintained on the two-dimensional characteristic of the tasks when using transformational strategies.

Discussion and Implications for Instruction

In this section, students' repertoires of representations are discussed in terms of general characteristics of repertoires and specific characteristics of representations of units and unit relationships. The nature of students' representations of units of linear and area measurement as these relate to students' repertoires of representations are integrated within the discussion. Implications for instruction are discussed where relevant throughout the section.

Repertoires of Representations

Characteristics of a repertoire of representations in either the whole number multiplication or common fraction context were defined by the forms of representations a student included as well as by the potential forms of representations a student excluded as means of explaining mathematical tasks.[2] As such, a repertoire of representations reflects a set of physical images which constitute part of a person's conception of a mathematical operation or relationship.

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2. In order to facilitate a general discussion about characteristics of repertoires of representations of whole number multiplication as well as repertoires of representations of common fractions and comparisons of common fractions without having to continually repeat these qualifiers, the use of "repertoires" or "repertoires of representations" should be assumed to mean repertoires in either of the mathematical contexts. It does not refer to a single repertoire encompassing representations in both mathematical contexts.

Repertoire of Representations as a Construct.

A repertoire bears similarities to other constructs in which multiple representations are thought to constitute a person's conception of a mathematical relationship as in the case of the concept image explored by Vinner and colleagues (Davis & Vinner, 1986; Hershkowitz, 1987; Hershkowitz, & Vinner, 1984; Tall & Vinner, 1981; Vinner, & Hershkowitz, 1980). A concept image comprises "all mental pictures and associated properties and processes of a concept built up by a person over the years through experiences of all kinds" (Tall & Vinner, 1981, p. 152). A repertoire of representations, defined to include only manipulative or diagrammatic representation of a person's mathematical ideas, would constitute only a part of a concept image.

Mathematical tasks in which a person need only recognize or identify representations have been shown to be less difficult than those in which a person must construct representations (Clements & Del Campo, 1987; Sinclair, 1971b). Hence, there may well be other forms of representations which a student might recognize, and interpret as a representation of the symbolic expressions, but which they would not evoke as a means of explaining the mathematics. In this sense, a repertoire should be considered to reflect a subset of a person's "concept image." Evoked concept images, defined as external expressions of a concept image (Tall & Vinner, 1981), would include a repertoire of representations as well as representations which a person may recognize and identify with the mathematical ideas but not evoke independently.

Structure of Repertoires of Representations

There were several characteristics common to all repertoires of representations, regardless of differences in mathematical contexts and variations among students' representations in particular contexts. A repertoire was exemplified by a dominant form of representation of units, and by the complexity of the relationships among units that were represented. Within a repertoire, representations based on different forms of units were related through two common characteristics. The first was the common system of measurement which underlay the different forms of units (either numerosity or area). The second was the two-dimensional characteristic of the different forms of representations. The measurement system provided a source of common meaning ascribed to units in representations and served to structure the quantitative relationships among representations in a repertoire. The two-dimensional characteristic of the representations provided a source of physical analogy between the different forms of units. As a result, discrete or contiguous representations could be derived, one from the other, through simple transformations while the system of measurement and the two-dimensional spatial framework remain invariant. The different forms of representations of units within a repertoire as well as the common characteristics among these representations are assumed to derive from a person's experiences of representations of the mathematical relationships or operations. The dominance of a particular form of representation in a repertoire and the system of measurement upon which the representations are based reflect a person's prevailing interpretation and reconstruction of these experiences.

Forms of Representations

Mathematical symbols and notation are considered to represent layers of meaning often expressed as dichotomies such as "syntactic and semantic" meanings (Resnick & Omanson, 1987), "surface structure and deep structure" (Skemp 1982), "form and understanding" (Hiebert, 1984), or "form and content" (Byers & Erlwanger, 1984). The same may be said of physical images which represent mathematical relationships. In this regard, "form and function" have been used as major categories to characterize layers of meaning associated with students' mathematical representations.

Content of the Form

As Byers and Erlwanger (1984) observed, the form of mathematical notation could be analysed further in terms of the content of the form. Mathematical notation is a form of expression of mathematical content in terms of mathematical relationships or operations it serves to represent, but it also has its own content in terms of the structure and meanings which underlie it. For example, an algebraic equation is a form of representation. It represents specific mathematical content in terms of relationships among specific variables. Other forms of representation such as a graph could also serve to represent the same mathematical content. The meaning and rules of manipulation of the algebraic notation in general terms constitutes another layer of content which must be understood before specific mathematical relationships can be intelligibly expressed. The general meanings and rules of manipulation associated with algebraic notation would be considered the content of this form of representation. Fischbein (1977) similarly argued that graphical forms of representations have content independent of the mathematical relationships among the variables being represented. There is therefore a distinction between the "content of the form of a representation," and the "form of the representation of specific mathematical content." Taking the relationship between variables as the mathematical content, this content can be represented in at least two forms: one form would be an algebraic equation, the other form would be a graph. However, each of these two forms of representation have their own content which must be understood in terms of the conventions and rules generally associated with their use in order for the mathematical relationships among variables to be represented in a manner which is universally recognizable and interpretable.

There are many interpretations which one might place on the form of a symbolic or physical representation which may not conform to the rules and conventions normally associated with them (E.g., Anghileri, 1989; Bell & Janvier,1981; Clay & Kolb,1983; Dufour-Janvier et al.,1987; Ernest, 1985; Hart, 1987, 1989; Hershkowitz, 1987; Hiebert, & Wearne, 1986; Novillis-Larson,1987; Peck & Jencks, 1981; Vergnaud et al., 1980; Vinner & Hershkowitz, 1980). The same physical form of a representation may be imbued by different people with different structural meaning independent of the mathematical content it is meant to represent.

In the case of the representation of units in the whole number multiplication or common fraction contexts, a form of representation having particular physical characteristics may be interpreted in terms of different systems of measurement. For example, a partitioned regions may be interpreted as units of area or numerosity. The form of a representation, itself, involves a dichotomy between the overt, physical characteristics of the representation which are describable and copiable, and the system of measurement which imbues these physical features with potential quantitative meanings. The latter is what, in this study, is considered to be the "content of the form." As such, a simple dichotomy of form and function is an insufficient framework within which to characterize the complexities in the layers of meaning in such mathematical representations. There is also the dichotomy between the physical form of a representation and the content of that form. This means that a simple analysis cannot be made which would assume that a student's representations containing discrete units are based necessarily on measures of numerosity, or that a student's representations containing congruent, contiguous units are necessarily based on measures of area. The system of measurement which imbues the physical features of units with potential quantitative meaning may be otherwise from the student's point of view. In this study, an assumption that discrete units as a form of representation were based on measures of numerosity as the content of that form, or that contiguous units as another form of representation were based on continuous measures as the content of the form, would have led to an inappropriate conclusion with regard to the system of measure underlying the student's thinking about these forms of representation of units. At times, even a student's description of attributes considered to be important in his or her representation was insufficient evidence on which to base such an inference. Students differentiated between their representations with regard to physical characteristics of the forms of these representations, and in many instances considered contiguous and discrete representations to be "completely different," but their differentiation did not relate to the measurement system on which they based the quantitative meaning of the units. Because the system of measure being considered by the student could not be inferred through the discrete and contiguous spatial characteristics of the units, the relationship between the physical characteristics of forms and the content of the forms of units often was equivocal from an observer's point of view.

Implications for Instruction. If the possibility of differences in the personal constructions placed by students on the content of the form of a representation is not considered, then the observation of consistencies between the physical forms of teacher's and student's representations may result in a belief that meanings of mathematical representations are shared when they are not. Because there are mathematical contexts in which students' thinking about the content of the form of representations is not easily illuminated, false-positive evaluations are possible in a variety of educational contexts, including informal teacher observations, paper & pencil tests, or diagnostic interviews.

<u>Measurement of Numerosity or Area as Content of the Form: Mathematical or Cultural Imperative</u>

Conventionally, whole number multiplication and common fractions are seldom represented in a continuous form even though units may be based on measures of continuous quantities. As Ohlsson (1988) stated,

> Partitioning replaces a [continuous] quantity with a set, namely the set of its parts. By reducing the continuous to the discrete case, partitioning enables us to assign numerical values to continuous quantities through counting. (p.74)

Once continuous quantities are partitioned to represent a mathematical concept or relationship, the continuous attribute of the units may not necessarily be most salient. The potential separateness of the contiguous units may be more pronounced. By implication, the units would become interpretable as units of numerosity which are incidentally contiguous, or interpretable as contiguous units of area.[3] Similarly, with the common use of discrete units which are congruent, the congruence or the discreteness could be the more salient attribute. In the former case, the units would be interpretable as units of area which are incidentally discrete, and in the latter case the units would be interpretable as discrete units of numerosity.

The possibilities of these alternative interpretations of discrete and contiguous units generally are not assumed in school mathematics. When instructional representations are based on partitioned regions, the contiguous units are meant to be interpreted as measures of a continuous quantity, that is as a measure of area. Similarly, with sets of objects, the discrete units are meant to be interpreted as measures of numerosity. However, the physical characteristics of units, whether discrete or contiguous, do not automatically mean that units must be interpreted as measures of numerosity or measures of area in order for the unit relationships embodied in the representation to remain mathematically valid. For example, contiguous representations of whole number multiplication need not have units of equal area to embody the unit relations between factors and product, nor must the units of equal area be interpreted as units of area measurement in order for a person to comprehend the mathematical

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3. For Ohlsson (1988), the term discrete refers to all units regardless of their spatial relationships. In this study, units are distinguished by their spatial relationships as either discrete or contiguous.

relationships embodied in the representation. Similarly, when representing a common fraction with contiguous units, the units may be interpreted as measures of the discrete or continuous attributes of the partitioned region. "A out of B pieces of pie" may mean A/Bths of the pieces (of pie) or may mean A/Bths of the pie. In the former case, the fraction expressed a measure of numerosity and the contiguity of the units is incidental to its meaning. In a situation with discrete units, A out of B congruent, discrete pieces of tile may be thought of as A/Bths of the pieces of tile or A/Bths of the amount of tiling. In the latter case, the fraction expressed a measure of the continuous quantity and the discreteness is incidental to its meaning. Either of these interpretations of the content of the form of the units representing A/B could apply when comparing common fractions with like-denominators, and adding, subtracting or dividing with common fractions with like-denominators without invalidating the mathematical sense of the representations. In both mathematical contexts, therefore, there are many situations where the units in a contiguous representation may be thought of as units of numerosity. A mathematical imperative for the content of contiguous forms of units to be thought of by students as measures of area often would occur only if area measurement were a critical characteristic of a problem setting.

The mathematical validity of alternative interpretations of the content of the form of units is supported further by the ambiguous language often associated with representations of discrete or contiguous units. The language associated with representations based on units of either length or area does not necessarily reference the attributes of the unit on which the quantification is based. For example, neutral phrases such as "a parts out of b," "a of b parts," or "a out of b" are associated with contiguous, or discrete and contiguous representations of common fractions (e.g., Eicholz, O'Daffer, Fleenor, 1974; Skypeck, 1984). Students also use this language when common fractions were being represented as measures of numerosity or measures of area. This language supports the imagery of the take-away interpretation of common fractions; that is, the physical image of a number of contiguous or discrete units being reduced by a specified amount, with the common fraction represented either by the units that were removed or those that remained (Peck & Jencks, 1981). As well, discrete units of area are used to represent common fraction relationships in the form of manipulative materials such as fraction squares, fraction strips, or pieces of "pies." Hence, incidentally contiguous units of numerosity and incidentally discrete units of area in students' repertoires not only may derive from mathematically credible interpretations of representations in particular mathematical contexts, but also may be given further credence by the ambiguous manner in which discrete and contiguous representations of units may be presented and discussed in schools.

Students' thinking about the form of their representations reflected the ambiguity of the quantitative significance of physical attributes of units. First, when students generated both discrete and contiguous representations they interpreted the attributes of the units and the unit relationships through a single measurement framework. The contiguity and discreteness of units were reversible characteristics which did not alter "the content of the form" nor the relationships among the units being represented. Some of the students' common interpretations of different forms of units could be explained by the student adopting mathematically valid, alternative interpretations of contiguous or discrete units. Second, students set limits on which physical characteristics of representations were admissible as a means of explaining the mathematics. Some of these limits were not essential to the problem of representing the mathematical properties of the task nor relevant to the quantitative properties the students ascribed to the units in their representations. However, these limits set by the students often reflected characteristics common to instructional representations. For example, characteristics such as equal parts played no quantitative role in some students' representations in either mathematical context, but were considered by the students to be important in any case. In some of these cases students expressed a subtle distinction between normally preferred characteristics of units in one mathematical context and mathematically necessary characteristics of units in another. They universally rejected diagrams with units of unequal area, yet they used aesthetic criteria in the multiplication context and quantitative criteria in the common fractions context to justify their decisions. When students place limits on the acceptability of different characteristics of representations, while maintaining an alternative interpretation of the content of the form of representations, they may be reflecting a school-mathematics culture which is only partially related to logical, mathematical imperatives.

Implications for instruction. The ambiguity regarding the quantitative significance of physical characteristics in a representation has several implications regarding students' use and interpretation of representations in the classroom. First, characteristics of units such as contiguity, discreteness, or congruence, do not necessarily define the physical attributes essential to the quantifications embodied in a representation unless the measurement context is explicit. Second, such characteristics of units do not necessarily define the attributes to which a student must attend in order to make a mathematically valid interpretation of a representation. Third, students who attend to the congruence of contiguous units when constructing or interpreting a representation need not be attending to length or area as the quantitative attribute. To assume that they are considering continuous measurement as a mathematically significant attribute in such cases may be totally invalid. Unless continuous measurement plays an explicit and significant role in problem settings, students may have extensive experiences with representations of contiguous units which they consistently interpret in terms of measures of numerosity without encountering any contradictions.

Dominance of Forms of Representations in Repertoires

Whether the commonalities in the dominant forms of representations in students' repertoires derive from shared experiences of similar curricular content and instructional practices, and/or from shared experiences of informal mathematical situations is an open question. It may be that characteristics of instructional representations reflect manifestations of a school-mathematics culture which students interpret as limiting the acceptable characteristics of representations within their repertoire. Such characteristics might reflect pedagogical rather than mathematical exigencies for a "good representational form." An example would be instructional representations of discrete units meant to be interpreted as measures of numerosity containing only congruent objects. The congruency of the objects may be assumed by students to be of importance in a manner similar to the imposition that parts of regions be congruent regardless of which system of measure the representation is based on. However at present, the experiences through which children construct their repertoires of representations can only be hypothesized.

There are similarities between patterns in students' responses in this study and other studies which suggest that some forms of representations of mathematical relationships may be generally more dominant in peoples' repertoires of representations. The general dominance of discrete representations in the whole number context in this study harkens to the observation of Lay (1982) that, in his experience, when school students and adults are asked to draw a diagram to represent a number such as "5"..."the image that seems first to come to mind is that of an array of five separate but similar objects" (p.260). This may relate to children's experiences of counting, comparing and partitioning sets of discrete objects which form the primary basis for the development of whole number concepts and operations (Bergeron & Herscovics, 1990; Fuson, 1988; Gelman & Gallistel, 1978; Piaget, 1965; Steffe, 1988; Steffe & von Glasersfeld, 1983). It is therefore possible that a dominant representation of whole number generally might involve representations of relationships among discrete units, regardless of the forms of representations of whole number which might have been encountered later. Such a case would imply that a person's dominant form of representation in a particular context is most likely to be that which is encountered first and most frequently.[4]

In a similar sense, the common use of regions (and particularly circles) as a spatial framework for the representation of unit relationships in the common fraction context harkens to a number of reports in which regions in general and circles in particular were the most common, and in many instances the only form of representation of common fractions evoked by children and adults (Clements & Lean, 1988; Lay, 1982; Peck & Jencks, 1982; Silver, 1983, 1986). Whether units within a region were based invariably on a person's conceptions of area measurement is not clear. However, cases reported by Peck & Jencks and in this study would suggest that the physical image of regions may dominate regardless of the system of measure on which a representation is based. In addition, others report that representations of common fractions based on parts of regions were less difficult to interpret and

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4. The dominant form of representation in a repertoire bears similarities to the notion of prototypes of concept images (Hershkowitz 1987; Hershkowitz & Vinner 1984; Vinner & Hershkowitz 1980). But, whereas a prototypical concept image was an example of a geometric concept selected or created most frequently by collections of people, a dominant form of representation is the one generated most frequently by an individual.

reconstruct than those based on other spatial frameworks (Behr, et al., 1983; Payne, 1975; Lesh, Landau & Hamilton, 1983). However, it is ironic that the circle plays such a prominent role as the region of preference when it is the region most difficult to partition.

The apparent universal dominance of regions as a form of representation of common fractions would suggest a significant sharing of experiences in which such representations may prevail over others. Payne (1975, 1984) argues that the partitioning of regions is an early experience of children in which the idea of fair shares motivates the concept of equal parts in the part-whole relationships of common fractions. However, why "fair shares" would not apply equally as a motivation for partitioning sets of discrete objects to represent common fractions is not explained. It may be that informal experiences with the partitioning of discrete sets are associated more directly with the whole number division operation. The idea of expressing this partitive action with discrete units as a representation of a common fraction may not be relevant to such a fair shares problem because the result can as easily be interpreted in terms of whole numbers. The observations of young children's partitioning strategies with discrete and continuous quantities by Hiebert and Tonneson (1978) would support this possible interpretation. On the other hand, the proverbial pie or more "nutritionally correct" pizza may be simply the representation for common fractions most frequently evoked by teachers and thereby the most common form of representation experienced by students.

<u>Implications for instruction</u>. Two implications may flow from the dominance of a form of representation in a repertoire and the analogous relationship among forms of units in a repertoire derived from a common system of measurement. First, it may be that when students are presented with a new, different form of representation of related mathematical relationships, they may interpret the units in terms of their dominant form of units and thereby interpret the mathematical relationships being represented in a manner different from that which was intended. Second, it may be that a restriction on the forms of representations in a student's repertoire limits the measurement framework through which problem situations may be interpreted. When faced with a related mathematical problem which evokes a figurative representation, the visualization of the problem situations involving other systems of measurement may be limited. In this regard, Silver (1983) found that young adults who were more

capable in a common fractions context "reported a greater variety of fraction images," and were "often able to explain procedures using several different fraction models" (p.113,114). Similarly, Lay (1982) suggests that some difficulties encountered in school science may relate back to students' dominant image of whole number units being discrete measures of numerosity. In such circumstances the students may not be able to accommodate to problem settings involving continuous measure. The possible interactive relationship between a student's repertoire of representations and their representation of new problem situations should be the subject of future research.

<u>The Differentiation of Forms of Representations of Units</u>

There are three comparisons which could be considered in regard to the ways students' representations of units might have been differentiated. The first would be to compare and contrast the forms of units in students' repertoires of representations in both arithmetical contexts with student's interpretation of units in the continuous measurement contexts. The second would be to compare and contrast the repertoires of representations in different arithmetic contexts in terms of the forms of units in representations; that is, similarities and differences between units used to represent multiplication and units used to represent comparisons of common fractions. The third would be to compare and contrast the forms of units in the representations within a repertoire. This discussion in regard to differentiation will focus primarily on the first two comparisons, that is patterns in the forms of units across arithmetic and measurement contexts in general, as well as patterns across arithmetic contexts in particular.

<u>Forms of representations in the arithmetical contexts compared to interpretation of units in the continuous measurement contexts</u>. Theoretically, the characteristics of the forms of representations in a person's repertoire are limited only by (1) a person's knowledge of units and unit relationships in different systems of measurement, (2) the facility with which any one system of measurement can serve to represent the mathematics of interest, and (3) the recognition by the person that a particular form or representation will serve to represent the mathematics of interest. In this study, it was the latter condition which appeared to influence the characteristics of a person's repertoire of representations most significantly. Neither students' rejections of a form of representation nor their interpretation of units

within their representations were related necessarily to their conceptions of units of length or area. For example, students who consistently represented and interpreted units of linear measurement appropriately as congruent line segments did not include linear measurement as a form of representation in the multiplication or common fraction contexts. Instead, students rejected the line as a spatial framework, interpreted units along the line to be discrete points, or transformed the line into partitioned regions. Students' responses to the prospect of using the line as a spatial framework for their representations in the arithmetic contexts could not be predicted by their conceptions of linear units. Students' ideas of what constituted mathematical representations in the multiplication or common fraction contexts appear to exist independent of the students' alternative knowledge of other systems of measurement. Repertoires of representations are not generalized to include all potential forms of representations. The development of their repertoires of representations appears to be context specific.

Similarly, students' ideas of what constituted a common fraction representation bore no relationship to their relative facility at partitioning different regions. Partitioned circles took precedence even though students generally were less successful at partitioning a circle compared to quadrilaterals. Although some students consistently used the comparisons of areas to represent the comparison of common fractions, part-whole relationships seldom were considered as a means through which to compare areas of fractional units in the area measurement context. Direct comparisons through perceptual or transformational reasoning strategies which did not incorporate notions of units and number were used more commonly. Bergeron and Herscovics (1987) also observed that, when non-congruent but equal fractional units, both of which had already been identified by the students to be 1/n, were compared in terms of their area, the students used a transformation reasoning strategy rather than a numerical reasoning strategy. Although a student might reason in the common fractions context that if a region is partitioned into n parts of equal area, then each part is 1/n of the region, the converse in the area measurement context, that if each part is 1/n of the same sized region, then the parts, regardless of their congruency, are equal in area does not necessarily follow. Experiences of representations of comparisons of common fractions usually would not require a person to consider this generalization regarding areas of fractional units because the same configuration usually is used to partition both regions. The parts within both regions would be of the same general shape.

It would appear that the students' "world of continuous measurement" and their "world of representations of numerical relationships" are not well integrated. The manner in which students might construct a representation in an arithmetic context which they consider to be based on linear or area measurement would be influenced by their conceptions of units of length or area. However, students who reasoned with area measurement in the common fractions context generally did not reason numerically when comparing fractional units in the area measurement context. As well, one can conclude nothing about a person's conceptions of units of continuous measurement from their contiguous representations when such representations are considered by the person to be based on measures of numerosity. The relationship between the forms of representations in students' repertoires and students' representation and interpretations of units in continuous measurement contexts is tenuous.

Linear measurement as content of the form: a special case. Just as concept images, as defined by Vinner and Hershkowitz (1980), often included restrictive characteristics which were not relevant to the formal definition of a geometric concept, so also students' repertoires were restricted in terms of the forms of representations in ways that were not relevant to a formal explanation of the mathematical operations or relationships. Most significant in this respect were the limitations associated with a one-dimensional framework, and the absence of measurement of length as bases for representing units. The exclusion of linear measurement appears to be anomalous for two reasons. First, length measurement is, from a geometric point of view, the least complex system of continuous measurement (Beilen & Franklin, 1962). Second, a number of students in this study were more consistent in appropriately reasoning with and representing units of length than units of area measurement; however, the converse was never true. Why, despite their greater facility with linear measurement, would students generate representations based on area measurement and not length measurement, represent units as discrete points on a line, and construct parts of a region surrounding a line when confronted with a partitioned line? There are several factors which may contribute to these patterns of responses.

Children's earliest experiences with concrete representations of number and operations of number involve discrete or contiguous objects in a two dimensional framework. The one dimensional characteristic of length would be an abstraction beyond these concrete experiences. As well, these early experiences would involve counting discrete or contiguous objects as units of numerosity alone, with the contiguity of units an incidental characteristic unrelated to this quantification process. The generalization of these experiences to a partitioned line may logically involve the interpretation of units as measures of numerosity, and the identification of points on the line as analogous to features of the representations of number encountered previously.

The general absence of linear measurement as a framework for representations should be considered in the context of the associations between discrete sets and contiguous regions in repertoires of representations in both mathematical contexts. The one-dimensional framework of the line stands in direct contrast to the two-dimensional characteristics of discrete sets or contiguous regions. Thus, students transformed the line into two-dimensional representations, identified and used the dots on the line as analogous to discrete objects in sets, or rejected the line altogether in favour of sets or regions. In all cases, their avoidance or rejection of a line as a spatial framework and line segments as units could be related to a general two-dimensional characteristic of their repertoires of representations, regardless of the mathematical context.

Even if students did identify line segments as representations of units, the perceptual complexity of linear representations might limit the extent to which this form of representation would be chosen by students. As Bright et al. (1988) indicated, units in a linear representation are not separated nor separable within a mathematical representation. When units are nested, the primary and aggregate units are not as distinguishable as when a second dimension or a discrete separation of units are characteristics of a representation. This perceptual complexity of linear representations was harkened to when students stated that rectangular arrays were preferable because the representations along a line were "confusing." A similar contrast could be made between partially contiguous and fully contiguous representations in either mathematical context. Partially contiguous representations were the norm rather than the exception, and the reasons for this form of representation being more common in the students' repertoire may also rest with the need to reduce perceptual complexity and enhance the features which would clearly define the primary and aggregate units within the representation. A second dimension provides a structure for keeping track of aggregate and primary units. Perceptual clarity is enhanced even more when the aggregate units are separated as discrete rows or columns. Compared to two-dimensional representations, whether discrete, partially contiguous or fully contiguous, the linear representations are perceptually more complex when units are nested within a representation.

The particular difficulties which children encounter when asked to represent mathematical relationships on a number line have been widely reported (Behr, et al., 1983; Bright et al., 1988; Dufour-Janvier et al., 1987; Ernest, 1985; Hart, 1981; Novillis Larson, 1980, 1987; Payne, 1975, 1984; Sowder, 1976: Vergnaud, 1983b). These difficulties may partially relate to a disjunction between the content of the forms of representations in a student's repertoire and the use of a representation in which the content of the form is linear measurement. Vergnaud, Errecalde, et al. (1980), investigating 10 to 13 year old students' understanding of the use of scales and axes, found that the status of intervals in terms of both order and continuous magnitudes was not established for many students at the age of 13 or 14. Some younger subjects either focused on dots along the line without reference to magnitudes of length, while others focused on magnitudes of length represented serially, not represented as nested intervals. In this study, scaling relationships between aggregate and primary units, which requires a nesting of scaled intervals along the line was also not apparent. The dominant form of units in students' repertoires generally predicted the form of units students represented when presented with a line, namely discrete points or parts of regions. Vergnaud (1983b) concluded that younger students' ability to produce graduations of their ruler upon a line gives an illusion that they understand number line properties. Considering the results of this study and that of Vergnaud, Errecalde, et al. (1980),it could be concluded that, even with deliberate instruction and extensive experience, linear representations and their properties may not become an integral part of a student's repertoire of representations during the elementary grades.

<u>Comparisons of representations of units in both repertoires of representations.</u> The forms of representations in students' repertoires were circumscribed in all cases. Students did not consider units

in the general case in which different systems of measurement were potential bases for representations. They did not use a multiplicity of measurement frameworks between which the physical attributes of the units changed. Instead, their representations were formulated around a central physical image of units which were incidentally contiguous or discrete, but which primarily referenced a single system of measurement, either that of numerosity or area.

Comparing both repertoires of representations generated by students, some students' repertoires differed from one mathematical context to another quite markedly. They considered only measures of numerosity as a means of representing whole number multiplication, and only measures of area as a means of representing comparison of common fractions. Furthermore, some of these students restricted all representations of common fraction comparisons to parts of regions while whole number multiplication was represented dominantly as relationships among discrete units. In contrast, there were students for whom representations of units across mathematical contexts were not clearly differentiated. Their representations were comprised dominantly of discrete units as measures of numerosity. Taking the question a step further, among the students in the latter case were some who also used discrete points to define units in the linear measure context, partitioned regions initially by counting lines rather than spaces, and, in at least one instance, based their comparison of the areas within partitioned regions on the numerosity of the units regardless of the differences in the size of units. Their marked inconsistency in their attention to attributes of linear or area measurement indicates an instability in their differentiation of the characteristics of units which measure discrete or continuous attributes in general. Their emphasis on discrete units apparent in the arithmetic contexts may be part of a more general conception of units as discrete objects or characteristics which correspond one-to-one within the counting process.

There was a sense of progression with regard to differentiation of forms of representations in a repertoire from two-dimensional, discrete forms to two-dimensional, contiguous forms, as well as a sense of progression from measures of numerosity to measures of area, without attention to linear measurement as an alternative form of representation. These progressions would appear to be related to experiences of representing of numerical relationship in different forms, rather than from experiences related to the representation of units of continuous measurement. That is, contiguous representations may derive from experiences of putting together discrete units rather than derive from experiences of partitioning areas of regions. It would seem that, although a knowledge of systems of continuous measurement would be prerequisite to the appropriate use of such units to represent arithmetical relationships, such a knowledge does not directly influence the systems of measurement which children use as content of the forms of such representations. However, as far as the results of this study is concerned, a progression in the differentiation of forms of representations in a repertoire can only be proposed as a conjecture to be clarified in future research.

The importance of differentiation of representations within a repertoire in terms of the systems of measure has been suggested by Silver (1983) and Lay (1982). In a similar sense, Bruner (1968) reports the "tentative conclusion" from a series of studies done in collaboration with Dienes and others that it is probably necessary for a child to have "a good stock of visual images" for embodying mathematical abstractions. Others have lent support to these arguments (Freudental, 1983; Lesh, 1979; Streefland, 1978). However, under what circumstances someone might come to differentiate forms of representations in a repertoire in terms of systems of measure as opposed to differentiating only in terms of overt physical characteristics of the units is an open question.

Implications for Research

In contrast to the present study in which students' current representations of units and unit relation were explored, questions arising from this study concern the dynamic, interactive processes through which students interpret and reconstruct such representations. The questions fall into areas of interest concerning:

1. Repertoires of representations as a construct in the mathematics learning process.
2. The conceptualization of units of continuous quantities.
3. The integration measurement of continuous quantities as content of the forms of representations of other mathematics.

Questions relating to each of these areas of interest will be discussed in turn.

<u>Repertoires of Representations as a Construct in the Mathematics Learning Process</u>

Positing the construct of personal repertoires of representations suggests implication for instruction which need to be researched further. Investigations are needed into the processes through which children (1) come to construct repertoires of representations, (2) may come to have a dominant form of representation characterizing their repertoires, and (3) may come to differentiate forms of representations in their repertoires in terms of different systems of measurement need to be investigated. The role that instructional experiences may play in these processes is not known. Of particular interest is the question of the interactive relationship between a child's dominant form of representation and the child's interpretation and reconstruction of alternative forms of representations used in instruction. In what ways might children assimilate alternatives form of representations to the forms of representations in their repertoire, and in what ways might children accommodate new and different forms of representations within their repertoires?

The characteristics of repertoires of representations have been derived from an investigation into students' representations of symbolic mathematical expressions in an interview setting. The context in which these students constructed representations was restricted in significant ways. The stability and dynamics of a repertoire need to be explored in a naturalistic setting where children are involved in representing mathematical problems other than symbolic expressions. Whether there is a relationship between a student's general repertoire as characterized in this study and the forms of representations which the students might generate as a means of explaining mathematical relationships in real world problem settings is not known. Whether or not the characteristics of their repertoires of representations would persist, or whether other forms of representations might be generated is a question for future research. Of particular interest is whether the content of the form of their representations of problems set in a continuous measurement context would differ from the content of the form of representations of similar mathematical relationships set in a discrete measurement context.

Considering the innumerable occasions in which teachers spontaneously represent mathematical relationships with materials or diagrams, teachers' primary repertoires of representations may contribute significantly to the structuring of children's representational experiences. As an example, Anghileri (1989) described a number line representation of multiplication to elementary students as jumps across stepping stones. Her decision to use this analogy may arise from her conception of children's understandings of attributes of units. But it may also arise from a way in which she thinks about the representation in a whole number context. The characteristics of teachers' repertoires of representations need to be studied. Of particular interest is the question of the interactive relationship between the teachers' repertoire, the representational experiences structured for children, and children's construction of a repertoire of representation. The National Council of Teachers of Mathematics (1989) has recognized the importance of mathematics being taught as a means of communication, and that communication includes being able to "model situations using oral, written, concrete, pictorial, graphical and algebraic methods" (p.78). The role of the teacher is central to the realization of such a recommendation. The results of studies into the characteristics of teachers' repertoires of representation may have significant implications for pre-service and in-service teacher education as a means of achieving this goal.

Finally, students' interpretation of instructional representations of complex unit relationships, their interpretation of representations of nested and double-nested units needs to be studied further. In the current study, there was some evidence to suggest that students may not perceive the nested relationship among units, particularly when the units are embedded in contiguous representations, be they one- or two-dimensional. Generally students used spatial separation of aggregate units to represent such nested relationships Even then the process of constructing such representations required a great deal of analysis of the mathematical situation. The results of such studies may have significant implications of the manner in which curricular materials are developed.

The Conceptualization of Units of Continuous Quantities

There are three issues arising from this study which suggest the need for further research into students' representations and interpretations of units in continuous measurement contexts. The first is a question of the manner in which children develop an alternative understanding of counting in the context of constructing and interpreting units of linear and area measurement. In contrast to the process of counting discrete objects, there is not a consistent one-to-one relationship between attributes counted in the construction and interpretation of linear and area units and the number of units implied by the count. It is insufficient to conclude, as Hirstein, Lamb, & Osborne. (1978) do, that a child who assumes that the count of points determines the number of linear units, "had no sense of a linear unit" (p.16). There is clearly a need for children to construct a conception of counting which admits to relationships between what is counted, how it is counted and the implication on such on the number units that differ from their conception of counting in a discrete context. The processes by which students come to differentiate between these two conceptions of counting in the context of constructing and interpreting units of area and length need to be researched. Although the linear measurement context has been used as a means to explicate the counting problem, a similar problem applies to the construction of units of area measurement.

The second is a question of the development and inter-relationship between perceptual, transformational and numerical reasoning strategies in the context of comparing either lengths or areas. There appears to be a process whereby perceptual reasoning strategies persist in children's thinking about comparisons, but alternative strategies of analysis eventually take precedence when judgements based on perceptual and other thinking strategies are in conflict. That is, all three types of reasoning strategies may continue to be employed rather than one more sophisticated strategy supplanting another. Conclusions such as those made regarding limitation of children's reasoning about units of area in arithmetical contexts when transformational reasoning strategies are employed may need to be revised (e.g., Bergeron & Herscovics,1987). The use of transformational reasoning strategies may not imply a lack of numerical reasoning strategies. Rather, it may be that in a particular problem context, transformational strategies are efficient and sufficient to resolve comparison problems.

Third is the question of the instructional processes which enhance children's understanding of the comparisons of quantities of continuous attributes and their construction of conceptions of units and unit relationships beyond the simple iteration of an object as a unit. Curricular materials on the measurement of length and area commonly begin with the physical iteration of non-standard units and then move directly to the counting of standard units of measurement. By implication, instruction of linear measurement then focuses on the use of the ruler and instruction of area measurement focuses on the counting of square units. The partitioning and construction of non-standard units, as well as the variety of geometric, spatial and number-theoretic properties of these processes are seldom if ever a focus of instruction. Teaching experiments are needed to elaborate upon the complexities of the conceptual and instructional processes which necessarily intervene between simple unit iteration and the application of standard units of measurement if students are to develop a rich and generalizable conception of measurement with linear and area units.

Integrating Measurement of Continuous Quantities as Content of Forms of Representations

The trichotomy of discrete-contiguous-continuous attributes of units is one that applies to a much wider mathematical domain than that explored in this study. Because the continuous attribute necessarily is reduced to contiguous units, all mathematical representations which display relationships among units are open to an interpretation based on the measurement of numerosity or measurement of continuous attributes. Indeed, it may well be that "mathematical experts" fluctuate between interpretations of representations of units as measures of numerosity in a discrete sense and as measure of continuous quantities in a contiguous sense. What is clearly suggested by this study is that knowledge of a continuous measurement system in terms of the nature of its units and relationships among units does not determine the ways in which one represents mathematical ideas. However, such knowledge clearly would be prerequisite to the representation of mathematical relationships based on such measurement systems.. The forms of representations of a mathematical domain are related to the ways in which one thinks about that domain of mathematics. In order to develop a fuller picture of the interplay between knowledge of continuous measurement systems and the forms of representation of units used as means of explicating other mathematical situations, studies exploring such relationships need to be conducted with students at higher grades, and with students at the intermediate grades in other domains of mathematics.

Concluding Comment

It is the belief of the investigator that the complexities of students' mathematical representations illuminated in this study are only a beginning to the quest for understanding the representational processes involved in a person's sense-making and representing of mathematical ideas. The construct of a repertoire of representations with its internal representational structure, as well as the acknowledgment of the considerable ambiguities in the meaning of units evident in the forms of personal and instructional representations, provide a perspective through which these representational processes might be explored.

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APPENDIX A

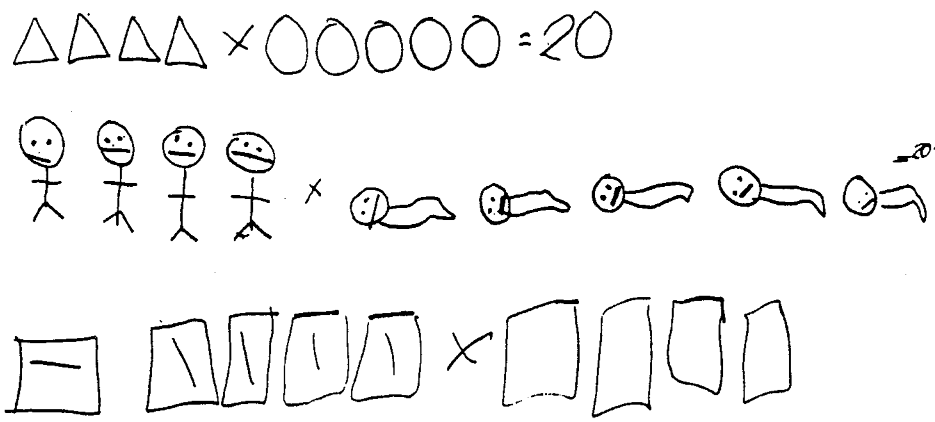
SELECTION OF INTERVIEW TRANSCRIPTS

UNCUED GENERATIVE INTERVIEW

Whole Number Multiplication

Marlene Grade 5

Task: 4 X 5 = _



Marlene: I don't know half my times tables.

Interviewer: You don't? Well I forget my times tables too, some of my times tables. I forget them and then I have to figure them out. Do you sometimes have to figure it out?

M: Yah. Count.

I: Yup. But that's okay as long as you can figure it out somehow, eh. Now what I want you to do is to imagine that you are a teacher.

M: Okay. Student!

I: ...and you want to teach some imaginary students some arithmetic - and when teachers teach some new ideas in arithmetic- if this student hasn't learned this stuff before- they often use rough pictures and diagrams to help them understand.

M: Yup.

I: So, you know what I mean ...

M: Like you make circles or ...

I: Yah, all those kind of things.

M: Yah, and then equals, plus, or times- and then you doing 2 or 1 more- and then you put an equals and you have to try it.

I: That's right. So what I'm going to ask you to do then is to imagine you are a teacher and you are teaching this imaginary student, and you're going to use rough pictures or diagrams to help the student understand the arithmetic. Now the first one that we are going to try to do is this multiplying. Four times five. What I want you to do is to think about teaching this to this imaginary student who doesn't know anything about multiplying, and see if there is a rough picture or diagram that you could draw that would help explain that.

M: Okay. [*drew four triangles*] ...four times... [*drew five circles*] There! (*see record sheet 1*)

I: Okay. And if you draw this picture, how are you going to explain to this student what this all means?

M: Four time how many there are in that what you put down... [*indicated the set of 5 circles*] ... and then you times it together... [*skip counted by fives using the 4 triangles as counters*] ...there's 20 there.

I: There's 20 there! How did you figure that out?

M: Count like 5... you count by five's like 5, 10, 15, 20. [*touches each triangle as she counts*]

I: Oh, I see.

M: And you could make this like .. 4... I don't know my fours so I count by fives.

I: It's hard to count by fours isn't it?

M: Yah, so I count by fives like 5, 10, 15, 20, 25, there! [*counted 5 fives, touching each circle with each count*] ...30, 35, 40, 45 [*continued onto the triangles with each count*] ... 45 altogether!

I: 45 altogether?

M: All these and these have 45 and these add 20. [*wrote 20 to right of diagram*]

I: Add 20, right. Is there any other completely different diagram or rough picture that you can draw in case I'm having trouble understanding my arithmetic?

M: Oh, make people.

I: Sure, what would that look like?

M: [*drew 4 stick people and 5 babies*] That's a baby in a sleeper ... it looks like a ... (unintelligible) ... that one looks like a sock. Four, five, ... [*puts faces on all people then laughs*]

I: Okay, now I'm a struggling student and I'm really having trouble understanding. How would you as a teacher explain that to me?

M: *[turned the page around to face interviewer]* Five *[points to babies]* times four *[points to people]* equals 20

I: It does?

M: It does.

I: I know it does, but I don't see 20 things there!

M: *[counts the sets of 4 and 5]*

I: How do we get 20?

M: 20 things here.

I: How do we get the answer 20?

M: You add them up with these ones. *[traces finger across the sets of 4 and 5]*

I: Can you show me how you do that? 'Cause I'm really slow.

M: *[laughs]*

I: I am. You have no idea how slow I am.

M: Five - you've got 5 here - and you times these together like ...*[points to set of 4]*... 5, 10, 15, 20.

I: Oh, I see. So these help you to count by the fives - how many times do you have to do it.

M: Five times 4 equals 20!

I: Well, I think that I've maybe learned that. Do you think so?

M: Yup.

I: All right, now my question is, "Is there any other completely different rough picture or diagram that you could draw to help this poor struggling student to understand this arithmetic?"

M: *[pause]* ... They make 20. ...*[re-examining second diagram]*

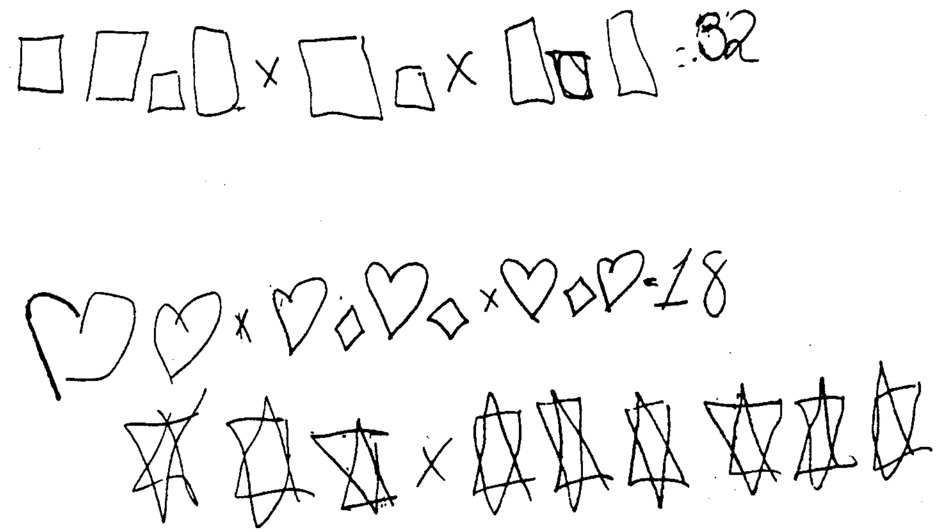
I: They make 20 and that's that.

M: Aha. Like you go ...*[drew third diagram of squares with and without a line inside]* Five times 4 equals 20.

I: Equals 20 just like that. Well let's try another question.

Task: 2 X 3 X 4 = _

$2 \times 3 \times 4 =$



M: Two times 3 times 4.

I: Now this is what the student is trying to learn- what it means to do 2 times 3 times 4.

M: Four times 3, ...[*tapped the numeral "2" twice then the numeral "3" three times*] ... 2 times 3 is 6 ... [*tapped her pencil on the numeral "4" four times*] ... 14 ...

I: It's 14. Could you draw a rough picture or diagram that would explain that to this poor struggling student.

M: [*drew set of 4, set of 2, and set of 3 boxes, each separated by a multiplication sign*]

I: Oh, what have we got here?

M: Little squares and big squares.

I: Yah, and how does this multiply?

M: Four times 2 times 3 equals 14.

I: Equals 14? Oh dear, how did you get 14?

M: Is that the answer?

I: I don't know what the answer is 'cause I don't understand multiplying.

M: Okay. You add ... well I don't know my four times.

I: Can you figure them out?

M: [*wrote 4, 8, 11, 16, 20, 32, 36 - while counting-on on her fingers*]

I: There, and how did you figure that out?

M: Added by fours. Four, then I say, 5, 6, 7, 8, then 8, 9, 10, 11, 12, - 13, 14, 15, 16 ... [*pauses on 8, 12, & 16*]

I: Oh, I see. Okay. So we've got 4, 8, 11, ...

M: Wow! [*correct 11 to 12*] ... 13, 14, 15, 16, ... yup, 17, 18, 19, 20, ... I did these too - ...[*changed the numbers from 20 onwards*] How far do you want me to go?

I: I don't know. You were doing your four times tables for something - 'cause you wanted to use it for explaining this, didn't you?

M: Yes. I'll go up to 32 - Put 4 there [*pointed to the set of 4 squares*] ... count 4 there ... [*count on the times table*] ... 1, 2, 3, 4 you have 16. You get 2 more [*two more fours from the times table, double counted the 16*] ... you've got 20 now ... add 3 ...[*three more fours from her times table*] and you've got 32 right there.

COMMENT: THE PREVIOUS PROCESS WAS THAT 4 X 2 X 3 IS EQUAL TO (4 X 4) + (4 X 2) + (4 X 3) BUT SHE DOUBLE COUNTED ON THE 16 TO GET THE FINAL RESULT OF 32 INSTEAD OF 36.

I: Oh, can you explain that to me again, because I think I kind of understood, but I'm not sure.

M: You count fours, oh 33, 34, 35, 36 ...[*extended the times table*] ...Four, 4, 8 ...[*skip count on the times table*] ... 4 times 2 [*on the diagram*] ... 4 times 2 is 8.

I: Oh, I see, okay.

M: Four times 2 is 8, okay and then you ... not that ... times 3 - 8 - 3, 6, 9 ... No I don't get it.

I: You don't get it. Well, 4 times 2 is 8 you said.

M: Yes, and 2 times 3 is 8, and 8 times 8 ...[*pause*]...

I: Is it getting a bit confusing?

M: Aha. The answer is 36. ... 8, 8, 16 ... 8. Ah, yes that's what you get. It's a mess.

I: Okay, is there any other completely different diagram, that is completely different, that you could draw for me to help me to understand, that is completely different.

M: [*drew sets of 2, 4, and 3 with hearts and diamonds*] Haven't got any room. Okay student, 2 times 4, 2 times 4 is 6, I think, and 6 times 3 ...[*skip counts on fingers*]... and it's 18.

I: How did you figure out the 18? I sort of thought I understood, but I'm not sure.

M: Aha?

I: How do we know that 3 times 6 is 18?

M: You count 3, 6, 9, 12, 15, 18!

I: Is there any rough picture or diagram that would explain that?

M: [*drew 3 stars and 6 stars*] I know lots of pictures. Times 6- 1, 2, 3, 4, 5, one more. Three times 6- 3, 6, 9, 12, 15, 18 [*skip counts on the set of 6 stars*].

I: Oh, I see.

M: You count by threes. 3, 6, 9, 12, 18, 21, 24, 27, 30, 33, ...

I: Do we have to go higher?

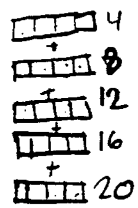
M: Yah to 50!

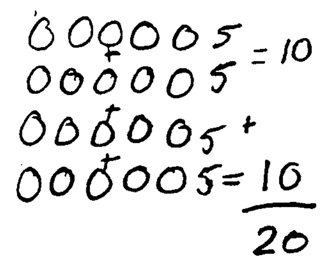
Coran Grade 7

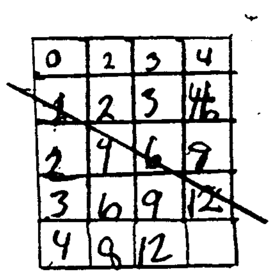
Task: 4 X 5 = _

$4+4+4+4+4=20$

20







Interviewer: [*Long introductory discussion on problems with teaching*] Lets go on to the first question and I think that you'll get the idea as we go along. Now the first question that we want to explain to this imaginary student is 4 times 5.

Coran: Twenty! [*laughs*]

I: Well, how did that work out? I mean, there is 4 and there is 5, how did you get 20?

C: Oh ... [*pause*] ...Five fours. Just put 4 plus 4 plus 4 plus 4 plus 4.

I: Okay. Well, put it on paper so that we could see what you would show the student. Imagine that the student needs to see lots and lots of examples.

C: [*wrote 4 + 4 + 4 + 4 + 4 = 20*]

I: Okay, so what would you say to the student to explain what you have just done?

C: Well, 4 times 5 means 4 sets of 5 ... 5 sets of 4 ... so you get 5 ... the long way would be to put 5 set of 4, like 4 plus 4 ...and that's, just add them together and that's 20. Then you just memorize it and then you just see 4 X 5 and you just automatically go "20."

I: Right, but that would be way down the road - but eventually that's what you do. You just memorize it, don't you.

C: Yah.

I: Now I am a really slow learner.

C: Oh, no!

I: It takes me a lot of examples. Is there some kind of rough picture or diagram that you could draw.

C: A rough picture or diagram.

I: Or diagram.

C: A picture ... um ... I don't know.

I: Well, just have a think. We've got lots of time.

C: Four times 5 is 20. ... This kid's slow!

I: Yup, so using numbers even can sometimes be hard to understand.

C: It's easy to do dividing, but I don't know.

I: Could you draw a picture of 4 plus 4 ... plus 4 equals 20?

C: Well, I could draw a little box.

I: Sure, that would help. That might help this student.

C: [*drew* | | | | *then* |⊥⊥|, *added on a fourth box to make* |⊥⊥⊥|] Make the 4 here. 1, 2, 3, 4 ... [*counted spaces*] ... five lines, four spaces.

I: All right. You didn't realize that the first time, eh?

C: Yah, I only did 3. It's just like my music bars. Five lines, four spaces. [*continued drawing other boxes while this conversation took place. Always drew 5 vertical bars first then the two horizontal bars*]

I: Now, how are you going to explain it?

C: There are four groups of 1, 2, 3, 4, blocks. 1, 2, 3, 4; 4 plus 4 is 8, plus 1, 2, 3, 4- plus 4 would be 12 [*wrote in the 4, 8, 12 down the side*] and plus 4- 1, 2, 3, 4 would be 16 plus 1, 2, 3, 4 would be 20.

I: That would explain it really well, wouldn't it.

C: I suppose so. [*doubtful tone*]

I: Sure.

C: It's not the neatest drawing.

COMMENT: CORAN WAS A PERFECTIONIST, AT ONE POINT HE DECLARED THAT HE WAS NOT ALLOWED TO MAKE MISTAKES.

I: Well, that's okay. We don't have rulers and things to help us do a neat drawing. Is there any other completely different diagram or rough picture that you could draw- supposing I needed lots of different examples in order for me to really understand.

C: Circles. Would circles be all right?

I: Let's see.

C: [*drew* O O O O, *then paused*] ... or else we could do the other way around. [*added a* "O" *to the set*] We could do 5 times 4. [*completed the array*] ... and that. Okay, five, 1, 2, 3, 4, 5 and then you'd have 5 is ten. [*wrote* "5" and "5"] Now, what's the question. A Grade 2 question?

I: Something like that.

C: Well, they should know what 10 plus 10 should be.

I: Oh yes, Grade 2's would know that.

C: [drew in + signs] ...So add. ...[*wrote* "= 10"] ... Put that there, and then you have 5 plus 5 equals another 10, and you just add these two together and you get 20.

I: Sure that would show it. Would there be any other completely different diagram that you could think of if you wanted to teach this?

C: Completely different?

I: Yes, completely different.

C: Totally different?

I: Like as totally different as you can totally make something different.

C: A square and then you split it up into total groups. But that would almost be dividing.

I: Can I see what you mean, because I'm not too sure.

C: Can you pass me that ruler over there? It would make it a bit neater to draw. ...[*drew a 4 by 4 square*] These are boxes ... Okay, that looks like 20. [*counts them*] Oh, God! There are 16 there! Let's see, a box with 20 in it. ... [d*rew a bottom row of 4 to make 20*] ... Let's see, a box with 20 in it ... [*skip counted the columns*] ... Okay, you see with these little squares ... you can count up these little squares, there's 1, 2, 3, 4 ...[*wrote numerals across in the squares across the top row*] ...then you could ... Could you put? ... Would this work if I put 2, 3 ...[*writing numerals down the 1st column*] ...Maybe that should be zero, and that should be 1? ...[*changed the number in the 1st column*] ... Would that work?

I: I don't know.

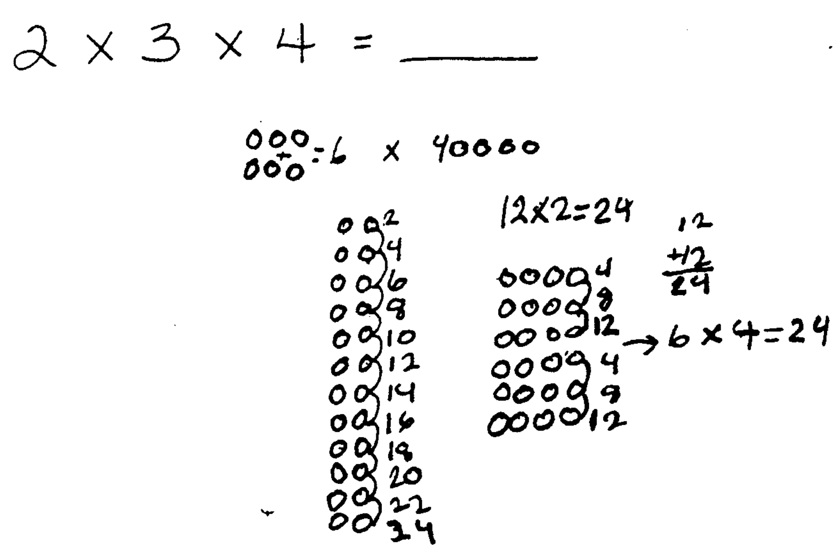
C: Let's see ... 3, 4 [*numerals in 1st column*] 1 times 2 ... Wait a minute! Yah! That would be ... [*wrote in* "2" - *making a multiplication square*] ... and then 2 times 2 would be 4 ... This isn't going to work. 3, 6, 9, 12 ... that would be 16 ...[*i.e., bottom right square*] This won't work! [*Very frustrated. drew a line through the diagram*]

I: That's all you can think of?

C: Yah.

I: Okay, that's fine.

Task: 2 X 3 X 4 = _



I: How would you explain what 2 times 3 times 4 would be....

C: What grade would that be? Three maybe?

I: Probably. It would be at least Grade 3.

C: Four?

I: Maybe even four.

C: Yah, I think I can do this one. Two times 3 times is ...[pause]... times 4 is 24.

I: All right. What kind of a rough picture or diagram can you draw to explain what 2 times 3 times 4 is?

C: Well, first you could start 2 times 3.... We have 3 circles here, two sets of them and 4 over here..... They should know what 3 plus 3 is, shouldn't they?

I: Oh, I think they should be able to figure that one out.

C: [*Laughs*] ... Three sets of 3 which is 3 plus 3 ... whatever that would be, and there would be 4 here, and all you do is 6 times 4 is 24 ... and that's going a bit fast in getting the answer. [*no diagram drawn for 6 times 4*]

I: It is, isn't it? Yah, for somebody who is trying to understand what it's all about, I think you'd have to show the steps in between.

C: And then you'd just have ... um ... 6 fours ... [*drew a 2 by 12 array*] and count that. Okay they should be able to add by two's probably, shouldn't they?

I: Oh, I should think so.

C: Okay, they would just go ... [*drew in loops from row to row and wrote in the numerals* "2, 4, 6, *etc. while counting to 24*] ... 24! So now I should split them into groups of 4. We know that 12 sets of two's is 24, and we just have to reduce that, so ... we have to say 6 sets of 4, which is what it says. ... [*drew the 4 by 6 array*] ... and that's ... okay, do they know their 4 times table by this time?

I: Hmm. They can add.

C: Okay, so there's 1, 2, 3, 4, there and 1, 2, 3, 4, and 4 plus 4 is 8. [*wrote the sums cumulatively on the side*] ... 1, 2, 3, 4 and 8 plus 4 is 12 ...[*repeated the same explanation for the next group of 12*] ...And there is 6 and there is 1, 2, 3, 4 in each, and then they should be able to add 12 and 12.

I: Oh, I should think so.

C: So you just add your 12 plus your 12 is, ... 3 groups of 4 is 12, and 3 groups of 4 is 12 ... plus 12 is 24.

I: Good, that would really help them understand the whole process. Would there be any other completely different diagram that would explain 2 X 3 X 4?

C: Not that I can think of.

Task: 3 X (4 + 5) = (3 X 4) + (3 X 5)

$3 \times (4+5) = (3\times4) + (3\times5) =$ $\qquad \begin{array}{r} 15 \\ +12 \\ \hline 27 \end{array}$

3×9

9
18
27

$\underbrace{9+9}_{18}+9 = 27 = 12 + 15 = 27$

9 18 27

4
4 = 8
4 = 12

$= 3+3+3=9 \qquad 3+3+3+3=12$

6 9 12

5
10
15

or $5+5+5=15$

10 15

I: Now this is another multiplying question.

C: Oooooh! Now I have to explain that?

I: Do you understand what that is saying?

C: Well, you do the brackets first. That's 20 ...[*4 X 5 rather than 4 + 5*]... that's 12, that's 15 ...[*3 X 4 and 3 X 5*]- 12 and 15 is ... um .. What's 12 plus 15?

I: Okay, so what you are doing- you're doing 4 plus 5, multiplying by 3 for the second ...so here you are adding first and then multiplying, and here you are multiplying first and then adding, and you want to explain to somebody why this will end up being equal to that.

C: Okay, well first of all you tell the kid that you make sure you do the brackets first.

I: Aha.

C: They should be able to ... What grade would they be?

I: Oh.

C: Four at least

I: At least 4.

C: Probably 4.

I: It might even be Grade 5. I can't remember exactly.

C: Getting pretty close to 5.

I: Yah, I would think so.

C: [*wrote out the question*]

I: Okay, so we want a rough picture or diagram that would explain that to the student.

C: Well they should know what 4 plus 5 is. Four plus 5 is easy.

I: Well, teachers often have to assume that the students don't know very much of anything.

C: [*drew a row of 5 circles below a row of 4 circles*] ... Okay.

I: Okay, there is 4 plus 5.

C: Four plus 5 equals 9, and you bring your 3 down ... times ...[*wrote "3 X 9"*]... and then you have 3 sets of 9 ...*[drew 3 rows of 9 circles*] ... Okay, 9 plus 9 is 18 plus 9 is 19, 20, 21, 22, 23, 24, 25, 26, 27 ...[*counting last row of 9*]... Or else you could so it this way. You could do it 9 plus 9 plus 9 ...[*wrote "9 + 9 + 9" and successive sums beneath the numerals*]... Do you think that they would get stuck on that being 27 by now?

I: Oh, I think that they may have a pretty good idea about that.

C: Okay, equals full house ...[*wrote "= 27 = , to continue numerically with the right hand side*]... Ummm... Three times 4, okay. I'll do this down here. ... That's 3 sets of 4 ... 4 ... 4 ...[*drew 3 rows of 4 circles*]... Four plus 4 is 8 plus 4 is 12 ...[*wrote numerals on the right side of the array*]

I: Three times 4 plus 3 times 5, is that what you are working on?

C: I'm doing my 3 times 4 - equals 9... um... [*wrote expression "3 + 3 + 3 = 9"*]... 3 plus 3 plus 3 equals 9 ... 6 plus ... Oh, no! What am I doing! ...[*wrote "3 + 3 + 3 + 3 = 9*]... 3 plus 3 is 6 plus3 is 9 plus 2 is 12. ... And then 27 equals 12 ...[*wrote "= 27 = 12 + ..." in the expression beside 3 groups of 9*]... And then I'd better do the 5 ...[*drew 3 rows of 5 circles*]... 5 , 10, 15 ...[*wrote numerals down the right-hand side of the array*]... or 5 plus 5 plus 5 ...[*wrote "5 + 5 + 5 = 15"*]... 5 plus 5 is 10 plus 5 is 15 and you add it up here ...[*added on to the expression to right of 3 X 9 array*]... Now this is okay. Here we go. Oh no, you don't have to put that there, 15 plus 12 ...[*wrote 15 + 12 as column addition*]... They must be past that grade.

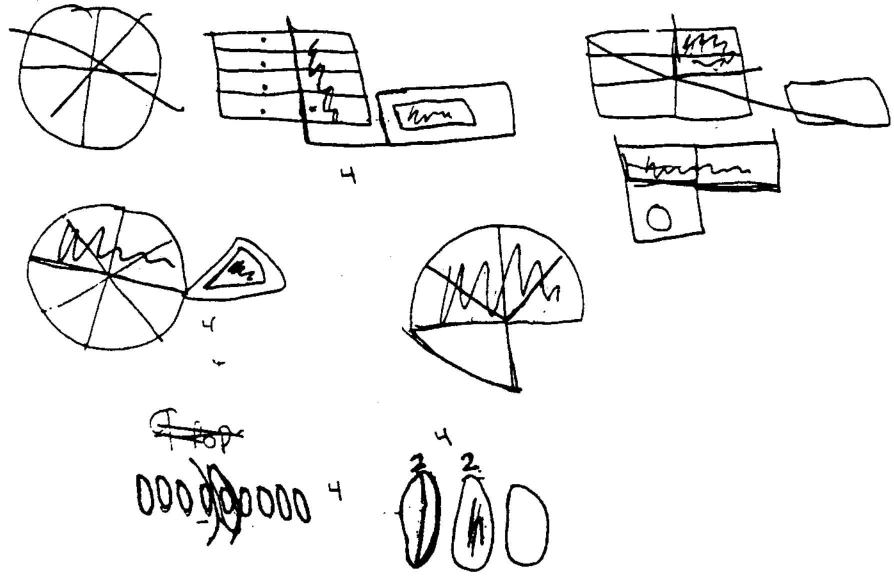
I: Oh yah, they can add.

C: They can add. So, that will be 7, that will be 2, so that is equal.

Common Fractions

Pete Grade 7

Task: Compare 5/9 and 2/3



I: Now the student is going to to do some learning about fractions.

P: Fractions? I know that.

I: You know that. Okay. Supposing you want to explain to a student how you can compare those fractions to decide whether they are the same or whether one is bigger or smaller. You want to use rough pictures or diagrams to explain that to the student.

P: I'll use pies. [*Drew a circle and began partitioning it by a successive halving process.*] Just wait! [*Crossed out the circle.*] Chocolate bars! [*Partitioned a square into 8, counted, and drew a small piece to the side Shaded 5 parts.*] You have 9 chocolate bars. Suppose I eat 5 of those chocolate bars there would be 4 left. And if I had 3 chocolate bars ... 3 large chocolate bars, and if I was supposed to eat 2 of them I would have this fraction. [*Partitioned a square into 6, and drew a smaller piece to the side to match diagram for 5/9. Shaded 2 parts.*] And I'd have this much left which would be more than this. [*Comparing the unshaded amounts in both diagrams, i.e., more was left with 2/3.*]

I: Okay. So this is 2 chocolate bars here, and where is the 3 chocolate bars? I'm just wanting to make sure I understand.

P: It looks like I did this diagram wrong. I put 7 in this one. [*Crossed it out and drew diagram with 3 large contiguous rectangles.*] There, that's better. And he'd eat two thirds of the chocolate bars which would leave the other person this chocolate bar which is less than this. [*Compared the amount unshaded or left over, i.e., less left over with 2/3.*] So he'd say that 5 ninths is more than 2 thirds ... He'd have ... If he had the 5 ninths ... If he had the fraction 5 ninths out of the 9 chocolate bars, he'd eat the most ... more chocolate bars than the 2 thirds fraction.

I: How many, how much more would he eat?

P: He'd eat one more.

I: One more?

P: [*Hesitates, measures with finger, the unit for 1/3 against the units for ninths.*] Yah, he'd eat one more.

I: Oh. How does that work? Could you explain how you decided that it would be one more?

P: Because a half of this (thirds) would equal one of these (ninths). If I halve this large chocolate bar in half, it would equal two chocolate bars (ninths). [*Measuring with fingers back and forth as he talks.*] I have two groups of these (thirds) so that would equal 4 chocolate bars (ninths), and I have eaten 5 ninths so it would equal 1 left over.

I: Is there any different diagram that you could use to explain the same thing?

P: Same thing. I could use a pie except I could ... I could use a pie like this. [*Drew a circle partitioned into 8 parts, and a small piece to the side.*] The I'd have 9 pies and who ever was eating the 9, 5 pies, 5 pieces of pies out of 9 pieces of pies would have eaten this much of the pie, except for the person who has to eat the 4 pieces of pie, okay, the left-overs. [*Shaded 5 pieces.*] And then that (2/3) would be something like that. [*Drew 3 contiguous segments of a circle.*] This is a jumbo pie, and the person out of 2 thirds pie would eat only 2 of the 3 pies. Those two. [*Shaded 2 parts.*] And then he'd take these and he'd ... Those (thirds) are bigger than these (ninths), right. [*Measured both segments with fingers.*] So I'd ask him to half the pie (thirds), and he'd go up ... [*Partitioned the thirds.*] And those, one would be equal to two of the smaller pies, and he'd do the same for this one. So he's eaten 4 ... and he'd only eaten half the pie over here, (i.e. 4 would be the same as half of the pie drawn for the ninths). But the person who ate 5 ninths of the pie had one more slice. So he'd have more.

I: Is there any other different sort of diagram besides your chocolate bars and your pies that you could draw. Sometimes students need to look at all kinds of different diagrams in order to really understand what you are saying.

P: Give and take.

I: What would that be?

P: I'd give the student 9 sticks, popsicle sticks. Do I have to draw them?

I: It would help me to remember. I'm not very good at remembering when I'm interviewing so many different students.

P: [*Drew 10 popsicle sticks, counted and rubbed one out.*] Yup, that would be 9 popsicle sticks. And I'd go up to the student I gave 5 popsicle sticks. I'd say I am going to take 5 of them, and I'd ask the boy how many he had left. And he'd, or the girl, and he'd say 4. And then I'd give him the same share of the popsicle sticks except those large kinds the doctors use, like those. [*Drew 3 large sticks.*]

I: Oh. Like the tongue depressors.

P: Yah. Then I'd say to him, "I'm going to take two of those." And then I'd ask him to measure one of the small popsicle sticks inside the big popsicle stick. And then he'd get 2 inside one of those tongue depressors. Okay, so they get 2, so this one would equal 2, and this one would equal 2, and he'd only have 4. [*Labelled the tongue depressors.*] He'd only have eaten this fraction of the one, I mean taken of the one. But this kid over here who took this popsicle stick, had 5 ninths popsicle, would have had one more popsicle. Would have had one more popsicle than this boy who had the jumbo popsicle sticks.

I: Because the measure was 2 of these to 1 of those, right?

P: Yup.

I: Anything else? Any other completely different diagrams?

P: No.

<u>Task</u>: Compare 5/8 and 7/12

I: Supposing you had this. It is a similar question but just different fractions. Supposing you wanted to explain to a student how to compare 5/8 and 7/12?

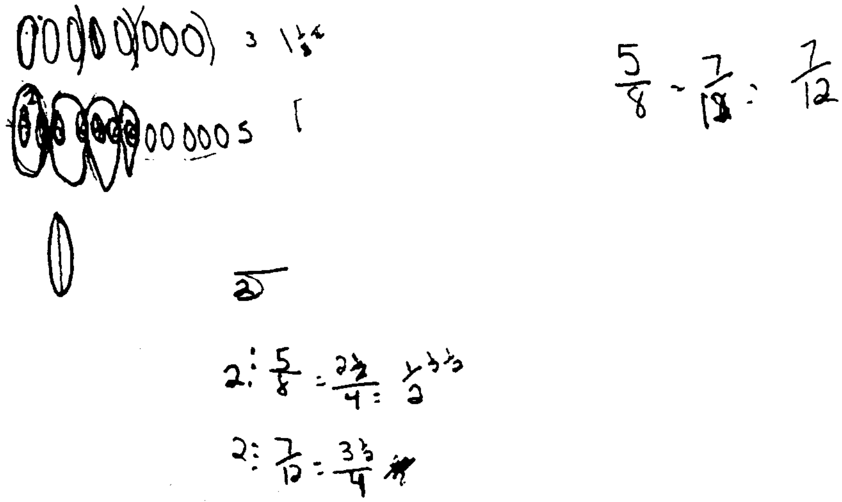
P: Like if they are equal, or different, or the same?

I: Yah, what kind of diagrams would you use?

P: I'd use the same method as I used before.

I: Okay. Could I see what that would look like with these two different fractions?

P: Okay. [*Drew 8 sticks*] And I would take 5 popsicle sticks, or I would give 5 popsicle sticks from to another student from that student, student A to student B. [*Marked off 5 and 3 popsicle sticks.*] The student I gave those 8 popsicle sticks to, I'd ask him to say how many he had left, and he'd say 3. Then he'd put down 3 left. [*Labelled diagram with a 3.*] ... Um ... And then I'd give another student 12 popsicle sticks but larger ... No, smaller ... smaller ... yah. [*Drew 12 small sticks.*] And I'd take 7 popsicle sticks from the student I gave 12 in the first place. [*Shaded 2 sticks.*] And then he'd total up how much he had left. He'd have 5 small ones. But then I'd compare on the blackboard the big one, and to the twelfths. You know, the small ones right here. [*Partitioned a large stick into 2 parts.*] So 2 of these would equal 1 big one. One big one, 1 big one, 1 big one, and a half. [*Circled 5 small sticks by twos.*] So you would have eaten 3 of these and one half (eighths). The student with the 5 popsicle sticks I gave to would have aten more, cause he would have aten 1 and 1/2 more.



I: So this fraction here which is the 5/8, and this fraction here, which is the 7/12, when you compare them you find which one is bigger?

P: That the 5 eighths is bigger.

I: The 5 eighths is bigger, and again you are saying that these 2 little popsicles sticks are the same as the big ...

P: Jumbo one.

I: Why did you decide that the little popsicle sticks should be used to show the twelfths, and that the big popsicle sticks should be used to show the eighths?

P: I don't know. The teacher did it that way. It was explained in grade 5.

I: In grade 5 you did it like that, did you?

P: Yah, except now we do it a different way.

I: Can you show me how you do it the different way that you do now?

P: Well we divide them. Like you divide the 5 eighths, right. We divide it into 4. Two divided 5 eighths. [*Wrote 2 - 5/8*] First, I'd divide this. It would equal 2 and 1/2 over 4, which I would divide again. ... No I wouldn't. ... Yes I would. Which I'd get one, two, and a half of a half. It's complicated, but ... And then 2 divided by 7 twelfths. [*Wrote 2 - 7/12.*] Which I'd get 3 and 1/2, 4. And I'd divide again. No, I couldn't divide again. So this would equal 3 and a half over 4 quarters. This (7/12) would be more! It's been since last year that I learned this.

I: Yah, it's hard to remember these things isn't it. Is there any other completely different diagram or picture that you could draw to describe the same thing.

P: No. Yah, but it would be sort of the same idea.

I: What? Using the "give and take" or could you do something completely different from that?

P: No. Oh, I could if the student was really, you know, dah!

I: Yah. We've got to assume that.

P: [*Wrote 5/8 = 7/12*] And he'd probably just say that 7 was bigger than the 5, and the 12 was bigger than the 8, and he'd presume that that would be that.

I: Okay, now if he was doing it that way, what kind of rough picture or diagram could you draw to show him that you have to think about it in another way.

P: Then I'd use this diagram right here.

I: You'd use the "give and take?"

P: I'd have to keep showing him this idea 'til it got through.

CUED GENERATIVE INTERVIEW

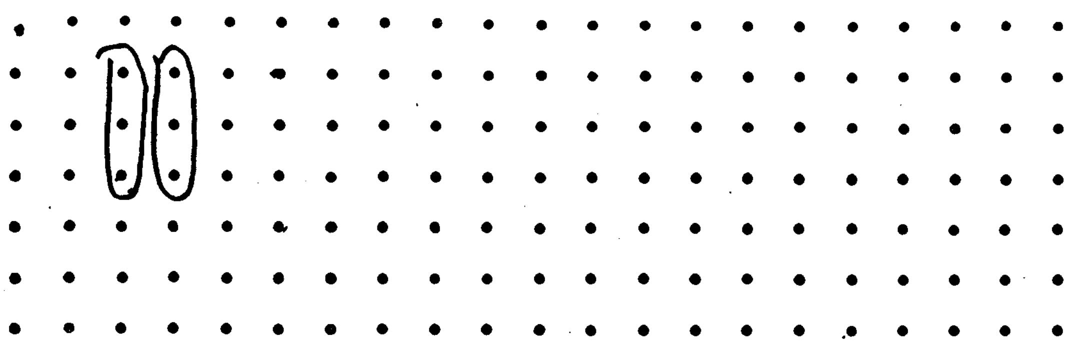
Whole Number Multiplication

Kasey (Grade 7)

Task: 2 X 3 X 4 = _

Interviewer: [Preamble about the similarity with previous interview, teacher role and survey of types of materials available.] The first one that we are going to start with is a multiplying one, and it is 2 times 3 times 4. So you want to think about teaching this to somebody who is just beginning to learn, and you can use anything at all that is on the table.

Kasey: [*long pause*] ...I'll use this. ... [*selected dotted paper and circles 2 vertical sets of 3*] ... [*pause*] ...

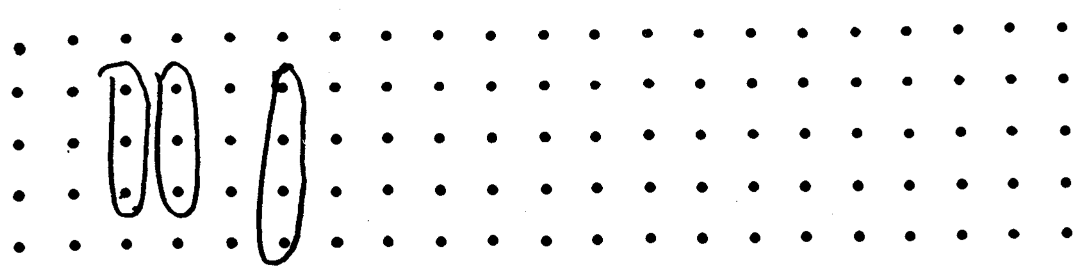


I: What were you thinking of trying?

K: I was trying to put them in groups and ...

I: Well, it might work and if it doesn't you can try something else.

K: [*circled a vertical set of 4*] ... [*pause*] ... No, I don't think that would work.



I: What's the problem?

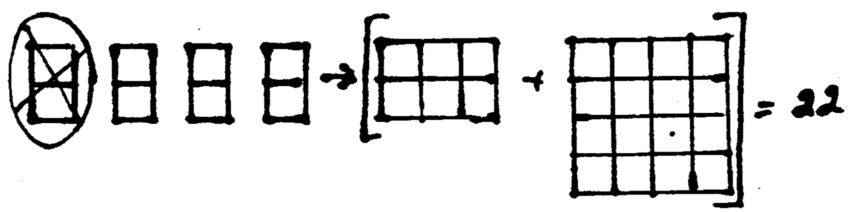
K: Well, it's just that if I have 2 with 3 in each group, and then 4, there is 3 numbers there and I wouldn't know how many to put in the 4 group.

I: Oh, right because it is 2 times 3 [*pause*] times 4.

K: Let's see if you can use the blocks.

I: Okay, sure. Let's move these out of your way.

K: Ummmm. [selected multilink] ...2 ones... [*made 4 sets of 2 linked units, removed one set, and pushed the 3 sets of 2 links together*] ... [pause] ... [*made 4 more sets of 4 linked units, counts the total 7 sets*] ...(unintelligible statement)



I: I beg your pardon?

K: I'm just trying to figure out how to do this. [*counted the blocks again*] ... There's just 22.

I: Hmm. What do you want? What are you trying to get?

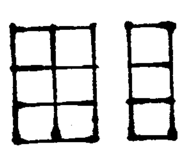
K: 24.

I: I see. How did you figure out that it was 24 when you did it in your head?

K: Two times 3 is 6, and 6 times 4 is 24.

I: Oh, I see.

K: Ummm. Maybe if I went ummm ... Okay, 2 times, 2 groups of 3; - 3 groups of 3 I mean. [*links of 2 rows of 3*] ... Maybe I'd better make it the same as the question. ...[*undid the blocks and formed links of 3 rows of 2*] ... Okay, that's 6 and then 6 times 4. ...[*made another group of 3*] ... Hmmm.

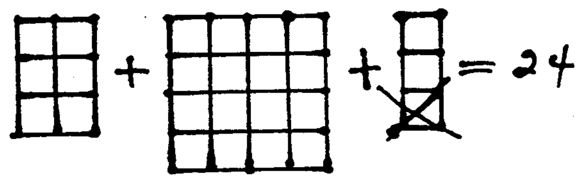


I: Would that give you 24?

K: Yah, but how would you use the blocks?

I: How would you do 6 times 4? How would you do that?

K: [*keeping 2 vertical links of 3, formed 4 vertical links of 4*] ...[long pause] ...[*made an additional vertical link of 3 units, undid a the link of 3 to form a link of 2, added the 2 to the previous links to give a total of 24. However, this did not represent the multiplication. Removed the last link of 2 and recounted the 2 links of 3 and 4 links of 4. 22 units still remained. Note: no verbal expression.*]

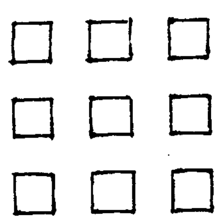


I: So, you are wanting to do 6 times 4, is that right?

K: Yah, I'm trying to figure out how to do it.

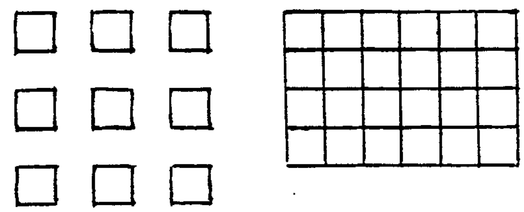
COMMENT: KASEY WAS PERFECTLY CAPABLE OF REPRESENTING 6 TIMES 4. INSTEAD SHE WAS TRYING TO CONSTRUCT A SINGLE REPRESENTATION FOR THE WHOLE MULTIPLICATIVE EXPRESSION, A GOAL WHICH SHE WAS HAVING DIFFICULTY ACHIEVING.

K: [*undid the linked 2 groups of 3 and displayed them in a discrete rectangular array*] ... I don't know how to do it.



I: It's getting a bit confusing isn't it? What if you just wanted to do 6 times 4 without having it related to the 2 times 3 times 4, what would you do?

K: [*built extra groups of 4 to make 6 vertical links of 4 units*]

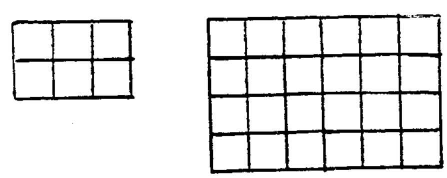


I: So this part would show 6 times 4, would it?

K: Well, maybe I'd do the first step first, the one that has brackets around it. ...[note: Kasey added the brackets around 2 X 3] ... and I'd do that, and then I'd change to 6 times 4 after I'd done the 2 times 3 is 6. ... I could do that.

I: That would be one way of doing it, wouldn't it?

K: [*changed to base 10 centimetre cubes, made 2 rows of 3 cubes*] Okay, that would give you 6. ... and then 6 into here ... [*made 6 vertical columns of 4 units*]

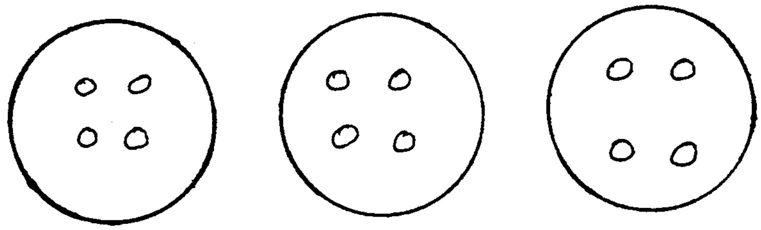


I: Okay, and how would you explain this to the imaginary student, if you want to tell him what you have been doing?

K: Okay. First I'd have the 3 times 2, and then I'd just take it down the 6 times 4, and tell them maybe they were building a building or something that would be 6 wide and 4 down.

I: That would certainly help, wouldn't it? Now is there a completely different way of explaining the same thing, using anything else that is on the table, that is <u>completely different</u>, completely different from the two things that you have already done?

K: [*selected small red disks and 3 circular filter papers*] and 1, 2, 3 candy dishes with 4 candies in each, and you count all the candies and you get 12, ...and then ... ummm

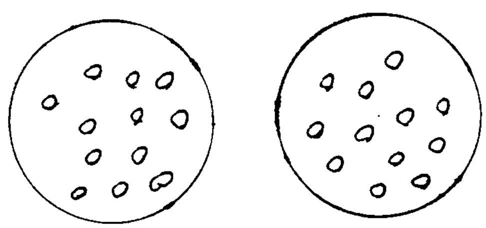


I: So which part are you doing at the moment?

K: This part.

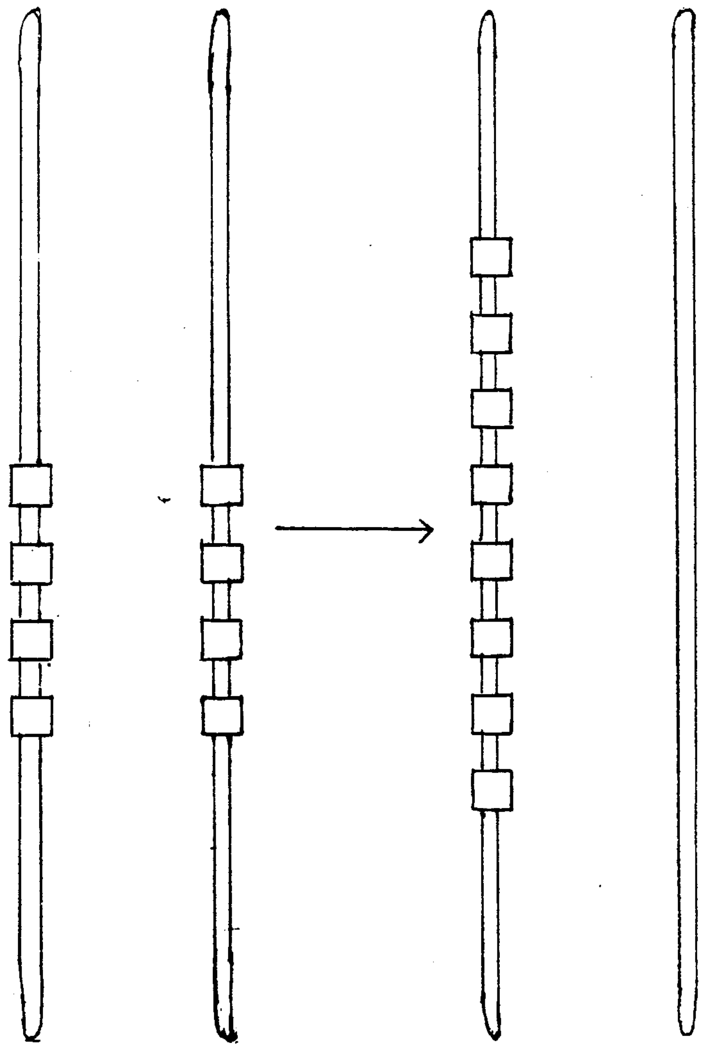
I: Oh, 3 times 4. Right!

K: And then if you were to pile all the candies into one dish, and then you go and get another dish and pile more candies in it - 12 - then you would have 24 candies. [i.e. 2 X (3 X 4)]



I: Aha. Sure. That's showing the 2 times 12. Yes that would be a good way of showing it. Would there be anything else that would be completely different, using anything on the table, - that would be completely different from anything you have done so far?

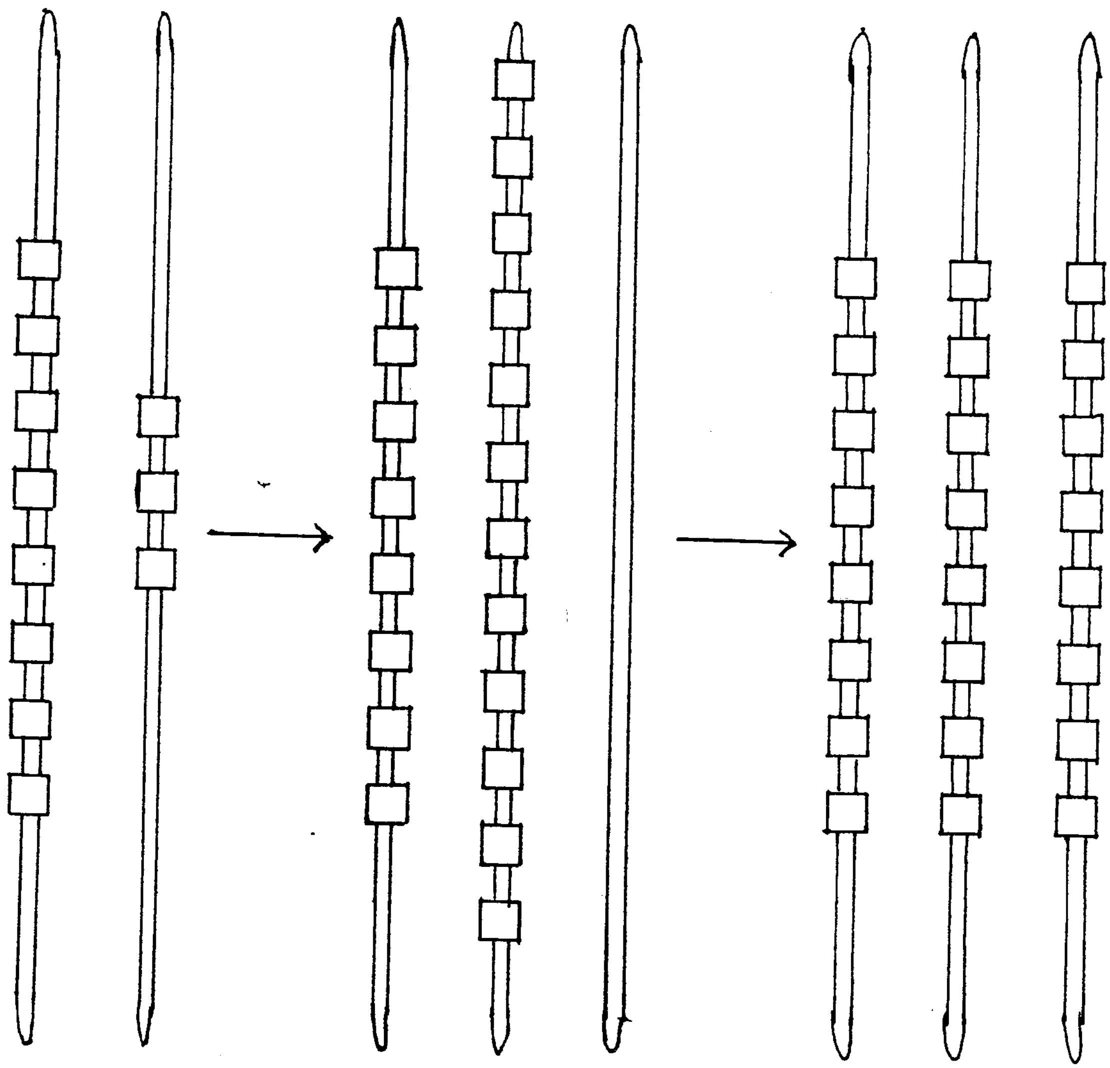
K: Okay. [*stretched out two pieces of string, and placed 4 hexagon blocks along each string*] ...Maybe a ladder with 4 steps on it ... Okay ... You want to see how many steps there are so you count them all [*i.e., 2 X 4*] ... and maybe you are short and you can't reach that step, so maybe you want to slide these all over ... [*moved the 4 hexagon blocks from the right-hand string to the left-hand string to make a group of 8*] ... and you don't have any steps for this ladder, so you might want to form this one. [*takes more hexagons for the right-hand string*]



I: At that point we have done ... when we put the 4 on each - what part of the question are we doing?

K: The 2 times 4 - so now we've got 8 ... and then we should have ... hmmmm.

I: So we did the 2 times 4 which was 8.



K: Now we have 3 ... [*8 on left string and 3 hexagons on the right string*] ... [long pause] ... I need ... so this one there is 12 - [placed 12 on the right string. Note 12 from 3 times 4] ... and you realize that you have too many ... [*took a third string*] ... so I'm just going to go a make another ladder. ...[*now 3 strings in place, with 8, 12, and 0 hexagons*] ... Oh, 1, 2, 3, 4, 5, 6, 7, 8 ... move these [*the extra 4 of the 12 on the second string moved to the third string*] ... 8 on that ladder, and we need 1, 2, 3, 4, 5, 6, 7, 8 ... [on the third ladder] ... and then you have 24 steps.

I: So what did you start out with?

K: Two times 4.

I: Two times 4, which was on your two ladders with 4 on each, and then you combined that on one ladder, and it was 8 steps, and then you have 3 ladders. Is there anything else that would be completely different?

K: No, not that I can think of.

I: Are there any things on the table that you could not use to help show 2 times 3 times 4 - in any way at all - that just can't be used?

K: The ruler.

I: Why wouldn't you be able to use the ruler? Do you have some reason for that?

K: Umm ... I guess you could if you wanted to draw squares or something - have 2 squares with 3 in each.

I: Oh, I see. You could use the ruler to draw the squares bu you couldn't use the ruler on its own?

K: NO.

I: Is there anything else on the table that you could not use to help explain 2 X 3 X 4?

K: I guess you could use the colour of the felts or something.

I: Could you use ... So you've used the multilink blocks, and these, and you've used all the blocks actually. Could you use ... are there any of these papers here that you could use?

K: Not the dots.

I: You couldn't use the dots. Could you use the line?

K: Not for this.

I: But you could use the squared paper can you?

K: You might be able to use the dots.

I: You <u>might</u> be able to.

K: I tried it but I don't think I could - but I don't know - someone else could.

I: Oh, I see. You're not too sure how it would work, but you think someone maybe could figure it out.

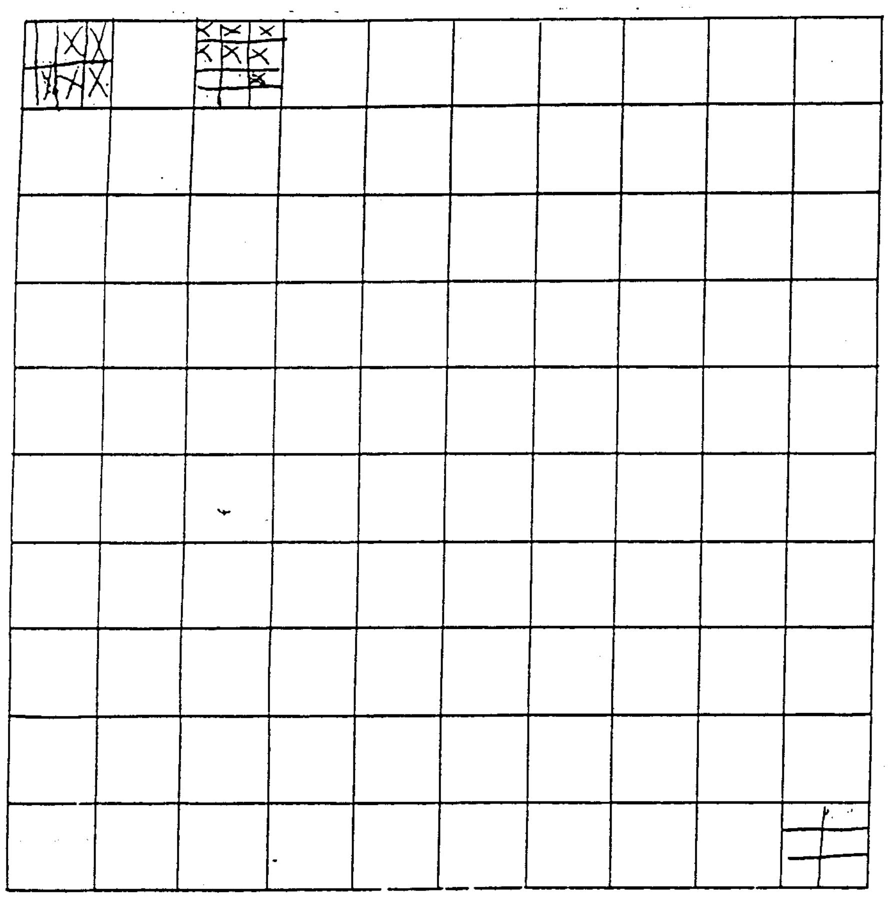
Common Fractions

Dahlia (Grade 7)

Task: Compare 5/8 & 7/12

Interviewer: Now this is a fractions question. What we want you to do is to explain to this imaginary student how you could compare those two fractions and decide whether they are the same amount or whether one would be smaller. Use any of the materials in front of you.

Dahlia: [*took inch square paper, partitioned an inch square in to 6 parts, then halved 2 of the sixths to get 8* parts] ... Okay, we have divided this and this one into eighths. ...Ummm ...[*long pause*]



I: Hmmm. Do you want to start again?

D: [*turns the paper* 180^0, *and begins partitioning the top left-hand square*] ... Okay, that's 3... [*draws 2 vertical lines*] ... six ... [*drew 1 horizontal line, counts, drew second horizontal line, counts*] ... just wait ...[*rubs out partitions, draws 3 vertical and 1 horizontal line*] ... There! and you've eaten 5 of these pieces, and they are all eaten....[*draws a cross in 5 parts*] ... and you take this one ... [*partitions second square, first into ninths with 2 vertical and 2 horizontal lines,* *counts, then added another horizontal line*] ... There's 12 and you've eaten 7 of these ...[*put cross in 7 parts*] ...Okay, and then these pieces are much bigger ...[*indicates the eighths*] ...and these are smaller ...[*indicates twelfths*]... and you've eaten more of that [*twelfths*] and less of that [*eighths*]... and if you put these together, like this one would be more [*5/8 > 7/12*], cause like it is only divided into 8 pieces and this one is 12.

I: So if I ate the 7/12 I would eat more than if I ate the 5/8?

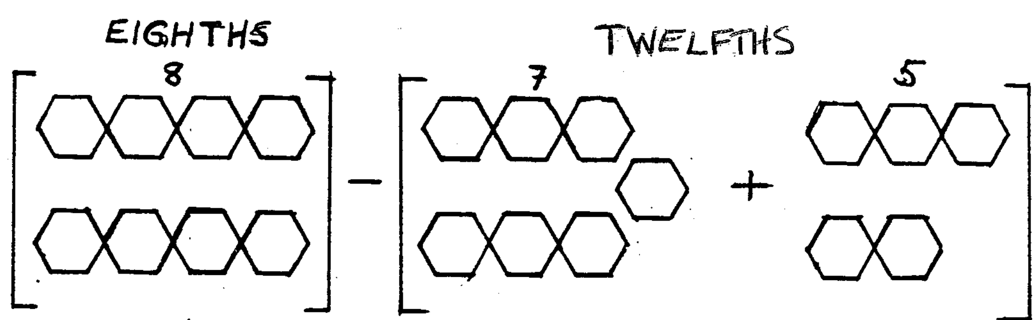
D: Less.

I: I would eat less?

D: Like if you ate all that [*7/12*] it would seem like you would have eaten more, but if you eat this [*5/8*] you've eaten more because the pieces are bigger.

I: Right, I get it. There are a smaller number of pieces but the pieces are bigger, so you actually get more altogether. Sure, that would explain it. Is there any other completely different way of explaining this using any of the materials we have in front of us?

D: [*nods yes*] See if this works. ...[*takes 8 yellow hexagons and forms 2 rows of 4 contiguous units*]... These are this great big pie with all these pieces. Okay ...[*took multilink*]... These are smaller ... Maybe we should, yah! ...[*replaced multilink with 7 more yellow hexagons as 2 rows of 3 and 1 unit beside, counts the 7 units, and adds 5 more to make a group of 12- now has sets of 8 & 12*] ...Okay, remove 5 of these ...[*5 from the set of 8*]... Okay, three- need 7- this one would be ... No, we can't use these.

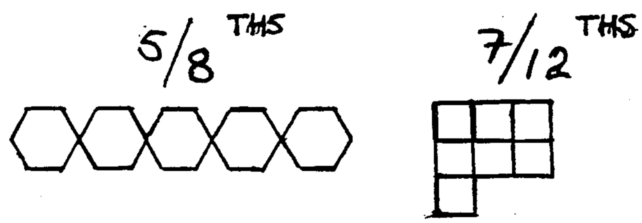


I: Why not?

D: These are the same size as these ...[*i.e., same units for both fractions*]

I: Oh, I see. They have to be a different size, would they?

D: Yup. Okay, well you could use those... [*replaced 12 hexagons for centimetre cubes because they are much smaller*]... Okay, a guy had this one ...[*5 hexagons*]... and I want to eat - he has 7 of this ...[*centimetre cubes*]...Okay, so this guy says, "Oh well, I had 7 pieces, so I had more than this one," but really this guy had more. ...[*moved cubes and hexagons together to compare amounts, and 5 hexagons were much larger in size*]



I: Oh, right, because the pieces are so much bigger.

D: Yes, he was in a family, and he had 5. His Mom and Dad or someone didn't like it. And this guy had 12 in his family, and 7 people in his family didn't like it, so he got all of those. But this is bigger ...[*5/8*]

I: Oh, I get it. Okay, is there anything else that you could do that was completely different, completely different from what you have just explained.

D: No.

I: Are there any things on the table that you could not use to explain the fact that 5/8 > 7/12?

D: The string, the ruler, and the line. That's about it I would think.

I: You could use the circle and the dotted paper?

D: Yes, because it would be like a cake.

I: Oh, right. And what about with the dotted paper?

D: I don't think so.

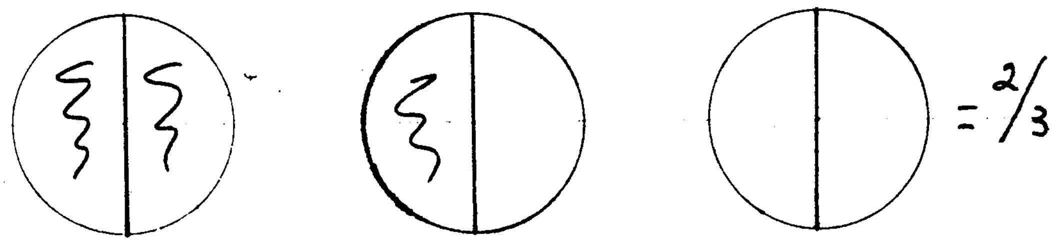
I: Not at all?

D: No.

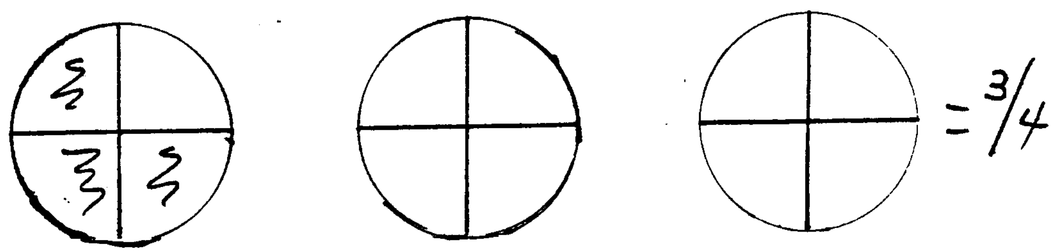
Task: Compare 2/3 & 3/4

I: This is another fractions one, okay. Supposing you wanted to explain to a student how you would compare those two fractions, two-thirds and three quarters. What would be most useful on the table to help explain it.

D: I'll try this ...[*took large circular filter papers*]... Okay, two thirds, that would be - you would have 3 of these all divided into 2 ...[*3 filter papers halved with a vertical line in each*]... Okay, and you have eaten 3 pieces. You've eaten that, and that, and that. So you have eaten a whole cake and a half of one ...[*shaded 3 halves*]... So that would be one half, okay ... and then you'd have a whole one ...[*referring to the amount left over*].



Two-thirds means "3 of these all divided into 2."



Three quarters means "3 cakes and you divide it (each) into 4."

I: That was for explaining the two-thirds?

D: Yah. [*labels each one*] ...and the next one would be - you'd still have 3 cakes ...[*took 3 more filter papers*]... and divide it into 4 ...[*partitioned each filter paper into fourths*]... and then you'd eat 3 pieces. So, you you've eaten that, and that, and that. And this guy has 4 people the next day- and they don't like it because they have to share the cake. And then the people don't like it so they give it to this fellow ...[*i.e., gave 3 pieces of fourths to one person because the others did not like the cake*]... and then this guy is going, "Oh good, I got three pieces! I think I got more of the thing." In fact, this guy [*2/3*] got more because it is only divided into 2, and that's 4. So this guy would only eat half of what he ate, probably. ...[*Her representation of 1 1/2 filter papers = 2/3, and 3/4ths of a filter paper = 3/4 is now compared directly. 3/4ths is half of the representation of 2/3*]

I: Yah, he would eat only half. So the person who ate 2/3 would eat more than the person who ate 3/4.

D: yah.

I: Is there anything else that you could do to explain which is bigger, 3/4 or 2/3. Anything else that you could use of the table to help explain that to a student. Say that they were having difficulties.

D: Well ... not really.

I: Not really. Okay. What things could you not use?

D: Sting, line, that's about all.

I: So you could use the dots?

D: No.

I: Square paper?

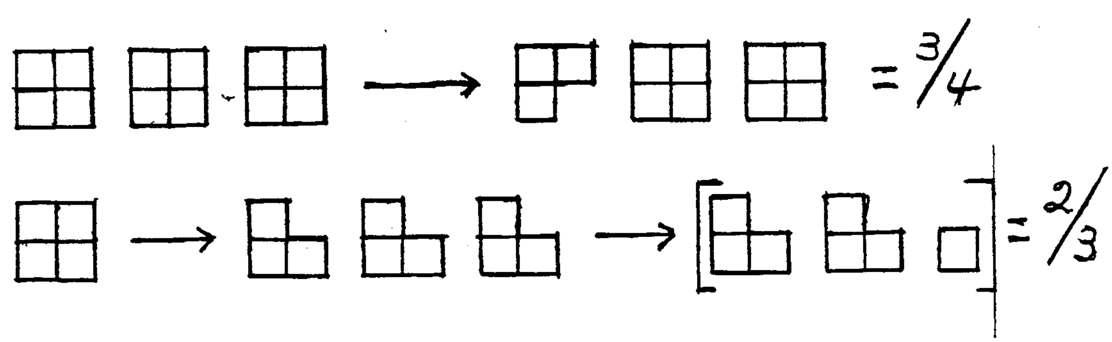
D: Yes.

I: What about the counters?

D: Could use that. [*point to the multilink blocks*]

I: How would you use those ones?

D: Okay, this is going to be a cake ...[*built 3 squares, each with 4 contiguous units*]... here, this guy has eaten 2 pieces - it's divided in - well ... let's do this one ...[*points to 3/4*]... It is divided into 4 and this guy has eaten 3 ...[*removed 1/4 from one of the 3 squares, leaving the 3 that were eaten*]... And then for that one [2/3] it would be divided into 3 ...[*built 3 three-link regions*]... But this would be bigger I think ...[*comparing the 3 unit region to the 4 unit region, dissatisfied with the size of the 3 unit region*]... the cake, since it's the same size divided into 3, it would be ... and this guy has eaten 2 of them [*removed 2 links from 1 of the regions, leaving 2 regions with 3 links and 1 additional link*]... But the cake would be bigger ...[*means the 3 links should be the same size as the 4 link*]... and then he's, he'd still have one piece left, like the other guy - The first guy got more since the cake was bigger... or the same size divided in half, or yah ...



I: So when you did the 2/3 you divided it into bigger pieces and the 3/4ths into smaller pieces?

D: Yah.

I: And so, if you had 1/3 left and 1/4 left, which would have eaten the most cake.

D: Two-thirds.

I: The two-thirds guy?

D: 'cause, say this had 3 pieces and you had 2 people, and they're really crazy over cakes so they take half a cake each. -and then there is 4 people and only, like, 3 cakes, and they have to divide it equally, so they just get smaller pieces from a lot of cakes.

I: Oh, I see. So what you've got is 3 cakes being shared among 4 people - with the 3 quarters.

D: Aha.

I: And the other one is?

D: 3 cakes divided by 2.

I: Divided by 2 people - aha, well you would get more wouldn't you.

James Grade 5

Task: Explain 3/4

Interviewer: Is there any way that you could use the things on the table to explain what 3 quarters means?

James: Hmmm. ...[*long pause*]... Hmmm ...[*long pause*]...

I: Is there anything at all that you could do to explain what 3 quarters means?

J: Not that I know of.

I: Not that you know of. Are there any kind of even rough ideas that you can think of that could explain what 3 quarters mean?

J: Hmmmm.

I: I know that you haven't been taught this yet, but I was just wondering if you had any ideas about it?

J: [*pause*] ... Hmmmm ...[*long pause*]

Task: Explain 1/2

I: No? Okay. Is there anything that you could use on the table that would explain what a half means.

J: Well, um ... two of these and a half of ...um ... a half of 2 is one. [*took two yellow hexagons*]

I: Aha. That would, wouldn't it. Is there anything else that you could do that was different that would explain what a half means - using anything on the table.

J: They'd be similar.

I: What things on the table - or are there things on the table that would not be useful for explaining what a half means?

J: The ruler, unless you count and use the lines.

I: You could use the lines to show a half, could you? How would you do that? Could you show me?

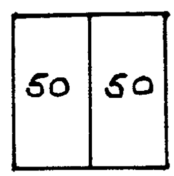
J: Well, you count 2, and what's a half of 2? One.

I: Oh, I see. Sure, that would explain it. Is there anything else on the table that you might not be able to use to explain a half?

J: [*pause*] ... Hmmm ...[*long pause*]... These, because you can't split them in half ...[*a flat of the base 10 blocks*]... like one side won't be even.

I: You couldn't?

J: No, you could split it. That would be even. ...[*partition the flat vertically with 50 on each side*]

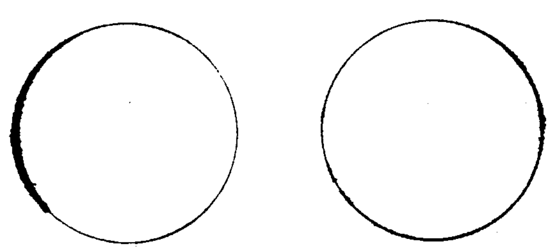


I: Right, that would show it, wouldn't it. What about the line or the round circles?

J: [*Nods yes*]

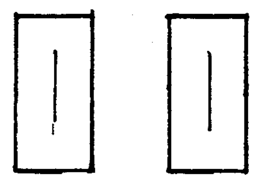
I: How would you use the round circles?

J: [*took 2 filter papers*] Use them like that and one is half.



I: Oh, I see. So, it would be similar to what you have already done. What about the line? Could you ...

J: The same. Take 2 of these and one is one half. [*i.e., two sheets of paper, each with a blank line on it*]



Task: Explain 1/3

I: All right. What about if we wanted to explain to somebody what one-third means.

J: Hmmm ... Hmmm ...[*long pause*]

I: What does one-third mean? Could you tell me?

J: What's 1 out of 3.

I: Oh. Well is there no way of showing that with anything on the table?

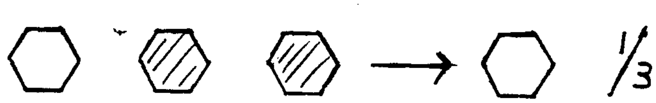
J: There is, but I don't know.

I: Can you show me 3?

J: [*took 3 yellow hexagons*]

I: Okay, what is one-third of 3?

J: [*put out 3 hexagons, and removed 2 to leave 1 as 1/3*] That's one third.



I: Aha. That would show it.

J: Hmmm. All of the rest would be the same.

I: All the rest would be the same.

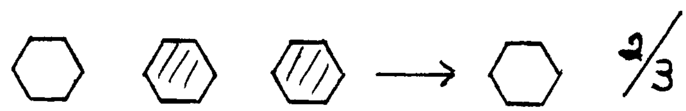
Task: Explain 2/3

I: What if we wanted to show somebody that fraction, two-thirds.

J: [*took 3 hexagons in a pile, lifted one and then 2 hexagons*]... Two-thirds, I don't know ...[*replaced all hexagons on the pile*]

I: What were you thinking of just then?

J: I was just thinking that it would be the same as before. Just take 2 away. [*pile of 3 hexagons, remove 2 hexagons, and leave 1 hexagon*]

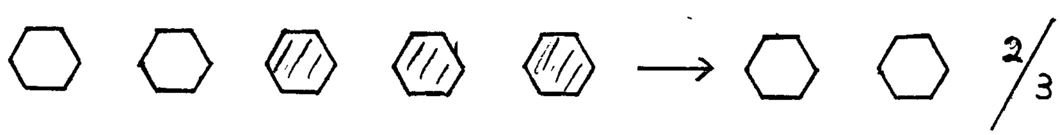


I: And what part would be one ... what would the one that's left be called?

J: One-third.

I: One -third. And how would that show two-thirds?

J: [*took the hexagons, laid out 5 and removed 3*] Five take-away 3 leaves 2.



I: Aha.

J: The rest would be basically the same.

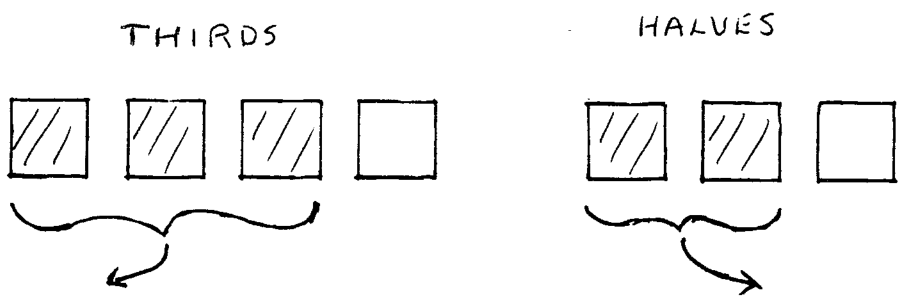
Task: Compare 1/3 & 1/2

I: Okay, now supposing we wanted to compare 2 fractions. Supposing we wanted to compare 1/3 and 1/2, and we wanted to decide which one would be bigger or smaller or whether they would be the same size. Is there anything we could do with the stuff on the table to figure that one out - to show how to figure it out?

J: [*long pause*]... They'd be the same.

I: Aha. How would you explain that to the student that they would be the same.

J: Well, we take 4 for this one [1/3], and 3 for this one [1/2], and I take 2 of these ...[*2 from the set of 3 to represent 1/2*]...and I take 3 of these ...[*3 from the set of 4 to represent 1/3*]



I: And then what you have left is the same. Aha. That would explain it. Would there be anything that you could do with things on the table that would be completely different to explain that same thing?

J: No. They'd be all the same.

I: Okay. Would you be able to use the squared paper, or the dotted paper, or the line?

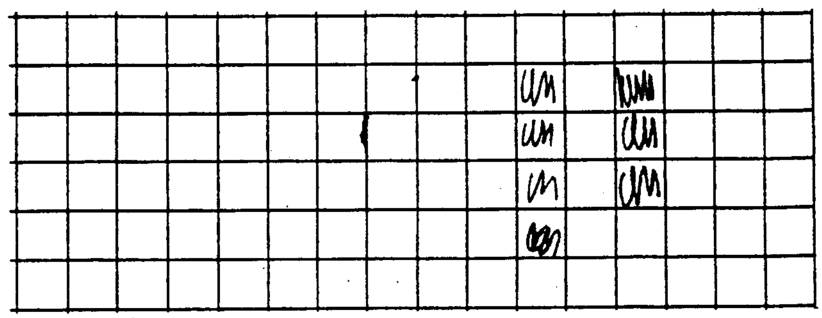
J: You could use the line if you had a whole bunch of these.

I: If you had a whole bunch of the lines you could. Could you use the squared paper on its own?

J: Yah.

I: What would it look like?

J: Three ... shade in these ...[*shaded a line of 3 squares and a line of 4 squares*]... and tell them to take 3 from this one ...[*covered 3 of the 4 squares with his finger*]... and ...(unintelligible)...



J: [*points to the numeral "1/2" on the card*]

I: In your picture, in the thing you just drew, which one is showing the one half, and which one is showing the one-third?

J: This one is this ...[*1/3 = 4 squares*]... and this is the one half...[*1/2 = 3 squares*]

I: And that's the one half that would explain it. Okay. Would there be anything else that you would use differently on the table, do you think? Or would they be basically the same?

J: Basically the same.

J: And you could use everything else on the table?

J: Aha.

I: Okay.

LINEAR INTERPRETIVE INTERVIEW

Whole Number Multiplication

Tammy (Grade 7)

Task: 2 X 3 X 4 = _

Beginning Diagram 1

Interviewer: The first one I want you to think about is this arithmetic question, 2 times 3 times 4. All right, I want you to think about teaching it to somebody, and the question is whether this is a good beginning diagram for teaching that arithmetic.

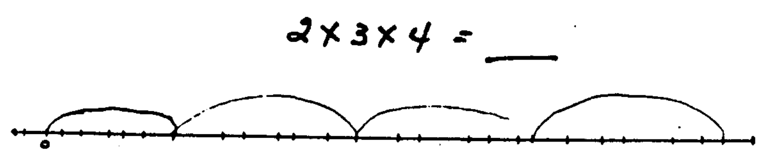
Tammy: Umm. It could be. Yah.

I: Yuh. Okay. How would you use it?

T: You could use it. You could have 2 groups of 3.

I: Could you show me with a pencil?

T: [*counted 1, 2, 3 including the zero marks, then stopped and counted the next 3 marks*]... 1, 2, 3 ...[*drew a jump from the 3rd mark to the zero mark, counted 3 more marks and drew a jump from the 3rd to the 6th*]... and that would be 2 groups of 3, and then you have 6 groups, and then if you said 6 times 4 - then you'd put - that - oh, no - then you'd have to erase ...[*erases the 2 groups of 3*]



I: You have to erase, do you?

T: Yah. To have one group of 6 you have to go up like that ...[*drew a jump from zero to the 6th mark, i.e., 6 spaces*]... That could be one group of 6, another group ...[*counted 6 marks and drew a jump backwards*], another group and another group. Then you leave it and when you are all finished there is 24 spaces.

I: Oh, right! And you're counting the spaces.

Beginning Diagram 2

I: Would this be a good beginning diagram for explaining 2 times 3 times 4?

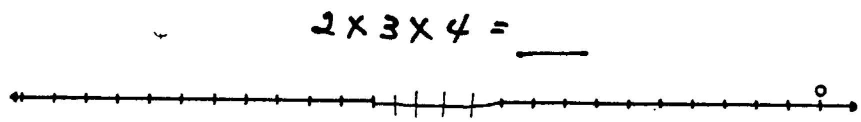
T: Hmmm. It might not be because there might not be enough spaces.

I: Oh, I see. Is there something you might like to do to fix it?

T: Hmmm ...

I: To change?

T: I'll try ...[*drew a connecting line and put marks in the centre*]... now you could probably do it the same way.



Beginning Diagram 3

I: Is this a good beginning diagram?

T: Yes, because it's got spaces- all the spaces. [*ran pencil up and down the line*]

I: Okay.

Beginning Diagram 4

I: Now is this a good beginning diagram for showing 2 times 3 times 4?

T: No, because it would be hard to find the groups, and probably there wouldn't be enough.

I: Okay, so it would be hard to see?

T: Yah, because it's all messy.

Beginning Diagram 5

I: What about this. Is this a good beginning diagram for explaining 2 times 3 times 4?

T: Hmmm. Yah, 'cause it would help them, 'cause you can always go like that. [*showed jumps along the diagram with her pencil*]

I: So you could use this. Would you use it in any way different from what you have already done?

T: Well, I'd have ... hmmm ... I can't find a way.

I: No. It would be the same way.

Task: Compare 5/9 & 2/3

Beginning Diagram 1

Interviewer: Supposing we wanted to teach somebody how you can compare two fractions. We want to be able to compare 5/9 and 2/3 to see whether they are the same size, or whether one is bigger than the other. Is this a good beginning diagram to help explain that to a student?

Tammy: No.

I: No, why not?

T: Because it, you don't really have ninths in there or nines, and you don't really have 3 in there. It is just spaces.

I: It is just a bunch of spaces.

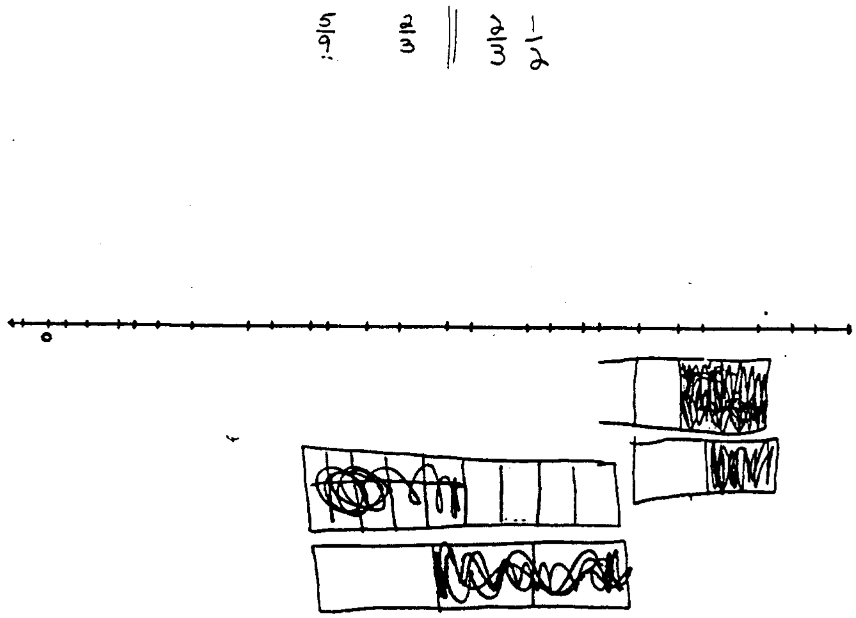
T: It's just a bunch of lines, just lines like that, and it would be hard to figure out nine, and ... really hard.

I: It would be hard. Is it impossible or just ...

T: Just hard.

I: Just hard. How would you do it, at all?

T: Okay, you could ...[*Drew 2 rectangles, partitioned bottom into 3 and the top into 9. Did not align the thirds and ninths*]... Okay, you have 5 ninths ...[*counted 5 from the* *left*]... 5 ninths , and you'd have 2 ...[*shaded 2 from the* *right*]... and this would be a bigger piece [*2/3*].



I: How come?

T: Because, a ... well a fifth is smaller than a half ... a second.

I: A fifth is smaller than a second.

T: Because the smaller they are, the bigger they are. The smaller they are, the bigger amount you have, and the bigger the number, the smaller the amount.

I: Oh, okay. What if you were comparing 2/3 and 1/2?

T: [*Drew 2 rectangles, partitioned one into thirds, the other in half*]... Okay. That has 2 in it and ... This one would be ... I guess it would be the same ... Oh, it's a tich smaller, see.

I: Which one was smaller?

T: This one is bigger [*2/3*].

I: Which one is that?

T: The two-thirds.

I: The two-thirds, but you just said that when the number get smaller, they get bigger?

T: Well, hmmm ... I guess, what it is, is that it depends what the number is.

I: Aha.

T: Like, the bigger number on the bottom maybe ...[points to 9]... makes it smaller, and if it is less, or if it's a smaller number, then it's bigger. I don't know.

I: You don't know. Okay. Now is it possible to explain those things using the line?

T: Yah, but I can't.

I: You can't figure out how to do it. All right.

Beginning Diagram 2

I: What about this. Is this a good beginning diagram for explaining 5 ninths and 2 thirds?

T: No, but it's not impossible. You can do it.

I: You can do it. How would you do it?

T: I don't know, but somebody could find a way.

I: Oh, I see. You can't think of anything, eh?

Beginning Diagram 3

I: What about this one?

T: It would be the same as the others.

I: It would be the same as the others. There would be no way that we could show the 5 ninths and the 2 thirds on this one?

T: I can't think of anything.

Beginning Diagram 4

I: Okay, what about this one? Is this a good beginning diagram?

T: Hmmm ... I don't know [*sighs*]. I can't think of anything.

Beginning Diagram 5

I: What about this one.

T: No. I don't know.

I: No. Okay.

<u>Task</u>: Compare 2/4 & 4/8

<u>Beginning Diagram 1</u>

I: Now, supposing we were just thinking about these 2 fractions here, 2 quarters and 4 eighths. We want to compare these two fractions.

T: They're the same.

I: They're the same, are they. Oh. Is this a good beginning diagram for explaining that?

T: No.

I: Is it possible at all?

T: Aha.

I: It is possible. Can you show me how it is possible?

T: No.

I: How do you know that it is possible then?

T: Because - there - I don't know - but somebody - cause it's got spaces.

I: So you think that because it's got spaces, you think that there should be some way of doing it. Is that what you're saying?

T: Aha.

I: Have you ever seen fractions put on a line like that?

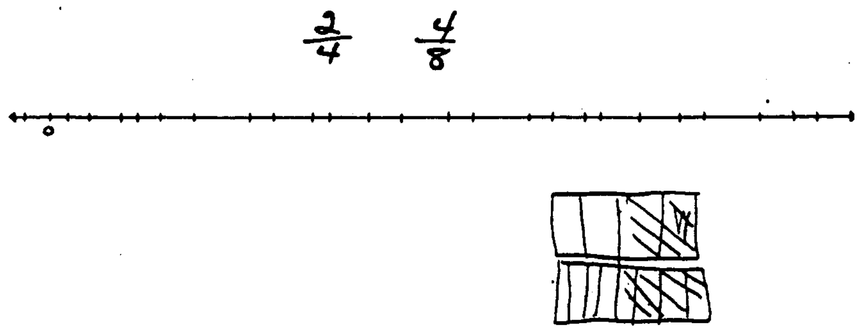
T: [Nods no]

I: No. All right. So, how would you explain that those things are the same? You said they were the same.

T: Well. Well you see, it's like that one. This would just be ... if you were ... Oh no, what was it called? If you were ... making it smaller because when you have 4 eighths you divide this into what ever that will go into to make it ... to make it 2 and then 4 [*referring to dividing the numerator and denominator of 4/8 to make 2/4*], and then if you were going to make it even smaller you'd do 1 half. So it would ... like that's how you sometimes do it in math when you have fractions. 'Cause that's how I learned it in Grade 6.

I: What kind of diagram would you draw to explain that to somebody.

T: [*Drew 2 rectangles, partitioned 4ths right to left, then 8ths right to left. Drew 6 lines for 8ths, counted, then inserted 7th line*]... Okay, 2 ...[*shaded 2 fourths, counted 4 of the 8 and shaded them*]... It's both a half, see.



I: Oh, I see! Sure, all right, that would explain it, wouldn't it.

AREA INTERPRETIVE INTERVIEW

Whole Number Multiplication

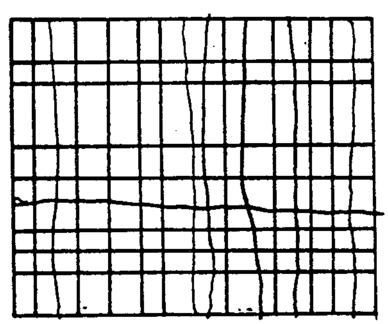
Edwin (Grade 5)

Task: 2 X 3 X 4 = _

Beginning Diagram 1

Interviewer: [*Preamble*] Is this a good beginning diagram for explaining 2 time 3 time 4.

$2 \times 3 \times 4 = \_\_\_$



Edwin: It's not good. You could use it.

I: But it's not a good idea?

E: No.

I: Is there something you would like to change in that diagram to make it into a good beginning diagram?

E: Well, no.

I: No.

E: It's just not a good diagram.

I: Okay. Why is it not a good diagram?

E: Well ... Okay, it's not complete.

I: What would make it complete?

E: If you put all the lines in.

I: Oh, where are we missing lines?

E: Here, here, and here.

I: Oh, I see. How do you know where the lines need to go?

E: Well, see, this and this here, it's a small space.

I: Oh, I see. So we're making them all ... so we are judging which ones go there.

E: Well, it has to be small like that ... small, small, small like that.

I: Okay, all right.

Beginning Diagram 2

I: Now is this a good diagram to show ...?

E: No way! [*Turns the page* 180^0, 90^0, *then back to original position*]

I: Is it impossible, or just not a good idea?

E: It's not a good idea.

I: Not a good idea, but it's not impossible?

E: Not impossible.

I: What's wrong with this one?

E: Well, you could use it.

I: Aha.

E: But it's like this and this - it's all like that ...[*holds hands parallel, with palms opposite, and angles both while keeping them parallel*]... It should be a plain, old square!

I: A plain old square.

E: Yah.

I: Right. Okay.

Beginning Diagram 3

I: Is this a good beginning diagram...?

E: Definitely not!!

I: Definitely not? Is it impossible to use it?

E: Yah, well it's not impossible, but it's too wiggley.

I: And why is being wiggley a problem?

E: Well, like ... well, it's not that big a problem ... but it is.

I: Aha.

E: Well, it should be a square.

I: Aha.

<u>Beginning Diagram 4</u>

I: Is that a good beginning diagram then?

E: Aha.

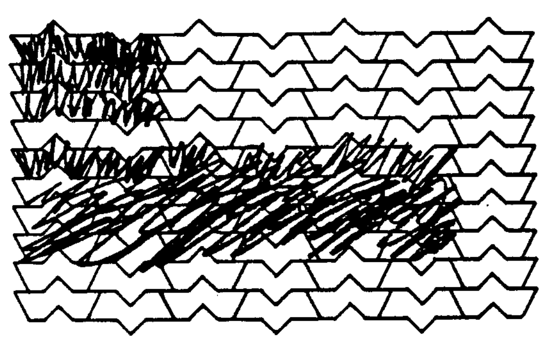
I: That is?

E: Yah. It's just not square, but all these things are nice and orderly.

I: Ah. Nice and orderly. How would you use it if you wanted to explain 2 times 3 times 4?

E: Okay. You have one of these, two right here ...[*shaded 2 in first row*]... 2 here, 2 here, 2 here - now you've got this - 1, 2, 3, 4, 5, 6 - that part right here is 6 ...[*2 by 3 contiguous units*]... okay. Then 6 times 4 ...[*drew around the first row of 6 units, then shaded in all 4 rows of 6*]

$$\overset{6}{(2 \times 3)} \times 4 = \_\_$$



I: Okay, now what have you done here?

E: I've got 1, 2, 3, 4, 5, 6, ... I've got 6 across and 4 down ... that's 6 times 4.

I: Oh, I see. That would show it, wouldn't it.

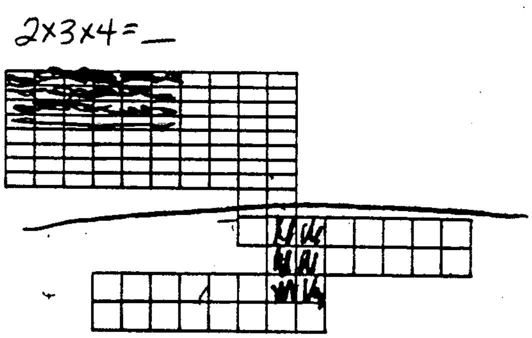
Beginning Diagram 5

I: What about this? Is this a good beginning diagram for explaining 2 times 3 times 4?

E: Nope.

I: No. How come?

E: Well, it should be a square. If it was a square - like this ...[*Indicated the top rectangle of the diagram*]... this would be something for - like - okay, if this thing was in ...[*re-align the bottom parts*]... and if there was a space like - you could have a space right here ...[*Drew a horizontal line between the top part and the rest for a separation*]... You should have a space right here, and this should move in ...[*the two parts at the bottom should move horizontally to form a single rectangle*].



I: OH, You mean this bottom thing should move in, or this thing should move in ...[*the tops part*]?

E: Well, this thing should stay right here ...[*top*]..., and this thing should move in and you should have some more over here ...[*move bottom part horizontally and fill in spaces so that it forms a rectangle*].

I: Oh, I see.

E: Then you could do 6 here [*top*], and 4 here [*bottom*]. Well, you could do 2 times 3 here ...[*shaded in bottom parts*]... 1, 2; 1, 2; 1, 2; 6, and then do 1, 2, 3, 4, 5, 6 ...drew 4 rows of 6 in the top parts] ... There!

I: Oh, I see, sure. So you could do the 2 times 3 down here, and the 4 times 6 up there. There'd be no problem with that?

E: Yah.

I: But you'd still want that all moved about, do you.

E: Aha, yup.

Task: Explain 3 2/3

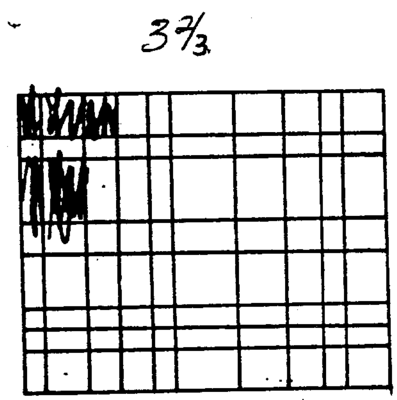
Beginning Diagram 1

I: Supposing you wanted to explain to somebody what 3 and 2 thirds means. Would this be a good beginning diagram to help explain that?

E: No.

I: No. Why not?

E: Well, it could be if you do ... Okay, 3, 3, 3 ...[*shaded 3 rectangles in the 1st row*]... Okay, you got 1, 2 and there's supposed to be another one right here ...[*shaded 2 in the 3rd row, and pointed to the 3rd blank rectangle*]



I: Aha.

E: But, somebody took it away.

I: So this, what is this showing? [*indicated the 2 in the 3rd row*]

E: Well, this is showing 2 in thirds.

I: Okay, and what is this thing up here? [*indicated the 3 in the 1st row*]

E: Three.

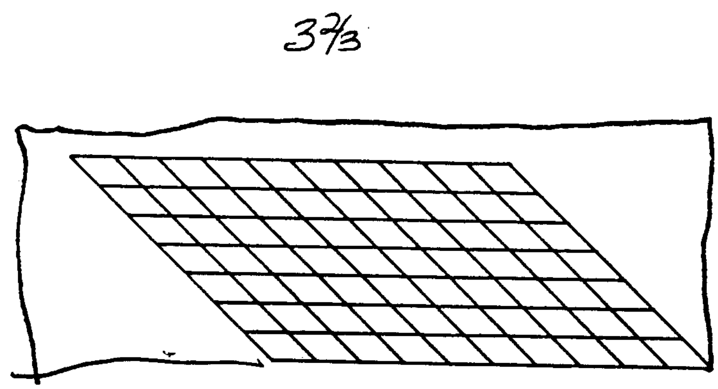
Beginning Diagram 2

I: Is this a good beginning diagram for explaining 3 and 2 thirds?

E: Well, you could- but no.

I: You could- but no.

E: No, it should be some ...[*drew a rectangle around the outside of the rhombus*]



I: You want it to be like that do you?

E: Yah.

Beginning Diagram 3

I: What about this one?

E: No way!

I: No way?

E: No way!

I: Is it impossible, or just not a good idea?

E: Just not a good idea.

I: But it's a really bad idea.

E: A really bad idea. It's woogley.

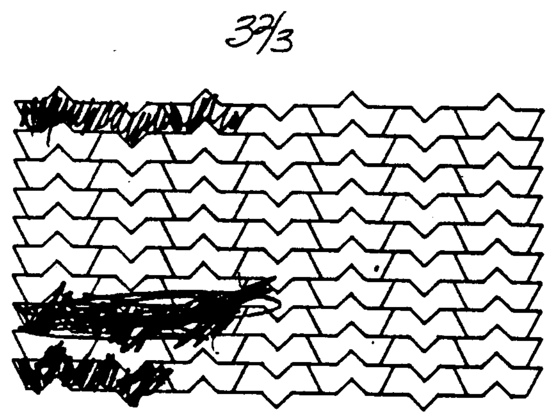
Beginning Diagram 4

I: What about this?

E: It's good!

I: That's good.

E: [*shades 3 in the top row*]... Bottom is showing the 2 thirds. Top is showing the 3. ...[*shades 2 in bottom row and 3 in two rows above*]... and this, the 3 ...[*shading the 2nd row of 3 near bottom*]... and then it goes off, okay. First you have 3 and a third.



I: Where is that?

E: Right here, this is 3 ...[*indicates the top row of 3*]... and a third ...[*indicates the row of 3 near the bottom*]... First you have 3 and a third, and then somebody takes one off you ...[*scribbles on the 3rd unit in the row of 3 near the bottom*]... and here is 2 of the 3 ...[*indicates the row of 2 at the bottom*]....

I: Oh, 3 thirds.

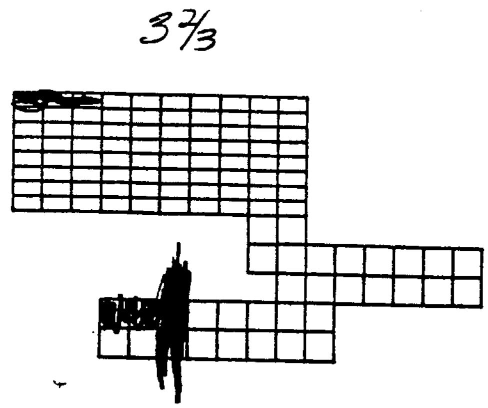
E: Yah.

I: ...and someone takes on off of you, and then you have 2 thirds left.

Beginning Diagram 5

I: Is this a good beginning diagram for showing 3 and 2 thirds?

E: Yup ...[*shaded 2 in the* cm^2 *at bottom*]... that would be ... Here's the 3 first ...[*shaded row of 3 in top rectangle*]... Right here, okay, you have 3 thirds ...[*shaded 3rd* cm^2 *at bottom*]...Okay, someone takes one off you, and you have ... There! Like that!



I: Hmmm ... Which is bigger, 2 thirds or 3?

E: Well, 3 thirds is bigger than 3 two's.

I: Three thirds is bigger than 2 thirds?

E: Yah.

I: Is 3 bigger than 3 thirds?

E: Yah, 3 is bigger than 3 thirds.

I: Is it? Oh, how come it looks so small in the picture?

E: What?

I: Your 3.

E: Well, it's just a different space...[pause]

I: Aha.

E: 'Cause this number right here is first so I should put it up here.

I: Oh, I see.

E: ... and this is second, so I put it down there.

I: So you put it down there.

E: Yup.

I: Oh, I see. Could you draw me a diagram to show how much bigger 3 is compared to 2 thirds?

E: No, I can't.

I: No? Okay.

<u>Task</u>: Explain 5/6

<u>Beginning Diagram 1</u>

I: Supposing we wanted to explain what 5/6 means. Could we use that?

E: Well, it's not that good. See the spaces.

I: The spaces what?

E: Well, the spaces are too wide ...[*indicates the 3rd row down*]

I: Oh.

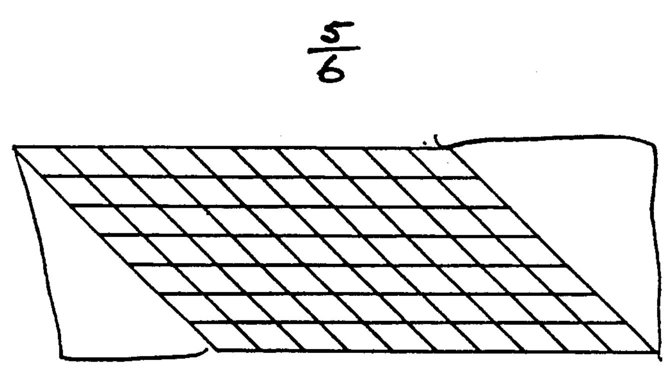
E: This kind.

I: Oh, too wide

<u>Beginning Diagram 2</u>

I: What about this one? Could you explain what 5/6 means with this?

E: No, it should be a square.



I: Should be a square, okay.

E: Or a rectangle.

Beginning Diagram 3

E: Too wiggley for that.

I: Too wiggley for that one.

Beginning Diagram 4

I: What about this one? Could we explain what 5/6 is?

E: Well - no, I can't.

I: No, what would you like to have in order to explain what 5/6 means?

E: A blank piece of paper ...[*turns paper over*].

Beginning Diagram 5

I: Just a minute, let's try this one first.

E: No.

I: Is it impossible to explain what 5/6 is using this?

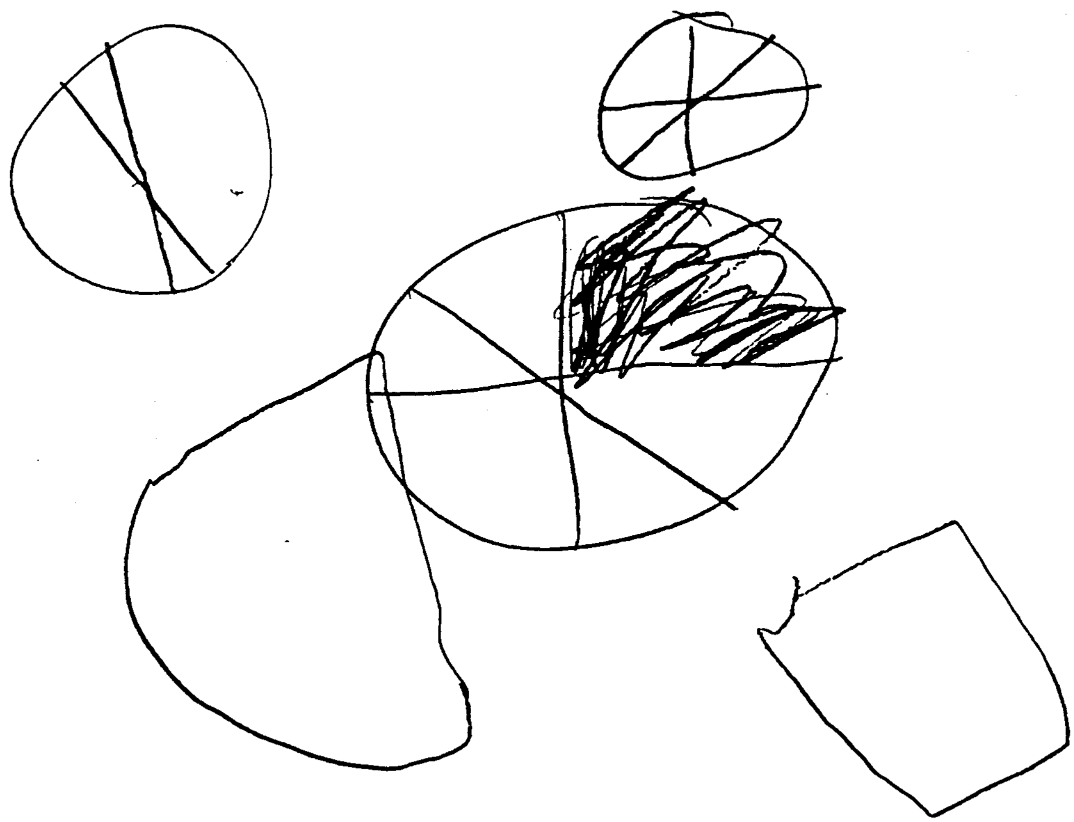
E: Well, I can't.

I: Okay.

<u>Blank Paper</u>

I: What blank piece of paper would you like?

E: Okay. ...[drew a large circle in the middle of the paper]... Mmmm ... I can't remember how to do it. ...[*partitioned into 4 with a vertical and horizontal line*]... the order. [*drew a diagonal to form 4/8ths and 2/4ths*]... Well, okay. It was a pie and somebody took one pieces of it - and he started to eat - started eating a piece of pie ...[*began drawing a semi-circle at the bottom left*]...



I: Well, if I drew this pie, and took one piece off does that show 5/6 [*referencing the first partitioned circle*] ?

E: [*drew and partitioned circle to the top right*]... Nope ...[*completed semi-circle to the left*]

I: Oh, you said you just have a pie and you take one piece off.

E: Well.

I: So, what else do you have to do?

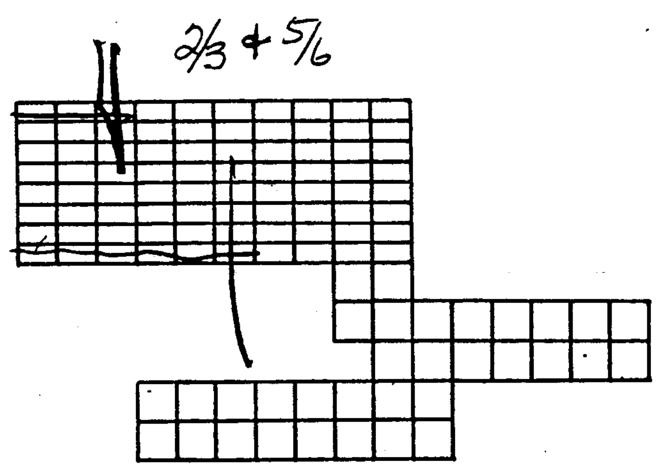
E: Well, you have to make all the other lines.

I: How many do you have to make?

E: Well, you have to make 6 lines - like that, that, and that ...[*uttered "that" for each diagonal drawn in the circle to the top right*]

I: Yah, okay. ...[*pause*]... Okay, we'll come back to this thing. How did you show me two-thirds on that diagram? [Diagram 5]

E: Well, I should go like this. Okay, you have there ...[*drew a line through 3 boxes in a row*]... and you take one off, one of that line pieces off.



I: Which part shows you two-thirds altogether?

E: That.

I: But you still can't show 5/6 on this diagram. eh?

E: Oh, yah!! You could ...[*drew a line through 6 boxes in a row*]... and just take one off.

I: Oh, oh my goodness me.

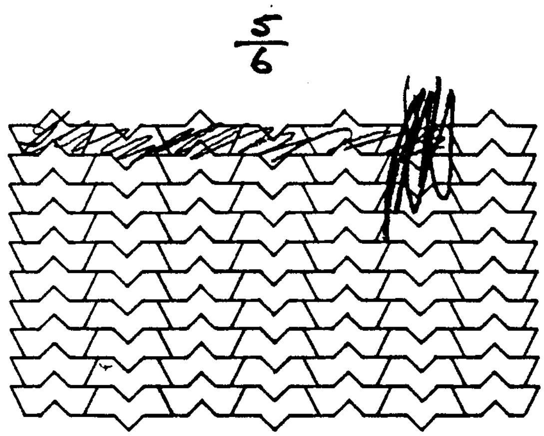
E: And you could do it with that one ...[*4th beginning diagram*]

I: So, you can do it with this one too?

E: Yah.

I: The same sort of thing, would you do?

E: Aha...[*shaded 6*]... and you take one off.



I: Okay.

E: But the other ones, you couldn't. ...[*1st, 2nd, & 3rd beginning diagrams*].

<u>Task</u>: Compare 2/4 & 4/8

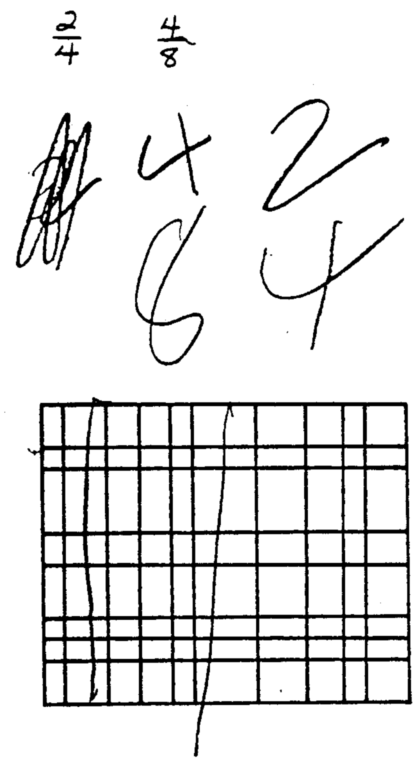
<u>Beginning Diagram 1</u>

Interviewer: Supposing you wanted to compare these two fractions, 2 fourths and 4 eighths. You want to compare them to see is they're the same size, or if one is bigger than the other.

E: Well, 4/8 is much bigger.

I: It's much bigger is it?

E: See - 4 is bigger than 2, and 8 is bigger than 4 ...[*wrote each numerator on the paper*]



I: Yah.

E: Okay, so this is much bigger [*4/8*].

I: That's much bigger.

E: Yup.

I: Okay, is that a good beginning diagram to use to explain that?

E: No

I: Why not?

E: Well, there should be no gaps ...[*drew vertical lines to reduce the width of some columns*]

I: There should be no gaps, okay.

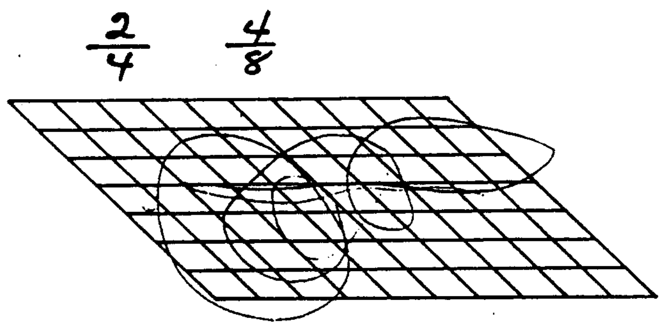
Beginning Diagram 2

I: What about this, is this a good beginning diagram ...

E: No.

I: No? What's wrong?

E: Well, this thing should be a square or a rectangle. Anything but that.



I: Anything but that. Okay.

Beginning Diagram 3

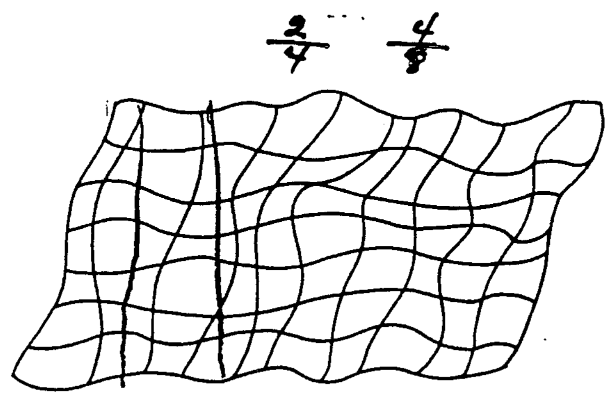
E: Wiggley, much too wiggley.

I: Much too wiggley. Is it impossible?

E: Well ... no ... the lines.

I: But, not a good idea.

E: Well, if it's like this, the lines should be straight, straight square.



I: Okay.

Beginning Diagram 4

I: What about this one. Is this a good beginning diagram ....

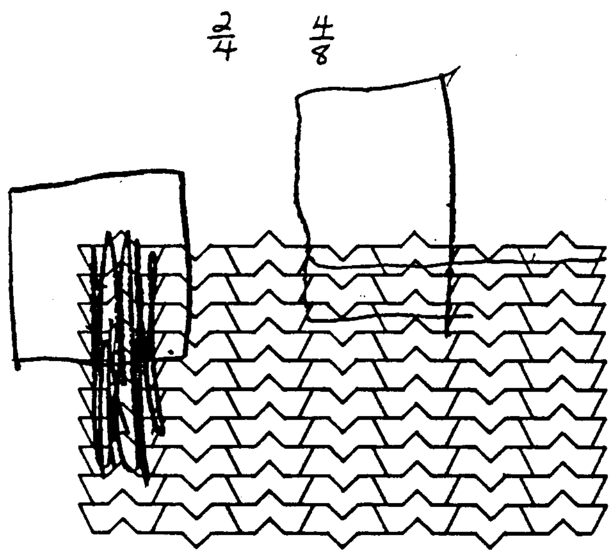
E: 1, 2, 3, 4, 5, 6, 7, ...[*counts regions along first row*]... No.

I: No. How come.

E: Not enough room.

I: Not enough room? There's no way to show the 8 except across there?

E: Yah, if you go down. 1, 2, 3, 4, 5, 6, 7, hmm..8. [*counted down first column*] I've got room here. ...[*shaded 8 contiguous regions*]... Take-away 4 here ...[*drew rectangle around top 8 regions*]... You give 4 to your friend, so you've got 4. There's the 4 out of 8 ... and ... that ...[*drew a line through 4 in a row*]...Give 2 to your friend ...[*completed a rectangle of 2 row of 2 contiguous regions*]



I: So, where is the 2 fourths, and where are the 4 eighths?

E: This is the 2 fourths ...[*2 units in the rectangle*]... here's the 2 right here, and here's the 8 through here, and here's the four ...[*4 units in the large rectangle*]

I: I see. So you are saying that ... which one is bigger?

E: Definitely this one [*4/8*].

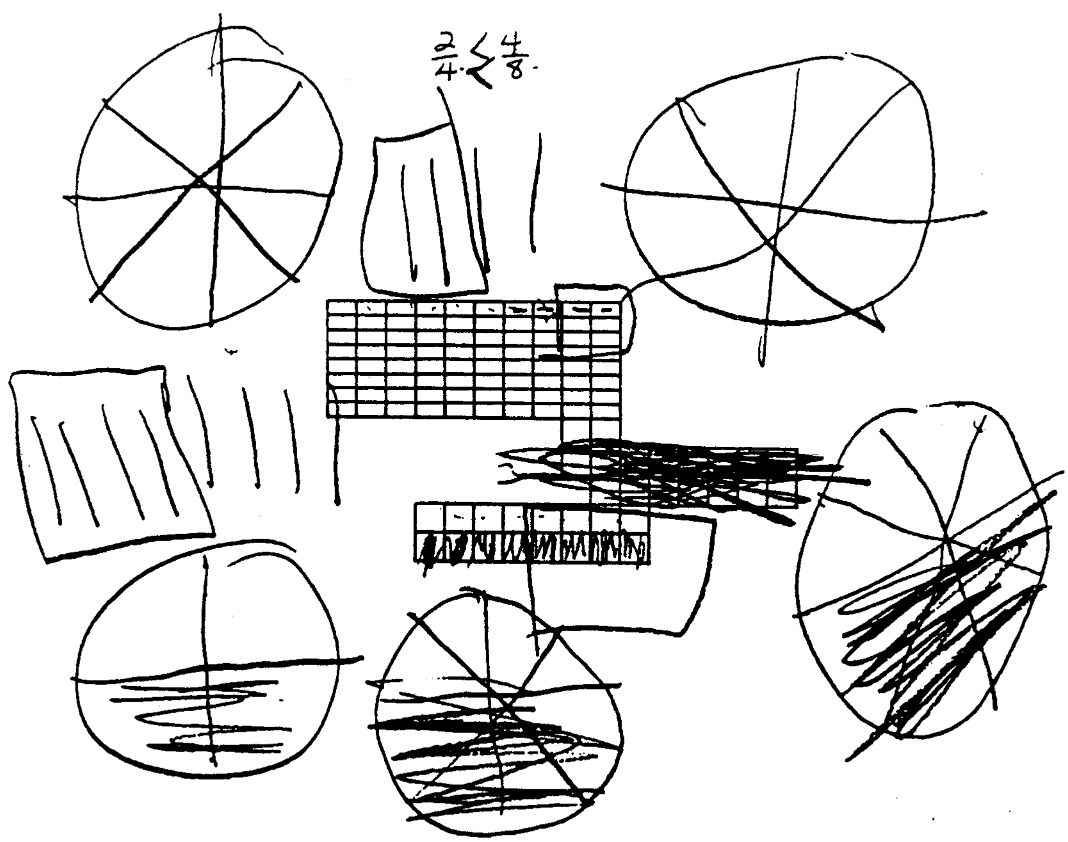
I: Definitely 4/8, all right.

<u>Beginning Diagram 5</u>

E: You could do it.

I: You could do that one?

E: Well ... Yup. 1, 2, 3, 4, 5, 6, 7, 8 ...[*counts the top row*]... 1, 2, 3, 4, 5, 6, 7, 8 ...[*counts* cm^2 *in the bottom rectangle*] I'll turn this around instead... [*rotated paper* 180^0] ... This way, I'll take this out ...[*scribbles out the centre block*]... Okay, you go ...[*shaded 8* cm^2]... Okay, you got 8 and you lose 4 - and you can't find it, so you've got 4/8.



I: That's 4 eighths, what about 2 fourths?

E: Oh, 2 fourths. Okay, you have 12, ... you got- you got 4 power pucks - like the 4 power pucks in the game, and a packman, and you eat up two of them - so, there's only 2 left [*drew lines in 4 small rectangles, and boxed off 2 of them*]

I: Oh, so 2 fourths is bigger or smaller than 4 eighths?

E: Smaller.

I: How do we know that it is smaller? What are we judging?

E: Well, okay. Anyone knows that this is much bigger ...[*put a "<" sign between 4 and 8*]... and this is much bigger ...[*put a "<" sign between 2 and 4*]... Okay- that - okay, like you start out with

8 - Okay you start - Okay, I remember. Well, okay, I remember - Well, this is a pattern too. It goes 2, 4, and 2, 4, 4, 8 ...[*looking at the numerals in both fractions*]... or ... it's sort of a pattern.

I: It is sort of a pattern, isn't it.

E: Yah.

I: What if I - if I had 2 quarters of a pie, and I had 4 eighths of a pie. Would my 2 quarters of a pie be bigger or smaller that my 4 eighths of a pie?

E: Smaller.

I: Smaller eh.

E: Two, like - okay ...[*drew a circle in the top left hand corner, partitioned it into 8*]... a lousy 8 pieces, okay.

I: Okay. If I have 2 fourths?

E: [*drew a circle in the bottom right-hand corner, partitioned it into 8 again, and shaded 4*]... There, see. ...[*drew circle in the bottom left-hand corner, partitioned it into 4, shaded 2*] You're going to have 4 here, and out of these 4 pieces, they will be bigger pieces ...[*i.e. fourths*]...These will be smaller pieces ...[*eighths*]...[*drew the pie centre bottom, partitioned 8 again, and shaded 4*]...These things would be smaller pieces of pie. You would have smaller pieces of pie. The pie would be the same. Like one pie is like this and another pie is like this ...[*measured circles with thumb & forefinger*]... But you have 8 pieces in one pie, and 4 in the other. The 4 pieces pies would be bigger than that ...[*i.e., 1/4 > 1/8*]...but ...

I: So if I eat 2 fourths of one pie and 4 eighths of the other pie, would I have eaten more when I eat the 4 eighths, or would I have eaten less, or would I have eaten the same amount?

E: The same amount. 'Cause you would have 4 left, and this and this is the same amount ...[*indicate equality of fourths with 4 eighths in diagrams*]

I: But you just told me that 4 eighths is bigger than 2 fourths?

E: It is but ... if the pie is the same it's a quantity. Well you have more of the quantity of 8. Well ... Well, you have 8 pieces but if the pie is the same, and 8 pieces, it really doesn't matter.

I: It doesn't matter.

E: Yah, it could be like ... it could be like 4 in a pie, and 8 in a pie. It doesn't really matter if the pies are the same. That's all.

I: But you could still say that 4 eighths is bigger than 2 fourths?

E: Aha.

I: So, it just depends?

E: Yah, if it's a pie - but if it's different, it would be bigger.

I: It would be bigger?

E: If it's not food.

I: If it's not food?

E: Yah.

I: So what kinds of things would you use to show when it's bigger?

E: Okay, you could use lines ...[*drew 8 lines*]... Okay, take-away 4, cause you use 4 of them to make something. So, you have 4 here. And the 2- 1, 2, 3, 4 ...[*drew 4 lines*]... and then you would take-away 2 because you made something.

I: Oh, I see.

LINEAR MEASUREMENT INTERVIEW

Lolande (Grade 7)

Task: Irregular Path D

Interviewer: Here are some measurement questions. What we have is Ant A and Ant B walking along different paths, and we want to know if one path is longer than the other, or whether they are the same length.

Lolande: [*measured the length of a segment in Path A as distance between 2 fingers, approximated the iteration of some of the segments in Path A, then marked that Path A was longer*]

I: Okay, can you explain to me how you decided that?

L: If you straightened them out, if you straightened them both out, this one [*Path A*] goes down so deep that if you straightened it out, it would be longer than B.

I: Is there any other way that you could figure that out? Any other way that you could do it to double check your decision?

L: You could go 1, 2, 3, 4, 5, 6, 7, 8 ...[*touching line segments in Path A*]... 1, 2, 3, 4, 5, 6, 7, 8 ...[*touching line segments in Path B*]... They're the same! Ha!

I: You've changed you mind have you?

L: Ya.

I: How come?

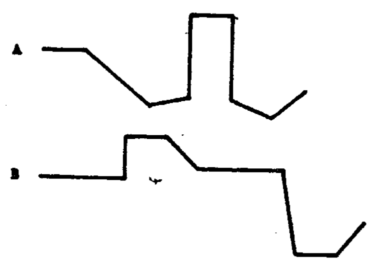
L: I just counted the points, like where they'd be, and they both have 8 ...[*indicated the end-points of the segments, not the beginning-point*].

I: You're sure of that are you?

L: [Nods, yes]

<u>Task</u>: Irregular Path E

L: [*moved on to this question on her own initiative, counts the end-points of Path A, and the beginning-point and end-points of Path B 9i.e., Path A = 8 points, Path B = 9 points). Repeats exactly the same counting process, then marks Path B as longer*]



ANT '<u>A</u>' path is <u>longer</u> ___
ANT '<u>B</u>' path is <u>longer</u> <u>B</u>
They are the <u>same</u> length ___

I: Okay, how did you decide that one?

L: Because I counted the sides again, and this on [Path B] looks so much longer ...[*does a stretching action with her hands*]

I: Show me how you counted the sides. I really wasn't paying close enough attention.

L: 1, 2, 3, 4, 5, 6, 7, 8, 9 ...[*counts the beginning-point and end-points of Path B*]... 1, 2, 3, 4, 5, 6, 7, 8, 9, ...[*counts the beginning-point and end-points of Path A*]... Oh! They are the same! But I think that if you stretch them out the bottom one would be longer.

I: So, does that mean that you can't always tell by counting them?

L: Nope

I: But you figured this one out by counting [*Irregular Path Task D, above*]

L: I don't think it is that. It would be A. [*Path A in the Irregular Path Task D is longer*]

I: And you think that that is a better reason?

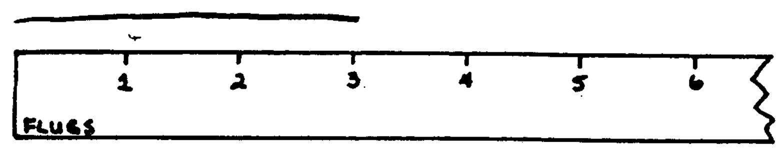
L: Yes.

I: It looks like it would be longer if you stretched in out, doesn't is.

<u>Task</u>: Ruler

I: Now, what we've got here is an old fashion ruler that measures length in flugs. Here is you ruler and it's got flugs marked on it. One flug is the same as 2 centimetres. So one flug is 2 centimetres. They want you to draw a line somewhere in here that would measure 6 centimetres.

L: [*counted 1, 2, 3, marks on ruler, and drew a line from 0 to 3 flugs*]



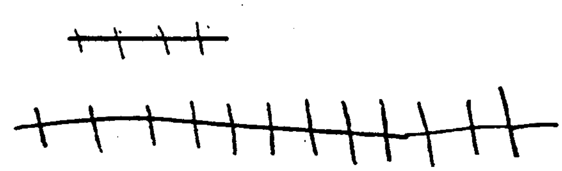
I: Okay, how did you decide that that's where you would draw the line.

L: Because, if one flug is 2 centimetres long, you will go 1, 2; 3, 4; 5, 6 ...[*pointing to the numerals 1, 2, & 3 with each pair counted*].

<u>Task</u>: Aggregate Unit

I: Now in the next question, the line below is 4 units long. It's some kind of unit. It's not centimetres. It's a different unit and the line is 4 units long. Draw another line that is 12 units long.

L: [*drew 4 marks at intervals on the line, drew a long line, positioned marks at intervals, counted the marks, extended the line further, and completed the placement of 12 marks along the line.*]



I: Could you show me how you count the units on this line.

L: 1, 2, 3, 4, ...[*counted the lines*]... I just did lines.

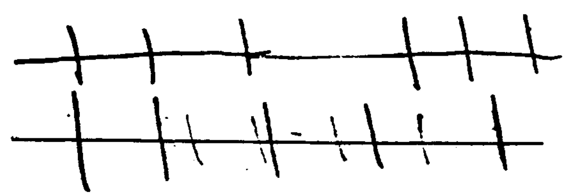
<u>Task</u>: Partitioning

I: In this one here, the path is 5 units long, that's this path here.

L: Aha.

E: And they want you to mark the units on the path.

L: [*placed 5 marks on the left 2 thirds of the path, rubbed them out and replaced the marks more evenly along the path*]



I: Okay, could you show me how you would count those 5 units?

L: 1, 2, 3, 4, 5 ...[*counted each mark*]

I: Now, they want you to draw another line underneath that is 3 units long.

L: [*drew a shorter line below and to the left, placed 3 marks on the line without reference to her 5 units above*]

I: We want it to be the same units as was on this path ...[*indicated the path with 5 units*]

L: Oh, I see.

I: Using this path as your ruler.

L: So do we draw on it?

I: No, just draw it underneath, but 3 units.

L: [*drew a line beneath and to the right, placed 3 marks along the line directly beneath the marks on the line above*]... It will probably be about that, because there are the 3 units ...[*points to the 3 marks on the line above*].

AREA MEASUREMENT INTERVIEW

Kit (Grade 5)

Task: Partition Regions

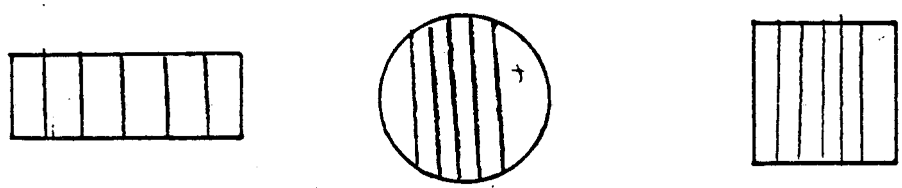
Kit: Show how you would divide each figure below into 6 equal parts. Do we have a ruler?

Interviewer: No.

K: No?

I: Well, you can approximate it, okay.

K: [*Rectangle*] [*took an eraser as a straight-edge, drew vertical lines from right to left*]... *No, five ...[rubbed and adjusted the size of the 6 units*]... There!



I: Okay.

K: [*Circle*] Okay, that - there's this, there's this, and there's 3, there's 4, and there's 5. And they are not even. ...[*drew vertical lines from right to left, rubbed them out and adjusted the placement of the lines*]... 1, 2, 3, 4, 5 ...[*counting the lines drawn*] ...[*pause*] ...

I: And each of those parts will have the same amount?

K: Aha, because these ones are longer than these...[*drew the outer parts to right and left wider than the inner parts to compensate differences in length of the parts*].

K: [*square*] There's 1, 2, 3, 4, 5, ...[*partitioned with vertical lines as with the rectangle*]

I: Okay.

<u>Task</u>: Cake 2

K: Imagine you are going to eat a piece of Cake A and a piece of Cake B. Each piece of cake is shaded. Is one of the pieces of cake bigger than the other, or is each piece of cake the same amount? Oh, okay. Now ... same.

Piece 'A' is bigger ___
Piece 'B' is bigger ___
They are the same amount ___

I: How did you figure that?

K: Well, they are both divided into four, and they are both the same size.

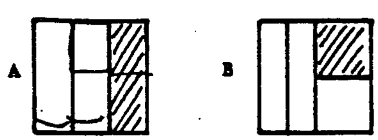
I: Aha, okay.

<u>Task</u>: Cake 3

K: B is bigger.

I: How did you figure that one?

K: Well, these ones are skinnier than this 'cause this one comes up to here on the ruler, and this one is here ...[*measures against eraser the horizontal width of the parts in A and B, hence B < A*]... That's the bigger one twice ...[*still B < A*]...but then this is hummungous ...[*horizontal width of the shaded part in Cake B*]... absolutely huge, and that's how big my piece of cake is ...[*piece in Cake B, implies B > A*]



Piece 'A' is bigger ___
Piece 'B' is bigger ___
They are the same amount ___

I: Aha.

K: But then again, if you break half of this up, here ...[partition piece in Cake A at the vertical mid-point, move bottom half to position contiguous with side of the top half]...you have that, the same! ...[i.e. the transformed part in Cake A is the same as the part in Cake B]

I: It's the same?

K: Yup.

<u>Task</u>: Cake 5

I: What about those ones?

K: [counted the columns across Cake A and Cake B]... If you cut this off, and put it down here and ... just pretend that that's plain, that would be 4, and would be, ... ooooh, ...B is bigger. ...[*partition part of Cake B at the vertical mid-point, transpose top of shaded part to a position contiguous with the side of the other part in B. Then the top half of the 2 columns in B are not shade* ("just pretend that is plain"), *there are then 4 parts in the two columns in Cake B, as there are in the two columns of Cake A*]

I: How did you figure that B was bigger again?

K: Well first of all you have this as your cake ...[*indicated bottom halves of the 2 columns in B*]...what, then this is not shaded in ...[*top halves of the 2 columns in B*]... then you've got, 1, 2, 3, 4, practically 5 sticks left ...[*indicating the vertical parts in Cake B which are now not shaded*]...but if you go over here, you have 1, 2, 3, 4, 5, 6.

I: Six? Where did you get the 6?

K: Okay, well, 1, 2, 3, ...[*counting the vertical parts in the right half of Cake A, which are sixths of the cake*]... and then you divide this ...[*partition the left half of Cake A into another 3 parts,- drew vertical lines to indicate this*]... 1, 2, 3, 4, 5, ...[*full columns*]...and if you divide that, its's 6.

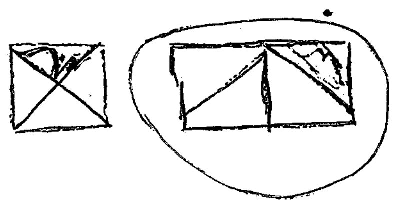
<u>Task</u>: Cake 6

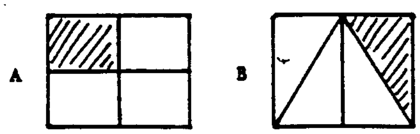
K: Okay, ha! Same amount.

I: How did you figure out that those were the same?

K: They are both divided into 4.

I: Aha. What if I had one like this and then one like this ...[*drew to rectangles of different sized, each partitioned into 4*]?





Piece '**A**' is **bigger** ___
Piece '**B**' is **bigger** ___
They are the **same** amount ___

K: That would not count.

I: It wouldn't? But they are both divided into 4.

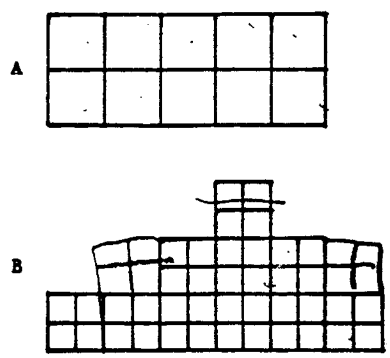
K: Yah, but this one is bigger ...[*circles the larger rectangle*].

I: Oh, I see. Okay.

Task: Tile 5

I: Two playrooms have tile on their floor. Does one playroom have more space to play, or do they have the same amount of space to play?

K: Okay, I'd say this isn't here and it comes there. ...[*assuming that a unit is 4 small squares, moved top unit in Room B to the right-hand space in the second row of units*]... Okay, then this isn't here, and this comes up here. ...[*moves the far-left unit in the bottom row to the left-hand space in the second row*]...and 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ...[*counting units in Room A*]... 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ...[*counting units in Room B*]... the same.

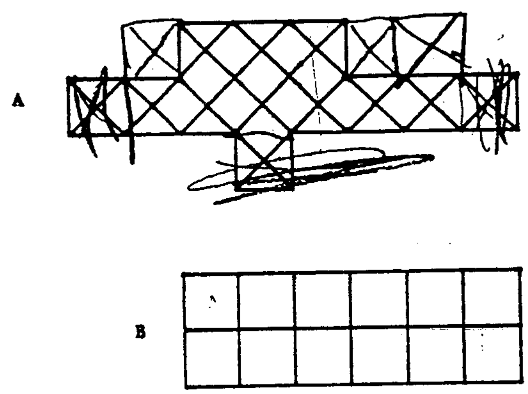


Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ✓

Task: Tile 6

I: What about this one?

K: Okay, this here is going to come up here ...[*assuming 1 unit = 4 triangles, or 1 cm^2 in Room A, move the farthest unit to the left in 2nd row to the left-hand space in the 1st row*]...and this part here is not there any more, then this part is going to come up here ...[*move unit at the far right of the second row to a space immediately to the right of the units in the 1st row*]...then this part is going to come up here ...[*move the units alone at the bottom to the space at the far right of the units in the 1st row*]...and then we are left with 2, 4, 6, 8, 10, 12 ...[*counting units in Room A*]... 2, 4, 6, 8, 10, 12 ...[*counting units in Room B*].



Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ✓

APPENDIX B

MEASUREMENT CONCEPTS TEST

MEASUREMENT CONCEPTS TEST

SCHOOL ______________________ GRADE ______________

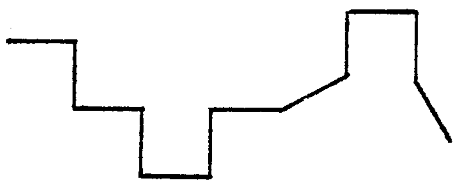
NAME ________________________

TEACHER ______________________

ANT 'A' and ANT 'B' walk along different paths. Their paths are shown below.
Is one path longer than the other? OR
Are the paths the same length?

PUT AN "X" BESIDE YOUR ANSWER FOR EACH PAIR OF PATHS.

A 

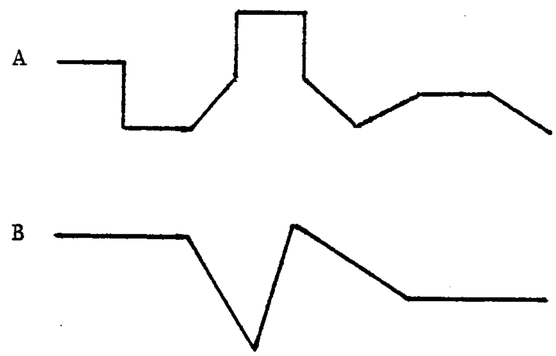
B 

ANT 'A' path is longer ____
ANT 'B' path is longer ____
They are the same length ____

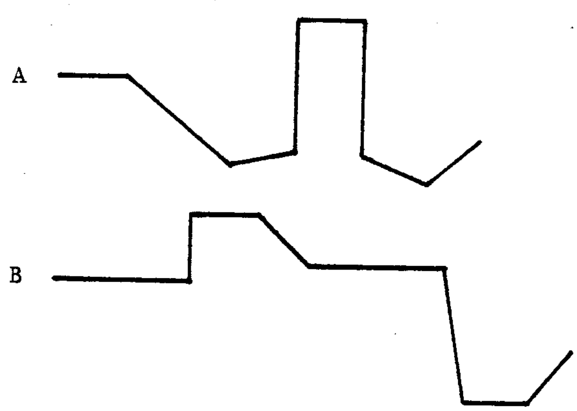
A 

B 

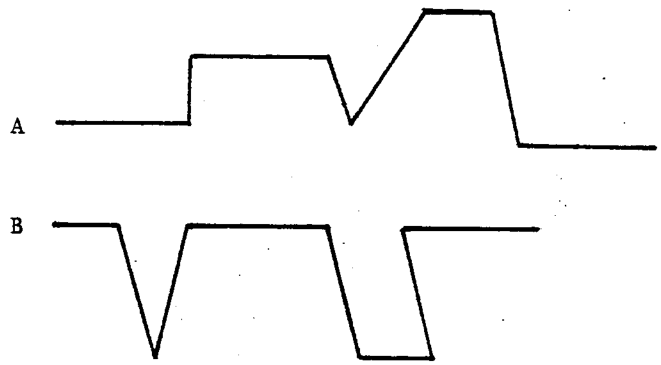
ANT 'A' path is longer ____
ANT 'B' path is longer ____
They are the same length ____



ANT '<u>A</u>' path is longer ____
ANT '<u>B</u>' path is longer ____
They are the same length ____



ANT '<u>A</u>' path is longer ____
ANT '<u>B</u>' path is longer ____
They are the same length ____



ANT 'A' path is longer ____

ANT 'B' path is longer ____

They are the same length ____

This path is six units long.

a) Mark the six units on the path.

b) Draw another path 5 units long.

This path is 5 units long.

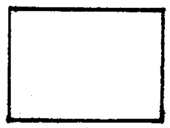
a) Mark the 5 units on the path.

b) Draw another path 3 units long.

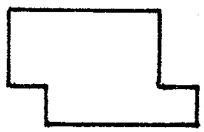
Room 'A' and room 'B' have wall to wall carpeting.
Does one room have more carpet than the other? OR
Do they have the same amount of carpet?

PUT AN "X" BESIDE YOUR ANSWER FOR EACH PAIR OF ROOMS.

A

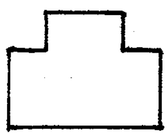


B

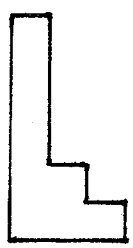


Room 'A' has more carpet ____
Room 'B' has more carpet ____
They have the same amount ____

A

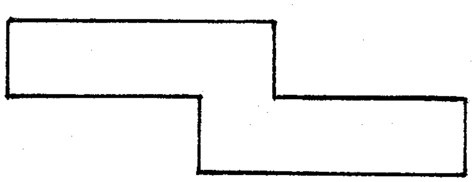


B

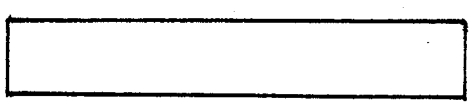


Room 'A' has more carpet ____
Room 'B' has more carpet ____
They have the same amount ____

A

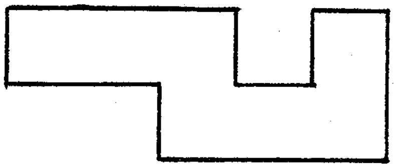


B



Room 'A' has more carpet ____
Room 'B' has more carpet ____
They have the same amount ____

A

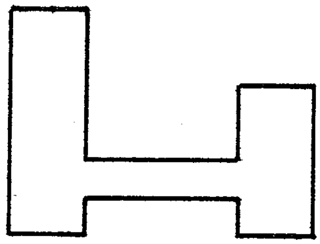


Room '<u>A</u>' has <u>more</u> carpet ____

Room '<u>B</u>' has <u>more</u> carpet ____

They have the <u>same</u> amount ____

B

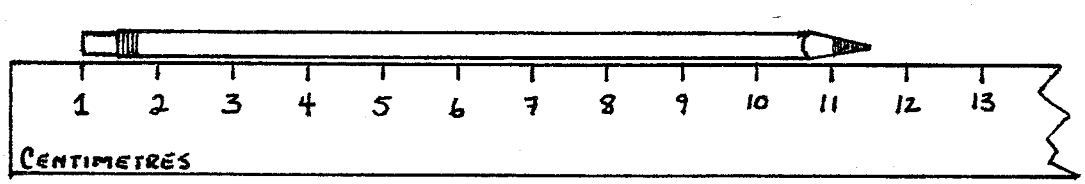


Draw a line that is twice as long as the one shown below.

Draw a line that is 5 times as long as the one shown below.

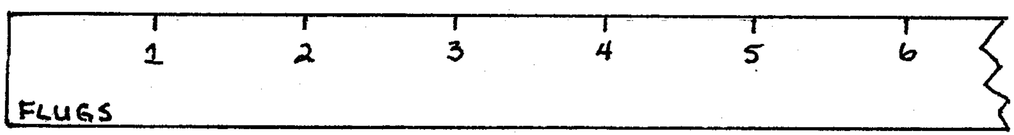
The line below is 4 units long. Draw a line that is 12 units long.

How long is this pencil?



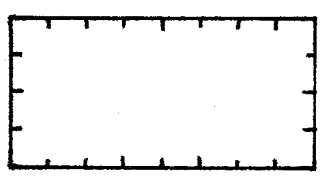
Answer __________________

This ancient ruler measures length in "FLUGS". One "FLUG" is the same as two centimetres. Draw a line above the ruler that is 6 centimetres long.



What is the area of RECTANGLE A?

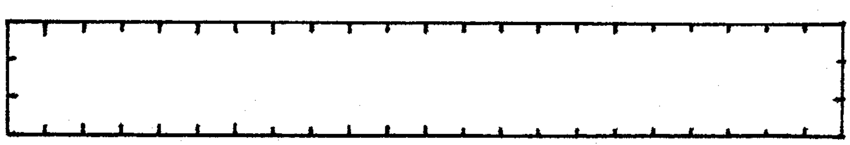
A



Answer __________________

Draw a line to cut this rectangle into two pieces so that one of the pieces has the SAME AREA as RECTANGLE A above.

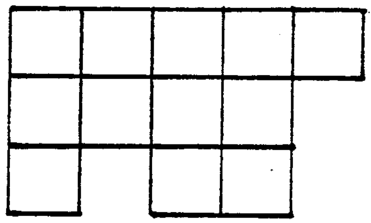
B



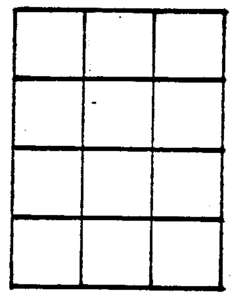
Two playrooms have tiles on their floor.
Does one playroom have more space to play? OR
Do they have the same amount of space to play?

PUT AN 'X' BESIDE YOUR ANSWER FOR EACH PAIR OF PLAYROOMS.

A

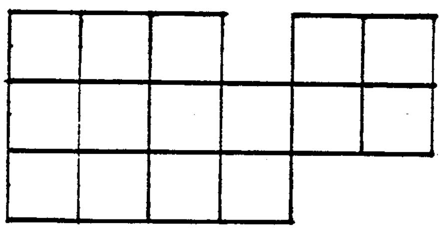


B

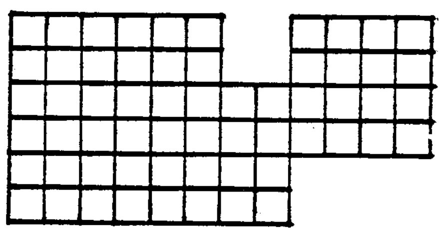


Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ____

A

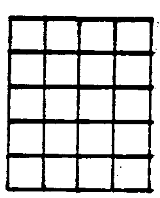


B

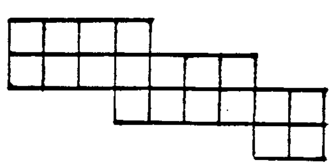


Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ____

A

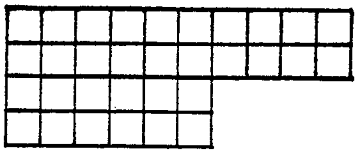


B

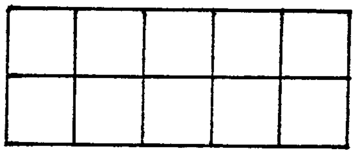


Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ____

A

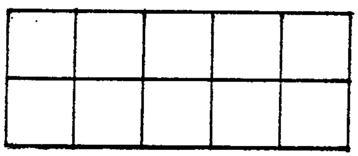


B

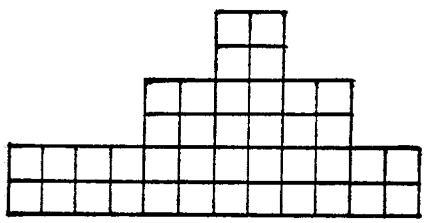


Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ____

A

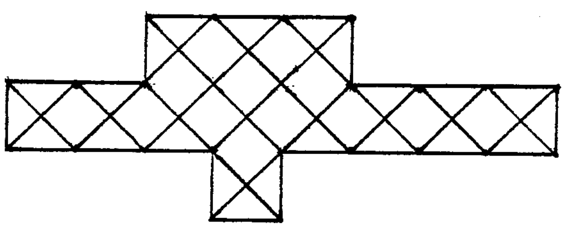


B

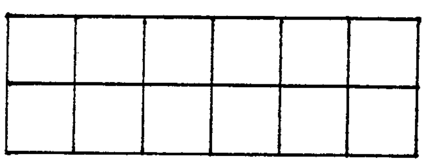


Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ____

A



B



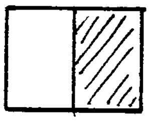
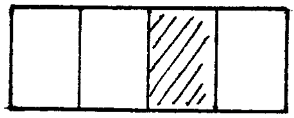
Room 'A' has more space ____
Room 'B' has more space ____
They have the same amount ____

Imagine you are going to eat a piece of cake 'A' and a piece of cake 'B'.
Each piece of cake is shaded.

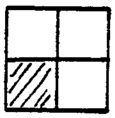
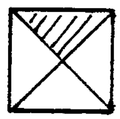
Is one piece of cake bigger than the other? OR

Is each piece the same amount of cake?

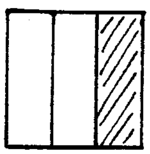
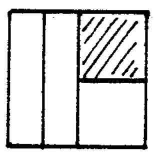
PUT AN 'X' BESIDE YOUR ANSWER FOR EACH QUESTION.

A  B 

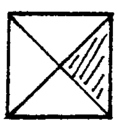
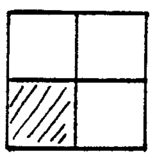
Piece 'A' is bigger ____
Piece 'B' is bigger ____
They are the same amount ____

A  B 

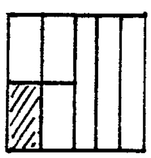
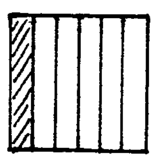
Piece 'A' is bigger ____
Piece 'B' is bigger ____
They are the same amount ____

A  B 

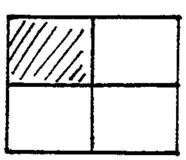
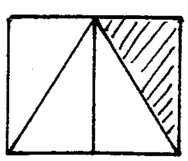
Piece 'A' is bigger ____
Piece 'B' is bigger ____
They are the same amount ____

A  B 

Piece 'A' is bigger ____
Piece 'B' is bigger ____
They are the same amount ____

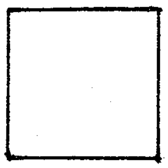
A  B 

Piece 'A' is bigger ____
Piece 'B' is bigger ____
They are the same amount ____

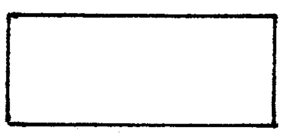
A  B 

Piece 'A' is bigger ____
Piece 'B' is bigger ____
They are the same amount ____

Below is a square figure. Draw another square which is twice as big as the one shown below.



Draw a rectangle like the one shown below, but make it twice as big.



Show how you would divide each figure below into 6 equal parts.