AN EVALUATION OF AN ALTERNATIVE ORGANIZATION FOR CURRICULUM EMPHASIZING THE INTERCONNECTEDNESS OF MATHEMATICS

BY

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ABSTRACT

This study was initiated as a response to the call of the National Council of Teachers of Mathematics (NCTM) and British Columbia Ministry of Education for a mathematics curriculum which would enable students to formulate connections among key mathematical ideas. As well, the study addresses the problem of students' errors in selecting appropriate strategies to complete a given algebraic task. Authorities such as Fey, Barbeau, Cooney, McKnight, and Manhard believe that a traditional curriculum organized around isolated mathematical objects (i.e., Real Numbers, Exponents, Radicals, Polynomials, etc.) tends to produce fragmented teaching and learning. The study investigated an alternative organization of the curriculum which emphasizes the main processes of mathematics (i.e., Factor, Simplify, and Solve). The research questions focused on the effects this reorganization would have on: students' understanding about the interconnectedness of mathematics, students' abilities in selecting appropriate strategies, students' achievement scores on standard tests, and amount of class time needed to cover the learning outcomes in the curriculum.

Two Mathematics 11 classes were selected to participate in the study. One was taught using the traditional organization of curriculum emphasizing mathematical objects, while the other was taught using an alternative

organization of curriculum emphasizing mathematical processes. The various research questions required both quantitative and qualitative methods in the acquisition and analysis of data. The teacher's journal was used to record classroom observations for the duration of the study. Tests containing open-ended items were given at the beginning and the end of the study to determine students' abilities in selecting appropriate strategies and to evaluate students' understanding of the interconnectedness of mathematics. These tests were followed by interviews with five students from each class to clarify their responses on the written tests and to provide further information with regard to the research questions. Students' achievement on standard tests was determined through an analysis of covariance.

From observations recorded in the teacher's journal it was noted that the process-organized class dealt with general ideas and concepts during introductory lessons rather than at the end of the unit as in the object-organized class. As well, the process class had numerous discussions about the main ideas of mathematics which occurred on a regular basis from the beginning of a unit till the end. From the journal and the acetate rolls used during instruction, it was determined that the process class required fewer class periods than the object class to cover the same portion of the curriculum.

Although there was no statistically significant difference between the classes in terms of achievement scores on the standard tests, the process class

was better able to identify the inappropriate use of strategies in a given solution and was better able to provide an appropriate strategy of their own in responding to algebraic tasks.

Findings from this study suggest that teachers should rely less on the traditional mathematical object organization as shown in curriculum guides and textbooks to structure their units and lessons, and more on their own organization of curriculum emphasizing what they believe to be most important mathematical ideas. An organization of the curriculum should not only highlight the main mathematical concepts and ideas, but it should also indicate how these ideas and concepts are related and connected. This study provides some evidence that when students experience a curriculum that is organized so as to emphasize processes which are used with different mathematical objects, their understanding of the interconnectedness of mathematics is improved.

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DEDICATION

This thesis is dedicated to my wonderful wife Mhairi and

our precious son Royden.

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CHAPTER 1: THE PROBLEM

Research results (British Columbia Ministry of Education, 1989; Paul & Richbart, 1985; Robitaille, 1992), government decisions (British Columbia Ministry of Education, 1990), and societal pressures have compelled teachers to reflect upon their own educational practices and question their current approaches to classroom instruction. The push for more practical, real-life situations to be incorporated in classroom instruction (Atchison et al., 1984; House, 1988) along with an emphasis on student-centered approaches have contributed to an immense pressure on educators to examine alternative forms of education for the future. In British Columbia changes have already been taking place. These include more practical and real-life skills being offered to students such as problem solving, graphing, and statistics found in the 1988 revisions of the curriculum guide (B.C. Ministry of Education, 1988), along with an emphasis on more student-centered approaches to teaching as outlined in the Year 2000 documents (B.C. Ministry of Education, 1990). Even the discipline of mathematics has felt the pressures to alter its approaches toward education and student development. Two statements from the National Council of Teachers of Mathematics (NCTM) Standards suggest the following directions for mathematics education.

> We need to shift toward mathematical reasoning -- away from memorizing procedures.
> We need to shift toward connecting mathematics, its ideas, and its applications -- away from treating mathematics as a body of isolated concepts and procedures. (NCTM, 1991, p. 3)

Teaching mathematics should involve more than the presentation of isolated topics and structures. It should engage students in a wide array of tasks that better enable them to formulate ideas regarding the interconnectedness of mathematical concepts and processes (Borko et al., 1992; Brutlag & Maples, 1992; Davis, 1987). It is the study of this interconnectedness of mathematics that should be at the heart of any organizational theme involving the teaching of mathematics. Cooney states that the organizational theme in teaching mathematics must address the needs of students and must enable students to better understand mathematical concepts and principles.

> The object of any organizational theme in teaching mathematics is to facilitate the students' learning of mathematics. Students' ability to learn mathematics is directly related to their understanding of how mathematical concepts and principles are related-- in short an understanding of the structures of mathematics. ... These considerations should receive primary attention in any strategy a teacher uses for organizing classroom instruction.
> (Cooney, 1985, p. 150)

With the influx of new, more sophisticated technology and industry's increasing demands for a more highly skilled labour

force, mathematics education needs to change accordingly (House, 1988). This would entail reexamining the content, organization, and emphasis of curriculum. Teachers themselves should become more adept at organizing and presenting mathematics as a network of interconnected concepts and procedures. The NCTM Standards call for teachers to teach mathematics in a manner that better enables students to see and understand connections among the principal concepts and procedures.

> The teacher should demonstrate a deep understanding of mathematical concepts and principles, connections between concepts and procedures, connections across mathematical topics. ... [M]aking frequent mathematical mistakes, using limited or inappropriate representations, or presenting mathematics as a static subject whose meaning is derived solely from symbolic representations suggests that the teacher does not have an acceptable command of mathematics.
>
> The teacher should engage the student in a series of tasks that involve interrelationships among mathematical concepts and procedures. The acquisition of mathematical concepts and procedures means little if the content is learned in an isolated way in which connections among various mathematical topics are neglected. ... [T]he teacher should emphasize communication with the intent of expanding students' understanding of mathematical content and connections. (NCTM, 1991, p. 89)

Student Difficulties

An important factor in aiding students to understand mathematics better is the determination of where their difficulties

lie. Analyzing common student errors can provide useful information for both teachers and students (Davis, 1984). In particular, a number of student errors involving algebra questions may be due to the presentation and organization of the current curriculum (Manhard, 1985). Many students have difficulties deciphering which concepts and procedures pertain to a given algebraic task. They may have a tendency to focus on the symbolic representations of the question rather than the question itself, relying on their ability to recall a previously attempted question containing similar symbolic representations (Booth, 1988; Kent, 1979). An example of such a student error is shown in Figure 1.

Simplify the following:

$$\frac{x+4}{x-2} - \frac{x+1}{x-3}$$

Possible Student Error:

$$C.D.=(x-2)(x-3)$$

$$(x-2)(x-3)\frac{x+4}{x-2} - \frac{x+1}{x-3}(x-2)(x-3)$$

$$(x-3)(x+4) - (x-2)(x+1)$$

$$x^2 + x - 12 - x^2 + x + 2$$

$$\underline{2x - 10}$$

Figure 1: An Example of a Student Error; Relying on the Symbolic Representation of the Question

In this example the symbolic representation involves two algebraic fractions. The student probably has some recollection of encountering algebraic fractions in other questions and proceeds to carry out a similar set of procedural steps. In the first step the student appears to be on the right track, as subtracting fractions requires a common denominator (C.D.). However in the second step the student multiplies both fractions by this common denominator. This may indicate the student has observed this procedure before, in the process of responding to a "solve" example. In other words, the student appears to be using a "solve" process in a "simplify" exercise. By overlooking what is meant by the term simplify and concentrating only on the two algebraic fractions, this student has made an error in selecting an appropriate strategy.

Equally as important as this student selecting an appropriate strategy is the connections he/she may be making about certain mathematical concepts and procedures. In Figure 2, an example is given to show what possible connections may have been utilized to respond to the task given in Figure 1.

If students are asked to choose which connection is most like the original example, students who make errors like the one shown in Figure 1 may be more likely to choose Connection 1, even though Connection 2 follows a very similar set of procedures. This would be a case where the student relies

primarily on the symbolic representations or the mathematical objects within the example to relate to other similar examples.

Simplify the following:

$$\frac{x+4}{x+2} - \frac{x+1}{x-3}$$

Possible Connection 1	**Possible Connection 2**
-connection more likely to lead to an **inappropriate** selection of strategy.	-connection more likely to lead to an **appropriate** selection of strategy.
Solve for x:	**Simplify the following:**
$\frac{x+4}{x+2} - \frac{x+1}{x-3} = 1$	$\frac{3}{4} - \frac{1}{2}$

Figure 2: An Example of the Possible Connections a Student Might Make on a Given Algebraic Task

Teachers may be contributing to these student difficulties, of selecting appropriate strategies and formulating meaningful connections in mathematics, by presenting a fragmented sequence of daily lessons (Artzt & Newman, 1991; Manhard, 1985). Consider how the mathematical concepts may have been presented to a student who responds as in Figure 1. This student may have received instruction on how to simplify rational expressions, followed later by instruction on how to solve rational equations, both taught under the unit heading "Rational Expressions." Thus it may seem logical to the student that the mathematical objects

(i.e., algebraic fractions) rather than the meaning of the directions (i.e., simplify) are more important, as the procedures for simplifying rational expressions and solving rational equations were likely taught consecutively under the same general topic. Upon reexamining the response from Figure 1, we see that the student demonstrates a certain degree of skill in following a set of mathematical procedures. However, skill or no skill, logic or no logic, this student will likely receive negative feedback from her/his teacher orally or on written evaluations.

> It is nice when what you do is right, but its much better when it is also appropriate. Knowing the right method won't do you much good if you don't get around to using it. It is no wonder some students may hold certain reservations about mathematics, particularly if they are given little opportunity to formulate meaningful connections among the required concepts and procedures. The problem of student alienation is compounded by the lack of linkages to other representations that might provide informative feedback on the appropriateness of actions taken. (Kaput, 1989, p. 172)

Students need to be presented with many opportunities to formulate meaningful connections in mathematics. They need to be presented with alternate ways of relating mathematical concepts and procedures to what is important. Students having difficulties may need to receive some direction in deciding what is important. The organization of any curriculum should reflect what is important -- in mathematics, is it the mathematical objects or

the mathematical processes?

Object Vs. Process

Mathematical objects refer to the unit headings found in the tables of contents of many algebra textbooks, such as Real Numbers, Exponents, Radicals, Polynomials, Rational Expressions, etc. Students having the curriculum presented to them in this object-organized format may be more likely to classify exercises and examples according to the symbolic representations (mathematical objects) of a given task, rather than the process required to respond appropriately to the directions. They may be more likely to classify an example as an "exponent" or a "rational expression" question, rather than as a "simplify" or "solve" question. Although each particular mathematical object does possess its own set of concepts and procedures, it is imperative that students be shown which sets of concepts and procedures are transferable to other mathematical objects. A means of enabling students to see the interconnectedness of mathematics across mathematical objects may be to reorganize the curriculum emphasizing mathematical processes.

Mathematical processes refer to the directions found in most algebraic exercises; these are given in the imperative: Simplify, Solve, Factor, Graph, etc. Under this alternative organizational scheme students may observe, for example, a series of "simplify" exercises and realize that the same concepts and procedures apply

regardless of the symbolic representations being used. As well, in attempting future algebraic tasks, students may be drawn more to the meaning of the direction itself, enabling them to select more appropriate strategies. In particular, when they are responding to a "simplify" exercise, they are relating it to another previously completed "simplify" task and not to a task involving a different mathematical process.

The organization of any curriculum should not be taken for granted or merely accepted as that found in the curriculum guide or text. The organization not only determines the order and focus of instruction, but may also play a major role in how students make and recognize mathematical connections. For a mathematical object organization of the curriculum, major headings are determined by the main symbolic representation. Each unit is further broken down into a sequence of instructional topics, based upon the operations and processes which pertain to the particular symbolic representation (mathematical object). The unit could also be broken down by teaching the main mathematical objects as a series of minor mathematical objects. An illustration of this traditional organization can be seen from a simplified version of the table of contents from the Mathematics 11 textbook (Alexander, Atkinson, Kelly, & Swift, 1989), as shown in Fig. 3.

1 THE REAL NUMBERS (m.o.)
- 1-1 The Natural Numbers (m.o.)
- 1-2 The Integers (m.o.)
- 1-3 The Rational Numbers (m.o.)
- 1-4 The Irrational Numbers (m.o.)
- 1-5 Operations With Radicals (p.o.)
- 1-6 Rationalizing the Denominator (p.o.)
- 1-7 Radical Equations (m.o.)

2 POWERS (m.o.)
- 2-1 Definition of Exponents and Powers (m.o.)
- 2-2 Positive Integral Exponents (m.o.)
- 2-3 Integral Exponents (m.o.)
- 2-4 Rational Exponents (m.o.)
- 2-5 Simplifying Exponential Expressions (p.o.)
- 2-6 Solving Exponential Equations (p.o.)

3 POLYNOMIALS AND RATIONAL EXPRESSIONS (m.o.)
- 3-1 Operations With Monomials (p.o.)
- 3-2 Operations With Polynomials (p.o.)
- 3-3 Factoring Trinomials (p.o.)
- 3-4 Factoring Difference of Squares (p.o.)
- 3-5 Factoring Sum and Difference of Cubes (p.o.)
- 3-6 Factoring by Grouping (s.p.)
- 3-7 Simplifying Rational Expressions (p.o.)
- 3-8 Multiplying and Dividing Rat. Exp. (p.o.)
- 3-9 Adding and Subtracting Rat. Exp. (p.o.)
- 3-10 Solving Rational Equations (p.o.)

m.o. - mathematical object
p.o. - specific process with a specific object
s.p. - specific process (involving only one set of procedures.)

Figure 3: An Example of the Units and Possible Lesson Topics Using an Organization of the Curriculum Emphasizing Mathematical Objects

Regardless of organization, it is good educational practice

for the teacher to draw students' attention to the similarities and differences among the processes involved in manipulating different mathematical objects. Despite these attempts, students may find it easier to see stronger connections between simplifying rational expressions and solving rational equations when they are taught in succession, as opposed to being taught at different times under different unit headings. The way the content of the curriculum is organized and presented in the form of daily lessons may play a key role as to what connections students see and make in mathematics (Artzt & Newman, 1991; Manhard, 1985). The organization of the content in some textbooks tends to suggest that the connections within the objects and symbols of mathematics are more important than the processes used in dealing with these objects (Ediger, 1986; Howson, Keitel, & Kilpatrick, 1981). This in itself may not be wrong. However, it needs to be asked whether there is an alternative organization of the curriculum that would encourage students to make more meaningful connections in mathematics. An organization of the curriculum emphasizing the processes and procedures involved in carrying out mathematical tasks may prove to be a beneficial alternative. This new approach to organizing the curriculum would use the main directions found in most algebraic tasks as the major unit headings: Simplify, Factor, Solve, Graph and Prove. These unit headings would be further subdivided into a series of sub-processes or strategies.

An outline for this organization is shown in Figure 4.

1 FACTOR (g.p.)
- **1-1 Prime Factoring (s.p.)**
- **1-2 Finding G.C.F. and L.C.M. (s.p.)**
- **1-3 Factoring Out The G.C.F. (s.p.)**
- **1-4 Factoring Two Terms (g.p.)**
- **1-5 Factoring Three Terms (g.p.)**
- **1-6 Factoring Four or More Terms (g.p.)**
- **1-7 Finding G.C.F and L.C.M. For Algebraic Expressions (g.p.)**

2 SIMPLIFY (g.p.)
- **2-1 Evaluating Arithmetic Expressions (g.p.)**
- **2-2 Utilizing Brackets (s.p.)**
- **2-3 Utilizing Exponents (g.p.)**
- **2-4 Utilizing Radicals (g.p.)**
- **2-5 Reducing (s.p.)**
- **2-6 Multiplication and Division (s.p.)**
- **2-7 Addition and Subtraction (s.p.)**

3 SOLVE (g.p.)
- **3-1 Isolating the Variable (s.p.)**
- **3-2 Factoring (s.p.)**
- **3-3 Utilizing the Quadratic Formula (p.o.)**
- **3-4 Eliminating Powers (p.o.)**
- **3-5 Eliminating Radicals (p.o.)**
- **3-6 Making the Bases the Same (p.o.)**
- **3-7 Solving for More Than One Variable (g.p.)**

p.o.- specific process with a specific object
s.p.- specific process (involving only one set of procedures.)
g.p.- general mathematical process (involving more than one set of procedures.)

Figure 4: An Example of the Units and Lesson Topics Using an Alternative Organization of the Curriculum Emphasizing Mathematical Processes

In this organizational scheme the prescribed curriculum still provides the content for instruction, but the order and focus of the instruction emphasizes the processes and strategies involved in actually doing the mathematics. As well, using mathematical processes as major headings may enable teachers to discuss with their classes the different types of relationships that exist among these terms. They may ask the students, "Can you think of an example in factoring where you need to simplify or solve?" or "Can you think of an example in simplifying where you need to solve or factor?" or "Can you think of an example in solving where you need to simplify or factor?" Students may see that at times certain processes are incorporated within other processes and that at other times, when two processes such as simplifying and factoring are inverses of one another, each provides a way of checking the other. Both organizational schemes may have their advantages and disadvantages; however, allowing for more than one representation of the curriculum over the duration of students' experiences in mathematics may better enable them to see alternative ways concepts and procedures relate to each other.

Organizing the curriculum to emphasize mathematical processes is intended to highlight the interrelationships that exist among the concepts, procedures, and ideas of mathematics, but how will it affect the time needed to cover an extensive

curriculum? And how will it affect students' performance on standard tests? Reorganizing the curriculum may require a great deal of effort, as materials and lessons would need to be developed to reflect a new curricular structure. Not only may this reorganization need to be seen as beneficial to students' understanding, but may also need to be seen as feasible for those who must implement this change. The teacher should be involved with any changes occurring to the content or the structure of the curriculum. If any curriculum change is to occur it must be seen as necessary and logical by teachers who must implement this change. Rachlin (1989) supports this argument in the following statement:

> Any proposed change must be understood, accepted as necessary, and considered feasible by teachers who will implement the change. ... Teachers play a large decision making role in the implementation of curriculum. A better understanding of the role of students' cognitions and behaviors in their learning is necessary but not sufficient to improve the learning of algebra. To implement change in curriculum we must understand the nature of teachers' beliefs and cognitions and the roles these beliefs and cognitions play in the decisions teachers make as they present the new curriculum to their students. ... In addition to research that helps us understand the diversity in students' thinking we also need to understand the diverse range of prior knowledge, experience, beliefs, attitudes, and ways of processing information that teachers bring to the change process. (Rachlin, 1989, p. 261)

Constraints Facing Teachers

Secondary mathematics teachers are faced with two major constraints. One is the amount of class time necessary to cover the intended learning outcomes of the curriculum; the other is student performance on school-wide, district-wide, or province-wide examinations. Within the Mathematics 11 curriculum for British Columbia, there are no fewer than fifty new learning objectives which must be covered over the course of the school year (British Columbia Ministry of Education, 1990). Many alternative approaches to mathematics education may be easily dismissed, if teachers perceive them as causing too great a burden on class time. An alternative approach should provide evidence to the teacher that it allows for time to cover the curriculum and more importantly time to address students' needs.

It may be argued that judging a teacher on student performance is unjust, yet this is sometimes done at the secondary level. It creates a tremendous pressure on the teacher to ensure that students have adequately covered test material necessary to perform well on common examinations. This also creates additional stresses on class time, as teachers must cover a certain amount of the curriculum by a certain point in time. Teachers need to be assured that any alternative approach will maintain or improve students' achievement on common tests or exams. At times, change occurs slowly at the secondary level due

in part to the pressures and constraints placed on the teachers. Therefore, for meaningful change to occur the constraints facing teachers must be taken into consideration.

Statement of the Problem

Many algebra students have difficulties making meaningful connections among the concepts and procedures within mathematics. As well, many of these same students make errors in selecting appropriate strategies necessary to respond correctly to given algebraic tasks. The traditional organization of the curriculum emphasizing mathematical objects may be a contributing factor to these student difficulties, as it provides an outline from which many teachers tend to structure the sequence of daily lessons (Robitaille, 1992; O'Shea, 1987). An organization of the curriculum which presents mathematics as a set of isolated and fragmented topics may tend to lead teachers to organize units of instruction as sets of isolated and fragmented lessons. An alternative organization of the curriculum emphasizing mathematical processes may provide an outline for teachers to arrange their daily lessons in a way that better enables students to see the interconnectedness of mathematics.

Teachers of Mathematics 11 are faced with many constraints due to a full curriculum with many new learning outcomes for students, along with standard common examinations to assess student achievement. One of the most important of these

constraints is time. In order to be attractive, an alternative organization of the curriculum should be seen by teachers as being potentially beneficial to students without causing tremendous burdens on class time. Therefore, it is necessary to demonstrate that an alternative organization of the curriculum is not only feasible, but may offer some potential benefits to students' understanding of mathematics.

Research Questions

The research questions of this study are designed to determine whether or not an alternative organization of the curriculum emphasizing mathematical processes is feasible and whether or not it offers any benefits to student understanding. What exactly are the effects of reorganizing the current Mathematics 11 curriculum, emphasizing mathematical processes rather than mathematical objects? In particular, will reorganizing the curriculum significantly affect:

1. the development of the students' understanding of and appreciation for the interconnectedness of mathematics?
2. the nature and number of students' errors in selecting an appropriate strategy to complete a given task?
3. the level of students' achievement on standard school- or district- wide tests?
4. the amount of class time spent on covering the intended learning outcomes of the curriculum?

The study examines a new approach to curriculum organization, testing its efficacy (i.e., the effects on student learning) and its feasibility (i.e., the amounts of time and effort it requires). These tests of effectiveness and feasibility are based on a small sample of 44 students with the teacher acting as researcher. Therefore, the answers to the research questions will not be generalizable beyond the study. The results will be used to provide evidence that this type of reorganization can occur without adverse effects on class time and student achievement, as well to explore some potential benefits for students' understanding of mathematics.

Limitations of the Study

The greatest limitation of this study was that the teacher who taught both classes also conducted the research and wrote the paper. This person from now on will be referred to as the teacher/researcher.

The generalizability of this study is restricted by the selection, number, and background of the students, by the length of time the study was conducted, and by the fact that the teacher was acting as the researcher. All the students in the two selected mathematics classes were assigned to those classes by normal school selection procedures. These procedures proved to be a threat to the validity of the study, as two out of a total of 11 Mathematics 11 classes in the school were reserved for the

students with "above average" ability. Neither of the classes involved in the study was among these. Therefore, the majority of the students in the study could be categorized as "average" to "below average" in ability. The results of the study can only be generalized with respect to these types of students. Another threat to the validity of the study was the factor of student mortality, since 48 students were initially assigned to these classes, but due to rescheduling only 44 took part for the entire project. The teacher/researcher had originally been assigned four Mathematics 11 classes, but one class was not assigned until two weeks into the school year, and another class consisted primarily of students repeating the course. Thus these two classes were not included in the study, and only two classes took part.

The study involved a small sample and took place over a period of ten weeks. These limitations were put in place to allow for a triangulation of methods needed to determine students' understanding about the interconnectedness of mathematics. The research questions involving achievement tests and class time can only be answered on a small scale. These questions were asked to test the feasibility of using this new organization of curriculum against the traditional one. This study is presented as an examination of an innovative approach to the organization of curriculum, and it is hoped that it will provide a guide for further research in this area.

CHAPTER TWO: REVIEW OF LITERATURE

This chapter begins with a brief sketch of the history of curricular change in British Columbia leading up to an analysis of the present mathematics curriculum guide. Next, the discussion will consider teachers' dependence on the curriculum guide and corresponding textbooks, followed by an evaluation of the traditional organization of the curriculum. The chapter then examines the effects of computer technology and problem solving on the mathematics curriculum, as well as the effects of various learning models and error analyses. Finally, recommendations made by various authorities are used to suggest how a new organization of the curriculum might look.

History of Curricular Change in British Columbia

In the 1820s, North American universities and colleges made algebra a part of their entrance requirements (Atchinson et al, 1984; Rachlin, 1989). From 1820 to 1900, the topics included equations and formulae, fundamental operations with rational expressions, powers and roots, and factoring polynomial expressions (Osborne & Crosswhite, 1970). British Columbia followed the rest of North America in establishing algebra as part of its mathematics curriculum.

In 1884, the Department of Education in British Columbia published Course of Study for Graded and Common Schools. The

document provided some guidance regarding the content of the mathematics curriculum. By 1890, the department started to suggest how certain content ought to be taught. Elsewhere, at this time, Cajori conducted a survey of mathematics teachers and found that many of them saw an over-emphasis on manipulative skills in the curriculum and were calling for a more meaningful treatment of mathematics (Rachlin, 1989).

In 1911, the mathematics curriculum was outlined in some detail for all grades and was linked to a prescribed textbook. During this same time, graphing was introduced into the curriculum as a way to integrate the fields of algebra and geometry. From this point on there has always been a gradually increasing emphasis on this topic and a gradually decreasing emphasis on factoring, powers and roots. In 1936, The British Columbia Department of Education produced a three volume guide containing specified content for all grade levels and providing much information on how mathematics teaching should be conducted based on student interest and motivation. This guide was considered to be somewhat revolutionary compared to the other curriculum guides in North America (Osborn & Crosswhite, 1970). Despite this revolutionary curriculum guide, in 1947 the prescribed textbooks constituted much of the course requirements for the school year (O'Shea, 1987).

From the 1960s to the beginning of the 1970s, the invasion

of "new math" in North America was to significantly alter the content of school mathematics. Some attribute this change to the launch of Sputnik in October of 1957 (Paul & Richbart, 1985; Usiskin, 1985). The "new math" introduced the concepts of sets and relations into the curriculum along with an emphasis on understanding. Still much of the content for the curriculum guide came directly out of the prescribed textbook (O'Shea, 1987).

By the early 1970s, the public perception of the new math was that it was a failure, for it was seen not to address the needs of weaker mathematics students (Usiskin, 1985). There was a push for curriculum to emphasize basic computational and procedural skills. In British Columbia at this time there were three prescribed textbooks, and teachers were asked to use their own judgement in selecting materials they felt were most appropriate in covering the intended learning outcomes in the curriculum guide. In 1976, a province-wide survey was carried out involving a random sample of 200 mathematics teachers in British Columbia (Beattie & Steblin, 1977). Seventy-one percent of the teachers said they used the curriculum guide in planning, and most commented that lack of time was the major obstacle to their participating in curriculum development.

1988 British Columbia Curriculum Guide

The present mathematics curriculum guide for British Columbia (British Columbia Ministry of Education, 1988) contains

most of the traditional algebraic topics, but there is an increased emphasis on problem solving, transformations of graphs of functions, and statistics. In the Mathematics 12 course there is a new unit introducing calculus to the students. The new curriculum guide provides a detailed outline for the content and intended learning outcomes to be taught and thus would have a certain amount of influence on lesson structure and unit organization.

In a survey conducted by Johnson (1986), 403 secondary mathematics teachers were asked to comment on the draft of the British Columbia Curriculum Guide. Most responses were either "no changes" indicating an acceptance of the proposed draft or "no comment" which was also taken to mean acceptance. Although there was an overall acceptance of the guide, some general concerns were expressed about the amount of content covered in the curriculum. A year later, Overgaard (1987) conducted a similar survey with 306 secondary mathematics teachers in British Columbia and found that 62% felt there was too much content dealt with in Mathematics 11 and 12. As well, 29% felt the level of difficulty was too high, and many of the teachers commented directly about the strict time limitations they were placed under while teaching these courses.

The Mathematics 11 section of the 1988 Mathematics 7-12 curriculum guide (British Columbia Ministry of Education, 1988)

has five main topics: Algebra, Relations and Functions, Geometry, Trigonometry, and Data Analysis. There are fifty new intended learning outcomes for students along with limiting examples and prerequisite skills. Under the topic of Algebra, the curriculum guide lists the new factoring required for grade 11 (i.e., sum and difference of cubes, and grouping as perfect square trinomials) and then presents objectives for simplifying rational expressions, followed by solving rational equations, and word problems relating to rational expressions. The curriculum guide then contains simplifying radicals followed by solving radical equations. Finally, it lists changing radicals to exponents and solving exponential equations (B.C. Ministry of Education, 1988). The guide's outline of the algebra section appears to emphasize mathematical objects, (i.e. rational expressions, radicals, and exponents).

The 1990 British Columbia Mathematics Assessment (Robitaille, 1992) was conducted in part to obtain information about the degree to which the 1988 curriculum had been implemented and to gain insights about the instructional practices employed by mathematics teachers. All students in Grades 4, 7 and 10 in B.C. were part of the study along with over 4000 teachers. Over 90% of the teachers felt the curriculum guide significantly assisted them in planning mathematics instruction. As well, 83% of the teachers found that the prescribed texts

corresponded well with the curriculum. An area in need of improvement was found in students' responses to the open-ended problems (Szetela, 1992). Students did not utilize a wide variety of strategies. As well, the inability of students to evaluate an already solved problem successfully suggested that students tend to do little monitoring of strategy implementation (Szetela, 1992).

Other areas of concern for the mathematics students of B.C. came from the Report to Schools (B.C. Ministry of Education, 1989). The markers of the 1989 Mathematics 12 January and June Provincial Examinations made the following observations about the areas students most needed improvement:

1. Arithmetic and process errors were extremely common;
2. Students made many errors from their inability to understand reciprocals;
3. Students showed a lack of understanding of binomial expansion;
4. Students were extremely weak in simplifying answers;
5. Students were often confused by the word "solve"; and
6. Many students did not recognize factoring as a quick and easy way to solve equations.

(British Columbia Ministry of Education, 1989, p. 4)

Observations 4 through 6 may suggest that students need to develop a broader understanding about mathematical processes.

Influence of Curriculum and Text on Instruction

There are strong indications that teachers rely heavily on the curriculum guide and the prescribed textbooks for constructing lesson plans (Begle, 1973; Goodlad, 1984; Howson, Keitel, & Kilpatrick, 1981; McKnight et al., 1987; Rowan and Morgan, 1980; Willoughby, 1990). Goodlad (1984) conducted a study of 1000 classrooms in 38 schools in the United States. The impression he received of curriculum from the topics, materials, and texts was that mathematics is presented as a body of fixed facts and skills to be acquired. Although teachers wanted their students to become logical thinkers, they tended to use rote methods of teaching, relying heavily on the textbook (Goodlad, 1984).

Ediger (1986) carried this reliance on textbooks one step further; she commented that not only do teachers rely heavily on the text, but they tend to use its table of contents as a framework for their units of instruction.

> Numerous mathematics teachers lean heavily upon the adopted single or series of textbooks to ascertain scope. The table of contents may then provide a generalized framework for what is to be taught and in what sequence. ... The above-named topics may well suggest unit titles in the mathematics curriculum. (Ediger, 1986, p. 14)

Glidden (1991) discussed the implications of the Second International Mathematics Study where eighth and twelfth grade teachers completed a questionnaire involving various teaching decisions. Both groups of teachers reported that certain teaching methods were not used because they were not treated in the textbook or they did not appear in the curriculum guide. Twelfth grade teachers also reported that they did not include certain topics as they were not covered in the textbook, the curriculum guide, or on external examinations. Glidden concluded that textbooks, curriculum guides, and external exams exert a powerful limiting influence over teachers.

Rowan & Morgan (1980) found that mathematics teachers who define curriculum in terms of the prescribed textbooks also tended to use standard tests as part of their definition. But Rowan & Morgan argued that curriculum should be based upon the rationale of the program, the nature of mathematics, and a knowledge of how students learn. After such a curriculum is established, texts and tests should be selected based upon how well they reflect the curricular goals.

The Traditional Approach to Curriculum Organization

Ediger (1986) discusses how the table of contents from the prescribed text can directly influence curriculum organization in the classroom. The prescribed text in British Columbia for Mathematics 11 along with the curriculum guide tend to organize

topics and learning outcomes emphasizing mathematical objects (British Columbia Ministry of Education, 1988). As well, publishers and teachers tend to divide mathematics up into tiny compartments and to teach one compartment at a time (Willoughby, 1990). Fey summarizes the typical organization of algebra emphasizing mathematical objects as follows, "Algebra in secondary schools is primarily the study of polynomial, rational, and exponential expressions over various subsets of the complex numbers" (Fey, 1989, p. 212).

The way mathematics curriculum is organized has a direct effect on how teachers plan mathematics instruction (Robitaille, 1992; Manhard, 1985). From an article discussing the implementation of the Curriculum and Evaluation Standards (NCTM, 1989), Brutlag and Maples (1992) comment on mathematics teaching in the past.

> In the past, most mathematics concepts were presented to students as separate ideas, each living in a different section of the textbook. Students were expected either to make connections on their own or to wait until the ideas were used together in another course. The result was that for many students, the connections were never made, nor was the significance of the mathematics apparent. (Brutlag & Maples, 1992, p. 230)

The traditional treatment of mathematics as a set of isolated topics allows for few interconnections between concepts and ideas. (Atchison et al, 1984; Manhard, 1985; NCTM, 1991). Artzt and

Newman (1991) suggest that presenting mathematics as a field of study separated by artificial boundaries results in fragmented learning.

Effects of Computer Technology on Mathematics

A curriculum that breaks down these artificial boundaries should include a means for adapting to changes from outside education, and should present mathematics to students and teachers in a way that promotes a better understanding about the interconnectedness of mathematics. Computer technology has brought about certain questions as to what should be emphasized in high school mathematics. Business and industry are requiring workers with better problem-solving ability, and this too has influenced how mathematics is taught in the classroom. Although this study did not use computers, a number of the features involved with computer instruction are similar to features used in the experimental treatment. Working with computers requires a broader understanding of how things work and relate to one another. Students need to be able to see and work with broader concepts and relationships as opposed to manipulating tiny isolated components (Fey, 1989). The structure of the mathematics curriculum should be designed to accommodate these types of changes and shifts in emphasis.

From hand-held calculators to desktop workstations, computer technology is poised and ready to impact the content and

presentation of mathematics in schools. Technology will decrease the value of many computational and procedural skills traditionally found in many mathematics curricula and will make possible tools for teaching and learning of mathematics in a variety of sophisticated ways (Mathematical Sciences Education Board, 1990). Two examples of such tools are the computer programs EXPRESSIONS and muMATH.

In an article investigating Artificial Intelligence in the teaching and learning of algebra, Thompson (1989) introduces readers to a computer program called EXPRESSIONS. This program was designed to emphasize the cognitive and structural features of algebra, both of which can be a major source of students' difficulties. The program is based on the premise that multiple representations of an object, along with student actions and immediate computer feedback, will develop the students' relational understanding of content (Thompson, 1989). Students can work their way through an algebraic expression or equation under the direct guidance of an expert algebraic performer, i.e. the computer. An appropriate strategy is selected by the student clicking a button on the right hand side of the screen. The part of the expression or equation being acted upon is similarly selected by the student clicking a button on the desired location. Students are able to see the immediate effects of the strategies they have selected and can evaluate their selections on their own

or with computer assistance. EXPRESSIONS has been used in a pilot study with seventh graders and has shown promising results in increasing students' awareness of selecting appropriate strategies (Thompson, 1989).

Fey (1989) discusses a similar program in his article on school algebra in the year 2000. He states that symbol manipulation computer programs such as muMATH can offer assistance to students much like the skills of an algebra expert. The instructions for major algebraic processes such as FACTOR, SOLVE, or SIMPLIFY provide simple procedures for formal transformations of algebraic expressions and equations. Using the computer to carry out the symbol manipulation allows the student to focus on strategy selection and implementation as well as the connections among the given information, the chosen strategy, and the results. These computer programs can be used to enrich students' understanding of fundamental concepts and rules.

It has been argued that the advancement of technology in calculators, computers and other electronic information processing necessitates a change in the goals and teaching of mathematics (Mathematical Sciences Education Board, 1990). The increased use of computer technology in society as well as in the schools provides an excellent opportunity to re-balance and reassess the important relationships among mathematical skills and concepts within the curriculum (Fey, 1989). Computer technology makes it

more important for students to be presented mathematics in a way that emphasizes connections among key mathematical concepts and procedures.

Effects of Problem Solving in Mathematics

Along with the affects of using computers in the mathematics classroom, the pressures from business and industry to hire employees with adequate problem-solving skills also needs to be addressed in the way mathematics is taught. The mathematics curriculum must allow for the integration of problem solving throughout. From the NCTM's Curriculum and Evaluation Standards for Mathematics, comes the following statement:

> Mathematical problem solving in its broadest sense is nearly synonymous with doing mathematics. Thus, whereas it is useful to differentiate among conceptual, procedural and problem-solving goals for students in the early stages of mathematical learning, these distinctions should begin to blur as students mature mathematically. In grades 9-12, the problem-solving strategies learned in earlier grades should have become increasingly internalized and integrated to form a broad basis for the students approach to doing mathematics, regardless of the topic at hand. (NCTM, 1989, p. 137)

This sentiment is echoed in the B.C. Ministry of Education's draft proposal for the Mathematics strand in the Graduation Year 2000 Program.

> Mathematical problem solving is not a distinct topic, but a process of inquiry that provides the context for constructing and applying mathematical knowledge. As such, it permeates all topics within the field of mathematics. (British Columbia Ministry of Education, 1992, p. 25)

Part of the reason for this emphasis on problem solving is that it requires a number of highly valued mathematical skills, such as: making a table to determine the relationship between two variables; working backwards as a way of checking a certain procedural step; and using a simpler numerical task to test out a particular method before performing the given algebraic task (Frederiksen, 1984). Simon (1980) and Schoenfeld (1985) believe these problem-solving skills can be taught and methods for teaching these skills need to be acquired by all mathematics teachers. As well, the importance of these problem-solving skills needs to be reflected in the present structure of the mathematics curriculum.

Theoretical Models for Understanding of Mathematics

In addition to determining what mathematical skills students should possess, it is equally important to determine how students acquire these skills and how they understand other mathematical concepts and relations. Three theoretical models have been presented, to provide a reference for investigating how individuals acquire and process knowledge in mathematics. The first, metacognition, is based upon students' awareness of their

own cognitive processes and their ability to monitor and regulate these processes. The last two learning models, referred to as the "frames" model and "schema" model, involve developing knowledge networks.

Metacognition

Garofalo and Lester (1985) suggest that many of the skills necessary in metacognition are also necessary for completing algebraic tasks successfully. Metacognitive knowledge involves the awareness and the implementation of strategies in carrying out certain mathematical tasks. Metacognitive knowledge is necessary when information about where, when, and how various strategies are applied, is required. As students develop in their own metacognitive abilities, retrieval of appropriate strategies comes more easily and adjustments to strategies are made (Garofalo & Lester, 1985).

The regulatory aspect of metacognition is concerned with the decisions and ongoing evaluation of activities one might engage in during the process of performing difficult mathematical tasks. These activities may include planning a course of action, selecting appropriate strategies, monitoring the implementation of the strategies, and evaluating the outcome of these strategies. These metacognitive skills are seen as crucial in solving problems and understanding mathematical relationships. (Schoenfeld, 1985; Silver, 1982).

The Frames Model

The first model of knowledge networks utilizes the concept of "frames," which Davis (1984) defines as a vast collection of knowledge representation structures. Intellectual growth occurs as new, more complex frames are built across the foundations of previously constructed frames. Although each individual is said to possess his/her own unique set of frames, in mathematics there are many commonly shared features of frames (Davis, 1984). A list of these features is presented below.

1. Frames serve as a structured outline for organizing new data.
2. Frames can be identified through analysis of student errors.
3. Frames may be legitimately based upon correct earlier learning.
4. Frames will not function correctly unless all the necessary information is provided.
5. Frames are persistent, in that changes to them are often not permanent.
6. Frames follow a definite set of rules.
7. Frames are usually retrieved when the learner is presented with specific cues and general situational similarities.

Students' memory capacity for frames is limited and therefore the more links among them, the easier the retrieval process becomes. (Davis, 1984; Simon, 1980; Young, 1982)

Perkins and Simmons (1988) have characterized frames

into four interlocked types of knowledge: the content frame, the problem-solving frame, the epistemic frame, and the inquiry frame. The content frame contains facts, definitions, and algorithms associated with the content of the subject matter. Retrieval of the content frame brings facts and vocabulary necessary to interpret the question. The problem-solving frame incudes general problem-solving strategies and metacognitive processes. Functions of this frame may include breaking a task down into smaller parts, regulating the time spent on any one particular strategy, and seeking alternative strategies. The validity of content and problem frames are measured according to the norms established in the epistemic frame. Retrieval of the epistemic frame is necessary to make sense of a mathematical concept, to prove equivalence from formal axioms, to connect real world situations to mathematics, to give evidence, or to explain rationales. The inquiry frame contains general beliefs and strategies that work to extend and challenge knowledge within a certain domain. Retrieval of the inquiry frame is necessary for students to construct their own tasks, to ask other questions related to a given task, to generalize findings, to challenge elements of theory, and to think creatively and critically (Perkins & Simmons, 1988).

These four frames of knowledge distinguish the difference between expertise and understanding of a particular subject

matter. Students may be master manipulators of textbook algebra problems, but may possess no understanding of algebra. In other words, they may have well developed content and problem-solving frames without similarly well developed epistemic or inquiry frames (Anderson, 1983). Students need knowledge from all four frames to understand mathematical concepts fully (Perkins & Simmons, 1988).

Perkins & Simmons use these four frames as a rationale for teaching with understanding. They say that instruction should include all four frames wherever possible, frames should not be treated in isolation, and instruction should present students with the substance of these frames and their interrelationships. The organization of curriculum along with lesson structure should also emphasize connections among these frames.

The Schema Model

The second model based upon knowledge networks is essentially another way of looking at individual frames. According to Marshal (1988), there are three types of knowledge: declarative knowledge, procedural knowledge, and schema knowledge. Declarative knowledge refers to the facts and concepts an individual requires to operate within a given domain. Procedural knowledge is the set of skills an individual must possess to function within a given domain. Schema knowledge is the ability of an individual to make sense of a new set of

circumstances and experiences. It is "a set of knowledge that relates a set of declarative facts to a set of procedural rules" (Marshal, 1988, p. 163).

Schema knowledge can be further broken down into four components (Marshal, 1990). The first component is feature recognition: this refers to the body of facts, features and characteristics that enable one to recognize which schema applies. The second component is constraint: this is the set of conditions, rules, and limitations that pertain to the operation of a particular schema. The third component is planning: this refers to the mechanisms within a schema that set appropriate subgoals and formulate a sequence of steps to be followed in order to complete a given task. The fourth component is implementation: this is the collection of procedural rules and algorithms that are applied to carry out the response to the given task.

The four components are not a set of hierarchically arranged levels of knowledge; rather they represent a network of knowledge consisting of many links and chains. The accessing of a schema is the manner in which knowledge is retrieved from an individual's long-term memory. A highly developed schema consists of many links among mathematical concepts and procedures. Thus when any single piece of information involves any aspect of a given schema, the entire network is activated. If several distinct components of knowledge exist in different, isolated

schema networks, then each network will need to be activated separately. In this circumstance, if any component has not been activated, then necessary knowledge may not be retrieved from an individual's long term memory to adequately respond to the task at hand (Marshall, 1988).

Through the study of schema development and from careful observation of individual differences in schema knowledge, valuable extensions of current theory about students' understanding could be obtained (Silver, 1982), in particular, studies involving comparisons of the schema knowledge utilized by experts and novices during work on mathematical tests. Schema theory has the potential to offer powerful explanations of students' knowledge and understanding regarding the interrelationships among mathematical concepts and ideas (Silver, 1982).

Skemp (1986) talks about the functions of schema. "It integrates existing knowledge, it acts as a tool for future learning, and it determines the type of understanding" (Skemp, 1986, p. 49). The integrative function of schemata comes into play when a concept is recognized on two levels: as itself and as a member of a class. This class concept is linked by mental schemata containing a vast number of other concepts which in turn influences the way this concept is adapted into the schema framework. As well, schemata from other fields of experience may be brought to the surface by existing links and may further

affect the interpretation and adaptation of the particular concept. The more schemata that are available, the better the chance of understanding the concept (Skemp, 1986).

Existing schemata are also valuable tools for the acquisition of future knowledge. Almost everything one learns depends on some sort of prior knowledge. Higher learning, therefore, depends on previously acquired basic schemata. Schemata which are built up during early learning of mathematics are crucial elements in determining the ease or difficulty with which a student understands new concepts. If a schema is acquired through meaning and understanding, it becomes an efficient learning device as future schemata are likely to be developed through a similar pattern. A new experience that fits into an existing schema is much better remembered, than one that does not. Unsuitable schemata are a major handicap to future learning as what is understood temporarily may be quickly forgotten (Skemp, 1986).

A schema can be just as powerful a hindrance as a help in acquiring new knowledge. It can be seen as a major instrument of adaptability serving both as an organizer of existing knowledge and an acquirer of new knowledge. If situations are encountered where the available schema is not adequate, the stability of the schema becomes an obstacle to adaptability. At this point reconstruction is required before the new situation can be

understood. Reconstruction of schemata in mathematics is essential as students must continually accept broader number systems. First they learn the natural number system, then they must adjust their schemata to include integers, and adjust their schemata again once rational numbers have been introduced. It is far easier to understand the defensive nature of students' reactions to new ideas, when one attributes it to the process of having to reconstruct a new schema. If time is taken in providing meaning and understanding during this time of reconstruction, the new schemata will also serve as useful tools for future learning (Skemp, 1986).

According to the schema model, to understand something means to assimilate it into an appropriate schema. Skemp (1978) describes two types of understanding: instrumental and relational. Instrumental understanding involves a knowledge of a set of well-delineated but isolated rules that necessitates the memorization of the type of task to which each rule applies. Instrumental understanding produces basic individual schemata that become enlarged as more tasks are performed. Relational understanding involves knowledge of general principles which can be extended or adopted to different situations. Schemata become connected to form broader networks of understanding. Relational understanding requires a combination of knowledge about "how to" do something with knowledge pertaining to the reasons underlying a particular

action. While schema acquired through instrumental understanding are considered to be compartmentalized bits of knowledge, schema acquired through relational understanding are well integrated networks (Skemp, 1978).

Metacognition, frames, and schema provide models for studying how students understand; the models share common notions about the value of connecting different and related pieces of knowledge through meaningful learning. The organization of mathematics material should provide students and teachers with examples of a sound network of knowledge. Ideally it should highlight a variety of relationships and connections that exist in mathematics. Presenting a lesson topic should be like activating a well-developed schema; it should trigger previously acquired schemata and form new links and chains among them (Kaput, 1989; Kirshner, 1989). Gaining insight about how students learn and understand mathematics could provide an important framework for developing an appropriate structure for the curriculum.

Analysis of Student Errors

If it is important to gain information about students' understanding, then it must be equally important to gain information about their misunderstanding. Davis (1984) supports the importance of studying patterns in students' errors as providing a means for improving instruction and learning. He states that there are two kinds of regularities observed in

mathematical errors. The first kind occurs with those extremely common errors made by a large number of students. He uses examples like $3 \times 5 = 8$ or $3^2 = 6$. An explanation for these types of errors is that students tend to apply the mathematics they have last understood while seeking solutions to given tasks (Davis, 1984; Manhard, 1985).

The second kind of regularity occurs when an individual consistently responds incorrectly to a certain type of question. There are a number of different explanations offered for these errors. Erlwanger (1973) noted in his study involving a sixth grade student, that a malfunction usually occurs in the same location of the student's cognitive machinery. He describes it as a "super-procedure" selecting the wrong "sub-procedure." Once the super-procedure has been activated by some sort of visual cue in the question, it then calls upon a series of sub-procedures that the student has either invented or misinterpreted (Erlwanger, 1973).

Davis offers other explanations for these errors committed by individuals. One explanation is the over-generalization of a previously learned rule or procedure to contexts where the rule is inappropriate or requires modification. Figure 5 shows an example of a student solving a quadratic equation by factoring without setting one side of the equation equal to zero.

Solve for x.

CORRECT APPLICATION OF A PROCEDURE	OVER GENERALIZATION OF A PROCEDURE
$x^2 - 5x + 6 = 2$	$x^2 - 5x + 6 = 2$
$x^2 - 5x + 6 - 2 = 2 - 2$	$(x - 2)(x - 3) = 2$
$x^2 - 5x + 4 = 0$	$(x - 2) = 2$ or $(x - 3) = 2$
$(x - 4)(x - 1) = 0$	$x = 4$ or $x = 5$
$x = 1$ or $x = 4$	

Figure 5: An Example of an Over Generalization of a Mathematical Procedure (Davis, 1984)

In order to correct these errors in process, students must first become aware of the nature of their errors. They must develop an understanding of the particular task at hand rather than relying on rotely memorized procedures to provide solutions. It is most probable that this lack of understanding about the nature of the given mathematical task has led to the error in the first place (Davis, 1984).

Booth (1988) and Kent (1979) both attribute many of students' systematic errors to a misfocus of attention. Booth in her analysis of the Strategies and Errors in Secondary Mathematics (SESM) project, conducted in the United Kingdom with students ages 13 to 15, states that students tend to focus their attention on finding an answer to an algebraic question,

rather than focusing on the procedural steps involved in obtaining an answer. Students are so intent about getting an answer that they overlook what the task is asking them to do (Booth, 1988). Many of students' errors in performing algebraic tasks may be attributed to this unidimensional focus on just looking for the answer. This may indicate a need to develop educational materials and lessons which help students focus upon the key concepts within a given task.

Gliner (1991) suggests that not only do students focus attention toward getting an answer, but they tend to focus on irrelevant characteristics of a question, leading them to make false generalizations. He says that often students focus solely on the surface structures of symbols which are devoid of meaning. Students need to become aware of what provides meaning in a question or task in order to respond appropriately (Gliner, 1991; Kent, 1979).

An Alternative Approach to Curriculum Organization

The organization of the curriculum and daily lessons should aid students in focusing upon what is most important in mathematics. Perhaps the present organization is placing too much emphasis on the symbolic representations and objects of mathematics and not enough on procedures and processes. A curriculum organization emphasizing mathematical processes and procedures may contribute to alleviating persistent student

errors. There is a definite need for a continuing assessment of mathematical topics assisted by analysis of errors in making decisions regarding students understanding of mathematics. (Booth, 1988)

The curriculum establishes the content for mathematics courses, and the organization of the curriculum should establish the emphasis and focus for students and teachers. At present, this organization could emphasize computer technology and problem-solving skills that are deemed desirable in business and industry (Willoughby, 1990). As well, this organization could emphasize the ways in which students learn and understand mathematics. It could also aid them in avoiding misunderstanding by highlighting the most important concepts and ideas. Curriculum organization could better enable students to see and understand the interconnectedness of mathematics. This notion of curriculum connecting key mathematical concepts is discussed in Curriculum and Evaluation Standards for Mathematics.

> Mathematics must be approached as a whole, concepts, procedures and intellectual processes are interrelated in a significant sense. The curriculum should include deliberate attempts through specific instructional activities, to connect ideas and procedures both among different mathematical topics and with other content areas. (NCTM, 1989, p. 11)

Similarly, the fourth curriculum intention from the 1992 British Columbia Ministry of Education's proposed mathematics

program reads as follows:

> The learner will have opportunities to develop mathematical power by making mathematical connections.
>
> *****
>
> Making mathematical connections enables learners to broaden their perspective to view mathematics as an integrated whole rather than as a set of isolated topics. Opportunities need to be provided for both students and teachers to develop a broader understanding of significant mathematical concepts and how they are related to other parts of the curriculum. These opportunities must include time to develop a substantial overview of the present mathematics curriculum. (B.C. Ministry of Education, 1992, pp. 26-27)

Cobb, Yackel, and Wood (1992) argue that curriculum should present mathematical meanings and structures in a readily apprehensible form, and that this approach to developing a curriculum would ensure that the connections among mathematical concepts students make are essentially correct and well thought out. A curriculum needs to stress the relationships among the topics in the major areas of mathematics. It should make use of certain key mathematical ideas that permeate all of mathematics (Artzt & Newman, 1991; Fey & Good, 1985; Young, 1982). These ideas need to be highlighted and used to demonstrate the interrelationships between apparently different mathematical topics (Artzt & Newman, 1991; Fey & Good, 1985).

Davis (1984) suggests that a major decision in presenting a

curriculum to students is the "size of bits and pieces" with which the student is being asked to deal. Any size chunk can be defensible as long as the key ideas are apparent. A curriculum consisting of only bite-sized pieces, may fail to produce the synthesis of the key ideas. Knowledge must eventually be integrated into larger wholes (Davis, 1984).

Sonnabend states that the mathematics curriculum should change its focus from isolated specific topics to the main ideas of mathematics. This change in focus needs to occur at all levels of mathematics in order to emphasize conceptual understanding. Focusing on the main ideas of mathematics allows room for other applications which may involve the employment of computers and calculators (Sonnabend, 1985). "By using a broad range of topics one improves the odds of getting a reasonable representative picture of the kind of mental information processing that mathematics requires" (Davis, 1984, p. 310).

Manhard (1985) discusses the benefits in transferring from a traditional organization of the curriculum to a more integrated one. He suggests taking the same topics, that had been used in the algebra-geometry-algebra sequenced curriculum, and reorganizing them to reflect a change in emphasis. In the traditional sequence, classes tended to study distinct subjects with few connections between them (Manhard, 1985). The proposed integrated program was designed to stress the important

underlying themes of mathematics and exemplify them across various topics. He suggests that important skills and concepts should be spiraled within such a program. It should also provide a built-in flexibility in that topics may be easily interchanged, lengthened or shortened. Concurrently this provides for more creative curriculum development and enables students to develop a greater understanding of the interrelationships among the different topics in mathematics. "An integrated program can demonstrate the interconnectedness of mathematics in a way other programs cannot" (Manhard, 1985, p. 196).

It is usually easier to develop and plan a curriculum than it is to implement and evaluate one. From the research end of the spectrum New York State had adopted an alternate integrated program on a trial basis (Paul and Richbart, 1985). They made every effort to include all the essential aspects of the traditional secondary curriculum. Certain topics had to be reorganized under the new structure and others had to be de-emphasized. In 1983 this integrated program was tested against the traditional program. The study involved 365 students from the integrated program and 935 students from the traditional program. The study concluded that students who were enrolled in the integrated program had no practical disadvantage on traditional forms of assessment, but showed some evidence of having a deeper level of understanding about certain key mathematical concepts than their

counterparts enrolled in the traditional program (Paul & Richbart, 1985).

Summary

The organization of curriculum should be adaptable to change with the impending future impact of computer technology. It should encompass the general mathematical skills found in the area of problem solving. The organization should also include an emphasis on how students learn and understand mathematics, as well as enabling students to see the bigger picture of mathematics. Authorities feel a need for the mathematics curriculum to engulf a broader spectrum of key mathematical ideas and provide a means for connecting the present isolated topics (B.C. Ministry of Education, 1992; NCTM, 1991). The answers pertaining to how best to organize the mathematics curriculum may lie within the interconnectedness of mathematics itself.

CHAPTER 3: METHODS

Due to the varying nature of the research questions two different methodological approaches were used in the study. The questions involving students' selection of strategies and understanding of the interconnectedness of mathematics lent themselves to a qualitative approach, whereas the questions involving student achievement and amount of class time seemed to require a quantitative approach. This chapter begins with a discussion of the general design of the study including population, sample selection, and treatments. Next, the methods and sources for collecting the data are presented, followed by the details of how the data were analyzed.

Population and Sample Selection

The study was conducted at a secondary school in the interior of British Columbia. The school has an enrollment of approximately 1200 students in Grades 8 through 12. Two different Mathematics 11 classes containing a total of 44 students were involved in the study. Selection of the students was carried out by normal timetabling procedures and the discretion of the teacher/researcher as outlined in Chapter 1. All students in the study wrote two pretests and two posttests. In addition, five students from each class were chosen to take part in follow-up interviews. The individual interviews were conducted to clarify

answers to the open-response questions and to solicit further data pertaining to students' understanding about the interconnectedness of mathematics. Students were selected for the interviews to represent the general make-up of the class in terms of age, previous mathematics courses, and on the basis of the likelihood that they would reveal insightful information, as judged from their pretests.

Treatments

The study provided a comparison between two different organizations of the curriculum. It was conducted from the beginning of the school year for approximately ten weeks until the end of the first term. One class was taught using a traditional organization of the curriculum emphasizing mathematical objects (i.e., Real Numbers, Exponents, Radicals and Rational Expressions), while the other class was taught using an alternative organization of the curriculum emphasizing mathematical processes (i.e., Factor, Simplify, and Solve). Both classes were taught by the same instructor and covered the same portion of the curriculum as mandated by the Ministry of Education for British Columbia (1988).

Data Collection Procedures

The collection of data came from a variety of different sources. These included students' responses on standard achievement tests and open-ended tests. Classroom observations

were recorded in a teacher journal and verified by colleagues' notes and by the acetate rolls used during instruction. Audio tapes of the follow-up interviews with selected individuals were used after the pretests and again after the posttests.

A pretest-posttest quasi-experimental design was used to deal with prior differences between the two classes regarding the students' abilities and knowledge of the interconnectedness of mathematics. Within the first two weeks of the school year all students were given two pretests. Pretest 1 (see Appendix A) was used to determine students' abilities with respect to standard school and district-wide tests. Pretest 2 (see Appendix B) was used to determine students' abilities in selecting appropriate strategies and to determine students' understanding about the interconnectedness of mathematics. After the two pretests, individual follow-up interviews were conducted with students to clarify their written answers and to gain additional information of their understanding about the interconnectedness of mathematics.

A journal was maintained by the teacher/researcher for the duration of the project, the first term of the school year. The journal was used to monitor the structure, content, and time of instruction. The teacher/researcher also used an overhead projector to present written notes and daily quizzes to the classes. The acetate rolls retained from the overhead projector provided a way to verify the number of periods of instruction

needed to cover the required portion of the curriculum. Wherever possible the same instructional material handed out in the lesson was given to both classes. The differences occurred in the order and organization of the material, and how it was presented to the students.

Over the last two weeks of the term, two posttests were given to the students. Posttest 1 (see Appendix C) was the district-wide term test written by all Mathematics 11 students. Posttest 2 (see Appendix D) was a series of open-ended items used to determine students' abilities in selecting appropriate strategies and to determine students' level of understanding. Both posttests were similar in nature and form to the corresponding pretests. However, the pretests utilized material from previous mathematics courses, while the posttests utilized material from the first term of Mathematics 11. Follow-up interviews were conducted after the posttests, involving the same five individuals from each class. All tests were incorporated into the regular classroom routine, and all interviews were scheduled before or after school hours.

Teacher's Journal

Descriptions of classroom observations were recorded by the teacher/researcher in his journal as close in time to the actual events as possible. Particular attention was given to any events that might have led to insightful information regarding students'

understanding about the interconnectedness of mathematics. Approximately once a week, throughout the project, colleagues would attend a class or a portion of a class to record their own observations or to verify the observations of the teacher/researcher by making comments in the teacher's journal. Any points of interest from these observations and comments were discussed and clarified. The journal and the acetate rolls were used to verify the number of class periods required to cover the designated part of the curriculum and the type of instruction each of the two classes received.

Audio-Taped Interviews

Follow-up interviews were conducted immediately following the two sets of pretests and posttests. The interviews were used to provide clarification for the responses of the students on the open-ended items. In addition, at the end of each interview another open-ended question was used to solicit additional information regarding students' understandings about the interconnectedness of mathematics. Figure 6 shows an example of an open-ended task used at the end of an interview.

Roberta the Robot has three questions for you. The only problem is her video display terminal has malfunctioned so she can only respond orally answering "yes" or "no" to all your questions. Your task is to find out as specifically as possible what the questions were.

The instructor would have three algebra questions written down on a piece of paper (not seen by the student) such as:

1. Solve $3^x = 27$,

2. Factor $x^2 - 7x + 12$, or

3. Simplify $\frac{x + 1}{x + 2} - \frac{3}{x - 1}$

The student would ask a series of "yes" or "no" questions to which the instructor would respond accordingly. The process continued until the students had confidence that she/he discovered the hidden algebra task or until she/he ran out of questions to ask.

Figure 6: An Example of an Open-Ended Interview Question

This question was designed to elicit the elements of an algebraic task that the student regards as most important or least important. As well, to determine how the student relates one aspect of a question to another.

Pretest 2 and Posttest 2

Pretest 2 and Posttest 2 were designed to assess selected higher order thinking skills, in this case students' selection of strategies and students' understandings of the interconnectedness

of mathematics. The tests included a variety of open-ended questions. Pretest 2 contained material from previous mathematics courses and Posttest 2 contained material from the first term of Mathematics 11. Pretest 2 included two presolved algebra questions, requiring students to evaluate a given solution for a particular algebraic task. As well, one explanatory question, two compare and contrast questions, and a non-routine problem were also included. All the pretests and posttests are included in the appendices.

The presolved algebra questions were adapted from the presolved problems used in the British Columbia Mathematics Assessment (1992). These items were used to assess metacognitive actions in problem solving, where deciding on an appropriate strategy plays a fundamental role. As Szetela has observed in commenting on these items.

> One way to assess critical analysis and metacognitive actions in problem solving is to present an already solved problem where the task is to evaluate the solution rather than solve a problem. Although the given problem is already solved, the student must acquire a suitable representation, be able to monitor the solution procedure, and decide on the appropriateness of the procedure and reasonableness of the answer. (Szetela, 1992, p. 194)

In addition to evaluating the given responses for the presolved algebra questions the students were asked to give their own responses to the question. Figure 7 shows a presolved algebra question containing a significant error in the procedural steps. All but one of the presolved questions contained both a major and a minor error. A major error refers to a procedural error in the solution indicating the strategy used was inappropriate. A minor error refers to an error in calculation or in symbol manipulation. There were three types of major errors used in questions. One was the implementation of a simplify strategy for a factor question, another was the implementation of a simplify strategy for a solve question, and the other was the implementation of a solve strategy for a simplify question.

The explanatory questions required students to write a list of procedural steps that they determined would be necessary to complete a given algebraic task. Figure 8 shows an explanatory question where the student is placed in a position of peer tutor.

Simplify the following:

$$\frac{x + 2}{x^2 - 9x + 20} + \frac{3x + 1}{x^2 - 7x + 12}$$

Tom's solution for this question is:

$$\frac{x + 2}{(x - 4)(x - 5)} + \frac{3x + 1}{(x - 3)(x - 4)}$$

$$C.D.=(x-4)(x-5)(x-3)$$

$$(x-4)(x-5)(x-3)\frac{x + 2}{(x - 4)(x - 5)} + \frac{3x + 1}{(x - 3)(x - 4)}(x-4)(x-5)(x-3) \quad *$$

$$(x-3)(x+2)+(3x+1)(x-5)$$

$$x^2 - x - 6 + 3x^2 - 16x - 5 \quad **$$

$$4x^2 - 17x - 11$$

Is anything wrong with Tom's solution? Explain.

Is anything right with Tom's solution? Explain.

Show how you would you have solved this problem?

* Major Error - multiplied by value other than 1.
**Minor Error - multiplied second set of brackets incorrectly.

Figure 7: Example of a Presolved Algebra Item: A Simplify Question With a Solve Type Strategy

Imagine you are talking to a friend on the phone about a difficulty they are having with simplifying rational expressions. They provide you with the following example:

$$\frac{x + 4}{2x^2 - 2x} - \frac{5}{2x - 2}$$

What would you say to guide them through the question step by step, so that they will better understand how to deal with similar questions? Anticipate their responses.

Figure 8: An Example of an Explanatory Question

The compare and contrast question shown in Figure 9 was used to determine what connections students make among different mathematical processes. For this item only the students' reasons were marked, as there was considered to be no one correct answer.

Column A	Column B
Add/Subtract	Factor
Multiply	Simplify
Divide	Solve

a)Match items from Column A to those which are most <u>similar</u> in Column B and explain why you made your choice.

b)Using the same columns, match items from Column A to those which are most <u>dissimilar</u> in Column B. Explain why you made your choice.

Figure 9: An Example of a Compare & Contrast Question

Finally, a non-routine algebra problem was given to determine how well students initiated a strategy on their own. Non-routine problems were not readily familiar to the students, but they could be solved with the knowledge students should possess. Figure 10 shows an example of a non-routine algebra problem.

Show that the product of the sum of three positive real numbers and the sum of their reciprocals is always at least 9. And when can the product be exactly 9? SHOW ALL WORK AND EXPLAIN!

Figure 10: An example of a Non-Routine Algebra Problem.

Posttest 2 contained three pre-solved algebra questions, one explanatory question, one non-routine algebra question and one process definition question where students were asked to list what they interpreted as being the important procedural steps for certain processes such as Reduce, Simplify, Factor etc.

Pretest 1 and Posttest 1

For the purposes of this study the effects of the treatment on student achievement were determined from a pretest-posttest analysis. Pretest 1 was given at the beginning of the term to estimate the students' ability on standard algebra tests. This test did not contain any material that had not been previously covered in prerequisite courses. It followed a similar structure and format as Posttest 1 so it could provide a means of accounting for any prior differences between the classes.

Posttest 1 was the Term 1 Test used by the school district to assess all Mathematics 11 students for the first term. This test was team marked and graded according to cutscores. The marking team consisted of all the Mathematics 11 teachers in the district who marked the exams in groups of two or three using a previously agreed upon answer key. The test consisted of 15 multiple choice questions and eight long answer questions totaling 35 marks. The content of the exam covered most of the algebra portion of the curriculum excluding relations and functions. These two tests were used to answer the research question about differences between the two classes on standard achievement tests.

Analysis of Data

Just as it was necessary to collect a wide range of data for the study, it was also necessary to have a wide range of analyses to interpret the data. Interpretations of the classroom observations and of the taped interviews, along with heuristic scoring scales were used to address the research questions regarding students' understanding about the interconnectedness of mathematics and students' abilities to select appropriate strategies. An ANCOVA was used to interpret the results on the standard tests. The teacher's journal and the acetate rolls were used to determine the number of class periods.

Interpretations of Observations

The interpretations of the observations focused on those events which showed any evidence of students' understandings about the interconnectedness of mathematics or of students' ability in selecting appropriate strategies. The questions and responses of the students and the teacher were examined along with their behaviors, attitudes, and beliefs. Any patterns that proved to be prevalent with regard to classroom atmosphere or lesson directions were noted. Discussion with colleagues helped to verify the observations and brought to the surface other interesting happenings in the class. Journal notes were constantly checked for consistency against the acetate rolls used during instruction.

Interpretations of Interviews

The interviews were conducted in two parts; the first part was to enable students to clarify their written responses on the open-ended test items. The subsequent transcripts were checked for information regarding students' reasons for a response, desires to change a response, mental processes required to obtain a response, and difficulties encountered in obtaining a response. Again the interpretation of this information explored possible causal relationships and meanings with respect to this study.

The second part of each interview involved the task of having students generate a set of "yes" or "no" type questions to unveil three likely algebra tasks in the possession of the

interviewer. This exercise examined those aspects of an algebraic task that students focused on or felt were the most important. How the students used the interviewer's responses to formulate their next question was also noted. The final sets of questions generated by the students were further examined to gain possible insights as to what connections each individual was making, as well as any correlations or discrepancies with other individuals interviewed.

Heuristic Scoring Scales

The results from Pretest 2 and Posttest 2 were classified according to heuristic scoring scales. These results were used to characterize students' ability in selecting an appropriate procedure and to gain insight regarding students' understanding about the interconnectedness of mathematics. These scales provided a convenient way of comparing all the responses given on the open-ended items in the two classes. All scores were arrived at by the consensus of the teacher/researcher and another mathematics instructor. Each instructor marked one of the sets of exams, then they traded and marked the other set. No marks appeared on the exams until the second marking, so as not to influence the second marker. Any discrepancies were discussed with other mathematics teachers until a consensus was reached.

In Figure 11 the heuristic scoring scale used to evaluate the presolved algebra questions is shown. It utilizes three

different aspects of the task: first, whether or not students could identify a major or a minor error in a previously solved algebra task; second, what level of logic and understanding students used in describing the errors they discovered; and third, whether or not students could select an appropriate strategy to complete the given task.

SCORING SCALE FOR PRESOLVED ALGEBRA QUESTIONS		
Identification of Errors	**0 -**	**Did not identify any errors or identified something that was not an error**
	1 -	**Identified minor error only**
	2 -	**Identified major error only**
	3 -	**Identified both major and minor errors**
Reasons for Errors	**0 -**	**No explanation**
	1 -	**Illogical or irrelevant explanation**
	2 -	**Simplistic or unfocused explanation**
	3 -	**Correct & Logical explanation**
Students' Own Solution	**0 -**	**Did not offer a solution or offered same solution as given**
	1 -	**Offered a different solution but still contained a major error**
	2 -	**Offered a different solution but still contained a minor error**
	3 -	**Offered a complete and correct solution**

Figure 11: The Scoring Scale for the Presolved Algebra Questions

For the explanatory test items, the heuristic scale shown in

Figure 12 was used to determine the students' level of understanding about a certain algebraic process. Allowances were made in the scale for those students who proceeded to answer the task directly rather than writing an explanation of the procedures in their own words.

SCORING SCALE FOR EXPLANATORY QUESTIONS

0 - No advice offered or answered the task directly with a major error.
1 - Advice offered, but containing major error.
2 - Advice offered, but it is confusing, oversimplified, or inconsistent.
3 - Advice offered containing essentially correct information for certain aspects of the question. Could not be used alone to obtain a correct solution or answered task directly with a minor error.
4 - Sound advice offered with only a few flaws in clarification or explanation of steps or answered task directly with correct answer.
5 - Clear and sound advice offered with no flaws. Could be used to obtain a correct solution.

Figure 12: The Scoring Scale for Explanatory Questions

In Figure 13, the heuristic scale used for the compare and contrast items in Pretest 2 are shown. This scale emphasizes the ability of students to make meaningful connections in mathematics based on logical reasoning. The scale relies heavily on the judgement of the evaluators, and any comparisons made must take this into account.

SCORING SCALE FOR COMPARE AND CONTRAST QUESTION	
0 -	No logical reasoning for making connections
1 -	Simplistic or poor reasoning for making connections
2 -	Some minor flaws in reasoning for making connections
3 -	Clear and logical reasoning for making connections

Figure 13: The Scoring Scale for Compare and Contrast Questions

The process definition question from Posttest 2 required students to make a list of important procedural steps that they thought were applicable to a certain mathematical process. The heuristic scale, shown in Figure 14, was designed to measure the quality and detail of the students' list.

SCORING SCALE FOR PROCESS DEFINITION QUESTION	
0 -	Incorrect steps listed
1 -	Only a few basic steps listed for all processes
2 -	Some important steps listed for each or most processes
3 -	Most of the important steps listed for each process

Figure 14: The Scoring Scale for Process Definition Question

The scoring scale for the non-routine algebra problems, shown in Figure 15, was designed to determine the students' level of ability in selecting or developing an appropriate strategy. This scoring scale attempted to isolate just the development and implementation of a strategy in response to the second research question.

SCORING SCALE FOR NON-ROUTINE PROBLEM

0 - No attempt.
1 - Attempted, but with no logical strategy.
2 - Attempted with a strategy that would not lead to a correct solution.
3 - Attempted with a strategy that could lead to a correct solution.

Figure 15: The Scoring Scale for Non-Routine Algebra Problems

Analysis of Covariance

The Posttest 1 results were analyzed with an ANCOVA using Pretest 1 as the covariate. This analysis was done to minimize the effects of any prior differences in ability between the two classes and to minimize the effects of the non-random school selection procedures. The null hypothesis for the ANCOVA was as follows:

> There is no significant difference between the adjusted mean Posttest 1 scores of the students in the two classes.

As mentioned previously, two threats to internal validity were the selection procedure followed by the school and student mortality. Threats to external validity or interaction effects were minimized by incorporating the tests into the regular classroom routine and spacing the tests ten weeks apart. Interaction between the classes could not be controlled. As this is a feasibility study involving a small sample, the ANCOVA results are only used to show whether there were adverse effects on

students' scores on standard tests. Any other interpretations from the ANCOVA must be made with caution.

Summary

Due to the nature of the research questions it was seen as necessary to use a variety of instruments and subsequent forms of analysis to address each one adequately. Although classroom observations and follow-up interviews were used to gain insight into students' abilities in strategy selection, their main purpose was to address the question about students' understanding of the interconnectedness of mathematics. Any themes or patterns pertaining to this question that presented themselves in the journal were noted and explored further as to their possible cause and meaning. The follow-up interviews were conducted to obtain additional information about students' understandings and perceptions of the interrelationships within mathematics.

The effect on the students' ability to select an appropriate strategy was assessed primarily from the scores for the open-ended tests obtained by heuristic type scales. These results were tabulated and used to compare the two different classes.

The effects of reorganizing the curriculum with respect to students' achievement on standard tests was addressed by the use of an ANCOVA comparing students' results on a district-wide test using a similarly structured pretest as the covariate. The effect on class time was obtained by counting the number of periods

necessary to cover the required portion of the curriculum as recorded in the teacher's journal and on the acetate rolls used in instruction.

CHAPTER 4: RESULTS

As described in Chapter 3, the results of the study are organized into two major sections. The first section presents the qualitative aspect and attempts to answer the first research question which involves students' understanding about the interconnectedness of mathematics. The classroom observations as recorded in the teacher's journal, colleague's notes, and acetate rolls provided evidence for a description of the two different classes. Certain aspects of these observations were examined in some detail including class discussions and examples of students' difficulties in understanding, along with some possible explanations for these difficulties. Following the classroom observations are the results from the individual interviews, which allowed for an analysis in greater depth of students' difficulties in understanding and communicating about mathematics. In addition, the open-ended oral task part of the interview provided results which indicated how these students had organized their own thoughts about mathematical concepts and relationships.

Next the chapter presents the quantitative aspects of the study with the results from the open-ended test scores. Although these results were also intended to address the research question relating to students' understanding, they seemed better able to respond to the question regarding the students' abilities to select

appropriate strategies. Each task from Pretest 2 and Posttest 2 was examined in detail utilizing many figures and tables to allow for the reader's own interpretation of the results. A comparison between these two open-ended tests is presented in order to highlight any signs of improvement that had taken place over the duration of the project.

The research question regarding students' achievement on standard tests was answered by the use of an ANCOVA comparing the results for the two classes on Posttest 1, using Pretest 1 as a covariate. Finally, the research question of class time was answered through the examination of the teacher's journal and the acetate rolls used during instruction.

Classroom Observations

Classroom observations recorded in the teacher's journal provided a sketch of the lesson and unit structure used for both classes. Any observable differences between the two classes or general tendencies that occurred within each class were recorded. These observations provided evidence concerning students' understandings of the interconnectedness of mathematics and students' errors with strategy selection. Possible causes for these errors are presented and discussed. Specific incidents from the classes are recorded with the lesson number and its setting within the unit. The class instructed under the mathematical object organization of the curriculum had Real Numbers, Exponents and

Radicals, and Rational Expressions as unit headings, while the class instructed under the mathematical process organization of the curriculum had Factor, Simplify and Solve as unit headings.

Mathematical Object Classroom

The organization of the curriculum emphasizing mathematical objects arranged lesson topics in sequence according to the different features of the main mathematical object. For example, a sequence of lesson topics in the Exponents and Radicals Unit was: Powers with Natural Number Exponents, Powers with Integral Exponents, Powers with Literal Exponents, Powers with Rational Exponents, and Exponential Equations. Each lesson usually focused on one particular aspect of the main object, in this case whether the exponent was an integer or a rational, etc. It wasn't until the latter lessons of the unit or the review lessons that the specific concepts and procedures were connected and discussed. The review lessons provided the class the opportunity to develop an outline for the unit indicating key concepts and procedures along with general rules. An example of such an outline is shown in Figure 16.

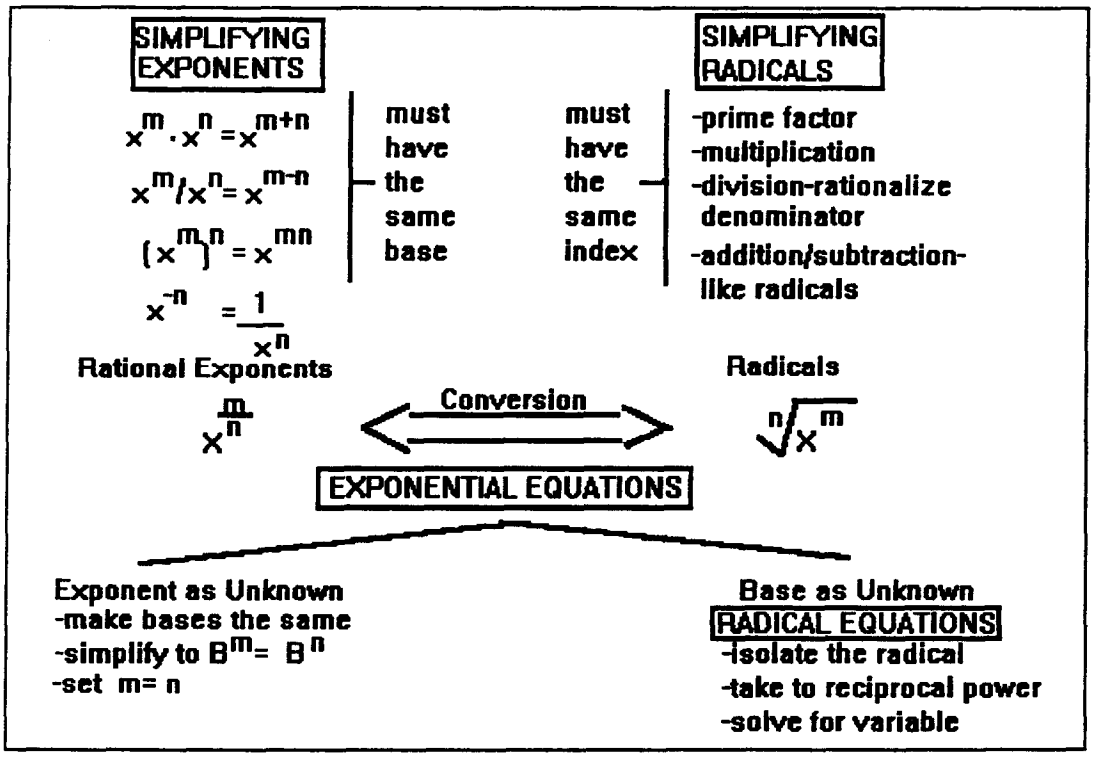


Figure 16: An Example of an Outline Produced During a Review Lesson in the Mathematical Object Class

This particular outline occurred in lesson 9 of 10 in the second unit, on Exponents and Radicals. The class was asked certain questions about what they could recall from the unit and what connections they could make among the main concepts. The results from this inquiry were illustrated on the blackboard by the instructor. An overall picture was created by linking together each key component of the unit. At the same time, there was a discussion about the major rules that applied to a number of different components such as keeping the base the same for simplifying exponents or only multiplying by one (multiplying the

numerator and denominator by the same number) when simplifying radicals. Other similarities between radicals and exponents were also discussed, for instance, pointing out for students that solving equations where the base is the unknown requires the same set of procedures as solving radical equations. For many of the students this was the first time within the unit that they were given the opportunity to see and discuss the interconnectedness of mathematics. They were presented with a graphic diagram that highlighted the main properties of exponents and illustrated how these properties were related to the main properties of radicals.

Review lessons almost always occurred before a major test or assignment; therefore cautions were usually thrown out to the students by the teacher/researcher about possible places for errors. For example, during lesson 4 of 10 in the second unit, on Exponents and Radicals, the teacher/researcher put an example on the board:

$$(x + y)^4 = x^4 + y^4$$

He asked, "How many people think this is OK?" A couple of hands were raised. "All right," the teacher/researcher continued, "What's wrong with it?" One student volunteered the answer "You can't do that when you're adding." The teacher/researcher then asked "What would make this equation correct?" Another student said "Make them both multiplications." The teacher/researcher

made this adjustment on the board (i.e., $(xy)^4 = x^4y^4$) and then put up a similar error with radicals:

$$\sqrt{x^2 + y^2} = x + y$$

A similar type of discussion took place; however, for this example the teacher/researcher asked a student to provide a numerical example that would disprove this equation. Errors were also examined and discussed after an assignment or test was handed back to the students.

Mathematical Process Classroom

The mathematical process organization of the curriculum was structured in such a way that the teacher/researcher felt compelled to deal with the whole picture at the beginning of the unit. At least a sketch of the whole picture was presented and subsequent lessons provided the details to be filled in. For example, when the teacher/researcher was conducting the first lesson for the Simplify Unit he first had to discuss what was meant by this term and what general concepts and procedures could be included in this unit. The teacher/researcher said "The most important rules to remember when you are simplifying are to follow the order of operations, BEDMAS [brackets, exponents, division and multiplication, and addition and subtraction], and to only multiply by one - or you change the question." In

subsequent lessons these two rules came up in most examples and questions. It reached the point where the teacher/researcher would ask "OK, this is a simplify question, what am I supposed to remember?" and the majority of the class would respond "You must follow BEDMAS and you can only multiply by one." Similarly at the beginning of the Solve Unit after the first initial lesson dealing with the properties of equality, the teacher/researcher said "These all come down to one important rule for solving equations - you must do the same thing to both sides of the equation." Again this rule came up constantly in each of the following lessons. Although these general statements were also used in the class where the curriculum was organized by mathematical objects, they did not occur as consistently from the beginning of the unit till the end, simply because these rules did not always apply to everything in the unit.

In one of the introductory lessons, lesson 2 of 8 in the first unit, Factor, a discussion developed about the general set of procedures involved with factoring. As a result of this discussion the class with the guidance of the teacher/researcher produced a list of procedures needed to complete a task on factoring. The following set of procedures was written on the blackboard for the students to copy:

1. Identify the number of terms.

2. Identify any common factors among the terms.

3. Factor these common factors out.

4. Identify any other similarities among the terms.

5. Use the appropriate factoring strategy

6. Check to see if answer only has one term or anything else that could be factored.

7. Check answer by simplifying i.e. multiplying out the brackets.

Each subsequent lesson referred back to this set of procedures while elaborating on the specific types of strategies and methods that could be used. These general sets of procedures produced as a result of class discussions at the beginning of a unit made it very evident when a certain lesson was out of place. In one such incident, during lesson 7 of 10 in the second unit, Simplify, the teacher/researcher had planned to teach a lesson involving undefined values. It became abundantly clear that the students could not find the undefined values while following the general procedural steps for simplifying. The lesson did not fit anywhere in the unit. The teacher/researcher recognized this midway through the lesson and said "I've made a mistake, we don't have to cover this topic until the next unit. Will somebody please make a note of this so I do not forget to deal with this topic in the next unit?" He then proceeded to

conduct a review lesson on what they had covered up to that point in the Simplify unit. This sort of situation never came up in the object class, as each unit dealt with more than one set of general procedures.

Classroom Discussions

Both classes were involved in important discussions dealing with the interconnectedness of mathematics. One difference was that in the class with the organization of curriculum emphasizing mathematical objects most of the discussions about the various relationships among concepts and procedures were held near the end of each unit during review lessons. This was due to the way the teacher/researcher arranged the sequence of the daily lessons in accordance with the outline of the learning outcomes in the curriculum guide. Usually a specific aspect of a mathematical object was dealt with during a lesson and the review lesson was the time when all the specific aspects or pieces were connected. In the class with the mathematical process organization of curriculum these discussions tended to be initiated in the introductory lessons of the unit and continued through until the end. In this case, the teacher/researcher had arranged the sequence of daily lessons where the introductory lessons dealt with the whole picture of a general process and each subsequent lesson added more detail to this picture. Holding these discussions at the beginning of unit also allowed for opportunities

to relate the concepts and procedures of the previously taught general process to the concepts and procedures of the general process about to be taught. For example at the beginning of the Solve unit, in lesson 1 of 7 the teacher/researcher wrote down the words Simplify and Solve on the blackboard and asked "What are the major differences between these two processes?" The students came up with the following lists of concepts and procedures:

Simplify	Solve
BEDMAS	Do the same thing to both sides
Only multiply by 1	Isolate the variable
Must have common denominator	Set = to 0
Reduce factors, not term	Factor
Half of an equation	Trial and Error
	Inspection

Later the students were asked to come up with examples where they had to use a number of the concepts and procedures from the Simplify list in order to solve an equation. At first there appeared to be a great deal of confusion, so the teacher/researcher wrote one such example on the board and told the students they could use other examples from their notes or their textbooks. Eventually most students were able to come up with an example and a few were selected to present theirs to the class. A similar discussion was carried out with the mathematical

object class, but not until the end of the first term at which point the teacher/researcher had presented them with all the pieces of the puzzle.

Another classroom discussion from lesson 2 of 7 in the Solve Unit involved the Properties of Equality. One student was complaining about how teachers always try to make mathematics more confusing than it already is. In response the teacher/researcher said "You're right. Why should we have all the fun?" He then wrote a linear equation on the board and stated "I want to see who can write the most possible steps in solving this equation." After about ten minutes one student was declared the winner. She was instructed to put her solution on the blackboard. As a class, they evaluated each step as to its correctness, and then decided which property it represented and whether or not it was a step under the Simplify process or the Solve process. Due to the sequence of the daily lessons chosen by the teacher/researcher no such discussion took place in the object-organized class, as Properties of Equality and Linear Equations had been taught at the beginning of the school year before general differences between solve and simplify had been presented to the students. Similarly, as a result of how the teacher/researcher had arranged the sequence of lessons, the class discussions that did take place in the object-organized class seemed to be under heavy time constraints due to their close

proximity with the tests at the end of the unit.

Examples of Student Difficulties

The examples of student difficulties in selecting appropriate strategies that were observed in the classes were similar to those discussed in Chapter 1. Both of the examples which follow come from the class taught with the organization of curriculum emphasizing mathematical objects. This is not intended to show a difference between the two classes, as similar difficulties were evident in both, but these two examples were the most prominent.

In lesson 2 of 11 for the third unit, on Rational Expressions, the following example was written on the blackboard.

Factor: $x(m + 2n) + x^2(m + 2n)$

The teacher/researcher began by asking "What should we do first here?" One student said, "Get rid of the brackets." The teacher/researcher inquired, "How many people would agree that is a good first step?" Approximately seven or eight hands went up. At this point the teacher/researcher wrote another question on the board:

Simplify: $x(m + 2n) + x^2(m + 2n)$

This time a different student interjected "What's the difference?" After a moment or two a few students began to recognize where the confusion lay, as witnessed from the nodding of heads and a couple of "Oh yeah's", finally a student from the back of the class said "This is part of that study of yours isn't it?"

A more intense situation developed while the class was going over the unit test for Rational Expressions, given at the completion of the unit (i.e., lesson 11 of 11). The teacher/researcher was explaining how to eliminate choices on the following multiple choice item:

7. Simplify the following completely:

$$\frac{5}{n^2} + \frac{3}{nw}$$

A. $5w + 3n$

B. $5nw + 3n^2$

C. $n(5w + 3n)$

D. $\frac{5w + 3n}{n^2}$

E. **None of these**

The teacher/researcher said, "You can eliminate the first three choices here, because they do not have denominators." One of the students, who had apparently circled one of these choices and who must have been experiencing a certain degree of frustration exclaimed, "That's not right! You have to get rid of denominators!" The teacher/researcher began to explain, but before he could get another word out the student jumped out of her seat, grabbed the chalk out of the teacher/researcher's hand and said "Here, I'll show you." She proceeded to write her solution on the blackboard that corresponded to choice A on the

item. It appeared from the nodding of heads that others were in agreement with her or they had also circled choice A. Once the teacher/researcher recovered his chalk he wrote the following question down using the student's same steps:

$$\frac{5}{n^2} = \frac{3}{nw}$$

$$(n^2w)\ \frac{5}{n^2} = \frac{3}{nw}\ (n^2w)$$

$$5w = 3n$$

The student who had written the solution said "See you have gotten rid of the denominator too." A couple of students from the front of the class looked back at her, one drew her attention to the equality sign. Once the student realized her mistake she appeared to be extremely embarrassed. The teacher/researcher attempted to reduce this by commenting on the commonness of such mistakes and began a discussion on how to avoid such errors in the future. These are not just examples about students selecting inappropriate strategies but are examples of how certain misunderstandings in mathematics are deeply entrenched within the minds of some students.

A Potential Source of Student Difficulties

One apparent source of students' difficulties in understanding certain mathematical concepts was their dependence on certain "short-cut" methods. Again, two examples are given

from the teacher's journal, this time both examples are from the process-organized class.

At the end of a class on solving by trial and error (lesson 3 of 7 in the third unit, Solve) the teacher/researcher put the following problem on the blackboard:

> If you had just won $1000 and you decided to invest it in a bank at an interest rate of 7% compounded annually how long would it take you to double your money?

A few students asked questions about what "compounded annually" meant. One student asked "Is that $1000 an 'is' or an 'of' ?" The teacher/researcher was somewhat perplexed by this question so he walked over to the student's desk and asked the student to show him what he meant. Apparently the student had been taught a short-cut method to solve percentage problems. He showed an example to the teacher/researcher:

17 is 65% of what number?

$$\frac{\%}{100} = \frac{\text{is}}{\text{of}} \qquad \frac{65}{100} = \frac{17}{x}$$

The "is" is the number closest in proximity to the word is and the "of" is the number closest to that particular word. The teacher/researcher thanked the student for showing him the method and proceeded with great difficulty to explain to the student that this method only works for specific types of

percentage problems written in a certain manner.

Another case occurred in lesson 8 of 10, in the second unit, Simplify, when the teacher/researcher called for student volunteers to answer questions on the blackboard. One student displayed her solution, as shown below, in response to a question involving the subtraction of two rational expressions:

$$\frac{x-3}{x^2-9x+20} - \frac{x-5}{x^2-7x+12}$$

$$\frac{(x-3)}{(x-4)(x-5)} - \frac{(x-5)}{(x-3)(x-4)}$$

$$\frac{1}{x-4} - \frac{1}{x-4}$$

$$0$$

The student was required to explain her solution and answer any questions. In this case the teacher/researcher asked her to describe the second step. She said "Oh, that's just cross-canceling." "Cross-canceling!" said the teacher/researcher. "What on earth is cross-canceling?" She explained it was where you crossed things out that were the same diagonally. The teacher/researcher continued the conversation in hopes of gaining some insight as to where this method came from and why she felt it was appropriate in this situation. It resulted only in the student giving the previous teacher's name and holding firm in her beliefs about the validity of this method. Following that

particular incident, cross-canceling, cross-reducing, and cross-multiplying came up in a dozen or more conversations.

Students' understanding of mathematical concepts and procedures was hindered not only by their beliefs about the validity of these "short-cut" methods, but also by short-cuts taken by the teacher/researcher in leaving out certain procedural steps for the sake of saving time or space on the blackboard. In lesson 9 of 10 in the Simplify unit for the process class, an incident arose where the teacher/researcher was providing answers and explanations to questions on a quiz he had just handed back. One question was a complex fraction where he went through the steps orally while writing down those he considered to be the most crucial. One student commented "I never understand the way you do these questions." The teacher/researcher repeated the steps again orally while pointing to the blackboard. The student responded "I still don't get it." The discussion continued as the class appeared to become more restless and the teacher/researcher less patient. Finally, the teacher/researcher erased the solution he had written down and began writing the steps down again, this time including all the steps he was saying orally but had not written down before. The student said "Now, I get it. Thank-you!" The teacher/researcher apologized for not having written down all the steps from the beginning and thanked the student for her persistence. These

and similar incidents that occurred throughout the period of observation lead to the following questions: "What steps should be included when modeling work for the students'?"; and "Who should decide?"

Individual Interviews

In the interview process students were asked to explain orally the written responses they had given previously on the open-ended tests. This was intended to provide additional information regarding the students' understandings about the interconnectedness of mathematics. In the following section the pseudonyms used to preserve the students anonymity contain a mnemonic device to associate each student with their particular class (This feature of the report was used by Hovdebo, 1987). Peter, Patty, Pam, Paul and Polly are students from the process class, while Olly, Othello, Olga, Olive and Oprah are from the object class. Although it was difficult to determine differences in the level of understanding between these two groups of students, the interviews did highlight some of the difficulties students encounter in making meaningful connections in mathematics.

Clarification of Written Responses

Most of the students had difficulty speaking about mathematics, which was evident not only to the instructor (teacher/researcher) but to the students themselves. This realization seemed to cause many students to back out of

conversations rather than to expand on their responses. These students tended to bring discussions to an end by expressing their lack of confidence in attempting to understand mathematics. The following are two typical responses from students, one from each class.

Instructor: Why did you say he should put all the denominators on the same side?
Polly: I don't know. I'm not very good at these questions. I never was very good at math.

Instructor: What did you mean when you said he wasn't done?
Othello: I'm not sure what I meant. Maybe he is done. I don't understand this stuff.

On many occasions students had to rely on the use of rudimentary or informal language. Often this language would describe the physical aspects of their written responses. The following is an example of a student's use of this type of physically descriptive language:

Instructor: What do you think the word "simplify" meant?
Olly: Squeezing everything together. Make it into one large number.

Next is an example of a student not being able to come up with the proper terminology.

Instructor: In your answer you use the terms top and bottom, what were you referring to?
Peter: The top and bottom of the fraction.

Finally, an example where a student could not come up with

the mathematical procedure used in a solution, so he invented one of his own.

Instructor: How did he make this error?
Paul: He didn't fully apply the common denominator to both sides.

The term "cross out" or "cancel" was used by a number of students to describe the mathematical process of simplifying fractions. Again these are examples where the students depended solely on the physical features or mathematical objects involved in the procedure, rather than making sense out of the directions stated in the task.

Instructor: What exactly did she do wrong in that step?
Olga: He screwed up in canceling.
Instructor: What do you mean by canceling?
Olga: You know, crossing out.
Instructor: You mean reducing?
Olga: Yeah, reducing.

Instructor: What exactly was the error in this question?
Patty: He crossed out wrong. He should have left them over the same thing.
Instructor: What do you mean by crossed out?
Patty: Canceling terms.

Instructor: Was there anything at all you didn't like about Frank's solution?
Oprah: It's a good solution, but he didn't get to cross anything out.
Instructor: What do you mean he didn't get to cross anything out?
Oprah: That's the only part of math I like, when you get to cross out all the things in common to get an answer.

Other students used the term "factor" to mean the opposite,

to describe an expansion or elimination of brackets. This would seemingly cause a great deal of confusion in relating other concepts and procedures to the process of factoring. The following are two situations where students used the term factor in this way:

Instructor: What exactly did Ernest do wrong?
Olly: He factored out the brackets.

Instructor: Could you describe to me again the error Ernie made?
Patty: He should have squared the x+2 and then factored the 9 into the trinomial.

Despite problems with terminology some students were able to gain further understanding during the course of the interview. One student, Pam, was able to identify a minor calculation error in one of the solutions, but she was not able to identify the major error of the wrong strategy being used. After she was directed to focus on the word in the question, she discovered this strategic error and conveyed a certain understanding about the relationship between the processes of factoring and simplifying.

Instructor: Could you describe the error Ernie makes?
Pam: He did not distribute the 9 through the brackets. He didn't follow BEDMAS.
Instructor: Are there any other errors Ernie makes?
Pam: Not really.
Instructor: What was the question asking him to do?
Pam: Factor. Oh, I see. He has two terms in his answer, so he must have simplified instead of factored.

Another example was that of a student who didn't reach a

complete understanding of the word simplify, but was able to gain insight about difficulties she encountered in the past.

Instructor: You agreed with Valerie's solution; could you just go through her steps again and explain each one to me?
Olive: OK. First she found a common denominator, then she multiplied both sides by it.
Instructor: Both sides of what?
Olive: The equation. [Pause] But there's no equals sign. This is one of those questions I keep getting wrong.
Instructor: That's OK. At least you've recognized it. Now, what should she have done?
Olive: Well, I know she needs to keep her denominator for the answer.
Instructor: Can you explain why she needs to keep her denominator?
Olive: I'm not sure.

It was difficult to assess students' understandings about the interconnectedness of mathematics as students had difficulty in formulating their ideas in words. It seemed evident that their ability to speak about mathematics was not truly indicative of their understanding. Yet, the problems in communicating about mathematics in these interviews may suggest that this is a contributing factor compounding the difficulties students have in understanding various mathematical relationships.

Oral Open-Ended Task

The results of the open-ended task that took place during the second part of the interviews are presented along with interpretations suggesting how each student may have arrived at his or her response. Each student was instructed to generate a

series of "Yes-No" type questions that would eventually lead to the discovery of a particular algebra task, seen by the instructor, but not seen by the student. For example, the instructor might be looking at an algebraic task involving exponential equations. The students had to ask enough questions so that they felt they had a good picture of what the task involved or until they could not come up with any more questions. Students' responses are presented with their percentile rank, indicating how they represent the range of performance in their entire class. Their percentile rank from Pretest 1 is given for the first set of questions, while their percentile rank from Posttest 1 is given for the second set.

This interview task was designed to elicit information about how students relate and connect key mathematical concepts. It was hoped that the student-generated list of "Yes or No" type questions would result in prioritized outline indicating what the students felt were the most important facets of an algebraic task. However, the results from this task were less than satisfactory and inconclusive, as most of the student-generated lists of questions were not as long or as revealing as anticipated. Consequently, only a few examples from each class are presented to show the range of the results. The student-generated lists of questions are analyzed according to the teacher/researcher's own interpretation of the results and any conclusions must be drawn

with caution.

The following are two examples of students that had a difficult time with this task. The first student is from the object class.

Olga (34th Percentile) First Interview:

Student's Questions:	Instructor's Response:
Does it have brackets?	No
Does it have numbers?	Yes
Does it have letters?	Yes
Do you have to multiply?	Yes
Are there fractions?	Yes
Do you find a common denominator?	No

Olga (32nd Percentile) Second Interview:

Student Questions:	Instructor's Response:
Does it have brackets?	No
Are there variables?	Yes
Do you solve for x?	Yes
Do you find a common denominator?	No
Do you make one side equal zero?	No
Do you have to factor?	No

In both interviews Olga attempted to discover the algebra question by piecing together its physical components i.e., asking about brackets, numbers and variables. From this aspect she does not seem to possess an in-depth schema about the hierarchical structure of mathematics. In her second interview she made a significant breakthrough and asked a question regarding process. Afterward she inquired about some possible procedures associated with the process. Therefore, she has demonstrated a certain increase in understanding about the different procedures associ-

ated with these types of questions.

Next is the second example of a student, from the process class, who experienced difficulties with the interview task.

Polly (19th percentile) First Interview:

Student's Questions:	Instructor's Response:
Is one of the questions a quadratic?	No
Are there any perfect squares?	No
Any perfect cubes?	Yes
Any higher than three?	No
Is there more than one term?	Yes
Is there more than three terms?	No
Two terms?	Yes
Are there any coefficients?	No

Polly (33rd percentile) Second Interview:

Student's Questions:	Instructor's Response:
Does it contain perfect squares?	Yes
Is it a fraction?	No
Is it a multiplication with fractions?	No
Is there more than one perfect square?	Yes
Is it a solve question?	No
Is it a simplify question?	No

In the first interview Polly had a methodical way of identifying the physical components of the question. Slow as it was, there seemed to be a definite system: identify the exponent or degree, identify the number of terms, and identify the coefficients of the terms. In her second interview Polly had applied a less systematic approach to the task, however, she did include processes as well as mathematical objects in her list of questions. Although she seemed to expand her knowledge base it is not clear that she has an in-depth understanding of the

relationships among processes, objects, and procedures.

The next two interviews are examples of students that seem to have developed a successful list of "Yes or No" type questions necessary to unveil the particular algebraic task within the possession of the instructor. Each of these two students had generated very different sets of questions, and this may be due to the different organizational schemes presented in each of their respective classes. The first set of interviews presented is with a student from the object class.

Olive (88th percentile) First Interview:

Student's Question	Instructor's Response
Are there any x's?	Yes
Do you have to factor anything?	No
Do you have to distribute?	No
Are there any fractions?	Yes
Do you have to get a common denominator?	Yes

Olive (88th percentile) Second Interview:

Student's Question	Instructor's Response
Is it an equation?	Yes
Do you have to isolate a square root?	No
Is it a quadratic equation?	No
Is it a rational equation?	Yes
Does it involve substitution method or addition and subtraction method?	No
Do you have to factor?	Yes

In the first interview, Olive seemed to possess a certain knowledge about the relationships of mathematical concepts and procedures. Once she found out factoring was not required she then inquired about its inverse operation. As she discovered it

was a fraction, her next question involved a major procedure involving fractions. In her second interview she seemed to have enhanced her understanding even further, as she identified the main type of question first, then proceeded with a logical series of hierarchically arranged questions. She established the fact that it was an equation, she determined the type of equation, and what method she should employ. She appears to have broadened her understanding about the interconnectedness of mathematics.

The following is taken from the second set of interviews with a student from the process class who had a more successful experience with this oral task than her classmates.

Pam (71st percentile) First Interview:

Student's Question	Instructor's Response
Is it a factoring question?	No
Do you have to simplify?	No
Do you have to solve?	Yes
Is it with like terms?	No
Is [sic] there brackets?	No

Pam (76th percentile) Second Interview:

Student's Question	Instructor's Response
Is it a simplify question?	No
Is it a factor question?	Yes
Is there a GCF?	Yes
Is [sic] there two terms?	Yes
Is it a difference of squares?	No
Is it a difference of cubes?	Yes

Although Pam discovered the main process early in her first interview, she was somewhat lost as to what to ask next. In her second interview again she identified the main process of the

question and then systematically uncovered what she needed to do with the question. First she determined it was a factor question, then she indicated that she needed to factor out a GCF; next she determined that there were two terms; and they could be factored as a difference of cubes. This indicates a definite development of understanding about how various concepts and procedures relate to one another.

On the whole, in the first set of interviews students tended to rely on questions that would lead to the unveiling of the various physical aspects of the algebraic task, as opposed to asking questions pertaining to the possible procedures or processes that may be involved with the question itself. For the process group, although three students did inquire about the major process involved in the question, it is likely that those inquiries were prepared by the students prior to the interview. As the questions following the process questions usually related back to discovering the physical features of the question.

The second set of interviews indicated that in both organizations of the curriculum, students could display a higher level of understanding, the same level or even a lower level of understanding. Thus the results were considered to be inconclusive. For those students whose understanding appeared to increase over the duration of the study, their organizational scheme for generating a set of questions seem to reflect the

organizational scheme of the curriculum from their respective classes. It was difficult to determine the level of a student's understanding from this open-ended interview task as it could not be determined where each student's set of questions originated, i.e., their own schemata or previously studied information.

Open-Ended Test Results

This section concerns itself with the results from both classes on the open-ended tests. These tests were designed to answer the research questions regarding students' understanding and students' abilities to select an appropriate strategy. These tests were much more successful at addressing the latter of these two questions. First, frequency tables are presented with the scores for each class on the open-ended tests. Then each item from these tests is further evaluated to highlight any differences between the two classes and to display any improvement that had occurred from the pretest to the posttest.

Pretest 2

The items on Pretest 2 are used to assess the students' ability in selecting an appropriate strategy and the student's understanding about the interconnectedness of mathematics. The results for each class on Pretest 2 are presented in Tables 1 and 2.

Table 1: Mathematical Object Class' Results on Pretest 2, in Number of Students

ITEM	OPEN-ENDED TASKS	S 0	C 1	O 2	R 3	E 4	S 5
1	PRESOLVED FACTOR [a]						
	-Identification of Errors	2	19	1	1		
	-Reasons for Errors	2	2	19	0		
	-Students' own Solutions	1	20	1	1		
2	EXPLANATORY [b]	2	4	7	4	5	1
3	COMPARISON [c]	3	11	9	0		
4	CONTRAST [c]	6	13	4	0		
5	PRESOLVED SIMPLIFY [a]						
	-Identification of Errors	5	11	6	1		
	-Reasons for Errors	4	4	15	0		
	-Students' own Solutions	4	13	5	1		
6	NON-ROUTINE [d] ALGEBRA PROBLEM	12	10	1	0		

[a] - Refers to Scoring Scale in Figure 11 on p. 65

[b] - Refers to Scoring Scale in Figure 12 on p. 66

[c] - Refers to Scoring Scale in Figure 13 on p. 67

[d] - Refers to Scoring Scale in Figure 15 on p. 68

Table 2: Mathematical Process Class' Results on Pretest 2, in Number of Students

ITEM	OPEN-ENDED TASKS	S 0	C 1	O 2	R 3	E 4	S 5
1	PRESOLVED FACTOR [a]						
	-Identification of Errors	7	12	1	1		
	-Reasons for Errors	5	7	8	1		
	-Students' own Solutions	7	12	1	1		
2	EXPLANATORY [b]	5	4	6	4	2	0
3	COMPARISON [c]	5	10	6	0		
4	CONTRAST [c]	13	7	1	0		
5	PRESOLVED SIMPLIFY [a]						
	-Identification of Errors	8	8	4	1		
	-Reasons for Errors	8	5	8	0		
	-Students' own Solutions	8	9	3	1		
6	NON-ROUTINE [d] ALGEBRA PROBLEM	14	5	1	1		

[a] - Refers to Scoring Scale in Figure 11 on p. 65

[b] - Refers to Scoring Scale in Figure 12 on p. 66

[c] - Refers to Scoring Scale in Figure 13 on p. 67

[d] - Refers to Scoring Scale in Figure 15 on p. 68

Presolved Algebra Items

The results for items involving presolved algebra questions give the assessment of three separate tasks: the students' ability to identify the errors within the given solution; the students' reasoning for identifying those errors; and the students' ability to provide their own error-free solution. For these particular

items there seemed to be evidence that many of the students received the same heuristical score on the tasks of identifying the errors and providing their own solution. In other words a student who received a score of 1 for identification of errors had a high probability of also receiving a score of 1 for their own solution. This possible relationship was investigated and discussed for each of these items.

The first presolved algebra question on Pretest 2 involved a factor question with the given solution displaying a simplify-type strategy. The assessment of the three tasks for this question are presented in Figure 17, as a comparison between the two different classes.

For the identification of errors, the results for both classes appear to be similar although fewer students in the process class were able to identify any of the errors within the given solution. Similarly in students' own solutions there were fewer students in the process class that provided a solution. The quality of the reasons given for the errors, with exception of one process student, seemed to slightly favour the students in the object class. The majority of students in both classes gave reasons that were simplistic or illogical.

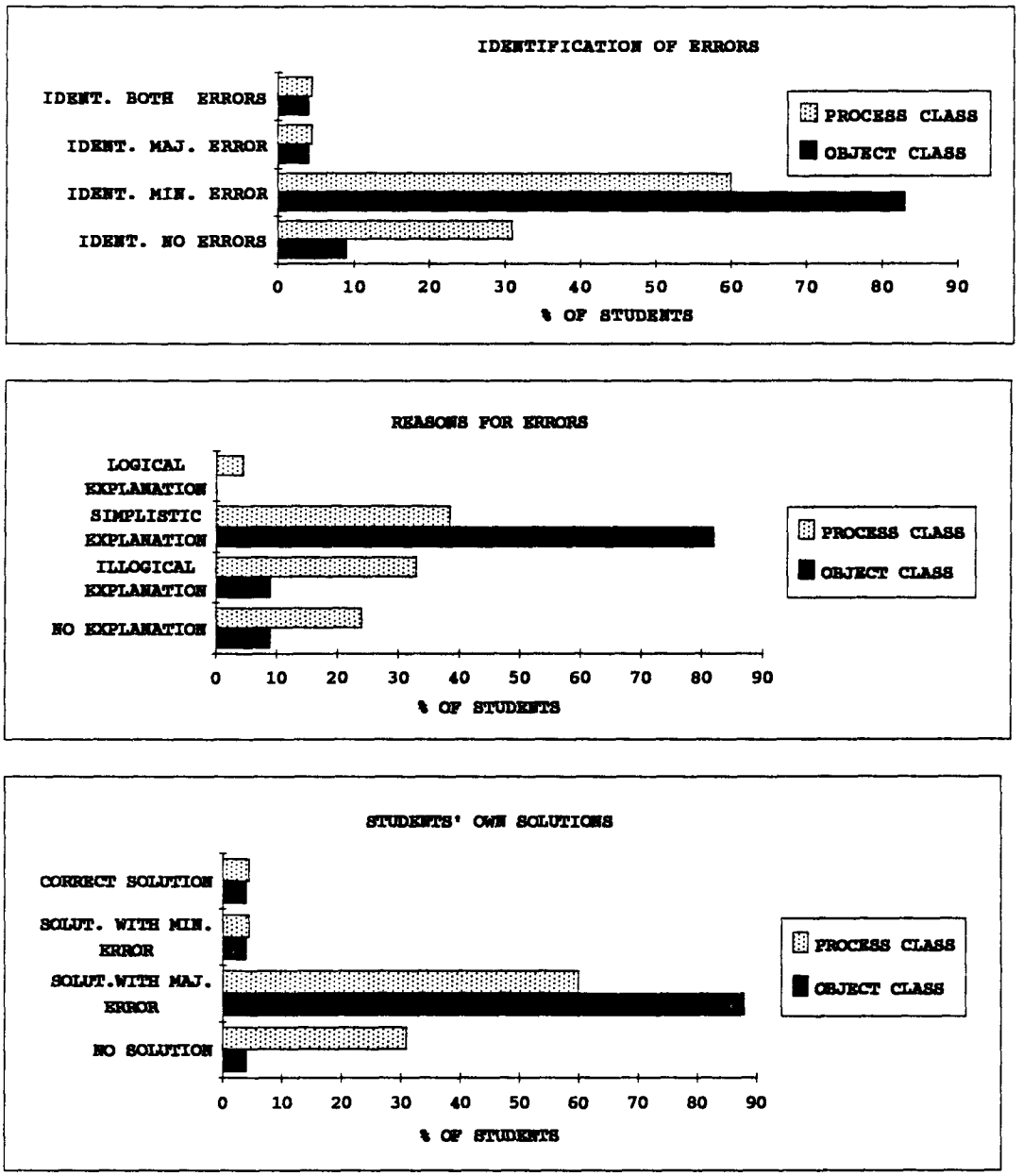


Figure 17: Students' Results for Item 1 of Pretest 2: Presolved Factor Question With Simplifying Strategy

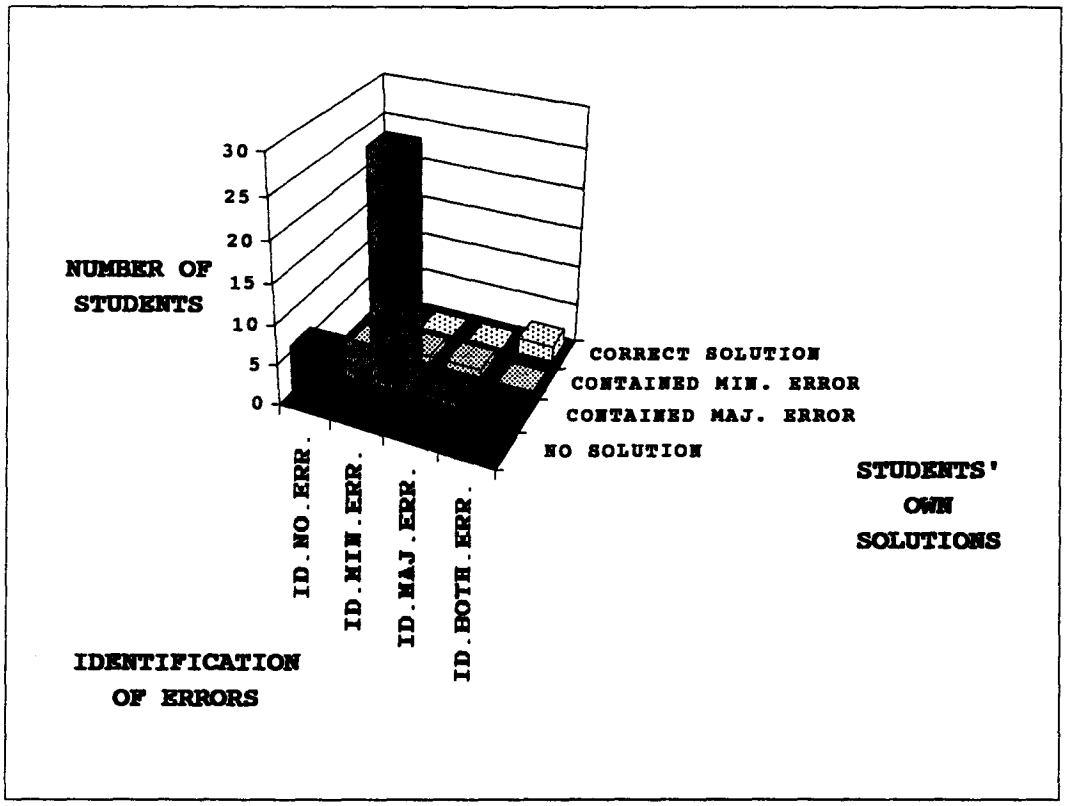


Figure 18: Cross-Tabulation Involving Identification of Errors and Students' Own Solutions on Item 1 of Pretest 2 (n=44)

As can be seen in Figure 18, there is a strong relationship between the identification of errors and the students' own solutions. The large vertical columns on the diagonal represent those students who received the same score on both. This question was difficult for most students, so the largest column is located in the (1,1) square. This column represents students that either identified the minor error only or could not identify any error, and whose own solution contained at least one major error.

Only two students from each class could identify the major error and provide a solution without a major error. This finding reinforces the argument that students have difficulties in identifying appropriate strategies and implementing them.

Figure 19 shows the percentage of students from each class that were able to identify the major error. This percentage is quite small for both classes.

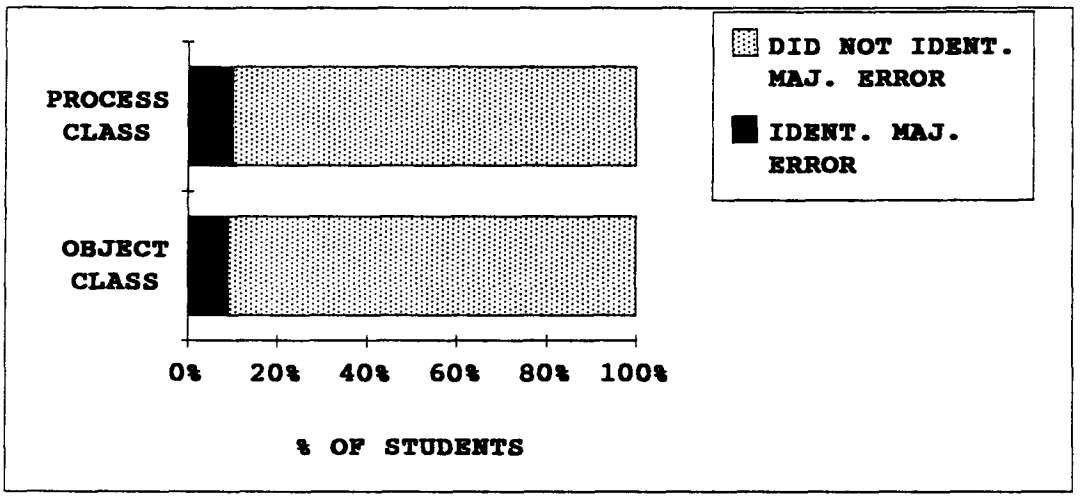
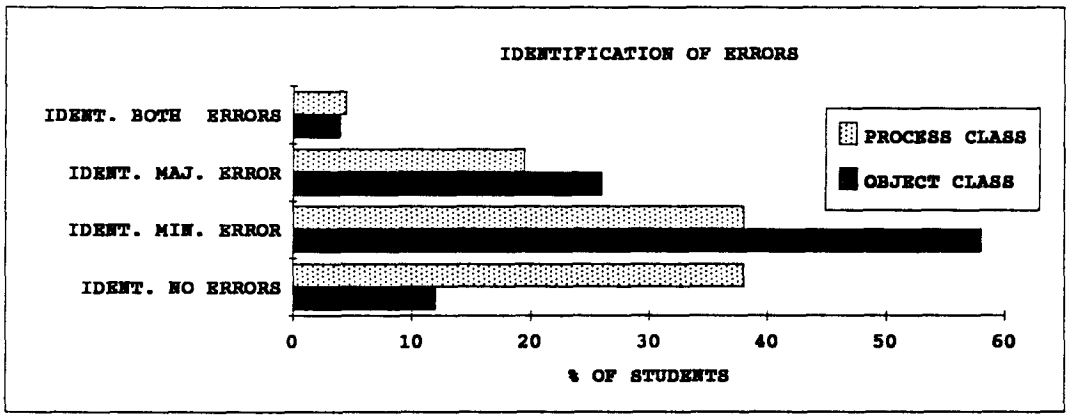
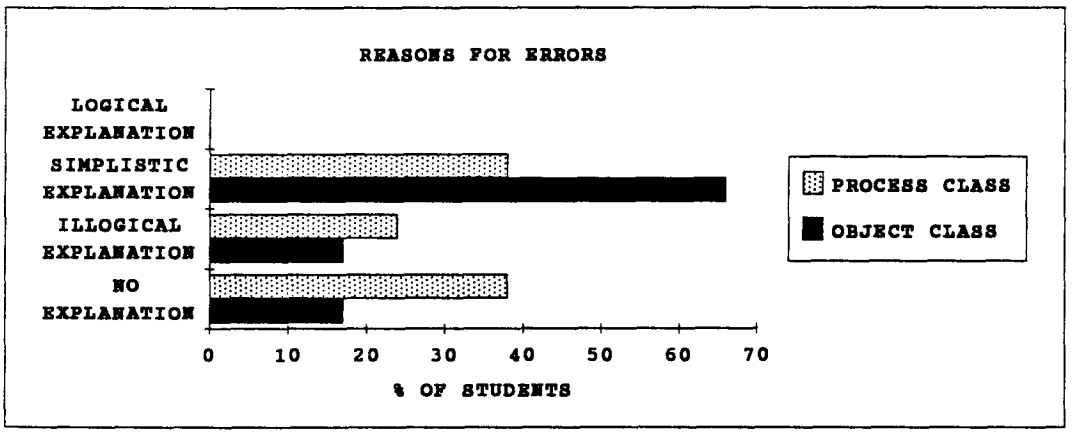


Figure 19: Percentage of Students Who Identified the Major Error on Item 1 of Pretest 2

The other presolved algebra question presented a Simplify question and worked it with a solve-type strategy. Figure 20 displays the results on this question for students in each class.





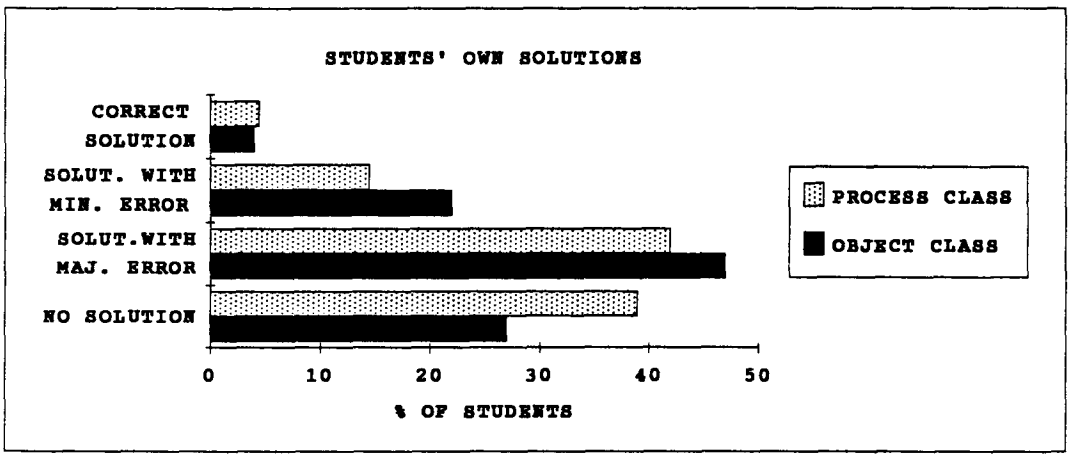


Figure 20: Students' Results on Item 5 of Pretest 2: Presolved Simplify Question With Solve Strategy

Both classes had similar results on the identification of errors, but fewer students in the process class were able to detect the major error. Similarly, there were fewer students in the process class who were able to provide their own solution to the question without a major error. The students' reasons for errors show a fairly decisive edge for the object class, as the process class had more students not supplying a reason or giving an illogical reason.

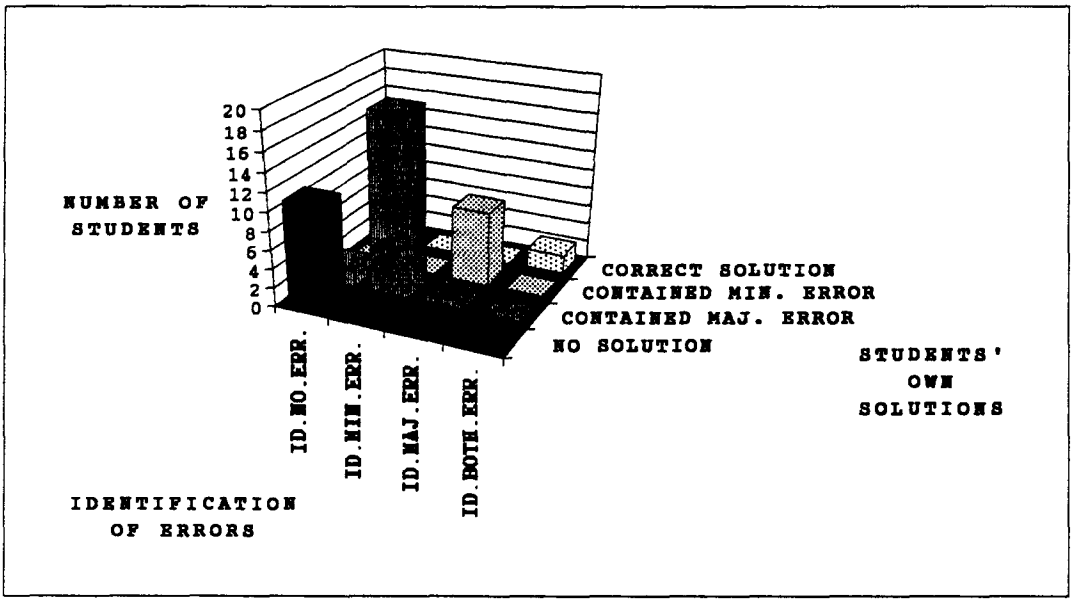


Figure 21: Cross-Tabulation Involving Identification of Errors and Students' Own Solutions on Item 5 of Pretest 2 (n=44)

Again there is a direct relationship as shown in Figure 21 between students' being able to identify errors in a given solution and students' being able to provide a solution without errors. This question appears to be somewhat less difficult for the

students than the first presolved question, but it still indicates that the majority of students were unable to identify the major error or provide a solution which does not contain a major error.

Figure 22 shows the percentage of students able to identify the major strategic error for the given solution. Both classes scored better on this item than on the first presolved algebra question.

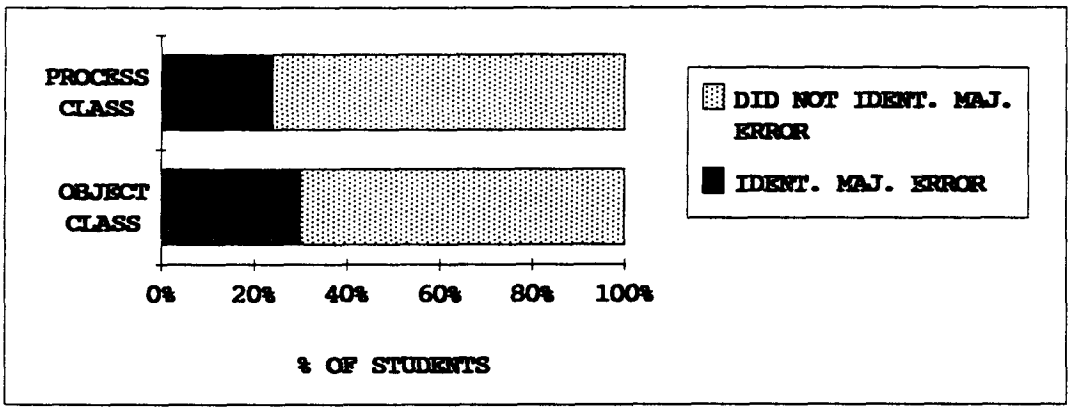


Figure 22: Percentage of Students Who Identified the Major Error on Item 5 of Pretest 2

Difficulties were encountered as the markers attempted to reach consensus on scoring the reasons portion of the presolved questions. The scores given for this portion are highly interpretive and based upon the judgement of the two markers. Therefore any claims made about this portion of these questions must be accepted with caution. Likewise, the following three test items, which all involve students' explanations and reasons and are scored using similar heuristic scales, must also be interpreted

with caution.

Explanatory Item

The explanatory item for Pretest 2 gave students the task of recording a made up conversation with a friend about how to subtract two rational expressions. They were expected to write in their own words the procedural steps necessary to arrive at a completed solution. It was discovered upon marking this item that many students just provided a solution of their own for the example given in the question. Allowances had to be made for these types of responses, and these are evident in the scoring scale shown in Figure 12. The results of the scoring for this item are displayed in Figure 23.

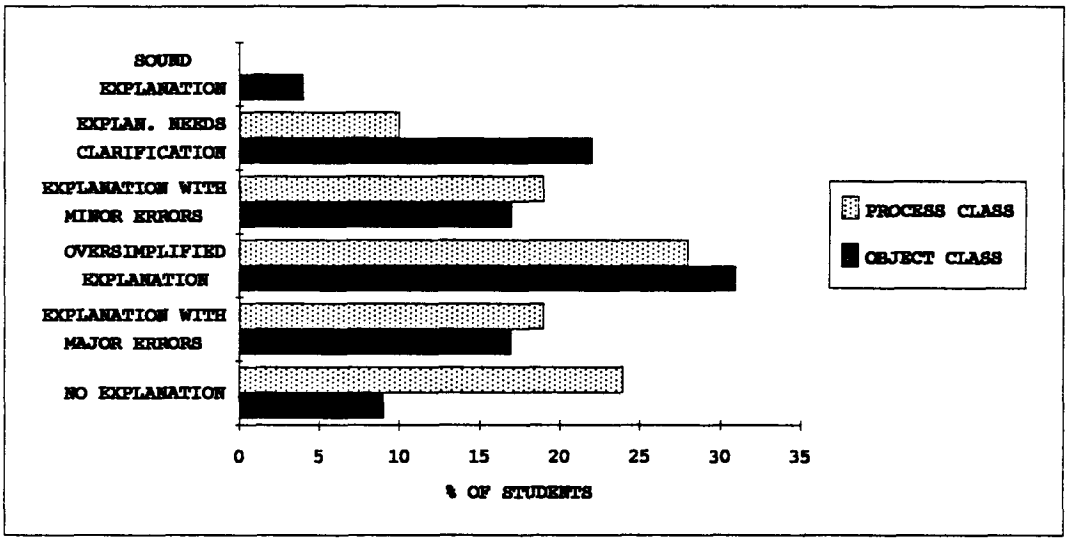


Figure 23: Students' Results on Item 2 of Pretest 2: Explanatory Question Involving the Subtraction of Two Rational Expressions

The object class performed slightly better in explaining a process, as they had four more students attempt the question than in the process class and three more students offering sound advice to their friend. The four middle categories had a fairly even distribution of students. Only one student from either class gave a response that could have been used to obtain a correct solution.

Compare and Contrast Items

The first of these two items required students to choose two among six possible processes that they felt were most similar to each other, while the second required students to choose two processes that they felt were most dissimilar to each other. For both items the students had to provide reasons for their choice. Only the reasons given were scored, since there were no completely right or wrong choices. Both items were designed to test for the students' understanding about the interconnectedness of mathematics, as they provided them with an opportunity to demonstrate how they could relate two different concepts. Figure 24 displays the results for the item requiring students to choose the most similar processes, and Figure 25 displays the results for the item requiring them to choose the most dissimilar processes.

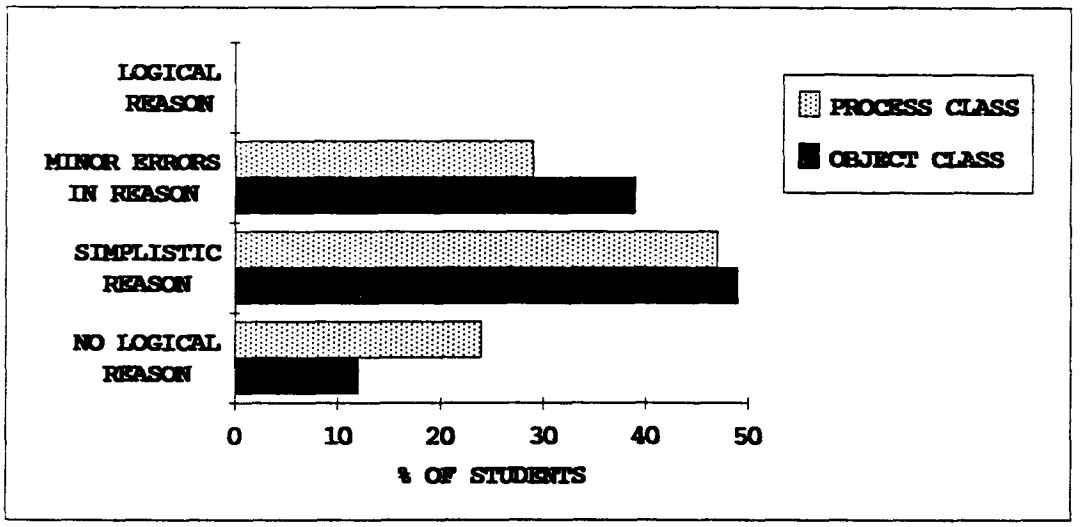


Figure 24: Students' Results on Item 3 of Pretest 2: Comparison Question Requiring Students to Choose the Two Most Similar Processes and Explain

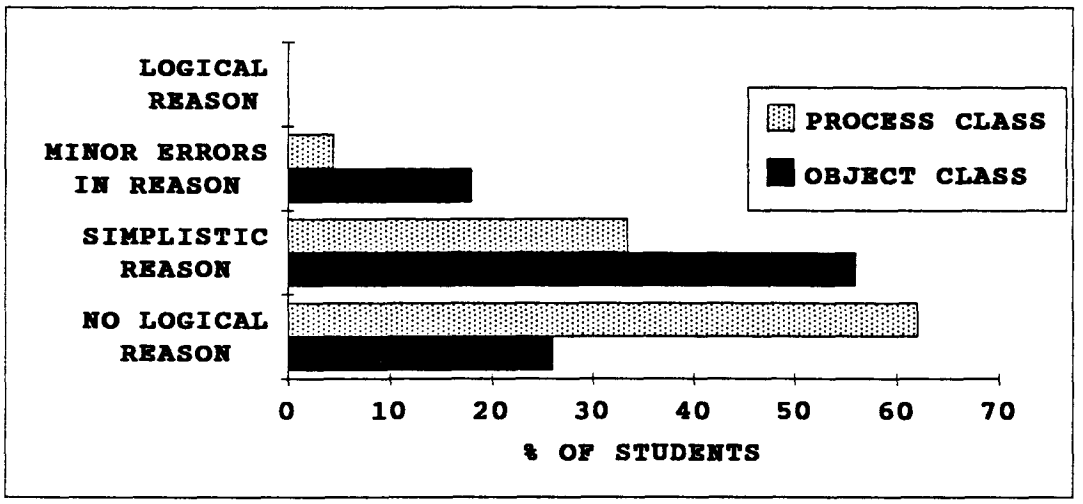


Figure 25: Students' Results on Item 4 of Pretest 2: Contrast Question Requiring Students to Choose the Two Most Dissimilar Processes and Explain

The students in both classes found these two tasks very difficult, as 62% of them either did not bother to respond or did not provide a logical reason for making their choice. On the comparison question over 60% of the students gave reasons that were at best simplistic, and over 62% did so on the contrast question. As well, not one student received the top score of three on either item. These two tasks seemed to be too demanding for the students involved. As a result of the difficulties faced by students in responding to these tasks and the difficulties faced by the markers in marking this task, it was decided to not include items like these on Posttest 2. They were replaced by another pre-solved algebra question that did seem to provide useful results and a process definition question requiring the students to list a set of procedures which are related to a given process. This last item was added in an attempt to find a task that would provide some evidence about the students' understanding about the interconnectedness of mathematics.

Although these two items were dropped from the posttest, they did raise two important points. The first is that it seems very apparent that students had a difficult time in understanding various connections and relationships that exist in mathematics. The second is that assessing students' understanding about the interconnectedness of mathematics is extremely difficult.

Non-Routine Algebra Problem

The last item on Pretest 2 was a non-routine algebra problem, i.e., an algebraic task that cannot readily be answered without some attention to strategy selection and implementation. The problem was marked on the basis of whether the students tried to implement a strategy and whether that strategy could eventually lead to a correct response. The results for this item are shown in Figure 26.

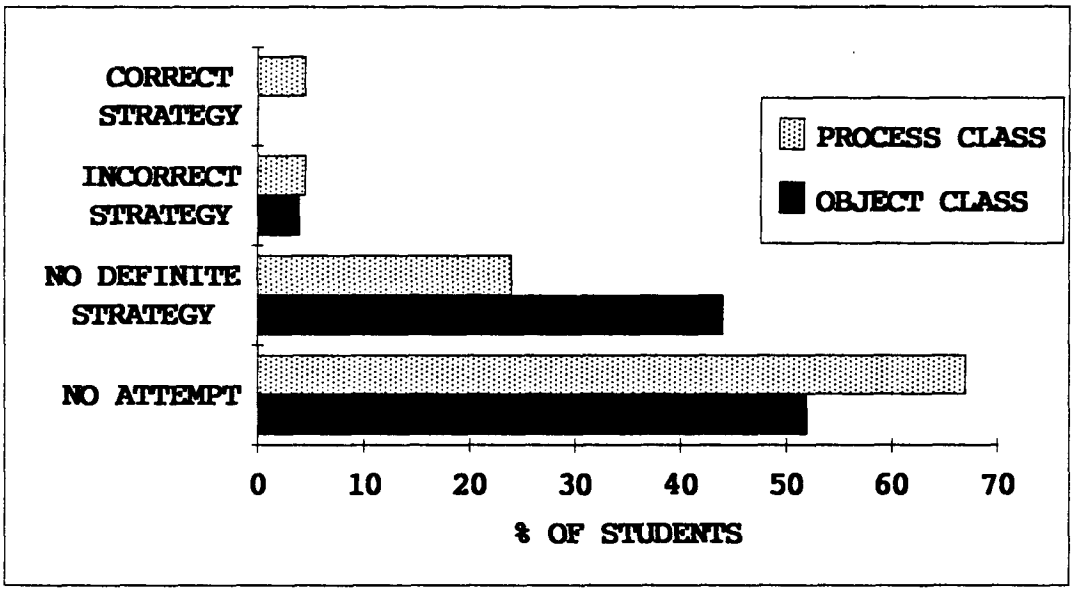
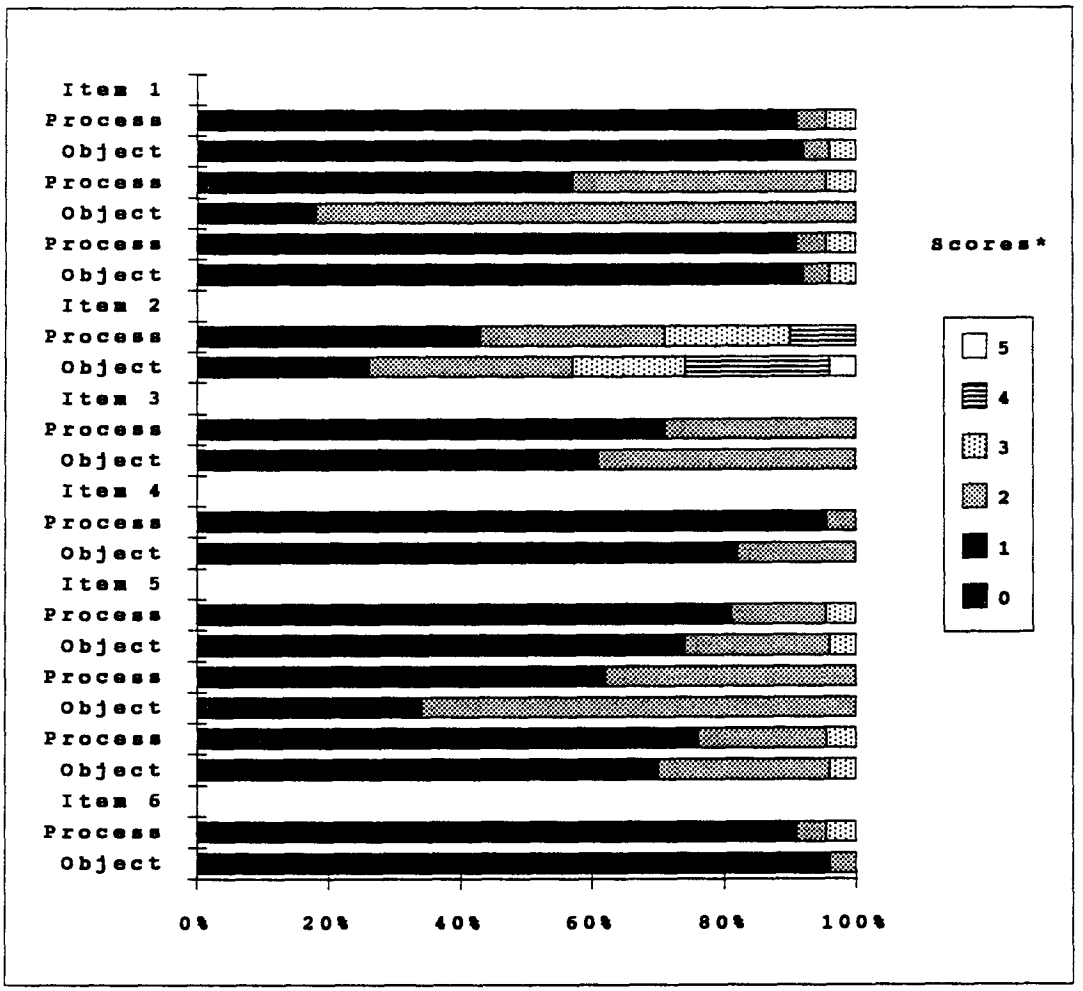


Figure 26: Students' Results on Item 6 of Pretest 2: The Non-Routine Algebra Problem

This item proved to be at least as difficult as the previous two, but in this case that was expected from the outset. Over 50% of the students in each class did not attempt to respond, and only three people tried to implement a strategy. This item was used to investigate similarities between students' difficulties in the two classes, and provide some bench mark to monitor any improvement.

Summary of Pretest 2 Results

The purpose of Pretest 2 was to determine whether there were any important differences between the two classes on the specific open-ended tasks; the pretest would act as a check for any differences observed in Posttest 2 at the end of the study. On most of the items in Pretest 2 the classes were very similar as they both seemed to struggle with the tasks given. In order to test for any statistically significant differences the Kolmogorov-Smirnoff two sample test was employed. The test compares the differences in cumulative percent of the results for students in the two different classes on each open-ended item. Figure 27 displays the cumulative percentages for both classes on the items from Pretest 2.



* Refer to Scoring Scales used for Table 1 and Table 2

Figure 27: Students' Results in Cumulative Percent Comparing The Two Classes' Responses on the Items From Pretest 2

The null hypothesis for the test was as follows:

Null Hypothesis: There is no significant difference between the distributions of results on the open-ended items of Pretest 2 for students in the process-organized class and students in the object-organized class.

Because this was an exploratory study with sample sizes of 21 and 23, a maximum difference in cumulative percentage of students of 37% is needed for the difference to be significant at the 0.10 level. The results are displayed in Table 3.

Table 3: Results of Using the Kolmogorov-Smirnoff Two Sample Test With the Cumulative Percents From the Students' Results on the Open-Ended Items of Pretest 2

	OPEN-ENDED ITEMS									
STATISTIC:	**1a**	**1b***	**1c**	**2**	**3**	**4**	**5a**	**5b**	**5c**	**6**
Max. Difference	.22	.39	.27	.17	.12	.36	.26	.28	.12	.15
Probability	>.2	.07	>.2	>.2	>.2	.12	>.2	>.2	>.2	>.2

a - identification of errors
b - reasons for errors
c - students' own solution
* - the null hypothesis can be rejected i.e., there is a statistically significant difference on this item.

For all other items, the null hypothesis cannot be rejected.

On only one item was there a statistically significant difference between the two classes. It was determined from the results that the object class was better able to provide reasons for the errors contained in the given solution of the presolved factor item. For all the other items on Pretest 2 there were no statistically significant differences between the two classes. Therefore, the results in Posttest 2 can be used to show any

indications of differences between the two classes, as well as any significant improvements on parallel items evident at the end of the study.

Posttest 2

Posttest 2, as previously mentioned, did not include two items from Pretest 2, i.e., the compare and contrast questions. Therefore, in the following section, Comparison Between Pretest 2 and Posttest 2, the improvement for the two replacement items is not part of the results. The purpose of these two replacement items is the same as the other four and that is to assess the students' ability in selecting an appropriate strategy and the students' understanding about the interconnectedness of mathematics. Tables 4 and 5 display the results from all the items on Posttest 2 for the object class and the process class respectively.

Presolved Algebra Items

The first presolved algebra question on Posttest 2 took a Solve question and presented it to the students with a simplify-type strategy. Although this specific type of presolved question was not used in Pretest 2, these questions seemed to provide interesting student responses. The assessment of the three tasks: identifying the error, giving reasons for the error, and providing their own solution, is given in Figure 28 for both classes.

Table 4: Mathematical Object Class' Results on Posttest 2, in Number of Students

ITEM	OPEN-ENDED TASKS	S 0	C 1	O 2	R 3	E 4	S 5
1	PRESOLVED SOLVE [a]						
	-Identification of Errors	3	7	5	8		
	-Reasons for Errors	2	3	13	5		
	-Students' Own Solutions	2	8	5	8		
2	EXPLANATORY [b]	3	7	3	6	3	1
3	PRESOLVED FACTOR [a]						
	-Identification of Errors*	16		7			
	-Reasons for Errors	13	5	4	1		
	-Students' own Solutions	12	4	6	1		
4	PROCESS DEFINITION [c]	3	13	5	2		
5	PRESOLVED SIMPLIFY [a]						
	-Identification of Errors	12	4	6	1		
	-Reasons for Errors	11	4	8	0		
	-Students' own Solutions	11	6	6	0		
6	NON-ROUTINE [d] ALGEBRA PROBLEM	4	14	5	0		

* - Presolved question contained only major error.

[a] - Refers to Scoring Scale in Figure 11 on p. 65

[b] - Refers to Scoring Scale in Figure 12 on p. 66

[c] - Refers to Scoring Scale in Figure 14 on p. 67

[d] - Refers to Scoring Scale in Figure 15 on p. 68

Table 5: Mathematical Process Class' Results on Posttest 2, in Number of Students

ITEM	OPEN-ENDED TASKS	S 0	C 1	O 2	R 3	E 4	S 5
1	PRESOLVED SOLVE [a]						
	-Identification of Errors	2	7	4	8		
	-Reasons for Errors	2	2	14	3		
	-Students' Own Solutions	2	7	7	5		
2	EXPLANATORY [b]	1	8	3	2	5	2
3	PRESOLVED FACTOR [a]						
	-Identification of Errors*	6		15			
	-Reasons for Errors	5	3	8	5		
	-Students' own Solutions	5	2	6	8		
4	PROCESS DEFINITION [c]	0	7	9	5		
5	PRESOLVED SIMPLIFY [a]						
	-Identification of Errors	3	5	9	4		
	-Reasons for Errors	3	1	13	4		
	-Students' own Solutions	3	5	12	1		
6	NON-ROUTINE [d] ALGEBRA PROBLEM	4	10	6	1		

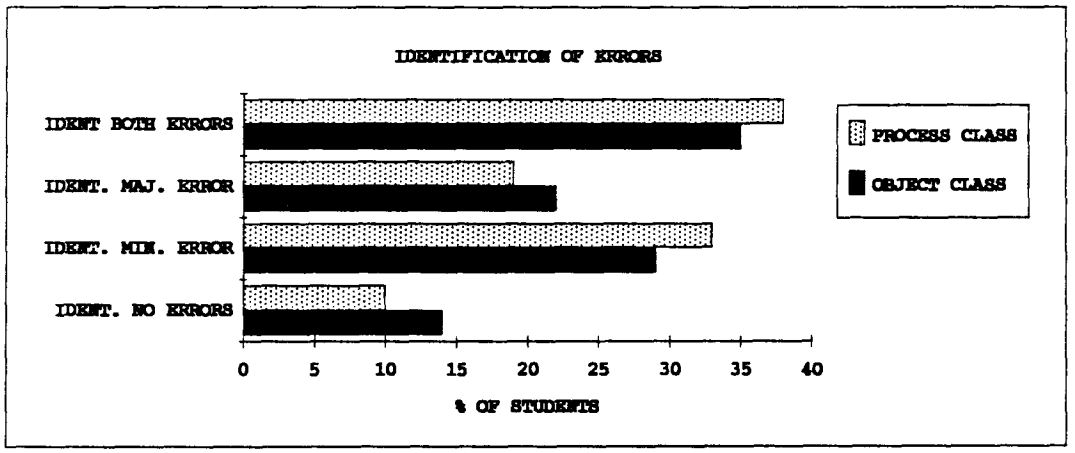
* - Presolved question contained only major error.

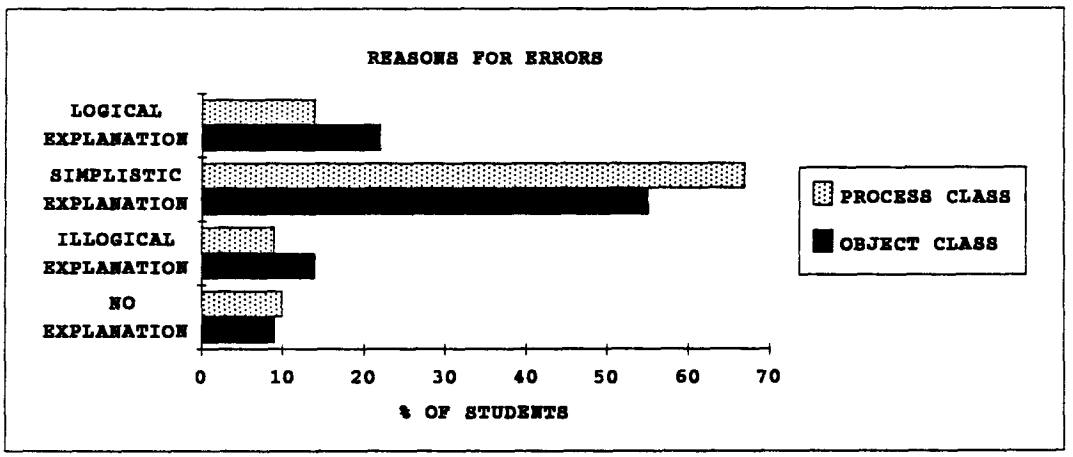
[a] - Refers to Scoring Scale in Figure 11 on p. 65

[b] - Refers to Scoring Scale in Figure 12 on p. 66

[c] - Refers to Scoring Scale in Figure 14 on p. 67

[d] - Refers to Scoring Scale in Figure 15 on p. 68





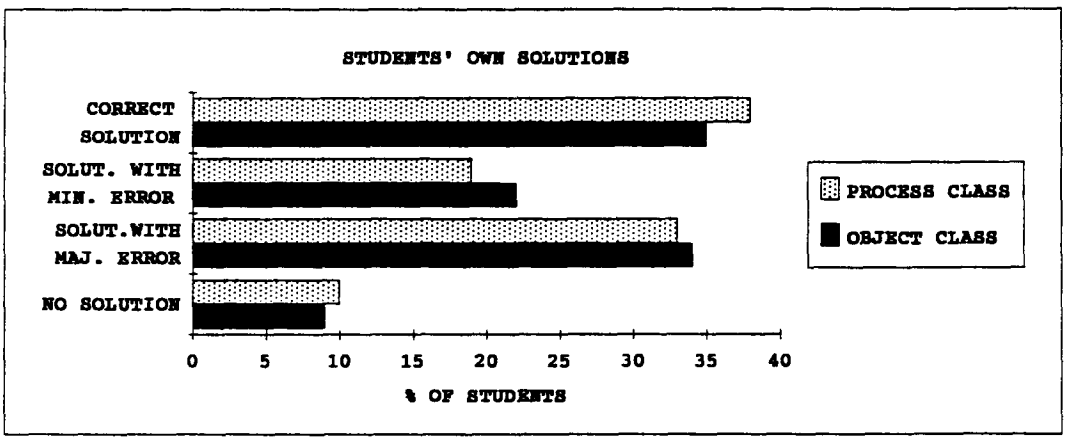


Figure 28: Students' Results on Item 1 of Posttest 2: Presolved Solve Question With Simplify Strategy

Both classes yielded very similar results for each part of this item. As with the presolved questions from Pretest 2, there was evidence of a strong relationship between the identification of the errors and the students own solution, as can be seen in Figure 29. The relationship for this item showed an even stronger relationship, as 13 of the 16 students able to identify both the major and minor errors were also able to provide correct solutions of their own. Conversely, 13 of 14 students that were only able to identify the minor error gave solutions of their own that still contained a major error.

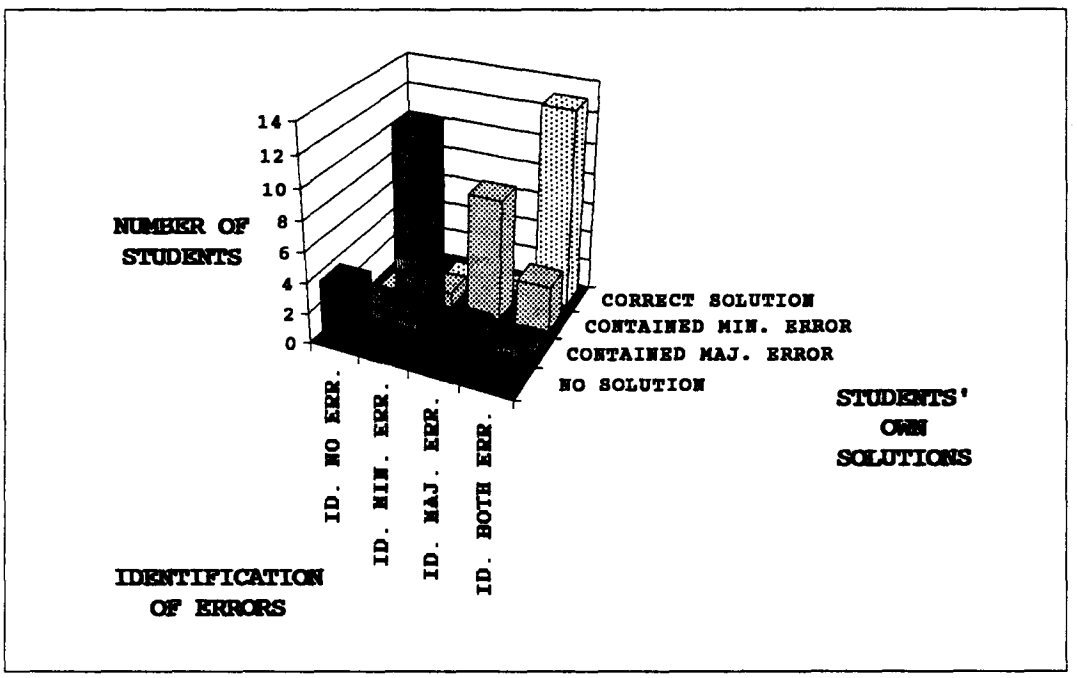


Figure 29: Cross-Tabulation Involving the Identification of Errors and Students' Own Solutions on Item 1 of Posttest 2 (n=44)

Most students found this task easier than the other

presolved algebra tasks from Pretest 2. Students may have found it easier to identify the major error in a Solve question as opposed to a Simplify or a Factor question, both given in Pretest 2. There were only four students who could not identify any errors and come up with a solution other than the one presented. As well, Figure 30 shows that over half the students in both classes were able to identify the major error.

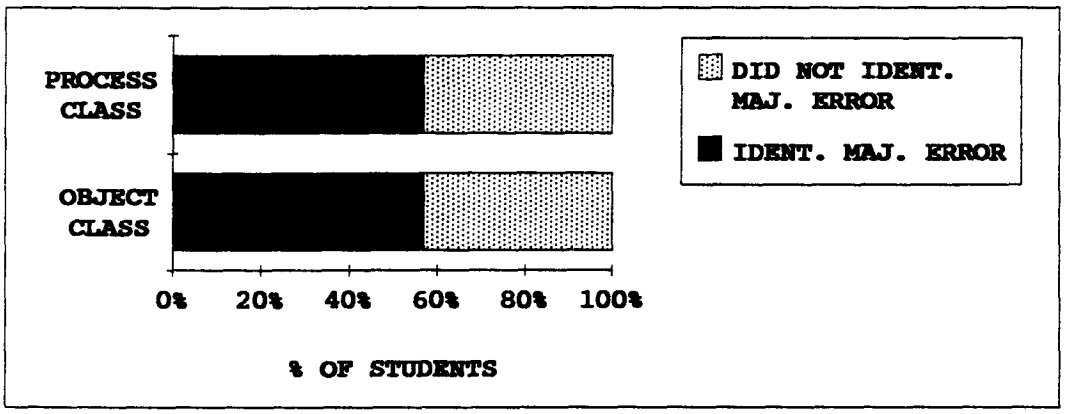
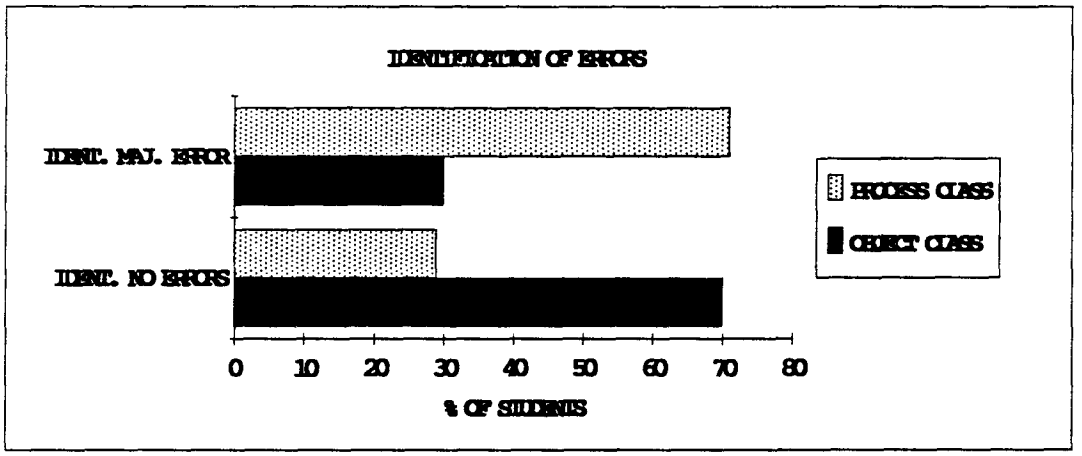
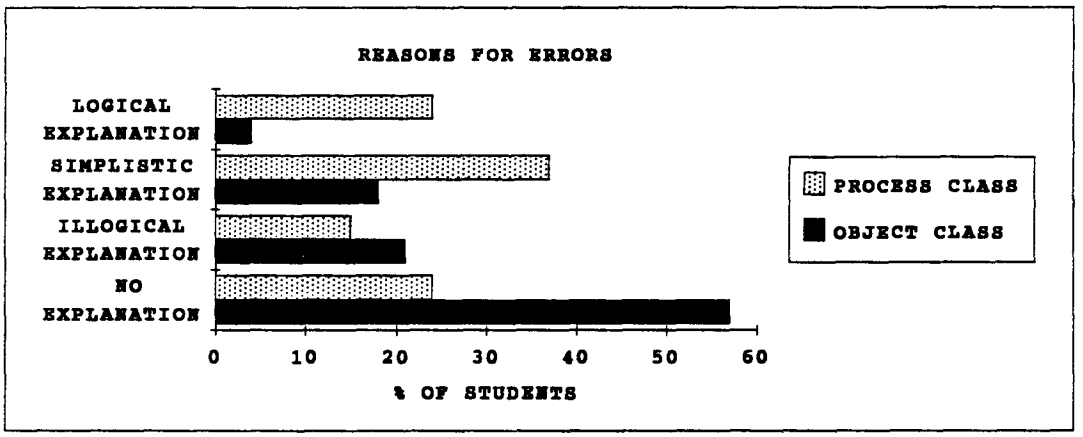


Figure 30: Percentage of Students Who Identified the Major Error on Item 1 of Posttest 2

The second presolved question parallels Item 1 from Pretest 2, i.e., it took a factor question and presented it with a simplify -type strategy. This particular presolved question had a given solution containing only a major strategic error. The reason for this was to determine whether or not the students stopped trying to identify errors once the first one was discovered. The results for this presolved posttest question are shown in Figure 31.



*Note there are no categories 1 and 3 as there was no minor error for this item.



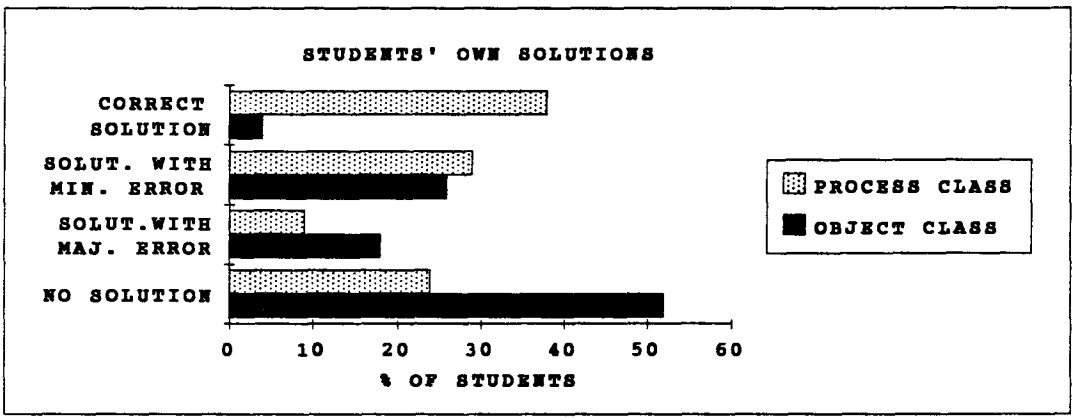


Figure 31: Students' Results on Item 3 of Posttest 2: Presolved Factor Question With Simplify Strategy

For this particular item, there is an observable difference between the two classes. The process group had more students able to identify the major error in question. Figure 32 highlights this difference, showing that 41% more students in the process group were able to identify the major error than in the object group. This difference is also apparent in the reasons given by students for the errors in the given solution, as five students in the process class gave reasons that were categorized as being correct and logical as opposed to only one student in the object class. Fourteen students in the process class provided their own solutions that contained no errors or only minor errors, while only seven students in the object class provided solutions of this quality. Eight students in the process class provided their own solution with no errors as opposed to only one in the object class. This was the first indication of any substantial differences between the two classes on any of the open-end items.

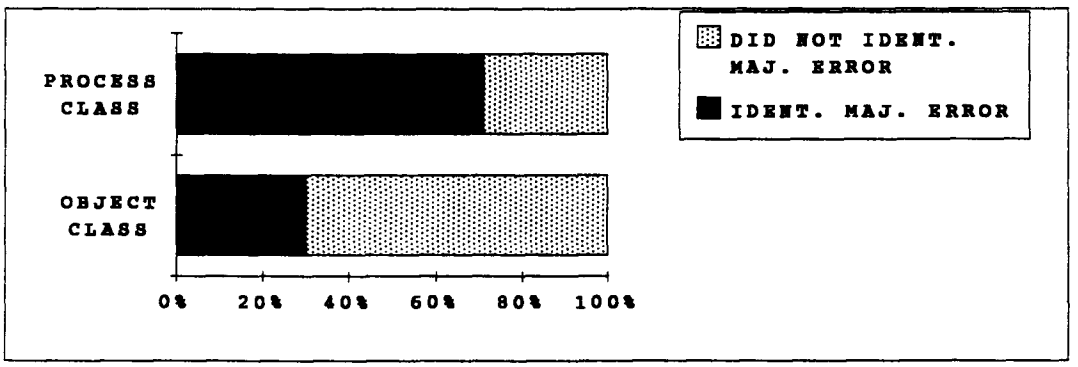


Figure 32: Percentage of Students Who Identified the Major Error on Item 3 of Posttest 2

Looking at these results another way shows that 20 of the 22 students who were able to identify the major error were also able to provide a solution without a major error. All 22 students who were not able to identify the major error in the given solution provided solutions that also contained a major error.

As Figure 33 shows, there are only two tall columns on the diagonal. This was due to this item containing no minor error. In order to observe the relationship between the identification of errors and the students' own solutions, categories 0 and 1 were combined under students' own solutions, as were categories 2 and 3. This made sense as students in categories 0 and 1 gave solutions with a major error and students in categories 2 and 3 gave solutions containing no major error. The fact that the two tallest columns are on the positive sloped diagonal indicates that there is a very strong relationship between these two tasks. There were only two students that were not part of these two columns.

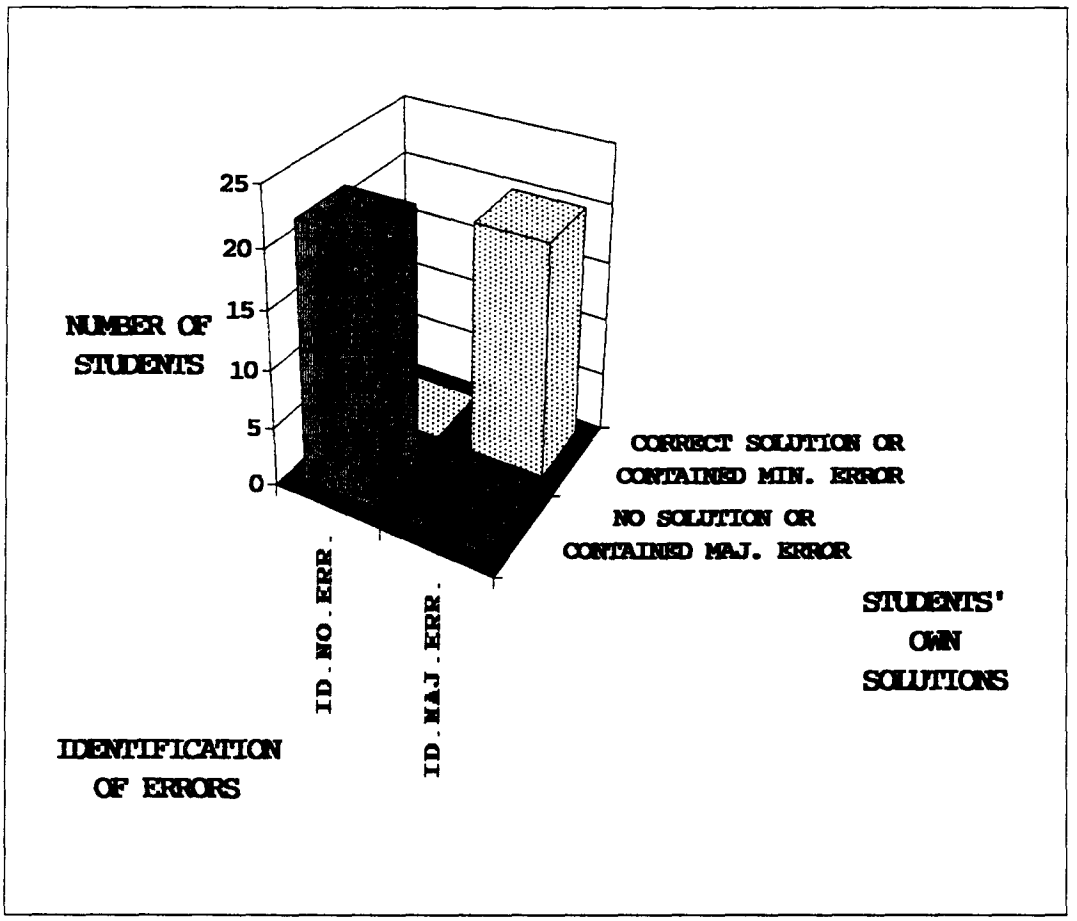


Figure 33: Cross-Tabulation Involving the Identification of Errors and Students' Own Solutions on Item 3 of Posttest 2 (n=44)

The last presolved algebra question on Posttest 2 paralleled Item 5 on Pretest 2 i.e., it took a simplify question and presented a given solution with a solve-type strategy. Figure 34 displays the students' results for this item.

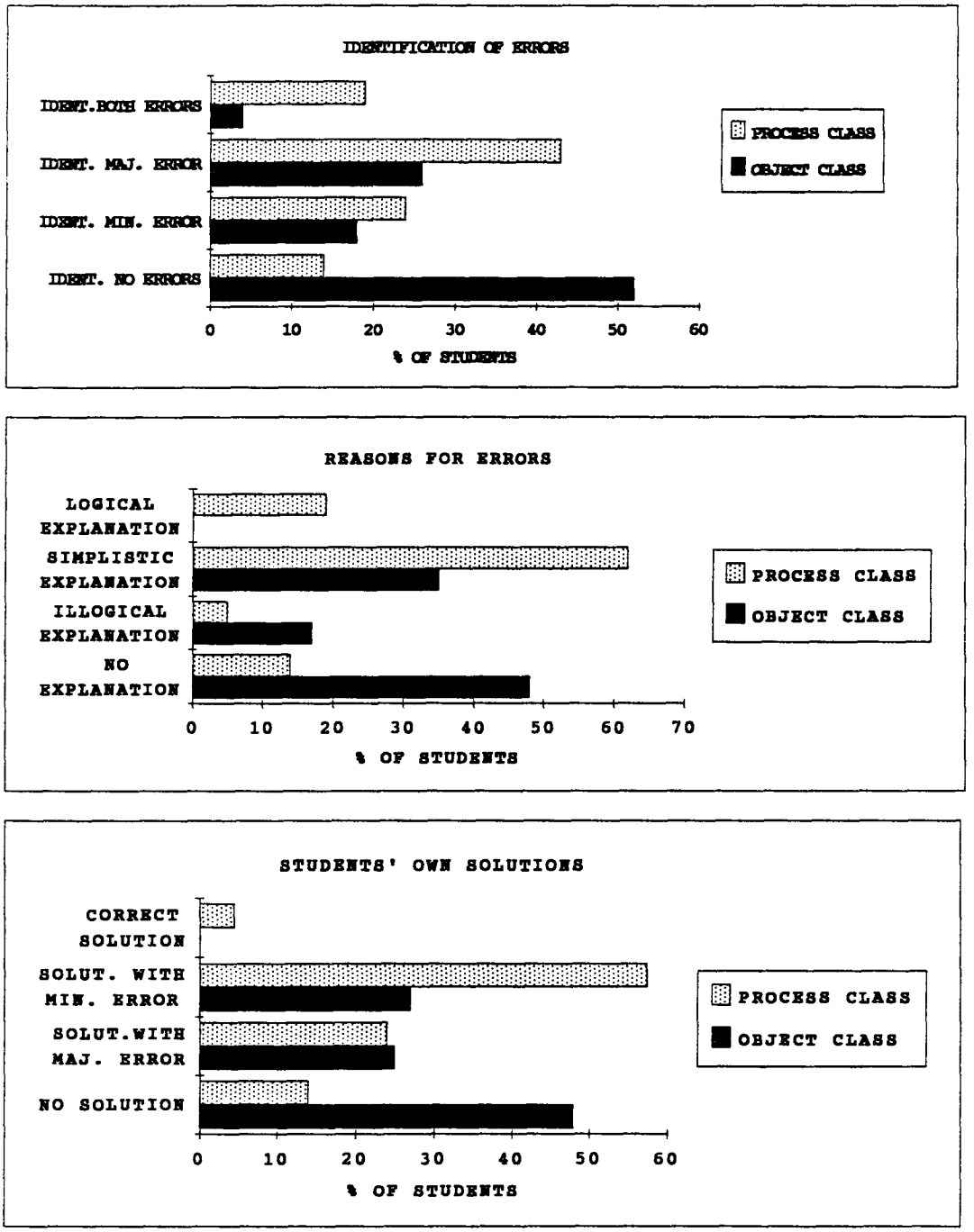


Figure 34: Students' Results on Item 5 of Posttest 2: Presolved Simplify Question With Solve Strategy

As in the previous presolved question there is a noticeable difference between the two classes that is consistent across all three tasks. As highlighted in Figure 35, 32% more students in the process class were able to identify the major error. For the reasons regarding the errors, four students in the process class were able to provide reasons categorized as correct and logical while none of the reasons given by the students in the object class attained this category. Thirteen students in the process class provided their own solutions with no major error, as compared with only six in the object class. Although this particular item was parallel to Item 5 of Pretest 2, it involved two rational expressions that were more difficult to factor and reduce. As a result, only one student from either class was able to provide a completely correct solution to the question. Therefore, most students felt the question part of this item was difficult.

As stated in Chapter 3, the items from Pretest 2 were based upon students' prior knowledge i.e., knowledge from Mathematics 10. Similarly, items from Posttest 2 were also based upon students' prior knowledge and this included knowledge from Mathematics 11. This may account for some students having a more difficult time with the tasks of Posttest 2 than with the similar tasks given in Pretest 2.

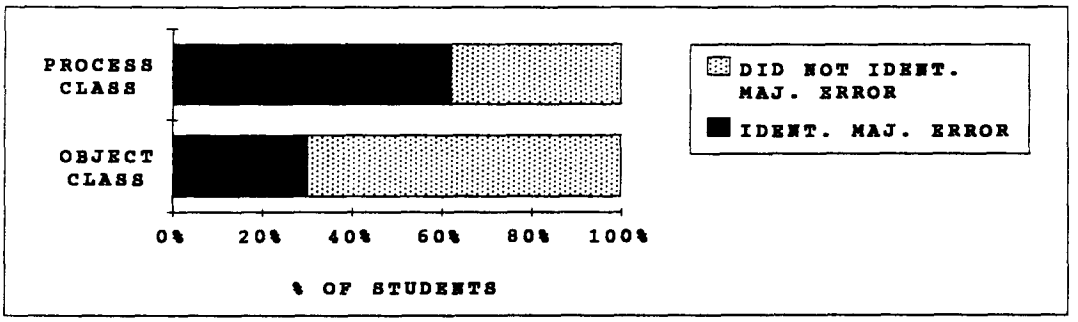


Figure 35: Percentage of Students Who Identified the Major Error on Item 5 of Posttest 2

Figure 36 shows a relatively strong relationship between the identification of errors and the students' own solutions; however, there is a column just below the upper portion of the diagonal that represents four students who were able to identify both the major and minor errors in the given solution but who had written their own solution still containing a minor error.

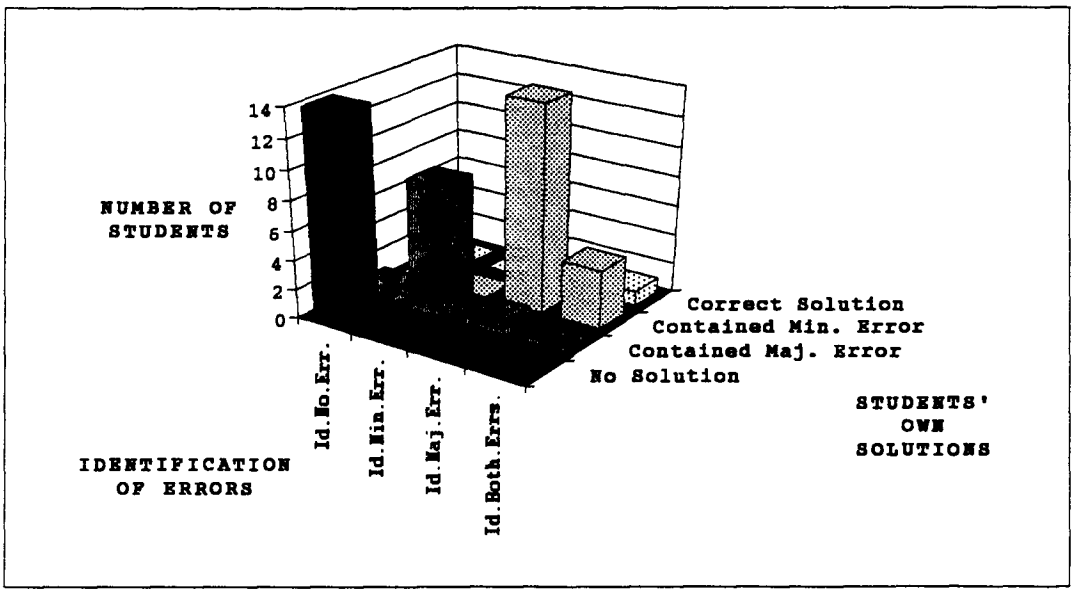


Figure 36: Cross-Tabulation Involving the Identification of Errors and Students' Own Solutions on Item 5 of Posttest 2 (n=44)

Even though this question seemed to be more difficult for all the students, the process class still managed to outperform the object class according to the scoring scale used as a greater percentage of students in the process class identified the major error.

Explanatory Item

The explanatory question in Posttest 2 parallels the explanatory question in Pretest 2, but it involves an addition of two rational expressions rather than subtraction, and it provided the first step for the students. The reason for providing this first step was to demonstrate what was expected in the question and to get more students to write down the procedural steps in their own words. The results for the scoring of this explanatory question are given in Figure 37.

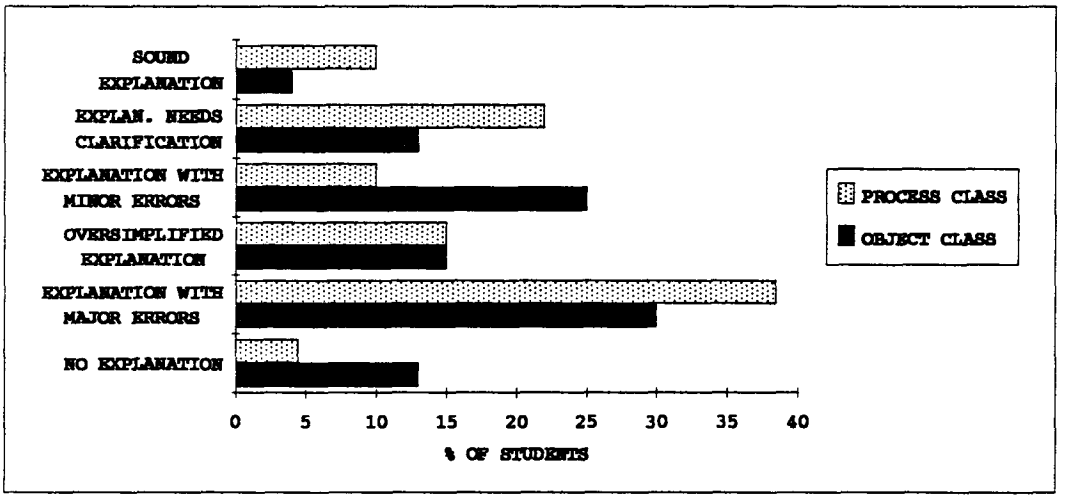


Figure 37: Students' Results on Item 2 of Posttest 2: Explanatory Question Involving The Addition of Two Rational Expressions

Due to the interpretation required in scoring this item, it is difficult to see any major differences between the two classes as most of the categories contain similar numbers of students in each. Part of the difficulty with this question may be that there are too many categories for such a small sample size. Even so, it is still apparent that most students had difficulty with this task, as over half the students in both classes wrote procedural steps that were confusing, oversimplified, or contained major errors, or no work was shown. Only three students from either class wrote down a set of procedural steps that could be used to achieve a correct solution.

Non-Routine Algebra Problem

The non-routine algebra problem on Posttest 2 was intended to parallel the problem from Pretest 2, however, it was essentially just a very difficult algebra question. It contained no words other than solve in the directions, this was in an attempt to appear less intimidating to students than the non-routine problem from Pretest 2. (Only one third of the students attempted that problem.) The results for this item are shown in Figure 38.

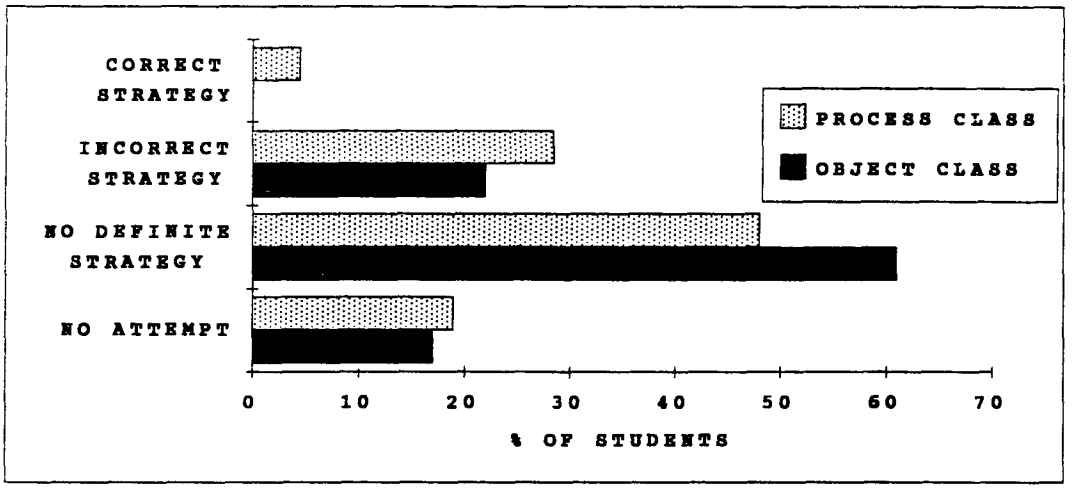


Figure 38: Students' Results on Item 6 of Posttest 2: The Non-Routine Algebra Problem

The results from both classes are similar as most of the students had difficulty with this item. Only one student from both classes was able to implement a strategy that could have led to a correct solution. Thirty-two out of the 44 students either did not attempt the question or made an attempt with no logical strategy. The attempt to make this task appear less intimidating obviously failed.

Process Definition Item

The last item on Posttest 2, the process definition question was not on Pretest 2. Its intent was to see how students connected certain procedures and operations to a given process. Upon re-examination of this question it appeared to be heavily biased towards the process class as they had already received materials and outlines organized around these processes.

Therefore, it may not be fair to use the results of this question to determine any differences in the levels of understanding between the two classes. The results for this question are displayed in Figure 39.

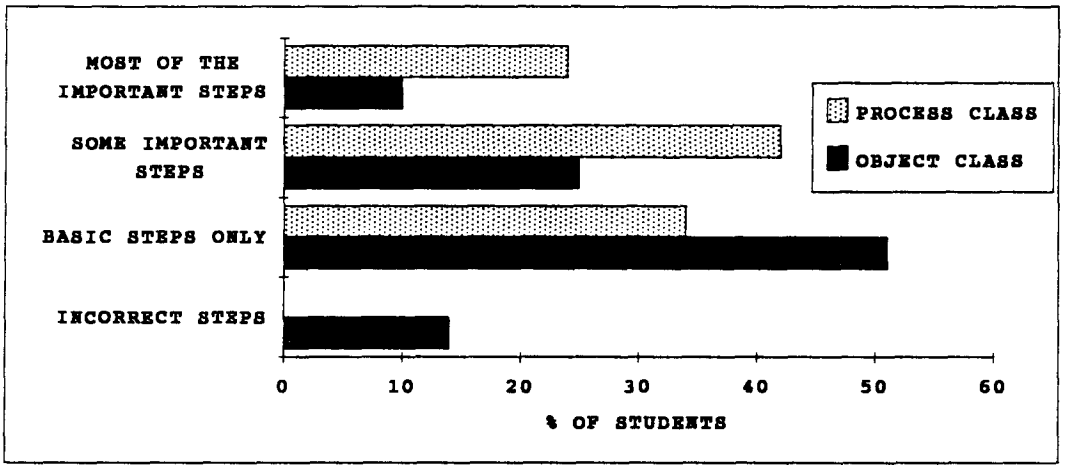
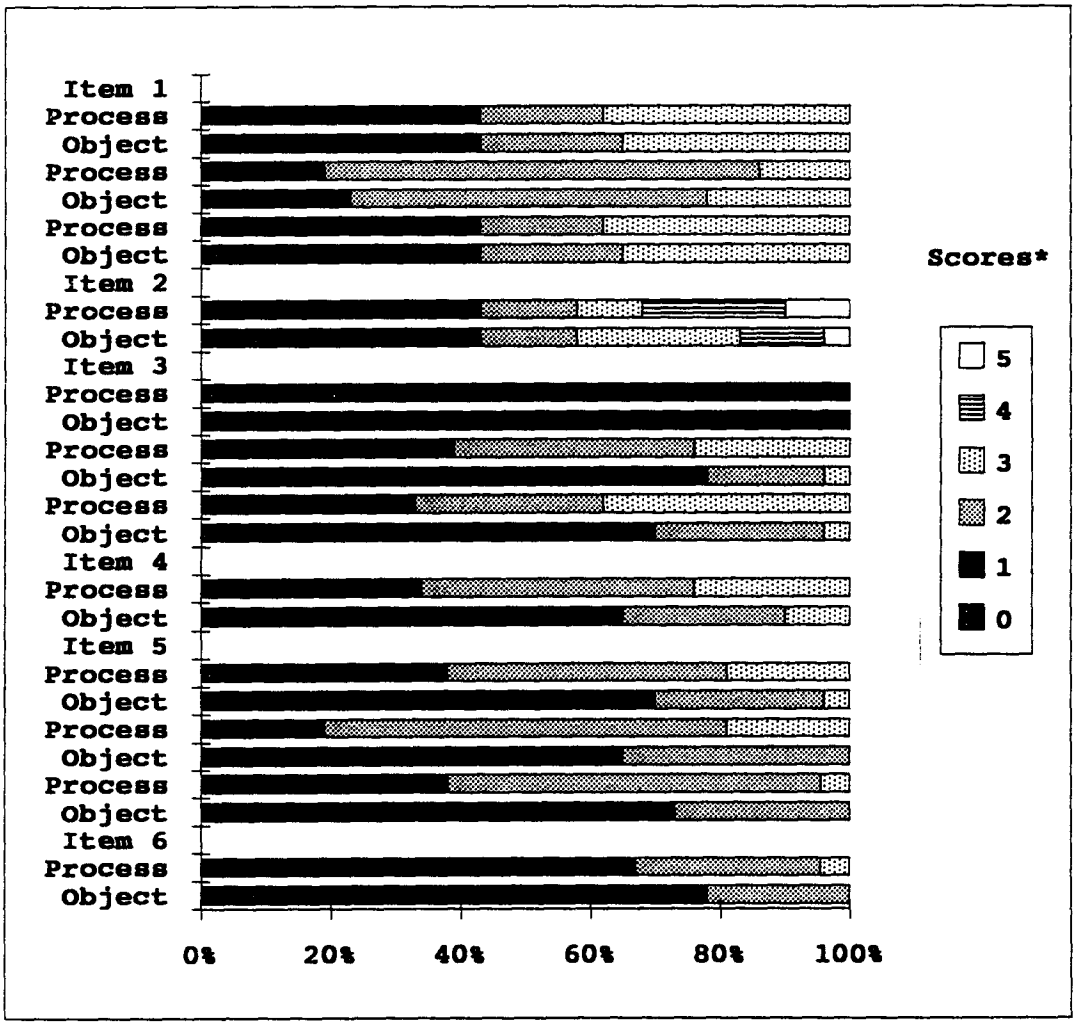


Figure 39: Students' Results on Item 4 of Posttest 2: Process Definition Question Involving Creating a List of Procedures for a Given Process

The results did show an observable difference between the two classes, as expected. All the students in the process class could identify at least a few basic procedures for each process. Although no comparisons can be made regarding levels of understanding between the two classes, this question does seem to verify that the treatment did cause the intended effect.

Summary of Posttest 2 Results

For Posttest 2 the results of the first presolved algebra question showed no indication of any differences in strategy selection or in understanding between the two classes, but the other two presolved questions did. There appeared to be a clear difference between the two classes on how well the students were able to identify errors in the given solution and how well they were able to provide solutions of their own without errors. Although the process definition question also produced observable differences in the results of the two classes, no conclusions can be drawn regarding these differences as the question was biased towards the process class. The explanatory question and the non-routine algebra problem suggested that both classes had difficulties in writing mathematical procedures in their own words and in implementing a strategy on tasks with which they were not familiar. Posttest 2 provided evidence of clear, observable differences between the two classes on some of the open-ended tasks. Again the Kolmogorov-Smirnoff test was employed to see if there were any statistically significant differences between the two classes on any of the items of Posttest 2. Figure 40 displays the students' results in percent; cumulative percentages are required to conduct the test.



* Refer to Scoring Scales used for Table 4 and Table 5

Figure 40: Students' Results in Cumulative Percent Comparing The Two Classes' Responses on the Items From Posttest 2

The null hypothesis for the test was as follows:

Null Hypothesis: There is no significant difference between the distributions of results on the open-ended items of Posttest 2 for students in the process-organized class and students in the object-organized class.

The probability of rejecting the null hypothesis was set at 0.10 and the results are displayed in Table 6.

Table 6: Results of Using the Kolmogorov-Smirnoff Two Sample Test With the Cumulative Percents From the Students' Results on the Open-Ended Items of Posttest 2

	OPEN-ENDED ITEMS											
STATISTIC	**1a**	**1b**	**1c**	**2**	**3a***	**3b***	**3c**	**4**	**5a***	**5b***	**5c**	**6**
Max.Diff.	.04	.12	.08	.15	.41	.39	.28	.36	.38	.46	.35	.11
Probability	>.2	>.2	>.2	>.2	.05	.07	>.2	.12	.09	.02	.14	>.2

a - identification of errors
b - reasons for errors
c - students' own solution
* - the null hypothesis can be rejected

For all other items, the null hypothesis cannot be rejected.

For Item 3, the Presolved Factor question, and Item 5, the Presolved Simplify question, there were statistically significant differences between the two classes, in favour of the process-organized class. On both items the process class was better able to identify the errors within the given solution, and was better able to provide reasons for the errors contained in the solution. As stated previously, the results for the reasons part of the presolved questions must be interpreted with caution.

Comparison Between Pretest 2 and Posttest 2

In order to evaluate the improvement shown by the classes on the open-ended items, the results from Pretest 2 and Posttest

2 were compared. With the exclusion of two pretest questions that were not included on the posttest and the decided differences between the two non-routine algebra problems, only three of the items could legitimately be called parallel. These included two presolved algebra questions and one explanatory question. It should be mentioned again that the scoring for the reasons portions of the presolved questions and the explanatory questions tended to require more subjectivity and must be interpreted with caution. For this reason the improvement comparisons for these two tasks will not be highlighted by their appearance in any of the figures.

Presolved Factor Question

First the results for the Presolved Factor Question with Simplify Strategy are examined for both the pretest and the posttest. Both classes showed evidence of improvement on this item, in being able to identify the major errors in the given solution and being able to provide solutions of their own containing no major errors. Yet there was a discrepancy between the two classes as shown in Figure 41. The object class had five more students able to identify the major error while the process class had 13 more students. Similarly, the object class had five more students able to provide solutions without a major error while the process class had 12. For the reasons portion of this item, the process class had eight more students in the top two

categories, while the object class had four more students in the top two categories. Overall the process class improved more than the object class on this particular item.

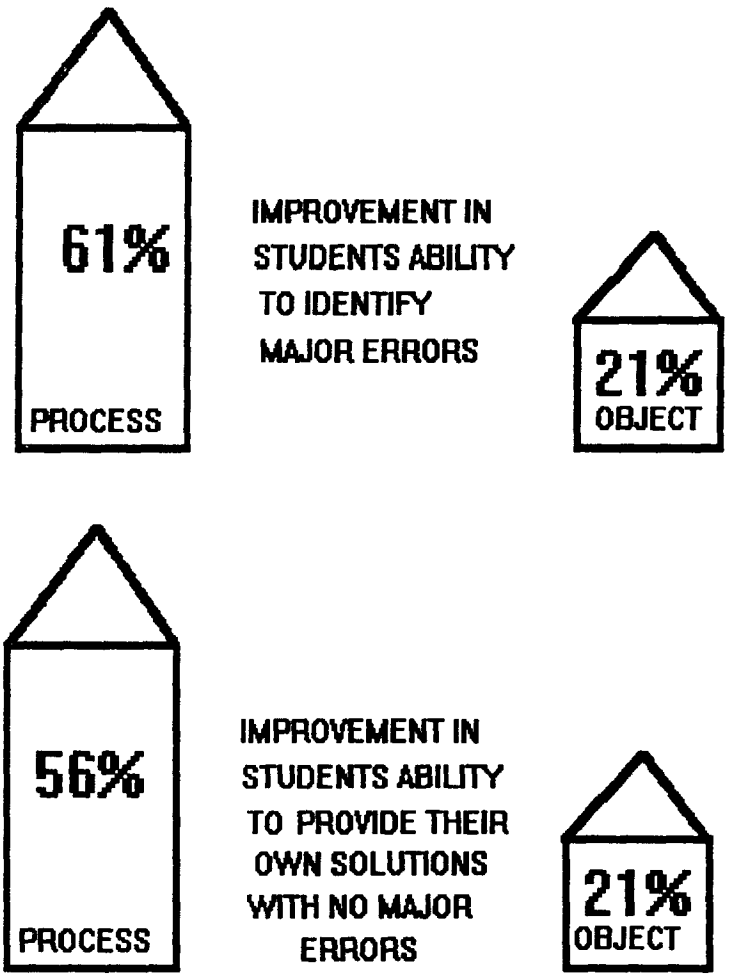


Figure 41: Improvement Shown as Percentage of Students for the Presolved Factor Question With Simplify Strategy

Presolved Simplify Question

The same sort of comparisons were made with the Presolved Simplify Question containing a Solve Strategy. In this item there was no evidence of improvement for the object class, as exactly the same number of students were able to identify the major error on the pretest question as on the posttest question. As well, the same number of students were able to provide their own solutions with no major errors. This could be due to the increased difficulty in the factoring required for the item on the posttest.

On the other hand the process class had eight more students able to identify the major error and nine more students provide solutions containing no major errors. Similarly, there were nine more students from the process class that gave reasons which were scored in the top two categories, while the object class had no more students in the top two categories. Figure 42 shows the improvement comparison between the two classes on this item.

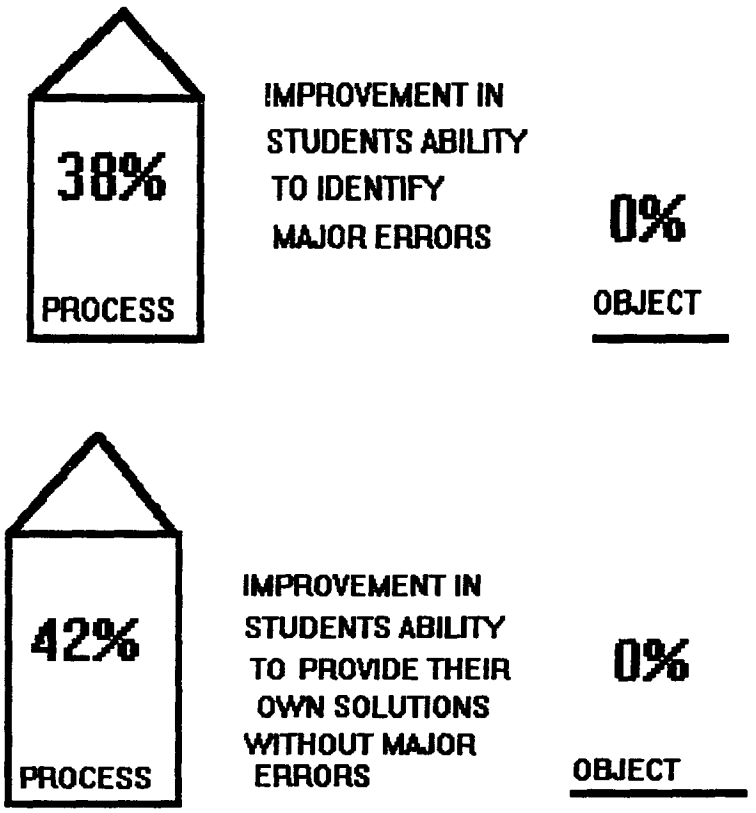


Figure 42: Improvement Shown as Percentage of Students for the Presolved Simplify Question With Solve Strategy

Explanatory Question

It was not so easy to measure improvement for the two classes' results on the explanatory question. Therefore, comparisons are made from an overall impression of the results and counting the number of students in the lower and the upper categories.

The object class showed very little evidence of any change

while the process class seemed to show minor improvements in their ability to write procedural processes in their own words. The object class had one fewer student attempt this task and two fewer students provide explanations that could be categorized as sound and logical. A reason for this poorer showing might be as a result of the directions given for this item in Posttest 2. They made it clear to students that they needed to write statements in their own words rather than just answering the question algebraically, as many had done on Pretest 2. For the process class there were 4 more students attempting this task and 4 more students having explanations that were categorized as sound and logical.

Summary of Comparison

The differences in improvement between the two classes as evidenced from the three parallel items on Pretest 2 and Posttest 2 was evident. The process class was better able to identify the major error in the given solution and was better able to provide their own solutions without any major strategic errors. As well, the process class seemingly had more students improve in their ability to write reasons and explanations regarding algebraic errors and procedures. The object class showed no measurable improvement in these areas.

The purpose of analyzing the results from these open-ended items was to determine any changes over the course of the study

in the students' ability to select appropriate strategies and in the students' understanding about the interconnectedness of mathematics.

Achievement Test Results

The purpose of analyzing the results for Pretest 1 and Posttest 1 was to determine whether there were any differences between the two classes in the students' achievement on standard algebra tests. The reasons for looking for differences in achievement was to provide a measure of the feasibility of organizing the curriculum around mathematical processes rather than mathematical objects.

Analysis of covariance was used to analyze the students' test scores on Posttest 1, the Term 1 Test, with Pretest 1 as the covariate. The ANCOVA increases the precision of the analysis of the results by limiting the effects of initial differences between the two classes. Thus it can more clearly identify the differences attributed to one of the two organizational schemes for the curriculum (McMillan & Schumacher, 1989). Although the differences between the two groups on Pretest 1 were not statistically significant as shown in Table 7, an ANCOVA was used to minimize the effects of the non-random school selection procedures. The probability of rejecting the null hypothesis was set at $p < 0.05$.

Table 7: t-Test Comparing the Means for Both Classes on Pretest 1

Null Hypothesis: There will be no significant difference between the mean scores on Pretest 1 of students in the process class and the students in the object class.

	Process Class	Object Class
Sample Size	21	23
Means (in Percent)	36.19	36.04
Variance	271.26	218.68
Sum of Squares	5425.24	4810.96

Observed t value: to = 0.007 Critical t value: tc =0.681

$to < tc$

Therefore, the null hypothesis cannot be rejected.

The summary of data for the scores of both classes on Posttest 1 is presented in Table 8.

Table 8: Summary of data for Posttest 1

	Process Class	Object Class
Mean	56.38	55.39
Standard Deviation	14.22	14.47
Range	51	63

The null hypothesis and the results of the Analysis of Covariance are presented in Table 9.

Table 9: Analysis of Covariance for the Mean Scores on Posttest 1 (Covariate: Pretest 1)

Null Hypothesis Tested: There is no significant difference between the mean scores on the Term 1 Test for the students of the process class and the students in the object class. ($p<.05$)

Sources of Variation	df	SS	MS	F	p
Between the classes	1	8.68	8.68	0.07	0.769
Within each class	41	5229.04	127.53		

df=degrees of freedom
SS=sum of squares
MS=mean square
P=probability of F

$p=0.769 > .05$

Therefore the null hypothesis cannot be rejected.

The analysis of the results indicates that there was no statistically significant difference between the adjusted means of the classes on Posttest 1, the Term 1 Test for the district.

Class Time

The amount of class time needed to cover a designated portion of the curriculum was also used as a measure of the feasibility of organizing the curriculum around mathematical processes rather than mathematical objects. The acetate rolls and the teacher's journal completed during instruction were used to determine the amount of class time spent on covering the designated portion of the curriculum. The object-organized class began their instructional lessons on September 3, 1991 while the

process-organized class began their lessons on September 4, 1991. During the study, intended learning outcomes 11.01 - 11.15 were covered as outlined in the mathematics curriculum guide (British Columbia Ministry of Education, 1988). The last lesson covering these learning outcomes for the object class occurred October 31, 1991 so a total of 29 class periods was required. For the process class the last lesson occurred on October 24, 1991 so a total of 25 class periods was spent. The process-organized class required four fewer classes to cover the same portion of the curriculum.

Summary of Results

The classroom observations as recorded in the teacher's journal provided examples of students' difficulties in selecting appropriate strategies and in understanding the interconnectedness of mathematics. The observations showed that there was a difference between the two classes in the way units were structured by the instructor. In the process class the introductory lessons dealt with broad ideas and concepts and general procedures, whereas in the object class the majority of the lessons tended to deal with specific topics. Most of the time it was not until the review lessons at the end of the unit that general ideas, concepts, and procedures were dealt with. Similarly the class discussions involving the interconnectedness of mathematics seemed to occur naturally during the introductory lessons for the process class and carried through till the

conclusion of the unit.

Outside of class time, individual interviews were conducted, in which the difficulties students had with strategy selection and understanding mathematics became very apparent. The students had many problems dealing with the language of mathematics as well as dealing with the mathematical meanings of certain concepts and procedures. The open-ended task conducted at the end of each interview suggested that either organization of the curriculum could add to or subtract from these student difficulties. There were indications from the open-ended task that students who increased their level of understanding tended to organize their perceptions of mathematics in ways similar to the way the curriculum was organized for them.

Students' difficulties were also very apparent from the results on various written open-ended items. On Pretest 2 students from both classes had difficulty identifying errors in strategy implementation, providing their own solutions, explaining procedural steps, and attempting to implement strategies on an unfamiliar task. Posttest 2 reflected many of these findings while at the same time showing definite differences between the two classes on two presolved algebra questions. Some of the differences on these two items were statistically significant. The process class had more students than the object class able to identify the strategic error in a given solution and able to

provide solutions of their own without any such strategic errors. The process class also had more students who improved in their abilities to write explanations and reasons pertaining to mathematics.

The analysis of students' results on standard tests used to assess overall achievement showed no such differences. It was determined by an ANCOVA that there was no statistically significant difference between the means of the two classes on Posttest 1.

The teacher's journal along with the acetate rolls used during instruction, provided a means of measuring the amount of class time needed by each class to cover the prescribed set of learning outcomes in the curriculum (British Columbia Ministry of Education, 1988). It was determined that the process class required four fewer class periods than the object class to cover the same learning outcomes.

CHAPTER 5: CONCLUSIONS

This study has investigated the effects of reorganizing the Mathematics 11 curriculum emphasizing mathematical processes rather than mathematical objects. A need for change in the current mathematics curriculum is expressed in the NCTM's Curriculum and Evaluation Standards (1989) as well as in the draft document for British Columbia's future mathematics curriculum within the Year 2000's Graduation Program (1992). A perceived need also arises from the prevalence of students' errors in selecting appropriate strategies and students' difficulties in understanding the interconnectedness of mathematics. In order for any curriculum change to occur it must be seen both as necessary and feasible by the teachers who must implement the change. Constraints facing teachers such as class time and success of the students on external exams must be considered. Once the need for curriculum change is established and the change is proven not to be detrimental then it becomes imperative to start implementing and evaluating this change.

The history of curricular change in British Columbia indicates that once a curriculum guide and textbook have been established for a course, change occurs very gradually (O'Shea, 1987). Many teachers tend to have an over reliance on the organizational schemes found in the curriculum guides and

prescribed textbooks in developing unit and lesson plans (Ediger, 1986; McKnight, 1987) The current Mathematics 11 curriculum guide is generally acceptable to most mathematics teachers yet there is some concern about the extensive amount of content to be covered (Johnson, 1987; Overgaard, 1987). The 1990 BC Mathematics Assessment (Robitaille, 1992) along with the comments made from markers in the Report to Schools (British Columbia Ministry of Education, 1989) seem to indicate that British Columbian students needed to develop a broader understanding of mathematics. From an overview of the curriculum guide Mathematics Grades 7-12 (British Columbia Ministry of Education, 1988) one could assess the organization as being traditional, emphasizing mathematical objects.

This organizational scheme of the curriculum has come under a great deal of scrutiny from critics as it tends to lead to fragmented teaching and learning of mathematics. Barbeau makes the following statement.

> When the school curriculum is subdivided into short units and the class moves quickly from one topic to the next without permeating very deeply into any of them, students are in danger of coming away without any sense of purpose or content. Although facts and techniques may be covered, their significance and power may be unrealized so that students quickly forget. (Barbeau, 1991, p. 522)

Researchers and educators are calling for a change to the

present mathematics curriculum (Artzt & Newman, 1991; Barbeau, 1991; Davis, 1984; Manhard, 1985; Skemp, 1986). In consideration of most of the necessary factors, researchers and authorities would seem to support an organization of the curricu-lum that highlights the major mathematical processes (Cobb, Yackel, & Wood, 1992; Davis, 1984; Young, 1982). Such a curriculum would emphasize the interconnectedness of mathematics.

In this study, attempting to investigate the effects of reorganizing the curriculum on students' understanding and students' ability to select appropriate strategies seemed to require a qualitative approach, while attempting to measure the effects on students' achievement and class time seemed to require a quantitative approach and a quasi-experimental design. However, the non-randomness of selection, small sample size, and the teacher acting as researcher meant that the results from these quantitative analyses must be interpreted with caution.

The study involved two Mathematics 11 classes, one was taught using the traditional organization of curriculum emphasizing mathematical objects, while the other class was taught using the alternative organization of curriculum emphasizing mathematical processes. At the beginning of the study students were given two pretests, one to determine their present level of ability on open-ended tasks and the other to determine their present performance on standard achievement tests. The pretests

were used to account for any prior differences in ability between the two classes. No substantial differences were found.

During the course of the study the teacher/researcher kept a journal as a record of any observations which could pertain to the research questions. At the completion of the instructional phase of the study the students were given two posttests, one was used to measure student achievement on standardized tests, and the other was a set of open-ended questions primarily used to determine students' abilities in selecting appropriate strategies, as well as, to determine any insights as to the students' understanding involving the interconnectedness of mathematics. Interviews were conducted after the pretests and posttests with previously selected students to clarify their responses on these tests as well to gain any additional information regarding their understanding about the interconnectedness of mathematics.

Any persistent differences between the two classes were commented upon along with any difficulties students had in understanding mathematics or selecting appropriate strategies. Similarly any differences in difficulties were noted for the clarification portion of the interviews. An additional open-ended task was conducted at that end of the interview where students had to formulate a series of "Yes or No" questions to uncover hidden algebraic tasks. These student-generated questions were recorded and analyzed according to their structure and level of

sophistication.

The open-ended items from the pretest and posttest were assessed according to heuristic scoring scales. The scoring scales were used to classify student responses from both classes. An analysis of covariance was used to measure any statistical differences between the means of the two classes on the district-wide term test. The parallel pretest was used as its covariate. The measurement of class time was expressed in terms of the number of periods required by each class to cover the required portion of the curriculum. This measurement was acquired through the use of the teacher's journal and confirmed by acetate rolls which were used regularly during instruction.

Answers to Research Questions

The answers to the research questions will be made in the inverse order from which they first appeared in Chapter 1. This inverted order represents the increasing levels of difficulty in obtaining each answer.

The class taught using the mathematical process organization of curriculum needed four periods fewer than the class taught from the mathematical object organization of curriculum to cover the designated portion of the curriculum.

From the analysis of covariance it was determined that there was no statistically significant difference between the means of the two classes on their Term 1 Test, Posttest 1.

Classroom examples were provided for both classes regarding students' difficulties in selecting appropriate strategies. The open-ended items from the pretest and posttest also indicated that both classes had difficulty in writing the procedural steps for a strategy in their own words. Students had difficulties supplying reasons to explain why certain strategies were inappropriate. In particular, on the pretest most students could not identify major errors in a previously implemented strategy nor were they able to provide solutions of their own using a correct strategy.

On Posttest 2, however, the process class showed signs of improvement on identifying major strategy errors and on selecting appropriate strategies of their own. There were also indications from the heuristic scales that more students in the process class improved in their abilities to provide reasons for the inappropriateness of strategies on previously solved questions, and to provide explanations for a set of procedural steps. Overall the process class showed more improvement on tasks requiring them to identify inappropriate strategies and select appropriate strategies of their own than the object class.

As far as assessing any differences between the two classes, on levels of understanding about the interconnectedness of mathematics was concerned, it was difficult to say if one class was better than the other. As a consequence of the organization

of the curriculum the unit and lesson structure chosen by the instructor provided the process class with a greater number of in-depth discussions than the object class. This would seem to indicate that the process class had more opportunities to develop and demonstrate their understanding of mathematics (Steen, 1989). As well, the lessons in the process class seemed to hit on the main mathematical concepts of a given unit on a more regular basis.

From the interviews it was concluded that the majority of the students had difficulty in communicating about mathematics. Some students used a low level descriptive form of communication while others often used mathematical terms like factor or reduce, inappropriately. Those students that apparently increased their understanding tended to utilize the curricular organization from their respective class for their own personal organization of mathematical concepts and procedures. There were students from both classes that improved in understanding, did not improve in understanding, or even showed signs of having less of an understanding than they had before.

Other Findings

One apparent discrepancy from the results is the process class improved fairly substantially on selecting appropriate strategies, yet there was no statistically significant difference between the two classes on Posttest 1. Part of this discrepancy

can be accounted for by the differences in what the two tests were measuring. The open-ended items on Posttest 2 were used to categorize student responses according to levels of logic and understanding. The algebraic items on Posttest 1 were used to measure the correctness of carrying out procedural steps, symbol manipulations, and numerical calculations. All algebraic items were marked out of one or two marks, and even though students from the process class were somewhat more adept at identifying major strategic errors and implementing appropriate strategies of their own, the majority of them still made minor errors in symbol manipulation or numerical calculations. Another reason for this discrepancy may be that before the Term 1 Test the teacher/researcher, out of professional obligation, had shared findings with both classes in hopes of better preparing them both for their upcoming test. The teacher/researcher would have been quite uncomfortable if the results on the Term 1 Test had indicated a statistically significant difference, as assurances had to be given to parents and students that there would be no adverse effects to students' grades in either class for taking part in the study.

The presolved questions on the open-ended tests proved to be a useful way of measuring students abilities in selecting appropriate strategies. There was an observable relationship between students ability to identify an inappropriate strategy

used in the given solution and ability to select and implement an appropriate strategy on their own to complete the given algebraic task. These questions were also relatively easy to score as students either identified the errors or they did not. Although such open-ended items worked reasonably well in assessing students' abilities in strategy selection, no such items were found in assessing students' understandings about the interconnectedness of mathematics; however, certain observations were made that may account for why students have difficulties in this area.

Many students had problems with the definitions of various mathematical terms; in fact, many had made up their own definitions which are not equivalent to the accepted ones. Some students come up with definite ideas about certain short-cut methods they have used in previous mathematics classes with a certain amount of success but with very little understanding. Students' problems with the language of mathematics and beliefs about short-cut methods are difficult to overcome and provide a hindrance to their understanding about the interconnectedness of mathematics.

Implications

According to this study, reorganizing the Mathematics 11 curriculum to emphasize mathematical processes is feasible. There appeared to be no detrimental effects on students' achievement scores on standard tests, and it required less class time than the

traditional organization of curriculum emphasizing mathematical objects. This alternative organization of curriculum lending itself to a definite lesson and unit structure also seemed to provide students greater opportunities to develop an understanding about certain mathematical concepts and ideas. They were confronted with general mathematical processes at the beginning of a unit which brought about many class discussions. The key ideas from these processes were emphasized throughout the majority of lessons within a given unit. From this emphasis on key processes and the greater opportunities to develop an understanding about the interconnectedness of mathematics, the students from the process class were somewhat better able to select appropriate strategies.

Despite the process class displaying superior performance in strategy selection, there was no evidence of such improvement on the common Term 1 Test, Posttest 1. This may be due to the evaluation techniques utilized in secondary mathematics i.e., paper and pencil achievement tests (Usiskin, 1985; Willoughby, 1990). If understanding mathematics and selecting appropriate strategies is important, there needs to be a way of assessing these skills and abilities. No effort to change curricula or teaching practices will succeed unless the instruments of assessment are aligned with educational goals (NCTM, 1989; Romberg, Zarinna, & Collis, 1990). Authentic assessment techni-

ques and instruments should be selected to reflect the type of information sought, and the information sought should focus on understanding, processes, and attitudes, as well as achievement. Curriculum, instruction and assessment need to change together in order to improve mathematics education (Kulm, 1990; Pandey, 1990).

From the results of this study it would appear that there is a great deal of usefulness in reorganizing curricula to emphasize the important ideas of mathematics. This can be done in a way that does not jeopardize students' scores on common external examinations or place any additional burdens on class time. Connections between various ideas need to be shown to students so that the students can formulate meaningful connections of their own. The reorganization should not only be designed to emphasize the main ideas of mathematics but should address difficulties students have in understanding mathematics. Through such an organization the classroom may become an open forum for discussing important ideas about mathematics and provide greater opportunities for understanding and learning mathematics.

Suggestions for Future Research

Obviously, the first suggestion for any future research on curriculum organization would be to eliminate the greatest limitation of this study. In other words conduct the same study without the teacher being the researcher and observe more than

two classes.

There is a need to include what educators and society view as important in the framework of the mathematics curriculum. Once this framework has been established an organization of curriculum along with suitable materials needs to be developed that reflects this importance. This reorganization of the curriculum could be tested against the more traditional organizations on a wide scale involving many different schools and utilizing common examinations. These examinations should include a variety of questions including traditional algebraic questions along with items used to assess a students' understanding about mathematical strategies and processes. Results could then be compared and analyzed on a much broader scale than was the case with this study.

Research should also be conducted on a much smaller scale, to demonstrate how individual students utilize a curriculum design in developing their own understanding of and appreciation for mathematics. During such a study the researcher could monitor individual students and conduct in-depth interviews to determine the level of student's understanding about key concepts and ideas in mathematics. Within such a study the students themselves could demonstrate how they think the major ideas of mathematics are and can be connected.

Not only should the effects of a new curriculum be looked

at from a student's point of view but also from a teacher's point of view. Teachers with definite teaching styles could be observed for a certain period of time using a traditional organization of curriculum. Later, the teachers could be made aware of an alternative organization and asked to implement it in using their particular teaching style. Do they spend the same amount of time lecturing? Do they conduct more meaningful discussions with their class? Have they developed a more effective teaching style? As well questions could raised about how the teachers found the whole process of organizing the curriculum to suit their needs.

Closing Comments

Teachers do not have to rely on the curriculum guide and the prescribed text book to organize their classroom lessons. They should search for ways of organizing the content of the curriculum that would emphasize those ideas in mathematics they feel are most important. Standards for measuring the effects of this organization must not be lowered, they should stand up to standard tests of achievement and constraints of class time. If it is a truly valuable organization there should also be other evidence to indicate students' improvement in understanding the interconnectedness of mathematics.

REFERENCES

Alexander, B., Atkinson, P., Kelly, B., & Swift, J. (1989). Mathematics 11. Don Mills, Ontario: Addison-Wesley.

Anderson, J. (1983). The architecture of cognition. Cambridge, MA: Harvard University Press.

Artzt, A. & Newman, C. (1991). Equivalence: A unifying concept. Mathematics Teacher, 84 (2), 128-132.

Atchison, W., Fey, J., Good, R., Heid, M., Johnson, J., Kantowski, M., & Rosen, L. (1984). Computing and mathematics. Reston, VA: National Council of Teachers of Mathematics.

Barbeau, E. (1991). A holistic approach to algebra. Mathematics Teacher, 84 (7), 522-525.

Beattie, I., & Steblin, V. (1977). Mathematics education in British Columbia - local curriculum development: Questionnaire results. Vector, 19 (1), 8-14.

Begle, E. (1973). Some lessons learned from SMSG. Mathematics Teacher, 66 (3), 207-214.

Booth, L. (1988). Children's difficulties in beginning algebra. In T. J. Cooney & C. R. Hirsch (Eds.), The ideas of algebra K-12: 1988 Yearbook (pp. 20-27). Reston, VA: National Council of Teachers of Mathematics.

Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? Jounal for Research in Mathematics Education, 23 (3), 194-222.

British Columbia Ministry of Education. (1988). Mathematics Grades 7-12. Victoria, BC: Program Development Branch.

British Columbia Ministry of Education. (1989). Report to schools. Victoria, BC: Program Development Branch.

British Columbia Ministry of Education. (1990). Year 2000: A framework for learning. Victoria, BC: Program Development Branch.

British Columbia Ministry of Education. (1992). Curriculum/ assessment framework: Mathematics (Draft). Victoria, BC: Program Development Branch.

Brutlag, D. & Maples, C. (1992). Making connections beyond the surface. Mathematics Teacher, 85 (3), 230-233.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics. Jounal for Research in Mathematics Education, 23 (1), 7-33.

Cooney, T. (1985). Organizing instruction: Logical considerations in organizing instruction for mathematics. In C. R. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum: 1985 Yearbook (pp. 147-167). Reston, VA: National Council of Teachers of Mathematics.

Davis, R. (1987). The changing curriculum: Mathematics. Washington, DC: Association for Supervision and Curriculum Development.

Davis, R. (1984). Learning mathematics: The cognitive science approach to mathematics education. Norwood, NJ: Ablex.

Ediger, M. (1986). Scope in the mathematics curriculum. Delta-k, 26 (1), 14-17.

Erlwanger, S. (1973). Benny's conception of rules and answers in IPI mathematics. Journal of Childrens Mathematical Behavior, 1 (2), 7-26.

Fey, J. & Good, R. (1985). Rethinking the sequences and priorties of high school mathematics curricula. In C. R. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum: 1985 Yearbook (pp. 43-52). Reston, VA: National Council of Teachers of Mathematics.

Fey, J. (1989). School Algebra for the year 2000. In S. Wagner & C. Keiran (Eds.), Research issues in the learning and teaching of algebra (pp. 199-213). Reston, VA: National Council of Teachers of Mathematics.

Frederiksen, N. (1984). Implications of cognitive theory for instruction in problem solving. Review of Educational Research, 54 (3), 363-406.

Garofalo, J. & Lester, F. (1985). Metacognition, cognitive monitoring and mathematical performance. Journal for Research in Mathematics Education, 16 (3), 163-176.

Glidden, P. (1991). Teacher's reasons for instructional decisions. Mathematics Teacher, 84 (8), 610-614.

Gliner, G. (1991). College students' organization of mathematics word problems in terms of mathematical structure vs. surface stucture. School Science and Mathematics, 91 (3), 105-110.

Goodlad, J. (1984). A place called school. New York: McGraw-Hill.

House, P. (1988). Reshaping school algebra: Why and how? In T. J. Cooney & C. R. Hirsch (Eds.), The ideas of algebra K-12: 1988 Yearbook (pp. 1-7). Reston, VA: National Council of Teachers of Mathematics.

Hovdebo, L. (1987). Portraits of understanding: Thought processes exhibited during a task-based interview. Unpublished master's thesis, University of Calgary, AB.

Howson, G., Keitel, C. & Kilpatrick, J. (1981). Curriculum development in mathematics. New York: Cambridge University Press.

Johnson, V. (1986). Summary report on the mathematics curriculum. Vector, 27 (4), 37-40.

Kaput, J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Keiran (Eds.), Research issues in the learning and teaching of algebra (pp. 167-194). Reston, VA: National Council of Teachers of Mathematics.

Kent, D. (1979). Some processes through which mathematics is lost. Educational Researcher, 21 (1), 27-35.

Kirshner, D. (1989). Critical issues in current representation system theory. In S. Wagner & C. Keiran (Eds.), Research issues in the learning and teaching of algebra (pp. 195-199). Reston, VA: National Council of Teachers of Mathematics.

Kulm, G. (1990). Assessing higher order mathematical thinking: What we neeed to know and be able to do. In G. Kulm (Ed.), Assessing higher order thinking in mathematics (pp. 1-6). Washington, DC: American Association for the Advancement of Science.

Manhard, W. (1985). Let's teach mathematics: A case for integrated mathematics programs. In C. R. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum: 1985 Yearbook (pp. 189-200). Reston, VA: National Council of Teachers of Mathematics.

Marshal, S. (1988). Assessing problem solving: A short term remedy for a long term solution. In R. Charles & E. Silver (Eds.), The teaching and assessing of problem solving (pp. 159-177). Reston, VA: National Council of Teachers of Mathematics.

Marshal, S. (1990). The assessment of schema knowledge for arithmetic story problems. In G. Kulm (Ed.), Assessing higher order thinking in mathematics (pp. 155- 168). Washington, DC: American Association for the Advancement of Science.

Mathematical Sciences Education Board. (1990). Reshaping school mathematics. Washington, DC: National Academy Press.

McKnight, C., Crosswhite, F., Dossey, J., Kifer, E., Swafford, J., Travers, K., & Cooney, T. (1987). The underachieving curriculum. Champaign,IL: Stipes.

McMillan, J., & Schumacher, S. (1989). Research in education (2nd ed.). New York: Harper Collins.

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston,VA: Author.

National Council of Teachers of Mathematics. (1991). Professional standards for teaching mathematics. Reston,VA: Author.

O'Shea, T. (1987). The new B.C. mathematics curriculum: A sheep in Wolfe's clothing? Vector, 29 (1), 32-37.

Osborne, A. & Crosswhite, F. (1970). Forces and issues related to curriculum and instruction, 7-12. In P. S. Jones (Ed.), A history of mathematics education in the United States and Canada: 32nd Yearbook (pp.155-196). Washington,DC: National Council of Teachers of Mathematics.

Overgaard, V. (1987). Summary report on the mathematics curriculum guide 7-12. Vector, 29 (1), 38-47.

Pandey, T. (1990). Power items and the alignment of curriculum and assessment. In G. Kulm (Ed.), Assessing higher order thinking in mathematics (pp. 39-52). Washington, DC: American Association for the Advancement of Science.

Paul, F. & Richbart, L. (1985). New York State's new three-year program for high school mathematics. In C. R. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum: 1985 Yearbook (pp. 201-210). Reston, VA: National Council of Teachers of Mathematics.

Perkins, D. & Simmons, R. (1988). Patterns of misunderstanding: An integrative model for science math and programing. Review of Educational Research, 58 (3), 303-326.

Rachlin, S. (1989). The research agenda in algebra: A curriculum development perspective. In S. Wagner & C. Keiran (Eds.), Research issues in the learning and teaching of algebra (pp. 257-265). Reston, VA: National Council of Teachers of Mathematics.

Robitaille, D. (1992). The 1990 British Columbia mathematics assessment. Victoria, BC: Ministry of Education.

Romberg, T., Zarinina, E., & Collis, K. (1990). A new world view of assessment in mathematics. In G. Kulm (Ed.), Assessing higher order thinking in mathematics (pp. 21-38). Washington, DC: American Association for the Advancement of Science.

Rowan, T. & Morgan, C. (1980). Designing a plan for change. In J. Price & J. Grawonski (Eds.), Changing school mathematics: A responsive process (pp. 103-127). Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. (1985). Mathematical problem solving. Orlando, FL: Acedemic Press.

Silver, E. (1982). Knowledge organization and mathematical problem solving. In F. Lester & J.Garofalo (Eds.), Mathematical problem solving: Issues in research (pp. 15-25). Phildelphia: Franklin Institute Press.

Simon, H. (1980). Problem solving and education. In D. Tuma & R. Reif (Eds.), Problem solving and education: Issues in teaching and research (pp. 81-96). Hillsdale, NJ: Erlbaum.

Skemp, R. (1986). Psychology of learning mathematics. Middlesex, England: Penguin Books Ltd.

Skemp, R. (1978). Relational understanding and instrumental understanding. Arithmetic Teacher, 26 (3), 9-15.

Sonnabend, T. (1985). Noncareer mathematics: The mathematics we all need. In C. R. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum: 1985 Yearbook (pp. 107-118). Reston, VA: National Council of Teachers of Mathematics.

Steen, L. (1989). Teaching mathematics for tomorrow's world. Educational Leadership, 10 (4), 18-22.

Szetela, W. (1992). Performance on open-ended problems. In D. Robitaille (Ed.), The 1990 British Columbia mathematics assessment (pp. 181-209). Victoria, BC: Ministry of Education.

Thompson, P. (1989). Artificial intelligence, advanced technology and the learning and teaching of algebra. In S. Wagner & C. Keiran (Eds.), Research issues in the learning and teaching of algebra (pp. 135-161). Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (1985). We need another revolution in secondary school mathematics. In C. R. Hirsch & M. Zweng (Eds.), The secondary school mathematics curriculum: 1985 Yearbook (pp. 1-21). Reston, VA: National Council of Teachers of Mathematics.

Willoughby, S. (1990). Mathematics education for a changing world. Alexandra, VA: Association for Supervision and Curriculum Development.

Young, S. (1982). The mental representation of geometrical knowledge. Journal of Mathematical Behavior, 3 (2), 123-144.

Appendix A

Pretest 1

Name _____

Block _____

Mathematics 11 Pretest #1

A. Multiple choice: Circle the best answer on the answer sheet provided.

1. If y is a real number then $\frac{y}{0}$ =

A. 0
B. y
C. any real number
D. undefined
E. none of these

2. If m + 2 = 10 then 3(m+2) = 3(10) is an example of:

A. The Distributive Property of Multiplication over Addition
B. The Multiplication Property of Equality
C. The Inverse Property of Multiplication
D. The Associative Property of Multiplication
E. None of these

3. Simplify

$$5 + 8 \times 6 + 3 \times 2 - 1$$

A. 36
B. 12
C. 51
D. 20
E. None of these

4. Simplify

$$1\frac{5}{9} - 4\frac{1}{3}$$

A. $-5\frac{8}{9}$

B. $-3\frac{2}{9}$

C. $-2\frac{7}{9}$

D. $-2\frac{2}{9}$

E. None of these

5. The square root of $\frac{16x^2y^{16}}{b^4}$ is

A. $\frac{4xy^4}{b^4}$

B. $\frac{4y^4}{b^2}$

C. $\frac{4xy^8}{b^2}$

D. $\frac{8xy^8}{b^2}$

E. None of these

6. The L.C.M. for $4m^2np$, $6mp^2$ and $9np^3$ is

A. mp

B. $72m^2np^3$

C. $36m^3n^2p^6$

D. $36m^2np^3$

E. None of these

7. In the equation $3^{n+1} = 81$, n must be:

A. 3
B. 4
C. 26
D. 77
E. None of these

8. Simplify

$$\sqrt{144 + 25}$$

A. 7
B. 13
C. 15
D. 17
E. None of these

9. Simplify

$$\sqrt{48} - \sqrt{75}$$

A. $-\sqrt{3}$
B. $-3\sqrt{3}$
C. $6 - 3\sqrt{5}$
D. $2\sqrt{12} - 5\sqrt{3}$
E. None of these

10. Simplify

$$(\sqrt{5} + 2)(2\sqrt{5} - 1)$$

A. 48
B. 8
C. $5\sqrt{2} - 3\sqrt{5}$
D. $8 + 3\sqrt{5}$
E. None of these

11. Factor completely

$$(x + 1)^2 - 9$$

A. $x^2 + 2x - 8$
B. $(x - 2)(x + 4)$
C. $(x - 8)(x + 10)$
D. $x^2 - 8$
E. None of these

12. The square of $x - 3$ is:

A. $x^2 - 9$
B. $x^2 + 9$
C. $x^2 - 3x + 9$
D. $x^2 - 6x + 9$
E. None of these

13. Simplify

$$(x + 2)(x^2 - 2x + 1)$$

A. $x^3 - 3x + 2$
B. $(x + 2)\ (x - 1)$
C. $x^3 - 4x + 1$
D. $(x + 2)(x - 1)^2$
E. None of these

14. A solution for

$$x^2 - x = 20 \text{ would be } x =$$

A. $\frac{20}{x-1}$
B. 4
C. 5
D. All real numbers except 1
E. None of these

15. Simplify

$$\frac{5}{n^2} + \frac{3}{nw}$$

A. $5w + 3n$

B. $5nw + 3n^2$

C. $n(5w + 3n^2)$

D. $\frac{5w + 3n}{n^2w}$

E. None of these

16. Simplify

$$n - n^{-1}$$

A. $\frac{n + 1}{n}$

B. $\frac{n^2 - 1}{n}$

C. 0

D. $2n$

E. None of these

17. Simplify

$$\frac{2x^2 + 7x - 15}{2x^2 - x - 3}$$

A. -2

B. $8x - 5$

C. $\frac{(2x + 3)(x - 5)}{(2x - 3)(x + 1)}$

D. $\frac{x + 5}{x + 1}$

E. None of these

B) <u>Long Answer</u> SHOW ALL WORK

1. Write 2.1555...as a reduced fraction

/2

1. _____

2. Simplify

$$\frac{2^{n+4} \cdot 2^{2n-3}}{2^{n-2}}$$

/2

2. _____

3. Simplify

$$\frac{\sqrt{2}}{\sqrt{27}} - \frac{\sqrt{27}}{\sqrt{2}}$$

/2

3. _____

4. Factor completely

$$4(5 - k) - 4k(k - 5)$$

/2

4. _____

5. Solve

$$\frac{5}{n - 2} + \frac{2}{n - 4} = -\frac{4}{n^2 - 6n + 8}$$

/2

5. _____

6. Simplify

$$\frac{3}{a-3} - \frac{5}{a+3} + \frac{1}{a^2-9}$$

/2

6. _____

7. Find the G.C.F. for

$$9 - c^2,\ 2c^2 - 10c + 12,\ c^2 - 6c + 9$$

/2

7. _____

8. Solve for x

$$ax + b = \frac{c}{d}$$

/2

8. _____

9. Henry can eat a tub of plums in 3 hours and Frank can eat the same size tub of plums in 4 hours. How long would it take them to eat one tub of plums together?

/3

9. _____

Appendix B

Pretest 2

Name:_____

Mathematics 11 **Pretest #2**

For this test it is your explanations which are most important, therefore answer as specifically as you can. For most questions there is more than one correct answer.

1. Factor the following:

$$(2x - 1)^2 - 9(x + 2)^2$$

Ernie's solution to this question is:

Is Ernie's solution correct?_____

If it is correct, briefly explain his steps._____

_____

_____

_____

If it is incorrect, state what is wrong and expain why it is wrong._____

_____

_____

_____

Show how you would answer this question.

2. Imagine you are talking to a friend on the phone about a difficulty they are having with simplifying rational expressions. They provide you with the following example:

$$\frac{x + 4}{2x^2 - 2x} - \frac{5}{2x - 2}$$

What would you say to guide them through the question step by step, so that they will better understand how to deal with similar questions? Anticipate their responses.

3.

Column A	Column B
Add/Subtract	Factor
Multiply	Simplify
Divide	Solve

3. Match items from Column A to those which are most similar in Column B and explain why you made your choice.

4. Using the same columns, match items from Column A to those which are most dissimilar in Column B. Explain why you made your choice.

5. Simplify the following:

$$\frac{x + 2}{x^2 - 9x + 20} + \frac{3x + 1}{x^2 - 7x + 12}$$

Frank's solution for this question is:

Is anything wrong with Frank's solution?

Explain._____

_____

_____

Is anything right with Frank's solution?

Explain._____

_____

_____

Show how you would you have answered this question?

6. Show that the product of the sum of three positive real numbers by the sum of their reciprocals is always at least 9. And when can the product be exactly 9? SHOW ALL WORK AND EXPLAIN!

Appendix C

Posttest 1

TERM 1 TEST MATHEMATICS 11

NAME _____

Block _____ Teacher _____

A. MULTIPLE CHOICE: Select the best answer for each question and circle the letter on the separate answer sheet.

1. Factor completely: $(x - 1)^3 + 27$

   a. $(x + 2)^3$
   b. $(x - 4)^3$
   c. $(x - 4)(x^2 + x + 7)$
   d. $(x + 2)(x^2 - 5x + 13)$
   e. $(x + 2)(x - 4)^2$

2. The multiplicative inverse of -2 is:

   a. 2
   b. $-\frac{1}{2}$
   c. 0
   d. 1
   e. 2^{-1}

3. If x, y and z are real numbers then what property is illustrated in the following example: $(x + 7) + z = z + (x + y)$

   a. associative property of addition
   b. cummutative property of addition
   c. distributive property of multiplication over addition
   d. identity property of addition
   e. inverse property of addition

4. The solution for x in the equation $2(2x - 3) - 2 = 4(x - 2)$ is:

   a. 0
   b. 2
   c. -3
   d. undefined
   e. any real number

5. Expand and simplify: $(x - \sqrt{2})^3$

   a. $x^3 - 8$
   b. $x^3 - 2\sqrt{2}$
   c. $x^3 - 3\sqrt{2}\,x^2 + 6x - 2\sqrt{2}$
   d. $x^3 - \sqrt{2}\,x^2 - 2x + 2\sqrt{2}$
   e. $x^3 - 4x^2 + 4x - 2\sqrt{2}$

6. Factor completely: $48x^2 - 6y^2 - 68xy$

   a. $(8x + 3)(6x - 2)$
   b. $2(8x + 3)(3x - 1)$
   c. $2(12x + 1)(2x - 3)$
   d. $(4x - 3)(12x + 2)$
   e. $2(4x - 3)(6x + 1)$

7. Solve for y. $\sqrt[3]{y^{-1}} = -3$

   a. $-\frac{1}{27}$
   b. -27
   c. $\frac{1}{27}$
   d. 27
   e. None of these

8. Simplify: $\left(\frac{729}{64}\right)^{-\frac{5}{6}}$

   a. $\frac{3}{2}$
   b. $\frac{-8}{27}$
   c. $\frac{32}{243}$
   d. $-\frac{3645}{128}$
   e. $\frac{2187}{320}$

9. Simplify completely: $\frac{2 - 3\sqrt{6}}{3\sqrt{2} + 4\sqrt{3}}$

   a. $\frac{6\sqrt{2} - 8\sqrt{3} - 9\sqrt{12} + 12\sqrt{18}}{30}$
   b. $\frac{6\sqrt{2} - 24\sqrt{3}}{66}$
   c. $\frac{-10\sqrt{3} - 30\sqrt{2}}{66}$
   d. $\frac{26\sqrt{3} - 42\sqrt{2}}{30}$
   e. $\frac{-10\sqrt{3} + 30\sqrt{2}}{108}$

10. Solve over the set of real numbers: $22 - \sqrt{2x - 5} = 13$

a. 615
b. 43
c. 27
d. 4
e. No real solution

11. For the expression $\frac{x - 4}{x^2 - 4x}$ x can be any real number except:

a. 0
b. 0, 4
c. 0, 2
d. 0, 2, -2
e. 0, 4, -2

12. Simplify: $\frac{3x^2 + 10 - 8}{4 - 4x - 3x^2}$

a. $\frac{-5x - 2}{2}$

b. $\frac{5x + 4}{2}$

c. -2

d. $\frac{-x - 4}{x + 2}$

e. $\frac{x + 4}{x - 2}$

13. Simplify: $\sqrt[3]{16} - \sqrt[3]{54}$

a. $-\sqrt[3]{2}$
b. $-\sqrt[3]{38}$
c. $5\sqrt[3]{2}$
d. $4 - \sqrt[3]{6}$
e. -1

14. Simplify: $\frac{9^{\frac{x}{2} - 3} \cdot 9^{x}}{27^{x + 2}}$

a. 3
b. $\frac{1}{3}$
c. $3^{\frac{1x}{2} - 5}$
d. 1
e. $\frac{1}{3^{12}}$

15. Simplify: $y^m (y^m + y^n)$

a. $y^{m^2} + y^{mn}$

b. $y^{m^2 n}$

c. $y^{2m} + y^{m+n}$

d. y^{3m+n}

e. $2y^m + y^{mn}$

B. LONG ANSWER: Show all work! Put final answer in space provided.

1. Write $1.3\overline{54}$ as an equivalent fraction.

(2)

2. Simplify: $\left(\frac{12c^2b^3}{20c^7b^2}\right)^{-3}$

(2)

3. Simplify: $\frac{4}{4x^2-1} + \frac{1}{2x+1} + \frac{3}{1-x-2x^2}$

(3)

4. Factor: $16x^2 - 49y^2 + 8x + 1$

(2)

5. Solve for r:

$$a = \frac{M}{\frac{n}{3} + \frac{x}{r^2}}$$

(3)

6. Solve for x: $9^{2x} \cdot 81^{x-2} - \left(\frac{1}{27}\right)^{1-2x} = 0$

(3)

7. Simplify: $\frac{32y^3 - 4x^3}{6x^2 - 8x^2y - 8xy^2}$

(2)

8. Simplify: $\sqrt[3]{\frac{9}{16}} + \sqrt[3]{\frac{32}{3}}$

(2)

Appendix D

Posttest 2

Name:_____

Mathematics 11 **Posttest #2**

For this test it is your explanations which are most important, therefore answer as specifically as you can. For most questions there is more than one correct answer.

1. Solve the following for x:

$$\frac{x + 1}{x^2 - 7x + 6} - \frac{2}{x - 6} = \frac{3}{x - 1}$$

Bert's solution to this question is:

Is Bert's solution correct?_____

If it is correct, briefly explain his steps._____

_____

_____

_____

If it is incorrect, state what is wrong and expain why it is wrong._____

_____

_____

_____

Show how you would answer this question.

2. You are tutoring a friend in mathematics. She has been instructed by her teacher to factor wherever possible. They have done this, as shown below, but they have no idea what to do next. Write down the steps she would have to follow, **IN YOUR OWN WORDS**, so that she could complete this question correctly.

Simplify the following:

$$\frac{x^2 - 3x - 18}{x^2 - 3x - 10} - \frac{x^2 - 2x - 8}{x^2 - 2x - 15}$$

Her first step is:

$$\frac{(x - 6)(x + 3)}{(x - 5)(x + 2)} + \frac{(x - 4)(x + 2)}{(x - 5)(x + 3)}$$

What should she do next? (IN YOUR OWN WORDS, EXPLAIN)

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

_____

3. Factor the following:

$$(x + 1)^3 - 8(x - 2)^3$$

Ernest's solution to this question is:

Is anything wrong with Ernest's solution?

Explain._____

_____

_____

Is anything right with Ernest's solution?

Explain._____

_____

_____

Show how you would you have answered this question?

4. For each of the following directional terms write down the most important steps necessary to correctly respond to each statement.

Reduce: _____

_____

_____

_____

Simplify: _____

_____

_____

_____

Evaluate: _____

_____

_____

_____

Factor: _____

_____

_____

_____

Rationalize: _____

_____

_____

_____

Solve: _____

_____

_____

_____

5. Simplify the following:

$$\frac{x - 5}{18x^2 + 7x - 8} - \frac{2x + 1}{6x^2 + 5x - 4}$$

Valerie's solution for this question is:

Is anything wrong with Valerie's solution?

Explain._____

_____

_____

Is anything right with Valerie's solution?

Explain._____

_____

_____

Show how you would you have answered this question?

6. Solve the following for x:

$$2^x - \frac{12}{2^x} = 1$$