SOME CONSEQUENCES OF TIME-REVERSAL SYMMETRY

by

DAVID PETER MAROUN

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Department of Physics

The University of British Columbia,
Vancouver 8, Canada

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ABSTRACT

The purpose of this work is to discuss the symmetry, or lack of it, under reversal of motion in physical objects, states and processes. Considerations of such symmetry are made in both classical and quantum physics, notably in the problem of reconciling the assumed time-reversal symmetry of microscopic processes with the observed asymmetry of macroscopic processes. In the case of classical mechanics, a simple model of a free particle colliding with a series of almost stationary or stationary particles of smaller mass is introduced in order to show how a friction-like phenomenon can arise from processes all of which have symmetry under reversal of motion.

It is maintained throughout that symmetry under reversal of motion is a property of all fundamental states and processes in nature.
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PREAMBLE

Suppose that while some physical process occurs, the entire process, from the initial state of the system to the final state inclusive, is photographed by a motion picture camera and the film later projected onto a motion picture screen. Suppose that this film is then run backwards so to show a process that is the reverse of the original one. If the original process was an elementary one involving few bodies—such as the collision of two billiard balls—then the observer will likely experience no surprise upon seeing the reversed process, and (as has been verified by actual experiment) may not be able to tell which process was the original and which was obtained by a projectionist’s trick. On the other hand, if the process was a complex one involving macroscopic phenomena—such as a body sliding on a rough surface, or an explosion—then the observer will experience a feeling of surprise upon seeing the reversed process; the latter will seem to him unnatural, and he will be able to tell the original from the reversed process.

It is a curious fact that while for elementary processes the reverse process can usually be found in nature, this is not so for processes in which statistical considerations play a role. The general belief is that the statistical laws are themselves responsible for this asymmetry with respect to the direction of time. An interesting and open question is: what consequences would result on the macroscopic level if elementary processes were not symmetrical with respect to the direction of time? In any case, thermodynamics and statistical mechanics do not give any information as to whether such time-reversal symmetry exists for elementary processes; thus it is interesting to inquire into the question of its existence in these cases.
CHAPTER I: INTRODUCTION

1. Terminology

Terms that will be used in this thesis will now be introduced and defined.

First of all, it is necessary to distinguish between reversibility, as understood in thermodynamics, and reversality. A process having reversibility is one for which the entropy of the entire closed physical system concerned (where a closed system is one that does not interact with other systems) remains constant; that is, the degree of randomness of the entire closed system remains constant. Such a process cannot actually be found in nature, though it may be approximated by some natural processes. An example of a reversible process would be the motion of a piston separating two gases at equal pressures; actually, it would not move, but by an infinitesimal alteration of the pressure of either gas, one could make the piston move in either direction so that the system passes through a continuous series of states of equilibrium, and so the original degree of randomness of the system is not destroyed. We shall refer to such a process as a reversible process. On the other hand, reversality may be defined as follows: A process or state referring to an object or objects has reversality if, and only if, reversal of all motion in that process or state yields a physically possible process or state involving the same actual objects. For example, the motion of two bodies $m_1$ and $m_2$ under mutual gravitational attraction is a process having reversality, since the process having initially $m_1$ and $m_2$ at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively, and having finally $m_1$ and $m_2$ at $\mathbf{r}'_1$ and $\mathbf{r}'_2$ with momenta $\mathbf{p}'_1$ and $\mathbf{p}'_2$ respectively, is just as consistent with the equations of motion as is the process which has initially $m_1$ and $m_2$ at positions $\mathbf{r}'_1$ and $\mathbf{r}'_2$ with momenta $-\mathbf{p}'_1$ and $-\mathbf{p}'_2$ respectively and has finally $m_1$ and $m_2$ at $\mathbf{r}_1$ and $\mathbf{r}_2$ with momenta $-\mathbf{p}_1$ and $-\mathbf{p}_2$ respectively. Since the motions of $m_1$ and $m_2$ are
completely determined by the equations of motion and the initial conditions, the notion of randomness does not apply to this situation and hence neither does the notion of reversibility.

There are also processes which are not reversible but have reversality. For example, the escape of a gas from a container into a vacuum is a process having reversality, since return of the gas into the container is physically possible; however, the escape involves passage of the system to a much more random state, so that the process is not reversible.

Next, the distinction will be made between the reverse process and the inverse process. If a physical process involves taking a system from a state A to a state B, then the inverse process takes the system from B to A, while the reverse process takes the system from -B to -A, where -B is the time-reversed state of B and -A is the time-reversed state of A. For instance, suppose that a process involves m particles with positions \( \vec{r}_1 \), momenta \( \vec{p}_1 \) and spins \( \vec{s}_1 \) in the initial state, and has n particles at positions \( \vec{r}_1' \) with momenta \( \vec{p}_1' \) and spins \( \vec{s}_1' \) in the final state. Then the inverse process has n particles with positions \( \vec{r}_1 \), momenta \( \vec{p}_1 \) and spins \( \vec{s}_1 \) in the initial state, and in the final state has m particles with positions \( \vec{r}_1' \), momenta \( \vec{p}_1' \) and spins \( \vec{s}_1' \). However, the reverse process begins with n particles having positions \( \vec{r}_1 \), momenta -\( \vec{p}_1 \) and spins -\( \vec{s}_1 \), and ends with m particles at positions \( \vec{r}_1' \) with momenta -\( \vec{p}_1' \) and spins -\( \vec{s}_1' \).

It will be said that reciprocity holds for a process if, and only if, the process and its corresponding reverse process have equal probabilities of occurrence. Similarly, it will be said that detailed balance holds for a process if, and only if, the process and its corresponding inverse have equal probabilities of occurrence.
Consider any physical system together with the functions and equations which describe its behaviour, and let t be the time. Then reversal in time of the motion of the system is formally equivalent to replacing t by \(-t\) in the functions. This is evident when one considers that any instant may arbitrarily be chosen as \(t = 0\), and by considering the reversed motion to begin at \(t = 0\), one considers motion in the negative direction of time.

Consider, for example, a system having a coordinate \(x\) and the corresponding velocity \(\frac{dx}{dt}\) given by the equations

\[
\begin{align*}
  x(t) &= A \sin bt \\
  \frac{dx}{dt}(t) &= A b \cos bt
\end{align*}
\]  

where \(A\) and \(b\) are constants. Then the reversed motion is given by

\[
\begin{align*}
  x(-t) &= A \sin b(-t) = -A \sin bt \\
  \frac{dx}{dt}(-t) &= -\frac{dx}{dt}(t) = +A b \cos bt
\end{align*}
\]  

provided one chooses \(x(0) = 0\) for both motions.

Functions obtained by substituting \(-t\) for \(t\) will be referred to as time-reversed functions. If the time-reversed functions satisfy the same equations of motion as do the original functions, then the reverse motion is physically possible, and so the motion has reversality. Returning to the example shows that both \(x(t)\) and \(x(-t)\) satisfy

\[
\frac{d^2 x}{dt^2} = -b^2 x
\]  

so that the motion described by \((I.5)\) has reversality.

---

2. Considerations Permitting One in Principle
To Check Whether Reversality, Reciprocity
Or Detailed Balance Holds
Evidently, if the time-reversed functions do not satisfy the same equations of motion as do the original functions, then the motion does not have reversality. For example, if a motion is described by the equation

\[
\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = 0 \tag{I.6}
\]

then the equation for \(y(-t)\) is

\[
\frac{d^2 y(-t)}{d(-t)^2} + \frac{dy(-t)}{d(-t)} + y(-t) = 0 \tag{I.7}
\]

\[
\frac{d^2 y(-t)}{dt^2} - \frac{dy(-t)}{dt} + y(-t) = 0 \tag{I.8}
\]

Hence the set of values through which \(y(t)\) and \(dy(t)/dt\) pass in going from the values \(y_1, (dy/dt)_1\) to the values \(y_2, (dy/dt)_2\) respectively is not the same as the set of values through which \(y(-t)\) and \(-dy(-t)/dt\) pass in going from \(y_2, -(dy/dt)_2\) to \(y_1, -(dy/dt)_1\) respectively. Thus the motion described by (I.6) lacks reversibility.

In order to make an experimental check on whether a process exhibits time-reversal symmetry, the process must have a definite sequence of states in time. An experimental check on whether reciprocity held would involve observing a system whose components passed through such a definite sequence, and then observing whether the process occurred as frequently as the reverse process. If it did, one could conclude that the probabilities for the two processes were equal and hence that reciprocity held; if it did not, then one could conclude that the probabilities for the two processes were not the same and hence that reciprocity did not hold. A test for detailed balance could be carried out similarly, but one would consider the inverse process instead of the reverse process.

An example of a situation which - if it existed - would violate reciprocity is shown in Figure 1. A particle decays, giving off an electron and gamma ray, both of which are at right angles to the spin of the particle,
so that the direction of travel of the gamma ray and the spin form a right-handed system. Reversal of the state yields a left-handed system. The objects in the original state cannot be put into this time-reversed state by a rotation nor by a reflection, so that the reverse state cannot be obtained without the intervention of other processes. This situation has never been observed experimentally.

An example of a case in which detailed balance would not hold is shown in Figure 2. Initially B is at rest while A moves to the right with momentum \( \vec{p} \); after collision, B moves to the right with momentum \( \vec{p} \) while A is at rest. For the inverse process, B must move right with momentum \( \vec{p} \), collide with A (which is initially at rest) and then remain at rest while A moves right with momentum \( \vec{p} \). Due to the triangular shapes of A and B, the inverse process cannot take place. Hence detailed balance will not hold; other processes are required before even an overall balance can occur. Such a situation, in which detailed balance does not hold, occurs when one has a classical gas of non-spherical molecules.

The notion of overall balance, which was mentioned previously, will now be made more precise. Consider a system of particles occupying various states. Let the probability of finding a particle in the \( i \)th state be \( P_i \) and let the probability per unit time of a transition from state \( i \) to state \( j \) be \( L_{ij} \) (\( L_{ii} = 0 \)). It is found that the irreversible approach of such a system to statistical equilibrium is given by the so-called master equations

\[
\frac{dP_i}{dt} = \sum_j \left( P_j L_{ji} - P_i L_{ij} \right). \tag{I.9}
\]

At equilibrium, for every \( i \) and \( j \), \( P_i = P_j \) and \( dP_i/dt = 0 \). In order that these two conditions be consistent with (I.9), one must have either of two conditions: In one case, \( L_{ij} = L_{ji} \), which yields detailed balance; in the other case, \( \sum_j L_{ij} = \sum_j L_{ji} \), and this will be called overall balance.
Figure 1. A hypothetical process which would violate time-reversal symmetry.
Figure 2. A process for which detailed balance would not hold.
As a particular example of a system for which detailed balance does not hold but overall balance does, the case will be considered in which there are particles in three states. The master equations, with values chosen for the L's to give overall balance, are

$$\frac{dP_1}{dt} = \frac{1}{4} P_1 + \frac{1}{2} P_3 - \frac{1}{2} P_1 - \frac{1}{4} P_2$$

$$\frac{dP_2}{dt} = \frac{1}{2} P_1 + \frac{3}{16} P_3 - \frac{1}{4} P_2 - \frac{7}{16} P_2$$

$$\frac{dP_3}{dt} = \frac{1}{4} P_1 + \frac{7}{16} P_2 - \frac{1}{2} P_3 - \frac{3}{16} P_3 .$$

(I.10)

It is readily seen that if $P_1 = P_2 = P_3$, then the right-hand side of each of equations (I.10) vanishes in agreement with the desired equilibrium conditions.

3. Further Definitions

It is convenient to introduce definitions of terms describing the nature of quantities with respect to time-reversal. It will be said that a quantity is even under time-reversal if, and only if, substitution of $-t$ for $t$ (where $t$ is time) leaves that quantity unchanged. Also, a quantity is odd under time-reversal if, and only if, that quantity changes sign upon substitution of $-t$ for $t$, becoming the negative of what it was originally. For example, the quantity given by $f(t) = t^2$ is even under time-reversal, since $f(-t) = (-t)^2 = f(t)$; and the quantity given by $g(t) = t^3$ is odd under time-reversal, since $g(-t) = (-t)^3 = -t^3 = -g(t)$. 

CHAPTER II. TIME-REVERSAL SYMMETRY IN CLASSICAL PHYSICS


For a particle of mass \( m \) and position \( \vec{r} \) which is acted upon by a force \( \vec{F} \), the Newtonian equation of motion is

\[
 m \frac{d^2 \vec{r}}{dt^2} = \vec{F} .
\]  

This equation determines \( \vec{r} \) as a function of time. If \( t \) is replaced by \(-t\), and if \( \vec{F} \) is invariant under time-reversal, then since the left-hand side is a second derivative, therefore the time-reversed function \( \vec{r}(-t) \) obeys the same equation as does \( \vec{r}(t) \), so that the system described has reversality. Thus it is seen that there is no inherent asymmetry with respect to reversal of motion in Newtonian mechanics; any such asymmetry arises only in special cases.

When frictional forces are present in a system, the equations of motion involve terms proportional to first derivatives with respect to time. In this case, time-reversal symmetry is destroyed. However, it is generally believed that friction is a purely macroscopic phenomenon resulting from the application of statistical laws to fundamental processes which themselves have reversality. An attempt will be made here to show how this may be the case.

First of all, consider a system of particles having coordinates \( x_1, x_2, \ldots, x_n \), masses \( m_1, m_2, \ldots, m_n \) (for convenience, the mass corresponding to coordinate \( x_1 \) is written as \( m_1 \), although a given particle may have more than one coordinate and hence some of the \( m_i \)'s may be identical to each other) and described by the Newtonian equations of motion

\[
 m_1 \ddot{x}_1 = F_1 (x_1, x_2, \ldots, x_n, t) 
\]

\[
 m_2 \ddot{x}_2 = F_2 (x_1, x_2, \ldots, x_n, t) 
\]

\[
 \ldots 
\]

\[
 m_n \ddot{x}_n = F_n (x_1, x_2, \ldots, x_n, t) 
\]  

(II.2)
where the F's are invariant with respect to the substitution of \(-t\) for \(t\).

It might be supposed that if \(x_2, x_3, \ldots, x_n\) were eliminated from the first of equations (II.2), thus leaving all terms in that equation as explicit functions of \(x_1\), its time derivatives, and \(t\), then there might result an equation for \(x_1\) lacking time-reversal invariance. That this cannot be so can be shown as follows: The equations (II.2) are invariant under substitution of \(-t\) for \(t\); hence the motion described by these equations has reversality, and consequently any other equations which correctly describe the motion must also have time-reversal invariance.

Another approach will now be tried. Suppose there is a free particle originally having velocity \(V\) in the \(x\)-direction, and while continuing to move in the \(x\)-direction - colliding at intervals of distance \(\delta\) with successive particles each having a mass \(m\) and a small velocity \(v\) which will be left undetermined. \(V\) and \(v\) are here defined to be positive if the corresponding particle travels in the positive \(x\)-direction, and negative if it travels in the negative \(x\)-direction; the positive \(x\)-direction is taken as the initial direction of travel of the free particle. Let \(M\) be the mass of the free particle and suppose that the collisions are all elastic. Taking \(x = 0\) as the initial position of the particle, its position prior to the first collision is given by

\[
x = Vt .
\]

(II.3)

Suppose that the first collision takes place. Since the collision is elastic, both linear momentum and kinetic energy are conserved. Thus

\[
mV + MV = m'V' + M'V' \tag{II.4}
\]

and

\[
\frac{m}{2}V^2 + \frac{M}{2}V^2 = \frac{m}{2}V'^2 + \frac{M'}{2}V'^2 \tag{II.5}
\]
where the primed quantities $v'$ and $V'$ are the velocities after the collision of $m$ and $M$ respectively. Solving (II.4) and (II.5) for $V'$ yields
\[
V' = \frac{(M - m)}{M + m} V + \frac{2Mv}{M + m}.
\]
(II.6)

Assume that $|v|$ is much less than $|V|$ and that $m$ is less than $M$. Then $V'$ is in the same direction as $V$ and is given by
\[
V' = \frac{(M - m)}{M + m} V.
\]
(II.7)

Since the particle of mass $M$ travelled a distance $S$ before the first collision, its position between the first and second collisions is given by
\[
x = (\frac{M - m}{M + m}) V (t - \frac{S}{V}) + S.
\]
(II.8)

Continuing in this manner yields for the position between the $n$th and the $(n + 1)$th collisions:
\[
x = (\frac{M - m}{M + m})^n V (t - \frac{S}{V}) \left\{ 1 + \frac{1}{(\frac{M - m}{M + m})} + \frac{4}{(\frac{M - m}{M + m})^2} + \cdots + \frac{1}{(\frac{M - m}{M + m})^{n-1}} \right\} + nS
\]
(II.9)

Using the formula for the sum of a geometric progression yields
\[
x = (\frac{M - m}{M + m})^n V (t - \frac{S}{V}) \left[ 1 - (\frac{M + m}{M - m})^n \right] + nS, \tag{II.10}
\]
\[
x = (\frac{M - m}{M + m})^n V \left( t + \frac{S}{V} \left[ \frac{M - m}{2m} \right] \left[ 1 - (\frac{M + m}{M - m})^n \right] \right) + nS, \tag{II.11}
\]
\[
+ nS
\]

The velocity between collisions is $(\frac{M - m}{M + m})^n V$ and it decreases as time goes on. Thus the free particle of mass $M$ slows down as does a body moving in the presence of friction; the particles of mass $m$ and small speed arranged regularly and close to each other are like particles in a solid. However,
all the processes involved in the slowing-down have reversality. A reverse process is conceivable but unlikely due to the difficulty of getting all the particles of mass \( m \) to behave in the exact reverse manner to that obtained in the original process. Indeed, if these particles behave like those in a solid, they will return after collision to their approximate original positions and velocities so that if the particle \( M \) is sent in the negative \( x \)-direction with a given velocity, it will simply slow down, as in the original process.

The relaxation time can be found by evaluating the time elapsed before the second term on the right-hand side of (II.6) is no longer negligible. Suppose that this happens at the \((n + 1)\) th collision. Then the velocity of \( M \) previous to this collision is \( \left( \frac{M - m}{M + m} \right)^n V \), and its velocity after this collision is, by analogy with (II.6),

\[
V' = \left( \frac{M - m}{M + m} \right)^{n+1} V + \frac{2mV}{M + m}
\]

and the condition that the second term is appreciable is

\[
\left( \frac{M - m}{M + m} \right)^{n+1} |V| \approx \frac{2m}{M + m} |v|,
\]

where some suitable kind of average value is used for \( |v| \). Using the fact that \( V \) is positive,

\[
\left( \frac{M + m}{M - m} \right)^{n+1} \approx \frac{\left( \frac{M + m}{2m} \right) V}{|v|}.
\]

As can be seen from the derivation used to get (II.11), the time required for \((n + 1)\) collisions is

\[
T = -\frac{\delta}{V} \left[ \frac{M - m}{2m} \right] \left[ 1 - \left( \frac{M + m}{M - m} \right)^{n+1} \right].
\]

From (II.14) and (II.15),

\[
T \approx \frac{\delta}{V} \left[ \frac{M - m}{2m} \right] \left[ \frac{V}{|v|} \left( \frac{M + m}{2m} \right) - 1 \right]
\]

is the relaxation time.
Suppose that the particles of mass \( m \) constitute a solid, and suppose that the solid has a simple cubic crystalline structure. Then (II.16) can be put into a more convenient form by expressing \( \delta \) in terms of the density of the solid. This can be seen as follows: Since the particles of mass \( m \) are separated by a distance \( \delta \), there is one particle per \( \delta^3 \) of volume. Hence the mass per unit volume - and so the density - is

\[
\rho = \frac{m}{\delta^3} \quad \text{(II.17)}
\]

Hence

\[
\delta = \sqrt[3]{\frac{m}{\rho}} \quad \text{(II.18)}
\]

and (II.16) becomes

\[
\tau \sim \frac{1}{\sqrt{\frac{3}{\rho}} \left[ \frac{M-m}{2m} \right] \left[ \frac{\sqrt{V}}{2m} \left( \frac{M+m}{m} \right) - 1 \right]} \quad \text{(II.19)}
\]

The discussion of this section, which was inspired by Blatt, 1959, merits further investigation, since it seems somewhat unfamiliar to physicists, and yet it provides an elementary and simple means of seeing how apparent time-reversal asymmetry can arise from processes having reversality.

2. Classical Electrodynamics

In rationalized MKS units, Maxwell's equations are

\[
\text{curl } \vec{H} = \vec{J} + \frac{2}{\希腊 \epsilon} \vec{D} \quad \text{(II.20)}
\]

\[
\text{curl } \vec{E} = - \frac{2}{\希腊 \epsilon} \vec{B} \quad \text{(II.21)}
\]

\[
\text{div } \vec{D} = \rho \quad \text{(II.22)}
\]

\[
\text{div } \vec{B} = 0 \quad \text{(II.23)}
\]

where \( \vec{H} \) is the magnetic field intensity, \( \vec{D} \) is the electric displacement, \( \vec{E} \) is the electric field, \( \vec{B} \) is the magnetic field, \( \vec{J} \) is the current density, and
\( \rho \) is the charge density. In a fairly general case, the components of \( \mathbf{D} \) can be taken as linear functions of the components of \( \mathbf{E} \), and the components of \( \mathbf{B} \) can be taken as linear functions of the components of \( \mathbf{H} \).

In order to find out the properties of these equations under time-reversal, \( t \) can be replaced by \(-t\) in them. Assume that \( \rho \) is an even function under this operation. Since \( \mathbf{J} \) will behave as a time derivative of the charge density under this transformation, it is an odd function of \( t \). Then Maxwell's equations remain invariant under time-reversal provided that \( \mathbf{D} \), and hence \( \mathbf{E} \), is an even function, and that \( \mathbf{H} \), and hence \( \mathbf{B} \), is an odd function.

However, the choice of \( \rho \) as even is quite arbitrary as far as Maxwell's equations are concerned; \( \rho \) could also be chosen as an odd function of \( t \). In that case, \( \mathbf{E} \) would have to be an odd function and \( \mathbf{B} \) an even function in order to preserve the form of these equations. Further information beyond Maxwell's equations is required to decide whether \( \rho \) is odd or even.

At any rate, it is evident that Maxwell's equations do not require any general asymmetry between the past and the future.

Nevertheless, irreversibility is observed in the phenomenon of radiation reaction. When a charged particle is accelerated, it loses energy by sending out electromagnetic radiation. This loss is regarded as due to a force acting on the particle given by

\[
e F_{\nu\mu} \dot{a}^\mu = \left( \frac{\hbar}{3} e^2 \right) \left( \dot{a}_n \ddot{a}_\mu - \ddot{a}_n \dot{a}_\mu \right) \dot{a}^\mu,
\]

where \( e \) is the electric charge, \( F_{\nu\mu} \) is a field tensor, \( a^n \) is a space-time coordinate on the world-line of the particle (\( a^4 \) is the product of the velocity of light and the time elapsed between a certain zero time and the moment of observation), and a dot indicates differentiation with respect to the proper cotime \( \alpha \), which is given by the product of the velocity of light \( c \) and the proper time, so that \( (\dot{\alpha})^2 = c^2 \left( \text{time interval} \right)^2 - \left( \text{space interval} \right)^2 \).
Lowering of indices takes place according to the convention that space components of a four-vector do not change at all under this operation, while time components simply change sign. \( d \omega \) has the same sign as \( d a \). Equation (II.24) gives the \( n \)-component of the force of radiation reaction; to get the equation of motion, the expression in (II.24) must be added to the \( n \)-components of all other forces that act on the particle and this sum must then be equated to the product of the mass of the particle and the second derivative of \( a_n \) with respect to \( \omega \).

In the case of a particle moving slowly relative to the speed of light, (II.24) can be simplified. For this case, \( d \omega \) can be approximated as \( c \, dt \).

The first space component of (II.24) can then be evaluated by noting that \( \dot{a}_n \) is of the order of the ratio of the particle's velocity to \( c \) and so is negligible; that \( \dot{a}_\mu \cdot \dot{a}^\nu = 1 \); and that \( \ddot{a}_n \) is the product of \( c^{-3} \) and the time rate of change of the given component of the acceleration \( \dot{A} \). Thus the right-hand side of (II.24) reduces to

\[
\frac{2}{3} \frac{e^2}{c^3} \frac{d}{d t} \dot{A}^\mu
\]

for the damping force.

The force of radiative reaction violates time-reversal symmetry, due to the odd-order derivatives which appear in it. Wheeler and Feynman (1945) used the absorber theory of radiation to reconcile the existence of the radiation reaction with the view that all the fundamental processes involved are symmetric with respect to the direction of time. Their argument runs as follows:

It is assumed that

1. An accelerated point charge in otherwise charge-free space does not radiate electromagnetic energy. Hence the problem of the existence of an irreversible process is disposed of in the absence of an absorber, and one concludes immediately that the irreversibility of the radiation reaction arises from the presence of an absorber.
The fields which act on a given particle arise only from other particles; that is, no particle interacts directly with itself.

These fields are represented by one-half the retarded plus one-half the advanced Lienard-Wiechart solutions of Maxwell's equations. This law of force is symmetric with respect to the past and the future.

Sufficiently many particles are present to absorb completely the radiation given off by the source.

Then the source of radiation is taken to be an accelerated charge located in the absorbing system. Since the absorption is to be complete, a test charge outside the absorbing medium will not be affected by the fields produced by the source. Thus, outside the absorber at all times

\[ \sum_{\mathbf{k}} \left( \frac{1}{2} F_{\text{RET}}^{(\mathbf{k})} + \frac{1}{2} F_{\text{ADV}}^{(\mathbf{k})} \right) = 0 \]  \hspace{1cm} (II.26)

where \( F_{\text{RET}}^{(\mathbf{k})} \) and \( F_{\text{ADV}}^{(\mathbf{k})} \) are respectively the retarded and the advanced fields due to the \( \mathbf{k} \)th particle in the system.

At large distances, \( \sum_{\mathbf{k}} F_{\text{RET}}^{(\mathbf{k})} \) represents an outgoing wave while \( \sum_{\mathbf{k}} F_{\text{ADV}}^{(\mathbf{k})} \) represents an incoming wave. Since these two waves cannot interfere destructively, each of these two sums must vanish separately in order that (II.26) be satisfied. Consequently, outside the absorber at all times,

\[ \frac{1}{2} \left( \sum_{\mathbf{k}} F_{\text{RET}}^{(\mathbf{k})} \right) - \frac{1}{2} \left( \sum_{\mathbf{k}} F_{\text{ADV}}^{(\mathbf{k})} \right) = 0 \]  \hspace{1cm} (II.27)

Now the left-hand side of (II.27) has no singularities within the absorber and is a solution of Maxwell's equations in free space. Thus, since it always vanishes outside the absorber, it must also be always zero inside it.

Let the \( \mathbf{a} \)th particle be the source. From the assumptions made previously, the total field acting on it is

\[ \sum_{\mathbf{k} \neq \mathbf{a}} \left( \frac{1}{2} F_{\text{RET}}^{(\mathbf{k})} + \frac{1}{2} F_{\text{ADV}}^{(\mathbf{k})} \right) \].  \hspace{1cm} (II.28)
Rewriting this expression yields
\[
\mathcal{E}_{\not{\lambda}} \equiv \sum_{\not{\lambda} \neq \alpha} F_{\text{RET}}^{(\not{\lambda})} + \left( \frac{1}{2} F_{\text{RET}}^{(\alpha)} - \frac{1}{2} F_{\text{ADV}}^{(\alpha)} \right)
\]
\[
- \sum_{\not{\lambda}} \left( \frac{1}{2} F_{\text{RET}}^{(\not{\lambda})} - \frac{1}{2} F_{\text{ADV}}^{(\not{\lambda})} \right).
\]
From (II.27), this becomes
\[
\sum_{\not{\lambda} \neq \alpha} F_{\text{RET}}^{(\not{\lambda})} + \left( \frac{1}{2} F_{\text{RET}}^{(\alpha)} - \frac{1}{2} F_{\text{ADV}}^{(\alpha)} \right).
\]

The second term represents a force on the source particle having as its \(n\)-component
\[
\mathcal{E}_{\not{\lambda}} \left( \frac{1}{2} F_{\eta \alpha \text{ RET}}^{(\alpha)} - \frac{1}{2} F_{\eta \alpha \text{ ADV}}^{(\alpha)} \right) \dot{\alpha}^\alpha
\]

where \(\mathcal{E}_{\not{\lambda}}\) is the charge on the particle. Dirac has shown that this reduces to
\[
\frac{2}{3} \mathcal{E}_{\not{\lambda}} \left( \dot{\alpha}_n \ddot{\alpha}_\alpha - \ddot{\alpha}_n \dot{\alpha}_\alpha \right) \dot{\alpha}^\alpha
\]

which is the force of radiative reaction (Dirac, 1938). Thus it is the second term in (II.30) which yields this force. Hence, using (II.30) and (II.32), the equation of motion of the source particle is
\[
c^2 m_a \ddot{\alpha}_n = \mathcal{E}_{\not{\lambda}} \sum_{\not{\lambda} \neq \alpha} F_{\eta \alpha \text{ RET}}^{(\not{\lambda})} \dot{\alpha}^\alpha
\]
\[
+ \left( \frac{2}{3} \mathcal{E}_{\not{\lambda}} \left( \dot{\alpha}_n \ddot{\alpha}_\alpha - \ddot{\alpha}_n \dot{\alpha}_\alpha \right) \dot{\alpha}^\alpha
\]

where \(m_a\) is the mass of the source particle. This result is in accord with experience.

Contrary to what might at first be thought, it is not the property of complete absorption, which was assumed to characterize the absorber, but rather statistical mechanical considerations which explain the irreversibility of the radiation reaction. The condition for complete absorption, (II.27), is symmetrical in advanced and retarded fields. By interchanging the roles of the advanced and retarded fields in the derivation following (II.27), the equation of motion of the source particle is obtained as
This equation is just as valid as (II.33) and is consistent with it, but the force of radiative reaction has a different sign in front of it. Thus the paradox of irreversibility remains.

However, in the case being considered, the absorber consisted of particles that were either at rest or in random motion at the initial time of acceleration of the source; thus their retarded fields were zero or of negligible effect at the source at that time, and in (II.33) the radiation reaction term is dominant. But the situation differs when advanced fields are considered. The absorber particles begin to move, in response to the acceleration of the source, just in time to contribute to \[ \sum_{\mathbf{k} \neq 0} F_{\mathbf{k}}^{(A)} \text{A}_{\mathbf{k} \mathbf{A}}. \]

Equating the right-hand sides of (II.33) and (II.34) at the initial time of acceleration, when the \( F_{\mathbf{k}}^{(R)} \) are negligible but the \( F_{\mathbf{k}}^{(A)} \) are not, shows that this contribution has twice the magnitude of the radiation reaction. Thus the second term on the right-hand side of (II.34) is cancelled out at \( t = 0 \), and a force of the expected sign and magnitude remains.

If the absorber particles undergo some sort of acceleration which is not due to the motion of the source and which takes place prior to \( t = 0 \), this may have some resultant effect on the source particle, but this effect is neglected in this calculation. If the motion of the absorber particles is sufficiently random, this effect will be very small anyway.

In order to see that it is indeed probability considerations which determine the irreversibility of radiation, one can imagine the reverse of the process just described. In this case, chaotic motion in the absorber causes each particle to receive at the proper moment just the right impulse to generate a disturbance converging on the source at the instant of its acceleration. The source receives energy and the particles of the absorber
lose velocity. This situation is just as consistent with the equations of motion as is the original process, and only the small probability of the initial conditions serves to exclude it.
CHAPTER III: TIME-REVERSAL IN QUANTUM MECHANICS

1. The Need To Represent Reversal Of Motion By An Anti-Unitary Operator

Let $\theta$ be the operator of reversal of motion so that if $\ket{b}$ is a quantum mechanical state vector, then $\theta \ket{b}$ is the state vector for the time-reversed state. Since the descriptions by $\ket{b}$ and $\theta \ket{b}$ must be equivalent, $\theta$ must satisfy one of two conditions: Either it is a unitary operator, in which case it satisfies

$$\theta \left( a_1 \ket{b} + a_2 \ket{c} \right) = a_1 \theta \ket{b} + a_2 \theta \ket{c}$$  \hspace{1cm} (III.1)

for any numbers $a_1$ and $a_2$ and any state vectors $\ket{b}$ and $\ket{c}$; or it is anti-unitary, and then it satisfies

$$\theta \left( a_1 \ket{b} + a_2 \ket{c} \right) = a_1^* \theta \ket{b} + a_2^* \theta \ket{c}$$  \hspace{1cm} (III.2)

for any numbers $a_1$ and $a_2$, and any state vectors $\ket{b}$ and $\ket{c}$. The symbol '*' represents the operation of taking the complex conjugate. It is necessary that either (III.1) or (III.2) be satisfied in order that the expectation values and transition probabilities not be affected by the transformation by $\theta$.

It is intuitively evident that since $\theta$ represents reversal of motion, therefore the Hamilton operator $H$, the coordinate operator $Q_i$, the operator for the conjugate linear momentum component $P_1$, and the angular momentum operator $J$ must transform as

$$\theta H \theta^{-1} = H$$
$$\theta P_1 \theta^{-1} = -P_1$$
$$\theta Q_i \theta^{-1} = Q_i$$
$$\theta J \theta^{-1} = -J$$  \hspace{1cm} (III.3)
Furthermore, these operators satisfy the equations (in natural units):

$$ H |b> = i \frac{\partial}{\partial t} |b> $$  \hspace{1cm} (III.4)

$$ [ P_i , Q_i ] = i I $$  \hspace{1cm} (III.5)

$$ \vec{J} \times \vec{J} = i \vec{J} $$  \hspace{1cm} (III.6)

where \( [ P_i , Q_i ] = P_i Q_i - Q_i P_i \) and \( I \) is the unit operator.

Suppose that \( \Theta \) is a unitary operator. Multiplying (III.4) on the left by \( \Theta \) gives

$$ \Theta H |b> = \Theta i \frac{\partial}{\partial t} |b> $$

$$ \Theta H \Theta^{-1} \Theta |b> = i \frac{\partial}{\partial t} \Theta |b> = -i \frac{\partial}{\partial t} (\Theta |b>) . $$

Using (III.3),

$$ H(\Theta |b>) = -i \frac{\partial}{\partial t} (\Theta |b>) . $$  \hspace{1cm} (III.7)

Thus \( \Theta |b> \) does not satisfy the same equation as does \( |b> \). Furthermore, application of \( \Theta \) to the left and \( \Theta^{-1} \) to the right of both sides of (III.5) and (III.6) yields

$$ [ P_i , Q_i ] = -i I $$  \hspace{1cm} (III.8)

$$ \vec{J} \times \vec{J} = -i \vec{J} $$  \hspace{1cm} (III.9)

which are inconsistent with (III.5) and (III.6).

This situation can be remedied by letting \( \Theta \) be an anti-unitary operator, so that

$$ \Theta = T K $$  \hspace{1cm} (III.10)

where \( T \) is a unitary operator and \( K \) is the operator of complex conjugation.

Application of \( \Theta \) to (37) then yields

$$ \Theta H |b> = \Theta i \frac{\partial}{\partial t} |b> $$
\[ \Theta H \Theta^{-1} |b\rangle = (-1)^\frac{2}{\tau} (\Theta |b\rangle) \]

so that \( \Theta |b\rangle \) satisfies the same equation as does \( |b\rangle \). Furthermore, application of \( \Theta \) on the left and \( \Theta^{-1} \) on the right leaves (III.5) and (III.6) invariant.

Thus in order to incorporate symmetry between past and future into the formalism of quantum mechanics, it is necessary to represent reversal of motion by an anti-unitary operator. The argument given here has not shown that quantum mechanics has symmetry with respect to time-reversal; rather, this is given here as a hypothesis needing experimental verification. However, this assumption is consistent with the rest of quantum mechanics, and no exception to it has been discovered up to the present time. Hence quantum mechanics itself does not require any asymmetry with respect to reversal of motion.

A discussion of this problem from a group-theoretical point of view may be found in Wigner, 1959.

Since \( \Theta \) is an anti-unitary operator, it is meaningless to speak of its eigenstates as far as single particle states are concerned. Indeed, multiplication of the state vector of such a state by an arbitrary phase factor will affect the result of application of \( \Theta \) to it, although it will not affect the physical significance of the vector. However, something analogous to an eigenstate of a linear operator can arise in the case of a state which is a direct product of single particle states and their reverse states. Thus if a state \( |c\rangle \) can be expressed as

\[ |c\rangle = |b\rangle \otimes |b\rangle^T \]  

(III.12)
where $|b^T\rangle$ is the time-reverse of $|b\rangle$, then multiplication of $|b\rangle$ by a phase $\exp(i\Omega)$ results in $|b^T\rangle$ being multiplied by $\exp(-i\Omega)$, so that $|c\rangle$ remains the same and the application of $\Theta$ to $|c\rangle$ gives the same result as if the multiplication by $\exp(i\Omega)$ had not been carried out. Thus the equation

$$\Theta |c\rangle = \mathcal{E} |c\rangle,$$

(III.13)

where $\mathcal{E}$ is a number, is meaningful. A state $|c\rangle$ for which (III.12) holds may be called a state of specific reversality. It is not clear whether such states are realized in nature. Some speculations have been made that there exists a second lepton number whose operator anticommutes with $\Theta$, so that (for example) a state consisting of one ordinary neutrino and one muon neutrino would be a candidate for having specific reversality.

Despite the absence of obvious selection rules analogous to those following from conservation of parity, time-reversal symmetry does lead to some experimentally accessible consequences whenever transitions take place involving a definite sequence of events in time. These consequences are connected with the so-called principle of reciprocity, which will be discussed in the following section.

2. Proof Of Reciprocity From Time-Reversal Symmetry

Consider the cases in which the probability amplitude for a transition from a state $|\tau\rangle$ to a state $|\tau'\rangle$ is given by the matrix element

$$|\langle \tau' | S | \tau \rangle|$$

(III.14)

of the scattering operator $S$, which is defined as the limit

$$S = \mathcal{U}(+\infty, -\infty)$$

(III.15)
of a unitary operator $U(t^2, t^1)$ which satisfies
\[ i \frac{\partial}{\partial t^2} U(t^2, t^1) = H_{\text{int}}(t^2) U(t^2, t^1) \]  
(III.16)

with $U(t^1, t^1) = 1$, where $H_{\text{int}}(t^2)$ is the interaction Hamiltonian in the interaction picture (see Jauch and Rohrlich, pages 117 ff.). If $|\tau\rangle$ is the reverse state of $|\tau\rangle$, and $|\tau'\rangle$ is the reverse of $|\tau\rangle$, then the probability amplitude for the process which is the reverse of $(III.14)$ is $<\tau|s|\tau'>$. The principle of reciprocity then states that

\[ <\tau'|s|\tau> = \frac{1}{t} <\tau|s|\tau'>. \]  
(III.17)

The reverse process $|\tau'| \rightarrow |\tau\rangle$ should not be confused with the inverse process $|\tau'\rangle \rightarrow |\tau\rangle$ in which only the sequence of states is interchanged.

In order to prove this principle, the anti-unitary property of $\Theta$ will be represented by the operation of transposition in the space of occupation numbers. Thus if, with a suitable choice of the phase, the vacuum state transforms as

\[ \Theta |0\rangle = <01, \]  
(III.18)

then any occupation state obtained from the vacuum by application of a given number of creation operators as

\[ |\tau\rangle = a^{+}(\tau_1) \ldots a^{+}(\tau_n) |0\rangle \]  
(III.19)

will transform as

\[ \Theta |\tau\rangle = \epsilon_n <0 | a^{\dagger}[\tau_n] \ldots a^{\dagger}[\tau_1] |0\rangle = \epsilon_n <\tau|1 \]  
(III.20)

where $\epsilon_n$ is a phase factor $\pm 1$ which can be specified arbitrarily for a given number $n$ of particles without violating the commutation (anti-commutation) relations between the operators $a(\tau)$ and $a^{+}(\tau)$. In order to
provide for the effect of $\Theta$ on all the dynamical variables, the time-reversed quantum numbers $\tau_\tau$ have been substituted for the original quantum numbers $\tau$. Similarly,

$$<\tau'|\Theta^{-1} = <0|a(\tau'_m)\ldots a(\tau'_2)\Theta^{-1}$$

$$= \epsilon_m a^+[<\tau'_1>\ldots a^+[<\tau'_m>]_{\tau'}|0>$$

$$= \epsilon_m /t'_\tau > ,$$

(III.21)

Treating the effect of $\Theta$ in this way, one can transform (III.14) as

$$<\tau'|S/\tau> = <\tau'|\Theta^{-1} \Theta S \Theta^{-1} \Theta |\tau>$$

$$= \epsilon_n \epsilon_m <\tau_\tau /S_\tau /\tau'_\tau > ,$$

(III.22)

where

$$S_\tau = \Theta S \Theta^{-1}$$

(III.23)

is the time-reversed scattering operator. It remains to prove that

$$S_\tau = S .$$

(III.24)

The adjoint of (49) is

$$-i \frac{2}{\partial t_2} U^+(t_2, t_1) = U^+(t_2, t_1) H_{INT}^+(t_2) .$$

(III.25)

Since $H_{INT}(t_2)$ is Hermitean, $H_{INT}^+(t_2) = H_{INT}(t_2)$; also,

$$U^+(t_2, t_1) = U(t_1, t_2).$$

Hence (III.25) becomes

$$-i \frac{2}{\partial t_2} U(t_1, t_2) = U(t_1, t_2) H_{INT}(t_2) .$$

(III.26)

Interchange of $t_1$ and $t_2$ yields

$$-i \frac{2}{\partial t_2} U(t_2, t_1) = U(t_2, t_1) H_{INT}^c(t_2) .$$

(III.27)

Adding (III.16) and (III.27) and setting $t_2 = t$, $t_1 = -t$ yields

$$2 i \frac{2}{\partial t} U(t, -t) = H_{INT}(t) U(t, -t)$$

$$+ U(t, -t) H_{INT}(-t) .$$

(III.28)
Time-reversal invariance requires that, from (III.28),
\[ 2 \frac{i}{\hbar} \frac{\partial}{\partial t} U(t, -t) = U_T(t, -t) \left[ H_{\text{INT}}(t) \right]_T + \left[ H_{\text{INT}}(-t) \right]_T U_T(t, -t) \]
(III.29)

where a time-reversed function is indicated by the subscript \( T \), so that (for example) \( U_T = \Theta U T \). The anti-unitary property of \( \Theta \) results in the reversal of the order of the factors in products on the right-hand side of the equation (III.28) in going to (III.29), since both \( U \) and \( H \) contain creation and annihilation operators.

From the definition of time-reversal,
\[ \left[ H_{\text{INT}}(t) \right]_T = H_{\text{INT}}(-t) \]
(III.30)

Hence (III.29) can be written
\[ 2 \frac{i}{\hbar} \frac{\partial}{\partial t} U_T(t, -t) = H_{\text{INT}}(t) U_T(t, -t) + U_T(t, -t) H_{\text{INT}}(-t) \]
(III.31)

Comparison of (III.28) and (III.31) shows that \( U_T(t, -t) \) satisfies the same equation as does \( U(t, -t) \). Hence
\[ U_T(t, -t) = U(t, -t) \]
(III.32)

Taking the limits of the functions in (III.32) as \( t \) goes to \( -\infty \), and using the definition of \( S \) in terms of \( U \),
\[ S_T = S \]
(III.33)

Substituting from (III.33) into (III.22) yields (III.17), so the principle of reciprocity holds.

Experimentally, for a process characterized by \( \langle \tau | S | \tau' \rangle \), one often does not observe the reverse process (which is characterized by the matrix element \( \langle \tau' | S | \tau \rangle \) but rather the inverse process...
(characterized by $\langle \tau' / S / \tau \rangle$). An equality between the probabilities for a process and its inverse is useful in the statistical analysis of states in thermal equilibrium and the approach to equilibrium, since it means that the transitions from state $\tau$ to a state $\tau'$ can be balanced directly by the inverse transitions from the state $\tau'$ to state $\tau$ without invoking any intermediate states through which such a balance might be effected. Under some conditions such an equality - in the form of the principle of detailed balance - is implied by reciprocity, as will be shown in the next section.

3. Conditions Under Which Detailed Balance Holds

The principle of detailed balance states that the probability for a transition is equal to the probability for the inverse transition:

$$|\langle \tau' / S / \tau \rangle|^2 = |\langle \tau / S / \tau' \rangle|^2,$$  \hspace{1cm} (III.34)

As was pointed out previously, this principle is not generally valid. However, there are special cases in which the principle does hold, and some of these will be given here in the form of theorems.

**Theorem 1.** If $S$ can be represented as the time integral of $H_{NT}$ (as can be done in cases of weak interaction wherein $S$ can be given by the first approximation of a perturbation expansion), then the principle of detailed balance holds.

**Proof:** From the Hermitean property of $H_{NT}$,

$$\langle \tau' / H_{NT} / \tau \rangle = \langle \tau / H_{NT} / \tau' \rangle.$$  \hspace{1cm} (III.35)

Since $S$ is approximately the time integral of $H_{NT}$,

$$\langle \tau' / S / \tau \rangle = \langle \tau / S / \tau' \rangle.$$  \hspace{1cm} (III.36)

Hence

$$|\langle \tau' / S / \tau \rangle|^2 = |\langle \tau / S / \tau' \rangle|^2.$$  \hspace{1cm} (III.37)
Theorem II. If the process is invariant under coordinate inversion, and if only those quantum numbers are measured which change sign under time reversal and also under coordinate inversion, then detailed balance holds.

Proof: Let the process occur between states characterized by momenta \( \mathbf{k} \) and spins \( \mathbf{s} \). By the principle of reciprocity,

\[
\langle \mathbf{k}', \mathbf{s}', \ldots | S | \mathbf{k}, \mathbf{s}, \ldots \rangle = \pm \langle -\mathbf{k}', -\mathbf{s}', \ldots | S | -\mathbf{k}, -\mathbf{s}, \ldots \rangle.
\]  

(III.38)

From invariance under inversion of coordinates,

\[
\langle -\mathbf{k}, -\mathbf{s}, \ldots | S | -\mathbf{k}', -\mathbf{s}', \ldots \rangle = \langle \mathbf{k}, \mathbf{s}, \ldots | S | \mathbf{k}', \mathbf{s}', \ldots \rangle.
\]  

(III.39)

From (III.38) and (III.39),

\[
\langle \mathbf{k}', \mathbf{s}', \ldots | S | \mathbf{k}, \mathbf{s}, \ldots \rangle = \pm \langle \mathbf{k}, \mathbf{s}, \ldots | S | \mathbf{k}', \mathbf{s}', \ldots \rangle.
\]  

(III.40)

Ignorance of the values of the spin variables is equivalent to having a process in which all values of the spins occur. Hence the probability for the transition is the sum over all values of all spins of the probability for transition with given momenta and spins:

\[
\sum_{\text{spins}} \frac{1}{|S|} |\langle \mathbf{k}', \mathbf{s}', \ldots | S | \mathbf{k}, \mathbf{s}, \ldots \rangle|^2.
\]  

(III.41)
From (III.40), this last is given by
\[
\sum_{\text{spins}} \langle -\vec{r}_A, -\vec{s}_A, \ldots | -\vec{r}_C, -\vec{s}_C, \ldots \rangle^2
\]
where the last step follows from the fact that each \( s_i \) takes on the same values as does \(-s_i\). (III.73) is the principle of detailed balance for the case when only the momenta are known.

**Theorem III.** Assume that a reaction involves two particles in the initial state and in the final state; that the spins of the particles lie in the reaction plane; and that the reaction is invariant under rotations in space. Then, even if parity is not conserved in the reaction, detailed balance holds.

**Proof:** Consider the diagram of the process in Figure 3. None of the spins \( \vec{s}_A, \vec{s}_B, \vec{s}_C, \vec{s}_D \) have any components perpendicular to the plane formed by the momentum vectors \( \vec{k}_A, \vec{k}_B \). By conservation of momentum, this plane is the same as the plane formed by \( \vec{k}_C \) and \( \vec{k}_D \). Next consider the reverse process as drawn in Figure 4. If the process is invariant under rotations in space, then the reverse process has an amplitude equal to the amplitude for the process which is obtained by rotating every vector in the reverse process by an angle \( \pi \) around an axis perpendicular to the plane of the reaction. The result of this rotation is shown in Figure 5. Comparison of Figure 5 with Figure 3 shows that Figure 5 gives the inverse process. Thus the amplitudes for the process and its inverse are equal, and therefore detailed balance holds. This conclusion is correct even if parity is not conserved in the reaction, since invariance under inversion of coordinates has not been assumed.
Figure 3. The original process.
Figure 4. The reverse process.
Figure 5. The reverse process with vectors rotated through $\pi$. 
Theorem IV. If a reaction is such that both the initial and final states are characterized only by the quantum numbers $j, m$ of the total angular momentum and by sets of other scalar quantum numbers $\tau_A, \tau_B, \ldots$ and $\tau_c, \tau_D, \ldots$ which change neither magnitude nor sign under time-reversal, and if the reaction is invariant under rotations in space, then the principle of detailed balance holds.

Proof: Under the conditions given, only the quantum number $m$ changes its value upon reversal of motion, and this change is only a change in sign. Then the principle of reciprocity implies

$$< \tau_c, \tau_D, \ldots, j, m \mid S \mid \tau_A, \tau_B, \ldots, j, m> = \pm < \tau_A, \tau_B, \ldots, j, -m \mid S \mid \tau_c, \tau_D, \ldots, j, -m>.$$  

(III.43)

If $S$ is invariant under rotations, the matrix element on the right-hand side is equal to the one obtained from it by a rotation of coordinates which transforms $m$ into $-m$ but leaves the scalars $\tau, j$ invariant. Hence

$$< \tau_c, \tau_D, \ldots, j, m \mid S \mid \tau_A, \tau_B, \ldots, j, m> = \pm < \tau_A, \tau_B, \ldots, j, m \mid S \mid \tau_c, \tau_D, \ldots, j, m>.$$  

(III.44)

Taking the square of the absolute value of both sides of (III.44) gives the principle of detailed balance.
A number of experiments have been performed which invoke, or test the validity of, detailed balance or reciprocity. For instance, consider the reactions

\[ H + p^+ \leftrightarrow D + \pi^+ \]  \hspace{1cm} (IV.1)

where H is a hydrogen atom, \( p^+ \) is a proton, D is a deuteron, and \( \pi^+ \) is a pion. These reactions are inverses of each other. For this case, theorem II of the preceding section is applicable, and detailed balance may be assumed to hold. Let \( p_p \) be the momentum of the \( p^+ \) relative to the H, while \( p_{\pi} \) is the momentum of the pion relative to the D. Then, using the center of mass coordinate system, the equality of the probabilities for the two reactions can be stated as

\[
\frac{1}{2} (2I_p + 1)^2 \frac{p_p^2}{p} \sigma_{p \rightarrow D} = (2I_D + 1) (2I_{\pi} + 1) p_{\pi}^2 \sigma_{AB} \]  \hspace{1cm} (IV.2)

Here the factor \( \frac{1}{2} \) on the left-hand side is due to the indistinguishability of the two protons in the reaction producing the pion, \( \sigma_{p \rightarrow D} \) is the cross-section for the production of the pion, and \( \sigma_{AB} \) is the cross-section for the inverse process, while the I's are the quantum numbers of the intrinsic angular momenta for the various particles. Thus

\[
2I_{\pi} + 1 = \frac{2}{3} \left( \frac{p_p}{p_{\pi}} \right)^2 \frac{\sigma_{p \rightarrow D}}{\sigma_{AB}} \]  \hspace{1cm} (IV.3)

This relation predicts the spin of the positive pion. Measurement of the momenta and cross-sections for the reactions then provides a means of determining experimentally this spin (see Elton, page 247).

It was pointed out by Jackson et al. (1957) that in the decay of oriented nuclei a correlation of the form
\[ 1 + D \left( \langle \mathcal{J} \rangle / J \right) \cdot \left( \frac{P_e}{E_e} \right) \times \left( \frac{P_\nu}{E_\nu} \right) \]  

(IV.4)

will exist among the electron momentum \( P_e \), the antineutrino momentum \( P_\nu \), and the polarization \( \langle \mathcal{J} \rangle / J \) of the nucleus if, and only if, beta decay is the result of an interaction that is not invariant under time-reversal. A search for this correlation was made for the case of free polarized neutrons by Burgy et al. (1958), who sought an upper limit for \( D \). Their results indicate that \( D \) is zero or little different from zero. Thus the existence of time-reversal symmetry is supported by their results.

Tests have also been made for strong interactions, based on comparison of the polarization \( P \) produced in the scattering of unpolarized protons and the asymmetry (or depolarization) \( e \) produced when fully polarized protons are scattered. If parity conservation is assumed, then invariance under time-reversal requires that \( P = e \). Also, in the case of proton-proton scattering, if there is any time-reversal asymmetry, then, at angles near \( 45^\circ \) in the center of mass coordinate system, \( |P - e| \) is a maximum and of the same order of magnitude as the ratio between the coefficients of the two parts of the scattering matrix which are respectively non-invariant and invariant under time-reversal. Hillman et al. (1958) obtained results for high energy scattering from hydrogen, lithium, beryllium and aluminum which indicate that time-reversal invariance holds to within a few percent. After similar experiments on proton-proton scattering, Abashian and Hafner (1958) concluded that the term of the scattering matrix which is not time-reversal invariant is no more than a few percent of the average magnitude of the invariant terms.

The possibility has also been considered that time-asymmetric events may occur in the decay of strange particles. Strange particles may decay in two ways: in leptonic decay, they give off light particles such as
electrons or neutrinos, while in non-leptonic decay they do not give off light particles. Sachs (1963) has suggested that tests for time-reversal symmetry be made for the non-leptonic modes of decay of strange particles independently of tests in other cases, and has presented the following considerations for such decay: The dominant decay mode of the $\Lambda$-hyperon is the non-leptonic mode

$$\Lambda \rightarrow p + \pi^-; \quad (IV.5)$$

here $p$ is a proton and $\pi^-$ is a negatively charged pion. One can define the polarization as the fraction of the $\Lambda$-hyperons whose spins are oriented one way or the other, and one can measure polarization quantitatively as the difference $P$ between the fraction spinning clockwise and the fraction spinning counterwise, these being the only possible spins for a $\Lambda$-particle. Then the rate of disintegration of a $\Lambda$-particle with polarization $P$ into a proton and a pion with certain definite velocities may be denoted by $n(P)$. While $n(P)$ also is a function of the velocities of the products, this dependence will not be made explicit here.

Let $P'$ be the time-reverse of $P$, while $n'$ is the time-reverse function corresponding to $n$. Then invariance under reversal of motion requires that

$$n'(P') = n(P'); \quad (IV.6)$$

The rate $n'(P')$ is obtained from $n(P)$ by expressing $P$ in terms of $P'$. Because of the way in which $P$ depends on spins (which themselves are odd under time-reversal), it changes sign under time-reversal, so

$$P' = -P. \quad (IV.7)$$
Suppose that any non-invariance under time-reversal revealed itself by a dependence of the decay rate on P in the manner

\[ n = n_1 + \rho n_2 \]  

where \( n_1 \) and \( n_2 \) satisfy (IV.6). By expressing the right-hand side of (IV.8) in terms of \( P' \), one obtains

\[ n' = n_1 - \rho n_2 \]

So \( n(P) \) does not satisfy (IV.6), and hence is not invariant under time-reversal. Actually, one can establish the lack of symmetry by showing that (IV.8) holds and that \( n_2 \) satisfies (IV.6).

The above argument can be modified to take account of the quantum mechanical nature of the problem. It yields a means of testing experimentally for a time-reversal invariance in the decay of strange particles. Cronin and Overseth (1962) have carried out such experiments and have found that they did not observe the term \( n_2 \), although the errors involved are comparable to the rather small value of \( n_2 \) calculated from data on pion-nucleon scattering upon assumption of time-reversal invariance. Thus their results favor time-reversal symmetry in the non-leptonic decay of strange particles.

Much work remains to be done to establish time-reversal symmetry experimentally in nuclear physics. As Henley and Jacobsohn (1959) pointed out, results of many of the experiments performed thus far have included rather large experimental errors, so that they have not conclusively established the invariance except as a rough approximation.

Some authors (see Sachs, loc. cit.) think that the problem of determining the direction of time may be linked to an as yet undiscovered fundamental process which lacks time-reversal symmetry and which exists on
the microscopic level; experimental evidence is not yet sufficient to rule out this possibility. However, nobody has yet shown that if such a process existed on the microscopic level, this would imply any observable asymmetry on the macroscopic level. The problem of the macroscopic consequences of time-reversal asymmetric microscopic processes is still an open one.
CHAPTER V: ARE THERE PARTICLES WHICH ARE THE TIME-REVERSED COUNTERPARTS OF OTHERS?

One may wonder whether, if the operation of reversal of motion is applied to a particle, the result may be a particle of a different kind than the original. Feynman (1949) suggested that a positron be regarded as an electron travelling along a world-line that leads backward in time. This further suggests the view that the antiparticle of any lepton (or light particle) is its time-reversed counterpart. Since the characteristic which distinguishes a lepton from its antiparticle is the lepton number $L$, which is $+1$ for a lepton and $-1$ for an antilepton, this means that $L$ is odd under time-reversal. However, discovery was later made of a lepton - namely the neutrino - which has a specific handedness while its antiparticle has the opposing handedness. Since handedness does not change under reversal of motion, one cannot regard $L$ as the characteristic which removes the degeneracy with respect to $\theta$ that arises if the time-reverse of a particle is the same as the original particle. Thus the change in lepton number required in going from an electron state to a positron state cannot be obtained by the operation of time-reversal, and hence the positron is not the time-reverse of an electron, contrary to Feynman's suggestion.

Of course, the above argument would fail if it were shown that $L$ did not have unique transformation properties under time-reversal, but this would amount to abandoning lepton number as a genuine attribute of light particles.

Nevertheless, there exists a second kind of neutrino, the muon neutrino, which differs from an ordinary neutrino by the intrinsic attribute $L^\mu$, known as the muon number; $L^\mu$ has the value $+1$ for one of these particles, and $-1$ for the other. From the decay of $\mu$ mesons, it is known that the muon neutrino has the same handedness as does an ordinary neutrino (see Danby et al., 1962, and Feinberg and Gürsey, 1962), so it is not inconsistent with
present knowledge to assume tentatively that the muon neutrino state is the
time-reverse of the ordinary neutrino state. This would mean that $L_\mu$ is
the attribute which removes the degeneracy of neutrino and muon neutrino
states, because $L_\mu$ is odd under time-reversal. Then the direct product of
these two states can be a state of specific reversality because the expectation
value of $L_\mu$ is zero for it so that the fact that $L_\mu$ anticommutes with $\Theta$
does not interfere with the assignment of a specific reversality. One can
speculate on the possibility of selection rules arising from the invariance
under $\Theta$ of the direct product state, in a manner similar to what happens in
the analogous case of a positronium state having specific conjugality.
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