AIRLINE YIELD MANAGEMENT
— A DYNAMIC SEAT ALLOCATION MODEL

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Abstract

Suppose an airplane has a seat capacity of C, we have time T left before the airplane will take off, the fare structure is given, and the arrival process of booking requests is stochastic, we want to know if there is an optimal policy to control booking process in order to maximize total expected revenue from this particular airplane. We formulate the problem as a continuous time Markov decision problem. Under certain conditions the existence and some of the properties of an optimal policy are shown. In the case where the arrival process is a nonhomogenous Poisson it is shown that the optimal policy has a very simple structure and that an ϵ-optimal policy can be easily computed. It is also shown that in general 'Littlewood-type' formula, even being used continuously overtime, does not protect enough seats for full fare passengers and results in less total revenue.

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Chapter 1

Introduction

The deregulation of North American Airlines allows airlines to practice price competition. On the one hand, this helps to stimulate the demand for air travelling because of the resulting proliferation of discount fare booking classes; from managerial point of view it's beneficial to sell those otherwise empty seats at a discount fare. On the other hand, the deregulation challenges airlines with a significant operational problem — the problem of determining an optimal booking policy: a booking policy which allocates optimally the seats of a particular airplane among the various fare classes. In other words, on the one hand, we have the freedom to segment market through price differentiation. On the other hand, we want to control the booking process by selling the right seats to the right passengers at the right prices and time to maximize total revenue(Smith et al. 1992).

Prior work on this problem falls into one of two categories and correspondingly, there are two distinct approaches to the problem. First, those work which attemp network optimality, the corresponding approach incorporates some or all complications, for example, multiple-flight itineraries, cancellations, overbookings, etc., via network flow and mathematical programming(Mayer 1976; Glover et al. 1982; Wollmer 1986; Dror, Trudeau and Ladany 1988). Second, those work which studies the problem in isolated settings, the corresponding approach is based on some restrictive assumptions (Rothstein 1971; Littlewood 1972; Bhatia and Parekh 1973; Richter 1982; Alstrup et al. 1986; Belobaba 1987;

Curry 1990; Wollmer 1990a 1990b; Brumelle and McGill 1991). This latter approach is the approach of this work.

It seems there are problems for the first approach to fully take into consideration the stochastical nature of the problem to achieve global optimization; while the second approach may not produce globally optimal solution it does produce easily implemantable solutions which are optimal under the assumptions they are based.

Those works which fall into the second category used either explicitly or implicitly some or all of the following assumptions:

1. *single flight leg*: Bookings are made on the basis of a single departure and landing.
2. *independent demand*: The demands for the different fare classes are mutually independent.
3. *low fare booking first*: The lowest fare reservations requests arrive first, followed by the next lowest, etc.
4. *no cancellations*: Cancellations, 'no-show' and overbooking are not considered.
5. *limited information*: The decision to close a class is based only on the number of current bookings.
6. *nested classes*: Any fare class can be booked into seats not taken by bookings in lower fare classes.

While assumption 6 is a common practice in airline reservation system today, assumptions 1 through 5 are restrictive. These sometimes overly restrictive assumptions serve the purpose of making the problem tractable.

In 1972, Littlewood considered the case where two classes of fares are offered. He proposed that an airline should continue to reduce the protection level for class 1 (full fare) seats as long as the fare for class 2 (discount) seats satisfy

$$\rho_2 \geq \rho_1 P[X_1 > p_1], \tag{1.1}$$

where ρ_i denotes the fare or average revenue from the ith fare class, $P[.]$ denotes probability, X_1 is full fare demand, and p_1 is the full fare protection level, the number of seats protected for the full fare passengers. The intiution here is clear—accept the immediate return from selling an additional discount seat as long as the discount revenue equals or exceeds the expected full fare revenue from the seat. In 1982, Richter gave a marginal analysis which proved that (1.1) gives an optimal allocation.

Wollmer(1990b), Curry(1990), and Brumelle(1991) extended Littlewood's formula to multiple fare case. In Brumelle's terms, under the Assumption 1 through 6 listed above the optimal protection levels $p_1^*, p_2^*, \ldots$, can be obtained by finding the solutions to the following system of equations

$$\begin{aligned} \rho_2 &= \rho_1 P[X_1 > p_1^*] \\ \rho_3 &= \rho_1 P[X_1 > p_1^* \cap X_1 + X_2 > p_2^*] \\ &\vdots \\ \rho_{k+1} &= \rho_1 P[X_1 > p_1^* \cap X_1 + X_2 > p_2^* \cap \cdots \cap X_1 + X_2 + \cdots + X_k > p_k^*]. \end{aligned} \tag{1.2}$$

where $p_i^*, i = 1, 2, \ldots$, is the optimal protection level, the optimal number of seats protected, for fare class 1 through i. $X_i, i = 1, 2, \ldots$, is the demands of ith class booking requests. Note that the first of these equations is just Littlewood's formula expressed as an equation. We will call (1.2) Littlewood's type formula.

Those works which fall into second category have another thing in common: they consider aggregate demands. In many cases, this is equivalent to eliminating time dimension from the problem.

In practice, airlines realize that the Assumptions on which Littlewood's formula is based do not always hold, especially the assumption that low fare demand occurs before high fare demand. To compensate for the consequence of the failure of this assumption, airlines use Littlewood's formula repeatedly during a booking period. However, Littlewood's formula, even being used continuously over a booking period, does not protect enough seats for higher fare passengers, and results in less total expected revenue. Intuitively, the assumption that low fare books first is equivalent to closing discount fare booking permanently when full fare booking begins. So Littlewood's formula 'makes sure' to accept enough discount fare passengers before closing the discount fare class, and in doing so it leaves too few seats to full fare passengers.

This thesis shows that a model which takes the time remaining until departure and which allows fare classes to reopen after being closed is computationally feasible. In comparison to Littlewood's formula, our model results considerable improvement in revenue per flight.

This work resembles the work of Rothstein(1971) and Alstrup et al.(1986) in the sense that the problem is formulated as a nonhomogenous Markovian sequential decision process. Rothstein considered one class of passengers and Alstrup considered two classes of passengers. Our model accomodates finite number of classes of passengers without conceptual or computational difficulty. Both Rothstein and Alstrup discretized time dimension on daily basis; hence to actually compute an optimal policy they aggregate booking requests on daily basis as well. The discretization is practical but from the viewpoint of modelling the problem it is somewhat arbitrary. We treat time as a continuum and prove the existence of an optimal policy via Contraction and Monotonicity Assumption. Further, in the case that the arrival processes are nonhomogenous Poisson we will show that an optimal policy has a very simple structure and that an ϵ-optimal policy can be found by solving a system of differential equations. The existence of an ϵ-optimal policy guarantees us a practical way to approach optimal at any prespecified degree of precision.

Chapter 2

Fomulation

2.1 Assumptions

We will make the following assumptions:

- Assumption 1: *single flight leg.* Booking requests are begun to be accepted on the basis of a single departure and landing.
- Assumption 2: *arrival process.* The arrival process is a semi-Markov process which does not depend on any booking policy.
- Assumption 3: *no cancellations, no overbookings.* Cancellations, 'no-shows' and overbookings are not considered.

Assumption 2 means that given a predetermined booking policy the state the system will enter next depends on its past only through the current state of the system. We will elaborate on this in the next section. While Assumption 1 and Assumption 3 serve the purpose of simplifying the problem; Assumption 2 serves a double purpose: it make the problem tractable and in many practical cases it is true; it contains nonhomogenous Poisson process as a special case.

Suppose we have m classes of booking requests and let the fare of ith class be ρ_i, $i = 1, 2, \ldots, m$, without loss of generality we assume that $\rho_1 > \rho_2 > \ldots > \rho_m$.

2.2 Markov Decision Process

The control of booking process can be formulated as a Markov decision process.

Arrival Process: The arrival process can be described by a sequence of random vectors $(\tau_i, \phi_i), i = 1, 2, \ldots$; where τ_i denotes the time of the ith booking request and ϕ_i the fare class.

Suppose the seat inventory is α_i when the ith booking request arrives, then system state can be described by random vector $\omega_i = (\alpha_i, \tau_i, \phi_i), i = 1, 2, \ldots$.

State Space: Suppose booking requests are begun to be accepted time T before departure and airplane has a seat capacity of C. Let initial state be $\omega_0 = (\alpha_0, \tau_0, \phi_0) = (C, T, 0)$, fare class 0 generates no revenue and is introduced only as a device to start the process; then $\omega_i, i = 1, 2, \ldots$, will take value from S defined as the following.

$$S = \{0, 1, \ldots, C\} \times \{t : 0 \leq t \leq T\} \times \{1, 2, \ldots, m\}.$$

Let $s = (n, t, f) \in S$. S is our state space and s is a point of S.

Let $T \geq \tau_1 \geq \tau_2 \geq \cdots$, by Assumption 2 we have

$$\begin{aligned} P_{(t,f)}(t', f') &= Pr[\tau_{i+1} \leq t', \phi_{i+1} = f' \mid (\tau_i, \phi_i) = (t, f)] \qquad (2.3) \\ &= Pr[\tau_{i+1} \leq t', \phi_{i+1} = f' \mid (\tau_i, \phi_i) = (t, f), (\tau_{i-1}, \phi_{i-1}), \ldots, (\tau_0, \phi_0)] \\ i &= 0, 1, \ldots \end{aligned}$$

Policy: Let $\pi(s) = 0 \,\text{or}\, 1$ for all $s = (n, t, f) \in S$ where $n > 0$ and $\pi(s) = 0$ for all $s = (0, t, f) \in S$. π is a policy which maps S onto D. $D = \{0, 1\}$ is the action set; an action 1 means to accept a booking request at a certain state and action 0 reject it. The restriction of $\pi(s) = 0$ for all $s = (0, t, f)$ is because of Assumption 3.

If the current state is $s = (n, t, f)$, then the next state will be $s' = (n - \pi(n, t, f), \tau', \phi')$; where given s, (τ', ϕ') has the distribution of (2.3) independent of π.

Transition Probability: Let

$$\begin{aligned}
P_{s,a}(s') &= Pr(s' \mid s, a) \\
&= Pr(\omega' = s' \mid \omega = s, \pi(\omega) = a) \qquad (2.4) \\
&= Pr(\alpha' = n - a, \tau' \leq t', \phi' = f' \mid \omega = (n, t, f), \pi(\omega) = a) \\
i &= 0, 1, \ldots
\end{aligned}$$

$P_{s,a}(s')$ is the probability distribution when the system is in state s, a policy π is used, and the system will transition into state s' next. Given π, (2.4) is identical with (2.3).

2.3 The Revenue Function

Define a real valued function r on $S \times D$ as the following

$$r(s, a) = \begin{cases} \rho_f & n > 0, a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The system starts off at the initial state $s_0 = (C, T, 0)$. In any state $s = (n, t, f)$, when an action $a \in \{0, 1\}$ is taken, as a joint result of s and a two things happen:

1. we receive a return $r(s, a)$.

2. the system moves to a new state s' according to probability distribution $P_{s,a}(s')$.

then the above evolution starts all over again begining from state s'. More specifically, if we accept a booking request, i.e. take action 1, we get revenue ρ_f and the seat inventory will be reduced by one; if we reject a booking request, i.e. take action 0, we don't get any revenue and the seat inventory remains unchanged. So we have

$$\begin{aligned} r(s,0) &= 0 \\ r(s,1) &= \rho_f. \end{aligned}$$

The revenue function of the system can be defined as

$$\overline{r}_\pi(\omega_0, \omega_1, \ldots) = \sum_{j=0}^{\infty} r(\omega_j, \pi(\omega_j))$$

This is just the summation of revenues we get in the booking process of a particular plane. Define

$$\begin{aligned} v_\pi(s_0) &= E[\overline{r}_\pi(\omega_0, \omega_1, \ldots) \mid \omega_0 = s_0] \\ &= E[\sum_{j=0}^{\infty} r(\omega_j, \pi(\omega_j)) \mid \omega_0 = s_0] \end{aligned} \tag{2.5}$$

$v_\pi(s_0) = v_\pi(C, T, 0)$ is the expected revenue with initial state $s_0 = (C, T, 0)$, i.e. the state that there are C empty seats and T time before departure, and a policy π is followed.

Similarly we define

$$\overline{r}_\pi(\omega_i, \omega_{i+1}, \ldots) = \sum_{j=i}^{\infty} r(\omega_j, \pi(\omega_j))$$

$$\begin{aligned} v_\pi(s) &= E[\overline{r}_\pi(\omega_i, \omega_{i+1}, \ldots) \mid \omega_i = s] \\ &= E[\sum_{j=i}^{\infty} r(\omega_j, \pi(\omega_j)) \mid \omega_i = s] \end{aligned} \tag{2.6}$$

$v_\pi(s) = v_\pi(n,t,f)$ is the expected revenue with current state $s = (n,t,f)$, i.e. the state that there are n empty seats and t time before departure, a booking request of class f just arrived, and a policy π is followed.

Lemma 1 $\quad 0 \le v_\pi(n,t,f) \le n\rho_1 \quad \forall (n,t,f) \in S.$

This is trivially true because we cannot sell more than all the seats we have and ρ_1 is the highest revenue a seat can generate; and if we don't sell we will not get any revenue. The significance of **Lemma 1** is that it implies that $v_\pi(s)$ is well defined for all $s \in S$.

Our objective is to find a policy π^* which maps S onto D and generates

$$v^*(s_0) = \sup_\pi v_\pi(s_0),$$

the maximal expected revenue where s_0 is fixed at $(C, T, 0)$.

2.4 The Functional Equation

Suppose π^*, an optimal policy, exists in the sense that

$$\begin{aligned} v^*(n,t,f) &= v_{\pi^*}(n,t,f) = \sup_\pi v_\pi(n,t,f) \qquad (2.7) \\ n &= 0,1,\ldots,C \\ 0 &\le t \le T \\ f &= 1,2,\ldots,m \end{aligned}$$

where $v_\pi(n,t,f)$, $(n,t,f) \in S$, is defined by (2.6). $v^*(n,t,f)$ is the expected revenue when the system is at state $s = (n,t,f)$ and an optimal policy is followed.

Lemma 2 *$v^*(n,t,f)$ is nondecreasing in seat inventory n for each fixed time t and fare type f.*

proof: We need to show that $v^*(n+k,t,f) \geq v^*(n,t,f)$ for each $k \geq 0$. Suppose π^* is an optimal policy and π^0 is the policy defined by

$$\pi^0(m,t,f) = \begin{cases} \pi^*(m-k,t,f) & \text{for } m \geq k; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$v^*(n+k,t,f) \geq v_{\pi^0}(n+k,t,f) = v^*(n,t,f).$$

Lemma 2 simply says that under an optimal policy we can expect a larger revenue if we have more seats — a very important fact to get a simple structured optimal policy in our context.

Let $v(s) = v(n,t,f)$ be the solution to the following functional equation

$$\begin{aligned} v(s) &= \max_{a\in\{0,1\}}\{r(s,a) + \int v(s')\,dP_{s,a}(s')\} \\ s &= (n,t,f) \\ s' &= (n-a,q,k) \\ s,s' &\in S \end{aligned} \tag{2.8}$$

In order to reveal the relationship between $v(s)$ and $v^*(s)$ we introduce some terminology. Let B be the collection of all bounded funtions from S to reals, and define a metric d on B as

$$\begin{aligned} d(u,v) &= \sup_{s\in S} \mid u(s) - v(s) \mid \\ u &\in B \\ v &\in B \end{aligned}$$

The space B is complete in this metric. Define

$$h(s,a,v) = r(s,a) + \int v(s')\,dP_{s,a}(s'). \tag{2.9}$$

We will assume $h(s,a,v)$ satisfies 'Contraction Assumption' in the sense that for some constant c such that $0 \leq c < 1$, we have

$$\begin{aligned} \mid h(s,a,u) - h(s,a,v) \mid &\leq \int \mid u(s') - v(s') \mid dP_{s,a}(s') \\ &\leq cd(u,v) \end{aligned} \tag{2.10}$$

for $u \in B$, $v \in B$, $a \in \{0,1\}$.

To satisfy the Contraction Assumption it is sufficient to have

$$c = \int dP_{s,a}(s') < 1 \qquad s, s' \in S \qquad a \in D.$$

$h(s,a,v)$ 'inherently' satisfies 'Monotonicity Assumption' in the sense that

$$h(s,a,u) \geq h(s,a,v) \tag{2.11}$$

for $u \in B$, $v \in B$, $u \geq v$,$a \in \{0,1\}$, and $s \in S$.

In terms of (2.9), a maximization operator A can be defined as

$$Av(s) = \max_{a \in \{0,1\}} h(s,a,v) \tag{2.12}$$

and (2.8) becomes

$$\begin{aligned} Av(s) &= v(s) \\ s &\in S. \end{aligned} \tag{2.13}$$

(2.13) has a unique solution $v(s)$ if Contraction Assumption is met. Further Denardo asserts that if Monotonicity Assumption is met as well, then

$$\begin{aligned} v(s) &= v^*(s) \\ s &\in S \end{aligned} \tag{2.14}$$

where v^* is defined by (2.7) and v is the solution to (2.13).

Denardo has also shown that Contraction Assumption requirement can be relaxed to 'N-stage Contraction Assumption' to gurantee (2.14) to hold. In our context 'N-stage Contraction Assumption' means that there is positive probability that the Nth booking request will be the last one before the plane will take off. N can be very large as long as it's finite. This makes sense in reality because we cannot have infinite number of booking requests.

Because of (2.14), we can find $v^*(s)$ by finding $v(s)$ or vice versa; and because of the simplicity of the action set D, if we can find $v(s)$,for some $s \in S$ we can easily figure out the corresponding value of the optimal policy $\pi(s)$.

2.5 Some Properties of an Optimal Policy

The functional equation (2.8) can be written out fully as the following

$$v(n,t,f) = \max\left\{\begin{array}{l} \sum_{k=1}^{m}\int_0^t v(n,q,k)dP[(n,q,k) \mid (n,t,f);0]; \\ \rho_f + \sum_{k=1}^{m}\int_0^t v(n-1,q,k)dP[(n-1,q,k) \mid (n,t,f);1] \end{array}\right\} \quad (2.15)$$

$$v(0,t,f) = 0 \quad (2.16)$$

$$v(n,0,f) = \rho_f \quad (2.17)$$

$$v(0,0,f) = 0 \quad (2.18)$$

$$n = 1,2,\ldots,C$$

$$0 < t \leq T$$

$$f = 1,2,\ldots,m$$

Define

$$v^0(n,t,f) = \sum_{k=1}^{m} \int_0^t v(n,q,k) dP[(n,q,k) \mid (n,t,f); 0]; \tag{2.19}$$

$$v^1(n,t,f) = \rho_f + \sum_{k=1}^{m} \int_0^t v(n-1,q,k) dP[(n-1,q,k) \mid (n,t,f); 1] \tag{2.20}$$

where $v^0(n,t,f)$ is the expected revenue if the class f booking request is not accepted and follow an optimal policy after t; $v^1(n,t,f)$ is the expected revenue if the class f booking request is accepted and follow an optimal policy after t. Let's enhance Assumption 2 by the following

- Assumption 4: *independent demand.* The demands for the different fare classes are mutually independent.

Assumption 2 and Assumption 4 together means that at an arrival epoch, whatever action is taken the arrival process will start stochastically anew. It's understood that a policy is assumed not to affect arrival process but it can affect seat inventory and hence affect the system state. Under Assumption 2 and Assumption 4 we will have

$$\begin{aligned} P[(n-a,q,k) \mid (n,t,f); a] &= P[(n-a,q,k) \mid t] \\ a &\in \{0,1\} \\ 0 &\leq q < t \\ k &= 1,2,\ldots,m \end{aligned} \tag{2.21}$$

Define

$$v^0(n,t) = \sum_{k=1}^{m} \int_0^t v(n,q,k) dP[(n,q,k) \mid t]; \tag{2.22}$$

then

$$\begin{aligned} v^1(n,t,f) &= \rho_f + \sum_{k=1}^{m} \int_0^t v(n-1,q,k) dP[(n-1,q,k) \mid t] \\ &= \rho_f + v^0(n-1,t), \end{aligned} \tag{2.23}$$

the system of equations (2.15)-(2.18) is the following equivalent

$$
\begin{aligned}
v(n,t,f) &= \max\{v^0(n,t); \rho_f + v^0(n-1,t)\} && (2.24)\\
v^0(n,t) &= \sum_{k=1}^{m} \int_0^t v(n,q,k)dP[(n,q,k) \mid t] && (2.25)\\
v^0(0,t) &= 0 && (2.26)\\
v^0(n,0) &= 0 && (2.27)\\
v^0(0,0) &= 0 && (2.28)\\
n &= 1,2,\ldots,C\\
0 &< t \leq T\\
f &= 1,2,\ldots,m
\end{aligned}
$$

from the system of equations (2.24)-(2.28) we can infer the follwoing

Corollary 1 *At the state $(n,t,f), n > 0$, the optimal policy will accept fare f if $v^0(n,t) \leq \rho_f + v^0(n-1,t)$; and reject f otherwise.*

Note that if $v^0(n,t) \leq \rho_f + v^0(n-1,t)$, then $v^0(n,t) \leq \rho_i + v^0(n-1,t)$, $i = 1,2,\ldots,f-1$, the following can be inferred

Corollary 2 *Under an optimal policy the fare classes should be nested; i.e. if we accept f at state $(n,t,f), n > 0$, we accept the fare class $i, i = 1,2,\ldots,f-1$ as well should they come.*

Chapter 3

Nonhomogenous Poisson Process

3.1 Further Assumptions

To be more specific let's make the following Assumption 5.

- Assumption 5: *arrival processes.* The arrival process of ith, $i = 1, 2, \ldots, m$, fare class is a nonhomogenous Poisson process $\{N_i(t) : t \leq T\}$ with intensity function $\lambda_i(t)$, $0 \leq \lambda_i(t) < +\infty$, which is piecewise continuous and has directional limits at each point $t, 0 \leq t \leq T$; also these processes are independent of any policy being used in the booking process. Let $\Lambda_i(t) = \int_t^T \lambda_i(q)\, dq, i = 1, 2, \ldots, m$, and

$$\begin{aligned} N(t) &= \sum_{i=1}^{m} N_i(t) \\ \lambda(t) &= \sum_{i=1}^{m} \lambda_i(t) \\ \Lambda(t) &= \sum_{i=1}^{m} \Lambda_i(t). \end{aligned}$$

  $\{N(t) : t \leq T\}$ is the Poisson process with intensity $\lambda(t)$, the superposition of arrival processes of all fare classes. If in some time interval where $\lambda(t) = 0$ we can remove this period of time from the overall booking period T beforehand, so we will assume without loss of generality that $\lambda(t) > 0$.

We shall inherit all the assumptions we made in **Chapter 1**.

Now we can write $P_{s,a}(s')$ explicitly in differential form

$$\begin{aligned} dP_{s,a}(s') &= \lambda_{f_1}(q)e^{[\Lambda(t)-\Lambda(q)]}d(q) \\ f_1 &= 1,2,\ldots,m \\ 0 &\leq t \leq T \\ 0 &\leq q \leq t \\ a &\in \{0,1\} \end{aligned} \tag{3.29}$$

(3.29) is consistent with the assumption that an action taken will not affect arrival process but will affect the state the system will enter next. The following verifies that the Contraction Assumption is met

$$\begin{aligned} c = \int dP_{s,a}(s') &\leq \int_0^t \lambda(q)e^{[\Lambda(t)-\Lambda(q)]}dq < 1 \\ 0 &\leq t \leq T \end{aligned}$$

3.2 The Solution of the Functional Equation

In this section we will solve the functional equation (2.8) for $v(s_0)$ in the nonhomogenous Poisson case as the following

$$v(n,t,f) = \max \left\{ \begin{array}{l} \sum_{k=1}^{m} \int_0^t v(n,q,k)\lambda_k(q)e^{[\Lambda(t)-\Lambda(q)]}dq; \\ \rho_f + \sum_{k=1}^{m} \int_0^t v(n-1,q,k)\lambda_k(q)e^{[\Lambda(t)-\Lambda(q)]}dq \end{array} \right\} \tag{3.30}$$

$$v(0,t,f) = 0 \tag{3.31}$$

$$v(n,0,f) = \rho_f \tag{3.32}$$

$$v(0,0,f) = 0 \tag{3.33}$$

$$\begin{aligned} n &= 1,2,\ldots,C \\ 0 &< t \leq T \\ f &= 1,2,\ldots,m \end{aligned}$$

Define

$$v^0(n,t) = \sum_{k=1}^{m}\int_0^t v(n,q,k)\lambda_k(q)e^{[\Lambda(t)-\Lambda(q)]}dq; \quad (3.34)$$

$$v^1(n,t,f) = \rho_f + \sum_{k=1}^{m}\int_0^t v(n-1,q,k)\lambda_k(q)e^{[\Lambda(t)-\Lambda(q)]}dq \quad (3.35)$$

In terms of (3.34) the system of equations (3.30)-(3.33) is the following equivalent

$$\begin{aligned}
v(n,t,f) &= \max\{v^0(n,t); v^1(n,t,f)\} && (3.36)\\
v^0(n,t) &= \sum_{k=1}^{m}\int_0^t v(n,q,k)\lambda_k(q)e^{[\Lambda(t)-\Lambda(q)]}dq && (3.37)\\
v^1(n,t,f) &= \rho_f + v^0(n-1,t) && (3.38)\\
v^1(0,t,f) &= 0 && (3.39)\\
v^0(0,t) &= 0 && (3.40)\\
v^1(n,0,f) &= \rho_f && (3.41)\\
v^0(n,0) &= 0 && (3.42)\\
v^1(0,0,f) &= 0 && (3.43)\\
v^0(0,0) &= 0 && (3.44)\\
n &= 1,2,\ldots,C\\
0 &< t \leq T\\
f &= 1,2,\ldots,m
\end{aligned}$$

from the system of equations (3.36)-(3.44) we can infer the same corollaries we did in **Chapter 1**. It's clear from the system of equations (3.36)-(3.44) that we only need to solve the following reduced system.

$$\begin{aligned}
v(n,t,f) &= \max\{v^0(n,t); \rho_f + v^0(n-1,t)\} && (3.45)\\
v^0(n,t) &= \sum_{k=1}^{m}\int_0^t v(n,q,k)\lambda_k(q)e^{[\Lambda(t)-\Lambda(q)]}dq && (3.46)
\end{aligned}$$

$$\begin{aligned} v^0(0,t) &= 0 && (3.47) \\ v^0(n,0) &= 0 && (3.48) \\ v^0(0,0) &= 0 && (3.49) \\ n &= 1,2,\ldots,C \\ 0 &< t \leq T \\ f &= 1,2,\ldots,m \end{aligned}$$

We make the following derivation from (3.46)

$$\begin{aligned} v^0(n,t) &- v^0(n,t-\Delta t) = \\ &= \sum_{k=1}^{m} \int_{t-\Delta t}^{t} v(n,q,k)\lambda_k(q)e^{[\Lambda(t)-\Lambda(q)]}dq - \\ & \quad v^0(n,t-\Delta t)[1-e^{[\Lambda(t)-\Lambda(t-\Delta t)]}] \end{aligned} \tag{3.50}$$

Divide both side of (3.50) by Δt, let Δt approaches 0, and apply mean-value theorem to the first term of the righthand side we get

$$\begin{aligned} [v^0(n,t)]' &= \sum_{k=1}^{m} v(n,t,k)\lambda_k(t) - v^0(n,t)\lambda(t) \\ &= \sum_{k=1}^{m} \max\{v^0(n,t); \rho_k + v^0(n-1,t)\}\lambda_k(t) - \sum_{k=1}^{m} v^0(n,t)\lambda_k(t) \\ &= \sum_{k=1}^{m} [\max\{v^0(n,t); \rho_k + v^0(n-1,t)\} - v^0(n,t)]\lambda_k(t) \\ &= \sum_{k=1}^{m} [\max\{0; \rho_k + v^0(n-1,t) - v^0(n,t)]\lambda_k(t) \\ &= \sum_{k=1}^{m} [\rho_k + v^0(n-1,t) - v^0(n,t)]^+\lambda_k(t) && (3.51) \\ n &= 1,2,\ldots,C \\ 0 &< t \leq T \end{aligned}$$

where

$$[v^0(n,t)]' = \frac{\partial v^0(n,t)}{\partial t}$$

and we have used the notation that $a^+ = \max(a, 0)$.

It's routine to verify that $v^0(n,t), n = 0, 1, \ldots, C, 0 \leq t \leq T$, is continuous from the definition (3.34) under the Assumption 5. This means that $v^1(n,t,f), n = 0, 1, \ldots, C; 0 \leq t \leq T; f = 1, 2, \ldots, m$, is continuous and so is $v(n,t,f), n = 0, 1, \ldots, C; 0 \leq t \leq T; f = 1, 2, \ldots, m$; and hence the mean-value theorem is applicable.

From (3.51) we conclude

$$[v^0(n,t)]' \geq 0, \forall 0 \leq t \leq T, n = 0, 1, \ldots, C. \tag{3.52}$$

and from (3.34),(3.35) we get the following

Corollary 3 *$v(n,t,f)$ is nondecreasing in $t, 0 \leq t \leq T, \forall n = 0, 1, \ldots, C, f = 1, 2, \ldots, m$.*

If at time t where $\lambda_i(t), i = 1, 2, \ldots, m$ are/is discontinuous they/it will approach(es) their/its appropriate directional limit(s) which we have assumed exist(s) as Δt approaches 0. At such t the directional derivative of $v^0(n,t)$ satisfies (3.51).

Take (3.52) into account and if we can ever accept a booking request we should have $v^0(n,t) - v^0(n-1,t) \leq \rho_1$. **Lemma 2** implies that we should have $v^0(n,t) - v^0(n-1,t) \geq 0$ and hence we have the following

Corollary 4 $0 \leq v^0(n,t) - v^0(n-1,t) \leq \rho_1 \quad \forall n = 1, 2, \ldots, C$ *and* $0 \leq t \leq T$.

It's clear now that we only need to solve (3.51) with the following initial condition to solve the function equation (3.30)-(3.33).

$$v^0(0,t) = 0 \tag{3.53}$$

$$v^0(n,0) = 0 \tag{3.54}$$

$$v^0(0,0) = 0 \tag{3.55}$$

Taking into account **Corollary 4** we can write (3.51) as

$$\begin{aligned}
[v^0(n,t)]' &= [\rho_1 + v_0(n-1,t) - v_0(n,t)]\lambda_1(t) + \\
&\quad [\rho_1 + v_0(n-1,t) - v_0(n,t)]^+\lambda_2(t) + \\
&\quad [\rho_1 + v_0(n-1,t) - v_0(n,t)]^+\lambda_3(t) + \\
&\quad \cdots + [\rho_1 + v_0(n-1,t) - v_0(n,t)]^+\lambda_m(t) \qquad (3.56) \\
n &= 1,2,\ldots,C \\
0 &< t \leq T,
\end{aligned}$$

3.3 An Algorithm

For ease of exposition let's concentrate on the case that only two types of fares are offered, i.e. $m = 2$. The multiple fare (more than 2 types of fares are offered) case is analogous.

In the two fare case equation (3.56) becomes

$$\begin{aligned}
[v^0(n,t)]' &= [\rho_1 + v^0(n-1,t) - v^0(n,t)]\lambda_1(t) + \\
&\quad [\rho_2 + v^0(n-1,t) - v^0(n,t)]^+\lambda_2(t) \qquad (3.57) \\
n &= 1,2,\ldots,C \\
0 &< t \leq T.
\end{aligned}$$

Define

$$V(t) = \begin{bmatrix} v^0(0,t) \\ v^0(1,t) \\ \vdots \\ v^0(n,t) \\ \vdots \\ v^0(C,t) \end{bmatrix} \qquad \mathcal{V}(t, V(t)) = \begin{bmatrix} \bar{v}_0(t,U) \\ \bar{v}_1(t,U) \\ \vdots \\ \bar{v}_n(t,U) \\ \vdots \\ \bar{v}_C(t,U) \end{bmatrix} \tag{3.58}$$

$$\bigcirc = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ \vdots \\ u_C \end{bmatrix} \qquad W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \\ \vdots \\ w_C \end{bmatrix} \tag{3.59}$$

where

$$\begin{aligned} \bar{v}_0(t,U) &= 0 \\ \bar{v}_n(t,U) &= [\rho_1 + u_{n-1} - u_n]\lambda_1(t) + \\ & \quad [\rho_2 + u_{n-1} - u_n]^+\lambda_2(t) \\ n &= 1, 2, \ldots, C \\ 0 &< t \leq T \end{aligned} \tag{3.60}$$

and

$$\begin{aligned} -\infty &< u_n < +\infty \\ -\infty &< w_n < +\infty \\ n &= 0, 1, \ldots, C \end{aligned}$$

Using (3.58)-(3.60) we can write (3.57), (3.53),(3.54),(3.55) as

$$
\begin{aligned}
V'(t) &= \mathcal{V}(t, V(t)) && (3.61)\\
V(0) &= \bigcirc && (3.62)\\
0 &< t \leq T
\end{aligned}
$$

The existence and uniqueness of (3.61)-(3.62) has been established by the Contraction Assumption; however, to develop an algorithm to solve it practically and to prove the convergence of the algorithm we would like to use the following theorem from the theory of differential equation which states that

Theorem 1 *The initial-value problem (3.61)-(3.62) has a unique solution $V(t)$ on $[0, T]$ if $\mathcal{V}(t, V(t))$ is bounded and continuous on the strip $I = \{(t; V) \mid 0 \leq t \leq T, V \in \Re^{C+1}\}$ and satisfies Lipschitz condition*

$$\| \mathcal{V}(t, U) - \mathcal{V}(t, W) \|_\infty \leq L \| U - W \|_\infty \tag{3.63}$$

for all $t \in [0, T]$ and all $U, W \in \Re^{C+1}$, L is a constant and

$$\| U \|_\infty = \max_{0 \leq n \leq C} \mid u_n \mid .$$

Suppose $\lambda_1(t), \lambda_2(t)$ are continuous then $\mathcal{V}(t, V(t))$ is continous and bounded and we need to verify that Lipschitz condition

$$
\begin{aligned}
\| \mathcal{V}(t, U) - \mathcal{V}(t, W) \|_\infty &= \max_{0 \leq n \leq C} \mid \bar{v}_n(t, U) - \bar{v}_n(t, W) \mid \\
&\leq L \| U - W \|_\infty && (3.64)\\
&= L \max_{0 \leq n \leq C} \mid u_n - w_n \mid
\end{aligned}
$$

is met.

For any function g, if $g(U)$ is a linear function of U, then

$$| g(U) - g(W) | \leq L' \parallel U - W \parallel_\infty \tag{3.65}$$

for some constant L'. And for any two functions g_1, g_2, if they satisfy (3.65), then $\max(g_1, g_2)$, $g_1 + g_2$ also satisfy (3.65) for some constant L''. This kind of reasoning can be extended to finite number of functions; and because (3.60) only involves addition and maximization operations of some functions which is linear in U we can conclude that Lipschitz condition (3.64) is met indeed.

In the case where $\lambda_1(t), \lambda_2(t)$ are piecewise continuous and their appropriate directional limits exist as we have assumed (Assumption 5) Lipschitz condition is still satisfied and $\mathcal{V}(t, V(t))$ is bounded, but piecewise continuous and its appropriate directional limits exist as well. Noting that $V(t)$ is continuous over the interval $[0, T]$, we can divide the interval $[0, T]$ into subintervals such that in each of these subintervals $\mathcal{V}(t, V(t))$ is continuous; while the above theorem of ordinary differential equation asserts the existence and uniqueness of the solution to (3.61)-(3.62) in each of these subinterval the following theorem of ordinary differential equation asserts that the solution to (3.61)-(3.62) exists, is unique over the interval $[0, T]$. So we can solve (refeq265)-(3.62) in each subinterval of $[0, T]$ where the conditions of **Theorem 1** is satisfied and then pieces together a solution to (3.61)-(3.62).

Theorem 2 *Let $V(t), a < t < b$, be a complete solution*[1] *to the differential equation (3.61)-(3.62). If $b \neq +\infty$ and t_i is a sequence such that $t_i \to b, (t_i < b)$, and the sequence $V(t_i)$ has a limit e, then the point (b, e) is a boundary point.*

[1]A solution $V(t), (a < t < b)$ of the differential equation (3.61)-(3.62) is a complete solution if there does not exist solution $V_1(t)$ defined in a larger interval and coinciding with $V(t)$ for $a < t < b$.

To solve (3.61)-(3.62) numerically we can discretize them as the following

$$
\begin{aligned}
V(t) &- V(t-\Delta t) = \Delta t \mathcal{V}(t-\Delta t, V(t-\Delta t)) + o(\Delta t) \\
V(0) &= \bigcirc \\
0 &< t \leq T
\end{aligned}
\tag{3.66}
$$

where

$$
o(\Delta t) = \begin{bmatrix} \circ \\ \circ(\Delta t) \\ \cdots \\ \circ(\Delta t) \\ \cdots \\ \circ(\Delta t) \end{bmatrix}
\tag{3.67}
$$

(3.61)-(3.62) can be solved numerically by the 'one-step' method defined as the following

$$
\begin{aligned}
t &= k\Delta t \\
t_i &= i\Delta t \\
V_i &= V_{i-1} + \Delta t \mathcal{V}(t_{i-1}, V_{i-1}) \\
V_0 &= \bigcirc \\
i &= 1, 2, \ldots, k
\end{aligned}
\tag{3.68}
$$

where

$$V_0 = \begin{bmatrix} v_0(0) \\ v_0(1) \\ \vdots \\ v_0(n) \\ \vdots \\ v_0(C) \end{bmatrix} \qquad V_i = \begin{bmatrix} v_i(0) \\ v_i(1) \\ \vdots \\ v_i(n) \\ \vdots \\ v_i(C) \end{bmatrix} \qquad i = 1, 2, \ldots, k \tag{3.69}$$

Theorem 3 *The one step method defined by (3.68)-(3.69) is convergent in the sense*

$$\lim_{\Delta t \to 0;\, t_k = t\ fixed} V_k = V(t). \tag{3.70}$$

Proof: Subtracting (3.69) from (3.66) we have the following

$$\begin{aligned}
\| V(t_k) - V_k \| &= \| (V(t_{k-1}) - V_{k-1}) + \\
&\quad \Delta t(\mathcal{V}(t_{k-1}, V(t_{k-1})) - \mathcal{V}(t_{k-1}, V_{k-1})) + o(\Delta t) \|_\infty \\
&\le \| (V(t_{k-1}) - V_{k-1}) \|_\infty + \\
&\quad \Delta t \| (\mathcal{V}(t_{k-1}, V(t_{k-1})) - \mathcal{V}(t_{k-1}, V_{k-1})) \|_\infty + \| o(\Delta t) \|_\infty \\
&\le \| (V(t_{k-1}) - V_{k-1}) \|_\infty + \\
&\quad \Delta t L \| V(t_{k-1}) - V_{k-1} \|_\infty + \circ(\Delta t) \\
&= (1 + \Delta t L) \| V(t_{k-1}) - V_{k-1} \|_\infty + \circ(\Delta t) \\
&\le (1 + \Delta t L)^k \| V(0) - V_0 \|_\infty + \\
&\quad \circ(\Delta t)[1 + (1 + \Delta t L) + \cdots + (1 + \Delta t L)^k] \\
&= \circ(\Delta t) \frac{(1 + \Delta t L)^k - 1}{\Delta t L} \\
&\le \begin{cases} \frac{\circ(\Delta t)}{\Delta t} \frac{e^{Lt} - 1}{L} & L \ne 0 \\ \frac{\circ(\Delta t)}{\Delta t} t & L = 0 \end{cases}
\end{aligned}$$

i.e.

$$\lim_{\Delta t \to 0;\, t_k = t\ fixed} \| V(t_k) - V_k \|_\infty = \lim_{\Delta t \to 0;\, t_k = t\ fixed} \| V(t) - V_k \|_\infty = 0$$

It should be mentioned that the analysis of the multiple fare case is tedious but completely analoguous to what we have done so far.

3.4 Incremental Revenue and the Optimal Policy

To further study the structure of the optimal policy we make the following derivation.

From (3.57) we can have

$$\begin{aligned}
[v^0(n-1,t)]' &= [\rho_1 + v^0(n-2,t) - v^0(n-1,t)]\lambda_1(t) + \\
&\quad [\rho_2 + v^0(n-2,t) - v^0(n-1,t)]^+\lambda_2(t) \qquad (3.71) \\
n &= 2,3,\ldots,C \\
0 &< t \leq T.
\end{aligned}$$

Define

$$\begin{aligned}
\Delta v^0(n,t) &= v^0(n,t) - v^0(n-1,t) \qquad (3.72) \\
0 &< t \leq T \\
n &= 1,2,\ldots,C
\end{aligned}$$

Subtracting (3.71) from (3.57) we get

$$\begin{aligned}
[\Delta v^0(n,t)]' &= \lambda_1(t)\{\Delta v^0(n-1,t) - \Delta v^0(n,t)\} + \\
&\quad \lambda_2(t)\{[\rho_2 - \Delta v^0(n,t)]^+ - [\rho_2 - \Delta v^0(n-1,t)]^+\} \qquad (3.73) \\
0 &< t \leq T \\
n &= 2,3,\ldots,C
\end{aligned}$$

where

$$[\Delta v^0(n,t)]' = \frac{\partial[\Delta v^0(n,t)]}{\partial t}$$

and

$$
\begin{aligned}
[\Delta v^0(1,t)]' &= [v^0(1,t)]' \\
&= \lambda_1(t)[\rho_1 - \Delta v^0(1,t)] + \\
&\quad \lambda_2(t)[\rho_2 - \Delta v^0(1,t)]^+ \\
0 &< t \leq T.
\end{aligned} \tag{3.74}
$$

Define

$$
\begin{aligned}
\Delta v^0(n,0) &= 0 \\
\Delta v^0(0,t) &= \rho_1 \\
\Delta v^0(0,0) &= \rho_1 \\
0 &< t \leq T \\
n &= 1,2,\ldots,C
\end{aligned} \tag{3.75}
$$

(3.73)-(3.75) are the following equivalent

$$
\begin{aligned}
[\Delta v^0(n,t)]' &= \lambda_1(t)\{\Delta v^0(n-1,t) - \Delta v^0(n,t)\} + \\
&\quad \lambda_2(t)\{[\rho_2 - \Delta v^0(n,t)]^+ - [\rho_2 - \Delta v^0(n-1,t)]^+\} \\
\Delta v^0(n,0) &= 0 \\
\Delta v^0(0,t) &= \rho_1 \\
\Delta v^0(0,0) &= \rho_1 \\
0 &< t \leq T \\
n &= 1,2,\ldots,C
\end{aligned} \tag{3.76}
$$

Because $v^0(n,t), 0 \leq t \leq T, n = 0,1,\ldots,C$, is continuous we conclude

Corollary 5 *$\Delta v^0(n,t), 0 \leq t \leq T, n = 0,1,\ldots,C$, is continuous.*

From (3.76) we conclude

Corollary 6 *$\Delta v^0(n,t), 0 \le t \le t, n = 0,1,\ldots,C$ is nondecreasing in t with n fixed if and only if $\Delta v^0(n,t), 0 \le t \le t, n = 0,1,\ldots,C$ is nonincreasing in n with t fixed.*

Define

$$\Delta V(t) = \begin{bmatrix} \Delta v^0(0,t) \\ \Delta v^0(1,t) \\ \vdots \\ \Delta v^0(n,t) \\ \vdots \\ \Delta v^0(C,t) \end{bmatrix} \qquad \bigcirc = \begin{bmatrix} \rho_1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.77}$$

$$\Delta \mathcal{V}(t,U) = \begin{bmatrix} \Delta \bar{v}_0(t,U) \\ \Delta \bar{v}_1(t,U) \\ \vdots \\ \Delta \bar{v}_n(t,U) \\ \vdots \\ \Delta \bar{v}_C(t,U) \end{bmatrix} \tag{3.78}$$

where

$$\Delta \bar{v}_0(t,U) = \rho_1 \tag{3.79}$$

$$\begin{aligned} \Delta \bar{v}_n(t,U) &= \lambda_1(t)\{u_{n-1} - u_n\} + \\ &\quad \lambda_2(t)\{[\rho_2 - u_n]^+ - [\rho_2 - u_{n-1}]^+\} \end{aligned} \tag{3.80}$$

$$0 < t \le T$$

$$n = 1,2,\ldots,C$$

Using (3.77)-(3.80) we can write (3.76) as

$$[\Delta V(t)]' = \Delta \mathcal{V}(t, \Delta V(t)) \tag{3.81}$$

$$\begin{aligned} \Delta V(0) &= \bigcirc \\ 0 &< t \le T \end{aligned} \tag{3.82}$$

Since equation (3.81)-(3.82) are derived from (3.61)-(3.62), all the statements we made about equation (3.61)-(3.62) are also valid in terms of (3.81)-(3.82); especially the solution to (3.81)-(3.82) exists, is unique in the interval $[0, T]$, and can be computed by the one-step method defined as the following.

$$\begin{aligned} t &= k\Delta t \\ t_i &= i\Delta t \\ \Delta V_i &= \Delta V_{i-1} + \Delta t \mathcal{V}(t_i, \Delta V_{i-1}) \\ \Delta V_0 &= \bigcirc \\ i &= 1, 2, \ldots, k \end{aligned} \tag{3.83}$$

where

$$\Delta V_0 = \begin{bmatrix} \Delta v_0(0) \\ \Delta v_0(1) \\ \vdots \\ \Delta v_0(n) \\ \vdots \\ \Delta v_0(C) \end{bmatrix} \qquad \Delta V_i = \begin{bmatrix} \Delta v_i(0) \\ \Delta v_i(1) \\ \vdots \\ \Delta v_i(n) \\ \vdots \\ \Delta v_i(C) \end{bmatrix} \qquad i = 1, 2, \ldots, k \tag{3.84}$$

The one step method defined by (3.83)-(3.84) is convergent in the sense

$$\| \Delta V(t_k) - \Delta V_k \| \le \begin{cases} \frac{o(\Delta t)}{\Delta t} \frac{e^{Lt}-1}{L} & L \neq 0 \\ \frac{o(\Delta t)}{\Delta t} t & L = 0 \end{cases}$$

i.e.

$$\lim_{\Delta t \to 0;\, t_k = t\ fixed} \Delta V_k = \Delta V(t). \tag{3.85}$$

We are now in a position to prove the following

Theorem 4 *$\Delta v^0(n,t), 0 \le t \le T, n = 0,1,\ldots,C$, is nondecreasing in t given n and nonincreasing in n given t.*

Proof: From (3.76) we have

$$\Delta v^0(0,t) = \rho_1 \qquad 0 \le t \le T$$

i.e. $\Delta v^0(0,t)$ is nondecreasing in t; and

$$\begin{aligned} \Delta v^0(1,t) &= v^0(1,t) - v^0(0,t) = v^0(1,t) \\ \Delta v^0(1,0) &= 0 \\ 0 &< t \le T. \end{aligned}$$

Since $v^0(1,t)$ is nondecreasing in t and not greater than ρ_1, we get $\Delta v^0(0,t) \ge \Delta v^0(1,t)$ and $\Delta v^0(0,t), \Delta v^0(1,t)$ are nondecreasing in t.

Suppose $k-1, k \ge 1$ is the least integer that the theorem holds hence for some t we have the following

$$\Delta v^0(k,t) > \Delta v^0(k-1,t) \tag{3.86}$$

where $\Delta v^0(k-1,t)$ is nondecresing in t. From (3.76) again we have

$$\begin{aligned} [\Delta v^0(k,t)]' &= \lambda_1(t)\Delta t\{\Delta v^0(k-1,t) - \Delta v^0(k,t)\} + \\ &\quad \lambda_2(t)\Delta t\{[\rho_2 - \Delta v^0(k,t)]^+ - [\rho_2 - \Delta v^0(k-1,t)]^+\} \qquad (3.87) \\ \Delta v^0(k,0) &= 0 \\ 0 &< t \le T \\ n &= 1,2,\ldots,C \end{aligned}$$

Because $\Delta v^0(k,t)$ is continuous in the interval $[0,T]$, we can divide the interval $[0,T]$ into subintervals in each of which $\Delta v^0(k,t)$ is monotone in such a way that $\Delta v^0(k,t)$ either nondecreasing or decreasing. Let the subintervals be $[0, t_1), [t_1, t_2), \ldots, [t_l, T]$, where $0 < t_1 < t_2 < \ldots < t_l < T, l < \infty$.

(3.85) and (3.87) can not be true at the same time. If they can, we assert that $\Delta v^0(k,t)$ can not be monotone in $[0, t_1)$. For suppose $\Delta v^0(k,t)$ is nondecreasing in t, from (3.87) we shall have $[\Delta v^0(k,t)]' \geq 0$; so $\Delta v^0(k,t) \leq \Delta v^0(k-1,t)$ and this contradicts (3.85). Suppose $\Delta v^0(k,t)$ is decreasing, from (3.87) we will get $[\Delta v^0(k,t)]' < 0$ hence $\Delta v^0(k,t) < 0$ in the interval $[0, t_l]$ and this contradicts **Corollary 4**. We have reached a contradiction and the theorem is proven.

Because of the continuity and monotonicity of $\Delta v^0(n,t)$, the $n-t$ plane is divided into two disjoint 'regions':

$$\begin{aligned} R^1 &= \{(n,t) : \rho_2 - \Delta v^0(n,t) < 0, 0 \leq t \leq T, n = 0, 1, \ldots, C\} \\ R^2 &= \{(n,t) : \rho_2 - \Delta v^0(n,t) \geq 0, 0 \leq t \leq T, n = 0, 1, \ldots, C\}. \end{aligned}$$

Corollary 7 *The optimal policy will accept booking requests from class 1 only in the 'region' R^1 and accept booking requests from either classes in the 'region' R^2.*

An analoguous analysis will lead to the conclusion that in the multiple fare case, the $n-t$ plane will be divided into m disjoint 'regions':

$$\begin{aligned} R^1 &= \{(n,t) : \rho_2 - \Delta v^0(n,t) < 0, 0 \leq t \leq T, n = 0, 1, \ldots, C\} \\ R^2 &= \{(n,t) : \rho_2 - \Delta v^0(n,t) \geq 0, \rho_3 - \Delta v^0(n,t) < 0, \\ &\quad\; 0 \leq t \leq T, n = 0, 1, \ldots, C\} \\ &\quad\; \ldots \\ R^{m-1} &= \{(n,t) : \rho_{m-2} - \Delta v^0(n,t) \geq 0, \rho_{m-1} - \Delta v^0(n,t) < 0, \end{aligned}$$

$$
\begin{aligned}
& 0 \le t \le T, n = 0, 1, \ldots, C\} \\
R^m &= \{(n,t) : \rho_{m-1} - \Delta v^0(n,t) \ge 0, 0 \le t \le T, n = 0, 1, \ldots, C\}
\end{aligned}
$$

and

Corollary 8 *In the multiple fare case the optimal policy will accept booking requests from class* 1 *only in the 'region'* R^1 *and accept booking requests from class* 1 *through class* k*, in the 'region'* $R^k, k = 2, 3, \ldots, m$.

Because $\Delta v^0(n,t)$ may not be strictly increasing in t we want the conditions in the above braces be met at the minimum t.

3.5 The Proof of the Convergence

Define the 'distance' between any two policy π^1, π^2 as the following

$$
\begin{aligned}
\| \pi^1 - \pi^2 \| &= \int_S \mid \pi^1(n,t,f) - \pi^2(n,t,f) \mid dt \\
&= \sum_{n=0}^{C} \sum_{f=1}^{m} \int_0^T \mid \pi^1(n,t,f) - \pi^2(n,t,f) \mid dt \qquad (3.88)
\end{aligned}
$$

Proposition 1 *For any two policy* π^1 *and* π^2 *if* $\| \pi^1 - \pi^2 \| \to 0$ *then* $\mid v_{\pi^1}(s_0) - v_{\pi^2}(s_0) \mid \to 0$.

To prove the proposition we define two regions as the following

$$
\begin{aligned}
D_1 &= \{(n,t,f) : \pi^1(n,t,f) = \pi^2(n,t,f), (n,t,f) \in S\} \\
D_2 &= \{(n,t,f) : \pi^1(n,t,f) \ne \pi^2(n,t,f), (n,t,f) \in S\}
\end{aligned}
$$

and obviously we have

$$S = D_1 + D_2.$$

Further define

$$\begin{aligned} I_{n,f} &= \{t : (n,t,f) \in D_2, (n,t,f) \in S\} \\ I &= \bigcup_{n,f} I_{n,f} \end{aligned}$$

Proof: For any policy π, let $L' = \max_{s_1 \in S} p_{s_0,\pi(s_0)}(s_1)$, Assumption 5 implies that $L' < +\infty$. Suppose $\| \pi^1 - \pi^2 \| \leq \epsilon$; from (2.21) and (3.29) we can have

$$\begin{aligned} \mid v_{\pi^1}(s_0) &- v_{\pi^2}(s_0) \mid \leq \int_S \mid v_{\pi^1}(s_1) - v_{\pi^2}(s_1) \mid dP_{s_0,\pi^1(s_0)}(s_1) \\ &\leq \rho_1 \cdot C \cdot \int_{D_2} dP_{s_0,\pi^1(s_0)}(s_1) \\ &\leq \rho_1 \cdot C \cdot L' \cdot \int_I dt \\ &= \rho_1 \cdot C \cdot L' \cdot \| \pi^1 - \pi^2 \| \\ &\leq \rho_1 \cdot C \cdot L' \cdot \epsilon. \end{aligned} \tag{3.89}$$

Following **Section 3.4** we define

$$\begin{aligned} \Delta\hat{v}(n,t) &= \Delta v_{i-1}(n) \qquad (3.90) \\ n &= 0, 1, \ldots, C \\ (i-1)\Delta t &\leq t \leq i\Delta t \\ i &= 1, 2, \ldots, k \\ T &= k\Delta t \end{aligned}$$

or in matrix form

$$\begin{aligned}
\Delta\hat{V}(t) &= \begin{bmatrix} \Delta\hat{v}(0,t) \\ \Delta\hat{v}(1,t) \\ \vdots \\ \Delta\hat{v}(n,t) \\ \vdots \\ \Delta\hat{v}(C,t) \end{bmatrix} = \begin{bmatrix} \Delta v_{i-1}(0) \\ \Delta v_{i-1}(1) \\ \vdots \\ \Delta v_{i-1}(n) \\ \vdots \\ \Delta v_{i-1}(C) \end{bmatrix} = \Delta V_{i-1} \qquad (3.91) \\
(i-1)\Delta t &\le t \le i\Delta t \\
i &= 1,2,\ldots,k \\
T &= k\Delta t
\end{aligned}$$

We have proven in **Section 3.4** that the monotone $\Delta v^0(n,t)$ specifies an optimal policy π^*. We also proven that $\Delta\hat{v}(n,t)$ uniformly converges to $\Delta v^0(n,t)$. By choosing Δt sufficiently small we can make $\Delta\hat{v}(n,t)$ monotone and such a $\Delta\hat{v}(n,t)$ will specify a policy $\hat{\pi}$.

Theorem 5 $\| \pi^* - \hat{\pi} \| \to 0$ *as* $\Delta t \to 0$.

Proof: Because $\Delta\hat{V}(t)$ uniformly converges to $\Delta V(t)$; for $\Delta v^0(n,t), n = 0,1,\ldots,C, 0 \le t \le T$ we can choose Δt sufficiently small so that whenever $\Delta v^0(n,t) < \rho_1$ we can have $\Delta\hat{v}(n,t) < \rho_1$; so $\Delta v^0(n,t)$ and $\Delta\hat{v}(n,t)$ will specify an identical policy for first class booking requests.

Similarly, from the uniform convergence we can choose Δt sufficiently small so whenever $\Delta v(n,t) < \rho_2$ we can have $\Delta\hat{v}(n,t) < \rho 2$; hence we need only to consider the case where $\Delta v^0(n,t) = \rho_2, \Delta\hat{v}(n,t) \ge \rho_2$.

Uniform convergence implies (see **Section** 3.4)

$$\begin{aligned} \mid \rho_2 - \Delta\hat{v}(n,t) \mid &\leq \mid \rho_2 - \Delta v^0(n,t) \mid + \mid \Delta v^0(n,t) - \Delta\hat{v}(n,t) \mid \\ &\leq \mid \rho_2 - \Delta v^0(n,t) \mid + \circ(\Delta t) \\ n &= 0,1,\ldots,C \\ 0 &\leq t \leq T \end{aligned}$$

Suppose at t^* we have $\Delta v^0(n,t^*) = \rho_2$, then $\mid \rho_2 - \Delta\hat{v}(n,t^*) \mid\leq \circ(\Delta t)$. Let $\hat{t}$ be the time point at which we first have $\rho_2 - \Delta\hat{v}(n,\hat{t}) \leq 0$, apparently we have $\mid t^* - \hat{t} \mid\leq \Delta t$; hence $\int_0^T \mid \pi^*(n,t,2) - \hat{\pi}(n,t,2) \mid dt \leq \Delta t$, and

$$\parallel \pi^* - \hat{\pi} \parallel\leq m \cdot C \cdot \Delta t.$$

The result follows.

Corollary 9 $\mid v_{\pi^*}(s_0) - v_{\hat{\pi}}(s_0) \mid\rightarrow 0$ *as* $\Delta t \rightarrow 0$*, where* $s_0 = (C, T, 0)$.

3.6 Littlewood's Formula vs. Optimality

The optimality can be achieved by solving the differential equation either (3.61)-(3.62) or (3.81)-(3.82) and following the optimal policy indicated thereof. In **Section 3.4** we have mentioned that $t - n$ plane be divided into two regions and the dividing 'line' is:

$$\begin{aligned} \rho_2 &- \Delta v^0(n,t) = 0 \qquad (3.92) \\ 0 &< t \leq T \\ n &= 0,1,\ldots,C \end{aligned}$$

It's understood that the optimality is achieved by comparing the incremental optimal expected revenue of accepting requests from class 1 passengers only with that of accepting requests from both classes. Along the dividing 'line' the two incremental expected revenues are equal.

On the other hand, if we use Littlewood's formula instead at time t and we should protect n seats for the booking requests of first class passengers as long as n is the maximal integer which solves the following inequality

$$\begin{aligned} \rho_2 &< \rho_1 P[N_1(t) \geq n] \\ 0 &\leq t \leq T \\ n &= 0, 1, \ldots, C \end{aligned} \tag{3.93}$$

where

$$\begin{aligned} P[N_1(t) \geq n] &= \sum_{j=n}^{\infty} \frac{[\Lambda_1(0) - \Lambda_1(t)]^j e^{-[\Lambda_1(0) - \Lambda_1(t)]}}{j!} \\ 0 &\leq t \leq T \\ n &= 0, 1, \ldots, C. \end{aligned} \tag{3.94}$$

Let

$$\begin{aligned} \Delta \underline{v}(n,t) &= \rho_1 P[N_1(t) \geq n] \\ 0 &\leq t \leq T \\ n &= 0, 1, \ldots, C \end{aligned} \tag{3.95}$$

Using (3.95) we can write (3.93) as the following

$$\begin{aligned} \rho_2 &< \Delta \underline{v}(n,t) \\ 0 &\leq t \leq T \\ n &= 0, 1, \ldots, C \end{aligned} \tag{3.96}$$

It's easy to see that $\Delta \underline{v}(n,t)$ is nonincreasing in n and nondecreasing in t. Suppose we use Littlewood's formula continuously overtime the following equation

$$\rho_2 = \Delta \underline{v}(n,t) \tag{3.97}$$

$$\begin{aligned} 0 &\leq t \leq T \\ n &= 0, 1, \ldots, C \end{aligned}$$

divides the $n - t$ plane into two disjoint 'regions' $R^i_\sigma, i = 1, 2$:

$$\begin{aligned} R^1_\sigma &= \{(n,t) : \rho_2 - \Delta \underline{v}(n,t) < 0, 0 \leq t \leq T, n = 0, 1, \ldots, C\} \\ R^2_\sigma &= \{(n,t) : \rho_2 - \Delta \underline{v}(n,t) \geq 0, 0 \leq t \leq T, n = 0, 1, \ldots, C\}. \end{aligned}$$

where we have used the subscript σ to denote the policy determined by using Littlewood's formula continuously.

The policy σ will accept booking requests from class 1 only in the 'region' R^1_σ and accept booking requests from either classes in the 'region' R^2_σ.

Let $t_i, i = 0, 1, \ldots$ be the dividing points, the dividing 'line' of the two regions can be found directly from (3.97), i.e.

$$\begin{aligned} \rho_2 &= \rho_1 P[N_1(t_1) \geq 1] = \rho_1 (1 - e^{-[\Lambda_1(0) - \Lambda_1(t_1)]}) \\ \rho_2 &= \rho_1 P[N_1(t_2) \geq 2] = \rho_1 (1 - e^{-[\Lambda_1(0) - \Lambda_1(t_2)]} - [\Lambda_1(0) - \Lambda_1(t_2)] e^{-[\Lambda_1(0) - \Lambda_1(t_2)]}) \\ &\ldots \\ \rho_2 &= \rho_1 P[N_1(t_n) \geq n] = \rho_1 (1 - \sum_{j=0}^{n-1} \frac{[\Lambda_1(0) - \Lambda_1(t_n)]^j e^{-[\Lambda_1(0) - \Lambda_1(t_n)]}}{j!}) \\ 0 &\leq t_i \leq T \\ i &= 1, 2, \ldots, n \\ n &= 1, 2, \ldots, C \end{aligned}$$

Define $t_0 = 0$. To further compare the optimal policy π^* given by optimality equation and the policy σ given by using Littlewood's formula continuously overtime, we take the derivative of (3.95) with respect to t and get the following system of differential equations

$$[\Delta \underline{v}(n,t)]' = [\Delta \underline{v}(n-1,t) - \Delta \underline{v}(n,t)] \lambda_1(t) \quad (3.98)$$

$$\begin{aligned} 0 &< t \le T \\ n &= 1, 2, \ldots, C. \end{aligned}$$

Define

$$\begin{aligned} \Delta \underline{v}(0,t) &= \rho_1 \\ \Delta \underline{v}(n,0) &= 0 \\ \Delta \underline{v}(0,0) &= \rho_1 \\ 0 &< t \le T \\ n &= 1, 2, \ldots, C \end{aligned} \tag{3.99}$$

Comparing (3.98)-(3.99) with (3.76) we conclude

$$\begin{aligned} [\Delta \underline{v}(n,t)]' &\le [\Delta v^0(n,t)]' \\ \Delta \underline{v}(n,t) &\le \Delta v^0(n,t) \\ 0 &\le t \le T \\ n &= 0, 1, \ldots, C \end{aligned}$$

i.e. the dividing 'line' computed by Littlewood's formula will lie below that computed by optimality equation. In other words, Littlewood's formula does not protect enough seats for the first class passengers.

Let $v_\sigma(n,t,f), n = 0, 1, \ldots, C, 0 \le t \le T, f = 1, 2$ be the expected revenue when there are n empty seats, time t before the plane will take off, a booking request of class f passengers just arrived, and the policy σ is followed. To compute $v_\sigma(n,t,f)$ we just evaluate policy σ. From **Section 2.3** we have

$$v_\sigma(n,t,f) \;=\; \rho_f \sigma(n,t,f) + \sum_{i=1}^{2} \int_0^t v_\sigma(n - \sigma(n,t,f), q, i) \lambda_i(q) e^{[\Lambda(t) - \Lambda(q)]} dq$$

$$\begin{aligned} v_\sigma(0,t,f) &= 0 \\ v_\sigma(n,0,f) &= \rho_f \\ v_\sigma(0,0,f) &= 0 \\ n &= 1,2,\ldots,C \\ 0 &< t \leq T \\ f &= 1,2. \end{aligned} \tag{3.100}$$

(3.100) can be written out as the following

$$v_\sigma(n,t,1) = \rho_1 + \sum_{i=1}^{2} \int_0^t v_\sigma(n-1,q,i)\lambda_i(q)e^{[\Lambda(t)-\Lambda(q)]}dq \tag{3.101}$$

and if $\rho_2 \geq \rho_1 P[N_1(t) \geq n]$

$$v_\sigma(n,t,2) = \rho_2 + \sum_{i=1}^{2} \int_0^t v_\sigma(n-1,q,i)\lambda_i(q)e^{[\Lambda(t)-\Lambda(q)]}dq \tag{3.102}$$

if $\rho_2 < \rho_1 P[N_1(t) \geq n]$

$$v_\sigma(n,t,2) = \sum_{i=1}^{2} \int_0^t v_\sigma(n,q,i)\lambda_i(q)e^{[\Lambda(t)-\Lambda(q)]}dq. \tag{3.103}$$

Define

$$\begin{aligned} v_\sigma(n,t) &= \sum_{i=1}^{2} \int_0^t v_\sigma(n,q,i)\lambda_i(q)e^{[\Lambda(t)-\Lambda(q)]}dq \\ v_\sigma(0,t) &= 0 \\ v_\sigma(n,0) &= \rho_f \\ v_\sigma(0,0) &= 0 \\ n &= 1,2,\ldots,C \\ 0 &< t \leq T. \end{aligned} \tag{3.104}$$

In terms of (3.104), (3.101)-(3.103) can be written as

$$v_\sigma(n,t,1) = \rho_1 + v_\sigma(n-1,t)$$

$$v_\sigma(n,t,2) = \begin{cases} \rho_2 + v_\sigma(n-1,t) & \text{if } \rho_2 \geq \rho_1 P[N_1(t) \geq n] \\ v_\sigma(n,t) & \text{if } \rho_2 < \rho_1 P[N_1(t) \geq n]. \end{cases}$$

From (3.104) we can derive that

$$\begin{aligned} v_\sigma(n,t) - v_\sigma(n,t-\Delta t) &= \sum_{i=1}^{2} \int_{t-\Delta t}^{t} v_\sigma(n,q,i) e^{\Lambda(t)-\Lambda(q)} dq - \\ & \quad v_\sigma(n,t-\Delta t)[1 - e^{\Lambda(t)-\Lambda(t-\Delta t)}] \end{aligned}$$

so we have

$$\begin{aligned} [v_\sigma(n,t)]' &= \sum_{i=1}^{2} v_\sigma(n,t,i)\lambda_i(t) - v_\sigma(n,t)\lambda(t) \\ &= \begin{cases} [\rho_1 + v_\sigma(n-1,t) - v_\sigma(n,t)]\lambda_1(t) & \text{if } \rho_2 \geq \rho_1 P[N_1(t) \geq n]. \\ \sum_{i=1}^{2}[\rho_i + v_\sigma(n-1,t) - v_\sigma(n,t)]\lambda_i(t)] & \text{if } \rho_2 < \rho_1 P[N_1(t) \geq n]. \end{cases} \\ v_\sigma(0,t) &= 0 \\ v_\sigma(n,0) &= \rho_f \qquad (3.105) \\ v_\sigma(0,0) &= 0 \\ n &= 1,2,\ldots,C \\ 0 &< t \leq T. \end{aligned}$$

To actually compute $v_\sigma(n,t,f)$ we only need to solve the system of differential equations (3.105). Comparing (3.105) with (3.57) and the boundary condition (3.53)-(3.55) we conclude

$$\begin{aligned} [v_\sigma(n,t)]' &\leq [v^0(n,t)]' \\ v_\sigma(n,t) &\leq v^0(n,t) \\ 0 &\leq t \leq T \\ n &= 0,1,\ldots,C \end{aligned}$$

i.e. The derivative of the expected revenue from policy σ is small than that from optimal policy. Because two systems of differential equations have the same initial condition, we conclude that Littlewood's formula, even being used continuously overtime, underestimates the maximal expected revenue.

The above conclusions also hold in the multiple fare case.

Chapter 4

The Directions of Further Research

It seems immediate that the model we have developed can be extended in two directions. First, we may be able to take cancellation and overbooking into consideration possibaly at the cost of a more complicated state space. Second, we may remove Assumption 1, i.e. consider multileg problem, a problem which is more realistic; but the structure of the optimal policy, if exists, will probably be complicated to interprete.

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