PLANETARY WAVES IN A POLAR OCEAN

by

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ABSTRACT.

The dynamics of the Arctic ocean are studied on a polar projection of the sphere. The density structure is idealized as a two-layer system, and a general formulation is developed which allows inclusion of latitudinal and longitudinal depth variations as well as asymmetries in the boundaries of the ocean. For simplicity, the density structure is neglected when depth variations are present. Time dependent displacements from equilibrium levels are assumed to be waves of constant zonal wave number; no radial propagation is considered. Amplitude equations are derived for these displacements, subject to the assumption that the polar basin is small enough to keep only a first approximation to the curvature of the Earth.

A semi-qualitative investigation of the possible solutions is made in the case of a symmetrical basin, using the Method of Signatures, and existence criteria are found for the solutions in the presence of radial depth variations. Concentrating thereafter on planetary waves, explicit solution for such motions in the simplest case (depth constant, symmetrical boundaries) allows comparison with the results of other investigators (Longuet-Higgins, 1964 b; Goldsbrough, 1914 a). It is found that the polar projection

and first approximation to the curvature give quite good results, so that this method may be applied to polar regions in the same way as the β-plane is used in mid-latitudes.

The general effects of radial bottom slopes are discussed and a simple example treated more explicitly. Some theorems of Ball (1963) on the motions of shallow rotating fluids in paraboloidal basins are found to hold for such basins in the polar plane approximation to the sphere.

G.L. Pickard

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I. INTRODUCTION.

Considerable progress has been made in recent years in the descriptive aspects of Arctic oceanography. It seems only logical that the next step in Arctic oceanographic research should be a study of the dynamics of the Arctic ocean. There being however no observations of long-period, large scale, phenomena in those regions, it is impossible to analyse the situation from an observational basis. The problem will therefore be tackled from a theoretical point of view, so as to obtain an idea of what to look for in a future observational program. This is not an uncommon approach in oceanographic research: data collection and reduction are essential, but difficult and time consuming, and it is necessary to have some definite observational aims before going to sea.

The problem of finding the properties of the characteristic free motions of the Arctic ocean is here formulated so as to allow the inclusion of some of the main topological features of the Arctic basin; both latitudinal and longitudinal bottom and boundary variations can be formally included in the model. Most of the work applies to motions within a very wide frequency range, but one type of oscillation was chosen for closer scrutiny : the long period vorticity waves called planetary or Rossby waves.

This choice was motivated not only by a particular interest in that type of motions, but also because they can be considered as good test material for some of the approximation methods used in the analysis. It would be useful, when studying oceanographic phenomena in the polar regions, to be able to consider the influence of the Earth's curvature only parametrically, as is done in mid-latitudes in the so-called β -plane (Veronis, 1963). The analysis would be greatly simplified and the physics more easily extracted from the mathematics. The problem is thus transferred from the surface of the sphere to a plane polar projection of the Arctic regions, the curvature of the Earth being only kept as a small correction. Since planetary waves depend intimately upon that curvature (through its effect on the local component of the rotation vector) for their existence and in their properties, comparison between results obtained for Rossby waves in the polar projection with similar results derived entirely in the spherical geometry will give an indication of the applicability of this approximation.

Before tackling the problem of the dynamics of the Arctic ocean, it will be useful to review some of the information available on Arctic oceanography and bathymetry (section II), and to look into the background of investigations related to planetary waves (section III). The mathematical formulation of the problem is developed in section IV,

where the assumptions leading to simplification of the problem (linearization, inviscid flow, hydrostatic pressure) are examined in detail. In section V, the polar plane approximation is introduced, and amplitude equations are derived for solutions corresponding to free zonal waves. The physics of planetary waves are also re-examined in more detail.

Some general conclusions as to the existence and properties of the solutions in a symmetrical ocean are obtained in section VI. These apply not only to planetary waves but to all modes of free zonal oscillations in the simplified model. The results of this last section are used as a guiding beacon in the search for explicit analytical solutions for planetary waves, which are investigated in basins of increasingly complex bathymetry. In section VII planetary eigensolutions are found for a symmetric basin with a flat bottom, first by solving the strict amplitude equation in terms of confluent hypergeometric functions, then solving approximately in terms of Bessel functions. The eigenfrequencies are more accurately computed through the second solution, and comparing them with their equivalents in spherical geometry (Longuet-Higgins, 1964 b), it appears that the polar plane approximation gives good results for small enough polar basins.

The general effect of radial bottom slopes is discussed in section VIII, and a simple example treated more explicitly. The applicability of some theorems of Ball (1963) (on the finite motion of shallow fluid in a rotating paraboloid) is investigated and found to be complete in the polar plane projection. Finally, a few remarks are made in section IX about the general asymmetric case.

II. BATHYMETRY AND OCEANOGRAPHY OF THE ARCTIC OCEAN.

The exploration of the Arctic, which for centuries had been motivated by the search for a Northwest or a Northeast passage, took in the past few decades a new orientation. The areas explored have extended from the continental shelves northward into the deep basins, while the interest has broadened from geography into geophysics. The bathymetric picture of the Arctic ocean has become more detailed at the same time as the distribution of the main oceanographic variables was mapped and an idea of the oceanography obtained.

Soundings from the ice have revealed many hitherto unsuspected features of the bottom topography (Gordienko, 1961; Ostenso, 1962). The main discovery was that of the Lomonosov ridge which divides the ocean into two deep basins, the Eurasian and the Canadian basins on the Russian and the Canadian sides of the ridge respectively. They have a mean depth of around 4 km, the Eurasian basin being slightly deeper than its counterpart; the sill depth of the bisecting ridge is about 1500 m. The only deep connection with other bodies of water is through the relatively narrow straits to the Greenland Sea. It must also be noted that the ocean is not bounded symmetrically, deep water reaching as far as 72°N in the Beaufort Sea, north of Alaska, but

only to 85°N at the northern tip of Greenland. A glance at the chart (Figure 1) will reveal all the important features.

A model pretending to include the topographic features having some influence on the large scale kinematics of the Arctic ocean will have to include a ridge and asymmetrical boundaries; a simpler symmetrical model will however be studied first, in order to gain some insight into the physics of the problem. The simplified model will have only latitudinal depth variations and will be bounded at a given parallel of latitude.

The narrow but deep opening to the Atlantic will not be included in this study, and the Arctic basin will be considered closed; this is justified on the basis that such a narrow opening should have only a small kinematic influence, although it cannot be neglected when forced motions are studied, and the effect of the Atlantic tide included.

Oceanographic sampling in the Arctic has revealed that, as in other oceans, a number of different water massed can be recognized in a vertical column of water, these water masses being characterized mainly by their temperature and salinity (Coachman, 1962). We are here interested mainly in the density structure of the Arctic waters, which depends mostly on surface salinity variations produced by melting or freezing of the ice cover. Figure 2 a) shows the observed range of salinities in the upper 300 metres in the

winter and the summer. The deep waters (below 50 m) are then almost homogeneous and will be considered of uniform density. The density structure is idealized by a two-layer system (Figure 2b). The top layer is shallow (50 m) and has an average density of 1.025 gm cm^{-3} (σ_t = 25), corresponding to a temperature of $0^{o}C$ and a salinity of 32 ‰; the bottom layer is much thicker, extending to the bottom of the ocean, and is slightly denser: 1.028 gm cm^{-3} ($0^{o}C$ and 35 ‰; $\sigma_t = 28$).

The stratification is most pronounced in the summer, when the ice is melting; it is also bound to vary in intensity from place to place because of the non-uniformity of the ice cover. These variations in time and space will not be taken into account, and the stratification will be considered uniform and constant.

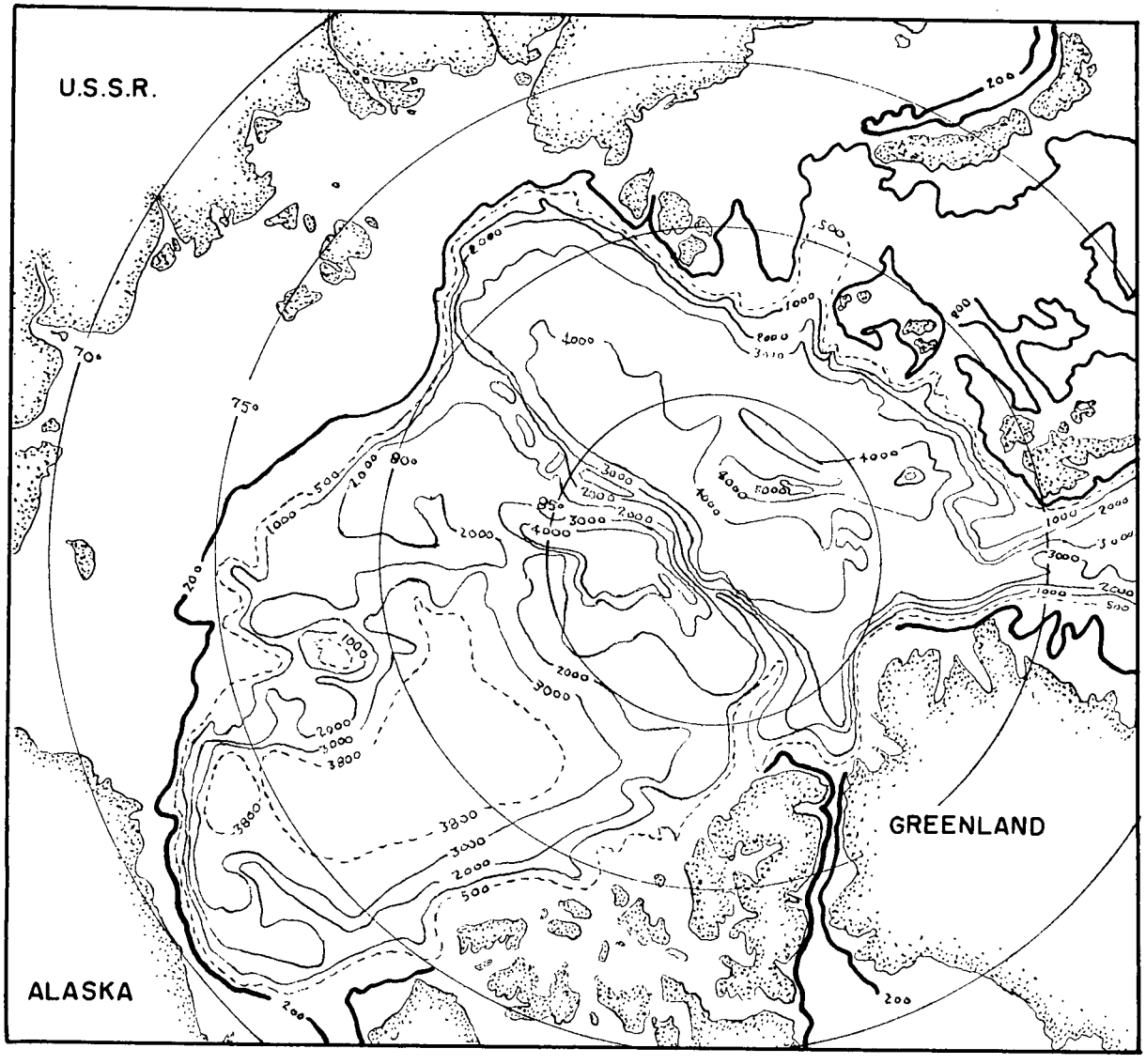
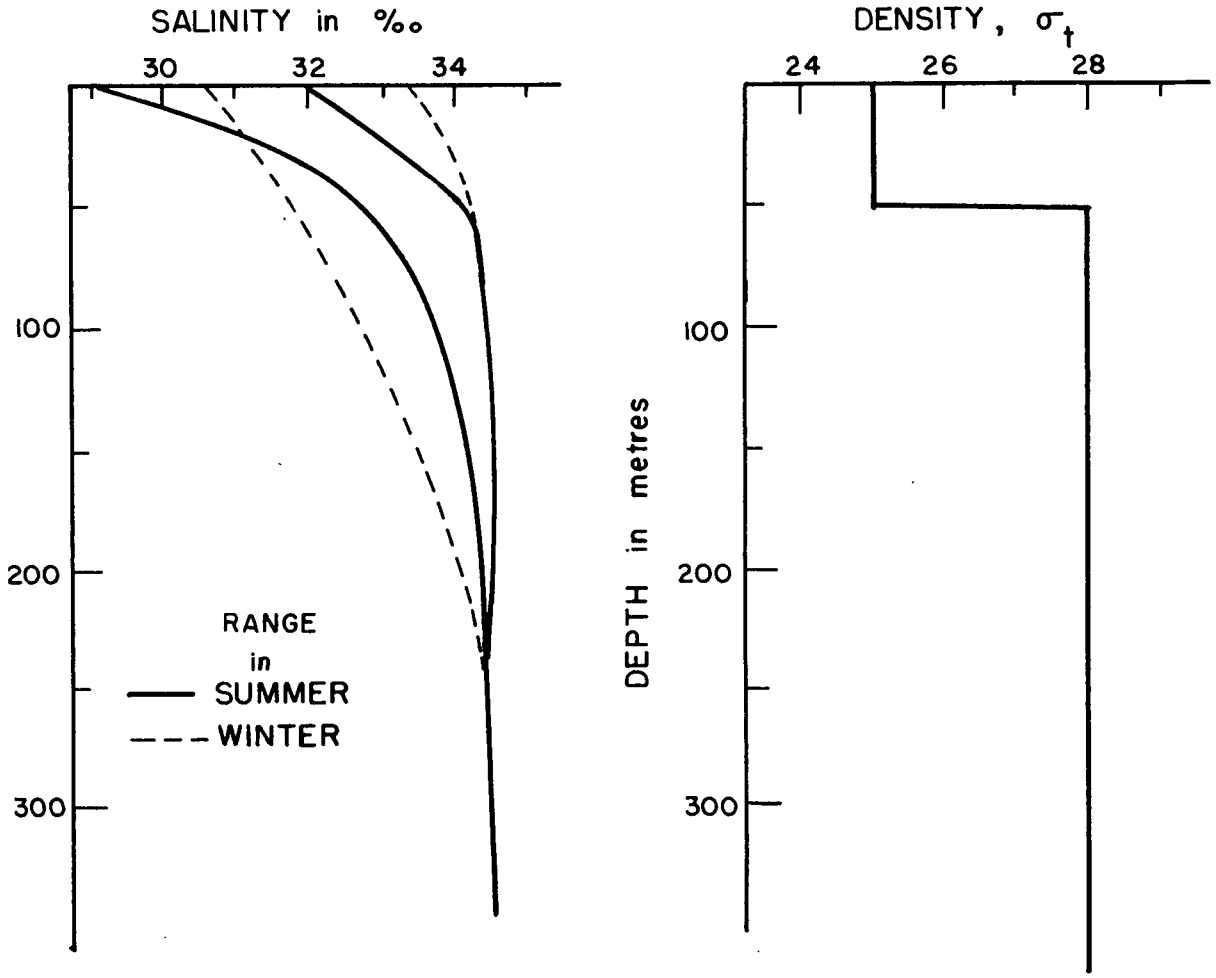


FIGURE I. BATHYMETRY OF THE ARCTIC OCEAN

(AFTER OSTENSO, 1961)

DEPTHS in METRES.



OBSERVED SALINITY STRUCTURE

a)

IDEALIZED DENSITY STRUCTURE

b)

FIGURE 2. ARCTIC DENSITY STRUCTURE.

III. PLANETARY WAVES.

The term planetary wave was coined by C.G. Rossby who encountered them in the study of time-dependent motions in a barotropic atmosphere (Rossby 1939). He identified them as vorticity waves associated with the sphericity of the Earth, and gave for them a semi-descriptive definition as "quasi-horizontal.... wave motions whose shape, wave-length and displacements are controlled by the variation of the Coriolis parameter with latitude." (Rossby, 1949).

Rossby first explained the dynamics of planetary waves by studying the effect of a velocity perturbation on a zonal current. The zonal stream does not have to be included however, and the vorticity equation for a fluid without any net transport of matter will account very well for the mechanisms involved in planetary waves. A closer look at the physics of these oscillations will be taken after the problem has been mathematically formulated, (section V). For the moment, they can be considered roughly analogous to short gravity waves, their motion being in the horizontal plane and about some latitude of equilibrium vorticity rather than in the vertical plane about a free equilibrium surface.

Planetary waves were first studied by meteorologists, and their importance in atmospheric circulation has been

closely investigated; they are now recognized as playing a major role in the heat and momentum balance of the atmosphere, as efficient large scale exchangers of these properties between low and high latitudes (Rossby, 1959). Their presence is immediately apparent on any high level (500 mb.) synoptic chart, and they have to be taken into account in any numerical forecasting scheme.

The study of planetary waves has not reached such a degree of completeness and sophistication in oceanography. This is mostly due to the presence of continents, which do not allow very large scale recognizable waves to develop, as in the atmosphere, and also complicate considerably any theoretical studies. The time scale between consecutive oceanographic measurements is also usually so large that many time dependent phenomena are not observed.

Some theoretical work has however been done on planetary waves. One should mention the studies of Veronis and Stommel (Veronis, 1956; Veronis and Stommel, 1956) who investigated the response of an infinite ocean to variable wind stresses, and found that for periods of more than one pendulum day, a considerable portion of the energy is transferred into quasi-geostrophic planetary waves. Also, since it was discovered by Stommel (Stommel, 1948) that the western intensification of wind-induced ocean currents depends on the curvature of the Earth, as do the Rossby waves, some work has been done on the influence of Rossby

waves on ocean circulation (Moore, 1963) and on the intensified boundary currents themselves (Warren, 1963). Although planetary waves might possibly not play as important a role in oceanic circulation as they do in atmospheric circulation, they seem to be of great importance in many time-dependent oceanographic phenomena, and this importance will probably be realized more fully as more research is done on the subject.

It was recognized some time after Rossby's work (Stommel, 1957) that there were antecedents in the literature for the semi-geostrophic planetary waves; oscillations of very much the same nature had been treated by tidal theorists under the appellation of Oscillations of the Second Class. This classification of the Oscillations of an ocean of constant depth on a rotating sphere into two classes was made by Hough (1898) after he found that his integration of the Laplace tidal equation under these conditions yielded two categories of solutions, characterized by the asymptotic behaviour of their periods as the frequency of rotation of the Earth decreases to zero. The Oscillations of the First Class have frequencies which tend to finite values as the rotation of the Earth vanishes; in those of the Second Class the frequencies tend to zero and they become steady motions on a non-rotating globe (see also Poincaré, 1910).

This classification has by some authors (e.g. Eckart,

1960; Chapter XVII, page 275) been attributed to Laplace, but Laplace's three species of tides were something entirely different, reflecting the longitudinal symmetry of the components of the tide producing forces, and had nothing to do with the asymptotic limit at vanishing rotation rates which Hough investigated. The label Laplace Oscillations of the Second Class is therefore erroneous, and it would be preferable to attach Hough's name to planetary waves.

The work done in the theory of tides was, then, available to oceanographers in their investigations of planetary waves; unfortunately, all of it was done in spherical geometry. This is quite appropriate to the study of phenomena on a sphere, but makes the analysis extremely difficult for enclosed oceans of variable depth. This difficulty was evident from the beginning, and Laplace himself, speaking of the general tidal amplitude equation, said that "L'intégration de l'équation dans le cas général où n (Earth's rotation) n'est pas nul, et où la mer a une profondeur variable, surpasse les forces de l'analyse;" (Laplace, 1799; Première partie, Livre IV, Chapitre I). Little progress has been made in that respect since Laplace's time. Some special cases have been studied by Goldsbrough for an ocean of constant depth bound by meridians (Goldsbrough, 1933), or by two parallels of latitude (Goldsbrough, 1914 b), or of rectangular form

(Goldsbrough, 1931). Love (1913) studied the case of a small circular ocean, and the only case of sloping bottom was treated by Proudman (1916), the depth varying only with latitude. It is then not surprising that tidal theory was no more useful to the study of planetary waves in actual oceans than it had been in predicting the amplitudes of the tides in these same oceans, and for the same reasons.

Rossby's analysis was very much simplified by his assumption that the characteristic effects of the terrestrial curvature could be preserved by making the Coriolis parameter vary linearly with latitude over a short distance. This approximation has been termed the β -plane approximation, and has been used extensively, both in oceanography and in meteorology, since its introduction by Rossby (Rossby, 1939). The theoretical work done on planetary waves in oceanography has nearly all been performed on the β -plane; the mathematics are thus quite simplified and a clear idea of the physical happenings over a restricted band of latitudes can be arrived at without the use of spherical geometry.

Concurrently with this work, Longuet-Higgins (1964 b) has introduced another very useful simplification which allows even clearer physical understanding of planetary waves. Taking advantage of the quasi-horizontal nature of the waves, he neglects the influence of surface displacements in the vorticity balance, reducing the problem to a two-dimensional one, which can be solved in terms of a stream

function. This approximation is valid for wave lengths smaller than the radius of the Earth. Longuet-Higgins uses it with the β -plane to examine the behaviour of planetary waves in basins of a variety of shapes, and shows that in the limit of small wave lengths the frequencies of the oscillations in the β -plane are identical to those on the sphere. This approximation is also used to advantage on the sphere; in particular, waves between two parallels of latitude are considered, of which waves in a polar ocean are just a special case. The problem is not treated in detail however, and since it is essentially two-dimensional, no depth variations can be included.

Bottom variations are included in my work, and the analysis is performed in a special projection about the pole to avoid the complexities of spherical geometry. Longuet-Higgins' approximation cannot be made if the bottom is not flat, so that the formulation of the problem is intermediate in complexity between the strict tidal problem of Laplace and Longuet-Higgins' assumption that the motions are purely two-dimensional.

IV. FORMULATION OF THE PROBLEM.

The motions representative of the Arctic ocean will be those depending, either for their existence or in modifications of their properties, on the geometry or the density structure of the Arctic ocean. Short gravity waves certainly do not fall into that category, and we can assume that the wave motions which will reflect in their characteristics the properties of the Arctic will be of periods of at least many hours and of scales of more than a few kilometres; this includes planetary waves at the lower frequency limit. With this fundamental assumption and a few secondary ones the mathematical formulation can be considerably simplified.

The movements of an incompressible fluid in a rotating frame of reference are described by the Navier-Stokes equation:

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} + \frac{1}{\rho}\nabla P - \nu\nabla^2 \underset{\sim}{u} + \underset{\sim}{g} = 0 \qquad (1)$$

and the continuity equation

$$\nabla \cdot \underset{\sim}{u} = 0 \quad , \qquad (2)$$

in which $\underset{\sim}{u}(\underset{\sim}{x},t)$ is the fluid velocity relative to the coordinate system $\underset{\sim}{x}$ rotating with an angular velocity Ω , P is the pressure, ρ the fluid

density, ν the kinematic viscosity of the fluid and $\underset{\sim}{g}$ the net acceleration due to gravity and the centrifugal force due to the Earth's rotation.

Assuming that the relative motion has a velocity scale U, a time scale τ and a horizontal length scale L, the relative magnitudes of the successive terms of (1) are, after division by ΩU,

$$\frac{1}{\Omega\tau}, \quad \frac{U}{\Omega L}, \quad 1, \quad \frac{P}{\rho\Omega UL}, \quad \frac{\nu}{\Omega L^2}, \quad \frac{g}{\Omega U} .$$

The horizontal balance in most oceanic flows is between pressure and Coriolis forces. Are we justified here in keeping only these terms and neglecting the other ones? The time derivative term will be comparable to the Coriolis term for periods of the order of a day, but even when we are interested in time-dependent motions of much longer periods, we cannot neglect that term, however small its influence in the horizontal momentum balance, because it contains one of our unknowns, the period of the motion. This importance of the time derivative term is made more evident by looking at the vorticity equation:

$$\frac{\partial \underset{\sim}{\zeta}}{\partial t} - \nabla\times\left[\underset{\sim}{u}\times(\underset{\sim}{\zeta} + 2\underset{\sim}{\Omega})\right] - \nu\nabla^2\underset{\sim}{\zeta} = 0,$$

where $\underset{\sim}{\zeta} = \nabla\times\underset{\sim}{u}$. When, as will be assumed below, the viscous and non-linear terms are of little importance,

the only terms left in the vorticity equation are the time rate of change term and the Coriolis term.

If, as originally assumed, the scale of motion, L, is large enough, the viscous terms will be of little influence in the Navier-Stokes equation:

$$\frac{\nu}{\Omega L^2} \approx \frac{\nu \text{ sec}}{10^{-4} L^2} << 1$$

This is certainly true if ν is the molecular viscosity (0.01 cm^2 sec^{-1} for water); even for an eddy viscosity as high as 10^6 $cm^2 sec^{-1}$, the inequality holds strongly for $L \geq 10$km. The influence of viscous forces will therefore be ignored and the fluid treated as inviscid. There are no dissipative processes and, if the ocean is bounded, no energy can be radiated away so that the total energy of the water mass will be constant.

The amplitude of the motions to be studied will be assumed small enough to make the non-linear terms of negligible influence: the Rossby number is small and

$$\frac{U}{\Omega L} \approx \frac{U \text{sec}}{10^{-4} L} << 1$$

Since Ω varies very little over the area considered, the ratio of velocity to horizontal scale (a measure of the relative importance of the local vertical component of the vorticity of the fluid to the planetary vorticity) is the most critical parameter in this study; it may depend on position, and if so, it must remain finite everywhere in

the domain in which we use the linearized equations. Also, since the total energy content of the system is an arbitrary parameter and has a direct influence on the amplitude of the velocities encountered, the Rossby number is determined only within an arbitrary multiplier. Linearization is thus appropriate to low energy systems, and its validity will have to be vindicated a posteriori by showing that the Rossby number remains analytic over the whole Arctic ocean.

A further assumption which can be made when the periods are long enough is to neglect vertical accelerations of the fluid compared to gravity. Using the criteria given by Proudman (1953; Chapter XI, p 223) for the validity of this simplification, one must have, in a homogeneous column of water,

$$\frac{H \dfrac{\partial^2 \eta}{\partial t^2}}{g\eta} = \frac{H}{g\tau^2} << 1$$

in which η is the displacement of the surface from its equilibrium position, g is gravity, and H the depth (here 4 km.). This will hold for periods longer than 10 minutes, so that the condition is not very restrictive. The equivalent criterion for the motions of the interface of a two-layer system is (Proudman, 1953; Chapter XV, p 336)

$$\rho H_2 \frac{\partial^2 \eta_2}{\partial t^2} \Big/ g\eta_2 \Delta\rho = \frac{\Delta\rho}{\rho} \frac{H_2}{g\tau^2} << 1$$

in which η_2 is the displacement of the interface, H_2 the thickness of the lower layer (again nearly 4 km.), the density of the lower layer, and $\Delta\rho$ the density difference between the two-layers (3.10^{-3} gm cm^{-3}). This condition is more binding than the previous one, but internal waves are in general of longer period than surface waves of the same dimensions; the period in that case must be longer than 12 hours.

In order to take the pressure as completely hydrostatic, it is also necessary that the vertical component of the Coriolis force be much smaller than gravity:

$$\frac{\Omega\, U \sin\theta}{g} << 1$$

in which θ is as defined in Figure 3. This depends on the total energy, as does the Rossby number, but the condition that it remains analytic at the pole is less stringent since $\sin\theta$ vanishes there.

After having been subjected to all the above simplifications, equations (1) and (2) appear as below; they are written out in the spherical polar coordinates shown in Figure 3.

$$\frac{\partial v}{\partial t} + 2\Omega u\cos\theta - 2\Omega w\sin\theta = \frac{-1}{\rho r\sin\theta}\frac{\partial P}{\partial\lambda} \qquad (3)$$

$$\frac{\partial u}{\partial t} - 2\Omega v\cos\theta = \frac{-1}{\rho r}\frac{\partial P}{\partial\theta} \qquad (4)$$

$$\frac{\partial P}{\partial r} = -\rho g \tag{5}$$

$$\frac{1}{r\sin\theta}\frac{\partial v}{\partial \lambda} + \frac{1}{r\sin\theta}\frac{\partial u\sin\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

The assumptions now about to be made are based on the geometry and situation of the Arctic ocean. In any ocean, the maximum depth is extremely small compared to the radius of the Earth. In the above equations, r will then be replaced by a constant radius, R, and d/dr by d/dz, where z is the local vertical, measured upwards from the

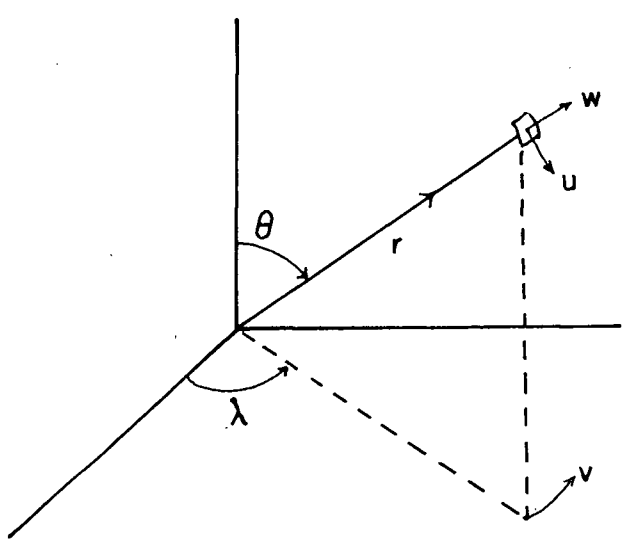


Figure 3. The spherical coordinates and the corresponding velocity components.

water surface. Furthermore, since we are working close enough to the pole for $\tan\theta$ to be quite small, and since the horizontal velocities are in general much larger than the vertical velocities for long waves, we will assume that

$$U\cos\theta \gg W\sin\theta \quad ,$$

so that only one Coriolis term is retained in (3).

The pressure forces can be replaced by equivalent expressions containing the gradients of the surface ($z = \eta_1$) and interface ($z = \eta_2$) displacements from an equilibrium level. If a two-layer structure as illustrated in Figure 4 is adopted, and the constant atmospheric pressure neglected, integration of the hydrostatic equation, (5), gives for the pressures in the top and bottom strata respectively

$$P_1 = g\,\rho_1(\eta_1 - z) \tag{7}$$

$$P_2 = g\left\{\rho_1(\eta_1 - \eta_2 + H_1) + \rho_2(\eta_2 - H_1 - z)\right\} \tag{8}$$

The momentum equations for the two layers are as follows when displacement gradients are substituted for pressure gradients; the continuity equation (6) is unchanged and has the same form in both layers ($\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ are the velocities in the upper and lower layer).

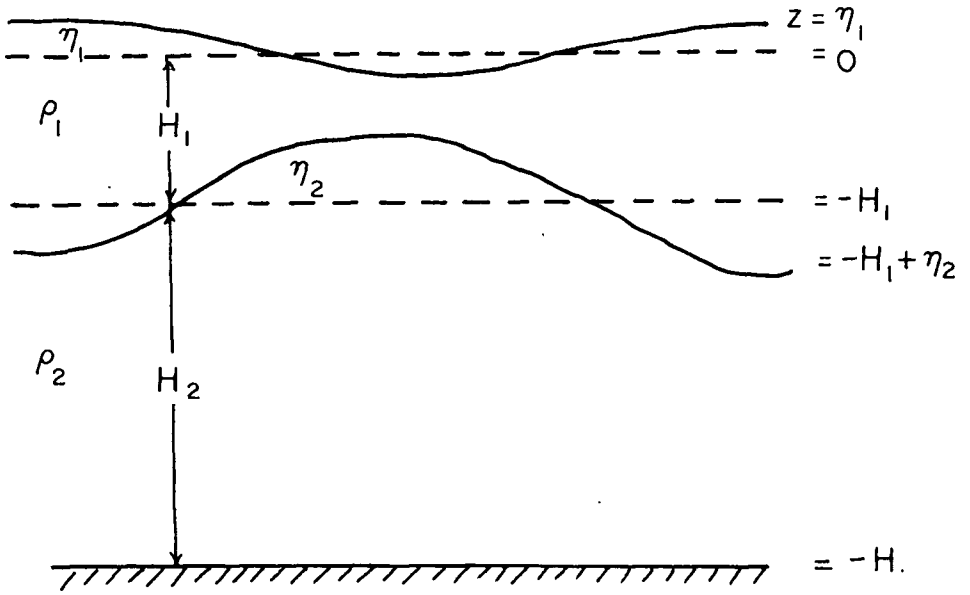


Figure 4. The two-layer system adopted for the density structure of the Arctic ocean.

$$\frac{\partial v_1}{\partial t} + 2\Omega u_1 \cos\theta = \frac{-g}{R\sin\theta}\frac{\partial \eta_1}{\partial \lambda} \tag{9}$$

$$\frac{\partial u_1}{\partial t} - 2\Omega v_1 \cos\theta = \frac{-g}{R}\frac{\partial \eta_1}{\partial \theta} \tag{10}$$

$$\frac{\partial v_2}{\partial t} + 2\Omega u_2 \cos\theta = \frac{-g}{R\sin\theta}\left[\frac{\rho_1}{\rho_2}\frac{\partial \eta_1}{\partial \lambda} + \frac{\Delta\rho}{\rho_2}\frac{\partial \eta_2}{\partial \lambda}\right] \tag{11}$$

$$\frac{\partial u_2}{\partial t} - 2\Omega v_2 \cos\theta = \frac{-g}{R}\left[\frac{\rho_1}{\rho_2}\frac{\partial \eta_1}{\partial \theta} + \frac{\Delta\rho}{\rho_2}\frac{\partial \eta_2}{\partial \theta}\right] \tag{12}$$

The following boundary conditions complete the description of the physical situation. The velocity

component perpendicular to a fixed boundary (with normal $\underset{\sim}{n}$) will vanish at that boundary: ($\underset{\sim}{u} \cdot \underset{\sim}{n} = 0$).
In particular, at the bottom,

$$w_2 \Big|_{z=-H} = \frac{-1}{R} \frac{\partial}{\partial \theta} H u_2 \Big|_{z=-H} = \frac{-1}{R \sin\theta} \frac{\partial}{\partial \lambda} H v_2 \Big|_{z=-H} , \tag{13}$$

and at a vertical wall at latitude θ_1,

$$u = \frac{v}{\sin\theta_1} \frac{\partial \theta_1}{\partial \lambda} . \tag{14}$$

The average depth of the interface, H_1, is considered constant: since it is determined by surface phenomena and by any mean currents (here zero), it will not be influenced by depth variations.

A linear surface boundary condition is used:

$$w_1 \Big|_{z=\eta_1} = \frac{\partial \eta_1}{\partial t} \tag{15}$$

and a similar condition at the interface, together with continuity of vertical velocities:

$$w_1 \Big|_{z=-H_1+\eta_2} = w_2 \Big|_{z=-H_1+\eta_2} = \frac{\partial \eta_2}{\partial t} \tag{16}$$

No non-slip conditions have been imposed, either at the boundaries or at the interface, since there is no viscosity in the model. A further general requirement is that all variables remain finite over the entire ocean, and especially at the pole, where singularities are likely to occur.

The equations (9)-(12) together with the continuity equation (6) are now integrated over their respective layers, subject to the above conditions. One then has for the top layer:

$$\frac{\partial v_1}{\partial t} + 2\Omega u_1 \cos\theta = \frac{-g}{R\sin\theta}\frac{\partial \eta_1}{\partial \lambda} \tag{17}$$

$$\frac{\partial u_1}{\partial t} - 2\Omega v_1 \cos\theta = \frac{-g}{R}\frac{\partial \eta_1}{\partial \theta} \tag{18}$$

$$\frac{1}{R\sin\theta}\frac{\partial v_1}{\partial \lambda} + \frac{1}{R\sin\theta}\frac{\partial u_1 \sin\theta}{\partial \theta} + \frac{1}{H_1}\frac{\partial}{\partial t}(\eta_1 - \eta_2) = 0 \tag{19}$$

and for the bottom layer:

$$\frac{\partial v_2}{\partial t} + 2\Omega u_2 \cos\theta = \frac{-g}{R\sin\theta}\left[\frac{\rho_1}{\rho_2}\frac{\partial \eta_1}{\partial \lambda} + \frac{\Delta\rho}{\rho_2}\frac{\partial \eta_2}{\partial \lambda}\right] \tag{20}$$

$$\frac{\partial u_2}{\partial t} - 2\Omega v_2 \cos\theta = \frac{-g}{R}\left[\frac{\rho_1}{\rho_2}\frac{\partial \eta_1}{\partial \theta} + \frac{\Delta\rho}{\rho_2}\frac{\partial \eta_2}{\partial \theta}\right] \tag{21}$$

$$\frac{1}{R\sin\theta}\frac{\partial v_2 H_2}{\partial \lambda} + \frac{1}{R\sin\theta}\frac{\partial u_2 H_2 \sin\theta}{\partial \theta} + \frac{\partial \eta_2}{\partial t} = 0 \tag{22}$$

These are the equations I shall attempt to solve, although in a different coordinate system. Before performing the transformation, a few words should be said concerning the influence of an ice cover on surface waves. Such a situation has been studied by Ewing and Crary (1934); their results show that the effect is negligible when the wave length is very large compared with the thickness of the ice. We are already limited by the hydrostatic assumption to periods longer than 10 minutes; waves of such periods will have phase speeds slightly less than 200 m sec^{-1} in a 4 km deep ocean, and correspondingly a scale of over a hundred kilometers. There is therefore no reason to worry about distortion due to the ice cover in the rest of this work. For shorter periods however, flexural-gravity waves will be observed to differ from pure gravity waves; observation of such shorter period motions has been made by Hunkins (1962) from floating research stations in the Arctic.

A tabulation of the values of the physical parameters used in this study will be quite useful, and I finish this section with such a list (Table I).

TABLE I. Values of the basic physical parameters used in this work.

QUANTITY.	SYMBOL.	NUMERICAL VALUE.
Angular velocity of the Earth.	Ω	0.7×10^{-4} sec^{-1}
Radius of the Earth.	R	6370 km.
Gravitational attraction.	g	10^{3} cm sec^{-2}
Depth of Arctic ocean (average).	H	4 km.
Thickness of surface layer.	H_1	50 m.
Radius of Arctic basin (average).	r_1	1500 km.
Density structure.	$\frac{\Delta\rho}{\rho}$	3×10^{-3}

V. THE POLAR PROJECTION.

i) Transformation of the equations.

We have now formulated the general problem of the dynamics of a polar basin on a rotating sphere, as per equations (17)-(22) and (14). We have also seen however in section III that the work of four generations of tidal theorists vindicates Laplace's opinion of the difficulty of such a general problem. The problem in its present form seems a bit hopeless, but any simplification should not be so drastic as to hide interesting physical phenomena. The analytical intractability stems from the combined influences of the spherical geometry and the variable topography; Rossby (1939) introduced the β-plane approximation to circumvent the difficulty. In the β -plane (Veronis, 1963), the surface of the sphere is transposed by a Mercator projection, so that parallels of latitude become straight lines, and, if a narrow band of latitudes is considered, the main influence of the curvature of the sphere can be reproduced by making the Coriolis parameter a linear function of latitude.

It seems therefore natural to try and project the polar regions on to such a plane, where only first approximations to the curvature would be kept. The β-plane being a Mercator projection, it is not applicable near the pole, and some other projection will have to be used in the present problem. If we are able to

show that the analysis of the motions on such a projection gives results not too different from those which have been obtained on the sphere, we then have a tool which can be used in the polar regions with the same confidence as the β-plane is used in mid-latitudes. Judging from the amount of light shed by the β -plane analyses on the meteorological and oceanographic situations in mid-latitudes, it would indeed be very valuable to have such a method available in the polar regions.

There are many ways of projecting the surface of the sphere in the polar regions on to a plane; I choose here what I think is the simplest one. An orthographic projection is made on to a plane tangent to the sphere at the pole (Figure 5). If plane polar coordinates r and ϕ are used on the plane of projection, the geometry of the mapping imposes the following relationship with the spherical coordinates:

$$dr = R \cos\theta \, d\theta \; ; \quad r = R \sin\theta$$
$$d\phi = d\lambda \qquad (23)$$

The working approximation, which will be introduced after all differentiations have been performed, will consist of neglecting terms of order $(r/R)^2$ with respect to unity. $\cos\theta$ then becomes equal to unity when not differentiated, and $d(\cos\theta)/dr$ is approximately $-r/R^2$. Only first approximations to the curvature are then kept, and this

will evidently be valid only for basins of small latitudinal extent about the pole. The southernmost corner of the Arctic ocean is in the Beaufort Sea, at 72°N: θ is then less than 18°, and $\sin^2\theta$ less than 0.095, so that the terms neglected are smaller than unity by at least an order of magnitude over the whole Arctic basin.

The main difference between this plane projection and the conventional β -plane is that the derivative of the Coriolis parameter is not a constant, but varies linearly with latitude.

Transfer of equations (17)-(22) to the polar plane by means of (23) gives in the new geometry:

$$\frac{\partial v_1}{\partial t} + 2\Omega u_1 \cos\theta = \frac{-g}{r}\frac{\partial \eta_1}{\partial \phi} \tag{24}$$

$$\frac{\partial u_1}{\partial t} - 2\Omega v_1 \cos\theta = -g \cos\theta \frac{\partial \eta_1}{\partial r} \tag{25}$$

$$\frac{1}{r}\frac{\partial v_1}{\partial \phi} + \frac{\cos\theta}{r}\frac{\partial u_1 r}{\partial r} = \frac{1}{H_1}\frac{\partial}{\partial t}(\eta_2 - \eta_1) \tag{26}$$

$$\frac{\partial v_2}{\partial t} + 2\Omega u_2 \cos\theta = \frac{-g}{r}\left[\frac{\rho_1}{\rho_2}\frac{\partial \eta_1}{\partial \phi} + \frac{\Delta\rho}{\rho_2}\frac{\partial \eta_2}{\partial \phi}\right] \tag{27}$$

$$\frac{\partial u_2}{\partial t} - 2\Omega v_2 \cos\theta = -g \cos\theta\left[\frac{\rho_1}{\rho_2}\frac{\partial \eta_1}{\partial r} + \frac{\Delta\rho}{\rho_2}\frac{\partial \eta_2}{\partial r}\right] \tag{28}$$

$$\frac{1}{r}\frac{\partial v_2 H_2}{\partial \phi} + \frac{\cos\theta}{r}\frac{\partial u_2 H_2 r}{\partial r} + \frac{\partial \eta_2}{\partial t} = 0 \quad . \tag{29}$$

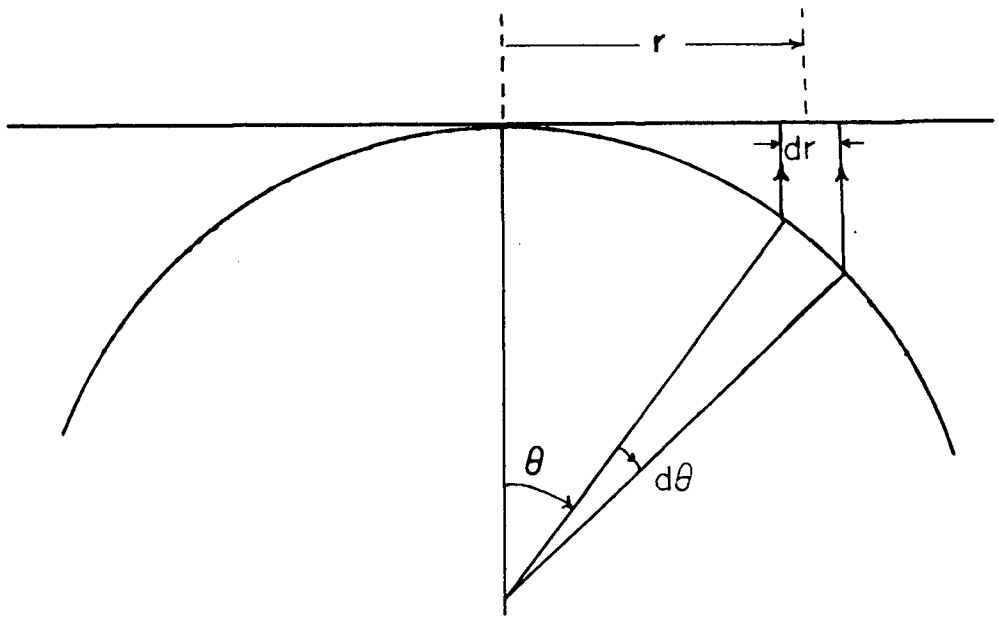


Figure 5. Orthographic projection of the surface of the sphere on to a plane.

Note that two additional terms involving cosθ appear, besides those associated with the Coriolis parameter; they account for two geometrical characteristics of the mapping. Equal areas on the sphere map on to progressively more exiguous areas of the plane as the latitude decreases; also, whereas on the sphere the angle between two meridians decrease steadily from the pole to lower latitudes, it remains constant on the plane of projection.

ii) Time dependent solutions.

Now let us look for time dependent solutions of (24)-(29) of the form

$$\eta_j = F_j(r,\phi)\, e^{i(\omega t - s\phi)} \tag{30}$$

in which ω is the frequency, s the constant azimuthal wave number, $F_j(r,\phi)$ an amplitude function to be determined, and the subscript j is used with the values 1 or 2 to denote surface or interface displacements, respectively. The derivatives of the displacements are written as

$$\frac{\partial \eta}{\partial t} = i\omega\eta \tag{31}$$

$$\frac{\partial \eta}{\partial r} = -i\sigma\eta \quad ; \sigma = \frac{i}{F}\frac{\partial F}{\partial r} \tag{32}$$

$$\frac{\partial \eta}{\partial \phi} = -i\gamma\eta \quad ; \gamma = s + \frac{i}{F}\frac{\partial F}{\partial \phi} \tag{33}$$

The subscripts have been left out of the above since these relations are independent of j. With the help of the relations (30)-(33), the momentum equations can be solved explicitly for the velocities in terms of displacements: the pairs (24)-(25) and (27)-(28) become

$$u_1 = g\, G_1\, \eta_1 \quad ; \quad v_1 = g\, J_1\, \eta_1 \tag{34}$$

$$u_2 = g\, G_1 \frac{\rho_1}{\rho_2} \eta_1 + g\, G_2 \frac{\Delta\rho}{\rho_2} \eta_2$$

$$v_2 = g\, J_1 \frac{\rho_1}{\rho_2} \eta_1 + g\, J_2 \frac{\Delta\rho}{\rho_2} \eta_2 \tag{35}$$

The functions G and J are abbreviations used for temporary convenience and defined by

$$G_j = \frac{(-\omega\, \sigma_j \cos\theta + i\, \gamma_j 2\Omega \cos\theta / r)}{(4\Omega^2 \cos^2\theta - \omega^2)} \tag{36}$$

$$J_j = \frac{(-\gamma_j\, \omega / r - i\, \sigma_j 2\Omega \cos^2\theta)}{(4\Omega^2 \cos^2\theta - \omega^2)} \tag{37}$$

Complete elimination of the velocities is achieved by substitution of (34) and (35) into the continuity equations (26) and (29). In terms of the abbreviated notation, G and J, the top layer equation becomes

$$\left\{ \cos\theta \left(i \frac{\partial G_1}{\partial r} + \sigma_1 G_1 + \frac{i G_1}{r} \right) + \gamma_1 \frac{J_1}{r} \right\} \eta_1 = \frac{\omega}{g H_1} (\eta_1 - \eta_2) \tag{38}$$

and the bottom layer equation

$$\left\{\cos\theta\left(\frac{i\,\partial G_1}{\partial r} + \sigma_1 G_1 + \frac{iG_1}{r}\right) + \frac{\gamma_1 J_1}{r}\right\}\frac{\rho_1}{\rho_2}\,\eta_1$$

$$+\left\{\cos\theta\left(i\,\frac{\partial G_2}{\partial r} + \sigma_2 G_2 + \frac{iG_2}{r}\right) + \frac{\gamma_2 J_2}{r}\right\}\frac{\Delta\rho}{\rho_2}\,\eta_2$$

$$+\left\{\frac{i\cos\theta}{H_2}\,\frac{\partial H}{\partial r}\,G_1 + \frac{iJ_1}{H_2\,r}\,\frac{\partial H}{\partial\phi}\right\}\frac{\rho_1}{\rho_2}\,\eta_1 \tag{39}$$

$$+\left\{\frac{i\cos\theta}{H_2}\,\frac{\partial H}{\partial r}\,G_2 + \frac{iJ_2}{H_2 r}\,\frac{\partial H}{\partial\phi}\right\}\frac{\Delta\rho}{\rho_2}\,\eta_2$$

$$= \frac{\omega\,\eta_2}{g\,H_2}$$

Let us look first at the case where the depth is constant and the ocean symmetric; the stratification will be retained only for this simple geometry. Even though the terms containing derivatives of the depth disappear from (39), it is not formally possible to eliminate F_1 or F_2 from (38) and (39). If, however, one considers that the ratio of the amplitudes of the surface and interface displacements depends only upon the density structure (relative depths and densities of layers), so that for a constant depth and uniform stratification this ratio is not a function of position, the subscripts are

no longer necessary in (38) and (39). Since now $F_1(r)/F_2(r)$ is a constant, the logarithmic derivative of either of these amplitudes is the same, and the variables σ_j and γ_j and J_j and G_j assume the same value for $j = 1$ and $j = 2$. Equation (39) then reduces to

$$\left\{\cos\theta\left(\frac{i\,dG}{dr} + \sigma G + \frac{iG}{r}\right) + \frac{sJ}{r}\right\}\left(\frac{\rho_1}{\rho_2}\eta_1 + \frac{\Delta\rho}{\rho_2}\eta_2\right) = \frac{\omega\eta_2}{g\,H_2} \qquad (40)$$

Eliminating the terms containing G and J between (38) and (40), a quadratic in the ratio of the amplitudes is obtained:

$$\left(\frac{F_1}{F_2}\right)^2 + \left(\frac{F_1}{F_2}\right)\left[\frac{\Delta\rho}{\rho_2} - \frac{H}{H_2}\right] - \frac{\Delta\rho}{\rho_2} = 0 \qquad (41)$$

Provided, as is the case here, that $\rho_1/\rho_2 \simeq 1$, and that $H/H_2 \simeq 1$, good approximations to the roots of (41) are

$$\frac{F_1}{F_2} = \frac{H}{H_2} \quad ; \quad \frac{F_1}{F_2} = -\frac{\Delta\rho}{\rho} \quad .$$

The first root corresponds to oscillations of the water mass as if it were homogeneous; the two displacements are in phase and nearly equal in amplitude. The second root can be identified with internal oscillations at the interface; the surface amplitude is much smaller and out of phase with the interface amplitude.

Note that it is possible to derive this result without the above mild assumption on the constancy of the

ratio of the amplitudes; if, following Rattray (1964), one eliminates the displacements rather than the velocities from (24)-(29), it is possible to separate formally the homogeneous and internal oscillations by introducing new dependent variables

$$u' = u_1 + u_2$$

$$u'' = \frac{H_2}{H} u_1 - \left[\frac{H_1}{H} + \frac{\Delta\rho}{\rho} \frac{H_1^2}{H^2} \right] u_2$$

This is indeed an ideal method when the bathymetric effects are neglected, but since u" implicitly contains the depth, I prefer to retain the other formulation as more convenient in studying the effects of bottom variations.

Substituting for the constant roots of (41), (38) can be written as

$$\cos\theta \left(i \frac{dG}{dr} + \sigma G + \frac{iG}{r} \right) + \frac{sJ}{r} = \frac{\omega}{gH_1} (1 - \eta_2/\eta_1) , \qquad (42)$$

in which η_1/η_2 is the constant appropriate either to the homogeneous or to the internal mode; the right hand side takes the form ω/gH and $\frac{\omega}{gH_1} \frac{\rho}{\Delta\rho}$ respectively in those two cases. Equation (42) is an ordinary differential equation in only one variable, F(r), and will be solved in section VII; it needs however be put in a more explicit form, and it is more convenient to do so right now.

Replacing G and J by their definition in (36), an intermediate equation follows:

$$\left(i\omega\sigma \cos^2\theta + 2\Omega\cos^2\theta \frac{s}{r} \right)\left(- 8\Omega^2 \frac{\sin\theta}{r} \right)$$

$$+ \left(4\Omega^2\cos^2\theta - \omega^2 \right)\Bigg[- i\omega \cos\theta \frac{d\sigma}{dr} + \frac{i\omega\sigma}{R}\tan\theta - \omega\frac{s^2}{r^2}$$

$$+ 2\Omega\frac{\tan\theta}{R}\frac{s}{r} - \omega\sigma^2\cos\theta + 2\Omega\cos\theta\,\sigma\frac{is}{r} - \frac{i\omega\sigma}{r}\cos\theta \qquad (43)$$

$$- 2\Omega\cos^2\theta \frac{is\sigma}{r}\Bigg]$$

$$= \left(4\Omega^2\cos^2\theta - \omega^2 \right)\frac{\omega}{gH_1}\left(1 - \eta_2/\eta_1 \right)$$

It is at this point that the approximation of the β-plane type is made; all differentiations have been performed, and no information will be lost by writing

$$\cos\theta = 1;$$

$$\sin\theta = \tan\theta = r/R.$$

Hence, the Coriolis parameter, f, and its derivative with respect to r, β, which, when expanded in r/R, are

$$f = 2\Omega\cos\theta = 2\Omega\left(1 - r^2/R^2\right)^{\frac{1}{2}}$$

$$\beta = -\frac{2\Omega}{R}\tan\theta = -\frac{2\Omega r}{R^2}\left(1 - r^2/R^2\right)^{-\frac{1}{2}}$$

become

$$f \simeq 2\Omega$$

$$\beta \simeq \frac{-2\Omega r}{R^2} \quad .$$

Neglecting terms in $(r/R)^2$ with respect to 1, (43) simplifies to

$$(4\Omega^2 - \omega^2)\left[-\sigma^2 - \frac{id\sigma}{dr} - \frac{i\sigma}{r} + \frac{i\sigma r}{R^2} - \frac{s^2}{r^2}\right] - 8\Omega^2 \frac{ir\sigma}{R^2} \tag{44}$$

$$-\frac{2\Omega s}{\omega R^2}(4\Omega^2 + \omega^2) = \frac{(4\Omega^2 - \omega^2)}{gH_1}(1 - \eta_2/\eta_1) \quad ,$$

which, after writing σ in terms of the amplitude function, F, through (32), becomes

$$\frac{d}{dr}\left(\frac{1}{F}\frac{dF}{dr}\right) + \frac{1}{F}\frac{dF}{dr}\left(\frac{1}{F}\frac{dF}{dr} + \frac{1}{r}\frac{(4\Omega^2 + \omega^2)r}{(4\Omega^2 - \omega^2)R^2}\right)$$

$$-\frac{s^2}{r^2} - \frac{2\Omega s}{R^2}\frac{(4\Omega^2 + \omega^2)}{(4\Omega^2 - \omega^2)} \tag{45}$$

$$-\frac{(4\Omega^2 - \omega^2)}{gH_1}(1 - \eta_2/\eta_1) = 0 \quad .$$

Note that although $(r/R)^2$ has consistently been considered much smaller than 1, $(4\Omega^2 + \omega^2)/(4\Omega^2 - \omega^2)\, r^2/R^2$ has been kept since it may not be negligible for suitable values of ω . This will probably not be the case for Rossby waves, which I expect to be of low frequency, but

may well be so for other types of motion to which this equation applies. We have in (45) an equation governing the amplitudes of waves in a two-layer system in the absence of any bottom variations and asymmetries.

When there are bottom slopes, in view of the complexity of equation (39), we will not consider the internal oscillations and limit our study to the motions of a homogeneous ocean. The stratification is then dropped, and only equation (39) remains; when $H_1 = 0$, (39) becomes

$$\cos\theta \left(i \frac{\partial G}{\partial r} + \sigma G + \frac{iG}{r} \right) + \frac{\gamma J}{r} + \frac{iJ}{rH} \frac{\partial H}{\partial \phi} + \frac{iG \cos\theta}{H} \frac{\partial H}{\partial r} = \frac{\omega}{gH} \tag{46}$$

An amplitude equation can be derived from (46) by following the same procedure used to obtain (45) from (42): the abbreviations G and J are rewritten in terms of their explicit definitions, (36) and (37), the approximation $(r/R)^2 << 1$ is made, and σ and γ are expanded in terms of F . A further assumption is now that the amplitude function F and the depth H are periodic functions of latitude,

$$F \propto e^{i(p\phi)}, \tag{47}$$

where p may be a function of r. Expressions of the form $\frac{i}{F} \frac{\partial F}{\partial \phi}$ and $\frac{i}{H} \frac{\partial H}{\partial \phi}$ are therefore real; the ensuing

amplitude equation is

$$\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial r}\left[\frac{1}{r} + \frac{1}{H}\frac{\partial H}{\partial r} + \frac{(4\Omega^2+\omega^2)}{(4\Omega^2-\omega^2)}\frac{r}{R^2} + \frac{2i\Omega}{\omega r}\left(\frac{1}{F}\frac{\partial F}{\partial \phi} + \frac{1}{H}\frac{\partial H}{\partial \phi}\right)\right]$$

$$+ \frac{\partial F}{r\,\partial \phi}\left[\frac{1}{rF}\frac{\partial F}{\partial \phi} + \frac{1}{rH}\frac{\partial H}{\partial \phi} - \frac{2is}{r} - \frac{i}{H}\frac{\partial H}{\partial r}\frac{2\Omega}{\omega} - \frac{2\Omega}{\omega}\frac{ir(4\Omega^2+\omega^2)}{R^2(4\Omega^2-\omega^2)}\right]$$

$$+ F\left[-\frac{s^2}{r^2} - \frac{2\Omega}{\omega}\frac{(4\Omega^2+\omega^2)}{(4\Omega^2-\omega^2)}\frac{s}{R^2} - \frac{(4\Omega^2-\omega^2)}{gH} - \frac{2s\Omega}{\omega rH}\frac{\partial H}{\partial r} - \frac{is}{r^2 H}\frac{\partial H}{\partial \phi}\right]$$

$$- \frac{2i\Omega}{\omega r}\frac{\partial^2 F}{\partial \phi \partial r} = 0 \qquad (48)$$

The description of the motions is not complete without boundary conditions appropriate to the idealized situation represented by equation (48). A general prescription, applicable to all cases, is that the function $F(r,\phi)$ satisfying (48) be continuous and analytic over the whole extent of the projection of the Arctic ocean, including the boundary, $r = r_1(\phi)$. The restriction on the velocity at a vertical wall must also be satisfied whenever such a situation arises; neglecting r_1^2/R^2 with respect to unity, (14) becomes, in the polar projection:

$$u(r_1) = v(r_1) \frac{\partial r_1}{r_1 \partial \phi} \tag{49}$$

and, in terms of the amplitude function, F, using (34) and (35), (49) becomes

$$\left.\frac{\partial F}{\partial r}\right|_{r=r_1} \left(\frac{-i\omega}{2\Omega} - \frac{\partial r_1}{r_1 \partial \phi}\right) + \left.\frac{\partial F}{\partial \phi}\right|_{r=r_1} \left(\frac{-1}{r_1} + \frac{i\omega}{2\Omega r_1^2} \frac{\partial r_1}{\partial \phi}\right)$$

$$+ F(r_1)\left(\frac{is}{r_1} + \frac{\omega s}{2\Omega r_1^2} \frac{\partial r_1}{\partial \phi}\right) = 0 \tag{50}$$

The rest of this work is concerned with solving the amplitude equations, first in the simplest cases (45), then in more complicated situations, (48). Although more general solutions will be kept in mind, specific calculations are done only for planetary waves.

iii) Physics of Rossby waves.

Now that the formulation has been established and the problem stated in stricter form, it becomes easier to study the effects of the various forces at work to produce or sustain oscillations in the basin.

Consideration of the momentum equations, (24) and (25), shows that local accelerations are produced by two causes only: Coriolis and gravity forces. The effect of

gravity is well known, and will tend to eliminate any deviations from the free equilibrium surface of the fluid; if gravity alone acts, familiar gravity waves result. The action of the Coriolis force is also well known: in the northern hemisphere it acts as a force pulling a moving body to the right of its direction of motion.

Both fundamental forces may be of similar importance to the motions; when this happens in a steady state, geostrophic currents result, in which pressure gradients exactly balance the Coriolis force. This is still simple and easily visualized. What happens when the Coriolis parameter is not constant, but varies with latitude, as it does on the Earth, and in general on any curved rotating surface ? Instead of studying the problem on the Earth, let us examine it on the polar projection introduced above. Although it is somewhat unrealistic to have a plane characterized by different rotation rates at different positions, the construct is useful in that it reproduces the properties of the spherical surface with simpler symbolism. In order to bring the variation of the Coriolis parameter into the equations and see what its influence is, let us examine the equation for the vertical component of vorticity, ξ . When viscosity is neglected but the non-linear terms retained, the following vorticity equation can be derived by cross-differentiation of the non-linearized momentum equations (Stommel, 1960; ch. VIII, p. 108):

$$\frac{d(\xi + f)}{dt} = \frac{f+\xi}{H+\eta} \frac{d(H+\eta)}{dt} \quad . \tag{51}$$

To obtain (51), use is also made of the integrated continuity equation in its non-linearized form:

$$(H+\eta)\left[\frac{\cos\theta}{r}\frac{\partial ur}{\partial r} + \frac{\partial v}{r\,\partial\phi}\right] = \frac{d(H+\eta)}{dt} \quad . \tag{52}$$

Equation (51) applies on the polar plane; f is the variable Coriolis parameter ($2\Omega\cos\theta$), H the total equilibrium depth, η the surface displacement from equilibrium, and ξ the vorticity component in the vertical direction, defined by

$$\xi = \frac{1}{r}\left[\cos\theta\,\frac{\partial vr}{\partial r} - \frac{\partial u}{\partial\phi}\right] \quad . \tag{53}$$

The vorticity equation (51) can be immediately integrated to yield

$$\frac{f+\xi}{H+\eta} = \text{constant} \quad . \tag{54}$$

This will be recognized as a form of the potential vorticity conservation theorem, of frequent use in

meteorology and oceanography. It states that the potential vorticity of a given volume of water, as defined by the left hand side of (54), does not change as that volume of water moves about in the fluid. This is of course only a special statement of the theorem of conservation of angular momentum in the absence of applied torques.

Now, in most oceanic flows, the local component of vorticity due to water movements (ξ) is much smaller than the planetary vorticity at that location (f); exceptions occur near the equator, where f vanishes, and in boundary currents like the Gulf Stream, where considerable shears are found. Barring these special cases, we can assume that $f \gg \xi$ at all positions and times, and that the motions studied will not disrupt this state of affairs. This is also justified by the assumption of small Rossby number, $U/L\Omega \ll 1$, which has been made above. U/L can be regarded as a measure of the local vorticity ξ . Equation (54) is then used in the approximate form $f/(H+\eta)$ const.

Let us see what constraints the potential vorticity equation imposes on the motions of the fluid when $f \gg \xi$. First consider for simplicity a situation where the depth is constant, so that it does not affect the potential vorticity balance. If a perturbation is introduced in the system without the application of torques, the potential vorticity will remain constant, and the approximate form of (54) still hold. Suppose we do this by changing the

surface elevation η ; there must then be an equivalent change in the denominator to keep the ratio constant. That can be achieved only by changing f, that is changing the latitude.

The fluid thus moves to a different latitude, where f has just the right value for the potential vorticity balance to be reestablished. The fluid is however gifted with inertia, and it will overshoot that latitude and find itself in the same kind of imbalance on the other side. The situation is then analogous to gravity waves about a free surface: we have instead vorticity waves about a critical latitude.

Rossby proceeded along such general lines in his first investigation of planetary waves(1939). He considered the waves as purely two-dimensional, so that (51) reduces to

$$\frac{d\xi}{dt} = -\beta v , \qquad (55)$$

where β is the rate of change of f with latitude, a constant in the plane projection used by Rossby (the β - plane). In the β -plane, and in Rossby's notation, u and v are velocities to the east (x) and north (y) and $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Assuming motions independent of latitude, and linearizing (55), a simple wave equation applicable to the study of planetary waves in the β-plane results:

$$\frac{\partial^2 \xi}{\partial x \, \partial t} = -\beta \xi \quad . \qquad (56)$$

For a simple vorticity wave of the form $\xi = \sin k(x-ct)$, (56) gives a phase velocity of $c = -\beta/k^2$.

Although many of the simplifications present in Rossby's analysis have not been made in deriving the amplitude equations (45) and (48) so that they are considerably more complicated than the simple wave equation (56), their solutions which depend on the variation of the Coriolis parameter will satisfy expression (54) expressing the conservation of potential vorticity. Extra complications over Rossby's problem will be due to β not being a constant, the amplitude of the waves varying with latitude, and the presence of boundaries and variable bottom topography. We can then expect planetary wave solutions in the present problem to be considerably more complicated than those found by Rossby.

Closer consideration of (54) shows that depth variations can have exactly the same kind of influence as variations of planetary vorticity; motions of a nature similar to planetary waves can then occur in the presence of depth gradients alone. Such waves on continental shelves have recently been investigated by Robinson (1964). When both f and H are allowed to vary, as will be the case here, their respective effects can reinforce or cancel each other.

There will then be two main forces acting on the fluid in the model used: gravity and Coriolis force. The influence of gravity is well known; we have seen however,

following an analysis similar to Rossby's (1939), that if the Coriolis parameter or the depth of the basin varies with position, there can arise oscillations governed mainly by the variation of f and/or H. These are the so-called planetary waves to which we will pay particular attention.

VI. SYMMETRIC OCEAN; GENERAL CONSIDERATIONS.

Before attempting to find explicit solutions for the equations describing the variations of the wave amplitude with position and for different values of the wave parameters, it would be wise to examine these equations together with the relevant boundary conditions and to see under what circumstances they admit solutions. This will prevent looking for solutions where they cannot be found, and also give some qualitative idea of their aspect. The analysis can be done much more simply when the problem has complete symmetry around the pole (except for the $e^{is\phi}$ part), as only ordinary differential equations remain.

In the absence of all longitudinal variations, (45) and (48) become so similar in form that they will be analysed as one:

$$\frac{d^2F}{dr^2} + \left\{ \frac{dF}{dr} \frac{4\Omega^2 + \omega^2}{4\Omega^2 - \omega^2} + \frac{1}{r}\left(1 + \frac{r}{H}\frac{dH}{dr}\right) \right\} \tag{57}$$

$$- F \left\{ \frac{s^2}{r^2} + \frac{2\Omega s}{\omega R^2} \frac{4\Omega^2 + \omega^2}{4\Omega^2 - \omega^2} + \frac{(4\Omega^2 - \omega^2)}{gH} + \frac{2\Omega s}{\omega r H} \frac{dH}{dr} \right\} = 0$$

The first term of (45) has been expanded, bringing in a

slight simplification; when there is no bottom slope, dH/dr vanishes in (57); the constant gH must be changed to $g\,H_1\Delta\rho/\rho$ if internal oscillations are to be studied.

Some non-dimensional variables and abbreviations are now introduced so as to make (57) more compact:

$$\begin{aligned}
\omega' &= \omega/2\Omega\ , \\
H' &= H/H_o\ ;\ H_o = \text{depth at the pole.} \\
x &= r/r_1\ , \\
\epsilon &= \frac{1+\omega'^2}{1-\omega'^2}\cdot\frac{r_1^2}{R^2}\ , \\
M &= \frac{4\Omega^2R^2}{gH}\ , \\
\delta &= \frac{s\epsilon}{\omega'} + (1-\omega'^2)M\,r_1^2/R^2\ .
\end{aligned} \tag{58}$$

In this modified notation, (57) is how

$$x^2\frac{d^2F}{dx^2} + \frac{xdF}{dx}\left(\epsilon x^2 + 1 + \frac{x}{H'}\frac{dH'}{dx}\right) - F\left(s^2 + \delta x^2 + \frac{xs}{\omega' H'}\frac{dH'}{dx}\right) = 0\ . \tag{59}$$

This last form will be useful when we come to looking for explicit analytical solutions. The problem is completed by a statement of the boundary conditions: F must be finite and analytic everywhere in the basin and on its boundaries, and at a vertical wall at $x = 1$, the following must hold for u to vanish (by (33),(34),

(35)).

$$\left.\frac{dF}{dx}\right|_{x=1} = \frac{s}{\omega'} F(1) \quad . \tag{60}$$

Note that gravitational effects are represented by M, and curvature effects by ϵ ; the relative magnitudes of expressions containing these quantities will decide which of the two, curvature or gravity, is the most influential for a particular type of motion.

In order to find for what values of frequency, wave number or bottom slopes the system (59)-(60) admits solution, it will be necessary to recast the problem in a different form. A new variable, ζ , is introduced and defined by

$$\zeta = 1/x \quad .$$

Like any linear ordinary differential equation of second order, (59) can be rewritten as a system of two first order equations in the original dependent variable (F) and a new one Z defined by the equations of the system

$$\frac{dF}{d\zeta} = \left\{-\frac{1}{\zeta H'} \exp(-\epsilon/2\zeta^2)\right\} Z = A(\zeta, \omega') Z \tag{61}$$

$$\frac{dZ}{d\zeta} = -\left\{\frac{H'}{\zeta^3}\left(s^2\zeta^2 + \delta + \frac{s\zeta}{\omega' H'} \frac{dH'}{dx}\right)\exp(\epsilon/2\zeta^2)\right\} F$$

$$= - B(\zeta, \omega', s) F \quad . \tag{62}$$

The boundary condition (60) is similarly reformulated, in terms of F and Z, as

$$\frac{F(1)}{Z(1)} = \frac{\omega'}{sH'} \exp(-\epsilon/2) \tag{63}$$

The method used to study existence conditions is sometimes called the "Method of Signatures", and is widely used in the Hamiltonian formulation of classical mechanics; I will briefly review the principles involved, following Eckart's presentation (Eckart, 1960; Chapter XIV).

Consider a system of two dependent and one independent variables as follows:

$$\frac{dF}{d\zeta} = A(\zeta, \lambda) Z , \tag{64}$$

$$\frac{dZ}{d\zeta} = -B(\zeta, \lambda) F . \tag{65}$$

This is exactly the form in which the present problem has been transformed in (61) and (62). The parameter λ stands for one or more independent parameters of the problem. Equations (64) and (65) are said to be in canonic form, and this may be brought out by formally defining a Hamiltonian, $H = \frac{1}{2}(AZ^2 + BF^2)$, in terms of which the above system takes a form identical to the canonical Hamiltonian equations of mechanics.

The (Z,F) space will be called the phase space, and a phase path is defined as the curve traced in phase space by a solution of (64)-(65) as ζ varies; the instantaneous position of the phase path is called a phase point. Some theorems can be established about the behaviour of the phase paths according to the signs of the functions A and B (see Eckart); they will not be proven here, but, except for the first one, they are all fairly obvious. It is understood in the following theorems that the functions A and B are finite and continuous in ζ .

THEOREM I. The phase paths are continuous curves. Let Z_1, F_1 be any point in phase space other than the origin, and ζ_1 an arbitrary finite value of ζ ; then there exists only one phase path passing through the point Z_1, F_1 for $\zeta = \zeta_1$. Moreover, no phase path goes through the origin for finite values of ζ , but some may approach it as $\zeta \to \pm \infty$.

THEOREM II. All phase paths intersect the coordinate axes at right angles.

THEOREM III. If ζ_1 is a root of A (or B), then the phase path has a horizontal (or vertical) tangent for that value of ζ .

These last two theorems are implicit in equations (64) and (65). The theorems to follow will be made more evident by writing (64)-(65) in plane polar coordinates:

$$Z = R \cos\theta ,$$

$$F = R \sin\theta ,$$

$$\frac{d\theta}{d\zeta} = A \cos^2\theta + B \sin^2\theta \tag{66}$$

$$\frac{1}{R}\frac{dR}{d\zeta} = (A - B)\sin\theta \cos\theta \tag{67}$$

Defining the signature of a segment of phase path as the particular combination of signs of A and B along that segment, and expressing it symbolically as (sign of A, sign of B), it becomes apparent that much of the behaviour of phase paths can be deduced from knowledge of the signatures along the path. Note that (A), as given by (61), is always negative.

THEOREM IV. In (+,+) segments, the angular velocity ($d\theta/d\zeta$) of the phase point is positive; in (-,-) segments it is negative. Furthermore, for such signatures, once the phase path has entered any quadrant, its distance to the axis perpendicular to the one it has just crossed must not increase.

THEOREM V. In (+,-) segments, the phase point moves away from the origin (and from both axes) in the first or third quadrants, and towards it in the other two. The situation is reversed in (-,+) segments.

These last two theorems are implicit in equations (64) to (67). The (+,+) and (-,-) segments are often called oscillatory segments, and the other ones non-oscillatory. Behaviour of the phase paths under different signatures is illustrated in Figure 6.

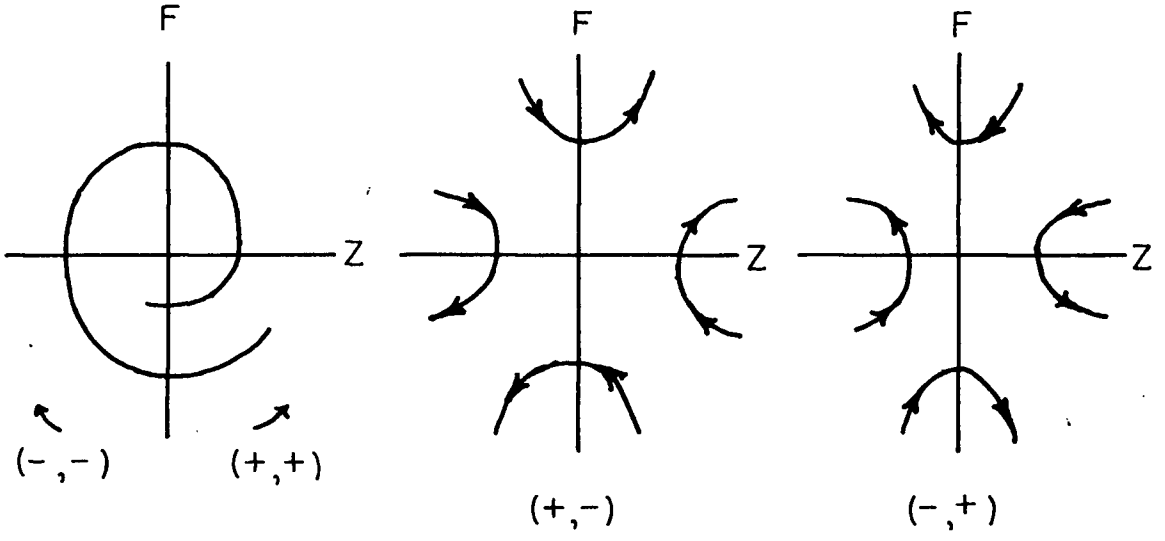


Figure 6. Phase paths in segments of various signatures; the phase point moves in the direction of the arrow.

What happens when ζ tends to infinity? If the path is oscillatory, it will spiral in or out, or the angular velocity may vanish, or the path may become a circle or an ellipse, depending on the signs and the functional behaviour of A and B at large ζ. When the segment is non-oscillatory, the phase paths have asymptotic directions along lines given by

$$\tan^2\theta_f = \lim_{\zeta \to \infty} (-A/B)$$

This is of course subject to the stipulations of theorem V; Figure 7 shows what happens to segments of signature (+,-); the situation is reversed for (-,+) segments, or when $\zeta \to -\infty$.

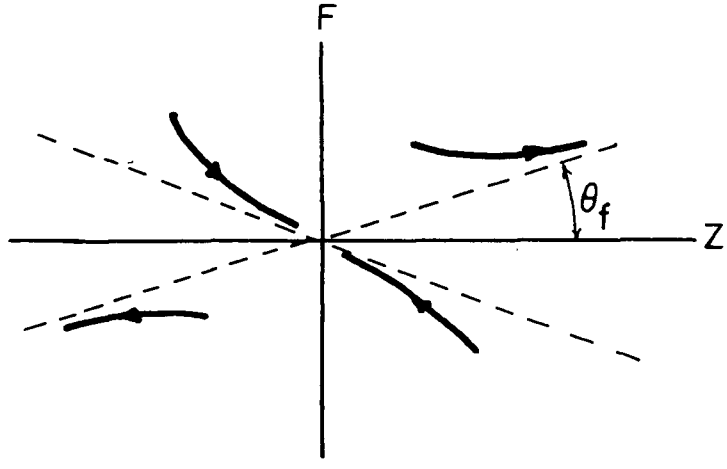


Figure 7. Asymptotic behaviour of phase paths of signature (+,-) as $\zeta \to +\infty$.

If A and/or B have roots in ζ , the signatures will change along a path; segments of different signatures will be joined together where A and/or B have roots; according to theorem III, this is at horizontal and vertical tangents respectively. An example of a path with compound signature is given in Figure 8.

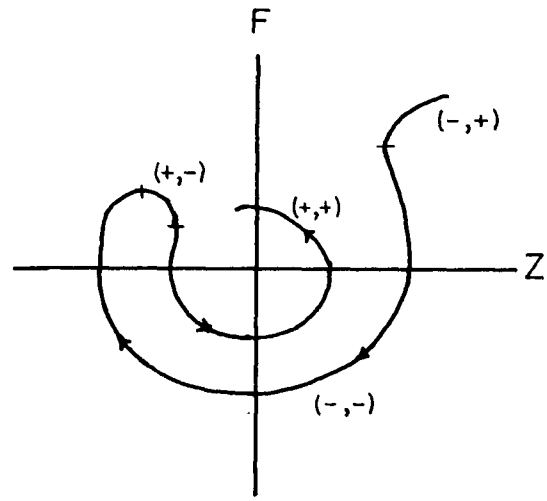


Figure 8. Phase path with compound signature: (-,+) (-,-) (+,-) (+,+).

This is all the theory needed for the intended analysis of the amplitude equations (61) and (62); its application can be explained in a few words. The functions A and B (and therefore the signatures) are known explicitly in terms of ζ and the parameters grouped under λ . Boundary conditions impose a starting point on the phase path and the restriction that the solution be finite for all values of ζ ; solutions will then exist only for those values of the parameters resulting in the right combination of signatures, allowing the phase path to remain in regions where the amplitude is finite. Although this method is very simple, it is quite powerful, and will give clear indications as to where the solutions can be found; it

even reveals what the solutions will look like: zeros and extrema of the amplitude can be read directly from the phase paths.

It will be convenient, in order to systematize the analysis to divide the wave number-frequency (s, ω') space into four domains, according to the sign of s and the magnitude of ω'; existence criteria will then be examined in turn in the four regions of the diagnostic diagram (as the s- ω' space is called). Even though we are concerned primarily with planetary waves, which are of low frequency, and hence presumably found in areas I and III, it is easy to extend the analysis to the whole diagnostic diagram.

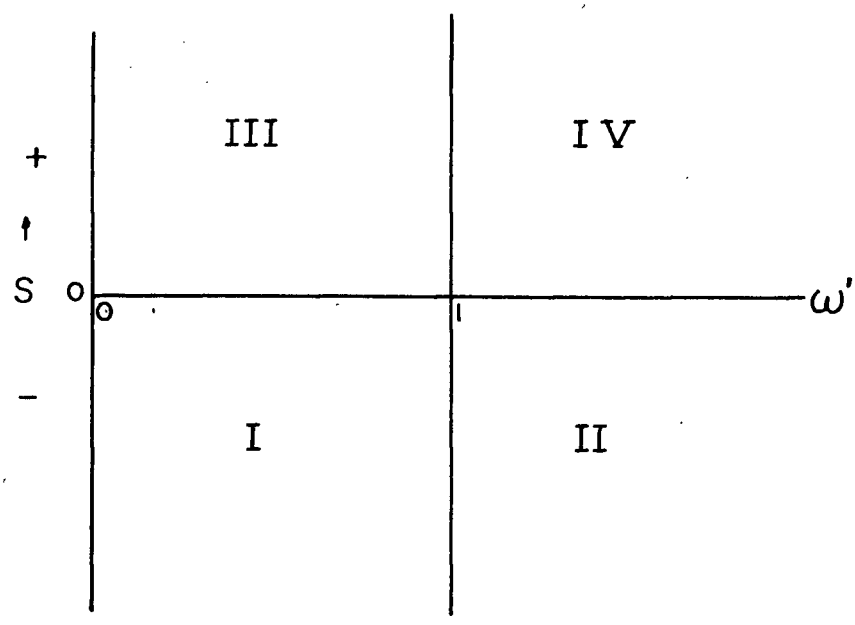


Figure 9. Subdivision of the diagnostic diagram into four regions.

Before we proceed to investigate the existence of solutions in the subdivions of the diagnostic diagram, we should make a survey of the properties of phase paths

which are invariant over the whole (s, ω') space. The first one is the asymptotic angle to which non-oscillating segments tend at large values of ζ. The functions A and B, as given by (61) and (62) yield for the final angle, according to its definition in (68),

$$\tan^2\theta_f = \lim_{\zeta \to +\infty} \left\{ \frac{1}{\left[s^2 + \frac{s}{\zeta\omega' H'} \frac{dH'}{dx} \right] H'^2} \right\}$$

The final angle is thus defined for all areas of the diagnostic diagram. Furthermore, if the non-dimensional depth, H', and its derivative are finite and continuous near the pole, $(\zeta \to +\infty)$, so that it can be reduced to a MacLaurin series around $x=0$, the final angle then simplifies to

$$\tan^2\theta_f = 1/s^2 \tag{69}$$

This is easily shown to be valid under the above stipulations: as $x \to 0$, $H' \to 1$, and $dH'/dx \propto n.x^{n-1}$, where n is positive and the lowest power of x in the series expansion of H'. The second term of the denominator therefore varies as x^n when $x \to 0$, so that it vanishes in the limit.

Another more special result can be derived concerning the behaviour of phase segments at large values of ζ ; only the magnitude of the final angle is defined by (69), and we

can obtain more detailed information as follows. Let us expand $d\theta/d\zeta$ through (61) and (62); at large values of ζ, and for a constant depth ($H' = 1$), we find that

$$\frac{d\theta}{d\zeta} = \frac{\cos^2\theta}{\zeta}\left(-1 + s^2\tan^2\theta + \tan^2\theta\,\frac{\delta}{\zeta^2}\right) \quad . \tag{70}$$

In (70), terms in ζ^{-n} have been neglected for $n > 2$. It should be remarked that since the terms neglected in deriving the expression for the asymptotic angle are of order ζ^{-2}, $(s^2\tan^2\theta - 1)$ is of order ζ^{-2} and has the same ζ dependence as the last term of (70).

It is useful in the present discussion to divide phase space in two regions as illustrated in Figure 9a. In region I, $\tan\theta < 1/|s|$; in region II, $\tan\theta > 1/|s|$.

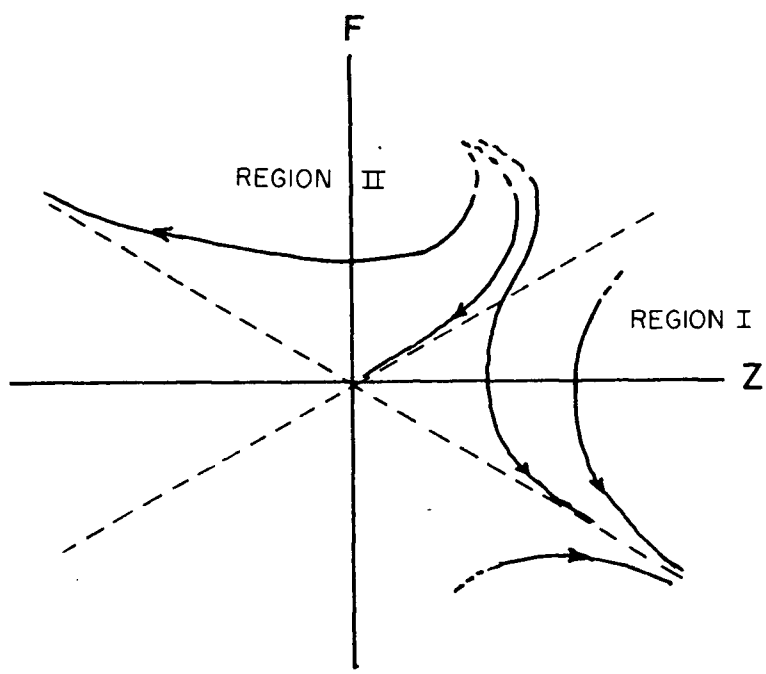


Figure 9a. Phase paths at large ζ .

Since ζ is very large, the signature of the phase paths discussed will be assumed to be (-,+). For brevity, we will also denote $(s^2\tan^2\theta - 1)$ by m/ζ^2, without worrying about the exact form of the constant m, except that it will be positive in region II, negative in region I . In terms of m , (70) becomes

$$\frac{d\theta}{d\zeta} = \frac{\cos^2\theta}{\zeta^3}\left(m + \frac{\delta}{s^2}\right) \quad . \tag{71}$$

I have used (69) to simplify the last term of (70).

Only the half space $Z > 0$ will be discussed, the situation being similar on the other side of the F axis. From theorem V, all phase paths with signature (-,+) in the fourth quadrant will depart from the origin and both axes, being asymptotic to the line $F = -Z/|s|$, so that they do not correspond to any eigensolutions.

In region I of the first quadrant, phase paths with signature (-,+) will tend to the origin along the line $F = Z/|s|$ provided they have $d\theta/d\zeta > 0$ at large ζ . As we will see later when we consider phase paths which start $(\zeta = 1)$ in that region, $d\theta/d\zeta < 0$ for $\zeta = 1$ for all frequencies not too near the inertial frequency. If the phase path is to go back to the asymptotic line in the first quadrant, there must then be a horizontal tangent: this implies a root of A, of which there is none. No eigensolutions will exist for such phase paths either.

Finally, phase paths in region II of the first quadrant will tend to the origin provided $d\theta/d\zeta$ is negative at large ζ, which is the case when δ is negative and large enough. If $|\delta|$ is too small, the phase path will go into the second quadrant; if it is too large, the phase path will cross the asymptotic line and wander in the fourth quadrant. Eigensolutions will then exist only for those values of ω' which allow the phase paths to approach the asymptotic line so closely that they do not depart from it again, but never crossing that line. For any given s and ω' there is but one such phase path.

AREA I. In this region of the diagnostic diagram, propagation is towards the west ($s < 0$) and frequency less than inertial ($\omega' = 1$); from the boundary condition (63) the phase path will start in the fourth or the second quadrant, since $F(1)/Z(1)$ is < 0 there. It is immaterial for the analysis which quadrant is chosen, and the starting point will be taken in the fourth. Again from (63), that point will be located between the lines $F = -Z/s$ and $F = 0$, unless the depth at the boundary is much less than at the pole ($H'(1) = 1$). The signature will be either (-,-) or (-,+) since A is always negative and only B can change sign. When $B > 0$, theorem V states that the phase path will move away from the origin and both axes. There are no solutions in that area of the diagnostic diagram corresponding to such values of wave number and frequency

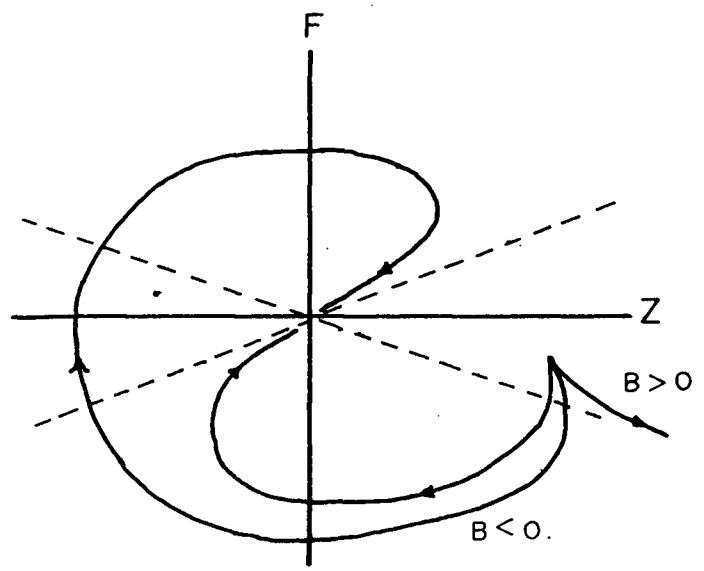


Figure 10. Phase paths for area I.

such that $B > 0$ for all values of ζ .

In order that the phase path remain in regions of phase space where F and Z are finite, B must be negative for a wide enough range of values of ζ to have an oscillatory segment of length sufficient to allow the phase point to leave the fourth quadrant. For a solution to exist, the phase path must be in the first or third quadrant when B changes sign.

B is negative when

$$s^2\zeta^2 + \frac{s}{\omega'}\left(\epsilon + \frac{\zeta}{H'}\frac{dH'}{dx}\right) + M(1 - \omega'^2)\frac{r_1^2}{R^2} < 0 . \quad (73)$$

This inequality will be satisfied either for very low frequencies or, considering the definition of ϵ,(58), for nearly inertial frequencies. We anticipate that the low frequency solutions will be planetary waves. It is doubtful that the present linearized formulation can treat the nearly inertial solutions adequately, since the velocities are not regular at that frequency in the present model.

It appears at once from (73) and theorem IV that if the inequality is satisfied for one value of frequency, $\omega'_{s,o}$ say, then it is going to be satisfied by an infinity of successively lower frequencies. Let us say that $\omega'_{s,o}$ is the first frequency that is found to produce a segment of signature (-,-) long enough to allow B to change sign in the third quadrant, in which theorem V rules that the phase path will move towards the origin as $\zeta \to +\infty$, and thus allow solutions to exist. Clearly, if a lower frequency, $\omega'_{s,1}$ is chosen such that B remains negative for larger values of ζ (i.e., it takes longer for the $s^2 \zeta^2$ term in (73) to catch up with the negative terms), the phase path will remain of oscillating nature for an extended range of ζ and, if $\omega'_{s,1}$ is properly chosen, it will reach the first quadrant before B changes sign. An infinite series of solutions can then exist for a same value of wave-number, s,: the system is degenerate in s. Denoting the solutions by $F_{s,n}(x)$, with frequencies $\omega'_{s,n}$ (with $n = 0,1,2....$), the phase path will wind around in a

clockwise direction requiring an extra π radians to reach the origin between each successive solution, and the frequencies will steadily decrease: $\omega'_{s,n+1} < \omega'_{s,n}$. The solution $F_{s,n}(x)$ will have n zeros between the pole and the boundary since the phase path crosses the F axis n times; n is then an index number denoting the number of radial nodes of the solution.

The influence of bottom topography on the solutions can be examined through their effect on the inequality (73). Let us consider nearly uniform bottom variations for the moment; the slope having the same sign for all x and being nearly constant. If the depth of the basin decreases from the pole to the boundary ($dH'/dx < 0$), then, as $|dH'/dx|$ increases, the negative part of B is made more negative, and it will take a smaller frequency to bring the phase path to the same point for the same value of ζ as when there is no bottom slope. Negative slopes (with respect to the variable x) have therefore the effect of reducing the frequency of planetary waves; slopes of the opposite sign will of course have the opposite effect. If $|dH'/dx|$ is large enough so that $\frac{\zeta}{H'} dH'/dx + \epsilon < 0$ for all values of ζ, the inequality (73) will not hold for any value of frequency in the first region of the diagnostic diagram. There is then a maximum uniform slope ($dH'/dx < 0$) allowing westward propagating planetary waves to exist in a symmetric polar basin. It is given by

$$|dH'/dx| > \epsilon H'(1).$$

If we do not insist that the bottom slope be uniform in x, solutions can be found for bathymetries that will include local slopes greater than the maximum uniform slope. The only condition to be satisfied is after all that the phase path escape from the initial quadrant and that B has its last root when the phase point is in the third or the first quadrant. The function B can have more than one root if the depth varies strongly enough; a simple example of such a situation, illustrated below, also shows the effect of sloping walls at the same time.

Consider a basin bounded by walls with slopes greater than the maximum average slope allowing westward propagating waves; the bottom is flat beyond the sloping walls and remains so until the pole is reached (Fig. 11). The signature will initially be (-,+) and the radius of the phase point will increase. When a value of ζ corresponding to the rapid decrease in slope is reached, B will become negative if the frequency is small enough, and the phase path can be carried

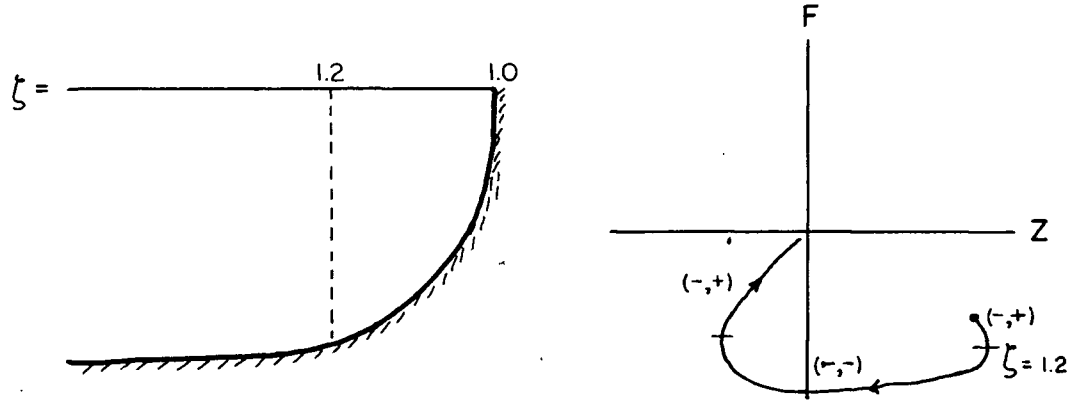


Figure 11. Bottom profile and phase paths for sloping walls in Area I of the diagnostic diagram.

to the third or to the first quadrant, where B will have its second root. The solutions will therefore be similar to those for a flat bottom, but with lower frequencies. There is then a wide class of bathymetries allowing westward propagating planetary waves in a symmetric polar basin, the general restriction being that for some wave number s and bottom configuration $H'(x)$ there exists a real value of frequency which allows B to have its last root in a quadrant of the phase space where the phase path will tend to the origin for large values of ζ .

AREA II. The propagation is still westward, but the frequency is now greater than inertial. The starting point is again taken in the fourth quadrant, but may now very likely be located below the line $F = -Z/k$ (Fig. 12). Once more, there will not exist any solutions for $B > 0$; the inequality $B < 0$ is still expressed by (73), but since ω' is now > 1, (making $\epsilon < 0$) this will hold either when $(1-\omega'^2)M\, r_1^2/R^2$ dominates or when $dH'/dx > 0$ and large. Gravity terms in M, or bottom slopes, are now more important than curvature terms ($1/R^2$ in ϵ). The frequency necessary to make the inequality hold is however not as large as one might think; for $s = -1$, (73) is already satisfied for $\omega' = 1.5$. This may then be a type of motion for which it is not justifiable to neglect $\epsilon\, x^2$ as compared to 1, in (59). The lowest frequency which makes $B < 0$ will increase with increasing k ($= |s|$) and r_1; the influence of bottom

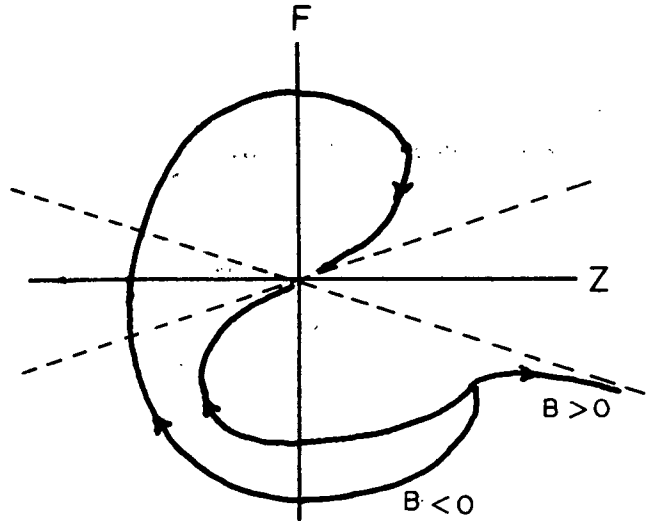


Figure 12. Phase paths for area II.

slopes in the inequality (73) is as in area I, but the frequency is altered in a different manner since B is now made negative by terms in which ω' is large instead of terms in which it is small, as in area I. A depth decreasing radially now causes an increase instead of a decrease in frequency over the corresponding flat bottom solution. The solutions are also subjected to an infinite degeneracy in wave number; they are controlled mostly by gravity effects, and cannot be called planetary waves.

AREA III. Propagation is now to the east, while frequency is less than inertial. The starting point will be taken in the first quadrant, and, unless $H'(1) << 1$, it will be found under the line $F = Z/s$ (Fig. 13). According to the explicit expression for B (62), the signature will

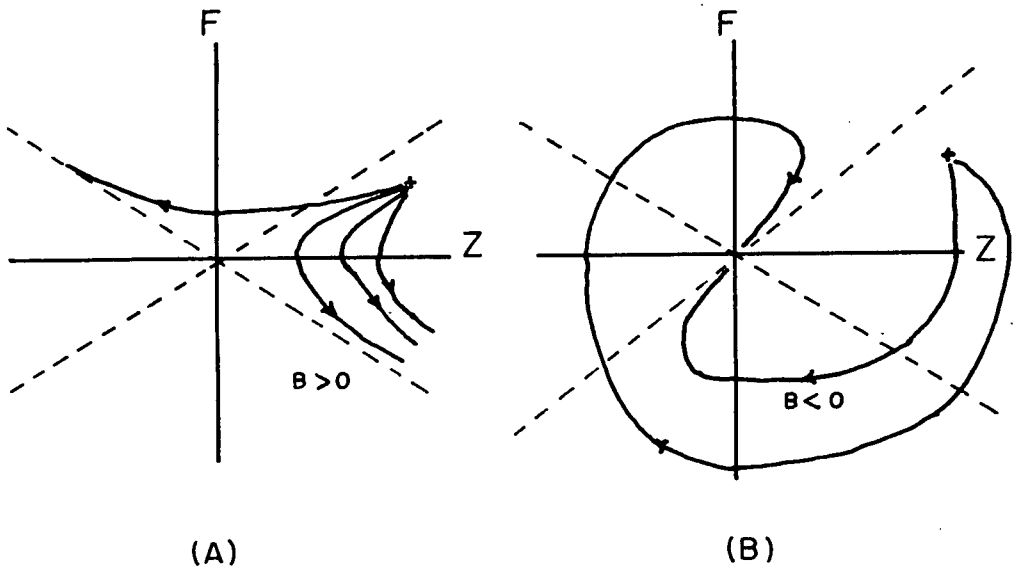


Figure 13. Phase paths for area III. (A). Flat bottom. (B) Large positive depth gradients: $dH'/d\zeta > 0$.

always be (-,+) unless there is a sufficiently large negative depth gradient, $\frac{dH'}{dx}$.

Let us first look for flat bottom solutions. The signature is (-,+) and, according to theorem V, the phase path moves towards the origin for that signature in the first quadrant. There will then be solutions if the phase path does not escape to the neighbouring quadrants. There will not be any solutions if the phase path moves into the second quadrant and stays there (theorem V); but once in quadrant III

the phase path cannot loop back into the first quadrant, because this would imply a root of B, of which there is none. For frequencies not too near the inertial frequency, the angular velocity of the phase path, $d\theta/d\zeta'$, is for $\zeta = 1$ of the sign of $(s^2+\delta)\,\omega'^2/s^2 - 1$; this is negative for $\omega' < 0.97$. According to the analysis of page 59a , there will then be no eigensolutions for $s > 0$ and $\omega' < 1$ in the absence of bottom slopes. There are then no eastward propagating planetary waves in a symmetrical polar ocean in the absence of bottom slopes.

When large negative radial depth derivatives are present, it is possible to choose a frequency low enough so that there is a segment of phase path of oscillating nature enabling the first or third quadrant to be reached in a manner consistent with the existence of solutions. The same multiplicity of solutions is found as in areas I and II. The influence of bottom slopes is now such that frequencies of equivalent solutions will now increase as the slope increases.

AREA IV. The frequency is here larger than inertial ($\omega' > 1$), and propagation to the east ($s > 0$). The starting point is now found (again in the first quadrant) above the line $F = Z/s$, provided $H'(1)$ is not much larger than unity. For $B > 0$ for all ζ , there may be eigensolutions as shown in Figure 14. There is then only one eigensolution for each wave number and no multiplicity as in the previous

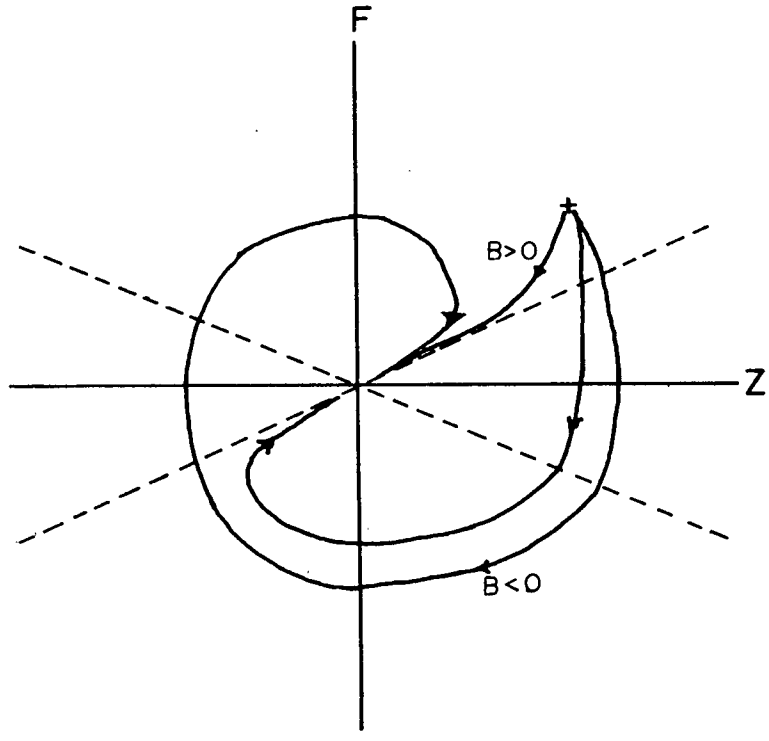


Figure 14. Phase paths for Area IV.

instances. When the frequency is large enough to make the third term of (73) dominant, and the signature (-,-) over a segment of phase path, solutions of the same general nature as those encountered in areas I and II will be found. Depth variations have the same influence on the sign of B as they have in area III; as in area II, the waves are controlled mostly by gravity and are not classifiable as planetary waves.

I have obtained in this section, by the semi-quantitative Method of Signatures, valuable information

as to where in the diagnostic diagram solutions compatible with the approximations and the boundary conditions can be found. Much information about the qualitative aspect of solutions is revealed by the behaviour of phase paths, and extensive degeneracy in wave number has been discovered in the possible solutions. In particular, it has been discovered that low frequency oscillations controlled mostly by curvature effects and presumably identifiable with planetary waves can be found only in area I of the diagnostic diagram in the absence of bottom slopes.

Little more information can be obtained from the Method of Signatures, and the amplitude equation must now be solved explicitly to find the form of the solutions and their frequencies.

VII. FLAT BOTTOM SOLUTIONS: SYMMETRIC BASIN.

i) Solution in terms of confluent hypergeometric functions.

Starting with the simplest case, which has already been studied in different formulations (Goldsbrough, 1914 a, Longuet-Higgins, 1964 b) and through which a check on the polar plane approximation can be made, I study first the oscillations of a basin with a flat bottom and symmetrical boundaries. In the absence of depth variations, the symmetric amplitude equation (59) reduces to

$$x^2\frac{d^2F}{dx^2} + x\frac{dF}{dx}(\epsilon x^2 + 1) - F(s^2 + \delta x^2) = 0 \quad , \qquad (74)$$

while the boundary condition (60) is unaltered.

If we concentrate our attention on long period planetary waves, in which ω' is so small that we can safely neglect ϵx^2 with respect to 1, (74) is considerably simplified and can be solved in terms of Bessel functions. We have seen however that in area II of the diagnostic diagram there were motions for which ϵx^2 might not be entirely negligible, and for which (74) should be kept in its entirety. It will then be useful, in view of possible future extension of this work to types of motion other than planetary waves, to solve the more exact

equation (74), partly to illustrate the method used and its possible application in other cases, partly as an <u>a priori</u> justification of the assumption to be made in (74) when planetary waves are studied. In the following pages, I will then find solutions of the more exact equation, (74); these pages may be omitted without solution of continuity and the reader may, if he wishes so, proceed to page 88 , where the approximation to (74) with $\epsilon x^2 << 1$ is treated, as appropriate to the study of planetary waves.

Equation (74) can be transformed into the standard form of the confluent hypergeometric equation (also called Kummer's equation) by the change of variables

$$F = x^{\pm s} u(y) \quad ; \; y = -\epsilon x^2/2 \quad . \tag{75}$$

The resulting equation for u(y) is

$$yu'' + u'(1 \pm s - y) - u\left(\frac{\pm s\epsilon - \delta}{2\epsilon}\right) = 0 \quad , \tag{76}$$

in which primes indicate differentiation with respect to y.

The appropriate solution of (76) must be such that $F(x) = x^{\pm s} u(y)$ vanishes, or at least remains finite, at the pole ($x = y = 0$). Furthermore, in order to insure continuity of the solutions around the pole, one must have $e^{i(2\pi + \phi)s} = e^{is\phi}$, so that s must be zero or an integer, positive or negative.

Writing Kummer's equation in general form, with parameters a and b, it is

$$yu'' + u'(b - y) - ay = 0 \quad . \tag{77}$$

Compariton of (76) and (77) shows that here, $b = 1 \pm s$, and $a = (\pm s - \delta)/2$. The parameter b is therefore an integer; in such a case (77) has two independent solutions: the first one is finite at the pole and is ${}_1F_1(a;\ b;\ y)$, and the second one is a logarithmic solution which diverges at $y = 0$ (Slater, 1960; Chapter I). The only solution of (74) which will remain finite at the pole for integral values of s is then

$$F(x) = cx^k \, {}_1F_1 \left(\frac{k\epsilon - \delta}{2}\right);\ 1 + k;\ -\epsilon x^2/2 \quad , \tag{78}$$

in which c is an arbitrary constant and k is $|s|$. The function ${}_1F_1(a;\ b;\ y)$ is the standard hypergeometric notation for the series

$$1 + \frac{a}{b} y + \frac{a(a+1)}{b(b+1)} \frac{y^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{y^3}{3!} + \ldots \tag{79}$$

The boundary condition (60) can also be expressed in terms of the solution of the problem. Using the differentiation formula for ${}_1F_1$ (Slater, 1960; Chapter 2)

$$\frac{d}{dy}\,{}_1F_1(a;\ b;\ y) = \frac{a}{b}\,{}_1F_1(a\ \ 1;\ b\ \ 1;\ y)\ , \qquad (80)$$

and a recursion formula

$$a\,{}_1F_1(a+1;b+1;\ y) = (a-b)\,{}_1F_1(a;\ b+1;\ y) + b\,{}_1F_1(a;\ b;\ y) \qquad (81)$$

the boundary condition (60) becomes

$$\frac{{}_1F_1\left(\frac{(k\epsilon-\delta)}{2\epsilon};\ k+2;\ -\epsilon/2\right)}{{}_1F_1\left(\frac{(k\epsilon-\delta)}{2\epsilon};\ k+1;\ -\epsilon/2\right)} = \frac{(k+1)(s/\omega'-k+\epsilon)}{(k'/2+1)\epsilon+\delta/2} \qquad (82)$$

Confluent hypergeometric functions are tabulated only for positive values of the argument y, so that when $\epsilon > 0$, it will be more convenient to express the solution in terms of functions of $+\epsilon x^2/2$. This is done by using Kummer's first theorem:

$$e^{-y}\,{}_1F_1(a;\ b;\ y) = {}_1F_1(b-a;\ b;-y) \qquad (83)$$

Instead of (78) we then have for solution

$$F(x) = cx^k\exp(-\epsilon x^2/2)\ {}_1F_1(\ k/2+1+\delta/2\epsilon\ ;\ k+1;\ \epsilon x^2/2). \qquad (84)$$

and for the boundary condition

$$\frac{{}_1F_1\left(k/2+1+\delta/2\epsilon;\ k+2;\ \epsilon/2\right)}{{}_1F_1\left(k/2+1+\delta/2\epsilon;\ k+1;\ \epsilon/2\right)} = \frac{2(k+1)(k - s/\omega')}{(k\epsilon - \delta)} \tag{85}$$

Before attempting to extract anything from the boundary conditions (82) or (85), it will be instructive to review a few of the descriptive properties of the series ${}_1F_1$. (Slater, 1960; Chapter 6. Jahnke and Emde, 1945; Chapter X). The variable y will be assumed positive; if it is not, the function can always be transformed via (83) to make it so.

1- There will not exist any roots of ${}_1F_1(a;\ b;\ y)$ unless either a or b or both are negative. In this work, b $(= k+1)$ is always positive.

2- If $b > 0$ and $a = -n+\theta$, where $n = 1,2,3\ldots$ and $0 \leq \theta < 1$, then ${}_1F_1(a;\ b;\ y)$ has exactly n zeros, and if y_n is the root at the largest value of y, $y_n \to \infty$ as $n \to \infty$.

3- It is evident from (79) that if a and b are > 0,

a) ${}_1F_1(a+1;\ b;\ y) > {}_1F_1(a;\ b;\ y) > 0$

b) ${}_1F_1(a;\ b+1;\ y) < {}_1F_1(a;\ b;\ y)$.

4- If y_1 and y_2 are roots of ${}_1F_1(a_1;\ b_1;\ y)$ and ${}_1F_1(a_2;b_2;y)$ respectively, and a_1 and a_2 are both negative,

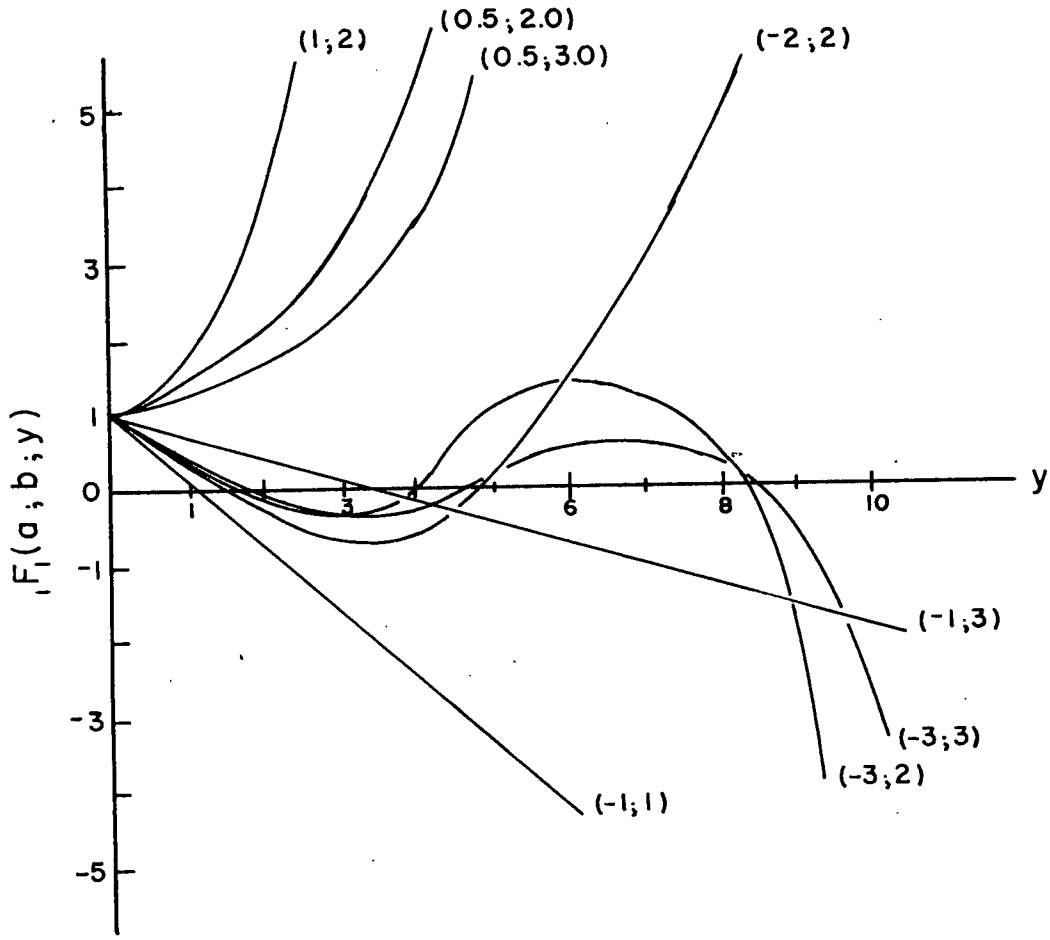


Figure 15. The confluent hypergeometric function $_1F_1(a;\ b;\ y)$ as a function of y for a few values of a and b.

then,

a) If $|a_1| > |a_2|$ while $b_1 = b_2$, then $y_1 < y_2$;

b) if $a_1 = a_2$ and $0 < b_1 < b_2$, then $y_1 < y_2$.

c) If y_m is the mth root (from the origin of y) of ${}_1F_1(a; b; y)$ then, when $b > 0$, as $a \to -\infty$, $y_m \to 0$.

5- At $y = 0$, (from (79)) the confluent hypergeometric series is equal to unity for all values of a and b.

6- As $y \to +\infty$, the series tends exponentially to $+\infty$ if there is an even number of roots, to $-\infty$ otherwise.

Figure 15 illustrates some of the above points and gives an idea of some aspects of the confluent hypergeometric series.

We found in section VI that in a symmetric ocean with a flat bottom low frequency solutions corresponding to planetary waves could exist only in area I of the diagnostic diagram (Figure 9). Concentrating our attention on the planetary wave solutions, we see that since in area I of the diagnostic diagram $s < 0$, $\omega' < 1$, then $\epsilon > 0$, and the eigenvalue problem consists in finding the value of frequency ω' which will satisfy the boundary condition in the form (85). Such a frequency equation is often called a characteristic equation. If we wished to study the high frequency solutions which can be found in areas II and IV of the diagnostic diagram (where $\omega' > 1$, and consequently $\epsilon < 0$) the same method as will be used below would be followed, but starting from equation (82).

Since in (85) the parameters of the confluent hypergeometric function as well as the right hand side depend on frequency, it will be wise to examine the boundary condition as a function of frequency. The frequency dependent expression on the right hand side of (85), hereafter referred to as (85 rhs), is plotted in Figure 16 against frequency for values of s corresponding to area I of the diagnostic diagram ($s < 0$). For k ($= |s|$) > 1, (85 rhs) is always positive and at least of order 10^2 for values of frequency not too near the inertial frequency ($\omega' = 1$). This is also true for $s = -1$, except in a range of values of frequencies where (85 rhs) is large and negative. From section VI, we expect that the planetary waves will be of low frequency so that for $k > 1$, the ratio of the two confluent hypergeometric functions will be large and positive; for $s = -1$, that ratio will be large also ($> 10^2$), but its sign will depend on whether the eigenfrequency is to the right or to the left of the singularity in (85 rhs).

The variation of the third variable of the confluent hypergeometric functions on the left hand side of (85), $\epsilon/2$, is quite simple and is illustrated in Figure 17. $\epsilon/2$ becomes large only near $\omega' = 1$, and is small and varies only very slowly near $\omega' = 0$.

The first variable, 'a', in each of the functions of the left hand side of (85) is, when δ is expanded in

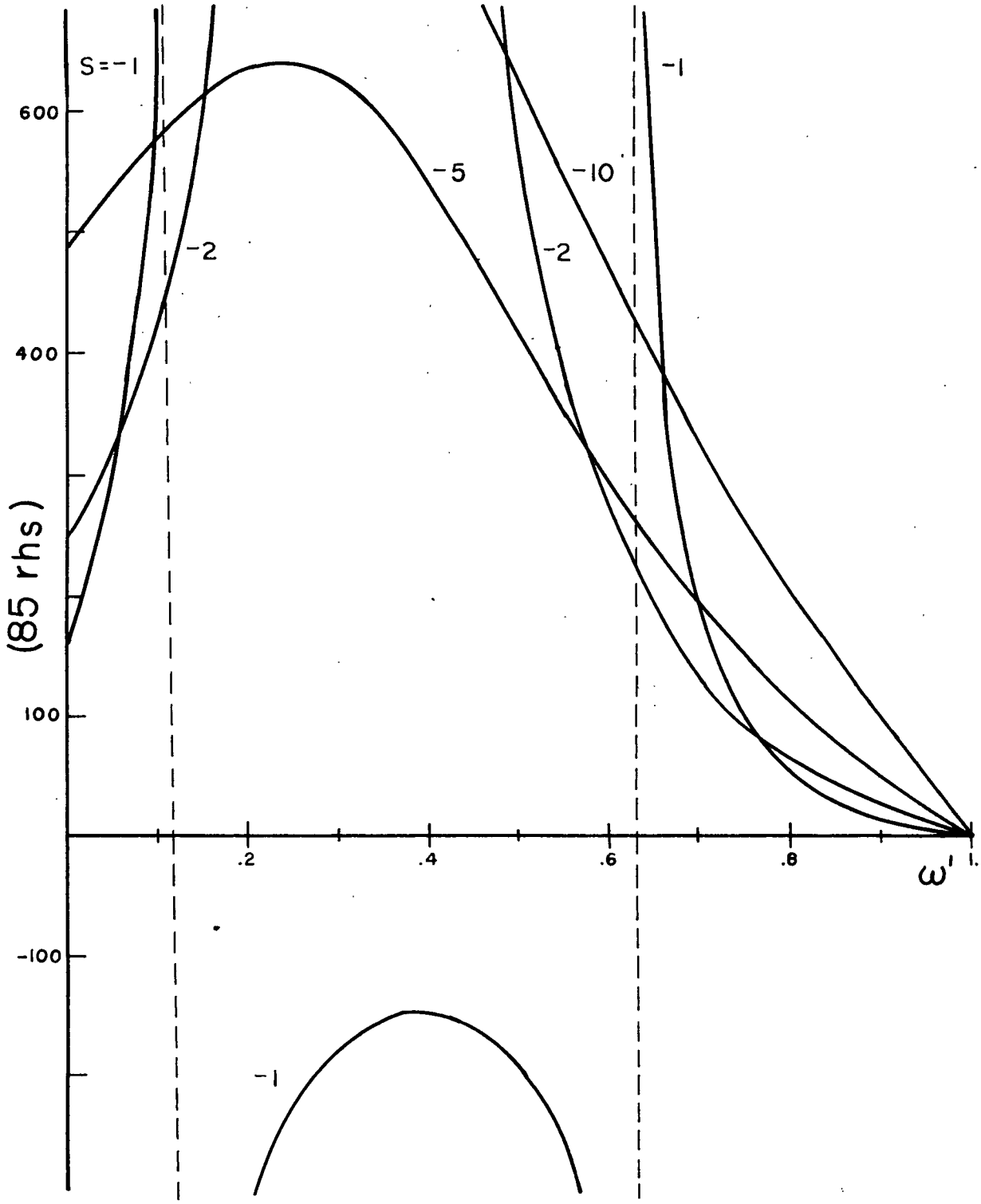


Figure 16. The expression (85 rhs) as a function of frequency, ω'.

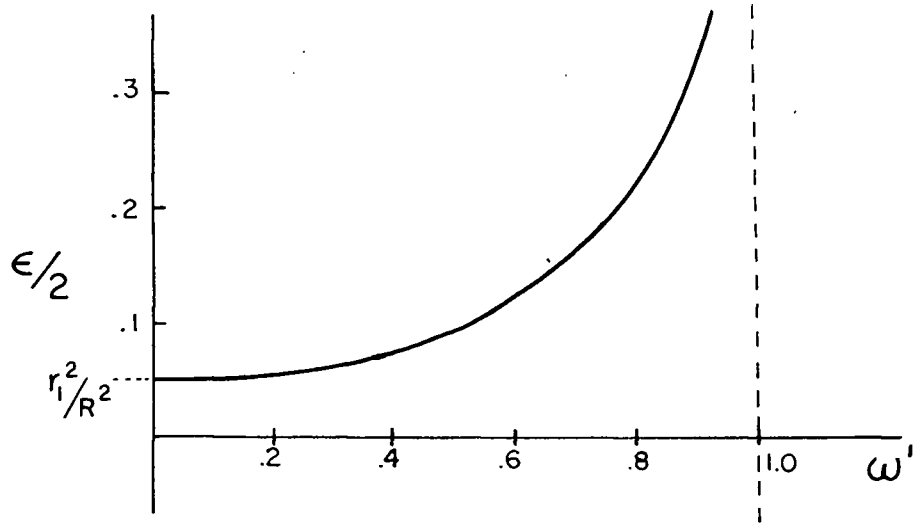


Figure 17. The expression $\epsilon/2$ as a function of frequency ω'.

terms of its definition in (58),

$$'a' = 1 + \frac{k}{2}(1 - 1/\omega') + \frac{M}{2}\left(\frac{1 - \omega'^2}{1 + \omega'^2}\right) \quad . \tag{86}$$

This expression is plotted against frequency for a few negative values of s (Figure 18). The value of M (= 20) appropriate to the homogeneous mode of oscillation has been used, but the shape of the curves is very similar, although on a different scale, when internal oscillations are studied. With $M = 0.53 \times 10^6$, the vertical scale is multiplied by about 10^4 and the roots moved to the extreme

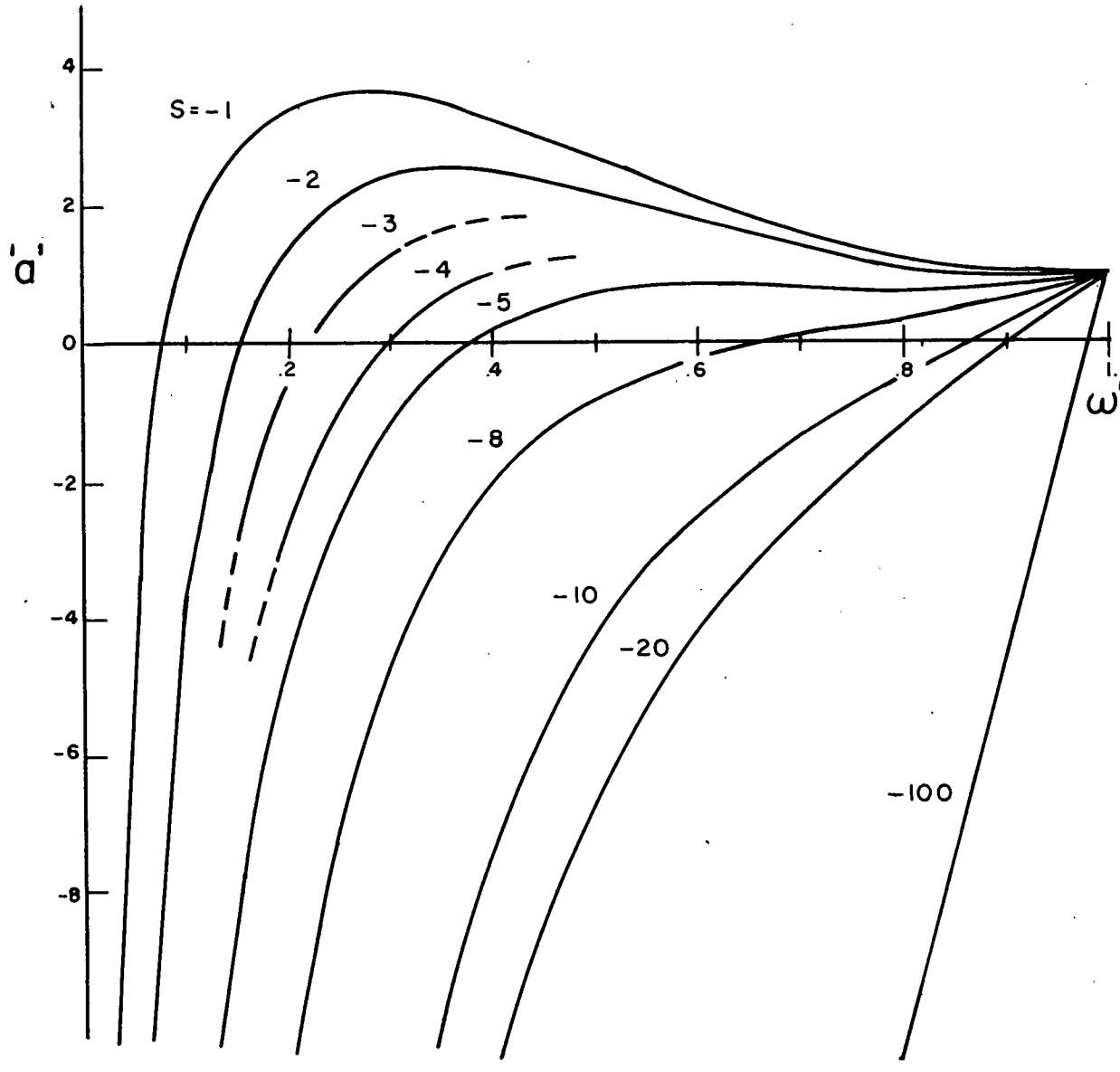


Figure 18. The parameter 'a' (86) plotted versus frequency ω' for a few values of s ($s < 0$).

left of the frequency scale.

From Figure 18 and the descriptive considerations on the properties of confluent hypergeometric functions, this much can be concluded about the left hand side of (85). For all frequencies higher than the root of (86), $a > 0$, $b > 0$, and from property 3b the ratio will be positive but less than unity. For ω' between zero and the root, $a < 0$, $b > 0$ and thus, from property 2, the ratio can assume either sign and amplitudes from zero to ∞. Consideration of (85 rhs) has already revealed that the ratio of confluent hypergeometric functions will be at least of order 10^2 in magnitude, so that it is only in the latter region (between the root of 'a' in Figure 18 and $\omega' = 0$) that solutions of (85) will be possible.

It is to be noticed that for $s = -1$ the root of (86) (Figure 18) occurs at a lower value of frequency than the first singularity in (82 rhs) (Figure 16), so that in the frequency range where solutions can exist the ratio of the confluent hypergeometric functions must be large and positive.

Curves of the functions ${}_1F_1(a; b\ 1; y)$ and ${}_1F_1(a; b; y)$ will follow each other fairly closely when plotted against y (Figure 19a), and the functions will have the same number of roots. The ratio of two such functions, as in (85), can be of order 10^2 or greater only near a root of the denominator. The left hand side of (85) will be large and positive at the positions

y_j ($j = 1,2..$) just to the left of zeros of ${}_1F_1(a; b; y)$ (Figure 19a). The precise value of the positions y_j will of course depend on the value of the other parameters ('a' and 'b') of the confluent hypergeometric functions, i.e., will depend on the frequency ω' through the definition of 'a' in (86). At low frequencies, the variable y (= $\epsilon/2$) is nearly constant: $\epsilon/2 \simeq r_1^2/_{2R^2}$. Although the number of positions y_j is equal to the number of roots of ${}_1F_1$, only that position $y_j(\omega') = r_1^2/_{2R^2}$ will give an eigenfrequency.

It is apparent from descriptive properties 1 and 4c) that, as ω' decreases and 'a' becomes more negative (Figure 18), the mth root (m fixed) of ${}_1F_1(a;b;y)$ tends to the origin of y. But as each mth root 'first' appears (when 'a' decreases to -m), it does so at a higher value of y than for the $(m-1)^{th}$ root. Hence, as ω' decreases, these roots must in turn, in order m, successively move to the left along the y-axis and pass through $y = r_1^2/_{2R^2}$. For a series of decreasing values of frequency the left hand side of the boundary condition (85) passes through a high positive value equal to the right hand side; this happens at the positions indicated as eigenvalues in Figure 19b and 19c. We then recognize the degeneracy predicted by the Method of Signatures, there being an infinite number of eigenfrequencies for each value of wave number s.

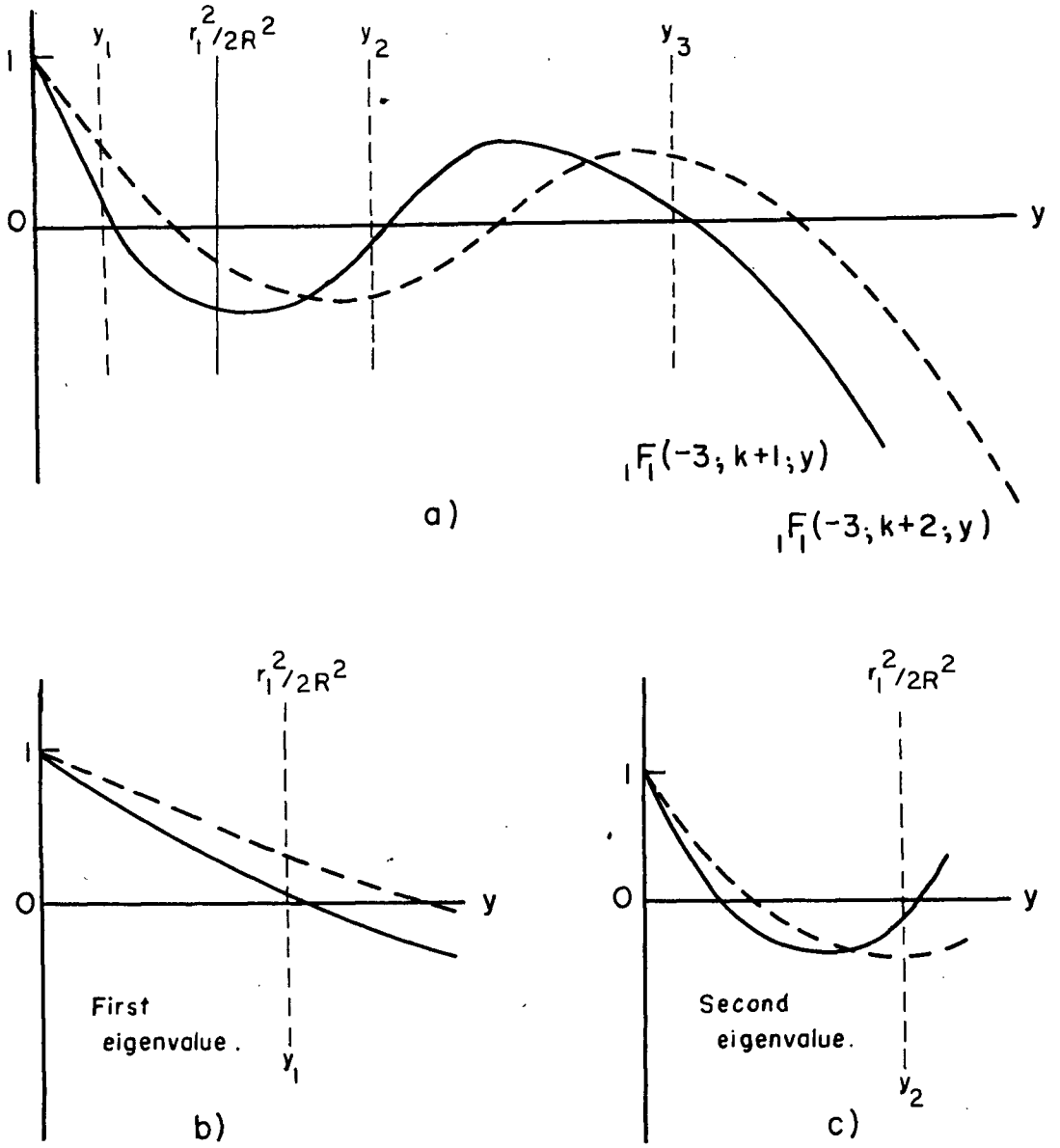


Figure 19. The relative positions of the confluent hypergeometric functions at the eigenvalues. a) The positions y_j where the ratio of the two functions is large and positive. b) The first eigenvalue. c) The second eigenvalue.

The eigenfrequencies will be calculated on the assumption that we are close enough to the root of the denominator of the left hand side of (85) to approximate the nth eigenfrequency by the nth root of ${}_1F_1(k/2+1+\delta/2\epsilon\ ;\ k+1;\ \epsilon/2)$. A first approximation to the nth root of ${}_1F_1(a;\ b;\ y)$ is (Slater, 1960; Chapter 6)

$$y_n = \frac{\pi^2(n + b/2 - 3/4)^2}{(2b - 4a)} \tag{87}$$

This is valid when $(-a + b/2) \gg 1$. It will be verified <u>a posteriori</u> that this approximation is a good one here. Substitution for the dummy variables a,b,y in terms of the actual variables gives for the root

$$\epsilon/2 = \frac{-\mu_{s,n}}{2+2\delta/\epsilon} \tag{88}$$

where $\mu_{s,n} = \pi^2/_{16}(4n+2k-1)^2$. When δ and ϵ are expanded in terms of frequency, by means of (58), a 5th degree algebraic equation in ω' results. If however the frequency is small enough, $\omega'^2 \ll 1$, a linear expression gives the eigenfrequency $\omega'_{s,n}$ explicitly:

$$\omega'_{s,n} = \frac{k}{1+M+\mu_{s,n}\dfrac{R^2}{r_1^2}} \tag{89}$$

A few eigenfrequencies, as calculated from this formula, are listed in Table II. It is certainly true that $\omega'^2 = 1$, so that we may safely pass from (88) to (89). Also, for small values of ω', the expression $(-a + b/2)$, which has to be large for approximation (87) to be valid, is found to be equal to $\mu_{s,n} R^2/r_1^2$; the adopted value of r_1^2/R^2 in the case of a symmetrical basin is 1/20 (Table I), so that the relevant expression is large, even for the smallest value of $\mu_{s,n}$, which is $\mu_{1,1}$ (= 15.33).

It can finally be verified, by actual computation of the ratio of confluent hypergeometric series on the left hand side of (85), that approximating the eigenfrequencies by the roots of the denominator gives quite good results, at least for small n and k. For example, a more precise calculation gives $\omega'_{1,1} = 0.00317$ ('a' = - 146), which differs only by about 3% from the approximate value given by (89) : 0.00306 ('a' = - 152.3). The exact value of 'a' which satisfies the boundary condition (85) is between -146 and -147, so that for small n and k, approximating the eigenfrequencies by the roots of ${}_1F_1(a; b; y)$ is a better approximation than that used to find those roots (87). This approximation however gradually looses its validity as k increases.

Although a long chain of approximations is necessary to find the eigenvalues ($\omega'_{s,n}$) when the solution is expressed in the form of confluent hyper-

TABLE II. Eigenfrequencies of Rossby waves in a homogeneous symmetrical ocean with a flat bottom, as calculated from the confluent hypergeometric solution. $\omega'_{s,n}$ given by (89).

s =	-1	Period (days)	-2	-3
n = 1	.00306	(164)	.00319	.00294
2	.00098	(510)	.00132	.00142
3	.00048	(1040)	.00072	.00084
4	.00028	(1785)	.00045	.00055
5	.00018	(2775)	.00031	.00039

geometric functions, the above example shows that it is possible to obtain approximations to the eigenfrequencies for the case of a flat bottom even when ϵx^2 is not neglected compared to unity. This is not necessary when planetary waves are under investigation, but the method has been developed in prevision of further work including types of oscillations where it is impossible to simplify (74) into (90) below. We have seen in section VI that there may be such time dependent motions in areas II and IV of the diagnostic diagram.

The approximation $\epsilon x^2 << 1$ will now be made, and the problem will acquire much clarity by so doing.

ii) Approximate solution in terms of Bessel functions.

Neglecting ϵx^2 in (74), the amplitude equation for a symmetrical ocean with a flat bottom becomes

$$x^2 \frac{d^2F}{dx^2} + x \frac{dF}{dx} - F(s^2 + \delta x^2) = 0 \quad , \tag{90}$$

which has solutions in terms of Bessel functions, (Z), of the general form (Whittaker and Watson, 1927; Chapter XVII)

$$F(x) = Z_{\pm s}(i\sqrt{\delta}\, x) \quad . \tag{91}$$

The boundary condition (60) is unchanged; at a vertical wall at $x = 1$,

$$\frac{Z'_{\pm s}(i\sqrt{\delta}\, x)}{Z_{\pm s}(i\sqrt{\delta}\, x)} = \frac{s}{i\,\omega'\sqrt{\delta}} \quad , \tag{92}$$

in which the prime indicates differentiation with respect to the argument of the Bessel function. Equation (92) becomes, using the formula

$$Z'_p(y) = -\frac{p}{y} Z_p(y) + Z_{p-1}(y)$$

which is applicable to all Bessel functions,

$$\frac{Z_{\pm s-1}(i\sqrt{\delta}\, x)}{Z_{\pm s}(i\sqrt{\delta}\, x)} = \frac{k}{i\sqrt{\delta}}\left(1 + \frac{s'}{k\omega'}\right) \quad . \tag{93}$$

When ω' is not nearly equal to 1 (which is the condition for the simplification leading to (90)) δ can be approximated by

$$\delta = \left[\frac{s}{\omega'} + M(1 - \omega'^2)\right] \frac{r_1^2}{R^2} \quad . \qquad (94)$$

The parameter δ, as given by (94), will be negative in area I of the diagnostic diagram ($s < 0$, $\omega' < 1$), provided ω' is small enough: $\omega' < 0.05$ for $s = -1$, and $\omega' < 0.1$ for $s = -2$, and so on. This is illustrated in Figure 20. In area II, $s < 0$, $\omega' > 1$, and δ is always negative; in area III, $s > 0$, $\omega' < 1$, and δ is always positive. Finally, in area IV ($s > 0$, $\omega' > 1$), δ will be negative unless ω' is very near unity.

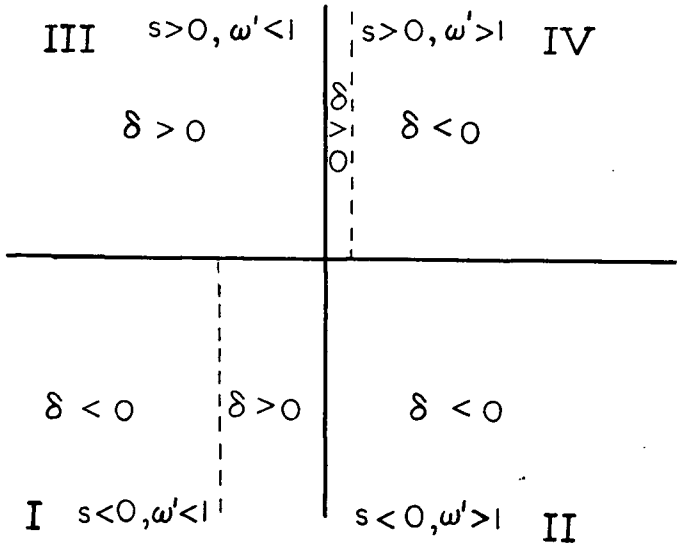


Figure 20. The sign of δ (as given by (94)) in the subdivisions of the diagnostic diagram.

For negative values of δ , the Bessel function $J_k(\sqrt{|\delta|}\ x)$ is real and will be used where the general expression for a Bessel function appears above. The magnitude of s, k, is chosen as the order of the Bessel function: since s is an integer, in which case Z_s and Z_{-s} are not linearly independent, it does not matter which one of these two functions is chosen for the solution.

When δ is positive, the function $I_k(\sqrt{\delta}\ x)$ is real and will be chosen for the solution of (90). This function is related to the ordinary Bessel function as follows:

$$I_k(y) = i^{-k} J_k(iy) \quad .$$

It is not necessary to include Bessel functions of the second kind to complete the solution (as should be done when s is an integer) because these diverge at $x = 0$, where we want our solution to be regular.

For $\omega' < 1$, the parameter δ is negative only when $s < 0$ (area I of the diagnostic diagram). We shall now see that solutions of (90) in that case correspond to the planetary wave solutions of (74) which we found above.

Let us look first at the completely symmetric case: $s = 0$. We can show that there will be no planetary waves exhibiting such symmetry. By (94), when $s = 0$, δ becomes

$$\delta = (1 - \omega'^2)\ M \frac{r_1^2}{R^2} \quad ,$$

which is positive for frequencies less than inertial. The boundary condition (93) then takes the form

$$I_1(\sqrt{\delta}\ ) = 0 . \tag{95}$$

$I_k(y)$ being a positive monotonic increasing function of y, (Jahnke and Emde, 1945; Chapter VIII), (95) can be satisfied only when $\delta = 0$, i.e., when $\omega' = 1$. This solution is however entirely inconsistent with the assumption $\epsilon x^2 << 1$, which led to (90), so that the case s = 0 cannot be properly discussed by the approximate equation (90). We must then go back (74) and find what the confluent hypergeometric solution predicts when s (and k) = 0; from (89), $\omega'_{0,n}$ is zero for all n, so that there are no planetary waves with zero wave number.

It should be noted that if higher frequencies are considered ($\omega' > 1$) solutions of (90) can be found for s = 0. The boundary condition then becomes

$$J_1(\sqrt{|\delta|}\ ) = 0, \tag{96}$$

which is satisfied when

$$\omega' = (1 + \beta^2_{1,n}\ ) . \tag{97}$$

The constant $\beta_{1,n}$ is the nth root of $J_1(y)$. This is of course only a limiting case of the gravity controlled oscillations of areas II and IV of the diagnostic diagram. We limit ourselves here to a discussion of planetary waves

in a polar ocean, so that nothing more will be said about these gravity waves.

For non-zero negative values of s, δ is negative when ω' is small enough (Figure 20), and the solution of (90) is

$$F(x) = J_k(\sqrt{|\delta|}\, x) . \tag{98}$$

The boundary condition is given by (93), with J_k replacing Z_k. Even when ω' is not very small, the right hand side of (93) will be considerably larger than unity, and one finds that the value of δ which satisfies the boundary condition differs from the root of the denominator of the left hand side of (93) only in the third significant figure. The eigenfrequencies will then be approximately given by

$$\omega'_{s,n} = \frac{k}{M + R^2/r_1^2\, \beta^2_{k,n}} \tag{99}$$

The constant $\beta_{k,n}$ is the nth root of the Bessel function J_k; k takes the values 1, 2, 3,... corresponding to s = -1, -2, -3,... . From Table I (R = 6370 km, r_1 = 1500 km) $R^2/r_1^2 = 20$. Some of the eigenfrequencies, as calculated from(99) are listed in Table III. They depart from the eigenfrequencies as calculated from the confluent hypergeometric solution at larger values of k. This is not

due to the approximation $\epsilon x^2 << 1$ becoming worse for large k (it becomes better, since ω' decreases) or to estimating the eigenvalue from the roots of the left hand side of (93) (this also improves as k increases), but stems from approximating the eigenvalues of the confluent hypergeometric solution by the roots of the denominator of (85). For a constant n, this last approximation gradually looses in exactitude as k increases. We will then adopt the eigenfrequencies listed in Table III, rather than those of Table II, as characteristic results of this analysis when performed in a polar plane where only a first approximation of the curvature of the Earth is retained. These results will later be compared with their equivalents as calculated in the spherical geometry.

The analysis is entirely similar when the internal mode of oscillation is investigated ; the constant M, which represents the effect of gravity, is now however changed from a value of 20 to a new and much higher value corresponding to the stratification adopted: $M = 0.53 \times 10^6$. The smaller effective gravity means that lower frequencies will be necessary to make the influence of the terrestrial curvature comparable in importance to gravity forces; this is best seen in the definition of δ (94), where a direct comparison of the two influences can be made. Some internal mode frequencies are tabulated in Table IV; the periods corresponding to these frequencies will be of the

TABLE III. Eigenfrequencies of Rossby waves in a homogeneous symmetrical ocean with a flat bottom, as calculated from the Bessel function solution.

$\omega'_{s,n}$ given by (99).

s =	-1	-2	-3	-5	-8
n = 1	.00324	.00366	.00359	.00320	.00261
2	.00099	.00139	.00155	.00163	.00152
3	.00048	.00073	.00088	.00101	
4	.00028	.00045	.00057	.00069	
5	.00018	.00031	.00040	.00051	

order of a millenium and more. Realizing that the surface layer is subjected to annual variations associated with atmospheric conditions, it is seen that the stratification will not remain constant during long period oscillations, and that it is not strictly correct to treat the depth of the interface as constant in time. If the conditions are purely periodic over a yearly period, and there is little secular change, it might be permissible to take the average stratification as constant, since the period of variations is so small compared with the period of the planetary waves. Needless to say, there is little hope of any direct measurement of the long-period internal waves.

TABLE IV. Eigenfrequencies of internal Rossby waves for a two-layer symmetrical ocean with a flat bottom, as calculated from the Bessel function solution. $\omega'_{s,n}$ given by (99). $M = 0.53 \times 10^{6}$.

s =	-1	Period (years)	-2	-3
n = 1	1.88×10^{-6}	(2915)	3.76×10^{-6}	9.40×10^{-6}
4	1.87x "	(2930)	3.74x "	9.30x "
9	1.83x "	(2995)	3.64x "	9.00x "

Before comparing the above eigenfrequencies with those obtained for a similar basin in the spherical geometry, let us see what the solutions look like, and what are the properties of those waves, the frequencies of which we just calculated. From (30) and (98), and (34), the surface displacement (for the homogeneous mode) and the corresponding velocities are explicitly given by

$$\eta = c\, J_k\left((k/\omega' - M)^{\frac{1}{2}} \frac{r_1 x}{R} \right) e^{i(\omega t - s\phi)} \tag{100}$$

$$u = \frac{-i\, cg\omega'}{2\Omega r_1} \left\{ \frac{dJ_k(\ )}{dx} - \frac{sJ_k(\ )}{\omega' x} \right\} e^{i(\omega t - s\phi)} \tag{101}$$

$$v = \frac{gc}{2\Omega r_1} \left\{ \frac{dJ_k(\quad)}{dx} - \frac{s\,\omega'}{x} J_k(\quad) \right\} e^{i(\omega t - s\phi)} \tag{102}$$

In the above expressions, the constant c has the dimensions of a length, and is determined by the total energy of the system. The argument of the Bessel functions is the same in (101) and in (102) as in (100), and has not been written in explicitly.

Besides varying in a periodic manner around the pole, the amplitude and velocities have nodes between the pole and the boundary. The velocity vector traces an ellipse as time varies, at any fixed location; the eccentricity of this ellipse is given by

$$1 - \frac{|\underline{u}|}{|v|} = 1 - \omega' \frac{\left| \frac{dJ_k(\quad)}{dx} - \frac{s}{\omega' x} J_k(\quad) \right|}{\left| \frac{dJ_k(\quad)}{dx} - \frac{s\,\omega'}{x} J_k(\quad) \right|}, \tag{103}$$

so that it will vary with distance from the pole, x, frequency, ω', and wave number, s. The direction in which the ellipse is traced will also vary from place to place (radially), and bands where the velocity vector rotates clockwise will alternate with bands where it rotates counter-clockwise, as one progresses from the pole towards the

boundary (Figure 22c). Sketches of amplitude and velocity contours for a few values of s and n, are found in figures 21 and 22.

Being of order x^k and x^{k-1} respectively at small values of x, the amplitude and velocities remain finite over the whole Arctic ocean and its boundaries provided $k > 0$, which is the case for the planetary waves studied now. The vorticity in the vertical direction, ξ , as given by (52), becomes in terms of functions of x

$$\xi = \frac{gc}{2\Omega r_1^2} \left\{ \frac{d^2 J_k(\quad)}{dx^2} + \frac{1}{x}\frac{dJ_k(\quad)}{dx} - \frac{s^2}{x^2} J_k(\quad) \right\} e^{i(\omega t - s\phi)} \tag{104}$$

At first sight, it seems that this expression would be of order x^{k-2} at small x; if however we replace the independent variable x by the argument of the Bessel functions, $\sqrt{|\delta|}\ x$, noting that $k^2 = s^2$, we find that

$$\xi = \frac{gc\ \delta^2}{2\Omega r_1^2} \left\{ J_k''(\quad) + \frac{1}{\sqrt{|\delta|}x} J_k'(\quad) - \frac{k^2}{|\delta| x^2} J_k(\quad) \right\} e^{i(\omega t - s\phi)} \tag{105}$$

Since J_k is a Bessel function, it satisfies Bessel's equation, whatever its argument:

$$y^2 J_k''(y) + y J_k'(y) + (y^2 - k^2)\ J_k(y) = 0 \quad .$$

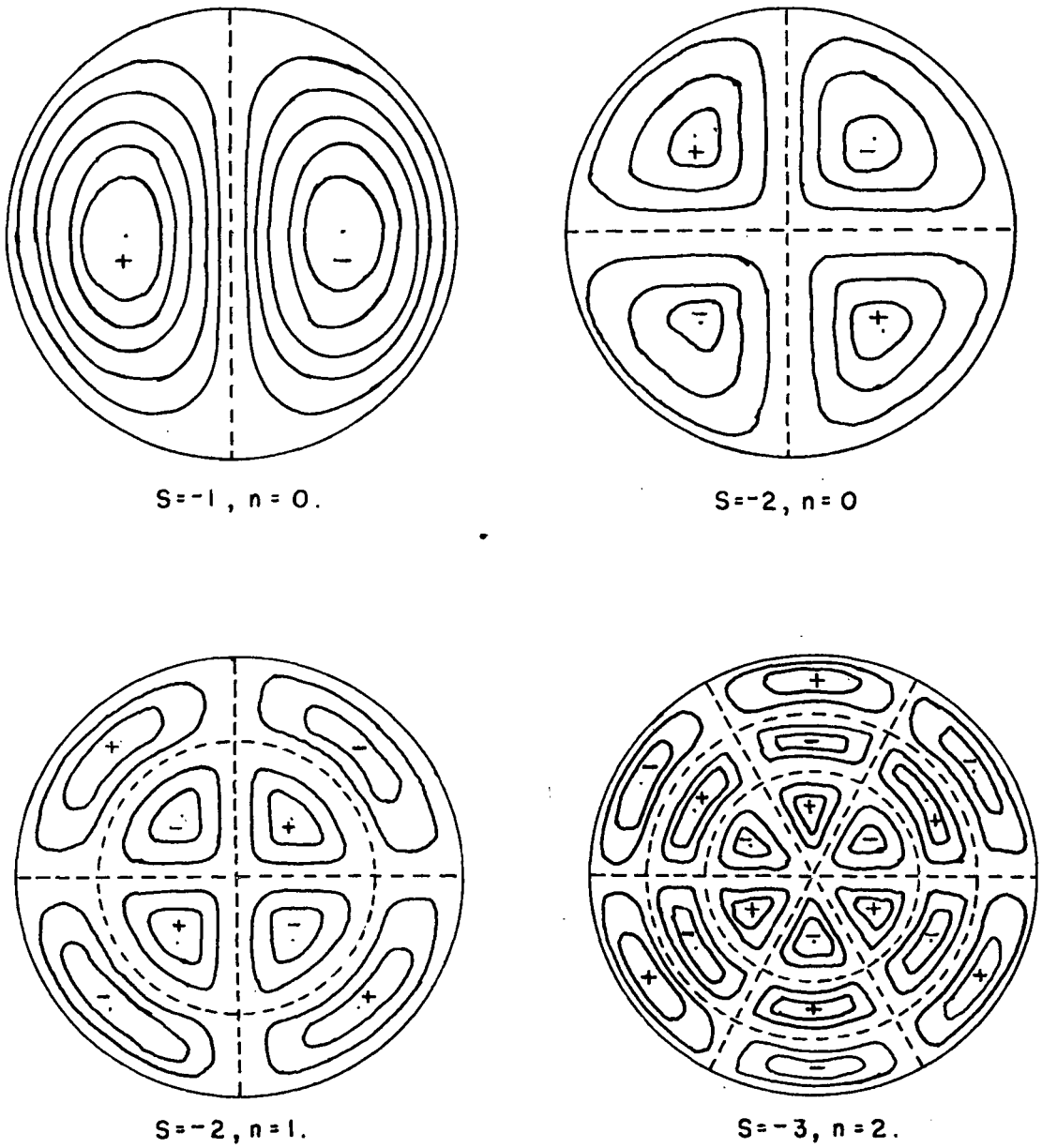


Figure 21. Sketches of surface amplitude contours (η , from (100)) for planetary waves in a symmetrical ocean with a flat bottom, for a few values of wave number, s, and index number, n. The patterns rotate clockwise with angular velocity $\omega'_{s,n}/s$. Dotted lines are nodal lines.

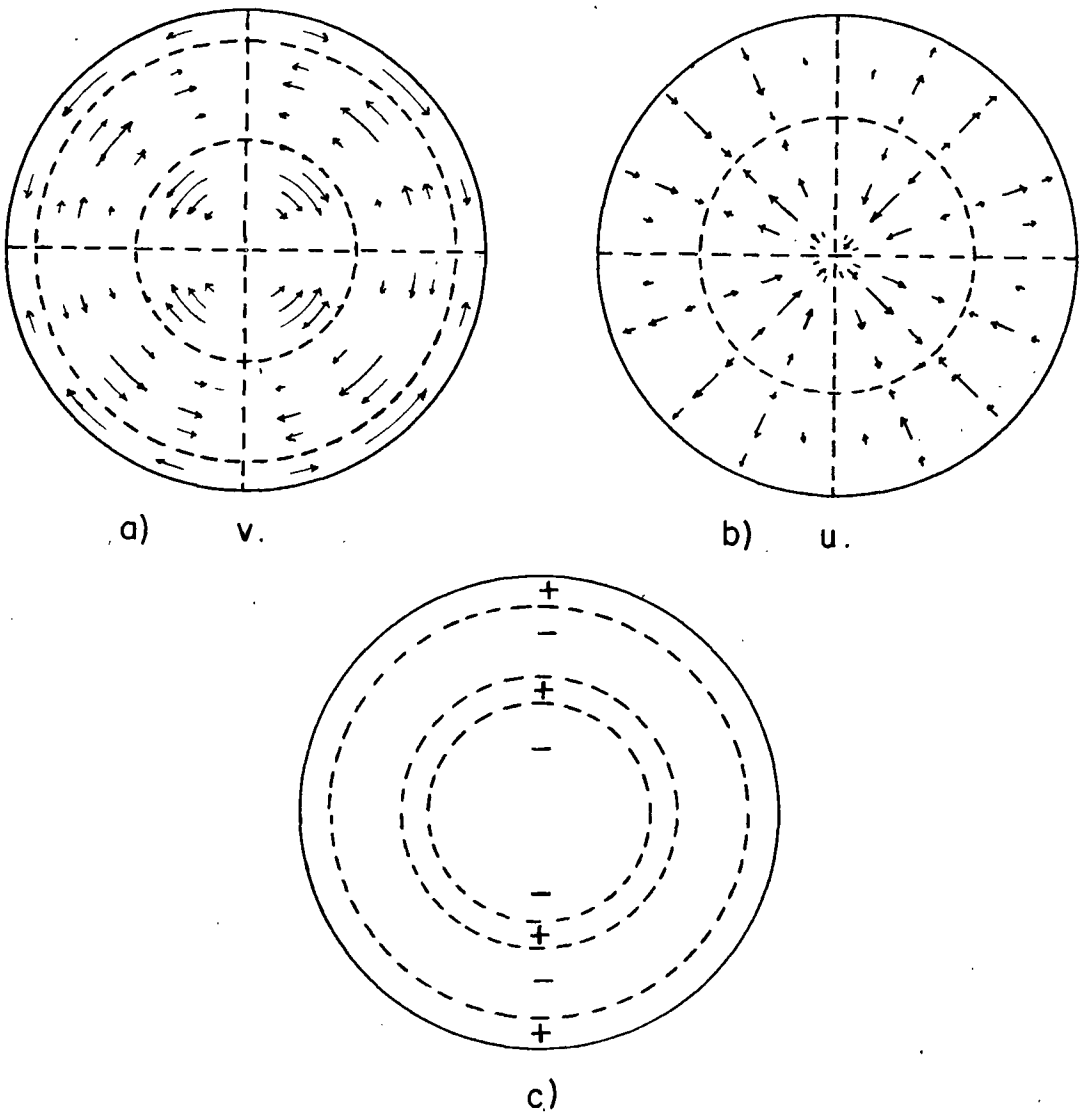


Figure 22. a) and b) Sketches of the zonal (v) and radial (u) components of the velocity field (from (101) and (102)) for s = -2, n = 1. The patterns rotate clockwise with angular velocity $\omega'_{-2,1}/2$. Dotted lines are nodal lines, and the velocities are larger where the arrows are longer. c) The direction in which the local velocity vector traces an ellipse : - for clockwise, + otherwise.

The primes in (105) and in Bessel's equation indicate differentiation with respect to the argument of the function. The three terms in brackets in (105) then reduce to $- J_k(\sqrt{|\delta|}\ x)$, and the vorticity becomes

$$\xi = \frac{-gc\delta^2}{2\Omega r_1^2} J_k(\sqrt{|\delta|}\ x)\ e^{i(\omega t - s\phi)} \qquad (106)$$

which is of order x^k at small x. In spite of the neglect of viscosity, there is no singularity in vorticity at the pole for the planetary waves considered, and the Rossby number remains finite everywhere.

To first order, the average energy transport due the wave motion in a vertical column of water is

$$< \int_z p\ \underset{\sim}{v}\ dz > \quad , \qquad (107)$$

in which the brackets indicate average over a cycle. When the pressure is hydrostatic, and the velocity does not depend on Z, (107) becomes

$$< \rho g \eta \underset{\sim}{v} H > \qquad (108)$$

The time average of the radial energy transport vanishes; the zonal component becomes proportional to

$$\frac{H}{(1 - \omega'^2)} \left[\frac{1}{2} \frac{dF^2}{dx} - \frac{s\,\omega' F^2}{x} \right] \quad . \tag{109}$$

The net energy transport in the zonal direction is the integral of (109) from the pole to the boundary; when the depth is constant, this is proportional to

$$F^2(1) - s\,\omega' \int_0^1 \frac{F^2}{x} dx \quad . \tag{110}$$

At the boundary, the amplitude is very small; as a matter of fact, we have approximated the eigenfrequencies by those values of frequency which make the amplitude vanish at $x = 1$. The second term of (110) will therefore dominate, and the energy will propagate on the average (over a cycle and over all values of the radial coordinate) in a direction opposite to that in which the phase moves. The energy transport so calculated differs from that given by the group velocity only by a non-divergent vector (Longuet-Higgins, 1964 a); it is more convenient to use the present method in closed basins.

The planetary waves just described correspond to the case $\delta > 0$ in (94). Let us look briefly at the case $\delta > 0$. From Figure 20, this can occur in area III and small bands of areas I and IV of the diagnostic diagram. The boundary condition (93) becomes, when $\delta > 0$,

$$\frac{I_{k-1}(\sqrt{\delta})}{I_k(\sqrt{\delta})} = \frac{k}{\sqrt{\delta}}\left(1+\frac{s}{k\,\omega'}\right). \qquad (111)$$

Since the functions $I_k(x)$ and $I_{k-1}(x)$ are monotonic increasing functions of x such that $I_{k-1}(x) > I_k(x) \geq 0$, the right hand side of (111) must be positive and greater than unity for the relation to be satisfied.

Oscillations of this kind will be very similar to Kelvin waves, in the sense that they will hug the sides of the basin, their amplitude increasing very rapidly near $x = 1$, according to the behaviour of $I_k(\sqrt{\delta}\ x)$. They are now being studied in more detail by H.G. Farmer, at the University of Washington (Farmer, 1964), and I therefore limit myself to a mention of their existence.

iii) Comparison with results on a sphere.

Let us now compare the results obtained above with their equivalents on the sphere. If the two are compatible, the validity of the analysis performed in the polar plane

approximation to the sphere will be established in the study of all possible motions of the contained fluid. This is so because planetary waves, being dependent on the curvature of the Earth for their existence will be more strongly affected by any departures from the exact curvature than any other type of motions. If the approximation works for planetary waves, it will then work and give reliable results for all other motions of the Arctic ocean.

The simplest basis of comparison is the work of Longuet-Higgins (1964 b). By assuming that the surface displacements have a negligible influence on the vorticity balance, Longuet-Higgins has been able to formulate the problem of planetary waves in two dimensions and to solve it in terms of a stream function. This approximation will give good results provided the wave length is smaller than the radius of the Earth. In particular, he gives for the stream function characterizing the planetary waves in a polar basin on the sphere

$$\psi = P_\nu^k (\cos\theta) \quad , \tag{112}$$

in which $P_\nu^k(\cos\theta)$ is the Legendre function of arguments k, ν , and $\cos\theta$; k is defined as above, and so is θ, while ν is a positive real number (not necessarily an integer) which allows the boundary condition to be satisfied:

$$P_\nu^k (\cos\theta_1) = 0 \; ; \qquad \theta_1 = \sin^{-1} r_1/R \tag{113}$$

The frequency of the planetary wave is then given by

$$\omega'_k = \frac{k}{\nu(\nu+1)} \quad . \tag{114}$$

Equation (114) is not very dissimilar in form to (99), but the evaluation of ν is much more complicated than finding $\beta_{k,n}$. The Legendre function $P_\nu^k(x)$, when x is real and k (but not ν) is an integer, is given by the expression

$$P_\nu^k(x)\frac{(-2)^k}{k!}\frac{\Gamma(\nu+k+1)}{\Gamma(\nu-k+1)}(1-x^2)^{k/2} \tag{115}$$

$$= {}_2F_1(1+k+\nu, k-\nu\,;\ k+1;\ \tfrac{1}{2}-\tfrac{1}{2}x) \quad ,$$

where ${}_2F_1$ is the standard notation for the usual hypergeometric series, $\Gamma(y)$ is the gamma function:

$$\Gamma(y) = \int_0^\infty e^{-t}t^{y-1}\,dt \ .$$

The expression (114) will have roots in ν only when the hypergeometric series vanishes; it is quite clear from the behaviour of hypergeometric series (Erdelyi et al., 1953; Chapter II) that there is an infinity of values of ν , of ever increasing magnitude, which makes the series zero for constant k and x. The same degeneracy then exists as

TABLE V. Computed eigenfrequencies of Rossby waves for a symmetric polar basin with a flat bottom on a sphere, according to Longuet-Higgins' model.

n=1.	s = 1	$\omega'_{s,1}$ =	0.00344
	2		0.00378
	3		0.00369
	5		0.00324
	8		0.00268
	12		0.00224
	16		0.00179

was found in the polar plane.

There is no analytical formula allowing calculation of the roots, and I have estimated the first root by computing the sum of the first 20 terms of the series for increasing values of ν until a change of sign occured. The cosine of the angle corresponding to $(r_1/R)^2 = 1/20$ is 0.975; the interval between successive values of ν was taken as 0.5 except for the last two estimates, where it was 1.0. The value of ν corresponding to the root was then estimated by linear interpolation. Only the first root was obtained this way, since only comparison in one direction was deemed necessary, and because of the

considerable time necessary for computing. The calculations were done on the University of British Columbia's IBM 1620, and the program written for this purpose is given in Appendix I. The results are listed in Table V.

Comparison of tables III and V shows that the eigenfrequencies calculated on a polar plane model differ from Longuet-Higgins' two-dimensional approximation on the sphere by small but appreciable values. To see whether this discrepancy arises from the polar plane assumption or from neglecting the influence of surface displacements, we can apply this last approximation to the polar plane, and see whether the results are closer to the stricter polar plane results or to the results on the sphere. In the first case, the discrepancy arises from the use of the polar plane, in the second, from the neglect of surface displacements.

For a two-dimensional problem in the angles θ and λ, we can write the velocities in terms of a stream function ψ ; in spherical polar coordinates,

$$u = \frac{1}{R \sin\theta} \frac{\partial \psi}{\partial \lambda}$$

$$v = \frac{-1}{R} \frac{\partial \psi}{\partial \theta} \qquad (116)$$

where u and v are radial and zonal velocities respectively. Elimination of the pressure gradients from the momentum equations (9) and (10) and of the velocities through the

continuity equation (6) gives a vorticity equation, which becomes in terms of the above defined stream function

$$\frac{\partial \nabla_H^2 \psi}{\partial t} - \frac{1}{R^2 \sin\theta} \frac{\partial f}{\partial \theta} \frac{\partial \psi}{\partial \lambda} = 0 \quad , \qquad (117)$$

which is the working equation of Longuet-Higgins. Transforming to the polar plane by means of the relations (23), (117) becomes in the new geometry

$$\frac{\partial}{\partial t} \left\{ \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial (r\cos\theta)}{\partial r} \right\} + 2\Omega \frac{\cos\theta}{R^2} \frac{\partial \psi}{\partial \phi} = 0. \qquad (118)$$

Let us look for vorticity waves of the form

$$\psi = \psi_o(r)\, e^{i(\omega t - s\phi)} \quad ; \qquad (119)$$

since this is of the same form as the postulated elevation η (3 0), the derivatives of ψ are obtained from (31)-(33) by replacing η by ψ . Defining x as in (58), substitution of (119) into (118) gives an amplitude equation for ψ_o:

$$x^2 \frac{d^2\psi_o}{dx^2} + x \frac{d\psi_o}{dx}\left(1 + \frac{r_1^2 x^2}{R^2}\right) - \psi_o\left(s^2 + \frac{s x^2 r_1^2}{\omega' R^2}\right) = 0 \qquad (120)$$

This equation resembles very much (74); in fact, when the frequency is low enough so that $\epsilon x^2 << 1$, the two differ only in the definition of the constant δ .

Putting now

$$\delta_1 = \frac{s\, r_1^2}{\omega'_R{}^2}$$

we see that in this case there is no gravity influence (as represented by the M term in (94)). This is of course a direct consequence of neglecting the surface elevations : the problem is treated as two-dimensional, and there are no departures from the equilibrium level on which gravity can act.

When δ_1 is negative, planetary wave solutions for the amplitude equation of the stream will exist analogous to those for the displacement amplitude in (94):

$$\psi_o(x) = J_k(\sqrt{|\delta_1|}\, x) \qquad (121)$$

The boundary condition is now slightly different, and is

$$\psi_o(1) = 0 = J_k(\sqrt{|\delta_1|}\ ) \qquad (122)$$

so that the eigenfrequencies are now given by the formula

$$\omega'_{s,n} = \frac{k}{20\, \beta^2_{k,n}} \qquad (123)$$

Comparison of (122) with (99) shows that for large values of k and/or n, the two formulae will give very similar results. Some values calculated from (122) are listed in Table VI, where they are compared with some of the eigenfrequencies derived by other methods.

TABLE VI. Comparison of eigenfrequencies of planetary waves in a symmetrical polar ocean with a flat bottom, as obtained from different methods. P-P means polar plane approximation; 2-D is Longuet-Higgins' two-dimensional approximation.

$s =$	$\omega'_{s,1} =$			
	P-P	2-D	P-P and 2-D	Goldsbrough
-1	.00324	.00344	.00342	.00494
-2	.00366	.00378	.00380	.00735
-3	.00359	.00369	.00369	
-5	.00320	.00324	.00326	
-8	.00261	.00268	.00268	
-12	.00222	.00224	.00225	
-16	.00179	.00179	.00179	
Calculated from equation	(99)	(114)	(122)	

It appears from Table VI that the eigenfrequencies $\omega'_{s,1}$ tend to the same values whatever the mode of calculation, provided k is large enough. Furthermore, the discrepancies between the solutions on the polar plane and the approximate two-dimensional solutions on the sphere cannot be attributed to the imperfection of the mapping on the plane, because when the problem is formulated as two-

dimensional on the plane, almost identical results are obtained as on the sphere. Since Longuet-Higgins' assumption that the surface displacements are of negligible influence on the vorticity balance is known to be valid only for wave lengths appreciably smaller than the radius of the Earth, the difference with the results on the polar plane would seemingly be caused by the inadequacy of the two-dimensional assumption at small wave numbers. The difference between results on the sphere and those on the polar plane using Longuet-Higgins approximation is not detectable, so that one must conclude that the results of column 1 in table VI are more precise than those of the following two columns.

When the diameter of the polar ocean is small enough to drop $(r_1/R)^2$ with respect to 1, the results provided by the polar plane approximation are as precise as those provided by Longuet-Higgins method on the sphere, and even more precise at low wave numbers. Another advantage is that the eigen-frequencies are much easier to calculate on the polar plane; finally, my formulation allows consideration of bathymetric variations in the model. This analysis is of course restricted to polar regions, and does not have the general applicability to all latitudes that Longuet-Higgins' method possesses.

Another basis of comparison is the work of Goldsbrough on the dynamics of tides in polar basins (Goldsbrough, 1914 a); his work is done entirely in

spherical polar coordinates, and the two frequencies which can be compared to the results of the present work are presented in the fourth column of Table VI. They depart considerably from the corresponding values in the other three columns. The process of calculating anything but a first approximation to Goldsbrough's frequencies is quite involved, since the eigenfrequencies are to be evaluated from an infinite determinant. A second approximation has been attempted, but does not yield values of ω' near those of interest. No apparent reason has been found for the discrepancy in the magnitude of the eigenvalues. Comparing Longuet-Higgins' simple and clear formulation with Goldsbrough's involved series solutions and infinite determinants one is tempted to give more faith to the results of the former.

In view of the good agreement of polar plane results with spherical geometry results as derived from Longuet-Higgins, and in spite of the not so good agreement with Goldsbrough's values, for which the basis of comparison is narrower (two frequencies), I then conclude that the polar plane will be quite useful in studying the motions of fluids in restricted polar basins, and give quantitatively precise results.

The polar plane, defined as the projection of Figure 5 together with the retention of only a first approximation to the Earth's curvature, can therefore be used in the Arctic regions in the same manner as the

β -plane is used in mid-latitudes.

This section has described the characteristic oscillations of the simplest polar basin: bounded along a parellel of latitude and without any depth variations. This is far from describing the actual Arctic bathymetry, and in the next section, an added degree of complexity will be introduced in the form of radial depth variations.

It may be asked whether such long period planetary waves as discovered above are of any dynamic significance, even in a simple symmetric basin. Veronis and Stommel (1956) have shown (in the β-plane formulation) that for winds acting over a period of more than half a pendulum day a significant portion of the total energy is transferred into long period semi-geostrophic planetary waves. This result does not depend on the particular projection used , and will hold just as well for the polar plane. Planetary waves can then be generated by fluctuating winds over such a symmetrical basin as studied above.

VIII. SYMMETRICAL OCEAN: RADIAL DEPTH VARIATIONS.

The next step in the scale of increasing complexities is the inclusion of radial depth variations: $dH'/dx \neq 0$; $dH'/d\phi = 0$. The water content of the polar basin is now considered vertically homogeneous; the amplitude is determined by equation (59) and the boundary condition (60). We have seen in section VI that it is possible for solutions to exist in the presence of a wide variety of bottom configurations; it is not easy however to solve explicitly the amplitude equation when $H'(x)$ is substituted in it. We will therefore have to be satisfied with the simplest bottom topography in order to obtain explicit solutions. This will suffice however to show the nature of the effects of the depth variations.

The following simple depth dependence illustrates very well the influence of bottom topography on planetary waves. Let us assume that the depth varies very little over the extent of the basin, so that it can be considered constant when not differentiated; its radial dependence is of the same form as that of the Coriolis parameter:

$$H' = (1 + px^2/2) \quad , \qquad (123)$$

in which $p/2 << 1$. When (123) is substituted into (59), the amplitude equation assumes the same form as when there are no depth variations; only some of the constants are changed:

$$x^2\frac{d^2F}{dx^2} + x\frac{dF}{dx}\left[1+(\epsilon+p)x^2\right] - F\left[s^2 + (\delta + \frac{ps}{\omega'})x^2\right] = 0 \ . \qquad (124)$$

This equation can be solved in terms of confluent hypergeometric functions, as (74) was, but this is not necessary when planetary waves are considered, since it is then possible to neglect $(\epsilon + p)x^2$ with respect to 1 and use the reduced equation

$$x^2\frac{d^2F}{dx^2} + x\frac{dF}{dx} - F\left[s^2 + (\delta + \frac{ps}{\omega'})x^2\right] = 0 \ , \qquad (125)$$

which is identical in form with (90), and has therefore similar solutions. Examining the planetary wave solutions, which, by analogy to (90), occur where $\delta + ps/\omega' < 0$, the amplitude is then

$$F(x) = J_k\left(\sqrt{\left|\delta + \frac{ps}{\omega'}\right|}\ x\right) \ . \qquad (126)$$

Expanding the constant $\delta + ps/\omega'$ in terms of frequency, and keeping only the first order terms in ω' because of the low frequencies of planetary waves, one has

$$\delta + \frac{ps}{\omega'} \simeq (p + \frac{r_1^2}{R^2})\frac{s}{\omega'} + M\frac{r_1^2}{R^2} < 0 \ . \qquad (127)$$

In equation (127) one can see the role played by a bottom configuration of the form (123): if p is positive, so that the depth increases towards the boundaries, then s must be negative and propagation to the west, in order to keep (127) negative. If p is negative, so that the depth decreases radially, and large enough to make $(p + r_1^2/_R2)$ negative also, then the wave number s will have to be positive for (127) to hold (for low frequencies, the frequency dependent part of (127) will dominate). Propagation is therefore to the east. This is exactly what the method of signatures predicted in section VI: planetary waves propagating towards the east can exist if dH'/dx is large and negative, corresponding to a negative p. One also observes in (127) that the variation of Coriolis parameter (the $r_1^2/_R2$ term) produces asymmetries between waves corresponding to equal but opposite depth gradients.

The boundary condition is

$$\frac{J_{k-1}\left(\sqrt{\left|\delta + \frac{ps}{\omega'}\right|}\right)}{J_k\left(\sqrt{\left|\delta + \frac{ps}{\omega'}\right|}\right)} = \frac{k\left(1 + \frac{s}{k\omega'}\right)}{\sqrt{\left|\delta + \frac{ps}{\omega'}\right|}}, \qquad (128)$$

and, as in the flat bottom case, the eigenfrequency is closely approximated by the root of the denominator of the left hand side. This gives for the frequencies

$$\omega'_{s,n} = \frac{-s(\ p + r_1^2/R^2)}{\beta^2{}_{k,n} + Mr_1^2/R^2} \qquad (129)$$

in which $\beta_{k,n}$ is defined as in (99). A few eigenfrequencies are tabulated in table VII.

The solutions corresponding to $(\ \delta + ps/\ \omega\ ') > 0$ have not been investigated, but they will be an extension of the results for a flat bottom when $\delta > 0$ (111).

Although the bathymetry adopted in the above example is very simple, it illustrates clearly the influence of bottom variations on the properties of planetary waves. For more complicated topographies, it may not be possible to find an explicit analytic solution of the amplitude equation; it may however be integrated numerically, given $H'(x)$, the frequency being adjusted until a value satisfying the boundary condition is found.

Some general theorems concerning the motion of shallow rotating liquids on a paraboloid have been demonstrated by Ball in a recent article (Ball, 1963); a special case has also been treated by Miles and Ball (1963). Do these theorems apply to an Arctic basin in the polar projection used in this study?

Ball's representation of the problem is slightly different from that used up to now; the reference level

TABLE VII. Some eigenfrequencies of planetary waves for a symmetrical ocean with a radial bottom slope. $H' = 1 + px^2/2$. $\omega'_{s,1}$, calculated from (129).

	$p = 0.1$	$p = -0.1$
	$s < 0$	$s > 0$
$k =$	$\omega'_{s,n} =$	
1	0.00954	0.00318
2	0.0110	0.00366
5	0.00963	0.00321

for the vertical coordinate is taken at the maximum depth of the basin, the elevation of the bottom above that location being Z, and the surface displacement and the local equilibrium depth being grouped under the same variable h. This is illustrated in Figure 23.

The first theorem proven by Ball is that the displacement of the centre of gravity of the liquid is independent of the motion that occurs within the liquid relative to the centre of gravity. His basic equations, with coordinates x, y and velocities u,v to the east and north respectively, are then

$$\frac{Du}{Dt} + g\frac{\partial}{\partial x}(h+Z) = fv \tag{130}$$

$$\frac{Dv}{Dt} + g\frac{\partial}{\partial y}(h+Z) = -fu \tag{131}$$

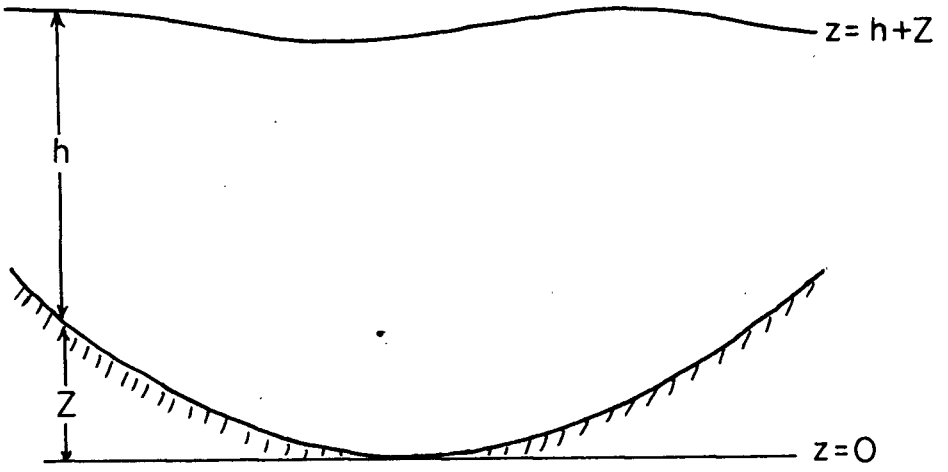


Figure 23. Ball's definition of vertical dimensions.

$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 . \qquad (132$$

In the above, f is the Coriolis parameter, $2\Omega \cos\theta$; the D /Dt are total time derivatives, the non-linear terms being included. The proof of the theorem involves multiplying the equations of motion by the depth and integrating to obtain expressions involving the coordinates

of the centre of gravity; performing these operations on the Coriolis acceleration terms, one has

$$\int fhv \, dS, \tag{133}$$

the integral being over an area enclosing a given amount of liquid, the boundaries moving with the liquid. If the Coriolis parameter is not a constant, but can be written as $f = f_o + f_1$, f_o being a constant, but not f_1, the integral (133) becomes (using Ball's notation, with $Q =$ total constant volume of fluid, X,Y, the coordinates of the centre of gravity)

$$Q \, f_0 \frac{dY}{dt} + \int f_1 vh \, dS \tag{134}$$

Ball implicitly assumes that either the Coriolis parameter is constant, or that the area of the basin is small enough so that it does not depart very much from some average value, the second term of (134) being therefore negligible. Since the variation of Coriolis parameter is of primordial importance in planetary waves, it would seem that the first theorem of Ball would not apply to them. We have seen however that as far as planetary waves are concerned a change in depth has an effect similar to a change in f (see the potential vorticity equation, (54)). The Coriolis parameter can therefore be considered constant, its variable part being represented by a depth variation. For example, if the Coriolis parameter were constant, but

a depth variation of the form (123) existed, with $p = r_1^2/R^2$,

$$H' = (1 + r_1^2/2R^2\, x^2), \quad f = \text{constant},$$

the free eigensolutions would be exactly those found for the flat bottom case in section VII. Ball's first theorem therefore applies in cases where f varies, so that

1) the displacement of the centre of gravity of the liquid is independent of the motion that occurs within the liquid relative to the centre of gravity, and

2) the equations governing the motion of the liquid relative to its centre of gravity have exactly the same form as the original equations of motion.

The other fundamental theorem demonstrated by Ball concerns the variations of the angular momentum; his equations are now put in polar form:

$$\frac{Du}{Dt} + g\frac{\partial}{\partial r}(h+Z) = (f + v/r)v \tag{135}$$

$$\frac{Dv}{Dt} + \frac{g}{r}\frac{\partial}{\partial \phi}(h+Z) = -(f + v/r)u \tag{136}$$

$$\frac{Dh}{Dt} + h\left[\frac{\partial v}{\partial \phi} + \frac{1}{r}\frac{\partial ur}{\partial r}\right] = 0 \,. \tag{137}$$

When (24), (25) and (29) (in which the indices are dropped) are formulated in the geometry of Figure 23, the resulting equations are identical to the linearized forms of the above except for a $\cos\theta$ multiplier attached to the second term of the first equation (135). The $\cos\theta$ which appears in the continuity equation can be neglected (to first order in r/R) since it is not differentiated. How will this slight difference affect the result?

The total energy of the liquid and its total absolute angular momentum about the polar axis are constants, as in Ball's work. With the moment of inertia, I, about a vertical axis through the origin defined as in Ball:

$$I = \int h\, r^2\, dS \quad , \qquad (138)$$

the theorem stating that if I and its time derivative are initially known, they are determined uniquely at all times thereafter still holds. The ensuing distension theory is valid, and it is therefore possible to separate the motion of the liquid in two parts: first a distension, defined as an isotropic two-dimensional dilatation and rotation (Figure 24), and second, motions superimposed on the distension.

To quote Ball, "the main effects of the distension on these (superimposed) motions are, first, a slowing down or speeding up of every aspect of the motion (whether vortices or gravity waves) according as the liquid as a

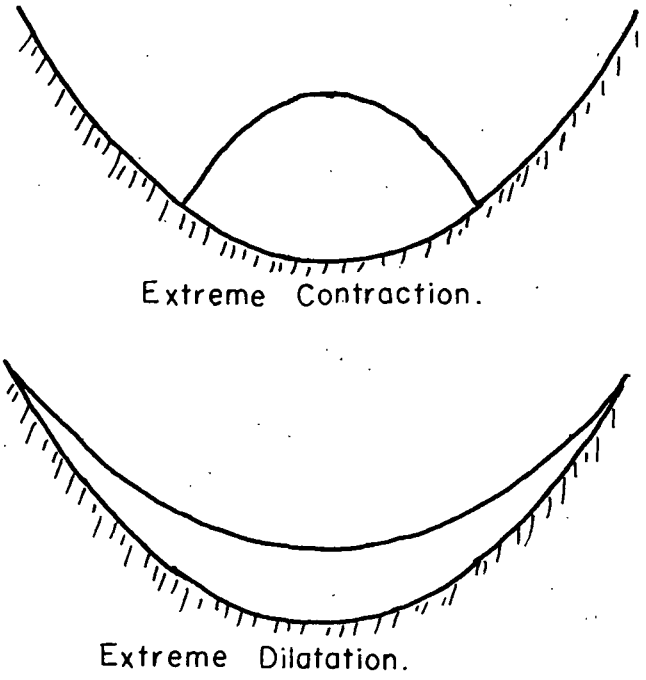


Figure 24. A 'distension' (Ball (1963)).

whole is stretched or contracted, and secondly, a general stabilization,....". This also applies to the motions studied in a paraboloidal polar basin: if planetary waves coexist with a distension, energy will be interchanged between the two modes of motion at a rate depending on the rate of dilatation associated with the distension. When the fluid contracts (water piles up at the pole), energy will be extracted by the waves from the distension and the frequencies will increase; when water is high at the edges, the frequencies will decrease.

It can be shown very simply that the special case concerning the influence of rotation on the frequencies of oscillations in a shallow rotating paraboloid, as studied by Miles and Ball (1963), will be applicable only when the influence of variations of the Coriolis parameter is negligible, so that the results therein cannot be applied to the study of planetary waves in the corresponding geometry.

The basin considered is a paraboloid of revolution, and the depth is given by

$$H' = (1 - x^2) \quad . \tag{139}$$

Elimination of all variables except surface amplitude in the shallow water linearized equations (Lamb, 1932; p 209) gives, in the notation of (58)

$$x^2(1 - x^2)\frac{d^2F}{dx^2} + x\frac{dF}{dx}(1 - 3x^2)$$

$$- F\left[s^2(1 - x^2) + \left(-\frac{2s}{\omega'} + \frac{M\ r_1^2}{R^2}\right)x^2\right] = 0 \tag{140}$$

The solution is found in term of a hypergeometric series, and the condition that this series converge at the boundaries yields a frequency condition from which Miles and Ball deduce that the frequencies of the dominant modes for azimuthal wave numbers $s = 0$ and $s = 1$ are independent of the frequency of rotation for an observer in a non-rotating frame of reference, and that the frequencies of

all other axisymmetric modes are decreased by rotation.

To see if these properties are also applicable to a basin in which the rotation varies with position, let us substitute the depth, as given by (139) into the amplitude equation (59):

$$x^2(1 - x^2)\frac{d^2F}{dx^2} + x\frac{dF}{dx}\left[\epsilon(x^2+1)(1 - x^2) - 2x^2\right]$$

$$- F\left\{(s^2+\frac{sx^2\epsilon}{\omega'})(1 - x^2) + \left[(1-\omega'^2)\frac{M\ r_1^2}{R^2} - \frac{2s}{\omega'}\right]x^2\right\} = 0 \quad (141)$$

Assuming that the frequencies are such that ϵx^2 can be neglected with respect to 1, and that $\omega'^2 << 1$ (141) becomes

$$x^2(1 - x^2)\frac{d^2F}{dx^2} + x\frac{dF}{dx}(1 - 3x^2)$$

$$- F\left\{(s^2+\frac{sx^2\epsilon}{\omega'})(1 - x^2) + \left[\frac{M\ r_1^2}{R^2} - \frac{2s}{\omega'}\right]x^2\right\} 0 \quad (142)$$

This will apply to the study of planetary motions of long periods; (142) would be identical to (140) were it not for the presence of a term in $sx^2\epsilon / \omega'$ in the first coefficient of F in (142). For the low frequencies encountered in planetary waves, this term is comparable

to s^2 and is not negligible. When the effects of terrestrial curvature (i.e., variation of the Coriolis parameter) cannot be neglected, there is a significant difference between motions in the polar plane and those in Miles and Ball's model. Equation (142) does not, like (140) have a solution in terms of a known series, and the condition for finiteness of the solutions will in general be different.

The Arctic basin does not of course have a simple paraboloidal bottom topography: the depth variations are not even symmetrical around the pole, so that results applicable to a flat bottom ocean or to an ocean with symmetrical bottom slopes are only of academic interest as far as the actual Arctic ocean is concerned. But since the contorted geometry of the Arctic does not yield easily to analysis, it is necessary to understand the situation in simplified situations before even looking at the more complex cases. It might also be possible to deduce some qualitative properties of the solutions in the more complex situations through knowledge of the physics in the simple topographies studied.

We have verified in this section that the eigensolutions predicted by the Method of Signatures (section VI) in the presence of sloping bottoms indeed exist and that the influence of the bottom slopes on the frequencies of the eigensolutions is as expected from the analysis of section VI. If the effect of the depth variations in the

potential vorticity balance is in the same direction as that of the Coriolis parameter ($dH'/dx > 0$), the frequency $\omega'_{s,n}$ is increased over that of the flat bottom solutions. If the influence of the depth is in the opposite direction ($dH'/dx < 0$), the frequency is decreased until it becomes zero for $dH'/dx = -r_1^2/_R 2x$. For steeper negative depth gradients the direction of propagation is reversed and planetary waves with positive wave number ($s > 0$) can exist.

The applicability of some recent theorems of Ball (1963) has been investigated; they have been found to apply quite generally, so that Ball's separation of the motions of the fluid in a shallow rotating paraboloid into three parts applies to the case of such basins in the polar plane. These three parts are as follows:

1) The motion of the centre of gravity, which is entirely independent of the motions relative to it.

2) An isotropic two-dimensional dilatation and rotation, which Ball calls a 'distension' (Figure 24).

3) The motions that remain after the removal of the velocity fields associated with the preceding motions. These theorems apply only to paraboloidal basins.

Concerning the effects of radial depth variations we can then make the following conclusions. As seen in the potential vorticity equation (54), depth variations have very much the same influence as variations in the Coriolis

parameter. Using the Method of Signatures (section VI), it can be determined whether eigensolutions will exist or not for any given radial depth variation $H'(x)$. Namely, solutions will exist when $H'(x)$, through its influence on the signatures, allows the phase path to terminate at the origin of the phase diagram. When $H'(x)$ is given explicitly, the signatures can be found for all values of ζ ($= 1/x$), and so can the position of the phase path. In principle, it is then possible to draw conclusions on the eigensolutions when the radial depth variation is known.

The actual explicit solution of the amplitude equations is possible only in very simple cases; one such simple case ($H' = (1 + px^2/2)$) has been examined above.

In the special case of paraboloidal basins, some general theorems due to Ball, and stated above, apply to the motions of the contained fluid, provided it is shallow.

The case of symmetric depth variations can then be considered to be resolved in principle, since even though it may not be possible to solve the amplitude equation explicitly, the existence (or non-existence) of solutions can be ascertained, and the amplitude equation solved numerically to find the eigenfrequencies.

IX. ASYMMETRICAL TOPOGRAPHY.

The problem becomes enormously more complex when asymmetries are allowed, either in the bathymetry or in the boundaries. A glance at Figure 1 shows that the asymmetries are very important and will in all probability play a dominant role in the dynamics of the Arctic ocean. The simple solutions of sections VII and VIII will then not be directly applicable to the actual Arctic ocean, and the more complete equation (48) must be solved. Using some of the abbreviations defined in (58), (48) becomes

$$r^2 \frac{\partial^2 F}{\partial r^2} + r \frac{\partial F}{\partial r} \left\{ 1 + \frac{r}{H} \frac{\partial H}{\partial r} + \frac{\epsilon r^2}{r_1^2} + \frac{i}{\omega'} \left[\frac{1}{F} \frac{\partial F}{\partial \phi} + \frac{1}{H} \frac{\partial H}{\partial \phi} \right] \right\}$$

$$+ \frac{\partial F}{\partial \phi} \left\{ \frac{1}{F} \frac{\partial F}{\partial \phi} + \frac{1}{H} \frac{\partial H}{\partial \phi} - 2is - \frac{ir}{H\omega'} \frac{\partial H}{\partial r} - \frac{i\epsilon}{\omega'} r_1^2/r^2 \right\}$$

$$- F \left\{ s^2 + \delta \frac{r^2}{r_1^2} + \frac{rs}{H\omega'} \frac{\partial H}{\partial r} + \frac{is}{H} \frac{\partial H}{\partial \phi} \right\}$$

$$+ \frac{\partial^2 F}{\partial r \partial \phi} \left(\frac{-ir}{\omega'} \right) = 0 \qquad (143)$$

The boundary condition is still as given by (50). It is however doubtful whether the assumed form for the surface displacements, (30), which is appropriate to the description of waves in a cylindrical basin, will be of any utility when important departures from cylindrical symmetry are present. It might be necessary in that case to reformulate the problem in a coordinate system more appropriate to the new symmetry, and in which it might be possible to separate the amplitude equation. Asymmetries in the boundaries will then not be included, in which case (50) reduces to

$$(i\,\omega')\left.\frac{\partial F}{\partial r}\right|_{r=r_1} + \frac{1}{r_1}\left.\frac{\partial F}{\partial \phi}\right|_{r=r_1} - \frac{is}{r_1}F(r_1) = 0 \qquad (144)$$

No solutions of (143) have been found, even in very special cases; it is not even known whether the system (143)-(144) has any solutions at all. In view of this uncertainty and of the non-linearity of the partial differential equation (143), it is doubtful whether one should pursue this line of attack any further. The problem might be more tractable with a less inclusive formulation, but on the other hand, it might be necessary to study the non-symmetrical situation with more qualitative arguments.

X. CONCLUSIONS.

The problem of the dynamics of the Arctic ocean has been formulated in a geometry appropriate to the polar regions by transforming the equations of motion and of continuity from the sphere to a polar plane. Although this mapping arises quite naturally in the study of geophysical phenomena, it seems that nobody had taken advantage of it in that respect previously.

The main advantage of the mapping is a considerable reduction in mathematical complexity; for a restricted polar cap, one can retain only a first approximation to the terrestrial curvature, thus analysing the problem in a modified beta-plane. Applying the transformed equations to the simple case of a symmetrical ocean with a flat bottom, one finds that the planetary wave eigensolutions compare reasonably well in frequency and appearance with their equivalents in a similar basin on the sphere, as derived by Goldsbrough and Longuet-Higgins. The solutions in the polar projection are furthermore easier to represent and their frequencies calculated with a minimum of labour.

Having found that the solutions for planetary waves were very similar in the polar plane and in the spherical geometry, we can conclude that the analysis in the approximate polar plane will yield almost undistorted results for all

possible modes of motion of a small Arctic ocean (the actual one is small enough). This is so because the planetary waves, depending in their existence and properties on the curvature of the Earth, are most likely to be distorted by any departures from the strict geoid. The polar plane projection can therefore be used as reliably as the ordinary beta-plane used in mid-latitudes.

The results concerning the symmetrical ocean with flat bottom are not new; this work goes beyond that of Goldsbrough and Longuet-Higgins in formulating the problem to include variable bathymetry and asymmetrical boundaries. The symmetrical ocean is of course much simpler to discuss, the amplitude then being determined by an ordinary differential equation. Using the Method of Signatures, some general criteria can be established concerning the existence and properties of the eigensolutions of a symmetrical basin. For a given bottom configuration, $H'(r)$, one can determine whether eigensolutions will be found or not; in particular, it is found that no planetary waves can propagate towards the east when the depth of the symmetric ocean is constant.

The real Arctic ocean is of course grossly asymmetrical, and if motions dependent on the topography are investigated, it can certainly not be approximated by a cylindrically symmetrical basin. No conclusions have been reached in this more general situation, mostly because of

the increased analytical complexity: the amplitude differential equation is now partial and non-linear. It might be necessary to treat the more general case by more qualitative methods since the mathematical difficulties have not so far been surmounted.

Needless to say, much work remains to be done before the dynamic oceanography of the actual Arctic ocean is understood in detail. I hope then that this work will serve as a basis as well as a stimulus for further developments and that the polar plane approximation here introduced will also prove useful in further research.

APPENDIX I.

FØRTRAN language program to find the first root in ν of the hypergeometric series

$$ {}_2F_1(1+k+\nu, k-\nu; k+1; \tfrac{1}{2} - \tfrac{1}{2}x) \quad . $$

The series is summed (first 20 terms) for larger and larger values of ν until the sign of the sum changes; the same procedure is then repeated for another value of k. The notation is not the same in the program below as in the above series: $1+k = C$, $k-\nu = B$, and $1+k+\nu = A$; ν is called RNU. The program below is the one used for finding the first root when k = 1,2,3,5 and 8; slight modifications are introduced for finding the root when k = 12 and 16.

```
$FØRTRAN
      RNU = 101
90    READ 100,C
100   FØRMAT( F12.0)
      D = C - 1.
      PRINT 110,D
110   FØRMAT(10X,5HM  = , F4.0)
      PRINT 120
120   FØRMAT(5X,2HNU,7X,6HSUMSER)
      X = 0.0125
190   A = C + RNU
```

```
      B = D - RNU
      DIMENSIØN Y(20)
      Y(1) = A*B*X/C
      SUMSER = 1. + Y(1)
      DØ 200 I = 2,20
      P = I
      Y(I) =  Y(I-1)*((A P-1.)*(B P-1.)*X)/((P D)*P)
200   SUMSER = SUMSER + Y(I)
      PRINT 210,RNU,SUMSER
210   FØRMAT(3X,F5.1,4X,F10.5)
      IF(SUMSER) 230,220,220
220   RNU = RNU + 0.5
      GØ TØ 190
230   GØ TØ 90
      END
$DATA
        2.
        3.
        4.
        6.
        9.
```

After calculating the first root for a value of k, RNU is not brought back to 10; the last value of RNU used is the first one used in calculating the series for the next value of k. This is possible because the value of the root increases monotonically with k. When the first root is calculated for k = 12 and 16, the program is started with RNU = 60.0 to save time. Also, statement 220

is changed to RNU = RNU + 1.0 for the same reason; the definition is decreased by increasing the width of the interval, but little percent precision is lost since RNU is much larger for those values of k.

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