THE STATISTICAL AND THERMODYNAMIC THEORY OF THERMAL RADIATION AND ITS APPLICATION TO DETECTOR SENSITIVITY

by

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ABSTRACT

The Bose-Einstein distribution is derived, and from this the mean values and fluctuations of the thermodynamic quantities describing a volume V of black body radiation at absolute temperature T, are calculated.

The problem of the energy fluctuation of a body of emissivity ε_ν in thermodynamic equilibrium with a volume of black body radiation, is considered from a statistical approach. The result $\text{var } E = kT^2C$, known to be correct from thermodynamics, is obtained.

The zero point energy difficulty in the mean energy of the radiation is discussed in detail. Arguments are presented supporting the inclusion of the zero point energy in the thermal radiation theory. The problem of the number of distinguishable levels that can be obtained from a certain signal power in a resonator is discussed in this section.

Finally the results of the theory above are employed to determine the ultimate sensitivity of radiation detectors. Care is taken to isolate factors which are not fundamental properties of the detector, from the treatment of the detector sensitivity. A bolometer and a phototube, energy and quantum detectors respectively, are discussed in detail.

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CHAPTER 1 - INTRODUCTION

The first objective of this thesis is the provision of a complete statistical treatment of the theory of thermal radiation. Beginning with the derivation of the Bose-Einstein distribution law we shall derive the mean values and the fluctuations of the thermodynamic functions describing black body radiation. Most of this groundwork, included in the thesis for completeness, can be found in statistical mechanics textbooks.

The problem of a body of emissivity $\varepsilon_\nu \neq 1$ placed in an enclosure of black body radiation is now considered from a statistical approach. Lewis (1947) has obtained an expression for the fluctuation of the number of photons absorbed in unit time by a body with $\varepsilon_\nu = 1$, while Fellgett (1949) has attempted the more general case of $\varepsilon_\nu \neq 1$. Our results are in agreement with Lewis for $\varepsilon_\nu = 1$, but as we shall mention in the text, we disagree with Fellgett's treatment. We then extend our analysis to obtain the well-known expression for the energy fluctuation of a body in thermodynamic equilibrium with its surroundings, var $E = kT^2C$.

The next topic which will be discussed is the zero point energy problem arising in the theory of thermal radiation. This difficulty is often glossed over or omitted completely by authors discussing the radiation energy. We shall outline the problem in detail and present arguments for the inclusion of the zero point energy in thermal radiation theory. The effect of the zero point energy on the number of distinguishable signal levels is discussed with reference to a result of Gabor (1950).

Finally we shall apply some of the results of the thermal radiation theory in a discussion of the fundamental limitations of the sensitivity of thermal radiation detectors. This question of the sensitivity of thermal radiation detectors has been widely discussed in the literature by such authors as Jones (1947, 1953), Fellgett (1949), Smith, Jones, and Chasmar (1957), and others, some of whom we shall mention in the text. In discussing the detectors we shall strive to restrict our treatment to the fundamental properties of the detector avoiding any purely technical factors. Noise reduction processes such as the observational technique and the use of filters, will be kept separate from the fundamental properties of the detector itself. We shall introduce several parameters which will give a quantitative evaluation of the detector performance. The fact that we have all the radiation fluctuation formulas and their derivations close at hand, should add considerably to the clarity of the detector discussion.

CHAPTER 2 - BLACK BODY RADIATION.

2.1 Properties of an Assembly of Bosons in an Isothermal Enclosure.

2.1.1 The number of distinguishable modes of vibration of electromagnetic radiation in a frequency interval ν to $\nu + d\nu$.

Consider a cubical box of volume V and length of side L. In order that standing waves be set up in the box, each side of the box must intersect with an integral number of half wave lengths of the radiation. Therefore, for standing waves to exist we must have:

$$\frac{L}{\lambda/2\cos\alpha} = n_1 \qquad \frac{L}{\lambda/2\cos\beta} = n_2 \qquad \frac{L}{\lambda/2\cos\gamma} = n_3$$

where λ is the wavelength of the radiation, n_1, n_2, and n_3, are positive integers, and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of the direction of propagation of the radiation. Since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$n_1^2 + n_2^2 + n_3^2 = \frac{4L^2}{\lambda^2}$$

In n space, co-ordinates n_1 n_2 n_3, this equation represents a sphere of radius $r = 2L/\lambda$. The total number of modes of vibration, that is the total number of possible standing wave arrangements, in the frequency interval ν to $\nu + d\nu$ can be represented in n space by the volume of the positive octant of a spherical shell of radius r and thickness dr. This result must be doubled to account for the two possible orientations of the polarization of the radiation. g_ν is defined as the number of modes of vibration in the frequency interval $d\nu$ about ν. Therefore:

$$g_\nu = (2)(\tfrac{1}{8})(4\pi r^2 dr)$$

where

$$r = \frac{2L}{\lambda} = \frac{2L\nu}{v}$$

$$dr = \frac{-2L\,d\lambda}{\lambda^2} = \frac{2L\,d\nu}{u}$$

v and u are the phase and group velocities respectively of the radiation. Therefore we have finally:

$$g_\nu = \frac{8\pi V \nu^2 d\nu}{v^2 u}$$

2.1.2 The number of photons in a volume V of black body radiation.

n_ν is defined as the number of photons in a volume V of black body radiation in a frequency interval $d\nu$ about ν. Now consider the number of different ways, W, in which n_ν indistinguishable photons can be arranged in g_ν distinguishable cells, referring to a mode of vibration as a cell, when there is no restriction on the number of photons allowed per cell:

$$W = \frac{g_\nu (g_\nu + n_\nu - 1)!}{g_\nu!\, n_\nu!}$$

This equation is the basis of Bose-Einstein statistics. Applying Stirling's approximation for factorials,

$$\ln N! \approx N \ln N - N$$

and neglecting the term unity with respect to g_ν, we obtain:

$$\ln W = n_\nu \ln\left(1 + \frac{g_\nu}{n_\nu}\right) + g_\nu \ln\left(1 + \frac{n_\nu}{g_\nu}\right)$$

We wish to determine the mean value of n_ν. The mean value of n_ν can be defined as the equilibrium value or most probable value of n_ν. It is known that the equilibrium state of a physical system corresponds to the state of greatest disorder. W, the number of different arrangements of the n_ν photons in the g_ν cells, can be thought of as a measure of the disorder of our system. Therefore the mean value of n_ν is the value of n_ν for which W, or ln W, is a maximum. In addition we require that:

$$E_\nu = n_\nu h\nu = \text{constant}$$

Applying Lagrange's method of undetermined multipliers, we obtain:

$$\frac{\partial(\ln W)}{\partial n_\nu} + \beta \frac{\partial E_\nu}{\partial n_\nu} = 0$$

$$\ln\left(\frac{g_\nu + \bar{n}_\nu}{\bar{n}_\nu}\right) + \beta h\nu = 0$$

$$\bar{n}_\nu = \frac{g_\nu}{e^{-\beta h\nu} - 1}$$

This is the well-known Bose-Einstein distribution function. In Appendix 1, β is shown to be equal to $-\frac{1}{kT}$. Replacing g_ν by the value obtained in section 2.1.1 we have:

$$\overline{n}_\nu = \frac{8\pi V \nu^2 d\nu}{v^2 \mu (e^{\frac{h\nu}{kT}} - 1)}$$

This is an expression for the mean number of photons with frequencies in an interval $d\nu$ about ν in a volume V of black body radiation at absolute temperature T.

The mean total number of photons in the volume V is obtained by integrating $\overline{n}_\nu$ over all frequencies:

$$\overline{n} = \int_0^\infty \frac{\overline{n}_\nu}{d\nu} \cdot d\nu = \int_0^\infty \frac{8\pi V \nu^2 d\nu}{v^2 \mu (e^{\frac{h\nu}{kT}} - 1)}$$

At this point the properties of the medium in which the radiation is confined must be considered. If the medium is dispersive v and u are functions of ν and the integral above cannot be evaluated for the general case. In a non-dispersive medium:

$$v = \mu = (\epsilon\mu)^{-\frac{1}{2}}$$

ϵ, the dielectric constant, and μ, the magnetic permeability of the medium, are frequency independent for a non-dispersive medium. Note that it is the product and not the individual values of these two quantities which determines the velocity of propagation of electromagnetic radiation in the medium. Throughout this work we shall consider the media carrying the radiation to be non-dispersive. The integral for the total number of photons can now be evaluated and we obtain:

$$\bar{n} = \frac{8\pi V k^3 T^3}{h^3 v^3}(2.404)$$

See Appendix 2 for the details of the integration.

2.1.3 Thermodynamical functions of black body radiation.

Energy, E.

The mean energy of the photons in the frequency interval $d\nu$ about ν is obtained from:

$$\bar{E}_\nu = \bar{n}_\nu h\nu$$

The zero point energy has been omitted in this expression and will be discussed in chapter 3. From the above relation we have directly:

$$\bar{E}_\nu = \frac{8\pi h V \nu^3 d\nu}{v^3(e^{\frac{h\nu}{kT}} - 1)}$$

Integrating over all frequencies to obtain the total radiation energy in the volume V, we obtain:

$$\bar{E} = \frac{8\pi^5 V k^4 T^4}{15 h^3 v^3} \qquad \text{(Appendix 2)}$$

Entropy, S.

Boltzmann and Planck have shown that the entropy, a measure of the disorder of a system, can be related to the number of possible states of the system by the expression:

$$S_\nu = k \ln W$$

For equilibrium this becomes:

$$\overline{S_\nu} = k \ln W_{max}$$

It should be noted that this Boltzmann Planck expression is a special case of the more general relation:

$$S_\nu = -k \sum_{i=1}^{W} P_i \ln P_i$$

in which P_i, the probability of a photon being in the i^{th} state, is equal to $\frac{1}{W}$, a constant.

Replacing ln W_{max} by the expression given in section 2.1.2, we have for the mean entropy of the radiation in a frequency band $d\nu$ about ν:

$$\overline{S_\nu} = k \left\{ \bar{n}_\nu \ln\left(1 + \frac{g_\nu}{\bar{n}_\nu}\right) + g_\nu \ln\left(1 + \frac{\bar{n}_\nu}{g_\nu}\right) \right\}$$

We replace g_ν and $\bar{n}_\nu$ by the expressions in sections 2.1.1 and 2.1.2 respectively and obtain:

$$\overline{S_\nu} = \frac{8\pi h V \nu^3 d\nu}{v^3 T\left(e^{\frac{h\nu}{kT}} - 1\right)} - \frac{8\pi V k}{v^3} \ln\left(1 - e^{\frac{h\nu}{kT}}\right) \nu^2 d\nu$$

The mean entropy for all the radiation in the volume V is obtained by integrating over all frequencies:

$$\bar{S} = \int_0^\infty \frac{\overline{S_\nu}}{d\nu} \cdot d\nu$$

$$\bar{S} = \frac{32\pi^5 V k^4 T^3}{45 h^3 v^3} = \frac{4\bar{E}}{3T}$$

(Appendix 2)

Helmholtz Free Energy, F.

The Helmholtz free energy is obtained directly from the thermodynamical equation:

$$F = E - TS$$

Therefore we can obtain directly:

$$\bar{F}_\nu = \bar{n}_\nu h\nu - kT\left\{\bar{n}_\nu \ln\left(1 + \frac{g_\nu}{\bar{n}_\nu}\right) + g_\nu \ln\left(1 + \frac{\bar{n}_\nu}{g_\nu}\right)\right\}$$

$$\bar{F} = -\frac{8\pi^5 V k^4 T^4}{45 h^3 v^3} = -\frac{\bar{E}}{3}$$

Radiation Pressure, P.

From thermodynamics: $P = -\left(\frac{\partial F}{\partial V}\right)_T$

Hence: $$\bar{P} = \frac{8\pi^5 k^4 T^4}{45 h^3 v^3} = \frac{\bar{E}}{3V}$$

Gibbs Free Energy, G.

Also from thermodynamics:

$$G = E + PV - TS$$

Therefore we see immediately:

$$\bar{G} = 0$$

2.1.4 The probability distribution function for the number of photons in the frequency interval $d\nu$ about ν.

For a system in which V and T are kept constant, the probability distribution of a suitable variable x describing the system, is given by:

$$P(x)\,dx = P(\bar{x}) \exp\left\{-\frac{[F(x) - F(\bar{x})]}{kT}\right\} dx$$

Let us consider the probability distribution of the number of photons n_ν in the frequency interval $d\nu$ about ν :

We have from thermodynamics:

$$\Delta F_\nu = \Delta E_\nu - T\Delta S_\nu$$

and we know:

$$\Delta E_\nu = h\nu\,\Delta n_\nu$$

$$\Delta S_\nu = k\,\Delta(\ln W)$$

Let us expand ln W in a Taylor series about ln W_{max} with respect to variations in n_ν:

$$\ln W = \ln W_{max} + \left(\frac{d\ln W}{dn_\nu}\right)_{n_\nu=\bar{n}_\nu} \cdot (n_\nu-\bar{n}_\nu) + \left(\frac{d^2\ln W}{dn_\nu^2}\right)_{n_\nu=\bar{n}_\nu} \cdot \frac{(n_\nu-\bar{n}_\nu)^2}{2}$$

Therefore:

$$\Delta F_\nu = \left[h\nu - kT\left(\frac{d\ln W}{dn_\nu}\right)_{n_\nu=\bar{n}_\nu}\right](n_\nu-\bar{n}_\nu) - \frac{kT}{2}\left(\frac{d^2\ln W}{dn_\nu^2}\right)_{n_\nu=\bar{n}_\nu} \cdot (n_\nu-\bar{n}_\nu)^2$$

F_ν can now be expanded in a Taylor series about $\bar{F}_\nu$:

$$F_\nu = \bar{F}_\nu + \left(\frac{dF_\nu}{dn_\nu}\right)_{n_\nu=\bar{n}_\nu} \cdot (n_\nu-\bar{n}_\nu) + \frac{1}{2}\left(\frac{d^2F_\nu}{dn_\nu^2}\right)_{n_\nu=\bar{n}_\nu} \cdot (n_\nu-\bar{n}_\nu)^2$$

At equilibrium F_ν is a minimum. Therefore:

$$\left(\frac{dF_\nu}{dn_\nu}\right)_{n_\nu=\bar{n}_\nu} = h\nu - kT\left(\frac{d\ln W}{dn_\nu}\right)_{n_\nu=\bar{n}_\nu} = 0$$

As we have seen in section 2.1.2, this equation leads to the Bose-Einstein distribution function for $\bar{n}_\nu$. Returning to ΔF_ν, we have:

$$\Delta F_\nu = -\frac{kT}{2}\left(\frac{d^2\ln W}{dn_\nu^2}\right)_{n_\nu=\bar{n}_\nu} \cdot (n_\nu-\bar{n}_\nu)^2$$

Therefore we can express the probability distribution function $P(n_\nu)$ as:

$$P(n_\nu) = P(\bar{n}_\nu)\exp\left[\frac{1}{2}\left(\frac{d^2\ln W}{dn_\nu^2}\right)_{n_\nu=\bar{n}_\nu} \cdot (n_\nu-\bar{n}_\nu)^2\right]$$

We know from the law of large numbers that the probability distribution function for a large number of photons will be Gaussian in form. For a Gaussian probability distribution:

$$P(n_\nu) = P(\bar{n}_\nu)\ \exp\left[-\frac{(n_\nu - \bar{n}_\nu)^2}{2\ \mathrm{var}\ n_\nu}\right]$$

Therefore by comparison we see that:

$$\mathrm{var}\ n_\nu = -\frac{1}{\left(\dfrac{d^2 \ln W}{dn_\nu}\right)_{n_\nu = \bar{n}_\nu}}$$

From section 2.1.2 we have for $g_\nu \gg 1$:

$$\ln W = n_\nu \ln\left(1 + \frac{g_\nu}{n_\nu}\right) + g_\nu \ln\left(1 + \frac{n_\nu}{g_\nu}\right)$$

$$\frac{d \ln W}{dn_\nu} = \ln\left(1 + \frac{g_\nu}{n_\nu}\right)$$

$$\frac{d^2 \ln W}{dn_\nu^2} = -\frac{1}{n_\nu\left(1 + \frac{n_\nu}{g_\nu}\right)}$$

Therefore, finally:

$$\mathrm{var}\ n_\nu = \bar{n}_\nu\left(1 + \frac{\bar{n}_\nu}{g_\nu}\right)$$

This result is obtained in a slightly different manner and is discussed fully in section 2.2.1.

2.1.5 The partition function.

Each distinguishable mode of vibration as defined in section 2.1.1 may be thought of as a resonator having a frequency ν and containing f_ν photons. ($f_\nu = 0, 1, 2 \cdots \infty$)

$$\bar{f}_\nu \equiv \frac{\bar{n}_\nu}{g_\nu} = \left(e^{\frac{h\nu}{kT}} - 1\right)^{-1}$$

The partition function for one resonator is defined as:

$$(Z_\nu)_1 \equiv \sum_{f_{\nu_1} = 0}^{\infty} \exp\left(-\frac{f_{\nu_1} h\nu}{kT}\right) = \left(1 - e^{-\frac{h\nu}{kT}}\right)^{-1}$$

As a result of there being an unlimited number of photons available, the number of photons in one resonator will be independent of the number of photons in any other resonator. Therefore the partition function for two resonators will be:

$$(Z_\nu)_2 = \sum_{f_{\nu_1}=0}^{\infty} \sum_{f_{\nu_2}=0}^{\infty} \exp\left(-\frac{(f_{\nu_1}+f_{\nu_2})h\nu}{kT}\right) = \left(1 - e^{-\frac{h\nu}{kT}}\right)^{-2}$$

It was shown in section 2.1.1 that there are $\frac{8\pi V \nu^2 d\nu}{v^3}$ resonators in the volume V of black body radiation in the frequency interval $d\nu$ about ν. Therefore the total partition function for the radiation in the frequency interval $d\nu$ will be:

$$Z_\nu = \left(1 - e^{-\frac{h\nu}{kT}}\right)^{-\frac{8\pi V \nu^2 d\nu}{v^3}}$$

$$\ln Z_\nu = -\frac{8\pi V \nu^2 d\nu}{v^3} \cdot \ln\left(1 - e^{-\frac{h\nu}{kT}}\right)$$

An integration of $\ln Z_\nu$ over all frequencies will give the natural logarithm of the total partition function for the radiation.

$$\ln Z = \int_0^\infty \frac{\ln Z_\nu}{d\nu} \cdot d\nu$$

$$\ln Z = \frac{8\pi^5 V k^3 T^3}{45 h^3 v^3} \qquad \text{(Appendix 2)}$$

The value of the partition function lies in the ease with which the thermodynamical functions can be obtained from it. This is shown by the following equations:

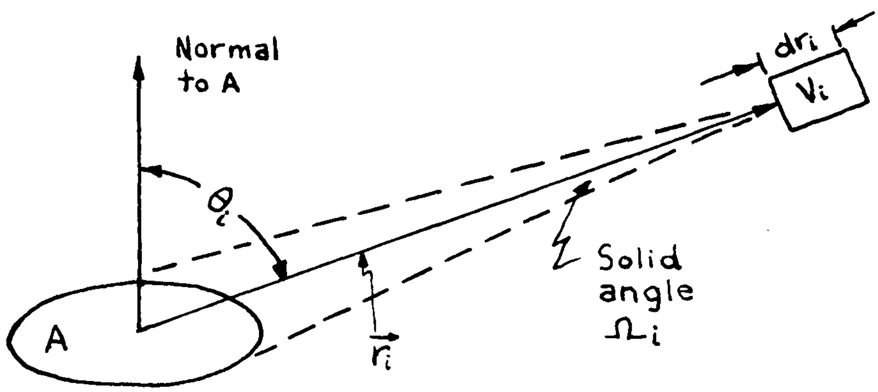
$$\bar{E} = kT^2 \left(\frac{\partial \ln Z}{\partial T}\right)_V \qquad \bar{F} = -kT \ln Z$$

$$\bar{P} = kT \left(\frac{\partial \ln Z}{\partial V}\right)_T \qquad \bar{S} = \left[\frac{\partial}{\partial T}\left(kT \ln Z\right)\right]_V$$

2.1.6 The relation between the photon flux and the photon density in a volume of black body radiation.

Let us place a small ring of area A in our volume V of black body radiation. We wish to calculate the flux of photons, m, passing through A in one direction.

We shall divide the volume V into many very small volumes V_i, each V_i containing n_i photons. The normal to the area A makes an angle θ_i with the vector $\vec{r}_i$ from A to V_i. Also A subtends a solid angle Ω_i at V_i. The situation is illustrated below:



Let b_i be the fraction of the photons in V_i which will pass through A in time $dt = dr/v$. From geometry:

$$\bar{b}_i = \frac{\Omega_i}{4\pi} = \frac{A\cos\theta_i}{4\pi r_i^2}$$

Define m as the photon flux in one direction through A, and m_i as the contribution to this flux from V_i. That is:

$$m = \sum_i m_i$$

From the definition of b :

$$m_i A\, dt = b_i n_i$$

$$m\,A\,dt = \sum_i b_i n_i$$

$$\overline{m}\,A\,dt = \sum_i \overline{b_i n_i} = \sum_i \overline{b_i}\;\overline{n_i}$$

since b_i and n_i are uncorrelated. However:

$$\frac{\overline{n}}{V} = \frac{\overline{n_i}}{V_i} = \text{photon density}$$

$$\overline{m}\,A\,dt = \sum_i \overline{b_i} \cdot \left(\frac{\overline{n_i}}{V_i}\right) \cdot V_i$$

We can express the infinitesimal element of volume V_i in spherical co-ordinates as:

$$V_i = 2\pi r_i^2 \sin\theta_i \; dr_i \; d\theta_i$$

Expressing the summation as an integral we obtain:

$$\overline{m}\,A\,dt = \int_0^{\pi/2} \frac{A\cos\theta}{4\pi r^2} \cdot \frac{\overline{n}}{V} \cdot 2\pi r^2 \sin\theta \; dr \; d\theta$$

The limits of 0 to $\frac{\pi}{2}$ on the integration of θ give us the flux in one direction only. From this integration we get:

$$\overline{m} = \frac{v}{4} \cdot \frac{\overline{n}}{V}$$

which relates the mean photon flux in one direction to the mean photon density for black body radiation in a non-dispersive medium. From section 2.1.2 we can obtain expressions for n_ν and n, from which we obtain:

$$\overline{m}_\nu = \frac{2\pi \nu^2 \, d\nu}{v^2 \left(e^{\frac{h\nu}{kT}} - 1\right)}$$

$$\overline{m} = \frac{2\pi k^3 T^3}{h^3 v^2} \cdot \quad (2.404)$$

$\overline{m}_\nu$ is the mean photon flux for photons having frequencies in a frequency interval $d\nu$ about ν for black body radiation. m is the mean total photon flux for all frequencies.

The photon energy flux is defined as:

$$\overline{H}_\nu = h\nu\,\overline{m}_\nu$$

This can also be written as:

$$\overline{H}_\nu = \frac{\upsilon}{4} \cdot \frac{\overline{E}_\nu}{V}$$

From the expressions for $\overline{m}_\nu$ or $\overline{E}_\nu$ we can obtain:

$$\overline{H}_\nu = \frac{2\pi h \nu^3 d\nu}{\upsilon^2 \left(e^{\frac{h\nu}{kT}} - 1\right)}$$

and upon integration over ν:

$$\overline{H} = \frac{2\pi^5 k^4 T^4}{15 h^3 \upsilon^2}$$

2.2 Fluctuations of the Parameters describing the Radiation.

2.2.1 The fluctuation of the number of photons.

Einstein has obtained in a nonrigorous manner, the following expression for the fluctuation of the number of photons in a system kept at constant temperature and volume:

$$\text{var } n = \frac{kT}{\left(\frac{d^2F}{dn^2}\right)_{n=\bar{n}}}$$

Refer to Appendix 3 for this derivation. In section 2.1.3 we saw that:

$$F_\nu = n_\nu h\nu - kT\left\{n_\nu \ln\left(1+\frac{g_\nu}{n_\nu}\right) + g_\nu \ln\left(1+\frac{n_\nu}{g_\nu}\right)\right\}$$

Differentiating twice with respect to n_ν, we get:

$$\left(\frac{d^2F_\nu}{dn_\nu^2}\right)_{n_\nu=\bar{n}_\nu} = \frac{kT}{\bar{n}_\nu\left(1+\frac{\bar{n}_\nu}{g_\nu}\right)}$$

This expression combined with Einstein's result leads to:

$$\text{var } n_\nu = \bar{n}_\nu + \frac{(\bar{n}_\nu)^2}{g_\nu}$$

This relation agrees with the result we obtained in section 2.1.4 for var n_ν from the probability distribution function for n_ν.

Up to this point in our discussion the wave approach and the corpuscular approach to radiation have been complementary. However, in the above equation for var n_ν, one can see what appears to be two independent contributions to the fluctuation. The first term,

$$\text{var } n_\nu = \bar{n}_\nu$$

is the fluctuation we would expect if the photons were distributed throughout the volume in a completely random manner. This term is referred to as the quantum or corpuscular contribution to the fluctuation.

Lorentz has shown that the second term,

$$\text{var } n_\nu = \frac{(\bar{n}_\nu)^2}{g_\nu}$$

is the result to be expected if the fluctuation were entirely due to the interference of a random mixture of harmonic waves. This term is often referred to as the classical or wave contribution. Therefore if we wish to retain the corpuscular approach to radiation we must realize that the photons are not randomly distributed in the volume but are distributed according to Bose-Einstein statistics. These statistics modify the random distribution to account for the wave effects mentioned above.

Return now to the expression for var n_ν and replace g_ν and n_ν by the values given in sections 2.1.1 and 2.1.2 respectively.

$$\text{var } n_\nu = \frac{8\pi V \nu^2 e^{\frac{h\nu}{kT}} d\nu}{v^3 (e^{\frac{h\nu}{kT}} - 1)^2}$$

This is an expression for the variance of the number of photons in volume V having frequencies in the interval $d\nu$ about ν for black body radiation in a non-dispersive medium. The variance in the total number of photons is:

$$\text{var } n = \frac{(2\pi kT)^3 V}{3h^3 v^3}$$

The integration of var n_ν to obtain var n is shown in Appendix 2.

At this point it is interesting to consider the ratio $\frac{\text{var } n}{\bar{n}}$ which is unity for a Poisson process and is greater than unity for a system obeying Bose-Einstein statistics. For the frequency interval $d\nu$ about ν :

$$\frac{\text{var } n_\nu}{\bar{n}_\nu} = \frac{1}{1 - e^{-\frac{h\nu}{kT}}} > 1$$

In the classical limit, $h\nu \ll kT$:

$$\frac{\text{var } n_\nu}{\bar{n}_\nu} \rightarrow \frac{kT}{h\nu} \gg 1$$

This represents a large departure from the random corpuscular theory of radiation. As was mentioned earlier, the wave properties of radiation dominate at low frequencies. In the quantum limit, $h\nu \gg kT$, $\frac{\text{var } n_\nu}{\bar{n}_\nu}$ approaches but is greater than one. This emphasizes the tendency of thermal radiation to behave as random particles when the mean energy per mode is small compared with the energy quantum $h\nu$.

2.2.2 The fluctuation of the energy of the radiation.

The fluctuation of the energy of the photons in a frequency band $d\nu$ about ν is entirely due to the fluctuation of the number of photons in the frequency interval. Therefore we can write:

$$\text{var } E_\nu = (h\nu)^2 \text{ var } n_\nu$$

or:

$$\text{var } E_\nu = \frac{8\pi V h^2 \nu^4 e^{\frac{h\nu}{kT}} d\nu}{v^3 \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^2}$$

The variance of the total energy of the radiation in the volume V is obtained by integrating var E_ν over all frequencies.

From Appendix 2: $$\text{var } E = \frac{(2\pi kT)^5 V}{15 h^3 v^3}$$

2.2.3 The fluctuation of the photon flux.

Recall the situation which was discussed in section 2.1.6. A ring of area A was placed in the volume of radiation. The volume V was divided into many small volumes V_i, each containing n_i photons. In section 2.1.6, from the definition of b_i we had:

$$m\,A\,dt = \sum_i b_i n_i = \sum_j b_j n_j$$

$$m^2 (A\,dt)^2 = \sum_i \sum_j b_i b_j n_i n_j$$

$$\overline{m^2} (A\,dt)^2 = \sum_i \sum_j \overline{b_i b_j n_i n_j} = \sum_i \sum_j \overline{b_i b_j}\ \overline{n_i n_j}$$

since b and n are statistically independent. Also:

$$\bar{m}\,A\,dt = \sum_i \bar{b}_i \bar{n}_i = \sum_j \bar{b}_j \bar{n}_j$$

Therefore:

$$(\bar{m})^2 (A\,dt)^2 = \sum_i \sum_j \bar{b}_i \bar{b}_j \bar{n}_i \bar{n}_j$$

Hence:

$$(A\,dt)^2 \operatorname{var} m = \sum_i \sum_j \left(\overline{b_i b_j}\ \overline{n_i n_j} - \bar{b}_i \bar{b}_j \bar{n}_i \bar{n}_j \right)$$

Since b_i and n_i are independent of b_j and n_j respectively:

$$\begin{aligned} \overline{b_i b_j} &= \bar{b}_i \bar{b}_j && \text{for } i \neq j \\ &= \overline{b_i^2} && \text{for } i = j \\ \overline{n_i n_j} &= \bar{n}_i \bar{n}_j && \text{for } i \neq j \\ &= \overline{n_i^2} && \text{for } i = j \end{aligned}$$

Therefore:

$$(A\,dt)^2 \operatorname{var} m = \sum_i \left(\overline{b_i^2}\ \overline{n_i^2} - (\bar{b}_i)^2 (\bar{n}_i)^2 \right)$$

Recall from section 2.1.6: $\bar{b}_i = \dfrac{A \cos \theta_i}{4\pi r_i^2}$

b_i represents a fraction of 4π steradians subtended by A at V_i. Any fraction of this solid angle is equivalent statistically to any other fraction of equal size. Therefore

b_i will have a Poissonian type of distribution which means that: $\overline{b_i^2} = (\overline{b_i})^2 + \overline{b_i}$

We are permitted to choose A very small and r_i very large, thus justifying the approximation:

$$\overline{b_i^2} \approx \overline{b_i} \qquad \text{since} \qquad \overline{b_i} \ll 1$$

In addition we may choose V_i very small so that n_i will be very much less than one, and:

$$\operatorname{var} n_i \approx \overline{n_i^2}$$

With these approximations:

$$(A\,dt)^2 \operatorname{var} m \approx \sum_i \overline{b_i} \operatorname{var} n_i$$

$$= \sum_i \frac{A\cos\theta_i}{4\pi r_i^2} \cdot \frac{\operatorname{var} n_i}{V_i} \cdot V_i$$

Recall: $V_i = 2\pi r_i^2 \sin\theta_i \, dr_i \, d\theta_i$

and note that var n is proportional to V in section 2.1.6.

Therefore: $\dfrac{\operatorname{var} n_i}{V_i} = \dfrac{\operatorname{var} n}{V}$

Now if we replace the summation over i by an integral over θ from 0 to $\frac{\pi}{2}$, and if we replace dr by $v \cdot dt$, we obtain:

$$(A\,dt)^2 \operatorname{var} m = \frac{(A\,dt)\,v \operatorname{var} n}{2V} \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta$$

The elemental time dt represents the time of observation of the fluctuations. For a continuous observation for a time t_o, we may replace dt by t_o. Therefore we obtain:

$$\operatorname{var} m = \frac{v}{4VAt_o} \operatorname{var} n$$

The preceding analysis has been in no way concerned with the frequencies of the photons. As a result the above expression will apply to both var m_ν, the variance of the flux of photons in a frequency band $d\nu$ about ν, and var m, the variance of the total photon flux. Therefore:

$$\text{var } m_\nu = \frac{\overline{m}_\nu}{At_o}\left(1 + \frac{\overline{n}_\nu}{g_\nu}\right)$$

or:

$$\text{var } m_\nu = \frac{2\pi \nu^2 \exp\left(\frac{h\nu}{kT}\right) d\nu}{\upsilon^2 At_o \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^2}$$

The variance of the total flux is:

$$\text{var } m = \frac{2(\pi k T)^3}{3 h^3 \upsilon^2 A t_o}$$

2.2.4 Fluctuation of the photon energy flux.

We have seen in section 2.2.2:

$$\text{var } E_\nu = (h\nu)^2 \text{ var } n_\nu$$

Similarly:

$$\text{var } H_\nu = (h\nu)^2 \text{ var } m_\nu$$

or:

$$\text{var } H_\nu = \frac{\upsilon}{4VAt_o} \text{ var } E_\nu$$

Therefore:

$$\text{var } H_\nu = \frac{2\pi h^2 \nu^4 \exp\left(\frac{h\nu}{kT}\right) d\nu}{\upsilon^2 At_o \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^2}$$

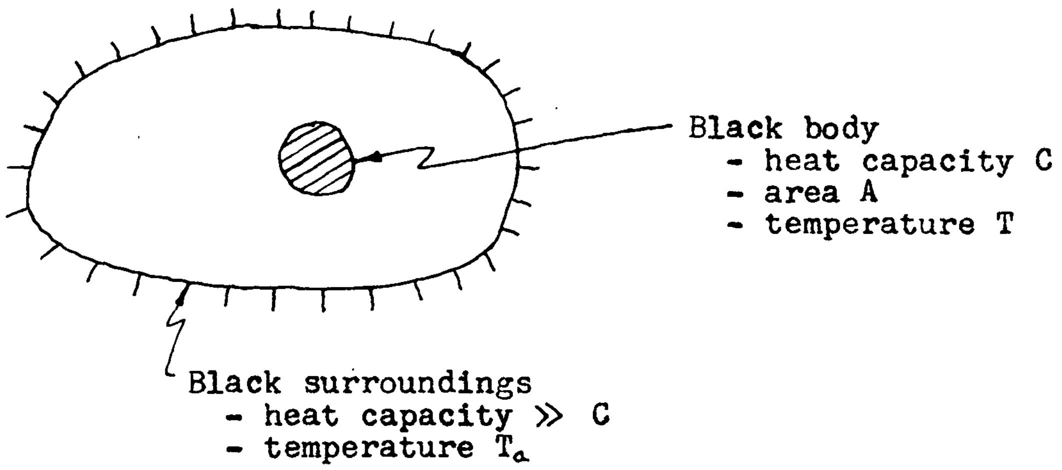
and:

$$\text{var } H = \frac{8(\pi k T)^5}{15 h^3 \upsilon^2 A t_o}$$

2.3 Energy Fluctuations of a Material Body in a Volume of Black Body Radiation.

2.3.1 A perfectly absorbing (black) body with radiative thermal coupling only between the body and the surroundings.

In our volume V of black body radiation, a small black body of heat capacity C is placed. The fluctuation in the photon fluxes incident on and emitted by the body will causes fluctuations in the energy content of the body. These energy fluctuations will produce fluctuations in the body temperature, a phenomenum known as temperature noise. The situation under discussion is:



In Appendix 4 it is shown that the heat capacity C of a volume of typical solid material is very much greater than the heat capacity C_R of an equal volume of black body radiation. Therefore we are justified in neglecting the heat capacity of the radiation.

We wish to obtain the energy or temperature response of the black body when a photon of energy $h\nu$ strikes the body at time $t = 0$. We shall assume the time required for the photon to transfer its energy to the lattice of the body is very short and may be neglected. The temperature of the body, T, can be written as:

$$T = T_a + \Delta T_\nu$$

where ΔT_ν is the temperature response of the body and is equal to $\frac{h\nu}{C}$ at $t = 0$.

From section 2.1.6 we can obtain an expression for the total power radiated by the small body:

$$\overline{P}_r = A\overline{H}_r = \frac{2\pi^5 k^4 A (T_a + \Delta T_\nu)^4}{15 h^3 v^2}$$

Similarly the power absorbed by the body from the surroundings is given by:

$$\overline{P}_a = A\overline{H}_a = \frac{2\pi^5 k^4 A T_a^4}{15 h^3 v^2}$$

The differential equation for the temperature response of the body is: $C\frac{d(\Delta T_\nu)}{dt} = \overline{P}_a - \overline{P}_r$

If we assume $\Delta T_\nu \ll T_a$

then $(T_a + \Delta T_\nu)^4 - T_a^4 \approx 4T_a^3 \Delta T_\nu$

and $C\frac{d(\Delta T_\nu)}{dt} + \lambda_R \Delta T_\nu = 0$

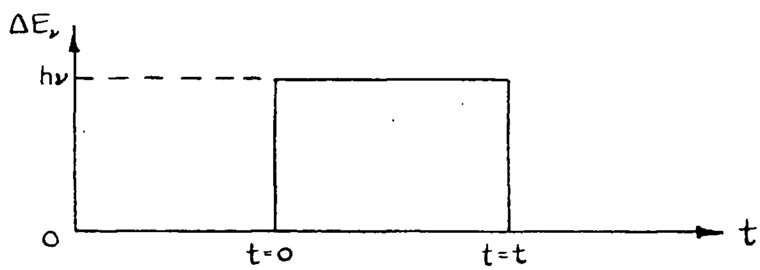
where $\lambda_R \equiv \frac{8\pi^5 k^4 T_a A}{15 h^3 v^2}$

λ_R is the thermal conductance between the body and the surroundings due to black body radiation and will have units of watts per degree absolute in the MKS system.

The solution of the differential equation is:

$$\Delta E_\nu = C\Delta T_\nu = (h\nu)\exp\left(-\frac{\lambda_R t}{C}\right)$$

The following objection might be raised against the previous discussion. It appears from the above equation for the energy response that the energy quantum $h\nu$ absorbed by the body is radiated in a continuous rather than a discrete manner. We know from the corpuscular theory of radiation that the smallest energy unit for radiation of frquency ν is $h\nu$ joules. The explanation of this apparent disagreement is that the energy response equation represents an average of the responses for a large number of events. For any single event a photon of energy $h\nu$ will be emitted at a time $t \geqq 0$ and the plot of ΔE_ν versus t will have the form:

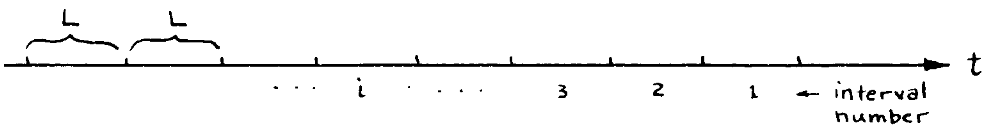


The average of a large number of responses of this type will lead to our energy response equation. In addition, for a single event:

$$P(t) = K\exp\left(-\frac{\lambda_R t}{C}\right)$$

where P(t)dt is the probability that the photon of energy $h\nu$ will be emitted from the body in the time interval between t and t + dt. K is a normalizing constant.

We now must modify Campbell's theorem to enable us to apply this theorem to our photon system which behaves according to Bose-Einstein statistics. Campbell's theorem, in its original form, expressed the mean and the variance of a parameter influenced by a Poissonian sequence of events, in terms of the response of this parameter to a single event. Let us divide the time scale into segments of length L:



Define; $h_\nu(t)$ - response of the system at $t=0$ to a photon striking the body a time t ago.

$$h_\nu(t) \equiv \Delta E_\nu$$

m_{ν_i} - the flux of photons in the frequency interval $d\nu$ about ν, arriving at the body during the i^{th} time interval.

A - area of the body.

The number of photons striking the body during the i^{th} time interval, having frequencies in a band $d\nu$ about ν, will be:

$$m_{\nu_i} \cdot A \cdot L$$

The energy response at $t=0$ to photons striking the body a time iL before will be:

$$E_\nu = \sum_{i=1}^{\infty} m_{\nu_i} \cdot AL \cdot h_\nu(iL)$$

$$\overline{E_\nu} = \sum_{i=1}^{\infty} \overline{m}_{\nu_i} \cdot AL \cdot h_\nu(iL)$$

$$\overline{E_\nu^2} = (AL)^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{m_{\nu_i} m_{\nu_j}}\, h_\nu(iL)\, h_\nu(jL)$$

$$\operatorname{var} E_\nu = (AL)^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\overline{m_{\nu_i} m_{\nu_j}} - \overline{m}_{\nu_i}\overline{m}_{\nu_j}\right) h_\nu(iL)\, h_\nu(jL)$$

Now $\quad \overline{m_{\nu_i} m_{\nu_j}} - \overline{m}_{\nu_i}\overline{m}_{\nu_j} = 0 \quad$ for $i \neq j$

$\qquad\qquad = \operatorname{var} m_{\nu_i} \quad$ for $i = j$

Therefore:

$$\operatorname{var} E_\nu = (AL)^2 \sum_{i=1}^{\infty} \operatorname{var} m_{\nu_i} \cdot h_\nu^2(iL)$$

As we let $L = dt \rightarrow 0$;

$$\mathrm{var}\, E_\nu \rightarrow (A\,dt)\cdot A \int_0^\infty \mathrm{var}\, m_\nu \cdot \hbar_\nu^2(t)\, dt$$

We have seen previously that:

$$\hbar_\nu(t) = (h\nu)\exp\left(-\frac{\lambda_R t}{c}\right)$$

Therefore integrating over t gives us:

$$\mathrm{var}\, E_\nu = \frac{A^2\, dt\cdot \mathrm{var}\, m_\nu \cdot (h\nu)^2}{2\lambda_R/c}$$

The elemental time dt represents the time of continuous observation and may be replaced by t_o. In this analysis we have considered only the incident flux, which, in this special case of a black body, is equal to the absorbed photon flux. The same argument can be applied to the emitted flux so that we can write finally the important relation:

$$\mathrm{var}\, E_\nu = \frac{Ac}{2\lambda_R}\cdot At_o\, \mathrm{var}\, m_{\nu_{total}} \cdot (h\nu)^2$$

where $(m_\nu)_{total}$ is the emitted photon flux plus the absorbed photon flux, or in other words, $(m_\nu)_{total}$ is the photon flux which contributes to the thermal exchange between the body and the surroundings. Considering photons with frequencies in the interval $d\nu$ about ν, this expression relates the variance of the energy of a body to the variance of the fluxes of photons contributing to the energy exchange between the body and its surroundings.

In section 2.2.3 the expression

$$\mathrm{var}\, m_\nu = \frac{2\pi\nu^2 \exp\left(\frac{h\nu}{kT}\right) d\nu}{v^2 At_o\left[\exp(h\nu/kT) - 1\right]^2}$$

was obtained. This is the variance of the flux of photons in one direction with frequencies in the interval $d\nu$ about ν for a stream of black body radiation. In our problem we have two independent streams of black body radiation, the absorbed radiation and the emitted radiation, which constitute the entire radiative thermal connection between the body and the surroundings. Since these two streams of radiation are statistically independent, the variance of the total flux contributing to the thermal exchange between the body and the surroundings will be:

$$\mathrm{var}(2m_\nu) = \frac{4\pi\nu^2 \exp\left(\frac{h\nu}{kT}\right)d\nu}{At_0 v^2\left(\exp\left(\frac{h\nu}{kT}\right)-1\right)^2}$$

If we put this expression into our equation for var E_ν and integrate over all frequencies to obtain var E, we arrive at:

$$\mathrm{var}\ E = \frac{2\pi ACh^2}{\lambda_R v^2}\int_0^\infty \frac{\nu^4 \exp\left(\frac{h\nu}{kT}\right)d\nu}{\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^2}$$

$$\mathrm{var}\ E = \frac{8(\pi kT)^5 AC}{15h^3 v^2 \lambda_R} \qquad \text{(Appendix 2)}$$

If we replace λ_R by its radiative value given earlier in this section, we obtain the familiar expression:

$$\mathrm{var}\ E = kT^2C$$

This result has been obtained from a statistical approach under the following conditions:

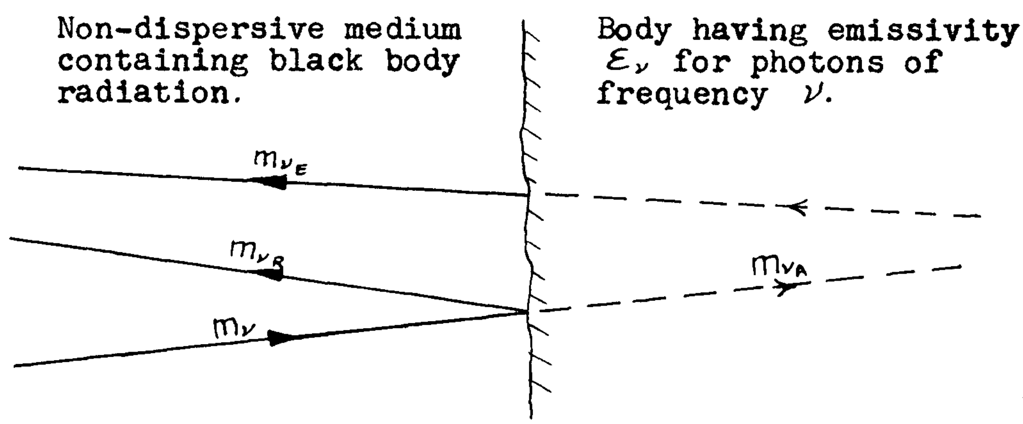
i) Black body and black surroundings.

ii) Radiative thermal coupling only.

iii) Thermal equilibrium between body and surroundings.

2.3.2 A statistical treatment of the energy fluctuations of a body of emissivity ε_ν in a volume of black body radiation.

Following previous notation the subscript ν will indicate that photons in a frequency band $d\nu$ about ν are being considered. In this problem, since the incident radiation is divided into reflected and absorbed radiation, we will have four radiation streams to consider.



m_ν - photon flux of incident black body radiation.

m_{ν_R} - photon flux of reflected radiation.

m_{ν_A} - photon flux of absorbed radiation.

m_{ν_E} - photon flux of emitted radiation.

Let us introduce M_ν, the number of photons incident upon the body in time t_o as;

$$M_\nu = m_\nu A t_o$$

in which A is the area of the body. In a similar manner M_{ν_R}, M_{ν_A}, and M_{ν_E}, may be defined.

The absorption process may be thought of as a binomial selection process with M_ν attempts, M_{ν_A} successes, and $\mathcal{E}_\nu$ the probability of a success per attempt. If M_ν were a nonfluctuating quantity, we could write:

$$\overline{M_{\nu_A}} = \mathcal{E}_\nu M_\nu$$

$$\overline{M^2_{\nu_A}} = \mathcal{E}_\nu M_\nu (1 - \mathcal{E}_\nu) + (\mathcal{E}_\nu M_\nu)^2$$

These are the expressions for the first and second moments of a quantity M_{ν_A} subject to fluctuations of a binomial nature with probability coefficient $\mathcal{E}_\nu$. Let $P(M_\nu)$ be the probability distribution function for M_ν. $P(M_\nu)$ is the probability that M_ν photons will arrive at the body in time t_o. Therefore, for any $P(M_\nu)$:

$$\overline{M_{\nu_A}} = \sum_{M_\nu=0}^{\infty} P(M_\nu) \sum_{M_{\nu_A}=1}^{M_\nu} \binom{M_\nu}{M_{\nu_A}} \mathcal{E}_\nu^{M_{\nu_A}} (1-\mathcal{E}_\nu)^{M_\nu - M_{\nu_A}} \cdot M_{\nu_A}$$

However: $$\sum_{M_{\nu_A}=1}^{M_\nu} \binom{M_\nu}{M_{\nu_A}} \mathcal{E}_\nu^{M_{\nu_A}} (1-\mathcal{E}_\nu)^{M_\nu - M_{\nu_A}} \cdot M_{\nu_A} = \mathcal{E}_\nu M_\nu$$

since both these expressions are equal to the first moment of M_{ν_A} which is subject to binomial type fluctuations.

Therefore $\overline{M_{\nu_A}} = \mathcal{E}_\nu \overline{M_\nu}$ as expected.

Similarly;

$$\overline{M_{\nu_A}^2} = \sum_{M_\nu=0}^{\infty} P(M_\nu) \underbrace{\sum_{M_{\nu_A}=1}^{M_\nu} \binom{M_\nu}{M_{\nu_A}} \mathcal{E}_\nu^{M_{\nu_A}} (1-\mathcal{E}_\nu)^{M_\nu - M_{\nu_A}} \cdot M_{\nu_A}^2}_{\text{Second moment of } M_{\nu_A}}$$

Therefore; $$\overline{M_{\nu_A}^2} = \mathcal{E}_\nu \overline{M_{\nu_A}} (1-\mathcal{E}_\nu) + \mathcal{E}_\nu^2 \overline{M_{\nu_A}^2}$$

and; $$\operatorname{var} M_{\nu_A} = \mathcal{E}_\nu^2 \operatorname{var} M_\nu + \mathcal{E}_\nu \overline{M_\nu} (1-\mathcal{E}_\nu)$$

From this expression for the variance of M_{ν_A} we can obtain directly the variance of M_{ν_R} by replacing $\mathcal{E}_\nu$ by $1-\mathcal{E}_\nu$, since the probability of a reflection per incident photon is $1-\mathcal{E}_\nu$. Therefore:

$$\operatorname{var} M_{\nu_R} = (1-\mathcal{E}_\nu)^2 \operatorname{var} M_\nu + \mathcal{E}_\nu \overline{M_\nu} (1-\mathcal{E}_\nu)$$

The principle of detailed balance requires that $\overline{M_{\nu_A}} = \overline{M_{\nu_E}}$. However it is important to realize that this principle of detailed balance does not require that the variance of M_{ν_A} equal the variance of M_{ν_E}. Let us now consider the fluctuation in the emitted radiation. As a result of our system being in thermal equilibrium, the total radiation incident on the body must be equal to the total radiation leaving the body. That is to say:

$$\overline{M}_\nu = \overline{M}_{\nu_E} + \overline{M}_{\nu_R}$$

In addition we would expect the fluctuations of the incident radiation to be equal to the fluctuations of the radiation leaving the body. That is:

$$\text{var } M_\nu = \text{var } (M_{\nu_E} + M_{\nu_R})$$

However since the emitted and reflected streams of radiation are statistically independent, we can write:

$$\text{var } M_\nu = \text{var } M_{\nu_E} + \text{var } M_{\nu_R}$$

Therefore, from our expression for var M_{ν_R} we can obtain:

$$\text{var } M_{\nu_E} = \varepsilon_\nu(2 - \varepsilon_\nu) \text{ var } M_\nu - \varepsilon_\nu \overline{M}_\nu (1 - \varepsilon_\nu)$$

The emitted radiation and the absorbed radiation constitute the entire radiative connection between the body and the surroundings. In addition these two streams of radiation are statistically independent. Therefore:

$$\begin{aligned} \text{var } (M_{\nu_E} + M_{\nu_A}) &= \text{var } M_{\nu_E} + \text{var } M_{\nu_A} \\ &= 2\varepsilon_\nu \text{var } M_\nu \end{aligned}$$

Returning to the flux notation:

$$\begin{aligned} \text{var } (m_{\nu_E} + m_{\nu_A}) &= 2\varepsilon_\nu \text{var } m_\nu \\ &= \frac{4\pi \varepsilon_\nu \nu^2 \exp\left(\frac{h\nu}{kT}\right) d\nu}{v^2 A t_0 \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^2} \end{aligned}$$

Recall the equation derived in section 2.3.1 expressing var E_ν in terms of var $m_{\nu_{total}}$. Replacing var $m_{\nu_{total}}$ by var $(m_{\nu_E} + m_{\nu_A})$, we have:

$$\text{var } E_\nu = \frac{2\pi A C h^2 \nu^4 \varepsilon_\nu \exp\left(\frac{h\nu}{kT}\right) d\nu}{v^2 \lambda_R \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^2}$$

The radiative thermal conductance, λ_R, for a body of emissivity ε_ν will be given by:

$$\lambda_R = \frac{\partial}{\partial T}\int_0^\infty \varepsilon_\nu \cdot h\nu \cdot \frac{\overline{m}_\nu}{d\nu} \cdot A\, d\nu$$

Assuming ε_ν is independent of temperature, and recalling that:

$$\overline{m}_\nu = \frac{2\pi\nu^2\, d\nu}{v^2\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}$$

we have:

$$\lambda_R = \frac{2\pi A h^2}{kT^2 v^2}\int_0^\infty \frac{\nu^4 \varepsilon_\nu \exp\left(\frac{h\nu}{kT}\right) d\nu}{\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^2}$$

Var E_ν is now integrated over all frequencies to obtain:

$$\mathrm{var}\ E = \frac{\dfrac{2\pi A h^2 C}{v^2}\displaystyle\int_0^\infty \frac{\nu^4 \varepsilon_\nu \exp\left(\frac{h\nu}{kT}\right) d\nu}{\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^2}}{\lambda_R}$$

Immediately it can be seen that the integrals in var E and in λ_R are equal. Therefore we have directly:

$$\mathrm{var}\ E = kT^2 C$$

This result, known to be correct from thermodynamics, has been obtained statistically for a body of emissivity ε_ν in thermal equilibrium with black body radiation.

Fellgett (1949) has obtained an expression for the variance of the number of photons absorbed by a body of emissivity ε_ν in a volume of black body radiation. His result is:

$$\mathrm{var}\ M_{\nu_A} = \varepsilon_\nu\, \mathrm{var}\ M_\nu .$$

compared with our result:

$$\mathrm{var}\ M_{\nu_A} = \varepsilon_\nu^2\, \mathrm{var}\ M_\nu + \varepsilon_\nu \overline{M}_\nu (1-\varepsilon_\nu)$$

Fellgett has not considered the fluctuation introduced by the absorption process. In addition the unjustified assumption that the variance of the absorbed photons is equal to the variance of the emitted photons is implicit in his argument.

2.3.3 The energy fluctuations of a black body in a volume of black body radiation in a nonequilibrium steady state condition.

The problem to be discussed here is similar to that discussed in section 2.3.1 with one difference. We wish to consider the situation when a nonfluctuating power P is applied to the body keeping the mean body temperature T_o appreciably higher than the temperature of the surroundings T_a.

As in section 2.3.1 we wish to obtain the energy response function when a photon of energy $h\nu$ strikes the body at $t = 0$. Let the body temperature be:

$$T = T_o + \Delta T_\nu$$

where $$\Delta T_\nu = \frac{h\nu}{C} \quad \text{at} \quad t = o.$$

The temperature response equation for this problem is:

$$C \frac{d(\Delta T_\nu)}{dt} = P + \overline{P}_a - \overline{P}_r$$

From section 2.3.1 we have:

$$\overline{P}_a = \frac{2\pi^5 k^4 A T_a^4}{15 h^3 \upsilon^2} \qquad \overline{P}_r = \frac{2\pi^5 k^4 A T^4}{15 h^3 \upsilon^2}$$

From the steady state form of the temperature response equation we obtain: $$P = \frac{2\pi^5 k^4 A \left(T_o^4 - T_a^4\right)}{15 h^3 \upsilon^2}$$

If we require $\Delta T_\nu \ll T$ then;

$$T^4 - T_o^4 \approx 4 T_o^3 \Delta T_\nu$$

and we can obtain directly:

$$C \frac{d(\Delta T_\nu)}{dt} + \lambda_R \Delta T_\nu = 0$$

where $$\lambda_R \equiv \frac{8\pi^5 k^4 T_o^3 A}{15 h^3 \upsilon^2}$$

This differential equation is identical to the one obtained in section 2.3.1, and we have shown there in detail the calculations leading to the result:

$$\text{var } E_\nu = \frac{AC}{2\lambda_R} \cdot At_o \text{ var } m_{\nu_{total}} (h\nu)^2$$

In this problem we have two statistically independent streams of black body radiation making up the radiative thermal connection between the body and the surroundings. These streams are the emitted radiation at temperature T and the incident radiation at temperature T_a. Therefore, recalling from section 2.2.3 the expression for the variance of the flux of a stream of black body radiation, we can write:

$$At_o \text{ var } m_{\nu_{total}} = \frac{2\pi\nu^2 \exp\left(\frac{h\nu}{kT}\right) d\nu}{v^2\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^2} + \frac{2\pi\nu^2 \exp\left(\frac{h\nu}{kT_a}\right) d\nu}{v^2\left[\exp\left(\frac{h\nu}{kT_a}\right)-1\right]^2}$$

Now if we substitute this value for $At_o \text{var } m_{\nu_{total}}$ into our equation for var E_ν, and integrate over all frequencies, or better still, compare with the similar calculation in section 2.3.1, we obtain:

$$\text{var } E = \frac{4\pi^5 k^5 AC}{15 h^3 v^2 \lambda_R} \left(T^5 + T_a^5\right)$$

or

$$\text{var } E = \frac{kT^2C}{2}\left(1 + \frac{T_a^5}{T^5}\right)$$

This result has been obtained for a black body of temperature $T > T_a$, the temperature of the surroundings, for a thermal conductance which is entirely radiative.

Notice that when $T = T_a$, var $E = kT^2C$ as expected. Also notice that when $T_a = 0$, var $E = \frac{1}{2}kT^2C$. This result is intuitively agreeable since by having $T_a = 0$, we are removing the incident radiation stream along with the energy fluctuations caused by this stream. Hence we would expect the energy fluctuations to be one-half of the equilibrium value.

2.3.4 The effect of conductive thermal connection between the body and the surroundings.

Up to now we have considered the thermal coupling between the body and the surroundings to be radiative only. ($\lambda = \lambda_R$). We now wish to consider the situation where $\lambda = \lambda_R + \lambda_c$, λ_c being the thermal conductance due to conduction by a medium connecting the body to the surroundings.

The expression for var E_ν obtained from the modified Campbell's theorem for a general λ will have the form:

$$\text{var } E_\nu = \frac{AC \cdot At_o \text{ var } m_{\nu_{total}} \cdot (h\nu)^2}{2(\lambda_R + \lambda_c)}$$

However $m_{\nu_{total}}$ will now consist of all corpuscular fluxes which contribute to the thermal exchange between the body and the surroundings. In the radiative λ case $m_{\nu_{total}}$ was the total photon flux only. In the conductive λ case we have a phonon flux, a phonon being a quantum of lattice vibration energy. It is necessary to realize that equal fluxes of photons and phonons are indistinguishable from the point of view of energy exchange or energy fluctuations. Phonons, like photons, behave according to Bose-Einstein statistics. Refer to Appendix 4 for an illustration of this photon phonon similarity.

In section 2.2.4, for black body radiation, we saw:

$$\text{var } m_\nu = \frac{\overline{m_\nu}}{At_0}\left(1 + \frac{\overline{n_\nu}}{g_\nu}\right)$$

Also, from section 2.3.2:

$$\lambda_R = \frac{\partial}{\partial T}\int_0^\infty h\nu \frac{\overline{m_\nu}}{d\nu} \cdot A\, d\nu \qquad \text{for } \varepsilon_\nu = 1$$

A similar equation will define λ_c only in this case $\overline{m_\nu}$ will be a phonon flux. From these two equations and the equation for var E_ν, it can be seen that if $\overline{m_\nu}$ is increased by a constant factor, there will be no change in the value of var E_ν. Therefore the relation var $E = kT^2C$ will not be affected by a change in $\overline{m_\nu}$.

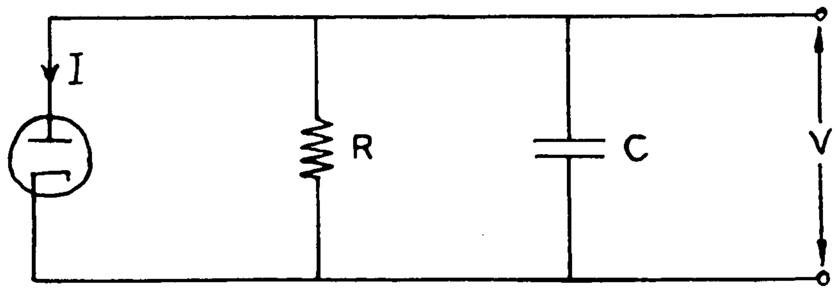
With this brief argument we have outlined a justification of the generalization of the equation var $E = kT^2C$ to include conductive as well as radiative thermal conductance.

2.3.5 The electrical analog of temperature noise.

Previously temperature noise was defined as the fluctuation of the temperature of a body in a volume of black body radiation. This temperature fluctuation is an observable effect of the quantized nature of radiation. In the preceding sections we found that temperature noise caused by photons in a frequency band $d\nu$ about ν is given by:

$$\text{var } T_\nu = \frac{\text{var } E_\nu}{C^2} = \frac{A^2 t_0 \text{ var } m_{\nu_{total}} \cdot (h\nu)^2}{2C\lambda}$$

We shall now point out an electrical analog of this temperature noise. Consider the following electrical circuit:



The variance of V as a result of the shot noise in the diode is obtained from Campbell's theorem in its original form, and is given by: $\mathrm{var}\, V = \frac{eIR}{2C} = \frac{e^2 \gamma R}{2C}$

e - electronic charge

γ - mean rate of arrival of electrons at the anode

The electrons arrive at the anode in a completely random manner. Therefore: $\mathrm{var}\,(\gamma t_o) = \gamma t_o$ from which we see: $\gamma = t_o \mathrm{var}\, \gamma$. Therefore:

$$\mathrm{var}\, V = \frac{e^2 \gamma R}{2C} = \frac{e^2 R t_o \mathrm{var}\, \gamma}{2C} .$$

The following table is a list of the electrical and radiative parameters which can be identified as analogs:

Electrical Parameters		Radiative Parameters	
C	capacitance	C	heat capacity
R	resistance	λ^{-1}	thermal resistance
γ	mean rate of arrival of electrons	$\overline{m}_\nu A$	mean rate of arrival of photons
e	electronic charge	$h\nu$	photon energy
I	current flow	P_ν	power flow
V	potential	T_ν	temperature
Q	charge	E_ν	energy

Substituting the appropriate radiation parameters into the equation for var V, we obtain:

$$\mathrm{var}\ T_\nu = \frac{(h\nu)^2 A^2 t_o\ \mathrm{var}\ m_\nu}{2\lambda c}$$

which is the result obtained from the statistical treatment of radiation. Since the charge on an electron is constant we can only apply this analogy to photons with constant energy, that is to photons in the frequency interval from ν to $\nu + d\nu$.

CHAPTER 3 - THE ZERO POINT ENERGY PROBLEM.

3.1 The Mean Energy.

The concept of a resonator in radiation theory was first mentioned in section 2.1.4 in connection with the partition function. There, a resonator was defined as a distinguishable mode of vibration. A resonator of this type may be represented electrically by an LC circuit with resonant frequency $\omega = (LC)^{-\frac{1}{2}}$. The radiation energy is represented by the thermal energy of the resonator.

The energy levels of such a resonator can be determined from Schroedinger's equation (Schiff 1954) as:

$$\epsilon_\nu = (f_\nu + \tfrac{1}{2})\,h\nu \qquad f_\nu = 0, 1, 2, \cdots$$

Notice that the energy of the lowest quantum state is $\frac{1}{2}h\nu$. This energy is often referred to as the zero point energy of the resonator.

In section 2.1.1 we derived an expression for the number of distinguishable standing waves, or resonators, in a frequency interval $d\nu$ about ν, in a volume V of electromagnetic radiation. We obtained the result:

$$g_\nu = \frac{8\pi V \nu^2 d\nu}{v^3}$$

Knowing the number of resonators and the energy per resonator in the frequency interval $d\nu$ about ν, we can obtain the mean radiation energy in this interval as:

$$\overline{E}_\nu = g_\nu \overline{\epsilon}_\nu = \left(\overline{f}_\nu + \tfrac{1}{2}\right) g_\nu h\nu$$

The mean number of photons per resonator, $\overline{f}_\nu$, has been determined previously in section 2.1.2 as:

$$\overline{f_\nu} \equiv \frac{\overline{n_\nu}}{g_\nu} = \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$

As before, the mean total radiation energy is obtained by integrating $\overline{E_\nu}$ over all frequencies. Immediately we see that this integration will lead to the result obtained for E in section 2.1.3 where the zero point energy was omitted, plus an infinite term arising from the zero point energy of the resonator. Note that this infinite term exists even when $T = 0$.

This infinite zero point energy of the radiation field is only one of several infinite additive terms which arise in quantum electrodynamics and have not been explained or satisfactorily avoided. One hesitates to omit the zero point energy as it arises from wave mechanics and is connected with the uncertainty principle. In fact, in view of the successes achieved by these theories one is almost forced to accept the zero point energy. In addition, there exist several observable effects of the zero point energy in other materials. One of these can be seen in the case of liquid helium where the zero point energy is sufficient to keep the helium from solidifying under its own vapour pressure in the region of $T = 0$. A second example is the scattering of X - rays by the zero point vibrations of a crystal lattice in the region of $T = 0$. Also several observable effects of the interaction between electrons and the zero point energy of the electromagnetic field have been discussed by Welton (1948). Among these is the displacement of the 2S energy level of hydrogen known as the Lamb shift.

Therefore it appears as though we must accept the zero point energy along with the resulting theoretically infinite mean energy of the radiation field. It should be noted that this infinite additive term included in the mean energy is a purely theoretical difficulty. As one would expect, this infinite energy is not observable because of the fact that any procedure used to measure this energy of the radiation field introduces a finite upper frequency cutoff. Welton (1948) and Weber (1956) both discuss the factors which determine this frequency cutoff.

3.2 Fluctuations.

3.2.1 Energy fluctuations.

Intuitively one would not expect a change in the zero point of the energy to affect the energy fluctuations. The Einstein-Fowler equation for the thermal energy fluctuations supports this intuitive reasoning. Recall the expression for the mean thermal energy of a resonator:

$$\bar{\epsilon}_\nu = \frac{h\nu}{2} + \frac{h\nu}{\exp(\frac{h\nu}{kT})-1} = \frac{h\nu}{2}\coth\frac{h\nu}{2kT}$$

The Einstein-Fowler equation is:

$$\operatorname{var} E = kT^2\frac{d\bar{E}}{dT}$$

Therefore we obtain for the thermal energy fluctuation of a resonator:
$$\operatorname{var}\epsilon_\nu = \frac{(h\nu)^2\exp(\frac{h\nu}{kT})}{\left[\exp(\frac{h\nu}{kT})-1\right]^2} = \left(\frac{h\nu}{2\sinh\frac{h\nu}{2kT}}\right)^2$$

Note that this result is not affected by the zero point energy but that the form of the relation between $\operatorname{var}\epsilon_\nu$ and $\bar{\epsilon}_\nu$ is:

$$\mathrm{var}\ \epsilon_\nu = \left(\bar{\epsilon}_\nu - \frac{h\nu}{2}\right)^2 e^{\frac{h\nu}{kT}} = \left(\bar{\epsilon}_\nu\right)^2 - \left(\frac{h\nu}{2}\right)^2$$

3.2.2 Quantum modification of Nyquist's theorem.

An interesting point arises in connection with the quantum modification of Nyquist's theorem. Nyquist has shown that the spectral density of the voltage fluctuation across a resistance R(f) at temperature T is given by:

$$S_v(f) = 4kTR(f) \qquad f \ll \frac{kT}{h}$$

Planck's modification of this theorem was to replace kT, the mean thermal energy of a resonator for classical frequencies, by $hf\left(e^{\frac{hf}{kT}}-1\right)^{-1}$, the Planck mean thermal energy of a resonator for quantum frequencies. However the suggestion has been put forth that kT be replaced by $hf\left(e^{\frac{hf}{kT}}-1\right)^{-1} + \frac{hf}{2}$ the mean thermal energy of a resonator including the zero point term, for the quantum case. Therefore:

$$S_v(f) = 4R(f)\left[\frac{hf}{2} + \frac{hf}{\exp(\frac{hf}{kT})-1}\right]$$

The variance of the voltage across R(f),

$$\mathrm{var}\ V = \int_0^\infty S_v(f)\,df$$

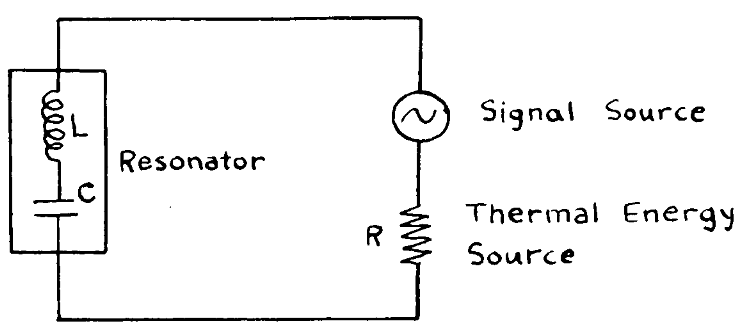
now contains an integral of the form; $\int_0^\infty f\,R(f)\,df$

From a purely theoretical standpoint we could choose R(f) to be frequency independent and the variance of V would diverge. However, as in the mean energy case, this theoretically infinite voltage fluctuation is not observable because of the finite cutoff frequency introduced by any procedure used to observe the fluctuations. In addition a frequency independent resistance is physically unattainable and it is possible that R(f) will vary as f^{-n} where n is greater than two. This frequency dependence of R(f) will result in a finite value of the voltage fluctuation.

This quantum modification of Nyquist's theorem is merely a different approach to the mean energy problem, since the mean energy is proportional to the mean square of the voltage. The fact that the voltage fluctuations depend upon the zero point energy is not in disagreement with the Einstein-Fowler equation which is concerned with energy fluctuations. We have shown that the energy fluctuations are not affected by the zero point energy.

3.2.3 The fluctuation of the energy of a resonator when both signal energy and thermal energy are present.

The following electrical representation of a resonator shall be used:



First consider the resonator with only the signal energy present. The signal source is such that a voltage of the form $V_s \sin \omega t$ is produced across the capacitor. Therefore the current through the resonator will be:

$$i_s = C\frac{dV}{dt} = C\omega V_s \cos \omega t$$

The instantaneous electromagnetic energy in the resonator is:

$$\epsilon = \tfrac{1}{2}Li^2 + \tfrac{1}{2}CV^2$$

However $\omega = (LC)^{-\frac{1}{2}}$, and therefore $\epsilon_s = \frac{1}{2}CV^2$

Now in a similar way let us represent the thermal energy in the resonator when the signal energy is zero. Let the instantaneous voltage across the capacitor as a result of the thermal energy be: $V_T \sin(\omega t + \phi)$. Hence, following the treatment of the signal voltage, we obtain :

$$i_T = C\omega V_T \cos(\omega t + \phi)$$

$$\epsilon_T = \tfrac{1}{2}CV_T^2$$

We have now considred the signal energy and the thermal energy in a resonator, each when the other is absent. Let us now consider the situation when both energies are present simultaneously. The principle of superposition may be applied to the voltage across the capacitor:

$$V = V_s \sin \omega t + V_T \sin(\omega t + \phi)$$

and as before:

$$i = C\omega\left[V_s \cos \omega t + V_T \cos(\omega t + \phi)\right.$$

$$\epsilon = \tfrac{1}{2}CV^2 + \tfrac{1}{2}Li^2$$

$$= \tfrac{1}{2}C\left[V_s^2 + V_T^2 + 2V_sV_T \sin \omega t \sin(\omega t + \phi) + 2V_sV_T \cos \omega t \cos(\omega t + \phi)\right]$$

The following notation shall be used for averages. An average over one cycle of ωt shall be written as:

$$\langle F \rangle = \frac{1}{2\pi}\int_0^{2\pi} F(\omega t)\, d(\omega t)$$

An average over all phase angles ϕ, <u>in addition to</u> an average over one cycle of ωt, shall be denoted by:

$$\overline{F} = \frac{1}{2\pi}\int_0^{2\pi} \langle F(\phi)\rangle \, d\phi$$

The total energy of the resonator when averaged over one cycle of ωt becomes:

$$\langle \epsilon \rangle = \frac{C}{2}\left[\langle V_s^2 \rangle + \langle V_T^2 \rangle + \langle 2 V_s V_T \cos\phi \rangle \right]$$

The phase angle ϕ between the signal voltage and the thermal voltage can have any value from 0 to 2π, all values having equal probability. Therefore we must average over ϕ to obtain the mean energy. This leads to:

$$\overline{\epsilon} = \frac{C}{2}\left[\overline{V_s^2} + \overline{V_T^2} \right]$$

or $\overline{\epsilon} = \overline{\epsilon_s} + \overline{\epsilon_T}$ which is the expected result.

Now consider the second moment of ϵ. If we average ϵ^2 over one cycle of ωt and then average over all ϕ, as was done in the mean energy case above, we get finally:

$$\overline{\epsilon^2} = \frac{C^2}{4}\left[\overline{V_s^4} + \overline{V_T^4} + 4\,\overline{V_s^2 V_T^2} \right]$$

Since the thermal and signal energies are independent we have:

$$\overline{\epsilon^2} = \overline{\epsilon_s^2} + \overline{\epsilon_T^2} + 4\,\overline{\epsilon_s}\,\overline{\epsilon_T}$$

$$\operatorname{var} \epsilon = \overline{\epsilon^2} - (\overline{\epsilon})^2$$

Therefore: $\operatorname{var} \epsilon = \operatorname{var} \epsilon_s + \operatorname{var} \epsilon_T + 2\,\overline{\epsilon_s}\,\overline{\epsilon_T}$

This result is somewhat surprising. First, if a noise-free signal is applied, the energy fluctuation in the resonator increases with the signal energy. Secondly, the magnitude of the energy fluctuation depends upon the mean thermal energy and therefore upon the zero point energy of the resonator.

A result of Gabor (1950) brings to light what appears to be a strange coincidence. Gabor obtains the result for

$\text{var}\,\epsilon$ as we have done. However, for the mean thermal energy Gabor uses the Planck mean energy which we shall denote by $\overline{\epsilon_T'}$. That is: $\overline{\epsilon_T'} = \overline{\epsilon_T} - \frac{h\nu}{2}$. It is easily shown that; $\text{var}\,\epsilon_T' = \text{var}\,\epsilon_T = h\nu\,\overline{\epsilon_T'} + (\overline{\epsilon_T'})^2$

Gabor's interpretation of the equation for $\text{var}\,\epsilon$ can be written as: $\text{var}\,\epsilon = \text{var}\,\epsilon_s + \text{var}\,\epsilon_T' + 2\,\overline{\epsilon_s}\,\overline{\epsilon_T'}$

Gabor now states that a noise-free signal will be free of classical or wave interference noise only and still will be subject to the quantum fluctuations. That is to say, for a noise-free signal: $\text{var}\,\epsilon_s = h\nu\,\epsilon_s$

and: $\text{var}\,\epsilon = h\nu\,\epsilon_s + h\nu\,\overline{\epsilon_T'} + (\overline{\epsilon_T'})^2 + 2\epsilon_s\overline{\epsilon_T'}$

which can be written as:

$$\text{var}\,\epsilon = h\nu\,\overline{\epsilon} + 2\,\overline{\epsilon}\,\overline{\epsilon_T'} - (\overline{\epsilon_T'})^2$$

However it is our contention that the expression for $\text{var}\,\epsilon$ should be interpreted, in terms of the Planck mean energy $\overline{\epsilon_T'}$, as:

$$\text{var}\,\epsilon = \text{var}\,\epsilon_s + \text{var}\,\epsilon_T' + 2\,\overline{\epsilon_s}\left(\overline{\epsilon_T'} + \frac{h\nu}{2}\right)$$

That is, the zero point energy of the resonator should be included. Now we may assume a noise-free signal to be free of both classical and quantum noise, that is $\text{var}\,\epsilon_s = 0$ for a noise-free signal. Therefore we have:

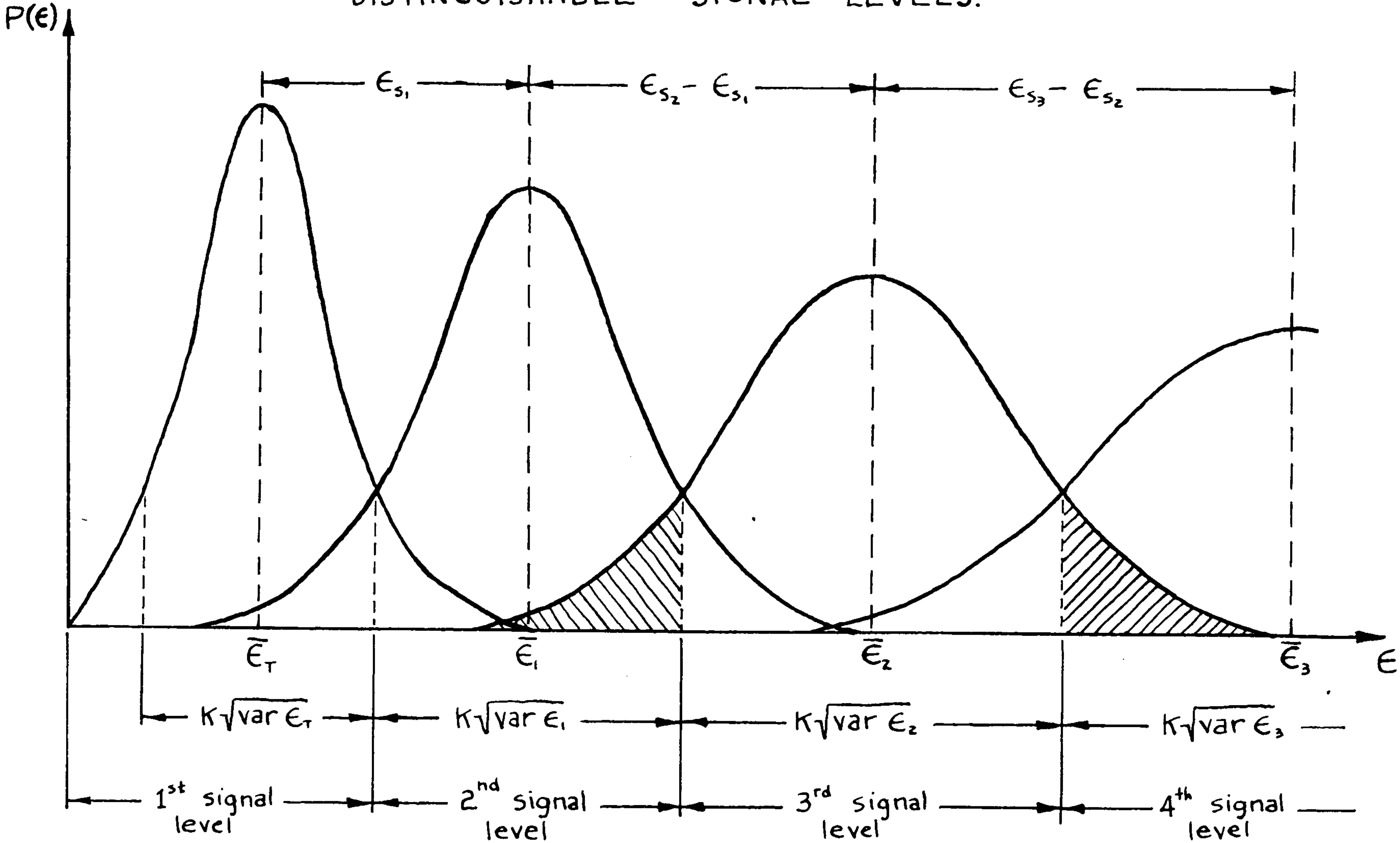
$$\text{var}\,\epsilon = h\nu\,\overline{\epsilon_T'} + (\overline{\epsilon_T'})^2 + 2\epsilon_s\left(\overline{\epsilon_T'} + \frac{h\nu}{2}\right)$$

or: $$\text{var}\,\epsilon = h\nu\,\overline{\epsilon} + 2\,\overline{\epsilon}\,\overline{\epsilon_T'} - (\overline{\epsilon_T'})^2$$

which is equivalent to the result obtained by Gabor. The fact that the term lost by omitting the zero point energy of the resonator, can be regained by assuming quantum noise to be present in the signal, appears to be a coincidence.

FIGURE 1.

DISTINGUISHABLE SIGNAL LEVELS.



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3.2.4 The signal energy required for L distinguishable signal levels.

The definition of a distinguishable signal level is somewhat arbitrary. It is generally agreed that the signal energy must be greater than or equal to the square root of the energy fluctuations in order that a distinguishable level exist. Figure 1 shows the first few distinguishable levels as we shall choose to define them. K is a constant which is determined by the allowable error. From figure 1 we see that:

$$\epsilon_{s_1} = \frac{1}{2}\left[K\sqrt{\operatorname{var}\epsilon_T} + K\sqrt{\operatorname{var}\epsilon_1}\right]$$

and in general:

$$\epsilon_{s_i} - \epsilon_{s_{i-1}} = \frac{1}{2}\left[K\sqrt{\operatorname{var}\epsilon_{i-1}} + K\sqrt{\operatorname{var}\epsilon_i}\right]$$

We have seen in section 3.2.3 that for a noise-free signal:

$$\operatorname{var}\epsilon_i = \operatorname{var}\epsilon_T + 2\,\overline{\epsilon}_T\,\epsilon_{s_i}$$

Using this expression for $\operatorname{var}\epsilon_1$ in the preceding equation for ϵ_{s_1}, we obtain:

$$\frac{2\,\epsilon_{s_1}}{\overline{\epsilon}_T} = K^2 + 2K\sqrt{\frac{\operatorname{var}\epsilon_T}{(\overline{\epsilon}_T)^2}}$$

and in general:

$$\frac{2\left(\epsilon_{s_i} - \epsilon_{s_{i-1}}\right)}{\overline{\epsilon}_T} = K^2 + 2K\sqrt{\frac{\operatorname{var}\epsilon_T}{(\overline{\epsilon}_T)^2} + \frac{2\,\epsilon_{s_{i-1}}}{\overline{\epsilon}_T}}$$

Recall the expressions for $\operatorname{var}\epsilon_T$ and $\overline{\epsilon}_T$ for a resonator from section 3.2.1:

$$\overline{\epsilon}_T = \frac{h\nu}{2}\coth\frac{h\nu}{2kT}$$

$$\operatorname{var}\epsilon_T = \left(\frac{h\nu}{2}\right)^2 \operatorname{cosech}^2\frac{h\nu}{2kT}$$

Therefore:

$$\frac{\operatorname{var}\epsilon_T}{(\overline{\epsilon}_T)^2} = \frac{1}{\cosh^2\frac{h\nu}{2kT}} \leq 1$$

With this result it is easily shown that:

$$\frac{2\epsilon_{s_1}}{\overline{\epsilon}_T} = \frac{K\left(K\cosh\frac{h\nu}{2kT} + 2\right)}{\cosh\frac{h\nu}{2kT}}$$

Similar expressions for $\epsilon_{s_2}\ \epsilon_{s_3}\cdots$ can be obtained in order. Finally a general expression for the energy required for L + 1 distinguishable levels can be obtained inductively;

$$\frac{\epsilon_{s_L}}{\overline{\epsilon}_T} = \frac{K^2L^2}{2} + \frac{KL}{\cosh\frac{h\nu}{2kT}}$$

In a signalling system a signal energy of zero can be used as a signal level. Hence we obtain L + 1 distinguishable signal levels from L nonzero signal energies. $\overline{\epsilon}_{s_L}$, the mean signal energy used when L + 1 distinguishable levels are available, can be calculated assuming that all signal levels have an equal probability of being used:

$$\frac{\overline{\epsilon}_{s_L}}{\overline{\epsilon}_T} = \frac{1}{L+1}\sum_{i=0}^{L}\frac{\epsilon_{s_i}}{\overline{\epsilon}_T}$$

This leads to the result:

$$\frac{\overline{\epsilon}_{s_L}}{\overline{\epsilon}_T} = \frac{KL}{12}\left[K(2L+1) + 6\,\mathrm{sech}\,\frac{h\nu}{2kT}\right]$$

K is determined by the allowable probability of an error. Chebyshev's inequality (Feller 1954) states that for a fluctuating variable x, with mean value $\overline{x}$, the probability per observation of finding x such that

$$|x - \overline{x}| \geq \frac{K}{2}\sqrt{\operatorname{var} x}$$

is less than $4/K^2$ no matter what the law of the fluctuations may be. This probability is, by our definition of distinguishable levels, just the probability of an error.

The probability of an error is represented in Figure 1. Consider the probability distribution for the energy of the resonator when a signal energy ϵ_{s_2} has been received. This probability distribution is the curve centred on $\overline{\epsilon}_2$ in Figure 1. The probability of an error, Q , is represented by the area of the shaded region under this curve. The area under the entire curve is equal to unity because of the normalization requirement. If the curve is Gaussian in form, that is if the probability distribution of the noise is Gaussian, we can calculate this probability of an error Q in terms of K.

$$P(\epsilon_2) = \frac{1}{\sqrt{2\pi \operatorname{var} \epsilon_2}} \exp\left[-\frac{(\epsilon_2-\overline{\epsilon}_2)^2}{2\operatorname{var}\epsilon_2}\right]$$

$$Q = 1 - 2\int_0^{\frac{K}{2}\sqrt{\operatorname{var}\epsilon_2}} \frac{1}{\sqrt{2\pi\operatorname{var}\epsilon_2}} \exp\left[-\frac{(\epsilon_2-\overline{\epsilon}_2)^2}{2\operatorname{var}\epsilon_2}\right] d(\epsilon_2-\overline{\epsilon}_2)$$

Put $x = \dfrac{\epsilon_2-\overline{\epsilon}_2}{\sqrt{2\operatorname{var}\epsilon_2}}$

$$Q = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{K}{2\sqrt{2}}} e^{-x^2}\,dx = \operatorname{cerf}\frac{K}{2\sqrt{2}}$$

It is interesting to compare the actual Q for Gaussian noise with the upper limit on Q given by the Chebyshev inequality, for several values of K.

K	$Q = \operatorname{cerf}\frac{K}{2\sqrt{2}}$ (Gaussian noise)	Upper bound on Q. (Chebyshev)
$2\sqrt{2}$	0.157	0.500
$4\sqrt{2}$	4.7×10^{-3}	0.125
$6\sqrt{2}$	2.2×10^{-5}	0.056

It is easily seen from this table that the probability of an error when the noise is Gaussian is well below the upper limit given by the Chebyshev inequality.

Shannon has developed the expression

$$L_{max} = \left(1 + \frac{\overline{\epsilon}_{s_L}}{\overline{\epsilon}_T}\right)^{\frac{1}{2}}$$

for the maximum number of distinguishable levels attainable with a negligible probability of an error per signal for a long message. $\overline{\epsilon}_{s_L}$ is the mean signal energy available and $\overline{\epsilon}_T$ is the mean noise energy present. This limit of Shannon's can be approached only with the optimum coding procedure and with a Gaussian probability distribution of the signal levels and of the noise.

For interest let us compare the number of distinguishable levels calculated on the basis of our definition with this upper limit given by Shannon. For $\frac{\overline{\epsilon}_{s_L}}{\overline{\epsilon}_T} \gg 1$:

$$L_{max} \cong \left(\frac{\overline{\epsilon}_{s_L}}{\overline{\epsilon}_T}\right)^{1/2} \approx \frac{KL}{\sqrt{6}}$$

where L is the number of levels calculated on the basis of our definition. We have neglected the extra level arising from the no signal condition since L has been assumed to be very much greater than one in the above approximation. Let us make the rather loose assumption that a Q of 4.7×10^{-3} constitutes a "negligible" probability of an error as considered by Shannon. We have seen earlier that if the noise is Gaussian, a Q of 4.7×10^{-3} can be attained with $K = 4\sqrt{2}$. Hence $L_{max} = \frac{KL}{\sqrt{6}} = 2.3\ L$ for $K = 4\sqrt{2}$. Therefore we see that even when we assume "negligible" to be 4.7×10^{-3}, and when we choose our noise to be Gaussian, the L we obtain is still considerably less than L_{max}, the maximum number of distinguishable levels attainable, as given by Shannon.

CHAPTER 4 - RADIATION DETECTORS

4.1 A Consideration of the Effects on the Detector Sensitivity of Factors which are not Fundamental Properties of the Detector.

4.1.1 The ideal energy detector.

In the above heading we refer to factors which are not fundamental properties of the detector. We intend to discuss three such factors; the procedure by which the detector output is observed, the signal waveform, and a low-pass filter between the detector and the observer. These factors will influence the noise level of the overall detection system. The expressions obtained for the sensitivity of the detection system will include quantitative measures of the smoothing effects of the low-pass filter and the observational procedure. However it is possible to obtain the sensitivity of the detector itself from these expressions by choosing the filter and the observational technique in such a way that they do not reduce the fluctuations of the detector output. The sensitivity of a detector is not discussed as such in the following work. Instead we have chosen to use var H, the uncertainty in our estimate of the incident flux, as a parameter to specify the detector performance. For real detectors we introduce the minimum detectable flux, defined as $(\text{var } H)^{\frac{1}{2}}$, as a performance specification for the detector. The sensitivity of the detector could very well be defined as $(\text{var } H)^{-\frac{1}{2}}$.

The ideal energy detector shall be used as an example for discussing these three factors mentioned above. The response of an energy detector depends upon the energy of the incident photons in contrast with a detector such as the photoelectric cell, the response of which depends upon the number of incident photons with energies above the thresh-hold level. An ideal energy detector is an energy detector in which the only source of noise is the temperature fluctuation of the sensitive element, known as temperature noise, caused by the fluctuations of the emitted and incident photon and phonon fluxes which comprise the thermal connection between the detector element and the surroundings. In addition we require that the detector element be in thermal equilibrium with the surroundings in a ideal energy detector.

We now shall obtain an expression for the response of the temperature of the detector element when a sinusoidally modulated power flux is incident upon the element. The notation which is to be used is:

C - heat capacity of the detector element. (joules deg.^{-1})

λ - total thermal conductance between the element and the surroundings. (watt-deg.^{-1})

A - sensitive area of the detector element. (meter^2)

$\tau = C/\lambda$ - thermal time constant of the detector. (sec.)

T - temperature of the detector element. (deg. Abs.)

T_a - temperature of the surroundings. (deg. Absolute)

$\Delta T = T - T_a$ - response of the element temperature to an input signal. (deg.)

$H(t)$ - signal power flux. (watts-meter^{-2})

ω_o - angular modulation frequency. (sec.$^{-1}$)

$F(t)$ - fluctuations of the total power incident upon and emitted by the detector element. (watts)

$\overline{F(t)} = 0$ (time average)

The differential equation for the temperature response of the element to a signal flux H(t) can be written as:

$$C\,\frac{d(\Delta T)}{dt} = -\lambda(\Delta T) + F(t) + AH(t)$$

If the signal power flux is of the form $H(t) = \frac{1}{2}H\,(1 + \cos\omega_o t)$ the solution of the differential equation is:

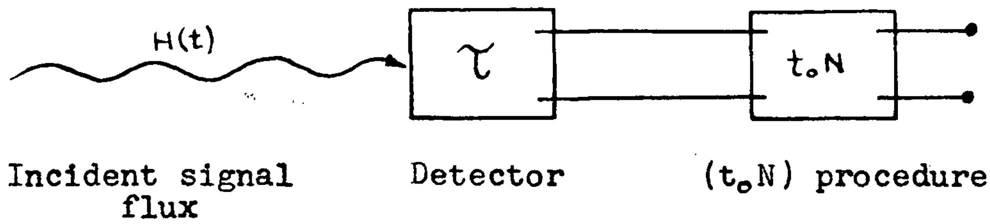
$$\Delta T = (\Delta T_o)e^{-\frac{t}{\tau}} - \frac{AH}{2\lambda}\,e^{-\frac{t}{\tau}}\left[1 + \frac{1}{1+\omega_o^2\tau^2}\right] + \frac{AH}{2\lambda}\left[1 + \frac{\cos(\omega_o t + \phi)}{\sqrt{1+\omega_o^2\tau^2}}\right] + e^{-\frac{t}{\tau}}\int_0^t \frac{F(\eta)}{C}\,e^{\eta/\tau}\,d\eta \qquad (4.01)$$

where $(\Delta T)_o$ is the value of ΔT at $t = 0$.

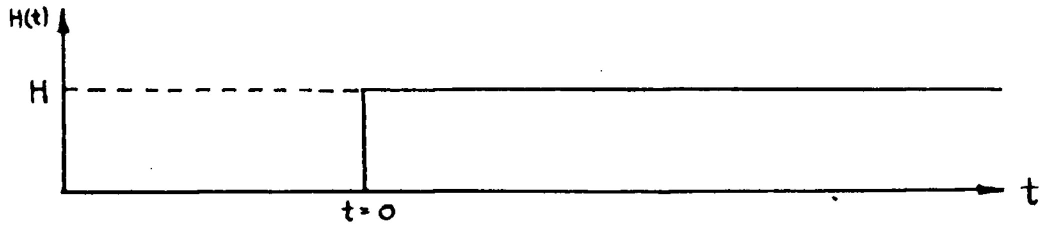
4.1.2 A step function signal and the $(t_o N)$ procedure.

The observational procedure to be used throughout our discussion consists of observing the detector output N times at equally spaced intervals, each of duration t_o/N. This sampling technique shall be referred to as the $(t_o N)$ procedure. In the ideal detector we assume that the temperature response ΔT can be observed directly. In a real detector it would be necessary to observe a secondary effect such as the thermoelectric voltage in a thermocouple or the temperature resistive effect used in a bolometer.

In this discussion the following arrangement shall be considered:



H(t) will have the form:



The readings obtained from the $(t_o N)$ procedure will be denoted by: $\quad (\Delta T)_i \qquad i = 0, 1, 2, \cdots N$

When $\omega_o = 0$, the general temperature response equation, 4.01, can be written:

$$\Delta T = (\Delta T_o)\, e^{-\frac{t}{\tau}} + \frac{AH}{\lambda}\left(1 - e^{-\frac{t}{\tau}}\right) + e^{-\frac{t}{\tau}} \int_0^t \frac{F(\eta)}{C}\, e^{\frac{\eta}{\tau}}\, d\eta$$

For the i^{th} reading:

$$(\Delta T)_i - (\Delta T)_o\, e^{-\frac{it_o}{N\tau}} = \frac{AH}{\lambda}\left(1 - e^{-\frac{it_o}{N\tau}}\right) + e^{-\frac{it_o}{N\tau}} \int_0^{\frac{it_o}{N\tau}} \frac{F(\eta)}{C}\, e^{\frac{\eta}{\tau}}\, d\eta \qquad (4.02)$$

From this i^{th} reading, the estimate of H which can be made is:

$$(H_i)_{est} = \frac{\lambda}{A} \frac{\left[(\Delta T)_i - (\Delta T)_o\, e^{-\frac{it_o}{N\tau}}\right]}{1 - e^{-\frac{it_o}{N\tau}}}$$

Now in our observational procedure we take N such readings and hence we could make N such estimates of H and average these for a best estimate of H. However, since the temperature fluctuation is present to the same extent for all the readings and the temperature response to the signal is increasing in proportion to $1 - e^{-\frac{it_o}{N\tau}}$, we can see that the estimates of H become more accurate as the time increases. Therefore we should give more weight to the later readings in our averaging system by a factor proportional to ΔT. Let us define the best estimate of H for the N readings taken in time t_o as:

$$H_N \equiv L \sum_{i=1}^{N} \left(1 - e^{-\frac{it_o}{N\tau}}\right)\left(H_i\right)_{est.}$$

where L is a normalization constant. If we were to repeat our process over many time intervals t_o and average our results, we would expect:

$$\begin{aligned} \overline{H_N} &= H \\ &= L \overline{\sum_{i=1}^{N} \frac{\lambda}{A}\left[(\Delta T)_i - (\Delta T)_o e^{-\frac{it_o}{N\tau}}\right]} \\ &= L \sum_{i=1}^{N} H \left(1 - e^{-\frac{it_o}{N\tau}}\right) \end{aligned}$$

from 4.02, since $\overline{F(t)} = 0$. Therefore:

$$L = \frac{1}{\sum_{i=1}^{N}\left(1 - e^{-\frac{it_o}{N\tau}}\right)}$$

Consider the second moment of H :

$$\begin{aligned} H_N^2 &= L^2 \overline{\sum_{i=1}^{N}\sum_{j=1}^{N} \left(1 - e^{-\frac{it_o}{N\tau}}\right)\left(H_i\right)_{est} \cdot \left(1 - e^{-\frac{jt_o}{N\tau}}\right)\left(H_j\right)_{est}} \\ &= L^2 \overline{\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\lambda^2}{A^2}\left[(\Delta T)_i - (\Delta T)_o e^{-\frac{it_o}{N\tau}}\right]\left[(\Delta T)_j - (\Delta T)_o e^{-\frac{jt_o}{N\tau}}\right]} \end{aligned}$$

From equation 4.02 we can write:

$$\begin{aligned} \overline{H_N^2} = {} & L^2 \sum_{i=1}^{N}\sum_{j=1}^{N} H^2 \left(1 - e^{-\frac{it_o}{N\tau}}\right)\left(1 - e^{-\frac{jt_o}{N\tau}}\right) + \\ & L^2 \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\lambda^2}{A^2C^2} e^{-\frac{(i+j)t_o}{N\tau}} \int_0^{\frac{it_o}{N}}\int_0^{\frac{jt_o}{N}} \overline{F(\eta)F(\xi)}\, e^{\frac{\xi+\eta}{\tau}}\, d\eta\, d\xi \end{aligned}$$

The first term reduces to $H^2 = (\overline{H_N})^2$. Therefore:

$$\text{var } H_N = L^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\lambda^2}{A^2C^2} e^{-\frac{(i+j)t_o}{N\tau}} \int_0^{\frac{it_o}{N}} \int_0^{\frac{jt_o}{N}} \overline{F(\eta)F(\zeta)}\, e^{\frac{\zeta+\eta}{\tau}} d\eta\, d\zeta$$

Uhlenbeck and Ornstein (1930) have shown:

$$\overline{F(\eta)F(\zeta)} = 2kT^2\lambda\, \delta(\eta-\zeta)$$

Since $\delta(\eta-\zeta)$ is equal to zero when $\zeta \neq \eta$ we need only integrate up to the smaller of the two limits for both integrations. Let us denote the smaller value of i or j by m. Introducing the new variables:

$$y = \eta + \zeta \qquad\qquad z = \eta - \zeta$$

the integral in the expression for var H becomes

$$\frac{1}{2}\int_0^{\frac{2mt_o}{N}} e^{\frac{y}{\tau}} dy \int_{\substack{\text{over the}\\ \delta \text{ function}}} 2kT^2\lambda\, \delta(z)\, dz = kT^2 C\left(e^{\frac{2mt_o}{N\tau}} - 1\right)$$

Therefore:

$$\text{var } H_N = L^2 \frac{kT^2\lambda^2}{A^2C} \sum_{i=1}^{N} \sum_{j=1}^{N} e^{-\frac{(i+j)t_o}{N\tau}} \left(e^{\frac{2mt_o}{N\tau}} - 1\right)$$

Since i and j are symmetrical we may write this expression as:

$$\text{var } H_N = \frac{L^2kT^2\lambda^2}{A^2C}\left[2\sum_{i=1}^{N}\sum_{j=1}^{i} e^{-\frac{(i+j)t_o}{N\tau}}\left(e^{\frac{2jt_o}{N\tau}} - 1\right) - \sum_{i=1}^{N}\left(1 - e^{-\frac{2it_o}{N\tau}}\right)\right]$$

Recalling the value of L, the normalization constant, and making use of the expression for the sum of a finite geometric progression, we can write the above equation as:

$$\text{var } H_N = \frac{kT^2\lambda^2}{A^2C}\left\{\frac{N\left(1-e^{-\frac{2t_o}{N\tau}}\right) - 2e^{-\frac{t_o}{N\tau}}\left(1+e^{-\frac{t_o}{N\tau}}\right)\left(1-e^{-\frac{t_o}{\tau}}\right) + e^{-\frac{2t_o}{N\tau}}\left(1-e^{-\frac{2t_o}{\tau}}\right)}{\left[N\left(1-e^{-\frac{t_o}{N\tau}}\right) - e^{-\frac{t_o}{N\tau}}\left(1-e^{-\frac{t_o}{\tau}}\right)\right]^2}\right\} \qquad (4.03)$$

In order to maximize the information obtained per reading of the $(t_o N)$ procedure, it is necessary to minimize the correlation between each of the individual readings. This is accomplished by choosing t_o and N so that $N \ll t_o/\tau$.

With this choice of t_o and N, equation 4.03 can be written:

$$\operatorname{var} H_N = \frac{kT^2\lambda^2}{A^2 C}\cdot\frac{1}{N} \qquad \left(N \ll \frac{t_o}{\tau}\right) \tag{4.04}$$

As N approaches infinity we have a continuous averaging process for which 4.03 becomes:

$$\operatorname{var} H_\infty = \frac{kT^2\lambda^2}{A^2 C}\left\{\frac{2t_o/\tau - 3 + 4e^{-\frac{t_o}{\tau}} - e^{-\frac{2t_o}{\tau}}}{\left[t_o/\tau - (1 - e^{-\frac{t_o}{\tau}})\right]^2}\right\} \tag{4.05}$$

In the continuous case it is desirable to have little or no correlation between the initial and final readings. This requires t_o to be $\gg \tau$ for which 4.05 becomes:

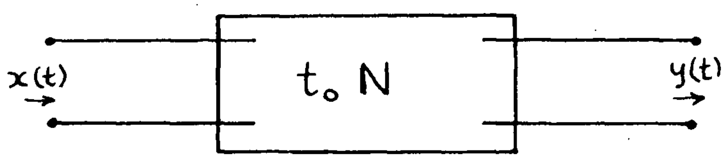
$$\operatorname{var} H_\infty = \frac{2kT^2\lambda}{A^2 t_o} \qquad (t_o \gg \tau) \tag{4.06}$$

Therefore we have obtained a general expression for the uncertainty in our estimate of H, given by 4.03, for a step function input signal and using the (t_oN) procedure. Equations 4.04, 4.05, and 4.06 give the uncertainty in our estimate of H for certain special cases of the (t_oN) procedure.

The problem we have just discussed has been considered by Dahlke and Hettner (1941) and Kappler (1946). The latter paper contains an expression for the detector sensitivity which becomes infinite as the observation time approaches zero. This is obviously an unsatisfactory tendency. We believe that our treatment, which has considered a more general observational procedure, is an improvement upon the analyses of the above authors.

4.1.3 The frequency response function $\alpha(f)$, and the equivalent bandwidth B_o, of the $(t_o N)$ sampling procedure.

In the previous section, because of the transient nature of the temperature response, we found it necessary to introduce a weighting factor in averaging the readings of our $(t_o N)$ procedure. Here we shall consider the $(t_o N)$ procedure for the case when all readings are given equal weight as would be the case in a steady state situation. Consider the following arrangement:

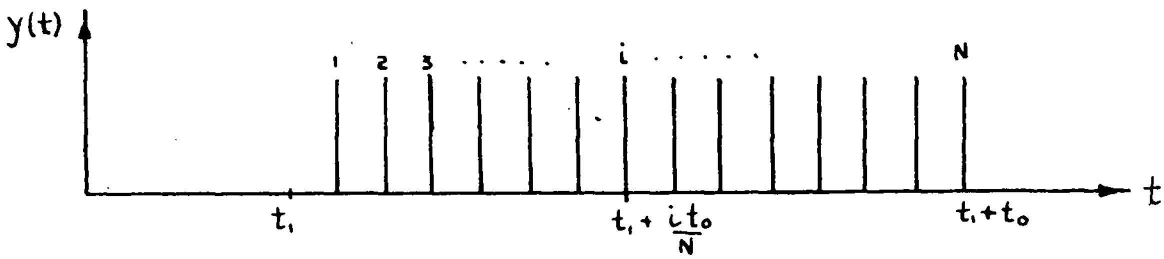


From the definition of our $(t_o N)$ procedure:

$$y(t) = \frac{1}{N} \sum_{n=1}^{N} x\left(t - \frac{nt_o}{N}\right)$$

Consider the response to an impulse, $x(t) = \delta(t - t_1)$

$$y(t) = \frac{1}{N} \sum_{n=1}^{N} \delta\left(t - t_1 - \frac{nt_o}{N}\right)$$



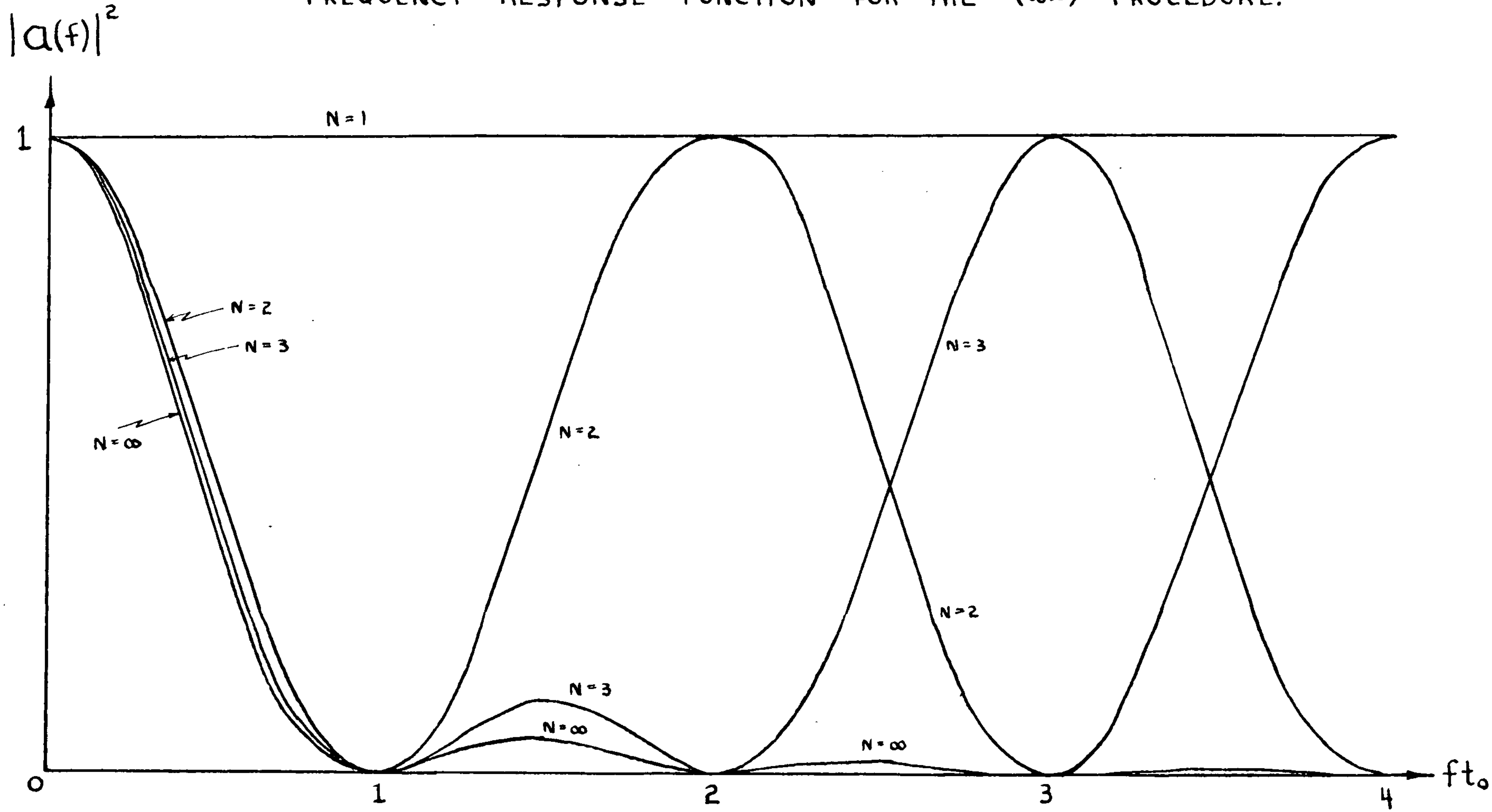
The frequency response of any system can be expressed as:

$$\alpha(f) = \int_0^\infty J(t) \exp(-2\pi i f t)\, dt$$

where J(t) is the response of the system to a unit impulse.

FIGURE 2

FREQUENCY RESPONSE FUNCTION FOR THE (t_0N) PROCEDURE.



Therefore for the $(t_o N)$ procedure:

$$a(f) = \sum_{n=1}^{N} \frac{1}{N} \int_0^{t_o} \delta\left(t - t_1 - \frac{nt_o}{N}\right) e^{-i2\pi ft}\, dt$$

$$= \frac{1}{N} \left\{ \frac{e^{-i2\pi ft_o} - 1}{e^{-\frac{i2\pi ft_o}{N}} - 1} \right\} e^{i2\pi ft_1}$$

$$|a(f)|^2 = \frac{1}{N^2} \cdot \frac{2 - 2\cos(2\pi ft_o)}{2 - 2\cos\left(\frac{2\pi ft_o}{N}\right)}$$

$$|a(f)|^2 = \left(\frac{\sin \pi ft_o}{N \sin \frac{\pi ft_o}{N}} \right)^2 \qquad (4.07)$$

Refer to Figure 2 for a plot of $|a(f)|^2$ against ft_o for $N = 1, 2$, and ∞.

The equivalent bandwidth of a system is defined as:

$$B \equiv \frac{\int_0^\infty |a(f)|^2\, df}{|a(f_o)|^2}$$

where f_o is the signal frequency. For a dc signal:

$$|a(f_o)|^2 = |a(0)|^2 = 1$$

for the $(t_o N)$ procedure. Therefore the bandwidth of the $(t_o N)$ procedure for a dc signal is :

$$B_o = \int_0^\infty \left(\frac{\sin \pi ft_o}{N \sin \frac{\pi ft_o}{N}} \right)^2 df \qquad (4.08)$$

This integral is infinite for finite values of N. However for the continuous observation procedure, that is as $N \to \infty$,

$$B_o = \int_0^\infty \left(\frac{\sin \pi ft_o}{\pi ft_o} \right)^2 df = \frac{1}{2t_o}$$

The infinite bandwidth for a finite number of observations can be realized by considering the averaging effect of this process on the set of frequencies:

$$f = \frac{iN}{t_o} \qquad (i = 1, 2, 3, \cdots \infty)$$

For these frequencies our observations will be separated by exactly i wavelengths in the time plane. Since each reading is taken at the same phase position on the waveform, there is no cancellation and

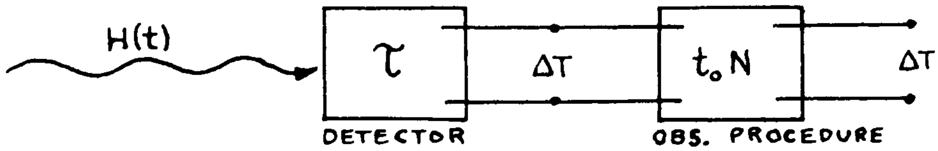
$$\left|a\left(\frac{it_o}{N}\right)\right|^2 = 1 \qquad (i = 0, 1, 2, \cdots \infty)$$

A filter which has unity transmission for an infinite number of discrete frequencies, must have an infinite bandwidth.

It should be noted that when a modulated signal is being observed by this $(t_o N)$ procedure, t_o, N, and f_o must be chosen so that $f_o = it_o / N$. (i is an integer). With this condition satisfied the signal will be passed undiminished whereas the amplitude of the random noise will be reduced.

4.1.4 A sinusoidally modulated signal being observed in a steady state condition.

In section 4.1.2 we studied the smoothing effect of the $(t_o N)$ procedure when the input signal H(t) was a step function, and we found that it was necessary to use a weighting factor when averaging the individual readings. In our present discussion the signal will be sinusoidally modulated and it will be assumed that the detector has been exposed to the signal for a time $t \gg \tau$. Therefore all transient effects may be disregarded and the aforementioned weighting factor will not be required. That is, the $(t_o N)$ procedure will have the frequency response function and the equivalent bandwidth given by 4.07 and 4.08 respectively. Once again we are considering:



where this time $H(t) = \frac{1}{2}H\cdot(1 + \cos\omega_o t)$.

We must now leave this specific situation briefly in order to calculate the spectral density of the temperature fluctuation of the detector element. In general, for a fluctuating variable x:

$$\text{var } x = \int_0^\infty S_x(f)df \equiv S_x(0)\cdot B_x$$

where B_x is the noise equivalent bandwidth. In addition, for a system characterized by a single time constant τ, it is easily shown that:

$$B_x = \frac{1}{4\tau} \qquad \text{and} \qquad S_x(f) = \frac{S_x(0)}{1+\omega^2\tau^2}$$

Therefore, for a single τ system:

$$S_x(f) = \frac{4\tau \text{ var } x}{1+\omega^2\tau^2}$$

We have seen earlier that the temperature fluctuation of the detector element is given by $\text{var } \Delta T = kT^2/C$. In addition we know that the temperature response is dependent upon the single time constant $\tau = C/\lambda$. Therefore the spectral density of the temperature fluctuation is:

$$S_{\Delta T}(f) = \frac{4kT^2/\lambda}{1+\omega^2\tau^2} \qquad (4.09)$$

Returning to our problem, let us define a smoothing factor for the ($t_o N$) procedure in the following way:

$$D \equiv \frac{\mathrm{var}\,\Delta T'}{\mathrm{var}\,\Delta T} = \frac{\int_0^\infty S_{\Delta T}(f)\cdot|a(f)|^2\,df}{\int_0^\infty S_{\Delta T}(f)\,df} \qquad (4.10)$$

$|a(f)|^2$ for the $(t_o N)$ procedure is given by equation 4.07 and $S_{\Delta T}(f)$ by 4.09. Therefore:

$$D_N = \frac{\int_0^\infty \frac{4kT^2}{\lambda}\cdot\frac{1}{1+\omega^2\tau^2}\cdot\left(\frac{\sin\pi f t_o}{N\sin\frac{\pi f t_o}{N}}\right)^2 df}{\int_0^\infty \frac{4kT^2}{\lambda}\cdot\frac{1}{1+\omega^2\tau^2}\cdot df}$$

Integration of this expression gives:

$$D_N = \frac{N\left(1-e^{\frac{-2t_o}{N\tau}}\right) - 2e^{\frac{-t_o}{N\tau}}\left(1-e^{\frac{-t_o}{\tau}}\right)}{N^2\left(1-e^{\frac{-t_o}{N\tau}}\right)^2} \qquad (4.11)$$

The integration of the numerator is shown in detail in Appendix 5. Equation 4.11 agrees with a result obtained by Burgess (1951).

If we require $t_o \gg \tau$, that is if we neglect all transient effects in the temperature response equation 4.01, we are left with:

$$\Delta T = \frac{AH}{2\lambda}\left[1 + \frac{\cos(\omega_o t + \phi)}{\sqrt{1+\omega_o^2\tau^2}}\right]$$

Recalling that the input signal was $H(t) = \frac{1}{2}H(1 + \cos\omega_o t)$, we can see that the frequency response function of the detector, $a_D(f)$, can be written:

$$a_D(f) \equiv \frac{\Delta T}{H/2} = \frac{\cos(\omega_o t + \phi)}{\sqrt{1+\omega_o^2\tau^2}}\cdot\frac{A}{\lambda}$$

and $$|a_D(f)|^2 = \frac{A^2}{\lambda^2}\left(1+\omega_o^2\tau^2\right)^{-1} \qquad (4.12)$$

From the definition of $a_D(f)$ and D_N, it is apparent that the uncertainty in our estimate of H is equal to:

$$\text{var}\, H_N = \frac{\text{var}\,\Delta T}{|a_o(f)|^2} \cdot D_N$$

That is:

$$\text{var}\, H_N = \frac{kT^2\lambda^2}{A^2C}\left(1+\omega_o^2\tau^2\right)\left[\frac{N\left(1-e^{-\frac{2t_o}{N\tau}}\right)-2e^{-\frac{t_o}{N\tau}}\left(1-e^{-\frac{t_o}{\tau}}\right)}{N^2\left(1-e^{-\frac{t_o}{N\tau}}\right)^2}\right] \qquad (4.13)$$

As in section 4.1.2, for as little correlation between individual readings as possible, we require $N \ll \frac{t_o}{\tau}$. When this condition is applied to 4.13 we obtain:

$$\text{var}\, H_N = \frac{kT^2\lambda^2}{A^2C}\left(1+\omega_o^2\tau^2\right)\cdot\frac{1}{N} \qquad \left(N \ll \frac{t_o}{\tau}\right) \qquad (4.14)$$

For the continuous averaging procedure we let N approach infinity and 4.13 becomes:

$$\text{var}\, H_\infty = \frac{kT^2\lambda^2}{A^2C}\left(1+\omega_o^2\tau^2\right)\left[\frac{2\tau^2}{t_o^2}\left(\frac{t_o}{\tau}-1+e^{-\frac{t_o}{\tau}}\right)\right] \qquad (4.15)$$

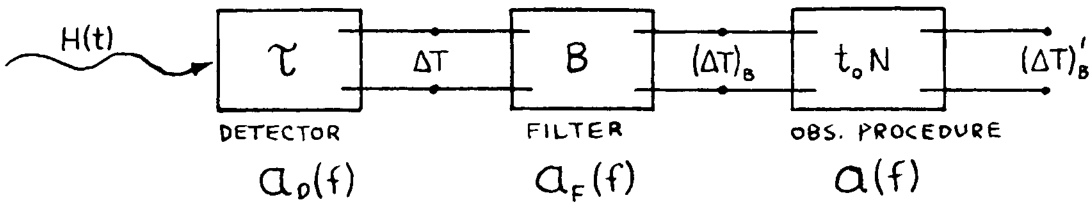
In the continuous case in order to have as little correlation between the readings taken at $t = 0$ and $t = t_o$, as possible, we require that $t_o \gg \tau$. Therefore from 4.15 we get:

$$\text{var}\, H_\infty = \frac{2kT^2\lambda}{A^2t_o}\left(1+\omega_o^2\tau^2\right) \qquad (t_o \gg \tau) \qquad (4.16)$$

When $\omega_o = 0$, equations 4.14 and 4.16 are equal to 4.04 and 4.06 respectively in section 4.1.2. Recall that in section 4.1.2 we considered a step function signal with the accompanying transient effects. However these transient effects are removed when we insist that $N \ll \frac{t_o}{\tau}$, or $t_o \gg \tau$ in the continuous case, as we have done in equations 4.04 and 4.06 respectively, and the problem is reduced to considering the response to a steady state dc signal. Similarly when $\omega_o = 0$ this present problem also is a consideration of a steady state dc signal. Hence the similarity between the results of this section and section 4.1.2 could have been expected.

4.1.5 The addition of a filter to the detection system.

The detection system to be considered in this section is the same as in the previous section except that a low-pass filter has been added between the detector and the observer.



The frequency response function of the filter is:

$$|a_F(f)|^2 = 1 \qquad 0 \leq f \leq B$$

$$|a_F(f)|^2 = 0 \qquad B < f < \infty$$

As our detector is characterized by the single time constant τ , the effective bandwidth of the detector is $B_\tau = 1/4\tau$. It is obvious that our filter will have the greatest smoothing effect on the detector output noise when $B \ll B_\tau$, and this is the situation we shall consider. The lower limit on B is usually determined by the modulation frequency f_o which is to be detected. If the detector is used primarily to observe dc signals the bandwidth B must be sufficiently large to permit the signal response to build up or decay in a reasonably short time.

We have shown in section 4.1.4 that the spectral density of the temperature fluctuation of the detector element is: $$S_{\Delta T}(f) = \frac{\frac{4kT^2}{\lambda}}{1+\omega_o^2\tau^2} \qquad (4.09)$$

The spectral density of the temperature fluctuation appearing at the output of the filter will be:

$$S_{(\Delta T)_B}(f) = \frac{4kT^2}{\lambda} \qquad 0 \le f \le B \quad . \quad \text{since} \quad B\tau \ll 1,$$

$$S_{(\Delta T)_B}(f) = 0 \qquad B < f < \infty$$

Therefore: $$\operatorname{var}(\Delta T)_B = \int_0^\infty S_{(\Delta T)_B}(f)\,df$$

$$= \frac{4kT^2B}{\lambda} = \frac{kT^2}{C}\cdot\frac{B}{B_\tau}$$

The smoothing effect of the ($t_o N$) procedure, defined by 4.10, will have the form:

$$(D_N)_B = \frac{\int_0^\infty S_{(\Delta T)_B}(f)\,|a(f)|^2\,df}{\int_0^\infty S_{(\Delta T)_B}(f)\,df}$$

which leads to:

$$(D_N)_B = \frac{\int_0^B \left(\frac{\sin \pi f t_o}{N \sin \frac{\pi f t_o}{N}}\right)^2 df}{B} \qquad (4.17)$$

From equations 4.10 and 4.12 it is apparent that:

$$\operatorname{var}(H_N)_B = \frac{\operatorname{var}(\Delta T)_B}{|a_D(f)|^2}\cdot (D_N)_B$$

From this we obtain:

$$\operatorname{var}(H_N)_B = \frac{kT^2\lambda^2}{A^2C}\cdot\frac{B}{B_\tau}\left(1+\omega_o^2\tau^2\right)\left[\frac{\int_0^B\left(\frac{\sin \pi f t_o}{N \sin\frac{\pi f t_o}{N}}\right)^2 df}{B}\right] \qquad (4.18)$$

Since we have required B to be $\ll B_\tau$, the requirement for very small correlation between the individual readings of our ($t_o N$) procedure becomes: $N \ll Bt_o$. When this condition is applied to 4.18 we have:

$$\operatorname{var}(H_N)_B = \frac{kT^2\lambda^2}{A^2C}\cdot\frac{B}{B_\tau}\cdot\left(1+\omega_o^2\tau^2\right)\cdot\frac{1}{N} \qquad (N \ll Bt_o) \qquad (4.19)$$

It should be noted that 4.19 is exact for:

$$\frac{Bt_o}{N} = \frac{i}{2} \qquad (i = 1, 2, 3, \cdots \infty)$$

For the case of continuous observation, that is as $N \to \infty$ in 4.18, we obtain:

$$\mathrm{var}\left(H_\infty\right)_B = \frac{kT^2\lambda^2 B}{A^2 C B_\tau}\cdot\left(1+\omega_o^2\tau^2\right)\left[\frac{1}{\pi t_o B}\left\{\frac{\cos(2\pi t_o B)-1}{2\pi t_o B} + Si(2\pi t_o B)\right\}\right]$$

where $Si(x) = \int_0^x \frac{\sin\alpha}{\alpha}\,d\alpha$ (4.20)

In order that there be little or no correlation between the initial and final readings in the continuous observation process we require $Bt_o \gg 1$. This condition reduces 4.20 to:

$$\mathrm{var}\left(H_\infty\right)_B = \frac{2kT^2\lambda}{At_o}\left(1+\omega_o^2\tau^2\right) \qquad (Bt_o \gg 1) \qquad (4.21)$$

Again we have obtained a general expression for the uncertainty in our estimate of H given by 4.18. Recall that the situation under discussion consisted of a sinusoidally modulated steady state signal, a low-pass filter at the detector output, and our (t_oN) procedure following the filter. In addition we have calculated var H for three special cases of the (t_oN) procedure; see equations 4.19, 4.20, and 4.21.

Notice that equation 4.21 is identical to 4.16, that is the filter has had no effect on the noise level. This stems from the requirement that $Bt_o \gg 1$. Recall the equivalent bandwidth of the continuous (t_oN) procedure from section 4.1.3 as $B_o = 1/2t_o$. Hence it can be seen that B_o, the bandwidth of the (t_oN) procedure, is much less than B, the bandwidth of the filter when $Bt_o \gg 1$, and the smoothing effect of the filter has been obscured by the much greater smoothing effect of the observational procedure.

4.2 The Specification of a Real Detector.

The discussion of detectors up to this point has been concerned with the effect on the sensitivity of an ideal energy detector of such factors as the detector output sampling technique, the inclusion of a low-pass filter, and the modulation of the input signal. The ideal energy detector, a most unrealistic device, was used as an example in order to confine the discussion as much as possible to the effects of these factors mentioned above, by avoiding the numerous additional problems encountered in a real detector. We will now restrict our treatment as much as possible to the fundamental properties of a real detector.

In order to give a quantitative evaluation of the performance of a real detector, we shall introduce several quantities which can be used as a basis for comparing different types of detectors. The first of these is the frequency response function which is defined as:

$$a(f_0) = \frac{\text{Detector Output}}{\text{Signal Input}}$$

where f_0 is the modulation frequency of the signal. The parameters describing the detector output and the signal input are chosen so the detector behaves in a linear fashion. The frequency response is determined basically by the detector time constant τ , which in the ideal energy detector we saw was equal to c/λ. In the photoemissive detector, to be discussed later, τ is determined by the stray capacitance in the circuitry of the detector. The dc gain $a(0)$ depends upon many factors as will be shown in the discussion of the bolometer and the phototube. The frequency response function of the detector does not include the effects of output filters or observational techniques.

A second parameter used to specify the performance of a real detector is the minimum detectable flux or minimum detectable power. The optical system focussing the radiation on the detector element determines which of the two parameters, flux or power, is more suitable for a measure of the detector sensitivity. If the incident radiation flux varies appreciably over the area of the detector element, the minimum detectable power must be used. However if the detector element is in a region of uniform radiation flux, the minimum detectable flux is the more informative quantity. In order to avoid duplication of equations we shall consider the minimum detectable flux H_{min} only. The minimum detectable power P_{min} can be obtained directly from the relation: $P_{min} = AH_{min}$. The minimum detectable flux is defined as the signal flux which will result in a signal to noise ratio of unity at the output of the detection system. That is:

$$H_{min} = \sqrt{\text{var } H}$$

Var H is the uncertainty of our estimate of H, the signal flux, as a result of all noise sources which are significant for the detector under consideration. This minimum detectable flux will be affected by the smoothing of our observational technique and of the output filtering system. However, as we pointed out earlier, the minimum detectable power flux of the detector itself can be obtained by requiring that $B \gg B_\tau$ and that the smoothing factor of the $(t_o N)$ procedure be equal to unity. When these conditions are satisfied the filter and the $(t_o N)$ procedure do not reduce the fluctuations of the detector output, and therefore the minimum detectable flux of the detector alone can be obtained. This quantity is of little practical value, however, since for the optimum performance of a detection system we require that the noise level be as small as possible and a filter with bandwidth $B \ll B_\tau$ will reduce the noise considerably. We have defined the minimum detectable flux from the point of view of an energy detector in which the parameter describing the input signal is H, an energy flux. As a result the minimum detectable flux mentioned will actually be the minimum detectable energy flux for an energy detector. For the quantum detector the parameter describing the input signal is m, a particle or photon flux, and therefore in this case we will be concerned with $\ddot{m}_{min}$ the minimum detectable photon flux.

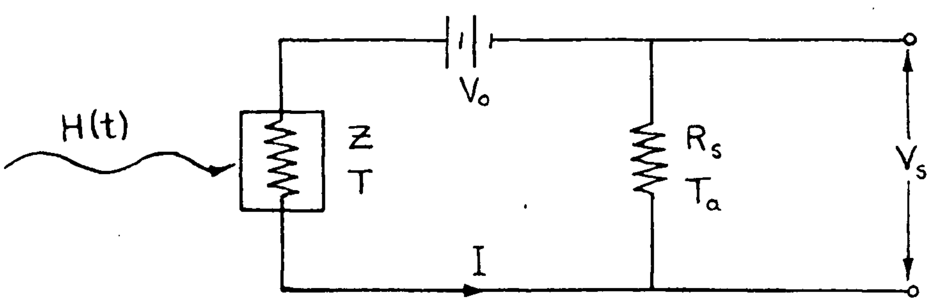
A third parameter which is used to specify the performance of a detector is the noise factor. We shall define the noise factor in the following way.

$$\eta = \frac{H_{min}\ (\text{real detector})}{H_{min}\ (\text{ideal detector})} > 1$$

In general, η will not depend on any factors such as observational procedure, which are not fundamental properties of the detector. However, the main disadvantage of the noise factor is that the minimum detectable flux which defines η will be different for an energy detector than for a particle detector. Therefore we shall consider η more fully in the discussion of the bolometer and the phototube.

4.3 The Bolometer.

The operation of the bolometer, an energy detector, depends upon the fact that the electrical resistance of most materials varies in an easily determined manner with the temperature of the material. The temperature change of the detector element as a result of the signal radiation, will cause a change in the electrical resistance of the element which can be observed as a voltage reading ΔV_s in the following circuit:



4.3.1 The frequency response function.

Let us choose an incident flux of the form:

$$H(t) = H_1 \exp(i\omega t)$$

This oscillatory flux will cause oscillations in T, Z, V_S, and I. Therefore let us write these quantities in the form:

$T = T_0 + T_1 \exp(i\omega t)$ $\qquad$ $Z = R_0 + R_1 \exp(i\omega t)$

$I = I_0 + I_1 \exp(i\omega t)$ $\qquad$ $V_S = V_{S_0} + V_{S_1} \exp(i\omega t)$

where we will assume that the amplitude of the oscillations is much less than the dc value in each case.

The dependence of Z upon T is given by the equation:

$$Z = R_a\left[1 + \alpha(T - T_a)\right]$$

R_a is the resistance of the bolometer element at the ambient temperature T_a. α is the temperature coefficient of resistance of the detector element and is defined by:

$$\alpha = \frac{1}{R}\frac{dR}{dT}$$

The dc and ac components of the above equation for Z give us respectively:

$$R_0 = R_a\left[1 + \alpha(T_0 - T_a)\right] \tag{4.22}$$

$$R_1 = \alpha R_a T_1 \tag{4.23}$$

Recall the form of the temperature response equation for the detector element:

$$C\frac{dT}{dt} + \lambda(T - T_a) = P$$

P is the total power dissipated in the detector element and consists of the radiation power plus the electrical heating power. $\qquad P = AH(t) + I^2 Z \qquad$ A is the sensitive area of the detector element. Note that we are representing the joule heating power by $I^2 Z$ and not the real part of $I^2 Z$, even though we shall see that both I and Z are complex. However the reactive components of I and Z are caused by the thermal inertia of the element combined with the feedback cycle $T \rightarrow Z \rightarrow I \rightarrow I^2 Z \rightarrow T$, and the detector element does not contain any inductive or capacitive elements capable of storing electromagnetic energy. Hence the joule heating power will be $I^2 Z$. We have:

$$I^2 Z = I_o^2 R_o + I_o^2 R_1 \exp(i\omega t) + 2I_o R_o I_1 \exp(i\omega t)$$

Therefore the dc and ac components of the temperature response equation are respectively:

$$\lambda(T_o - T_a) = I_o^2 R_o \qquad (4.24)$$

$$AH_1 + I_o^2 R_1 + 2I_o R_o I_1 = i\omega C T_1 + \lambda T_1 \qquad (4.25)$$

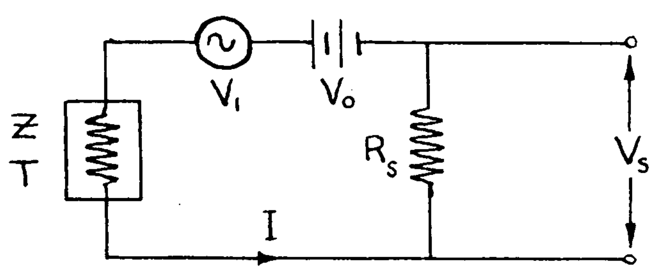
Also we have: $V_o = I(Z + R_s)$, the ac components of which are: $0 = I_o R_1 + I_1 (R_o + R_s)$ (4.26)

Finally we can see directly that: $V_{s_1} = I_1 R_s$ (4.27)

By combining equations 4.23, 4.25, 4.26, and 4.27 we can obtain the frequency response function for the detector.

$$a_o(f) = \frac{V_{s_1}}{H_1} = \frac{-\alpha I_o R_a A}{\lambda + i\omega C - I_o^2 R_a \alpha} \cdot \frac{R_s}{R_s + R_o \dfrac{(\lambda + i\omega C + I_o^2 R_a \alpha)}{\lambda + i\omega C - I_o^2 R_a \alpha}} \qquad (4.28)$$

The analysis we have just completed can be considered from a different point of view. The incident flux can be represented by a voltage generator V_1 in series with the bolometer as shown:

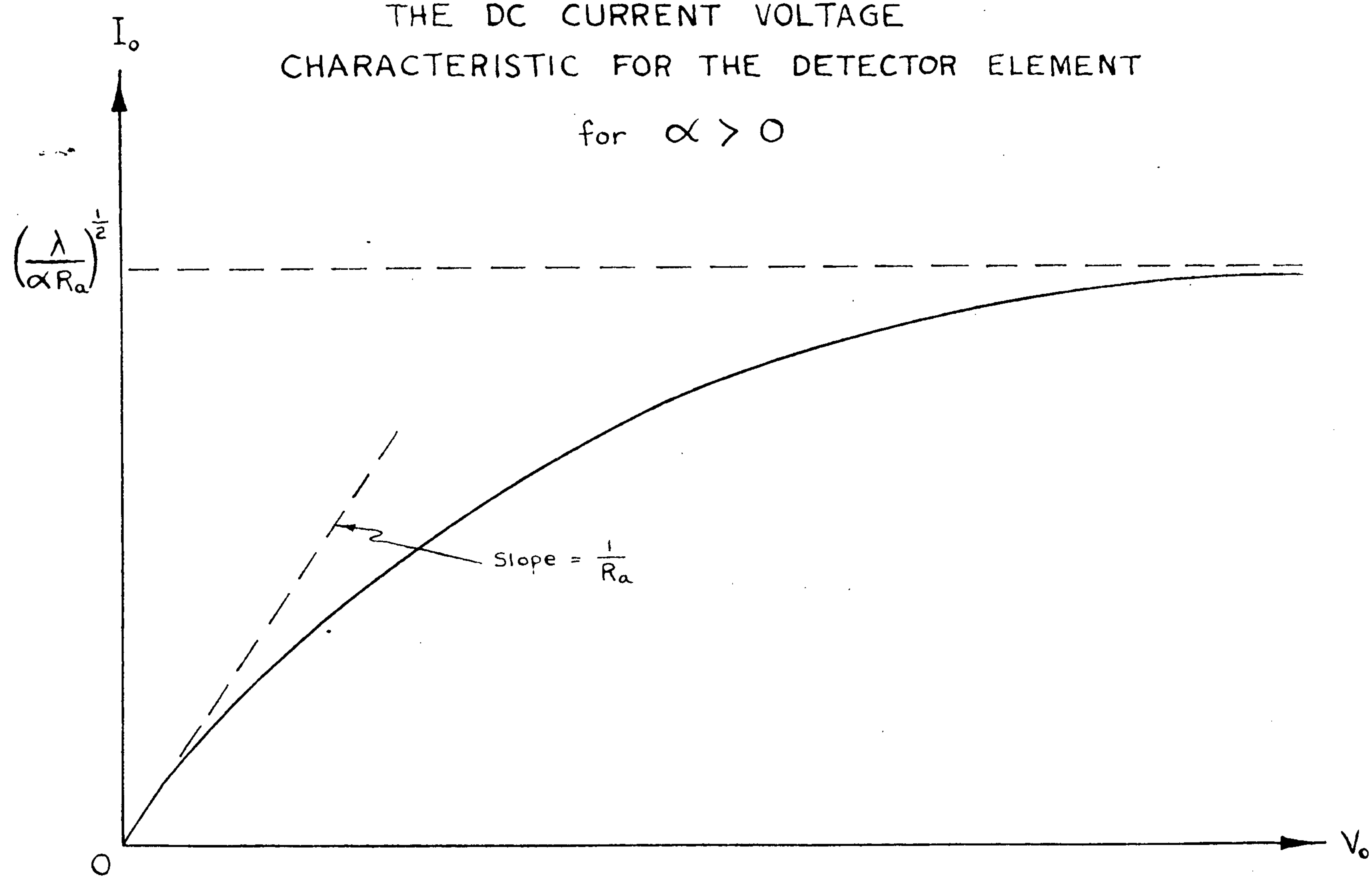


If $V_1 \ll V_o$, we have from circuit theory:

$$V_{s_1} = V_1 \cdot \frac{R_s}{R_s + Z}$$

FIGURE 3

THE DC CURRENT VOLTAGE CHARACTERISTIC FOR THE DETECTOR ELEMENT for $\alpha > 0$



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V_{S_1} is the response of the voltage V_S to the signal V_1. Comparing this expression with 4.28, the frequency response function, we can conclude that:

$$Z = \frac{R_o\left(\lambda + i\omega C + I_o^2 R_a \alpha\right)}{\lambda + i\omega C - I_o^2 R_a \alpha} \qquad (4.29)$$

In Appendix 6 the dc and ac impedances of the detector element are calculated using a direct circuit analysis approach. The ac impedance obtained is in agreement with equation 4.29. Figure 3 is a plot of the dc current voltage characteristic.

We may now write 4.28 as:

$$a_D(f) = -\frac{\alpha I_o R_a A}{\lambda + i\omega C - I_o^2 R_a \alpha} \cdot \frac{R_S}{R_S + Z}$$

$$|a_D(f)|^2 = \frac{(\alpha I_o R_a A)^2}{(\lambda - I_o^2 R_a \alpha)^2 + (\omega C)^2} \cdot \frac{R_S^2}{|R_S + Z|^2}$$

This expression can be written, using 4.22 and 4.24, in the form:

$$|a(f)|^2 = \frac{\alpha^2 R_o A^2 (T_o - T_a)}{\left[1 + \{1 + \alpha(T_o - T_a)\}^2 \omega^2 \tau^2\right] \lambda} \cdot \frac{R_S^2}{|R_S + Z|^2} \qquad (4.30)$$

4.3.2 The minimum detectable energy flux.

Let us first consider the output voltage fluctuation resulting from the temperature noise. From the definition of the frequency response function we can write:

$$S_{V_S}(f) = |a_D(f)|^2 \, S_H \qquad (4.31)$$

S_H is the spectral density of the fluctuation of the incident radiation energy flux. In section 2.2.3 we saw that:

$$\mathrm{var}\, H = \frac{8\pi^5 k^5 T^5}{15 h^3 v^2 A t_o}$$

This is an expression for the fluctuation of the black body radiation energy flux arriving at an area A and observed continuously for a time t_o. In section 4.1.3 we showed that the equivalent bandwidth of a continuous observation process of duration t_o is given by $B_o = 1/2t_o$.

At this point in order to simplify the analysis, we shall assume that the incident and emitted radiation streams are both at temperature T. We know that the incident radiation is at temperature T_a, the temperature of the surroundings, but the inclusion of this nonequilibrium condition in our analysis is complicated by the impending generalization of our formulas to include conductive as well as radiative thermal coupling. The dependence of var H upon T for the conductive thermal connection is no longer a fifth power relationship, and hence the ratio of the radiative coupling to the conductive coupling would have to be determined in order to include this nonequilibrium condition in our fluctuation formulas. The assumption that both streams of radiation are at temperature T will result in a pessimistic value for the semsitivity of the detector.

Returning to our equation for var H, we replace $1/2t_o$ by B_o, and multiply by two to account for emitted and incident radiation streams. Therefore:

$$\mathrm{var}\, H = \frac{32\pi^5 k^5 T^5 B_o}{15 h^3 v^2 A} \qquad \text{or} \qquad \mathrm{var}\, H = \frac{4kT^2 \lambda_R B_o}{A^2}$$

Appealing to the argument of section 2.3.4, we shall generalize this result to include conductive as well as radiative thermal coupling. $\text{var } H = \dfrac{4kT^2\lambda B_o}{A^2}$

The spectral density of this fluctuation can be written as:

$$S_H = \frac{4kT^2\lambda}{A^2} \tag{4.32}$$

From 4.31 and 4.32 we can obtain the spectral density of the fluctuation of V_S. We shall now attach a low-pass filter of bandwidth B to the detector output. The voltage fluctuation at the output of the filter will now be given by:

$$(\text{var } V_S)_{temp.} = \int_0^B S_{V_S}(f)df$$

For the case where $B \ll B_\tau$, that is when the filter is the dominant factor in the noise reduction system:

$$(\text{var } V_S)_{temp.} = 4kT^2BR_o\,\alpha^2(T_o - T_a)\cdot\frac{R_S^2}{|R_S + Z|^2} \tag{4.33}$$

At first the factor $T_o - T_a$ may seem surprising, since we know that the temperature fluctuations of the element exist when $T_o = T_a$. However the condition $T_o = T_a$ represents the trivial case of $V_S = 0$ since I_o must equal zero if there is to be no joule heating.

For the case where $B \gg B_\tau$, which is of theoretical interest only, we have:

$$(\text{var } V_S)_{temp.} = \left(\frac{kT^2}{C}\right)\left(\frac{\alpha^2[T_o - T_a]R_o}{1 + \alpha[T_o - T_a]}\right)\cdot\frac{R_S^2}{|R_S + Z|^2} \tag{4.34}$$

$(\text{var } V_S)_{temp.}$ is the fluctuation of the output voltage of the filter following the detector, resulting from the temperature fluctuation of the detector element.

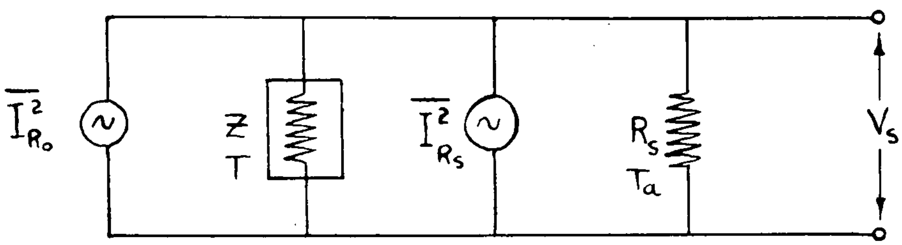
The second and only other major source of noise in the bolometer is the thermal or Johnson noise present in the detector element and in R_s. From Nyquist's theorem we know that the spectral density of the current fluctuations through an impedance $Z = R + iX$ will be given by:

$$S_I(f) = \frac{4kTR}{|Z|^2} \qquad f \ll \frac{kT}{h}$$

In our detection system the bandwidth of this Johnson noise will be limited by the filter at the detector output. Therefore:

$$\overline{I^2} = \text{var}\, I = \frac{4kTRB}{|Z|^2}.$$

The Johnson noise can be represented by the following circuit;



$$\overline{I^2_{R_o}} = \frac{4kTR_oB}{|Z|^2}$$

$$\overline{I^2_{R_s}} = \frac{4kT_aB}{R_s}$$

We choose an R_s with a large heat capacity thus enabling us to assume that the temperature of R_s remains equal to T_a. The total current fluctuation will be:

$$\overline{I^2} = 4kB\left[\frac{TR_o}{|Z|^2} + \frac{T_a}{R_s}\right]$$

The output impedance of the detector is $\dfrac{R_s Z}{R_s + Z}$. Therefore:

$$(\text{var}\, V_s)_{\text{Johnson}} = 4kB\left[\frac{TR_o}{|Z|^2} + \frac{T_a}{R_s}\right]\left|\frac{R_s Z}{R_s + Z}\right|^2 \qquad (4.35)$$

This is the fluctuation in the output voltage of our filter, caused by the Johnson noise.

The total output voltage fluctuation of the detection system is obtained by adding the temperature noise contribution to the Johnson noise contribution. That is:

$$\text{var } V_s = (\text{var } V_s)_{temp.} + (\text{var } V_s)_{Johnson}$$

These individual sources of noise can be added because the Johnson noise is completely independent of the temperature noise. In order to simplify the expressions somewhat we shall assume that $R_s \gg Z$. Therefore from 4.33 and 4.35 we have:

$$\text{var } V_s = 4kTBR_o\left[1 + \alpha^2 T(T_o - T_a)\right] \quad \text{for } B \ll B_\tau \tag{4.36}$$

and from 4.34 and 4.35:

$$\text{var } V_s = 4kTBR_o\left[1 + \frac{\alpha^2 T(T_o - T_a)}{1 + \alpha(T_o - T_a)} \cdot \frac{B}{B_\tau}\right] \quad \text{for } B \gg B_\tau \tag{4.37}$$

Gill (1958) states an equation for the output voltage fluctuations of a bolometer which is equivalent to our equation 4.36. He then questions the validity of the addition of the Johnson and temperature noise, comparing this situation to a galvanometer subject to Brownian fluctuations with a resistor across the terminals. We know that the Brownian fluctuations of the galvanometer indicator and the Johnson noise of the resistor are not added and that equipartition determines the galvanometer fluctuation for all values of the resistor. This results from the fact that the Brownian fluctuation spectral density is changed when a resistor is connected across the galvanometer terminals because of the damping effect which results. Therefore both the Brownian fluctuation and the Johnson noise depend upon the value of the resistance and hence are not independent fluctuations. However no such relationship exists between the temperature noise and the Johnson noise in a bolometer and we cannot see any justification for comparing the two situations.

It is interesting to consider the ratio of the temperature noise to the Johnson noise which can be easily obtained from 4.36 and 4.37:

$$\frac{\text{Temperature noise}}{\text{Johnson noise}} = \frac{(\mathrm{var}\ V_s)_{temp}}{(\mathrm{var}\ V_s)_{Johnson}} = \alpha^2 T(T_o - T_a) \quad \text{for } B \ll B_\tau$$

$$= \frac{\alpha^2 T(T_o - T_a)}{1 + \alpha(T_o - T_a)} \quad \text{for } B \gg B_\tau$$

Consider a metallic detector element in which $\alpha \approx \frac{1}{T_o}$:

$$\frac{\text{Temperature noise}}{\text{Johnson noise}} = \frac{T_o - T_a}{T_o} < 1 \quad \text{for } B \ll B_\tau$$

$$= \left(\frac{T_o - T_a}{2T_o - T_a}\right)\frac{B_\tau}{B} \ll 1 \quad \text{for } B \gg B_\tau$$

In both cases, $B \ll B_\tau$ and $B \gg B_\tau$, the Johnson noise predominates over the temperature noise in a metallic detector element. In the case $B \gg B_\tau$ the temperature noise may be neglected in comparison with the Johnson noise.

The minimum detectable energy flux can now be obtained directly from the relation:

$$H^2_{min} \equiv \mathrm{var}\, H = \frac{\mathrm{var}\ V_s}{|a_o(f)|^2}$$

From 4.30 and 4.36 we get:

$$H^2_{min} = \frac{4kT^2\lambda B}{A^2}\left[\frac{1}{\alpha^2 T(T_o - T_a)} + 1\right]\left[1 + \left\{1 + \alpha(T_o - T_a)\right\}^2 \omega^2\tau^2\right] \tag{4.38}$$

for $B \ll B_\tau$. The case for $B \gg B_\tau$ can be obtained directly from 4.30 and 4.37. For a metallic detector element in which $\alpha \approx \frac{1}{T_o}$, and when $\omega\tau \ll 1$, $T_o = 1.4\ T_a$ is the value of T_o for which H_{min} is a minimum in equation 4.38. It is apparent that the optimum value of λ is the smallest attainable value. However one must remember that it is desirable to have $\tau = C/\lambda \ll 1/\omega$, where ω is the modulation frequency of the signal. Therefore we must minimize C, the heat capacity of the detector element, and then minimize λ, keeping $\lambda \gg C_{min}\,\omega$. The dependence of C and λ upon A, the area of the detector element, prohibits us from making A large in an attempt to decrease H_{min}. A suitable detector area must be chosen as the first step in the detector design. A large value of α, the temperature coefficient of resistance of the bolometer element, is desirable and negative values of α are acceptable. In fact for frequencies at which $\omega\tau$ is comparable to or greater than unity, negative values of α are preferable. Note that the electrical resistance R, of the bolometer element, does not affect the sensitivity of the detector.

The smoothing effect of the observational procedure, which was discussed in detail earlier in this chapter, has been omitted in this treatment of the bolometer noise in an attempt to confine the discussion to the fundamental properties of the bolometer.

4.3.3 The noise factor.

The noise factor was defined previously as:

$$\eta \equiv \frac{H_{min}\text{ (real detector)}}{H_{min}\text{ (ideal detector)}}$$

In section 4.1.1 we defined the ideal energy detector and in sections 4.1.2, 4.1.4, and 4.1.5 we calculated var H = H^2_{min}

for this device with various input signals and smoothing arrangements. From equation 4.18, when the smoothing of the noise is done entirely by the low-pass filter of bandwidth B, we have for the ideal energy detector:

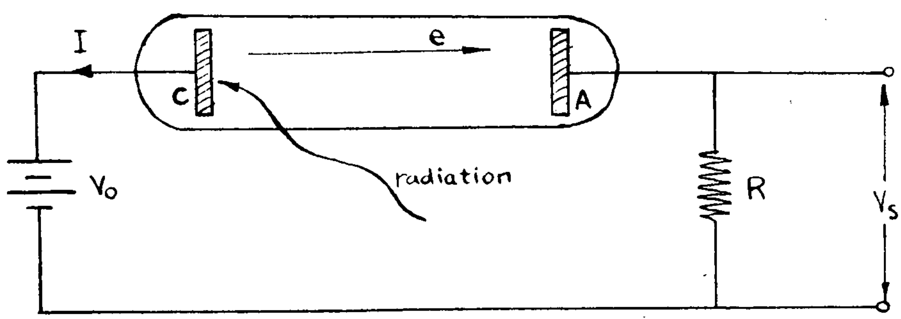
$$H^2_{min} = \frac{4kT^2\lambda B}{A^2}\left(1 + \omega^2\tau^2\right)$$

For the real energy detector, in this case the bolometer, we have equation 4.38 for H_{min} when the filter is the dominating factor in the noise reduction. Therefore immediately:

$$\eta^2 = \left(1 + \frac{1}{\alpha^2 T(T_o - T_a)}\right)\left(\frac{1 + [1 + \alpha(T_o - T_a)]^2\omega^2\tau^2}{1 + \omega^2\tau^2}\right) \qquad \text{for } B \ll B_\tau$$

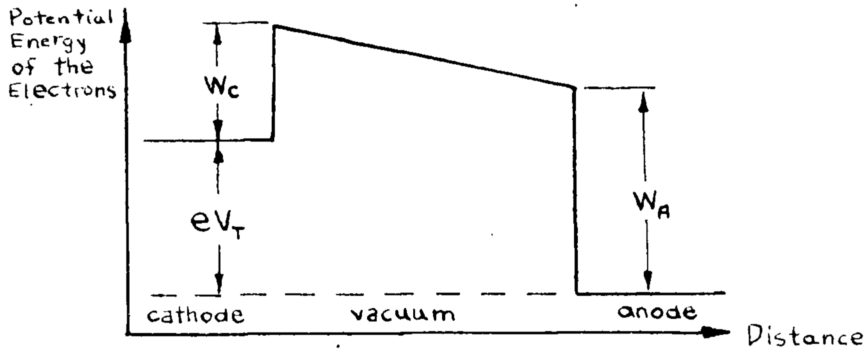
4.4 The Vacuum Phototube.

In contrast to the bolometer which is an energy detector, the vacuum phototube is a quantum or particle detector. That is its response depends upon the number of photons, not the energy of the photons, with energies above a certain threshold value. The vacuum phototube can be used as a radiation detector in the following way:



V_T is the voltage across the tube.

The greatest sensitivity will be obtained from this device when all the electrons emitted from the cathode, reach the anode. The currents involved in detecting the radiation will be small and there will be no space charge effect. The potential energy diagram for electrons in the phototube must have the following shape:



W_c is the work function of the cathode and W_A is the work function of the anode. The work function of a metal is the energy required by an electron to enable it to leave the metal. From the potential energy diagram we can see that if all electrons emitted by the cathode are to reach the anode, we require; $eV_T + W_c > W_A$. In addition; $V_o = V_T + IR$. Therefore we have an upper limit on the value of R:

$$R < \frac{V_o - \frac{W_A - W_c}{e}}{I}$$ e is the charge of an electron.

The dark current I_o, is the current flowing in the phototube in the absence of signal radiation. This current is caused mainly by thermionic emission of electrons from the cathode. The thermionic current shall be denoted by I_{oT}. Photoemission by the black body radiation from the surroundings at temperature T_a contributes a small current which we shall call I_{op}. Richardson's equation gives the total dark current from a perfect cathode surface in thermodynamic equilibrium with the surroundings:

$$I_o = I_{oT} + I_{oP} = \left(\frac{4\pi e m k^2}{h^3}\right) A T_a^2 \exp\left(-\frac{W_c}{kT_a}\right)$$

A is the area of the photocathode. Consider briefly the magnitude of I_{oP}. From section 2.1.6 we have an expression for the mean photon flux with frequencies in the interval $d\nu$ about ν, in one direction, for black body radiation at temperature T ;

$$\left(\overline{m}_\nu\right)_o = \frac{2\pi \nu^2 d\nu}{v^2\left(\exp\left(\frac{h\nu}{kT_a}\right) - 1\right)}$$

If $q(\nu)$ is the quantum efficiency of the photocathode;

$$I_{oP} = Ae \int_0^\infty q(\nu) \cdot \frac{(\overline{m}_\nu)_o}{d\nu}\, d\nu$$

Let us assume $q(\nu) = 1$ for $\nu \geq \frac{W_c}{h}$

$q(\nu) = 0$ for $\nu < \frac{W_c}{h}$

We shall carry this assumption through the entire discussion of the phototube. Therefore:

$$I_{oP} = \frac{2\pi A e}{v^2} \int_{\frac{W_c}{h}}^\infty \frac{\nu^2 d\nu}{\exp\left(\frac{h\nu}{kT_a}\right) - 1}$$

Since W_c is considerably larger than kT_a for room temperatures and for the lowest known work functions, we can approximate the integral and obtain finally:

$$I_{oP} \approx \frac{W_c^2}{2kT_a m v^2} \cdot I_o \ll I_o$$

Therefore the dark current is almost entirely due to the thermionic current.

4.4.1 A monochromatic radiation signal.

a) Frequency response function.

We define a monochromatic signal of frequency ν to have a bandwidth $\Delta\nu$ about ν, where $\Delta\nu \lll \nu$. In addition we require that $\nu > \frac{W_c}{h}$ so that $q(\nu)$ will equal unity. We can obtain from section 2.1.6 the mean photon flux for a monochromatic signal of frequency ν:

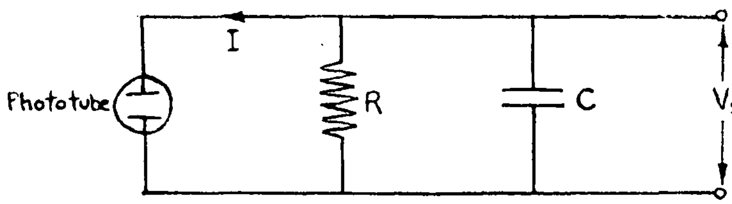
$$\overline{m}_s = \frac{2\pi\nu^2\Delta\nu}{v^2\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]} \tag{4.39}$$

Since $q(\nu) = 1$, the signal current will be:

$$I_s = eAq(\nu)\overline{m}_s = \frac{2\pi Ae\nu^2\Delta\nu}{v^2\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]} \tag{4.40}$$

The upper limit on the signal modulation frequency which can be used is determined either by the transit time of the electrons in the phototube or by the time constant τ = RC caused by the stray capacitance C, of the circuit. Some typical values for these quantities could be:

$R \sim 10^6$ ohms, $C \sim 10^{-11}$ farads. Therefore $RC \sim 10^{-5}$ sec. The transit time of the electrons in the tube will be of the order of 10^{-8} seconds. Therefore we shall neglect the transit time of the electrons and assume that the stray capacitance of the circuit determines the frequency response of the detector. The equivalent circuit of the detector will be:



$$|V_s| = I_s|Z| = \frac{I_s R}{1+(\omega CR)^2} \tag{4.41}$$

ω is the signal modulation frequency. The frequency response function for a quantum detector is defined as:

$$a_o(f) = V_s/\bar{m}_s \tag{4.42}$$

Therefore, from equations 4.39 to 4.42 we can obtain the frequency response function of the phototube as:

$$|a_o(f)| = \frac{eAR}{1+(\omega CR)^2} \tag{4.43}$$

for a monochromatic radiation signal.

b) Minimum detectable photon flux.

In the vacuum phototube we have three major sources of noise: i) Signal noise- The fluctuation of the signal photon flux which behaves according to Bose-Einstein statistics will cause a similar fluctuation in the emitted electrons. ii) Shot noise of the dark current- Since the dark current is almost entirely caused by thermionic emission we can assume the electrons are emitted randomly and the spectral density of the current fluctuations caused by this random emission will be given by the familiar Schottky relation: $S_I(f) = 2eI_o$. Note that the mean value of the dark current does not affect the noise level. iii) Johnson noise in R- Nyquist's theorem tells us that the Johnson noise may be represented by a voltage generator $\overline{V_R^2}$ in series with R having a spectral density of: $S_{V_R}(f) = 4kT_a R$. By choosing R with a large heat capacity, we can assume that the temperature of R will remain equal to T_a. In addition we arrange that the heat capacity of the cathode is very large so that there will be no appreciable fluctuation of the cathode temperature.

Let us consider the signal noise first of all. From section 2.2.3 we can obtain an expression for the fluctuation of a monochromatic signal flux:

$$\mathrm{var}\ m_{\nu_s} = \frac{4\pi\nu^2 B_0 \exp\left(\frac{h\nu}{kT_s}\right)\Delta\nu}{v^2 A\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]^2}$$

where we have replaced the bandwidth of a continuous observation of duration t_0 by $B_0 = 1/2t_0$. The spectral density of the flux fluctuation will be:

$$S_{m_{\nu_s}} = \frac{4\pi\nu^2 \exp\left(\frac{h\nu}{kT_s}\right)\Delta\nu}{v^2 A\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]^2}$$

or from 4.39:

$$S_{m_{\nu_s}} = \frac{2\,\overline{m}_s \exp\left(\frac{h\nu}{kT_s}\right)}{A\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]}$$

From 4.42 we have:

$$S_{V_s}(f) = |a_0(f)|^2 S_{m_s} = \frac{e^2 A R^2}{1+(\omega C R)^2}\cdot\frac{2\exp\left(\frac{h\nu}{kT_s}\right)\ \overline{m}_s}{\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]}$$

This is an expression for the spectral density of the output voltage fluctuations resulting from the signal noise.

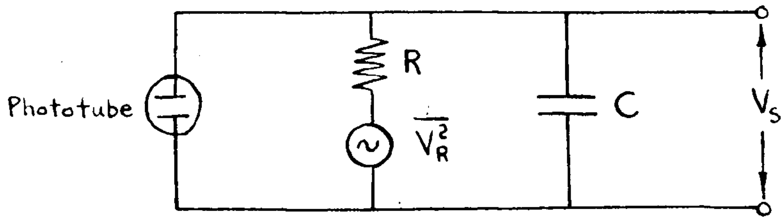
We have already seen that the spectral density of the current fluctuations as a result of the shot noise of the dark current is given by; $S_I(f) = 2eI_0$. Immediately from 4.41:

$$S_{V_s}(f) = 2eI_0\cdot\frac{R^2}{1+(\omega C R)^2}$$

This is the contribution of the dark current shot noise to the spectral density of the fluctuation of V_S.

The Johnson noise contribution to the fluctuation of

V_s may be determined from the following circuit:



C is the stray capacitance.

$$S_{V_R}(f) = 4kT_a R$$

$$S_{V_s}(f) = \frac{\left(\frac{1}{\omega C}\right)^2}{R^2 + \left(\frac{1}{\omega C}\right)^2} \cdot S_{V_R}(f)$$

$$S_{V_s}(f) = \frac{4kT_a R}{1 + (\omega C R)^2}$$

We have now calculated the contribution of each of the major sources of noise to the fluctuation of V_s. Since each of these noise sources is statistically independent of the other two, we may add the individual contributions to the fluctuation of V_s, to obtain the total fluctuation of V_s.

$$S_{V_s}(f) = \frac{R^2}{1 + \omega^2 C^2 R^2} \cdot \left[\frac{2e^2 A \overline{m}_s \exp\left(\frac{h\nu}{kT_s}\right)}{\exp\left(\frac{h\nu}{kT_s}\right) - 1} + 2eI_o + \frac{4kT_a}{R} \right]$$

Immediately we have:

$$\operatorname{var} V = R^2 B_\tau \left[\frac{2e^2 A \overline{m}_s \exp\left(\frac{h\nu}{kT_s}\right)}{\exp\left(\frac{h\nu}{kT_s}\right) - 1} + 2eI_o + \frac{4kT_a}{R} \right] \qquad (4.44)$$

where $B_\tau = 1/4RC$ is the bandwidth of the phototube. Notice that the bandwidth of the detector has a smoothing effect on the Johnson noise. Recall in the bolometer the detector bandwidth had no effect on the Johnson noise.

We now can obtain an expression for the minimum detectable photon flux:

$$m_{min}^2 \equiv \text{var } m_s = \frac{\text{var } V_s}{|a_0(t)|^2}$$

$$m_{min}^2 = \frac{B_\tau\left[1+\omega^2 C^2 R^2\right]}{(eA)^2}\left[\frac{2e^2 A(m_{min})\exp\left(\frac{h\nu}{kT_s}\right)}{\exp\left(\frac{h\nu}{kT_s}\right)-1} + 2eI_o + \frac{4kT_a}{R}\right] \tag{4.45}$$

Solving this quadratic equation for m_{min}, we obtain:

$$m_{min} = \frac{B_\tau\left[1+(\omega CR)^2\right]\exp\left(\frac{h\nu}{kT_s}\right)}{A\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]}\left\{1+\sqrt{1+\frac{\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]^2\left[2eI_o+\frac{4kT_a}{R}\right]}{e^2 B_\tau \exp\left(\frac{2h\nu}{kT}\right)\left[1+(\omega CR)^2\right]}}\right\} \tag{4.46}$$

This expression, and in fact the entire preceding analysis, can be simplified if we assume $\exp(\frac{h\nu}{kT_s}) \gg 1$. By requiring $\exp(\frac{h\nu}{kT_s})$ to be $\gg 1$, we are saying that radiation behaves according to the corpuscular theory rather than the wave theory, a condition which is reasonable for a photoemissive process. If this assumption were made at the beginning of our analysis we could have treated the signal noise using Schottky's relation; $S_I(f) = 2eI_S$, since the photons tend to arrive randomly when the corpuscular properties of the radiation are dominant.

c) Noise factor.

We define the noise factor for a quantum detector as:

$$\eta \equiv \frac{m_{min}\text{ (real detector)}}{m_{min}\text{ (ideal detector)}} > 1$$

An ideal quantum detector can be conveniently defined as one in which the only source of noise is the signal noise. Therefore from 4.45 we have immediately:

$$m_{min}(\text{ideal}) = \frac{2B_\tau\left(1+\omega^2 C^2 R^2\right)\exp\left(\frac{h\nu}{kT_s}\right)}{A\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]} \tag{4.47}$$

Therefore from 4.46 and 4.47 we have:

$$\eta = \frac{1}{2}\left\{1 + \sqrt{1 + \frac{\left[\exp\left(\frac{h\nu}{kT_s}\right) - 1\right]^2 \left[2eI_o + \frac{4kT_a}{R}\right]}{e^2 B_r \exp\left(\frac{2h\nu}{kT_s}\right)\left[1 + (\omega CR)^2\right]}}\right\} \qquad (4.48)$$

As expected, when we remove the dark current noise ($I_o = 0$), and the Johnson noise ($T_a = 0$), $\eta = 1$.

4.4.2 Black body radiation signal.

a) Frequency response function.

From section 2.1.6 we can obtain an expression for the flux of photons in the frequency interval $d\nu$ about ν for a stream of black body radiation at temperature T_s:

$$\left(\overline{m}_\nu\right)_s = \frac{2\pi\nu^2 d\nu}{v^2\left[\exp\left(\frac{h\nu}{kT_s}\right) - 1\right]}$$

Recall; $q(\nu) = 1$ for $\nu \geq \frac{W_c}{h}$

$q(\nu) = 0$ for $\nu < \frac{W_c}{h}$

Therefore the signal current will be:

$$I_s = \frac{2\pi eA}{v^2}\int_{\frac{W_c}{h}}^{\infty} \frac{\nu^2 d\nu}{\exp\left(\frac{h\nu}{kT_s}\right) - 1}$$

From equation 4.41 we have:

$$|V_s| = \frac{2\pi eAR}{v^2\left(1 + \omega^2 C^2 R^2\right)^{1/2}} \cdot \int_{W_c/h}^{\infty} \frac{\nu^2 d\nu}{\exp\left(\frac{h\nu}{kT_s}\right) - 1} \cdot$$

We can obtain the total photon flux for black body radiation by integrating $(\overline{m}_\nu)_s$ over all frequencies ν, as in section 2.1.6 to obtain:

$$\overline{m}_s = \frac{2\pi k^3 T_s^3}{h^3 v^2} (2.404)$$

Therefore immediately:

$$|a_D(f)| \equiv \frac{|V_s|}{\overline{m}_s} = \frac{eAR}{2.404\,(1+\omega^2C^2R^2)^{1/2}} \int_{\frac{W_c}{kT_s}}^{\infty} \frac{x^2\,dx}{\exp(x)-1} \qquad (4.49)$$

For the physically unrealizable case of $W_c = 0$, we see that:

$$|a_D(f)| = \frac{eAR}{\sqrt{1+(\omega CR)^2}} \qquad (4.50)$$

which we have seen is the frequency response function for the monochromatic signal (equation 4.43). This is to be expected since when $W_c = 0$, $q(\nu) = 1$ for all photons, and the quantum detector does not distinguish between the energies of the photons once the energy is sufficient to eject an electron from the photocathode.

For the physically more probable situation of $\exp\left(\frac{W_c}{kT_s}\right) \gg 1$:

$$\int_{\frac{W_c}{kT_s}}^{\infty} \frac{x^2\,dx}{e^x-1} \approx \int_{\frac{W_c}{kT_s}}^{\infty} x^2 e^{-x}\,dx = e^{-\frac{W_c}{kT_s}}\left[\left(\frac{W_c}{kT_s}\right)^2 + \frac{2W_c}{kT_s} + 2\right]$$

The value of this integral will be less than one for the values of W_c/kT_s for which the approximation is valid.

Therefore:

$$|a_D(f)| \approx \frac{eAR}{\sqrt{1+(\omega CR)^2}} \cdot \exp\left(-\frac{W_c}{kT_s}\right)\left[\left(\frac{W_c}{kT_s}\right)^2 + \frac{2W_c}{kT_s} + 2\right] \qquad (4.51)$$

As W_c increases, fewer photons are capable of ejecting electrons and the frequency response function becomes smaller.

b) Minimum detectable photon flux.

The Johnson noise and the dark current shot noise will be the same for the black body radiation signal as for the monochromatic signal. However the signal noise must be reconsidered. From section 2.2.3 we have an expressionn for the fluctuation of the flux of photons in the frequency interval $d\nu$ about ν:

$$\mathrm{var}\ m_{\nu_s} = \frac{4\pi\nu^2 B_0 \exp\left(\frac{h\nu}{kT_s}\right) d\nu}{v^2 A\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]^2}$$

where we have replaced $1/2t_0$ by B_0. Immediately the spectral density of this flux fluctuation will be:

$$S_{m_{\nu_s}} = \frac{4\pi\nu^2 \exp\left(\frac{h\nu}{kT_s}\right) d\nu}{v^2 A\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]^2}$$

The spectral density of the resulting current fluctuation will be:

$$S_{I_s} = \frac{4\pi e^2 A}{v^2}\int_{\frac{W_c}{h}}^{\infty} \frac{\nu^2 \exp\left(\frac{h\nu}{kT_s}\right) d\nu}{\left[\exp\left(\frac{h\nu}{kT_s}\right)-1\right]^2}$$

Now we can obtain the spectral density of the resulting output voltage fluctuations:

$$S_{V_s} = \frac{R^2}{1+(\omega CR)^2}\cdot S_{I_s}$$

$$S_{V_s} = \frac{4\pi e^2 A R^2 k^3 T_s^3}{v^2 h^3 (1+\omega^2 C^2 R^2)}\int_{\frac{W_c}{kT_s}}^{\infty} \frac{x^2 e^x dx}{(e^x-1)^2}$$

Let us first consider the practical case where $\exp\left(\frac{W_c}{kT_s}\right) \gg 1$:

$$S_{V_s}(f) \cong \frac{4\pi e^2 A R^2 k^3 T_s^3}{v^2 h^3 (1+\omega^2 C^2 R^2)}\cdot \exp\left(-\frac{W_c}{kT_s}\right)\left[\left(\frac{W_c}{kT_s}\right)^2 + \frac{2W_c}{kT_s} + 2\right]$$

$$= \frac{R^2}{1+(\omega CR)^2}\cdot 2eI_s$$

Recalling that $S_V(f) = \frac{R^2}{1+(\omega CR)^2}\cdot S_I(f)$, we can see that the signal noise can be represented by the Schottky formula; $S_I(f) = 2eI_s$. As we mentioned previously, the requirement

$\exp(\frac{W_c}{kT_s}) \gg 1$ is equivalent to assuming that all photons capable of causing photoemission are behaving according to the corpuscular theory of radiation. Therefore we can use a special case of equation 4.46, for $\exp(\frac{W_c}{kT_s}) \gg 1$, to obtain the minimum detectable photon flux for a black body radiation signal:

$$m_{min} = \frac{B_\tau(1+\omega^2C^2R^2)}{A}\left\{1+\sqrt{1+\frac{2eI_o+\frac{4kT_a}{R}}{e^2B_\tau(1+\omega^2C^2R^2)}}\right\}$$

For the unrealistic but theoretically interesting case of $W_c = 0$; we have for the signal noise contribution to $S_{V_s}(f)$;

$$S_{V_s}(f) = \frac{4\pi^3e^2AR^2k^3T_s^3}{3v^2h^3(1+\omega^2C^2R^2)}$$

$$= \frac{R^2}{1+(\omega CR)^2}\cdot 2eI_s\cdot\frac{\pi^2}{3(2.404)}$$

Notice that the signal noise cannot be represented by Schottky's formula in this case, as the wave properties of the radiation have an appreciable effect on the fluctuation. Following the procedure used in the case of the monochromatic signal we can obtain the minimum detectable photon flux for a black body radiation signal when $W_c = 0$ as:

$$m_{min} = \frac{\pi^2B_\tau(1+\omega^2c^2R^2)}{3(2.404)A}\left\{1+\sqrt{1+\frac{\pi^4\left(2eI_o+\frac{4kT_a}{R}\right)}{9(2.404)^2e^2B_\tau(1+\omega^2C^2R^2)}}\right\}$$

Let us consider the optimum value of some of the parameters of the phototube. Obviously it is desirable to have the dark current I_o as small as possible. The optimum value of R is the largest value of R that will allow $R \ll 1/\omega C_{min}$, where C_{min} is the smallest stray capacitance that can be

attained. The value of the voltage source used in the detector must satisfy the condition; $V_0 > IR + \frac{W_A - W_c}{e}$
A large detector area is beneficial but it must be remembered that both I_0 and C vary directly with A.

Throughout this discussion of the phototube, in order to confine our analysis to the phototube itself, we have not considered the smoothing effects of filters or observational procedures. If a filter of bandwidth $B \ll B_\tau$ is used to reduce the noise, our equations will still apply if B_τ is replaced by B.

It is interesting to note that for a phototube of this type $4kT_a/R$ is usually much larger than $2eI_0$. In addition $4kT_a/Re^2B$ is usually very much greater than one, and the expression for m_{min} can be approximated as:

$$m_{min}^2 \approx 4kTRB \cdot \left(\frac{1 + \omega^2 C^2 R^2}{e^2 A^2 R^2}\right)$$

That is, as in the bolometer, the Johnson noise of R is usually the most significant noise source in the phototube.

c) Noise factor.

Following the treatment of the monochromatic radiation signal, we have directly:

$$\eta = \frac{1}{2}\left\{1 + \sqrt{1 + \frac{2eI_0 + \frac{4kT_a}{R}}{e^2 B_\tau (1 + \omega^2 C^2 R^2)}}\right\} \qquad \exp\left(\frac{W_c}{kT_s}\right) \gg 1$$

$$\eta = \frac{1}{2}\left\{1 + \sqrt{1 + \frac{\pi^4 \left(2eI_0 + \frac{4kT_a}{R}\right)}{9(2.404)^2 e^2 B_\tau (1 + \omega^2 C^2 R^2)}}\right\} \qquad W_c = 0$$

APPENDIX 1

To Show that β equals $-1/kT$.

Consider a small change in energy of the system δE_ν: $\delta E_\nu = \delta(n_\nu h\nu) = n_\nu \delta(h\nu) + h\nu\,\delta n_\nu$

The first term is the change in energy as a result of the change in volume: $n_\nu \delta(h\nu) = -P\,\delta V$

whereas the second term represents the change in energy resulting from externally added heat: $h\nu\,\delta n_\nu = \delta Q$

From the Lagrangian maximization procedure in section 2.1.2 we have: $\delta(\ln W) = -\beta\, h\nu\,\delta n_\nu = -\beta\,\delta Q$ (1)

Recall the Boltzmann equation: $S = k \ln W$

from which we have: $\delta(\ln W) = \delta S/k$ (2)

The second law of thermodynamics is $\delta Q = T\delta S$ (3)

Equations 1, 2, and 3 lead immediately to:

$$\beta = -\frac{1}{kT}$$

APPENDIX 2

Integrals.

1. $$\int_0^\infty \frac{x^m\,dx}{e^x-1}$$

We know $$\int_0^\infty x^m e^{-ax}\,dx = \frac{m!}{a^{m+1}}$$

Now: $$\int_0^\infty \frac{x^m\,dx}{e^x-1} = \int_0^\infty \frac{x^m e^{-x}\,dx}{1-e^{-x}}$$
$$= \int_0^\infty x^m e^{-x}\left(1+e^{-x}+e^{-2x}+\cdots\right)dx$$
$$= m!\left(1+\frac{1}{2^{m+1}}+\frac{1}{3^{m+1}}+\cdots\right)$$
$$= m!\,\zeta(m+1)$$

$\zeta(n)$ is the Riemann zeta function tabulated in the Jahnke-Emde tables.

In the integral for n, the total number of photons, $m = 2$ and the integral is equal to 2.404 since $\zeta(3) = 1.202$. In the integral for E, the total energy of the photons, $m = 3$ and the integral equals $\pi^4/15$ since $\zeta(4) = \pi^4/90$

2. $$-\int_0^\infty x^m \ln\left(1-e^{-x}\right)dx$$

Recall $$-\ln(1-e^{-x}) = \sum_{\ell=1}^{\infty} \frac{e^{-\ell x}}{\ell}$$

Therefore: $$-\int_0^\infty x^m \ln(1-e^{-x})\,dx = \int_0^\infty \sum_{\ell=1}^{\infty} \frac{x^m e^{-\ell x}}{\ell}\,dx$$
$$= \sum_{\ell=1}^{\infty} \frac{m!}{\ell^{m+2}}$$
$$= m!\,\zeta(m+2)$$

In the integral for S, the entropy of the radiation, $m = 2$ and the integral equals $\pi^4/45$ since $\zeta(4) = \pi^4/90$.

APPENDIX 2 (continued)

3. $$\int_0^{\infty} \frac{x^{m+1} e^x \, dx}{(e^x - 1)^2}$$

Integrating by parts:

$$\int_0^{\infty} \frac{x^{m+1} e^x \, dx}{(e^x - 1)^2} = -\frac{x^{m+1}}{e^x - 1}\Bigg|_0^{\infty} + \int_0^{\infty} \frac{(m+1)\, x^m \, dx}{e^x - 1}$$

$$= (m+1)! \left(1 + \frac{1}{2^{m+1}} + \frac{1}{3^{m+1}} + \cdots \right)$$

$$= (m+1)! \, \zeta(m+1)$$

The integral for var n equals $\pi^2/3$ since m = 1 in this integral and $\zeta(2) = \pi^2/6$. The integral for var E equals $\frac{4\pi^4}{15}$ since m = 3 in this integral and $\zeta(4) = \frac{\pi^4}{90}$.

APPENDIX 3

Einstein's Treatment of Fluctuations in n_ν.

We know that at equilibrium the Helmholtz free energy is a minimum. Therefore:

$$(F_\nu)_{n_\nu=\bar{n}_\nu} = F_{\nu_{min}} \qquad \left(\frac{\partial F_\nu}{\partial n_\nu}\right)_{n_\nu=\bar{n}_\nu} = 0$$

Expand F_ν in a Taylor series about the equilibrium value with respect to variations in n_ν:

$$F_\nu = (F_\nu)_{n_\nu=\bar{n}_\nu} + (n_\nu-\bar{n}_\nu)\left(\frac{\partial F_\nu}{\partial n_\nu}\right)_{n_\nu=\bar{n}_\nu} + \frac{(n_\nu-\bar{n}_\nu)^2}{2}\left(\frac{\partial^2 F_\nu}{\partial n_\nu^2}\right)_{n_\nu=\bar{n}_\nu} + \cdots$$

Neglecting terms beyond the second order in $(n_\nu - \bar{n}_\nu)$, we are left with:

$$F_\nu - \bar{F}_\nu = \frac{(n_\nu-\bar{n}_\nu)^2}{2}\left(\frac{\partial^2 F_\nu}{\partial n_\nu^2}\right)_{n_\nu=\bar{n}_\nu}$$

In a system in which V and T are kept constant, the probability distribution of a suitable variable describing the system is given by:

$$P(x)\,dx = P(\bar{x})\exp\left\{-\frac{[F(x)-\bar{F}(x)]}{kT}\right\}dx$$

In this discussion we are interested in the variable n_ν.

Therefore: $$P(n_\nu)dn_\nu = P(\bar{n}_\nu)\exp\left\{-\frac{(F_\nu-\bar{F}_\nu)}{kT}\right\}dn_\nu$$

From the Taylor expansion of F_ν we can write $P(n_\nu)dn_\nu$ in the form:

$$P(n_\nu)dn_\nu = P(\bar{n}_\nu)\exp\left\{-\frac{(n_\nu-\bar{n}_\nu)^2}{2kT}\cdot\left(\frac{\partial^2 F_\nu}{\partial n_\nu^2}\right)_{n_\nu=\bar{n}_\nu}\right\}dn_\nu$$

Now we can write:

$$\text{var}\, n_\nu = \overline{(n_\nu-\bar{n}_\nu)^2} = \frac{\int_{-\infty}^{+\infty} P(n_\nu)\,(n_\nu-\bar{n}_\nu)^2\,d(n_\nu-\bar{n}_\nu)}{\int_{-\infty}^{+\infty} P(n_\nu)\,d(n-\bar{n}_\nu)}$$

APPENDIX 3 (continued)

Substituting into this expression for $P(n_\nu)$ and integrating we obtain directly:

$$\text{var } n_\nu = \frac{kT}{\left(\frac{\partial^2 F_\nu}{\partial n_\nu^2}\right)_{n_\nu = \bar{n}_\nu}}$$

The integrals encountered are of the form:

$$\int_{-\infty}^{+\infty} x^2 e^{-ax^2} dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

APPENDIX 4

A Comparison between the Energy Content of a Volume of Photons and an Equal Volume of Phonons.

In section 2.1.3 we saw that the mean energy density of a volume of black body radiation was given by:

$$\frac{\overline{E}}{V} = \frac{8\pi h}{v^3}\int_0^\infty \frac{\nu^3\,d\nu}{e^{\frac{h\nu}{kT}}-1}$$

The Debye model of the lattice heat capacity of a solid leads to the expression (Kittel 1956) :

$$\frac{\overline{E_o}}{V} = \frac{3}{2}\cdot\frac{8\pi h}{v_o^3}\int_0^{\nu_m} \frac{\nu^3\,d\nu}{e^{\frac{h\nu}{kT}}-1}$$

for the mean energy density of a solid material. In the Debye equation, ν is the lattice vibration frequency and v_o the propagation velocity of the lattice vibration waves. We have assumed that this propagation velocity is equal for transverse and longitudinal waves.

$$\nu_m = v_o\left(\frac{3N}{4\pi V}\right)^{\frac{1}{3}}$$

where N is the number of atoms in the volume V of solid. A quantum of electromagnetic energy is known as a photon while a quantum of lattice vibrational energy is known as a phonon. These expressions for $\overline{E_o}/V$ and $\overline{E}/V$ illustrate the similarity in the behaviour of photons and phonons.

The factor 3/2 in the phonon mean energy equation arises from the fact that there are three possible polarizations for the lattice vibrational wave; one longitudinal and two transverse, whereas the longitudinal polari-

APPENDIX 4 (continued)

zation does not exist in the electromagnetic radiation. The finite upper limit of the integral in the phonon case arises from the requirement that the total number of modes of vibration of the lattice waves be equal to 3N, N being the number of atoms in the material. In the photon case, all frequencies and hence an infinite number of modes of vibration, are allowed. Consider the ratio of the energy densities: $\overline{E}_0/\overline{E} < 3v^3/2v_0^3$. The upper limit of the integral in the phonon energy expression gives rise to the nonequality sign. However this nonequality is overridden by the very large value of the ratio v^3/v_0^3. v, the photon velocity or the propagation velocity of the electromagnetic radiation, is of the order of 10^8 meters per second, whereas v_0, the phonon velocity or the propagation velocity of the lattice vibration wave, is of the order of 5×10^3 meters per second. Therefore the ratio v^3/v_0^3 will be in the vicinity of 10^{13}. Physically, this means that there are approximately 10^{13} times as many modes of vibration per unit volume for phonons as there are for photons. As a result $\overline{E}_0 \gg \overline{E}$, and also the heat capacity of a volume of typical solid material will be very much greater than the heat capacity of an equal volume of radiation. For example:

$C/V = 8.2 \times 10^{-8}$ joules deg.^{-1} meter^{-3} for radiation at $T = 300^\circ K$.

$C/V = 3.4 \times 10^{6}$ joules deg.^{-1} meter^{-3} for copper at $T = 300^\circ K$.

APPENDIX 5

The Integration of D_N:

$$D_N = 4\tau \int_0^\infty \frac{df}{1+4\pi^2 f^2 \tau^2} \cdot \frac{\sin^2 \pi f t_0}{N^2 \sin^2 \frac{\pi f t_0}{N}}$$

Consider:

$$\frac{\sin^2 Nx}{\sin^2 x} = \frac{1 - \cos 2Nx}{1 - \cos 2x} = \frac{(e^{i2\pi x} - 1)^2 e^{i2x}}{(e^{i2x} - 1)^2 e^{i2Nx}}$$

Realizing that:

$$\frac{e^{i2Nx} - 1}{e^{i2x} - 1} = \sum_{a=0}^{N-1} e^{i2ax}$$

we can obtain the result:

$$\frac{\sin^2 Nx}{\sin^2 x} = \left(\sum_{a=1}^{N} 2a \cos 2(N-a)x \right) - \frac{1}{N}$$

Returning to the integral for M_N; if we put $2\pi f\tau = y$,

we have:

$$D_N = \frac{2}{\pi N^2} \int_0^\infty \frac{dy}{1+y^2} \cdot \frac{\sin^2 \frac{t_0 y}{2\tau}}{\sin^2 \frac{t_0 y}{2\tau N}}$$

$$= \frac{2}{\pi N^2} \int_0^\infty \frac{dy}{1+y^2} \left\{ \left[\sum_{a=1}^{N} 2a \cos \frac{2(N-a) t_0 y}{2N\tau} \right] - N \right\}$$

$$= \frac{2}{N^2} \exp\left(-\frac{t_0}{\tau}\right) \sum_{a=1}^{N} a \exp \frac{a t_0}{N\tau} - \frac{1}{N}$$

With some algebraic manipulations it can be shown that:

$$\sum_{a=1}^{N} a \exp\left(\frac{a t_0}{N\tau}\right) = \frac{N \exp\left(\frac{t_0}{\tau}\right)\left[1 - \exp\left(\frac{-t_0}{N\tau}\right)\right] - \exp\left(\frac{-t_0}{N\tau}\right)\left[\exp\left(\frac{t_0}{\tau}\right) - 1\right]}{\left[1 - \exp\left(\frac{-t_0}{N\tau}\right)\right]^2}$$

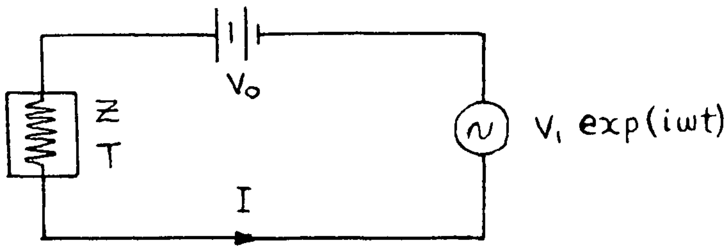
Therefore we obtain finally:

$$D_N = \frac{N\left(1 - \exp\left(\frac{-2t_0}{N\tau}\right)\right] - 2\exp\left(\frac{-t_0}{N\tau}\right)\left[1 - \exp\left(\frac{-t_0}{\tau}\right)\right]}{N^2 \left[1 - \exp\left(\frac{-t_0}{N\tau}\right)\right]^2}$$

APPENDIX 6

The Impedance of the Bolometer Element.

The circuit to be considered is:



When $I = 0$, $T = T_\alpha$, and $Z = R_\alpha$. The oscillator will cause oscillations in I, T, and Z, so we shall write:

$$V = V_o + V_1 \exp(i\omega t) \qquad Z = R_o + R_1 \exp(i\omega t)$$

$$T = T_o + T_1 \exp(i\omega t) \qquad I = I_o + I_1 \exp(i\omega t)$$

We shall assume the amplitude of the oscillation is much smaller than the dc value for all quantities. The dependence of Z upon T is given by: $Z = R_\alpha[1 + \alpha(T - T_\alpha)]$ from which we have the dc and ac components respectively:

$$R_o = R_\alpha[1 + \alpha(T_o - T_\alpha)] \tag{1}$$

$$R_1 = R_\alpha \alpha T_1 \tag{2}$$

Also we know that; $V = IZ$, the dc and ac components of which are;

$$V_o = I_o R_o \tag{3}$$

$$V_1 = I_o R_1 + I_1 R_o \tag{4}$$

The temperature response equation is:

$$C \frac{dT}{dt} = -\lambda(T - T_\alpha) + IV$$

Once again we use IV instead of the real part of IV for the reasons discussed in section 4.3.1. The dc and ac components of the temperature response equation are respectively:

APPENDIX 6 (continued)

$$\lambda(T_o - T_a) = V_o I_o \qquad (5)$$

$$(i\omega C + \lambda)\, T_1 = I_o V_1 + V_o I_1 \qquad (6)$$

From equations 2, 4, and 6, we can obtain:

$$Z \equiv \frac{V_1}{I_1} = \frac{R_o\left[\lambda + i\omega C + \alpha I_o^2 R_a\right]}{\lambda + i\omega C - \alpha I_o^2 R_a}$$

which is identical to equation 4.29.

From equations 1, 3, and 5, we have:

$$V_o = \frac{I_o R_a}{1 - \dfrac{\alpha I_o^2 R_a}{\lambda}}$$

A plot of V_o versus I_o is given in Figure 3.

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